On the geometry of the Thurston metric on Teichmüller spaces: geodesics that disobey an analogue of Masur's criterion.

by

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Abstract

We construct a counterexample for an analogue of Masur's criterion in the setting of Teichmüller space with the Thurston metric. For that, we find a minimal, filling, non-uniquely ergodic lamination λ on the seven-times punctured sphere with uniformly bounded annular projection distances. Then we show that a geodesic in the corresponding Teichmüller space that converges to λ, stays in the thick part for the whole time.

Dedication

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Chapter 1

Introduction

The Thurston metric is an asymmetric Finsler metric on Teichmüller space that was first introduced by Thurston in [50]. The distance between marked hyperbolic surfaces X and Y is defined as the log of the infimum over the Lipschitz constants of maps from X to Y, homotopic to the identity. Thurston showed that when S has no boundary, the distance can be computed by taking the ratios of the hyperbolic lengths of the geodesic representatives of simple closed curves (s.c.c.):

$$d_{Th}(X, Y) = \sup_{\alpha \text{ - s.c.c.}} \log \frac{\ell_\alpha(Y)}{\ell_\alpha(X)}. \tag{1.1}$$

A class of oriented geodesics for this metric called *stretch paths* was introduced in [50]. Given a complete geodesic lamination λ on a hyperbolic surface X, a stretch path starting from X is obtained by stretching the leaves of λ and and extending this deformation to the whole surface. The stretch path is controlled by the *horocyclic foliation*, obtained by foliating the ideal triangles in the complement of λ by horocyclic arcs and endowed with the transverse measure that agrees with the hyperbolic length along the leaves of λ. That is, the projective class of the horocyclic foliation is invariant along the stretch path.

Thurston showed that there exists a geodesic between any two points in Teichmüller space equipped with this metric that is a finite concatenation of stretch path segments. In general, geodesics are not unique: the length ratio in Equation (1.1) extends continuously to the compact space of projective measured laminations $\mathbb{P}\mathcal{ML}(S)$ and the supremum is usually (in a sense of the word) realized on a single point which is a simple closed curve, thus leaving freedom for a geodesic.

The following is our main theorem:

Theorem 1.0.1. *There are Thurston stretch paths in a Teichmüller space with minimal, filling, but not uniquely ergodic horocyclic foliation, that stay in the thick part* *for the whole time.*

The theorem contributes to the study of the geometry of the Thurston metric in comparison to the better studied Teichmüller metric. Namely, our result is in contrast with a criterion for the divergence of Teichmüller geodesics in the moduli space, given by Masur:

Theorem 1.0.2 (Masur's criterion, [31])**.** *Let $\mathfrak{q}$ be a unit area quadratic differential on a Riemann surface X in the moduli space $\mathcal{M}(S)$. Suppose that the vertical foliation of $\mathfrak{q}$ is minimal but not uniquely ergodic. Then the projection of the corresponding Teichmüller geodesic X_t to the moduli space $\mathcal{M}(S)$ eventually leaves every compact set as $t \to \infty$.*

Remark 1.0.3. The horocyclic foliation is a natural analogue of the vertical foliation in the setting of the Thurston metric, see [36], [7].

Remark 1.0.4. Compare Theorem 1.0.1 to a result of Brock and Modami in the case of the Weil-Petersson metric on Teichmüller space [5]: they show that there exist Weil-Petersson geodesics with minimal, filling, non-uniquely ergodic ending lamination, that are recurrent in the moduli space, but not contained in any compact subset. Hence our counterexample disobeys Masur's criterion even more than in their setting of the Weil-Petersson metric.

Despite being asymmetric, and in general admitting more than one geodesic between two points, the Thurston metric exhibits some similarities to the Teichmüller metric. For example, it differs from the Teichmüller metric by at most a constant in the thick part[1] and there is an analog of Minsky's product region theorem [8]; every Thurston geodesic between any two points in the thick part with bounded combinatorics is cobounded[2] [26]; the shadow of a Thurston geodesic to the curve graph is a reparameterized quasi-geodesic [27].

Nevertheless, the Thurston metric is quite different from the Teichmüller metric. For one, the identity map between them is neither bi-Lipschitz [28], nor a quasi-isometry [8]. In the Teichmüller metric, whenever the vertical and the horizontal foliations of a geodesic have a large projection distance in some subsurface, the boundary of that subsurface gets short along the geodesic[3] [45]. However, it follows from our construction that the endpoints of a cobounded Thurston geodesic do not necessarily have bounded combinatorics. The reason behind it is that a condition equivalent to a curve getting short along a stretch path that is expressed in terms of the subsurface

[1] here the constant $C(\varepsilon)$ depends on the thick part $\mathcal{T}_\varepsilon(S)$.

[2] for every $x, y \in \mathcal{T}_\varepsilon(S)$ with $K-$bounded combinatorics (Defnition 2.2 in [26]), every $\mathcal{G}(x, y)$ is in the $\varepsilon'(\varepsilon, K, S)-$thick part.

[3] for every $\varepsilon > 0$ there exists K such that $d_W(\mu_+, \mu_-) > K$ implies $\inf_t \ell_{\partial W}(\mathcal{G}(t)) < \varepsilon$.

projections of the endpoints is more restrictive than in the case of the Teichmüller metric [46], and involves only the annular subsurface of α (see Theorem 2.0.9 for a precise and more general statement). This allows us to produce our counterexample by constructing a minimal, filling, non-uniquely ergodic lamination with uniformly bounded annular subsurface projections.

The structure of this thesis is as follows. In Chapter 2 we provide a brief background on the objects that we study. In Chapter 3 we provide an exposition on Sections 1-3 of the preprint "Minimal stretch maps between hyperbolic surfaces" [50]. In Chapter 4 we present our results. We make a further comment on the content of Chapter 4.

The construction will be done on the seven-times punctured sphere. First, in Section 4.1 we construct a minimal, filling, non-uniquely ergodic lamination λ using a modification of the machinery developed in [25]. Namely, we choose a partial pseudo-Anosov map τ supported on a subsurface Y homeomorphic to the three-times punctured sphere with one boundary component. We pick a finite-order homeomorphism ρ, such that the subsurface $\rho(Y)$ is disjoint from Y, and the orbit of the subsurface Y under ρ fills the surface. Then we set $\varphi_r = \tau^r \circ \rho$ and provided with a sequence of natural numbers $\{r_i\}_{n=1}^{\infty}$ and a curve γ_0, define

$$\Phi_i = \varphi_{r_1} \circ \ldots \circ \varphi_{r_i}, \;\; \gamma_i = \Phi_i(\gamma_0).$$

We show that under a mild growth condition on the coefficients r_i, the sequence of curves γ_i forms a quasi-geodesic in the curve graph and converges to an ending lamination λ in the Gromov boundary. In Section 4.2, we introduce a Φ_i-invariant bigon track and provide matrix representations of the maps τ and $\tau \circ \rho$. In Section 4.3, we use them to produce coarse estimates for the intersection numbers between the pairs of curves in the sequence γ_i. In Section 4.4, we exploit the asymmetry between the intersection numbers to show the non-unique ergodicity of λ and we find all ergodic transverse measures on λ. In Section 4.5, we prove that λ has uniformly bounded annular subsurface projections. Finally, in Section 4.6 we show that there are Thurston stretch paths whose horocyclic foliation is λ, that stay in the thick part of the Teichmüller space for the whole time.

Chapter 2

Background

2.0.1 Notation.

We adopt the following notation. Given two quantities A and B, we write $A \asymp_{K,C} B$ if $\frac{1}{K}B - C \leqslant A \leqslant KB + C$. Further, unless explicitly stated, by the following notation we will mean that there are universal constants $K \geqslant 1, C \geqslant 0$ such that

- $A \overset{+}{\prec} B$ means $A \leqslant B + C$.
- $A \overset{+}{\asymp} B$ means $A - C \leqslant B \leqslant A + C$.
- $A \overset{*}{\prec} B$ means $A \leqslant KB$.
- $A \overset{*}{\asymp} B$ means $\frac{1}{K}B \leqslant A \leqslant KB$.
- $A \overset{*}{\underset{+}{\asymp}} B$ means $\frac{1}{K}B - C \leqslant A \leqslant KB + C$.

2.0.2 Curves and markings.

Let $S = S_{g,n,m}$ be the oriented surface of genus $g \geqslant 0$ with $n \geqslant 0$ punctures, with $m \geqslant 0$ boundary components, and with negative Euler characteristic; we will write $S_{g,n}$ when $m = 0$ for brevity. A simple closed curve on S is called *essential* if it does not bound a disk or a punctured disk; and *non-peripheral* if it can not be homotoped to a boundary component of S. We will call a *curve* on S the free homotopy class of an essential non-peripheral simple closed curve on S. Given two curves α and β on S, we will denote the minimal geometric intersection number between their representatives by $i(\alpha, \beta)$. A *multicurve* is a collection of disjoint curves on S. A *pants decomposition* $\mathcal{P}$ on S is a maximal multicurve on S, i.e. its complement in S is a disjoint union of three-times punctured spheres. A collection of curves Γ is called *filling* if for any curve β on S: $i(\alpha, \beta) > 0$ for some $\alpha \in \Gamma$. A *marking* μ on S is a filling collection of curves. The intersection of a marking μ with a curve α is defined as

$$i(\mu,\alpha)=\sum_{\gamma\in\mu} i(\gamma,\alpha).$$

2.0.3 Curve graph.

The *curve graph* $\mathcal{C}(S)$ of a surface S is a graph whose vertex set $\mathcal{C}_0(S)$ is the set of all curves on S. Two vertices α and β are connected by an edge if the underlying curves realize the minimal possible geometric intersection number for two curves on S. This means that $i(\alpha,\beta)=0$, i.e. the curves are disjoint, unless S is one of the few exceptional surfaces: if the interior of S is the punctured torus, then $i(\alpha,\beta)=1$, and if the interior of S is the four-times punctured sphere, then $i(\alpha,\beta)=2$. The curve graph is the 1-skeleton of the curve complex, introduced by Harvey in [18]. The metric d_S on the curve graph is induced by letting each edge have unit length. Masur and Minsky showed in [32] that the curve graph is Gromov hyperbolic using Teichmüller theory.

Theorem 2.0.1. *[32] The curve graph $\mathcal{C}(S)$ is Gromov hyperbolic.*

For proofs using other techniques see for example [9], [19].

Although the compact annulus $\mathcal{A}$ is not a surface of negative Euler characteristic, it is crucial for us to consider it and we separately define its curve graph. Let the vertices of $\mathcal{C}(\mathcal{A})$ be the arcs connecting two boundary components of $\mathcal{A}$, up to homotopies that fix the boundary pointwise. Two vertices are connected by an edge of length 1 if the underlying arcs have representatives with disjoint interiors. It is easy to check that $\mathcal{C}(\mathcal{A})$ is quasi-isometric to $\mathbb{Z}$ with the standard metric, hence also Gromov hyperbolic (see Section in 2.4 [33] for more details).

2.0.4 Measured laminations and measured foliations.

For the background on (measured) geodesic laminations we refer to Chapter 8 in [29]. We provide some additional definitions and facts that we use later in the paper. A geodesic lamination is *complete* if its complementary regions in S are ideal triangles. A geodesic lamination λ is *chain-recurrent* if it is in the closure of the set of multicurves in $\mathcal{GL}(S)$. The *stump* of a geodesic lamination is its maximal compactly supported sublamination that admits a transverse measure of full support.

There is a forgetful map from the space of projective measured laminations $\mathbb{P}\mathcal{ML}(S)$ to the set of *measurable* laminations (the laminations admitting some transverse measure with full support) that forgets the transverse measure. Consider the subset of filling measurable (hence minimal) laminations, and give it the quotient subspace topology of $\mathbb{P}\mathcal{ML}(S)$. The resulting space is the space of *ending laminations* $\mathcal{EL}(S)$.

Theorem 2.0.2. *[23] The Gromov boundary of the curve graph $\mathcal{C}(S)$ is homeomorphic to the space of ending laminations $\mathcal{EL}(S)$. If a sequence of curves $\{\alpha_i\}$ is a quasi-geodesic in $\mathcal{C}(S)$ that converges to $\lambda \in \mathcal{EL}(S)$, then any limit point of $\{\alpha_i\}$ in $\mathbb{P}\mathcal{ML}(S)$ projects to λ under the forgetful map.*

For the background on measured foliations we refer to [14]. The spaces $\mathcal{MF}(S)$ and $\mathcal{ML}(S)$ are canonically identified, and we will sometimes not distinguish between measured laminations and measured foliations; similarly for their projectivizations $\mathbb{P}\mathcal{ML}(S)$ and $\mathbb{P}\mathcal{MF}(S)$. A lamination is *uniquely ergodic* if it supports a unique transverse measure up to scaling. Otherwise it is *non-uniquely ergodic*.

2.0.5 Teichmüller space and Thurston boundary.

A *marked hyperbolic surface* is a complete finite-area Riemannian surface of constant curvature -1 with a fixed homeomorphism from the underlying topological surface S. Two marked hyperbolic surfaces X and Y are called equivalent if there is an isometry between X and Y in the correct homotopy class. The collection of equivalence classes of marked hyperbolic surfaces is called the *Teichmüller space* $\mathcal{T}(S)$ of the surface S. By $\ell_\alpha(X)$ we denote the hyperbolic length of the unique geodesic representative of the curve α on the surface X. For $\varepsilon > 0$, the *ε-thick part* $\mathcal{T}_\varepsilon(S)$ of the Teichmüller space is the set of all marked hyperbolic surfaces with no curves shorter than ε. A *Bers constant* of S is a number $B(S)$ such that for every $X \in \mathcal{T}(S)$, there exist a pants decomposition on X such that the length of each curve in it is at most $B(S)$. We recall that the Teichmüller space can be compactified via the *Thurston boundary* homeomorphic to $\mathbb{P}\mathcal{ML}(S)$ so that the compactification is homeomorphic to the disc. For the details of the construction using the space of geodesic currents we refer to Chapter 8 in [29].

2.0.6 The action of the mapping class group on the Thurston boundary.

The *mapping class group* of a surface S is the group of the isotopy classes of self-homeomorphisms of S. We refer to [14], [13] for the background on pseudo-Anosov homeomorphisms.

The mapping class group acts continuously on the space of projective measured laminations $\mathbb{P}\mathcal{ML}(S)$. We recall some results concerning the action of a partial pseudo-Anosov map on the Thurston boundary, following Ivanov ([21], §3 and the

Appendix). Let f be a homeomorphism of the surface S, such that there exist subsurfaces $S_1, \ldots, S_n \subset S$ with disjoint interiors, such that f preserves each S_i and the restriction maps $f|_{S_i}$ are pseudo-Anosov, and further such that f restricts to a homeomorphism homotopic to the identity in the complement $S \setminus \sqcup S_i^{\circ}$. We remark that in [21] more general mapping classes are considered, which are called *pure*, but we will not need this generality. Let μ_i^s, μ_i^u be the stable and unstable laminations of $f|_{S_i}$, viewed as laminations on S. Define the following subsets of $\mathbb{P}\mathcal{ML}(S)$:

$$\Delta_f^u = \left\{ \left[\sum_{i=1}^n m_i \mu_i^u \right] : m_i \geqslant 0, \sum_{i=1}^n m_i > 0 \right\}, \Psi_f^s = \{[\nu \neq 0 : i(\nu, \mu_i^s) = 0, i = 1, \ldots, n]\}.$$

Similarly define Δ_f^s and Ψ_f^u. Clearly, we have $\Delta_f^u \subset \Psi_f^u, \Delta_f^s \subset \Psi_f^s$. The following holds:

Theorem 2.0.3 ([21], Theorem A.1 in the Appendix)**.** *For all $x \in \mathbb{P}\mathcal{ML}(S) \setminus \Psi_f^s$, the limit* $\lim_{m\to\infty} f^m(x)$ *exists and is contained in Δ_f^u. Similarly, if $y \in \mathbb{P}\mathcal{ML}(S) \setminus \Psi_f^u$, then* $\lim_{m\to\infty} f^{-m}(y) \in \Delta_f^s$.

2.0.7 Subsurface projections.

By a *subsurface* $Y \subset S$ we mean an isotopy class of a proper, closed, connected, embedded subsurface, such that its boundary consists of curves or boundary components of S and its punctures agree with those of S. If Y is an annular subsurface, we assume that its core curve is a curve in S. We assume Y is not a pair of pants.

The subsurface projection is a map $\pi_Y : \mathcal{GL}(S) \to 2^{\mathcal{C}_0(Y)}$ from the space of geodesic laminations on S to the power set of the vertex set of the curve graph of Y. Equip S with a hyperbolic metric. Let $\widetilde{Y}$ be the Gromov compactification of the cover of S corresponding to the subgroup $\pi_1(Y)$ of $\pi_1(S)$ with the hyperbolic metric pulled back from S. There is a natural homeomorphism from to $\widetilde{Y}$ to Y, allowing to identify the curve graphs $\mathcal{C}(\widetilde{Y})$ and $\mathcal{C}(Y)$. For any geodesic lamination λ on S, let $\widetilde{\lambda}$ be the closure of the complete preimage of λ in $\widetilde{Y}$. Suppose that $Y \subset S$ is a nonannular subsurface. An arc $\beta \in \widetilde{\lambda}$ is *essential* if no component of $\widetilde{Y} \setminus \beta$ has closure which is a disk. For each essential arc $\beta \in \widetilde{\lambda}$, let $\mathcal{N}_\beta$ be a regular neighborhood of $\beta \cup \partial\widetilde{Y}$. Define $\pi_Y(\lambda)$ to be the union of all curves which are either curve components of $\widetilde{\lambda}$ or curve components of $\partial\mathcal{N}_\beta$, where β is an essential arc in $\widetilde{\lambda}$. If $Y \subset S$ is an annular subsurface, define $\pi_Y(\lambda)$ to be the union of all arcs β in $\widetilde{\lambda}$ that connect two boundary components of $\widetilde{Y}$.

