GABOR FRAMES AND CONTACT GEOMETRY:
FROM MODELS OF THE PRIMARY VISUAL CORTEX TO HIGHER DIMENSIONAL SIGNAL
ANALYSIS ON MANIFOLDS

by

Vasiliki Liontou

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Abstract

This thesis has two objectives: first, to provide a model of the functional architecture of the primary visual cortex ($V_1$) in terms of both geometry and signal analysis and second to provide a mathematical framework for signal analysis on certain classes of contact manifolds. It is organized in three main parts.

Firstly, we introduce a model of the primary visual cortex ($V_1$), which allows the compression and decomposition of a signal by a discrete family of orientation and position dependent receptive profiles. We show in particular that a specific framed sampling set and an associated Gabor system is determined by the Legendrian circle bundle structure of the 3-manifold of contact elements on a surface (which models the $V_1$—cortex), together with the presence of an almost complex structure on the tangent bundle of the surface (which models the retinal surface). We identify a maximal area of the signal planes, deter-
mained by the retinal surface, that provides a finite number of receptive profiles, sufficient for good encoding and decoding. We consider the extension of this model for receptive fields dependent on position, orientation, frequency and phase.

Moreover, we provide a construction of Gabor Frames that encode local linearizations of a signal detected on a curved smooth manifold of arbitrary dimension. In particular we use Gabor Filters that can detect higher-dimensional boundaries on the manifolds. We describe an application in configuration spaces in robotics with sharp constrains. The construction is a generalization of the geometric framework, developed for the study of the visual cortex.
Finally, we present a general construction of Gabor analysis on manifolds with coorientable contact distribution, equipped with a Legendrian fibration and an almost CR-Structure. This construction is suitable for studying the stability of Gabor frames under contact transformations of the manifold. We prove that Gabor frames with a specific class of window functions are stable under a certain class of contact transformations.
To my wonderful parents...
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Chapter 1

Introduction

This doctoral thesis has dual purpose:

- the first goal is to build a model of the functional architecture of the primary visual cortex ($V_1$) in terms of both geometry and signal analysis as well as extending this model into higher dimensional spaces such as configuration spaces in robotics.

- the second goal is to prove results that explain the connection between geometry and signal analysis in these models.

For signals on manifolds, in general, there is no good construction of associated filters for signal analysis, although partial results exist involving splines discretization, diffusive wavelets, or special geometries such as spheres and conformally flat manifolds, see for instance [BK17], [EW11], [Pes04]. The geometric modelling of the primary visual cortex in terms of contact geometry and the description of receptive profiles in terms of Gabor filters suggest a general way of performing signal analysis on unit cotangent bundles of Riemannian manifolds, a specific class of contact manifolds.

The problem can be described as follows. Suppose that a signal $\mathcal{I} : B \rightarrow \mathbb{R}$ is detected on a space $B$ that is not a flat vector space but it is rather a curved manifold. This space could be the retina, which is a sphere without the point where the optic nerve is connected or a the space of configurations of a robotic arm. The filters for signal analysis, on the other hand, necessarily live on a vector space, where the convolution products happen that generate the coefficients encoding the signal. Thus, over the curved manifold one needs a consistent system of linearizations of the signal and filters that can be used to encode $\mathcal{I}$, in such a way that also accounts for the underlying geometry of $B$. The key observation to this purpose is that the combined presence of the contact structure on the Legendrian circle bundle, which is the visual cortex, and a complex structure on the base surface determine an associated bundle of framed lattices, which in turn provides the
required discrete sampling set for the Gabor frames.

**Motivation**  A recurring question in mathematical modeling in neuroscience is whether it is possible to reconstruct visual stimuli by the activity of the visual cortex ($V_1$). Alternatively stated, could one use firing neurons of the visual cortex as elementary signals to represent and reconstruct the visual stimulus? Although there is a variety of methods in signal analysis that answer this question in $\mathbb{R}^n$, they cannot be applied to describe the activity of the visual cortex which has a more complicated geometry. More specifically, the neurons of visual cortex are arranged in columns over the points of the retina they are connected to, forming a fiber bundle over $\mathbb{R}^2$. For instance, orientation sensitive neurons are arranged into a circle bundle over the retina, parameterized by the angle $\theta \in S^1$ which they fire to. Thus, the need for the construction of a signal analysis framework on manifolds, and in particular on fiber bundles, arises.

The answer to this problem is suggested by the functional architecture of $V_1$. Since the 1980s, two seemingly unrelated mathematical models have been used to describe the functional architecture of $V_1$. The first model describes the connectivity between neurons of the visual cortex, organized in hypercolumns over the retina. This connectivity is dictated by the sensitivity of neurons in specific orientations and is mathematically modeled by a co-oriented contact structure, namely a distribution of tangent hyperplanes $\xi \subset TM$, defined as the kernel of a differential 1-form $a$ on $M$ which is maximally non-degenerate, meaning

$$a \wedge (da)^n \neq 0$$

or equivalently that the 2-form $da$ is non-degenerate on $\xi$. Considering $V_1$ as a fiber bundle equipped with a contact structure describes both its hypercolumnar structure and the connections between neurons of different columns ([Hof89],[PT99],[SCP02]).

On the other hand, in order to reconstruct a visual signal out of the activity of the visual cortex, it is necessary to represent it using a discrete set of simple cells. As argued by Daugman in 1985 ([Dau85]), simple cells of $V_1$ try localize at the same time the position $(x, y)$ and the frequency of $(\eta_1, \eta_2)$ of a signal detected on the retina. However, it is impossible to detect both position and frequency of a signal with arbitrary precision due to space-frequency uncertainty principle. To solve this problem of signal analysis, Dennis Gabor (1946) suggested a method to represent signals in space as a signal with spatial and frequency variables. In particular, Gabor proposed to expand a signal function $I : \mathbb{R}^2 \rightarrow \mathbb{R}$ using space and frequency shifts of the same mother function, the Gaussian

$$g(x, y) = e^{-\pi |(x,y)|^2}.$$
The composition of countable space and frequency shifts, respectively
\[ T_{(w_1, w_2)} g(x, y) = e^{-\pi |(x, y) - (w_1, w_2)|^2} \text{ and } M_{(\eta_1, \eta_2)} g(x, y) = e^{-\pi |(x, y)|^2} e^{2\pi i (\eta_1 \cdot x + \eta_2 \cdot y)} \]
creates a countable family of elementary functions
\[ \{ M_{\eta} T_{w} g : (w, \eta) \in \Lambda \text{ where } \Lambda = A\mathbb{Z}^4 \text{ and } A \in GL(4, \mathbb{R}) \} \]
which can be used to represent the signal \( I \) as a series
\[
I(x, y) = \sum_{(w, \eta) \in \Lambda} C_{(w, \eta)} M_{\eta} T_{w} g(x, y).
\] (1.0.1)

Here its coefficient \( C_{(w, \eta)} \) represents a quantum of information, associated to an area of the space-frequency plane covered by \( M_{\eta} T_{w} g \). The mother function \( g \) can be replaced by another function \( f \in L^2(\mathbb{R}^2) \), however the choice of \( g(x, y) = e^{-\pi |(x, y)|^2} \) is not arbitrary. The function \( g \) and its space-frequency shifts minimize the uncertainty principle. Hence, choosing Gabor functions to model receptive profiles of simple cortical cells describes the optimal efficiency with which simple cells process spatiotemporal information, since Gabor functions minimize the uncertainty principle (\cite{Dau80},\cite{Dau85},\cite{Mar80}). The series (1.0.1) does not necessarily converge. In order for the series to converge, the family of elementary Gabor functions \( \{ M_{\eta} T_{w} g : (w, \eta) \in \Lambda \} \) satisfies a ”weak Parseval’s identity”
\[
A ||I||^2 \leq \sum_{(w, \eta) \in \Lambda} | < I, M_{\eta} T_{w} g > |^2 \leq B ||I||^2
\] (1.0.2)
for some \( A, B > 0 \), which is called the frame condition. This condition provides a lot more freedom than the convergence requirement for Fourier series does, making Gabor representations a more suitable tool for biological models.

The same neurons in the visual cortex appear as Gabor elementary functions \( M_{w} T_{\eta} g \) and as points of a circle bundle equipped with a coorientable contact structure. Recent work of Petitot and Tondut \cite{PT99}, Citti and Sarti \cite{CS14b} and Sarti, Citti, Petitot \cite{SCP92} has shown that the simple cell’s profile shape as Gabor elementary functions and their hypercolumnar arrangement together with their connectivity with respect to orientation sensitivity are governed by the same action or the roto-translation group \( SE(2) \), as well as by a principle of selectivity of maximal response. Thus, the co-existence of both structures is not coincidental and the signal analysis properties of \( V_1 \) are determined by its geometric structure as a contact fiber bundle. This observation leads to the question: can we use the visual cortex as motivation to examine which classes of contact manifolds carry a Gabor analysis framework determined by their geometry? In the following paragraphs a signal analysis framework for unit cotangent bundles of Riemannian manifolds \((B, g)\) is shortly
presented. Firstly, such a framework for the 3-dimensional Legendrian fiber bundle underlying the model of $V_1$ is summarized. Finally, the main points of the generalization of this framework on the unit cotangent bundles of arbitrary dimension are listed and the boundary detection property of the filters in use is highlighted.

**Main results**  In previous models of the visual cortex, where the retina is considered to be $\mathbb{R}^2$, a signal is a function from the retina that takes values in $\mathbb{R}$. The signal is integrated against Gabor elementary functions, namely the receptive profiles, which depend on the point of the retina where the signal is received and the orientation sensitivity of the neuron processing the signal. When taking into account that the base manifold $B$, where the signal is detected, is non-flat, one must distinguish the parametrizing variables for position $(x, y) \in B$, which are points on the base of the fiber bundle $\pi : S(T^*B) \to B$ and the domain of the signal, which is the tangent space $T_{(x,y)}B$ of the surface $B$ at the point where the signal is received. Thus the retinal signal, denoted as $I$, is a collection of compatible signals on planes $T_{(x,y)}B$ parametrized by $S(T^*B)$. We consider $I$ to be an $\mathbb{R}$-valued function from the pull-pack bundle of $TB$ through the projection $p : S(T^*B) \to B$ denoted by $E$, namely $E := p^*(TB)$. Additionally, $I$ is square integrable with respect to the measure given by the volume form of $S(T^*B)$ and the norm on the fibers of $E$ induced by the inner product on $TB$ through the pullback map. For a compact manifold $B$, the exponential map $\exp_g : TB \to B$, induced by the metric $g$, from $TB$ to $B$ allows for a comparison between the description of signals in terms of the linear variables of $TB$ and the nonlinear variables of $B$. The linear variables of $TB$ are the ones to which the Gabor filter analysis applies. Let $f : B \to \mathbb{R}$ be a function in $L^\infty(B)$ with the measure given by the volume form of the Riemannian metric. Then $f$ determines a signal $I(f) \in L^2(E, \mathbb{R})$, with the property that $f$ can be recovered from the restrictions $I(f) \mid_{B(E_{(b,x)})}$ (Lemma 3.3.6).

One of our main observations is that the Legendrian circle bundle structure of $S(T^*B)$, together with the existence of an almost complex structure on $B$, provide a natural choice of a framed lattice (a lattice together with the choice of a basis) on the bundle $E \oplus E^\vee$ over the contact 3-manifold manifold $S(T^*B)$, where $E^\vee$ is the dual of $E$. In particular, the contact 3-manifold $S(T^*B)$ is equipped with two contact 1-forms $a$ and $a_J$. The contact 1-form $a$ is the restriction of the Liouville 1-form $\lambda$ of $T^*B$ to the bundle of unit covectors of $B$, and the 1-form $a_J$ is obtained by the action of the almost complex structure $J : TS \to TS$, $J^2 = -1$. These two contact 1-forms have the property that the circle fibers of $S(T^*B)$ are Legendrian to both, while the trajectories of the Reeb field of each contact structure are Legendrian to the other contact structure. Thus one can consider the following bundle of lattices.
The bundle of lattices
\[ \Lambda = \left( a \mathbb{Z} R_\alpha + \beta \mathbb{Z} R_\beta \right) \oplus \left( \gamma \mathbb{Z} a + \delta \mathbb{Z} a_1 \right) \]
(1.0.3)
on the fibers of \( E \oplus E^\vee \), where \( \alpha, \beta, \gamma, \delta \in \mathbb{R} \), is spanned by the contact 1-forms and their Reeb fields and it is completely determined by the geometric data of the contact 1-form, the Reeb vector field and the complex structure of the surface \( B \). This lattice together with a mother function \( \Psi_0 : E \rightarrow \mathbb{R} \) determine an associated Gabor system. We first show how to obtain an appropriate Gabor window, which has the general form of the Gabor filters considered in [SCP92] and [Dau80]. In particular, these filters consist of a Gaussian window \( g(x, y) = e^{-\pi \langle (x, y) \rangle^2} \) and a harmonic modulation \( M_{(r_1, r_2)} \) \( g = e^{2\pi i (r_1, r_2) \cdot (x, y)} g \). Because of the Fourier transform relation between functions on the tangent and cotangent bundle of a Riemannian manifold, introduced by Landsman in [Lang93], the covectors of a Riemannian manifold are often referred to as spatial frequencies. Under this lens we introduce the following window function on the vector bundle \( TB \oplus T^*B \).

**Definition 1.0.1** (=Definition 3.1.5). A window function on the bundle \( TB \oplus T^*B \) over \( B \) is a smooth real-valued function \( \Phi_0 \) of the form
\[ \Phi_0, (x, y)(V, \eta) := \exp \left( -V^i A_{(x, y)} V - i \langle \eta, V \rangle_{(x, y)} \right), \]
(1.0.4)
defined on the total space of \( TB \oplus T^*B \), where \( A \) is a smooth section of \( T^*B \otimes T^*B \) that is symmetric and positive definite as a quadratic form on the fibers of \( TB \), with the property that at all points \( (x, y) \) in each local chart \( U \) in \( B \) the matrix \( A_{(x, y)} \) has eigenvalues uniformly bounded away from zero, \( \text{Spec}(A_{(x, y)}) \subset [\lambda, \infty) \) for some \( \lambda > 0 \).

We show that the restriction of \( \Phi_0 : TB \oplus T^*B \rightarrow TB \oplus S(T^*B) \) induces a window function \( \Psi_0 \) on the total spaces of \( E \) and that this window function together with the lattice \( \Lambda \) from 1.0.3 determine a Gabor system
\[ \mathcal{G}(\Psi_0, \Lambda_{\alpha, J} \oplus \Lambda_{\alpha, J}^\vee) \]
which consists, at each point \( (x, y, \theta) \in M \) of the Gabor system
\[ \mathcal{G}(\Psi_0, (x, y, \theta), \Lambda_{\alpha, J}, (x, y, \theta) \oplus \Lambda_{\alpha, J}^\vee, (x, y, \theta)) \]
in the space \( L^2(E_{(x, y, \theta)}) \).

In general, whether a Gabor system in \( L^2(\mathbb{R}^n) \) satisfies the frame condition is a very subtle property that depends crucially on the lattice, and also depends on the choice of the window function. In the case of Gabor systems in one dimension with a Gaussian window function and lattices of the form \( \Lambda = a \mathbb{Z} + \beta \mathbb{Z} \) the frame condition can be
completely characterized as in [Lyu1992], [Seki2008] as the set \( \{(\alpha, \beta) \in \mathbb{R}_+^2 | \alpha \beta < 1\} \). When the section \( A \) of \( T^*B \otimes T^*B \) is diagonal in the basis \( \{R_a, R_{a_j}\} \) in the local charts of \( \mathcal{E} \), the Gabor systems \( \mathcal{G}(\Psi_0, \Lambda_{a,J} \oplus \Lambda_{a_j}^\vee) \) split into one-dimensional Gabor systems, leading to the following result.

**Theorem 1.0.2** (=3.1.12 and 3.1.13). The Gabor systems \( \mathcal{G}(\Psi_0, b\Lambda_{a,J} \oplus \Lambda_{a_j}^\vee) \) with scaling \( b : S(T^*B) \rightarrow (0,1) \) are frames. The Gabor systems \( \mathcal{G}(\Psi_0, \Lambda_{a,J} \oplus \Lambda_{a_j}^\vee) \) are not frames.

On the other hand, the fundamental question of Gabor analysis, is to identify, for a given window function \( \phi \) the set of lattices \( \Lambda \in \mathbb{R}^{2n} \) for which \( \mathcal{G}(\Lambda, \phi) \) is a frame and it is widely open. Some results for higher dimensional Gabor systems were obtained in [Grö11a], [GL20]. When the section \( A \) of \( T^*B \otimes T^*B \) is not diagonal in the basis of the Reeb fields, the Gabor system \( \mathcal{G}(\Psi_0, \Lambda_{a,J} \oplus \Lambda_{a_j}^\vee) \) cannot be reduced to two 1-dimensional subsystems and therefore one needs to employ methods that have been used in the study of higher dimensional Gabor systems, such as the complex analytic method of Bargmann transform.

In section 3.1.9 we investigate when the frame condition for \( \mathcal{G}(\Psi_0, \Lambda_{a,J} \oplus \Lambda_{a_j}^\vee) \) with arbitrary \( A \) is satisfied. To answer this question we generalize selected results from [Grö11b] for Gabor systems with complex lattices, to make them applicable within our specific context. For this purpose, we introduce a geometric Bargmann transform of a function \( f \in L^2(\mathcal{E}, \mathbb{C}) \) onto the global Bargmann-Fock space \( \mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee) \)

\[
\mathcal{B} : L^2(\mathcal{E}, \mathbb{C}) \rightarrow \mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)
\]

and we prove that it is an isomorphism between the two spaces. Using the geometric Bargmann transform we prove the following theorem for \( \Psi_0 \) with arbitrary \( A \).

**Theorem 1.0.3** (= Theorem 3.1.22). There exist a window function \( \Psi_0 : \mathcal{E} \rightarrow \mathbb{R} \) such that the family \( \mathcal{G}(\Psi_0, \Lambda) = \{M_\lambda T_\lambda^\vee \Psi_0 : (\lambda, \lambda^\vee) \in \Lambda\} \) is a frame.

The cortical simple cells are organized in hypercolumns, over each point \( (x, y) \) of the retina, with respect to their sensitivity on a specific value of a visual feature, which is not necessarily the orientation. These features include color, spatial frequency, etc. In this context, the hypercolumnar architecture of \( V_1 \), for more than one visual feature, is modeled by a fiber bundle of dimension higher than 3 over the retina. Each visual feature considered adds one more dimension to the fibers of the bundle. Thus, for the process of signals from an extended model, which includes more features than the three-dimensional orientation-selectivity framework, it is essential that higher dimensional models have optimal signal analysis properties. In [BSC20], Baspinar, Citti and Sarti extend the orientation selective model to include spatial frequency and phase. However, we show that the lift of the window function, proposed in [SCP92] for the 3-dim model, to the 5-dimensional
contact manifold given by the contactization of the symplectization of $M$, denoted by $\mathcal{CS}(M)$, in the form proposed in [BSC20], only defines a Gabor system in a distributional sense, and cannot satisfy the frame condition even distributionally. We show that a simple modification of the proposed window function of [BSC20] restores the desired Gabor frame property and allows for good signal analysis in this higher dimensional model. We consider the bundle of signal planes to be 3-dimensional vector bundle $\tilde{\mathcal{E}}$ over the 5-dimensional $\mathcal{CS}(M)$.

**Theorem 1.0.4** ( = Theorem 3.1.29). There exists a bundle of framed lattice $\tilde{\Lambda} \subset \tilde{\mathcal{E}} \oplus \tilde{\mathcal{E}}^\vee$ such that together with the lift $\tilde{\Psi}_0$ of the window function $\Psi_0$ on $\mathcal{CS}(M)$ they form a Gabor frame $\mathcal{G}(\tilde{\Psi}_0, \tilde{\Lambda})$.

The second part of the thesis is devoted to extending the framework described above for Riemann surfaces with Riemannian metric to arbitrary closed Riemannian manifolds and addresses a broader type of question: given a signal on a curved smooth manifold of dimension $n$, provide a construction of Gabor frames that consistently encodes local linearizations (local mapping to vector spaces) with Gabor filters that are adapted to the detection of higher-dimensional boundaries. By higher dimensional boundaries we mean here $(n - 1)$-dimensional hypersurfaces inside the $n$-dimensional manifold where the signal has either a jump discontinuity or undergoes very rapid change (a smooth approximation to a jump discontinuity).

**Theorem 1.0.5** ( = Theorem 3.3.15 ). Let $B$ be an $n$-dimensional smooth compact manifold. There exists a collection of $n$ contact 1-forms on $M = S(T^*B)$ and a window function $\Psi : \pi^*(TB) \to \mathbb{R}$ which determine a Gabor system $\mathcal{G}(\Psi, \Lambda) = \{ M_{\lambda} T_{\lambda'} \Psi : (\lambda, \lambda') \in \Lambda \}$ on $B$.

The Gabor filters $M_{\lambda} T_{\lambda'} \Psi$ are especially suitable to detect $(n - 1)$-dimensional boundaries in a signal $f : B \to \mathbb{R}$. Consider a signal $f : B \to \mathbb{R}$ that is a characteristic function $f = \chi_U$ of a bounded open set $U \subset B$ with smooth boundary $\Sigma = \partial U$ given by an $(n - 1)$-dimensional smooth hypersurface $\Sigma$ in $B$. Let $\mathcal{I}(f)\varepsilon_m : \varepsilon_m \to \mathbb{R}$ denote the lifted signals on the fibers of the bundle $\mathcal{E}$ of signal spaces. The output of the signal $\mathcal{I}(f)$ after being filtered by the Gabor filter $\Psi_{(b,p)}$ where $(b, p) \in S(T^*B)$ has the following form

$$\mathcal{O}_b(f, \eta_p) := \int_{\varepsilon_{(b,p)}} \mathcal{I}(f) |\varepsilon_{(b,p)}(V) : \Psi_{(b,p)}(V) \, dV.$$ 

For fixed $(b, p) \in S(T^*B)$, the filter $\Psi_{(b,p)}$ is given by the formula

$$\Psi_{(b,p)}(V) = \exp(-V^tA_bV - i\langle \eta_p, V \rangle_b),$$

where $\eta_p$ is just the point $p \in S^{n-1} \simeq S(T^*_bB)$ seen as a cotangent vector. The output function attains its local maximum when $\eta_p \in T^*_bB \simeq T_bB$ is the normal vector to $\Sigma$ at the
point \( b \in \Sigma \). Therefore, by the method of non-maximal suppression one can locate the boundary of the signal \( f : B \to \mathbb{R} \).

Theorem 1.0.6 (= Theorem 3.3.16). For a given signal \( f : B \to \mathbb{R} \) of the form \( f = \chi_U \), with corresponding lift \( I(f) : \mathcal{E} \to \mathbb{R} \), and for fixed \( b \in \Sigma \subset B \), the output function \( O_b(\eta_p) \) has a local maximum at the normal vector \( v_b(\Sigma) \) at \( b \) to the boundary hypersurface \( \Sigma = \partial U \),

\[
\argmax_{p \in S^{n-1}} O_b(\chi_U, \eta_p) = v_b(\Sigma).
\]

Similarly as in the lower dimensional case, one can use a version of the Bargmann transform to obtain necessary and sufficient conditions for the Gabor system \( \mathcal{G}(\Psi, \Lambda) \) to be a frame.

An important property of Gabor filters in two dimensions with Gaussian window is their “sensitivity to direction”, which makes them especially useful in boundary detection in image analysis. This property together with their ability to detect higher dimensional boundaries make them good candidates for encoding and transmitting information for motion planning. Consider a mechanism, such as a robot \( R \), whose possible movements in the ambient 3-dimensional space are parameterized by a configuration space \( \mathcal{M}(R) \), which is usually describable as a manifold of some higher dimension \( N = \dim \mathcal{M}(R) \). For a general account of geometric and topological robotics see for instance [Far08], [Sel05]. The configuration space \( \mathcal{M}(R) \) describes the possibilities and the constraints on motion that are intrinsic to the mechanism itself. In addition to that, one may need to consider further constraints that come from the interaction with the environment. These constraints can be thought of as probability distributions over the manifold of configurations. In section 3.3.5 we present how Gabor filters can be used to encode and transmit motion constraints.

Lastly, in section 3.4 we present a framework for Gabor frames on contact manifolds from a different perspective. The objective of this section is to provide a more convenient approach for studying the stability of Gabor frames under contact transformations of the underlying contact manifold. The functional architecture of the visual cortex serves as the fundamental framework for Gabor frames on specific classes of contact manifolds, as discussed in Sections 3.1 and 3.3. While this model entails certain constraints imposed by the distinctive structure of the visual cortex, in section 3.4, we introduce a framework specifically designed to address the following problem:

Let \((K, \tau)\) be a contact manifold with coorientable contact distribution \( \tau \). If \( \mathcal{G}(\phi, \Lambda) \) is a Gabor system with an appropriate choice of window function \( \phi : \tau \to \mathbb{R} \) and a family of horizontal lattices (namely a section \( \Lambda \) of a lattice bundle over \( K \)),

\[
\Lambda = \sum_i \mathbb{Z} V_i \subset \tau
\]
where \( \{V_i\}_{i=1}^{2n} \) is a local frame for \( \tau \) satisfying the frame condition) for which contact transformations \( F : (K, \tau) \to (K, \tau) \) does the Gabor system \( G(F^*\phi, dF\Lambda) \) satisfy the frame condition?

An important observation is that the contact transformation \( F : (K, \tau) \to (K, \tau) \) transforms the lattice \( \Lambda_q \) at each point \( q \) of the manifold \( K \) through multiplication with the conformally symplectic matrix \( dF|_{\tau_q} \). The properties of Gabor frames in \( L^2(\mathbb{R}^n) \) under symplectic and Hamiltonian deformations have been studied by M. De Gosson and K. Gröchenig and JL Romero in [GGR16] and [Gos15] following a beneficial approach to redefine Gabor frames by employing the Heisenberg-Weyl operators, which are commonly known in the fields of harmonic analysis and quantum mechanics.

Motivated by the redefinition of Gabor frames in terms of Heisenberg-Weyl operators we introduce the notion of Gabor \( h- \) Bundles over a contact manifold \( (K, \tau) \) with coorientable contact distribution \( \tau \) equipped with a Legendrian fibration \( K \to B \). The Legendre fibration induces a Lagrangian subbundle \( L \) of the symplectic bundle \( \tau \). Given a horizontal local frame \( \{V_i\}_{i=1}^{2n} \subset \tau \) compatible with the fibration and a family of functions \( \phi_b : L_b \to \mathbb{R} \) varying smoothly in \( b \) a Gabor \( h- \) Bundle is the set of functions

\[
G_\tau(\phi, \Lambda) := \bigsqcup_{b \in B} \{ T^{h(b)}(\lambda)\phi_b : \lambda \in \Lambda_b \subset \tau_q \}. \tag{1.0.5}
\]

We present one possible construction of a geometric setting for Gabor frames on a compact, connected contact manifold \( (K, \tau) \) with coorientable contact distribution \( \tau \) equipped with a Legendre fibration \( K \to B \). The construction consists of taking the fibers of \( p_*(\tau) \subset TB \) as “position variables” and the fibers of \( TL \subset \tau \) as the associated momenta. More precisely, one can take the restriction of the window function \( \phi : TB \to \mathbb{R} \) to \( p_*(\tau) \) and the pullback \( \psi = p^*(\phi|_{p_*(\tau)}) \) as the resulting window function on \( \tau \). Finally for the lattice we consider a local frame \( \ell = \{\ell_1, ..., \ell_n\} \) of the Legendrian bundle \( L = ker(dp) \) and using the almost CR-structure \( J : \tau \to \tau \) compatible with the conformal symplectic structure of \( \tau \).

