Nonlinear Schrödinger Evolutions from Low Regularity Initial Data

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1 Cubic NLS on $\mathbb{R}^2$

2 High-Low Fourier Truncation

3 Bilinear Strichartz Estimate

4 The $I$-Method of Almost Conservation
1. Cubic NLS Initial Value Problem on $\mathbb{R}^2$
We consider the initial value problems:
\[
\begin{cases}
(i \partial_t + \Delta) u = \pm |u|^2 u \\
u(0, x) = u_0(x).
\end{cases}
\]

The $+$ case is called **defocusing**; $-$ is **focusing**. $NLS_{3}^{\pm}(\mathbb{R}^2)$ is ubiquitous in physics. The solution has a dilation symmetry
\[
u^\lambda(\tau, y) = \lambda^{-1} u(\lambda^{-2} \tau, \lambda^{-1} y).
\]
which is invariant in $L^2(\mathbb{R}^2)$. This problem is **$L^2$-critical**.

(This talk mostly addresses the defocusing case.)
Mass = $\int_{\mathbb{R}^d} |u(t, x)|^2 dx$.

Momentum = $2\Im \int_{\mathbb{R}^2} \overline{u}(t) \nabla u(t) dx$.

Energy = $H[u(t)] = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u(t)|^2 dx \pm \frac{1}{2} |u(t)|^4 dx$.

- Mass is $L^2$; Momentum is close to $H^{1/2}$; Energy involves $H^1$.
- Dynamics on a sphere in $L^2$; focusing/defocusing energy.
- Local conservation laws express how quantity is conserved: e.g., $\partial_t |u|^2 = \nabla \cdot 2\Im(\overline{u}\nabla u)$. Frequency Localizations?
The solution of the linear Schrödinger initial value problem

\[
\begin{cases}
(i \partial_t + \Delta) u = 0 \\
 u(0, x) = u_0(x).
\end{cases} \quad (LS(\mathbb{R}^d))
\]

is denoted \( u(t, x) = e^{it\Delta} u_0 \). The solution can be given explicitly

- Fourier Multiplier Representation:

\[
e^{it\Delta} u_0(x) = c_\pi \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{-it|\xi|^2} \hat{u}_0(\xi) d\xi.
\]

- Convolution Representation:

\[
e^{it\Delta} u_0(x) = c_\pi^{1} \frac{1}{(it)^{d/2}} \int_{\mathbb{R}^d} e^{i \frac{|x-y|^2}{4t}} u_0(y) dy.
\]
The solution of the linear Schrödinger initial value problem

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u(0, x) &= u_0(x).
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(\textit{LS}(\mathbb{R}^d))

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- **Convolution Representation:**

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\]
The solution of the linear Schrödinger initial value problem

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- Convolution Representation:

\[
e^{it\Delta} u_0(x) = c_\pi^{1/2} \frac{1}{(it)^{d/2}} \int_{\mathbb{R}^d} e^{i \frac{|x-y|^2}{4t}} u_0(y) dy.
\]
Estimates for Linear Schrödinger Propagator

- Fourier Multiplier Representation $\implies$ Unitary in $H^s$:
  \[ \| D_x^s e^{it\Delta} u_0 \|_{L_x^2} = \| D_x^s u_0 \|_{L_x^2}. \]

- Convolution Representation $\implies$ Dispersive estimate:
  \[ \| e^{it\Delta} u_0 \|_{L_x^\infty} \lesssim \frac{C}{td^{1/2}} \| u_0 \|_{L_x^1}. \]

- Spacetime estimates? **Strichartz estimates** hold, for example,
  \[ \| e^{it\Delta} u_0 \|_{L^4(\mathbb{R}_t \times \mathbb{R}_x^2)} \leq C \| u_0 \|_{L^2(\mathbb{R}_x^2)}. \]

(Strichartz estimates linked to Fourier restriction phenomena.)
Local-in-time theory for $\text{NLS}^3_3(\mathbb{R}^2)$

- $\forall \ u_0 \in L^2(\mathbb{R}^2) \ \exists \ T_{lwp}(u_0)$ determined by
  \[ \| e^{it\Delta} u_0 \|_{L^4_t([0, T_{lwp}] \times \mathbb{R}^2)} < \frac{1}{100} \]
such that
  \[ \exists \ \text{unique} \ u \in C([0, T_{lwp}]; L^2) \cap L^4_t([0, T_{lwp}] \times \mathbb{R}^2) \] solving $\text{NLS}^3_3(\mathbb{R}^2)$.

