Approximate Ramsey Methods in Functional Analysis

by

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Abstract

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We study various aspects of *approximate Ramsey theory* and its interactions with functional analysis. In particular, we consider approximate versions of the structural Ramsey property and the amalgamation property within the context of multi-seminormed spaces, Fréchet spaces and other related structures from functional analysis. Along the way, we develop the theory of Fraïssé limits of classes of finite-dimensional Fréchet spaces, and we prove a version of the Kechris-Pestov-Todorčević correspondence relating the approximate Ramsey property to the topological dynamics of the isometry groups of certain infinite-dimensional Fréchet spaces. Motivated by problems regarding the structural Ramsey theory of Banach spaces, we study various generalizations of the Dual Ramsey Theorem of Carlson and Simpson. Specifically, using techniques from the theory of topological Ramsey spaces we obtain versions of the Dual Ramsey Theorem where ω is replaced by an arbitrary countable ordinal. Moving toward block Ramsey theory, we prove an infinite-dimensional version of Gowers' approximate Ramsey theorem concerning the oscillation stability of $S(c_0)$, the unit sphere of the Banach space c_0. We then show that results of this form can be parametrized by products of infinitely many perfect sets of reals, and we use this result to obtain a parametrized version of Gowers' c_0 theorem.

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Contents

Chapter 1

Introduction

Ramsey's Theorem, originally obtained by F. P. Ramsey in 1929 as a lemma in order to prove a result in decidability theory, is the following remarkable result:

Theorem 1.0.1 (Ramsey [58])**.** *For all positive integers k, l, m there is an integer n such that for every l-colouring c of the set $[N]^k$ of all k-element subsets of some set N of cardinality n, there is $M \subseteq N$ of cardinality m such that c is constant when restricted to the set $[M]^k$ of all k-element subsets of M. In other words, $[M]^k$ is* monochromatic *for c.*

Above, an *l-colouring* c of a set X is simply a mapping $c : X \to l$. The conclusion of Ramsey's theorem is often denoted with the Erdős-Rado arrow notation: $n \to (m)^k_l$. The fact that such an integer n exists for any k, l, m is an immediate consequence of the so-called *Infinite Ramsey Theorem* (also proved in [58]) where one colours the set $[\omega]^k$ of all k-element subsets of ω:

Theorem 1.0.2 (Ramsey [58])**.** *For all positive integers k, l, and every l-colouring of $[\omega]^k$, there is an infinite $M \subseteq \omega$ such that $[M]^k$ is monochromatic.*

Using the above arrow notation, the Infinite Ramsey Theorem can be compactly expressed as

$$(\forall k)\,(\forall l)\,\omega \to (\omega)^k_l.$$

The Finite and Infinite Ramsey Theorems are, of course, the prototypical results in the field now known as *Ramsey theory*, which attempts to address questions of the following kind: Given a finite partition of a collection $\mathcal{R}$ of mathematical structures, how large of a subcollection $\mathcal{R}' \subseteq \mathcal{R}$ can be found in such a way that all members of $\mathcal{R}'$ belong to the same piece of the partition? It is often useful to frame such statements in terms of n-colourings of $\mathcal{R}$; one then looks for a *monochromatic* subcollection $\mathcal{R}'$ with certain desired properties.

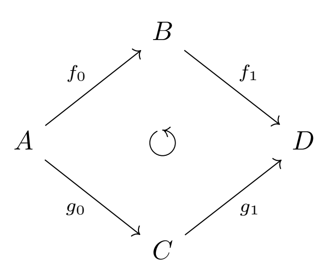
These two theorems have lead to many developments in finite and infinite combinatorics since Ramsey's original paper [58], and – together with their many variations, generalizations, and related techniques – have seen applications in areas of mathematics as diverse as Banach space theory [22, 23] and computability theory [10]. The extensions of Ramsey's theorem which are addressed in this thesis fall into two broad subcategories, each of which corresponds to a possible generalization of the Infinite Ramsey Theorem.

1.1 Structural Ramsey theory

The first subcategory of Ramsey theory which is featured in this thesis is rooted in *structural* Ramsey theory, which is studied in Chapter 2 and is concerned with colouring finite substructures of a given structure in the sense of first-order model theory, so that a *structure* is a set equipped with some collection of functions and relations. This structural framework for Ramsey theory was developed independently by Abramson and Harrington [1] and Nešetřil and Rödl [52, 53], and has since seen many applications and extensions. From this point of view, ω corresponds to a countable set with no further structure and the operation of taking a finite subset (of a fixed size) corresponds to taking a substructure. Thus, one can ask if versions of the Infinite Ramsey Theorem hold when ω and its substructures are replaced with a more complicated structure, such as a countably-infinite graph and its finite subgraphs, or a vector space of countably-infinite dimension over a finite field and its finitely-generated subspaces. It is useful to frame these properties in terms of *embeddings* between structures, i.e. injective mappings which preserve all functions and relations. In this approach, one says that a class $\mathcal{K}$ of finite structures has the *Ramsey property* if for any pair of structures $A, B \in \mathcal{K}$ and any number of colours r, there is a large enough $C \in \mathcal{K}$ such that the following holds: For any r-colouring of the set $\mathrm{Emb}(A, C)$ of embeddings of A into C (which one can view as all possible "copies" of A in C), there is a an embedding φ of B into C such that the colouring is constant when restricted to the set

$$\varphi \circ \mathrm{Emb}(A, B) := \{\varphi \circ \psi \in \mathrm{Emb}(A, C) : \psi \in \mathrm{Emb}(A, B)\}.$$

This approach is closely linked with model theory and in particular with *Fraïssé theory*, which allows one to take a "limit" of certain classes of finite (or finitely-generated) structures. To describe this in general, we say that a class $\mathcal{K}$ of structures has the *amalgamation property* if for any pair of embeddings $f_0 : A \to B$ and $g_0 : A \to C$ between objects in $\mathcal{K}$, there is $D \in \mathcal{K}$ together with embeddings $f_1 : B \to D$ and $g_1 : C \to D$ such that $f_1 \circ f_0 = g_1 \circ g_0$. The following commutative diagram summarizes this situation:



Assuming $\mathcal{K}$ is a *Fraïssé class* (which amounts to it satisfying a few more mild properties beyond amalgamation) the *Fraïssé limit* of the class $\mathcal{K}$ can be formed – this is the unique (up to isomorphism) countably infinite structure whose collection of finitely-generated substructures is exactly $\mathcal{K}$, and which, moreover, satisfies a property known as *ultrahomogeneity*: Every partial isomorphism between finitely-generated substructures can be extended to an automorphism of the entire structure. Thus, the automorphism group of an ultrahomogeneous structure is "rich" in a precise sense. The classical model-theoretic *Fraïssé correspondence* provides a connection between countable ultrahomogeneous structures and Fraïssé classes of finitely-generated first-order structures. Since Fraïssé's original paper [17] (see also [31]), many other versions of the Fraïssé correspondence have been developed; one of the more prominent instances of this is the development the Fraïssé theory of *metric structures* (i.e. metric spaces with additional compatible structure) due to Ben Yaacov [7]. Other presentations of Fraïssé theory in the context of structures from functional analysis, including "non-commutative" structures, have been developed recently in [16, 43]. When working with Fraïssé limits in the context of functional analysis, it is often helpful to work with notions of ultrahomogeneity which allow for an "error". The motivating example is due to Gurarij [26], who constructed a separable Banach space $\mathbb{G}$, now known as the *Gurarij space*, which is universal for separable Banach spaces and which is now known to satisfy the following additional property: For every finite-dimensional subspace $X \subseteq \mathbb{G}$, every $\varepsilon > 0$ and every isometric embedding $f : X \to \mathbb{G}$, there is a surjective linear isometry $g : \mathbb{G} \to \mathbb{G}$ such that $\|g(x) - f(x)\| \leq \varepsilon\|x\|$ for each $x \in X$. In other words, $\mathbb{G}$ is *approximately ultrahomogeneous.* Lusky [45] later showed that the Gurarij space is unique up to isometry, in the sense that any other space which is of almost universal disposition for all finite-dimensional Banach spaces must be isometrically isomorphic to $\mathbb{G}$. (A simpler proof of the uniqueness and universality of the $\mathbb{G}$ can be found in [41].)

Interest the structural Ramsey property has recently been renewed due to a newly-discovered connection between structural Ramsey theory and the topological dynamics of certain automorphism groups. The topological dynamical property which is relevant here is that of *extreme amenability*, which can be seen as a strong fixed-point property of topological groups. Precisely, a topological group G is extremely amenable if, for every continuous action of G on a compact Hausdorff space X, there is $x \in X$ such that $g \cdot x = x$ for all $g \in G$. *A priori*, this property seems rare and initially few examples of extremely amenable groups were known. Important early examples of such groups included the unitary group of the infinite-dimensional separable Hilbert space, as discovered by Gromov and Milman [25]. This motivated the following question:

Problem 1.1.1. Find naturally occurring examples of extremely amenable groups and investigate the corresponding Ramsey-theoretic and topological dynamical phenomena.

The aforementioned connection between structural Ramsey theory and topological dynamics, known as the *Kechris-Pestov-Todorčević correspondence* (henceforth referred to as the KPT correspondence), provides a plentiful source of natural examples of extremely amenable groups:

Theorem 1.1.2 (Kechris-Pestov-Todorčević [38])**.** *Let $\mathcal{K}$ be a Fraïssé class, let $\mathbb{K}$ be its Fraïssé limit, and let G be the automorphism group of $\mathbb{K}$ equipped with the topology of pointwise convergence. Then G is extremely amenable if and only if $\mathcal{K}$ has the Ramsey property.*

Part of the recent renaissance in structural Ramsey theory is motivated by the discovery of principles similar to the KPT correspondence in contexts which lie outside the domain of first-order model theory – in particular, to structures which come from functional analysis – and hence obtaining new examples of extremely amenable groups. For instance, Melleray and Tsankov [48] extended the KPT correspondence to the setting of *metric Fraïssé structures*; the theory of such structures was developed by Ben Yaacov [7] in the context of *continuous logic*, in which one replaces the equality symbol with a metric. These developments spurred an interest in *approximate* Ramsey properties, i.e. structural Ramsey properties which hold up to an arbitrarily small error. Recent successful endeavours in this direction were achieved in [6] and [16] where the authors prove an approximate Ramsey property for the class of all finite-dimensional Banach spaces and for the finite-dimensional ℓ_p spaces, respectively, and then use such results to prove extreme amenability of certain automorphism groups. Interestingly, the former result makes crucial use of the Dual Ramsey Theorem of Graham and Rothschild [24] (which will be discussed in Chapter 3). The ARP for the class of all finite-dimensional Banach spaces was then used to show that the group of surjective linear isometries of $\mathbb{G}$ is extremely amenable. In this thesis, this approach will be applied to study classes of finite-dimensional *multi-seminormed spaces*, i.e. vector spaces equipped with a finite or infinite sequence of seminorms. Part of the motivation for studying these spaces is their close relation to Fréchet spaces, i.e. locally convex topological vector spaces which are completely metrizable via a translation-invariant metric. After defining an appropriate notion of a Fraïssé class, we prove the following result in Chapter 2.

Theorem 1.1.3 (Fraïssé correspondence). *Let $\mathcal{K}$ be a class of finite-dimensional multi-seminormed spaces such that each $X \in \mathcal{K}$ is equipped with a finite sequence of seminorms. The following are equivalent:*

(1) $\mathcal{K}$ is a Fraïssé class.

(2) There is a unique separable Fraïssé Fréchet space E such that $\mathcal{K}$ is precisely (up to isometries) the collection of all linear subspaces of E equipped with the first n seminorms from E, for various $n \in \mathbb{N}$.

This result then motivates the development of the KPT correspondence in the setting of multi-seminormed spaces, which is also obtained in Chapter 2 after defining an appropriate version of the approximate Ramsey property (ARP).

Theorem 1.1.4 (Kechris-Pestov-Todorčević correspondence). *Suppose E is an infinite-dimensional multi-seminormed space which is approximately ultrahomogeneous. The following are equivalent:*

(i) For each n, the collection of all linear subspaces of E equipped with the first n seminorms from E has the ARP.

(ii) The group of all surjective seminorm-preserving linear mappings $g : E \to E$ is extremely amenable when endowed with the topology of pointwise convergence.

Naturally, the next course of action is to look for examples of classes of finite-dimensional multi-seminormed spaces with the ARP. It turns out that classes of normed spaces with the ARP give rise to many such examples. Of particular interest is the ARP for the class $\mathcal{M}_n$ of all finite-dimensional Fréchet spaces with a sequence of seminorms of length n; the following result is established and used to obtain new examples of extremely amenable groups.

Theorem 1.1.5. *For every $n \in \omega$, every $X, Y \in \mathcal{M}_n$, every number of colours r, and every $\varepsilon > 0$, there is a sufficiently large $Z \in \mathcal{M}_n$ such that every r-colouring of $\mathrm{Emb}(X, Z)$ has an ε-*monochromatic *set of the form $\varphi \circ \mathrm{Emb}(X, Y)$ for some $\varphi \in \mathrm{Emb}(Y, Z)$. In other words, there is a colour $j \in \{1, \ldots, r\}$ such that every embedding of the form $\varphi \circ \psi$ for $\psi \in \mathrm{Emb}(X, Y)$ is ε-close to an embedding with colour j.*

The notion of nearness in this setting is determined by a collection of operator norms associated to each seminorm and will be defined in Chapter 2. These results are then used to prove extreme amenability of the group of linear self-isometries of certain infinite-dimensional separable Fréchet spaces – including those which arise as Fraïssé limits of classes of finite-dimensional Fréchet spaces – thus obtaining new examples of extremely amenable topological groups.

1.2 Dual Ramsey theory

The second subcategory of Ramsey theory featured below is *infinite-dimensional* Ramsey theory, studied in Chapters 3 and 4. The development of this field was motivated by a question of Dana Scott, who asked the following during a seminar at Stanford in the 1960s:

(∗) Given an arbitrary finite colouring of the set $[\omega]^\omega$ of infinite subsets of ω, is there an infinite subset X such that all further infinite subsets of X have the same colour?

If we view the integer k in the statement of the Infinite Ramsey Theorem as the *dimension* of the objects being coloured, then this question can be viewed as a statement about an *infinite-dimensional version* of the Infinite Ramsey Theorem. As is well known, it turns out that such an extension is false – one can construct colourings of $[\omega]^\omega$ which act as counterexamples to (∗). However, the construction of any such counterexample makes essential use of the Axiom of Choice and so they are never Borel in the natural topology on $[\omega]^\omega$. Thus, the problem then shifts to determining which types of colourings are admissible in the sense that the statement (∗) becomes true when restricted to such colourings. Infinite-dimensional Ramsey theory, then, studies such conditions together with other examples beyond that of colourings of $[\omega]^\omega$. Galvin and Prikry [19] showed that such a result holds for Borel colourings of $[\omega]^\omega$. Silver [63] extended this to analytic colourings; Ellentuck [15] later obtained a complete characterization using a finer topology known as the *Ellentuck topology*. Broad generalizations of this phenomenon (with $[\omega]^\omega$ replaced by more general spaces) have been obtained using the general theory of *topological Ramsey spaces*, which are spaces on which one can develop Ramsey-type results and which admit an "optimal" version of the infinite-dimensional Ramsey theorem. The study of such spaces was initiated by Carlson and Simpson [12, 14] and eventually fully developed by Todorčević [65]. Part of the contributions of this thesis consist of finding examples of topological Ramsey spaces via the framework from [65] as well as extending this framework in various ways in order to obtain new examples of infinite-dimensional Ramsey-type results.

The main focus of Chapter 3 is *dual* Ramsey theory, which is concerned with colouring spaces of equivalence relations on a given set. One of the most profound generalizations of the Infinite Ramsey Theorem is due to Carlson and Simpson [13] who proved a *dual* version of Ramsey's theorem, henceforth referred to as the Dual Ramsey Theorem. In this realm, the space of k-element subsets of ω in the statement of the Infinite Ramsey Theorem is replaced with the space of all equivalence relations on ω with k equivalence classes. To precisely state the Carlson-Simpson theorem, we first establish some notation: Given $k < \omega$, let $\mathcal{E}_k$ be the set of all equivalence relations on ω with exactly k-many equivalence classes. We also let $\mathcal{E}_\infty$ be the set of all equivalence relations on ω with infinitely many equivalence classes. Given $X \in \mathcal{E}_\infty$ and $\alpha \in \omega \cup \{\infty\}$, we write $\mathcal{E}_\alpha \upharpoonright X$ for the set of all $Y \in \mathcal{E}_\alpha$ which are *coarsenings* of X in the sense that each equivalence class of Y is a union of equivalence classes of X. Carlson and Simpson proved the following results for Borel colourings, but we state them in their stronger forms as found in [65, Chapter 5.6].

Theorem 1.2.1 (Dual Ramsey Theorem)**.** *For every finite Souslin measurable colouring c of $\mathcal{E}_k$, there is $X \in \mathcal{E}_\infty$ such that c is constant on $\mathcal{E}_k \upharpoonright X$.*

Theorem 1.2.2 (Dual Silver Theorem)**.** *For every finite Souslin measurable colouring c of $\mathcal{E}_\infty$, there is $X \in \mathcal{E}_\infty$ such that c is constant on $\mathcal{E}_\infty \upharpoonright X$.*

The notion of Souslin measurability will be defined in Chapter 3. A natural question is whether or not dual Ramsey-type results exist for partitions of other structures in place of the natural numbers:

Problem 1.2.3. Find and study the more general frameworks that support the corresponding versions of the Dual Ramsey Theorem.

It turns out that it is possible to extend the Dual Ramsey Theorem to an arbitrary countable ordinal α in place of the first countable ordinal, which is identified with ω in the Carlson-Simpson theorem. It turns out that, while the result is false if one considers all possible equivalence relations on α, there is an optimal collection of "allowable" equivalence relations on α for which the Dual Ramsey Theorem holds, in the sense that all conditions in the definition of the allowable equivalence relations are necessary. The following is the main result of Chapter 3:

Theorem 1.2.4. *For every countable ordinal α there is an optimal class of equivalence relations on α, denoted $\mathcal{E}(\alpha)$, such that the following holds: For every Souslin measurable finite colouring c of $\mathcal{E}(\alpha)$, there is an equivalence relation E on α such that the family of all coarsenings of E with exactly k equivalence classes is monochromatic for c.*

Precise definitions are given in Chapter 3. The proof of this result makes use of the general theory of topological Ramsey spaces and proceeds by constructing a space of equivalence relations on ω which code certain equivalence relations on a given ordinal α.

1.3 Block Ramsey theory

Another aspect of infinite-dimensional Ramsey theory which arises in this thesis is *block* Ramsey theory, in which one studies Ramsey-theoretic properties of infinite block sequences of vectors in normed spaces. Block sequences are often useful for understanding the geometry of a separable Banach space with a Schauder basis, as passing to a subspace spanned by a block sequence often captures many aspects of the geometry of such a space; this approach is, for instance, applied to the problem of oscillation stability. In Chapter 4 we study the block Ramsey theory of space $\mathrm{FIN}_{\pm k}$ of all functions

$$p : \omega \to \{0, \pm 1, \ldots, \pm k\}$$

such that the support $\mathrm{supp}(p)$ of p is finite, and at least one of $\pm k$ is included in the range of p. These spaces are extensions of the space of finite subsets of ω and, in fact, act as nets of the unit sphere of c_0, the Banach space of all real sequences converging to 0 equipped with the supremum norm $\|\cdot\|_\infty$. In this setting, vector addition in c_0 corresponds to the coordinate-wise sum $p+q$ of disjointly supported elements of $\mathrm{FIN}_{\pm k}$, while scalar multiplication is captured by the mapping $T : \mathrm{FIN}_{\pm k} \to \mathrm{FIN}_{\pm(k-1)}$ defined by

$$T(p)(n) := \begin{cases} p(n) - 1 & \text{if } p(n) > 0, \\ 0 & \text{if } p(n) = 0, \\ p(n) + 1 & \text{if } p(n) < 0. \end{cases}$$

Gowers [22] showed that this space supports an "approximate" Ramsey theorem: For every r-colouring c of $\mathrm{FIN}_{\pm k}$, there is a sequence $P = (p_n)_{n\in\omega}$ of elements of $\mathrm{FIN}_{\pm k}$ such that the subspace generated by P, defined by

$$\begin{aligned}[P]_{\pm k} :=& \{\varepsilon_0 T^{j_0}(p_{n_0}) + \cdots + \varepsilon_m T^{j_m}(p_{n_m}) : m \in \omega, n_0 < \cdots < n_m < \alpha, \\ & \varepsilon_0, \ldots, \varepsilon_m \in \{\pm 1\}, j_0, \ldots, j_m < k \text{ and } \min j_i = 0\},\end{aligned}$$

is *almost* monochromatic for c: There is a colour $j \in \{1, \ldots, n\}$ such that every element of the above subspace has distance at most 1 from an element with colour j. Furthermore, the sequence P is a *block sequence* in the sense that $\max \operatorname{supp}(p_n) < \min \operatorname{supp}(p_m)$ whenever $n < m$. This result can be seen as a generalization of Hindman's finite sums theorem [29], which states that for any finite colouring of the set FIN of finite subsets of ω, there is an infinite block sequence B such that the set of all finite unions of elements of B is monochromatic. Interestingly, Gowers' motivation for extending Hindman's theorem comes from the geometry of Banach spaces. As it turns out, Gowers' theorem implies that the unit sphere $S(c_0)$ of c_0 is *oscillation stable*: Every real-valued Lipschitz function on $S(c_0)$ becomes constant – up to a specified error – when restricted to an appropriate subspace of c_0. This result was then used by Odell and Schlumprecht [54] to solve the famous *distortion problem* in Banach space theory. An exact version of Gowers' theorem exists for FIN_k, the set of all finitely-supported functions $p : \omega \to \{0, 1, \ldots, k\}$ such that $k \in \operatorname{range}(p)$; this can be used to prove an oscillation stability result for Lipschitz functions on the positive part of the unit sphere of c_0.

One of the main contributions of this thesis is a proof of a natural infinite-dimensional version of Gowers' $\mathrm{FIN}_{\pm k}$ theorem. Such an analogue cannot be obtained by standard methods of topological Ramsey spaces, and so the proof uses techniques from the theory of ultrafilters. This result appears in Chapter 4 and it exhibits the first example of an approximate infinite-dimensional Ramsey result:

Theorem 1.3.1. *For every Souslin measurable r-colouring of the space $\mathrm{FIN}_{\pm k}^{[\infty]}$ of all infinite block sequences, there are $B \in \mathrm{FIN}_{\pm k}^{[\infty]}$ and $j \in \{1, \ldots, r\}$ such that the following holds: For every block sequence $A = (a_n)_{n \in \omega}$ with the property that $a_n \in [B]_{\pm k}$ for each n, there is $\widetilde{A} = (\widetilde{a}_n)_{n < \omega} \in \mathrm{FIN}_{\pm k}^{[\infty]}$ such that*

$$c(\widetilde{A}) = j \text{ and } ||a_n - \widetilde{a}_n||_\infty \leq 1 \text{ for all } n.$$

Hindman's theorem was extended in another direction by Milliken [51] who proved an *infinite-dimensional* version of Hindman's theorem: For every analytic colouring of the set $\mathrm{FIN}^{[\infty]}$ of all infinite block sequences of finite subsets of ω, there is an infinite block sequence B such that the set

$$\{C \in \mathrm{FIN}^{[\infty]} : \text{ every } X \in C \text{ is a union of sets from } B\}$$

is monochromatic. Todorčević then showed [65] that Milliken's theorem also holds for FIN_k in place of FIN. Given such an infinite-dimensional result, one natural way to strengthen it is to *parametrize* it by some space of interest. One line of research in this direction is to parametrize such results by perfect subsets of the Cantor space 2^ω with its standard metrizable topology, in the sense that we colour a product of the form $X \times 2^\omega$ and look for monochromatic subsets of the form $Y \times P$ where Y is a "nice" subset of X and P is a perfect subset of 2^ω. The first result of this kind is due to Miller and Todorčević [50, p. 183] and involves a parametrization of the Galvin-Prikry theorem [19]; Pawlikowski [56] later showed that Ellentuck's theorem [15] can be parametrized by perfects subsets of 2^ω. The result of Pawlikowski was then generalized by Mijares and Nieto [49] who attempted to prove that the abstract Ramsey theorem of Todorčević [65] can be parametrized as above.

Instead of asking for a parametrization involving one perfect subset of 2^ω, one can look for a *sequence* of perfect subsets of 2^ω. For instance, Milliken's theorem was parametrized by Todorčević [65, Theorem 5.45] using sequences of perfect subsets of 2^ω in the following way; the proof makes use of an infinite-dimensional version of the Hales-Jewett theorem [65, Corollary 5.42].

Theorem 1.3.2 (Parametrized Milliken Theorem)**.** *For every finite Souslin measurable colouring of* $\mathrm{FIN} \times (2^\omega)^\omega$ *there are* $B \in \mathrm{FIN}^{[\infty]}$ *and a sequence* $(P_i)_{i<\omega}$ *of non-empty perfect subsets of* 2^ω *such that* $[B]^{[\infty]} \times \prod_{i<\omega} P_i$ *is monochromatic.*

Results of this form have been applied, for example, to give proofs of results in Banach space theory, including some famous results of Rosenthal [60, 61]. More generally, one can ask which spaces admit a parametrization using sequences of perfect subsets of 2^ω. In her thesis, Zheng [68] isolated a necessary and sufficient condition for the existence of such a parametrization within the context of topological Ramsey space theory; we refer the reader there for more information and for applications of parametrized Ramsey theory. The second main contribution of Chapter 4 is the following pair of results:

Theorem 1.3.3 (Parametrized Milliken-Todorčević Theorem)**.** *For every finite Souslin measurable colouring of* $\mathrm{FIN}_k^{[\infty]} \times (2^\omega)^\omega$ *there are* $B \in \mathrm{FIN}_k^{[\infty]}$ *and a sequence* $(P_i)_{i<\omega}$ *of non-empty perfect subsets of* 2^ω *such that* $[B]_k^{[\infty]} \times \prod_{i<\omega} P_i$ *is monochromatic.*

Theorem 1.3.4. *For every Souslin measurable* r*-colouring of* $\mathrm{FIN}_{\pm k}^{[\infty]} \times \mathbb{R}^\omega$*, there are* $B \in \mathrm{FIN}_{\pm k}^{[\infty]}$*, a sequence* $(P_i)_{i\in\omega}$ *of non-empty perfect sets of reals, and* $j \in \{1, \ldots, r\}$ *such that the following holds: For every block sequence* $(a_n)_{n\in\omega}$ *with* $a_n \in [B]_{\pm k}$ *for each* n *and every sequence* $(p_i)_{i\in\omega} \in \prod_{i\in\omega}^{\infty} P_i$*, there is* $(\widetilde{a}_n)_{n\in\omega} \in \mathrm{FIN}_{\pm k}$ *such that*

$$c((\widetilde{a}_n), (p_i)) = j \quad \textit{and} \quad \max_{n\in\omega} ||a_n - \widetilde{a}_n||_\infty \leq 1.$$

1.4 Layout

We now give a brief summary of the contents of each chapter. Chapter 2 is concerned with the development of the Fraïssé and KPT correspondences for Fréchet spaces. The chapter begins with the relevant preliminaries regarding Fraïssé theory and the Kechris-Pestov-Todorčević correspondence in the setting of Banach spaces. We then consider amalgamation and Fraïssé properties of finite-dimensional multi-seminormed spaces. Among other examples, we show that the class of all finite-dimensional vector spaces equipped with a finite sequence of seminorms is Fraïssé. We then we develop a notion of a *Fraïssé Fréchet space* and we obtain examples of such Fréchet spaces by taking *Fraïssé limits* of classes of certain finite-dimensional multi-seminormed spaces. We also show that the separable Fréchet spaces of almost universal disposition constructed in [4] can be realized as Fraïssé limits. An analogue of the KPT correspondence for multi-seminormed spaces is then established; this involves isolating a useful version of the approximate Ramsey property for such spaces. The chapter concludes by applying this correspondence to obtain new examples of extremely amenable groups. Chapter 3 contains proofs of generalizations of the Dual Ramsey Theorem to arbitrary countable ordinals. After giving an overview of the theory of topological Ramsey spaces, we define a Ramsey space of equivalence relations on ordinals of the form $\omega \cdot n$ for $n < \omega$; the proof uses the left-variable Hales-Jewett theorem. This is then used to obtain a Dual Ramsey Theorem for ordinals of the form $\omega \cdot n$. We then prove a version of the left-variable Hales-Jewett theorem for infinite alphabets before extending the aforementioned results to arbitrary countable ordinals. Chapter 4 begins with a review of Gowers' theorems as well as the necessary techniques from the theory of ultrafilters. This is then used to give a proof of an infinite-dimensional version of the $\mathrm{FIN}_{\pm k}$ theorem. The chapter then shifts its focus to *parametrized* Ramsey theory; in particular, we show that the latter result (as well as the associated infinite-dimensional result concerning FIN_k) can be strengthened to a parametrized version.

Most of the material in Chapter 2 is based on joint work with Jordi López-Abad [36]. Chapter 3 is based on joint work with Stevo Todorčević [37]. Chapter 4 incorporates material from the published work [34] [1] as well as the paper [35]. [2]

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[2]Accepted for publication in *Ann. Pure Appl. Logic.*

Chapter 2

Approximate Ramsey properties of Fréchet spaces

In this chapter we study Ramsey-theoretic properties of *Fréchet spaces*, which are locally convex topological vector space whose topology is metrizable by a complete, translation-invariant metric. More concretely, Fréchet spaces can be viewed as topological vector spaces with a compatible sequence of seminorms. Fréchet spaces (in both of the previous senses) arise naturally in many areas of functional analysis and partial differential equations; examples include Banach spaces, spaces of infinitely-differentiable functions, spaces of holomorphic functions, and many other objects from analysis. Our main goal in this chapter is to extend the results of Ferenczi, López-Abad, Mbombo and Todorčević [16] (which primarily deals with Banach spaces) to spaces equipped with a finite or infinite sequence of seminorms and in particular to Fréchet spaces with strong universality and homogeneity properties. In particular, we develop Fraïssé theory for multi-seminormed spaces together with the relevant versions of the KPT correspondence and the approximate Ramsey property. A similar approach was used in [5] to study different classes of exact operator spaces, where a sequence of norms is also an essential part of the structure. In the present chapter we provide a Fraïssé-theoretic proof of the existence of the spaces constructed in [4], which were originally defined using properties of a universal (Fraïssé) operator on $\mathbb{G}$ originally considered in [20]. Such an operator can be seen as a "Gurarij" version of the Rota universal operator on the separable infinite-dimensional Hilbert space (for which we refer the reader to, e.g., [43, Section 4.1]). In addition to the known examples discussed above, we also obtain many new examples of Fréchet spaces with strong forms of approximate ultrahomogeneity, including examples which naturally arise from various combinations of Hilbert spaces and L_p spaces. As a result, we obtain many new examples of extremely amenable groups; in particular, we show that the group $\mathrm{Iso}(\mathbb{G}^\omega, (\|\cdot\|_n)_{n<\omega})$ of all surjective linear seminorm-preserving isometries of $(\mathbb{G}^\omega, (\|\cdot\|_n)_{n<\omega})$ is extremely amenable.

2.1 Preliminaries

2.1.1 Fraïssé theory for Banach spaces

In this section we will review some concepts from the Fraïssé theory of Banach spaces developed in [16], which is (in a precise sense) a more refined version of the general theory of metric Fraïssé structures as in [7]. Throughout the chapter, we consider topological vector spaces over $\mathbb{R}$ or $\mathbb{C}$. Given two normed spaces X, Y and $\varepsilon \geq 0$, let $\mathrm{Emb}_\varepsilon(X, Y)$ denote the set of all *ε-isometric embeddings* from X into Y, i.e. linear mappings $f : X \to Y$ such that

$$\frac{1}{1+\varepsilon}\|x\| \leq \|f(x)\| \leq (1+\varepsilon)\|x\|$$

for each $x \in X$. When $\varepsilon = 0$, we simply refer to such a mapping as an *isometric embedding* and we denote the collection of all such mappings by $\mathrm{Emb}(X, Y)$. If there is a surjective linear isometry from X to Y, we say that X and Y are *isometric.* We let $\mathrm{Iso}(X)$ denote the set of all surjective linear isometries from X to itself and we equip it with the corresponding strong operator topology, i.e. the topology of pointwise convergence.

It is a basic fact of linear algebra that any two finite-dimensional normed spaces are isomorphic, in the sense that there is a linear bijection mapping on space onto the other. Thus, in order to obtain a more refined classification of finite-dimensional normed spaces for the same dimension, we work with the *Banach-Mazur pseudometric* on the space of all n-dimensional normed spaces defined by

$$d_{\mathrm{BM}}(X, Y) := \log\left(\inf\{\|T\| \cdot \|T^{-1}\| : \ T \text{ is an isomorphism from } X \text{ to } Y\}\right)$$

where $\|T\|$ is the *operator norm* of the linear mapping T between normed spaces, defined by $\|T\| := \sup\{\|T(x)\|_Y : \|x\|_X = 1, x \in X\}$. Observe that X and Y are isometric precisely when $d_{\mathrm{BM}}(X, Y) = 0$.

We now introduce the notion of a *Fraïssé Banach space*, which is one of the main concepts of [16] and which captures a strong from of the concept of (approximate) ultrahomogeneity encountered in [7, 48].

Definition 2.1.1. Let E be a Banach space and let $\mathcal{K}$ be a class of finite-dimensional normed spaces.

(a) E is *$\mathcal{K}$-universal* if every $X \in \mathcal{K}$ embeds into E via an isometric embedding.

(b) E is *$\mathcal{K}$-ultrahomogeneous* if for every $X \in \mathcal{K}$ and $\gamma, \eta \in \mathrm{Emb}(X, E)$, there is $g \in \mathrm{Iso}(E)$ such that $g \circ \gamma = \eta$.

(c) E is *approximately $\mathcal{K}$-ultrahomogeneous* if for every $\varepsilon > 0, X \in \mathcal{K}$ and $\gamma, \eta \in \mathrm{Emb}(X, E)$, there is $g \in \mathrm{Iso}(E)$ such that $\|g \circ \gamma - \eta\| \leq \varepsilon$.

(d) E is *$\mathcal{K}$-Fraïssé* if for every dimension $k \in \mathbb{N}$ and every $\varepsilon > 0$, there is a $\delta > 0$ such that the following holds: For every $X \in \mathcal{K}$ of dimension k and every $\gamma, \eta \in \mathrm{Emb}_\delta(X, E)$, there is $g \in \mathrm{Iso}(E)$ such that $\|g \circ \gamma - \eta\| \leq \varepsilon$.

When $\mathcal{K}$ is the *age of* E, i.e. the set $\mathrm{Age}(E)$ of all finite-dimensional subspaces of E, we omit the reference to $\mathcal{K}$ in the above definitions.

Note that a $\mathcal{K}$-Fraïssé space is automatically approximately $\mathcal{K}$-ultrahomogeneous. The ultrahomogeneity concepts above can be viewed more naturally in terms of canonical actions of the group $\mathrm{Iso}(E)$ on spaces of the form $\mathrm{Emb}_\delta(X, E)$, where $g \in \mathrm{Iso}(E)$ acts on $\gamma \in \mathrm{Emb}_\delta(X, E)$ by $g \cdot \gamma := g \circ \gamma$. More precisely, E is $\mathcal{K}$-ultrahomogeneous if the above action is transitive for every $X \in \mathcal{K}$, while E is approximately $\mathcal{K}$-ultrahomogeneous if and only if for every $X \in \mathcal{K}$ and every $\varepsilon > 0$, the natural action of $\mathrm{Iso}(E)$ on $\mathrm{Emb}(X, E)$ is *ε-transitive*, i.e. for every $\gamma, \eta \in \mathrm{Emb}(X, E)$ there is $g \in \mathrm{Iso}(E)$ such that $\|g \circ \gamma - \eta\| \leq \varepsilon$. Similarly, E is $\mathcal{K}$-Fraïssé if and only if for every $k \in \mathbb{N}$ and every $\varepsilon > 0$, there is a $\delta > 0$ such that the natural action of $\mathrm{Iso}(E)$ on $\mathrm{Emb}_\delta(X, E)$ is ε-transitive for every $X \in \mathcal{K}$ with dimension k.

Example 2.1.2.

- The *Gurarij space* $\mathbb{G}$ is a Fraïssé Banach space. This is the unique (up to isometry) separable Banach space which is of *almost universal disposition*: For every pair of finite-dimensional Banach spaces $X \subseteq Y$, every $\varepsilon > 0$ and every isometric embedding $\gamma : X \to \mathbb{G}$, there is an ε-isometric embedding $\eta : Y \to \mathbb{G}$ which extends γ. The Gurarij space was, of course, first constructed by Gurarij [26], and uniqueness was proved by Lusky [45]. The fact that $\mathbb{G}$ is Fraïssé in the above sense is essentially due to Kubiś and Solecki [41]. However, $\mathbb{G}$ is not ultahomogeneous (see [16, Example 2.5]).
- A Hilbert space is Fraïssé ([16, Example 2.4]).
- The classical $L_p[0,1]$ spaces, for $p \neq 4, 6, 8, \dots$ are Fraïssé. The fact that such spaces are Fraïssé is a recent result of Ferenczi, López-Abad, Mbombo and Todorčević [16], but such spaces were known to be approximately ultrahomogeneous; the latter result was shown by Lusky [46] in the late 1970s. Interestingly, the spaces $L_p[0,1]$ are *not* approximately ultrahomogeneous (and hence not Fraïssé) when $p = 4, 6, 8, \dots$; this is a consequence of work of Randrianantoanina [59] (see also [16] for more details).
- While not all $L_p[0,1]$ spaces are Fraïssé for their age, it *is* the case that each $L_p[0,1]$ is $\{\ell_p^n\}_n$-Fraïssé (see [16, Section 3]).

Surprisingly, the first three families of examples above are the only known examples of separable Fraïssé Banach spaces.

An internal characterization of the Fraïssé property of Banach spaces is given by the notion of a *Fraïssé class*, which is the Banach space version of the classical notion of a Fraïssé class from model theory. More precisely, we will be interested in classes satisfying some (or all) of the following properties:

Definition 2.1.3. Let $\mathcal{K}$ be a class of finite-dimensional normed spaces.

1. $\mathcal{K}$ has the *hereditary property* (HP) if $X \in \mathcal{K}$ whenever $Y \in \mathcal{K}$ and $\mathrm{Emb}(X, Y) \neq \varnothing$.
2. $\mathcal{K}$ has the *joint embedding property* (JEP) if for every $X, Y \in \mathcal{K}$ there is $Z \in \mathcal{K}$ such that $\mathrm{Emb}(X, Z)$ and $\mathrm{Emb}(Y, Z)$ are non-empty.
3. $\mathcal{K}$ has the *near amalgamation property* (NAP) if for every $\varepsilon > 0$, every $X, Y, Z \in \mathcal{K}$ and every pair of isometric embeddings $f_0 : X \to Y, f_1 : X \to Z$, there is $W \in \mathcal{K}$ together with isometric embeddings $g_0 : Y \to W, g_1 : Z \to W$ such that $\|g_0 \circ f_0 - g_1 \circ f_1\| \leq \varepsilon$.
4. $\mathcal{K}$ is an *amalgamation class* if $(\{0\}, \|\cdot\|) \in \mathcal{K}$ and for every dimension $k \in \mathbb{N}$ and every$\varepsilon > 0$ there is a $\delta > 0$ such that the following holds: For every $X, Y, Z \in \mathcal{K}$ with $\dim X = k$ and every pair of δ-isometric embeddings $f_0 : X \to Y, f_1 : X \to Z$, there is $W \in \mathcal{K}$ together with isometric embeddings $g_0 : Y \to W, g_1 : Z \to W$ such that $\|g_0 \circ f_0 - g_1 \circ f_1\| \leq \varepsilon$.
5. $\mathcal{K}$ is *Fraïssé* if it is a hereditary d_{BM}-closed amalgamation class.

The canonical example of a Fraïssé class is the age of a separable Fraïssé Banach space. Interestingly, this idea also goes the other way: Given a Fraïssé class $\mathcal{K}$, there is a unique (up to isometry) separable $\mathcal{K}$-Fraïssé space E such that every $X \in \mathcal{K}$ is isometric to an element of $\mathrm{Age}(E)$ and vice versa (so that $\mathrm{Age}(E)$ and $\mathcal{K}$ are equal "modulo isometric types"). This is the *Fraïssé correspondence* for Banach spaces obtained in [16]; the space E above is called the *Fraïssé limit* of $\mathcal{K}$ and is denoted $\mathrm{Flim}(\mathcal{K})$. More generally, one can construct a Fraïssé limit for amalgamation classes which are not necessarily Fraïssé. In this case, one obtains a $\mathcal{K}$-Fraíssé space with the property that every $X \in \mathcal{K}$ is isometric to an element of Age(Flim($\mathcal{K}$)) (but not necessarily vice versa).

2.1.2 The approximate Ramsey property and the KPT correspondence

Let G be a Hausdorff topological group. (From now on all topological groups we consider are assumed to be Hausdorff.) A *G-flow* is a continuous action $G \times X \to X : (g, x) \mapsto g \cdot x$ on a non-empty, compact, Hausdorff topological space X. In this case we write $G \curvearrowright X$; we often identify X with the flow when G and its action on X are understood. A topological group is *extremely amenable* if every G-flow X has a common fixed point: There is $x \in X$ such that for every $g \in G$ we have $g \cdot x = x$.

In the seminal work of Kechris, Pestov and Todorčević [38] it is shown that the extreme amenability of automorphism groups of first-order ultrahomogeneous structures can be characterized via Ramsey-theoretic properties. This correspondence, together with its many iterations and generalizations, has come to be known as the *Kechris-Pestov-Todorčević correspondence* and has spurred a significant amount of interest in Fraïssé theory and structural Ramsey theory in recent years. The KPT correspondence in the context of metric structures (as in [7]) was proved in [48]; part of the work involved in establishing the KPT correspondence in this setting has to do with finding the "correct" Ramsey property, which in this case happens to be an "approximate" Ramsey property. In [6] or [16], many different approximate Ramsey properties of classes of Banach spaces are studied and used to establish a KPT correspondence for Banach spaces. The main Ramsey property of interest for us will be the following:

Definition 2.1.4. A collection $\mathcal{K}$ of finite-dimensional normed spaces has the *approximate Ramsey property* (ARP) if for every $X, Y \in \mathcal{K}, r \in \mathbb{N}$ and $\varepsilon > 0$ there is $Z \in \mathcal{K}$ such that every r-colouring of $\mathrm{Emb}(X, Z)$ *ε-stabilizes* on a set of the form $\gamma \circ \mathrm{Emb}(X, Y)$ for some $\gamma \in \mathrm{Emb}(Y, Z)$, i.e. there is $i < r$ such that $\gamma \circ \mathrm{Emb}(X, Y)$ is contained in the set

$$(c^{-1}\{i\})_\varepsilon := \{\xi \in \mathrm{Emb}(X, Z) : \exists \eta \;\; c(\eta) = i \text{ and } \|\xi - \eta\| \leq \varepsilon\}.$$

In this case we will also say that $\gamma \circ \mathrm{Emb}(X, Y)$ is *ε-monochromatic.*

The following classes of Banach spaces have the ARP:

Example 2.1.5.