The construction consists of taking the fibers of \( p_*(\tau) \subset TB \) as “position variables” and the fibers of \( TL \subset \tau \) as the associated momenta. More precisely, one can take the restriction of the window function \( \phi : TB \to \mathbb{R} \) to \( p_*(\tau) \) and the pullback \( \psi = p^*(\phi|_{p_*(\tau)}) \) as the resulting window function on \( \tau \). Finally for the lattice we consider a local frame \( \ell = \{\ell_1, ..., \ell_n\} \) of the Legendrian bundle \( L = ker(dp) \) and using the almost CR-structure \( J \) we construct a bundle of framed lattices \( \Lambda \subset \tau \) of the form

\[
\Lambda_{\ell}^J = \Lambda_L \oplus \Lambda_L^\perp \tag{1.0.6}
\]

where \( \Lambda_L \) is a subset of the vertical bundle \( L \) and \( \Lambda_L^\perp \) is the orthogonal with respect to the metric compatible with \( J \) and the conformal symplectic structure of \( \tau \). In the specific instance when the window function \( \psi : JL \to \mathbb{R} \) is of the form

\[
\psi_\sigma(V) = \exp(-\pi\sigma(V, JV))
\]

where \( \sigma \) is in the conformal symplectic structure of \( \tau \), we show prove the following
equivalence relation between Gabor frames over $K$.

**Theorem 1.0.7** (= Theorem 3.4.22). The Gabor System Bundle $\mathcal{G}(\psi_\sigma, \Lambda^J)$ is a Gabor–$\hbar$–frame bundle if and only if the Gabor System Bundle $\mathcal{G}(\mathcal{J}^*_\mathcal{J}^\psi, \psi_\sigma, \mathcal{J}(\Lambda^J))$ is a frame bundle, where $\mathcal{J}^*_\mathcal{J}^\psi$ is pull-back of $\psi_\sigma$ under the bundle map

$$\mathcal{J}^*_\mathcal{J}^\psi : \mathcal{J}L \to L.$$ 

While the preceding construction provides a Gabor system convenient for establishing equivalence relations between different Gabor frames, this is not yet the one that we need for signal analysis, generalizing the case of the visual cortex to higher dimensions. In the last part of Chapter 3, we present a modification of the construction above which accommodates the necessary properties and serves as an analogue of the Gabor systems of Sections 3.1 and 3.3.

**Outline** The structure of the thesis is the following. In chapter 2 an introduction to the main notions and theorems of Gabor analysis and contact geometry that are used throughout the thesis, as well as a review of the models of the visual cortex are displayed. The aspects of Gabor analysis and contact geometry that are related to the models of the visual cortex are highlighted. In chapter 3 one can find the main results presented in three sections. In section 3.1 the results of [LM23] are presented In section 3.2 one can find selected results and formulas from section 3 on flat tori and on the hyperbolic half plane. Section 3.3 is a generalization of the geometric set up, developed for the study of the visual cortex in 3.1. A construction of Gabor Frames that encode local linearizations of a signal detected on a curved smooth manifold of arbitrary dimension is presented. In section 3.4, one can find a general construction of Gabor frames over manifolds with coorientable contact structure, a Legendrian fibration and an almost CR-structure. Finally, chapter 4 includes a description of the Fourier transform for tangent and cotangent bundles which serves as an inspiration for the construction in Chapter 3 as well as a selection of definitions and statements that have been used.
Chapter 2

Preliminaries

2.1 Time-Frequency Analysis

The purpose of this section is to present the essential definitions and notation from frame theory that are used in chapter 3 as well as some selected results about Gabor frames that highlight the properties that make them a good choice for models of biological systems. The resources used for composing this section are Gröchenig’s book [Grö01] as well as the survey papers of Christensen [Chr01] and Heil [Hei07]. The original references are cited when possible.

2.1.1 Frame Theory

Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle -, - \rangle$.

**Definition 2.1.1.** A family of elements $\{f_n\}_{n \in \mathbb{N}}$ is a frame if there exist positive constants $A$ and $B$ such that

$$A \|f\|_\mathcal{H}^2 \leq \sum_{n \in \mathbb{N}} |\langle f, f_n \rangle|^2 \leq B \|f\|_\mathcal{H}^2, \forall f \in \mathcal{H}$$

(2.1.1)

The constants $A$ and $B$ are called frame bounds and they are not unique. The optimal frame bounds are $A^{\text{op}} := \sup_A \{A > 0 : A \text{ lower frame bound of } \{f_n\}_{n \in \mathbb{N}}\}$ and $B^{\text{op}} := \inf_B \{B > 0 : B \text{ upper frame bound of } \{f_n\}_{n \in \mathbb{N}}\}$.

The two inequalities in the frame condition ensure that both the encoding map that stores information about signal $f$ into the coefficients $c_n := \langle f, f_n \rangle$ for $n \in \mathbb{N}$ and the decoding map that reconstructs the signal from these coefficients are bounded linear operators. The lower bound of 2.1.1 implies that the sequence $\{f_n\}_{n \in \mathbb{N}}$ is complete and therefore the coefficients $\langle f, f_n \rangle$ uniquely determine $f$. This ensures good encoding and
decoding, even though frames do not form an orthonormal basis, unlike in Fourier analysis.

In fact, a frame can be overcomplete, meaning that we can remove an arbitrary $f_{n_0}$ and the sequence $\{f_n\}_{n \neq n_0}$ will still be complete. The elements $f_n$ are not linearly independent, hence they do not form an orthonormal basis. However orthonormal bases are special cases of frames.

Example 2.1.2. An orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ satisfies Parseval’s identity

$$\sum_{n \in \mathbb{N}} |\langle e_n, f \rangle|^2 = \|f\|^2, \quad \forall f \in \mathcal{H}$$

and therefore it is a frame with optimal frame bounds $A = B = 1$.

When a family $\{f_n\}_{n \in \mathbb{N}}$ attains the upper bound of (2.1.1), it is called a Bessel sequence. If a frame $\{f_n\}_{n \in \mathbb{N}}$ does not satisfy the inequalities (2.1.1) after an arbitrary element $f_{n_0}$ is removed, then it is called an exact frame. In the particular case of an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$, when an arbitrary element $e_{n_0}$ is removed, the upper bound continues to exist

$$\sum_{n \neq n_0} |\langle e_n, f \rangle|^2 \leq \|f\|^2.$$ 

On the contrary, the lower bound ceases to exist and the orthonormal set $\{e_n\}_{n \neq n_0}$ is not complete, namely there exists some non-zero $f \in \mathcal{H}$ such that $\langle e_n, f \rangle = 0$ for every $n \neq n_0$.

Definition 2.1.3. For a frame $\{f_n\}_{n \in \mathbb{N}}$ in $\mathcal{H}$, the operator

$$T : \{e_n\} \mapsto \sum_{n \in \mathbb{N}} e_n f_n$$

from $\ell^2$ into $\mathcal{H}$ is called the pre-frame operator.

Remark 2.1.4. The operator (2.1.2) is well-defined if and only if $\{f_n\}$ is a Bessel sequence, namely exactly when the upper bound of (2.1.1) exists. From the Banach-Steinhaus theorem it follows that $T$ is also bounded, if $T$ is well-defined in $\ell^2$. The adjoint is given by the coefficient operator

$$T^* : f \mapsto \{\langle f, f_n \rangle\}_{n \in \mathbb{N}}.$$ 

Theorem 2.1.5 (Frame Decomposition). Given a frame $\{f_n\}_{n \in \mathbb{N}}$ with pre-frame operator $T$ and adjoint $T^*$, the operator

$$S : \mathcal{H} \to \mathcal{H}, \quad S f = TT^* f = \sum_{n \in \mathbb{N}} \langle f, f_n \rangle f_n$$
is bounded, positive and surjective. Additionally, every \( f \in \mathcal{H} \) can be represented as

\[
f = SS^{-1}f = \sum_{n \in \mathbb{N}} (f, S^{-1}f_n) f_n
\]  

(2.1.3)

For a proof of this theorem we refer the interested reader to [HW89].

### 2.1.2 Gabor Frames

In 1946, D. Gabor, in “Theory of Signal communication” ([Gab46]), stressed the utility of Gaussian wave packets

\[
\phi_{a,b}^c(t) = 2^{1/4}c^{1/2}e^{2\pi ia t}e^{-\pi c^2(t-b)^2}
\]

as building blocks for signals, namely elements of \( L^2(\mathbb{R}) \). The wave packets \( \phi_{a,b}^c \) can be obtained by time translations and modulations of the Gaussian \( \phi_0^c = e^{-\pi c^2 t^2} \). For every \((x, w) \in \mathbb{R}^2\) the operators

\[
T_x \phi_0^c(t) := \phi_0^c(t-x) \quad \text{and} \quad M_w \phi_0^c(t) := e^{2\pi i wt} \phi_0^c(t)
\]

translate the signal on the time-domain (here represented by \( x \)) and translate the signal on the frequency domain (here represented by \( w \)), respectively. The operators \( T_x M_w \), called time-frequency shifts, which result from the composition of translations and modulations are isometries on \( L^p \) for each \( 1 \leq p \leq \infty \) and satisfy the following commutation relations

\[
T_x M_w = e^{-2\pi iwx} M_w T_x.
\]  

(2.1.4)

Furthermore, the translation and modulation operators are dual to each other with respect to the Fourier transform \( \mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \),

\[
\mathcal{F}(T_x f) = M_{-x} \mathcal{F}(f) \quad \text{and} \quad \mathcal{F}(M_w f) = T_w \mathcal{F}(f)
\]  

(2.1.5)

These formulas explain why the modulation operators are called frequency-shifts and lead to the following formula

\[
\mathcal{F}(T_x M_w f) = e^{-2\pi iwx} T_w M_{-x} \mathcal{F} f
\]  

(2.1.6)

Gabor suggested to consider the time-frequency domain \( \mathbb{R}^2 \) as the union of countably many rectangles of side length \( c \) and \( c^{-1} \) centered at points \((kc, ℓc^{-1})\), \( k, ℓ \in \mathbb{Z} \). To every such rectangle he associated a function \( M_w T_x \phi_0^c \), which has the property of minimizing both the essential support in time and the essential bandwidth, simultaneously. More specifically, time-frequency shifts of the Gaussian \( \phi_0^c \) minimize the uncertainty principle.
**Theorem 2.1.6 (Uncertainty Principle).** If \( f \in L^2(\mathbb{R}) \),

\[
\left( \int_{-\infty}^{+\infty} (t - x)^2 \mid f(t) \mid^2 \, dt \right)^{1/2} \left( \int_{-\infty}^{+\infty} (y - w)^2 \mid f(y) \mid^2 \, dy \right)^{1/2} \geq \frac{1}{4\pi} \| f \|^2. \tag{2.1.7}
\]

The equality is attained if and only if \( f \) is a multiple of \( M_w T_x \phi_c^0 \) for some \((x, w) \in \mathbb{R}^2\) for some \( c > 0 \).

The previous theorem is the classical uncertainty principle in dimension \( n = 1 \), which is often referred to as the Heisenberg-Pauli-Weyl inequality. For a more detailed discussion on the uncertainty principle and its connections with Gabor analysis we refer the interested reader to [Grö01].

The uncertainty principle implies that each \( M_w T_x \phi_c^0 \) represents a "quantum of information" over a rectangle of area 1 and every \( f \in L^2(\mathbb{R}) \) can be expanded to a superposition of these elementary signals. However, the expansion with respect to \( M_w T_x \phi_c^0 \) is not an orthonormal expansion like the Fourier series so one needs to use frame theory which leads to the following definition.

**Definition 2.1.7.** Given a non-zero function \( g \in L^2(\mathbb{R}^n) \) and parameters \( a, b > 0 \) the set of time-frequency shifts

\[
\mathcal{G}(g, a, b) = \{ T_x M_w g : (x, w) \in \Lambda = a \mathbb{Z}^n \times b \mathbb{Z}^n \}
\]

is called a **Gabor System**. The function \( g \) is called the **window function** and \( \Lambda \) the **lattice** of the system. If \( \mathcal{G}(g, \Lambda) \) satisfied the frame condition (2.1.1) it is called a **Gabor frame** and the operator

\[
Sf = \sum_{(x, w) \in \Lambda} \langle f, T_x M_w g \rangle T_x M_w g.
\tag{2.1.9}
\]

is called the **frame operator**.

The convergence of the frame operator is unconditional, namely the rearrangement of the indices \( x \) and \( w \) or interchanging the summation with the action of a linear operator is permitted.

**Definition 2.1.8.** Let \( \{ e_i : i \in I \} \) be a countable set in a Banach space \( B \). The series \( \sum_{i \in I} e_i \) is said to converge **unconditionally** to \( f \in B \) if for every \( \epsilon > 0 \) there exists a finite set \( F_0 \subset I \) such that

\[
\| \sum_{i \in F} e_i - f \|_B < \epsilon, \quad \forall F \subset F_0 \text{ finite}.
\]

Unconditional convergence together with the fact that a Gabor frame remains a frame
after removing an at least finite elements ensure that the decomposition of a signal $f$ using Gabor functions is "stable".

**Proposition 2.1.9.** [Grö01] If $G(g,a,b)$ is a frame for $L^2(\mathbb{R}^n)$, every $f \in L^2(\mathbb{R}^n)$ possesses expansions

$$f = \sum_{(x,w) \in \Lambda} \langle f, T_x M_w g \rangle S^{-1}(T_x M_w g) = \sum_{(x,w) \in \Lambda} \langle f, S^{-1}(T_x M_w g) \rangle (T_x M_w g)$$

which converge unconditionally on $L^2(\mathbb{R}^n)$. Additionally, the frame operator $S^{-1}$ commutes with the time-frequency shifts $T_x M_w$, and therefore $S^{-1}(T_x M_w g)$ is a Gabor frame with window $S^{-1}g$.

**Density Theorems**

Whether a Gabor System $G(g,\Lambda)$ is a frame depends, in general, both on the choice of the window function $g$ and the discrete set $\Lambda$. Window functions are typically assumed to have a Gaussian shape. It is in general an interesting and highly nontrivial problem of signal analysis to characterize the lattices $\Lambda$ for which the frame condition (2.1.1) holds, for a given choice of window function, see [Grö11a] for more details.

The Density Theorem for Gabor Frames states the essential conditions (and for some specific cases the sufficient conditions) for a Gabor system to be a frame. In particular, the Density theorem states that necessary conditions can be formulated in terms of the set $\Lambda$, independently of the window $g$. Roughly, an appropriate index set $\Lambda$ should not be "too sparse" or locally "too dense". The purpose of this paragraph is to present the versions of the Density Theorem that are used through this thesis. The main source is the survey "History and Evolution of the Density Theorem for Gabor Frames" ([Hei07]).

The first version of the Density Theorem provides a necessary condition for a Gabor system with a lattice of the form $a\mathbb{Z} \times b\mathbb{Z}$ for some $a, b \in \mathbb{R}^*$ to be a frame. These lattices are called rectangular lattices. In particular, when the window function is Gaussian, the necessary condition is also sufficient.

**Theorem 2.1.10 (Rectangular Lattice $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$).** Let $G(g,\Lambda)$ be a Gabor system with $g \in L^2(\mathbb{R})$ and rectangular lattice $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ for some $a, b \in \mathbb{R}^*$.

1. If $G(g,\Lambda)$ is a frame for $L^2(\mathbb{R})$, then $0 < ab < 1$.

2. For the Gaussian function $g(t) = 2^{1/4}e^{-\pi t^2}$, the system $G(g,\Lambda)$ is a frame if and only if $0 < ab < 1$.

The first part of the theorem was proved by Jessen in [Jan94b]. The second part is proved by Lyubarskii in [Lyu92] and by Seip and Wallsten in [Sei92b].
Although Theorem 2.1.10 can be generalized to Gabor systems $\mathcal{G}(g, \Lambda)$ with a rectangular lattice of the form $\Lambda = a\mathbb{Z}^n \times b\mathbb{Z}^n$ for arbitrary $n$, establishing a sufficient condition is significantly more complicated in dimensions higher than 1, even for Gaussian windows of the form $g(t) = 2^{d/4}e^{-\pi t^4}, t \in \mathbb{R}^n$.

Proposition 2.1.11. Let $\Phi \in L^2(\mathbb{R}^n)$ be a separable function, namely there exist $n$ functions $\Phi_i, i = 1, ..., n$ such that $\Phi(t_1, ..., t_n) = \Phi_1(t_1) \cdot ... \cdot \Phi_n(t_n)$ and let $\Lambda = \prod_{i=1}^n a_i\mathbb{Z} \times b_i\mathbb{Z}$ for $a_i, b_i \in \mathbb{R}^*$. The Gabor system $\mathcal{G}(\Phi, \Lambda)$ is a frame if and only if the Gabor system $\mathcal{G}(\Phi_i, a_i\mathbb{Z} \times b_i\mathbb{Z})$ is a frame, for every $i \in \{1, ..., n\}$.

The last proposition indicates that the products $a_i b_i$ cannot provide a sufficient condition in higher dimensions. The Density Theorem for higher dimensions is formulated in terms of the volume of the lattice.

Definition 2.1.12. For a matrix $A \in GL(2n, \mathbb{R})$ the volume of the general lattice $\Lambda = AZ^{2n}$ is defined as

$$\text{vol}(\Lambda) := \text{det}(A)$$

The Density Theorem for general lattices in higher dimensions is formulated in terms of the volume of the lattice.

Theorem 2.1.13 (General Lattice). Let $A$ be an invertible $2n \times 2n$ real matrix and let $\Lambda$ be a general lattice of the form $\Lambda = AZ^{2n}$. Then the following statements hold

1. If $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^n)$, then $0 < \text{vol}(A) < 1$.

2. If $\text{vol}(\Lambda) > 1$, then the Gabor system $\mathcal{G}(g, \Lambda)$ is incomplete and therefore it cannot be a frame.

In contrast to dimension $n = 1$, even for a Gaussian window, the frame property is no longer characterized by the volume of the lattice. However, some sufficient conditions can be given for lattices with certain symmetries such as symplectic or complex lattices. Complex lattices were introduced in 2010 by Gröchenig [Grö11b].

Definition 2.1.14. A complex lattice in $\mathbb{C}^n$ is a lattice of the form

$$\Lambda = M \bigoplus_{j=1}^n L_i$$

for some $M \in GL(n, \mathbb{C})$ and $L_i = z_1^i\mathbb{Z} + z_2^i\mathbb{Z}$ for some $z_1^i, z_2^i \in \mathbb{C}$ such that $\text{Im}(z_1^i \overline{z_2^i}) = -1$.

Finding sufficient conditions for Gabor Frames with complex lattices is equivalent to a sampling problem in Bargmann-Fock space. Restricting our interest to complex lattices
allows us to use complex variable methods and apply techniques used in dimension $1$ to Gabor Frames in higher dimensions. In particular, when the window function is a Gaussian $g(t) = e^{-\pi t^2}, t \in \mathbb{R}$ one can solve the sampling problem in the Bargmann-Fock space and “transfer” the results to $L^2(\mathbb{R}^n)$, using the Bargmann Transform.

**Definition 2.1.15.** The Bargmann transform of a function $f$ in $L^2(\mathbb{R}^n)$ is defined by

$$Bf(z) = \int_{\mathbb{R}^n} f(t)e^{2\pi it \cdot z - \pi t^2 - \frac{y^2}{2}} dt,$$  \hspace{1cm} (2.1.10)

where, for $z \in \mathbb{C}^{2n}$ we write $z = x + iw$ for some $x, w \in \mathbb{R}^n$ and $z^2 = (x + iw) \cdot (x + iw) = x \cdot x - y \cdot y + i2x \cdot w$ and $|z|^2 = x^2 + w^2$.

The Bargmann–Fock space $\mathcal{F}^2_\mathbb{R}$, consists of entire functions of $z \in \mathbb{C}^n$ with finite norm

$$\|F\|_{\mathcal{F}^2_\mathbb{R}}^2 = \int |F(z)|^2 e^{-\pi |x|^2} dz < \infty,$$  \hspace{1cm} (2.1.11)

induced by the inner product

$$\langle F, G \rangle_{\mathcal{F}^2_\mathbb{R}} = \int_{\mathbb{C}^n} F(z) \overline{G(z)} e^{-\pi |z|^2} dz.$$

Moreover, we are going to use the Bargmann-Fock space Bargmann–Fock space $\mathcal{F}^\infty_\mathbb{R}$, which is the space of all entire functions on $\mathbb{C}^n$ satisfying

$$|F(z)| \leq Ce^{-\frac{\pi |z|^2}{2}}, \text{ for all } z \in \mathbb{C}^n.$$  \hspace{1cm} (2.1.12)

**Theorem 2.1.16.** The Bargmann Transform $B$ is a unitary operator from $L^2(\mathbb{R}^n)$ onto $\mathcal{F}^2_\mathbb{R}(\mathbb{C})$.

It follows from the previous theorem that there is an equivalence between sets of sampling for $\mathcal{F}^2_\mathbb{R}$, namely discrete sets $\Lambda \subset \mathbb{C}^n$ such that there exist $A, B > 0$ for which

$$A\|f\|_{\mathcal{F}^2_\mathbb{R}} \leq \sum_{\lambda \in \Lambda} |f(z)|^2 e^{-\pi |\lambda|^2} \leq B\|f\|_{\mathcal{F}^2_\mathbb{R}}$$  \hspace{1cm} (2.1.13)

and Gabor frames in $L^2(\mathbb{R}^n)$ with Gaussian window.

**Corollary 2.1.17.** [Grö11a] Let $\Lambda$ be a lattice in $\mathbb{R}^n$ and $\phi(t) = e^{-\pi t^2}$. The following are equivalent:

1. $\Lambda$ is a set of sampling for $\mathcal{F}^2_\mathbb{R}$
2. $\mathcal{G}(\phi, \Lambda)$ is a Gabor frame.

**Irregular Gabor Systems** Finding a Gabor system with a lattice as its index set might be challenging in biological applications. Even if we perturb one point of a regular lattice
Λ = AZ^{2n}, the resulting set is no longer a lattice. For instance, such perturbations can be a result of noise. Hence, in order to use Gabor analysis for signal transmission in the visual cortex we cannot expect that the sampling sets will have a lot of structure. We consider the index set Λ of an irregular Gabor system $\mathcal{G}(g, \Lambda)$ to be uniformly discrete, namely

$$\inf \{|\mu - \lambda|: \lambda \neq \mu\} > 0,$$

and we need to replace the volume of the lattice, which provides sufficient and necessary conditions in the regular case, with a new tool called Beurling Density. Beurling density provides lower and upper bounds for the number of points of Λ inside of unit cube.

**Definition 2.1.18.** Let $Q_c(x)$ denote the cube in $\mathbb{R}^{2n}$ with side length $c \in \mathbb{R}_+$ and center $x \in \mathbb{R}^{2n}$. Let, also, $|E|$ denote the cardinality of a set $E$. For a sequence $\Lambda$ in $\mathbb{R}^{2n}$, the lower and upper Beurling densities are

$$D^-(\Lambda) = \liminf_{c \to \infty} \inf_{x \in \mathbb{R}^{2n}} \frac{|\Lambda \cap Q_c(x)|}{c^{2n}}$$

and

$$D^+(\Lambda) = \limsup_{c \to \infty} \sup_{x \in \mathbb{R}^{2n}} \frac{|\Lambda \cap Q_c(x)|}{c^{2n}}.$$

**Theorem 2.1.19** (Density theorems for irregular Gabor Systems). ([Sei92a]) Let $g \in L^2(\mathbb{R}^n)$ and $\Lambda \subset \mathbb{R}^{2n}$.

1. If the Gabor system $\mathcal{G}(g, \Lambda)$ is complete, then $0 \leq D^-(\Lambda) \leq D^+(\Lambda) \leq \infty$
2. If $\mathcal{G}(g, \Lambda)$ is a frame, then $1 \leq D^-(\Lambda) \leq D^+(\Lambda) < \infty$
3. If $g$ is a Gaussian window function and $\Lambda \subset \mathbb{R}^2$ is uniformly discrete, then the Gabor system $\mathcal{G}(g, \Lambda)$ is a frame if and only if $D^-(\Lambda) > 1$.

### 2.2 Contact Geometry

The purpose of this section is to present the definitions and notation from contact geometry that are used in models of the functional architecture of the visual cortex ([Hof89], [Pet97] and [SCP92]) as well as in chapter 3. The resources used for composing this section are [Gei08] and appendix 4 from [Arn13].

Let $M$ a smooth manifold and $TM \to M$ its tangent bundle. A smooth subbundle $H \subset TM$ of codimension 1 is called a hyperplane distribution of $M$. A hyperplane $H_m$ at $m$ together with the point $m \in M$ is called a contact element of $M$. Each $H_m$ determines a covector $a_m \in T^*M - \{0\}$ up to multiplication by a non-zero scalar,

$$H_m = \ker(a_m) = \ker(\lambda a_m), \; \lambda \neq 0.$$
Thus, for every \( m \in M \) there exists some neighbourhood \( U \subset M \) or \( m \) such that \( H_{|U} = \ker (a_{|U}) \) for some 1-form \( a \), called locally defining 1-form for \( H \). Any 1-form obtained from \( a \) through multiplication with a non-zero scalar is a locally defining 1-form.

**Definition 2.2.1.** A contact structure \( \tau \subset TM \) on a manifold \( M^{2n+1} \) is a smooth distribution of tangent hyperplanes which are maximally non-integrable, namely for any locally defining 1-form \( a \) it holds that

\[
a \wedge (da)^n \neq 0.
\]

The condition \( a \wedge (da)^n \neq 0 \) for a locally defining 1-form is equivalent to \( da \) being non-degenerate on \( \tau \) and therefore the dimension of \( M \) has to be odd. A contact structure \( \tau \) is coorientable if \( (TM/\tau)^* \) is a trivial line bundle. In this case, \( (TM/\tau)^* \) admits a global non-zero section and therefore one can define a globally defining 1-form \( a \) for \( \tau \), \( \tau = \ker (a) \), by pulling back the non-zero section of \( (TM/\tau)^* \) to \( TM^* \) through the bundle projection \( TM \to TM/\tau \).

In correspondence with Lagrangian submanifolds of symplectic manifolds, isotropic submanifolds of maximal dimension are of high importance in contact geometry.

**Definition 2.2.2.** Let \( (M^{2n+1}, \tau) \) be a contact manifold. A submanifold \( L \subseteq M \) is isotropic if \( T_q L \subset \tau_q \) for every \( q \in L \). An isotropic submanifold of dimension \( n \) is a called a Legendrian submanifold.

**Definition 2.2.3.** A Legendre Fibration of a contact manifold \( (K, \tau) \) is a fibration \( K \xrightarrow{p} B \) of \( K \) over \( B \), such that the fiber \( p^{-1}(b) \) for every \( b \in B \) is a Legendrian submanifold,

\[
T_q p^{-1}(b) \subset \tau_q, \text{ for every } q \in K \text{ and } b = p(q).
\]

The vector bundle \( L := \bigsqcup_{q \in K} T_q p^{-1}(b) \) is subbundle of \( \tau \).