We say that a lamination λ intersects the subsurface Y *essentially* if $\pi_Y(\lambda)$ is nonempty. The projection distance between two laminations $\lambda, \lambda' \in \mathcal{GL}(S)$ that intersect

Y essentially is

$$d_Y(\lambda, \lambda') = \mathrm{diam}_{\mathcal{C}(Y)}(\pi_Y(\lambda) \cup \pi_Y(\lambda')).$$

If Y is an annular subsurface with the core curve α, we will write $d_\alpha(\lambda, \lambda')$ instead of $d_Y(\lambda, \lambda')$ for convenience (when the quantity makes sense). We also define the subsurface projections for markings by taking the union of the projections of the individual curves in a marking. We state the *Bounded geodesic image theorem* proved by Masur and Minsky in [33].

Theorem 2.0.4. *[33] Given a surface S there exists $\mathsf{M} = \mathsf{M}(\delta)$ such that whenever Y is a subsurface and $g = \{\gamma_i\}$ is a geodesic in $\mathcal{C}(S)$ such that γ_i intersects essentially Y for all i, then $d_Y(g) \leqslant \mathsf{M}$.*

See also [51] for another proof. We state a corollary of Theorem 2.0.4, which follows from the stability of quasi-geodesics in Gromov hyperbolic spaces (Theorem 1.7, Chapter III.H in [4]):

Corollary 2.0.5. *Given $k \geqslant 1$ and $c \geqslant 0$, there exists a constant $\mathsf{A} = \mathsf{A(S)}$ such that the following holds. Let $\{\gamma_i\}$ be a 1-Lipschitz (k, c)-quasi-geodesic in $\mathcal{C}(S)$ and let Y be a subsurface of S. If every γ_i intersects Y essentially, then for every i and j:*

$$d_Y(\gamma_i, \gamma_j) \leqslant \mathsf{A}.$$

We also state a useful lemma on the convergence of the projection distances (we note that the definition of the projection distance in [5] is slightly different from ours, but this only results in a bounded change of the additive error compared to their statement).

Lemma 2.0.6 ([5], Lemma 2.7)**.** *Suppose that a sequence of curves $\{\alpha_i\}$ converges to a lamination λ in the Hausdorff topology on $\mathcal{GL}(S)$. Let Y be a subsurface, so that λ intersects Y essentially. Then for any geodesic lamination λ' that intersects Y essentially we have*

$$d_Y(\alpha_i, \lambda') \asymp_{1,8} d_Y(\lambda, \lambda')$$

for all i sufficiently large.

Finally, we state the following proposition:

Proposition 2.0.7 ([35], p. 121-122)**.** *Let $\nu_\pm$ be the unstable and stable laminations of the pseudo-Anosov map Ψ on the surface S. Then the subsurface projection distances $d_Y(\nu_+, \nu_-)$ are uniformly bounded over all subsurfaces $Y \subset S$, i.e.* $\sup_{Y \subset S} d_Y(\nu_+, \nu_-) < \infty$.

2.0.8 Relative twisting.

In Section 2.0.7, the projection distances between laminations for the annular subsurfaces were defined. Here we extend the definition to us allow to compute projection distances between a lamination and a hyperbolic structure, and between two hyperbolic structures. We will refer to any of these quantities as the relative twisting about a curve α.

Suppose α is a curve, X is a hyperbolic metric and λ is a geodesic lamination on S. Suppose that λ intersects α essentially. Consider the Gromov compactification of the annular cover X_α that corresponds to the cyclic subgroup $\langle\alpha\rangle$ in the fundamental group $\pi_1(S)$, with the hyperbolic metric pulled back from X. Consider the complete preimage $\widetilde{\lambda}$ of λ in X_α. Let $\alpha^\perp$ be a geodesic arc in X_α that is perpendicular to the geodesic in the homotopy class of the core curve. Define $d_\alpha(X,\lambda)$ to be the maximal distance between $\widetilde{\omega}$ and $\alpha^\perp$ in $\mathcal{C}(X_\alpha)$, where $\widetilde{\omega}$ is any arc of $\widetilde{\lambda}$ that connects two boundary components of X_α and $\alpha^\perp$ is any perpendicular.

Lastly, we define $d_\alpha(X,Y)$, where X,Y are two hyperbolic metrics on S. Let S_α be the compactification of the annular cover that corresponds to α. Let X_α, Y_α be the compactified covers with the hyperbolic metrics defined as before. Using the first metric, construct a geodesic arc $\alpha^\perp_X$, perpendicular to the geodesic in the homotopy class of the core curve. Similarly, construct a geodesic arc $\alpha^\perp_Y$. Define $d_\alpha(X,Y)$ to be the maximal distance between $\alpha^\perp_X$ and $\alpha^\perp_Y$ in $\mathcal{C}(S_\alpha)$, over all possible choices of the perpendiculars.

2.0.9 Thurston metric on Teichmüller space.

For a background on the Thurston metric we refer to [50] and [41], while here we mention the necessary notions and state the results that we will use.

In [50], Thurston showed that the best Lipschitz constant is realized by a homeomorphism from X to Y. Moreover, there is a unique largest chain-recurrent lamination $\Lambda(X,Y)$, called the *maximally stretched lamination*, such that any map from X to Y realizing the infimum in Equation (1.1), multiplies the arc length along the lamination by the factor of $e^{d_{Th}(X,Y)}$. Generically, $\Lambda(X,Y)$ is a curve ([50], Section 10).

For a complete lamination ν, Thurston constructed a homeomorphism $\mathcal{F}_\nu : \mathcal{T}(S) \to \mathcal{MF}(\nu)$, where $\mathcal{MF}(\nu)$ is the subspace of measured foliations transverse to ν. The image of a point X in the Teichmüller space under $\mathcal{F}_\nu$ is the horocyclic foliation of the pair (X,ν). The space $\mathcal{MF}(\nu)$ has a natural cone structure given by the *shearing coordinates* which produce an embedding $s_\nu : \mathcal{T}(S) \to \mathbb{R}^{\dim \mathcal{T}(S)}$ such that the image is an open convex cone. We refer to [2], [48] for the details of the construction. The stretch paths with non-empty horocyclic foliation form open rays from the origin in the image of s_ν. Namely, given any X in Teichmüller space $\mathcal{T}(S)$, a complete lamination ν, and $t \in \mathbb{R}$, we let $\mathrm{stretch}(X, \nu, t)$ be a unique point in $\mathcal{T}(S)$, such that

$$s_\nu(\mathrm{stretch}(X, \nu, t)) = e^t s_\nu(X).$$

Every stretch path with non-empty horocyclic foliation converges to the projective class of the horocyclic foliation in the Thurston boundary as $t \to \infty$ ([40], Theorem 5.1). Every stretch path such that the stump of ν is uniquely ergodic converges to the projective class of the stump of ν as $t \to -\infty$ [49]. We summarize these results in one theorem.

Theorem 2.0.8 ([40],[49])**.** *The stretch path stretch(X, ν, t) with non-empty horocyclic foliation $\mathcal{F}_\nu(X)$ converges to the projective class of the horocyclic foliation $[\mathcal{F}_\nu(X)]$ in the Thurston boundary as $t \to \infty$. Every stretch path stretch(X, ν, t) such that stump(ν) is uniquely ergodic converges to the projective class of the stump $[$stump$(\nu)]$ in the Thurston boundary as $t \to -\infty$.*

2.0.10 Twisting parameter along a Thurston geodesic.

We introduce the notions necessary to state Theorem 2.0.9. We say that a curve α *interacts* with a lamination λ if α is a leaf of λ or if α intersects λ essentially. We call $[a, b]$ the $\varepsilon-$*active interval* for α along a Thurston geodesic $\mathcal{G}(t)$ if $[a, b]$ is the maximal interval such that $\ell_\alpha(a) = \ell_\alpha(b) = \varepsilon$. We use the notation $\mathrm{Log}(x) = \max(1, \log(x))$. Denote $X_t = \mathcal{G}(t)$.

Theorem 2.0.9 ([12], Theorem 3.1)**.** *There exists a constant ε_0 such that the following statement holds. Let $X, Y \in \mathcal{T}_{\varepsilon_0}(S)$ and α be a curve that interacts with $\Lambda(X, Y)$. Let $\mathcal{G}$ be any geodesic from X to Y and $\ell_\alpha = \min_t \ell_\alpha(t)$. Then*

$$d_\alpha(X, Y) \overset{*}{\underset{+}{\asymp}} \frac{1}{\ell_\alpha} Log \frac{1}{\ell_\alpha},$$

If $\ell_\alpha < \varepsilon_0$, then $d_\alpha(X, Y) \overset{+}{\asymp} d_\alpha(X_a, X_b)$, where $[a, b]$ is the ε_0-active interval for α. Further, for all sufficiently small ℓ_α, the relative twisting $d_\alpha(X_t, \Lambda(X, Y))$ is uniformly bounded for all $t \leqslant a$ and $\ell_\alpha(t) \overset{*}{\asymp} e^{t-b} \ell_\alpha(b)$ for all $t \geqslant b$. All errors in this statement depend only on ε_0.

Remark 2.0.10. We note that the statement of Theorem 2.0.9 remains true if the condition $X, Y \in \mathcal{T}_{\varepsilon_0}(S)$ is replaced with a weaker condition $\ell_\alpha(X), \ell_\alpha(Y) \geqslant \varepsilon_0$. The proof is identical. This will be crucial for us to make Corollary 4.6.3.

Chapter 3

Exposition on “Minimal stretch maps between hyperbolic surfaces”

The content of this chapter is not original. We follow the preprint [50] in an attempt to make the exposition more accessible, and we cover the first three sections. We advise the reader to use the expository part of this thesis as follows: if while going through Thurston's original preprint the reader feels an urge to see more details, we advice to have a look at this manuscript and check whether the needed claim is addressed here. We hope that it could be (slightly) useful for graduate students interested in the Thurston metric.

We also hoped to cover [50] more substantially, but the time constraints did not allow this to happen. We plan to continue this work elsewhere in the future.

3.0.1 Introduction.

Definition 3.0.1. Let X and Y be metric spaces. The map $f : X \to Y$ is called *Lipschitz*, if there exists a number $C \geqslant 0$, such that $d_Y(f(x_1), f(x_2)) \leqslant C d_X(x_1, x_2)$ for all $x_1, x_2 \in X$. The least such value of C is called the *Lipschitz constant* of f, denoted by $L(f)$.

Definition 3.0.2. Let X and Y be points in Teichmüller space $\mathcal{T}(S)$. The *Lipschitz distance* from X to Y is the following quantity:

$$L(X, Y) = \inf_{\varphi \simeq \mathrm{id}} \log L(\varphi),$$

where φ is a Lipschitz homeomorphism homotopic to identity.

Claim 3.0.3. The set of C-Lipschitz homeomorphisms from X to Y in a given homotopy class is non-empty for C large enough.

Proof. For every homeomorphism $\varphi : X \to Y$ there exists a diffeomorphism $\varphi' : X \to Y$ homotopic to φ (see [39]). First, suppose S is compact. For any two distinct points $x, y \in X$, consider the minimizing geodesic between them: a smooth map $\gamma : [0,1] \to X$ with $\gamma(0) = x, \gamma(1) = y$ such that $\int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)} dt = d_X(x,y)$. Since X is compact, $\sup_{x \in X} \|D\varphi'(x)\| = C < \infty$. Then

$$d_Y(\varphi'(x), \varphi'(y)) \leqslant \int_0^1 \left\| \frac{d}{dt}\varphi'(\gamma(t)) \right\|_{\varphi'(\gamma(t))} dt \leqslant \int_0^1 C\|\dot{\gamma}(t)\|_{\gamma(t)} dt = C d_X(x,y).$$

Hence the map φ' is C-Lipschitz. If S has punctures, choose disjoint simple closed horocycles of length 1 around each cusp in X and Y. Map the horocycles of X to the horocycles of Y by an isometry; extend the map to the compact part in the complement of the cusp neighborhoods by a diffeomorphism; map the corresponding cusp neighborhoods isometrically. The resulting map is a Lipschitz diffeomorphism by the above argument. □

Claim 3.0.4. For any points X and Y in $\mathcal{T}(S)$ there exists a continuous map that realizes the least Lipschitz constant in a given homotopy class. In other words, $L(X,Y)$ is the log of the *best* Lipschitz constant in a given homotopy class.

Proof. Suppose S is compact. The set of C-Lipschitz maps from X to Y homotopic to identity is non-empty for C large enough by Claim 3.0.3. Denote the non-empty set of all such maps by Z_C for a fixed value of C. Let $f_n \in Z_C$ be a sequence of maps whose Lipschitz constants $L(f_n)$ converge to $e^{L(X,Y)}$. Since the set Z_C is equicontinuous, pointwise relatively compact and closed, by the Arzelà–Ascoli theorem it is a compact subset of $C(X,Y)$ (equipped with the compact-open topology). Since X is compact and Y is a metric space, the space $C(X,Y)$ is metrizable. Thus, the sequence f_n has a convergent subsequence. It is easy to check that the limiting map f realizes the least Lipschitz constant.

If S has punctures, to apply the Arzelà–Ascoli theorem we need to show that Z_C is pointwise relatively compact. It is enough to show this for one point x in X. Choose x to be a point on a simple closed geodesic α in X. Then the image of α under any element of Z_C has to stay in some compact subset of Y. Finally, since X is hemicompact and Y is a metric space, the space $C(X,Y)$ is metrizable. □

Definition 3.0.5. Let X and Y be points in Teichmüller space $\mathcal{T}(S)$. The *curve length ratio distance* from X to Y is the following quantity:

$$K(X,Y) = \sup_{\alpha} \log \frac{\ell_\alpha(Y)}{\ell_\alpha(X)},$$

where α is a closed curve in S (not necessarily simple) that can not be homotoped to a point or a neighborhood of a puncture.

Claim 3.0.6. $K(X,Y) \leqslant L(X,Y)$.

Proof. Since $\log(x)$ is continuous and monotone, it is enough to prove that for every closed curve α and every C-Lipschitz map $f : X \to Y, f \simeq \mathrm{id}$, we have $\ell_\alpha(Y) \leqslant C\ell_\alpha(X)$. Notice that since $f \simeq \mathrm{id}$, we have $f(\alpha) \simeq \alpha$ in Y. Let $\alpha : [0,r] \to X$ be a natural parametrization of the geodesic representative of α in X. Then the map $f \circ \alpha : [0,r] \to Y$ is C-Lipschitz. In particular, the velocity of the curve $f \circ \alpha$ is at most C, when exists. Then by Theorem 2.7.6 in [6], we have:

$$\ell_\alpha(Y) \leqslant L(f \circ \alpha) = \int_0^r v_{f\circ\alpha}(t)\, dt \leqslant C \int_0^r 1\, dt = C\ell_\alpha(X).$$

□

3.0.2 Elementary properties of the Lipschitz constant.

Claim 3.0.7. $L(X,Y) \geqslant 0$, and $L(X,Y) = 0$ if and only if $X = Y$.

Proof. We essentially follow the proof of Lemma 6.1 in [43] with some added clarifications and details. Suppose that $L(X,Y) \leqslant 0$ for some $X, Y \in \mathcal{T}(S)$. Pick a sequence of homeomorphisms $\varphi_n : X \to Y$, such that their Lipschitz constants $L(\varphi_n)$ converge to $e^{L(X,Y)} \leqslant 1$. By Claim (previous section), up to subsequence φ_n converges to a continuous map $\varphi : X \to Y$ with $L(\varphi) = e^{L(X,Y)} \leqslant 1$.

First, we show that φ is surjective. Pick a point $y \in Y$; since each φ_n is a bijection, we can define $x_n = \varphi_n^{-1}(y) \in X$. Up to subsequence, $x_n \to x \in X$ (for compact X it is immediate from the compactness, and if X has cusps then the preimages have to be trapped in the compact set because otherwise there is a non-trivial closed curve through y or arbitrary small length). We show that $\varphi(x) = y$. Fix $\varepsilon > 0$. We have

$$d_Y(\varphi(x), y) = d_Y(\varphi(x), \varphi_n(x_n)) \leqslant d_Y(\varphi(x), \varphi_n(x)) + d_Y(\varphi_n(x), \varphi_n(x_n)).$$

Since $\varphi_n \to \varphi$ uniformly on the compact sets, there is N such that for all $n \geqslant N$: $d_Y(\varphi_n(x), \varphi(x)) < \varepsilon/2$. By the equicontinuity of φ_n, there is $\delta > 0$ such that $d_X(x_1, x_2) < \delta$ implies $d_Y(\varphi_m(x_1), \varphi_m(x_2)) < \epsilon/2$ for all m. Since $x_n \to x$, there is N' such that $d_X(x_n, x) < \delta$ for all $n > N'$. Pick $n > \max(N, N')$. Then $d_Y(\varphi(x), y) < \varepsilon$. Since this holds for all $\varepsilon > 0$, we have $d_Y(\varphi(x), y) = 0$, i.e. $\varphi(x) = y$.

Next, we cover X by a collection of hyperbolic disks $\mathcal{D}_X$ with disjoint interiors, such that their total area equals the area of X. This can be done as follows: choose a pants decomposition of X, then subdivide each pair of pants into two ideal triangles via three infinite geodesics, then cover each ideal triangle by disks in the same fashion as in the Apollonian gasket, see Figure 3.1[1]. It follows from [22] that the areas match (they actually prove it for the Euclidean plane, but the hyperbolic case follows).