- The class $\{\ell_\infty^n\}_n$ has the ARP ([6, Theorem 2.15]). Interestingly, the proof makes use of the Dual Ramsey Theorem of Graham and Rothschild [24] which will be discussed in the next chapter.
- The class of all finite-dimensional *polyhedral* spaces, i.e. Banach spaces X such that the unit ball of X is a polyhedron ([6, Section 2.4]).
- The class of all finite-dimensional Banach spaces ([6, Corollary 2.16]).
- The classes $\{\ell_p^n\}_n$ for every $p \in [1, \infty)$ ([16, Example 5.4]).

The following is the KPT correspondence for Banach spaces (proved in [6, 16]) which, for an approximately ultrahomogeneous space E, characterizes the extreme amenability of $\mathrm{Iso}(E)$ in terms of the approximate Ramsey property of $\mathrm{Age}(E)$.

Theorem 2.1.6 (Kechris-Pestov-Todorčević correspondence)**.** *Suppose E is an infinite-dimensional Banach space approximately ultrahomogeneous. The following are equivalent:*

(i) $\mathrm{Age}(E)$ *has the ARP.*

(ii) $\mathrm{Iso}(E)$ *with the strong operator topology is extremely amenable.*

Since, by universality, the age of the Gurarij space $\mathbb{G}$ is the class of all finite-dimensional Banach spaces, the KPT correspondence implies that the group $\mathrm{Iso}(\mathbb{G})$ of surjective linear isometries of $\mathbb{G}$ is extremely amenable (see [6]). Furthermore, the ARP of the classes $\{\ell_p^n\}_n$ (for $p \in [1, \infty)$) imply the extreme amenability of $\mathrm{Iso}(L_p)$ with the strong operator topology (see [16]). It is worth mentioning that the extreme amenability of such groups was already known via purely analytic techniques; the case $p \in [1, \infty) \setminus \{2\}$ is due to Giordano and Pestov [21], while the case $p = 2$ (i.e. the Hilbertian case) is a classical result of Gromov and Milman [25].

2.2 Fraïssé classes of finite-dimensional Fréchet spaces

Let X be a Fréchet space, i.e. a locally convex Hausdorff topological vector space which is completely metrizable via a translation-invariant metric. A *fundamental system of seminorms* on X is a sequence of seminorms $(\|\cdot\|_{X,n})_{n<\alpha}$ on X, where $\alpha \leq \omega$, such that the sets

$$B_{n,\varepsilon}(x) = \{y \in X : \max_{m<n} \|x - y\|_{X,m} < \varepsilon\},\ x \in X, n < \alpha, \varepsilon > 0$$

form a basis for the topology of X. It is a standard fact that every Fréchet space admits a (non-unique) fundamental system of seminorms and so we can view a Fréchet space as a *multi-seminormed space* of the above form, i.e. a (finite or infinite) tuple consisting of a topological vector space equipped with a sequence of seminorms. By definition, a fundamental system of seminorms is always *separating*: For every non-zero $x \in X$ there is a seminorm $\|\cdot\|_{X,n}, n < \alpha$ such that $\|x\|_{X,n} \neq 0$; in this case we also say that X is *separated*, which happens precisely when the sequence of seminorms induces a Hausdorff topology on X. We will work more generally with multi-seminormed spaces of the form $\mathbf{X} = (X, (\|\cdot\|_{X,n})_{n<\lambda_{\mathbf{X}}})$, where X is a vector space, $1 \leq \lambda_{\mathbf{X}} \leq \omega$ is the *length* of $\mathbf{X}$, which is defined as the length of the associated sequence of seminorms (which we always assume is non-empty), and where $(\|\cdot\|_{X,n})_{n<\lambda_X}$ is a (finite or infinite) sequence of seminorms on X. We will always conflate the above tuple $\mathbf{X}$ with its underlying space X, and so we will always write λ_X for the length when the sequence of seminorms is understood. We point out that in general we do not require a multi-seminormed space to be separated. With this in mind, we reserve the term "Fréchet space" for a separated multi-seminormed space $(X, (\|\cdot\|_{X,n})_{n<\lambda_X})$ such that the topology generated by the seminorms $\|\cdot\|_{X,n}$ is complete. It is clear that in this case the sequence of seminorms $(\|\cdot\|_{X,n})$ forms a fundamental system of seminorms on X. Finally, following the terminology in [4, 67], we say that a multi-seminormed space $(X, (\|\cdot\|_{X,n})_{n<\lambda_X})$ is *graded* if, for all $x \in X$, $\|x\|_{X,n} \leq \|x\|_{X,m}$ whenever $n \leq m < \lambda_X$. For more information about (graded) Fréchet spaces, we refer the reader to [28, 47, 67]. The reader is also referred to [42, 64] for more information about the general theory of multi-seminormed spaces (which are also sometimes called seminormed spaces or multinormed spaces in the literature).

Throughout this chapter we will work with various subclasses of the class $\mathcal{M}$ of finite-dimensional multi-seminormed spaces equipped with a fundamental system of seminorms. Of particular interest will be the subclass $\mathcal{G}$ consisting of all graded $X \in \mathcal{M}$. Given any subclass $\mathcal{K} \subseteq \mathcal{M}$, and $\alpha \leq \omega$, we let $\mathcal{K}_{<\alpha}, \mathcal{K}_{\leq\alpha}$ and $\mathcal{K}_{=\alpha}$ denote the collections of all $X \in \mathcal{K}$ such that $\lambda_X < \alpha, \lambda_X \leq \alpha$ and $\lambda_X = \alpha$, respectively. For any such $\mathcal{K}$ we also let $\mathcal{K}^{\mathrm{sep}}$ denote the subclass of $\mathcal{K}$ consisting of all separated members of $\mathcal{K}$.

Given two multi-seminormed spaces X and Y and a linear mapping $f : X \to Y$, we say that a linear mapping $f : X \to Y$ is *multi-bounded* when $\lambda_X \leq \lambda_Y$ and when

$$\|f\|_{\mathrm{mb}} := \sup_{m<\lambda_X} \|f\|_{(X,\|\cdot\|_{X,m}),(Y,\|\cdot\|_{Y,m})} < \infty,$$

and where as usual $\|f\|_{(X,\|\cdot\|_{X,m}),(Y,\|\cdot\|_{Y,m})}$ is the (semi)norm of f defined as $\sup_{\|x\|_{X,m}\leq 1} \|f(x)\|_{Y,m}$. When working with the operator norm, we will usually suppress the notation and omit the reference to the underlying spaces when this is clear from context. Observe that since we are dealing with seminorms, this quantity might be infinite, even when X is finite-dimensional. A *multi-isomorphism* $f : X \to Y$ is linear bijection such that both f and f^{-1} are multi-bounded.

Much like in the class of normed spaces and others, we define a Banach-Mazur-like pseudometric, quantifying distances between isometric types. (The relevant notion of an isometry in this setting will be defined below.) We define

$$d_{\mathrm{BM}}(X,Y) := \log \inf_f \|f\|_{\mathrm{mb}} \|f^{-1}\|_{\mathrm{mb}},$$

where f runs over all multi-isomorphisms (if any) between X and Y.

Proposition 2.2.1. *$\mathcal{M}_{<\omega}$ and $\mathcal{G}_{<\omega}$ are σ-compact, and consequently separable.*

Proof. $\mathcal{G}_{<\omega}$ is closed in $\mathcal{M}_{<\omega}$, so it suffices to prove that $\mathcal{M}_{<\omega}$ is σ-compact. As in the case of the Banach-Mazur pseudometric on the class of finite-dimensional normed spaces, it is not difficult to see that the closed d_{BM}-balls on $\mathcal{M}_{<\omega}$ are compact. But unlike for the normed space case, where for each dimension k the diameter is finite (hence the Banach-Mazur compactum is compact), this is not the case for multi-seminormed spaces. Given $(X, (\|\cdot\|_{X,k})_{k<\lambda_X}) \in \mathcal{M}_{<\omega}$, let $\overline{\alpha}_X = (\alpha_s^X)_{s\subseteq\lambda_X}$ be defined for $s \subseteq \lambda_X$ by $\alpha_s^X := \dim \bigcap_{k\in s} \ker \|\cdot\|_{X,k}$ and $\alpha_\varnothing^X := \dim X$.

Claim 2.2.2. The classes of finite distances (i.e. isomorphic classes) are exactly, given a sequence $\bar{\alpha} = (\alpha_s)_{s\subseteq n}$ of positive integers, the sets $\mathcal{A}_{\bar{\alpha}} := \{X \in \mathcal{M}_{<\omega} : \bar{\alpha}_X = \bar{\alpha}\}$.

This fact easily implies that $\mathcal{M}_{<\omega}$ is σ-compact. We prove now the claim. The sets $\mathcal{A}_{\bar{\alpha}}$ are closed under isomorphic images because if $f : X \to Y$ is a multi-isomorphism, then $\overline{\alpha}_X = \overline{\alpha}_Y$, because

$$f(\bigcap_{k\in s} \ker \|\cdot\|_{X,k}) = \bigcap_{k\in s} \ker \|\cdot\|_{Y,k} \text{ for every } s \subseteq \lambda_X = \lambda_Y.$$

This follows from the fact that f is multi-bounded. The converse is also true: for suppose that $\overline{\alpha}_X = \overline{\alpha}_Y$ and let $\lambda = \lambda_X = \lambda_Y$. First, observe that $\alpha_\varnothing^X = \dim X$, so we have that $\dim X = \dim Y$. The argument is by induction on the number $N = N_X = N_Y$ of $s \subseteq \lambda$ such that $\alpha_s^X > 0$. If $N = 0$, this means that all seminorms are in fact norms, so the desired result follows from the well-known fact that the norms on finite-dimensional spaces of the same dimension are all equivalent. Suppose now that $N > 0$. Let us choose $k_0 < \lambda$ such that $\alpha_{\{k_0\}} > 0$. For suppose that $X, Y \in \mathcal{A}_{\overline{\alpha}}$. Let $X_0 := \ker \|\cdot\|_{X,k}$, $Y_0 := \ker \|\cdot\|_{Y,k}$, and let $X_1 \subseteq X$ and $Y_1 \subseteq Y$ be complementary subspaces of X_0 and Y_0 respectively. Then $N_{X_1} = N_{Y_1} < N$, so there is some multi-isomorphism $f : X_1 \to Y_1$, and similarly

$X_0' := (X_0, (\|\cdot\|_{X,k})_{k<\lambda_X, k\neq k_0})$ and $Y_0' := (Y_0, (\|\cdot\|_{Y,k})_{k<\lambda_X, k\neq k_0})$ satisfy that $N_{X_0'} = N_{Y_0'} < N$, so there is a multi-isomorphism $g : X_0' \to Y_0'$. Then $h : X \to Y$ defined by $h(x_0 + x_1) := f(x_0) + g(x_1)$ for $x_0 \in X_0$ and $x_1 \in X_1$ is a multi-isomorphism. □

Given $\varepsilon \geq 0$, let $\mathrm{Emb}_\varepsilon(X, Y)$ denote the set of all injective *multi-ε-isometric embeddings* from X into Y, i.e. linear mappings $f : X \to Y$ such that

$$\frac{1}{1+\varepsilon}\|x\|_{X,m} \leq \|f(x)\|_{Y,m} \leq (1+\varepsilon)\|x\|_{X,m}$$

for each $x \in X$ and $m < \lambda_X$. When $\varepsilon = 0$, we simply refer to such a mapping as a *multi-isometric embedding*; the collection of all such mappings $f : X \to Y$ will be denoted $\mathrm{Emb}(X, Y)$. Note that, unlike in the setting of normed spaces, a seminorm-preserving mapping need not be injective. A *multi-isometry* (resp. multi-ε-isometry) is a surjective multi-isometric embedding (resp. multi-ε-isometric embedding).

Below we will make use of the notion of a *modulus* relative to a set S, i.e. is a function $\varpi : S \times [0, \infty[\to [0, \infty[$ such that, for every $s \in S$, the function $\varpi(s, \cdot)$ is increasing and continuous at 0 with value 0. In our case, S will be the product $\mathbb{N} \times (\mathbb{N} \cup \{\infty\})$.

Definition 2.2.3. Let $\mathcal{K}$ be a class of finite-dimensional multi-seminormed spaces.

(1) $\mathcal{K}$ has the *hereditary property* (HP) if $X \in \mathcal{K}$ whenever $Y \in \mathcal{K}$ and $\mathrm{Emb}(X, Y) \neq \varnothing$.

(2) $\mathcal{K}$ has the *joint embedding property* (JEP) if for every $X, Y \in \mathcal{K}$ there is $Z \in \mathcal{K}$ such that $\mathrm{Emb}(X, Z)$ and $\mathrm{Emb}(Y, Z)$ are non-empty.

(3) $\mathcal{K}$ has the *near amalgamation property* (NAP) if for every $\varepsilon > 0$, every $X, Y, Z \in \mathcal{K}$ and every pair of multi-isometric embeddings $f_0 : X \to Y, f_1 : X \to Z$, there is $W \in \mathcal{K}$ together with multi-isometric embeddings $g_0 : Y \to W, g_1 : Z \to W$ such that $\|g_0 \circ f_0 - g_1 \circ f_1\|_n \leq \varepsilon$ for each $n < \lambda_X$.

(4) $\mathcal{K}$ is an *amalgamation class with modulus of stability* ϖ if $(\{0\}, \|\cdot\|) \in \mathcal{K}$ and for every $d \in \mathbb{N}$, $l \in \mathbb{N} \cup \{\infty\}$. $\varepsilon > 0$ and $\delta \geq 0$, every $X, Y, Z \in \mathcal{K}$ such that $\dim X = d$ and $\lambda_X = l$, and every pair of multi-δ-isometric embeddings $f_0 : X \to Y, f_1 : X \to Z$, there is $W \in \mathcal{K}$ together with multi-isometric embeddings $g_0 : Y \to W, g_1 : Z \to W$ such that:

   (i) $\|g_0 \circ f_0 - g_1 \circ f_1\|_n \leq \varpi(d, l, \delta) + \varepsilon$ for each $n < \lambda_X$.

   (ii) W is separated.

(5) $\mathcal{K}$ is *Fraïssé* if it is a hereditary d_{BM}-closed amalgamation class such that $\mathcal{K} \subseteq \mathcal{M}_{<\omega}$.

Observe that every amalgamation class $\mathcal{K}$ automatically has the JEP since one can simply amalgamate over the trivial normed space. We remark that condition (4)(ii) does not appear in the usual presentation of metric Fraïssé theory (and in particular the Fraïssé theory of Banach spaces developed in [16]). However, it appears to be necessary when working with arbitrary multi-normed spaces in order to guarantee injectivity of certain seminormed-preserving mappings encountered when constructing a Fraïssé limit. This condition will be trivially satisfied by all collections $\mathcal{K}$ of interest. Conditions (4)(i) and (4)(ii) together imply that for any X belonging to an amalgamation class $\mathcal{K}$, there is some separated $Y \in \mathcal{K}$ such that $\mathrm{Emb}(X, Y) \neq \varnothing$. We will make use of this fact without reference.

We now proceed to show that the classes considered above are amalgamation classes. The following is a particular case of a more general result explained in Proposition 2.2.7, where it is shown that classes of multi-seminormed spaces defined from amalgamation classes of normed spaces are also amalgamation classes. We present the proof here since a version of the relevant construction (which is essentially a pushout in the category of multi-seminormed spaces) will be used in the sequel.

Proposition 2.2.4. *For each $\mathcal{K} \in \{\mathcal{M}, \mathcal{G}\}$ and each $\alpha \leq \omega$, the classes $\mathcal{K}^{\mathrm{sep}}_{=\alpha}, \mathcal{K}^{\mathrm{sep}}_{\leq\alpha}$ and $\mathcal{K}^{\mathrm{sep}}_{<\alpha}$ are amalgamation classes with modulus of stability $\varpi(d, l, \delta) = 2\delta$. Furthermore, $\mathcal{K}_{<\omega}$ is an amalgamation class with the same modulus of stability.*

Proof. First we will show that $\mathcal{K}^{\mathrm{sep}}_{\leq\alpha}$ is an amalgamation class with modulus $\varpi(d, l, \delta) = \delta$ with respect to *expansive* multi-δ-isometric embeddings, i.e. multi-δ-isometric embeddings $f : X \to Y$ which additionally satisfy $\|f(x)\|_{Y,n} \geq \|x\|_{X,n}$ for each $x \in X$ and $n < \lambda_X$. To this end, fix $\varepsilon > 0, \delta \geq 0, d \in \mathbb{N}$ and $l \in \mathbb{N} \cup \{\infty\}$. Let $X, Y, Z \in \langle\bar{\mathcal{K}}\rangle$ where $\dim X = d$ and $\lambda_X = l$, and fix contractive multi-δ-isometric embeddings $f : X \to Y$ and $g : X \to Z$. We can suppose, without loss of generality, $\lambda_Y \leq \lambda_Z$. Consider the sum $Y \oplus Z$ together with the canonical inclusion mappings $i : Y \to Y \oplus Z$ and $j : Z \to Y \oplus Z$. Equip $Y \oplus Z$ with a sequence $(\|\cdot\|_n)_{n<\lambda_Z}$ of seminorms in the following way:

(i) For each $n \in [0, \lambda_X)$, let

$$\|(y, z)\|_n := \inf\{\|u\|_{Y,n} + \|v\|_{Z,n} + (\delta+\varepsilon)\|x\|_{X,n} : x \in X, u \in Y, v \in Z, y = u + f(x), z = v - g(x)\}.$$

(ii) Suppose $n \in [\lambda_X, \lambda_Y)$. If $\mathcal{K} \neq \mathcal{G}$, let $\|(y, z)\|_n := \max\{\|y\|_{Y,n}, \|z\|_{Z,n}\}$. Otherwise, assuming inductively that $\|(y, z)\|_{n-1}$ has been defined, let $\|(y, z)\|_n := \max\{\|(y, z)\|_{n-1}, \|y\|_{Y,n}, \|z\|_{Z,n}\}$.

(iii) Suppose $n \in [\lambda_Y, \lambda_Z)$. If $\mathcal{K} \neq \mathcal{G}$, let $\|(y, z)\|_n := \|z\|_{Z,n}$. Otherwise, assuming inductively that $\|(y, z)\|_{n-1}$ has been defined, let $\|(y, z)\|_n := \max\{\|(y, z)\|_{n-1}, \|z\|_{Z,n}\}$.

Note that the alternative in the last two parts of the above construction is needed to guarantee that $(\|\cdot\|_n)_{n<\lambda_Z}$ is a graded sequence of seminorms whenever X, Y and Z are graded. Furthermore, $Y \oplus Z$ is separated whenever Y and Z are both separated. In the case where we are working in $\mathcal{K}_{<\omega}$, the above space can be turned into a separated space by extending the above sequence of seminorms by a norm.

First we check that i is a multi-isometric embedding. Suppose first that $n < \lambda_X$. Then for each $y \in Y$, $\|i(y)\|_n = \|(y, 0)\|_n \leq \|y\|_{Y,n}$ by definition of $\|\cdot\|_n$. On the other hand, given any $x \in X, u \in Y$ and $v \in Z$ such that $y = u + f(x)$ and $0 = v - g(x)$, we have

$$\begin{aligned}
\|u\|_{Y,n} + \|v\|_{Z,n} + (\delta+\varepsilon)\|x\|_{X,n} &= \|u\|_{Y,n} + \|g(x)\|_{Z,n} + (\delta+\varepsilon)\|x\|_{X,n} \\
&\geq \|u\|_{Y,n} + \|x\|_{X,n} + (\delta+\varepsilon)\|x\|_{X,n} \\
&= \|u\|_{Y,n} + (1+\delta+\varepsilon)\|x\|_{X,n} \\
&\geq \|u\|_{Y,n} + \|f(x)\|_{Y,n} \\
&\geq \|u + f(x)\|_{Y,n} = \|y\|_{Y,n}
\end{aligned}$$

and so $\|i(y)\|_n \geq \|y\|_{Y,n}$. Thus i preserves the first λ_X seminorms. If $\lambda_X \leq n < \lambda_Y$, then $\|i(y)\|_n = \|y\|_{Y,n}$ by definition and so i is a multi-isometric embedding.

Next we check that j is a multi-isometric embedding. The case where $n < \lambda_X$ is exactly as before. In the case where $\lambda_X \leq n < \lambda_Y$, we have $\|j(z)\|_n = \|z\|_{Z,n}$ by definition. Otherwise, $n \geq \lambda_Y$. In the graded case we have

$$\|j(z)\|_n = \max\{\|(0, z)\|_{\lambda_Y-1}, \|z\|_{Z,n}\} = \max\{\|z\|_{Z,\lambda_Y-1}, \|z\|_{Z,n}\} = \|z\|_{Z,n}$$

since Z is graded. In the non-graded case we simply have $\|j(z)\|_n = \|z\|_{Z,n}$ by definition, and so j is a multi-isometric embedding.

It remains to check $\|i \circ f - j \circ g\| \leq \delta + \varepsilon$. To see this, note that for each $x \in X$ and $n < \lambda_X$ we have

$$\|i(f(x)) - j(g(x))\|_n = \|(f(x), -g(x))\|_n \leq (\delta + \varepsilon)\|x\|_{X,n}$$

by taking $u = 0$ and $v = 0$ in the definition of $\|\cdot\|_n$. Thus $i : Y \to Y \oplus Z$ and $j : Z \to Y \oplus Z$ are the desired multi-isometric embeddings.

Now suppose that f and g are merely multi-δ-isometric embeddings. Define a new sequence of seminorms $(\|\cdot\|'_{X,n})_{n<\lambda_X}$ on X by setting $\|x\|'_{X,n} := \frac{1}{1+\delta}\|x\|_{X,n}$. Then f and g become expansive multi-δ'-isometric embeddings from X (equipped with the new sequence of seminorms) into Y and Z, respectively, where $\delta' = 2\delta + \delta^2$. Thus we can apply the above argument to find $W \in \mathcal{K}^{\mathrm{sep}}_{\leq\alpha}$ and multi-isometric embeddings $i : Y \to W, j : Z \to W$ such that

$$\|i(f(x)) - j(g(x))\|_{W,n} \leq (\delta' + \varepsilon)\|x\|'_{X,n} \text{ for all } n < \lambda_X \text{ and } x \in X \text{ such that } \|x\|'_{X,n} \leq 1.$$

Now fix $n < \lambda_X$ and $x \in X$ such that $\|x\|_{X,n} \leq 1$. Then $\|x\|'_{X,n} \leq \frac{1}{1+\delta} \leq 1$ and so

$$\|i(f(x)) - j(g(x))\|_{W,n} \leq (\delta' + \varepsilon)\|x\|'_{X,n} \leq \frac{\delta'}{1+\delta} + \varepsilon = \frac{\delta + \delta(1+\delta)}{1+\delta} + \varepsilon \leq 2\delta + \varepsilon.$$

This shows that $\mathcal{K}^{\mathrm{sep}}_{\leq\alpha}$ is an amalgamation class with modulus $\varpi(d, l, \delta) = 2\delta$. In the case where $\mathcal{K} = \mathcal{G}$, note that the space $Y \oplus Z$ constructed above is graded whenever X, Y and Z are graded. This proofs for $\mathcal{K}^{\mathrm{sep}}_{<\alpha}, \mathcal{K}^{\mathrm{sep}}_{=\alpha}$ and $\mathcal{K}_{<\omega}$ are similar. □

Remark 2.2.5. The above proof can easily be adapted to prove amalgamation properties for classes of other "norm-like" structures from functional analysis. For instance, one can similarly show that the class of all finite-dimensional F-(semi)-normed spaces (as in [32]) is an amalgamation class with respect to δ-isometric embeddings.

Another source of examples of amalgamation classes of multi-seminormed spaces will come from the following proposition, which will make use of known amalgamation properties of classes of normed spaces. (The relevant definitions are analogous to our case; the reader is referred to [16] for more details, examples and discussion.) The following is standard notation: Given a seminormed space $X = (X, \|\cdot\|)$, we denote by $X_{\|\cdot\|}$ the normed space $(X/\ker\|\cdot\|, \|\cdot\|)$ where $\|[x]\| := \|x\|$. Note that this norm is well-defined.

Definition 2.2.6. Let $\bar{\mathcal{K}} := \{\mathcal{K}_n\}_{n<\alpha}$, $\alpha \leq \omega$, be a collection of classes $\mathcal{K}_n$ of normed spaces. We define $\langle\bar{\mathcal{K}}\rangle_{\leq\alpha}$, $\langle\bar{\mathcal{K}}\rangle_{<\alpha}$, $\langle\bar{\mathcal{K}}\rangle_{=\alpha}$ as the classes of separated multi-seminormed spaces $(X, (\|\cdot\|_n)_{n<\lambda_X}) \in \mathcal{M}_{\leq\alpha}$, $\mathcal{M}_{<\alpha}$ and in $\mathcal{M}_{=\alpha}$, respectively, such that $X_{\|\cdot\|_n} \in \mathcal{K}_n$ for all $n < \lambda_X$.

Proposition 2.2.7. *Let $\{\mathcal{K}_n\}_{n<\omega}$ be a sequence of families of finite-dimensional normed spaces.*

(a) If each $\mathcal{K}_n$ is hereditary, then so are $\langle\bar{\mathcal{K}}\rangle_{<\alpha}$ and $\langle\bar{\mathcal{K}}\rangle_{\leq\alpha}$ for each $\alpha \leq \omega$.

(b) Suppose each $\mathcal{K}_n$ is an amalgamation class with modulus of stability $\varpi^{\mathcal{K}_n} : \mathbb{N} \to [0, \infty[$. Then for each $\alpha \leq \omega$ the class $\langle\bar{\mathcal{K}}\rangle_{<\alpha}$ is also an amalgamation class with modulus of stability

$$\varpi(d, l, \delta) := \max_{n<l} \varpi^{\mathcal{K}_n}(d, \delta).$$

Furthermore, for each $n < \omega$ the classes $\langle \bar{\mathcal{K}} \rangle_{\leq n}$ and $\langle \bar{\mathcal{K}} \rangle_{=n}$ are amalgamation classes with the same modulus of stability.

Proof. The proof of (a) is relatively straightforward, so we leave the details to the reader. We only check (b) for $\langle \bar{\mathcal{K}} \rangle_{<\alpha}$ since the proofs for the other classes are similar. Fix $\varepsilon > 0, \delta \geq 0$ together with integers d, l. Let $X, Y, Z \in \langle \bar{\mathcal{K}} \rangle_{<\alpha}$ where $\dim X = d$ and $\lambda_X = l$, and fix multi-δ-isometric embeddings $f_0 : X \to Y$ and $g_0 : X \to Z$. Suppose, without loss of generality, $\lambda_Y \leq \lambda_Z$. To construct the required element of $\langle \bar{\mathcal{K}} \rangle_{<\alpha}$, we will first define a sequence $(W_i)_{i<\lambda_Z}$ of finite-dimensional normed spaces. For each integer $i < l$, define mappings $f_0^i : X_{\|\cdot\|_i} \to Y_{\|\cdot\|_i}$ and $g_0^i : X_{\|\cdot\|_i} \to Z_{\|\cdot\|_i}$ by setting

$$f_0^i([x]_i) = [f_0(x)]_i \text{ and } g_0^i([x]_i) = [g_0(x)]_i.$$

Then each f_0^i and g_0^i is a δ-isometric embedding between elements of $\mathcal{K}_i$ and so, since $\mathcal{K}_i$ is an amalgamation class for each $i < n$, there are $W_i \in \mathcal{K}_i$ and isometric embeddings $f_1^i : Y_{\|\cdot\|_i} \to W_i, g_1^i : Z_{\|\cdot\|_i} \to W_i$ such that

$$\|f_1^i \circ f_0^i - g_1^i \circ g_0^i\|_{X_{\|\cdot\|_i}, W_i} \leq \varpi^{\mathcal{K}_i}(d, \delta) + \varepsilon.$$

For $i \in [l, \lambda_Y)$, let W_i be obtained from an application of the JEP to the pair $Y_{\|\cdot\|_i}, Z_{\|\cdot\|_i}$; let f_1^i and g_1^i, respectively, denote the corresponding multi-isometric embeddings. Finally, for $i \in [\lambda_Y, \lambda_Z)$, simply let $W_i = Z_{\|\cdot\|_i}$.

Let $W = \prod_{i<\lambda_Z} W_i$ and equip W with a family of seminorms defined by setting

$$\|(w_0, \dots, w_{\lambda_Z - 1})\|_i = \|w_i\|_{W_i} \text{ for } i < \lambda_Z.$$

Observe that $W \in \langle \bar{\mathcal{K}} \rangle_{<\alpha}$ since W is separated and $W_{\|\cdot\|_i} \cong W_i$ for each $i < \lambda_Z$. Define $f_1 : Y \to W$ and $g_1 : Z \to W$ by

$$f_1(y) = \left(f_1^0([y]_0), \dots, f_1^{\lambda_Y - 1}([y]_{\lambda_Y - 1}), 0, \dots, 0 \right),$$

$$g_1(z) = \left(g_1^0([z]_0), \dots, g_1^{\lambda_Y - 1}([z]_{\lambda_Y - 1}), [z]_{\lambda_Y}, \dots, [z]_{\lambda_Z - 1} \right).$$

Then it is straightforward to check that f_1 and g_1 are multi-isometric embeddings which witness the amalgamation property for $\mathcal{K}$ for the above parameters. Injectivity follows from the fact that Y and Z are separated. ☐

Example 2.2.8. (1) The class $\mathcal{M}_{<\omega}$ of all finite-dimensional multi-seminormed spaces X such that $\lambda_X < \omega$ is a Fraïssé class. In this case, observe that we do not need to restrict to separated multi-seminormed spaces since an arbitrary finite sequence of seminorms can always be extended by a norm. The same is true of the subclass $\mathcal{G}_{<\omega}$ of all graded $X \in \mathcal{M}_{<\omega}$.

(2) For each $\alpha \leq \omega$, the classes $\mathcal{M}^{\text{sep}}_{<\alpha}, \mathcal{M}^{\text{sep}}_{\leq\alpha}$ and $\mathcal{M}^{\text{sep}}_{=\alpha}$ of all finite-dimensional *separated* multi-seminormed spaces such that $\lambda_X < \alpha, \lambda_X \leq \alpha$ and $\lambda_X = \alpha$, respectively, are amalgamation classes. Note, however, that these are not Fraïssé classes since it is possible for a non-separated multi-seminormed space to embed into a separated space according to our definition of a multi-isometric embedding. The same is true for the corresponding graded versions $\mathcal{G}^{\text{sep}}_{<\alpha}, \mathcal{G}^{\text{sep}}_{\leq\alpha}$ and $\mathcal{G}^{\text{sep}}_{=\alpha}$.

(3) For each infinite $I \subseteq \omega$, let $\mathcal{M}^I_{<\omega}$ denote the collection of all $(X, (\|\cdot\|_{X,n})_{n<\lambda_X}) \in \mathcal{M}_{<\omega}$ such that $\|\cdot\|_{X,n}$ is a norm for each $n \in I \cap [0, \lambda_X[$. Then $\mathcal{M}^I_{<\omega}$ is a Fraïssé class. (This follows from the proof of Proposition 2.2.4.) The amalgamation classes $\mathcal{M}^I_{<n}$ and $\mathcal{M}^I_{=n}$ can be defined similarly.

(4) Let $\mathcal{M}^{\mathcal{H}}_{<\omega}$ denote the class of all finite-dimensional Fréchet-Hilbert spaces with a finite sequence of seminorms, where $(X, (\|\cdot\|)_{n<\lambda_X})$ is a *Fréchet-Hilbert space* if $\|\cdot\|_n$ is a Hilbertian seminorm, i.e. a seminorm induced by a semi-inner product on X. Equivalently, X is Fréchet-Hilbert if each quotient space $X_{\|\cdot\|_n}$ is a Hilbert space. Then $\mathcal{M}^{\mathcal{H}}_{<\omega}$ is a Fraïssé class. (We refer the reader to [47] for more on Fréchet-Hilbert spaces and related concepts.)

(5) Given a sequence $(p_n)_{n<\omega} \subseteq [1,\infty[$ with $p_n \notin \{4,6,8,\dots\}$, consider the class $\mathcal{M}^{(p_n)}_{<\omega}$ of all multi-seminormed spaces $(X, (\|\cdot\|_n)_{n<\lambda_X})$ such that for each $n < \lambda_X$ the normed space $X_{\|\cdot\|_n}$ can be isometrically embedded into $L_{p_n}[0,1]$. Then this is a Fraïssé class for any such sequence (p_n). This follows from the fact that $\mathrm{Age}(L_p[0,1])$ is a Fraïssé class of finite-dimensional normed spaces for each $p \in [1,\infty[$ such that $p \notin \{4,6,8,\dots\}$ (see [16]). In this case, the corresponding modulus of stability ϖ associated to $\mathrm{Age}(L_p[0,1])$ depends on the dimension for each p, and hence so does the corresponding modulus of stability associated to $\mathcal{M}^{(p_n)}_{<\omega}$.

(6) In general, one can combine the various classes of normed spaces considered above to form new amalgamation classes of multi-seminormed spaces. For instance, given any sequence $(E_n)_{n<\alpha}$ of Fraïssé Banach spaces, one can set $\mathcal{K}_n = \mathrm{Age}(E_n)$ for each n (noting that these are all Fraïssé classes of normed spaces) and form the associated class $\langle\bar{\mathcal{K}}\rangle_{\leq\alpha}$ of multi-seminormed spaces, where $\vec{\mathcal{K}} = (\mathcal{K}_n)$.

In all but the last two examples, the corresponding modulus of stability is always independent of both the "dimension" and the "length" parameters in the sense that modulus only depends on the third variable. In the setting of normed spaces, such classes have been studied in [43] and are called *stable* Fraïssé classes. Thus, stable Fraïssé classes of finite-dimensional normed spaces give rise to stable Fraïssé classes of finite-dimensional multi-seminormed spaces. In the last two families of examples, however, it is unknown if the modulus of stability depends on the dimension. This observation inspires the following:

Question 2.2.9. Are there any examples of amalgamation classes of finite-dimensional seminormed spaces such that the corresponding modulus of stability depends on all three parameters?

2.3 Fraïssé Fréchet spaces

For a multi-seminormed space X, let $\mathrm{Iso}(X)$ be the group of multi-isometries $f : X \to X$ equipped with the topology generated by basic open sets of the form

$$\{g \in \mathrm{Iso}(X) : \max_{m<n} \|(f-g) \restriction Y\|_m < \varepsilon\}$$

where $f \in \mathrm{Iso}(X)$, Y is a finite-dimensional linear subspace of X, $\varepsilon > 0$ and $n < \lambda_X$. This is the analogue of the strong operator topology in the setting of multi-seminormed spaces; in particular, a sequence (f_k) in $\mathrm{Iso}(X)$ converges to f if and only if $\|f_k(x) - f(x)\|_n$ converges to 0 as $k \to \infty$ for each $x \in X$ and each $n \in \mathbb{N}$. For the sake of brevity, we will simply refer to this topology as the topology of pointwise convergence on $\mathrm{Iso}(X)$. We also equip $\mathrm{Iso}(X)$ with its *left uniform structure*, i.e. the uniformity generated by entourages of the diagonal of the form

$$\{(f,g) \in \mathrm{Iso}(X)^2 : f^{-1} \circ g \in U\}$$

where U is a neighbourhood of the identity.

For a multi-seminormed space $(E, (\|\cdot\|_{E,n})_{n<\lambda_E})$, let $\mathrm{Age}(E)$ denote the *age of* E, which is defined as the class of all finite-dimensional multi-seminormed spaces of the form

$$(X, (\|\cdot\|_{E,n})_{n<\lambda_X})$$

where X is a linear subspace of E and $\lambda_X \leq \lambda_E$. (Warning: This is not the standard definition of the age, which is usually defined as the collection of all finitely-generated substructures of a given structure.) We also let $\mathrm{Age}_\alpha(E) := (\mathrm{Age}(E))_{=\alpha} = \mathrm{Age}(E) \cap \mathcal{M}_{=\alpha}$ for each $\alpha \leq \omega$. The collections $\mathrm{Age}_{\leq\alpha}(E)$ and $\mathrm{Age}_{\leq\alpha}(E)$ are defined similarly.

Definition 2.3.1. Let $(E, (\|\cdot\|_n)_{n<\lambda_E})$ be a multi-seminormed space and let $\mathcal{K}$ be a class of finite-dimensional multi-seminormed spaces.

(a) E is $\mathcal{K}$*-universal* if every $X \in \mathcal{K}$ embeds into E via a multi-isometric embedding.

(b) E is *approximately* $\mathcal{K}$*-ultrahomogeneous* if for every $\varepsilon > 0, X \in \mathcal{K}$ and $\gamma, \eta \in \mathrm{Emb}(X, E)$, there is $g \in \mathrm{Iso}(E)$ such that $\|g \circ \gamma - \eta\|_n \leq \varepsilon$ for each $n < \lambda_X$.

(c) E is $\mathcal{K}$*-Fraïssé with modulus of stability* ϖ if for every $d \in \mathbb{N}, l \in \mathbb{N} \cup \{\infty\}, \varepsilon > 0$ and $\delta \geq 0$, every $X \in \mathcal{K}$ and every $\gamma, \eta \in \mathrm{Emb}_\delta(X, E)$, there is $g \in \mathrm{Iso}(E)$ such that $\|g \circ \gamma - \eta\|_n \leq \varpi(d, l, \delta) + \varepsilon$ for each $n < l$.

When $\mathcal{K} = \mathrm{Age}(E)$, we will occasionally omit the reference to $\mathcal{K}$ in the above definitions. We also omit the reference to the modulus of stability when working with a general Fraïssé Fréchet space.

In order to transfer properties of an amalgamation class to a relevant Fraïssé Fréchet space, we will need a slight modification of the given modulus. To this end, for a modulus $\varpi : \mathbb{N}^2 \times [0, \infty[\to [0, \infty[$ we define a new modulus

$$\varpi^*(d, l, \delta) := \inf_{\delta' > \delta} \varpi(d, l, \delta').$$

Note that ϖ^* is indeed a modulus which furthermore satisfies $\varpi(d, l, \delta) \leq \varpi^*(d, l, \delta)$ for any $d, l \in \mathbb{N}$.

Our main goal in this section is to show that when $\mathcal{K}$ is a Fraïssé class with modulus ϖ, then there is a unique (up to a multi-isometry) separable, $\mathcal{K}$-universal and $\mathcal{K}$-Fraïssé Fréchet space with modulus ϖ^* whose finite-dimensional subspaces are precisely those from $\mathcal{K}$. Note that any $\mathcal{K}$-Fraïssé space is automatically approximately $\mathcal{K}$-ultrahomogeneous. Furthermore, if E is Fraïssé, then $\mathrm{Age}_{<\omega}(E)$ is a Fraïssé class; the amalgamation property follows from the Fraïssé property together with the fact that any finite-dimensional subspace $X \in \mathrm{Age}_{<\omega}$ is eventually separated by the sequence of seminorms from E.

Below we will need to make use of an inductive limit construction for spaces with a finite sequence of seminorms. Suppose $(X_n, I_n)_{n<\omega}$ is a sequence such that:

(1) $X_n \in \mathcal{M}_{<\omega}$ for each n.

(2) $(\lambda_{X_n})_{n<\omega}$ is non-decreasing and converges to a fixed $\lambda \leq \omega$.

(3) $I_n \in \mathrm{Emb}(X_n, X_{n+1})$ for each n.

We define the *inductive limit* $\lim_n(X_n, I_n)$ as follows: First, for each $m \leq n$, define $I_{m,n} \in \mathrm{Emb}(X_m, X_n)$ by setting $I_{m,m} = \mathrm{id}_{X_m}$ and $I_{m,n+1} = I_n \circ I_{m,n}$. Then let V be the linear subspace of the product space

$\prod_n X_n$ defined by declaring $(x_n)_{n<\omega} \in V$ if and only if there is some m such that $x_n = I_{m,n}(x_m)$ for all $n \geq m$. For each $k < \lambda$, let N_k be the linear subspace of V consisting of all $(x_n)_n$ such that there is m such that $k < \lambda_{X_m}$ and $\|x_n\|_k = 0$ for all $n \geq m$. Let $N = \bigcap_{k<\omega} N_k$ and let $V_0 = V/N$.

Define a sequence of seminorms $(\|\cdot\|_k)_{k<\lambda}$ on V_0 by

$$\|(x_n)_n + N\|_k = \|x_m\|_{X_m,k}$$

where m is the least integer such that $k < \lambda_{X_m}$ and $x_n = I_{m,n}(x_m)$ for all $n \geq m$. Note that these seminorms are well-defined and they form an increasing sequence whenever each X_n is graded since each I_n is a multi-isometric embedding.

To take a completion, we proceed as in the case of normed spaces. To this end, define an equivalence relation $\sim$ on the space of all Cauchy sequences in V_0 by declaring

$$(\sigma_n) \sim (\tau_n) \iff (\forall k < \lambda) \lim_n \|\sigma_n - \tau_n\|_k = 0,$$

where a sequence $(\sigma_n) \subseteq V_0$ is Cauchy if it is Cauchy with respect to each seminorm $\|\cdot\|_k$ on V_0. Let $\lim_n(X_n, I_n)$ denote the resulting quotient space equipped with the sequence of seminorms of length λ given by

$$|||[(\sigma_n)]_\sim|||_k = \lim_n \|\sigma_n\|_k \text{ for each } k < \lambda.$$

Observe that $(\lim_n(X_n, I_n), (|||\cdot|||_k)_{k<\lambda})$ is a complete, separated multi-seminormed space, and so it is a Fréchet space. Given $x \in V_0$, let $\mathcal{C}(x)$ denote the equivalence class (in $\lim_n(X_n, I_n)$) of the Cauchy sequence with constant value x. Then for each m the canonical mapping

$$I_m^{(\infty)} : X_m \to \lim_n(X_n, I_n) : x \mapsto \mathcal{C}\big((\overbrace{0, \ldots, 0}^{(m)}, x, I_{m,m+1}(x), I_{m,m+2}(x), \ldots)\big)$$

is a seminorm-preserving linear mapping. When X_m is separated, it is routine to check that $I_m^{(\infty)}$ is injective and hence a multi-isometric embedding. Note that union $\bigcup_n I_n^{(\infty)}(X_n)$ is dense in $\lim_n(X_n, I_n)$, in the sense that for every $[\sigma]_\sim \in \lim_n(X_n, I_n)$, there is a sequence (σ_m) belonging to the union which converges to $[\sigma]_\sim$ with respect to each $|||\cdot|||_k$. In particular, such a sequence converges to $[\sigma]_\sim$ with respect to the pseudometric induced by $\max_{l<k} |||\cdot|||_l$ for any given $k < \lambda$. From now on, whenever we are working with an inductive limit of separated spaces X_n, we will identify X_n with its image $I_n^{(\infty)}(X_n)$ in $\lim_n(X_n, I_n)$. In this way, each $I_n : X_n \to X_{n+1}$ becomes the corresponding inclusion mapping, so that (X_n) is an increasing sequence of finite-dimensional subspaces of $\lim_n(X_n, I_n)$ and X_n is equipped with the first λ_{X_n} seminorms induced by the inductive limit. Furthermore, $\bigcup_n X_n$ is dense in the inductive limit.

