Advanced Problem Solving Seminar

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How can the Dirichlet’s Box Principle be applied to solve each of the problems below.

a) Is there any set $S \subseteq \{1,2,3,4,\ldots,2018\}$ such that $|S| = 1010$ and $\forall a, b \in S, a \nmid b$?

b) Let $P_1, P_2,\ldots, P_9$ be any 9 distinct points in $\mathbb{Z}^3$. Can we affirm that there exist $i,j \in \{1,2,3,\ldots,9\}$ such that the open segment $P_iP_j$ contains at least one point $Q \in \mathbb{Z}^3$?

c) Is it true that any sequence $s_1, s_2, \ldots, s_{2018}$ consisting of the 2018 positive integers 1, 2, ..., 2018 written in any order contains a monotonic subsequence that has at least 45 terms?
How can the Invariance Principle be applied to solve each of the problems below.

a) Let the ordered triples \((x_n, y_n, z_n)\) be recursively defined by \((x_0, y_0, z_0) = \left(\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right)\) and 

\[ \forall n \in \mathbb{N}, (x_{n+1}, y_{n+1}, z_{n+1}) = (x_n - y_n, y_n - z_n, z_n - x_n). \]

What is the value of \(\sum_{k=1}^{\infty} \frac{1}{x_k^2 + y_k^2 + z_k^2}\)?

b) A rectangular table is randomly filled with real numbers. At any stage, we are allowed to select a row or a column of the table and multiply each of its entries by \(-1\). By repeating this operation, can we always reach a stage in which the sum of all the entries on the table is non-negative?

c) To each vertex of a pentagon, a random integer \(a_i\) with \(a_1 + a_2 + \cdots + a_5 > 0\) is assigned. If for 3 consecutive vertices \((x, y, z)\) we have \(y < 0\), then we replace \((x, y, z)\) by \((x + y, -y, y + z)\).

This step is repeated as long as there is a \(y < 0\). Can we affirm that this process will always stop?
How can the Extremal Principle be applied to solve each of the problems below.

a) Is there any \((a, b, c, d) \in \mathbb{Z}^4\) for which \(a^2 + b^2 = 3(c^2 + d^2)\)?

b) Seventeen segments are given on a line. Prove that some five of the segments have a common point or some five of the segments are pairwise disjoint.

c) In a tournament, draws are not possible and every participant plays against every other participant exactly once. After the tournament, each player makes a list with the names of all the players that were beaten by him as well as the names of all the players beaten by those beaten by him. Can we affirm that one of these lists contains the names of all the players in the tournament?