We show that the images of the interiors of any two distinct disks in $\mathcal{D}_X$ are disjoint, and that the image of each disk in $\mathcal{D}_X$ is a disk of the same area. Recall that $\text{Area}(X) = \text{Area}(Y) = 2\pi|\chi(S)|$. Let $\mathcal{D}_Y$ be a collection of disks in Y obtained as follows: for every disk in $\mathcal{D}_X$ centered at $p \in X$, consider the disk centered at $\varphi(p) \in Y$ with the same radius. The union of all such disks is $\mathcal{D}_Y$. Since φ is $1-$Lipschitz, the image of a disk in $\mathcal{D}_X$ centered at $p \in X$ is contained in the disk of the same radius centered at $\varphi(p) \in Y$. In particular, $\varphi(\mathcal{D}_X) \subset \mathcal{D}_Y$. Since φ is Lipschitz and the set $X \setminus \mathcal{D}_X$ has measure zero, it follows from the definition of a measure zero set that the set $\varphi(X \setminus \mathcal{D}_X)$ also has measure zero. Thus any two distinct disks in $\mathcal{D}_Y$ have disjoint interiors, otherwise the areas do not match. Further, each disk in $\mathcal{D}_X$ surjects onto the corresponding disk in $\mathcal{D}_Y$, otherwise the areas do not match again (since the image of a disk is compact).

It follows then that for a point of the boundary of a disk in $\mathcal{D}_Y$ there is a point in a disk in $\mathcal{D}_X$ that maps to it. Since φ is $1-$Lipschitz, it follows that this point is also a boundary point, and also that φ sends radii to radii, and that it sends them isometrically. Since φ does not increase angles between radii when they are small enough, it follows by dissecting the disk that the angles have to be preserved. It follows that φ restricts to an isometry on the disks in $\mathcal{D}_X$. From the gluing pattern of the disks and since they form a dense subset, these isometries then extend to isometries between the ideal triangles in X and Y. Since φ is Lipschitz, the shears between adjacent triangles have to be preserved. This produces a global isometry between X and Y. □

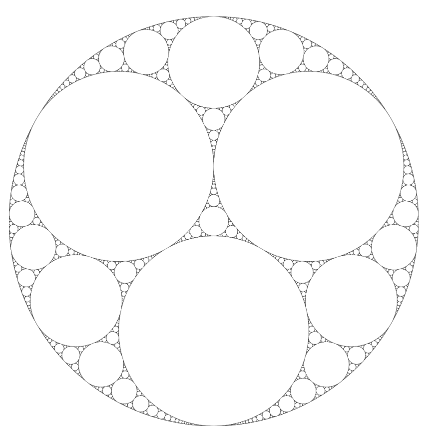

Figure 3.1: The Apollonian gasket.

Claim 3.0.8 (Triangle expands)**.** For every $K \geqslant 1$, there is a K-Lipschitz homeomorphism of the filled ideal hyperbolic triangle (i.e. the convex hull of three distinct points in $\partial\mathbb{H}^2$), which maps the sides to themselves and multiplies the arclength of the sides by K.

Proof. Foliate each corner by the horocycles until they meet at the midpoints of the ideal triangle. Map every point that does not lie on a horocycle to itself. Each foliated corner can be isometrically represented in the upper half-plane model of $\mathbb{H}^2$ as the set of points $C = \{(x, e^t) \in \mathbb{R}^2 \,|\, x \in [0,1], t \geqslant 0\}$ foliated by the horizontal segments. Map the point that corresponds to $(x, e^t) \in C$ to the point that corresponds to $(x, e^{Kt}) \in C$. Note that the obtained map that we denote by f_K is well-defined and is a homeomorphism. A schematic picture of the homeomorphism f_K from Thurston's preprint is in Figure 3.2. It is straightforward to see that f_K multiplies the arclength

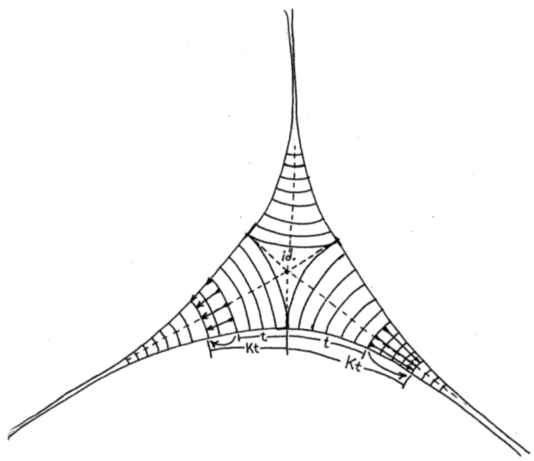

Figure 3.2: A sketch of the homeomorphism f_K from [50].

of the sides by K. Next, observe that for each corner C, the map f_K preserves a pair of orthogonal foliations given by the vertical geodesic rays and the horizontal horocyclic segments, respectively. Further, f_K is K-Lipschitz on the leaves of the vertical foliation and contracting on the leaves of the horizontal foliation. Since the foliations are orthogonal, the norm of the differential of f_K is bounded by K from above over C (see Section 5 in [44] for details). Hence the restriction of f_K to each corner is K-Lipschitz.

We deduce that the local Lipschitz constant of f_K at every point of the ideal triangle is at most K. For the interior points of the corners it follows from the above argument, and it is immediate for the complement of the corners. For the boundary of a corner one can smoothly extend $f_K|_C$ to the set $\tilde{C} = \{(x, e^t) \in \mathbb{R}^2 \,|\, x \in [0,1], t \in \mathbb{R}\}$ by the same formula: in a small neighborhood of each point $(x, 1), x \in (0, 1)$ the new map stretches at least as f_K, but it is K-Lipschitz by the above argument. Now, the convexity of the corners and the ideal triangle combined with Lemma 2.9 in [16] imply the claim. □

Claim 3.0.9. There exists a pair of points X and Y in $\mathcal{T}(S)$, such that

$$L(X, Y) \neq L(Y, X).$$

Proof. Suppose S is compact. Choose a pants decomposition of S, and extend it to some Fenchel-Nielsen coordinates on $\mathcal{T}(S)$. Let X be the point in $\mathcal{T}(S)$ such that all curves in the pants decomposition have the same length $a > 0$, and all the twist coordinates are zero. Let $Y \in \mathcal{T}(S)$ be a similarly defined surface where $a > 0$ is replaced with ka for some $k > 1$. There is a $k-$Lipschitz homeomorphism from X to Y obtained by picking three disjoint infinite geodesics in each pair of pants that spiral around the boundary so that the geodesics spiral in the *opposite* direction around each pants curve, and mapping the ideal triangles in the complement using the map constructed in Claim 3.0.8. Thus $L(X, Y) \leqslant \log k$. Moreover, since the pants curves in Y are k times longer than in X, by Claim 3.0.6 it follows that $L(X, Y) = \log k$.

We want to show that $L(Y, X) < L(X, Y)$ (it is true for all $a > 0, k > 1$). However, an easy bound that we can provide for $L(Y, X)$ is a lower bound using Claim 3.0.6. Therefore we need to produce explicit Lipschitz maps from Y to X. For this we refer to the construction from [43], where optimal Lipschitz maps between symmetric right-angled hexagons are described. Indeed, it follows from the construction in [43] that $L(Y, X)$ equals the log of the ratio of lengths of the seam curves. This ratio can be computed using a Lambert quadrilateral with an angle $\frac{\pi}{3}$ (see Lemma 4.1 in [43]). Assuming further that $ka \leqslant 4$ and using the estimates on hyperbolic functions (see Lemma 3.3 in [12]), we obtain:

$$L(Y, X) = \log \frac{\operatorname{arcsinh} \frac{1}{2 \sinh \frac{a}{4}}}{\operatorname{arcsinh} \frac{1}{2 \sinh \frac{ka}{4}}} \leqslant \log \frac{\log \frac{6}{a}}{\log \frac{2}{ka}}.$$

Now let $a = 0.1, k = 2$. Then $L(Y, X) \leqslant \log \frac{\log 60}{\log 10} < \log \frac{\log 100}{\log 10} = \log 2 = L(X, Y)$, thus $L(X, Y) \neq L(Y, X)$.

The proof works in case of punctures too by repeating the same argument and by adapting the construction in [43] replacing symmetric right-angled hexagons with symmetric pentagons with four right angles and one zero angle where needed. □

3.0.3 Properties of ratios of lengths.

Proposition 3.0.10. *For any two distinct points X and Y in $\mathcal{T}(S)$, we have:*

$$K(X,Y) > 0.$$

Proof. Recall that X and Y have the same hyperbolic area: $\text{Area}(X) = \text{Area}(Y) = 2\pi|\chi(S)|$. By [38], [15] there exists an area-preserving diffeomorphism between them: $\varphi : X \to Y$. Further, if S has punctures, the diffeomorphism φ can be chosen so that it restricts to an isometry on the cusp neighborhood. Let $\widetilde{\varphi} : \mathbb{H}^2 \to \mathbb{H}^2$ be a lift of φ to the universal cover. Then we have

$$\widetilde{\varphi} \circ g = \varphi_*(g) \circ \widetilde{\varphi} \tag{3.1}$$

for all $g \in \pi_1(X)$, where $\varphi_* : \pi_1(X) \to \pi_1(Y)$ is the induced homomorphism. Choose a point p in the domain, and define the function $D : \mathbb{H}^2 \to \mathbb{R}$ as follows:

$$D(q) = d(\widetilde{\varphi}(p), \widetilde{\varphi}(q)) - d(p,q). \tag{3.2}$$

Lemma 3.0.11. *If the function D has an upper bound, then $X = Y$.*

Proof. Since φ is a diffeomorphism, it follows from the proof of Claim 3.0.3 that $\widetilde{\varphi}$ is bi-Lipschitz. Then $\widetilde{\varphi}$ extends to a homeomorphism between the compactified hyperbolic planes $\bar{\varphi} : \bar{\mathbb{H}}^2 \to \bar{\mathbb{H}}^2$ (see Section 5.2 in [29]). Further, Equation (3.1) extends to $\partial\mathbb{H}^2$ by continuity.

Consider two infinite rays r_1, r_2 from p at an angle $\alpha < \pi/2$ and the wedge W_α between them. Then the image of the wedge $\bar{\varphi}(W_\alpha)$ is an "approximate" wedge bounded by two quasi-geodesic rays $\bar{\varphi}(r_1), \bar{\varphi}(r_2)$. By Morse lemma (see Lemma 5.2.10 in [29]), the quasi-geodesic rays are bounded Hausdorff distance away from some geodesic rays R_1, R_2, respectively, in the image. Suppose that the angle between R_1 and R_2 is β, and that the Hausdorff distance between $\bar{\varphi}(r_i)$ and R_i is at most K. Suppose also that $C > 0$ is an upper bound for the function D.

We show that there is a universal upper bound on the ratio of angles $\frac{\alpha}{\beta}$, which implies that the map $\bar{\varphi}^{-1}_{\partial\mathbb{H}^2}$ is Lipschitz with respect to the angular metric on $\partial\mathbb{H}^2$ obtained from the Poincaré disk model. Parametrize the rays $r_1(t), r_2(t)$ by arclength so that $r_1(0) = r_2(0) = p$. The area of the circular sector of W_α at time t equals $2\alpha \sinh^2(t/2)$. Suppose that $\bar{\varphi}(r_1)$ travels distance K in the direction perpendicular to R_1 away from the wedge between R_1 and R_2, and then fellow travels R_1 along the hypercycle. At time t, the distance $d(\bar{\varphi}(p), \bar{\varphi}(r_1(t)))$ is at most $t + C$ by the bound on the function D. By the triangle inequality, we have $d(\bar{\varphi}(p), \bar{\varphi}(r_1(t))) \geqslant$

$d(\bar{\varphi}(p), \mathrm{proj}_{R_1} r_1(t)) - K$. Hence $d(\bar{\varphi}(p), \mathrm{proj}_{R_1} r_1(t)) \leqslant t + C + K$ (draw a picture!). Let the ray R_1 be a vertical ray in the upper-half plane $\mathbb{H}^2$ and let γ be the angle that a hypercycle at distance K from R_1 makes with $\partial\mathbb{H}^2$. By the angle of parallelism, we have $\cosh K = \frac{1}{\sin\gamma}$. We find an upper bound for the area of the domain E_t whose boundary is given by the trajectory of $\bar{\varphi}(r_1(t))$ by the time t, its orthogonal projection onto R_1 and the perpendicular from $\bar{\varphi}(r_1(t))$ to R_1:

$$\begin{aligned} \mathrm{Area}(E_t) &= \iint_{E_t} \frac{dxdy}{y^2} \leqslant \int_{\arcsin\frac{1}{\cosh K}}^{\pi/2} \int_1^{e^{t+C+K}} \frac{rdrd\theta}{r^2\sin^2\theta} \\ &= \int_1^{e^{t+C+K}} \frac{dr}{r} \int_{\arcsin\frac{1}{\cosh K}}^{\pi/2} \frac{d\theta}{\sin^2\theta} = (t+C+K)\sinh K. \end{aligned} \tag{3.3}$$

The area of the $K-$neighborhood of the point $\bar{\varphi}(p)$ equals to $4\pi\sinh^2(K/2)$. Since the map $\widetilde{\varphi}$ is area-preserving, we obtain the following upper bound on the area of the circular sector of W_α at time t by making an overestimate of the area of its image:

$$2\alpha\sinh^2\left(\frac{t}{2}\right) \leqslant 4\pi\sinh^2(K/2) + 2\beta\sinh^2\left(\frac{t+C+K}{2}\right) + 2(t+C+K)\sinh(K). \tag{3.4}$$

It follows that for every $\varepsilon > 0$ we can find t large enough so that $\frac{\alpha}{\beta} \leqslant e^{C+K} + \varepsilon$. Since the constants K, C are universal, the map $\bar{\varphi}^{-1}_{\partial\mathbb{H}^2}$ is Lipschitz. Since Lipschitz functions are absolutely continuous and $\bar{\varphi}^{-1}_{\partial\mathbb{H}^2}$ is monotone on $\partial\mathbb{H}^2$ (since it is a homeomorphism), it follows from the formula $f(x) = f(a) + \int_a^x f'(t)\,dt$ for absolutely continuous functions that $\bar{\varphi}^{-1}_{\partial\mathbb{H}^2}$ has positive derivative on a subset of positive Lebesgue measure.

It follows then from [24] that $\bar{\varphi}^{-1}_{\partial\mathbb{H}^2}$ is the extension of an isometry ψ^{-1} of $\mathbb{H}^2$ (see also [1] for related remarks). Hence

$$\psi \circ g = \varphi_*(g) \circ \psi \tag{3.5}$$

holds for all $g \in \pi_1(X)$ on $\partial\mathbb{H}^2$. Both terms in Equation (3.5) are isometries, and isometries are determined by their boundary extension, hence Equation (3.5) holds everywhere in $\mathbb{H}^2$. Thus ψ descends to an isometry $\psi : X \to Y$. A homotopy between $\widetilde{\varphi}$ and ψ can be obtained by taking convex combination of them, since it is also π_1-equivariant and hence descends to a map between surfaces. Thus $X = Y$. □

Lemma 3.0.12 (Approximate orbits transitive)**.** *Let X be a volume-preserving vector field on a manifold M of finite volume which generates a flow for all time. Then for any compact set $A \subset M$ and for any $\varepsilon > 0$ there is a constant C such that any two points in A are connected by an ε-approximate flow line in time less than C, where*

ε-approximate flow line is a curve $\gamma(t)$ such that $\|T\gamma(t) - X(\gamma(t))\| \leqslant \varepsilon$.

Proof. Pick a point $x \in A$. Since the vector field X generates a flow for all time, there exists a neighborhood $\mathcal{N}_x$ of x where X can be smoothly straightened. Consider the Euclidean metric in the local coordinates on $\mathcal{N}_x$, and choose some $\delta-$neighborhood of x (in the Euclidean metric) that lies inside of $\mathcal{N}_x$. Consider the closed $\delta/2-$neighborhood $B_{\delta/2}(x)$ of x. For every $y \in B_{\delta/2}(x)$, there is the largest such $\varepsilon_y > 0$ such that $\|v - X(y)\|_E \leqslant \varepsilon_y$ implies $\|v - X(y)\| \leqslant \varepsilon$. Since this dependence is smooth and $B_{\delta/2}(x)$ is compact, it attains positive minimum $\varepsilon' > 0$ on $B_{\delta/2}(x)$.

It follows that the set of points in $B_{\delta/2}(x)$ that can be reached by an $\varepsilon-$approximate flow line from x before leaving the set, contains a circular cone with angle ε' (in the Euclidean metric). Since it has positive volume, by Poincaré recurrence theorem, there is a point z in the cone that will eventually come back to it along the trajectory of X. It follows that as soon as it comes back to the boundary of $B_{\delta/2}(x)$, we can alter its trajectory so that it reaches x by an $\varepsilon'-$approximate flow line (in the Euclidean metric, hence by $\varepsilon-$approximate flow line). Since by Poincaré recurrence theorem the set of points in the cone that will come back infinitely often has full measure, we can extend this argument to show that any point in $B_{\delta/4}(x)$ can be reached from x, and vice versa. Hence we can find an open neighborhood of x so that any point can reach any other point by an $\varepsilon-$approximate flow line in some uniformly bounded time. By the compactness of A, there is a finite open cover with these properties, for some time T. If there are N neighborhoods in the cover, then in time NT any point in A can reach any other point by an $\varepsilon-$approximate flow line. □

By Lemma 3.0.11 we can assume that the function D has no upper bound. Let $q \in \mathbb{H}^2$ be a point such that $D(q) = D > 0$, where D is some sufficiently large constant that we will specify below. We can assume that q projects to some fixed compact subsets of X and Y. Indeed, since φ is an isometry in a neighborhood of each cusp, any upper bound for D in the complete preimage in $\mathbb{H}^2$ of the complement to these neighborhoods gives (the same) upper bound for D in the whole $\mathbb{H}^2$. Assume that p projects to the same compact subsets of X and Y.