We will need the following consequence of the amalgamation property, the proof of which can be found in [16, Lemma 2.32] in the case of normed spaces. The proof is identical, but we include the details for the sake of completeness.

Lemma 2.3.2. *Suppose $\mathcal{K}$ is an amalgamation class with modulus ϖ. Fix $\varepsilon > 0$, $\Delta \subseteq \mathbb{R}^+$ finite and $\mathcal{A} \cup \{Y\} \subseteq \mathcal{K}$ finite. Then there is some $Z \in \mathcal{K}$ and $I \in \mathrm{Emb}(Y, Z)$ such that for every $X \in \mathcal{A}$, every*

$\delta \in \Delta$ and every $\gamma, \eta \in \mathrm{Emb}_\delta(X, Y)$, there is $J \in \mathrm{Emb}(Y, Z)$ such that

$$\max_{l<\lambda_X} \|I \circ \gamma - J \circ \eta\|_l \leq \varpi(\dim X, \lambda_X, \delta) + \varepsilon.$$

Proof. Fix enumerations $\mathcal{A} = (X_j)_{j=1}^m$ and $\Delta = (\delta_j)_{j=1}^n$. For each pair (j, k) such that $1 \leq j \leq m$ and $1 \leq k \leq n$, fix a finite $\varepsilon/3$-dense subset $(\gamma_r^{(j,k)})_{r=1}^s$ of $\mathrm{Emb}_{\delta_k}(X_j, Y)$. By repeatedly applying the amalgamation property of $\mathcal{K}$, we can find a sequence $(Z_r)_{r=1}^{s^2mn+1} \subseteq \mathcal{K}$ and a sequence $(I_r)_{r=1}^{s^2mn}$ such that:

1. $Z_1 = Y$.
2. $I_r \in \mathrm{Emb}(Z_r, Z_{r+1})$.
3. For every $1 \leq r_0, r_1 \leq n$, every $1 \leq j \leq m$ and $1 \leq k \leq n$, setting $\bar{r} := s^2(j-1)(k-1) + r_0 r_1 + 1$, there is some $J \in \mathrm{Emb}(Y, Z_{\bar{r}+1})$ such that

$$\|J \circ \gamma_{r_0}^{(j,k)} - I_{\bar{r}} \circ I_{\bar{r}-1} \circ \cdots \circ I_1 \circ \gamma_{r_1}^{(j,k)}\|_l \leq \varpi(\dim X_j, \lambda_{X_j}, \delta_k) + \varepsilon$$

   for each $l < \lambda_{X_j}$.

Then setting $Z = Z_{s^2mn+1}$ and $I := I_{s^2mn} \circ I_{s^2mn-1} \circ \cdots \circ I_1$ works. □

Before proceeding, we require one more piece of notation. Given two classes $\mathcal{K}$ and $\mathcal{K}'$ of finite-dimensional multi-seminormed spaces, write $\mathcal{K} \preceq \mathcal{K}'$ when for every $X \in \mathcal{K}$ there are $Y \in \mathcal{K}'$ and a multi-isometry of X onto Y, and write $\mathcal{K} \equiv \mathcal{K}'$ when $\mathcal{K} \preceq \mathcal{K}' \preceq \mathcal{K}$.

Theorem 2.3.3. *If $\mathcal{K}$ is an amalgamation class with modulus ϖ, then there is a separable $\mathcal{K}$-Fraïssé Fréchet space E with modulus ϖ^* such that $\mathcal{K} \preceq \mathrm{Age}_{<\omega}(E)$. Furthermore, if $\mathcal{K}$ is a Fraïssé class then $\mathrm{Age}_{<\omega}(E) \equiv \mathcal{K}$.*

Proof. Let $\{Z_n\}_{n<\omega}$ be a countable d_{BM}-dense subset of $\mathcal{K}$. Fix an enumeration (δ_n) of $\mathbb{Q} \cap [0, 1]$ such that $\delta_0 = 0$. Using Lemma 2.3.2, we find a sequence $(X_n, I_n)_{n<\omega}$ of separated spaces $X_n \in \mathcal{K}$ and multi-isometric embeddings $I_n \in \mathrm{Emb}(X_n, X_{n+1})$ with the following properties:

(a) Let $\lambda_{\mathcal{K}} := \sup_{X \in \mathcal{K}} \lambda_X$.

   (a) If $\lambda_{\mathcal{K}} = \lambda$, then $\lambda_{X_n} \geq n$ for all n.

   (b) If $\lambda_{\mathcal{K}} < \lambda$, then $\lambda_{X_n} = \lambda_{\mathcal{K}}$ for all n.

(b) For every $k \leq n$, every $X \in \{Z_j\}_{j \leq n} \cup \{X_j\}_{j \leq n}$ and every $\gamma, \eta \in \mathrm{Emb}_{\delta_k}(X, X_n)$, there is $J \in \mathrm{Emb}(X_n, X_{n+1})$ such that

$$\max_{l<\lambda_X} \|I_n \circ \gamma - J \circ \eta\|_l \leq \varpi(\dim X, \lambda_X, 2^{-n}) + 2^{-n}.$$

(c) $\mathrm{Emb}(Z_m, X_n) \neq \varnothing$ for each pair $m < n$.

Note that (c) can be arranged by applying the JEP of $\mathcal{K}$. Let $E = \lim_n (X_n, I_n)$. For simplicity, we assume from now on that (X_n) is an increasing sequence of subspaces of E, I_n is the inclusion mapping $X_n \subseteq X_{n+1}$, and $I_n^{(\infty)}$ is the inclusion mapping $X_n \subseteq E$. In particular, we work exclusively with the sequence of seminorms $(\|\cdot\|_k)_{k<\lambda_{\mathcal{K}}}$ associated to E.

The proof of the theorem will be complete once we prove the following two claims.

Claim 2.3.4. E is $\mathcal{K}$-Fraïssé.

Proof of Claim. Fix $X \in \mathcal{K}$ together with $\varepsilon > 0, \delta' > \delta \geq 0$ and $\gamma, \eta \in \mathrm{Emb}_\delta(X, E)$. Choose a large enough n such that $2^{-(n-1)} < \varepsilon$ and such that there are $j, k \leq n$ and a sufficiently small $\delta'' \geq 0$ with the following properties:

(i) $\delta < \delta_j < \delta'$.

(ii) $(\varpi(\dim X, \lambda_X, \delta_j) + \varepsilon)\delta'' < \varepsilon/3$.

(iii) There are $\theta \in \mathrm{Emb}_{\delta''}(X, Z_k)$ and $\widetilde{\gamma}, \widetilde{\eta} \in \mathrm{Emb}_{\varpi(\dim X, \lambda_X, 2^{-n})}(Z_k, X_n)$ such that

$$\max_{l<\lambda_X} \|\widetilde{\gamma} \circ \theta - \gamma\|_j \leq \varepsilon/3 \text{ and } \max_{l<\lambda_X} \|\widetilde{\eta} \circ \theta - \eta\|_j \leq \varepsilon/3.$$

Using the definition of the sequence (X_n, I_n), recursively construct sequences of embeddings

$$J_s \in \mathrm{Emb}(X_{n+2s}, X_{n+2s+1}) \ (s \geq 0) \text{ and } L_s \in \mathrm{Emb}(X_{n+2s-1}, X_{n+2s}) \ (s \geq 1)$$

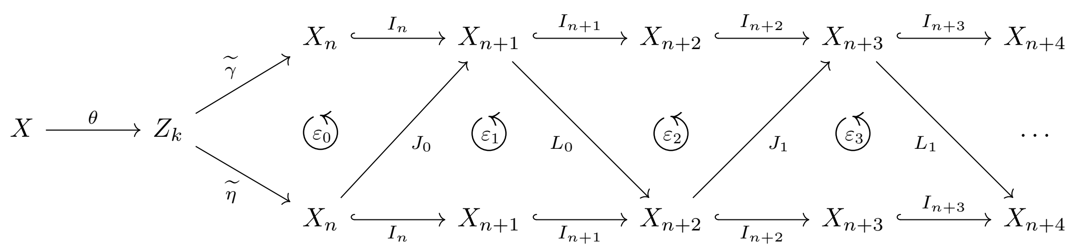
such that:

(iv) $\|J_0 \circ \widetilde{\eta} - \widetilde{\gamma}\|_l \leq \varpi(\dim X, \lambda_X, \delta_j) + 2^{-n}$ for each $l < \lambda_X$.

(v) $\|J_s \circ L_s - \mathrm{id}_{X_{n+2s-1}}\|_l \leq 2^{-(n+2s)}$ for each $s \geq 1$ and each $l < \lambda_{X_{n+2s-1}}$.

(vi) $\|L_{s+1} \circ J_s - \mathrm{id}_{X_{n+2s}}\|_l \leq 2^{-(n+2s+1)}$ for each $s \geq 0$ and each $l < \lambda_{X_{n+2s}}$.

Letting $\varepsilon_0 = \varpi(\dim X, \lambda_X, \delta_j) + 2^{-n}$ and $\varepsilon_m = 2^{-n+m}$ for $m \geq 1$, the above situation can be summarized by the following approximately commutative diagram:



Now, according to (v) and (vi), for each fixed $x \in X_{n+2s}$ and $l < \lambda_{X_{n+2s}}$ we have

$$\|J_{s+1}L_{s+1}J_s(x) - J_s(x)\|_l \leq 2^{-(n+2s+2)}\|J_s(x)\|_l = 2^{-(n+2s+2)}\|x\|_l.$$

On the other hand, we also have

$$\|J_{s+1}L_{s+1}J_s(x) - J_{s+1}(x)\|_l = \|J_{s+1}(L_{s+1}J_s(x) - x)\|_l = \|L_{s+1}J_s(x) - x\|_l \leq 2^{-(n+2s+1)}\|x\|_l$$

and so an application of the triangle inequality then yields

$$\|J_{s+1}(x) - J_s(x)\|_l \leq (2^{-(n+2s+2)} + 2^{-(n+2s+1)})\|x\|_l = 3 \cdot 2^{-(n+2s+2)}.$$

Then for every $s, t < \omega$ and every we have

$$\begin{aligned}\|J_{s+t}(x) - J_s(x)\|_l &\leq \sum_{0\leq i\leq t-1} \|J_{s+i+1}(x) - J_{s+i}(x)\|_l \leq \sum_{0\leq i\leq t-1} 3\cdot 2^{-(n+2(s+i)+2)}\|x\|_l \\ &= \frac{3}{2^{n+2s+2}} \sum_{0\leq i\leq t-1} 2^{-(2i)}\|x\|_l \leq \frac{3}{2^{n+2s+2}}\cdot 2\|x\|_l \leq \frac{3}{2^{n+2s+1}}\|x\|_l.\end{aligned} \tag{2.1}$$

In particular, this implies that for each $x \in \bigcup_{n<\omega} X_n$ the sequence $(J_s(x))_s$ is Cauchy with respect to each seminorm $\|\cdot\|_{E,l}$ for $l < \lambda_E$; indeed, given $l < \lambda_E$, simply choose a large enough N such that $l < \lambda_{X_n}$ and $x \in X_n$ for all $n > N$, noting that inequality (1) holds for all such n. Thus $(J_s)_{s<\omega}$ is pointwise Cauchy in E, so by completeness we can define a linear mapping $J : \bigcup_{n<\omega} X_n \to E$ by setting $J(x) = \lim_{s\geq k} J_s(x)$ where k is least such that $x \in X_k$; we then extend J to a mapping $J : E \to E$. Note that J is a seminorm-preversing linear mapping. To see that J is a bijection, we define as before a seminorm-preserving linear mapping $L : \bigcup_{n<\omega} X_n \to E$ by $L(y) = \lim_{s\geq k} L_s(y)$ where k is least such that $y \in X_k$, and we extend it to a mapping $L : E \to E$. Then, since E is separated, (v) and (vi) imply $L \circ J = J \circ L = \mathrm{id}_E$. Thus J and L are both multi-isometries. Finally, note that for each $l < \lambda_X$ we have

$$\|J_s \circ \widetilde{\eta} - \widetilde{\gamma}\|_l \leq \varpi(\dim X, \lambda_X, \delta_j) + \sum_{0\leq i\leq 2s} 2^{-(n+i)} \leq \varpi(\dim X, \lambda_X, \delta_j) + 2^{-(n-1)}.$$

Taking the limit as $s \to \infty$ we see that $\|J \circ \widetilde{\eta} - \widetilde{\gamma}\|_l \leq \varpi(\dim X, \lambda_X, \delta_j) + 2^{-(n-1)}$ and so

$$\begin{aligned}\|J\circ\eta - \gamma\|_l &\leq \|J\circ\eta - J\circ\widetilde{\eta}\circ\theta\|_l + \|J\circ\widetilde{\eta}\circ\theta - \widetilde{\gamma}\circ\theta\|_l + \|\widetilde{\gamma}\circ\theta - \gamma\|_l \\ &\leq \frac{\varepsilon}{3} + (\varpi(\dim X, \lambda_X, \delta_j) + 2^{-(n-1)})\|\theta\|_l + \frac{\varepsilon}{3} \\ &\leq \varpi(\dim X, \lambda_X, \delta_j) + \varepsilon \leq \varpi(\dim X, \lambda_X, \delta') + \varepsilon.\end{aligned}$$

Thus E is $\mathcal{K}$-Fraïssé with modulus ϖ^*. □

Claim 2.3.5. $\mathcal{K} \preceq \mathrm{Age}_{<\omega}(E)$. Furthermore, $\mathrm{Age}_{<\omega}(E) \preceq \mathcal{K}$ whenever $\mathcal{K}$ is Fraïsse.

Proof of Claim. Fix $X \in \mathcal{K}$. Since $\mathcal{K}$ is an amalgamation class, X isometrically embeds into a separated space X'. Thus we can assume without loss of generality that X itself is separated. Now, find a decreasing positive sequence $(\delta_n)_n$ such that $\varpi(\dim X, \lambda_X, \delta_n) \leq 2^{-n}$ for each n. Using the definition of the sequence $\{Z_n\}$ together with property (c), for each n we find some $\gamma_n \in \mathrm{Emb}_{\delta_n}(X, E)$. Since E is $\mathcal{K}$-Fraïssé, for each n we can choose $g_n \in \mathrm{Iso}(E)$ such that $\|g_n \circ \gamma_{n+1} - \gamma_n\|_l \leq 2^{-(n-1)}$ (where we take $\varepsilon = 2^{-n}$) for each $l < \lambda_X$. Define a sequence $(\eta_n)_n$ of multi-δ_n-isometric embeddings of X into E by setting $\eta_0 = \gamma_0$ and

$$\eta_{n+1} = g_0 \circ \cdots \circ g_n \circ \gamma_{n+1} \text{ for each } n > 0.$$

Then $(\eta_n)_n$ is pointwise Cauchy, since

$$\|\eta_{n+k} - \eta_n\|_l \leq \sum_{j=n}^{n+k-1} 2^{-(j-1)} \leq 2^{-(n-2)}$$

and so the limit $\eta : X \to E$ is a multi-isometric embedding; injectivity follows from the assumption that X is separated.

Finally, note that the construction of E implies $\mathrm{Age}_{<\omega}(E) \preceq \overline{\mathcal{K}}^{\mathrm{BM}}$, where the latter collection is the d_{BM}-closure of $\mathcal{K}$ in $\mathcal{M}_{<\omega}$. Thus, if $\mathcal{K}$ is a Fraïssé class, i.e. a d_{BM}-closed amalgamation class with the hereditary property, then the latter collection is precisely $\mathcal{K}$. Thus $\mathrm{Age}_{<\omega}(E) \preceq \mathcal{K}$ and so $\mathrm{Age}_{<\omega}(E) \equiv \mathcal{K}$ in this case. □

This completes the proof of the two claims and hence of the existence of a separable, $\mathcal{K}$-universal, $\mathcal{K}$-Fraïssé Fréchet space. □

We will henceforth refer to the space E constructed in Theorem 2.3.3 as the *Fraïssé limit* of the class $\mathcal{K}$ and we will denote it by $\mathrm{Flim}(\mathcal{K})$. Next we show that such a space is unique whenever $\mathcal{K}$ is a Fraïssé class. More generally, we have:

Proposition 2.3.6. *Suppose E and F are separable approximately ultrahomogeneous Fréchet spaces such that $\lambda_E = \lambda_F$ and $\mathrm{Age}_{<\omega}(E) \equiv \mathrm{Age}_{<\omega}(F)$. Then E and F are multi-isometric.*

Proof. Recursively define increasing sequences (X_n) and (Y_n) of elements of $\mathrm{Age}(E)$ and $\mathrm{Age}(F)$, respectively, sequences (k_n) and (l_n) of integers, and sequences of multi-isometric embeddings $(\gamma_n : X_n \to Y_n)$ and $(\eta_n : Y_n \to X_{n+1})$ such that the following conditions hold:

(i) (k_n) and (l_n) are non-decreasing and converge to $\lambda_E = \lambda_F$.

(ii) $X_n \in \mathrm{Age}_{k_n}(E)$ and $Y_n \in \mathrm{Age}_{l_n}(F)$.

(iii) $\|\eta_n \circ \gamma_n - \mathrm{id}_{X_n}\|_{E,m} \leq 2^{-n}$ for all $m < k_n$.

(iv) $\|\gamma_{n+1} \circ \eta_n - \mathrm{id}_{Y_n}\|_{F,m} \leq 2^{-n}$ for all $m < l_n$.

(v) $\bigcup_{n<\omega} X_n$ and $\bigcup_{n<\omega} Y_n$ are dense in E and F, respectively.

To start, let $X_0 = Y_0 = \{0\}, \gamma_0 = 0$ and $k_0 = 1$. Now assume we have defined $X_n, Y_n, \gamma_n, \eta_{n-1}, k_n$ and l_n for $n \geq 0$. Using the fact that $\mathrm{Age}_{<\omega}(E) \equiv \mathrm{Age}_{<\omega}(F)$, fix $\theta \in \mathrm{Emb}(Y_n, E)$. Since E is approximately ultrahomogeneous, we can find $g \in \mathrm{Iso}(E)$ such that

$$\|g \circ \theta \circ \eta_n - \mathrm{id}_{X_n}\|_{E,m} \leq 2^{-n} \text{ for every } m < k_n.$$

Let $\eta_n := g \circ \theta \in \mathrm{Emb}(Y_n, X)$, let $k_{n+1} = \min\{l_n + 1, \lambda_E\}$ and let X_{n+1} be a finite-dimensional subspace of E containing $X_n \cup \eta_n(Y_n)$, appropriately enlarged so that condition (iv) will eventually hold, and equipped with the first k_{n+1} seminorms from E. The construction of $Y_{n+1}, \gamma_{n+1} : Y_{n+1} \to X_{n+1}$ and l_{n+1} is similar. This complete the inductive construction.

Now, note that for each $x \in \bigcup_{n<\omega} X_n$ the sequence $(\gamma_n(x))_n$ is Cauchy with respect to each seminorm $\|\cdot\|_{F,m}$: Given $m < \omega$, we can choose a large enough N such that $m < \lambda_{X_n}$ and $x \in X_n$ for all $n > N$. Then $\|\gamma_{n+1}(x) - \gamma_n(x)\|_{F,m} \leq 2^{-(n-1)}$, which implies that the sequence $(\gamma_n)_{n<\omega}$ is pointwise Cauchy with respect to $\|\cdot\|_{F,m}$. Thus $(\gamma_n)_{n<\omega}$ is pointwise Cauchy in F, so by completeness we can define a linear mapping $\gamma : \bigcup_{n<\omega} X_n \to F$ by setting $\gamma(x) = \lim_{n\geq k} \gamma_n(x)$ where k is least such that $x \in X_k$. Similarly, we can define a multi-isometric embedding $\eta : \bigcup_{n<\omega} Y_n \to E$ by $\eta(y) = \lim_{n\geq k} \eta_n(y)$ where k is least such that $y \in Y_k$, and we extend it to a mapping $\eta : F \to E$. Then, since E and F are both separated, (iii) and (iv) imply $\gamma \circ \eta = \mathrm{id}_F$ and $\eta \circ \gamma = \mathrm{id}_E$, and so γ is a multi-isometry. □

Corollary 2.3.7 (Fraïssé correspondence)**.** *Let $\mathcal{K}$ be a class of finite-dimensional multi-seminormed spaces such that $\mathcal{K} \subseteq \mathcal{M}_{<\omega}$. The following are equivalent:*

(1) $\mathcal{K}$ is a Fraïssé class.

(2) $\mathcal{K} \equiv \mathrm{Age}_{<\omega}(E)$ for a unique separable Fraïssé Fréchet space $E = \mathrm{Flim}(\mathcal{K})$.

Proof. Suppose $\mathcal{K}$ is a Fraïssé class. By the previous two results, $\mathrm{Flim}(\mathcal{K})$ exists and is unique. By construction, we have $\mathcal{K} \equiv \mathrm{Age}_{<\omega}(\mathrm{Flim}(\mathcal{K}))$. To prove the other direction of the corollary, assume E is s separable Fraïssé Fréchet space such that $\mathcal{K} \equiv \mathrm{Age}_{<\omega}(E)$. Then $\mathcal{K}$ is hereditary, d_{BM}-compact class. Furthermore, the fact that E is separated implies that the class of separated elements of $\mathrm{Age}_{<\omega}(E)$ is cofinal in $\mathrm{Age}_{<\omega}$. Indeed, given $(X, (\|\cdot\|_n)_{n<m}) \in \mathrm{Age}_{<\omega}(E)$, we can use the fact that X is finite-dimensional to find a sufficiently large N such that $(X, (\|\cdot\|_{n<N}))$ becomes a separated subspace of E. From this observation, the amalgamation property of $\mathcal{K}$ follows from the Fraïssé property of E. Thus $\mathcal{K}$ is a Fraïssé class. □

Example 2.3.8. (1) Let $\mathbb{G}^\omega$ be the product of countably many copies of the Gurarij space $\mathbb{G}$. In [4] it is shown that there is a sequence $(\|\cdot\|_n)_{n<\omega}$ of seminorms on $\mathbb{G}^\omega$ such that $(\mathbb{G}^\omega, (\|\cdot\|_n)_{n<\omega})$ is a separable graded Fréchet space which is Fraïssé and universal for the class of all finite-dimensional graded multi-seminormed spaces with an infinite sequence of seminorms. It is also shown that there is a sequence $(\|\cdot\|'_n)_{n<\omega}$ of seminorms such that $(\mathbb{G}^\omega, (\|\cdot\|'_n)_{n<\omega})$ is a separable Fréchet space which is Fraïssé and universal for the class of all finite-dimensional multi-seminormed spaces with an infinite sequence of seminorms. (The authors of [4] do not use Fraïssé-theoretic terminology; however, this follows from [4, Proposition 4.1] and [4, Proposition 5.5], respectively.) Below we will show that these two spaces can be obtained as Fraïssé limits of the classes $\mathcal{G}_{<\omega}$ and $\mathcal{M}_{<\omega}$, respectively.

(2) For each $n \geq 1$, the spaces $\mathrm{Flim}(\mathcal{M}^{\mathrm{sep}}_{\leq n})$ and $\mathrm{Flim}(\mathcal{G}^{\mathrm{sep}}_{\leq n})$ can be seen as separated n-seminormed versions of the spaces considered in the previous example. (See Example 2.2.8 for the relevant notation.) Note that $\mathrm{Age}_{<\omega}(\mathrm{Flim}(\mathcal{M}^{\mathrm{sep}}_{\leq n}))$ is strictly larger than $\mathcal{M}^{\mathrm{sep}}_{\leq n}$, since the former collection contains non-separated multi-seminormed spaces. An analogous fact holds for $\mathrm{Flim}(\mathcal{G}^{\mathrm{sep}}_{\leq n})$.

(3) Let E be the space $(\mathbb{G}^\omega, (\|\cdot\|_n)_{n<\omega})$ considered in Example 2.3.8(1). For each $k \in \mathbb{N}$, let E_k be the multi-seminormed space $(\mathbb{G}^\omega, (\|\cdot\|_n)_{n<k})$ obtained by truncating the associated sequence of seminorms. Then using the properties of $(\mathbb{G}^\omega, (\|\cdot\|_n)_{n<\omega})$ it follows that E_k is universal and Fraïssé for $\mathcal{M}_{<k}$. An interesting special case occurs when $n = 1$, since E_1 can be seen as a seminormed version of the Gurarij space. In fact, if we let $\|\cdot\|$ be the seminorm on E_1, then the quotient $E_1/\ker\|\cdot\|$ equipped with the corresponding quotient norm is separable, universal and approximately ultrahomogeneous for the class of all finite-dimensional Banach spaces, and so it is isometric to the Gurarij space. Note that the spaces E_k are not separated, and so it is unclear if they are unique up to isometry.

(4) $E = \mathrm{Flim}(\mathcal{M}^{\mathcal{H}}_{<\omega})$ is the unique separable Fréchet-Hilbert space which is $\mathcal{M}^{\mathcal{H}}_{<\omega}$-Fraïssé. To see that E is indeed Fréchet-Hilbert, observe that each quotient space $E_{\|\cdot\|}$ (where $\|\cdot\|$ is a seminorm belonging to the sequence of seminorms associated to E) is approximately ultrahomogeneous for the class of all finite-dimensional Hilbert spaces, and hence is isometric to a Hilbert space.

(5) For each sequence $(p_n) \subseteq [1, \infty[$ with $p_n \notin \{4, 6, 8, \dots\}$, $\mathrm{Flim}(\mathcal{M}^{(p_n)}_{<\omega})$ is a Fraïssé Fréchet space. In particular, if $p_n = p$ for each n, then $\mathrm{Flim}(\mathcal{M}^{p}_{<\omega})$ can be seen as a *Fréchet-L_p-space*, which is the L_p analogue of the space considered in the previous example.

To conclude this section, we will use Proposition 2.3.6 to show that the spaces considered in Example 2.3.8(1) can be obtained as Fraïssé limits; we will make use of the fact that E is universal and Fraïssé for the class of all finite-dimensional (graded) multi-seminormed spaces with an infinite sequence of seminorms. From now until the end of the section, E will denote one of these two spaces. Before proceeding, we need some new terminology: Given $n < \omega$ and two multi-seminormed spaces X and Y such that $\lambda_X = \lambda_Y = \omega$, an *multi-isometric n-embedding* from X to Y is an injective linear mapping $f : X \to Y$ such that $\|f(x)\|_{Y,m} = \|x\|_{X,m}$ for each $x \in X$ and each $m < n$.

The following is a version of the near amalgamation property in the context of multi-isometric n-embeddings for a fixed n.

Lemma 2.3.9. *Suppose X, Y and Z are finite-dimensional separated multi-seminormed spaces with $\lambda_X = \lambda_Y = \lambda_Z = \omega$ and $f : X \to Y, g : X \to Z$ are multi-isometric n-embeddings for a fixed $n < \omega$. Then for every $\varepsilon > 0$ there is a finite-dimensional multi-seminormed space W with $\lambda_W = \omega$ and multi-isometric embeddings $i_Y : Y \to W$ and $i_Z : Z \to W$ such that $\|i_Y \circ f - i_Z \circ g\|_m \le \varepsilon$ for all $m < n$. Furthermore, $W \in \mathcal{G}$ whenever $X, Y, Z \in \mathcal{G}$.*

Proof. Fix all parameters and consider the sum $Y \oplus Z$ together with the canonical inclusion mappings $i_Y : Y \to Y \oplus Z$ and $i_Z : Z \to Y \oplus Z$. Equip $Y \oplus Z$ with a sequence $(\|\cdot\|_m)_{m<\omega}$ of seminorms defined by declaring

$$\|(y, z)\|_m := \inf\{\|u\|_{Y,m} + \|v\|_{Z,m} + \varepsilon\|x\|_{X,m} : x \in X, u \in Y, v \in Z, y = u + f(x), z = v - g(x)\}$$

for each $m < n$, and $\|(y, z)\|_m := \|y\|_{Y,m} + \|z\|_{Z,m}$ for each $m \ge n$. Let W be the sum $Y \oplus Z$ equipped with this sequence of seminorms. Note that W is a graded multi-seminormed space whenever X, Y and Z are graded. As in the proof of Lemma 2.2.4, it is straightforward to check that the inclusion mappings $i_Y : Y \to Y \oplus Z$ and $i_Z : Z \to Y \oplus Z$ are multi-isometric embeddings which satisfy $\|i_Y \circ f - i_Z \circ g\|_m \le \varepsilon$ for all $m < n$. □

Lemma 2.3.10. *Let X be a finite-dimensional subspace of E such that $\lambda_X = \omega$. For every $\varepsilon > 0, n \in \mathbb{N}$, finite-dimensional multi-seminormed space Y and multi-isometric n-embeddings $\eta : X \to E$ and $\gamma : X \to Y$, there is a multi-isometric embedding $f : Y \to E$ such that $\|f \circ \gamma - \eta\|_m \le \varepsilon$ for each $m < n$.*

Proof. Apply the previous lemma to $\varepsilon/2$, $\gamma : X \to Y$ and $\eta : X \to Z := \eta(X)$ to find the corresponding W, i_Y and i_Z. Since E is approximately ultrahomogeneous, there is $g \in \mathrm{Iso}(E)$ such that $\|g(i_Z(z)) - z\|_m \le \frac{\varepsilon}{2}\|z\|_m$ for all $z \in Z$ and $m \in \mathbb{N}$. Let $f = g \restriction_W \circ i_Y$. Then for each $x \in X$ and $m < n$ we have

$$\begin{aligned}\|f(\gamma(x)) - \eta(x)\|_m &\le \|g(i_Y(\gamma(x))) - g(i_Z(\eta(x)))\|_m + \|g(i_Z(\eta(x))) - \eta(x)\|_m \\ &\le \|g(i_Y(\gamma(x)) - i_Z(\eta(x)))\|_m + \frac{\varepsilon}{2}\|\eta(x)\|_m \\ &\le \frac{\varepsilon}{2}\|x\|_m + \frac{\varepsilon}{2}\|x\|_m \le \varepsilon\|x\|_m.\end{aligned}$$

Thus f is the desired multi-isometric embedding. □

Corollary 2.3.11. *$(\mathbb{G}^\omega, (\|\cdot\|_n)_{n<\omega})$ and $(\mathbb{G}^\omega, (\|\cdot\|'_n)_{n<\omega})$ are approximately ultrahomogeneous for $\mathcal{G}_{<\omega}$ and $\mathcal{M}_{<\omega}$, respectively. In particular, $(\mathbb{G}^\omega, (\|\cdot\|_n)_{n<\omega}) = \mathrm{Flim}(\mathcal{G}_{<\omega})$ and $(\mathbb{G}^\omega, (\|\cdot\|'_n)_{n<\omega}) = \mathrm{Flim}(\mathcal{M}_{<\omega})$.*

Proof. We only prove the result for $\mathcal{G}_{<\omega}$; the proof for $\mathcal{M}_{<\omega}$ is identical. Fix $X \in \mathcal{G}_{<\omega}$ together with multi-isometric embeddings $\gamma, \eta : X \to E$ for a fixed $\varepsilon > 0$. Extend the sequence of seminorms $(\|\cdot\|_{X,n})_{n<\lambda_X}$

to an infinite sequence in the natural way by declaring $\|\cdot\|_{X,m} = \|\cdot\|_{X,\lambda_X-1}$ for all $m \geq \lambda_X$. Then γ and η become multi-isometric λ_X-embeddings. Let $Y = \gamma(X) \subseteq E$ be equipped with the sequence of seminorms from E and apply the previous lemma to find a multi-isometric embedding $f : Y \to E$ such that $\|f(\gamma(x)) - \eta(x)\|_m \leq \frac{\varepsilon}{2}\|x\|_m$ for each $x \in X$ and $m < \lambda_X$. By the approximate ultrahomogeneity of E, there is $g \in \mathrm{Iso}(E)$ such that $\|g(y) - f(y)\|_m \leq \frac{\varepsilon}{2}\|y\|_m$ for each $y \in Y$ and $m \in \mathbb{N}$. Then

$$\|g(\gamma(x)) - \eta(x)\|_m \leq \|g(\gamma(x)) - f(\gamma(x))\|_m + \|f(\gamma(x)) - \eta(x)\|_m \leq \varepsilon\|x\|_m$$

for each $x \in X$ and $m < \lambda_X$, as required. □

2.4 The approximate Ramsey property

As in [6] or [16], we can characterize the extreme amenability of the group of multi-isometries of certain Fréchet spaces in terms of an approximate Ramsey property. Before stating the relevant Ramsey properties, we need some terminology. Given $r \in \mathbb{N}$, an *r-colouring* of a set X is simply a mapping $X \to r$. If X is equipped with a finite sequence of seminorms $(\|\cdot\|_m)_{m<\lambda_X}$ and $n \leq \lambda_X$, an *n-continuous colouring* of X is a mapping $c : X \to [0,1]$ such that

$$|c(x) - c(y)| \leq \max_{m<n} \|x - y\|_m \text{ for all } x, y \in X.$$

Thus, an n-continuous colouring is simply a mapping of X into $[0,1]$ which is 1-Lipschitz with respect to the pseudometric induced by $\max_{m<n} \|\cdot\|_m$.

Definition 2.4.1. Let $\mathcal{K}$ be a collection of finite-dimensional multi-seminormed spaces.

(a) $\mathcal{K}$ has the *discrete approximate Ramsey property* (discrete ARP) if for every $X, Y \in \mathcal{K}, r \in \mathbb{N}$ and $\varepsilon > 0$ there is $Z \in \mathcal{K}$ such that every r-colouring of $\mathrm{Emb}(X,Z)$ *ε-stabilizes* on a set of the form $\gamma \circ \mathrm{Emb}(X,Y)$ for some $\gamma \in \mathrm{Emb}(Y,Z)$, i.e. there is $i < r$ such that $\gamma \circ \mathrm{Emb}(X,Y)$ is contained in the set

$$(c^{-1}\{i\})_\varepsilon := \{\xi \in \mathrm{Emb}(X,Z) : \exists \eta \;\; c(\eta) = i \text{ and } \|\xi - \eta\|_m \leq \varepsilon \text{ for all } m < \lambda_X\}.$$

In this case we will also say that $\gamma \circ \mathrm{Emb}(X,Y)$ is *ε-monochromatic*.

(b) $\mathcal{K}$ has the *continuous approximate Ramsey property* (continuous ARP) if for every $X \in \mathcal{K}_{<\omega}, Y \in \mathcal{K}$ and $\varepsilon > 0$ there is $Z \in \mathcal{K}$ such that for every λ_X-continuous colouring c of $\mathrm{Emb}(X,Z)$ there is $\gamma \in \mathrm{Emb}(Y,Z)$ such that the *oscillation* of c on $\gamma \circ \mathrm{Emb}(X,Y)$, defined as

$$\mathrm{osc}(c \restriction \gamma \circ \mathrm{Emb}(X,Y)) := \sup\{|c(\gamma \circ \eta) - c(\gamma \circ \eta')| : \eta, \eta' \in \mathrm{Emb}(X,Y)\},$$

is at most ε. In this case we will say that c *ε-stabilizes* on $\gamma \circ \mathrm{Emb}(X,Y)$.

We will abbreviate the conclusions of the discrete and continuous ARP by writing $Z \to_\varepsilon (Y)^X_r$ and $Z \to_\varepsilon (Y)^X$, respectively. Exactly as in [6], it turns out that these two notions are equivalent.

Lemma 2.4.2. *Let $\mathcal{K} \subseteq \mathcal{M}_{<\omega}$. Then $\mathcal{K}$ satisfies the discrete ARP if and only if it satisfies the continuous ARP.*

Proof. Suppose first that $\mathcal{K}$ has the discrete ARP. Fix $X, Y \in \mathcal{K}$ and $\varepsilon > 0$. Let $D \subseteq [0,1]$ be a finite ε-dense set. Apply the discrete ARP with $|D|$-many colours to find $Z \in \mathcal{K}$ such that $Z \to_\varepsilon (Y)^X_{|D|}$. We claim that Z witnesses the continuous ARP for the above parameters. Indeed, given a λ_X-continuous colouring $c : \mathrm{Emb}(X, Z) \to [0,1]$, define a $|D|$-colouring $\widetilde{c} : \mathrm{Emb}(X, Z) \to D$ by the condition $|c(\varphi) - \widetilde{c}(\varphi)| \leq \varepsilon$ for every $\varphi \in \mathrm{Emb}(X, Z)$. Then there is $\gamma \in \mathrm{Emb}(Y, Z)$ such that $\widetilde{c}$ ε-stabilizes on $\gamma \circ \mathrm{Emb}(X, Y)$. It follows that c 4ε-stabilizes on $\gamma \circ \mathrm{Emb}(X, Y)$.

Now suppose $\mathcal{K}$ has the continuous ARP. We prove that $\mathcal{K}$ has the discrete ARP by induction on r, the number of colours. When $r = 1$ this is trivial, so suppose inductively that $\mathcal{K}$ satisfies the discrete ARP for r-colourings. Fix $X, Y \in \mathcal{K}$ and $\varepsilon > 0$. By the inductive hypothesis, there is $Z_0 \in \mathcal{K}$ such that $Z_0 \to_\varepsilon (Y)^X_r$. Since $\mathcal{K}$ has the continuous ARP, there is $Z \in \mathcal{K}$ such that $Z \to_\varepsilon (Z)^X$. We claim that Z witnesses the discrete ARP for the parameters X, Y, ε and $r + 1$. Indeed, fix a colouring $c : \mathrm{Emb}(X, Z) \to r + 1$ and define $\widetilde{c} : \mathrm{Emb}(X, Z) \to [0,1]$ by setting

$$\widetilde{c}(\varphi) = \min\left\{1, \inf_{\psi \in c^{-1}\{r\}} \max_{m < \lambda_X} \|\varphi - \psi\|_m\right\}.$$

It is routine to check that $\widetilde{c}$ is an λ_X-continuous colouring, so there is $\gamma \in \mathrm{Emb}(Z_0, Z)$ such that $\widetilde{c}$ ε-stabilizes on $\gamma \circ \mathrm{Emb}(X, Z_0)$. If there is some $\varphi \in \mathrm{Emb}(X, Z_0)$ such that $c(\gamma \circ \varphi) = r$, then $\gamma \circ \mathrm{Emb}(X, Z_0) \subseteq (c^{-1}\{r\})_\varepsilon$ and so we are done since then c ε-stabilizes on $\gamma \circ \gamma_0 \circ \mathrm{Emb}(X, Y)$ for any choice of $\gamma_0 \in \mathrm{Emb}(Y, Z_0)$. If no such φ exists, we can define an r-colouring d of $\mathrm{Emb}(X, Z_0)$ by setting $d(\varphi) = c(\gamma \circ \varphi)$. By definition of Z_0, there is $\gamma_0 \in \mathrm{Emb}(Y, Z_0)$ such that d ε-stabilizes on $\gamma_0 \circ \mathrm{Emb}(X, Y)$. It then follows that c ε-stabilizes on $\gamma \circ \gamma_0 \circ \mathrm{Emb}(X, Y)$. □

The next lemma is a particular instance of a more general phenomenon which rephrases the Ramsey property of an age in terms of its limit. The proof is similar to that of [48, Proposition 3.4]. Recall that $\mathrm{Age}_n(E)$ denotes the collection of all finite-dimensional linear subspaces $X \subseteq E$ equipped with the first n seminorms from E.

Lemma 2.4.3. *Suppose E is an approximately ultrahomogeneous multi-seminormed space. Then $\mathrm{Age}_n(E)$ has the continuous ARP if and only if for every $X, Y \in \mathrm{Age}_n(E), \varepsilon > 0$ and λ_X-continuous colouring c of $\mathrm{Emb}(X, E)$, there is $\gamma \in \mathrm{Emb}(Y, E)$ such that $\mathrm{osc}(c \restriction \gamma \circ \mathrm{Emb}(X, Y)) \leq \varepsilon$.*

Proof. The left-to-right direction is straightforward and will not be used in what follows, so we will only show the right-to-left direction. We will prove the contrapositive. First, let $H = \{\eta_1, \dots, \eta_k\}$ be a finite $\varepsilon/3$-dense subset of $\mathrm{Emb}(X, Y)$, where the latter set is equipped with the pseudometric induced by $\max_{m<\lambda_X} \|\cdot\|_m$. We will need the following claim, the proof of which is routine.

Claim 2.4.4. Suppose there is $Z \in \mathrm{Age}_n(E)$ such that for every λ_X-continuous colouring of $\mathrm{Emb}(X, Z)$, there is $\gamma \in \mathrm{Emb}(Y, Z)$ such that $\mathrm{osc}(c \restriction \gamma \circ H) \leq \varepsilon/3$. Then $Z \to_\varepsilon (Y)^X$.