A diffeomorphism \( F : K_1 \to K_2 \) between two contact manifolds \( (K_1, \tau_1) \) and \( (K_2, \tau_2) \) such that \( F_*(\tau_1) = \tau_2 \), where \( F_* \) denotes the differential \( F_* : TK_1 \to TK_2 \) is called a contactomorphism/contact transformation. Two contact manifolds \( (K_1, \tau_1) \) and \( (K_2, \tau_2) \) are contactomorphic if there exist a contactomorphism \( F : K_1 \to K_2 \).

**The manifold of oriented contact elements**

Given a manifold \( M \), the manifold of oriented contact elements of \( M \) is

\[
C_M = \{ (m, H_m) : H_m \text{ hyperplane of } T_m M \text{ equipped with an orientation } \}
\]

and it is equipped with a natural projection to \( M \),

\[
\pi_C : C_M \to M, \ (m, H_m) \mapsto m
\]
, and with a contact structure $\tau$ by the following skating condition:

$$\tau = \{ X \in T^*M : \pi_C^*(X) \in H \}.$$  

On the other hand, the cotangent sphere bundle

$$S(T^*M) = T^*M - \{0\} / \sim$$

where $(m,a) \sim (m,a')$ if and only if $a = \lambda a'$ for some $\lambda > 0$ is isomorphic to $C_M$ as a bundle over $M$. Indeed, if we denote elements of $S(T^*M)$ by $(m,[p])$, with $[p]$ being the equivalence class of $p \in T^*M - \{0\}$, the map

$$h : (m,[p]) \mapsto (m,\ker(p_m))$$

is a diffeomorphism from $S(T^*M)$ to $C_M$. Additionally, if we denote by $\pi_S$ the projection $\pi_S : S(T^*M) \to M$, then $\pi_C \circ h = \pi_S$.

If $M$ has a metric $S(T^*M)$ can be identified with the space of unit covectors. Thinking $S(T^*M)$ as the space of unit covectors of $M$, one can restrict the canonical $1$-form of $T^*M$ to $S(T^*M)$ to obtain a $1$-form whose kernel defines a contact structure on $S(T^*M)$. More specifically, let $\pi$ denote the projection $\pi : T^*M \to M$, the Liouville $1$-form on $T^*M$ is defined as

$$\lambda_{(m,p)} = d\pi^*(p).$$  \hfill (2.2.1)

Restricted to $S(T^*M)$, $\lambda$ induces a $1$-form which is locally expressed in canonical coordinates as

$$a := \lambda_{|S(T^*M)} = dm_1 + \sum_{i>1} p_i dm_i,$$  \hfill (2.2.2)

which satisfies the condition $a \wedge (da)^n = dm_1 \wedge \ldots \wedge dm_n \wedge dp_2 \wedge \ldots \wedge dp_n \neq 0$. The contact structure on $S(T^*M)$ is

$$\xi := \ker(a)$$

and $dh(\xi_{(m,[p])}) = \tau_{(m,\ker(p_m))}$, namely $(C_M, \tau)$ and $(S(T^*M), \xi)$ are contactomorphic. We are going to use the term manifold of oriented contact elements to refer to both contact manifolds, when no confusion arises.

### 2.2.1 Symplectization and contactization

Given a contact manifold $(M, \alpha)$, with $\alpha$ a given contact $1$-form, one can always form a symplectic manifold $(M \times \mathbb{R}, \omega)$ with $\omega = d(e^s \cdot \alpha)$ with $s \in \mathbb{R}$ the cylinder coordinate. Setting $w = e^s \in \mathbb{R}_+^*$, one has $\omega = dw \wedge \alpha + w \, da$ on $M \times \mathbb{R}_+^*$. In particular, the symplectization of the manifold of contact elements $S(T^*S)$ is the complement of the zero section
\[ T^*S_0 := T^*S \setminus \{0\} \] with symplectic form written in a chart \((U, z)\) of \(S\) with \(z = x + iy\) in the form
\[
\omega = dw \wedge \alpha + w \, d\alpha = du \wedge dx + dv \wedge dy
\] (2.2.3)

Given a symplectic manifold \((Y, \omega)\), if the symplectic form is exact, \(\omega = d\lambda\), then one can construct a contactization \((Y \times S^1, \alpha)\) with \(\alpha = \lambda - d\phi\), where \(\phi\) is the angle coordinate on \(S^1\). When the symplectic form is not exact, it is possible to construct a contactization if there is some \(\bar{h} > 0\) such that the differential form \(\omega/\bar{h}\) defines an integral cohomology class, \([\omega/\bar{h}] \in H^2(Y, \mathbb{Z})\). In this case there is a principal \(U(1)\)-bundle \(S\) on \(Y\) with Euler class \(e(S) = [\omega/\bar{h}]\), endowed with a connection \(\nabla\) with curvature \(\nabla^2 = \omega/\bar{h}\). This is also known as the prequantization bundle. This connection determines a \(U(1)\)-invariant 1-form \(\alpha\) on \(S\). The non-degeneracy condition for the symplectic form \(\omega\) implies the contact condition for the 1-form \(\alpha\). Different choices of the potential \(\alpha\) of the connection \(\nabla\) lead to equivalent contact manifolds up to contactomorphisms, see section 2.2 for a brief summary of symplectization and contactization.

**Lemma 2.2.4.** The contactization of the symplectization of the contact 3-manifold \(M = S(T^*S)\) is the 5-manifold \(T^*S_0 \times S^1\) with the contact form
\[
\tilde{\alpha} = \lambda - d\phi = w\alpha - d\phi.
\]

*Proof.* The symplectization of a contact manifold is an exact symplectic manifold, hence it admits a contactization in the simpler form described above. Thus, starting with the contact manifold \(M = S(T^*S)\) for a 2-dimensional compact surface \(S\), endowed with the contact form \(\alpha\) as in (3.1.1) that makes \(M\) a Legendrian circle bundle, one obtains the symplectization \(T^*S_0\) with Liouville form \(\lambda = w\alpha\), \(w = e^s \in \mathbb{R}_+\), and the contactization of the resulting exact symplectic manifold \((T^*S_0, \omega = d\lambda)\) is given by \(T^*S_0 \times S^1\) with the contact form \(\tilde{\alpha} = \lambda - d\phi = w\alpha - d\phi\).

**Remark 2.2.5.** We will use the notation
\[
S(M) := T^*S_0 \quad \text{and} \quad CS(M) := T^*S_0 \times S^1,
\] (2.2.4)
for the symplectization \(S(M)\) of \(M = S(T^*S)\) with symplectic form \(\omega = dw \wedge \alpha + \omega da\) (2.2.3), and the contactization \(CS(M)\) for this symplectization, endowed with the contact 1-form \(\tilde{\alpha} = \lambda - d\phi = w\alpha - d\phi\).
2.3 Models of the Primary Visual Cortex

The cortex is a thin surface layer of neuronal tissue that surrounds the brain, wherein much of what we see, hear and feel is perceived. The neurons are organized in three different ways, namely topographic, laminar and columnar. The topographic structure refers to the point-to-point correspondence between the retina and the visual cortex. In particular, each neuron in the visual cortex corresponds to an area of the retina where a visual stimulus can evoke a response of this neuron. This area is called **receptive field** of the neuron. The laminar structure describes the organization of neurons in layers according to their morphology. Lastly, the term columnar refers to the organization of neurons in tiny microcolumns that are transversal to the laminar structure. Simple cells of the visual cortex are piled into columns according to the position of their receptive field and their orientation response, see [HW59] and [HW62]. More specifically, a simple cell is excited by a picture if this picture is located in the area of the retina which corresponds to this cell and it has a specific orientation. In other words, any particular cortical neuron has both a **direction-field-like** response and an **areal** one. Simple cells that correspond to the same areal and orientation response form one microcolumn. In regard to this thesis, the shape of the receptive fields, their columnar organization as well as the interaction between them are the topics of interest. The works of Hoffman [Hof89], Daugmann [Dau85], [Dau80] and Sarti, Citti and Petitot [SCP92], [Pet17] address each of these topics respectively. In the following paragraphs a quick review of these works is attempted.

2.3.1 The Primary Visual Cortex is a Legendrian Fiber Bundle

A basic property of visual perception is the recognition of visual contours, which are curves that bound a perceived object. These curves are originally located on the retina and they are "lifted" to the visual cortex through the visual pathway. Motivated by this property of the visual cortex, Hoffman in [Hof89] presented a model of the primary visual cortex that would encode the orientation response and the contour lifting property. More specifically, he argued that the primary visual cortex $V_1$ is the manifold of contact elements of the retinal surface $\mathcal{R}$, $V_1 = \mathcal{C}_\mathcal{R}$, as described in 2.2, together with the projection $\pi : \mathcal{C}_\mathcal{R} \to \mathcal{R}$ which represents the point-to-point correspondence between neuron of $V_1$ and the center of its receptive field. Additionally, given a visual contour $\gamma(t) = (x(t), y(t)) \subset \mathcal{R}$ on the retina, the lift $\tilde{\gamma}(t) = (x(t), y(t), p(t))$ on the visual cortex has to satisfy the equation $p = -\frac{x}{y}$. Equivalently, the contour lifts are horizontal to the distribution of contact hyperplanes $\tau$ of (2.2) defined by the skating condition. Hence, $V_1$ is modeled by the fiber bundle $(\mathcal{C}_\mathcal{R}, \mathcal{R}, \pi)$ together with the contact structure $\tau$. The later dictates how the visual contours are per-
ceived. Finally, the fibers $\pi^{-1}(b)$ for each point $b \in \mathcal{R}$ are isomorphic to $S^1$ and each point of the fiber represents a neuron which has a receptive field centered at $b$ and reacts to a specific direction $\theta \in S^1$. These fibers are Legendrian submanifolds of the contact manifold $(C\mathcal{R}, \tau)$.

### 2.3.2 The receptive profiles of the primary visual cortex are elementary Gabor Filters

In the works of Daugman [Dau80], [Dau85] and Marcelja [Mar80], it is argued why Gabor filters are the right choice for the modeling of receptive profiles of visual neurons in $V_1$. In particular, simple cells of the primary visual cortex try to localize at the same time the position $(x, y)$ and the frequency $w$ of a signal detected in the retina. However, the uncertainty principle in signal analysis indicates that it is impossible to detect both position and frequency with arbitrary precision. Gabor filters minimize the uncertainty and therefore they process spatiotemporal information optimally. Thus, a receptive profile, centered at $(x_0, y_0)$, with preferred spatial frequency $w = \sqrt{u_0^2 + v_0^2}$ and preferred orientation $\theta = \arctan(\frac{v_0}{u_0})$ is efficiently modelled by a bivariate, real-valued Gabor function $f(x, y)$ of the form

$$
\exp(-\pi((x - x_0)^2 + (y - y_0)^2)) \exp(-2\pi i(u_0(x - x_0) + u_0(y - y_0))).
$$

Given a distribution $I(x, y)$, specifying the distribution of light intensity of a visual stimulus, the receptive profile generates the response to that distributed stimulus via integration

$$
\text{response} = \int \int_{-\infty}^{+\infty} I(x, y) f(x, y) dx dy.
$$

The integral representing the response of a receptive field is commonly used in time-frequency analysis, as short time Fourier transform. In a Euclidean space $\mathbb{R}^d$ of arbitrary dimension, the short time Fourier transform of a signal $I$ with respect to a window function $g$ is a linear and continuous, joint time-frequency representation defined as

$$
V_g I(x, w) = \int_{\mathbb{R}^d} I(t) g(t-x) e^{-2\pi i t \cdot w}, \text{ for } x, w \in \mathbb{R}^d.
$$

More specifically, in the plane $\mathbb{R}^2$, the response of a receptive profile to a visual signal $I$ is equal to the short time Fourier transform of the signal $I$ with respect to the Gaussian $g(x, y) = \exp(-\pi(x^2 + y^2))$, multiplied with a complex exponential

$$
\text{response} = e^{2\pi i(x_0 y_0 \cdot (u_0, v_0))} V_g I(x_0, y_0, u_0, v_0).
$$
2.3.3 Simple cell connectivity is determined by the shape of their receptive profiles

In [SCP92], a model of the connectivity between neurons of $V_1$ that combines the results of Hoffman and Daugman is presented and describes both co-axial and trans-axial excitatory connections. They consider the retinal surface to be $\mathbb{R}^2$ with coordinates $(x, y)$. The neuron with receptive fields centered on $(x, y)$, process the signal $\mathcal{I} : \mathbb{R}^2 \rightarrow \mathbb{R}$ with their receptive field $\Psi_{(x,y,\theta)} : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ which depends on the center of the receptive field and on the orientation sensitivity of each neuron parametrized by $\theta \in S^1$. It has been observed experimentally that the different receptive profiles are obtained by a mother receptive profile centered at $(0,0)$ with formula $\Psi_0(V, W) = \exp(-V^2 - W^2) \exp(2iW)$ via the action of the group $SE(2)$ on $\mathbb{R}^2$,

$$
\Psi_{(x,y,\theta)}(V, W) = \Psi_0(R_{\theta}(V, W) + (x, y)), \quad R_{\theta} \in SO(2).
$$

For a fixed point $(x, y)$ the receptive profiles $\{\Psi_{(x,y,\theta)} : \theta \in S^1\}$ constitute a fiber of the Legendrian fiber bundle describe in 2.3.1. When a visual stimulus is located on $(x, y)$ all the different neurons on the fiber are excited, however the ones that contribute to the lift of the stimulus from the retina to the space of parameters are those who maximize the output function

$$
\mathcal{O}_{\theta}(V, W) = \int \mathcal{I}(V, W) \Psi_{(x,y,\theta)}(V, W) dV dW.
$$

In particular, the detection of the boundary of a stimulus is performed by the odd simple cells that maximally fire when there is a discontinuity of the stimulus. The odd part of the filter $\Psi_{(x,y,\theta)}(V, W)$ is locally approximated by the Gaussian derivative $\partial_W \Psi_0$ and after integrating by parts and changing variables the output function becomes

$$
\mathcal{O}_{\theta}(V, W) = \int (-\sin(\theta) \partial_x + \cos(\theta) \partial_y) \mathcal{I}(R_{\theta}^{-1}(V, W) - (x, y)) \Psi_0 dV dW.
$$

It follows that the maximum output is obtained for $\theta_0$ such that $X = -\sin(\theta_0) \partial_x + \cos(\theta_0) \partial_y$ in the direction of $\nabla \mathcal{I}$ and therefore the boundary at $(x, y)$ lifts to a curve passing through $(x, y, \theta_0)$ which is tangent to the plane spanned by $\partial_\theta$ and $\cos(\theta) \partial_x + \sin(\theta) \partial_y$. These planes are locally described as the kernel of the contact 1-form $a = -\sin(\theta) dx + \cos(\theta) dy$. The last statement shows how the contact structure of the visual cortex is dictated by the shape of the receptive profiles of cortical cells. The vector field $\cos(\theta) \partial_x + \sin(\theta) \partial_y$ is the preferred direction of the diffusion of the visual signal, which is the direction of co-axial excitatory connections. The vector field $-\sin(\theta) \partial_x + \cos(\theta) \partial_y$ is the direction of trans-axial excitatory connections.

On the other hand the even cells, which are the real part of $\Psi_0$, are responsible for object detection. They detect both orientation and scale, which is denoted by $\sigma$. Their structure
is analogous to that of odd cells but have a symplectic instead of a contact structure.

**Theorem 2.3.1 ([SCP08]).** The output function, which now depends on the position given by \((x, y)\), the orientation parametrized by \(\theta\) and the scale \(\sigma\) reaches a local maximum at \((\theta_0, \sigma_0)\) where \(1/\sqrt{2}\sigma_0\) is the distance of \((x, y)\) from the nearest point on the boundary of the stimulus \(I\) and \(\theta_0\) is the orientation of the boundary at this point.

The space of parameters is now \(\mathbb{R}^2 \times S^1 \times \mathbb{R}_+\) which together with the 2-form \(w = d(\sigma^{-1} a)\) is the symplectization of \((\mathbb{R}^2 \times S^1, a)\).

Finally, in [BSC20] they present a model of the visual cortex orientation, frequency and phase selective behaviour of cortical cells. Each receptive profile in this context is obtained from the mother filter by rotation, frequency modulation and phase shift. The receptive profile of a simple cell is represented by the Gabor function

\[ \Psi_a(x, y, s) = e^{-i(r(x-q_1, y-q_2)-\phi(s-\phi))}e^{-|x-q_1|^2-|y-q_2|^2}, \quad a = (q_1, q_2, \phi, r, \nu) \in \mathbb{R}^6 \]

where \(w > 0\) denotes the spacial frequency, \(\theta\) the orientation and \(r = (w \cos(\theta), w \sin(\theta))\). The velocity of plane wave propagation is denoted by \(\nu\) and \(s-\phi\) is the phase centered at \(\phi\).
Chapter 3

Main Results

3.1 Gabor frames from contact geometry in models of the primary visual cortex

In this section, we introduce a model of the primary visual cortex \((V_1)\), which allows the compression and decomposition of a signal by a discrete family of orientation and position dependent receptive profiles. We show in particular that a specific framed sampling set and an associated Gabor system is determined by the Legendrian circle bundle structure of the 3-manifold of contact elements on a surface (which models the \(V_1\)-cortex), together with the presence of an almost complex structure on the tangent bundle of the surface (which models the retinal surface). We identify a maximal area of the signal planes, determined by the retinal surface, that provides a finite number of receptive profiles, sufficient for good encoding and decoding. We then consider a 5-dimensional model where receptive profiles also involve a dependence on frequency and scale variables, in addition to the dependence of position and orientation. In this case we show that the proposed window function does not give rise to frames (even in a distributional sense), while a natural modification of the same window generates Gabor frames with respect to the appropriate lattice determined by the contact geometry. This section is based upon the article [LM23].

3.1.1 Signals on manifolds of contact elements

The geometries underlying the models of [PT99], and of [Pet17], [SCP92], and the model of [BSC20], are respectively the 3-manifold \(M\) of oriented contact elements of a compact 2-dimensional Riemann surface, as described in section 2.2, or the 5-manifold given by the contactization of the symplectization of \(M\), as described in section 2.2.1.
CHAPTER 3. MAIN RESULTS

The main aspect of the geometry that will play a crucial role in our construction of the associated Gabor frames is the fact that these contact 3-manifolds are endowed with a pair of contact forms $\alpha, \alpha_J$ related through the (integrable) almost-complex structure $J$ of the tangent bundle $TS$ of the Riemann surface $S$. They have the property that the circle fibers are Legendrian for both contact forms, while the Reeb vector field of each is Legendrian for the other. This leads to a natural framing, namely a natural choice of a basis for the tangent bundle $TM$, completely determined by the contact geometry. It consists of the fiber direction $\partial_\theta$ and the two Reeb vector fields $R_\alpha, R_{\alpha_J}$.

Legendrian circle bundles

The results we discuss in this section apply, slightly more generally, to the case of a 3-manifold $M$ that is a Legendrian circle bundle over a two dimensional compact surface $S$.

The Legendrian condition means that the fiber directions $TS^1$ inside the tangent bundle $TM$ are contained in the contact planes distribution $\xi \subset TM$. Such Legendrian circle bundles over surfaces are classified, see [Lut83]. p. 179. They are all either given by the unit cosphere bundle $M = S(T^*S)$, with the contact structure induced by the natural symplectic structure on the cotangent bundle $T^*S$, or by pullbacks of the contact structure on $M$ to a $d$-fold cyclic covering $M' \to M$, that exists for $d$ dividing $2g - 2$, where $g = g(S)$ is the genus of $S$. The case of $M = S(T^*S)$ is the manifold of contact elements of $S$. In the following, we will restrict our discussion to this specific case.

In the geometric models of the $V_1$ cortex developed in [Pet17], [SCP92], [CS14a], the surface $S$ represents the retinal surface, while the fiber direction in the Legendrian circle bundle $M = S(T^*S)$ represents an additional orientation variable, which keeps track of how the tangent orientation in $TS$ of a curve in $S$ is lifted to a propagation curve in the visual cortex, where a line is represented by the envelope of its tangents rather than as a set of points.

The fibers of the sphere bundle $S(T^*S)$ are unit circles $S^1$, hence they can be seen as parametrizing direction, that is, (oriented) lines in the plane $\mathbb{R}^2 \simeq T^*_{(x,y)}S$. One can also identify the circles with copies of $\mathbb{P}^1(\mathbb{R})$ parametrizing lines in the plane. This would correspond to considering the projectivized cotangent bundle instead of the unit sphere bundle. While these two models are topologically equivalent in dimension $n = 2$, they differ when considering the sub-Riemannian geometry of the rototranslation group $SE(2)$ as model geometry for the neural connectivity of the $V_1$ cortex, as in [CS06] and [SCP92].
Liouville tautological 1-form and almost-complex twist

Given a manifold \( Y \) and its cotangent bundle \( T^*Y \), we denote the canonical Liouville 1-form (2.2.1), given in coordinates by \( \lambda = \sum_i p_i dx^i \), or intrinsically as \( \lambda_{(x,p)}(v) = p(d\pi(v)) \) for \( v \in T_xY \) and \( \pi : T^*Y \to Y \) the bundle projection. The canonical symplectic form on \( T^*Y \) is \( \omega = d\lambda \).

Given an almost complex structure \( J \) on \( Y \), namely a \((1,1)\) tensor \( J \) with \( J^2 = -1 \), written in coordinates as \( J = \sum_{k,\ell} J^k_{\ell} dx^\ell \otimes \partial_x^k \), the twist by \( J \) of the tautological Liouville 1-form on \( T^*Y \) is given by

\[
\lambda_J := \sum_{k,\ell} p_{J^k_{\ell}} dx^\ell,
\]

the 2-form \( \omega_J = d\lambda_J \) satisfies

\[
\omega_J(\cdot, \cdot) = \omega(\hat{J} \cdot, \cdot),
\]

where in local coordinates

\[
\hat{J} = \begin{pmatrix}
    J_i^j & 0 \\
    \sum_k p_k (\partial_x^i J_k^j - \partial_x^j J_k^i) & J_i^j
  \end{pmatrix},
\]

see for instance [Ber07].

In particular, in the case of a Riemann surface \( S \), with coordinates \( z = x + iy \) on \( S \) and \( p = (u,v) \) in the cotangent fiber, the tautological 1-form is locally of the form \( \lambda = u dx + v dy \), with \( \omega = du \wedge dx + dv \wedge dy \). The twisted tautological form with respect to \( J \) given by multiplication by the imaginary unit, \( J : (u,v) \mapsto (-v,u) \), given by \( \lambda_J = -v dx + u dy \), with \( \omega_J = -dv \wedge dx + du \wedge dy \).

**Proposition 3.1.1.** On the contact 3-manifold \( M_w = S_w(T^*S) \), given by the cosphere bundle of radius \( w \), consider the contact 1-form \( \alpha \), (2.2.2), induced by the tautological Liouville 1-form \( \lambda \) and the contact 1-form \( \alpha_J \) determined by the twisted \( \lambda_J \). The contact planes of these two contact structures intersect along the circle direction \( \partial_\theta \). The Reeb field \( R_\alpha \) of \( \alpha \) is Legendrian for \( \alpha_J \) and the Reeb field \( R_{\alpha_J} \) is Legendrian for \( \alpha \). The twist \( J \) fixes the \( \partial_\theta \) generator and exchanges the generators \( R_{\alpha_J} \) and \( R_\alpha \).

**Proof.** On the contact 3-manifold \( M_w = S_w(T^*S) \), given by the cosphere bundle of radius \( w \), the contact 1-form induced by the tautological Liouville 1-form \( \lambda \), written in a chart \((U,z)\) on \( S \) with local coordinate \( z = x + iy \), is given by

\[
\alpha = w \cos(\theta) dx + w \sin(\theta) dy,
\] (3.1.1)

where \((w,\theta)\) are the polar coordinates in the cotangent fibers, and the corresponding contact planes distribution on \( U \times S^1_w \) is generated by the vector fields \( \partial_\theta \) and \(- \sin(\theta) \partial_x + \)
\[
\cos(\theta)\partial_y, \text{ and with the Reeb vector field } \\
R_\alpha = w^{-1}\cos(\theta)\partial_x + w^{-1}\sin(\theta)\partial_y 
\] (3.1.2)

The contact structure on \(M_w\) induced by the twisted Liouville 1-form \(\lambda_J\) is given in the same chart \((U, z)\) by
\[
\alpha_J = -w\sin(\theta)dx + w\cos(\theta)dy 
\] (3.1.3)

with contact planes spanned by \(\partial_\theta\) and \(\cos(\theta)\partial_x + \sin(\theta)\partial_y\) and with Reeb vector field
\[
R_{\alpha_J} = -w^{-1}\sin(\theta)\partial_x + w^{-1}\cos(\theta)\partial_y. 
\] (3.1.4)

\[\square\]

### 3.1.2 The bundle of signal planes

In the model of receptive profiles in the visual cortex (see [Pet17], [SCP92]), signals are regarded as functions on the retinal surface and the receptive profiles are modelled by Gabor filters in these and dual variables. When taking into account the underlying geometric model, however, one needs to distinguish between the local variables \((x, y)\) on a chart \((U, z = x + iy)\) on the surface \(S\) (or the local variables \((x, y, \theta)\) on the 3-manifold \(M\) and the linear variables in its tangent space \(T_{(x,y)}S\). Thus, we think of the retinal signal as a collection of compatible signals in the planes \(T_{(x,y)}S\), as \((x, y)\) varies in \(S\). We consider a real 2-plane bundle on the 3-manifold \(M\) that describes this geometric space where retinal signals are mapped.

**Definition 3.1.2.** Let \(E\) be the real 2-plane bundle on the contact 3-manifold \(M = S(T^*S)\) obtained by pulling back the tangent bundle \(TS\) of the surface \(S\) to \(M\) along the projection \(\pi : S(T^*S) \to S\) of the unit sphere bundle of \(T^*S\),
\[
E = \pi^*TS. 
\] (3.1.5)

At each point \((x, y, \theta)\) \(\in M\), with \(z = x + iy\) the coordinate in a local chart \((U, z)\) of \(S\), the fiber \(E_{(x,y,\theta)}\) is the same as the fiber of the tangent bundle \(T_{(x,y)}S\). Also let \(E^\vee\) be the dual bundle of \(E\), namely the bundle of linear functionals on \(E\),
\[
E^\vee = \text{Hom}(E, \mathbb{R}).
\]

Locally the exponential map from \(TS\) to \(S\) allows for a comparison between the description of signals in terms of the linear variables of \(TS\) and the nonlinear variables of \(S\).
The linear variables of TS are the ones to which the Gabor filter analysis applies. Thus, in terms of the contact 3-manifold M, we think of a signal as a consistent family of signals on the fibers $E_{(x,y,\theta)}$, or equivalently a signal on the total space of the 2-plane bundle $E$. The filters in turn will depend on the dual linear variables of $E$ and $E^\vee$. We make this idea more precise in the next subsections. The definition and properties of the pull-back bundle are explained in more detail in section 4.2.1.