Let $\varepsilon = \frac{1}{2}$. Consider all extensions of the geodesic from $\widetilde{\varphi}(p)$ to $\widetilde{\varphi}(q)$ that have geodesic curvature at most ε. These extensions project to approximate flow lines for the geodesic flow on the unit tangent bundle T^1Y, therefore by Lemma 3.0.12 there is an extension of bounded length (say, C) that ends at $g(\widetilde{\varphi}(p))$ for some $g \in \pi_1(Y)$. We produced a closed curve in Y which we denote by α.

Next, we find a lower bound on the length of the geodesic representative of α in Y. Denote by γ the concatenation of the $g-$translates of the constructed path between

$\widetilde{\varphi}(p)$ and $g(\widetilde{\varphi}(p))$ in $\mathbb{H}^2$. By Lemma 2.3.13 in [20] (applied to a subdivision of γ into arcs of length 1), we obtain that γ is an infinite simple curve in $\mathbb{H}^2$ that is asymptotic to the axis of g. Moreover, by Lemma 2.3.13 in [20], we have that the Hausdorff distance between γ and the axis of g is at most 11. We have:

$$d(\widetilde{\varphi}(p), g(\widetilde{\varphi}(p))) \geqslant d(\widetilde{\varphi}(p), \widetilde{\varphi}(q)) - d(\widetilde{\varphi}(q), g(\widetilde{\varphi}(p))) \geqslant D + d(p,q) - C. \tag{3.6}$$

Then

$$\ell_\alpha(Y) \geqslant d(\widetilde{\varphi}(p), g(\widetilde{\varphi}(p))) - 22 \geqslant D + d(p,q) - C - 22. \tag{3.7}$$

Since $\widetilde{\varphi}$ is bi-Lipschitz, we have $\ell_\alpha(X) \leqslant d(p,q) + LC$ for some L that depends only on φ. Since C depends only on $\varepsilon = \frac{1}{2}$ and on the fixed compact subsets of X and Y, we can find $q \in \mathbb{H}^2$ so that $D(q) > LC + C + 22$. It follows that the curve α obtained this way satisfies $\ell_\alpha(Y) > \ell_\alpha(X)$, so we are done. □

Proposition 3.0.13. *The value of $K(X,Y)$ does not decrease if the supremum in the Definition 3.0.5 is restricted to simple closed curves.*

Proof. Given $X, Y \in \mathcal{T}(S)$ and a closed curve α (not necessarily simple), let $r_{XY}(\alpha) = \frac{\ell_\alpha(Y)}{\ell_\alpha(X)}$. We start by studying the hyperbolic pairs of pants.

Lemma 3.0.14 (Shrinking at the waist)**.** *Let X and Y be marked hyperbolic pairs of pants with geodesic boundary consisting of curves $\alpha_1, \alpha_2, \alpha_3$. If $r_{XY}(\alpha) > 1$ for some non-peripheral closed curve α, then*

$$r_{XY}(\alpha) \leqslant \max_{i=1,2,3} r_{XY}(\alpha_i). \tag{3.8}$$

Proof. First, we consider the case when $\ell_{\alpha_i}(X) = a$ and $\ell_{\alpha_i}(Y) = ka$ for some $a > 0, k \geqslant 1$. Decompose X and Y into two ideal hyperbolic triangles as in Claim 3.0.9 (see also Section 7.4 in [29]). There is a $k-$Lipschitz homeomorphism from X to Y obtained by applying the map from Claim 3.0.8 to each ideal triangle. Therefore the curve α is mapped to a curve that is at most k times longer than the original one, so this case follows.

For the general case, let $k = \max_{i=1,2,3} r_{XY}(\alpha_i)$. If $k > 1$, construct a $k-$Lipschitz homeomorphism from X to the hyperbolic pair of pants X' such that the lengths of the boundary curves of X' are the $k-$multiples of the ones of X. Attach the funnels to X' making it a complete hyperbolic surface without boundary $\widehat{X'}$. There is a $1-$Lipschitz map from $\widehat{X'}$ to $\widehat{Y}$, obtained by cutting a strip around an infinite simple geodesic that intersects the boundary curve which is shorter in Y than in X', and then gluing appropriately (see [42] for the details). The composition of these maps sends α to a curve homotopic to α whose length is at most $k \cdot 1$ times longer than the original one.

Finally, if $k \leqslant 1$, then it follows from the above construction that there is a $1-$Lipschitz map from $\widehat{X}$ to $\widehat{Y}$, therefore $r_{XY}(\alpha) \leqslant 1$, which is a contradiction with the assumption. □

Remark 3.0.15. Lemma 3.0.14 extends to the case when one or two boundary components are replaced with a cusp by letting $r_{XY}(\alpha_i) = 1$ at each cusp. The proof is identical.

In Proposition 3.0.10 we showed that $K(X, Y) > 0$ for any two distinct points X and Y in the Teichmüller space. It implies that $r_{XY}(\alpha) > 1$ for some closed curve α. We show that for any (primitive) non-simple closed curve α such that $r_{XY}(\alpha) > 1$ there is a curve $\widetilde{\alpha}$ with fewer self-intersections such that $r_{XY}(\widetilde{\alpha}) \geqslant r_{XY}(\alpha)$. It is clear that this is sufficient to prove the proposition, since then for a sequence of closed curves $\{\alpha_i\}$ realizing $K(X, Y)$ we have $r_{XY}(\alpha_i) > 1$ for all sufficiently large i, and thus it can be modified to a sequence of *simple* closed curves realizing $K(X, Y)$ by applying sufficiently many times the procedure that we describe. More specifically, we show that it is possible to choose $\widetilde{\alpha}$ to be (one of the components of) a smoothing of any self-intersection of the geodesic representative of α in X.

First, assume that α represents a primitive conjugacy class in the fundamental group of S, that is to say that it does not wrap around itself. It is clear that $K(X, Y)$ does not decrease if we restrict to primitive closed curves, since $\ell_{\alpha^k}(\cdot) = k\ell_\alpha(\cdot)$. Let $p \in \alpha$ be a point of self-intersection of (a small perturbation of) the geodesic representative of α in X. Since α is primitive, the self-intersection at p is transverse.

Consider the hyperbolic structure in a small neighborhood of α in X. This allows to construct a complete hyperbolic pair of pants without boundary $\widehat{X}$ with a "figure eight" geodesic $\widehat{\alpha}$ such that there is a local isometry from a neighborhood of $\widehat{\alpha}$ to a neighborhood of α that maps $\widehat{\alpha}$ to α and the self-intersection p' of $\widehat{\alpha}$ to p of α. It follows that the lengths of the three closed geodesics obtained by smoothing the self-intersection p of α equal to the lengths of the three simple closed geodesics of $\widehat{X}$. Repeat the same construction for the geodesic representative of α in Y: we can find a self-intersection q of α in Y such that the closed curves obtained by smoothing q are homotopic to the ones obtained from smoothing p of α in X (the representatives differ by a sequence of Reidemeister III moves, see Lemma 2.17 in [30] for details). It follows from Lemma 3.0.14 that one of the components $\widetilde{\alpha}$ of a smoothing of α satisfies $r_{XY}(\widetilde{\alpha}) \geqslant r_{XY}(\alpha)$, which finishes the proof. □

We conclude this section by mentioning the preprint [11], where another proof of

Proposition 3.0.10 and of Proposition 3.0.13 are presented, using dilogarithm identities on hyperbolic surfaces. For various results generalizing Thurston's theory to higher dimensions see [16].

Chapter 4

Results

4.1 Constructing the lamination

In this section we construct a quasi-geodesic $\{\alpha_i\}$ in the curve graph of the seven-times punctured sphere $S_{0,7}$ converging to the ending lamination λ in the Gromov boundary.

4.1.1 Alpha sequence

Denote by $S = S_{0,7}$ the seven-times punctured sphere, obtained by doubling a regular heptagon on the plane along its boundary. Consider four curves $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ on S as shown in Figure 4.1.

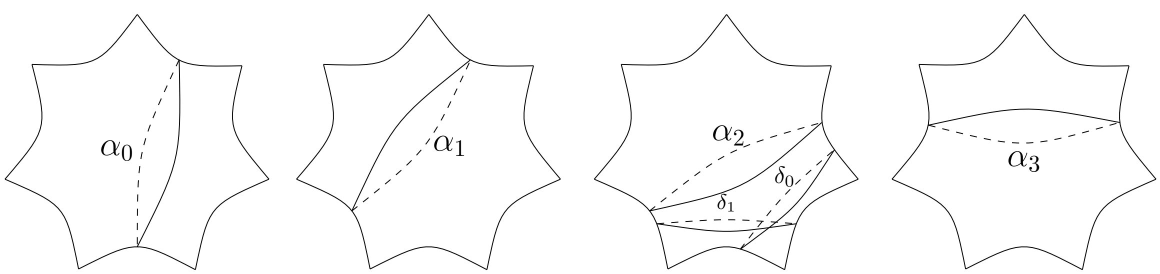

Figure 4.1: The curves $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ and δ_0, δ_1 on S.

Let ρ be the finite order homeomorphism of S which is realized by the counter-clockwise rotation along the angle of $\frac{6\pi}{7}$. In other words, the map ρ rotates S by 3 'clicks' counterclockwise. Let Y_2 be the subsurface of S with the boundary curve α_2 and with 3 punctures. Denote by τ the partial pseudo-Anosov map on S supported on the subsurface Y_2 and obtained as the composition of two half-twists $\tau = H_{\delta_1}^{-1} H_{\delta_0}$ (the core curves are shown in Figure 4.1).

For any $n \in \mathbb{N}$, let $\varphi_n = \tau^n \circ \rho$. Let $\{r_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of natural numbers. We assume for the remainder of the thesis that the sequence

$\{r_{n+1} - r_n\}$ is also strictly increasing[1]. Set

$$\Phi_i = \varphi_{r_1}\varphi_{r_2}\dots\varphi_{r_{i-1}}\varphi_{r_i}. \tag{4.1}$$

Observe that for any $a, b, c \in \mathbb{N}$:

$$\alpha_1 = \varphi_c(\alpha_0), \alpha_2 = \varphi_b\varphi_c(\alpha_0), \alpha_3 = \varphi_a\varphi_b\varphi_c(\alpha_0). \tag{4.2}$$

In particular, for $i = 1, 2, 3$ we have that $\Phi_i(\alpha_0) = \alpha_i$. Define the curves $\alpha_i = \Phi_i(\alpha_0)$ for every $i \in \mathbb{N}$.

Denote by Y_i the subsurface with boundary curve α_i and 3 punctures. We begin with the observations on the sizes of the subsurface projections between the curves in the sequence $\{\alpha_i\}$ and the local progress that the sequence makes in the curve graph of S:

Claim 4.1.1. There is a constant $\alpha > 0$, so that for every $i \geqslant 2$

$$d_{Y_i}(\alpha_{i-2}, \alpha_{i+2}) \geqslant \alpha\, r_{i-1} - 1.$$

Proof. First we expand the expression using Equation (4.1) and simplify it by applying Equation (4.2), the fact that the mapping class group acts on the curve graph by isometries, and the triangle inequality:

$$\begin{aligned} d_{Y_i}(\alpha_{i-2}, \alpha_{i+2}) &= d_{\Phi_{i-2}\varphi_{r_{i-1}}\varphi_{r_i}(Y_0)}(\Phi_{i-2}(\alpha_0), \Phi_{i-2}\varphi_{r_{i-1}}\varphi_{r_i}\varphi_{r_{i+1}}\varphi_{r_{i+2}}(\alpha_0)) \\ &= d_{Y_2}(\alpha_0, \varphi_{r_{i-1}}(\alpha_3)) = d_{Y_2}(\alpha_0, \tau^{r_{i-1}}(\rho\alpha_3)) \\ &\geqslant d_{Y_2}(\alpha_0, \tau^{r_{i-1}}(\alpha_0)) - d_{Y_2}(\tau^{r_{i-1}}(\alpha_0), \tau^{r_{i-1}}(\rho\alpha_3)) \\ &= d_{Y_2}(\alpha_0, \tau^{r_{i-1}}(\alpha_0)) - d_{Y_2}(\alpha_0, \rho\alpha_3) \\ &= d_{Y_2}(\alpha_0, \tau^{r_{i-1}}(\alpha_0)) - 1. \end{aligned} \tag{4.3}$$

Since the mapping class τ restricts to a pseudo-Anosov map on the surface Y_2, by Proposition 3.6 in [32] we have $d_{Y_2}(\alpha_0, \tau^n(\alpha_0)) \geqslant \alpha\, n$ for some $\alpha > 0$, so the result follows. □

Lemma 4.1.2. *There is a number $B \in \mathbb{N}$, such that the curves α_i and α_{i+5} fill the surface S for every $i \geqslant B$.*

Proof. By Equation (4.2), $\alpha_{i+5} = \Phi_i\varphi_{r_{i+1}}\varphi_{r_{i+2}}\varphi_{r_{i+3}}\varphi_{r_{i+4}}\varphi_{r_{i+5}}(\alpha_0) = \Phi_i\varphi_{r_{i+1}}\varphi_{r_{i+2}}(\alpha_3)$, therefore it is enough to consider the pair of curves α_0 and $\varphi_{r_{i+1}}\varphi_{r_{i+2}}(\alpha_3)$. Next, for all $a, b \in \mathbb{N}$ we trivially have: $\varphi_a\varphi_b = \tau^a\rho\tau^b\rho = \tau^a\rho\tau^b\rho^{-1}\rho^2$, and $\rho\tau^b\rho^{-1}$ is a partial

[1] this condition will be used in Claim 4.1.4, Lemma 4.3.3, Claim 4.4.2 and Lemma 4.4.5.

pseudo-Anosov map supported on the subsurface $\rho(Y_2) = Y_3$, thus it commutes with τ. Hence we can write

$$\varphi_a\varphi_b = \tau^a\rho\tau^b\rho^{-1}\rho^2 = \rho\tau^b\rho^{-1}\tau^a\rho^2. \tag{4.4}$$

Since Y_2 is a non-annular subsurface, the post composition with a mapping class supported on a disjoint subsurface does not affect the subsurface projections, hence by Equation (4.4)

$$\begin{aligned} d_{Y_2}(\alpha_0, \varphi_{r_{i+1}}\varphi_{r_{i+2}}(\alpha_3)) &= d_{Y_2}(\alpha_0, \rho\tau^{r_{i+2}}\rho^{-1}\tau^{r_{i+1}}\rho^2(\alpha_3)) \\ &= d_{Y_2}(\alpha_0, \tau^{r_{i+1}}(\rho^2\alpha_3)). \end{aligned} \tag{4.5}$$

Similarly, for the subsurface Y_3 we have

$$\begin{aligned} d_{Y_3}(\alpha_0, \varphi_{r_{i+1}}\varphi_{r_{i+2}}(\alpha_3)) &= d_{Y_3}(\alpha_0, \tau^{r_{i+1}}\rho\tau^{r_{i+2}}\rho(\alpha_3)) \\ &= d_{Y_3}(\alpha_0, \rho\tau^{r_{i+2}}\rho(\alpha_3)) \\ &= d_{Y_3}(\alpha_0, \rho\tau^{r_{i+2}}\rho^{-1}(\rho^2\alpha_3)). \end{aligned} \tag{4.6}$$

Since the sequence $\{r_i\}$ is strictly increasing, by the proof of Claim 4.1.1 there is a number B such that the quantities in Equation (4.5) and Equation (4.6) are at least 3 for all $i \geqslant B$.

Now suppose that the curves α_0 and $\varphi_{r_{i+1}}\varphi_{r_{i+2}}(\alpha_3)$ do not fill S. Thus there is a curve γ disjoint from both of them. Observe that disjoint curves that intersect the subsurface Y_i essentially, project to a set of diameter at most 1 in $\mathcal{C}(Y_i)$ ([33], Lemma 2.2). If γ intersects essentially either Y_2 or Y_3, we obtain a path of length at most 2 that connects the projections of α_0 and $\varphi_{r_{i+1}}\varphi_{r_{i+2}}(\alpha_3)$ to the respective curve graph, which is impossible for $i \geqslant B$. Since the complement to the union of subsurfaces Y_2 and Y_3 in S is homeomorphic to the three times punctured sphere $S_{0,3}$, the curve γ has to be either α_2 or α_3, but the curve α_0 intersects each of them essentially. □

Next, we prove the main result of the section.

Proposition 4.1.3. *The path $\{\alpha_i\}$ is a quasi-geodesic in the curve graph $\mathcal{C}(S)$.*

Proof. Let M be the constant associated with the bounded geodesic image theorem (Theorem 2.0.4). In the next two claims we show that for large enough R, the path $\{\alpha_i\}_{i>\mathsf{R}}$ is a J-local $(3, \frac{1}{3})$-quasi-geodesic for $\mathsf{J} = \mathsf{R} - \mathsf{M}$.

Claim 4.1.4. There is $C > \mathsf{M}$, so that for every $C < \mathsf{R} < j < h < k$ such that

$$k - j < \mathsf{J},\ k - h \geqslant 3,\ h - j \geqslant 2, \tag{4.7}$$

the curves α_j, α_k fill the surface S and

$$d_{Y_h}(\alpha_j, \alpha_k) \geqslant \mathsf{R} - (k - j). \tag{4.8}$$

Proof of Claim 4.1.4. The proof is by induction on $n = k - j$.

Base: $n = 5$. First, by Lemma 4.1.2, the curves α_j and α_{j+5} fill the surface S for all $j \geqslant B$. Notice that by Equation (4.2), $i(\alpha_{j+2}, \alpha_{j+5}) = i(\alpha_0, \alpha_3) \neq 0$, so the curve α_{j+5} intersects essentially the subsurface Y_{j+2}.