Now, if $\mathrm{Age}_n(E)$ does not have the continuous ARP, then there are $X, Y \in \mathrm{Age}_n(E)$ and $\varepsilon > 0$ such that no $Z \in \mathrm{Age}_n(E)$ witnesses $Z \to_\varepsilon (Y)^X$. Thus, by the Claim, for each such Z we can fix a bad λ_X-continuous colouring c_Z such that $\mathrm{osc}(c_Z \restriction \gamma \circ H) \geq \varepsilon$ for any choice of $\gamma \in \mathrm{Emb}(Y, Z)$. Fix an ultrafilter $\mathcal{U}$ on $\mathrm{Age}_n(E)$ such that

$$\{W \in \mathrm{Age}_n(E) : V \subseteq W\} \in \mathcal{U} \text{ for each } V \in \mathrm{Age}_n(E).$$

Define a mapping $c : \mathrm{Emb}(X, E) \to [0,1]$ by setting $c(\gamma) = \lim_{\mathcal{U}} c_Z(\gamma)$. Note that the ultralimit exists (by boundedness) and is well-defined since $\{W \in \mathrm{Age}_n(E) : \gamma(X) \subseteq W\} \in \mathcal{U}$. Furthermore, c is an λ_X-continuous colouring. We claim that c is a bad colouring of $\mathrm{Emb}(X, E)$. To this end, take any $\rho \in \mathrm{Emb}(Y, E)$ and note that $\{W \in \mathrm{Age}_n(E) : \rho(Y) \subseteq W\} \in \mathcal{U}$. Furthermore, for any such W we have $\rho \in \mathrm{Emb}(Y, W)$ and so by our initial hypothesis we know

$$|c_W(\rho \circ \eta_i) - c_W(\rho \circ \eta_j)| > \varepsilon \text{ for some } i, j \in \{1, \ldots, k\}.$$

Since $\mathcal{U}$ is an ultrafilter, it follows that there are $i, j \in \{1, \ldots, k\}$ such that

$$\{W \in \mathrm{Age}_n(E) : |c_W(\rho \circ \eta_i) - c_W(\rho \circ \eta_j)| > \varepsilon\} \in \mathcal{U}.$$

It then follows that $|c(\rho \circ \eta_i) - c(\rho \circ \eta_j)| > \varepsilon$. Since ρ was arbitrary, we see that c is a bad colouring of $\mathrm{Emb}(X, E)$. □

The following is the KPT correspondence for multi-seminormed spaces (cf. [38, 48]).

Theorem 2.4.5 (Kechris-Pestov-Todorčević correspondence). *Suppose E is an infinite-dimensional multi-seminormed space which is approximately ultrahomogeneous. The following are equivalent:*

(i) $\mathrm{Age}_n(E)$ *has the ARP for each $n \in \mathbb{N}$ such that $1 \leq n \leq \lambda_E$.*

(ii) $\mathrm{Iso}(E)$ *is extremely amenable when endowed with the topology of pointwise convergence.*

Proof. $(i) \to (ii)$: Fix an $\mathrm{Iso}(E)$-flow $\mathrm{Iso}(E) \curvearrowright K$ for K compact, $\varepsilon > 0, p \in K$, an entourage U and a finite $F \subseteq \mathrm{Iso}(E)$. By one of the well-known characterizations of extreme amenability (see [57] or [16, Claim 5.11.2]) it is enough to find $g \in \mathrm{Iso}(E)$ such that $F \cdot (g \cdot p)$ is U-small, i.e. such that

$$(f_0 \cdot (g \cdot p), f_1 \cdot (g \cdot p)) \in U \text{ for each } f_0, f_1 \in F.$$

Before proceeding, we will define a directed family of pseudometrics which generate the left uniformity of $\mathrm{Iso}(E)$: For each $n \in \mathbb{N}$ such that $n \leq \lambda_E$ and each finite-dimensional $X \subseteq E$, define a pseudometric d_X^n on $\mathrm{Iso}(E)$ by

$$d_X^n(g, h) = \max_{m<n} \|g \restriction X - h \restriction X\|_m.$$

We can assume without loss of generality that all entourages are symmetric. Fix an entourage V such that $V \circ V \circ V \circ V \subseteq U$. Since the mapping $\mathrm{Iso}(E) \to K : g \to g^{-1} \cdot p$ is left uniformly continuous, there are n, $X \subseteq E$ and $\delta > 0$ such that

$$d_X^n(g, h) \leq \delta \text{ implies } (g^{-1} \cdot p, h^{-1} \cdot p) \in V.$$

Equip X with the first n seminorms induced from E, so that $X \in \mathrm{Age}_n(E)$. Let Y be any member of $\mathrm{Age}_n(E)$ containing $\bigcup\{g(X) : g^{-1} \in F \cup \{\mathrm{id}\}\}$. Fix a finite set $\{x_i\}_{i<r} \subseteq K$ such that $K \subseteq \bigcup_{i<r} V[x_i]$, where $V[x] = \{y \in K : (x, y) \in V\}$. Apply the approximate Ramsey property of $\mathrm{Age}(E)$ to the parameters $n, X, Y, \delta/3$ and r to obtain $Z \in \mathrm{Age}_n(E)$ such that

$$Z \to_{\delta/3, n} (Y)_r^X.$$

Define a colouring $c : \mathrm{Emb}(X,Z) \to r$ by first choosing, for each $\gamma \in \mathrm{Emb}(X,Z)$, some $g_\gamma \in \mathrm{Iso}(E)$ such that $\max_{m<n} \|g_\gamma \restriction X - \gamma\|_m \le \delta/3$; such a choice is possible by the approximate ultrahomogeneity of E. Then define $c(\gamma) = i$ if $i < r$ is the least index such that $g_\gamma^{-1} \cdot p \in V[x_i]$. By definition of Z there are $\rho \in \mathrm{Emb}(Y,Z)$ and $i < r$ such that

$$\rho \circ \mathrm{Emb}(X,Y) \subseteq (c^{-1}\{i\})_{\delta/3,n}.$$

In particular, this implies that for each $\eta \in \mathrm{Emb}(X,Y)$ there is $h_\eta \in \mathrm{Iso}(E)$ such that $\max_{m<n} \|\rho\circ\eta - h_\eta \restriction X\|_m \le 2\delta/3$ and $h_\eta^{-1} \cdot p \in V[x_i]$. Choose $g \in \mathrm{Iso}(E)$ such that $\max_{m<n} \|g \restriction Y - \rho\|_m \le \delta/3$. Now, given $f_0, f_1 \in F$, let $\eta_j := f_j^{-1} \restriction X$ for $j = 0,1$ and note $\eta_j \in \mathrm{Emb}(X,Y)$. Then for each $j = 0,1$,

$$\begin{aligned} d_X^n(g \circ f_j^{-1}, h_{\eta_j}) &= \max_{m<n} \|g \circ \eta_j - h_{\eta_j} \restriction X\|_m \\ &\le \max_{m<n} \|g \circ \eta_j - \rho \circ \eta_j\|_m + \max_{m<n} \|\rho \circ \eta_j - h_{\eta_j} \restriction X\|_m \\ &\le \delta/3 + 2\delta/3 = \delta. \end{aligned}$$

By choice of δ, this implies $(f_j \circ g^{-1} \cdot p, h_{\eta_j}^{-1} \cdot p) \in V$ for each j. Since $(x_i, h_{\eta_j}^{-1} \cdot p) \in V$ for each j, our choice of V implies $(f_0 \circ g^{-1} \cdot p, f_1 \circ g^{-1} \cdot p) \in U$. Thus $F \cdot (g^{-1} \cdot p)$ is U-small.

$(ii) \to (i)$: To prove the ARP, we use the previous two lemmata together with the following characterization of extreme amenability in terms of the family of pseudometrics defined above. (See [57] or [48, Proposition 3.9].)

- $(*)$ $\mathrm{Iso}(E)$ is extremely amenable if and only if, for every finite $F \subseteq \mathrm{Iso}(E), \varepsilon > 0, X \subseteq E, n \in [1, \lambda_E] \cap \mathbb{N}$ and 1-Lipschitz map $f : (\mathrm{Iso}(E), d_X^n) \to [0,1]$, there is $g \in \mathrm{Iso}(E)$ such that $\mathrm{osc}(f \restriction g \circ F) \le \varepsilon$.

Fix $X, Y \in \mathrm{Age}_n(E)$ together with $\varepsilon > 0$ and an n-continuous colouring c of $\mathrm{Emb}(X,E)$. Let $H = \{\eta_1, \ldots, \eta_k\}$ be a finite ε-dense subset of $\mathrm{Emb}(X,Y)$ where the latter set is equipped with the pseudometric induced by $\max_{m<n} \|\cdot\|_m$. Apply the approximate ultrahomogeneity of E to find $F = \{g_1, \ldots, g_k\} \subseteq \mathrm{Iso}(E)$ such that $\|g_i \restriction X - \eta_i\|_m \le \varepsilon$ for all $m < n$ and all $i \le k$. Let $\widetilde{c} : (\mathrm{Iso}(E), d_n^X) \to [0,1]$ be the 1-Lipschitz mapping defined by $\widetilde{c}(g) = g \restriction X$ and use $(*)$ to find $g \in \mathrm{Iso}(E)$ such that $\mathrm{osc}(\widetilde{c} \restriction g \cdot F) \le \varepsilon$. Then for any $i, j \le k$, by the triangle inequality the term $|c(g \circ \eta_i) - c(g \circ \eta_j)|$ is bounded above by

$$|c(g \circ \eta_i) - c(g \circ g_i \restriction X)| + |c(g \circ g_i \restriction X) - c(g \circ g_j \restriction X)| + |c(g \circ g_j \restriction X) - c(g \circ \eta_j)|.$$

Since c is an n-continuous colouring, the first term is bounded above by

$$\max_{m<n} \|g \circ \eta_i - g \circ g_i \restriction X\|_m = \max_{m<n} \|\eta_i - g_i \restriction X\|_m \le \varepsilon$$

by our choice of g_i. Similarly, the third term is bounded above by ε. To bound the second term, note that

$$|c(g \circ g_i \restriction X) - c(g \circ g_j \restriction X)| = |\widetilde{c}(g \circ g_i) - \widetilde{c}(g \circ g_j)| \le \varepsilon$$

by our choice of g. Thus, if we let $\gamma = g \restriction Y$, we see that the oscillation of c on $\gamma \circ H$ is bounded by 3ε. Then, using the definition of the η_i, it follows that $\mathrm{osc}(\gamma \circ \mathrm{Emb}(X,Y)) \le 5\varepsilon$. □

Our next goal is to show that various classes of finite-dimensional multi-seminormed spaces have the ARP, which will allow us to apply the KPT correspondence in certain instances to obtain examples of extremely amenable multi-isometry groups. As in the case of the amalgamation property, our main source of examples of such classes will come from classes of finite-dimensional normed spaces which are known to have the ARP in the context of normed spaces. The key lemma is the following:

Lemma 2.4.6. *Suppose $\mathcal{K}_1, \ldots, \mathcal{K}_n$ are classes of finite-dimensional normed spaces with the ARP. Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ be finite-dimensional seminormed spaces such that $(X_i)_{\|\cdot\|}$ and $(Y_i)_{\|\cdot\|}$ belong to $\mathcal{K}_i$ for each $i \leq n$. Let $\varepsilon > 0$ and $r < \omega$. Then there are finite-dimensional seminormed spaces $Z_1, \ldots, Z_n$ such $(Z_i)_{\|\cdot\|} \in \mathcal{K}_i$ for each i and, for every colouring $c : \prod_{j=1}^n \mathrm{Emb}(X_j, Z_j) \to r$, there are $\rho_j \in \mathrm{Emb}(Y_j, Z_j)$, $j = 1, \ldots, n$, such that*

$$\prod_{j=1}^{n} \rho_j \circ \mathrm{Emb}(X_j, Y_j) \text{ is } \varepsilon\text{-monochromatic.}$$

The proof will involve a standard strategy for obtaining product Ramsey properties. First we will need:

Lemma 2.4.7. *Suppose $\mathcal{K}$ is a class of finite-dimensional normed spaces with the ARP. For any finite-dimensional seminormed spaces X and Y such that $X_{\|\cdot\|}$ and $Y_{\|\cdot\|}$ belong to $\mathcal{K}$, any $\varepsilon > 0$ and $r \in \mathbb{N}$, there is a finite-dimensional seminormed space Z such that $Z_{\|\cdot\|} \in \mathcal{K}$ and every colouring $c : \mathrm{Emb}(X, Z) \to r$ ε-stabilizes on a set of the form $\rho \circ \mathrm{Emb}(X, Y)$ for $\rho \in \mathrm{Emb}(Y, Z)$.*

Proof. Let X, Y be finite-dimensional seminormed spaces, $\varepsilon > 0$ and $r \in \mathbb{N}$. Let $\widetilde{X} = X/\ker\|\cdot\|_X$ and $\widetilde{Y} = Y/\ker\|\cdot\|_Y$ be equipped with the norms $\|[x]\| = \|x\|_X$ and $\|[y]\| = \|y\|_Y$ respectively. By the ARP of $\mathcal{K}$, there is a finite-dimensional normed space $Z \in \mathcal{K}$ such that

$$Z \to_\varepsilon (\widetilde{Y})_r^{\widetilde{X}}.$$

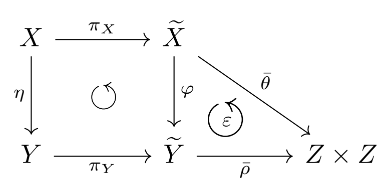
Consider the product $Z \times Z$ equipped with the seminorm $\|(z_1, z_2)\| := \|z_1\|_Z$. We claim that this space witnesses the ARP for the parameters X, Y, ε, r. Note that $Z \times Z \in \mathcal{K}_{\|\cdot\|}$ since the associated kernel is isomorphic to Z. Now, fix a colouring $c : \mathrm{Emb}(X, Z) \to r$ and define $\widetilde{c} : \mathrm{Emb}(\widetilde{X}, Z) \to r$ by $\widetilde{c}(\gamma) = c(\gamma \circ \pi_X)$ where $\pi_X : X \to \widetilde{X}$ is the canonical surjection. By definition of Z, there is $\rho \in \mathrm{Emb}(\widetilde{Y}, Z)$ such that

$$\rho \circ \mathrm{Emb}(\widetilde{X}, \widetilde{Y}) \subseteq (\widetilde{c}^{-1}\{i\})_\varepsilon \text{ for some } i < r.$$

Let $\bar{\rho} : \widetilde{Y} \to Z \times Z$ be defined by setting $\bar{\rho}(y) = (\rho(y), 0)$. We will show

$$(\bar{\rho} \circ \pi_Y) \circ \mathrm{Emb}(X, Y) \subseteq (c^{-1}\{i\})_\varepsilon.$$

To this end, fix $\eta \in \mathrm{Emb}(X, Y)$ and define a mapping $\varphi : \widetilde{X} \to \widetilde{Y}$ by $\varphi([x]) = \pi_Y(\eta(x))$; it is easy to check that φ is a well-defined multi-isometric embedding which satisfies $\varphi \circ \pi_X = \pi_Y \circ \eta$. Then by definition of ρ there is $\theta \in \widetilde{c}^{-1}\{i\}$ such that $\|\rho \circ \varphi - \theta\| \leq \varepsilon$. Let $\bar{\theta} : \widetilde{X} \to Z \times Z$ be given by $\bar{\theta}(x) = (\theta(x), 0)$. The situation is summarized by the following diagram:



Then $c(\theta \circ \pi_X) = i$ and, since

$$(\rho \circ \pi_Y) \circ \eta = \rho \circ (\pi_Y \circ \eta) = \rho \circ (\varphi \circ \pi_X),$$

we have

$$\|(\rho \circ \pi_Y) \circ \eta - \theta \circ \pi_X\| = \|(\rho \circ \varphi) \circ \pi_X - \theta \circ \pi_X\| \leq \|\rho \circ \varphi - \theta\| \leq \varepsilon.$$

Thus $\rho \circ \pi_Y$ is the desired embedding. □

Proof of Lemma 2.4.6. The proof is by induction on n. The case $n = 1$ follows from the previous lemma, so fix all parameters and suppose that we can find normed spaces $Z_1, \dots, Z_n$ such that every r-colouring of $\prod_{j=1}^n \mathrm{Emb}(X_j, Z_j)$ has a $\varepsilon/2$-monochromatic set of the form $\prod_{j=1}^n \rho_j \circ \mathrm{Emb}(X_j, Y_j)$. Let D be a finite $\varepsilon/2$-dense subset of $\prod_{j=1}^n \mathrm{Emb}(X_j, Z_j)$, where the seminorm on the product is the maximum of the given seminorms. Apply Lemma 2.4.7 to X_{n+1} and Y_{n+1} to find a seminormed space Z that works for the error $\varepsilon/2$, and the number of colours being the cardinality of ${}^D r$. We claim that $Z_1, \dots, Z_n, Z$ works. For suppose that $c : \prod_{j=1}^{n+1} \mathrm{Emb}(X_j, Z_j) \to r$. We have the induced colouring $\widehat{c} : \mathrm{Emb}(X_{n+1}, Z) \to {}^D r$,

$$\widehat{c}(\xi)(\eta_1, \dots, \eta_n) := c(\eta_1, \dots, \eta_n, \xi) \in r \text{ for every } (\eta_1, \dots, \eta_n) \in D.$$

By the choice of Z there is some $\rho \in \mathrm{Emb}(Y_{n+1}, Z)$ such that $\rho \circ \mathrm{Emb}(X_{n+1}, Y_{n+1})$ is $\varepsilon/2$-monochromatic for $\widehat{c}$ with colour $\theta \in {}^D r$. The mapping $\theta : D \to r$ defines an r-colouring $\widehat{\theta} : \prod_{j=1}^n \mathrm{Emb}(X_j, Z_j) \to r$ by $\varepsilon/2$-proximity. For each $j = 1, \dots, n$, let $\rho_j \in \mathrm{Emb}(Y_j, Z_j)$ be such that $\prod_{j=1}^n \rho_j \circ \mathrm{Emb}(X_j, Y_j)$ is $\varepsilon/2$-monochromatic for $\widehat{\theta}$ with colour $s \in r$. Then the set

$$\left(\prod_{j=1}^n \rho_j \circ \mathrm{Emb}(X_j, Y_j) \right) \times (\rho \circ \mathrm{Emb}(X_{n+1}, Y_{n+1})) \subseteq \prod_{j=1}^{n+1} \mathrm{Emb}(X_j, Z_j)$$

is ε-monochromatic for c with colour s: Let $(\gamma_j)_{j=1}^{n+1} \in \prod_{j=1}^{n+1} \mathrm{Emb}(X_j, Y_j)$. There is some $(\mu_j)_{j=1}^n \in \prod_{j=1}^n \mathrm{Emb}(X_j, Z_j)$ such that $\widehat{\theta}((\mu_j)_{j=1}^n) = s$ and $(\mu_j)_{j=1}^n$ is $\varepsilon/2$-close to $(\rho_j \circ \gamma_j)_{j=1}^n$. Choose $(\pi_j)_{j=1}^n \in D$ that is $\varepsilon/2$-close to $(\mu_j)_{j=1}^n$ such that $\theta((\pi_j)_{j=1}^n) = s$. Let $\mu \in \mathrm{Emb}(X, Z)$ be such that μ is $\varepsilon/2$-close to $\rho \circ \gamma$ and $\widehat{c}(\mu) = \theta$. This last equality means by definition that

$$s = \theta((\mu_j)_{j=1}^n) = \widehat{c}(\mu)((\mu_j)_{j=1}^n) = c(\mu_1, \dots, \mu_n, \mu).$$

Finally, observe that μ is $\varepsilon/2$-close to ρ, and each μ_j is ε-close to $\rho_j \circ \gamma_j$. □

For the next proposition, recall the definition of the classes $\langle \bar{\mathcal{K}} \rangle_{=n}$ from Definition 2.2.6.

Proposition 2.4.8. *Let $\bar{\mathcal{K}} = \{\mathcal{K}_n\}_{n<\omega}$ be a collection of finite-dimensional normed spaces such that each $\mathcal{K}_n$ has the ARP. For every $n \in \mathbb{N}, X, Y \in \langle \bar{\mathcal{K}} \rangle_{=n}$, $r \in \mathbb{N}$, and every $\varepsilon > 0$ there is $Z \in \langle \bar{\mathcal{K}} \rangle$ such that every r-colouring of $\mathrm{Emb}(X, Z)$ has an ε-monochromatic set of the form $\rho \circ \mathrm{Emb}(X, Y)$ for some $\rho \in \mathrm{Emb}(Y, Z)$. Furthermore, if $\mathcal{K}_n = \mathcal{K}$ for each n and $\mathcal{K}$ is closed under ℓ_∞-sums, then Z can be chosen to be graded when X and Y are graded.*

Proof. Fix all parameters and apply the previous lemma to $(X, \|\cdot\|_j)_{j=1}^n$, $(Y, \|\cdot\|_j)_{j=1}^n$, ε and r to find the corresponding $Z_1, \dots, Z_n$. Let $Z := \prod_{j=1}^n Z_j$, and for each $j = 1, \dots, n$, let

$$\|(z_1, \dots, z_n)\|_j := \|z_j\|_{Z_j}.$$

Note that $Z \in \langle \bar{\mathcal{K}} \rangle_{=n}$ by construction. We claim that Z witnesses the ARP for the given parameters. For suppose that $c : \mathrm{Emb}(X, Z) \to r$ is a colouring. Given a sequence $\vec{\gamma} = (\gamma_j)_j \in \prod_{j=1}^n \mathrm{Emb}((X, \|\cdot\|_j), Z_j)$, define a mapping $F(\vec{\gamma}) : X \to Z$ by

$$F(\vec{\gamma})(x) = (\gamma_1(x), \dots, \gamma_n(x)).$$

Observe that $F(\vec{\gamma}) \in \mathrm{Emb}(X, Z)$ since each γ_j is a multi-isometric embedding. Using this mapping, define an induced colouring $\widehat{c} : \prod_{j=1}^n \mathrm{Emb}((X, \|\cdot\|_j), Z_j) \to r$, by $\widehat{c}(\vec{\gamma}) = c(F(\vec{\gamma}))$ where $\vec{\gamma} = (\gamma_j)_j$. By definition of each Z_j, there are $\rho_j \in \mathrm{Emb}((Y, \|\cdot\|_j), Z_j)$ such that

$$\prod_{j=1}^n \rho_j \circ \mathrm{Emb}((X, \|\cdot\|_j), (Y_j, \|\cdot\|_j)) \text{ is } \varepsilon\text{-monochromatic.}$$

Define $\rho \in \mathrm{Emb}(Y, Z)$ by $\rho(y) = (\rho_1(y), \dots, \rho_n(y))$. Note that $\rho \circ \eta = F((\rho_j \circ \eta)_j)$ and so $c(\rho \circ \eta) = \widehat{c}((\rho_j \circ \eta)_j)$. In particular, it follows that $\rho \circ \mathrm{Emb}(X, Y)$ is ε-monochromatic.

In the case where $\mathcal{K}_n = \mathcal{K}$ for all n and $\mathcal{K}$ is closed under ℓ_∞-sums, we work with the same underlying space Z but we instead equip it with the sequence of seminorms given by

$$\|(z_1, \dots, z_n)\|_j := \max_{i \le j} \|z_i\|_{Z_i}.$$

Then $Z_{\|\cdot\|_j}$ is multi-isometric to the ℓ_∞-sum of $Z_i, i \le j$, and so $Z \in \langle \bar{\mathcal{K}} \rangle_{=n}$ by our additional assumption on $\mathcal{K}$. The rest of the proof is identical to that of the general case. □

By appealing to the various known approximate Ramsey properties of classes of finite-dimensional normed spaces as considered in [6, 16], we can apply Theorem 2.4.5 and Proposition 2.4.8 together to obtain the following result. The notation used below corresponds to that of Example 2.2.8.

Theorem 2.4.9. *The following groups are extremely amenable when equipped with the topology of pointwise convergence:*

(1) $\mathrm{Iso}(\mathbb{G}^\omega, (\|\cdot\|_n)_{n<\alpha})$ *and* $\mathrm{Iso}(\mathbb{G}^\omega, (\|\cdot\|'_n)_{n<\alpha})$ *for each* $\alpha \le \omega$. *In particular, the multi-isometry groups of the spaces constructed in [4] are extremely amenable.*

(2) $\mathrm{Iso}(\mathrm{Flim}(\mathcal{M}^{\mathcal{H}}_{<\omega}))$.

(3) $\mathrm{Iso}(\mathrm{Flim}(\mathcal{M}^{(p_n)}_{<\omega}))$ *for any sequence* $(p_n) \subseteq [1, \infty[$ *with* $p_n \notin \{4, 6, 8, \dots\}$.

The extreme amenability of the groups $\mathrm{Iso}(\mathbb{G}^\omega, (\|\cdot\|_n)_{n<\omega})$ and $\mathrm{Iso}(\mathbb{G}^\omega, (\|\cdot\|'_n)_{n<\omega})$ should naturally be compared to the extreme amenability of $\mathrm{Iso}(\mathbb{G})$. The following important questions remain open:

Question 2.4.10. Are $\mathrm{Iso}(\mathbb{G}^\omega, (\|\cdot\|_n)_{n<\omega})$ and $\mathrm{Iso}(\mathbb{G}^\omega, (\|\cdot\|'_n)_{n<\omega})$ topologically distinguishable from $\mathrm{Iso}(\mathbb{G})$? Are $\mathrm{Iso}(\mathbb{G}^\omega, (\|\cdot\|_n)_{n<\omega})$ and $\mathrm{Iso}(\mathbb{G}^\omega, (\|\cdot\|'_n)_{n<\omega})$ universal Polish groups?

Throughout this chapter we have adopted a very particular viewpoint in order to develop Fraïssé theory for Fréchet spaces. Specifically, we have been working with the category of multi-seminormed spaces with morphisms given by seminorm-preserving linear mappings. A natural question is whether or not a similar theory can be developed for more relaxed categories which capture the notion of a Fréchet space. For instance, the following general problem remains open:

Problem 2.4.11. Develop a theory of *metric* Fréchet spaces, i.e. pairs of the form (X, d) where X is a vector space and d is a translation-invariant metric inducing a Fréchet topology on X.

In the above setting, one may be able to use more general machinery in order to develop a Fraïssé theory for such spaces, e.g. as in [7, 40]. It is unclear if the use of such machinery is possible in the setting of multi-seminormed spaces, since in general one needs to work with an arbitrarily large (finite) number of seminorms.

One specific corollary of our construction is that the spaces $\mathrm{Flim}(\mathcal{M}_{<\omega})$ and $\mathrm{Flim}(\mathcal{G}_{<\omega})$ are – in addition to their defining properties – Fraïssé for the classes $\mathcal{M}_\omega$ and $\mathcal{G}_\omega$, respectively, by virtue of being multi-isometric to the spaces constructed in [4]. Since the construction of the Fraïssé limit makes use of the fact that the elements of a given Fraïssé class are finitely-seminormed, it is not clear that the aforementioned property must be true for general Fraïssé Fréchet spaces. More precisely, given a class $\mathcal{K} \subseteq \mathcal{M}_{<\omega}$, one can define a class $\mathcal{K}_\omega \subseteq \mathcal{M}_\omega$ by declaring that $(X, (\|\cdot\|_n)_{n<\omega}) \in \mathcal{K}_\omega$ exactly when

$$(X, (\|\cdot\|_n)_{n<m}) \in \mathcal{K} \text{ for all } m < \omega.$$

This motivates the following:

Problem 2.4.12. If E is a $\mathcal{K}$-Fraïssé Fréchet space, is E necessarily $\mathcal{K}_\omega$-Fraïssé?

The proof that $\mathbb{G}^\omega$ is $\mathcal{G}_\omega$-Fraïssé (with an appropriate sequence of seminorms) from [4] makes use of the existence of a universal operator $\pi : \mathbb{G} \to \mathbb{G}$ constructed in [20]. This operator can essentially be seen as a *Fraïssé* operator in a precise sense (see, e.g., [43]). The operator constructed in [20] is the Gurarij analogue of Rota's universal operator on ℓ_2, as in [11, 62]. Thus, in order to shed light on a possible affirmative answer to Problem 2.4.12, one may need to construct similar operators for an arbitrary Fraïssé Banach space in place of $\mathbb{G}$.

Chapter 3

Topological Ramsey spaces of equivalence relations

In this chapter we study various infinite versions of the following theorem of Graham and Rothschild [24], originally proved in 1971 and which started what is now known as Dual Ramsey Theory:

Theorem 3.0.1 (Graham-Rothschild)**.** *For every pair of integers $d < m$ and every $r \in \mathbb{N}$ there is $n > m$ such that the following property holds: For every r-colouring of the space of all rigid surjections from n onto d, there is a rigid surjection $\gamma : n \to m$ such that the set*

$$\{\sigma \circ \gamma : \ \sigma \text{ is a rigid surjection from } m \text{ onto } d\}$$

is monochromatic.

Here we view d, m and n as ordered sets, and we call a surjection $\gamma : n \to d$ *rigid* if

$$\min \gamma^{-1}(s_0) < \min \gamma^{-1}(s_1) \text{ for every } s_0, s_1 \in d \text{ such that } s_0 < s_1.$$

In this setting, a rigid surjection $\gamma : n \to d$ corresponds to a partition of $n = \{0, \ldots, n-1\}$ into d pieces, and so the previous result can be viewed as a Ramsey theorem regarding partitions of finite sets. The Graham-Rothschild theorem (which in this form is sometimes referred to as the finite Dual Ramsey Theorem) was used in [6] to prove the approximate Ramsey property (ARP) for the family $\{\ell_\infty^n\}_n$. Recall from the previous chapter that this means that for every d, m, r and every $\varepsilon > 0$, there is an n such that every r-colouring of $\mathrm{Emb}(\ell_\infty^d, \ell_\infty^n)$ ε-stabilizes on a set of the form $\gamma \circ \mathrm{Emb}(\ell_\infty^d, \ell_\infty^m)$ for some $\gamma \in \mathrm{Emb}(\ell_\infty^m, \ell_\infty^n)$. The above conclusion may be more succinctly expressed using the Erdős-Rado arrow notation

$$\ell_\infty^n \to_\varepsilon (\ell_\infty^m)_{r,1}^{\ell_\infty^d},$$

where the subscript indicates that only one colour is needed in order to find an ε-monochromatic set. Such a result also holds for $1 \leq p < \infty$: In [16] it is shown that for every such p, every d, m, r and every $\varepsilon > 0$,

$$\ell_p^n \to_\varepsilon (\ell_p^m)_{r,1}^{\ell_p^d}.$$

The proof of the latter result crucially depends on a variant of the dual Ramsey theorem for *equipartitions* originally conjectured in [39].

Two natural questions arise:

Question 3.0.2. Fix $p \in [1, \infty]$. Given $n, r \in \mathbb{N}$ and $\varepsilon > 0$, is there a $t \in \mathbb{N}$ such that $\ell_p \to_\varepsilon (\ell_p)^{\ell_p^n}_{r,t}$?

Question 3.0.3. Fix $p \in [1, \infty]$. Does $\ell_p \to_\varepsilon (\ell_p)^{\ell_p}_{r,1}$ for every $r \in \mathbb{N}$ and $\varepsilon > 0$, up to a "definable" restriction on the allowable colourings?

The first question can be seen as a "big Ramsey" version of the ARP of the family $\{\ell_p^n\}_n$ (in the sense of [38, Section 11]) while the second statement would be the corresponding infinite-dimensional version (and hence necessitates a restriction on the permitted colourings; this will be discussed below). Given the relation between the Graham-Rothschild theorem and the ARP for Banach spaces, it is natural to assume that a positive answer to either of the above questions would require a better understanding of infinite analogues of the Graham-Rothschild theorem.

Luckily, as we have seen in Chapter 1, an infinite dual Ramsey theorem already exists via the Carlson-Simpson theorem. Unluckily, it does not seem as though the Carlson-Simpson theorem would be of much help in proving the above generalizations of the ARP; the proof of the ARP of the family $\{\ell_\infty^n\}_n$, for instance, codes copies of ℓ_∞^d in ℓ_∞^n via products, and so it seems that one copy of ω would be insufficient for our purposes. The main goal of this chapter is to obtain version results of the Carlson-Simpson theorem relative to finite products $\omega \cdot n$ of ω; more generally, we achieve this goal for any countable limit ordinal in place of ω, up to a restriction on the placement of the minimal representatives of each equivalence relation. To do so, we use topological Ramsey spaces and code equivalence relations on a countable limit ordinal via certain "alternating" equivalence relations on ω.

3.1 Preliminaries

3.1.1 Topological Ramsey spaces

The exposition in this subsection closely follows that of [65, Chapter 5] and we refer the reader there for more information. Let $\mathcal{R}$ be a set which we will think of as a collection of infinite sequences of objects. We equip $\mathcal{R}$ with a quasiorder $\leq$ and a *restriction map*

$$r : \mathcal{R} \times \omega \to \mathcal{AR}$$

where

$$\mathcal{AR} = \bigcup_{n<\omega} \mathcal{AR}_n$$

and where $\mathcal{AR}_n$ (which we think of as the collection of n^{th} approximations to elements of $\mathcal{R}$) is the range of $r_n = r \restriction \mathcal{R} \times \{n\}$. We will be interested in structures $(\mathcal{R}, \leq, r)$ of the above form which moreover satisfy the following four axioms. The relevant notation will be explained after the statement of the axioms.

A.1. For all $A, B \in \mathcal{R}$ we have:

1. $r_0(A) = \emptyset$.
2. $A \neq B$ implies $r_n(A) \neq r_n(B)$ for some n.

3. $r_n(A) = r_m(B)$ implies $n = m$ and $r_k(A) = r_k(B)$ for all $k < n$.

A.2. There is a quasiorder $\leq_{\text{fin}}$ on $\mathcal{AR}$ such that:

1. $\{a \in \mathcal{AR} : a \leq_{\text{fin}} b\}$ is finite for all $b \in \mathcal{AR}$.
2. For all $A, B \in \mathcal{R}$ we have

$$A \leq B \text{ iff } (\forall n)(\exists m)\, r_n(A) \leq_{\text{fin}} r_m(B).$$

3. For all $a, b \in \mathcal{AR}$,

$$a \sqsubseteq b \wedge b \leq_{\text{fin}} c \to (\exists d \sqsubseteq c)\, a \leq_{\text{fin}} d.$$

A.3. Let $A, B \in \mathcal{R}$ and $a \in \mathcal{AR}$ be such that $d = \text{depth}_B(a) < \infty$. Then:

1. $[a, A'] \neq \emptyset$ for all $A' \in [r_d(B), B]$.
2. If $A \leq B$ and $[a, A] \neq \emptyset$, then there is $A' \in [r_d(B), B]$ such that $[a, A'] \subseteq [a, A]$.

A.4. If $[a, B] \neq \emptyset$ and $d = \text{depth}_B(a) < \infty$, then for every $\mathcal{O} \subseteq \mathcal{AR}_{|a|+1}$ there exists $A \in [r_d(B), B]$ such that $r_{|a|+1}[a, A] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, A] \subseteq \mathcal{O}^c$.

Above, we use the notation $|a|$ to denote the *length* of $a \in \mathcal{AR}$, which is the unique integer n such that $a = r_n(A)$ for some $A \in \mathcal{R}$. We also write $a \sqsubseteq b$ whenever there is $B \in \mathcal{R}$ and $n \leq m < \omega$ such that $a = r_n(B)$ and $b = r_m(B)$. For $a \in \mathcal{AR}$ and $B \in \mathcal{R}$, we define

$$[a, B] = \{A \in \mathcal{R} : A \leq B \text{ and } (\exists n)\, r_n(A) = a\}.$$

Also recall that the *depth* of a in B, for $a \in \mathcal{AR}$ and $B \in \mathcal{R}$, is defined by

$$\text{depth}_B(a) = \begin{cases} \min\{n < \omega : a \leq^{\circ}_{\text{fin}} r_n(B)\} & \text{if } (\exists n)\, a \leq^{\circ}_{\text{fin}} r_n(B), \\ \infty & \text{otherwise.} \end{cases}$$

We equip $\mathcal{AR}$ with the discrete topology, $\mathcal{AR}^\omega$ with the corresponding product topology, and $\mathcal{R}$ with the subspace topology from $\mathcal{AR}^\omega$. We say $(\mathcal{R}, \leq, r)$ is *closed* if $\mathcal{R}$ is closed when viewed as a subspace of $\mathcal{AR}^\omega$. The *Ellentuck topology* on $\mathcal{R}$ is the topology generated by basic open sets of the form $[a, B]$ for $a \in \mathcal{AR}, B \in \mathcal{R}$. Note that this topology refines the metrizable topology on $\mathcal{R}$ when considered as a subspace of $\mathcal{AR}^\omega$.

Definition 3.1.1. Let $\mathcal{X} \subseteq \mathcal{R}$.

(i) $\mathcal{X}$ has the *property of Baire* if $\mathcal{X} = \mathcal{O} \triangle M$ for some Ellentuck open $\mathcal{O} \subseteq \mathcal{R}$ and some Ellentuck meagre $M \subseteq \mathcal{R}$.

(ii) $\mathcal{X}$ is *Ramsey* if for every non-empty basic set $[a, B]$ there is $A \in [a, B]$ such that $[a, A] \subseteq \mathcal{X}$ or $[a, A] \cap \mathcal{X} = \emptyset$. If the second alternative always holds, then $\mathcal{X}$ is *Ramsey null*.

Theorem 3.1.2 (Abstract Ellentuck Theorem)**.** *Suppose $(\mathcal{R}, \leq, r)$ is closed and satisfies axioms A.1 to A.4. Then $\mathcal{X} \subseteq \mathcal{R}$ has the property of Baire if and only if it is Ramsey. Furthermore, $\mathcal{X} \subseteq \mathcal{R}$ is Ellentuck meagre if and only if it is Ramsey null.*

A structure of the above form which satisfies the conclusion of the Abstract Ellentuck Theorem is called a *topological Ramsey space*.

Recall that a *Souslin scheme* is a family of subsets $(X_s)_{s\in\omega^{<\omega}}$ of some underlying set which is indexed by finite sequences of non-negative integers. The *Souslin operation* turns a Souslin scheme $(X_s)_{s\in\omega^{<\omega}}$ into the set

$$\bigcup_{x\in\mathcal{N}}\bigcap_{n<\omega} X_{x\restriction n}$$

where $\mathcal{N}$ denotes the Baire space, i.e. the set of all infinite sequences in ω. Then $\mathcal{X}\subseteq\mathcal{R}$ is *Souslin measurable* if it belongs to the minimal field of subsets of $\mathcal{R}$ which contains all Ellentuck open sets and is closed under the Souslin operation. In particular, every analytic or coanalytic subset of $\mathcal{R}$ is Souslin measurable. Finally, we say finite colouring $c:\mathcal{R}\to n$ is Souslin measurable if each set $c^{-1}\{i\}$, $i<n$ is Souslin measurable.

We make note of the following useful consequence of the Abstract Ellentuck Theorem. From now on, all topological notions will refer to the Ellentuck topology on $\mathcal{R}$.

Corollary 3.1.3. *Suppose $(\mathcal{R},\leq,r)$ is closed and satisfies axioms A.1 to A.4. Then for every finite Souslin measurable colouring of $\mathcal{R}$ and every $B\in\mathcal{R}$, there is $A\leq B, A\in\mathcal{R}$ such that the set $\{A'\in\mathcal{R}:A'\leq A\}$ is monochromatic.*

3.1.2 The Hales-Jewett theorem

In proving that certain collections of equivalence relations are topological Ramsey spaces, we will make essential use of the *Hales-Jewett theorem*. Originally proved in [27], this is a powerful combinatorial result which involves colouring finite words over an alphabet. We will be concerned with a particular infinitary version of the Hales-Jewett theorem which we now proceed to describe. Let L be a finite set which we refer to as an *alphabet*, and let v be a symbol distinct from all members of L. We let W_L denote the set of all *words* formed from L, i.e. all functions $w:n\to L$ where $n<\omega$. Similarly, we let W_{Lv} denote all *variable words*, i.e. functions $x:n\to L\cup\{v\}$ such that $v\in\operatorname{range}(x)$. We will often write $|w|$ for the domain (or length) of the word w. For two words x and y we write $x^\frown y$ for the word obtained by concatenating x and y. If x is a variable word and $\lambda\in L\cup\{v\}$, we write $x[\lambda]$ for the element of W_L or W_{Lv} obtained by replacing every occurrence of the variable v in x by λ. Given a sequence $X=(x_n)_{n<\omega}$ of variable words over an alphabet L, we write $[X]_L$ for the *partial subsemigroup of W_L generated by X*, which consists of all words of the form

$$x_{n_0}[\lambda_0]^\frown\ldots^\frown x_{n_k}[\lambda_k]$$

where $k<\omega, n_0<\cdots<n_k<\omega$ and $\lambda_i\in L$ for each $i\leq k$. The partial subsemigroup $[X]_{Lv}$ of W_{Lv} is defined similarly.

We will make use of the following version of the infinite Hales-Jewett theorem originally considered in [13] and obtained in the form below in [30]; another proof can be found in [65, Chapter 2]. First, let us say that $x\in W_{Lv}$ is a *left-variable word* if the first letter of x is v.

Theorem 3.1.4 (Left-variable Hales-Jewett Theorem)**.** *Suppose L is a finite alphabet. Then for every finite colouring of W_L there is a sequence $X=(x_n)_{n<\omega}$ of left-variable words together with a variable-free word w_0 such that the translate $w_0{}^\frown[X]_L$ is monochromatic.*

3.2 Alternating equivalence relations for finite partitions

Let $\mathcal{P}$ be a partition of ω into finitely many infinite sets $P_0, \ldots, P_{l-1}$ such that $\min P_i < \min P_j$ whenever $i < j$. Let $\mathcal{E}_\infty^{\mathcal{P}}$ denote the set of all $\mathcal{P}$-*alternating* equivalence relations on ω, i.e. equivalence relations E on ω such that:

(a) E has infinitely many equivalence classes.

(b) If $(p_k(E))_{k<\omega}$ is an increasing enumeration of the set of minimal representatives of the equivalence classes of E, then $p_k(E) \in P_n$ where $n \equiv k \mod l$.

Given $E \in \mathcal{E}_\infty^{\mathcal{P}}$, a P_n-*class* is an equivalence class X of E such that $\min X \in P_n$. Note that condition (b) implies that each $E \in \mathcal{E}_\infty^{\mathcal{P}}$ has infinitely many P_n-classes for each $n < l$.

Before proceeding, we establish some notation and terminology which will be used throughout this chapter when working with equivalence relations. For any two equivalence relations E and F on the same set, write $E \leq F$ whenever every class of E is a union of a classes of F. In this case we also say that E is a *coarsening* of F, or that E is *coarser* than F. For any equivalence relation E on a well-ordered set, $p(E)$ will denote the set of all minimal representatives of the classes of E. The n^{th} *approximation* of an equivalence relation E on ω is given by

$$r_n(E) = E \restriction p_n(E)$$

where $p_n(E)$ is the n^{th} element of $p(E)$ when the latter set is enumerated in increasing order. Let $\mathcal{AE}_\infty^{\mathcal{P}}$ denote the set of all finite approximations of elements of $\mathcal{E}_\infty^{\mathcal{P}}$. For $a \in \mathcal{AE}_\infty^{\mathcal{P}}$, let $|a|$ denote the *length* of a, which is defined as the unique integer such that $a = r_n(E)$ for some E; equivalently, $|a|$ is the number of equivalence classes of a. $(\mathcal{AE}_\infty^{\mathcal{P}})_n$ will denote all approximations of length n. The *domain* of an approximation a is the set

$$\operatorname{dom}(a) = \{0, 1, \ldots, p_{|a|}(E) - 1\} = p_{|a|}(E)$$

where E is such that $a = r_n(E)$. The relation $\leq$ admits a finitization obtained by setting $a \leq_{\text{fin}} b$ if and only if $\operatorname{dom}(a) = \operatorname{dom}(b)$ and a is coarser than b.