### 3.1.3 Fourier transform relation and signals

Over a compact Riemannian manifold $Y$, functions on the tangent and cotangent bundles $TY$ and $T^*Y$ are related by Fourier transform in the following way. Let $S(TY,\mathbb{R})$ denote the vector space of smooth real valued functions on $TY$ that are rapidly decaying along the fiber directions, and similarly for $S(T^*Y,\mathbb{R})$. Let $(\eta,v)_x$ denote the pairing of tangent and cotangent vectors $v \in T_x Y, \eta \in T^*_x Y$ at a point $x \in Y$. One defines

$$F : S(TY,\mathbb{R}) \rightarrow S(T^*Y,\mathbb{R})$$

$$(F \varphi)_x(\eta) = \frac{1}{(2\pi)^{\dim Y}} \int_{T^*_x Y} e^{2\pi i (\eta,x)} \varphi_x(v) \, dvol_x(v), \varphi \in S(TY,\mathbb{R})$$ (3.1.6)

with respect to the volume form on $T_x Y$ induced by the Riemannian metric.

Because of this Fourier transform relation, cotangent vectors in $T^*Y$ are sometimes referred to as “spatial frequencies”.

In the model we are considering, the manifold over which signals are defined is the total space $E$ of the bundle of signal planes introduced in section 3.1.2 above, namely real 2-plane bundle $E = \pi^* TS$. We can easily generalize the setting described above, by replacing the pair of tangent and cotangent bundle $TY$ and $T^*Y$ of a manifold $Y$ with a more general pair of a vector bundle $E$ and its dual $E^\vee$. The variables in the fibers of $E^\vee$ are the spatial frequencies variables of the models of the visual cortex of [Pet17], [SCP92]. In this geometric setting a “signal” is described as follows.

**Definition 3.1.3.** A signal is a real valued function on the total space $E$ of the bundle of signal planes, with $I \in L^2(E,\mathbb{R})$, with respect to the measure given by the volume form of $M$ and the norm on the fibers of $E$ induced by the inner product on TS through the pullback map. A smooth signal is a smooth function that decays to zero at infinity in the fiber directions, $I \in C_0^\infty(E,\mathbb{R})$.

The assumption that $I$ is smooth is quite strong, as one would like to include signals that have sharp contours and discontinuous jumps, but we can assume that such signals are smoothable by convolution with a sufficiently small mollifier function that replaces sharp contours with a steep but smoothly varying gradient.
### 3.1.4 Signal analysis and filters

For signals defined over $\mathbb{R}^n$, instead of over a more general manifold, signal analysis is performed through a family of filters (wavelets), and the signal is encoded through the coefficients obtained by integration against the filters. Under good conditions on the family of filters, such as the frame condition for Gabor analysis, both the encoding and the decoding maps are bounded operators, so the signal can be reliably recovered from its encoding through the filters.

For signals on manifolds there is in general no good construction of associated filters for signal analysis, although partial results exist involving splines discretization, diffusive wavelets, or special geometries such as spheres and conformally flat manifolds, see for instance [BK17], [EW11], [Pes04]. One of our goals here is to show that geometric modelling of the visual cortex in terms of contact geometry and the description of receptive fields in terms of Gabor frames suggest a general way of performing signal analysis on a specific class of contact manifolds.

The signal analysis model we propose in the following relies on encoding a signal $f : S \to \mathbb{R}$ that is supported on a curved Riemann surface $S$ in terms of a function defined on the total space of the 2-plane bundle $E$ over the 3-dimensional contact manifold $M = S(T^*S)$. The restrictions to the fibers of $E$ provide a collection of signals defined on 2-dimensional linear spaces, which describe the lifts of the original signal $f : S \to \mathbb{R}$ to the local linearizations of $S$ given by the fibers of the tangent bundle $TS$. The presence of the additional circle coordinate $S^1$ in the 3-manifold $M = S(T^*S)$ will account for the fact that the Gabor filters used for signal analysis, which themselves live on the liner fibers of $E$ include a directional preference specified by the angle coordinate in the fibers of $S(T^*S)$.

In terms of modelling of the visual cortex, what we are presenting in section 3.1.5 below is a functional analytic model of the lifting of signals from the (curved) retinal surface to linear spaces where the Gabor filters corresponding to the receptive fields of the $V_1$ neurons act to encode the signal. In particular, as we discuss in section 3.1.6 below, we will introduce a version of geometric Bargmann transform. In our setting, since signals are lifted from $S$ to the bundle $E$, the appropriate Bargmann transform is defined in terms of the duality of the bundles $E$ and $E^\vee$ over the contact 3-manifold $M$. This version of geometric Bargmann transform differs from other versions previously considered in [BC15], [Bar+11], [Dui+18] constructed in terms of the geometry of the Lie algebra of $SE(2)$, or in [BHP15] where frame systems are generated using unitary actions of discrete groups. It also differs from other generalizations of the Bargmann transform such as the [Ant98].
3.1.5 Gabor filters on the manifold of contact elements

In this section we present a construction of a family of Gabor systems associated to the contact manifolds described in the previous sections. As above we consider a compact Riemann surface $S$, and its manifold of contact elements $M = S(T^*S)$ with the two contact 1-forms $\alpha$ and $\alpha_J$ described in section 3.1.1 above.

Gabor filters and receptive profiles

As argued in [Dau85], simple-cells in the $V_1$ cortex try to localize at the same time the position and the frequency of a signal, and the shape of simple cells is related to their functionality. However, the uncertainty principle in space-frequency analysis implies that it is not possible to detect, with arbitrary precision, both position and momentum. At the same time, the need for the visual system to process efficiently spatiotemporal information requires optimal extraction and representation of images and their structure. Gabor filters provide such optimality, since they minimize the uncertainty, and are therefore regarded as the most suitable functions to model the shape of the receptive profiles.

The hypothesis that receptive field profiles are Gabor filters is motivated by the analytic properties of Gabor frames. In addition to the minimization of the uncertainty principle mentioned above, the frame condition for Gabor systems provides good encoding and decoding properties in signal analysis, with greater stability to errors than in the case of a Fourier basis. It is therefore a reasonable assumption that such systems would provide an optimal form of signal analysis implementable in biological systems. We will be working here under the hypothesis that receptive field profiles in the $V_1$ cortex are indeed Gabor filters. In this section we show how to obtain such Gabor filters directly from the contact geometry described in the previous section, while in the next section we discuss the frame condition.

Gabor systems and Gabor frames

Given a point $\lambda = (s, \xi) \in \mathbb{R}^{2d}$, with $s, \xi \in \mathbb{R}^d$, we consider the operator $\rho(\lambda)$ on $L^2(\mathbb{R}^d)$ given by

$$\rho(\lambda) := e^{2\pi i (s, \xi)} T_s M_{\xi}$$

with the translation and modulation operators as in (2.1.2).

A Gabor system, for a given choice of a “window function” $g \in L^2(\mathbb{R}^d)$ and a 2$d$-dimensional lattice $\Lambda = AZ^{2d} \subset \mathbb{R}^{2d}$, for some $A \in \text{GL}_{2d}(\mathbb{R})$, we will use the notation

$$G(g, \Lambda) = \{\rho(\lambda)g\}_{\lambda \in \Lambda}.$$
We will consider in this section cases where the discrete set is a translate of a lattice by some vector. Although a translate of a lattice is no longer a lattice, it is a uniformly discrete set, see (2.1.2) and [Sei92a].

**Remark 3.1.4.** A Gabor system \( \mathcal{G}(g, \Lambda) \) as in (3.1.8) is a Gabor frame if and only if a Gabor system as in 2.1.8 is a frame, so we will use them interchangeably.

In the modelling of the \( V_1 \) cortex, receptive profiles are accurately modelled by Gabor functions, hence it is natural to consider the question of whether there is a lattice \( \Lambda \), directly determined by the geometric model of \( V_1 \), with respect to which the receptive profiles are organized into a Gabor frame system. This is the main question we will be focusing on in the rest of this paper.

**Window function**

The construction of Gabor filters we consider here follows closely the model of [SCP92], reformulated in a way that more explicitly reflects the underlying contact geometry described in the previous section. We first show how to obtain the mother function (window function) of the Gabor system and then we will construct the lattice that generates the system of Gabor filters.

Let \( V \) and \( \eta \) denote, respectively, the linear variables in the fibers \( V \in T_{(x,y)}S \simeq \mathbb{R}^2 \), \( \eta \in T^*_{(x,y)}S \simeq \mathbb{R}^2 \), with \( \langle \eta, V \rangle_{(x,y)} \) the duality pairing of \( T^*_{(x,y)}S \) and \( T_{(x,y)}S \). We write \( V = (V_1, V_2) \) and \( \eta = (\eta_1, \eta_2) \) in the bases \( \{\partial_x, \partial_y\} \) and \( \{dx, dy\} \) of the tangent and cotangent bundle determined by the choice of coordinates \( (x, y) \) on \( S \).

**Definition 3.1.5.** A window function on the bundle \( TS \oplus T^*S \) over \( S \) is a smooth real-valued function \( \Phi_0 \) defined on the total space of \( TS \oplus T^*S \), of the form

\[
\Phi_{0,(x,y)}(V, \eta) := \exp \left( -V^t A_{(x,y)} V - i \langle \eta, V \rangle_{(x,y)} \right),
\]

where \( A \) is a smooth section of \( T^*S \otimes T^*S \) that is symmetric and positive definite as a quadratic form on the fibers of \( TS \), with the property that at all points \( (x, y) \) in each local chart \( U \) in \( S \) the matrix \( A_{(x,y)} \) has eigenvalues uniformly bounded away from zero, \( \text{Spec}(A_{(x,y)}) \subset [\lambda, \infty) \) for some \( \lambda > 0 \).

**Lemma 3.1.6.** The restriction of a window function \( \Phi_0 \) as in (3.1.9) to the bundle \( TS \times S(T^*S) \) determines a real-valued function on the total space of the bundle \( E \), which in a local chart is of the form

\[
\Psi_{0,(x,y,\theta)}(V) := \exp \left( -V^t A_{(x,y)} V - i \langle \eta_{\theta}, V \rangle_{(x,y)} \right).
\]
Proof. Consider the restriction of $\Phi_0$ to the bundle $TS \times S_w(T^*S) \subset TS \oplus T^*S$, for some $w > 0$, over a local chart $(U, z = x + iy)$ of $S$. This means restricting the variable $\eta \in T^*_{(x,y)}S$ to $\eta = (\eta_1, \eta_2) = (w \cos(\theta), w \sin(\theta))$, with $\theta \in S^1$,

$$\Phi_{0,(x,y)}|_{TS \times S_w(T^*S)}(V, \theta) = \exp \left(-V^t A_{(x,y)} V - i \langle \eta_\theta, V \rangle_{(x,y)} \right), \tag{3.1.11}$$

with $\eta_\theta = (w \cos(\theta), w \sin(\theta))$. In particular, we restrict ourselves to the case $w = 1$.

We can identify the total space of the bundle $TS \times S(T^*S)$ with the total space of the bundle of signal planes $E$ over $M = S(T^*S)$. Indeed, the direct sum of two vector bundles $E_1, E_2$ over the same base space $S$ is given by

$$E_1 \oplus E_2 = \{(e_1, e_2) \in E_1 \times E_2 \mid \pi_1(e_1) = \pi_2(e_2)\}.$$ 

Similarly, when considering sphere bundles

$$E_1 \times S_w(E_2) = \{(e_1, e_2) \in E_1 \times S_w(E_2) \mid \pi_1(e_1) = \pi_2(e_2)\}$$

consider the projection onto the second coordinate, $P : E_1 \oplus E_2 \to E_2$. This projection has fibers $P^{-1}(e_2) = \pi_2^{-1}(e_2)$. Thus, the total space of the bundle $E_1 \oplus E_2$, endowed with the projection $P$, can be identified with the pullback $\pi_2^* E_1$ over $E_2$, with fibers $(\pi_2^* E_1)_{e_2} = \{e_1 \in E_1 \mid \pi_1(e_1) = \pi_2(e_2)\}$, and similarly when restricting to the sphere bundle of $E_2$.

Thus, we can write the function in (3.1.11) equivalently as a real-valued function $\Psi_0$ on the total space of the bundle $E$ over the contact 3-manifold $M$, which is of the form (3.1.10). \hfill \Box

This provides the reformulation of the Gabor profiles considered in [SCP92] in terms of the underlying geometry of the bundle $E$ over $M$.

**Lattices**

As above, consider the bundle of signal planes $E$ over $M = S(T^*S)$. The two contact forms $\alpha$ and $\alpha_j$ discussed in Section 3.1.1 determine a choice of basis for $TM$ given by the Legendrian circle fiber direction $\partial_\theta$, together with the two Reeb vector fields $R_\alpha$ and $R_{\alpha_j}$, each of which is Legendrian for the other contact form. Over a local chart $U$ of $S$, these two vector fields are given by (3.1.2), (3.1.4) and lie everywhere along the $TS$ direction, hence they determine a basis of the fibers $E_{(x,y,\theta)}$ of the bundle of signal planes for $z = x + iy \in U$.

We denote by $\{R_\alpha^\vee, R_{\alpha_j}^\vee\}$ the dual basis of $E^\vee$ (over the same chart $U$ of $S$) characterized by $\langle R_\alpha^\vee, R_\alpha \rangle = 1$, $\langle R_\alpha^\vee, R_{\alpha_j} \rangle = 0$, $\langle R_{\alpha_j}^\vee, R_\alpha \rangle = 0$, $\langle R_{\alpha_j}^\vee, R_{\alpha_j} \rangle = 1$. By the properties of
Figure 3.1: The Reeb fields $R_a$ and $R_{a_j}$ determine a basis of the tangent space $T_{(x,y)}S$ and therefore a basis of $E_{(x,y,\theta)}$.

Reeb and Legendrian vector fields, we can identify the dual basis with the contact forms, 
\[
\{ R^\vee_a, R^\vee_{a_j} \} = \{ \alpha, \alpha_j \}.
\]

Thus, the contact geometry of $M$ determines a canonical choice of a basis $\{ R_a, R_{a_j} \}$ for the bundle $E$ and its dual basis $\{ \alpha, \alpha_j \}$ for $E^\vee$.

This determines bundles of framed lattices (lattices with an assigned basis) over a local chart in $M$ of the form
\[
\Lambda_{a,j} := Z R_a + Z R_{a_j} \quad \text{(3.1.12)}
\]
\[
\Lambda^\vee_{a,j} := Z \alpha + Z \alpha_j \quad \text{(3.1.13)}
\]

where $\Lambda_{a,j}$ and $\Lambda^\vee_{a,j}$ here can be regarded as a consistent choice of a lattice $\Lambda_{a,j,(x,y,\theta)}$ (respectively, $\Lambda^\vee_{a,j,(x,y,\theta)}$) in each fiber of $E$ (respectively, of $E^\vee$). The bundle of framed lattices
\[
\Lambda_{a,j} \oplus \Lambda^\vee_{a,j} \quad \text{(3.1.14)}
\]
correspondingly consists of a lattice in each fiber of the bundle $E \oplus E^\vee$ over $M$. We will also equivalently write the bundle of lattices (3.1.14) in the form $\Lambda \oplus \Lambda_j$ with
\[
\Lambda = Z R_a \oplus Z \alpha, \quad \Lambda_j = Z R_{a_j} \oplus Z \alpha_j. \quad \text{(3.1.15)}
\]

In the following, we will often simply use the term “lattice” to indicate bundles of framed
lattices over $M$ as above.

**Lemma 3.1.7.** The choice of the window function $\Psi_0$ described in Section 3.1.5, together with the lattice (3.1.14), determine a Gabor system

$$\mathcal{G}(\Psi_0, \Lambda_{a,j} \oplus \Lambda_{a,j}^\vee)$$

which consists, at each point $(x, y, \theta) \in M$ of the Gabor system

$$\mathcal{G}(\Psi_{0,(x,y,\theta)}, \Lambda_{a,j,(x,y,\theta)} \oplus \Lambda_{a,j,(x,y,\theta)}^\vee)$$
in the space $L^2(E_{(x,y,\theta)})$.

**Proof.** The Gabor functions in $\mathcal{G}(\Psi_0, \Lambda \oplus \Lambda_J)$ are of the form

$$\rho(\lambda)\Psi_0 = \rho(\xi)\rho(W)\Psi_0 = e^{2\pi i \langle \xi, V \rangle} \Psi_0(V - W),$$

for $\lambda = (\xi, W)$ with $\xi \in \Lambda_{a,j}^\vee \subset E^\vee$ and $W \in \Lambda_{a,j} \subset E$.

**Injectivity radius function and lattice truncation**

In order to adapt this construction to a realistic model of signal processing in the $V_1$ cortex, one needs to keep into account the fact that in reality only a finite, although large, number of Gabor filters in the collection $\mathcal{G}(\Psi_0, \Lambda \oplus \Lambda_J)$ contribute to the analysis of the retinal signals. This number is empirically determined by the structure of the neurons in the $V_1$ cortex. This means that there is some (large) cutoff size $R_{\text{max}} > 0$ such that the part of the lattice that contributes to the available Gabor filters is contained in a ball of radius $R_{\text{max}}$.

There is also an additional constraint that comes from the geometry. Namely, we are using Gabor analysis in the signal planes determined by the vector bundle $E$ to analyze a signal that is originally stored on the retinal surface $S$. Lifting the signal from $S$ to the fibers of $E$ and consistency or results across nearby fibers is achieved through the exponential map

$$\exp_{(x,y)} : T_{(x,y)}S \to S$$

from the tangent bundle of $S$ (of which $E$ is the pullback to $M$) to the surface. At a given point $(x, y) \in S$ let $R_{\text{inj}}(x, y) > 0$ be the supremum of all the radii $R > 0$ such that the exponential map $\exp_{(x,y)}$ is a diffeomorphism on the ball $B(0, R)$ of radius $R$ in $T_{(x,y)}S$. For a compact surface $S$, we obtain a continuous *injectivity radius function* given by $R_{\text{inj}} : S \to \mathbb{R}_+^*$ given by $(x, y) \mapsto R_{\text{inj}}(x, y)$. 


Thus, to obtain good signal representations and signal analysis in the signal planes, we want that the finitely many available lattice points that perform the shift operators $T_W = \rho(W)$ in the Gabor system construction lie within a ball of radius $R_{inj}$ in the fibers of $E$.

It is reasonable to assume that the maximal size $R_{max}$, determined by empirical data on neurons in the visual cortex, will be in general very large, and in particular larger than the maximum over the compact surface $S$ of the injectivity radius function. This means that, in order to match these two bounds, we need to consider a scaled copy of the lattice $\Lambda_{a,J}$. We obtain the following scaling function.

**Lemma 3.1.8.** Let $b_M : M \rightarrow \mathbb{R}^*_+$ be the function given by

$$b_M(x, y, \theta) := \frac{R_{inj}(x, y)}{R_{max}}, \quad (3.1.16)$$

where $R_{inj}(x, y)$ is the injectivity radius function and $R_{max} > 0$ is an assigned constant. For $R_{max} > \max_{(x, y) \in S} R_{inj}(x, y)$, consider the rescaled lattice

$$\Lambda_{b,a,J} := b_M \Lambda_{a,J} = \mathbb{Z} b_M R_{a} \oplus \mathbb{Z} b_M R_{a \theta} \quad \text{and} \quad \left\{ \begin{array}{l}
\Lambda_{b} = \mathbb{Z} b_M R_{a} \oplus \mathbb{Z} \alpha,
\Lambda_{b,J} = \mathbb{Z} b_M R_{aJ} \oplus \mathbb{Z} \alpha_J.
\end{array} \right. \quad (3.1.17)$$

All the lattice points of the original lattice $\Lambda_{a,J}$ that are within the ball of radius $R_{max}$ correspond to lattice points of the rescaled $\Lambda_{b,a,J}$ that are within the ball of radius $R_{inj}(x, y)$ in $E_{(x,y,\theta)}$. In particular, for $B$ a ball of measure 1 in $E_{(x,y,\theta)}$, and $N(r) = \# \{ \lambda \in \Lambda_{b,a,J} \cap r \cdot B \}$, we have

$$D^-(\Lambda_{b,a,J}) = \lim_{r \rightarrow \infty} \frac{N(r)}{r} = b_M^{-1} > 1. \quad (3.1.18)$$

**Proof.** The first statement is clear by construction. Moreover, under the assumption that $R_{max} > \max_{(x, y) \in S} R_{inj}(x, y)$, the function $b_M$ of (3.1.16) is everywhere smaller than one,

$$b_M(x, y, \theta) < 1, \quad \forall (x, y, \theta) \in M, \quad (3.1.19)$$

so that the density $D^-(\Lambda_{b,a,J}) > 1$. \hfill \Box

**Remark 3.1.9.** Note that we only need to rescale the $\Lambda_{a,J}$ part of the lattice in $E$ and not the $\Lambda_{a,J}^\vee$ part of the lattice in $E^\vee$, since the $\Lambda_{a,J}^\vee$ part only contributes modulation operators $M_\xi$ that do not move the coordinates outside of the injectivity ball of the exponential map, unlike the translation operators $T_W$ with $W \in \Lambda_{a,J}$.

We can also make the choice here to scale both parts of the lattice by the same factor $b = b_M$, and work with the scaled lattice $\Lambda_{b,a,J} \oplus \Lambda_{b,a,J}^\vee$ even if the scaling of the modulation
Figure 3.2: The dark grey area represents the neurons which contribute to the analysis of the retinal signals. The light grey disk represents the area inside the injectivity radius $R_{inj}$. All the lattice points of the original lattice $\Lambda_{a,J}$ correspond to lattice points of the re-scaled lattice $\Lambda_{b,a,J}$ that are within the ball of radius $R_{inj}$.

part is not necessary by the observation of Remark 3.1.9 above. The difference between these two choices can be understood geometrically in the following way. One usually normalizes the choice of the Reeb vector field of a contact form by the requirement that the pairing is $\langle \alpha, R_a \rangle = 1 = \langle \alpha_J, R_{a,J} \rangle$. However, one can make a different choice of normalization. Scaling only the $\Lambda_{a,J}$ part of the lattice and not the $\Lambda_{a,J}^\vee$ corresponds to changing this normalization, while scaling both parts means that one maintains the normalization. As will be clear in the argument of Proposition 3.1.22, these two choices are in fact equivalent and give the same signal analysis properties.

3.1.6 The Gabor frame condition

In this section we check that the Gabor systems introduced above on the bundle of signal spaces $\mathcal{E}$ satisfy the frame condition. This condition is necessary for discrete systems
of Gabor filters to perform good signal analysis, in the sense that signals can be reconstructed from their measurements by the filters. In the usual setting of Gabor systems with Gaussian window on a single vector space \( \mathbb{R}^n \), the frame condition has been extensively studied. However, while in the 1-dimensional case the frame condition can be characterized in terms of a density property for the lattice ([Lyu92], [Sei92a]), in higher dimensions the question of whether a Gabor frame with Gaussian window in \( \mathbb{R}^n \) and a given lattice \( \Lambda \subset \mathbb{R}^{2n} \) satisfies the frame condition is generally open and very difficult to assess, see [Grö11a]. Since we are specifically interested here in the 2-dimensional case, we will follow the method developed in [Grö11a], based on the Bargmann transform, adapted to our geometric setting.

We discuss separately the case where, in a local chart \( U \) in \( S \), the quadratic form \( A \) in the window function \( \Psi_0 \) is diagonal in the basis \( \{ R_a, R_s \} \) and the general case where it is not diagonal. The first case has the advantage that it reduces to one-dimensional Gabor systems, for which we can reduce the discussion to a famous result of Lyubarskiǐ and Seip, [Lyu92], [Sei92a], after the slightly different form of the window function is accounted for. The more general case can be dealt with along the lines of the results of [Grö11a] for 2-dimensional Gabor systems. In particular, the analysis of the frame condition relies on the complex analytic technique of Bargmann transform and sampling.

As discussed in section 3.1.4 above, the notion of geometric Bargmann transform that we introduce here, for the purpose of investigating the frame condition, is defined in terms of the geometry of the dual pair of vector bundles \( E \) and \( E^\vee \) over the contact 3-manifold \( M = S(T^*S) \), since in our setting retinal signals \( f : S \to \mathbb{R} \) are lifted to signals that live on the linear fibers \( E \), with the angular coordinate of the circle fibers of \( S(T^*S) \) accounting for the directionality of the Gabor filters.

### 3.1.7 Gabor frame condition

Let \( E \) be the bundle of signal planes on the contact 3-manifold \( M \) as above. Let \( \Psi_0 \) be a window function, which we assume of the form (3.1.10). Suppose given a lattice bundle \( \Lambda \), namely a bundle over \( M \) with fiber isomorphic to \( \mathbb{Z}^4 \), where the fiber \( \Lambda_{(x,y,\theta)} \) is a lattice in \( (E \oplus E^\vee)_{(x,y,\theta)} \). We form the Gabor system \( G(\Psi_0, \Lambda) \) as in Lemma 3.1.7, with Gabor functions \( \rho(\lambda_{(x,y,\theta)})\Psi_0|_{E_{(x,y,\theta)}} \) for \( \lambda_{(x,y,\theta)} \in \Lambda_{(x,y,\theta)} \).

**Definition 3.1.10.** The Gabor system \( G(\Psi_0, \Lambda) \) satisfies the smooth Gabor frame condition on \( M \) if there are smooth \( \mathbb{R}_+^n \)-valued functions \( C, C' \) on the local charts of \( M \), such that the frame condition holds pointwise in \( (x, y, \theta) \),

\[
C_{(x,y,\theta)} \| f \|_{L^2(E_{(x,y,\theta)})}^2 \leq \sum_{\lambda_{(x,y,\theta)} \in \Lambda_{(x,y,\theta)}} |(f \rho(\lambda_{(x,y,\theta)}) \Psi_0)_{(x,y,\theta)}|^2 \leq C'_{(x,y,\theta)} \| f \|_{L^2(E_{(x,y,\theta)})}^2 ,
\]  

(3.2.20)
Note that, although, the manifold \( M \) is compact, so that globally defined continuous functions \( C, C' : M \to \mathbb{R}_+ \) would have a minimum and a maximum that are strictly positive and finite, in the condition above we are only requiring that the functions \( C, C' \) are defined on the local charts, without necessarily extending globally to \( M \). Indeed, since global vector fields on an orientable compact surface \( S \) necessarily have singularities (unless \( S = T^2 \)), the frame condition will not in general extend globally, while it holds locally within each chart, with not necessarily uniformly bounded \( C, C' \). If these functions extend globally to \( M \), then a stronger global frame condition

\[
C_{\min} \|f\|_{L^2(E(x,y,\theta))}^2 \leq \sum_{\lambda(x,y,\theta) \in \Lambda(x,y,\theta)} |\langle f, \rho(\lambda(x,y,\theta)) \Psi_0 \rangle|^2 \leq C_{\max}' \|f\|_{L^2(E(x,y,\theta))}^2
\]

would also be satisfied, but one does not expect this to be the case, except in special cases like the parallelizable \( S = T^2 \). In the case directly relevant to the modeling of the primary visual cortex, one assumes that the retinal surface is represented by a chart \( U \subset S \) with \( S = S^2 \) a sphere.

### 3.1.8 The diagonal case: dimensional reduction

Consider first the case where the quadratic form \( A \) in (3.1.10) is diagonal in the basis \( \{R_\alpha, R_{\alpha J}\} \) of the bundle \( E \).

First observe that, in a local chart \( U \) of \( S \), the unit vector \( \eta_\theta \in T^*S \) is in fact the vector \( \eta_\theta = (\cos(\theta), \sin(\theta)) \) in the basis \( \{dx, dy\} \), which is the dual basis element \( \alpha \), as in (3.1.1).