Next, observe that the curves α_2 and α_3 project to the same curve in Y_0. By the triangle inequality we have

$$\begin{aligned} d_{Y_{j+2}}(\alpha_j, \alpha_{j+5}) &\geqslant d_{Y_{j+2}}(\alpha_j, \alpha_{j+4}) - d_{Y_{j+2}}(\alpha_{j+4}, \alpha_{j+5}) \\ &= d_{Y_{j+2}}(\alpha_j, \alpha_{j+4}) - d_{Y_0}(\alpha_2, \alpha_3) \\ &= d_{Y_{j+2}}(\alpha_j, \alpha_{j+4}). \end{aligned} \tag{4.9}$$

Next, by Claim 4.1.1, we have $d_{Y_{j+2}}(\alpha_j, \alpha_{j+4}) \geqslant \alpha\, r_{j+1} - 1$. Since the sequence $\{r_{j+1} - r_j\}$ is strictly increasing, we can find a constant $C > \max\{B, \mathsf{M}\}$, so that for every $j > C$ we have $\alpha\, r_{j+1} - 1 \geqslant j$ and then

$$d_{Y_{j+2}}(\alpha_j, \alpha_{j+4}) \geqslant \alpha\, r_{j+1} - 1 \geqslant j > \mathsf{R} > \mathsf{R} - (k - j). \tag{4.10}$$

Step. Assume the triple $j < h < k$ satisfies Equation (4.7) and $k - j = n + 1 < \mathsf{J}$. Since $n + 1 \geqslant 6$, either $k - h \geqslant 4$ or $h - j \geqslant 3$. Let us consider the first case, a similar argument proves the second one.

The triple $j < h < k - 1$ satisfies Equation (4.7), therefore by the induction hypothesis

$$d_{Y_h}(\alpha_j, \alpha_{k-1}) \geqslant \mathsf{R} - (k - j - 1).$$

Notice that $i(\alpha_h, \alpha_k) \neq 0$ (for $k - h \geqslant 5$ it follows from the induction hypothesis, and for $k - h = 4$ it is a consequence of Claim 4.1.1), so α_k intersects essentially the subsurface Y_h. Then by triangle inequality and the disjointedness of any two successive curves in the path $\{\alpha_i\}$:

$$d_{Y_h}(\alpha_j, \alpha_k) \geqslant d_{Y_h}(\alpha_j, \alpha_{k-1}) - d_{Y_h}(\alpha_{k-1}, \alpha_k) \geqslant \mathsf{R} - (k - j).$$

Suppose the curves α_j and α_k do not fill the surface S. Since the curves α_j and α_{k-1} do fill by the induction hypothesis, the distance $d_S(\alpha_j, \alpha_{k-1})$ is at least 3. Hence by triangle inequality $d_S(\alpha_j, \alpha_k) = 2$. Let $\{\alpha_j, \alpha', \alpha_k\}$ be some geodesic connecting α_j with α_k in the curve graph $\mathcal{C}(S)$. Pick h such that $j < h < h + 1 < k$, $h - j \geqslant 2$ and

$k-(h+1) \geqslant 3$. Then

$$d_{Y_h}(\alpha_j, \alpha_k), d_{Y_{h+1}}(\alpha_j, \alpha_k) \geqslant \mathsf{R} - (k-j) > \mathsf{R} - \mathsf{J} = \mathsf{M},$$

so by Theorem 2.0.4 the curve α' has to miss both subsurfaces Y_h and Y_{h+1}. Like in the proof of Lemma 4.1.2, this means that the curve α' has to be either α_h or α_{h+1}, which is impossible, because the curves α_j and α_k intersect each of them essentially. □

Claim 4.1.5. For every pair or curves α_j, α_k with $\mathsf{R} < j < k$ and $k-j < \mathsf{J}$, we have:

$$d_S(\alpha_j, \alpha_k) \geqslant \frac{k-j}{3} - \frac{1}{3}.$$

Proof of Claim 4.1.5. By Theorem 2.0.4 and Claim 4.1.4, we know that for every h, such that the triple $j < h < k$ satisfies Equation (4.7), any geodesic $\mathcal{G}$ between the curves α_j and α_k in the curve graph $\mathcal{C}(S)$ must contain some vertex v that does not intersect essentially the subsurface Y_h. We show that for each vertex v there can be at most 3 such subsurfaces.

Notice that for every sextuple $j < h_1 < h_2 < h_3 < h_4 < k$ so that every triple $j < h_i < k, (i = 1, \dots, 4)$ satisfies Equation (4.7), the four subsurfaces $Y_{h_i}, (i = 1, \dots, 4)$ fill S. Indeed, if $h_4 - h_1 \geqslant 5$, it follows from Claim 4.1.4 as curves $\alpha_{h_1}, \alpha_{h_4}$ fill S. If $h_4 - h_1 = 3$, it follows from Equation (4.2) that it is enough to check that the subsurfaces Y_0, Y_1, Y_2, Y_3 fill S, which follows from examining Figure 4.1. If $h_4 - h_1 = 4$, the result follows from Equation (4.2) and the observation that any three non-consecutive subsurfaces among Y_0, Y_1, Y_2, Y_3 fill S.

For each $h \in \{j+2, \dots, k-3\}$ map the curve α_h to some vertex in $\mathcal{G}$ that does not intersect essentially the subsurface Y_h. It follows that this map is at most 3−to−1 and it omits the endpoints, so $\mathcal{G}$ has at least $2 + \frac{k-j-4}{3}$ vertices. This bounds the distance between curves α_j and α_k from below:

$$d_S(\alpha_j, \alpha_k) \geqslant 2 + \frac{k-j-4}{3} - 1 = \frac{k-j}{3} - \frac{1}{3}.$$

□

We proved that the path $\{\alpha_i\}_{i>\mathsf{R}}$ is a J-local $(3, \frac{1}{3})$-quasi-geodesic. By the hyperbolicity of the curve graph (Theorem 2.0.1) and by Theorem 1.4 in [10] ("local quasi-geodesic implies quasi-geodesic"), if we pick R and hence J large enough, we obtain that $\{\alpha_i\}_{i>R}$ is a global quasi-geodesic. Therefore, $\{\alpha_i\}$ is also a quasi-geodesic. □

We obtain an immediate corollary from Theorem 2.0.2:

Corollary 4.1.6. *There is an ending lamination λ on S representing a point in the Gromov boundary of $\mathcal{C}(S)$, such that*

$$\lim_{i\to\infty} \alpha_i = \lambda.$$

Furthermore, every limit point of $\{\alpha_i\}$ in $\mathbb{P}\mathcal{ML}(S)$ defines a projective class of transverse measure on λ.

4.2 Invariant bigon track

In this section, we introduce a maximal birecurrent bigon track that is invariant under the homeomorphisms Φ_i defined in Equation (4.1). We refer the reader to §3.4 in [17] for more details on bigon tracks. The bigon track T is shown in Figure 4.2:

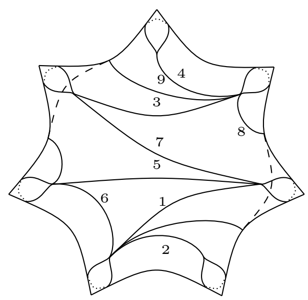

Figure 4.2: The bigon track T with a numbering of strands.

Let $V(T)$ be the convex cone consisting of all non-negative real assignments of weights to the branches of T, that satisfy the switch conditions. Pick the ordered subset $B(T)$ of 9 branches of T as on Figure 4.2, and denote by w_i the vector in $V(T)$, that assigns the weight 1 to the i-th branch ($i = 1, \dots, 9$) and the weight 0 to all other branches in $B(T)$. It is evident that $V(T)$ is the non-negative orthant in the vector space $W(T)$ of all real assignments of weights to the branches of T (that satisfy the switch conditions) with basis $w_1, \dots, w_9$. The dimension of the space of measured laminations on S equals to 8, and the natural map from $V(T)$ to $\mathcal{ML}(S)$ is not injective precisely because T has a bigon. One can check, by drawing a picture, that the vectors $4w_2 + 2w_6 + 2w_9$ and $4w_4 + 2w_8$ define the same curve on S. Therefore, we can identify the space of measured laminations, carried by T, with the linear quotient cone $V'(T) = V(T)/\sim$, where for $\mu_1, \mu_2 \in V(T)$ we let $\mu_1 \sim \mu_2$ when $\mu_1 - \mu_2 \in \mathrm{span}(2w_2 - 2w_4 + w_6 - w_8 + w_9)$.

Proposition 4.2.1. *The bigon track T is Φ_i-invariant.*

Proof. It is enough to check that T is invariant under maps τ and $\tau \circ \rho$. This can be done directly:

Figure 4.3: The action of τ on T.

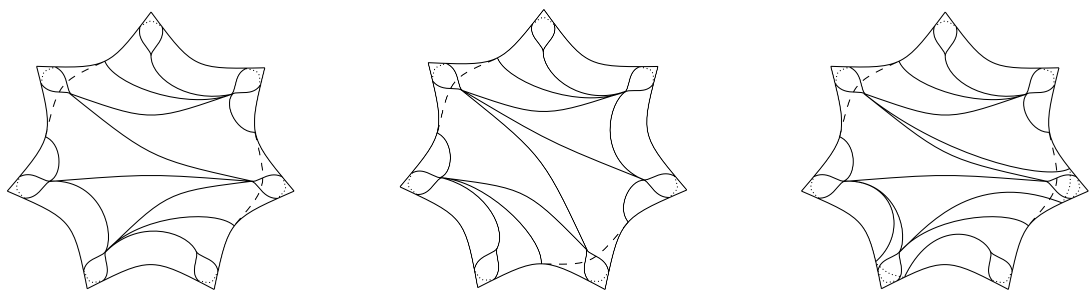

Figure 4.4: The action of ρ followed by τ on T.

□

We write down the matrices of the induced actions of τ and $\tau \circ \rho$ on the cone $V(T)$, that we denote by A and B, respectively:

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

To conclude the section, we define a sequence of curves on S that is carried by T.

4.2.1 Gamma sequence

Let γ_0 be the curve on S defined by the vector $2w_1$ in $V(T)$, as shown in Figure 4.5. Define the curves $\gamma_i = \Phi_i(\gamma_0)$. By Proposition 4.2.1, the sequence of curves γ_i is carried by the bigon track T, and since $d_S(\alpha_i, \gamma_i) = 2$ for all $i \in \mathbb{N}$, Corollary 4.1.6 holds for $\{\gamma_i\}$ with the same ending lamination in the limit.

We estimate the intersection numbers for pairs of curves in the sequence $\{\gamma_i\}$ in Section 4.3 and show that the ending lamination λ is not uniquely ergodic in Section 4.4. We use the sequence $\{\alpha_i\}$ again in Section 4.5.

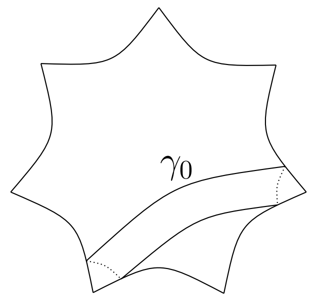

Figure 4.5: The curve γ_0 on S.

4.3 Estimating the intersection numbers

In this section we estimate the intersection numbers between pairs of curves in the sequence $\{\gamma_i\}$, introduced in Section 4.2.1, up to multiplicative errors. We prove:

Proposition 4.3.1. *Suppose the sequence $\{r_i\}$ satisfies the conditions of Lemma 4.3.5 for sufficiently large K. Then there is a constant $i_0 \in \mathbb{N}$, such that for all $i_0 < i < j$ with $i \equiv j \mod 2$*

$$i(\gamma_i, \gamma_j) \stackrel{*}{\asymp} \phi^{2r_{i+1}-2r_i} \cdot i(\gamma_{i-1}, \gamma_j) \stackrel{*}{\asymp} \prod_{\substack{k=i+1 \\ k \equiv i+1 \mod 2}}^{j-1} 2F_{2r_k-2},$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the Golden ratio, and F_i is the i-th Fibonacci number, such that $F_0 = 0, F_1 = 1$.

We start by introducing some notation. Let $s_i = r_i - 1$. Define the matrices $P_i = A^{s_i} B A^{s_{i+1}} B$. Notice that the matrix P_i represents the action of the mapping class $\varphi_{r_i}\varphi_{r_{i+1}}$ on $V(T)$. We will study the infinite products of the matrices P_i.

Claim 4.3.2. The matrix P_i is given by the following matrix:

$$\begin{pmatrix} F_{2s_i} & F_{2s_i+1}-1 & F_{2s_i+2} & F_{2s_i+2}-1 & F_{2s_i} & 0 & F_{2s_i} & F_{2s_i-1}-1 & F_{2s_i+2}-1 \\ F_{2s_i-1}-1 & F_{2s_i}+1 & F_{2s_i+1}-1 & F_{2s_i+1}-1 & F_{2s_i-1}-1 & 0 & F_{2s_i-1}-1 & F_{2s_i-2}+1 & F_{2s_i+1}-1 \\ 0 & 0 & F_{2s_{i+1}} & F_{2s_{i+1}+1}-1 & F_{2s_{i+1}+2} & F_{2s_{i+1}+2}-1 & F_{2s_{i+1}} & 0 & F_{2s_{i+1}-1}-1 \\ 0 & 0 & F_{2s_{i+1}-1}-1 & F_{2s_{i+1}}+1 & F_{2s_{i+1}+1}-1 & F_{2s_{i+1}+1}-1 & F_{2s_{i+1}-1}-1 & 0 & F_{2s_{i+1}-2}+1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Further, in every matrix norm the following convergence holds:

$$\lim_{i\to\infty} \frac{P_i}{2F_{2s_{i+1}}} = L,$$

where

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & \phi/2 & \phi^2/2 & \phi^2/2 & 1/2 & 0 & 1/2\phi \\ 0 & 0 & 1/2\phi & 1/2 & \phi/2 & \phi/2 & 1/2\phi & 0 & 1/2\phi^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is a rank 1 matrix with the largest eigenvalue 1.

Proof. Direct check. Notice that the vector $(0, 0, \phi, 1, 0, 0, 0, 0, 0)^t$ is an eigenvector of L with the eigenvalue 1. $\square$

We introduce more notation. Notice that the product of matrices $P_1P_3 \dots P_{2n-1}$ represents the map Φ_{2n}. Let $c_i = 2F_{2s_{i+1}}$ be the scaling factor in Claim 4.3.2. Define the matrices $R_i = P_i/c_i$ and $\varepsilon_i = R_i - L$. Denote by $\|\cdot\|$ the Frobenius norm on the space $\mathrm{End}(W(T))$ with a fixed basis $\{w_1, \dots, w_9\}$ on $W(T)$, turning it into a Banach algebra. We show:

Lemma 4.3.3. *The matrices*

$$R_iR_{i+2} \dots R_{i+2k}$$

converge to a non-zero matrix for all $i \in \mathbb{N}$ as $k \to \infty$. Further, there is a constant $C > 0$, so that for every $i, k \in \mathbb{N}$, the following estimate holds:

$$\|R_iR_{i+2} \dots R_{i+2k} - L\| \leqslant C(e^{C(\sum_{\ell=0}^{k} \|\varepsilon_{i+2\ell}\|)} - 1). \tag{4.11}$$

Proof. By Claim 4.3.2, we have $L^2 = L$ and if $L = S\widetilde{L}S^{-1}$ is a Jordan decomposition of L, then $\|\widetilde{L}\| = 1$. Define the matrices $\widetilde{\varepsilon}_j = S^{-1}\varepsilon_j S$. Let $C = \|S\| \cdot \|S^{-1}\|$. Then

$$\begin{aligned} \|R_i R_{i+2} \dots R_{i+2k} - L\| &\leqslant \|S\| \cdot \left\|(\widetilde{L} + \widetilde{\varepsilon}_i)(\widetilde{L} + \widetilde{\varepsilon}_{i+2}) \dots (\widetilde{L} + \widetilde{\varepsilon}_{i+2k}) - \widetilde{L}\right\| \cdot \left\|S^{-1}\right\| \\ &\leqslant C \sum_{j=0}^{k} \left(\|\widetilde{L}\|^j \prod_{0 \leqslant \ell_1 < \dots < \ell_{k+1-j} \leqslant k} \|\widetilde{\varepsilon}_{i+2\ell_1}\| \dots \left\|\widetilde{\varepsilon}_{i+2\ell_{k+1-j}}\right\| \right) \\ &= C((1 + \|\widetilde{\varepsilon}_i\|) \dots (1 + \|\widetilde{\varepsilon}_{i+2k}\|) - 1). \end{aligned} \tag{4.12}$$

Together with the above estimate, the estimate in Equation (4.11) follows from the inequalities $1 + \varepsilon < e^{\varepsilon}$ for $\varepsilon > 0$, and $\|\widetilde{\varepsilon}_j\| \leqslant C \|\varepsilon_j\|$. Observe that we have the following coarse equalities for the Frobenius norm of the matrices ε_i for all $i \geqslant 2$:

$$\|\varepsilon_i\| \overset{*}{\asymp}_{18} \frac{F_{2s_i}}{F_{2s_{i+1}}} = \frac{c_{i-1}}{c_i} \overset{*}{\asymp}_2 \phi^{2s_i - 2s_{i+1}}. \tag{4.13}$$

Since the sequence $\{s_{i+1} - s_i\}$ is strictly increasing, we have $\sum_{i=1}^{\infty} \|\varepsilon_i\| < \infty$. Therefore, if the matrix limit $\lim_{k\to\infty} R_i R_{i+2} \dots R_{i+2k}$ exists, we can guarantee that it is a non-zero matrix for all large enough i by Equation (4.11). Since each matrix R_j has full rank, it holds for all i.