Theorem 3.2.1. $(\mathcal{E}_\infty^{\mathcal{P}}, \leq, r)$ *is a topological Ramsey space.*

Proof. As in [65, Chapter 5.6], it is routine to check that $(\mathcal{E}_\infty^{\mathcal{P}}, \leq, r)$ is closed and satisfies the Ramsey space axioms A.1, A.2 and A.3. Hence it is enough to prove A.4; we follow the proof of [65, Lemma 5.69]. So let $[a, E] \neq \emptyset$ be a basic set, $n = |a|$ and $\mathcal{O} \subseteq (\mathcal{AE}_\infty^{\mathcal{P}})_{n+1}$. Note that we may assume $[a, E]$ has the property that $a = r_n(E)$, since then the arbitrary case follows from A.3(2) by taking an appropriate coarsening of a. Assuming $a = r_n(E)$, it follows that $\operatorname{depth}_E(a) = n$ and so we want to find $F \in [a, E]$ such that $r_{n+1}[a, F] \subseteq \mathcal{O}$ or $r_{n+1}[a, F] \subseteq \mathcal{O}^c$.

Consider an arbitrary end-extension $b \in r_{n+1}[a, E]$. Then b is an equivalence relation on a set of the form $p_m(E) = \{0, 1, \ldots, p_m(E) - 1\}$ for some $m > n$, such that b has one more equivalence class with minimal representative $p_n(E)$. Thus b joins the classes of E with minimal representatives among $p_{n+1}(E), \ldots, p_{m-1}(E)$ to a class with minimal representative $\leq p_n(E)$. Note that if $n \equiv k \mod l$ then $m \equiv k + 1 \mod l$ because of our condition for being a member of $\mathcal{E}_\infty^{\mathcal{P}}$. Let $\lambda := \frac{m-n-1}{l}$. Then any such b can be coded as a word w^b in the alphabet $L = (n+1)^l$ such that w^b has length λ: If we let $\pi_j(w^b(i))$

denote the j^{th} coordinate of the letter $w^b(i)$, then for each $i < \lambda$ and $j < l$, we set $\pi_j(w^b(i)) = k$ where $p_{(n+1)+il+j}(E)$ is joined to $p_k(E)$ via b. In other words, each block of the form

$$\big(p_{n+il+1}(E), p_{n+il+2}(E), \ldots, p_{n+(i+1)l}(E)\big)\,,\ \ i < \lambda$$

is associated to the letter $(k_0, \ldots, k_{l-1}) \in L$ where $p_{k_j}(E)$ is joined to $p_{(n+1)+il+j}(E)$. Conversely, any word $w \in W_L$ which has length λ' corresponds to a unique $b = b(w) \in r_{n+1}[a, E]$ where $\operatorname{dom}(b) = r_{n+1+l\cdot\lambda'}(E)$: For each $i < \lambda'$ and $j < l$, join the class $p_{(n+1)+il+j}(E)$ to $p_{\pi_j(w(i))}(E)$ and let $b(w)$ be the corresponding equivalence relation on the set $p_m(E) = \{0, 1, \ldots, p_m(E) - 1\}$.

Define a colouring $c : W_L \to 2$ by setting $c(w) = 0$ if and only if $b(w) \in \mathcal{O}$, and apply the left-variable Hales-Jewett theorem to obtain a variable-free word w_0 together with a sequence $X = (x_i)_{i<\omega}$ of left-variable words such that the translate $w_0{}^\frown[X]_L$ is monochromatic for the above colouring. In other words, either

1. $b(w) \in \mathcal{O}$ for every $w \in w_0{}^\frown[X]_L$, or
2. $b(w) \notin \mathcal{O}$ for every $w \in w_0{}^\frown[X]_L$.

For each $i < \omega$, the left-variable word x_i determines a variable word y_i of length $l \cdot |x_i|$ in the alphabet $\{0, \ldots, n\}$ which is obtained by replacing each letter $(k_1, \ldots, k_l) \in L$ with the string $k_1 \ldots k_l$ and replacing the variable v with the string $0 \ldots 0v0 \ldots 0$ of length l where v occurs in the $(i \mod l)^{\text{th}}$ place. Similarly, w_0 corresponds to a word u_0 of length $l \cdot |w_0|$ in the alphabet $\{0, \ldots, n\}$. Now form the infinite word

$$y = u_0{}^\frown y_0{}^\frown y_1{}^\frown \ldots{}^\frown y_k{}^\frown \ldots$$

out of u_0 and $(y_i)_{i<\omega}$. To define F, it suffices to say how it acts on $p(E)$. If at place i of y we find a letter $k \in \{0, \ldots, n\}$, we let $p_{n+1+i}(E)$ and $p_k(E)$ be F-equivalent. If we find a variable at places

$$i, i' \in [|u_0| + |y_0| + \cdots + |y_{j-1}|, |u_0| + |y_0| + \cdots + |y_{j-1}| + |y_j|)$$

(where we set $y_{-1} = \emptyset$) then we let $p_{n+1+i}(E)$ and $p_{n+1+i'}(E)$ be F-equivalent, with no other connections. In this way, the sequence of minimal representatives $(p_i(F))_{i<\omega}$ of F is a subsequence of $(p_i(E))_{i<\omega}$ obtained by removing blocks which have length a multiple of l, and so F is $\mathcal{P}$-alternating.

It remains to check that F satisfies the conclusion of axiom A.4. It is clear that $F \in [a, E]$. Now take an arbitrary end-extension $b \in r_{n+1}[a, F]$. By our condition for being a member of $\mathcal{E}_\infty^{\mathcal{P}}$, b must be an equivalence relation on a set of the form $p_m(F)$ where $m \equiv n + 1 \mod l$. In particular, this implies that b can be obtained from a word of the form

$$u_0{}^\frown y_0[\lambda_0]^\frown \ldots{}^\frown y_k[\lambda_k]$$

for some $k < \omega$ and $\lambda_0, \ldots, \lambda_k \in \{0, \ldots, n\}$, and so b corresponds to a word

$$w_0{}^\frown x_0[\lambda'_0]^\frown \ldots{}^\frown x_k[\lambda'_k]$$

for some $\lambda'_0, \ldots, \lambda'_k \in L$. Thus F satisfies the conclusion of A.4. □

Corollary 3.2.2. *Suppose c is a finite Souslin measurable colouring of $\mathcal{E}_\infty^\mathcal{P}$. Then for every $E \in \mathcal{E}_\infty^\mathcal{P}$ there is $F \leq E$ such that the family $\mathcal{E}_\infty^\mathcal{P} \upharpoonright F$ of all coarsenings of F is c-monochromatic.*

In the above space we only prescribed conditions on the minimal representatives associated to an equivalence relation. We can also prescribe conditions on *all* of elements belonging to an equivalence class rather than just the minimal representatives. One relevant instance of this will be the following: Let $\mathcal{P}$ be a partition of ω into finitely many sets $P_0, \dots, P_{l-1}$ such that $\min P_i < \min P_j$ whenever $i < j$ and let $\mathcal{E}_\infty^{\mathcal{P}'}$ denote the set of all $\mathcal{P}$-alternating equivalence relations E on ω which also satisfy:

(c) If X is a P_n-class, then $X \subseteq \bigcup_{m \geq n} P_m$.

Theorem 3.2.3. *$(\mathcal{E}_\infty^{\mathcal{P}'}, \leq, r)$ is a topological Ramsey space.*

Proof. As in the proof of Theorem 3.2.1, it is enough to check axiom A.4. So we let $[a, E] \neq \emptyset$ be a basic set, $n = |a|$ and $\mathcal{O} \subseteq (\mathcal{A}\mathcal{E}_\infty^{\mathcal{P}'})_{n+1}$. As before, we assume $a = r_n(E)$ so and we aim to find $F \in [a, E]$ such that $r_{n+1}[a, F] \subseteq \mathcal{O}$ or $r_{n+1}[a, F] \subseteq \mathcal{O}^c$. Consider an arbitrary end-extension $b \in r_{n+1}[a, E]$. Then b is an equivalence relation on a set of the form $p_m(E) = \{0, 1, \dots, p_m(E) - 1\}$ for some $m > n$, such that b has one more equivalence class with minimal representative $p_n(E)$. Thus b joins the classes of E with minimal representatives among $p_{n+1}(E), \dots, p_{m-1}(E)$ to a class with minimal representative $\leq p_n(E)$. Note that if $n \equiv k \mod l$ then $m \equiv k + 1 \mod l$ because of condition (b) in the definition of $\mathcal{P}$-alternating. Furthermore, condition (c) implies that each P_i-class can only be joined to an earlier P_j-class where $j \leq i$.

Exactly as in the proof of Theorem 3.2.1, any end-extension b as above can be coded as a word w^b in the alphabet $L = (n+1)^l$ such that w^b has length $\lambda := \frac{m-n-1}{l}$. Conversely, any word $w \in W_L$ which has length λ' corresponds to a unique $b = b(w) \in r_{n+1}[a, E]$ where $\mathrm{dom}(b) = r_{n+1+l\cdot\lambda'}(E)$. To describe this assignment, we first define for each $w \in W_L \cup W_{Lv}$ a word $\widetilde{w}$ of the same length as follows: If the i^{th} position of w is occupied by the variable v, then we let $\widetilde{w}(i) = v$. Otherwise, $w(i) \in L$ and we define $\widetilde{w}(i)$ by setting the j^{th} coordinate of $\widetilde{w}(i)$ to be

$$\pi_j(\widetilde{w}(i)) = \begin{cases} \pi_j(w(i)) & \text{if } p_{(n+1)+il+j}(E) \text{ can be joined to } p_{\pi_j(w(i))}(E), \\ 0 & \text{otherwise} \end{cases}$$

for each $j < l$. Then, given $w \in W_L$ as above, for each $i < \lambda'$ and $j < l$ join the class with minimal representative $p_{(n+1)+il+j}(E)$ to $p_{\pi_j(\widetilde{w}(i))}(E)$ and let $b(w)$ be the corresponding equivalence relation on the set $p_m(E) = \{0, 1, \dots, p_m(E) - 1\}$.

Define a colouring $c : W_L \to 2$ by setting $c(w) = 0$ if and only if $b(w) \in \mathcal{O}$, and apply the left-variable Hales-Jewett theorem to obtain a variable-free word w_0 together with a sequence $X = (x_n)_{n<\omega}$ of left-variable words such that either

1. $b(w) \in \mathcal{O}$ for every $w \in w_0{}^\frown[X]_L$, or
2. $b(w) \notin \mathcal{O}$ for every $w \in w_0{}^\frown[X]_L$.

Using w_0 and X, construct (exactly as in the proof of Theorem 3.2.1) a variable-free word u_0 and a sequence of left-variable words $(y_i)_{i<\omega}$ over the alphabet $\{0, \dots, n\}$ and then form the infinite variable word

$$y = u_0{}^\frown y_0{}^\frown y_1{}^\frown \dots{}^\frown y_k{}^\frown \dots$$

over the alphabet $\{0, \ldots, n\}$. Define an infinite variable word $\widetilde{y}$ by setting the i^{th} letter to be

$$\widetilde{y}(i) = \begin{cases} y(i) & \text{if } p_{n+1+i}(E) \text{ can be joined to } p_{y(i)}(E), \\ v & \text{if } y(i) = v, \\ 0 & \text{otherwise.} \end{cases}$$

and use it to define an equivalence relation F exactly as in the proof of Theorem 3.2.1. The placement of the variables in the infinite word $\widetilde{y}$ (which are the same as that of y) ensures that F is $\mathcal{P}$-alternating, while condition (c) in the definition of $\mathcal{E}_\infty^{\mathcal{P}'}$ is satisfied by the fact that we replaced y with $\widetilde{y}$. Then each end-extension $b \in r_{n+1}[a, F]$ can be obtained as $b = b(w)$ for some word

$$w = w_0{}^\frown x_0[\lambda_0']^\frown \ldots{}^\frown x_k[\lambda_k']$$

where $\lambda_0', \ldots, \lambda_k' \in L$. It then follows from our choice of w_0 and X that F satisfies the conclusion of A.4. □

Corollary 3.2.4. *Suppose c is a finite Souslin measurable colouring of $\mathcal{E}_\infty^{\mathcal{P}'}$. Then for every $E \in \mathcal{E}_\infty^{\mathcal{P}'}$ there is $F \leq E$ such that the family $\mathcal{E}_\infty^{\mathcal{P}'} \restriction F$ of all coarsenings of F is c-monochromatic.*

To conclude this section, we apply Corollary 3.2.4 to prove an analogue of the dual Ramsey theorem for ordinals of the form $\omega \cdot l$. In this setting it is natural to work with equivalence relations whose equivalence classes are precisely the fibres of a *rigid surjection* $\omega \cdot l \to \omega \cdot l$, i.e. a surjection $f : \omega \cdot l \to \omega \cdot l$ such that

$$\min f^{-1}(\alpha) < \min f^{-1}(\beta) \text{ for all } \alpha < \beta < \omega \cdot l.$$

When $l = 1$ there is a natural bijective correspondence between rigid surjections and equivalence relations, but for $l > 1$ this is no longer the case; in particular, it is easy to check that a rigid surjection $f : \omega \cdot l \to \omega \cdot l$ corresponds uniquely to an equivalence relation E on $\omega \cdot l$ such that $p(E)$ has infinite intersection with each copy of ω. Thus, we will work exclusively with equivalence relations on $\omega \cdot l$ which arise from rigid surjections.

Working with any partition $\mathcal{P}$ of ω as above, we view each P_i, $i < l$ as the i^{th} copy of ω in $\omega \cdot l$, i.e. P_i is identified with $\omega \times \{i\}$. In order to obtain a Ramsey theorem in this setting, we need to prescribe conditions on the set of minimal representatives of an equivalence relation on $\omega \cdot l$; in particular we need to ensure that there is only one possible "pattern" for the set of minimal representatives of an equivalence relation. A natural requirement is to ask that these representatives cycle between the different copies of ω when ordered according to the lexicographical ordering $\preceq$ on $\omega \cdot l$, i.e. $(n, i) \preceq (m, j)$ if and only if $n < m$, or $n = m$ and $i \leq j$. Additionally, it is necessary to ask that the minimal representatives (under the standard ordering on $\omega \cdot l$) agree with the $\preceq$-minimal representatives. Thus, we let $\mathcal{E}_\infty(\omega \cdot l)$ be the set of all equivalence relations E on $\omega \cdot l$ such that the following conditions hold:

(a) If $(p_k)_{k<\omega}$ is a $\preceq$-increasing enumeration of $p(E)$, then $p_k \in \omega \times \{k \mod l\}$ for each k.

(b) If $(q_k)_{k<\omega}$ is a $\preceq$-increasing enumeration of the set of $\preceq$-minimal representatives of E, then $q_k = p_k$ for all k.

In order to transfer equivalence relations from ω to $\omega \cdot l$ (and vice versa) we first define a mapping $\phi : \omega \to \omega \cdot l$ as follows: Given $n < \omega$, let $\theta(n)$ be the unique integer such that $n \in P_{\theta(n)}$, and define a mapping $\psi : \omega \to \omega$ by setting $\psi(n) = m$ if and only if n is the m^{th} element of $P_{\theta(n)}$. Then let $\phi(n) = (\psi(n), \theta(n))$ and note that ϕ is a bijection between ω and $\omega \cdot l$. Given $E = (E_n)_{n<\omega} \in \mathcal{E}_\infty^{\mathcal{P}'}$, define an equivalence relation on $\omega \cdot l$ by setting $\Phi(E) = (\phi'' E_n)_{n<\omega}$. The following lemma summarizes the main properties of Φ that we will need.

Lemma 3.2.5. *Φ is a bijective mapping between $\mathcal{E}_\infty^{\mathcal{P}'}$ and $\mathcal{E}_\infty(\omega \cdot l)$ which has the property that $F \leq E$ if and only if $\Phi(F) \leq \Phi(E)$.*

Proof. The second part of the lemma is immediate from the fact that ϕ is a bijection, while the first part will follow once we show that the image of Φ is contained in $\mathcal{E}_\infty(\omega \cdot l)$. To prove the latter it is enough to show that Φ preserves the set of minimal representatives of each equivalence relation $E \in \mathcal{E}_\infty^{\mathcal{P}'}$ in the sense that $\phi(p(E)) = p(\Phi(E))$. To this end, fix $E = (E_n)_{n<\omega} \in \mathcal{E}_\infty^{\mathcal{P}'}$. Then, for each $n < \omega$, $p_n = \min E_n$ belongs to $P_{\sigma(n)}$ where $\sigma(n) < l$ and $\sigma(n) \equiv n \mod l$, and so the definition of ϕ implies that $\phi(p_n) \in \omega \times \{\sigma(n)\}$. Since $E_n \subseteq \bigcup_{m \geq \sigma(n)} P_m$ it follows that for each $x \in E_n$ we have $\phi(x) \in \omega \times \{m\}$ for some $m \geq \sigma(n)$. If $m > \sigma(n)$ then $\phi(x) > \phi(p_n)$ in the usual ordering on $\omega \cdot l$, so assume $m = \sigma(n)$. By definition of p_n, we know $p_n \leq x$ in the ordering on ω and so $\psi(p_n) \leq \psi(x)$. Thus $\phi(p_n) \leq \phi(x)$ for any $x \in E_n$. Hence $\phi(p_n) = \min \phi'' E_n$ and so

$$p(\Phi(E)) = \{\phi(p_n) : n < \omega\} = \phi(p(E)).$$

Since $\phi(p_n) \in \omega \times \{\sigma(n)\}$ for each n, this implies $\Phi(E) \in \mathcal{E}_\infty(\omega \cdot l)$. Thus the image of $\mathcal{E}_\infty^{\mathcal{P}'}$ under Φ is contained in $\mathcal{E}_\infty(\omega \cdot l)$. The fact that $\Phi : \mathcal{E}_\infty^{\mathcal{P}'} \to \mathcal{E}_\infty(\omega \cdot l)$ is a bijection is now straightforward. □

We can equip $\mathcal{E}_\infty(\omega \cdot l)$ with the topology inherited from $\mathcal{E}_\infty^{\mathcal{P}'}$ via Φ, where the latter set is equipped with either the Ellentuck topology or the standard metrizable topology. Note that both of these topologies refine the topology induced from $2^{(\omega \cdot l)^2}$ when each equivalence relation in $\mathcal{E}_\infty(\omega \cdot l)$ is identified with a subset of $(\omega \cdot l)^2$ in the standard way, e.g. as in [13]. Thus, the notion of Souslin measurability in the next result can refer to either of these three topologies. The following is the extension of the dual Silver theorem to ordinals of the form $\omega \cdot l$.

Corollary 3.2.6. *Suppose c is a finite Souslin measurable n-colouring of $\mathcal{E}_\infty(\omega \cdot l)$. Then for every $E \in \mathcal{E}_\infty(\omega \cdot l)$ there is $F \leq E$ such that the family $\mathcal{E}_\infty(\omega \cdot l) \restriction F$ of all coarsenings of F is c-monochromatic.*

Proof. Fix c and E as in the statement of the corollary. Let $\widetilde{c} = c \circ \Phi$; then $\widetilde{c}$ is Souslin measurable and so by Corollary 3.2.4 there are $F \leq \Phi^{-1}(E)$ and $i < n$ such that $\widetilde{c}(F') = i$ for each $F' \in \mathcal{E}_\infty^{\mathcal{P}'}$ such that $F' \leq F$. Then $\Phi(F) \leq E$, and if $F' \in \mathcal{E}_\infty(\omega \cdot l)$ has the property that $F' \leq \Phi(F)$, then

$$c(F') = c(\Phi(\Phi^{-1}(F'))) = \widetilde{c}(\Phi^{-1}(F')) = i$$

since $\Phi^{-1}(F') \leq F$. Thus $\Phi(F)$ satisfies the conclusion of the corollary. □

We conclude this section with a description of the corresponding version of the dual Ramsey theorem for $\omega \cdot l$. For each $k < \omega$, let $\mathcal{E}_k(\omega \cdot l)$ denote the set of all equivalence relations E on $\omega \cdot l$ with exactly k equivalence classes such that if $(p_i)_{i<k}$ is a $\preceq$-increasing enumeration of $p(E)$ then

$$p_i \in \omega \times \{i \mod l\} \text{ for all } i < k,$$

and if $(q_i)_{i<k}$ is a $\preceq$-increasing enumeration of the set of $\preceq$-minimal representatives of E, then $q_i = p_i$ for all $i < k$. Equip $\mathcal{E}_k(\omega \cdot l)$ with the topology inherited from $2^{(\omega \cdot l)^2}$ by identifying each equivalence relation in $\mathcal{E}_k(\omega \cdot l)$ with a subset of $(\omega \cdot l)^2$. We then have the following extension of the dual Ramsey theorem to $\omega \cdot l$. The proof is the same as the corresponding result from [65, Corollary 5.72]; we include it here for the sake of completeness.

Corollary 3.2.7. *Suppose c is a finite Souslin measurable colouring of $\mathcal{E}_k(\omega \cdot l)$. Then for every $E \in \mathcal{E}_\infty(\omega \cdot l)$ there is $F \leq E$ such that the family $\mathcal{E}_k(\omega \cdot l) \upharpoonright F$ of all k-coarsenings of F is c-monochromatic.*

Proof. Define a mapping $\pi : \mathcal{E}_\infty(\omega \cdot l) \to \mathcal{E}_k(\omega \cdot l)$ by letting $\pi(E')$, for $E' \in \mathcal{E}_\infty(\omega \cdot l)$, be the equivalence relation obtained by joining each equivalence class with minimal representative $p_n(E'), n \geq k$ to the class with minimal representative 0. Then π is continuous and so $c \circ \pi$ is a finite Souslin measurable colouring of $\mathcal{E}_\infty(\omega \cdot l)$. Thus, by the previous result there is $F \in \mathcal{E}_\infty(\omega \cdot l), F \leq E$ such that $c \circ \pi$ is constant on $\mathcal{E}_\infty(\omega \cdot l) \upharpoonright F$. Let F^* be a coarsening of F such that every equivalence class of F^* contains infinitely many classes of F. Then every $F' \in \mathcal{E}_k(\omega \cdot l) \upharpoonright F^*$ can be expressed as $F' = \pi(G)$ for some $G \in \mathcal{E}_\infty(\omega \cdot l) \upharpoonright F$, and so it follows that $\mathcal{E}_k(\omega \cdot l) \upharpoonright F^*$ is c-monochromatic. □

3.3 A left-variable Hales-Jewett theorem for infinite alphabets

In order to extend the results of the previous section to the case where $\mathcal{P}$ is an infinite partition of ω, we will need access to an infinite alphabet L when coding end-extensions. While the natural extension of the left-variable Hales-Jewett theorem to an infinite alphabet is false (see, e.g., [65, Remark 2.38]), for our purposes we only need a weak version of such a result which we will prove in this section. To do so, we make use of the theory of idempotent ultrafilters; a brief overview is included below, but we refer the reader to the first few sections of [65, Chapter 2] for more details.

Suppose $L = \bigcup_{n<\omega} L_n$ is an infinite alphabet given as an increasing chain of finite subsets L_n. Let $S = W_L \cup W_{Lv}$ and consider the semigroup $(S, {}^\frown)$ and its extension $(S^*, {}^\frown)$, where $S^* = \beta S \setminus S$ and where βS is the Stone-Čech compactification of $(S, {}^\frown)$. We view S^* as the collection of all non-principal ultrafilters on S. Each $\mathcal{U} \in S^*$ corresponds to an *ultrafilter quantifier* in the following way: Given a first-order formula $\varphi(x)$ with a free variable x ranging over elements of S, we write

$$(\mathcal{U}x)\, \varphi(x) \text{ iff } \{x \in S : \varphi(x)\} \in \mathcal{U}.$$

It is easy to check that ultrafilter quantifiers commute with conjunction and negation of first-order formulas. Using these quantifiers, the extensions of the operations of concatenation and substitution to S^* can be characterized as follows: For $A \subseteq S$ and $\lambda \in L$,

$$A \in \mathcal{U}^\frown \mathcal{V} \text{ iff } (\mathcal{U}x)(\mathcal{V}y)\, x^\frown y \in A,$$

$$A \in \mathcal{U}[\lambda] \text{ iff } (\mathcal{U}x)\, x[\lambda] \in A.$$

An ultrafilter $\mathcal{U} \in S^*$ is *idempotent* if $\mathcal{U}^\frown \mathcal{U} = \mathcal{U}$. Given two idempotent ultrafilters $\mathcal{U}, \mathcal{V} \in S^*$, we write $\mathcal{U} \leq \mathcal{V}$ whenever

$$\mathcal{U}^\frown \mathcal{V} = \mathcal{V}^\frown \mathcal{U} = \mathcal{U}.$$

Finally, we say that an idempotent ultrafilter $\mathcal{U} \in I \subseteq S^*$ is *minimal* if it is minimal in I with respect to the partial order $\leq$. We will make use of the following standard facts about minimal idempotents in compact semigroups, which we state in our specific context; the reader is referred to [65, Chapter 2.1] for the general proofs, and to [8] or [18] for more details on the use of idempotent ultrafilters in Ramsey theory.

Below, a subset I of S^* is a *left-ideal* if I is non-empty and $\mathcal{U}^\frown\mathcal{V} \in I$ for every $\mathcal{U} \in S^*$ and every $\mathcal{V} \in I$. The notions of right-ideal and two-sided ideal are defined similarly.

Lemma 3.3.1. *1. Every closed subsemigroup of S^* contains a minimal idempotent.*

2. For every two-sided ideal $I \subseteq S^*$ and every idempotent $\mathcal{U}$, there is an idempotent $\mathcal{V} \in I$ such that $\mathcal{V} \leq \mathcal{U}$.

3. For every minimal idempotent $\mathcal{V}$ and every right-ideal $J \subseteq S^*$, there is an idempotent $\mathcal{U} \in J$ such that $\mathcal{V}^\frown\mathcal{U} = \mathcal{V}$ and $\mathcal{U}^\frown\mathcal{V} = \mathcal{U}$.

For a sequence $X = (x_n)_{n<\omega}$ in W_{Lv}, let $[X]^*_L$ denote the set of all words in W_L of the form

$$x_0[\lambda_0]^\frown \dots ^\frown x_k[\lambda_k]$$

where $k < \omega$ and $\lambda_i \in L_i$ for each $i \leq k$. We then have the following version of the left-variable Hales-Jewett theorem for infinite alphabets.

Theorem 3.3.2. *For every finite colouring of W_L there is a variable-free word w_0 together with an infinite sequence $X = (x_n)_{n<\omega}$ of left-variable words such that the translate $w_0{}^\frown[X]^*_L$ is monochromatic.*

Proof. We follow the proof of [65, Theorem 2.37]. Using the previous lemma, let $\mathcal{W}$ be a minimal idempotent belonging to the closed subsemigroup $\{\mathcal{X} \in S^* : W_L \in \mathcal{X}\}$ and let $\mathcal{V} \leq \mathcal{W}$ be a minimal idempotent belonging to the two-sided ideal $\{\mathcal{X} \in S^* : W_{Lv} \in \mathcal{X}\}$. Since $v^\frown W_{Lv}$ is a right-ideal of S, the collection

$$J = \{\mathcal{X} \in S^* : v^\frown W_{Lv} \in \mathcal{X}\}$$

is a right-ideal of S^* and so we can choose an idempotent $\mathcal{U} \in J$ such that $\mathcal{V}^\frown\mathcal{U} = \mathcal{V}$ and $\mathcal{U}^\frown\mathcal{V} = \mathcal{U}$. Note that since each substitution mapping $\mathcal{X} \mapsto \mathcal{X}[\lambda]$ ($\lambda \in L$) is a homomorphism, it follows that each ultrafilter $\mathcal{V}[\lambda]$ is an idempotent of S^* such that

$$\mathcal{V}[\lambda] \leq \mathcal{W}[\lambda] = \mathcal{W},$$

where the equality comes from the fact that the mapping $x \mapsto x[\lambda]$ is the identity on W_L. Thus $\mathcal{V}[\lambda] = \mathcal{W}$ by minimality of $\mathcal{W}$; in particular, this implies

$$\mathcal{W}^\frown\mathcal{U}[\lambda] = \mathcal{W} \text{ and } \mathcal{U}[\lambda]^\frown\mathcal{W} = \mathcal{U}[\lambda]$$

for all $\lambda \in L$.

Let P be the colour of the given colouring which belongs to $\mathcal{W}$. By recursion on $n < \omega$, we will build a sequence (x_n) of left-variable words from W_{Lv} and a sequence (w_n) of variable-free words from W_L such that

$$w_0{}^\frown x_0[\lambda_0]^\frown w_1{}^\frown x_1[\lambda_1]^\frown \dots ^\frown x_{k-1}[\lambda_k]^\frown w_k \in P$$

for all $k < \omega$ and $\lambda_i \in L_i, i \leq k$. Then taking $x'_n = x_n{}^\frown w_{n+1}$ for each n will give the required sequence X of left-variable words.

For arbitrary $Q \subseteq W_L, w \in W_L$ and $n < \omega$, define

$$Q/w = \{t \in W_L : w^\frown t \in Q\}$$

and

$$\partial_n Q = \{w \in Q : Q/w \in \mathcal{W} \text{ and } Q/w \in \mathcal{U}[\lambda] \text{ for all } \lambda \in L_n\}.$$

Using the equations $\mathcal{W}^\frown\mathcal{W} = \mathcal{W}$ and $\mathcal{W}^\frown\mathcal{U}[\lambda] = \mathcal{W}$ for all $\lambda \in L$, we see that $\partial_n Q \in \mathcal{W}$ whenever $Q \in \mathcal{W}$ since each L_n is finite.

To start the recursive construction, we use the fact that $P \in \mathcal{W}$ to get $\partial_0 P \in \mathcal{W}$; in particular $\partial_0 P$ is non-empty and so there is $w_0 \in W_L$ such that

$$(\mathcal{U}x)(\forall\lambda \in L_0)\, x[\lambda] \in P/w_0.$$

Combining this with the fact that $\mathcal{U}[\lambda]^\frown\mathcal{W} = \mathcal{U}[\lambda]$ for all $\lambda \in L$, we can find a left-variable word x_0 from W_{Lv} (since $\mathcal{U}$ concentrates on $v^\frown W_{Lv}$) and sets $P_{0,\lambda} \in \mathcal{W}$ ($\lambda \in L_0$) such that $x_0[\lambda] \in P/w_0$ and $x_0[\lambda]^\frown w \in P/w_0$ for all $w \in P_{0,\lambda}$. Let

$$P_0 = \bigcap\{P_{0,\lambda} : \lambda \in L_0\}.$$

Since $P_0 \in \mathcal{W}$, we also have $\partial_1 P_0 \in \mathcal{W}$; in particular $\partial_1 P_0$ is non-empty and so there is $w_1 \in W_L$ such that, for all $\lambda \in L_0$,

$$P/(w_0{}^\frown x_0[\lambda]^\frown w_1) \in \mathcal{W} \text{ and } (\forall\mu \in L_1)\, P/(w_0{}^\frown x_0[\lambda]^\frown w_1) \in \mathcal{U}[\mu].$$

Rewriting this fact in terms of $\mathcal{U}$, this means

$$(\mathcal{U}x)(\forall\mu \in L_1)(\forall\lambda \in L_0)\, x[\mu] \in P/(w_0{}^\frown x_0[\lambda]^\frown w_1)$$

and so, combining this with the fact that $\mathcal{U}[\lambda]^\frown\mathcal{W} = \mathcal{U}[\lambda]$ for all $\lambda \in L$, we can find a left-variable word x_1 in W_{Lv} and sets $P_{1,\mu} \in \mathcal{W}$ ($\mu \in L_1$) such that $x_1[\mu] \in P/(w_0{}^\frown x_0[\lambda]^\frown w_1)$ and $x_1[\mu]^\frown w \in P/(w_0{}^\frown x_0[\lambda]^\frown w_1)$ for all $w \in P_{1,\mu}$ and for all $\lambda \in L_0$. Now let

$$P_1 = \bigcap\{P_{1,\mu} : \mu \in L_1\}$$

and continue inductively to construct the required sequences (x_n) and (w_n). □

3.4 Alternating equivalence relations for infinite partitions

In this section we will prove an analogue of Theorem 3.2.1 in the case where ω is partitioned into infinitely many pieces. Let $\mathcal{P} = \{P_n : n < \omega\}$ be a partition of ω into infinite sets such that $\min P_i < \min P_j$ whenever $i < j$. In what follows, we use the same notation as in the case where $\mathcal{P}$ is a finite partition of ω, as it will be clear from context which partition $\mathcal{P}$ (and hence which space of equivalence relations) we are considering. Let $\mathcal{E}^{\mathcal{P}}_\infty$ denote the set of all equivalence relations E on ω such that:

(a) E has infinitely many equivalence classes.

(b) If $(p_k(E))_{k<\omega}$ is an increasing enumeration of $p(E)$, then $p_k(E) \in P_n$ if and only if there is $m < \omega$ such that $k = 2^n + 2^{n+1} \cdot m - 1$. Equivalently, $p_k(E) \in P_n$ if and only if $n < \omega$ is maximal such that 2^n divides $k + 1$. (Cf. OEIS A007814.)

Note that condition (b) implies that each $E \in \mathcal{E}_\infty^{\mathcal{P}}$ has infinitely many P_n-classes for each $n < \omega$. (Recall that these are equivalence classes X such that $\min X \in P_n$.) As before, we will say that such an equivalence relation is *$\mathcal{P}$-alternating.* Letting $p_k = p_k(E)$, condition (b) merely states that the first few minimal representatives of E must satisfy

$$p_0, p_2, p_4, \dots \in P_0,$$

$$p_1, p_5, p_9, \dots \in P_1,$$

$$p_3, p_{11}, p_{19}, \dots \in P_2,$$

$$\vdots$$

and so on, so that in general we have $p_k \in P_{\sigma(k)}$ where σ is the sequence defined by the condition in (b), i.e. $\sigma(k)$ is the largest power of 2 which divides $k + 1$. Note that for any $q < \omega$ the distance between any two minimal representatives $p_n, p_m \in P_q$ is a multiple of 2^{q+1}. In particular, any interval of length at least 2^{q+1} contains a minimal representative of a P_q-class.

Theorem 3.4.1. *$(\mathcal{E}_\infty^{\mathcal{P}}, \leq, r)$ is a topological Ramsey space.*

Proof. It is enough to check axiom A.4. Let $[a, E] \neq \emptyset$ be a basic set, $n = |a|$ and $\mathcal{O} \subseteq (\mathcal{A}\mathcal{E}_\infty^{\mathcal{P}})_{n+1}$. We can assume $a = r_n(E)$ and we aim to find $F \in [a, E]$ such that $r_{n+1}[a, F] \subseteq \mathcal{O}$ or $r_{n+1}[a, F] \subseteq \mathcal{O}^c$. As before, consider an arbitrary end-extension $b \in r_{n+1}[a, E]$. Then b is an equivalence relation on a set of the form $p_m(E) = \{0, 1, \dots, p_m(E) - 1\}$ for some $m > n$, such that b has one more equivalence class with minimal representative $p_n(E)$. Thus b joins the classes of E with minimal representatives among $p_{n+1}(E), \dots, p_{m-1}(E)$ to a class with minimal representative $\leq p_n(E)$. Suppose $p_m(E) \in P_q$ (and note that q only depends on n). By our condition for being a member of $\mathcal{E}_\infty^{\mathcal{P}}$, there is $\lambda < \omega$ such that $m = n + 1 + 2^{q+1} \cdot \lambda$.

Let $(q_i)_{i<\omega}$ be the sequence satisfying $p_{n+1+i} \in P_{q_i}$ for all i; in particular $q_0 = q$. Define a sequence $(t_i)_{i<\omega}$ recursively by

$$t_0 = 2^{q_0+1},$$

$$t_i = t_{i-1} \cdot 2^{q_i+1} \quad (i > 0)$$

and use this to define a nested sequence of finite alphabets L_i as follows:

$$L_0 = (n+1)^{t_0},$$

$$L_i = L_{i-1} \cup (n+1)^{t_i} \quad (i > 0).$$

Then let $L = \bigcup_{i<\omega} L_i$. Each $b \in r_{n+1}[a, E]$ of the above form can be coded by a word $w^b \in W_L$ of length λ, by associating each block of the form

$$\left(p_{n+i\cdot t_0+1}(E), p_{n+i\cdot t_0+2}(E), \dots, p_{n+(i+1)\cdot t_0}(E)\right), \ \ i < \lambda$$

to the letter $(k_0, \dots, k_{t_0-1}) \in L_0$ where $p_{k_j}(E)$ is joined to $p_{n+1+i\cdot t_0+j}(E)$.

Conversely, any word $w \in W_L$ determines an end-extension $b(w) \in r_{n+1}[a, E]$ as follows: First, for each $i < \omega$ let m_i be chosen such that $t_i = t_0 \cdot m_i$. Then each letter $l \in (n+1)^{t_{i(l)}}$ of w determines a word $z_l \in W_{L_0}$ of length $m_{i(l)}$, where the j^{th} letter of z_l is the sequence

$$\left(\pi_{j\cdot t_0}(l), \pi_{j\cdot t_0+1}(l), \dots, \pi_{j\cdot t_0+(t_0-1)}(l)\right).$$

By concatenating all such words z_l, w determines a word $z \in W_{L_0}$ of length

$$\lambda' = \sum_{l \text{ is a letter of } w} m_{i(l)}.$$

Now, for each $i < \lambda'$ and $j < t_0$, join the class $p_{n+1+i\cdot t_0+j}(E)$ to $p_{\pi_j(z(i))}(E)$ and let $b(w)$ be the corresponding equivalence relation on the set $p_m(E)$, where $m = n + 1 + t_0 \cdot \lambda'$.

Define a colouring $c : W_L \to 2$ by setting $c(w) = 0$ if and only if $b(w) \in \mathcal{O}$, and apply Theorem 3.3.2 to obtain a variable-free word w_0 together with a sequence $X = (x_i)_{i<\omega}$ of left-variable words such that the translate $w_0{}^\frown[X]^*_L$ is monochromatic for c. In other words, either

1. $b(w) \in \mathcal{O}$ for every $w \in w_0{}^\frown[X]^*_L$, or

2. $b(w) \notin \mathcal{O}$ for every $w \in w_0{}^\frown[X]^*_L$.

For each letter $l \in L$, write $\operatorname{dom}(l)$ for the domain of l, i.e. for the unique integer t_j such that $l \in (n+1)^{t_j}$. For each occurrence of v in x_i for some $i < \omega$, then we set $\operatorname{dom}(v) = t_i$ (note that the domain depends on i). Define a variable-free word u_0 together with a sequence of variable words $(y_i)_{i<\omega}$ as follows: First, let u_0 be the variable-free word of length

$$\sum_{j<|w_0|} \operatorname{dom}(w_0(j))$$

in the alphabet $\{0, \dots, n\}$ obtained by replacing each letter $(k_1, \dots, k_{t_j}) \in (n+1)^{t_j}$ from w_0 with the string $k_1 \dots k_{t_j}$. Then, assuming we have defined u_0 and $y_0, \dots, y_{i-1}$, let y_i be the variable word of length

$$\sum_{j<|x_i|} \operatorname{dom}(x_i(j))$$

in the alphabet $\{0, \dots, n\}$ which is obtained as follows:

(i) Replace each letter $(k_1, \dots, k_{t_j}) \in (n+1)^{t_j}$ in x_i with the string $k_1 \dots k_{t_j}$.

(ii) Replace the left-most variable v in x_i with the string $0 \dots 0v0 \dots 0$ of length t_i, where v occurs in the least place $N < t_i$ such that q_s is equal to q_i, where

$$s = |u_0| + |y_0| + \cdots + |y_{i-1}| + N.$$

Note that such an N exists since $t_i \geq 2^{q_i+1}$ and so the interval

$$[|u_0| + |y_0| + \cdots + |y_{i-1}|, |u_0| + |y_0| + \cdots + |y_{i-1}| + t_i)$$

must contain an index s such that $q_s = q_i$ by definition of the sequence σ.

(iii) Assume inductively that we have defined the first j letters of y_i, and consider the least occurrence of the variable v in x_i which has not been replaced. Then replace v with the string $0 \dots 0v0 \dots 0$ of length t_i, where v occurs in the least place $N < t_i$ such that q_s is equal to q_i, where

$$s = |u_0| + |y_0| + \cdots + |y_{i-1}| + j + N.$$

As before, such N exists since the interval

$$[|u_0| + |y_0| + \cdots + |y_{i-1}| + j, |u_0| + |y_0| + \cdots + |y_{i-1}| + j + t_i)$$

has length $t_i \geq 2^{q_i+1}$.

Now form the infinite word

$$y = u_0{}^\frown y_0{}^\frown y_1{}^\frown \dots{}^\frown y_k{}^\frown \dots$$

out of u_0 and $(y_i)_{i<\omega}$. To define F, it suffices to say how it acts on $p(E)$. If at place i of y we find a letter $k \in \{0, \dots, n\}$, we let $p_{n+1+i}(E)$ and $p_k(E)$ be F-equivalent. If we find a variable at places

$$i, i' \in [|u_0| + |y_0| + \cdots + |y_{j-1}|, |u_0| + |y_0| + \cdots + |y_{j-1}| + |y_j|)$$

(where we set $y_{-1} = \emptyset$) then we let $p_{n+1+i}(E)$ and $p_{n+1+i'}(E)$ be F-equivalent, with no other connections. Then, by our placement of each of the variables, it follows that F is $\mathcal{P}$-alternating. As in the proof of Theorem 3.2.1, F satisfies the conclusion of A.4. □

Corollary 3.4.2. *Suppose c is a finite Souslin measurable colouring of $\mathcal{E}_\infty^{\mathcal{P}}$. Then for every $E \in \mathcal{E}_\infty^{\mathcal{P}}$ there is $F \leq E$ such that the family $\mathcal{E}_\infty^{\mathcal{P}} \upharpoonright F$ of all coarsenings of F is c-monochromatic.*

We can also impose conditions on the elements of each class of an equivalence relation: Given a sequence $\mathcal{I} = (I_n)_{n<\omega}$ of subsets of ω such that $n \in I_n$ for all $n < \omega$, let $\mathcal{E}_\infty^{\mathcal{P},\mathcal{I}}$ be the set of all $\mathcal{P}$-alternating equivalence relations on ω which also satisfy:

$(c)_{\mathcal{I}}$ If X is a P_n-class, then $X \subseteq \bigcup\{P_m : m \in I_n\}$.