- $\forall \ u_0 \in H^s(\mathbb{R}^2), s > 0, T_{lwp} \sim \| u_0 \|_{H^s}^{-\frac{2}{s}}$ and regularity persists:
  $u \in C([0, T_{lwp}]; H^s(\mathbb{R}^2))$.

- Define the maximal forward existence time $T^*(u_0)$ by
  \[ \| u \|_{L^4_t([0, T^*-\delta] \times \mathbb{R}^2)} < \infty \]
  for all $\delta > 0$ but diverges to $\infty$ as $\delta \downarrow 0$.

- $\exists$ small data scattering threshold $\mu_0 > 0$
  \[ \| u_0 \|_{L^2} < \mu_0 \implies \| u \|_{L^4_t(\mathbb{R} \times \mathbb{R}^2)} < 2\mu_0. \]
What is the ultimate fate of the local-in-time solutions?

\textbf{L}^2\text{-critical Scattering Conjecture:}
\[ L^2 \ni u_0 \mapsto u \text{ solving } NLS_3^+(\mathbb{R}^2) \text{ is global-in-time and} 
\[ \|u\|_{L_{t,x}^4} < A(u_0) < \infty. \]

Moreover, \( \exists u_\pm \in L^2(\mathbb{R}^2) \) such that
\[ \lim_{t \to \pm \infty} \| e^{\pm i t \Delta} u_\pm - u(t) \|_{L^2(\mathbb{R}^2)} = 0. \]

Same statement for focusing \( NLS_3^- (\mathbb{R}^2) \) if \( \|u_0\|_{L^2} < \|Q\|_{L^2} \).

\textbf{Remarks:}

- Known for small data \( \|u_0\|_{L^2(\mathbb{R}^2)} < \mu_0 \).
- Known for \textbf{large radial data} [Killip-Tao-Visan 07].
**$NLS^{\pm}_{3}(\mathbb{R}^2)$: Present Status for General Data**

<table>
<thead>
<tr>
<th>regularity</th>
<th>idea</th>
<th>reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s &gt; \frac{2}{3}$</td>
<td>high/low frequency decomposition $H(Iu)$</td>
<td>[Bourgain98] [CKSTT02]</td>
</tr>
<tr>
<td>$s &gt; \frac{4}{7}$</td>
<td>resonant cut of 2nd energy $H(Iu)$ &amp; Interaction Morawetz</td>
<td>[CKSTT07] [Fang-Grillakis05] [CGTz07]</td>
</tr>
<tr>
<td>$s &gt; \frac{1}{2}$</td>
<td>$H(Iu)$ &amp; Interaction $I$-Morawetz</td>
<td></td>
</tr>
<tr>
<td>$s &gt; \frac{2}{5}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s &gt; \frac{1}{3}$</td>
<td>resonant cut &amp; $I$-Morawetz</td>
<td>[C-Roy08]</td>
</tr>
<tr>
<td>$s &gt; 0$?</td>
<td></td>
<td></td>
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</tbody>
</table>

- Morawetz-based arguments are only for defocusing case.
- Focusing results assume $\|u_0\|_{L^2} < \|Q\|_{L^2}$.
- Unify theory of focusing-under-ground-state and defocusing?
2. Bourgain’s High-Low Fourier Truncation
Summary

Consider the 2D IVP

\[
\begin{aligned}
&iu_t + \Delta u + \lambda |u|^2 u = 0 \\
&u(0) = \varphi \in L^2(\mathbb{R}^2).
\end{aligned}
\]

The theory on the Cauchy problem asserts a unique maximal solution

\[u \in C([1 - T^*_j, T^*_j]; L^2(\mathbb{R}^2)) \cap L^4([1 - T^*_j, T^*_j]; L^4(\mathbb{R}^2)).\]
Consider the Cauchy problem for defocusing cubic NLS on $\mathbb{R}^2$:

\[
\begin{aligned}
(i\partial_t + \Delta)u &= +|u|^2u \\
u(0, x) &= \phi_0(x).
\end{aligned}
\] (NLS$_3^+(\mathbb{R}^2)$)

We describe the first result to give global well-posedness below $H^1$.