Thus, the window function (3.1.10) used in [SCP92] is of the form

\[
\Psi_{0,(x,y,\theta)}(V) = \rho\left(\frac{1}{2\pi}(0,1)\right)\hat{\Psi}_{0,(x,y,\theta)}(V)
\]

where

\[
\hat{\Psi}_{0,(x,y,\theta)}(V) := \exp\left(-V^t A(x,y) V\right)
\]

and \((0, -1) \in \Lambda \) is the covector \(-\eta_\theta(x,y) = \alpha(x,y,\theta)\). Thus, the Gabor system can be equivalently described as

\[
\mathcal{G}(\Psi_0, \Lambda \oplus \Lambda_J) = \mathcal{G}(\hat{\Psi}_0, \hat{\Lambda} \oplus \Lambda_J)
\]

\[
\hat{\Lambda} = \hat{\xi}_0 + \Lambda = \{(W, \xi) \in \mathcal{E} \oplus \mathcal{E}^\vee | W \in \mathbb{Z}R_\alpha, \xi \in \xi_0 + \mathbb{Z}\alpha\} \quad \text{with} \quad \xi_0 := -\frac{1}{2\pi} \alpha \in \mathcal{E}^\vee.
\]

Note that \( \hat{\Lambda} \) is no longer a lattice (a discrete abelian subgroup in each fiber \( \mathcal{E}_{(x,y,\theta)} \oplus \mathcal{E}_{(x,y,\theta)}^\vee \) in the local chart): it is however a uniformly discrete set given by the translate \( \xi_0 + \Lambda \).
**Lemma 3.1.11.** If the quadratic form $A$ in (3.1.10) is diagonal, $A = \text{diag}(\kappa_1^2, \kappa_2^2)$, in the basis $\{R_\alpha, R_{\alpha_j}\}$ of $E$ in a local chart, then the Gabor frame condition for $G(\Psi_0, \Lambda \oplus \Lambda_j)$ is equivalent to the frame condition for two uncoupled problems for the one-dimensional Gabor systems $G(\psi_0, \Lambda)$ and $G(\phi_0, \Lambda_j)$, with $\psi_0(V_1) = \exp(-\kappa_1^2 V_1^2 - iV_1)$ and $\phi_0(V_2) = \exp(-\kappa_2^2 V_2^2)$.

*Proof.* Given the duality pairing relations between the contact forms $\alpha, \alpha_j$ and their Reeb vector fields $R_\alpha$ and $R_{\alpha_j}$, if we write the vectors $V \in E(x,y,\theta)$ in coordinates $V = V_1 R_\alpha + V_2 R_{\alpha_j}$ over the local chart, then the window function is written in the form

$$
\Psi_{0,(x,y,\theta)}(V_1, V_2) = \exp(-\kappa_1^2 V_1^2 - iV_1) \cdot \exp(-\kappa_2^2 V_2^2) = \psi_0(V_1) \cdot \phi_0(V_2),
$$

and the Gabor system is of the form

$$
(\rho(\lambda)\Psi_0)(V) = (\rho(\lambda_1)\psi_0)(V_1) \cdot (\rho(\lambda_2)\phi_0)(V_2)
$$

$$
\lambda_1 = (\xi_1, W_1) \in \Lambda \quad \text{and} \quad \lambda_2 = (\xi_2, W_2) \in \Lambda_j.
$$

This means that, in this case, the Gabor frame condition problem for $G(\Psi_0, \Lambda \oplus \Lambda_j)$ reduces to two uncoupled problems for the one-dimensional Gabor systems $G(\psi_0, \Lambda)$ and $G(\phi_0, \Lambda_j)$. The frame condition for $G(\Psi_0, \Lambda \oplus \Lambda_j)$ is satisfied if it is satisfied for $G(\psi_0, \Lambda)$ and $G(\phi_0, \Lambda_j)$, where the first problem, by the discussion above, is equivalent to the frame condition for the system $G(\hat{\psi}_0, \hat{\Lambda})$ with $\hat{\Lambda} = \xi_0 + \Lambda$ and $\hat{\psi}_0(V_1) = \exp(-\kappa_1^2 V_1^2)$. \hfill $\square$

**Theorem 3.1.12.** The functions in the Gabor system $G(\Psi_0, \Lambda \oplus \Lambda_j)$ are not frames.

*Proof.* The second case above is a one-dimensional Gabor system with a Gaussian window function $g(t) = e^{-\kappa^2 t^2}$ and the lattice $\mathbb{Z}^2$, while the first case is a one-dimensional Gabor system with a modified window function of the form $g(t) = e^{-\kappa^2 t^2 - iat}$ and the lattice $\mathbb{Z}^2$ or equivalently a window function $\hat{g}(t) = e^{-\kappa^2 t^2}$ and the discrete set $(0,a) + \mathbb{Z}^2$.

For a lattice $\Lambda = A\mathbb{Z}^d$ with $A \in \text{GL}_d(\mathbb{R})$ the density is given by $s(\Lambda) = |\det(A)|$. In particular it is $s(\Lambda) = 1$ for the standard lattice $\mathbb{Z}^2$. The *density theorem* for Gabor frames, [Jan94a] (see also Proposition 2 of [Grö11a]), states that if a Gabor system $G(g, \Lambda)$ is a frame in $L^2(\mathbb{R}^d)$ and the window is a rapid decay function $g \in \mathcal{S}(\mathbb{R}^d)$, then necessarily $s(\Lambda) < 1$. Thus, these one-dimensional Gabor systems are not frames, hence the original system $G(\Psi_0, \Lambda \oplus \Lambda_j)$ also does not satisfy the frame condition. \hfill $\square$

On the other hand, the situation changes when one takes into account the scaling of the lattice discussed in Section 3.1.5.

**Theorem 3.1.13.** Consider the rescaled lattices $\Lambda_{b,a_j}$, $\Lambda_b$, $\Lambda_{b,j}$ of (3.1.17). The system $G(\Psi_0, \Lambda_{b} \oplus \Lambda_{b,j})$ does satisfy the frame condition.
Proof. The Gabor frame question for the system $\mathcal{G}(\Psi_0, \Lambda_b + \Lambda_{b,f})$ reduces to the question of whether the one-dimensional systems $\mathcal{G}(\phi_0, \Lambda_b)$ and $\mathcal{G}(\hat{\phi}_0, \hat{\Lambda}_b)$ with $\hat{\Lambda}_b = \xi_0 + \Lambda_b$ are frames.

In the case of one-dimensional systems, there is a complete characterization of when the frame condition is satisfied, [Lyu92], [Sei92a], [Sei92b]. This characterization is obtained by reformulating the problem in terms of a complex analysis problem of sampling and interpolation in Bargmann–Fock spaces. In the case of a Gaussian window function $\psi$ and a uniformly discrete set $\Lambda \subset \mathbb{R}^2$, it is proved in [Sei92a] that the Gabor system $\mathcal{G}(\psi, \Lambda)$ is a frame if and only if the lower Beurling density satisfies $D^{-}(\Lambda) > 1$, where

$$D^{-}(\Lambda) = \lim_{r \to \infty} \inf \frac{N^{-}_\Lambda(r)}{r^2},$$

with $N^{-}_\Lambda(r)$ the smallest number of points of $\Lambda$ contained in a scaled copy $r\mathcal{I}$ of a given set $\mathcal{I} \subset \mathbb{R}^2$ of measure one, with measure zero boundary. The value $D^{-}(\Lambda)$ is independent of the choice of the set $\mathcal{I}$. In the case of a rank two lattice this corresponds to the condition $s(\Lambda) < 1$, which is therefore also sufficient.

Thus, the one-dimensional systems $\mathcal{G}(\phi_0, \Lambda_{b,f})$ and $\mathcal{G}(\hat{\phi}_0, \hat{\Lambda}_b)$ are frames if and only if $s(\Lambda_{b,f}) < 1$ and $s(\Lambda_b) < 1$, since the translate $\hat{\Lambda}_b$ and $\Lambda_b$ have the same lower Beurling density. Since the scaling function satisfies $b_M < 1$ everywhere on $M$, as in (3.1.19), we have seen in Lemma 3.1.8 that these conditions are satisfied. It follows that the Gabor system $\mathcal{G}(\Psi_0, \Lambda_b + \Lambda_{b,f})$ is a frame. 

### 3.1.9 The non-diagonal case: Bargmann transform

In the more general case where the quadratic form in $\Psi_0$ is not necessarily diagonal in the basis $\{ R_\alpha, R_{a,f} \}$ in a local chart, the question of whether the Gabor system $\mathcal{G}(\Psi_0, \Lambda_b + \Lambda_{b,f})$ satisfies the frame condition can still be reformulated in terms of sampling and interpolation in Bargmann–Fock spaces, see [Grö11a].

**Bargmann transform and Gabor frames**

We are going to use the relation between the Bargmann transform and Gabor systems with Gaussian window function, see for instance 2.1.16 and 2.1.17. In our setting, because of the form (3.1.21) of the window function, we need a simple variant of this relation between Gabor systems and Bargmann transform which we now illustrate.

A set $\Lambda \subset \mathbb{C}^n$ is a *sampling set* for $\mathcal{F}^2_n$ if there are constants $C, C' > 0$, such that, for all
\( F \in \mathcal{F}_n^2, \)
\[
C \cdot \|F\|_{\mathcal{F}_n^2}^2 \leq \sum_{\lambda \in \Lambda} |F(\lambda)|^2 e^{-\pi |\lambda|^2} \leq C' \cdot \|F\|_{\mathcal{F}_n^2}^2.
\]

A set \( \Lambda \subset \mathbb{C}^n \) is a set of uniqueness for \( \mathcal{F}_n^\infty \) if a function \( F \in \mathcal{F}_n^\infty \) satisfying \( F(\lambda) = 0 \) for all \( \lambda \in \Lambda \) must vanish indentically, \( F \equiv 0 \). For \( \Lambda \subset \mathbb{C}^n \), let \( \bar{\Lambda} = \{ \lambda \mid \lambda \in \Lambda \} \).

We consider as in [Grö07] the modulation spaces \( M^p(\mathbb{R}^n) \), which is defined as the space of tempered distributions \( f \in S'(\mathbb{R}^n) \) whose bounded Short-Time Fourier transform has bounded \( L^p \) norm, \( \|V_p f\|_p < \infty \), for all \( \varphi \in S(\mathbb{R}^d) \), where
\[
V_p f = \langle f, M_{\varphi} T_x \varphi \rangle = \int_{\mathbb{R}^d} f(t) \overline{\varphi(t-x)} e^{-2\pi i \xi \cdot t} \, dt.
\]

Similarly, the modulation space \( M^\infty(\mathbb{R}^n) \) is the space of tempered distributions \( f \in S'(\mathbb{R}^n) \) with \( \|V_p f\|_\infty < \infty \), for all \( \varphi \in S(\mathbb{R}^d) \).

**Proposition 3.1.14.** Let \( \Lambda \subset \mathbb{C}^n \) be a lattice and let \( \phi(x) = e^{-\pi |x|^2} e^{-2\pi i a \cdot x} \in L^2(\mathbb{R}^n) \), for some fixed \( a \in \mathbb{R}^n \). Then the following conditions are equivalent:

1. The Gabor system \( \mathcal{G}(\phi, \Lambda) \) is a frame.
2. The set \( \bar{\Lambda}_a := \bar{\Lambda} + ia \) is a sampling set for \( \mathcal{F}_n^2 \).
3. The set \( \bar{\Lambda}_a \) is a set of uniqueness for \( \mathcal{F}_n^\infty \).

**Proof.** For the proof of 1 \( \iff \) 2 it suffices to prove that
\[
|\langle f, M_{w} T_{x} \varphi \rangle| = |\mathcal{B}(x - i(w + a))| e^{-\pi (x - i(w + a))^2}. 
\]

We have
\[
V_p f(x, w) = \int_{\mathbb{R}^n} f(t) e^{-\pi (t-x)^2} e^{-2\pi i (a \cdot (t-x))} e^{-2\pi i (w \cdot t)} \, dt =
\]
\[
e^{2\pi i (a \cdot x)} \int_{\mathbb{R}^n} f(t) e^{-\pi t^2 + 2\pi tx - \pi x^2} e^{2\pi i (a \cdot w) \cdot t} \, dt =
\]
\[
e^{2\pi i a \cdot x} e^{-\pi i x \cdot (a + w)} e^{-\frac{1}{2} (x^2 + (a + w)^2)} \int_{\mathbb{R}^n} f(t) e^{-\pi t^2} e^{2\pi t \cdot (x - i(w + a))} e^{\frac{1}{2} (x - i(w + a))^2} \, dt.
\]

Moreover, for \( z' = x + i(w + a) \),
\[
V_p f(x, w) = e^{-\frac{1}{2} |z'|^2} e^{-\pi i x \cdot z'} e^{2\pi i (a \cdot x)} \mathcal{B}(z')
\]

Thus, \( |V_p f(x, w)| = |\mathcal{B}(z')| e^{-\frac{1}{2} |z'|^2} = |\mathcal{B} f(x - i(w + a))| e^{-\pi (x - i(w + a))^2} \). Thus, we obtain
\[
\sum_{\lambda \in \Lambda} |V_p f(\lambda)| = \sum_{z' \in \bar{\Lambda}_a} |\mathcal{B} f(z')| e^{-\frac{1}{2} |z'|^2},
\]
and \( \sum_{\lambda \in \Lambda} |V_\phi f(\lambda)| \leq \|f\|_{L^2(\mathbb{R}^n)} \) if and only if
\[
\sum_{z' \in \mathcal{A}_z} |B f(z')| e^{-\frac{\pi}{2} |z'|^2} \leq \|B f\|_{F^2_\mathcal{A}}.
\]

To prove 2 \( \iff \) 3, starting with the assumption that \( \mathcal{A}_\alpha \) is a set of sampling for \( \mathcal{F}^\infty_\mathcal{A} \), let \( F \in \mathcal{F}^\infty_\mathcal{A} \) be such that \( F(\lambda) = 0 \) for all \( \lambda \in \mathcal{A}_\alpha \). The Bargmann-Fock space \( \mathcal{F}^\infty_\mathcal{A} \) is related to the modulation space \( M^\infty(\mathbb{R}^n) \) through the Bargmann transform (3.3.17),
\[
\mathcal{F}^\infty_\mathcal{A} = \mathcal{B}(M^\infty(\mathbb{R}^n)).
\]
Thus, there exists an element \( f \in M^\infty(\mathbb{R}^n) \) such that \( B f = F \). Thus, we have \( B f(\lambda) = 0 \), for all \( \lambda \in \mathcal{A}_\alpha \), hence \( \langle f, \pi(\lambda) \phi \rangle = 0 \), for all \( \lambda \in \Lambda \). The equivalence \( 1 \iff 2 \) then implies that \( f \equiv 0 \), hence \( F \equiv 0 \).

Conversely, suppose that \( \mathcal{A}_\alpha \) is a set of uniqueness for \( \mathcal{F}^\infty_\mathcal{A} \). Theorem 3.1 of [Grö07] shows that the frame condition for the Gabor system \( \mathcal{G}(\phi, \Lambda) \), for a window \( \phi \in \mathcal{S}(\mathbb{R}^n) \), is equivalent to the condition that the Gabor transform map is one-to-one as a map
\[
V_\phi : M^\infty(\mathbb{R}^n) \to \ell^\infty(\Lambda), \quad V_\phi : f \mapsto V_\phi f|_\Lambda.
\]
Since we have \( \phi \in \mathcal{S}(\mathbb{R}^n) \), it suffices to prove that the Gabor transform \( f \mapsto V_\phi f|_\mathcal{A}_\alpha \) is one-to-one as a map \( M^\infty(\mathbb{R}^n) \to \ell^\infty(\mathcal{A}_\alpha) \).

Let \( D \) denote the map \( D : M^\infty(\mathbb{R}^n) \to \ell^\infty(\Lambda) \) given by
\[
D : f \mapsto \{B f(\lambda)\}_{\lambda \in \Lambda},
\]
and let \( T : \ell^\infty(\Lambda) \to \ell^\infty(\mathcal{A}_\alpha) \) be given by
\[
T : \{e_{\lambda}\}_{\lambda \in \Lambda} \mapsto \{e^{\pi i 1(\lambda_2 + a)} e^{-|\lambda + (0,a)|^2 / 2} e_{\lambda}\}_{\lambda + (0,a) \in \mathcal{A}_\alpha},
\]
The operator \( V_\phi \) of (3.1.24) is the composite \( V_\phi = T \circ D \), which is injective since both \( T \) and \( D \) are.

**Corollary 3.1.15.** The Gabor system \( \mathcal{G}(\phi, \Lambda) \) with window function \( \phi(x) = e^{-\pi x^2} \) is a frame if and only if \( \mathcal{G}(\tilde{\phi}, \Lambda) \) with window function \( \tilde{\phi}(x) = e^{-\pi x^2} e^{-2\pi i (a \cdot x)} \) is a frame.

**Proof.** For the window \( \tilde{\phi} \) the system \( \mathcal{G}(\tilde{\phi}, \Lambda) \) is a frame iff the system \( \mathcal{G}(\tilde{\phi}, \Lambda) \) is a frame and the latter is equivalent to
\[
\sum_{z \in \Lambda} |B f(z - ia)| e^{-\frac{\pi}{2} |z|^2} \leq ||B f||_{F^2_\mathcal{A}},
\]
which we have seen is equivalent to $\mathcal{G}(\phi, \Lambda)$ being a frame.

**Geometric Bargmann transform**

We apply this Bargmann transform argument to our geometric setting. The bundle $E$ is endowed with an almost complex structure, coming from the identification $E = \pi^*TS$ with $S$ a Riemann surface, hence the dual $E^\vee$ can also be endowed with an almost complex structure. However, for the purpose of applying the Bargmann transform argument in our setting, we just need to consider the bundle $E \oplus E^\vee$ as a complex 2-plane bundle over $M$. First note that the local bases $\{\alpha, \eta\}$ of $E$ and $\{\alpha, \eta\}$ of $E^\vee$ determine a local isomorphism between $E$ and $E^\vee$. For $(W, \eta) \in (E \oplus E^\vee)_{(x, y, \theta)}$, with $W = W_1R_\alpha + W_2R_\eta$, and $\eta = \eta_1\alpha + \eta_2\alpha$, we define $J : E \oplus E^\vee \rightarrow E \oplus E^\vee$ with $J^2 = -1$ by setting

$$J(W, \eta) := (\eta, -W) = \eta_1 R_\alpha + \eta_2 R_\eta - W_1 \alpha - W_2 \alpha.$$

We can then take $W + i\eta := (W, \eta)$ with scalar multiplication by $\lambda \in \mathbb{C}$, $\lambda = x + iy$ with $x, y \in \mathbb{R}$ given by $\lambda \cdot (W + i\eta) = (x + y)j(W, \eta)$. This gives a fiberwise identification

$$\mathcal{I} : (E \oplus E^\vee)_{(x, y, \theta)} \rightarrow \mathbb{C}^2 \quad (W, \eta) \mapsto z = (z_1, z_2) = (W_1 + i\eta_1, W_2 + i\eta_2). \quad (3.1.25)$$

Given the choice of a window function $\Psi_{0,(x, y, \theta)}(V)$ as in (3.1.10), with a quadratic form on the fibers of $E$ over the local chart, determined by a smooth section $A$ of $T^*S \otimes T^*S$ that is symmetric and positive definite, we consider an associated quadratic form

$$Q : E \oplus E^\vee \rightarrow \mathbb{C}, \quad Q_{(x, y, \theta)}(W + i\eta) := W^t A_{(x, y)} W + 2i\langle \eta, W \rangle_{(x, y, \theta)} - \eta^t \eta, \quad (3.1.26)$$

where $\langle \eta, W \rangle$ is the duality pairing of $E$ and $E^\vee$, and $\eta^t \eta$ denotes the pairing with respect to the metric in $E^\vee$ determined by the metric on $S$. We use the notation

$$Q(z) := Q \circ \mathcal{I}^{-1}(z) \quad \text{and} \quad V \bullet z := V^t A_{(x, y)} W + i\langle \eta, V \rangle. \quad (3.1.27)$$

We also define $\tilde{Q} : E \oplus E^\vee \rightarrow \mathbb{C}$ as

$$\tilde{Q}_{(x, y, \theta)}(W, \eta) := \frac{\pi}{2} W^t A_{(x, y)} W + (\eta + \frac{\eta_\theta}{2\pi})^t (\eta + \frac{\eta_\theta}{2\pi}). \quad (3.1.28)$$

We write $\tilde{Q}(z) := \tilde{Q} \circ \mathcal{I}^{-1}(z)$.

**Definition 3.1.16.** The Bargmann transform of a function $f \in L^2(E, \mathbb{C})$ is a function $Bf :
$\mathcal{E} \oplus \mathcal{E}^\vee \to \mathbb{C}$ defined fiberwise by

$$
(B f)|_{(\mathcal{E} \oplus \mathcal{E}^\vee)(x, y, \theta)}(W, \eta) := \int_{\mathcal{E}(x, y, \theta)} f|_{\mathcal{E}(x, y, \theta)}(V) e^{2\pi i V \bullet z - \pi V' A_{(x, y)} V + \frac{\eta}{2} Q(z)} d\text{vol}_{(x, y, \theta)}(V),
$$

(3.1.29)

with the notation as in (3.1.27) and with $d\text{vol}_{(x, y, \theta)}(V)$ the volume form on the fibers of $\mathcal{E}$ determined by the Riemannian metric on $S$.

**Lemma 3.1.17.** Consider the window function $\Psi_0$ as in (3.1.10). The Gabor functions

$$
\rho(W, \eta)\Psi_0(V) = e^{2\pi i (\eta \cdot V - W)}\Psi_0(V - W),
$$

with $(W, \eta) \in \mathcal{E} \oplus \mathcal{E}^\vee$, satisfy

$$
|\langle f, \rho(W, \eta)\Psi_0 \rangle| = |B f(W - i(\eta + \frac{\eta_0}{2\pi}))| e^{-\tilde{Q}(W, \eta)}. \tag{3.1.30}
$$

with $\tilde{Q}$ as in (3.1.28).

**Proof.** We have

$$
\langle f, \rho(W, \eta)\Psi_0 \rangle = \int_{\mathcal{E}(x, y, \theta)} f(V) e^{-\pi (V - W)^4(V - W) - 2\pi i (\eta_0 \cdot V - W) - 2\pi i (\eta_0 \cdot V)} d\text{vol}(V)
$$

$$
= e^{2\pi i (\eta_0 \cdot W)} \int_{\mathcal{E}(x, y, \theta)} f(V) e^{\pi V^4_0 - 2\pi V\eta + \frac{\eta_0}{2} V^2 - 2\pi i (\eta_0 \cdot V)} d\text{vol}(V)
$$

$$
= e^{2\pi i (\eta_0 \cdot W)} e^{-ir(\eta + \frac{\eta_0}{2\pi}, W)} e^{-\frac{1}{2} W^4 - \frac{1}{2} (\eta + \frac{\eta_0}{2\pi})^2} e^{-\frac{1}{2} Q(W - i(\eta + \frac{\eta_0}{2\pi})))} d\text{vol}(V)
$$

with $Q$ as in (3.1.26) and $\tilde{Q}$ as in (3.1.28).

**Remark 3.1.18.** Under the identification (3.1.25) we write (3.1.30) equivalently as

$$
|\langle f, \rho(W, \eta)\Psi_0 \rangle| = |B f(z)| e^{-\tilde{Q}(W, \eta)} \quad \text{for} \quad z = W + i(\frac{\eta_0}{2\pi} + \eta). \tag{3.1.31}
$$

**Definition 3.1.19.** The global Bargmann-Fock space $\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)$ is the space of functions $F : \mathcal{E} \oplus \mathcal{E}^\vee \to \mathbb{C}$ such that $F|_{(\mathcal{E} \oplus \mathcal{E}^\vee)(x, y, \theta)} \circ \mathcal{I}^{-1} : \mathbb{C}^2 \to \mathbb{C}$ is entire with

$$
\|F\|_{\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)}^2 = \int_M \int_{\mathbb{C}^2} \left| F|_{(\mathcal{E} \oplus \mathcal{E}^\vee)(x, y, \theta)} \circ \mathcal{I}^{-1}(z) \right|^2 e^{-2Q(z)} dz d\text{vol}(x, y, \theta) < \infty.
$$

The fiberwise Bargmann-Fock space $\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)(x, y, \theta)$ is the space of functions $F : (\mathcal{E} \oplus \mathcal{E}^\vee)(x, y, \theta) \to \mathbb{C}$.
Let $\Lambda$ be a bundle of lattices over $M$ where, over a local chart we have $\Lambda_{(x,y,\theta)}$ a lattice in $(\mathcal{E} \oplus \mathcal{E}^\vee)_{(x,y,\theta)}$. The bundle $\Lambda$ satisfies the smooth sampling condition for $F^2(\mathcal{E} \oplus \mathcal{E}^\vee)$ if there are $\mathbb{R}_+^*$-valued smooth functions $C, C'$ on the local charts of $M$, such that, for all $(x, y, \theta)$ in a local chart of $M$ and for all $F \in F^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{(x,y,\theta)}$, the estimates

$$
C_\mu \cdot \|F\|^2_{F^2(\mathcal{E} \oplus \mathcal{E}^\vee)_\mu} \leq \sum_{(W, \eta) \in \Lambda} \left| F_{(\mathcal{E} \oplus \mathcal{E}^\vee)_\mu} \left( e^{-2\mathcal{Q}_\mu(W, \eta)} \right) \right| \leq C'_\mu \cdot \|F\|^2_{F^2(\mathcal{E} \oplus \mathcal{E}^\vee)_\mu} \quad (3.1.32)
$$

are satisfied, for $\mu = (x, y, \theta)$ in a local chart of $M$, and with $\mathcal{Q}$ as in (3.1.28).

**Theorem 3.1.21.** For any $(x, y, \theta)$ in a local chart of $M$, the Bargmann transform $B$ of (3.1.29) is a bijection from $L^2(\mathcal{E}_{(x,y,\theta)})$ to $F^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{(x,y,\theta)}$, with

$$
\|B f\|_{F^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{(x,y,\theta)}} = K_{(x,y)} \cdot \|f\|_{L^2(\mathcal{E}_{(x,y,\theta)})} \quad (3.1.33)
$$

for a smooth $\mathbb{R}_+^*$-valued function $K$ over the local charts $U$ of $S$. Moreover, $G(\Psi_0, \Lambda)$ is a frame for $L^2(\mathcal{E}_{(x,y,\theta)})$ if and only if $\mathcal{N} + i\frac{\pi}{2\pi}$ is a set of sampling for $F(\mathcal{E} \oplus \mathcal{E}^\vee)_{(x,y,\theta)}$.