Modifying the proof of Theorem 3.4.1 in the natural way (as in the proof of Theorem 3.2.3) yields the following results.

Theorem 3.4.3. *$(\mathcal{E}_\infty^{\mathcal{P},\mathcal{I}}, \leq, r)$ is a topological Ramsey space.*

Corollary 3.4.4. *Suppose c is a finite Souslin measurable colouring of $\mathcal{E}_\infty^{\mathcal{P},\mathcal{I}}$. Then for every $E \in \mathcal{E}_\infty^{\mathcal{P},\mathcal{I}}$ there is $F \leq E$ such that the family $\mathcal{E}_\infty^{\mathcal{P},\mathcal{I}} \upharpoonright F$ of all coarsenings of F is c-monochromatic.*

Our next goal is to apply Corollary 3.4.4 to obtain dual Ramsey results for countable limit ordinals above or equal to ω^2. Unless otherwise specified, for the rest of this section we fix a countable limit ordinal $\alpha \geq \omega^2$. Using left division of ordinals, we find a countable ordinal $\beta \geq \omega$ such that $\alpha = \omega \cdot \beta$. (The fact that the remainder is 0 follows from the assumption that α is a limit.) Fix a bijection $f : \omega \to \beta$ such that $f(0) = 0$. Let $\mathcal{P}$ be any partition of ω as above, and identify each P_n with the $f(n)^{\text{th}}$ copy of ω in α, i.e. the set $\omega \times \{f(n)\}$. We remark that unlike the case for the ordinals $\omega \cdot l$, in general there is no natural way to "order" the copies of ω in α via a bijection f. However, for an ordinal $\alpha < \varepsilon_0$, there is a "default" choice for f given by first constructing a canonical bijection between ω and ε_0 using the

Cantor normal form and then restricting to α. Since we will not make use of this canonical bijection, we leave the details to the interested reader.

As before, we work with equivalence relations on α which arise from rigid surjections $\alpha \to \alpha$, and so it is necessary to prescribe conditions on the set of minimal representatives of such an equivalence relation. In this case, we ask that the representatives behave according to the sequence used to define the space $\mathcal{E}_\infty^{\mathcal{P}}$ introduced above. To formalize this requirement, let σ be the sequence used in the part (b) of the definition of the space $\mathcal{E}_\infty^{\mathcal{P}}$ and consider the bijection $\omega \to \alpha$ defined by

$$n \mapsto \big(|\{m < n : \sigma(m) = \sigma(n)\}|, f(\sigma(n))\big).$$

Then α inherits a linear ordering $\preceq_f$ via this bijection in such a way that $(\alpha, \preceq_f)$ has order type ω. If we enumerate the elements of α as $(\gamma_n)_{n<\omega}$ in $\preceq_f$-increasing order, then

$$\gamma_0, \gamma_2, \gamma_4, \cdots \in \omega \times \{f(0)\},$$

$$\gamma_1, \gamma_5, \gamma_9, \cdots \in \omega \times \{f(1)\},$$

$$\gamma_3, \gamma_{11}, \gamma_{19}, \cdots \in \omega \times \{f(2)\},$$

$$\vdots$$

etc., and in general $\gamma_n \in \omega \times \{f(\sigma(n))\}$, so that $(\alpha, \preceq_f)$ is ordered according to the sequence σ. Let $\mathcal{E}_\infty^f(\alpha)$ denote the set of all equivalence relations E on α such that:

(a) If $(p_k)_{k<\omega}$ is a $\preceq_f$-increasing enumeration of $p(E)$, then $p_k \in \omega \times \{f(\sigma(k))\}$ for each k.

(b) If $(q_k)_{k<\omega}$ is a $\preceq_f$-increasing enumeration of the set of $\preceq_f$-minimal representatives of E, then $q_k = p_k$ for all k.

For each n, let

$$I_n = \{m < \omega : f(m) \geq f(n)\}$$

and $\mathcal{I} = (I_n)_{n<\omega}$. As in the case for $\mathcal{E}_\infty(\omega \cdot l)$, we can define bijections

$$\varphi : \omega \to \alpha \text{ and } \Phi : \mathcal{E}_\infty^{\mathcal{P},\mathcal{I}} \to \mathcal{E}_\infty^f(\alpha)$$

which allow us to transfer equivalence relations on ω to equivalence relations on α. Note that condition $(c)_\mathcal{I}$ ensures that $\phi(p(E)) = p(\Phi(E))$ for each $E \in \mathcal{E}_\infty^{\mathcal{P},\mathcal{I}}$.

Below, the notion of Souslin measurability can refer to either the topology on $\mathcal{E}_\infty^f(\alpha)$ inherited from $\mathcal{E}_\infty^{\mathcal{P},\mathcal{I}}$ via Φ (where the latter set is equipped with the Ellentuck topology or the metrizable topology), or the topology induced from 2^{α^2} when each equivalence relation in $\mathcal{E}_\infty^f(\alpha)$ is identified with a subset of α^2. The following results extend the dual Silver and dual Ramsey theorems to the ordinal α. The proofs are exactly as in the previous section.

Corollary 3.4.5. *Suppose c is a finite Souslin measurable colouring of $\mathcal{E}_\infty^f(\alpha)$. Then for every $E \in \mathcal{E}_\infty^f(\alpha)$ there is $F \leq E$ such that the family $\mathcal{E}_\infty^f(\alpha) \restriction F$ of all coarsenings of F is c-monochromatic.*

To state the corresponding dual Ramsey theorem for α, fix $k < \omega$ and a bijection f as above. Let $\mathcal{E}_k^f(\alpha)$ denote the set of all equivalence relations E on α with exactly k equivalence classes such that if

$(p_i)_{i<k}$ is a $\preceq_f$-increasing enumeration of $p(E)$, then $p_i \in \omega \times \{f(\sigma(i))\}$ for all $i < k$, and if $(q_i)_{i<k}$ is a $\preceq_f$-increasing enumeration of the set of $\preceq_f$-minimal representatives of E, then $q_i = p_i$ for all $i < k$. Equip $\mathcal{E}_k^f(\alpha)$ with the topology inherited from 2^{α^2} by identifying each equivalence relation in $\mathcal{E}_k^f(\alpha)$ with a subset of α^2.

Corollary 3.4.6. *Suppose c is a finite Souslin measurable colouring of $\mathcal{E}_k^f(\alpha)$. Then for every $E \in \mathcal{E}_\infty^f(\alpha)$ there is $F \leq E$ such that the family $\mathcal{E}_k^f(\alpha) \upharpoonright F$ of all k-coarsenings of F is c-monochromatic.*

We conclude by noting that the methods of this chapter can also be used to obtain similar results for a countable successor ordinal α if one is willing to work with surjections $f : \alpha \to \alpha$ which are rigid on the "limit" part of α; that is, writing $\alpha = \beta + n$ for a limit ordinal β and $n < \omega$, we require $f \upharpoonright \beta : \beta \to \beta$ to be a rigid surjection. On the other hand, any rigid surjection $f : \beta + n \to \beta + n$ is necessarily equal to the identity when restricted to the interval $[\beta, \beta + n)$ and so the corresponding rigid surjection version of the dual Ramsey theorem follows from the dual Ramsey theorem for β.

Chapter 4

Ramsey theory of infinite block sequences

We now shift our focus to aspects of a subfield of infinite-dimensional Ramsey theory known as *block Ramsey theory.* In this setting, one considers spaces of infinite block sequences in a Banach space and studies the relevant Ramsey-theoretic aspects. The study of block sequences in a separable Banach space with a Schauder basis is often helpful for understanding its geometry. The utility of this approach is highlighted in, for instance, the problem of oscillation stability in a Banach space. In this chapter we will work entirely in the Banach space c_0 consisting of all real-valued sequences which converge to 0. The main goal of this chapter is twofold: First, we use ultrafilter methods to show that Gowers' $\mathrm{FIN}_{\pm k}$ theorem has a natural infinite-dimensional analogue which can be viewed as an "approximate" Ramsey theorem. We then show that the Parametrized Milliken Theorem holds when FIN is replaced with FIN_k, and holds "approximately" when FIN is replaced with $\mathrm{FIN}_{\pm k}$, thus obtaining new examples of parametrized infinite-dimensional Ramsey results. In particular, this will yield a second proof of the infinite-dimensional $\mathrm{FIN}_{\pm k}$. All proofs in this chapter will make use of *ultra-Ramsey theory*; this approach is advantageous since it allows for more flexibility when dealing with "approximate" Ramsey-theoretic results.

4.1 Preliminaries

4.1.1 Gowers' theorems

For convenience, we recall some of the basic definitions concerning Gowers' theorems from Chapter 1. Given $k \in \mathbb{N}$, let $\mathrm{FIN}_{\pm k}$ denote the set of all functions $p : \omega \to \{0, \pm 1, \ldots, \pm k\}$ such that

$$\operatorname{supp} p := \{n < \omega : p(n) \neq 0\}$$

is finite and such that p achieves at least one of the values $\pm k$. Given $p, q \in \mathrm{FIN}_{\pm k}$, write $p < q$ whenever $\max \operatorname{supp} p < \min \operatorname{supp} q$. In this case $p + q$ will denote the element of $\mathrm{FIN}_{\pm k}$ given by the coordinate-wise sum of p and q. This operation gives $\mathrm{FIN}_{\pm k}$ the structure of a partial semigroup.

We also have an operation between various FIN spaces: The *tetris operation* $T : \mathrm{FIN}_{\pm k} \to \mathrm{FIN}_{\pm(k-1)}$

is defined by

$$T(p)(n) := \begin{cases} p(n) - 1 & \text{if } p(n) > 0, \\ 0 & \text{if } p(n) = 0, \\ p(n) + 1 & \text{if } p(n) < 0. \end{cases}$$

It is easy to check that T is a surjective homomorphism of partial semigroups. For $\alpha \leq \omega$, a sequence $(p_n)_{n<\alpha}$ is a *block sequence in* $\mathrm{FIN}_{\pm k}$ if $p_n \in \mathrm{FIN}_{\pm k}$ and $p_n < p_m$ for all $n < m < \alpha$. Let $\mathrm{FIN}_{\pm k}^{[\infty]}$ denote the space of all infinite block sequences in FIN_k. Given a block sequence $P = (p_n)_{n<\alpha}$, the *partial subsemigroup of* $\mathrm{FIN}_{\pm k}$ *generated by* P is defined as

$$[P]_{\pm k} := \{\varepsilon_0 T^{j_0}(p_{n_0}) + \cdots + \varepsilon_m T^{j_m}(p_{n_m}) : m < \omega, n_0 < \cdots < n_m < \alpha,$$
$$\varepsilon_0, \ldots, \varepsilon_m \in \{\pm 1\}, j_0, \ldots, j_m < k \text{ and } \min j_i = 0\}.$$

If $Q = (q_n)_{n<\beta}$, $\beta \leq \alpha$ is another block sequence, write $Q \leq P$ and say Q is a *block subsequence of* P whenever $q_n \in [P]_{\pm k}$ for all $n < \beta$. We write $[P]_{\pm k}^{[\infty]}$ for the set of all infinite block subsequences of P.

For a subset $A \subseteq \mathrm{FIN}_{\pm k}$ and $\varepsilon > 0$, define

$$(A)_\varepsilon := \{p \in \mathrm{FIN}_{\pm k} : (\exists q \in A)\, ||p - q||_\infty \leq \varepsilon\}$$

where $||\cdot||_\infty$ denotes the ℓ_∞ norm. We can now state the following theorem of Gowers, originally proved in [22] using the theory of idempotent ultrafilters in order to show that every real-valued uniformly continuous function on the unit sphere of c_0 is oscillation stable.

Theorem 4.1.1 (Gowers)**.** *For every $k, r \in \mathbb{N}$ and every $c : \mathrm{FIN}_{\pm k} \to r$ there are $i < r$ and $P \in \mathrm{FIN}_{\pm k}^{[\infty]}$ such that $[P]_{\pm k} \subseteq \left(c^{-1}\{i\}\right)_1$.*

There is also an exact version of Gowers' theorem, which we now describe: Given $k \in \mathbb{N}$, let FIN_k denote the set of all functions $p : \omega \to k + 1$ such that $\operatorname{supp} p$ is finite and $k \in \operatorname{range}(p)$. The ordering $<$ on FIN_k and the sum $p + q$ of two elements of FIN_k are defined analogously. The corresponding tetris operation $T : \mathrm{FIN}_k \to \mathrm{FIN}_{k-1}$ is defined by

$$T(p)(n) := \begin{cases} p(n) - 1 & \text{if } p(n) > 0, \\ 0 & \text{if } p(n) = 0. \end{cases}$$

As before, a block sequence in FIN_k is a sequence $P = (p_n)_{n<\omega}$ such that $p_n < p_m$ whenever $n < m$. $\mathrm{FIN}_k^{[\infty]}$ will denote the space of all infinite block sequences in FIN_k. The partial subsemigroup of FIN_k generated by $P = (p_n)_{n<\omega} \in \mathrm{FIN}_k^{[\infty]}$ is

$$[P]_k := \{T^{j_0}(p_{n_0}) + \cdots + T^{j_m}(p_{n_m}) : m < \omega, n_0 < \cdots < n_m < \alpha,$$
$$j_0, \ldots, j_m < k \text{ and } \min j_i = 0\}$$

and the set of all infinite block subsequences of P will be denoted $[P]_k^{[\infty]}$. The following result was also proved by Gowers in [22].

Theorem 4.1.2 (Gowers)**.** *For every $k, r \in \mathbb{N}$ and every $c : \mathrm{FIN}_k \to r$ there is $P \in \mathrm{FIN}_k^{[\infty]}$ such that $[P]_k$ is monochromatic.*

We refer the reader to [2, 33, 65] for details and proofs of Gowers' theorems. The interested reader is also referred to [55, 66] for discussions and proofs of the finite versions of Gowers' theorems.

The case $k = 1$ of the following theorem is due to Milliken [51] while the general case is due to Todorčević [65]; they are the infinite-dimensional versions of Hindman's theorem [29] and Gowers' FIN_k theorem, respectively.

Theorem 4.1.3 (Milliken-Todorčević Theorem)**.** *For every finite Souslin measurable colouring of* $\mathrm{FIN}_k^{[\infty]}$ *there is* $P \in \mathrm{FIN}_k^{[\infty]}$ *such that* $[P]_k^{[\infty]}$ *is monochromatic.*

4.1.2 Ultra-Ramsey theory of block sequences

Recall that an *ultrafilter* on a set X is a collection $\mathcal{U}$ of subsets of X satisfying the following four properties:

1. $\varnothing \notin \mathcal{U}$.
2. $A, B \in \mathcal{U}$ implies $A \cap B \in \mathcal{U}$.
3. $A \in \mathcal{U}, B \supseteq A$ implies $B \in \mathcal{U}$.
4. For every $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.

Let βX denote the set of all ultrafilters on X; then βX is a compact Hausdorff space under the topology generated by basic open sets of the form

$$\overline{A} := \{\mathcal{U} \in \beta X : A \in \mathcal{U}\}$$

where A is a non-empty subset of X. It is useful to view ultrafilters as quantifiers (e.g. as in Blass [9]) in the following way. Let $\mathcal{U}$ be an ultrafilter on a set X. Given a first-order formula $\varphi(x)$ with a free variable x ranging over elements of X, we write

$$(\mathcal{U}x)\varphi(x) \iff \{x \in X : \varphi(x)\} \in \mathcal{U}.$$

Using the ultrafilter properties above it is easy to check that ultrafilter quantifiers commute with conjunction and negation of first-order formulas, i.e. we have

$$(\mathcal{U}x)\varphi(x) \wedge (\mathcal{U}x)\psi(x) \iff (\mathcal{U}x)(\varphi(x) \wedge \psi(x)),$$

$$\neg(\mathcal{U}x)(\varphi(x)) \iff (\mathcal{U}x)(\neg\varphi(x))$$

for any first-order formulas $\varphi(x)$ and $\psi(x)$.

Throughout this chapter we will prove various results using *ultra-Ramsey theory* as developed in [65]. We now proceed to describe a class of trees which form the basis for the required ultra-Ramsey theory in this section. To this end, for each $k \in \mathbb{N}$ we view the space $\mathrm{FIN}_{\pm k}^{[<\infty]}$ as a tree ordered by end-extension $\sqsubseteq$ and with root $\varnothing$, the empty sequence. Unless otherwise specified, for the rest of this section we fix $k \in \mathbb{N}$ together with the ultrafilter $\mathcal{U}$ on $\mathrm{FIN}_{\pm k}$ given by Lemma 4.1.8. The next two definitions are adapted from [65, Chapter 7.2].

Definition 4.1.4. A $\mathcal{U}$*-tree* is a downward closed subtree $U \subseteq \mathrm{FIN}_{\pm k}^{[<\infty]}$ such that

$$U_t := \{p \in \mathrm{FIN}_{\pm k} : (t, p) \in U\} \in \mathcal{U}$$

for all $t \in U$ which extend the stem of U, where the *stem* is the $\sqsubseteq$-maximal element of U which is comparable to every other node of the tree.

The stem of a $\mathcal{U}$-tree U will be denoted by $\mathrm{stem}(U)$. Note that each $\mathcal{U}$-tree U has root $\varnothing$ since it is downward closed under $\sqsubseteq$, while $\mathrm{stem}(U)$ can be any finite block sequence in $\mathrm{FIN}_{\pm k}$.

Given a $\mathcal{U}$-tree U, the set of infinite branches of U is denoted by

$$[U] := \{(p_n)_{n<\omega} \in \mathrm{FIN}_{\pm k}^{[\infty]} : (p_0, \ldots, p_m) \in U \text{ for all } m < \omega\}.$$

For $t \in U$ let $|t|$ denote the *length* of t, which is just the domain of t when viewed as a finite sequence in $\mathrm{FIN}_{\pm k}^{[<\infty]}$. For $m < \omega$, the m^{th} level $U(m)$ of U is the set of all $t \in U$ of length m.

In order to prove an infinite-dimensional version of Theorem 4.1.1 we work with a topology defined using $\mathcal{U}$-trees and which extends the usual metrizable topology on $\mathrm{FIN}_{\pm k}^{[\infty]}$. Working with this topology allows us to remedy the fact that the space $\mathrm{FIN}_{\pm k}$ lacks an exact pigeonhole principle.

Definition 4.1.5. Let $\mathcal{X} \subseteq \mathrm{FIN}_{\pm k}^{[\infty]}$. $\mathcal{X}$ is $\mathcal{U}$*-open* if for every $A \in \mathcal{X}$ there is a $\mathcal{U}$-tree U such that $A \in [U] \subseteq \mathcal{X}$. $\mathcal{X}$ is $\mathcal{U}$*-Ramsey* if for every $\mathcal{U}$-tree U there is a $\mathcal{U}$-subtree $U' \subseteq U$ with $\mathrm{stem}(U) = \mathrm{stem}(U')$ such that $[U'] \subseteq \mathcal{X}$ or $[U'] \subseteq \mathcal{X}^c$. If the second alternative always holds then we say $\mathcal{X}$ is $\mathcal{U}$*-Ramsey null*.

The collection of all $\mathcal{U}$-open subsets of $\mathrm{FIN}_{\pm k}^{[\infty]}$ forms a topology, called the $\mathcal{U}$*-topology*, which refines the metrizable topology of $\mathrm{FIN}_{\pm k}^{[\infty]}$. The next two results are adapted from [65, Chapter 7.2] by replacing the tree $\mathbb{N}^{[<\infty]}$ of finite subsets of $\mathbb{N}$ ordered by end-extension with the tree $\mathrm{FIN}_{\pm k}^{[<\infty]}$. We state them in our context without proof. First, recall that a subset A of a topological space X has the *property of Baire* if there is an open set $U \subseteq X$ such that the symmetric difference of A and U is meagre in X. We then have the following version of Todorčević's ultra-Ellentuck theorem, which builds on a theorem of Ellentuck [15] relating the notions of Baire and Ramsey in the setting of $\mathbb{N}^{[\infty]}$, the set of all infinite subsets of $\mathbb{N}$.

Theorem 4.1.6. *Let $\mathcal{X} \subseteq \mathrm{FIN}_{\pm k}^{[\infty]}$. Then $\mathcal{X}$ has the property of Baire relative to the $\mathcal{U}$-topology if and only if $\mathcal{X}$ is $\mathcal{U}$-Ramsey. Furthermore, $\mathcal{X}$ is meagre with respect to the $\mathcal{U}$-topology if and only if $\mathcal{X}$ is $\mathcal{U}$-Ramsey null.*

The next result uses a classical fact of Nikodym (see, e.g., [65, Chapter 4.1]) which says that, in any topological space, the property of Baire is preserved under the Souslin operation.

Corollary 4.1.7. *For every $r \in \mathbb{N}$ and every Souslin measurable $c : \mathrm{FIN}_{\pm k}^{[\infty]} \to r$ there are $i < r$ and a $\mathcal{U}$-tree U with stem $\varnothing$ such that $[U] \subseteq c^{-1}\{i\}$.*

Given two ultrafilters $\mathcal{U}, \mathcal{V} \in \beta\,\mathrm{FIN}_{\pm k}$, define the *sum* of $\mathcal{U}$ and $\mathcal{V}$ by declaring

$$A \in \mathcal{U} + \mathcal{V} \iff (\mathcal{U}p)(\mathcal{V}q)\,(p + q \in A)$$

for $A \subseteq \mathrm{FIN}_{\pm k}$. To ensure that this operation is always defined we restrict our attention to the set of all *cofinite* ultrafilters on $\mathrm{FIN}_{\pm k}$, i.e. ultrafilters $\mathcal{U} \in \beta\,\mathrm{FIN}_{\pm k}$ which satisfy

$$X_m := \{p \in \mathrm{FIN}_{\pm k} : p(n) = 0 \text{ for all } n < m\} \in \mathcal{U}$$

for all $m < \omega$. Let $\gamma\,\mathrm{FIN}_{\pm k}$ denote the set of all cofinite $\mathcal{U} \in \beta\,\mathrm{FIN}_{\pm k}$. Then $(\gamma\,\mathrm{FIN}_{\pm k}, +)$ is a compact right-topological semigroup. (We refer the reader to [65, Chapter 2] for details.) We also extend the tetris operation $T : \mathrm{FIN}_{\pm k} \to \mathrm{FIN}_{\pm(k-1)}$ to a map $T : \gamma\,\mathrm{FIN}_{\pm k} \to \gamma\,\mathrm{FIN}_{\pm(k-1)}$ by setting

$$A \in T(\mathcal{U}) \iff T^{-1}(A) \in \mathcal{U}$$

for each $A \subseteq \mathrm{FIN}_{\pm(k-1)}$; in other words, $T(\mathcal{U})$ is the *Rudin-Keisler image* of $\mathcal{U}$ under T. This extension of T is a continuous surjective homomorphism. Below we will consider the sign-flipped version of the tetris operation given by

$$-T : \mathrm{FIN}_{\pm k} \to \mathrm{FIN}_{\pm(k-1)} : p \mapsto -T(p)$$

together with its extension to $\gamma\,\mathrm{FIN}_{\pm k}$ (the definition of which is analogous to the extension of T to $\gamma\,\mathrm{FIN}_{\pm k}$ above).

Given $A \subseteq \mathrm{FIN}_{\pm k}$, let $-A := \{-x : x \in A\}$. We will need the following result, the proof of which uses the general theory of idempotents in compact semigroups.

Lemma 4.1.8. *There exists a cofinite ultrafilter $\mathcal{U}$ on $\mathrm{FIN}_{\pm k}$ such that*

$$\mathcal{U} + (-T)^j\mathcal{U} = (-T)^j\mathcal{U} + \mathcal{U} = \mathcal{U} \textit{ for all } j \in \{0, \dots, k\}.$$

Furthermore, $\mathcal{U}$ is subsymmetric*: For every $A \in \mathcal{U}$ we have $-(A)_1 \in \mathcal{U}$.*

The proof of the first part of the above result can be found in [2, Chapter III.5] or [33, Lemma 4]. The second part follows from the first (see [33, Lemma 11]) but we point out here that the theory of subsymmetric ultrafilters was first developed in [65, Chapter 2] (and in an earlier manuscript of Todorčević) and is used there to give an ultrafilter proof of Gowers' theorem. Note that the ultrafilter $\mathcal{U}$ given by Lemma 4.1.8 has the property that, for any $A \in \mathcal{U}$ and $j < k$,

$$(\mathcal{U}f)(\mathcal{U}g)\left(\{f, g, f + (-T)^j(g), (-T)^j(f) + g\} \subseteq A\right).$$

Since ultrafilter quantifiers commute with finite conjunctions it follows that

$$(\mathcal{U}f)(\mathcal{U}g)\left(\{f, g, f + (-T)^j(g), (-T)^j(f) + g : j < k\} \subseteq A\right)$$

for any $A \in \mathcal{U}$.

Theorem 4.1.9. *Let $\mathcal{X} \subseteq \mathrm{FIN}_{\pm k}^{[\infty]}$. Then $\mathcal{X}$ has the property of Baire relative to the $\mathcal{U}$-topology if and only if $\mathcal{X}$ is $\mathcal{U}$-Ramsey. Furthermore, $\mathcal{X}$ is meagre with respect to the $\mathcal{U}$-topology if and only if $\mathcal{X}$ is $\mathcal{U}$-Ramsey null.*

Corollary 4.1.10. *For every $r \in \mathbb{N}$ and every Souslin measurable $c : \mathrm{FIN}_{\pm k}^{[\infty]} \to r$ there are $i < r$ and a $\mathcal{U}$-tree U with stem $\emptyset$ such that $[U] \subseteq c^{-1}\{i\}$.*

In this brief section we define a class of subtrees which will allow us to inductively construct certain block sequences during the proof of Theorem 4.2.1. First, notice that if $p, q \in \mathrm{FIN}_{\pm k}$ satisfy $||p - q|| \leq 1$, then

$$n \in (\operatorname{supp} p \setminus \operatorname{supp} q) \cup (\operatorname{supp} q \setminus \operatorname{supp} p) \text{ implies } |p(n)|, |q(n)| \leq 1.$$

This motivates the following weak version of the tetris operation: Given $p \in \mathrm{FIN}_{\pm k}$ define $S(p) \in \mathrm{FIN}_{\pm k}$ by

$$S(p)(n) := \begin{cases} p(n) & \text{if } |p(n)| \neq 1 \\ 0 & \text{if } |p(n)| = 1. \end{cases}$$

We will repeatedly use the fact that $||p - S(p)|| \leq 1$ for all $p \in \mathrm{FIN}_{\pm k}$. In particular, notice that $||p - q|| \leq 1$ implies $\operatorname{supp} S(p) \subseteq \operatorname{supp} q$ and $||S(p) - q|| \leq 2$. This will allow us to control the supports of elements which are close to a fixed $q \in \mathrm{FIN}_{\pm k}$. Also note that S is *idempotent*, i.e. $S \circ S = S$. The following lemma allows us to replace a given $\mathcal{U}$-tree with one which behaves well with respect to S, at the cost of adding an approximate constant.

Lemma 4.1.11. *Suppose V is a $\mathcal{U}$-tree with* $\mathrm{stem}(V) = \emptyset$*. Then there is a $\mathcal{U}$-tree U with* $\mathrm{stem}(U) = \emptyset$ *such that $[U] \subseteq ([V])_1$ and such that U is S-*closed*: For every $t \in U$ and every $p \in \mathrm{FIN}_{\pm k}$, we have*

$$(t, p) \in U \to (t, S(p)) \in U.$$

Proof. Fix a well-ordering $<$ of $\mathrm{FIN}_{\pm k}^{[<\infty]}$. We construct, by induction on $n \geq 1$, each level $U(n)$ of U above $\emptyset$ together with projections $\pi_n : U(n) \to V(n)$ satisfying $||t - \pi_n(t)|| \leq 1$ for all $t \in U(n)$. To begin, take $U_\emptyset := V_\emptyset \cup S(V_\emptyset)$ and hence

$$U(1) := \{(p) \in \mathrm{FIN}_{\pm k}^{[1]} : p \in U_\emptyset\}.$$

The projection $\pi_1 : U(1) \to V(1)$ is defined by setting, for $t = (p) \in U(1)$,

$$\pi_1(t) := \begin{cases} (p) & \text{if } p \in V_\emptyset \\ \left(\min\left(V_\emptyset \cap S^{-1}(p)\right)\right) & \text{otherwise} \end{cases}$$

where the minimum is taken with respect to $<$. Furthermore, since $||q - S(q)|| \leq 1$ for all $q \in \mathrm{FIN}_{\pm k}$, we have $||t - \pi_1(t)|| \leq 1$ for all $t \in U(1)$. Now suppose we have constructed the first $m > 1$ levels $U(1), \ldots, U(m)$ of U above $\emptyset$ with their corresponding projections $\pi_1, \ldots, \pi_m$. For each $t \in U(m)$, set $U_t := V_{\pi_m(t)} \cup S(V_{\pi_m(t)})$. We then define

$$U(m+1) := \{(s, p) \in \mathrm{FIN}_{\pm k}^{[m+1]} : s \in U(m),\, p \in U_s\}.$$

The projection $\pi_{m+1} : U(m+1) \to V(m+1)$ is defined by setting, for $t = (s, p) \in U(m+1)$ with $s \in U(m)$ and $p \in U_s$,

$$\pi_{m+1}(t) := \begin{cases} (\pi_m(s), p) & \text{if } p \in V_{\pi_m(s)} \\ \left(\pi_m(s), \min\left(V_{\pi_m(s)} \cap S^{-1}(p)\right)\right) & \text{otherwise} \end{cases}$$

where the minimum is taken with respect to $<$. Inductively we have $||s - \pi_m(s)|| \leq 1$ and so by definition of S we have $||t - \pi_{m+1}(t)|| \leq 1$. This completes the inductive construction of U. The fact that U is S-closed follows easily from the above construction. To finish, we check that $[U] \subseteq ([V])_1$. Let

$P = (p_n)_{n<\omega}$ be an infinite block sequence corresponding to a branch of U. We define a projection $\pi_\infty : [U] \to [V]$ by setting

$$\pi_\infty(P) := (\pi_n \circ r_n(P))_{n\in\mathbb{N}}$$

where $r_n : [U] \to U(n)$ is the n^{th} restriction mapping given by

$$r_n(P) := (p_0, \dots, p_{n-1}).$$

Note that $\pi_\infty(P)$ is indeed a branch in V since $s \sqsubseteq t$ implies $\pi_{|s|}(s) \sqsubseteq \pi_{|t|}(t)$ for any $s, t \in U$. Since for every $P \in [U]$ we have $||P - \pi_\infty(P)|| \leq 1$ and $\pi_\infty(P) \in [V]$, we obtain that $[U] \subseteq ([V])_1$. □

4.2 A proof of the infinite-dimensional approximate Gowers theorem

In this section we will work with multi-dimensional versions of the $\mathrm{FIN}_{\pm k}$ spaces defined above. For each $m \in \mathbb{N}$, let $\mathrm{FIN}_{\pm k}^{[m]}$ be the set of all block sequences in $\mathrm{FIN}_{\pm k}$ of length m. We also let

$$\mathrm{FIN}_{\pm k}^{[<\infty]} := \bigcup_{m\in\mathbb{N}} \mathrm{FIN}_{\pm k}^{[m]}$$

be the set of all finite block sequences in $\mathrm{FIN}_{\pm k}$. Furthermore, recall that $\mathrm{FIN}_{\pm k}^{[\infty]}$ denotes the set of all infinite block sequences in $\mathrm{FIN}_{\pm k}$.

For each $\alpha \in \mathbb{N} \cup \{\infty\}$ we extend the ℓ_∞ norm to a metric on $\mathrm{FIN}_{\pm k}^{[\alpha]}$ by setting, for $P = (p_n)_{n<\alpha}$ and $Q = (q_n)_{n<\alpha}$,

$$||P - Q|| := \sup_{n<\alpha} ||p_n - q_n||.$$

Finally, for $\alpha \in \mathbb{N} \cup \{\infty\}$, $\mathcal{X} \subseteq \mathrm{FIN}_{\pm k}^{[\alpha]}$ and $\varepsilon > 0$, define

$$(\mathcal{X})_\varepsilon := \{P \in \mathrm{FIN}_{\pm k}^{[\alpha]} : (\exists Q \in \mathcal{X})\, ||P - Q|| \leq \varepsilon\}.$$

Let $\langle P \rangle_{\pm k}^{[\infty]}$ denote the set of all $Q \in \mathrm{FIN}_{\pm k}^{[\infty]}$ such that $Q \leq P$. In this section we give a proof of the following result.

Theorem 4.2.1. *For every $k, r \in \mathbb{N}$ and every Souslin measurable $c : \mathrm{FIN}_{\pm k}^{[\infty]} \to r$ there are $i < r$ and an infinite block sequence $P \in \mathrm{FIN}_{\pm k}^{[\infty]}$ such that $\langle P \rangle_{\pm k}^{[\infty]} \subseteq \left(c^{-1}\{i\}\right)_1$.*

To do so, we first need to consider the following modification of the usual notion of block subsequence. Given a block sequence $P = (p_n)_{n<\omega} \in \mathrm{FIN}_{\pm k}^{[\infty]}$, let $\langle P \rangle_{(-T)}$ be the partial subsemigroup consisting of all vectors of the form

$$(-T)^{j_0}(p_{n_0}) + \cdots + (-T)^{j_m}(p_{n_m})$$

where $m < \omega, n_0 < \cdots < n_m < \omega$ and $j_0, \dots, j_m < k$ are such that $\min j_i = 0$. If $Q = (q_n)_{n<\omega}$ is another block sequence, write $Q \leq_{(-T)} P$ to denote that $q_n \in \langle P \rangle_{(-T)}$ for every $n < \omega$. We define $\langle P \rangle_{(-T)}$ for finite block sequences $P = (p_n)_{n<m}$ similarly; in this case we write $\langle p_0, \dots, p_{m-1} \rangle_{(-T)}$ for the corresponding (finite) partial subsemigroup.

Lemma 4.2.2. *Let U be a $\mathcal{U}$-tree with stem $\emptyset$. There is $P = (p_n)_{n<\omega} \in \mathrm{FIN}_{\pm k}^{[\infty]}$ such that $Q \leq_{(-T)} P$ implies $Q \in [U]$.*

Proof. By induction on $n < \omega$ we define two sequences $A_0 \supseteq A_1 \supseteq \dots$ and $p_0 < p_1 < \dots$ such that, for all $n < \omega$,

1. $p_n \in A_n \in \mathcal{U}$,
2. $A_{n+1} \subseteq \{q \in \mathrm{FIN}_{\pm k} : \langle p_n, q\rangle_{(-T)} \subseteq A_n\}$, and
3. $A_n \subseteq U_t \cap -(U_t)_1$ for every $t \in U$ such that

$$\operatorname{supp} \sum t \subseteq \bigcup_{i<n} \operatorname{supp} p_i$$

where, for a node $t = (t_0, \dots, t_{m-1}) \in U$, $\sum t$ is the element $\sum_{i<m} t_i \in \mathrm{FIN}_{\pm k}$. To start, take $A_0 := U_\emptyset \cap -(U_\emptyset)_1$ and note that $A_0 \in \mathcal{U}$ since $\mathcal{U}$ is subsymmetric and $U_\emptyset \in \mathcal{U}$. By definition of $\mathcal{U}$ we have

$$(\mathcal{U}p)(\mathcal{U}q)\left(\langle p, q\rangle_{(-T)} \subseteq A_0\right)$$

and so we take any $p_0 \in \mathrm{FIN}_{\pm k}$ such that $(\mathcal{U}q)\left(\langle p_0, q\rangle_{(-T)} \subseteq A_0\right)$; in particular $p_0 \in A_0$ by definition of $\langle p_0, q\rangle_{(-T)}$. We then take A_1 to be the intersection of the set $\{q \in \mathrm{FIN}_{\pm k} : \langle p_0, q\rangle_{(-T)} \subseteq A_0\}$ with

$$\bigcap \left\{U_t \cap -(U_t)_1 : t \in U \text{ and } \operatorname{supp} \sum t \subseteq \operatorname{supp} p_0\right\}.$$

Note that $A_0 \supseteq A_1$ and $A_1 \in \mathcal{U}$ since there are only finitely many $t \in U$ satisfying $\operatorname{supp} \sum t \subseteq \operatorname{supp} p_0$, and since each $U_t \cap -(U_t)_1 \in \mathcal{U}$ using the fact that $\mathcal{U}$ is subsymmetric.

Now suppose $A_0, \dots, A_n$ and $p_0, \dots, p_{n-1}$ have been constructed. Since $\mathcal{U}$ is cofinite, pick any $p_n \in \mathrm{FIN}_{\pm k}$ such that $p_n > p_{n-1}$ and $(\mathcal{U}q)\left(\langle p_n, q\rangle_{(-T)} \subseteq A_n\right)$; in particular $p_n \in A_n$. Then take A_{n+1} to be the intersection of the set $\{q \in \mathrm{FIN}_{\pm k} : \langle p_n, q\rangle_{(-T)} \subseteq A_n\}$ with

$$\bigcap \left\{U_t \cap -(U_t)_1 : t \in U \text{ and } \operatorname{supp} \sum t \subseteq \bigcup_{i<n+1} \operatorname{supp} p_i\right\}.$$

As before, we have $A_{n+1} \in \mathcal{U}$ and $A_n \supseteq A_{n+1}$. This completes the induction.

To check that P is the desired block sequence, we prove the following properties:

(4) $\langle p_m, \dots, p_n\rangle_{(-T)} \subseteq A_m$ for all $m \leq n < \omega$.

(5) If $Q = (q_n)_{n<\omega} \leq_{(-T)} P$, then $(q_0, \dots, q_m) \in U$ for all $m < \omega$.

We check (4) by downward induction on $m \leq n$ for $n < \omega$ fixed. The case $m = n$ follows from (1), while the case $m = n-1$ follows using (1) and (2) to obtain $\langle p_{n-1}, p_n\rangle_{(-T)} \subseteq A_{n-1}$. Now suppose inductively that (4) holds for some $m \leq n$; we aim to show $\langle p_{m-1}, p_m, \dots, p_n\rangle_{(-T)} \subseteq A_{m-1}$. Take any

$$q = \sum_{i=m-1}^{n} (-T)^{j_i}(p_i)$$

with $j_{m-1}, \dots, j_n \in \{0, \dots, k\}$ and $\min j_i = 0$. We consider two cases: Suppose first that there is $i > m-1$ such that $j_i = 0$. Then $q' := \sum_{i=m}^{n} (-T)^{j_i}(p_i) \in \langle p_m, \dots, p_n\rangle_{(-T)} \subseteq A_m$ where the inclusion comes from the inductive hypothesis. Then $q' \in A_m$ and so

$$q \in \langle p_{m-1}, q \rangle_{(-T)} \subseteq A_{m-1}$$

by (2). Now suppose $j_i > 0$ for each $i > m-1$ (so that, in particular, $j_{m-1} = 0$). Let $l := \min\{j_m, \dots, j_n\} > 0$ and write

$$q = p_{m-1} + (-T)^l \left(\sum_{i=m}^{n} (-T)^{j_i - l}(p_i) \right).$$

By the inductive hypothesis we have

$$q'' := \sum_{i=m}^{n} (-T)^{j_i - l}(p_i) \in \langle p_m, \dots, p_n \rangle_{(-T)} \subseteq A_m,$$

and so $q \in \langle p_{m-1}, q'' \rangle_{(-T)} \subseteq A_{m-1}$ by (2). This completes the proof of (4).

To prove (5) (and thus the lemma), suppose $Q = (q_n)_{n<\omega} \leq_{(-T)} P$ and fix $q = (q_0, \dots, q_m)$. We prove $q \in U$ by induction on $m < \omega$. If $m = 0$ then $q = (q_0)$ and by definition of Q we can write

$$q_0 = \sum_{i<l} (-T)^{j_i}(p_{n_i})$$

for some $l < \omega, n_0 < \dots < n_{l-1} < \omega$ and $j_i \in \{0, \dots, k\}$ with $\min j_i = 0$. Then $q_0 \in \langle p_{n_0}, \dots, p_{n_{l-1}} \rangle_{(-T)}$ and so by (4) we have $q_0 \in A_{n_0} \subseteq A_0 \subseteq U_\emptyset$ where we use the definition of A_0 above. Thus $q = (q_0) \in U$. Now suppose $m > 0$ and write $t := (q_0, \dots, q_{m-1})$ so that $q = (t, q_m)$ and $t \in U$ by the inductive assumption. Again, by definition of Q we can write

$$q_m = \sum_{i<l} (-T)^{j_i}(p_{n_i})$$

for some $l < \omega, n_0 < \dots < n_{l-1} < \omega$ and $j_i \in \{0, \dots, k\}$ with $\min j_i = 0$. Then $q_m \in \langle p_{n_0}, \dots, p_{n_{l-1}} \rangle_{(-T)}$ and so by (4) we have $q_m \in A_{n_0}$. Since $q_{m-1} < q_m$ it must be the case that

$$\operatorname{supp} \sum t \subseteq \bigcup_{i<n_0} \operatorname{supp} p_i.$$

Then by (3) we obtain $q_m \in U_t$ and so $q = (t, q_m) \in U$. This finishes the inductive proof of (5) and hence the proof of the lemma is complete. □

In what follows, we will only need the following corollary of the above proof.

Corollary 4.2.3. *For every $\mathcal{U}$-tree U with stem $\emptyset$ there is $P = (p_n)_{n<\omega} \in \mathrm{FIN}_{\pm k}^{[\infty]}$ together with a sequence $A_0 \supseteq A_1 \supseteq \dots$ of subsets of $\mathrm{FIN}_{\pm k}$ such that:*

1. *$A_n \subseteq U_t \cap -(U_t)_1$ for every $t \in U$ such that $\operatorname{supp} \sum t \subseteq \bigcup_{i<n} \operatorname{supp} p_i$,*
2. *$\langle p_m, \dots, p_n \rangle_{(-T)} \subseteq A_m$ for all $m \leq n < \omega$.*

Recall that for a block sequence $P = (p_n)_{n<\omega}$ in $\mathrm{FIN}_{\pm k}$, $\langle P \rangle_{\pm k}^{[\infty]}$ denotes the set of all infinite block subsequences of P in $\mathrm{FIN}_{\pm k}$. We then have the following key lemma which makes use of the S-closed $\mathcal{U}$-trees defined in the previous section.

Lemma 4.2.4. *Let U be an S-closed $\mathcal{U}$-tree with* $\operatorname{stem}(U) = \emptyset$. *Then there is an infinite block sequence* $P = (p_n)_{n<\omega}$ *in* $\mathrm{FIN}_{\pm k}$ *such that* $\langle P \rangle_{\pm k}^{[\infty]} \subseteq ([U])_3$.