- NLS$_3^+(\mathbb{R}^2)$ is GWP in $H^s$ for $s > \frac{2}{3}$ [Bourgain 98].
- First use of Bilinear Strichartz estimate was in this proof.
- Proof cuts solution into low and high frequency parts.
- For $u_0 \in H^s$, $s > \frac{2}{3}$, Proof gives (and crucially exploits),

\[u(t) - e^{it\Delta}\phi_0 \in H^1(\mathbb{R}_x^2).\]
Fix a large target time $T$.

Let $N = N(T)$ be large to be determined.

Decompose the initial data:

$$\phi_0 = \phi_{\text{low}} + \phi_{\text{high}}$$

where

$$\phi_{\text{low}}(x) = \int_{|\xi| < N} e^{ix \cdot \xi} \hat{\phi}_0(\xi) d\xi.$$ 

Our plan is to evolve:

$$\phi_0 = \phi_{\text{low}} + \phi_{\text{high}}$$

$$u(t) = u_{\text{low}}(t) + u_{\text{high}}(t).$$
Low Frequency Data Size:

- Kinetic Energy:
  \[
  \| \nabla \phi_{low} \|_{L^2}^2 = \int_{|\xi|<N} |\xi|^2 |\hat{\phi}_0(\xi)|^2 \, dx \\
  = \int_{|\xi|<N} |\xi|^{2(1-s)} |\xi|^{2s} |\hat{\phi}_0(\xi)|^2 \, dx \\
  \leq N^{2(1-s)} \| \phi_0 \|_{H^s}^2 \leq C_0 N^{2(1-s)}. 
  \]

- Potential Energy: \( \| \phi_{low} \|_{L^4_x} \leq \| \phi_{low} \|_{L^2}^{1/2} \| \nabla \phi_{low} \|_{L^2}^{1/2} \)
  \[\implies H[\phi_{low}] \leq C N^{2(1-s)}.\]

High Frequency Data Size:

\[ \| \phi_{high} \|_{L^2} \leq C_0 N^{-s}, \| \phi_{high} \|_{H^s} \leq C_0. \]
The NLS Cauchy Problem for the low frequency data

\[
\begin{cases}
(i \partial_t + \Delta) u_{\text{low}} = +|u_{\text{low}}|^2 u_{\text{low}} \\
u_{\text{low}}(0, x) = \phi_{\text{low}}(x)
\end{cases}
\]

is well-posed on \([0, T_{lwp}]\) with \(T_{lwp} \sim \|\phi_{\text{low}}\|_{H^1}^{-2} \sim N^{-2(1-s)}\).

We obtain, as a consequence of the local theory, that

\[
\|u_{\text{low}}\|_{L^4_{[0, T_{lwp}], x}} \leq \frac{1}{100}.
\]
The NLS Cauchy Problem for the low frequency data

\[
\begin{cases}
(i\partial_t + \Delta)u_{\text{high}} &= +2|u_{\text{low}}|^2u_{\text{high}} + \text{similar} + |u_{\text{high}}|^2u_{\text{high}} \\

u_{\text{high}}(0, x) &= \phi_{\text{high}}(x)
\end{cases}
\]

is also well-posed on \([0, T_{\text{lwp}}]\).

**Remark:** The LWP lifetime of NLS evolution of \(u_{\text{low}}\) AND the LWP lifetime of the DE evolution of \(u_{\text{high}}\) are controlled by \(\|u_{\text{low}}(0)\|_{H^1}\).
The high frequency evolution may be written

\[ u_{\text{high}}(t) = e^{it\Delta} u_{\text{high}} + w. \]

The local theory gives \( \|w(t)\|_{L^2} \lesssim N^{-s} \). Moreover, due to smoothing (obtained via bilinear Strichartz), we have that

\[ w \in H^1, \quad \|w(t)\|_{H^1} \lesssim N^{1-2s+}. \quad \text{(SMOOTH!)} \]

Let’s postpone the proof of \( \text{(SMOOTH!)} \).
Universal $t \in [0, T_{lwp}]$, we have

$$u(t) = u_{low}(t) + e^{it\Delta_\phi} + w(t).$$

At time $T_{lwp}$, we define data for the progressive scheme:

$$u(T_{lwp}) = u_{low}(T_{lwp}) + w(T_{lwp}) + e^{iT_{lwp}\Delta_\phi}.$$

$$u(t) = u_{low}^{(2)}(t) + u_{high}^{(2)}(t)$$

for $t > T_{lwp}$. 
Hamiltonian Increment: $\phi_{\text{low}}(0) \longmapsto u_{\text{low}}^{(2)}(T_{lwp})$