Finally, we show that the sequence $R_i R_{i+2} \dots R_{i+2k}$ is a Cauchy sequence with respect to the Frobenius norm. Let $j, k > N$:

$$\begin{aligned} &\|R_i R_{i+2} \dots R_{i+2k} - R_i R_{i+2} \dots R_{i+2j}\| \\ &\leqslant \|R_i \dots R_{i+2k} - R_i \dots R_{i+2N} L\| + \|R_i \dots R_{i+2N} L - R_i \dots R_{i+2j}\|; \\ &\|R_i \dots R_{i+2k} - R_i \dots R_{i+2N} L\| \\ &\leqslant C \left\|(\widetilde{L} + \widetilde{\varepsilon}_i) \dots (\widetilde{L} + \widetilde{\varepsilon}_{i+2N})\right\| \cdot \left\|(\widetilde{L} + \widetilde{\varepsilon}_{i+2N+2}) \dots (\widetilde{L} + \widetilde{\varepsilon}_{i+2k}) - \widetilde{L}\right\| \\ &\leqslant C \left(\prod_{j=i}^{\infty} (1 + \|\widetilde{\varepsilon}_j\|) \right) \left(e^{C(\sum_{\ell=N+1}^{k} \|\varepsilon_{i+2\ell}\|)} - 1 \right). \end{aligned} \tag{4.14}$$

We have $\sum_{j=i}^{\infty} \|\widetilde{\varepsilon}_j\| < \infty$, hence $\prod_{j=i}^{\infty} (1 + \|\widetilde{\varepsilon}_j\|) < \infty$. Then for every $\varepsilon > 0$ we can choose N so that

$$\|R_i \dots R_{i+2\ell} - R_i \dots R_{i+2N} L\| < \varepsilon/2$$

for all $\ell > N$, which proves the lemma. □

Claim 4.3.4. In every matrix norm the following convergence holds:

$$\lim_{i\to\infty} \frac{P_i - c_i L}{c_{i-1}} = Q,$$

where

$$Q = \begin{pmatrix} 1/2 & \phi/2 & \phi^2/2 & \phi^2/2 & 1/2 & 0 & 1/2 & 1/2\phi & \phi^2/2 \\ 1/2\phi & 1/2 & \phi/2 & \phi/2 & 1/2\phi & 0 & 1/2\phi & 1/2\phi^2 & \phi/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is a rank 1 matrix with the largest eigenvalue 1. Further, the matrices $T_i = P_i - c_i L - c_{i-1} Q$ have uniformly bounded entries.

Proof. Direct check. Notice that the vector $(\phi, 1, 0, 0, 0, 0, 0, 0, 0)^t$ is an eigenvector of Q with the eigenvalue 1. □

Lemma 4.3.5. *Let $A > 1$ and let $\{r_i\}$ be any sequence as in Section 4.1.1. For any $N \in \mathbb{N}$ there exist $K, L \in \mathbb{N}$, so that if the sequence $\{r_i\}$ satisfies $(r_{i+1} - r_i) - (r_i - r_{i-1}) \geqslant K/2$ for all $i \geqslant 1$, then the following holds*

$$\frac{\|\varepsilon_i\|}{A^{\sum_{\ell=0}^{\infty}\|\varepsilon_{i+1+\ell}\|} - 1} \geqslant N \tag{4.15}$$

for all $i \geqslant L$.

Proof. Let $t_i = 2r_{i+1} - 2r_i$. Then we have $t_{i+j} - t_i \geqslant Kj$ for all $i, j \in \mathbb{N}$. By Equation (4.13), we have $\|\varepsilon_i\| \overset{*}{\asymp}_{36} \phi^{-t_i}$. Further, we have

$$\sum_{\ell=0}^{\infty} \|\varepsilon_{i+1+\ell}\| \overset{*}{\asymp}_{36} \sum_{\ell=0}^{\infty} \phi^{-t_{i+1+\ell}} \leqslant \sum_{\ell=0}^{\infty} \phi^{-t_{i+1}-K\ell} = \phi^{-t_{i+1}} \sum_{\ell=0}^{\infty} \phi^{-K\ell} = \frac{\phi^{-t_{i+1}}}{1-\phi^{-K}}. \tag{4.16}$$

Notice that given $a > 1$, then for all small enough $x > 0$ the following holds:

$$a^x \leqslant 1 + (\log(a) + 1)x. \tag{4.17}$$

Then we have

$$A^{\sum_{\ell=0}^{\infty}\|\varepsilon_{i+1+\ell}\|} - 1 \overset{*}{\prec}_{36} 1 + (\log(A)+1)\frac{\phi^{-t_{i+1}}}{1-\phi^{-K}} - 1 = \phi^{-t_{i+1}} \frac{\log(A)+1}{1-\phi^{-K}} \tag{4.18}$$

for all $i \geqslant L$ for some $L \in \mathbb{N}$. Finally, we have

$$\frac{\|\varepsilon_i\|}{A^{\sum_{\ell=0}^{\infty}\|\varepsilon_{i+1+\ell}\|}-1} \overset{*}{\succ}_{1296} \phi^{t_{i+1}-t_i}\frac{1-\phi^{-K}}{\log(A)+1} \overset{*}{\succ}_{2} \frac{\phi^K}{\log(A)+1}, \tag{4.19}$$

from which the lemma follows. □

We are ready to prove the proposition.

Proof of Proposition 4.3.1. Suppose that $j-i$ is even. Since $\varphi_n(\gamma_0)=\gamma_1$ for every $n\in\mathbb{N}$, we have:

$$i(\gamma_i,\gamma_j)=i(\Phi_{i-1}(\gamma_1),\Phi_{i-1}\varphi_{r_i}\dots\varphi_{r_j}(\gamma_0))=i(\gamma_1,\varphi_{r_i}\dots\varphi_{r_{j-1}}(\gamma_1)).$$

We can express the latter curve as a vector in $V(T)$:

$$P_iP_{i+2}\dots P_{j-2}(2w_3)=c_ic_{i+2}\dots c_{j-2}R_iR_{i+2}\dots R_{j-2}(2w_3). \tag{4.20}$$

By Lemma 4.3.3, the vector $R_iR_{i+2}\dots R_{j-2}(2w_3)$ can be approximated by the vector $L(2w_3)=w_3+\frac{1}{\phi}w_4$. Notice that $L(2w_3)$ defines the unstable lamination of the homeomorphism $\rho\tau\rho^{-1}|_{Y_3}$ (viewed as a lamination on S), which fills Y_3. Hence it has a positive intersection number with the curve γ_1 which is non-peripheral in Y_3. By Lemma 4.3.3 and the continuity of the intersection number, we can choose i large enough so that for all $j>i$, the intersection number between the curve γ_1 and the lamination defined by the vector $R_iR_{i+2}\dots R_{j-2}(2w_3)$ is bounded above and below from zero, where the bound is independent of i and j, as long as i is large enough. In other words, there exist a number $i_0\in\mathbb{N}$, so that for $i_0\leqslant i<j$, the intersection number $i(\gamma_i,\gamma_j)$ equals to $c_ic_{i+2}\dots c_{j-2}$ up to a fixed multiplicative constant. This proves the first comparison.

For the second comparison, we have

$$i(\gamma_{i-1},\gamma_j)=i(\Phi_{i-1}(\gamma_0),\Phi_{i-1}\varphi_{r_i}\dots\varphi_{r_j}(\gamma_0))=i(\gamma_0,\varphi_{r_i}\dots\varphi_{r_{j-1}}(\gamma_1)).$$

We can express the latter curve as a vector in $V(T)$ using Claim 4.3.4:

$$P_iP_{i+2}\dots P_{j-2}(2w_3)=(c_iL+c_{i-1}Q+T_i)c_{i+2}c_{i+4}\dots c_{j-2}R_{i+2}R_{i+4}\dots R_{j-2}(2w_3). \tag{4.21}$$

By Lemma 4.3.3, the matrix $R_{i+2}R_{i+4}\dots R_{j-2}$ can be approximated by the matrix L. Plugging it into Equation (4.21), we obtain

$$(c_iL+c_{i-1}Q+T_i)c_{i+2}c_{i+4}\dots c_{j-2}L(2w_3)=c_{i+2}c_{i+4}\dots c_{j-2}(c_iL+c_{i-1}QL+T_iL)(2w_3). \tag{4.22}$$

Notice that the lamination defined by the vector $L(2w_3)$ has zero intersection number with the curve γ_0 since they are supported in disjoint subsurfaces Y_3 and Y_2, respectively. Notice that $QL(2w_3) = \frac{\phi^3}{2}(w_1 + \frac{1}{\phi}w_2)$ defines the unstable lamination of the homeomorphism $\tau|_{Y_2}$ (viewed as a lamination on S), which fills Y_2. Hence it has positive intersection number with the curve γ_0. Since the matrices $T_i L$ have uniformly bounded entries by Claim 4.3.4, by the continuity of the intersection number we can choose i large enough, so that the intersection number between the lamination defined by Equation (4.22) and the curve γ_0 coarsely equals to $c_{i-1}c_{i+2}c_{i+4}\dots c_{j-2}$.

Next, we estimate the matrix norm error using Lemma 4.3.3:

$$\begin{aligned} &\|P_iP_{i+2}\dots P_{j-2} - c_{i+2}c_{i+4}\dots c_{j-2}P_iL\| \\ \leqslant\ & c_{i+2}c_{i+4}\dots c_{j-2}\cdot\|P_i\|\cdot\|R_{i+2}R_{i+4}\dots R_{j-2} - L\| \\ \overset{*}{\prec}_{18}\ & c_ic_{i+2}\dots c_{j-2}\cdot C(e^{C(\sum_{\ell=0}^{k}\|\varepsilon_{i+2+2\ell}\|)} - 1). \end{aligned} \tag{4.23}$$

By Equation (4.13), the ratio of the matrix norm error to the intersection number estimate is coarsely less than

$$\frac{c_ic_{i+2}\dots c_{j-2}\cdot(e^{C(\sum_{\ell=0}^{k}\|\varepsilon_{i+2+2\ell}\|)} - 1)}{c_{i-1}c_{i+2}c_{i+4}\dots c_{j-2}} = \frac{e^{C(\sum_{\ell=0}^{k}\|\varepsilon_{i+2+2\ell}\|)} - 1}{c_{i-1}/c_i} \overset{*}{\asymp}_{18} \frac{e^{C(\sum_{\ell=0}^{k}\|\varepsilon_{i+2+2\ell}\|)} - 1}{\|\varepsilon_i\|}.$$

By choosing sufficiently large K as in Lemma 4.3.5, we can guarantee that the error is negligible for all i large enough. Thus there exists a number $i_0 \in \mathbb{N}$, so that for $i_0 \leqslant i < j$, the intersection number $i(\gamma_{i-1}, \gamma_j)$ equals to $c_{i-1}c_{i+2}c_{i+4}\dots c_{j-2}$ up to a fixed multiplicative constant, which finishes the proof of the proposition. □

4.4 Non-unique ergodicity

In this section we show that the ending lamination λ constructed in Section 4.1 is not uniquely ergodic. Namely, we prove that there are two appropriately scaled subsequences of the sequence $\{\gamma_i\}$ with disjoint sets of indices that converge in $\mathcal{ML}(S)$ to non-zero laminations that project to two different points in $\mathbb{P}\mathcal{ML}(S)$. Further, we show that λ admits exactly two transverse ergodic measures up to scaling.

Let i_1 be an even number such that $i_1 > i_0$, where i_0 is the constant from Proposition 4.3.1. Let i_2 be an even number such that $i_2 > i_1$ and $\{\gamma_{i_1}, \gamma_{i_2}\}$ form a filling set of curves on S. Let $\mu_{\gamma_{i_{1,2}}}$ be the marking consisting of γ_{i_1} and γ_{i_2}.

Claim 4.4.1. The sequence $\{\gamma_{2n}/i(\gamma_{2n}, \mu_{\gamma_{i_{1,2}}})\}$ has a convergent subsequence with non-zero limit in $\mathcal{ML}(S)$. The same holds for the sequence $\{\gamma_{2n+1}/i(\gamma_{2n+1}, \mu_{\gamma_{i_{1,2}}})\}$.

Proof. We prove the claim for the first sequence, the other case is identical.

Notice that for every $n \in \mathbb{N}$:

$$i(\gamma_{2n}/i(\gamma_{2n}, \mu_{\gamma_{i_{1,2}}}), \mu_{\gamma_{i_{1,2}}}) = i(\gamma_{2n}, \mu_{\gamma_{i_{1,2}}})/i(\gamma_{2n}, \mu_{\gamma_{i_{1,2}}}) = 1.$$

Thus our sequence belongs to the following subset of geodesic currents:

$$\{\alpha \in \mathcal{GC}(S) \,|\, i(\alpha, \mu_{\gamma_{i_{1,2}}})) \leqslant 1\},$$

which satisfies the compactness criterion, since $\mu_{\gamma_{i_{1,2}}}$ is filling (see Prop. 8.2.25 in [29]). Since the space of geodesic currents is metrizable, our sequence has a convergent subsequence. It follows from the continuity of the intersection number that the limit belongs to $\mathcal{ML}(S)$ and also that it is non-zero. □

Let $\widehat{\lambda}, \widehat{\widehat{\lambda}} \in \mathcal{ML}(S)$ be the limits in Claim 4.4.1, respectively.

Claim 4.4.2. The following holds:

$$\frac{i(\gamma_{2n}, \widehat{\lambda})}{i(\gamma_{2n}, \widehat{\widehat{\lambda}})} \to \infty, \quad \frac{i(\gamma_{2n+1}, \widehat{\lambda})}{i(\gamma_{2n+1}, \widehat{\widehat{\lambda}})} \to 0.$$

Proof. Choose $2n > i_2$ and $k > n$. Then by Proposition 4.3.1 and since the sequence $\{r_{i+1} - r_i\}$ is strictly increasing[2]:

$$\begin{aligned}
\frac{i(\gamma_{2n}, \frac{\gamma_{2k}}{i(\gamma_{2k}, \mu_{\gamma_{i_{1,2}}})})}{i(\gamma_{2n}, \frac{\gamma_{2k+1}}{i(\gamma_{2k+1}, \mu_{\gamma_{i_{1,2}}})})} &= \frac{i(\gamma_{2n}, \gamma_{2k})}{i(\gamma_{2k}, \mu_{\gamma_{i_{1,2}}})} \cdot \frac{i(\gamma_{2k+1}, \mu_{\gamma_{i_{1,2}}})}{i(\gamma_{2n}, \gamma_{2k+1})} \\
&\overset{*}{\asymp} \frac{c_{2n}c_{2n+2}\cdots c_{2k-2}}{c_{i_2}c_{i_2+2}\cdots c_{2k-2}} \cdot \frac{c_{i_2}c_{i_2+3}c_{i_2+5}\cdots c_{2k-1}}{c_{2n}c_{2n+3}c_{2n+5}\cdots c_{2k-1}} \\
&= \frac{1}{c_{i_2}c_{i_2+2}\cdots c_{2n-2}} \cdot \frac{c_{i_2}c_{i_2+3}c_{i_2+5}\cdots c_{2n+1}}{c_{2n}} \\
&= \frac{c_{i_2+3}}{c_{i_2+2}} \cdot \frac{c_{i_2+5}}{c_{i_2+3}} \cdot \ldots \cdot \frac{c_{2n+1}}{c_{2n}} \\
&> \frac{c_{2n+1}}{c_{2n}} \overset{*}{\asymp} \phi^{2(r_{2n+2}-r_{2n+1})} \to \infty.
\end{aligned} \tag{4.24}$$

The above estimate holds for every $k > n$, thus also holds for $\widehat{\lambda}, \widehat{\widehat{\lambda}}$. For the second

[2]actually, the condition that $\{r_i\}$ is strictly increasing is sufficient here.

sequence, we similarly have:

$$\begin{aligned}
\frac{i(\gamma_{2n+1}, \frac{\gamma_{2k}}{i(\gamma_{2k},\mu_{\gamma_{i_{1,2}}})})}{i(\gamma_{2n+1}, \frac{\gamma_{2k+1}}{i(\gamma_{2k+1},\mu_{\gamma_{i_{1,2}}})})} &= \frac{i(\gamma_{2n+1}, \gamma_{2k})}{i(\gamma_{2k}, \mu_{\gamma_{i_{1,2}}})} \cdot \frac{i(\gamma_{2k+1}, \mu_{\gamma_{i_{1,2}}})}{i(\gamma_{2n+1}, \gamma_{2k+1})} \\
&\overset{*}{\asymp} \frac{c_{2n+1}c_{2n+4}c_{2n+6}\cdots c_{2k-2}}{c_{i_2}c_{i_2+2}\cdots c_{2k-2}} \cdot \frac{c_{i_2}c_{i_2+3}c_{i_2+5}\cdots c_{2k-1}}{c_{2n+1}c_{2n+3}\cdots c_{2k-1}} \\
&= \frac{c_{2n+1}}{c_{i_2}c_{i_2+2}\cdots c_{2n+2}} \cdot \frac{c_{i_2}c_{i_2+3}c_{i_2+5}\cdots c_{2n-1}}{1} \\
&< \frac{c_{i_2+3}}{c_{i_2+4}} \cdot \frac{c_{i_2+5}}{c_{i_2+6}} \cdot \ldots \cdot \frac{c_{2n+1}}{c_{2n+2}} \\
&< \frac{c_{2n+1}}{c_{2n+2}} \overset{*}{\asymp} \phi^{2(r_{2n+2}-r_{2n+3})} \to 0.
\end{aligned} \tag{4.25}$$

□

We have an immediate corollary:

Corollary 4.4.3. *The measured laminations $\widehat{\lambda}, \widehat{\widehat{\lambda}}$ are not multiples of each other, i.e. they project to different points in $\mathbb{P}\mathcal{ML}(S)$.*

Hence λ is not uniquely ergodic. Finally, we prove:

Proposition 4.4.4. *The measured laminations $\widehat{\lambda}$ and $\widehat{\widehat{\lambda}}$ define ergodic transverse measures on λ. Further, these are the only ergodic transverse measures on λ up to scaling.*

Proof. Endow the space $W(T)$ with the inner product such that the basis $\{w_1, \ldots, w_9\}$ is an orthonormal basis. We start with a lemma:

Lemma 4.4.5. *The columns of the matrices $P_iP_{i+2}\ldots P_{i+2k}, k \in \mathbb{N}$ can be put into two groups independently of $i, k \in \mathbb{N}$, such that the angles between the column vectors within each group decay at least exponentially as $k \to \infty$.*

Proof of Lemma 4.4.5. Let L and Q be the matrices introduced in Claim 4.3.2 and Claim 4.3.4, respectively. Notice that $LQ = 0$. Recall the decomposition of the the matrix P_i as the sum of the matrices

$$P_i = c_iL + c_{i-1}Q + T_i, \tag{4.26}$$

where T_i has uniformly bounded entries by Claim 4.3.4.