Proof. Find an infinite block sequence P as in Corollary 4.2.3. We claim that P satisfies the conclusion of the lemma. To see this, fix an infinite block subsequence $Q = (q_n)_{n<\omega}$ of P. For convenience, we fix some notation: For each $n < \omega$ let I_n be the smallest set of non-negative integers such that

$$q_n \in \langle p_i : i \in I_n \rangle_{\pm k}.$$

Notice that since Q is a block subsequence of P we have $\max I_n < \min I_m$ whenever $n < m$.

We will find a block sequence $Q' = (q'_n)_{n<\omega} \in [U]$ such that $||q_n - q'_n|| \leq 3$ and $\operatorname{supp} q'_n \subseteq \operatorname{supp} q_n$ for all $n < \omega$. Suppose, for some $n \geq 0$, we have defined $q'_0, \ldots, q'_{n-1} \in \mathrm{FIN}_{\pm k}$ such that $s := (q'_0, \ldots, q'_{n-1}) \in U$, $||q_i - q'_i|| \leq 3$ and $\operatorname{supp} q'_i \subseteq \operatorname{supp} q_i$ for all $i < n$. (In the case where $n = 0$ we simply have $s = \emptyset$.) Write

$$q_n = \sum_{i \in I_n} \varepsilon_i T^{j_i}(p_i)$$

for some $\varepsilon_i \in \{\pm 1\}$ and $j_i < k$ such that $\min j_i = 0$. Note that since

$$\operatorname{supp} q'_i \subseteq \operatorname{supp} q_i \subseteq \bigcup_{j \in I_i} \operatorname{supp} p_i,$$

we must have

$$\operatorname{supp} \sum s \subseteq \bigcup_{i < \min I_n} \operatorname{supp} p_i.$$

(When $s = \emptyset$ we let $\sum s = \emptyset$.) We consider the following two cases:

Case 1. There is $i \in I_n$ such that $\varepsilon_i = +1$ and $j_i = 0$.

For each $i \in I_n$, set $r_i := \varepsilon_i T^{j_i}(p_i)$ for convenience. We consider the following two subcases:

(a) $\varepsilon_i = +1$ and j_i is even, or $\varepsilon_i = -1$ and j_i is odd. In either case, set $r'_i := r_i$ and note that $r'_i = (-T)^{j_i}(p_i)$.

(b) $\varepsilon_i = +1$ and j_i is odd, or $\varepsilon_i = -1$ and j_i is even. In either case, set $r'_i := T(r_i)$ and note that $r'_i = (-T)^{j_i+1}(p_i)$.

We then set

$$q'_n := \sum_{i \in I_n} r'_i.$$

Note that $\operatorname{supp} q'_n \subseteq \operatorname{supp} q_n$ and $q'_n \in \langle p_i : i \in I_n \rangle_{(-T)}$ by the assumption given by Case 1. Since $||r_i - r'_i|| \leq 1$ for all $i \in I_n$ we have $||q_n - q'_n|| \leq 1$. Furthermore, by Corollary 4.2.3 we have $\langle p_i : i \in I_n \rangle_{(-T)} \subseteq A_{\min I_n}$ (using the notation of Corollary 4.2.3) and so $q'_n \in U_t$ for every $t \in U$ such that

$$\operatorname{supp} \sum t \subseteq \bigcup_{i < \min I_n} \operatorname{supp} p_i.$$

In particular, $q'_n \in U_s$ and so $(s, q'_n) \in U$.

Case 2. For every $i \in I_n$, if $j_i = 0$ then $\varepsilon_i = -1$.

Apply Case 1 to $-q_n$ to obtain $r \in \langle p_i : i \in I_n \rangle_{(-T)}$ such that $||(-q_n) - r|| \leq 1$ and $\operatorname{supp} r \subseteq \operatorname{supp}(-q_n)$. By Corollary 4.2.3 we have $\langle p_i : i \in I_n \rangle_{(-T)} \subseteq A_{\min I_n}$ and so $r \in U_t \cap -(U_t)_1$ for every $t \in U$ such that

$$\operatorname{supp} \sum t \subseteq \bigcup_{i < \min I_n} \operatorname{supp} p_i.$$

In particular, $-r \in (U_s)_1$ and so there is $r' \in U_s$ such that $||(-r) - r'|| \leq 1$. Since U is S-closed, we have $(s, S(r')) \in U$ and so we set $q'_n := S(r')$. Note that by definition of S we have $\operatorname{supp} q'_n \subseteq \operatorname{supp}(-r) = \operatorname{supp} r \subseteq \operatorname{supp} q_n$. Furthermore, using the fact that $||r' - S(r')|| \leq 1$ we have

$$||q_n - q'_n|| \leq ||q_n - (-r)|| + ||(-r) - r'|| + ||r' - S(r')|| \leq 3$$

and so q'_n satisfies our requirements. This completes the inductive construction of Q'. It is clear from the above construction that $Q' \in [U]$ and $||q_n - q'_n|| \leq 3$ for all $n < \omega$ and so $Q \in ([U])_3$. □

To finish the proof of Theorem 4.2.1 we will need the following mapping which was originally used in [33] to give an alternate proof of Gowers' theorem. Given $m \in \mathbb{N}$, let $\Phi_m : \mathrm{FIN}_{\pm 2m} \to \mathrm{FIN}_{\pm m}$ be defined by setting, for $p \in \mathrm{FIN}_{\pm 2m}$ and $n < \omega$,

$$\Phi_m(p)(n) := \begin{cases} \frac{p(n)}{2} & \text{if } p(n) \text{ is even,} \\ \frac{p(n)-1}{2} & \text{if } p(n) > 0 \text{ and } p(n) \text{ is odd,} \\ \frac{p(n)+1}{2} & \text{if } p(n) < 0 \text{ and } p(n) \text{ is odd.} \end{cases}$$

The following lemma is easy to check.

Lemma 4.2.5. *For each $m \in \mathbb{N}$, the mapping Φ_m has the following properties:*

(i) *Φ_m is a surjective homomorphism of partial semigroups which, in addition, satisfies $\Phi_m(-p) = -\Phi_m(p)$ for every $p \in \mathrm{FIN}_{\pm 2m}$.*

(ii) *For every $p_0 < p_1 \in \mathrm{FIN}_{\pm 2m}$ and every $j_0, j_1 < k+1$ with $\min\{j_0, j_1\} = 0$, we have*

$$\Phi_m\left(T^{2j_0}(p_0) + T^{2j_1}(p_1)\right) = T^{j_0}(\Phi_m(p_0)) + T^{j_1}(\Phi_m(p_1)).$$

(iii) *For every $p_0, p_1 \in \mathrm{FIN}_{\pm 2m}$ and every $l < \omega$, we have*

$$||p_0 - p_1|| \leq 2l \implies ||\Phi_m(p_0) - \Phi_m(p_1)|| \leq l.$$

Now, for $k \in \mathbb{N}$ fixed as in the previous sections, let $\Psi : \mathrm{FIN}_{\pm 4k} \to \mathrm{FIN}_{\pm k}$ be given by $\Psi := \Phi_k \circ \Phi_{2k}$. Using the properties listed in Lemma 4.2.5 it is easy to verify that Ψ is a surjective homomorphism which satisfies:

(a) For every $p_0 < p_1 \in \mathrm{FIN}_{\pm 4k}$ and every $j_0, j_1 < k+1$ with $\min\{j_0, j_1\} = 0$, we have

$$\Psi\left(T^{4j_0}(p_0) + T^{4j_1}(p_1)\right) = T^{j_0}(\Psi(p_0)) + T^{j_1}(\Psi(p_1)).$$

(b) For every $p_0, p_1 \in \mathrm{FIN}_{\pm 4k}$, if $||p_0 - p_1|| \leq 4$ then $||\Psi(p_0) - \Psi(p_1)|| \leq 1$.

We extend Ψ to $\mathrm{FIN}_{\pm 4k}^{[\infty]}$ by setting $\Psi((p_n)_{n<\omega}) := (\Psi(p_n))_{n<\omega}$. It is straightforward to check that Ψ is continuous with respect to the usual metrizable topologies. Furthermore, note that if P and P' are two block sequences in $\mathrm{FIN}_{\pm 4k}$ which satisfy $||P - P'|| \leq 4$, then $||\Psi(P) - \Psi(P')|| \leq 1$. We are now ready to finish the proof of the main theorem.

Proof of Theorem 4.2.1. Let $c : \mathrm{FIN}_{\pm k}^{[\infty]} \to r$ be Souslin measurable. We define a colouring $\widetilde{c} : \mathrm{FIN}_{\pm 4k}^{[\infty]} \to r$ by setting $\widetilde{c} := c \circ \Psi$. Since Ψ is continuous and c is Souslin measurable, it follows immediately that $\widetilde{c}$ is Souslin measurable. By Corollary 4.1.10 there are $i < r$ and a $\mathcal{U}$-tree V with stem $\emptyset$ such that $[V] \subseteq \widetilde{c}^{-1}\{i\}$. Applying Lemma 4.1.11, find an S-closed $\mathcal{U}$-tree U such that $[U] \subseteq ([V])_1$; in particular we get $[U] \subseteq \left(\widetilde{c}^{-1}\{i\}\right)_1$. Since $\mathcal{U}$ is S-closed, by Lemma 4.2.4 we can find an infinite block sequence $\widetilde{P} = (\widetilde{p_n})_{n<\omega}$ in $\mathrm{FIN}_{\pm 4k}$ such that $\langle \widetilde{P} \rangle_{\pm 4k}^{[\infty]} \subseteq ([U])_3$ and hence $\langle \widetilde{P} \rangle_{\pm 4k}^{[\infty]} \subseteq \left(\widetilde{c}^{-1}\{i\}\right)_4$.

Let $P := \Psi(\widetilde{P}) \in \mathrm{FIN}_{\pm k}^{[\infty]}$ so that $p_n := \Psi(\widetilde{p_n})$ for each $n < \omega$. We claim that P satisfies $\langle P \rangle_{\pm k}^{[\infty]} \subseteq \left(c^{-1}\{i\}\right)_1$. Indeed, if $Q = (q_n)_{n<\omega} \in \mathrm{FIN}_{\pm k}^{[\infty]}$ is an infinite block subsequence of P, then for each $n < \omega$ we have

$$q_n = \sum_{i<m} \varepsilon_i T^{j_i}(p_{n_i})$$

for some $\varepsilon_i \in \{\pm 1\}, n_0 < \cdots < n_{m-1}$ and $j_i < k$ such that $\min j_i = 0$. Then using property (a) of Ψ listed above we see that $q_n = \Psi(\widetilde{q_n})$, where

$$\widetilde{q_n} := \sum_{i<m} \varepsilon_i T^{4j_i}(\widetilde{p_{n_i}}) \in \langle \widetilde{P} \rangle_{\pm k}$$

and so, setting $\widetilde{Q} := (\widetilde{q_n})_{n<\omega}$, we see that $Q = \Psi(\widetilde{Q})$. Since $\widetilde{Q}$ is a block subsequence of $\widetilde{P}$, by our choice of $\widetilde{P}$ we can find $Q' \in \widetilde{c}^{-1}\{i\}$ such that $||\widetilde{Q} - Q'|| \leq 4$. Then, as observed above, property (b) of Ψ implies $||\Psi(\widetilde{Q}) - \Psi(Q')|| \leq 1$. Since $i = \widetilde{c}(Q') = c(\Psi(Q'))$ we obtain $\Psi(Q') \in c^{-1}\{i\}$ and so $Q \in \left(c^{-1}\{i\}\right)_1$ as required. □

In fact, we can do a bit better: Given an infinite block sequence P in $\mathrm{FIN}_{\pm k}$, the proof of Lemma 4.1.8 (from either [2] or [33]) can be adapted to show the existence of an ultrafilter $\mathcal{U}$ on the partial semigroup $\langle P \rangle_{\pm k}$ which has the properties listed in Lemma 4.1.8. One can then develop the theory of $\mathcal{U}$-trees on $\langle P \rangle_{\pm k}^{[<\infty]}$ and prove a corresponding analogue of Corollary 4.1.10. By equipping $\langle P \rangle_{\pm k}^{[\infty]}$ with its natural analogue of the metrizable topology and replacing $\mathrm{FIN}_{\pm k}^{[\infty]}$ with $\langle P \rangle_{\pm k}^{[\infty]}$ in the proof of the main result, we obtain the following relativized version of Theorem 4.2.1.

Theorem 4.2.6. *For every $k, r \in \mathbb{N}$, every infinite block sequence P in $\mathrm{FIN}_{\pm k}$ and every Souslin measurable $c : \mathrm{FIN}_{\pm k}^{[\infty]} \to r$ there are $i < r$ and an infinite block sequence $Q \leq P$ such that $\langle Q \rangle_{\pm k}^{[\infty]} \subseteq \left(c^{-1}\{i\}\right)_1$.*

The previous result can be used to "diagonalize" Theorem 4.2.1 as follows. First note that, for each $j < k \in \mathbb{N}$, the j^{th} iterate of the tetris operation $T^{(j)} : \mathrm{FIN}_{\pm k} \to \mathrm{FIN}_{\pm(k-j)}$ can be extended to $T^{(j)} : \mathrm{FIN}_{\pm k}^{[\infty]} \to \mathrm{FIN}_{\pm(k-j)}^{[\infty]}$ by setting $T^{(j)}((p_n)_{n<\omega}) := (T^{(j)}(p_n))_{n<\omega}$. We then have the following:

Corollary 4.2.7. *For every $k, r \in \mathbb{N}$ and every Souslin measurable (with respect to the disjoint union topology) colouring*

$$c : \bigcup_{j=1}^{k} \mathrm{FIN}_{\pm j}^{[\infty]} \to r$$

there are $i_1, \ldots, i_k < r$ *and* $P \in \mathrm{FIN}_{\pm k}^{[\infty]}$ *such that*

$$\langle T^{(k-j)}(P)\rangle_{\pm j}^{[\infty]} \subseteq \left(c^{-1}\{i_j\}\right)_1$$

for each $j = 1, \ldots, k$.

Proof. Note that each canonical inclusion

$$\iota_j : \mathrm{FIN}_{\pm j}^{[\infty]} \to \bigcup_{j=1}^{k} \mathrm{FIN}_{\pm j}^{[\infty]}$$

is continuous and so, as in the proof of Theorem 4.2.1, each Souslin measurable colouring of the union induces a Souslin measurable colouring of $\mathrm{FIN}_{\pm j}^{[\infty]}$ by composing with ι_j, for each $j \in \{1, \ldots, k\}$. Thus by Theorem 4.2.1 we can find $P_1 \in \mathrm{FIN}_{\pm 1}^{[\infty]}$ and $i_1 < r$ such that $\langle P_1\rangle_{\pm 1}^{[\infty]} \subseteq (c^{-1}\{i_1\})_1$. Take any $Q_2 \in \mathrm{FIN}_{\pm 2}^{[\infty]}$ such that $T(Q_2) = P_1$ and apply Theorem 4.2.6 to Q_2 to obtain $P_2 \leq Q_2$ and $i_2 < r$ such that $\langle P_2\rangle_{\pm 2}^{[\infty]} \subseteq (c^{-1}\{i_2\})_1$. Continue inductively to obtain $P_j \leq Q_j \in \mathrm{FIN}_{\pm j}^{[\infty]}$ and $i_j < r$, for $j = 2, \ldots, k$, such that $T(Q_j) = P_{j-1}$ and $\langle P_j\rangle_{\pm j}^{[\infty]} \subseteq (c^{-1}\{i_j\})_1$.

We claim that setting $P := P_k$ works. Indeed, for a fixed $j = 1, \ldots, k$ we have $T^{(k-j)}(P) \leq P_j$ by construction (and using the general fact that $T(P) \leq T(Q)$ whenever $P \leq Q$) and so the desired conclusion follows from the choice of P_j. □

As an application of Theorem 4.2.1, we conclude this section with a proof of the multi-dimensional version of Theorem 4.1.1. Recall that, for $d \in \mathbb{N}$, $\mathrm{FIN}_{\pm k}^{[d]}$ denotes the set of all block sequences in $\mathrm{FIN}_{\pm k}$ of length d. Given an infinite block sequence P let $\langle P\rangle_{\pm k}^{[d]}$ be the set of all $Q = (q_n)_{n<d} \in \mathrm{FIN}_{\pm k}^{[d]}$ such that $q_n \in \langle P\rangle_{\pm k}$ for each $n < d$.

Corollary 4.2.8. *For every* $k, d, r \in \mathbb{N}$ *and every colouring* $c : \mathrm{FIN}_{\pm k}^{[d]} \to r$ *there are* $i < r$ *and an infinite block sequence* $P \in \mathrm{FIN}_{\pm k}^{[\infty]}$ *such that* $\langle P\rangle_{\pm k}^{[d]} \subseteq \left(c^{-1}\{i\}\right)_1$.

Proof. Given a colouring c as above, let $\widetilde{c} : \mathrm{FIN}_{\pm k}^{[\infty]} \to r$ be given by $\tilde{c}((p_n)_{n<\omega}) := c((p_n)_{n<d})$. Then $\widetilde{c}$ is continuous and hence Souslin measurable. By Theorem 4.2.1 there are $i < r$ and $P \in \mathrm{FIN}_{\pm k}^{[\infty]}$ such that $\langle P\rangle_{\pm k}^{[\infty]} \subseteq \left(\widetilde{c}^{-1}\{i\}\right)_1$. Given $Q = (q_n)_{n<d} \in \langle P\rangle_{\pm k}^{[d]}$ extend Q arbitrarily to any $\widetilde{Q} \in \langle P\rangle_{\pm k}^{[\infty]} \cap [Q]$. By choice of P there is $Q' = (q'_n)_{n<\omega} \in \widetilde{c}^{-1}\{i\}$ such that $||q_n - q'_n|| \leq 1$ for all $n < \omega$. Then $c((q'_n)_{n<d}) = i$ and so $Q \in (c^{-1}\{i\})_1$. □

4.3 A parametrized Milliken-Todorčević theorem

Our next goal is to show that the Parametrized Milliken Theorem introduced in Chapter 1 still holds when FIN is replaced with FIN_k. On the other hand, while an exact Ramsey theorem is not possible in the setting of $\mathrm{FIN}_{\pm k}$, we will obtain an "approximate" parametrized Ramsey theorem for $\mathrm{FIN}_{\pm k}$. The proofs make use of ultra-Ramsey theory.

Theorem 4.3.1 (Parametrized Milliken-Todorčević Theorem). *For every finite Souslin measurable colouring of* $\mathrm{FIN}_k^{[\infty]} \times (2^\omega)^\omega$ *there are* $B \in \mathrm{FIN}_k^{[\infty]}$ *and a sequence* $(P_i)_{i<\omega}$ *of non-empty perfect subsets of* 2^ω *such that* $[B]_k^{[\infty]} \times \prod_{i<\omega} P_i$ *is monochromatic.*

To do so, we will need a combinatorial result which can be seen as a common infinite-dimensional generalization of Gowers' FIN_k theorem and the Hales-Jewett theorem. We remark here that a general framework for obtaining infinitary Gowers-Hales-Jewett theorems has been developed in [44]; other versions are considered in [3] and alluded to in [33]. Our approach is heavily inspired by that of [65].

Given an infinite alphabet $L = \bigcup_{n<\omega} L_n$ presented as an increasing union of finite subalphabets L_n, we fix a distinguished letter $0 \in L_0$ together with k distinct variables $v_1, \dots, v_k \notin L$. W_L will denote the set of all variable-free words over L and, for each $i \in \{1, \dots, k\}$, W_{Lv_i} will denote the set of all variable words x over L such that

$$i = \max\{j \le k : \ v_j \text{ appears in } x\}.$$

An element of W_{Lv_k} will be called a *v_i-variable word.*

Let

$$S = W_L \cup \bigcup_{1\le i\le k} W_{Lv_i}$$

and work in the semigroup $(S, {}^\frown)$ where ${}^\frown$ denotes the concatenation operator on pairs of words. Given $x \in W_{Lv_k}$ and a k-tuple $\vec{\lambda} = (\lambda_1, \dots, \lambda_k) \in L^k \cup \{\vec{v}\}$, let $x[\vec{\lambda}]$ be the word obtained by replacing each occurrence of v_i with λ_i, where $\vec{v} = (v_1, \dots, v_k)$. In addition to substitution, we also have a version of the tetris operation defined for v_k-variable words: Given $x \in W_{Lv_k}$, define $T(x) \in W_{Lv_{k-1}}$ by

$$T(x)(n) := \begin{cases} v_{i-1} & \text{if } x(n) = v_i \text{ for } i > 1, \\ 0 & \text{if } x(n) = v_1, \\ x(n) & \text{if } x(n) \in L. \end{cases}$$

Define $T(w) = w$ for each $w \in W_L$. Given a sequence $X = (x_n)_{n<\omega}$ of v_k-variable words, the *partial subsemigroup of W_{Lv_k} generated by X*, denoted by $[X]_{Lv_k}$, is defined to be the set of all v_k-variable words of the form

$$T^{j_0}(x_{n_0}[\vec{\lambda}_0])^\frown \dots {}^\frown T^{j_l}(x_{n_l}[\vec{\lambda}_l])$$

where $l < \omega$, $n_0 < \cdots < n_l < \omega$, $j_0, \dots, j_l \le k$ and $\vec{\lambda}_i \in L^k_{n_i} \cup \{\vec{v}\}$ for each $i \le l$; note that for such an expression to be a v_k-variable word, there must be some $i \le l$ such that $j_i = 0$ and $\vec{\lambda}_i = \vec{v}$. We also consider the *partial subsemigroup of W_L generated by X*, defined as

$$[X]_L = \{x_{n_0}[\vec{\lambda}_0]^\frown \dots {}^\frown x_{n_l}[\vec{\lambda}_l] \in W_L : n_0 < \cdots < n_l < \omega \text{ and } (\forall i \le l)\, \vec{\lambda}_i \in L^k_{n_i}\}.$$

Let $W^{[\infty]}_{Lv_k}$ denote the set of all infinite sequences $X = (x_n)_{n<\omega}$ in W_{Lv_k} which are *rapidly increasing*, i.e. sequences (x_n) such that

$$|x_n| > \sum_{i<n} |x_i| \text{ for all } n < \omega$$

where $|x|$ denotes the length (equivalently, the domain) of the word x. The notion of a finite rapidly increasing sequences is defined similarly. We equip $W^{[\infty]}_{Lv_k}$ with the *metrizable topology*, i.e. the Polish topology generated by sets of the form

$$\{(y_n)_{n<\omega} \in W^{[\infty]}_{Lv_k} : y_i = x_i \text{ for all } i \le m\}$$

where $(x_0, \dots, x_m)$ is a finite rapidly increasing sequence in W_{Lv_k}. Given $x \in [X]_{Lv_k}$, the *support* of x in

X, denoted $\operatorname{supp}_X(x)$, is the set $\{n_0 < \cdots < n_m\}$ of indices such that

$$x = T^{j_0}(x_{n_0}[\vec{\lambda}_0])^\frown \ldots ^\frown T^{j_l}(x_{n_l}[\vec{\lambda}_l])$$

for some choice of $n_0 < \cdots < n_l < \omega$, $j_0, \ldots, j_l \leq k$ and $\vec{\lambda}_i \in L^k_{n_i} \cup \{\vec{v}\}$. The requirement that our sequences be rapidly increasing is necessary to ensure that $\operatorname{supp}_X(x)$ is uniquely defined. Using this observation, we can define an ordering $\leq$ on $W^{[\infty]}_{Lv_k}$ by setting, for rapidly increasing sequences $X = (x_n)$ and Y, $X \leq Y$ if and only if $x_n \in [Y]_{Lv_k}$ for all $n < \omega$ and

$$\max \operatorname{supp}_Y(x_n) < \min \operatorname{supp}_Y(x_m) \text{ whenever } n < m.$$

In this case, we say that X is a *block subsequence* of Y; we denote by $[Y]^{[\infty]}_{Lv_k}$ the set of all infinite block subsequences of Y. The set of all finite block subsequences of a a finite rapidly increasing sequence $(y_0, \ldots, y_m)$ will be denoted by $[y_0, \ldots, y_m]_{Lv_k}$.

Our first goal is to prove the following theorem, which is a common generalization of Gowers' FIN_k theorem and the infinitary Hales-Jewett theorem.

Theorem 4.3.2. *For every finite Souslin measurable colouring of $W^{[\infty]}_{Lv_k}$, there is $X \in W^{[\infty]}_{Lv_k}$ such that $[X]^{[\infty]}_{Lv_k}$ is monochromatic.*

To prove such a result we will use ultra-Ramsey theory. Before we describe the relevant results, we need to construct an ultrafilter on W_{Lv_k} which will be used throughout this section. To this end, work in the Stone-Čech compactification $(\beta S, {}^\frown)$ of the semigroup $(S, {}^\frown)$. Using ultrafilter quantifiers, the extension of the concatenation operation to βS is characterized as follows:

$$A \in \mathcal{U}^\frown \mathcal{V} \iff (\mathcal{U}x)(\mathcal{V}y)\ x^\frown y \in A.$$

Similarly, the extension of the tetris operation to βS is determined by

$$A \in T(\mathcal{U}) \iff (\mathcal{U}x)\ T(x) \in A.$$

Let S^* denote the closed subsemigroup of βS consisting of all non-principal ultrafilters on S which are *cofinite*, i.e. ultrafilters $\mathcal{U}$ such that

$$\{x \in W_{Lv_i} : |x| > n\} \in \mathcal{U} \text{ for all } n < \omega.$$

Define

$$S^*_L = \{\mathcal{U} \in S^* : W_L \in \mathcal{U}\}$$

and, for each $i \in \{1, \ldots, k\}$,

$$S^*_{Lv_i} = \{\mathcal{U} \in S^* : W_{Lv_i} \in \mathcal{U}\}.$$

Then S^*_L and $S^*_{Lv_i}$ (for each $1 \leq i \leq k$) are closed subsemigroups of S^*. Let $\mathcal{W}$ be a minimal idempotent in S^*_L, and choose any idempotent $\mathcal{V}_1 \leq \mathcal{W}$ in $S^*_{Lv_1}$, where the order $\leq$ is defined by declaring

$$\mathcal{V} \leq \mathcal{W} \text{ if and only if } \mathcal{V}^\frown \mathcal{W} = \mathcal{W}^\frown \mathcal{V} = \mathcal{V}.$$

(We refer the reader to [65, Chapter 2] for information on the theory of minimal idempotents in compact semigroups.) Starting with $\mathcal{V}_1$, recursively construct a sequence $(\mathcal{V}_i)_{1\le i\le k}$ of idempotents such that for each $i < j$:

1. $\mathcal{V}_i$ is an idempotent in $S^*_{Lv_i}$.
2. $\mathcal{V}_i \geq \mathcal{V}_j$.
3. $T^{(j-i)}(\mathcal{V}_j) = \mathcal{V}_i$.

Assume $\mathcal{V}_1, \ldots, \mathcal{V}_{i-1}$ have been constructed and let

$$S_i = \{\mathcal{U} \in S^*_{Lv_i} : T(\mathcal{U}) = \mathcal{V}_{i-1}\}.$$

Since $T : S^*_{Lv_i} \to S^*_{Lv_{i-1}}$ is a continuous surjective homomorphism, it follows (as in the proof of [65, Lemma 2.24]) that

$$S_i^\frown \mathcal{V}_{i-1} = \{\mathcal{U}^\frown \mathcal{V}_{i-1} : \mathcal{U} \in S_i\}$$

is a non-empty closed subsemigroup of S_i. Thus there is an idempotent in $S_i{}^\frown \mathcal{V}_{i-1}$ of the form $\mathcal{U}^\frown \mathcal{V}_{i-1}$; then let

$$\mathcal{V}_i = \mathcal{V}_{i-1}{}^\frown \mathcal{U}^\frown V_{i-1}.$$

It is routine to check that $\mathcal{V}_i$ is an idempotent which satisfies the required properties.

We will also need the following:

Claim 4.3.3. For each $\vec{\lambda} \in L^k$ and each $i \leq k$, $\mathcal{V}_i[\vec{\lambda}] = \mathcal{W}$.

Proof. Since each mapping $\mathcal{U} \mapsto \mathcal{U}[\vec{\lambda}]$ for $\vec{\lambda} \in L^k$ is a homomorphism, it follows that $\mathcal{V}_i[\vec{\lambda}] \in S^*_L$ is an idempotent and $\mathcal{V}_i[\vec{\lambda}] \leq \mathcal{W}[\vec{\lambda}] = \mathcal{W}$, so that $\mathcal{V}_i[\vec{\lambda}] = \mathcal{W}$ by minimality of $\mathcal{W}$. □

Let $W^{[<\infty]}_{Lv_k}$ be the set of all finite rapidly increasing sequences in W_{Lv_k}. We view $W^{[<\infty]}_{Lv_k}$ as a tree ordered by end-extension and with root $\emptyset$, the empty word. The next two definitions are adapted from [65, Chapter 7.2] by replacing the the tree $\mathbb{N}^{[<\infty]}$ of finite subsets of $\mathbb{N}$ with $W^{[<\infty]}_{Lv_k}$.

Definition 4.3.4. A $\mathcal{V}_k$*-tree* is a downward closed subtree $U \subseteq W^{[<\infty]}_{Lv_k}$ such that

$$U_t := \{x \in W_{Lv_k} : (t, x) \in U\} \in \mathcal{V}_k$$

for all $t \in U$ which extend the stem of U, where the *stem* is the $\sqsubseteq$-maximal element of U which is comparable to every other node of the tree. The stem of a $\mathcal{V}_k$-tree U will be denoted by $\mathrm{stem}(U)$.

Given a $\mathcal{V}_k$-tree U, the set of infinite branches of U is denoted by

$$[U] := \{(x_n)_{n<\omega} \in W^{[\infty]}_{Lv_k} : (x_0, \ldots, x_m) \in U \text{ for all } m < \omega\}.$$

For $t \in U$ let $|t|$ denote the *length* of t, which is just the domain of t when viewed as a finite sequence in $W^{[<\infty]}_{Lv_k}$.

Definition 4.3.5. Let $\mathcal{X} \subseteq W^{[\infty]}_{Lv_k}$. $\mathcal{X}$ is $\mathcal{V}_k$*-open* if for every $A \in \mathcal{X}$ there is a $\mathcal{V}_k$-tree U such that $A \in [U] \subseteq \mathcal{X}$. $\mathcal{X}$ is $\mathcal{V}_k$*-Ramsey* if for every $\mathcal{V}_k$-tree U there is a $\mathcal{V}_k$-subtree $U' \subseteq U$ with $\mathrm{stem}(U) = \mathrm{stem}(U')$ such that $[U'] \subseteq \mathcal{X}$ or $[U'] \subseteq \mathcal{X}^c$.

The collection of all $\mathcal{V}_k$-open subsets of $W_{Lv_k}^{[\infty]}$ forms a topology, called the $\mathcal{V}_k$*-topology*, which refines the metrizable topology of $W_{Lv_k}^{[\infty]}$. The next two results are adapted from [65, Chapter 7.2] by replacing the tree $\mathbb{N}^{[<\infty]}$ of finite subsets of $\mathbb{N}$ ordered by end-extension with the tree $W_{Lv_k}^{[<\infty]}$. We then have the following version of Todorčević's ultra-Ellentuck theorem from [65, Chapter 7] together with its usual corollary.

Theorem 4.3.6. *Let $\mathcal{X} \subseteq W_{Lv_k}^{[\infty]}$. Then $\mathcal{X}$ has the property of Baire relative to the $\mathcal{V}_k$-topology if and only if $\mathcal{X}$ is $\mathcal{V}_k$-Ramsey.*

Theorem 4.3.7. *For every $r \in \mathbb{N}$ and every Souslin measurable $c : W_{Lv_k}^{[\infty]} \to r$ there are $i < r$ and a $\mathcal{V}_k$-tree U with stem $\emptyset$ such that $[U] \subseteq c^{-1}\{i\}$.*

Our next goal is to show that for any $\mathcal{V}_k$-tree U there is a rapidly increasing sequence Y with the property that $X \in [U]$ whenever $X \leq Y$. To this end, we have the following key lemma:

Lemma 4.3.8. *For every $\mathcal{V}_k$-tree U with stem $\emptyset$ there is $Y = (y_n)_{n<\omega} \in W_{Lv_k}^{[\infty]}$ together with a decreasing sequence $(A_n)_{n<\omega}$ of subsets of W_{Lv_k} such that:*

(a) $A_n \subseteq U_t$ for every $t \in U$ such that $t \leq (y_0, \dots, y_{n-1})$.

(b) $[y_m, \dots, y_n]_{Lv_k} \subseteq A_m$ for all $m \leq n < \omega$.

Proof. By induction on n, define a decreasing sequence $(A_n)_{n<\omega}$ together with a rapidly increasing sequence $(y_n)_{n<\omega}$ such that, for all $n < \omega$:

1. $y_n \in A_n \in \mathcal{V}_k$.
2. $A_{n+1} \subseteq \{z \in W_{Lv_k} : [y_n, z]_{Lv_k} \subseteq A_n\}$.
3. $A_n \subseteq U_t$ for every $t \in U$ such that $t \leq (y_i)_{i<n}$.

To start, take $A_0 := U_\emptyset$ and note that $A_0 \in \mathcal{V}_k$ since U is $\mathcal{V}_k$-tree. Using the properties of the sequence $(\mathcal{V}_i)_{1\leq i\leq k}$ of idempotents constructed above, we have

$$\mathcal{V}_k = T^j(\mathcal{V}_k)^\frown\mathcal{V}_k = \mathcal{V}_k{}^\frown T^j(\mathcal{V}_k)$$

for all $j \leq k$. Rewriting this fact in terms of the ultrafilter quantifier and using the fact that $A_0 \in V_k$, it follows that

$$(\mathcal{V}_k y)(\mathcal{V}_k z)\,([y, z]_{Lv_k} \subseteq A_0)$$

and so we take any $y_0 \in W_{Lv_k}$ such that $(\mathcal{V}_k z)\,([y_0, z]_{Lv_k} \subseteq A_0)$; in particular $y_0 \in A_0$ by definition of $[y_0, z]_{Lv_k}$. We then take A_1 to be the intersection of the set $\{z \in W_{Lv_k} : [y_0, z]_{Lv_k} \subseteq A_0\}$ with

$$\bigcap \{U_t : t \in U \text{ and } t \leq (y_0)\}\,.$$

Note that $A_0 \supseteq A_1$ and $A_1 \in \mathcal{V}_k$ since there are only finitely many $t \in U$ satisfying $t \leq (y_0)$ and since each $U_t \in \mathcal{V}_k$.

Now suppose $A_0, \dots, A_n$ and $y_0, \dots, y_{n-1}$ have been constructed. Since $\mathcal{V}_k$ is cofinite, it follows that there is $y_n \in W_{Lv_k}$ such that

$$|y_n| > \sum_{i=0}^{n-1} |y_i|$$

and $(\mathcal{V}_k z)\,([y_n, z]_{Lv_k} \subseteq A_n)$; in particular $y_n \in A_n$. Then take A_{n+1} to be the intersection of the set $\{z \in W_{Lv_k} : [y_n, z]_{Lv_k} \subseteq A_n\}$ with

$$\bigcap \{U_t : t \in U \text{ and } t \leq (y_0, \dots, y_n)\}\,.$$

Observe that the collection $[y_0, \dots, y_{n-1}]_{Lv_k}$ is finite since we only allow substitutions of the form $y_i[\vec{\lambda}_i]$ for $\vec{\lambda}_i \in L_i^k \cup \{\vec{v}\}$ and so there are only finitely many sets in the above intersection. Thus $A_{n+1} \in \mathcal{V}_k$ and $A_n \supseteq A_{n+1}$. This completes the inductive construction of the sequences (A_n) and (y_n). In particular, condition (a) is satisfied by (3).

We check condition (b) by downward induction on $m \leq n$ for $n < \omega$ fixed. The case $m = n$ follows from (1), while the case $m = n-1$ follows using (1) and (2) to obtain $[y_{n-1}, y_n]_{Lv_k} \subseteq A_{n-1}$. Now suppose inductively that (b) holds for some $m \leq n$; we aim to show $[y_{m-1}, y_m, \dots, y_n]_{Lv_k} \subseteq A_{m-1}$. Take any

$$z = T^{j_{m-1}}(y_{m-1}[\vec{\lambda}_{m-1}])^\frown \dots {}^\frown T^{j_n}(y_n[\vec{\lambda}_n])$$

with $j_{m-1}, \dots, j_n \leq k$ and $\vec{\lambda}_i \in L_i^k \cup \{\vec{v}\}$ are such that $\min j_i = 0$ and $\vec{\lambda}_i = \vec{v}$ for some $i \in \{m-1, \dots, n\}$. We consider two cases: Suppose first that there is $i > m-1$ such that $j_i = 0$. Then

$$z' := T^{j_m}(y_m[\vec{\lambda}_m])^\frown \dots {}^\frown T^{j_n}(y_n[\vec{\lambda}_n]) \in [y_m, \dots, y_n]_{Lv_k} \subseteq A_m$$

where the inclusion comes from the inductive hypothesis. Thus $z' \in A_m$ and so

$$z \in [y_{m-1}, z]_{Lv_k} \subseteq A_{m-1}$$

by (2). Now suppose $j_i > 0$ for each $i > m-1$ (so that, in particular, $j_{m-1} = 0$). Let $l := \min\{j_m, \dots, j_n\} > 0$ and write

$$z = y_{m-1}{}^\frown T^l\left(T^{j_m - l}(y_m[\vec{\lambda}_m])^\frown \dots {}^\frown T^{j_n - l}(y_n[\vec{\lambda}_n])\right).$$

By the inductive hypothesis we have

$$z'' := T^{j_m - l}(y_m[\vec{\lambda}_m])^\frown \dots {}^\frown T^{j_n - l}(y_n[\vec{\lambda}_n]) \in [y_m, \dots, y_n]_{Lv_k} \subseteq A_m,$$

and so $z \in [y_{m-1}, z'']_{Lv_k} \subseteq A_{m-1}$ by (2). This completes the proof of the lemma. □

Proof of Theorem 4.3.2. Let $c : W_{Lv_k}^{[\infty]} \to r$ be Souslin measurable. By Corollary 4.3.7 there is a $\mathcal{V}_k$-tree U with stem $\emptyset$ such that $[U] \subseteq c^{-1}\{i\}$ for some $i < r$. Let $Y = (y_n)$ be the rapidly increasing sequence given by applying Lemma 4.3.8 to U. To finish the proof of the theorem, it is enough to show $[Y]_{Lv_k}^{[\infty]} \subseteq [U]$. Let $X = (x_n) \leq Y$; we show $X \in [U]$ by induction on the length m of $s := (x_0, \dots, x_{m-1})$. When $m = 0$ we have $s = \emptyset$ which belongs to U by assumption. So we assume $s \in U$ and show $(s, x_m) \in U$. Since $x_m \in [Y]_{Lv_k}$, we can write

$$x = T^{j_0}(y_{n_0}[\vec{\lambda}_0])^\frown \dots {}^\frown T^{j_l}(y_{n_l}[\vec{\lambda}_l])$$

for some $n_0 < \dots < n_l < \omega, j_0, \dots, j_l \leq k$ and $\vec{\lambda}_i \in L_{n_i}^k \cup \{\vec{v}\}$ such that $j_i = k$ and $\vec{\lambda}_i = \vec{v}$ for some $i \leq l$, i.e. $x_m \in [y_{n_0}, \ldots, y_{n_l}]_{Lv_k}$. By definition of Y,

$$[y_{n_0}, \ldots, y_{n_l}]_{Lv_k} \subseteq A_{n_0} \subseteq U_t$$

for each $t \in U$ such that $t \leq (y_0, \ldots, y_{n-1})$. Since X is a block subsequence of Y, by definition we have $\max \operatorname{supp}_Y(x_{m-1}) < \min \operatorname{supp}_Y(x_m)$ and so $s \leq (y_0, \ldots, y_{n-1})$. Thus $x_m \in U_s$ and so $(s, x_m) \in U$, as required. □

We are now in a position to prove the main theorem of this section, which allows us to parametrize the Milliken-Todorčević theorem by a sequence of perfect subsets of 2^ω. Parts of the proof are similar to that of [65, Theorem 5.45], but we include the details for the sake of completeness.

Proof of Theorem 4.3.1. Fix a finite Souslin measurable colouring c of the product $\mathrm{FIN}_k^{[\infty]} \times (2^\omega)^\omega$. Let

$$L_n = \{\sigma \in 2^\omega : (\forall i > n)\, \sigma(i) = 0\}$$

and $L = \bigcup_{n<\omega} L_n$. Define a mapping $\varphi : W_{Lv_k}^{[\infty]} \to \mathrm{FIN}_k^{[\infty]}$ as follows: Given $(x_m)_{m<\omega} \in W_{Lv_k}^{[\infty]}$, let $\varphi((x_m)) = (a_m)$, where $a_m \in \mathrm{FIN}_k$ consists of all ordered pairs of the form $\langle |x_0| + \cdots + |x_{m-1}| + l, i\rangle$ where v_i occupies the l^{th} place in x_m, as well as all pairs $\langle n, 0\rangle$ for any other $n < \omega$. We also define a mapping $\psi : W_{Lv_k}^{[\infty]} \to 2^{\omega\times\omega}$ by $\psi((x_m))(n,i) = \sigma(i)$ if $\sigma \in L$ occupies the n^{th} place in the infinite variable word $x_0{}^\frown x_1{}^\frown x_2{}^\frown \ldots$ and where $\psi((x_m))(n,i) = 0$ if a variable occupies the n^{th} place in the above infinite word.

Define a colouring c^* of $W_{Lv_k}^{[\infty]}$ by setting

$$c^*((x_m)) = c\,(\varphi((x_m)), \psi((x_m)))$$

where $(2^\omega)^\omega$ and $2^{\omega\times\omega}$ are identified via the mapping

$$((\varepsilon_{n,i})_n)_i \mapsto (\varepsilon_{n,i})_{(n,i)}.$$

It is easy to check that φ and ψ are both continuous, from which it follows that c^* is Souslin measurable. Apply Theorem 4.3.2 to find $Y = (y_m) \in W_{Lv_k}^{[\infty]}$ such that $[Y]_{Lv_k}^{[\infty]}$ is monochromatic for c^*. Using Y, we define a block sequence $B = (b_m) \in \mathrm{FIN}_k^{[\infty]}$ where b_m consists of all ordered pairs of the form $\langle |y_0| + \cdots + |y_{2m}| + l, i\rangle$ where the l^{th} place of y_{2m+1} is occupied by v_i, as well as all pairs $\langle n, 0\rangle$ for any other $n < \omega$. Let P be the collection of all doubly-indexed sequences $(\varepsilon_{n,i})$ such that

$$(\varepsilon_{n,i}) = \psi((y_{2m}[\sigma_{2m}]{}^\frown y_{2m+1}))$$

for some sequence of letters $(\sigma_{2m}) \in \prod_{m<\omega} L_{2m}$. Note that P is contained in the image of $[Y]_{Lv_k}^{[\infty]}$ under ψ.