The Hamiltonian increment due to $w(T_{lwp})$ being added to low frequency evolution can be calculated. Indeed, by Taylor expansion, using the bound (SMOOTH!) and energy conservation of $u_{\text{low}}$ evolution, we have using

$$H[u_{\text{low}}^{(2)}(T_{lwp})] = H[u_{\text{low}}(0)] + (H[u_{\text{low}}(T_{lwp}) + w(T_{lwp})] - H[u_{\text{low}}(T_{lwp})] \sim N^{2(1-s)} + N^{2-3s+} \sim N^{2(1-s)}.$$ 

Moreover, we can accumulate $N^s$ increments of size $N^{2-3s+}$ before we double the size $N^{2(1-s)}$ of the Hamiltonian. During the iteration, Hamiltonian of “low frequency” pieces remains of size $\lesssim N^{2(1-s)}$ so the LWP steps are of uniform size $N^{-2(1-s)}$. We advance the solution on a time interval of size:

$$N^s N^{-2(1-s)} = N^{-2+3s}.$$ 

For $s > \frac{2}{3}$, we can choose $N$ to go past target time $T$. 
How do we prove (SMOOTH!)?

Bourgain’s Bilinear Strichartz Estimate: For (dyadic) $N \leq L$

$$\|e^{it\Delta} f_L e^{it\Delta} g_N \|_{L^2_{t,x}} \leq \frac{N^{\frac{2-1}{2}}}{L^{\frac{1}{2}}} \|f_L\|_{L^2_x} \|g_N\|_{L^2_x}.$$

**Corollary**

For $s \geq \frac{1}{2}$

$$\|D_x^s(u_1 u_2)\|_{L^2_{[0,\delta],x}} \leq C(\|u_1\|_{X^{s,1/2+}_{[0,\delta]}} \|u_2\|_{X^{0,1/2+}_{[0,\delta]}} + \|u_1\|_{X^{1/2,1/2+}_{[0,\delta]}} \|u_2\|_{X^{s-1/2,1/2+}_{[0,\delta]}}).$$

Thus, the Bilinear Estimate allows us move half a derivative off the high frequency part and instead onto of the low frequency part.
Treatment of a typical term in $w$

- Using the controls we have on $u_{low}, u_{high}$ from the local theory on $[0, T_{lwp}]$, we want to prove for

$$w = \int_0^t e^{i(t-t')\Delta} |u_{low}|^2 u_{high}(t') dt'$$

that $\sup_{t\in[0,T_{lwp}]} \| \nabla w \|_{L^2} < N^{1-2S^+}$.

- By Sobolev embedding, we have

$$\| w \|_{L^\infty_{[0,T_{lwp}]} H^1} \leq \| w \|_{X^{1,1/2+}_{[0,T_{lwp}]} \chi^{1,1/2+}}.$$

- The mapping $f \mapsto \int_0^t e^{i(t-t')\Delta}$ is formally $f \mapsto (i\partial_t + \Delta)^{-1}f$ which, due to time localization, is essentially $\hat{f} \mapsto \langle \tau + |\xi|^2 \rangle \hat{f}$. It suffices to control $\| D_x |u_{low}|^2 u_{high} \|_{X^{0,-1/2+}}$. Proceed by duality....
Treatment of a typical term in $w$

\[ \| w \|_{L^\infty_{[0, T_{\text{lwp}}]}} H^1 \leq \sup_{\| g \|_{X^{0,1/2-}} \leq 1} \langle g, D_x (|u_{\text{low}}|^2 u_{\text{high}}) \rangle. \]

\[ \lesssim \sup_g \langle g D_x u_{\text{low}}, u_{\text{low}} u_{\text{high}} \rangle + \sup_g \langle g u_{\text{low}}, D_x (u_{\text{low}} u_{\text{high}}) \rangle \]

\[ = \text{easier} + \sup_g \langle D_x^{1/2} (g u_{\text{low}}), D_x^{1/2} (u_{\text{low}} u_{\text{high}}) \rangle. \]