Let $E_1 = \{w_1, w_2, w_8\}, E_2 = \{w_3, w_4, w_5, w_6, w_7, w_9\}$ be two complementary subsets of basis vectors of $W(T)$. We accordingly divide the columns of the matrices $P_iP_{i+2}\ldots P_{i+2k}$ into two groups. First, we prove the lemma for the vectors in E_2. Let

$M = \max_i\{\|L\|, \|Q\|, \|T_i\|\}$, where $\|\cdot\|$ is the induced norm on $\mathrm{End}(W(T))$. Using Equation (4.26), we write

$$P_iP_{i+2}\dots P_{i+2k} = c_{i+2k}P_iP_{i+2}\dots P_{i+2k-2}L + P_iP_{i+2}\dots P_{i+2k-2}\left(c_{i+2k-1}Q + T_{i+2k}\right). \tag{4.27}$$

Notice that since L has rank 1 and every P_j is full rank, the images of the vectors in the set E_2 under the matrix $c_{i+2k}P_iP_{i+2}\dots P_{i+2k-2}L$ are collinear, non-zero and have comparable lengths. Notice that for every $w_\ell \in E_2$, $(L(w_\ell), w_3) \geqslant 1/2\phi$ and $(P_i(w_3), w_3) = c_i/2$ (see Claim 4.3.2 for the matrices L and P_i). Thus we have the following lower bound:

$$\|c_{i+2k}P_iP_{i+2}\dots P_{i+2k-2}L(w_\ell)\| \geqslant \frac{c_ic_{i+2}\cdots c_{i+2k}}{\phi\cdot 2^{k+1}} \tag{4.28}$$

for every $w_\ell \in E_2$. For the second term in Equation (4.27), we have the following upper bound:

$$\|P_iP_{i+2}\dots P_{i+2k-2}\left(c_{i+2k-1}Q + T_{i+2k}\right)(w_\ell)\| \leqslant 2\cdot 3^k\cdot M^{k+1}\cdot c_{i-1}c_{i+1}\dots c_{i+2k-1} \tag{4.29}$$

for every $w_\ell \in E_2$, using the equality $LQ = 0$ and since there are $2\cdot 3^k$ terms in the expansion obtained by applying Equation (4.26) such that each term has norm at most M^{k+1}. By Equation (4.13), we obtain an upper bound on the ratio of the right-hand sides in Equation (4.29) and Equation (4.28):

$$\frac{\phi\cdot\, 2^{k+2}\cdot 3^k\cdot M^{k+1}\cdot c_{i-1}c_{i+1}\cdots c_{i+2k-1}}{c_ic_{i+2}\cdots c_{i+2k}} < \frac{(12M)^{k+2}}{\phi^{2\sum_{j=i,j\equiv i \bmod 2}^{i+2k}(s_{j+1}-s_j)}}. \tag{4.30}$$

Since the sequence $\{s_{i+1} - s_i\}$ is strictly increasing, the right-hand side of Equation (4.30) decays at least exponentially with k, and this case of the lemma follows.

Similarly, we prove the lemma for the vectors in E_1. Notice that for every $w_\ell \in E_1$, $L(w_\ell) = 0$ and hence $c_{i+2k}P_iP_{i+2}\dots P_{i+2k-2}L(w_\ell) = 0$. Notice that for every $w_\ell \in E_1$, $(Q(w_\ell), w_1) \geqslant 1/2\phi$ and $(P_i(w_1), w_1) = c_{i-1}/2$. Thus we have

$$\|c_{i+2k-1}P_iP_{i+2}\dots P_{i+2k-2}Q(w_\ell)\| \geqslant \frac{c_{i-1}c_{i+1}\cdots c_{i+2k-1}}{\phi\cdot 2^{k-1}} \tag{4.31}$$

for every $w_\ell \in E_1$. We also similarly have $\|P_i\dots P_{i+2k-2}T_{i+2k}(w_\ell)\| \leqslant 3^k\cdot M^{k+1}\cdot c_ic_{i+2}\dots c_{i+2k-2}$ for every $w_\ell \in E_1$. Since the right-hand side of the expression

$$\frac{\phi\cdot 2^{k+1}\cdot 3^k\cdot M^{k+1}\cdot c_ic_{i+2}\cdots c_{i+2k-2}}{c_{i-1}c_{i+1}\cdots c_{i+2k-1}} < \frac{(12M)^{k+2}}{\phi^{2\sum_{j=i+2,j\equiv i \bmod 2}^{i+2k}(s_j-s_{j-1})}} \tag{4.32}$$

decays at least exponentially with k, the lemma follows.

□

Notice that the weight assignments to the branches in $B(T)$ of the bigon track T given by any transverse measure on the lamination λ belong to the cone $P_1P_3\dots P_{1+2k}(V(T))$ for every k. It follows from Lemma 4.4.5 that there are at most two ergodic transverse measures on λ up to scaling. Combining with Corollary 4.4.3, we conclude that there are exactly two ergodic transverse measures on λ up to scaling.

Writing $\widehat{\lambda}$ and $\widehat{\widehat{\lambda}}$ as weighted sums of these two ergodic measures, Claim 4.4.2 implies that each has zero weight on a different ergodic measure, implying that they are ergodic themselves. □

4.5 Relative twisting bounds

In this section we prove that the lamination λ constructed in Section 4.1 has uniformly bounded annular projection distances. To show this, we return to the sequence of curves $\{\alpha_i\}$, defined in Section 4.1.1.

We start with claims on the action of a partial pseudo-Anosov map on the Thurston boundary, and from them derive some bounds on the subsurface projections (see Section 2.0.6 for the notation).

Claim 4.5.1. A partial pseudo-Anosov map f acts cocompactly on the space $X = \mathbb{P}\mathcal{ML}(S) \setminus \Psi^u_f$, i.e. there is a compact set $K \subset X$ such that $\langle f\rangle \cdot K = X$.

Proof. Choose some hyperbolic metric on S. Consider the following continuous function on $\mathbb{P}\mathcal{ML}(S)$:

$$J^u(x) = \sum_{i=1}^{n} \frac{i(x,\mu^u_i)}{\ell(x)}.$$

Notice that $J^u(x) = 0$ if and only if $x \in \Psi^u_f$, i.e. X is precisely the set of all points in $\mathbb{P}\mathcal{ML}(S)$ where J^u does not vanish. Since $\mathbb{P}\mathcal{ML}(S)$ is compact, $J^u \leqslant M$ for some $M > 0$. Since $\Delta^s_f \subset X$ is compact, $J^u \geqslant \varepsilon$ on Δ^s_f for some $\varepsilon > 0$. Consider the preimage set $K_{\varepsilon/2} = (J^u)^{-1}([\frac{\varepsilon}{2}, M])$. It is closed in $\mathbb{P}\mathcal{ML}(S)$, thus compact. Further, $K_{\varepsilon/2} \subset X$. By Theorem 2.0.3, each $f-$orbit in X intersects $K_{\varepsilon/2}$, which proves the claim. □

We believe the following is true, however we couldn't find a proof for it. Luckily, this generality will not be necessary for our purposes.

Claim 4.5.2. The partial pseudo-Anosov map f acts cocompactly on the space $X = \mathbb{P}\mathcal{ML}(S) \setminus \{\Psi^u_f \cup \Psi^s_f\}$, i.e. there is a compact set $K \subset X$ such that $\langle f\rangle \cdot K = X$.

The following weaker claim will be sufficient for our needs.

Claim 4.5.3. A partial pseudo-Anosov map f supported on a connected subsurface $Y \subset S$ acts cocompactly on the space $X = \mathbb{P}\mathcal{ML}(Y) \setminus \{\Delta^u_f \cup \Delta^s_f\}$, i.e. there is a compact set $K \subset X$ such that $\langle f \rangle \cdot K = X$.

Proof. Notice that $\Delta^u_f = [\mu^u_f], \Delta^s_f = [\mu^s_f]$. Pick some metric on $\mathbb{P}\mathcal{ML}(Y)$. For $\varepsilon > 0$, let U^u_ε be the ε-neighborhood of $[\mu^u_f]$ and similarly define U^s_ε. Notice that $K_\varepsilon = \mathbb{P}\mathcal{ML}(Y) \setminus \{U^u_\varepsilon \cup U^s_\varepsilon\}$ is compact. Suppose that for every $\varepsilon > 0$ (small enough so that $U^u_\varepsilon \cap U^s_\varepsilon = \varnothing$), there is a point $x \in U^s_\varepsilon$, such that $f(x) \in U^u_\varepsilon$. By passing to the limit, it follows from the continuity of f that $f([\mu^s_f]) = [\mu^u_f]$, which is impossible. Combining with Theorem 2.0.3, it implies that there is some $\varepsilon' > 0$ such that every point $x \in U^s_{\varepsilon'}$, we have $f^n(x) \in K_{\varepsilon'}$ for some $n \in \mathbb{N}$. By repeating the same argument for U^u_ε, we obtain the claim. □

We prove the following corollary, that we will use in what follows. See also Lemma 7.4 in [37] for a similar result.

Corollary 4.5.4. *Let f be a partial pseudo-Anosov map supported on a connected subsurface $Y \subset S$. Let α and β be curves on S that intersect Y essentially. Then there is a constant E such that for every subsurface $W \subset Y$, which is not an annular subsurface with the core curve in ∂Y, we have*

$$d_W(f^{e_1}(\alpha), f^{e_2}(\beta)) \leqslant E, \tag{4.33}$$

for all $e_1, e_2 \in \mathbb{Z}$ such that $f^{e_1}(\alpha), f^{e_2}(\beta)$ intersect W essentially.

Proof. Since W is not an annular subsurface with the core curve in ∂Y, there is a curve γ in ∂W, such that $\gamma \notin \partial Y$. Let K be a compact set that satisfies Claim 4.5.3. By applying an appropriate power of f to Equation (4.33), we may assume that the projective class of some component γ of ∂W is in K (up to a change in powers e_1, e_2). Pick a hyperbolic metric on S. Suppose that there is a sequence of integers $\{n_i\}$ and curves γ_{n_i} on Y, such that there are connected closed arcs α_{n_i} of $f^{n_i}(\alpha)$ disjoint from γ_{n_i} whose lengths tend to infinity as $i \to \infty$. Then necessarily $\{n_i\} \to \infty$ or $\{n_i\} \to -\infty$. We consider the first case, the other one is similar. Since α intersects Y essentially, we have $\lim_{i\to\infty}[f^{n_i}(\alpha)] \in \Delta^u_f$ by Theorem 2.0.3. In our setting, Δ^u_f consists of a single point $[\mu^u_f]$. Since μ^u_f is minimal, the arcs α_{n_i} converge to the support of μ^u_f (as closed subsets of S). From the compactness of K, up to a further subsequence, the curves γ_{n_i} converge to a lamination ξ in Y with no transverse intersection with μ^u_f. Since μ^u_f is filling and uniquely ergodic, they coincide in $\mathbb{P}\mathcal{ML}(Y)$. This contradicts the fact that K is disjoint from $\{[\mu^u_f], [\mu^s_f]\}$. A uniform bound on the lengths of the arcs of $f^{e_1}(\alpha)$ in the complement $Y \setminus \partial W$ gives a uniform bound on the size of the projection to W. Applying the above argument to β finishes the proof. $\square$

Denote by $\mu_{\alpha_{i,j}}$ the collection of curves consisting of α_i and α_j. We prove the main result of the section.

Proposition 4.5.5. *There is a constant $D \in \mathbb{N}$, such that the relative twisting coefficients $d_\gamma(\mu_{\alpha_{D,D+5}}, \alpha_i)$ are uniformly bounded for $i \geqslant D+10$ and for all curves γ on S.*

Proof. It follows from the proof of Proposition 4.1.3 that there is a constant D, such that $D > B$ and for all $i, j \geqslant D$ with $|i-j| \geqslant 5$, the curves α_i and α_j fill S. Depending on the distance from the curve γ to the path $\{\alpha_i\}_{i\geqslant D}$, we consider the following three cases:

Case 1: $d_S(\gamma, \{\alpha_i\}_{i\geqslant D}) \geqslant 2$**.**

By Corollary 2.0.5, the relative twisting coefficients $d_\gamma(\mu_{\alpha_{D,D+5}}, \alpha_i)$ are bounded by 2A.

Case 2: $d_S(\gamma, \{\alpha_i\}_{i\geqslant D}) = 1$**.**

We start with the following useful claims:

Claim 4.5.6. For each $i \geqslant B$, there is a unique curve γ on S such that $i(\gamma, \alpha_i) = i(\gamma, \alpha_{i+4}) = 0$.

Proof of Claim 4.5.6. It follows from the proof of Lemma 4.1.2 that there is a unique curve on S disjoint from α_0 and $\varphi_a(\alpha_3)$ for all sufficiently large $a \in \mathbb{N}$, and further this curve is independent of a. Notice that this curve is $\beta = \rho(\delta_0)$. Thus $\Phi_i(\beta)$ is a unique curve disjoint from $\alpha_i = \Phi_i(\alpha_0)$ and $\alpha_{i+4} = \Phi_i(\varphi_{r_{i+1}}\alpha_3)$ for $i \geqslant B$. $\square$

Claim 4.5.7. For each $i \geqslant B+1$, there is a unique curve γ on S such that

$$i(\gamma, \alpha_i) = i(\gamma, \alpha_{i+3}) = 0,\ i(\gamma, \alpha_{i-1}) > 0,\ i(\gamma, \alpha_{i+4}) > 0. \tag{4.34}$$

Proof of Claim 4.5.7. It is easy to see from Figure 4.1 that there are exactly three curves on S disjoint from α_0 and α_3. These curves are β, $\rho^{-1}(\beta)$ and $\rho^3(\beta)$. Thus the only curves that satisfy the first two equalities in Equation (4.34) are the curves $\Phi_i(\beta)$, $\Phi_i(\rho^{-1}(\beta)) = \Phi_{i-1}(\beta)$ and $\Phi_i(\rho^3(\beta))$. By Claim 4.5.6, neither of the first two curves satisfy both inequalities in Equation (4.34). By Claim 4.5.6, the curve $\Phi_i(\rho^3(\beta))$ satisfies both inequalities in Equation (4.34) for $i \geqslant B+1$. $\square$

Notice that the curve γ can not be distance 1 from curves α_i, α_j with $|i-j| \geqslant 5$. Let n be the maximum among all values $|i-j|$ such that γ is disjoint from the curves α_i, α_j. Depending on n, we consider five subcases:

Case 2.1: $n = 4$**.** Suppose γ is disjoint from the curves α_i and α_{i+4}. Using Claim 4.5.6, we can uniquely identify γ. By Corollary 2.0.5, we can localize the twisting and restrict to $d_\gamma(\alpha_{i-1}, \alpha_{i+5})$. We have

$$d_\gamma(\alpha_{i-1}, \alpha_{i+5}) = d_{\Phi_i(\beta)}(\alpha_{i-1}, \alpha_{i+5}) = d_\beta(\varphi_{r_i}^{-1}\alpha_0, \varphi_{r_{i+1}}\varphi_{r_{i+2}}\alpha_3). \tag{4.35}$$

Observe that we can rewrite the curve $\varphi_{r_{i+1}}\varphi_{r_{i+2}}\alpha_3$ as follows: $\varphi_{r_{i+1}}\varphi_{r_{i+2}}\alpha_3 = \tau^{r_{i+1}}\rho\tau^{r_{i+2}}\rho^{-1}(\rho^2\alpha_3)$, where $\rho\tau^{r_{i+2}}\rho^{-1}$ is supported on the subsurface Y_3. Since the post composition with a mapping class supported on a disjoint subsurface affects the annular subsurface projection distance by at most 4 when the boundary is disjoint from the core curve ([34], Section 3), we can eliminate $\tau^{r_{i+1}}$:

$$d_\beta(\varphi_{r_i}^{-1}\alpha_0, \varphi_{r_{i+1}}\varphi_{r_{i+2}}\alpha_3) \asymp_{1,8} d_\beta(\rho^{-1}\tau^{-r_i}\rho(\rho^{-1}\alpha_0), \rho\tau^{r_{i+2}}\rho^{-1}(\rho^2\alpha_3)). \tag{4.36}$$

As $i \to \infty$, the curves $\rho^{-1}\tau^{-r_i}\rho(\rho^{-1}\alpha_0), \rho\tau^{r_{i+2}}\rho^{-1}(\rho^2\alpha_3)$ converge in Hausdorff topology to the laminations that contain the stable and unstable laminations of the maps $\rho^{-1}\tau\rho$ and $\rho\tau\rho^{-1}$, respectively. These sublaminations, which we denote by ν_1, ν_2, are filling in the subsurfaces Y_1 and Y_3, thus they intersect the curve $\beta \subset Y_1 \cap Y_3$ essentially. Finally, by Lemma 2.0.6, for all i large enough the relative twisting $d_\beta(\rho^{-1}\tau^{-r_i}\rho(\rho^{-1}\alpha_0), \rho\tau^{r_{i+2}}\rho^{-1}(\rho^2\alpha_3))$ differs from the relative twisting $d_\beta(\nu_1, \nu_2)$ by at most 8. This gives the uniform bound.