The proof of Theorem 4.3.1 will be complete once we prove the following two claims:

Claim 4.3.9. There is an infinite sequence (P_i) of perfect subsets of 2^ω such that $\prod_{i<\omega} P_i \subseteq P$.

Proof. Let y denote the infinite variable word $y_0{}^\frown y_1{}^\frown y_2{}^\frown \ldots$ and, for each $m > 0$, let I_{2m-1} be the interval

$$[|y_0| + \cdots + |y_{2m-1}|, |y_0| + \cdots + |y_{2m-1}| + |y_{2m}|).$$

For each $i < \omega$, let P_i be the set of all $\delta \in 2^\omega$ satisfying the following conditions:

(1) If $y(n) \in L$, then $\delta(n) = y(n)(i)$.

(2) If $n < |y_0| + \cdots + |y_{2i-1}|$ and $y(n)$ is a variable, then $\delta(n) = 0$.

(3) $\delta(n) = \delta(n')$ for all $n, n' \in I_{2m-1}$ such that $y(n)$ and $y(n')$ are variables.

Since P_i has no restrictions at the minimal place of each interval I_{2m-1} where a variable occurs, it follows that P_i is perfect. To show the required inclusion of sets, let $(\delta_i) \in \prod_{i<\omega} P_i$ and let $(\varepsilon_{n,i})$ be the doubly-indexed sequence such that $\varepsilon_{n,i} = \delta_i(n)$. For each m, let n_m be the least place in the interval I_{2m-1} where a variable occurs in y. Then for each m choose $\sigma_{2m} \in L_{2m}$ such that $\sigma_{2m}(i) = \delta_i(n_m)$ for each $i < \omega$. Then it is routine to check that the sequence (σ_{2m}) witnesses the fact that $(\delta_i) \in P$. This proves the claim. □

Claim 4.3.10. $[B]_k^{[\infty]} \times P \subseteq (\varphi \times \psi)[Y]_{Lv_k}^{[\infty]}$.

Proof. Let $(A, (\varepsilon_{n,i})) \in [B]_k^{[\infty]} \times P$. By definition of P, there is a sequence $(\sigma_{2m}) \in \prod_{m<\omega} L_{2m}$ such that

$$(\varepsilon_{n,i}) = \psi((y_{2m}[\sigma_{2m}]^\frown y_{2m+1})).$$

If we let $X = (x_m)_{m<\omega}$ be given by $x_m = y_{2m}[\sigma_m]^\frown y_{2m+1}$ then $\varphi(X) = B$. For each $l < \omega$, let I_l be the smallest interval of integers such that

$$a_l = \sum_{i \in I_l} T^{j_i}(b_i)$$

for some integers $j_i \le k$, and note that the sequence $(I_l)_{l<\omega}$ is a block sequence. Fix $l < \omega$ and let $\{p, p+1, \ldots, p+q\}$ be an enumeration of the interval $(\max(I_{l-1}), \max(I_l)]$ where we set $\max(I_{-1}) = -1$ for convenience. Then let

$$z_l = T^{r_0}(x_p[\vec{\lambda}_0])^\frown \ldots ^\frown T^{r_q}(x_{p+q}[\vec{\lambda}_q])$$

where the parameters are determined as follows:

(i) If $p + i \notin I_l$, then let $\vec{\lambda}_i$ be the k-tuple $(\vec{0}, \ldots, \vec{0})$ where $\vec{0} \in 2^\omega$ is the sequence which is constantly 0. In this case, let $r_i = 0$.

(ii) If $p + i \in I_l$, then let $\vec{\lambda}_i = \vec{v}$ and $r_i = j_{p+i}$.

Then $Z = (z_l)_{l<\omega}$ is a block subsequence of X and hence of Y. By construction, $\varphi(Z) = A$. Finally, note that $\psi(Z) = (\varepsilon_{n,i})$ since the infinite word $z_0{}^\frown z_1{}^\frown z_2{}^\frown \ldots$ is obtained from the infinite word $y_0[\sigma_0]^\frown y_1{}^\frown \ldots ^\frown y_{2m}[\sigma_{2m}]^\frown y_{2m+1}^\frown \ldots$ by replacing some occurrences of a variable with the constant sequence $\vec{0} \in L$. In particular, this shows $\psi(Z) = \psi((y_{2m}[\sigma_{2m}]^\frown y_{2m+1}))$. Thus $(A, (\varepsilon_{n,i})) = (\varphi(Z), \psi(Z))$ as required. □

This finishes the proofs of the two claims, and hence the proof of the theorem is complete. □

4.4 A parametrization of the infinite-dimensional approximate Gowers theorem

In this section we prove the following approximate Ramsey theorem, which parametrizes the infinite-dimensional version of Gowers' $\mathrm{FIN}_{\pm k}$ theorem from [34]. First, given two infinite block sequences $A = (a_n)$ and $B = (b_n)$ in $\mathrm{FIN}_{\pm k}$, let

$$||A - B|| = \sup_{n<\omega} ||a_n - b_n||_\infty.$$

Theorem 4.4.1 (Parametrized $\mathrm{FIN}_{\pm k}^{[\infty]}$ Theorem)**.** *For every finite Souslin measurable colouring* $c : \mathrm{FIN}_{\pm k}^{[\infty]} \times (2^\omega)^\omega \to n$*, there are* $B \in \mathrm{FIN}_{\pm k}^{[\infty]}$*, a sequence* $(P_i)_{i<\omega}$ *of non-empty perfect subsets of* 2^ω*, and* $j < n$ *such that the following holds: For every* $(A, (p_i)) \in [B]_{\pm k}^{[\infty]} \times \prod_{i<\omega}^{\infty} P_i$ *there is* $\widetilde{A} \in \mathrm{FIN}_{\pm k}^{[\infty]}$ *such that*

$$c(\widetilde{A}, (p_i)) = j \quad \textit{and} \quad ||A - \widetilde{A}|| \leq 1.$$

To prove this result, we will need to develop an infinite-dimensional version of the Gowers-Hales-Jewett theorem which can code information about $\mathrm{FIN}_{\pm k}$. As before, given an infinite alphabet $L = \bigcup_{n<\omega} L_n$ presented as an increasing union of finite subalphabets L_n, fix a distinguished letter $0 \in L_0$ together with variables

$$v_1, v_{-1}, v_2, v_{-2}, \ldots, v_k, v_{-k} \notin L.$$

W_L will denote the set of all variable-free words over L and, for each $i \in \{1, \ldots, k\}$, $W_{Lv_{\pm i}}$ will denote the set of all variable words x over L such that

$$i = \max\{j \leq k : \ v_j \text{ or } v_{-j} \text{ appears in } x\}.$$

Let

$$S = W_L \cup \bigcup_{1\leq i\leq k} W_{Lv_{\pm i}}$$

and work in the semigroup $(S, {}^\frown)$. Given $x \in W_{Lv_{\pm k}}$ and a $2k$-tuple

$$\vec{\lambda} = (\lambda_{-k}, \ldots, \lambda_{-1}, \lambda_1, \ldots, \lambda_k) \in L^{2k} \cup \{\vec{v}\},$$

let $x[\vec{\lambda}]$ be the word obtained by replacing each occurrence of v_j with λ_j for each $j \in \{\pm 1, \ldots, \pm k\}$, where

$$\vec{v} = (v_{-k}, \ldots, v_{-1}, v_1, \ldots, v_k).$$

The tetris operation $T : W_{Lv_{\pm k}} \to W_{Lv_{\pm(k-1)}}$ is defined as follows: Given $x \in W_{Lv_{\pm k}}$, define $T(x) \in W_{Lv_{\pm(k-1)}}$ by

$$T(x)(n) := \begin{cases} v_{i-1} & \text{if } x(n) = v_i \text{ for } i > 1, \\ v_{i+1} & \text{if } x(n) = v_i \text{ for } i < -1, \\ 0 & \text{if } x(n) \in \{v_1, v_{-1}\}, \\ x(n) & \text{if } x(n) \in L. \end{cases}$$

As before, set $T(w) = w$ for each $w \in W_L$. In this setting we also have a notion of reflection: Given

$x \in S$, let $-x$ be the word obtained by replacing each occurrence of a variable v_i with v_{-i} for each $i \in \{\pm 1, \dots, \pm k\}$. Note that the mapping $x \mapsto -x$ is a semigroup homomorphism which is equal to the identity when restricted to W_L.

Given a sequence $X = (x_n)_{n<\omega}$ in $W_{Lv_{\pm k}}$, the *partial subsemigroup of* $W_{Lv_{\pm k}}$ *generated by* X, denoted $[X]_{Lv_{\pm k}}$, is defined to be the set of all elements of $W_{Lv_{\pm k}}$ which are of the form

$$\varepsilon_0 T^{j_0}(x_{n_0}[\vec{\lambda}_0])^\frown \dots ^\frown \varepsilon_l T^{j_l}(x_{n_l}[\vec{\lambda}_l])$$

where $l < \omega$, $n_0 < \cdots < n_l, j_i \leq k, \varepsilon_i \in \{\pm 1\}, \min_{i \leq l} j_i = 0$ and $\vec{\lambda}_i \in L^{2k}_{n_i} \cup \{\vec{v}\}$ for each $i \leq l$.

Let $W^{[\infty]}_{Lv_{\pm k}}$ denote the set of all *rapidly increasing* sequences in $W_{Lv_{\pm k}}$, defined as in the previous section and equipped with its natural metrizable topology. Exactly as before, the notion of rapidly increasing allows us to uniquely define the *support* of a word $x \in [X]_{Lv_{\pm k}}$ relative to some rapidly increasing sequence X. Given $X = (x_n)_{n<\omega}$ and $Y \in W^{[\infty]}_{Lv_{\pm k}}$, write $X \leq Y$ if and only if $x_n \in [Y]_{Lv_{\pm k}}$ for all $n < \omega$ and

$$\max \operatorname{supp}_Y(x_n) < \min \operatorname{supp}_Y(x_m) \text{ whenever } n < m.$$

As before, when this happens we say that X is a *block subsequence* of Y and we write $[Y]^{[\infty]}_{Lv_{\pm k}}$ for the set of all infinite block subsequences of Y. As is the case for $\mathrm{FIN}_{\pm k}$, we cannot expect to obtain an exact Ramsey theorem in this setting; rather, we will only be able to prove an *approximate* version of such a theorem which will make use of a suitable metric. First, we need the following:

Definition 4.4.2. For a word $x \in W_{Lv_{\pm k}}$, define

$$L(x) = \{n < |x| : x(n) \in L \setminus \{0\}\}.$$

Two words $x, y \in W_{Lv_{\pm k}}$ are *compatible* if:

(i) $|x| = |y|$.

(ii) $L(x) = L(y)$ and $x(n) = y(n)$ for all $n \in L(x)$.

Note that compatibility is a transitive relation on the set of pairs of words. Now, define a metric on the set $\{v_{\pm 1}, \dots, v_{\pm k}\} \cup \{0\}$ by setting $d(v_i, v_j) = |i - j|$ for variables v_i and v_j, and $d(v_i, 0) = |i|$. Using this, define a metric d on $W_{Lv_{\pm k}}$ taking values in $\mathbb{R} \cup \{\infty\}$ by

$$d(x, y) = \begin{cases} \sup\{d(x(i), y(i)) : i \in |x| \setminus L(x)\} & \text{if } x \text{ and } y \text{ are compatible,} \\ \infty & \text{otherwise.} \end{cases}$$

We then extend this to a metric on $W^{[\infty]}_{Lv_{\pm k}}$, also denoted d, by setting

$$d((x_n), (y_n)) = \sup_{n<\omega} d(x_n, y_n).$$

For $\varepsilon > 0$, $A \subseteq W_{Lv_{\pm k}}$ and $\mathcal{X} \subseteq W^{[\infty]}_{Lv_{\pm k}}$, let

$$(A)_\varepsilon = \{x \in W_{Lv_{\pm k}} : (\exists y \in A)\, d(x, y) \leq \varepsilon\},$$

$$(\mathcal{X})_\varepsilon = \{X \in W^{[\infty]}_{Lv_{\pm k}} : (\exists Y \in \mathcal{X})\, d(X, Y) \leq \varepsilon\}.$$

Theorem 4.4.3. *For every $k, r \in \mathbb{N}$ and every Souslin measurable $c : W^{[\infty]}_{Lv_{\pm k}} \to r$ there are $i < r$ and an infinite block sequence $X \in W^{[\infty]}_{Lv_{\pm k}}$ such that $[X]^{[\infty]}_{\pm k} \subseteq \left(c^{-1}\{i\}\right)_1$.*

To prove Theorem 4.4.3, we use ultra-Ramsey theory. First we will construct an ultrafilter which behaves well with respect to the mapping

$$-T : W_{Lv_{\pm k}} \to W_{Lv_{\pm(k-1)}} : x \mapsto -T(x)$$

in a sense that we now make precise. Work in the closed subsemigroup $S^* \subseteq \beta S$ consisting of all non-principal cofinite ultrafilters on S, where *cofinite* is defined as before. Define

$$S^*_L = \{\mathcal{U} \in S^* : W_L \in \mathcal{U}\}$$

and, for each $i \in \{1, \ldots, k\}$,

$$S^*_{Lv_{\pm i}} = \{\mathcal{U} \in S^* : W_{Lv_{\pm i}} \in \mathcal{U}\}.$$

Then S^*_L and $S^*_{Lv_{\pm i}}$ (for each $1 \leq i \leq k$) are non-empty closed subsemigroups of S^*. Let $\mathcal{W}$ be a minimal idempotent in S^*_L, and choose any idempotent $\mathcal{V}_1 \leq \mathcal{W}$ in $S^*_{Lv_1}$. Exactly as in the previous section, recursively construct a sequence $(\mathcal{V}_i)_{1\leq i\leq k}$ of idempotents starting with $\mathcal{V}_1$ such that for each $i < j$:

1. $\mathcal{V}_i$ is an idempotent in $S^*_{Lv_{\pm i}}$.
2. $\mathcal{V}_i \geq \mathcal{V}_j$.
3. $(-T)^{(j-i)}(\mathcal{V}_j) = \mathcal{V}_i$.
4. For each $\vec{\lambda} \in L^k$, $\mathcal{V}_i[\vec{\lambda}] = \mathcal{W}$.

In particular, note that (4) implies $(-T)(\mathcal{V}_1) = \mathcal{W}$ since $(-T)(x) = x[\vec{0}]$ for each $x \in W_{Lv_{\pm 1}}$ and where $\vec{0} = (0, \ldots, 0)$. In addition to the above properties, we will also need the following useful fact. First, given $A \subseteq W_{Lv_{\pm k}}$, let $-A$ be the set of all words of the form $-x$ for $x \in A$.

Lemma 4.4.4. *The ultrafilter $\mathcal{V}_k$ is* subsymmetric*, i.e. $-(A)_1 \in \mathcal{V}_k$ whenever $A \in \mathcal{V}_k$.*

Proof. Since $\mathcal{V}_k \leq \mathcal{V}_{k-1}$ and $(-T)(\mathcal{V}_k) = \mathcal{V}_{k-1}$ by property (3) in the definition of the ultrafilters $(\mathcal{V}_i)_{1\leq i\leq k}$, we have

$$\mathcal{V}_k = (-T)(\mathcal{V}_k)^\frown \mathcal{V}_k = \mathcal{V}_k{}^\frown(-T)\mathcal{V}_k.$$

(When $k = 1$, define $\mathcal{V}_0 = \mathcal{W}$.) Thus, for each $A \subseteq W_{Lv_{\pm k}}$,

$$\begin{aligned} A \in \mathcal{V}_k &\iff (\mathcal{V}_k x)(\mathcal{V}_k y)\, (-T)(x)^\frown y \in A \\ &\implies (\mathcal{V}_k x)(\mathcal{V}_k y)\, (-x)^\frown T(y) \in (A)_1 \\ &\iff (\mathcal{V}_k x)(\mathcal{V}_k y)\, x^\frown(-T)(y) \in -(A)_1 \\ &\iff -(A)_1 \in \mathcal{V}_k{}^\frown(-T)\mathcal{V}_k = \mathcal{V}_k \end{aligned}$$

where we use the easy fact that $(-T)(x)^\frown y$ and $(-x)^\frown T(y)$ are compatible. ☐

View the space $W^{[<\infty]}_{Lv_{\pm k}}$ of finite rapidly increasing sequences as a tree ordered by end-extension and with root $\emptyset$. Fix the subsymmetric cofinite ultrafilter $\mathcal{V}_k$ define above. Exactly as in the previous section, we define the notions of $\mathcal{V}_k$-tree, $\mathcal{V}_k$-open and $\mathcal{V}_k$-Ramsey relative to the tree $W_{Lv_{\pm k}}^{[<\infty]}$. An application of the ultra-Ellentuck theorem in this setting then yields:

Corollary 4.4.5. *For every $r \in \mathbb{N}$ and every Souslin measurable $c : W_{Lv_{\pm k}}^{[\infty]} \to r$ there are $i < r$ and a $\mathcal{V}_k$-tree U with stem $\emptyset$ such that $[U] \subseteq c^{-1}\{i\}$.*

Given $\alpha \leq \omega$ and a sequence $X = (x_n)_{n<\alpha}$ in $W_{Lv_{\pm k}}$, let $[X]_{(-T)}$ denote the set of all words of the form

$$(-T)^{j_0}(x_{n_0}[\vec{\lambda}_0])^\frown \dots ^\frown (-T)^{j_l}(x_{n_l}[\vec{\lambda}_l])$$

where $l \geq 0, n_0 < \dots < n_l < \alpha, \vec{\lambda}_i \in L_{n_i}^k \cup \{\vec{v}\}$, and $j_0, \dots, j_l \leq k$ such that $\min j_i = 0$ and $\vec{\lambda}_i = \vec{v}$ for some $i \leq l$. When the sequence $X = (x_n)_{n<m}$ is finite, we will often write $[x_0, \dots, x_{m-1}]_{(-T)}$ for the above collection. If $\alpha \leq \omega$ and $X = (x_n)_{n<\alpha}$, Y are rapidly increasing sequences in $W_{Lv_{\pm k}}$, write $X \leq_{(-T)} Y$ whenever $x_n \in [Y]_{(-T)}$ for every $n < \alpha$ and

$$\max \operatorname{supp}_Y(x_n) < \min \operatorname{supp}_Y(x_m) \text{ whenever } n < m < \alpha.$$

Lemma 4.4.6. *For every $\mathcal{V}_k$-tree U with stem $\emptyset$ there is $Y = (y_n)_{n<\omega} \in W_{Lv_{\pm k}}^{[\infty]}$ together with a decreasing sequence $(A_n)_{n<\omega}$ of subsets of $W_{Lv_{\pm k}}$ such that:*

(a) $A_n \subseteq U_t \cap -(U_t)_1$ for every $t \in U$ such that $t \leq_{(-T)} (y_0, \dots, y_{n-1})$.

(b) $[y_m, \dots, y_n]_{(-T)} \subseteq A_m$ for all $m \leq n < \omega$.

Proof. By induction on n, define a decreasing sequence $(A_n)_{n<\omega}$ together with a rapidly increasing sequence $(y_n)_{n<\omega}$ such that, for all $n < \omega$:

1. $y_n \in A_n \in \mathcal{V}_k$.
2. $A_{n+1} \subseteq \{z \in W_{Lv_{\pm k}} : [y_n, z]_{(-T)} \subseteq A_n\}$.
3. $A_n \subseteq U_t \cap -(U_t)_1$ for every $t \in U$ such that $t \leq_{(-T)} (y_i)_{i<n}$.

To start, take $A_0 := U_\emptyset \cap -(U_\emptyset)_1$ and note that $A_0 \in \mathcal{V}_k$ since $\mathcal{V}_k$ is subsymmetric and $U_\emptyset \in \mathcal{V}_k$. The definition of $\mathcal{V}_k$ implies

$$(\mathcal{V}_k y)(\mathcal{V}_k z)\left([y, z]_{(-T)} \subseteq A_0\right)$$

and so we take any $y_0 \in W_{Lv_{\pm k}}$ such that $(\mathcal{V}_k z)\left([y_0, z]_{(-T)} \subseteq A_0\right)$; in particular $y_0 \in A_0$ by definition of $[y_0, z]_{(-T)}$. We then take A_1 to be the intersection of the set $\{z \in W_{Lv_{\pm k}} : [y_0, z]_{(-T)} \subseteq A_0\}$ with

$$\bigcap \left\{U_t \cap -(U_t)_1 : t \in U \text{ and } t \leq_{(-T)} (y_0)\right\}.$$

Note that $A_0 \supseteq A_1$ and $A_1 \in \mathcal{V}_k$ since there are only finitely many $t \in U$ satisfying $t \leq_{(-T)} (y_0)$, and since each $U_t \cap -(U_t)_1 \in \mathcal{V}_k$ using the fact that $\mathcal{V}_k$ is subsymmetric.

Now suppose $A_0, \dots, A_n$ and $y_0, \dots, y_{n-1}$ have been constructed. Since $\mathcal{V}_k$ is cofinite, it follows that there is $y_n \in W_{Lv_{\pm k}}$ such that

$$|y_n| > \sum_{i=0}^{n-1} |y_i|$$

and $(\mathcal{V}_k z)\left([y_n, z]_{(-T)} \subseteq A_n\right)$; in particular $y_n \in A_n$. Then take A_{n+1} to be the intersection of the set $\{z \in W_{Lv_{\pm k}} : [y_n, z]_{(-T)} \subseteq A_n\}$ with

$$\bigcap \left\{U_t \cap -(U_t)_1 : t \in U \text{ and } t \leq_{(-T)} (y_0, \ldots, y_n)\right\}.$$

Observe that the collection $[y_0, \ldots, y_{n-1}]_{(-T)}$ is finite since we only allow substitutions of the form $y_i[\vec{\lambda}_i]$ for $\vec{\lambda}_i \in L_i^k \cup \{\vec{v}\}$ and so there are only finitely many sets in the above intersection. Thus $A_{n+1} \in \mathcal{V}_k$ and $A_n \supseteq A_{n+1}$. This completes the inductive construction of the sequences (A_n) and (y_n). In particular, condition (a) is satisfied by (3). The verification of (b) is exactly the same as that of the corresponding condition in the statement of Lemma 4.3.8 after making the obvious adjustments. □

Lemma 4.4.7. *Let U be a $\mathcal{V}_k$-tree with* stem$(U) = \emptyset$. *Then there is an infinite rapidly increasing sequence $Y = (y_n)_{n<\omega}$ in $W_{Lv_{\pm k}}$ such that $[Y]_{\pm k}^{[\infty]} \subseteq ([U])_2$.*

Proof. Let Y be as in Lemma 4.4.6. We claim that Y satisfies the conclusion of the lemma. To see this, fix an infinite rapidly increasing block subsequence $X = (x_n)_{n<\omega}$ of Y. We will construct a rapidly increasing sequence $X' = (x'_n)_{n<\omega} \in [U] \cap [Y]_{(-T)}^{[\infty]}$ such that $d(x_n, x'_n) \leq 2$ for each $n < \omega$. Suppose, for some $n \geq 0$, we have defined $x'_0, \ldots, x'_{n-1} \in W_{Lv_{\pm k}}$ such that $s := (x'_0, \ldots, x'_{n-1}) \in U$ and $d(x_i, x'_i) \leq 2$ for each $i < n$. (In the case where $n = 0$ we simply have $s = \emptyset$.) Write

$$x_n = \varepsilon_0 T^{j_0}(y_{n_0}[\vec{\lambda}_0])^\frown \ldots ^\frown \varepsilon_l T^{j_l}(y_{n_l}[\vec{\lambda}_l])$$

where $l < \omega$, $n_0 < \cdots < n_l, j_i \leq k, \varepsilon_i \in \{\pm 1\}$ and $\vec{\lambda}_i \in L_{n_i}^{2k} \cup \{\vec{v}\}$ are such that $\min j_i = 0$ and $\vec{\lambda}_i = \vec{v}$ for some $i \leq l$. We consider the following two cases:

Case 1. There is $i \leq l$ such that $j_i = 0, \vec{\lambda}_i = \vec{v}$ and $\varepsilon_i = +1$.
For each $i \leq l$, set $z_i := \varepsilon_i T^{j_i}(y_{n_i}[\vec{\lambda}_i])$ for convenience. We consider the following two subcases:

(a) $\varepsilon_i = +1$ and j_i is even, or $\varepsilon_i = -1$ and j_i is odd. In either case, set $z'_i := z_i$ and note that $z'_i = (-T)^{j_i}(y_{n_i}[\vec{\lambda}_i])$.

(b) $\varepsilon_i = +1$ and j_i is odd, or $\varepsilon_i = -1$ and j_i is even. In either case, set $z'_i := T(z_i)$ and note that $z'_i = (-T)^{j_i+1}(y_{n_i}[\vec{\lambda}_i])$.

We then set $x'_n := z_0'^\frown \ldots ^\frown z'_l$. Note that x'_n is compatible with x_n, and $x'_n \in [y_{n_i} : i \leq l]_{(-T)}$ by the assumption given by Case 1. Since $d(z_i, z'_i) \leq 1$ for all $i \leq l$ we have $d(x_n, x'_n) \leq 1$. Furthermore, by the choice of the sequence Y we have

$$[y_{n_i} : i \leq l]_{(-T)} \subseteq A_{n_0}$$

(using the notation of Lemma 4.4.6) and so $x'_n \in U_t$ for every $t \in U$ such that $t \leq_{(-T)} (y_0, \ldots, y_{n_0-1})$. In particular, $x'_n \in U_s$ since

$$\max \operatorname{supp}_Y(x_{n-1}) < \min \operatorname{supp}_Y(x_n) = n_0$$

and so $(s, x'_n) \in U$.

Case 2. For every $i \leq l$, if $j_i = 0$ and $\vec{\lambda}_i = \vec{v}$, then $\varepsilon_i = -1$.
Apply Case 1 to $-x_n$ to obtain $z \in [y_{n_i} : i \leq l]_{(-T)}$ such that $d(-x_n, z) \leq 1$. By definition of Y we have

$$[y_{n_i} : i \leq l]_{(-T)} \subseteq A_{n_0}$$

and so $z \in U_t \cap -(U_t)_1$ for every $t \in U$ such that $t \leq_{(-T)} (y_0, \dots, y_{n_0-1})$. As before, this implies $-z \in (U_s)_1$ and so there is $z' \in U_s$ such that $d(-z, z') \leq 1$. Set $x'_n := z'$. Then x'_n is compatible with x_n and

$$d(x_n, x'_n) \leq d(x_n, -z) + d(-z, z') = d(-x_n, z) + d(-z, z') \leq 2$$

and so x'_n satisfies our requirements.

This completes the inductive construction of X'. It is clear from the above construction that $X' \in [U]$ and $d(x_n, x'_n) \leq 2$ for all $n < \omega$ and so $X \in ([U])_2$. $\square$

To minimize the "error" in the previous result, we will use the following family of mappings: For each $k \in \mathbb{N}$, let $\Phi_k : W_{Lv_{\pm 2k}} \to W_{Lv_{\pm k}}$ be defined by setting

$$\Phi_k(x)(n) := \begin{cases} v_{i/2} & \text{if } x(n) = v_i \text{ where } i \text{ is even,} \\ v_{(i-1)/2} & \text{if } x(n) = v_i \text{ where } i \text{ is odd and positive,} \\ v_{(i+1)/2} & \text{if } x(n) = v_i \text{ where } i \text{ is odd and negative,} \\ x(n) & \text{if } x(n) \in L. \end{cases}$$

The following properties of Φ_k are easy to check:

(i) Φ_k is a surjective homomorphism of partial semigroups which, in addition, satisfies $\Phi_k(-x) = -\Phi_k(x)$ for every $x \in W_{Lv_{\pm 2k}}$.

(ii) For every $x, y \in W_{Lv_{\pm 2k}}$ and every $i, j \leq k$ with $\min\{i, j\} = 0$,

$$\Phi_k\left(T^{2i}(x)^\frown T^{2j}(y)\right) = T^i(\Phi_k(x))^\frown T^j(\Phi_k(y)).$$

(iii) For every $x, y \in W_{Lv_{\pm 2k}}$, $d(x, y) \leq 2 \implies d(\Phi_k(x), \Phi_k(y)) \leq 1$. In particular, Φ_k preserves the compatibility relation between words.

We extend Φ to $W^{[\infty]}_{Lv_{\pm 2k}}$ by setting $\Phi((y_n)_{n<\omega}) := (\Phi(y_n))_{n<\omega}$. It is straightforward to check that Φ is continuous with respect to the usual metrizable topologies. Furthermore, note that if Y and Y' are two sequences in $W_{Lv_{\pm 2k}}$ which satisfy $d(Y, Y') \leq 2$, then $d(\Phi(Y), \Phi(Y')) \leq 1$. We are now ready to finish the proof of the approximate Gowers-Hales-Jewett theorem.

Proof of Theorem 4.4.3. Let $c : W^{[\infty]}_{Lv_{\pm k}} \to r$ be Souslin measurable and define a colouring $\widetilde{c} : W^{[\infty]}_{Lv_{\pm 2k}} \to r$ by setting $\widetilde{c} := c \circ \Phi$. Since Φ is continuous and c is Souslin measurable, it follows that $\widetilde{c}$ is Souslin measurable. By Corollary 4.4.5 there are $i < r$ and a $\mathcal{V}_k$-tree U with stem $\emptyset$ such that $[U] \subseteq \widetilde{c}^{-1}\{i\}$. Applying Lemma 4.4.7, find an infinite rapidly increasing sequence $\widetilde{Y} = (\widetilde{y_n})_{n<\omega}$ in $W_{Lv_{\pm 2k}}$ such that $[\widetilde{Y}]^{[\infty]}_{\pm 2k} \subseteq ([U])_2$.

Let $Y := \Phi(\widetilde{Y}) \in W^{[\infty]}_{Lv_{\pm k}}$ so that $y_n := \Phi(\widetilde{y_n})$ for each $n < \omega$. We claim that Y satisfies

$$[Y]^{[\infty]}_{\pm k} \subseteq \left(c^{-1}\{i\}\right)_1.$$

Indeed, if $X = (x_n)_{n<\omega} \in W^{[\infty]}_{Lv_{\pm k}}$ is an infinite rapidly increasing subsequence of Y, then for each $n < \omega$ we have

$$x_n = \varepsilon_0 T^{j_0}(y_{n_0}[\vec{\lambda}_0])^\frown \dots ^\frown \varepsilon_l T^{j_l}(y_{n_l}[\vec{\lambda}_l])$$

for some $\varepsilon_i \in \{\pm 1\}, n_0 < \cdots < n_l, \vec{\lambda}_i \in L^k_{n_i} \cup \{\vec{v}\}$ and $j_i \leq k$ such that $\min j_i = 0$ and $\vec{\lambda}_i = \vec{v}$ for some $i \leq l$. Then properties (i) and (ii) of Φ listed above imply $x_n = \Phi(\widetilde{x_n})$, where

$$\widetilde{x_n} := \varepsilon_0 T^{2j_0}(y_{n_0}[\vec{\lambda}_0])^\frown \ldots ^\frown \varepsilon_l T^{2j_l}(y_{n_l}[\vec{\lambda}_l]) \in [\widetilde{Y}]_{\pm k}$$

and so, setting $\widetilde{X} := (\widetilde{x_n})_{n<\omega}$, we see that $X = \Phi(\widetilde{X})$. Since $\widetilde{X}$ is a rapidly increasing subsequence of $\widetilde{Y}$, by our choice of $\widetilde{Y}$ we can find $X' \in \widetilde{c}^{-1}\{i\}$ such that $d(\widetilde{X}, X') \leq 2$. Then, as observed above, property (iii) of Φ implies

$$d(\Phi(\widetilde{X}), \Phi(X')) \leq 1.$$

Since $i = \widetilde{c}(X') = c(\Phi(X'))$ we obtain $\Phi(X') \in c^{-1}\{i\}$ and so $X \in \left(c^{-1}\{i\}\right)_1$ as required. □

We are now equipped to prove a parametrized version of the infinite-dimensional $\mathrm{FIN}_{\pm k}$ theorem.

Proof of Theorem 4.4.1. Fix a finite Souslin measurable colouring c of $\mathrm{FIN}^{[\infty]}_{\pm k} \times (2^\omega)^\omega$. As before, let

$$L_n = \{\sigma \in 2^\omega : (\forall i > n)\, \sigma(i) = 0\}$$

and $L = \bigcup_{n<\omega} L_n$. Define a mapping $\varphi : W^{[\infty]}_{Lv_{\pm k}} \to \mathrm{FIN}^{[\infty]}_{\pm k}$ by setting $\varphi((x_m)) = (a_m)$, where $a_m \in \mathrm{FIN}_{\pm k}$ consists of all ordered pairs of the form

$$\langle |x_0| + \cdots + |x_{m-1}| + l, i \rangle$$

where v_i occupies the l^{th} place in x_m, and where a_m takes the value 0 at all other points of ω. We also define a mapping $\psi : W^{[\infty]}_{Lv_{\pm k}} \to 2^{\omega\times\omega}$ by $\psi((x_m))(n,i) = \sigma(i)$ if $\sigma \in L$ occupies the n^{th} place in the infinite variable word $x_0{}^\frown x_1{}^\frown x_2{}^\frown \ldots$ and where $\psi((x_m))(n,i) = 0$ if a variable occupies the n^{th} place in the above infinite word.

Define a Souslin measurable colouring c^* of $W^{[\infty]}_{Lv_{\pm k}}$ by setting

$$c^*((x_m)) = c\left(\varphi((x_m)), \psi((x_m))\right)$$

and apply Theorem 4.4.3 to find $Y = (y_m) \in W^{[\infty]}_{Lv_{\pm k}}$ and a colour r such that

$$[Y]^{[\infty]}_{Lv_{\pm k}} \subseteq \left((c^*)^{-1}\{r\}\right)_1.$$

Using Y, we define a block sequence $B = (b_m) \in \mathrm{FIN}^{[\infty]}_{\pm k}$ where b_m consists of all ordered pairs of the form $\langle |y_0| + \cdots + |y_{2m}| + l, i \rangle$ where the l^{th} place of y_{2m+1} is occupied by v_i, and where b_m takes the value 0 at all other points of ω. As before, we let P be the collection of all $(\varepsilon_{n,i})$ such that

$$(\varepsilon_{n,i}) = \psi((y_{2m}[\sigma_{2m}]^\frown y_{2m+1}))$$

for some sequence $(\sigma_{2m}) \in \prod_{m<\omega} L_{2m}$.

Exactly as in the proof of Theorem 4.3.1 we can show there is an infinite sequence (P_i) of perfect subsets of 2^ω such that $\prod_{i<\omega} P_i \subseteq P$. We will also need the following:

Claim 4.4.8. $[B]^{[\infty]}_{\pm k} \times P \subseteq (\varphi \times \psi)[Y]^{[\infty]}_{Lv_{\pm k}}$.

Proof. Let $(A, (\varepsilon_{n,i})) \in [B]^{[\infty]}_{\pm k} \times P$. By definition of P, there is a sequence $(\sigma_{2m}) \in \prod_{m<\omega} L_{2m}$ such that

$$(\varepsilon_{n,i}) = \psi((y_{2m}[\sigma_{2m}]^\frown y_{2m+1})).$$

If we let $X = (x_m)_{m<\omega}$ be given by $x_m = y_{2m}[\sigma_m]^\frown y_{2m+1}$ then $\varphi(X) = B$. For each $l < \omega$, let I_l be the smallest interval of integers such that

$$a_l = \sum_{i \in I_l} s_i T^{j_i}(b_i)$$

for some integers $j_i \leq k$ and $s_i \in \{\pm 1\}$, and note that the sequence $(I_l)_{l<\omega}$ is a block sequence. Fix $l < \omega$ and let $\{p, p+1, \ldots, p+q\}$ be an enumeration of the interval $(\max(I_{l-1}), \max(I_l)]$ where we set $\max(I_{-1}) = -1$ for convenience. Then let

$$z_l = \rho_0 T^{r_0}(x_p[\vec{\lambda}_0])^\frown \ldots {}^\frown \rho_q T_{r_q}(x_{p+q}[\vec{\lambda}_q])$$

where the parameters are determined as follows:

(i) If $p + i \notin I_l$, then let $\vec{\lambda}_i$ be the $2k$-tuple $(\vec{0}, \ldots, \vec{0})$ where $\vec{0} \in 2^\omega$ is the sequence which is constantly 0. In this case, let $r_i = 0$ and $\rho_i = 1$.

(ii) If $p + i \in I_l$, then let $\vec{\lambda}_i = \vec{v}$, $r_i = j_{p+i}$ and $\rho_i = s_{p+i}$.

Then, exactly as before, one checks that $(A, (\varepsilon_{n,i})) = (\varphi(Z), \psi(Z))$. □

We now verify that B and (P_i) are as desired. To this end, fix

$$(A, (\varepsilon_{n,i})) \in [B]^{[\infty]}_{\pm k} \times \prod_{i<\omega} P_i \subseteq [B]^{[\infty]}_{\pm k} \times P$$

and apply the previous claim to find $Z = (z_n) \in [Y]^{[\infty]}_{Lv_{\pm k}}$ such that $(A, (\varepsilon_{n,i})) = (\varphi(Z), \psi(Z))$. By definition of Y, there is $\widetilde{Z} = (\widetilde{z}_n) \in W^{[\infty]}_{Lv_{\pm k}}$ such that $d(Z, \widetilde{Z}) \leq 1$ and $c^*(\widetilde{Z}) = r$. Using the definition of the metric d it must be that z_n is compatible with $\widetilde{z}_n$ for each n, and so it follows that $\psi(Z) = \psi(\widetilde{Z})$. Furthermore, note that

$$||\varphi(Z) - \varphi(\widetilde{Z})|| = d(Z, \widetilde{Z}) \leq 1$$

according to the definitions of d and φ. Let $\widetilde{A} = \varphi(\widetilde{Z})$; then $||A - \widetilde{A}|| \leq 1$ and

$$c(\widetilde{A}, (\varepsilon_{n,i})) = c((\varphi(\widetilde{Z}), \psi(\widetilde{Z}))) = c^*(\widetilde{Z}) = r.$$

This finishes the proof of theorem. □

As an easy consequence, we obtain a parametrized version of Gowers' $\mathrm{FIN}_{\pm k}$ theorem:

Corollary 4.4.9. *For every finite colouring $c : \mathrm{FIN}_{\pm k} \times (2^\omega)^\omega \to n$, there are $B \in \mathrm{FIN}^{[\infty]}_{\pm k}$, a sequence $(P_i)_{i<\omega}$ of non-empty perfect subsets of 2^ω, and $j < n$ such that the following holds: For every $(b, (p_i)) \in [B]_{\pm k} \times \prod_{i<\omega} P_i$ there is $\widetilde{b} \in \mathrm{FIN}_{\pm k}$ such that*

$$c(\widetilde{b}, (p_i)) = j \textit{ and } ||b - \widetilde{b}||_\infty \leq 1.$$

We conclude this chapter with an application of the previous result to the oscillation stability of uniformly equicontinuous families of real-valued functions on S_{c_0}, the unit sphere of the Banach space c_0. The following result can be seen as a parametrization of Gowers' c_0 theorem. The proof is similar to Gowers' original proof [22]; see also [23].

Theorem 4.4.10. *Let $\{f_\sigma : \sigma \in (2^\omega)^\omega\}$ be a family of functions $S_{c_0} \to \mathbb{R}$ which is uniformly bounded and uniformly equicontinuous. Then for every $\varepsilon > 0$ there are an infinite-dimensional subspace X of c_0 and a sequence $(P_n)_{n<\omega}$ of perfect subsets of 2^ω such that the oscillation of each mapping f_σ for $\sigma \in \prod_{n<\omega} P_n$ is at most ε when restricted to S_X, the unit sphere of X.*

Proof. Apply uniform equicontinuity to the given ε to find $\delta > 0$ such that

$$|f_\sigma(x) - f_\sigma(y)| \leq \varepsilon/5$$

for all $\sigma \in (2^\omega)^\omega$ and all $x, y \in S_{c_0}$ such that $||x - y||_\infty \leq \delta$. Fix k large enough such that $(1+\delta)^{1-k} < \delta$ and let $\Delta_{\pm k}$ be the subset of S_{c_0} consisting of all finitely-supported vectors with coordinates belonging to the set

$$\{\pm(1+\delta)^{i-k} : i = 1, \ldots, k\} \cup \{0\}.$$

Let $\varphi : \mathrm{FIN}_{\pm k} \to \Delta_{\pm k}$ be the bijection defined by

$$\varphi(p)(n) := \begin{cases} (1+\delta)^{i-k} & \text{if } p(n) = i > 0, \\ 0 & \text{if } p(n) = 0, \\ -(1+\delta)^{i-k} & \text{if } p(n) = -i < 0. \end{cases}$$

Since the family of functions (f_σ) is uniformly bounded, there is a partition of $\bigcup_{\sigma \in (2^\omega)^\omega} \mathrm{range}(f_\sigma)$ into finitely many disjoint intervals $I_0, \ldots, I_{l-1}$ such that the length of each interval is at most $\varepsilon/5$. Define a colouring $c : \mathrm{FIN}_{\pm k} \times (2^\omega)^\omega \to l$ by setting

$$c(p, \sigma) = j \iff f_\sigma(\varphi(p)) \in I_j$$

and find $B = (b_n) \in \mathrm{FIN}_{\pm k}^{[\infty]}$, a sequence $(P_i)_{i<\omega}$ of non-empty perfect subsets of 2^ω, and $j < l$ satisfying the conclusion of Corollary 4.4.9 with respect to c. Using the choice of k together with the implication

$$||p - q||_\infty \leq 1 \implies ||\varphi(p) - \varphi(q)||_\infty \leq \delta,$$

it follows from the choice of B and (P_i) that

$$|f_\sigma(\varphi(p)) - f_\sigma(\varphi(q))| \leq \frac{3\varepsilon}{5} \text{ for all } p, q \in [B] \text{ and all } \sigma \in \prod_{i<\omega} P_i.$$

Now let X be the linear span of the set $\{\varphi(b_n) : n < \omega\}$ in c_0. Then it is straightforward to check that the set $\{\varphi(b) : b \in [B]\}$ is a δ-net in S_X. Using the previous inequality, this implies

$$|f_\sigma(x) - f_\sigma(y)| \leq \varepsilon \text{ for all } x, y \in S_X \text{ and all } \sigma \in \prod_{i<\omega} P_i.$$

Thus the oscillation of each function f_σ for $\sigma \in \prod_{i<\omega} P_i$ is at most ε on S_X. □

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