The corollary and the available bounds then give (SMOOTH!).
3. Bourgain’s Bilinear Strichartz Estimate
3. Bourgain’s Bilinear Strichartz Estimate

- Recall the Strichartz estimate for \((i\partial_t + \Delta)\) on \(\mathbb{R}^2\):
  \[
  \| e^{it\Delta} u_0 \|_{L^4(\mathbb{R}_t \times \mathbb{R}^2_x)} \leq C \| u_0 \|_{L^2(\mathbb{R}^2_x)}.
  \]

- We can view this trivially as a bilinear estimate by writing
  \[
  \| e^{it\Delta} u_0 \ e^{it\Delta} v_0 \|_{L^2(\mathbb{R}_t \times \mathbb{R}^2_x)} \leq C \| u_0 \|_{L^2(\mathbb{R}^2_x)} \| v_0 \|_{L^2(\mathbb{R}^2_x)}.
  \]

- Bourgain refined this trivial bilinear estimate for functions having certain Fourier support properties.
Bourgain’s Bilinear Strichartz Estimate

**Theorem**

For (dyadic) $N \leq L$ and for $x \in \mathbb{R}^2$,

$$\| e^{it\Delta} f_L e^{it\Delta} g_N \|_{L^2_{t,x}} \leq \frac{N^{1/2}}{L^{1/2}} \| f_L \|_{L^2_x} \| g_N \|_{L^2_x}.$$

- Here $\text{spt} (\hat{f}_L) \subset \{ |\xi| \sim L \}$, $g_N$ similar.
- Observe that $\sqrt{\frac{N}{L}} \ll 1$ when $N \ll L$. 

Shrinks Constant
3. Bourgain’s Proof

Proof. Since the standard Strichartz inequality yields (112) without the 

\[
\left( \frac{M_1}{M_2} \right)^{\frac{1}{2}}
\]

-factor, we may assume \(M_2 \gg M_1\).

Writing

\[
(e^{it\Delta} \psi_1)(e^{it\Delta} \psi_2) = \int \hat{\psi}_1(\xi_1)\hat{\psi}_2(\xi_2) e^{i(\xi_1 + \xi_2) \cdot x + (|\xi_1|^2 + |\xi_2|^2) t} \, d\xi_1 \, d\xi_2,
\]

it follows from Parseval’s identity and Cauchy-Schwarz that

\[
\| (e^{it\Delta} \psi_1)(e^{it\Delta} \psi_2) \|_2^2 = \int d\xi d\lambda \left| \int \hat{\psi}_1(\xi_1)\hat{\psi}_2(\xi - \xi_1) \delta_0(|\xi_1|^2 + |\xi - \xi_1|^2 - \lambda) \, d\xi_1 \right|^2
\]

\[
\leq \|\psi_1\|_2^2 \|\psi_2\|_2^2 \left[ \sup_{\lambda, |\xi| \sim M_2} \text{mes}_{(1)} |\xi_1| \, |\xi_1| \sim M_1 \right.
\]

\[
\text{and } |\xi_1|^2 + |\xi - \xi_1|^2 = \lambda \left]
\]

\[
< C \frac{M_1}{M_2}.
\]
Proof Based on Change of Variables

Ideas from (Kenig-Ponce-Vega); see [C-Delort-Kenig-Staffilani].

Recall the Fourier multiplier representation of the propagator:

\[
e^{it\Delta} f(x) = c_{\pi} \int_{\mathbb{R}^2} e^{ix \cdot \xi} e^{-it|\xi|^2} \hat{f}(\xi) d\xi
\]

\[
= c_{\pi} \int_{\mathbb{R}^{1+2}} e^{i(x \cdot \xi + t\tau)} \delta_0(\tau + \|\xi\|^2) \hat{f}(\xi) d\tau d\xi.
\]

With \( f = f_L \) and \( g = g_N \), we wish to estimate

\[
\|e^{it\Delta} f \ e^{it\Delta} g\|_{L^2_{t,x}} = \| \mathcal{F}[e^{it\Delta} f \ e^{it\Delta} g]\|_{L^2_{\tau,\xi}}.
\]

Using Fourier tranform property, \( \mathcal{F}(ab) = \hat{a} \ast \hat{b} \), we find....
Fourier Manipulations; Dirac Evaluations