**Proof.** For the window function $\Psi_0$ as in (3.1.10), we have

$$
\|\Psi_0\|^2_{L^2(\mathcal{E}_{(x,y,\theta)})} = \int_{\mathcal{E}_{(x,y,\theta)}} |\Psi_0(V)|^2 dV = \int_{\mathcal{E}_{(x,y,\theta)}} e^{-2V^\dagger A(y) V} dV = \frac{\pi}{2\sqrt{\det(A(y))}}.
$$

as a standard Gaussian integral in 2 dimensions. Because we assumed that the matrices $A(x,y)$ in the window function $\Psi_0$ of (3.1.10) have spectrum bounded away from zero, and
that $S$ is compact, the quantity 

$$K_{(x,y)} := \frac{\pi}{2\sqrt{\det(A_{(x,y)})}}$$

determines a smooth real valued function $K : S \rightarrow \mathbb{R}$ with a strictly positive minimum and a bounded maximum. Moreover, by Theorem 3.2.1 and Corollary 3.2.2 of [Grö01], the orthogonality relation

$$\langle V_{\psi_1}f_1, V_{\psi_2}f_2 \rangle_{L^2(\mathbb{R}^{2n})} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^n)} \cdot \langle \psi_1, \psi_2 \rangle_{L^2(\mathbb{R}^n)}$$

for the short time Fourier transform

$$V_\psi f(x,\omega) = \int_{\mathbb{R}^n} f(t) \overline{\psi(t-x)} e^{-2\pi i t \cdot \omega} \, dt,$$

gives the identity

$$\| \langle f, \rho(W,\eta) \Psi_0 \rangle \|_{L^2(\mathcal{E}(x,y,\theta))} = \| f \|_{L^2(\mathcal{E}(x,y,\theta))} \cdot \| \Psi_0 \|_{L^2(\mathcal{E}(x,y,\theta))}.$$ 

Moreover, by (3.1.31) we have, for $z = \mathcal{I}(W,\eta)$,

$$\| \langle f, \rho(W,\eta) \Psi_0 \rangle \|_{L^2(\mathcal{E}(x,y,\theta))} = \int_{(\mathcal{E} \oplus \mathcal{E}^\vee)(x,y,\theta)} |\langle f, \rho(W,\eta) \Psi_0 \rangle|^2 \, d\text{vol}(W,\eta)$$

$$= \int_{\mathbb{R}^2} |B f(z)|^2 e^{-2\hat{\Psi}(z)} \, dz.$$ 

Injectivity then follows, while surjectivity follows by the same argument showing the density of $B(L^2(\mathbb{R}^n)) \subset \mathcal{F}_n^2$ in the proof of Theorem 3.4.3 of [Grö01], applied pointwise in $(x,y,\theta) \in M$.

The Gabor system $\mathcal{G}(\Psi_0,\Lambda)$ satisfies the smooth frame condition of Definition 3.1.10 if there are smooth functions $C_{(x,y,\theta)}, C'_{(x,y,\theta)} > 0$ on the local charts of $M$ such that

$$C_{(x,y,\theta)} \| f \|_{L^2(\mathcal{E}(x,y,\theta))}^2 \leq \sum_{\lambda = (W,\eta) \in \Lambda} |\langle f, \rho(\lambda) \Psi_0 \rangle|^2 \leq C'_{(x,y,\theta)} \| f \|_{L^2(\mathcal{E}(x,y,\theta))}^2.$$ 

By (3.1.31) we see that this is equivalent to the smooth sampling condition of Definition 3.1.20 for $\Lambda + i\frac{\eta}{2\pi}$.

**Theorem 3.1.22.** With the scaling by the function $b = b_M(x,y,\theta)$ of (3.1.16), the Gabor system $\mathcal{G}(\Psi_0,\Lambda_{b,n} \oplus \Lambda^\vee_{b,n})$ satisfies the frame condition.

**Proof.** We write here the window function $\Psi$ as $\Psi_0^4$ to emphasize the dependence on the
Finally, by \( \text{Gr"{o}} \) if the uniformly discrete set \( V \) is a set of sampling, for the matrix \( A \to bA \), we rewrite the above as

\[
\sum_{(n,m) \in \mathbb{Z}^2 \times \mathbb{Z}^2} b^2 \left| \int_{E_{(x,y)}} f(\sqrt{b} U) M_{\sqrt{b} m} T_{\sqrt{b} n} e^{-U(bA_{(x,y)})} d\text{vol}_{(x,y)}(V) \right|^2 = b^2 \sum_{\lambda \in \sqrt{b} \Lambda_0 \oplus \sqrt{b} \Lambda_0^\perp} \left| \langle f_{\sqrt{b}}, \rho(\bar{\lambda}) \Psi_{0}^A \rangle \right|^2,
\]

where \( f_{\sqrt{b}}(V) = f(\sqrt{b} V) \). Therefore, the Gabor system \( G(\Psi_{0}^A, \Lambda_{0,j} \oplus \Lambda_{0,j}^\perp) \) is a frame for \( L^2(E_{(x,y)}) \) if and only if \( G(\Psi_{0}^{bA}, \Lambda_{\sqrt{b},0,j} \oplus \Lambda_{\sqrt{b},0,j}^\perp) \) is a frame for \( L^2(E_{(x,y)}) \). Moreover, by Lemma 3.1.21, we know that \( G(\Psi_{0}^{bA}, \Lambda_{\sqrt{b},0,j} \oplus \Lambda_{\sqrt{b},0,j}^\perp) \) is a frame for \( L^2(E_{(x,y)}) \) if and only if the uniformly discrete set \( (\sqrt{b} \mathbb{Z}^2 + i \sqrt{b} \mathbb{Z}^2) + \frac{y}{2\pi} \) is a set of sampling for \( F(E \oplus E^\perp)_{(x,y)} \). Finally, by [Grö11a], \( (\sqrt{b} \mathbb{Z}^2 + i \sqrt{b} \mathbb{Z}^2) + \frac{y}{2\pi} \) is a set of sampling if and only if the complex lattice \( T(\mathbb{Z}^2 + i\mathbb{Z}^2) \) is a set of sampling, for the matrix

\[
T = \begin{pmatrix} \sqrt{b} & 0 \\ 0 & \sqrt{b} \end{pmatrix}.
\]

By Proposition 11 of [Grö11a], the latter condition is satisfied if and only if \( \sqrt{b} < 1 \), which we know is the case by Lemma 3.1.8.

3.1.10 Gabor frames: symplectization and contactization

As in section 2.2.1, we consider the contactization \( CS(M) \) of the symplectization \( S(M) \) of the manifold of contact elements \( M = S_{\omega}(T^* S) \) of a surface \( S \). In the context of geometric models of the \( V_1 \) cortex, the 5-dimensional contact manifold \( CS(M) \) corresponds to the model for the receptive fields considered in [BSC20] where an additional pair of dual variables is introduces, describing phase and velocity of spatial wave propagation. This model is motivated by the goal of describing visual perception based on neurons sensitive not only to orientation, but also to frequency and phase, with the frequency-phase and
the position-orientation uncertainty minimized by the Gabor functions profiles. From the point of view of this model, it is worth pointing out that, although higher dimensional, the 5-dimensional contact manifold \( CS(M) \) is completely determined by the contact 3-manifold \( M \) with no additional independent choices, being just the contactization of the symplectization.

Note that, while the contact structure of \( CS(M) \) is the natural extension of the contact structure of \( M \), this does not directly imply that modelling the visual cortex requires an increasing family of contact structures to account for different families of cells sensitive to different features, as different features may be described by the same geometry.

**Remark 3.1.23.** The twist \( \alpha \mapsto \alpha_J \) of \((3.1.3)\) of the contact structure on \( M = S(T^*S) \) induces corresponding twists of the symplectization \( \omega \) as in \((2.2.3)\)

\[
\omega \mapsto \omega_J = \omega \wedge da_J + a_J \wedge d\omega
\]

and of the contact form \( \tilde{\alpha} \) of \( CS(M) \) as in \(2.2.4\)

\[
\tilde{\alpha} \mapsto \tilde{\alpha}_J = w\alpha_J - d\phi.
\]

Given local charts on \( M \) with the choice of local basis

\[
X_\theta = \partial_\theta, \quad R_{a_J} = -w^{-1} \sin(\theta) \partial_x + w^{-1} \cos(\theta) \partial_y \tag{3.1.34}
\]

for the contact planes \( \xi \) of the contact structure \( \alpha \) on \( M \) and the Reeb field \( R_{\alpha} = w^{-1} \cos(\theta) \partial_x + w^{-1} \sin(\theta) \partial_y \), we obtain a basis of the contact hyperplane distribution of the five-dimensional contact manifold \((T^*S_0 \times S^1, \tilde{\alpha})\), in the corresponding local charts, given by

\[
\{X_\theta, R_{a_J}, R_{\phi,\alpha}, X_w\}.
\]

\[ R_{\phi,\alpha} := \partial_\phi + R_{\alpha} \quad \text{and} \quad X_w := \partial_w. \]

In the case of the twisted contact structure \( \alpha_J \), with the choice of basis

\[
X_\theta = \partial_\theta, \quad R_{\tilde{\alpha}} = w^{-1} \cos(\theta) \partial_x + w^{-1} \sin(\theta) \partial_y \tag{3.1.35}
\]

for the contact plane distribution \( \xi_J \), and the Reeb vector field \( R_{\tilde{\alpha},J} = -w^{-1} \sin(\theta) \partial_x + w^{-1} \cos(\theta) \partial_y \), we similarly obtain a basis for the contact hyperplanes \( \tilde{\xi}_J \) given by

\[
\{X_\theta, R_{\tilde{\alpha},J}, R_{\phi,\alpha,J}, X_w\}, \tag{3.1.36}
\]

\[ R_{\phi,\alpha,J} := \partial_\phi + R_{\tilde{\alpha},J}. \]
The bundle $\mathcal{E}$ of signal planes on $M$ determines the following bundles on the symplectization $S(M)$ and the contactization $CS(M)$.

**Definition 3.1.24.** Let $\hat{\mathcal{E}}$ denote the pullback of the bundle $\mathcal{E}$ of signal planes to $T^*S_0 \simeq M \times \mathbb{R}_+$ via the projection to $M$, and let $\tilde{\mathcal{E}}$ denote the vector bundle over $CS(M)$ given by $\tilde{\mathcal{E}} \oplus TS^1 = \pi^*_T S_0 \hat{\mathcal{E}} \oplus \pi^*_S TS^1$, with pullbacks taken with respect to the two projections of $CS(M) = T^*S_0 \times S^1$ on the two factors.

The signals in this setting will be functions $I : \hat{\mathcal{E}} \to \mathbb{R}$. The vector bundle $\tilde{\mathcal{E}}$ on $CS(M)$ is a rank 3 real vector bundle over a 5-dimensional manifold.

**Remark 3.1.25.** A basis of sections for $\tilde{\mathcal{E}}$ over a local chart is obtained by taking the vectors $\{R_a, R_{a_j}, \partial_{\phi}\}$. There are two other choices of basis directly determined by the contact forms $\tilde{\alpha}$ and $\tilde{\alpha}_j$, namely $\{R_a, R_{\phi,a_j}, R_{\tilde{\alpha}_j}\}$, where the first two vectors span the intersection $\tilde{\mathcal{E}} \cap \tilde{\xi}$ of the contact hyperplane distribution with the bundle $\tilde{\mathcal{E}}$ and the last vector is the Reeb field of $\tilde{\alpha}_j$, or $\{R_{a_j}, R_{\phi,a_j}, R_{\tilde{\alpha}_j}\}$, with the first two vector fields spanning $\tilde{\mathcal{E}} \cap \tilde{\xi}_j$ and the third the Reeb field of $\tilde{\alpha}$. The first basis has the advantage of providing consistent choices of basis for both $\mathcal{E}$ and $\hat{\mathcal{E}}$.

**Lemma 3.1.26.** In a local chart of $S(M)$ with coordinates $(x, y, w, \theta)$, the window function $\Psi_0$ as in (3.1.10) extends to a window function on $\hat{\mathcal{E}}$ given by

$$
\Psi_{0,(x,y,w,\theta)}(V) = \exp \left( -V^t A_{(x,y)} V - i \langle \eta_{(w, \theta)}, V \rangle_{(x,y)} \right),
$$

with $\eta_{(w, \theta)} = (w \cos(\theta), w \sin(\theta))$.

**Proof.** The window function $\Psi_0$ as in (3.1.10) is obtained as restriction to $TS \oplus S(T^*S)$ of a window function $\Phi_0$ on $TS \oplus T^*S$ defined as in (3.1.9) in Definition 3.1.5. By identifying $S(M) = T^*S_0$ and $\hat{\mathcal{E}}$ with the pullback of $TS$ to $S(M)$, we see that $\Phi_0$ induces a window function $\Psi_0$ on $\hat{\mathcal{E}}$ of the form (3.1.37).

We further extend the window function (3.1.37) to $\tilde{\mathcal{E}}$ so as to obtain a window function that is a modified form of the function considered in the model of [BSC20].

**Definition 3.1.27.** In a local chart of $CS(M)$ with coordinates $(x, y, w, \theta, \phi)$, window functions on $\tilde{\mathcal{E}}$ extending the window function (3.1.37) are functions on $\tilde{\mathcal{E}}$ of the form

$$
\Psi_{0,(x,y,w,\theta,\phi),\zeta_0}(V, v) = \exp \left( -V^t A_{(x,y)} V - i \langle \eta_{(w, \theta)}, V \rangle_{(x,y,w, \theta)} - \kappa_\phi^2 v^2 - i \langle \zeta_0, v \rangle_{\phi} \right),
$$

(3.1.38)
for \( \eta_{(w, \theta)} \) as in (3.1.37), and with \( \zeta_0 \in T^*_\phi S^1 \) and \( \nu \in T_\phi S^1 \). The two-dimensional Gabor systems of the form \( \{ \rho(W, \eta)\Psi_0 |_{\mathcal{E}_{(x, y, \theta, \phi)}} \} \) are then replaced by a three-dimensional system of the form

\[
\rho(W, \eta, \nu, \zeta)\Psi_0 |_{\mathcal{E}_{(x, y, \nu, \beta, \phi)}} (V, \nu),
\]

with \( (W, \eta, \nu, \zeta) \in (\tilde{\mathcal{E}} \oplus \tilde{\mathcal{E}}^\vee)_{(x, y, \nu, \beta, \phi)} \).

In the setting of [BSC20], the additional variables \( \phi \in S^1 \) (with its linearization \( \nu \in T_\phi S^1 \)) and the dual variable \( \zeta \in T^*_\phi S^1 \), which we view here as part of the bundle \( \tilde{\mathcal{E}} \) over the contact manifold \( \mathcal{CS}(M) \), represent a model of phase and velocity of spatial wave propagation. The window function \( \Psi_0 \) that we consider here differs from the function considered in [BSC20], which does not have the Gaussian term in the \( \nu \in T_\phi S^1 \) variable. While they consider the limit case where \( \kappa_\phi = 0 \), we argue here that one needs this additional term to be non-zero (though possibly small) in order to have good signal analysis properties for the associated Gabor system, in the presence of these additional variables. The Gaussian term in \( \nu \) can in principle be replaced by another rapid decay function, however, it seems more natural to use a Gaussian term, like we have for the variables in \( \mathcal{E} \), in order to maintain a similar structure for all the variables of \( \tilde{\mathcal{E}} \). We will return to discuss the case \( \kappa_\phi = 0 \) of [BSC20] in section 3.1.11.

Note that the goal of the model of [BSC20] is different, as they apply the Gabor transform to signals that are independent of the frequency and phase variables, so that the problem outlined above with the frame condition does not arise. It is only when the signal analysis is performed on the larger space given by the 3-dimensional linear fibers of the bundle \( \tilde{\mathcal{E}} \), rather than on the 2-dimensional bundle \( \tilde{\mathcal{E}} \), that one needs to modify the window function as described above.

Let \( \tilde{\mathcal{E}}^\vee \) denote the dual bundle of \( \tilde{\mathcal{E}} \), with the choice of local basis \( \{ R_a, R_a, t, t_\phi \} \) for \( \tilde{\mathcal{E}} \) and the dual local basis \( \{ a, a, \theta, \phi \} \). This determines bundles of framed lattices over the local charts of \( \mathcal{CS}(M) \)

\[
\tilde{\mathcal{E}}^\vee = \mathbf{Z} a + \mathbf{Z} a + \mathbf{Z} \theta + \mathbf{Z} \phi = \Lambda_{a, \theta} \oplus L^\vee, \quad (3.1.40)
\]

with \( \Lambda_{a, \theta} \) and \( \Lambda_{a, \theta}^\vee \) the bundles of framed lattices of (3.1.12) and (3.1.13).

We consider the bundle of framed lattices \( \tilde{\mathcal{E}} \oplus \tilde{\mathcal{E}}^\vee \), which has the property that, in a local chart, the fibers

\[
(\tilde{\mathcal{E}} \oplus \tilde{\mathcal{E}}^\vee)_{(x, y, w, \beta, \phi)} \subset (\tilde{\mathcal{E}} \oplus \tilde{\mathcal{E}}^\vee)_{(x, y, w, \beta, \phi)}
\]

are lattices in the fibers of the 3-plane bundle \( \tilde{\mathcal{E}} \oplus \tilde{\mathcal{E}}^\vee \) over the 5-dimensional contact manifold \( \mathcal{CS}(M) \).

The window function \( \Psi_0 \) and the bundle of framed lattices \( \tilde{\mathcal{E}} \oplus \tilde{\mathcal{E}}^\vee \) determine a Gabor
system
\[ \mathcal{G}(\Psi_0, \tilde{\Lambda} \oplus \tilde{\Lambda}) = \left\{ \rho(\lambda)\Psi_0 \mid \lambda = (W, \eta, v, \zeta) \in (\tilde{\Lambda} \oplus \tilde{\Lambda})_{(x, y, w, \beta, \phi)} \right\}. \]  

(3.1.41)

As in the case of the bundle of framed lattices \( \Lambda_{\alpha, j} \oplus \Lambda_{\alpha, j}^\vee \), we consider a scaling of the lattices in the fibers of \( \tilde{\Lambda} \oplus \tilde{\Lambda}^\vee \), for the same reasons discussed in § 3.1.5. We define \( R_{\text{max}} > 0 \) as in § 3.1.5. For the \( TS^1 \) direction of \( \tilde{E} \), the injectivity radius \( R_{\text{inj}}^{S^1} \) is constant and equal to half the length of the \( S^1 \) circle. Thus, we take, as in § 3.1.5, a scaling factor of the form \( \gamma = R_{\text{inj}}^{S^1} / R_{\text{max}} \). As discussed in § 3.1.5 we can assume that in our model \( R_{\text{max}} > R_{\text{inj}}^{S^1} \) so that \( \gamma < 1 \). We then consider the bundle of framed lattices determined by this choice of scaling on \( \Lambda \) and the previous choice of scaling on \( \Lambda_{\alpha, j} \).

**Definition 3.1.28.** Let \( \tilde{\Lambda}_{b, \gamma} \oplus \tilde{\Lambda}^\vee \) be the bundle of framed lattices of the form
\[ \tilde{\Lambda}_{b, \gamma} \oplus \tilde{\Lambda}^\vee = \Lambda_{b, \alpha, j} \oplus L_{\lambda} \oplus \Lambda_{\alpha, j}^\vee \oplus L^\vee, \]  

(3.1.42)

where \( \Lambda_{b, \alpha, j} \) is the scaled lattice of (3.1.17) with \( b = b_M \) the function of (3.1.16), while \( L_{\lambda} = \lambda L \) for the constant \( \lambda = R_{\text{inj}}^{S^1} / R_{\text{max}} \) as above. This has associated Gabor system
\[ \mathcal{G}(\Psi_0, \tilde{\Lambda}_{b, \gamma} \oplus \tilde{\Lambda}) = \left\{ \rho(\lambda)\Psi_0 \mid \lambda = (W, \eta, v, \zeta) \in \tilde{\Lambda}_{b, \gamma} \oplus \tilde{\Lambda} \right\}. \]  

(3.1.43)

where for simplicity of notation we have suppressed the explicit indication of the fibers of \( \tilde{E} \oplus \tilde{E}^\vee \) as in (3.1.41).

We then have the following result about the Gabor frame condition for the Gabor systems (3.1.41) and (3.1.43).

**Theorem 3.1.29.** The Gabor system \( \mathcal{G}(\Psi_0, \tilde{\Lambda}_{b, \gamma} \oplus \tilde{\Lambda}) \) of (3.1.43) is a frame. The Gabor system \( \mathcal{G}(\Psi_0, \tilde{\Lambda} \oplus \tilde{\Lambda}) \) of (3.1.41) is not a frame.

**Proof.** By construction the Gabor systems with window function \( \Psi_0 \) and lattice \( \tilde{\Lambda} \oplus \tilde{\Lambda}^\vee \) (or \( \tilde{\Lambda}_{b, \gamma} \oplus \tilde{\Lambda}^\vee \)) split as a product of a 2-dimensional system \( \mathcal{G}(\Psi_0, \Lambda_{\alpha, j} \oplus \Lambda_{\alpha, j}^\vee) \) (or \( \mathcal{G}(\Psi_0, \Lambda_{b, \alpha, j} \oplus \Lambda_{\alpha, j}^\vee) \)) and a 1-dimensional Gabor system \( \mathcal{G}(\psi_0, L \oplus L^\vee) \) or \( \mathcal{G}(\psi_0, L_{\lambda} \oplus L^\vee) \), where
\[ \psi_{0, \phi}(v) = \exp(-\kappa \phi v^2 - i \langle \zeta_0, v \rangle \phi) = \exp(-\kappa \phi v^2 - i \zeta_0 v). \]

Thus, the frame condition for \( \mathcal{G}(\Psi_0, \tilde{\Lambda} \oplus \tilde{\Lambda}^\vee) \) holds if and only if it holds for both \( \mathcal{G}(\Psi_0, \Lambda_{\alpha, j} \oplus \Lambda_{\alpha, j}^\vee) \) and \( \mathcal{G}(\psi_0, L \oplus L^\vee) \). Similarly, the frame condition for \( \mathcal{G}(\Psi_0, \tilde{\Lambda}_{b, \gamma} \oplus \tilde{\Lambda}^\vee) \) holds if and only if it holds for both \( \mathcal{G}(\Psi_0, \Lambda_{b, \alpha, j} \oplus \Lambda_{\alpha, j}^\vee) \) and \( \mathcal{G}(\psi_0, L_{\lambda} \oplus L^\vee) \).

For the 1-dimensional systems with a rapid decay function as window function, the frame
condition holds if and only if the lower Beurling density $D^-$ of the lattice is strictly greater than one. For the lattice $L \oplus L^\vee$ this condition is not satisfied (see Proposition 3.1.12) so the Gabor system is not a frame, while for the lattice $L_\Lambda \oplus L^\vee$ is satisfied since $\gamma < 1$ (see Proposition 3.1.13). Thus, in the case of the Gabor system $\mathcal{G}(\Psi_0, \Lambda_{b,\gamma} \oplus \Lambda^\vee)$ of (3.1.43) the question is reduced to the question of whether the 2-dimensional system $\mathcal{G}(\Psi_0, \Lambda_{b,a,l} \oplus \Lambda_{a,l})$ is a frame. We know this system is indeed a frame by Proposition 3.1.22.

3.1.11 Gelfand triples and Gabor frames

We return here to discuss the case of the profiles considered in [BSC20], with the term $\kappa_\phi = 0$. As mentioned above, the function $\tilde{\Psi}_0|_{\kappa=0}$ is not a window function for a Gabor system in the usual sense, as it is not of rapid decay (and not even $L^2$) along the fibers of $\tilde{\mathcal{E}}$. However, we can still interpret it as a tempered distribution on the fibers of $\tilde{\mathcal{E}}$. Thus, one can at least ask the question of whether this window function defines Gabor frames in a distributional sense. To formulate Gabor systems in such a setting, it is convenient to consider the formalism of Gelfand triples (also known as rigged Hilbert spaces, [GV14]).

We consider here the same setting as in [TTT18], [Tsc21] for distributional frames, with Gelfand triples given by

$$S(\tilde{\mathcal{E}}_{(x,y,w,\theta,\phi)}) \hookrightarrow L^2(\tilde{\mathcal{E}}_{(x,y,w,\theta,\phi)})d\text{vol}_{(x,y,w,\theta,\phi)} \hookrightarrow S'(\tilde{\mathcal{E}}_{(x,y,w,\theta,\phi)}),$$

where the space $S$ of tempered distributions is densely and continuously embedded in the $L^2$-Hilbert space, which is densely and continuously embedded in the dual space $S'$ of distributions. The pairing $\langle f, g \rangle$ of distributions $f \in S'$ and test functions $g \in S$ extends the Hilbert space inner product. We write the above triples for simplicity of notation in the form

$$S(\tilde{\mathcal{E}}) \hookrightarrow L^2(\tilde{\mathcal{E}}) \hookrightarrow S'(\tilde{\mathcal{E}}).$$

**Definition 3.1.30.** A distributional Gabor system $\mathcal{G}(\Phi_0, \tilde{\Lambda})$ on $\tilde{\mathcal{E}}$ is given by a window generalized-function $\Phi_0 \in S'(\tilde{\mathcal{E}})$ and a bundle of lattices $\tilde{\Lambda}$ with

$$\mathcal{G}(\Phi_0, \tilde{\Lambda}) = \{\rho(\lambda)\Phi_0 | \lambda \in \tilde{\Lambda}\} \subset S'(\tilde{\mathcal{E}}).$$

The distributional Gabor system $\mathcal{G}(\tilde{\Psi}_0, \tilde{\Lambda})$ is a distributional frame for the bundle $\tilde{\mathcal{E}}$ on $CS(M)$ if there are bounded smooth functions $C, C' : CS(M) \to \mathbb{R}_+$ with strictly positive $\inf_{CS(M)} C$ and $\inf_{CS(M)} C'$, such that, for all $f \in S(\tilde{\mathcal{E}})$

$$C_{(x,y,w,\theta,\phi)}\|f\|^2_{L^2(\tilde{\mathcal{E}}_{(x,y,w,\theta,\phi)})} \leq \sum_{\lambda \in \tilde{\Lambda}_{(x,y,w,\theta,\phi)}} |\rho(\lambda)\Phi_0(f)|^2 \leq C'_{(x,y,w,\theta,\phi)}\|f\|^2_{L^2(\tilde{\mathcal{E}}_{(x,y,w,\theta,\phi)})}.$$
Lemma 3.1.31. Let \( \tilde{\Phi}_0 = \Psi_0|_{\mathbb{R}_m} \), with \( \Psi_0 \) as in (3.1.38). The systems \( G(\tilde{\Phi}_0, \tilde{\Lambda} \oplus \tilde{\Lambda}^\vee) \) and \( G(\Phi_0, \Lambda_{b,\gamma} \oplus \Lambda^\vee_{a,J}) \) with the lattices as in Definition 3.1.28, are distributional Gabor systems that decompose into a product of a 2-dimensional ordinary Gabor system given by \( G(\Psi_0, \Lambda_{a,J} \oplus \Lambda^\vee_{a,J}) \) or \( G(\Psi_0, \Lambda_{b,a,J} \oplus \Lambda^\vee_{a,J}) \), respectively, and a 1-dimensional distributional Gabor system of the form \( G(\phi_0, L \oplus L^\vee) \) or \( G(\phi_0, L_{\gamma} \oplus L^\vee) \), respectively, with window generalized-function \( \phi_0(v) = \exp(-i\zeta_0 v) \in S'(\mathbb{R}) \). The distributional Gabor system \( G(\tilde{\Phi}_0, \tilde{\Lambda} \oplus \tilde{\Lambda}^\vee) \) does not satisfy the distributional Gabor frame condition if and only if the 1-dimensional distributional Gabor system \( G(\phi_0, L_{\gamma} \oplus L^\vee) \) satisfies the distributional frame condition.