Case 2.2: $n = 3$**.** Suppose γ is disjoint from the curves α_i and α_{i+3}. Using Claim 4.5.7, we can uniquely identify γ. By Corollary 2.0.5, we can localize the twisting and restrict to $d_\gamma(\alpha_{i-1}, \alpha_{i+4})$. We have

$$d_\gamma(\alpha_{i-1}, \alpha_{i+4}) = d_{\Phi_i(\rho^3(\beta))}(\alpha_{i-1}, \alpha_{i+4}) = d_{\rho^3(\beta)}(\varphi_{r_i}^{-1}\alpha_0, \varphi_{r_{i+1}}\alpha_3). \tag{4.37}$$

Using similar argument as in the previous Case 2.1, we can write

$$d_{\rho^3(\beta)}(\varphi_{r_i}^{-1}\alpha_0, \varphi_{r_{i+1}}\alpha_3) = d_{\rho^3(\beta)}(\rho^{-1}\tau^{-r_i}\rho(\rho^{-1}\alpha_0), \tau^{r_{i+1}}(\rho\alpha_3)) \asymp_{1,8} d_{\rho^3(\beta)}(\rho^{-1}\alpha_0, \tau^{r_{i+1}}(\rho\alpha_3)). \tag{4.38}$$

As $i \to \infty$, the curves $\tau^{r_{i+1}}(\rho\alpha_3)$ converge in Hausdorff topology to the lamination that contains the unstable lamination of the map τ, which we denote by ν_3. The lamination ν_3 fills the subsurface Y_2, thus intersects the curve $\rho^3(\beta) \subset Y_2$ essentially. Again, by Lemma 2.0.6, for all i large enough the relative twisting $d_{\rho^3(\beta)}(\rho^{-1}\alpha_0, \tau^{r_{i+1}}(\rho\alpha_3))$

differs from the relative twisting $d_\beta(\rho^{-1}\alpha_0, \nu_3)$ by at most 8. This gives the uniform bound.

Case 2.3: $n = 2$. Suppose that γ is disjoint from the curves α_i and α_{i+2}. It follows from Claim 4.5.6, Claim 4.5.7 and Figure 4.1 that γ can be any curve supported in the subsurface Y_{i+1}, except for $\Phi_{i-2}(\beta), \Phi_i(\beta)$ and α_{i+1}. Similarly, it is enough to bound the following relative twisting:

$$d_\gamma(\alpha_{i-1}, \alpha_{i+3}) = d_{\Phi_{i-1}^{-1}\gamma}(\alpha_0, \varphi_{r_i}\alpha_3) = d_{\Phi_{i-1}^{-1}\gamma}(\alpha_0, \tau^{r_i}(\rho\alpha_3)), \tag{4.39}$$

where $\Phi_{i-1}^{-1}\gamma$ is any curve in Y_2 except for $\rho^{-1}(\beta), \tau^{r_i}\rho(\beta)$ and α_2. In this case, Corollary 4.5.4 is applicable if we let f be τ, Y be Y_2, W be $\Phi_{i-1}^{-1}\gamma$, α be α_0, β be $\rho\alpha_3$, and $e_1 = 0, e_2 = r_i$.

Case 2.4: $n = 1$. If γ is disjoint from the curves α_i and α_{i+1}, then we can simply bound the relative twisting by the intersection number:

$$d_\gamma(\alpha_{i-1}, \alpha_{i+2}) \leqslant i(\alpha_{i-1}, \alpha_{i+2}) = 2.$$

Case 2.5: $n = 0$. If γ is disjoint from the curve α_i, then similarly

$$d_\gamma(\alpha_{i-1}, \alpha_{i+1}) \leqslant i(\alpha_{i-1}, \alpha_{i+1}) = 2.$$

Case 3: $d_S(\gamma, \{\alpha_i\}_{i \geqslant D}) = 0$.

Suppose $\gamma = \alpha_j$. The only curves in the path $\{\alpha_i\}$ disjoint from α_j are the curves α_{j-1} and α_{j+1}. Therefore we can restrict to $d_{\alpha_j}(\alpha_{j-2}, \alpha_{j+2})$. We have

$$d_{\alpha_j}(\alpha_{j-2}, \alpha_{j+2}) = d_{\alpha_2}(\alpha_0, \varphi_{r_{j-1}}\alpha_3) = d_{\alpha_2}(\alpha_0, \tau^{r_{j-1}}(\rho\alpha_3)). \tag{4.40}$$

To show that the relative twisting coefficients in Equation (4.40) are uniformly bounded, we use the definition of the map τ as the composition of two half-twists. Choose some marked complete hyperbolic metric X on S of finite volume, and let $\widetilde{\alpha_2}$ and $\widetilde{\rho\alpha_3}$ be intersecting (at point p) geodesic lifts in the universal cover $\widetilde{X} \cong \mathbb{H}^2$. Choose an orientation of $\widetilde{\rho\alpha_3}$, and let $\widetilde{\delta_0}$ be the first lift of δ_0 that $\widetilde{\rho\alpha_3}$ intersects after passing the point p. Choose a lift of the half-twist $\widetilde{H_{\delta_0}} : \mathbb{H}^2 \to \mathbb{H}^2$ that preserves the point p, and thus preserves the endpoints of the lifts $\widetilde{\alpha_2}$ and $\widetilde{\delta_0}$ (see Section 2 in [47] for more details on the lifts of Dehn twists). Let $\widetilde{\delta_1}$ be the first lift of δ_1 along $\widetilde{\delta_0}$ travelled in the left direction after the lift $\widetilde{\rho\alpha_3}$ intersects it. Orient $\widetilde{\delta_0}$ as described in the previous sentence, and orient $\widetilde{\delta_1}$ so that the concatenated oriented path along $\widetilde{\rho\alpha_3}, \widetilde{\delta_0}$ and $\widetilde{\delta_1}$ forms a zigzag. Let $\widetilde{H_{\delta_1}^{-1}} : \mathbb{H}^2 \to \mathbb{H}^2$ be a lift of the half-twist that preserves the point p too. Then the map $\widetilde{\tau} = \widetilde{H_{\delta_1}^{-1}}\widetilde{H_{\delta_0}}$ is a lift of the map τ. It follows that the image of the forward endpoint of $\widetilde{\rho\alpha_3}$ under the iterates of $\widetilde{\tau}$ belongs to the interval between the forward endpoints of the lifts $\widetilde{\delta_0}$ and $\widetilde{\delta_1}$ (draw a picture!). Repeat the same argument for the other endpoint of $\widetilde{\rho\alpha_3}$. It follows that the relative twisting coefficients are uniformly bounded. □

Together with Lemma 2.0.6, we obtain an immediate corollary.

Corollary 4.5.8. *Given a marking μ on S, there is a constant C such that $d_\alpha(\mu, \lambda) \leqslant C$ for all curves α on S.*

We remark that not all projection distances of λ are uniformly bounded.

Claim 4.5.9. Given a marking (or a curve) μ on S, for all sufficiently large i the following holds:

$$d_{Y_i}(\mu, \lambda) \overset{*}{\succ} r_{i-1}. \tag{4.41}$$

Proof. By Lemma 4.1.1 and Corollary 2.0.5, we have

$$d_{Y_i}(\mu_{\alpha_{D,D+5}}, \alpha_j) \geqslant \alpha\, r_{i-1} - 1 - 2A$$

for all $i \geqslant D+5$ and $j \geqslant i+2$. By Corollary 4.1.6 and Lemma 2.0.6, we have

$$d_{Y_i}(\mu_{\alpha_{D,D+5}}, \lambda) \geqslant \alpha\, r_{i-1} - 2A - 9$$

for all $i \geqslant D+5$. Finally, from the triangle inequality we have

$$d_{Y_i}(\mu, \lambda) \geqslant \alpha\, r_{i-1} - 2A - 9 - d_{Y_i}(\mu_{\alpha_{D,D+5}}, \mu) \geqslant \alpha\, r_{i-1} - 2A - 9 - i(\mu_{\alpha_{D,D+5}}, \mu)$$

for all $i \geqslant D+5$ (if μ is a curve, then possibly for a larger threshold). Since the sequence $\{r_i\}$ is strictly increasing, we can guarantee that $d_{Y_i}(\mu, \lambda) \geqslant \frac{\alpha}{2} r_{i-1}$ for all sufficiently large i. □

Corollary 4.5.10. *Suppose X_t is a Teichmüller geodesic such that λ is the support of its vertical foliation, and its horizontal foliation is given by a simple closed curve on S. Then the curves α_i get short along X_t for all sufficiently large i such that the minimal length of α_i along X_t is less than $\frac{1}{r_i}$, up to fixed multiplicative constant.*

Proof. The statement follows from Claim 4.5.9 and Theorem 6.1 in [45]. In particular, X_t does not stay in the thick part of the Teichmüller space. Moreover, it follows from Theorem 1.0.2 that X_t diverges in the moduli space as $t \to \infty$. □

4.6 Geodesics in the thick part

In this section we prove Theorem 1.0.1. We start with a lemma that relates the limit of a sequence in $\mathcal{T}(S)$ in the Thurston boundary and short curves along the sequence.

Lemma 4.6.1. *Let $X_n \in \mathcal{T}(S)$ be a sequence of hyperbolic metrics converging to $[\lambda]$ in the Thurston boundary, and let β_n be a curve in S such that $\ell_{\beta_n}(X_n) \leqslant B(S)$, where $B(S)$ is a Bers constant of S. Let $[\beta]$ be a limit point of the sequence $[\beta_n]$ in the Thurston boundary. Then $i(\lambda, \beta) = 0$.*

Proof. By definition, there is a sequence $\{c_n\}$ of positive numbers, such that $c_n X_n \to \lambda$ as geodesic currents. We have (Prop. 15 in [3]):

$$i(c_n X_n, c_n X_n) = c_n^2\, i(X_n, X_n) = c_n^2 \pi^2 |\chi(S)|.$$

By the continuity of the intersection number, $i(c_n X_n, c_n X_n) \to i(\lambda, \lambda) = 0$, since $\lambda \in \mathcal{ML}$. Hence $c_n^2 \to 0$, and in particular $c_n \to 0$. By definition, there is a sequence $\{b_n\}$ of non-negative numbers, such that $b_n \beta_n \to \beta$ as geodesic currents. Let γ be a filling collection of curves on S, then $i(\gamma, b_n\beta_n) = b_n i(\gamma, \beta_n) \geqslant b_n$. We also have $i(\gamma, b_n\beta_n) \to i(\gamma, \beta) < \infty$. Hence the sequence $\{b_n\}$ must be bounded from above, so suppose $b_n \leqslant N$. Then

$$i(c_n X_n, b_n \beta_n) = c_n b_n \ell_{\beta_n}(X_n) \leqslant c_n N B(S).$$

Since $i(c_n X_n, b_n \beta_n) \to i(\lambda, \beta)$, we obtain $i(\lambda, \beta) = 0$. □

Next, we prove:

Lemma 4.6.2. *Let $X_n, Y_n \in \mathcal{T}(S)$ be sequences of hyperbolic metrics converging to $[\lambda]$ and $[\mu]$ in the Thurston boundary, respectively. Suppose that α is a curve on S such that λ and μ intersect α essentially. Suppose that $\ell_\alpha(X_n), \ell_\alpha(Y_n) > B(S)$ for all large enough n. Then*

$$d_\alpha(X_n, Y_n) \stackrel{+}{\asymp} d_\alpha(\lambda, \mu)$$

for infinitely many values of n.

Proof. Since $\ell_\alpha(X_n) > B(S)$ for all large enough n, it follows from the definition of a Bers constant that each large enough n there exists a curve β_n on S such that β_n intersects α essentially and $\ell_{\beta_n}(X_n) \leqslant B(S)$. We show that the relative twisting coefficients $d_\alpha(X_n, \beta_n)$ are uniformly bounded. Let $\ell_n = \ell_\alpha(X_n)$. By the Collar Lemma ([13], Section 13.5), the w_n-neighborhood (collar) of α in X_n, where $\omega_n = \operatorname{arcsinh}\left(\frac{1}{\sinh(\ell_n/2)}\right) \stackrel{*}{\asymp}_4 e^{-\ell_n/2}$ is embedded in X_n. Consider an arc $\widehat{\beta}_n$ of β_n inside the collar of α in X_n with one endpoint on α and one endpoint on the boundary of the neighborhood. Since the collar is embedded, the length of $\widehat{\beta}_n$ in X_n is at most $B(S)$. From the right angle hyperbolic triangle with the hypotenuse $\widehat{\beta}_n$ and with one of the sides along α, we find a lower bound on the angle δ_n that $\widehat{\beta_n}$ makes with α in X_n:

$$\sin \delta_n \geqslant \frac{\sinh \omega_n}{\sinh(B(S))}. \tag{4.42}$$

Denote by L_n the length of the orthogonal projection of a lift of β_n on a lift of α in the universal cover of X_n, such that the lifts intersect at the angle δ_n. Then from the angle of parallelism formula we have $\cosh \frac{L_n}{2} \sin \delta_n = 1$. Then we find:

$$L_n \leqslant 2 \operatorname{arccosh} \frac{\sinh B(S)}{\sinh \omega_n} \stackrel{+}{\asymp} 2 \log(e^{\ell_n/2}) = \ell_n, \tag{4.43}$$

where the additive error is at most $2B(S) + 6 \log 2$. Finally, we estimate the relative twisting coefficients (see [34], Section 3):

$$d_\alpha(X_n, \beta_n) \stackrel{+}{\asymp}_2 \frac{L_n}{\ell_n} \stackrel{+}{\asymp}_3 1. \tag{4.44}$$

By Lemma 4.6.1, after passing to a subsequence, $[\beta_n]$ converges to a lamination β with no transverse intersection with λ. Since the curves β_n have uniformly bounded lengths for large enough n, the lamination β intersects α essentially. By repeating the same argument for Y_n and invoking Lemma 2.0.6, we obtain the claim. □

We are ready to prove a corollary of Theorem 2.0.9:

Corollary 4.6.3 (Bounded annular combinatorics implies cobounded)**.** *Let $\mathcal{G}(t), t \in \mathbb{R}$ be a stretch path in $\mathcal{T}(S)$ with the horocyclic foliation λ and the maximally stretched lamination μ, such that $\mathcal{G}(t) \to [\mu]$ as $t \to -\infty$. Suppose that λ and μ are minimal and filling. If there exists a number $K \in \mathbb{N}$ such that $d_\alpha(\mu, \lambda) < K$ for all curves α on S, then there exists $\varepsilon(K) > 0$ such that $\mathcal{G}(t)$ lies in the thick part $\mathcal{T}_\varepsilon(S)$ for all $t \in \mathbb{R}$.*

Proof. Suppose that there is a curve α on S that gets shorter than ε_0 along the geodesic $\mathcal{G}(t)$, where ε_0 is the constant in the statement of Theorem 2.0.9 — otherwise there is nothing to prove. Since $\mathcal{G}(t)$ is a stretch path and the maximally stretched lamination μ is filling, α interacts with μ. Let $[a, b]$ be the ε_0-active interval for α. Indeed, this interval in bounded: for example, if there is a sequence $t_i \to \infty$ such that $\ell_\alpha(\mathcal{G}(t_i)) \leqslant \varepsilon_0$, then by Lemma 4.6.1 we have $i(\alpha, \lambda) = 0$, which is impossible since λ is minimal and filling (we note that ε_0 is small enough, in particular less than Bers constant). Then it follows from Theorem 2.0.9 that $\ell_\alpha(\mathcal{G}(t)) \to \infty$ as $t \to \infty$. By a similar argument it can be shown that there are infinitely many numbers $m \in \mathbb{N}$ such that $\ell_\alpha(\mathcal{G}(-m)) > B(S)$. By choosing large enough n, so that the interval $[-n, n]$ contains the interval $[a, b]$ and Lemma 4.6.2 applies for $X_n = \mathcal{G}(-n), Y_n = \mathcal{G}(n)$, we conclude by combining Theorem 2.0.9 with the condition $d_\alpha(\mu, \lambda) < K$ that there is a lower bound on the minimal length of α along $\mathcal{G}(t)$ that depends only on K. $\square$

Finally, we prove the main result of the thesis.

Proof of Theorem 1.0.1. Let $[\lambda] \in \mathbb{P}\mathcal{ML}(S)$ be the projective class of some non-zero transverse measure on the lamination λ, constructed in Section 4.1 (as it was shown in Lemma 4.4.4, there is an interval of choices for $[\lambda]$). By the construction, λ is minimal and filling, and in Section 4.4 we have shown that λ is not uniquely ergodic. Let $\nu = \nu_+$ be the unstable foliation of some pseudo-Anosov map on S, and let $\widehat{\nu}_+$ be a completion of ν_+ obtained by adding finitely many leaves. Since ν_+ is uniquely ergodic, the set of measured foliations transverse to $\widehat{\nu}_+$ is $\mathcal{MF}(S) \setminus \{m\nu_+ \,|\, m > 0\}$, in particular it contains λ. Thus there is a point $X \in \mathcal{T}(S)$ such that $\mathcal{F}_{\widehat{\nu}_+}(X) = \lambda$ (see Section 2.0.9). Moreover, by Theorem 2.0.8, the stretch path $\text{stretch}(X, \widehat{\nu}_+, t)$ converges to $[\lambda]$ as $t \to \infty$ and to $[\nu_+]$ as $t \to -\infty$.

By Corollary 4.6.3, to prove that $\text{stretch}(X, \widehat{\nu}_+, t)$ stays in the thick part, it is sufficient to show that the relative twisting coefficients $d_\alpha(\nu_+, \lambda)$ are uniformly bounded for all curves α on S. Let μ be some marking on S. By the triangle inequality, we have

$$d_\alpha(\nu_+, \lambda) \leqslant d_\alpha(\nu_+, \mu) + d_\alpha(\mu, \lambda). \tag{4.45}$$

The uniform boundedness of the first term on the right hand side of Equation (4.45) follows from the proof of Proposition 2.0.7. The uniform boundedness of the second term follows from Corollary 4.5.8, which completes the proof. $\square$

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