We wish to estimate (in $L^2_{\tau, \xi}$) the expression

$$\int \delta_0(\tau_1 + |\xi_1|^2)\hat{f}(\xi_1)\delta_0(\tau_2 + |\xi_2|^2)\hat{g}(\xi_2).$$

$\tau = \tau_1 + \tau_2$
$\xi = \xi_1 + \xi_2$

Evaluating the $\delta$ functions, we find $\tau_j = -|\xi_j|^2$, so

$$\int \hat{f}(\xi_1)\hat{g}(\xi_2)$$

$\tau = -|\xi_1|^2 - |\xi_2|^2$
$\xi = \xi_1 + \xi_2$

We proceed by duality. Let's test this against $d(\tau, \xi)$. ...
Duality Reduces Matters to Certain Integral

\[ \| e^{it\Delta} f e^{it\Delta} g \|_{L^2_{t,x}} = \sup_{\|d\|_{L^2_{\tau,\xi}} \leq 1} \left\langle d(\tau, \xi), \int \hat{f}(\xi_1) \hat{g}(\xi_2) \right\rangle. \]

\[ \tau = -|\xi_1|^2 - |\xi_2|^2 \]
\[ \xi = \xi_1 + \xi_2 \]

\[ = \sup_d \int d(-|\xi_1|^2 - |\xi_2|^2, \xi_1 + \xi_2) \hat{f}(\xi_1) \hat{g}(\xi_2) d\xi_1 d\xi_2. \]

The preceding Fourier manipulations have reduced matters to bounding a certain integral. Our task is to show the integral above is bounded by

\[ \lesssim \sqrt{\frac{N}{L}} \| f \|_{L^2} \| g \|_{L^2} \| d \|_{L^2}. \]
Setting Up the Change of Variables

Let’s define a change of variables motivated by the arguments of $d$:

$$u = -|\xi_1|^2 - |\xi_2|^2, \quad v = \xi_1 + \xi_2.$$ 

- Note that $u \in \mathbb{R}$ and $v \in \mathbb{R}^2$. Thus, $dudv$ is a measure in 3d while $d\xi_1 d\xi_2$ is a measure in 4d.
- Note also that $\xi_2$ is the argument of $g = g_N$ so it is localized to the smaller dyadic shell $|\xi_2| \sim N \ll L$.
- Let’s denote the components of $\xi_j \in \mathbb{R}^2$ with superscripts:

$$\xi_j = (\xi_j^1, \xi_j^2).$$

- The full change of variables is the defined via

$$dudv \ d\xi_2 = |J| \ d\xi_1^1 d\xi_2^2 d\xi_2^1 \ d\xi_1^1.$$ 

We have an extra variable outside the changed integral.
The Jacobian

The Jacobian matrix $J$ is calculated as

$$J = \begin{bmatrix}
\frac{\partial u}{\partial \xi_1} & \frac{\partial v^1}{\partial \xi_1} & \frac{\partial v^2}{\partial \xi_1} \\
\frac{\partial u}{\partial \xi_2} & \frac{\partial v^1}{\partial \xi_2} & \frac{\partial v^2}{\partial \xi_2} \\
\frac{\partial u}{\partial \xi_1} & \frac{\partial v^1}{\partial \xi_1} & \frac{\partial v^2}{\partial \xi_1} \\
\frac{\partial u}{\partial \xi_2} & \frac{\partial v^1}{\partial \xi_2} & \frac{\partial v^2}{\partial \xi_2}
\end{bmatrix}. $$

The explicit forms for $u, v$ permit calculating

$$|J| = 2|\xi^1_1 - \xi^2_1|. $$

Since $|\xi_1| \sim L$, we may assume by rotation that $|J| \sim L$. 
Our task: Estimate, for $|\xi_1| \sim L$, $|\xi_2| \sim N$, the integral

$$\int d(-|\xi_1|^2 - |\xi_2|^2, \xi_1 + \xi_2) \hat{f}(\xi_1)\hat{g}(\xi_2) d\xi_1^1 d\xi_2^1 d\xi_1^2 d\xi_2^2.$$ 

We insert the Jacobian and reexpress inner integration as

$$\int d(-|\xi_1|^2 - |\xi_2|^2, \xi_1 + \xi_2) \frac{\hat{f}(\xi_1)\hat{g}(\xi_2)}{|J|} |J| d\xi_1^1 d\xi_2^1 d\xi_1^2 d\xi_2^2.$$ 