Proof. We view \( \tilde{\Phi}_0 \) as the distribution in \( S'(\mathbb{R}) \) that acts on test functions \( f \in S(\mathbb{R}) \) as

\[
\langle f, \tilde{\Phi}_0 \rangle_{x,y,w}(x,y,w) = \int_{\mathbb{R}^3} \tilde{\Phi}_0(x,y,w) (V, v) f(V,v) \, d\mathcal{V}_{x,y,w}(v).
\]

As in Proposition 3.1.29 we see that the distributions \( \rho(\lambda)\tilde{\Phi}_0 \) are products of a function \( \rho(\lambda')\Psi_0 \in S(\mathbb{R}) \) and a distribution \( \rho(\lambda'')\tilde{\Phi}_0 \) in \( S'(\mathbb{R}) \), with \( \lambda = (\lambda', \lambda'') \) for \( \lambda \in \tilde{\Lambda} \oplus \tilde{\Lambda}^\vee \) and \( \lambda' \in \Lambda_{a,J} \oplus \Lambda^\vee_{a,J} \) and \( \lambda'' \in L \oplus L^\vee \) (and similarly for the scaled versions of the lattices). Since these Gabor systems decouple, the distributional frame condition becomes equivalent to the ordinary frame condition for the part that is an ordinary frame and the distributional frame condition for the part that is a distributional frame. Thus, the distributional Gabor systems \( G(\tilde{\Phi}_0, \tilde{\Lambda} \oplus \tilde{\Lambda}^\vee) \) and \( G(\phi_0, \Lambda_{b,\gamma} \oplus \Lambda^\vee) \) are distributional Gabor frames if and only if the respective 2-dimensional ordinary Gabor systems are ordinary frames and the respective 1-dimensional distributional Gabor systems are distributional frames. In the first case we know that the frame condition already fails at the level of the 2-dimensional ordinary Gabor system. In the second case the 2-dimensional system satisfies the usual frame condition by Proposition 3.1.22, hence the question reduces to whether the 1-dimensional distributional system \( G(\phi_0, L_{\gamma} \oplus L^\vee) \) satisfies the distributional frame condition.

The following statement shows that, even when interpreted in this distributional setting the Gabor system generated by the window function \( \tilde{\Phi}_0 \) as in [BSC20] does not give rise to frames, hence it does not allow for good signal analysis.

Theorem 3.1.32. The distributional Gabor system \( G(\tilde{\Phi}_0, \tilde{\Lambda}_{b,\gamma} \oplus \tilde{\Lambda}^\vee) \) does not satisfy the distributional frame condition.

Proof. By Lemma 3.1.31 we can equivalently focus on the question of whether the one-dimensional distributional Gabor system \( G(\phi_0, \gamma \mathbb{Z} \oplus \mathbb{Z}) \) satisfies the distributional frame
condition. Given a signal \( f \in S(\mathbb{R}) \), we have, for \( \lambda = (\gamma n, m) \), and \( \phi_0(t) = e^{-i\tilde{\zeta}_0 t} \),

\[ \langle f, \rho(\lambda) \phi_0 \rangle = \int_{\mathbb{R}} e^{-2\pi imt} f(t) e^{i\tilde{\zeta}_0 (t-\gamma n)} \, dt = e^{-i\tilde{\zeta}_0 \gamma n} \int_{\mathbb{R}} e^{-2\pi i (m - \frac{\tilde{\zeta}_0}{2\pi}) t} f(t) \, dt = e^{-i\tilde{\zeta}_0 \gamma n} \hat{f}(m - \frac{\tilde{\zeta}_0}{2\pi}). \]

Note that when we take \( |\langle f, \rho(\lambda) \phi_0 \rangle|^2 \) the dependence on \( n \) disappears entirely so the sum over the lattice is always divergent.

\[ \Box \]

**Remark 3.1.33.** The window function \( \tilde{\Psi}_0 \mid_{\kappa_{\phi}=0} \) in [BSC20] is chosen so that the Lie group and Lie algebra structure underlying receptive profiles of this form (see [Pet17], [SCP92]) determines horizontal vector fields given by the basis \( \{ X_\theta, R_\alpha, R_{\phi,a,1}, X_w \} \) of (3.1.36) of the contact hyperplanes \( \tilde{\xi}_J \). However, if we replace this choice of window with our window \( \tilde{\Psi}_0 \) where \( \kappa_{\phi} \neq 0 \), the same Lie group of transformations acts on these types of profiles generating the same horizontal vector fields. Note that also the original goal of [BSC20] of describing receptive profiles of neurons sensitive to frequency and phase variables, with the frequency-phase uncertainty minimized is already satisfied by the Gabor system generated by our proposed window function \( \tilde{\Psi}_0 \), without the need to impose \( \kappa_{\phi} = 0 \).
3.2 Examples

In this section we present two examples of the construction in section 3.1, applied on complex tori with flat metric as well as on the hyperbolic half-plane. The previous section was written, having in mind that the underlying manifold should satisfy the sufficient conditions to represent the retina. In particular, the retina is the sphere \( S^2 \) without a point (the point where the optic nerve is attached). In this section, we apply the results of section 3.1 on manifolds which satisfy these conditions but are not the sphere without one point. Our objective is to see how different base manifolds \( B \) of the fiber bundle \( S(T^*B) \to B \) interact with the signals and the Gabor systems introduced in the previous section.

3.2.1 Hyperbolic Half Plane

**Definition 3.2.1.** The hyperbolic plane \( \mathbb{H} \) is the upper half plane

\[
\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : y > 0\}
\]

together with the Riemannian metric \( g_{ij} = \delta_{ij} \frac{1}{y^2} \).

The Christoffel symbols of the Riemannian connection are

\[
\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{22} = 0, \quad \Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}, \quad \Gamma^2_{11} = \frac{1}{y}
\]

The Riemannian metric \( g \) lifts to a Riemannian metric \( \tilde{g} \) on the cotangent bundle \( T^*\mathbb{H} \), according to (4.1.3) and (4.1.4). Expressed in coordinates \( (T^*\mathbb{H}, (x, y, p_1, p_2)) \) \( \gamma_{ij} \) from (4.1.3) become

\[
\gamma_{11} = \frac{1}{y} p_2, \quad \gamma_{12} = -\frac{1}{y} p_1, \quad \gamma_{21} = -\frac{1}{y} p_1, \quad \gamma_{22} = -\frac{1}{y} p_2
\]

and the components of \( \tilde{g} \) at \( (x, y, p_1, p_2) \) are given by the following matrix

\[
\tilde{g}_{(x,y,p_1,p_2)} = \begin{pmatrix}
\frac{1}{y^2} + p_1^2 + p_2^2 & 0 & -yp_2 & yp_1 \\
0 & \frac{1}{y^2} + p_1^2 + p_2^2 & yp_1 & yp_2 \\
-yp_2 & yp_1 & y^2 & 0 \\
yp_1 & yp_2 & 0 & y^2
\end{pmatrix}
\]

The Riemannian volume form on \( T^*\mathbb{H} \) induced by \( \tilde{g} \) is

\[
dvol_{T^*\mathbb{H}}(x, y, p_1, p_2) = dx \wedge dy \wedge dp_1 \wedge dp_2
\]
and written in polar coordinates \(d\text{vol}_{T^*H}(x,y,r,\theta) = rdr \wedge dy \wedge d\theta\). Let \(g^*\) denote the bundle metric on \(T^*H\) induced by \(g\). The unit cotangent bundle \(S(T^*H)\) is defined fiberwise by

\[
S(T^*_{(x,y)}H) = \{ p \in T^*_{(x,y)}H : g^*(p, p) = 1 \} = \{(p_1, p_2) \in T^*_{(x,y)}H : p_1^2 + p_2^2 = \frac{1}{y^2} \} = \{(r, \theta) \in T^*_{(x,y)}H : r^2 = \frac{1}{y^2} \}.
\]

We obtain the volume form

\[
d\text{vol}_{S(T^*H)}(x,y,\theta) = \frac{1}{y} dx \wedge dy \wedge d\theta \tag{3.2.1}
\]
on \(S(T^*H)\) by contracting \(d\text{vol}_{T^*H}(x,y,r,\theta)\) with the vector field \(\partial_r\) which is transversal to \(S(T^*H)\).

The bundle of signal planes is defined according to 3.1.2 as the pullback bundle \(\mathcal{E} = \pi^*TH\) of \(TH\) with respect to the projection \(\pi : TH \to H\). The projection onto the second component gives a map \(h : \mathcal{E} \to TH\) such that \(h|_{\mathcal{E}_{(x,y)}} \to T_{(x,y)}H\) is a linear isomorphism between the fibers of \(\mathcal{E}\) and \(TH\). Let \((TH, (x,y,v_1\partial_x + v_2\partial_y))\) be canonical coordinates on \(TH\). The Riemannian metric \(g\) on \(H\) induces the volume form \(d\text{vol}_{T_{(x,y)}H}\) on the fibers, expressed in coordinates

\[
d\text{vol}_{T_{(x,y)}H} = \sqrt{g_{ij}} dv_1 dv_2 = \frac{1}{y^2} dv_1 dv_2.
\]
The pullback of \(d\text{vol}_{T_{(x,y)}H}\) through the map \(h|_{\mathcal{E}_{(x,y)}}\) is a volume form on the fibers of \(\mathcal{E}\),

\[
d\text{vol}_{\mathcal{E}_{(x,y)}} := h^*_{\mathcal{E}_{(x,y)}}(d\text{vol}_{T_{(x,y)}H}) = \frac{1}{y^2} dv_1 dv_2 \tag{3.2.2}
\]

**Signal and Frames on the hyperbolic half plane** Following definition 3.1.3 we obtain the following formulas for signals in \(H\). A signal is a function \(I : \mathcal{E} \to \mathbb{R}\) such that

\[
|I|_{L^2} = \left( \int_{S(T^*H)} \left( \int_{\mathcal{E}_{(x,y,\theta)}} |I|^2 d\text{vol}_{\mathcal{E}_{(x,y,\theta)}} \right) d\text{vol}_{S(T^*H)} \right)^{1/2} = \left( \int_{S(T^*H)} \left( \int_{\mathcal{E}_{(x,y,\theta)}} I^2(v_1, v_2) \frac{1}{y^2} dv_1 dv_2 \right) \frac{1}{y} dx dy d\theta \right)^{1/2} < \infty \tag{3.2.3}
\]

**The frame condition** Suppose \(\Lambda\) is a lattice bundle, namely a bundle over \(S(T^*H)\) such that each fiber \(\Lambda_{(x,y,\theta)}\) is a lattice on \(\mathcal{E} \oplus \mathcal{E}^\perp_{(x,y,\theta)}\). Let also \(\Psi_0\) be a window function defined as in (3.1.10), the Gabor system \(\mathcal{G}(\Psi_0, \Lambda)\) is a frame if and only if it satisfies equation
using the volume form \((3.2.2)\) the Gabor frame condition for signals on \(\mathbb{H}\) takes the following form. A Gabor system \(\mathcal{G}(\Psi_0, \Lambda)\) satisfies the smooth Gabor frame condition on \(S(T^*\mathbb{H})\) if there exist smooth functions \(\mathbb{R}_+^n\) -valued functions on the coordinate charts of \(S(T^*\mathbb{H})\) such that

\[
C_{(x,y,\theta)} \int_{\mathcal{E}_{(x,y,\theta)}} |f(v_1, v_2)|^2 \frac{1}{y^2} dv_1 dv_2 \leq 
\]

\[
\leq \sum_{(\xi,W) \in \Lambda_{(x,y,\theta)}} \int_{\mathcal{E}_{(x,y,\theta)}} f(v_1, v_2) e^{-2\pi i \xi \cdot (v_1, v_2)} \Psi_{0,(x,y,\theta)}((v_1, v_2) - W) \frac{1}{y^2} dv_1 dv_2 \leq 
\]

\[
\leq C'_{(x,y,\theta)} \int_{\mathcal{E}_{(x,y,\theta)}} |f(v_1, v_2)|^2 \frac{1}{y^2} dv_1 dv_2
\]

Remark \(3.2.2\). From the previous formulas one can see that the norm \(L^2(\mathcal{E}_{(x,y,\theta)})\) of the fiber \(\mathcal{E}_{(x,y,\theta)}\) of the bundle of signal planes \(\mathcal{E} \to S(T^*\mathbb{H})\) depends on the \(y\)-coordinate. More specifically, a signal \(\mathcal{I}_{(x,y,\theta)} : \mathcal{E}_{(x,y,\theta)} \to \mathbb{R}\) lies on the weighted \(L^2\) space with weight \(w(x,y,\theta) = \frac{1}{y^2}\).

The Bundle of Lattices  As in \(3.1.12\) and \(3.1.13\) the basis \(\{a,a_j\}\) for \(\mathcal{E}^\vee\) and \(\{R_a, R_{a_j}\}\) determine a framed bundle of lattices \(\Lambda = \text{span}_\mathbb{Z}\{a,a_j\} \oplus \text{span}_\mathbb{Z}\{R_a, R_{a_j}\}\) for \(\mathcal{E} \oplus \mathcal{E}^\vee\). Let \(\{\mathbb{H} \times U_i\}_{i=1,2}\) coordinate charts of \(S(T^*\mathbb{H})\) with coordinate functions the angle functions \(\theta_j : \mathbb{H} \times U_j \to \mathbb{R}\) such that \(p_j = ye^{i\theta_j(p_j)}\) for \(p_j \in U_j\). Expressed in the coordinate chart \((\mathbb{H} \times U_j, (x, y, \theta_j))\) the basis \(\{a,a_j\}\) and \(\{R_a, R_{a_j}\}\) take the following form

\[
a = \frac{1}{y} (\cos(\theta_j)dx + \sin(\theta_j)dy) \quad \text{and} \quad a_j = \frac{1}{y} (-\sin(\theta_j)dx + \cos(\theta_j)dy)
\]

\[
R_a = y(-\sin(\theta_j)\partial_x + \cos(\theta_j)\partial_y) \quad \text{and} \quad R_{a_j} = y(\cos(\theta_j)\partial_x + \sin(\theta_j)\partial_y)
\]

The following corollary is a consequence of theorem \(3.1.13\).

Corollary \(3.2.3\). Let \(\Psi_0\) be the window function as in \((3.1.10)\) and \(\beta_i, \gamma_i : \mathbb{H} \to \mathbb{R}_+\) positive-valued functions on \(\mathbb{H}\) for \(i = 1, 2\). If the quadratic form \(A\) in \((3.1.10)\) is diagonal, \(A = \text{diag}(\kappa_1^2(x,y), \kappa_2^2(x,y))\), in the global frame \(\{R_a, R_{a_j}\}\) of \(\mathcal{E}\), the Gabor system \(\mathcal{G}(\Psi_0, \Lambda_{\beta,\gamma})\) with the bundle of scaled lattices

\[
\Lambda_{\beta,\gamma} = (\beta_1 R_a \mathbb{Z} + \beta_2 R_{a_j} \mathbb{Z}) \oplus (\gamma_1 a \mathbb{Z} + \gamma_2 a_j \mathbb{Z})
\]

satisfies the frame condition \(3.2.4\) if and only if \(\beta_i \gamma_i < 1\) for every \((x,y) \in \mathbb{H}\).

Proof. It follows by theorem \(3.1.13\) and lemma \(3.1.13\) that the Gabor system \(\mathcal{G}(\Psi_0, \Lambda_{\beta,\gamma})\) splits into two Gabor systems \(\mathcal{G}(\psi_1, \beta_1 R_a \mathbb{Z} + \gamma_1 a \mathbb{Z})\) and \(\mathcal{G}(\psi_2, \beta_2 R_{a_j} \mathbb{Z} + \gamma_2 a_j \mathbb{Z})\) with \(\psi_1(v_1) = \exp(-\kappa_1^2 v_1^2 - iv_1)\) and \(\psi_2(v_2) = \exp(-\kappa_2^2 v_2^2 - iv_2)\). The frame condition is satisfied for \(\mathcal{G}(\Psi_0, \Lambda_{\beta,\gamma})\) is it is satisfied for both subsystems and therefore it is satisfied if and
only if $\beta_i \gamma_i < 1$ for $i = 1, 2$. 

\[ \text{\bf 3.2.2 Flat tori} \]

Let $(a_1, a_2)$ be a basis of $\mathbb{C}$ (as a 2-dimensional vector space over $\mathbb{R}$) and $\Gamma$ the lattice associated to this basis $\Gamma = \{ \sum_{i=1}^{2} k_i a_i : k_i \in \mathbb{Z} \}$. The lattice $\Gamma$ acts on $\mathbb{C}$ by translations and the action is free and properly discontinuous. Additionally, for fixed $\gamma \in \Gamma$ the translation $T_\gamma : \mathbb{C} \to \mathbb{C}$ is biholomorphic and therefore the covering map $q : \mathbb{C} \to \mathbb{C}/\Gamma$ is holomorphic. Hence the quotient has a complex structure with holomorphic local charts given by the local sections of $q$. Finally,

$$p : \mathbb{C} \to T^2 = S^1 \times S^1, \quad p(\sum_{i=1}^{2} x_i a_i) = (e^{2\pi i x_1}, e^{2\pi i x_2})$$

yields, by quotient, a diffeomorphism $\tilde{p} : \mathbb{C}/\Gamma \to T^2$. The complex structure $J_\Gamma : T\mathbb{T}^2 \to T\mathbb{T}^2$ on $T^2$ is determined by the lattice $\Gamma$.

If $\langle - , - \rangle$ is the Euclidean metric on $\mathbb{C}$, the metric on the complex torus is the induced metric. Let $\{ \partial_{x_1}, \partial_{x_2} \}$ be the coordinate frame associated to the holomorphic local coordinates $(x_1, x_2)$, the induced metric $g_\Gamma$ expressed in coordinates is

$$g_\Gamma = \sum_{i,j} \langle a_i, a_j \rangle dx_i dx_j.$$

The metric $g_\Gamma$ lifts to a metric $\tilde{g}_\Gamma$ on the unit cotangent bundle $T^*\mathbb{T}^2 = \mathbb{T}^2 \times \mathbb{R}^2$ according to \ref{sec:4.1.3} and \ref{sec:4.1.4}. Namely, if $(U, (x_1, x_2, p_1, p_2))$ is a local coordinate chart of $T^*\mathbb{T}^2$, $\tilde{g}_\Gamma$ is given by

$$\tilde{g}_\Gamma = \sum_{ij} g_{\Gamma,ij} dx_i dx_j + \sum_{ij} g_{\Gamma,ij} dp_i dp_j$$

$$= \sum_{ij} \langle a_i, a_j \rangle dx_i dx_j + \frac{1}{\langle a_1, a_1 \rangle \langle a_2, a_2 \rangle - 2 \langle a_1, a_2 \rangle} (\langle a_2, a_2 \rangle dp_1 dp_1 - \langle a_1, a_2 \rangle dp_1 dp_2 + \langle a_1, a_1 \rangle dp_2 dp_2).$$

Let $g^*_\Gamma$ denote the bundle metric on $T^*\mathbb{T}^2$ induced by $g_\Gamma$. The unit cotangent bundle is defined according to $Q_\Gamma = 1$, that is

$$S(T^*\mathbb{T}^2) = \{ p \in T^*_{(x_1, x_2)} \mathbb{T}^2 : Q_\Gamma(p) = 1 \text{ for } (x_1, x_2) \in \mathbb{T}^2 \}$$
Signal and Frames on the flat torus  The space of signals on a flat torus depends on the basis $\Gamma$. For instance, if $\Gamma$ is an orthogonal basis, by definition (3.1.3) we obtain the following formulas for signals in $T^2_\Gamma$. A signal is a function $I : \mathcal{E} \to \mathbb{R}$ such that

$$
\|I\|_{L^2} = \left( \int_{S(T^*T^2)} \left( \int_{\mathcal{E}(x_1,x_2,\theta)} |I|^2 \sqrt{\det(g_{\Gamma ij})} dv_1 dv_2 \right) \right)^{1/2}
$$

$$
= \left( \int_{S(T^*T^2)} \left( \int_{\mathcal{E}(x_1,x_2,\theta)} I^2(v_1,v_2) \sqrt{\det(g_{\Gamma ij})} dv_1 dv_2 \right) \right)^{1/2}
$$

$$
\frac{\langle a_1, a_1 \rangle \langle a_2, a_2 \rangle}{\sqrt{\langle a_1, a_1 \rangle \sin(\theta) + \langle a_2, a_2 \rangle \cos(\theta)}} dx_1 dx_2 d\theta < \infty,
$$

where $dvol_{S(T^*T^2)}$ is written in polar coordinates.

**Remark 3.2.4.** From the previous formulas one can see that the norm $L^2(\mathcal{E}(x_1,x_2,\theta))$ of the fiber $\mathcal{E}(x_1,x_2,\theta)$ of the bundle of signal planes $\mathcal{E} \to S(T^*T^2)$ is independent of the coordinates $(x_1,x_2,\theta)$, in contrast to the hyperbolic half-plane.
3.3 Gabor frames and higher dimensional boundaries in signal analysis on manifolds

In this section we introduce a construction of Gabor Frames that encode local linearizations of a signal detected on a curved smooth manifold of arbitrary dimension. In particular we use Gabor Filters that can detect higher-dimensional boundaries on the manifolds. We describe an application in configuration spaces in robotics with motion constraints. The construction is a generalization of the geometric set up, developed for the study of the visual cortex.

3.3.1 Geometry of higher-dimensional signal analysis

We first recall some general facts of Riemannian geometry that will be useful in the rest of the section.

Let $S(T^*B)$ denote the unit sphere bundle of the cotangent bundle $T^*B$ of a Riemannian metric $(B, g)$ as defined in section 2.2. For simplicity of notation, we write

$$\pi_S : S(T^*B) \rightarrow B$$

for the induced projection map. When no confusion arises, we will just use $\pi$ for both the projection $\pi_S$ on $S(T^*B)$ and the projection of $T^*B$. Each fiber $\pi^{-1}(b) \subset S(T^*B)$, for $b \in B$, is isomorphic to the unit sphere $S^{n-1}$ through a isomorphism $j_b : \pi^{-1}(b) \rightarrow S^{n-1}$.

The 1-form $\alpha$ on $S(T^*B)$ denotes the contact 1-form induced by the Liouville form $\lambda$ on $T^*B$.

Since $B$ is a Riemannian manifold, the Riemannian metric $g_B$ provides a (non-canonical) isomorphism between tangent and cotangent bundles

$$g_B : TB \cong T^*B, \quad g_B : v \mapsto g_B(v, \cdot).$$

Thus, we can interpret the fiber $S^{n-1}$ of $M = S(T^*B)$ at a point $b \in B$ as parameterizing hyperplanes in either $T^*_bB$ or $T_bB$.

**Action by rotations** The group $SO(n)$ of orientation preserving orthogonal transformations of $(\mathbb{R}^n, (-, -))$ acts transitively on $S^{n-1} \subset \mathbb{R}^n$

$$SO(n) \times S^{n-1} \rightarrow S^{n-1}, \quad (A, p_1, ..., p_n) \mapsto A \cdot (p_1, ..., p_n).$$
Suppose $B$ is oriented, then $F_{SO(n)}(T^*B)$ denotes the principal $SO(n)$-bundle of positively oriented orthonormal frames of $T^*B$ with respect to the bundle metric

$$g^B_b(p_1, p_1) = g^B_b(g^{-1}_bp_1, g^{-1}_bp_2), \text{ for } p_1, p_2 \in T^*_bB,$$

The unit cotangent bundle $S(T^*B)$ is the bundle associated to the action of $SO(n)$ on $S^{n-1}$

$$F_{SO(n)}(T^*B) \times_{SO(n)} S^{n-1} / SO(n) = S(T^*B).$$

The bundle $F_{SO(n)}(T^*B)$ admits a left action of the group $SO(n),$

$$SO(n) \rightarrow Aut(F_{SO(n)}(T^*B)), \quad A \mapsto v_A$$

Then, each map $v_A \times id : F_{SO(n)}(T^*B) \times S^{n-1} \rightarrow F_{SO(n)}(T^*B) \times S^{n-1},$ with $A \in SO(n),$ induces a map on the quotients

$$F_A : S(T^*B) \rightarrow S(T^*B), \quad (b, p) \mapsto (b, A^{-1}(p)),$$

therefore one can consider the corresponding action of $SO(n)$ on the sphere bundle $S(T^*B),$ given by

$$SO(n) \times S(T^*B) \rightarrow S(T^*B), \quad (A, (b, p)) \mapsto F_A(b, p) = (b, A^{-1}(p)).$$

**Lemma 3.3.1.** The map

$$SO(n) \rightarrow \text{End}(S(T^*B)), \quad A \mapsto F_A$$

is a smooth right group action of $SO(n)$ on $S(T^*B).$ Additionally, the orbits of $SO(n)$ are exactly the fibers of $\pi_S : S(T^*B) \rightarrow B.$

**Proof.** Firstly, $SO(n)$ acts smoothly on $F_{SO(n)}(T^*B) \times S^{n-1}$ by $A \mapsto v_A \times id$ and the quotient map

$$q : F_{SO(n)}(T^*B) \times S \rightarrow S^{n-1}(T^*B), \quad (f, p) \mapsto [(f, p)]$$

is smooth. Therefore, the action map

$$SO(n) \times S(T^*B) \rightarrow S(T^*B), \quad (A, [(f, p)]) \mapsto F_A([(f, p)]) = q \circ (v_A \times id)(f, p)$$

is smooth.

To prove that the map $A \mapsto F_A$ is an antihomomorphism we consider the following: for $A_1, A_2 \in SO(n)$ and $(b, p) \in S(T^*B),$

$$F_{A_1A_2}(b, p) = (b, (A_1A_2)^{-1}(p)) = (b, A_2^{-1}A_1^{-1}(p)) = F_{A_2}F_{A_1}(b, p).$$
Finally, since the action of $SO(n)$ on $F_{SO(n)} \times S^{n-1}$ is transitive and $q$ is surjective, the orbits of $SO(n)$ on $S(T^*B)$ are exactly the fibers $\pi^{-1}(b)$ for all $b \in B$. \qed

**Multiple contact forms** The above action of $SO(n)$ induces an action on the sections of the cotangent bundle of $S(T^*B)$

$$SO(n) \times \Gamma(T^*(S(T^*B))) \to \Gamma(T^*(S(T^*B)))$$

$$(A, \alpha) \mapsto F^*_A \alpha,$$

where by $\Gamma(T^*(S(T^*B)))$ we denote the space of smooth co-vector fields. The maps $F_A$ have the property $\pi_S \circ F_A = \pi_S$. Abusing the notation slightly, we are going to consider $F_A$ to be $F_A = id \times A$ for the proof of the following lemma.

**Lemma 3.3.2.** For any map $F_A : S(T^*B) \to S(T^*B)$, with $A \in SO(n)$, the standard contact form $\alpha$ on $S(T^*B)$ has the property that

$$(F^*_A \alpha)_{(b,p)}(X_{(b,p)}) = \alpha_{(b,Ap)}(X_{(b,Ap)}), \ X \in \mathfrak{X}(S(T^*B))$$

and $F^*_A \alpha$ is a contact 1-form on $S(T^*B)$.