Changing variables, we observe this equals

$$\int d(u, v) H(u, v; \xi_2^2) |J| dudv$$

where

$$H(u, v; \xi_2^2) = \frac{\hat{f}(\xi_1)\hat{g}(\xi_2)}{|J|}.$$
We apply Cauchy-Schwarz in \( u, v \) to bound by

\[
\|d\|_{L^2} \left( \int_{u,v} |H(u, v; \xi_2^2)|^2 dudv \right)^{1/2}.
\]

We drop \( \|d\|_{L^2} \leq 1 \) by duality and change variables back. We get

\[
\left( \int_{\xi_1, \xi_2} \left| \frac{\hat{f}(\xi_1)\hat{g}(\xi_2)}{|J|} \right|^2 |J| d\xi_1 d\xi_2 d\xi_1 \right)^{1/2}.
\]

One factor of the Jacobian denominator remains! We gain \( L^{-1/2} \). We still have the extra outside integration....
Recalling what we must control, using what we have obtained....

\[
\int_{|\xi_2^2| \lesssim N} \left( \int_{\xi_1, \xi_2} \left| \frac{\hat{f}(\xi_1)\hat{g}(\xi_2)}{|J|} \right|^2 |J| d\xi_1 d\xi_2 \right)^{1/2} \, d\xi_2.
\]

Apply Cauchy-Schwarz in $\xi_2^2$ and pay the penalty in the numerator of $N^{1/2}$.

We gain over the trivial bilinear estimate by the factor

\[
\sqrt{\frac{(\text{measure of extra support})}{|J|}} = \sqrt{\frac{N}{L}}.
\]
4. The I-Method of Almost Conservation
4. The $I$-Method of Almost Conservation

Let $H^s \ni u_0 \mapsto u$ solve NLS for $t \in [0, T_{lwp}]$, $T_{lwp} \sim \|u_0\|_{H^s}^{-2/s}$.

Consider two ingredients (to be defined):

- A smoothing operator $I = I_N : H^s \mapsto H^1$. The NLS evolution $u_0 \mapsto u$ induces a smooth reference evolution $H^1 \ni lu_0 \mapsto lu$ solving $I(NLS)$ equation on $[0, T_{lwp}]$.

- A modified energy $\tilde{E}[lu]$ built using the reference evolution.

We postpone how we actually choose these objects.
**First Version of the $l$-method:** $\tilde{E} = H[lu]$

For $s < 1$, $N \gg 1$ define smooth monotone $m : \mathbb{R}_\xi^2 \to \mathbb{R}^+$ s.t.

$$m(\xi) = \begin{cases} 
1 & \text{for } |\xi| < N \\
\left(\frac{|\xi|}{N}\right)^{s-1} & \text{for } |\xi| > 2N.
\end{cases}$$

The associated Fourier multiplier operator, $\hat{(lu)}(\xi) = m(\xi)\hat{u}(\xi)$, satisfies $I : H^s \to H^1$. Note that, pointwise in time, we have

$$\|u\|_{H^s} \lesssim \|lu\|_{H^1} \lesssim N^{1-s}\|u\|_{H^s}.$$

Set $\tilde{E}[lu(t)] = H[lu(t)]$. Other choices of $\tilde{E}$ are mentioned later.
AC Law Decay and Sobolev GWP index

1. **Modified LWP.** Initial $v_0$ s.t. $\|\nabla lv_0\|_{L^2} \sim 1$ has $T_{lwp} \sim 1$.

2. **Goal.** $\forall u_0 \in H^s$, $\forall T > 0$, construct $u : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{C}$.

3. $\iff$ **Dilated Goal.** Construct $u^\lambda : [0, \lambda^2 T] \times \mathbb{R}^2 \rightarrow \mathbb{C}$.

4. **Rescale Data.** $\|I\nabla u_0^\lambda\|_{L^2} \lesssim N^{1-s} \lambda^{-s} \|u_0\|_{H^s} \sim 1$ provided we choose $\lambda = \lambda(N) \sim N^{\frac{1-s}{s}} \iff N^{1-s} \lambda^{-s} \sim 1$.

5. **Almost Conservation Law.** $\|I\nabla u(t)\|_{L^2} \lesssim H[Iu(t)]$ and

$$\sup_{t \in [0, T_{lwp}]} H[Iu(t)] \leq H[Iu(0)] + N^{-\alpha}.$$  

6. **Delay of Data Doubling.** Iterate modified LWP $N^\alpha$ steps with $T_{lwp} \sim 1$. We obtain rescaled solution for $t \in [0, N^\alpha]$.

$$\lambda^2(N)T < N^\alpha \iff T < N^\alpha + \frac{2(s-1)}{s} \quad \text{so} \quad s > \frac{2}{2 + \alpha} \quad \text{suffices.}$$
First Version of the $l$-method: $\tilde{E} = H[lu]$

A Fourier analysis established the almost conservation property of $\tilde{E} = H[lu]$ with $\alpha = \frac{3}{2}$ which led to...

**Theorem (CKSTT 02)**

$NLS_3^+(\mathbb{R}^2)$ is globally well-posed for data in $H^s(\mathbb{R}^2)$ for $\frac{4}{7} < s < 1$. Moreover, $\|u(t)\|_{H^s} \lesssim \langle t \rangle^{\beta(s)}$ for appropriate $\beta(s)$.

- The smoothing property $u(t) - e^{it\Delta}u_0 \in H^1$ is not obtained.
- Same result for $NLS_3^-(\mathbb{R}^2)$ if $\|u_0\|_{L^2} < \|Q\|_{L^2}$. Here $Q$ is the ground state (unique positive solution of $-Q + \Delta Q = -Q^3$).
- Fourier analysis leading to $\alpha = \frac{3}{2}$ in fact gives $\alpha = 2$ for most frequency interactions.
**Almost Conservation Law for $H[lu]$**

**Proposition**

Given $s > \frac{4}{7}$, $N \gg 1$, and initial data $\phi_0 \in C_0^\infty(\mathbb{R}^2)$ with $E(I_N u_0) \leq 1$, then there exists a $T_{lwp} \sim 1$ so that the solution $u(t, x) \in C([0, T_{lwp}], H^s(\mathbb{R}^2))$ of $NLS_3^+(\mathbb{R}^2)$ satisfies

$$E(I_N u)(t) = E(I_N u)(0) + O(N^{-\frac{3}{2} +}),$$

for all $t \in [0, T_{lwp}]$. 
Ideas in the Proof of Almost Conservation

- **Standard Energy Conservation Calculation:**
  \[
  \partial_t H(u) = \Re \int_{\mathbb{R}^2} \overline{u_t}(|u|^2 u - \Delta u) \, dx
  = \Re \int_{\mathbb{R}^2} \overline{u_t}(|u|^2 u - \Delta u - iu_t) \, dx = 0.
  \]

- For the smoothed reference evolution, we imitate....
  \[
  \partial_t H(\overline{u}) = \Re \int_{\mathbb{R}^2} \overline{\overline{u_t}}(|\overline{u}|^2 \overline{u} - \overline{\Delta u} - \overline{iu_t}) \, dx
  = \Re \int_{\mathbb{R}^2} \overline{\overline{u_t}}(|\overline{u}|^2 \overline{u} - \overline{I(|u|^2 u)}) \, dx \neq 0.
  \]

- The increment in modified energy involves a commutator,
  \[
  H(\overline{u})(t) - H(\overline{u})(0) = \Re \int_0^t \int_{\mathbb{R}^2} \overline{\overline{u_t}}(|\overline{u}|^2 \overline{u} - \overline{I(|u|^2 u)}) \, dx \, dt.
  \]

- Littlewood-Paley, Case-by-Case, (Bi)linear Strichartz, \(X_{s,b} \ldots \).
The almost conservation property
\[
\sup_{t \in [0, T_{lwp}]} \tilde{E} [Iu(t)] \leq \tilde{E}[Iu_0] + N^{-\alpha}
\]
leads to GWP for
\[
s > s_\alpha = \frac{2}{2 + \alpha}.
\]

The \( I \)-method is a \textit{subcritical method}. To prove the Scattering Conjecture at \( s = 0 \) via the \( I \)-method would require \( \alpha = +\infty \).

The \( I \)-method \textit{localizes the conserved density in frequency}. Similar ideas appear in recent critical scattering results.

There is a \textit{multilinear corrections algorithm} for defining other choices of \( \tilde{E} \) which yield a better AC property.