**Proof.** The contact 1-form $\alpha$ on $S(T^*B)$ is induced by the Liouville form $\lambda$ on $T^*B$ as $i^*(\lambda_{(b,1,p)}) = \alpha_{(b,p)}$, through the inclusion

$$i : S(T^*B) \to T^*B, \ (b_1, ..., b_n, p_2, ..., p_n) \mapsto (b_1, ..., b_n, 1, p_2, ..., p_n).$$

Then, if $A$ is any $A \in SO(n)$ and $F_A : S(T^*B) \to S(T^*B)$ its induced map, then for $X \in T_{(b,p)}S(T^*B)$ we have

$$(F^*_A \alpha)_{(b,p)}(X_{(b,p)}) = \alpha_{F(b,p)}(dF_A X_{|_{(b,p)}}) = i^*(\lambda_{(b,A(i(p)))})(dF_A X_{|_{(b,p)}}) \quad (3.3.1)$$

Additionally, the diagram

$$\begin{array}{ccc}
S(T^*B) & \xrightarrow{i} & T^*B \\
\downarrow\pi_S & & \downarrow\pi \\
B & \xrightarrow{=} & B
\end{array}$$

commutes and therefore $i^* \circ \pi^* = \pi^*_S$ and (3.3.1) becomes

$$i^*(\pi^*(A(i(p))_b))(dF_A X_{|_{(b,p)}}) = \pi^*_S(A(i(p))_b)(dF_A X_{|_{(b,p)}}) = (A(i(p))_b)(d\pi_S dF_A X_{|_{(b,p)}})$$

$$= (A(i(p))_b)(d\pi_S X_{|_{(b,Ap)}}) = \pi^*_S(A(i(p))_b)(X_{(b,Ap)}) = i^* \pi^*(A(i(p))_b)(X_{(b,Ap)})$$
Thus, $\tau^*(\lambda_{(b,A(i(p)))})(X_{(b,Ap)}) = \alpha_{F_{A}(b,p)}(X_{(b,Ap)})$.

Moreover, since $F_A$ is a diffeomorphism for every $A \in SO(n)$ it follows that $F_A^*a$ is a contact 1-form.

\textbf{Proposition 3.3.3.} On the co-sphere bundle $M = S(T^*B)$ of an $n$-dimensional manifold $B$ there is a collection $\{F_i^*\alpha\}_{i=1}^{n-1}$ of 1-forms which satisfy the conditions:

1. All the $F_i^*\alpha$, $i = 1, \ldots, n-1$, are contact 1-forms on $M$, with $F_0^*\alpha = \alpha$.
2. At each point $(b, p) \in M$, the cotangent vectors

$$\alpha_{(b,p)}, F_1^*\alpha_{(b,p)}, F_2^*\alpha_{(b,p)}, \ldots, F_{n-1}^*\alpha_{(b,p)}$$

are orthogonal with respect to the inverse Riemannian metric tensor on $B$ (the Riemannian metric on $T_{(b,p)}B$).

3. The fibers $\{\pi^{-1}(b), b \in B\}$ of the fiber bundle $\pi : S(T^*B) \to B$ are Legendrian submanifolds in each of the contact distributions $\xi_i$ induced by the contact 1-form $F_i^*\alpha$, for each $i = 0, \ldots, n-1$.

4. If $V \to M$ is the vertical tangent bundle of $\pi : M \to B$, $\Gamma(V)$ denotes the vertical vector fields of $M$ and $\Re_{F_i^*\alpha}$ is the Reeb field of the contact 1-form $F_i^*\alpha$ for $i = 0, \ldots, n$, then $\text{span}\{\Re_{F_i^*\alpha}\} \cap \Gamma(V) = \{0\}$. Moreover, for every $m \in M$ the vectors $\{\Re_{F_i^*\alpha}(m) : i = 0, \ldots, n-1\}$ are linearly independent.

\textbf{Proof.} Let $(b, p)$ a point in $M$ such that $p$ is in the fiber $\pi^{-1}(b)$ over $b \in B$ and $(U, \Phi)$ a coordinate chart around $(b, p)$. Since the action of $SO(n)$ on the fiber $\pi^{-1}(b)$ is transitive, there exist $R_1, \ldots, R_{n-1} \in SO(n)$ such that $\{\Phi(p), \Phi(R_1(p)), \ldots, \Phi(R_{n-1}(p))\}$ is an orthogonal basis of $\mathbb{R}^n$ with respect to the metric induced by the Riemannian metric on $T_{(b,p)}B$ and $\Phi^{-1} : \mathbb{R}^n \to U$. The 1-forms $F_i^*\alpha = F_{R_i}^*\alpha$, for $i = 0, \ldots, n-1$, that are induced by these rotations are mutually orthogonal contact forms, since by Lemma 3.3.2 we have that

$$F_i^*\alpha_{(b,p)} := \alpha_{(b,R_i(p))} = \pi_{(b,p)}^*(R_i(p)). \quad (3.3.2)$$

Now, we need to show that the fibers $\{\pi^{-1}(b), b \in B\}$ are Legendrian submanifolds in all the contact distributions $\xi_i$ associated to the contact forms $F_i^*\alpha$. Consider a vector $v_p \in T_p\pi^{-1}(b)$. Written in canonical local coordinates $(U, \Phi(b, p)) = (b_1, \ldots, b_n, q_1, \ldots, q_{n-1})$, $v_p$ takes the form

$$v_p = f_1(b, p)\partial_{q_1} + \ldots + f_{n-1}(b, p)\partial_{q_{n-1}}.$$ 

Thus,

$$F_i^*\alpha_{(b,p)}(v) = \alpha_{(b,R_i(p))}(dR_i v) = \pi^*(R_i p)(dR_i v) = R_i p(d\pi(dR_i v)) = R_i p(0) = 0.$$
for all \( i = 0, \ldots, n - 1 \) and therefore \( T_p \pi^{-1}(b) \subset \text{Ker}(F_i^* \alpha) \) for all \( i = 0, \ldots, n - 1 \).

Finally, it is left to prove part 4. On the coordinate chart \((U_i, (b_1, \ldots, b_n, q_1, \ldots, q_{n-1}))\) any contact 1-form \( F_i^* \alpha \) can be written in local coordinates as

\[
F_i^* \alpha = a_{i,1} (q_1, \ldots, q_{n-1}) \, db_1 + \cdots + a_{i,n} (q_1, \ldots, q_{n-1}) \, db_n,
\]

since

\[
\alpha = q_1 \, db_1 + \cdots + \sqrt{1 - q_1^2 - \cdots - q_{n-1}^2} \, db_n \quad \text{and}
\]

\[
(a_{i,1}, \ldots, a_{i,n}) = R_i(q_1, \ldots, q_{n-1}, \sqrt{1 - q_1^2 - \cdots - q_{n-1}^2}).
\]

Thus, from the defining property of the Reeb field, it follows that

\[
\iota_{\mathfrak{R}_{F_i^* \alpha}} d F_i^* \alpha(X) = 0 \quad \text{for all vector fields } X \text{ in } TM.
\]

Writing the Reeb fields \( \mathfrak{R}_{F_i^* \alpha} \) in coordinates \( \mathfrak{R}_{F_i^* \alpha} = R_i^{b_1} \, \partial b_1 + \cdots + R_i^{b_n} \, \partial b_n + R_i^{q_1} \, \partial q_1 + \cdots + R_i^{q_{n-1}} \, \partial q_{n-1} \), it follows that

\[
\begin{pmatrix}
\frac{\partial a_{i,1}}{\partial q_1} & \cdots & \frac{\partial a_{i,1}}{\partial q_{n-1}} \\
\vdots & \ddots & \vdots \\
\frac{\partial a_{i,n}}{\partial q_1} & \cdots & \frac{\partial a_{i,n}}{\partial q_{n-1}}
\end{pmatrix}
\begin{pmatrix}
R_i^{q_1} \\
\vdots \\
R_i^{q_{n-1}}
\end{pmatrix} = 0,
\]

and therefore

\[
\mathfrak{R}_{F_i^* \alpha} = R_i^{b_1} \, \partial b_1 + \cdots + R_i^{b_n} \, \partial b_n.
\]

\[
\square
\]

### 3.3.2 The bundle of signal spaces

The setting for signal analysis through Gabor filters requires the underlying linear space structure, as the Gabor system of filters is constructed via the linear operations of translation and modulation. When the signal \( f : B \rightarrow \mathbb{R} \) itself is defined on an \( n \)-dimensional manifold \( B \) rather than on a vector space \( \mathbb{R}^n \), applying Gabor signal analysis requires the use of local linearizations of the manifold \( B \), over which one can construct Gabor filters. These linearizations are provided by the tangent spaces (translation coordinates) and their duals (modulation coordinates). A special case of this kind of Gabor signal analysis on manifolds, motivated by models of the visual cortex in neuroscience, was introduced in [LM23], in the case where \( B \) is a Riemann surface. We generalize here the construction to arbitrary dimensions. The key observation of [LM23] is that it is important to maintain the distinction between the coordinates of the curved manifolds \( B \) and \( M = S(T^*B) \) and the linear coordinates in the fibers of the bundles \( TB \) and \( T^* B \). Making this dis-
tinction precise geometrically requires introducing a suitable bundle of signal spaces. This generalizes the bundle of signal planes introduced in [LM23].

**Definition 3.3.4.** The bundle of signal spaces $\mathcal{E}$ is the real n-space bundle on the contact $(2n - 1)$-dimensional manifold $M = S(T^*B)$ obtained by pulling back the tangent bundle $TB$ of the n-dimensional manifold $B$ to $M$ along the projection $\pi_S : S(T^*B) \to B$ of the unit sphere bundle of $T^*B$,

$$\mathcal{E} := \pi_S^* TB. \quad (3.3.3)$$

Let $\mathcal{E}^\vee$ denote the dual bundle of $\mathcal{E}$

$$\mathcal{E}^\vee = \text{Hom}(\mathcal{E}, \mathbb{R}) = \bigsqcup_{m \in M} \text{Hom}(\mathcal{E}_m, \mathbb{R}). \quad (3.3.4)$$

The real vector bundle $\mathcal{E}$ of rank $n$ and its dual determine a rank $2n$ vector bundle over the $2n - 1$ dimensional manifold $M$, given by their direct sum

$$\mathcal{E} \oplus \mathcal{E}^\vee. \quad (3.3.5)$$

The spatial frequencies (modulation operators) are represented by the fiber coordinates of the dual bundle $\mathcal{E}^\vee$, while the translation operators are provided by the fiber coordinates of the bundle $\mathcal{E}$, see section 3.3.2 below.

The $L^2$ space $L^2(\mathcal{E}, \mathbb{R})$ is determined by the condition

$$\left( \int_M \int_{\mathcal{E}_{(b,p)}} \mathcal{I}^2(u, b, p) \text{dvol}_{\mathcal{E}_{(b,p)}}(u) \text{dvol}_M(b, p) \right)^{1/2} < \infty, \quad (3.3.6)$$

where $(b, p)$ are the local coordinates of $M$, and $u = (u_1, ..., u_n)$ are the coordinates of the fibers $\mathcal{E}_{(b,p)}$. The norm on the fibers $\text{dvol}_{\mathcal{E}_{(b,p)}}$ is induced by the inner product on $TB$ through the pullback map and $\text{dvol}_M$ is the measure induced by the Riemannian volume form on $M$ determined by the Riemannian metric on $B$ (see paragraph 4.1.4 for more details).

**Definition 3.3.5.** A signal is a function

$$\mathcal{I} : \mathcal{E} \to \mathbb{R},$$

on the bundle of signal spaces, with $\mathcal{I} \in L^2(\mathcal{E}, \mathbb{R})$.

Given a Riemannian manifold $B$, the exponential map is a locally defined map $\exp : TB \to B$ from the tangent bundle of $B$ to the manifold $B$, where fiberwise $\exp_b : T_bB \to B$ is obtained by considering, for a vector $v \in T_bB$ the unique geodesic $\gamma_v$ in a neighborhood
of $b$ in $B$ starting at $b$ with tangent vector $v$ and setting $\exp_b(v) = \gamma_v(1)$. The domain of definition of $\exp_b$ in $T_bB$ is a sufficiently small ball $B(0, R)$ around $0 \in T_bB$ such that for all $v \in B(0, R)$ the point $\gamma_v(1) \in B$ is uniquely determined by the existence and uniqueness theorem applied to solutions of the geodesic equation for the Riemannian metric in $B$. The exponential map is defined on all of $T_bB$, for all $b \in B$, iff the manifold $B$ is geodesically complete.

At a given point $b \in B$ let $R_{inj}(b) > 0$ denote the supremum of all the radii $R > 0$ such that the exponential map $\exp_b$ is a diffeomorphism on the ball $B(0, R)$ of radius $R$ in $T_bB$ to its image in $B$. For a compact manifold $B$, this determines a continuous injectivity radius function $R_{inj} : B \to \mathbb{R}_+^*$. We denote by $B(T_bB)$ the ball $B(0, R_{inj}(b))$ in the tangent space $T_bB$. Under the pullback from $B$ to $M$ we obtain a collection of balls of radius $R_{inj}(b)$ in each fiber $\mathcal{E}(b, p)$ with $(b, p) \in M$. We denote these balls by $B(\mathcal{E}(b, p))$.

**Lemma 3.3.6.** Let $B$ be a compact smooth manifold. Let $f : B \to \mathbb{R}$ be a smooth function (or more generally a function in $L^\infty(B)$ with the measure given by the volume form of the Riemannian metric). Then $f$ determines a signal $\mathcal{I}(f) \in L^2(\mathcal{E}, \mathbb{R})$, with the property that $f$ can be recovered from the restrictions $\mathcal{I}(f)|_{B(\mathcal{E}(b, p))}$.

**Proof.** Consider then a smooth function $\chi : \mathcal{E} \to \mathbb{R}$, such that the restriction

$$\chi_{(b, p)} := \chi|_{\mathcal{E}(b, p)}$$

to the fiber $\mathcal{E}(b, p)$ is a rapidly decaying Schwartz function $\chi_{(b, p)} : \mathcal{E}(b, p) \to \mathbb{R}$, which satisfies $\chi_{(b, p)} \equiv 1$ inside the ball $B(\mathcal{E}(b, p))$.

Compact manifolds are geodesically complete, hence the exponential map of $B$ is defined on the full tangent spaces, not just on a neighborhood of the origin, hence the function $f : B \to \mathbb{R}$ determines a function $f \circ \exp : TB \to \mathbb{R}$ by precomposition. Since $f$ is bounded (respectively, essentially bounded) on $B$, the pullback is bounded (respectively, essentially bounded) on $TB$.

Consider then the pullback diagram

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{h} & TB \\
\downarrow{\pi_\mathcal{E}} & & \downarrow{\pi_{TB}} \\
M & \xrightarrow{\pi_{\mathcal{S}}} & B
\end{array}
$$

with $\pi_\mathcal{S} : M \to B$, $\pi_{TB} : TB \to B$, and $\pi_\mathcal{E} : \mathcal{E} \to M$ the projections, and define the function $\mathcal{I}(f) : \mathcal{E} \to \mathbb{R}$ as

$$\mathcal{I}(f) := \chi \cdot f \circ \exp \circ h.$$
The function $f \circ \exp \circ h$ is bounded (or essentially bounded) on the fibers $E_{(b,p)}$ and $\chi$ is rapidly decreasing, hence their product is in $L^2(E_{(b,p)})$. This fiberwise $L^2$ norm varies smoothly with the point $(b,p) \in M$, and is bounded on the compact manifold $M$, so that the integration along $M$ in (3.3.6) is also finite. Thus $I(f) \in L^2(E, \mathbb{R})$.

Moreover, since we have $\chi_{(b,p)} \equiv 1$ inside the ball $B(E_{(b,p)})$, the restrictions satisfy $I(f) |_{B(E_{(b,p)})} = f \circ \exp \circ h$. Since the map $h : E \to TB$ is the fiberwise identification $E_{(b,p)} \simeq T_b B$, for all $(b,p) \in \pi^{-1}_S(b)$, and the map $\exp \circ h$ is a diffeomorphism when restricted to $B(E_{(b,p)})$ since $\exp$ is a diffeomorphism on $B(0, R_{inj}(b)) \subset T_b B$. Thus, the restriction of $f$ to $\exp(B(0, R_{inj}(b)))$ can be fully reconstructed from $I(f) |_{B(E_{(b,p)})}$ and the collection of the functions $I(f) |_{B(E_{(b,p)})}$ fully determine $f$.

Lemma 3.3.6 shows that it is equivalent to think of a signal as a function $f : B \to \mathbb{R}$ defined over the $n$-dimensional manifold $B$, or as the corresponding $I(f) : E \to \mathbb{R}$, that is, as a function on the bundle of signal spaces as in Definition 3.3.5. Passing from $f : B \to \mathbb{R}$ to $I(f) : E \to \mathbb{R}$ corresponds to replacing a signal on a manifold with a consistent collection of signals on its local linearizations.

Remark 3.3.7. Note that for $p_1$ and $p_2$ in the same fiber $\pi_S^{-1}(b)$ over some $b \in B$, the lifted signals $I(f) |_{E_{(b,p_1)}}$ and $I(f) |_{E_{(b,p_2)}}$ are equal to $(f \circ \exp) |_{T_b B}$ and therefore the lift of a signal $f : B \to \mathbb{R}$ is constant along the fibers of $M$. This property has an important physical meaning in the case when $B$ is the retina and $M$ is the visual cortex. Namely, the retinal signal $f$, which depends only on the spatial variables $b \in B$ is lifted to the visual cortex $M$ where now one can use filters which depend on intrinsic variables like orientation or color to analyze the lifted signal. Although the retinal signal $f$ is lifted unchanged into the signal $I(f)$, the latter lives on a space where functions depending on more parameters can exist.
Figure 3.3: Passing from \( f : B \to \mathbb{R} \) to \( \mathcal{I}(f) : \mathcal{E} \to \mathbb{R} \) corresponds to replacing a signal on a manifold with a consistent collection of signals on its local linearizations.

**The lattice bundle** We use the data of the signal bundle \( \mathcal{E} \) and its dual \( \mathcal{E}^\vee \), along with the contact 1-forms \( F_i^* \alpha \) and their Reeb vector fields \( R_{F_i^* \alpha} \) discussed above, to obtain an additional structure on the bundle \( \mathcal{E} \oplus \mathcal{E}^\vee \) consisting of a “lattice bundle”, in the sense of the following definition.

**Definition 3.3.8.** Given a vector bundle \( F \) over a smooth manifold \( M \), a framed lattice bundle structure on \( F \) is a subbundle \( \Lambda \subset F \) with discrete fibers \( \Lambda_m = \pi_{\Lambda}^{-1}(m) \), for \( m \in M \), where \( \pi_{\Lambda} = \pi_F |_{\Lambda} \) is the restriction of the projection \( \pi_F : F \to M \), with the property that there is a dense open set \( U \subset M \) such that, for all \( m \in U \) the fiber \( \Lambda_m \subset F_m \) is a lattice in the vector space \( F_m = \pi_F^{-1}(m) \), endowed with a generating set (frame).

**Lemma 3.3.9.** The contact 1-forms \( \{ F_i^* \alpha \}_{i=0}^{n-1} \) on the manifold \( M = S(T^*B) \) of an \( n \)-dimensional smooth manifold \( B \) determine a framed lattice bundle structure on the rank 2n vector bundle \( \mathcal{E} \oplus \mathcal{E}^\vee \) of (3.3.5).

**Proof.** Consider the set of 1-forms \( \{ F_i^* \alpha \}_{i=0}^{n-1} \) as sections of \( T^*B \), and the set of associated Reeb vector fields \( R_{F_i^* \alpha} \) as sections of \( TB \). Under the identification of the fibers \( \mathcal{E}_{(b,p)} \cong T_bB \), for all \( (b,p) \in \pi_S^{-1}(b) \), induced by the pullback map \( h : \mathcal{E} \to TB \), we can identify the \( R_{F_i^* \alpha} \) as sections of \( \mathcal{E} \) by precomposition with the projection \( \pi_S : M \to B \) and postcomposition with the identification \( T_bB \cong \mathcal{E}_{(b,p)} \). Similarly, we can identify the \( F_i^* \alpha \) with sections of \( \mathcal{E}^\vee \). For simplicity, we maintain the same notation \( R_{F_i^* \alpha} \) and \( F_i^* \alpha \) for these resulting sections without writing the pullback maps explicitly. We can then consider the
spans
\[
\Lambda_E := \text{span}_Z \{R_{i^*\alpha}\} \subset \mathcal{E},
\Lambda_{E^\vee} := \text{span}_Z \{F_{i^*\alpha}\} \subset \mathcal{E}^\vee,
\Lambda := \Lambda_E \oplus \Lambda_{E^\vee} \subset \mathcal{E} \oplus \mathcal{E}^\vee. 
\]

The rank of the fibers \(\Lambda_m\) is \(2n\) on a dense open set \(\mathcal{U} = \pi_S^{-1}(\mathcal{V})\) where \(\mathcal{V} \subset B\) is the dense open set where none of the 1-forms \(F_{i^*\alpha}\) and of the vector fields \(R_{i^*\alpha}\) vanish. Thus, it is a framed lattice bundle structure on \(\mathcal{E} \oplus \mathcal{E}^\vee\) with frame given by the set \(\{R_{i^*\alpha}, F_{i^*\alpha}\}_{i=0}^{n-1}\).

Remark 3.3.10. Unlike the case where \(B\) is a 2-dimensional Riemann surface, considered in [LM23], in this more general case the Reeb fields \(R_{i^*\alpha}\) are in general not in the kernel of the forms \(F_{j^*\alpha}\) with \(j \neq i\), that is, one does not have the "dual-basis relation"
\[
\langle F_{j^*\alpha}, R_{i^*\alpha} \rangle = \delta_{ij}.
\]

It is possible to consider another framed lattice bundle structure on \(\mathcal{E} \oplus \mathcal{E}^\vee\), where the frame \(\{R_{i^*\alpha}, F_{i^*\alpha}\}_{i=0}^{n-1}\) of Lemma 3.3.9 is replaced by another frame \(\{\mathfrak{U}_i, F_{i^*\alpha}\}_{i=0}^{n-1}\), where the sections \(\mathfrak{U}_i\) of \(\mathcal{E}\) are taken to be the dual basis of the \(F_{i^*\alpha}\) (seen as sections of \(\mathcal{E}^\vee\)), so that the relation
\[
\langle F_{j^*\alpha}, \mathfrak{U}_i \rangle = \delta_{ij}
\]
holds on the dense open set \(\mathcal{U} = \pi_S^{-1}(\mathcal{V})\), with \(\mathcal{V} \subset B\) the dense open set where all the 1-forms \(F_{j^*\alpha}\) are non-trivial. In this case the lattice bundle structure on \(\mathcal{E} \oplus \mathcal{E}^\vee\) is given by
\[
\Lambda_E := \text{span}_Z \{\mathfrak{U}_i\} \subset \mathcal{E},
\Lambda_{E^\vee} := \text{span}_Z \{F_{i^*\alpha}\} \subset \mathcal{E}^\vee,
\Lambda := \Lambda_E \oplus \Lambda_{E^\vee} \subset \mathcal{E} \oplus \mathcal{E}^\vee. 
\]

We refer to the lattice bundle structure of (3.3.7) as the Reeb lattice bundle structure and to (3.3.8) as the dual-basis lattice bundle structure.

We use the following notation for sections of lattice bundles.

Definition 3.3.11. For a framed lattice bundle structure \(\Lambda = \Lambda \oplus \Lambda^\vee\) on a vector bundle \(\mathcal{E} \oplus \mathcal{E}^\vee\) over \(M\), we write \((\lambda, \lambda')\) for sections
\[
(\lambda, \lambda') \in \Gamma(M, \Lambda),
\]
that is, for sections of \(\mathcal{E} \oplus \mathcal{E}^\vee\) with values in the discrete subbundle \(\Lambda\).
The choice of window function In the usual setting of time-frequency analysis of signals $I \in L^2(\mathbb{R})$, the signal analysis is performed through linear transform using a family of wavelets $\{\phi_a\}_{a \in A}$.

In our setting, the construction of wavelets takes place on the $n$-dimensional linear spaces $E_{(b,p)}$ given by the fibers of the bundle $E$ of signal spaces introduced in Definition 3.3.4, with translation and modulation operators associated to the bundle of lattices introduced in section 3.3.2 above. We describe here the appropriate choice of window function and the construction of the resulting Gabor systems on the bundle $E$.

Let $V$ and $\eta$ denote the variables in the fibers $T_b B$ and $T_b^* B$ respectively, for $b \in B$, with $\langle \eta, V \rangle$ denoting the duality pairing of $TB$ and $T^* B$.

**Definition 3.3.12.** A window function on the bundle $TB \oplus T^* B$ is a smooth real-valued function $\Phi : TB \oplus T^* B \to \mathbb{R}$ from the total space of the vector bundle, defined fiberwise as

$$\Phi_b(V, \eta) = \exp \left( -V^t A_b V - i \langle \eta, V \rangle_b \right),$$

where $A$ is a symmetric, positive definite tensor field $A : B \to T^* B \otimes T^* B$, such that, for all points $b \in B$ the eigenvalues are uniformly bounded away from zero.

**Lemma 3.3.13.** The restriction of the function $\Phi$ of (3.3.9) to the bundle $TB \times S(T^* B)$ induces a smooth function $\Psi : E \to \mathbb{R}$ from the total space of the bundle $E$ of signal spaces,

$$\Psi_{(b,p)}(V) = \exp \left( -V^t A_b V - i \langle \eta_p, V \rangle_b \right),$$

where $\eta_p$ is just the point $p \in S^{n-1} \simeq S(T^*_b B)$ seen as a cotangent vector.

**Proof.** As discussed above, the map $h : E \to TB$ gives an identification of the fibers $E_{(b,p)} \simeq T_b B$, for all $(b, p) \in \pi^{-1}_E (b)$. Thus, we can identify the variables $V$ as variables in the fibers $E_{(b,p)}$. The choice of a point $m = (b, p) \in M$ corresponds to the choice of a point $b \in B$ and a unit vector $\eta_p$ in the unit cotangent sphere $S(T^*_b B)$. \hfill $\Box$

**Definition 3.3.14.** A Gabor system $G(\Psi, \Lambda)$ on an $n$-dimensional smooth compact manifold $B$ is determined by the data of a smooth window function $\Psi : E \to \mathbb{R}$ that is of rapid decay in the fiber directions $V \in E_{(b,p)}$, for all $(b, p) \in M$, and a lattice bundle $\Lambda$ over $M$. The Gabor system $G(\Psi, \Lambda)$ with these data consists of the collection

$$G(\Psi, \Lambda) = \{ M_{\lambda'} T_{\lambda} \Psi \mid (\lambda, \lambda') \in \Gamma(M, \Lambda) \},$$

where $(\lambda, \lambda')$ are sections as in Definition 3.3.11.

Lemma 3.3.9 and Lemma 3.3.13 immediately imply the following statement.
Theorem 3.3.15. Let $B$ be an $n$-dimensional smooth compact manifold. The collection $\{F_i^a\}_{i=0}^{n-1}$ of contact 1-forms on $M = S(T^*B)$ and the function $\Phi$ of (3.3.9) determine a Gabor system $\mathcal{G}(\Psi, \Lambda)$ on $B$, with window function $\Psi$ as in (3.3.10) and with lattice bundle structure $\Lambda$ on $\mathcal{E} \oplus \mathcal{E}^{\vee}$ as in Lemma 3.3.9 (with either the Reeb lattice bundle structure of (3.3.7) or the dual-basis lattice bundle structure of (3.3.8)).

We can regard the Gabor system $\mathcal{G}(\Psi, \Lambda)$ as a consistent collection of Gabor systems $\mathcal{G}(\Psi_m, \Lambda_m)$ in $L^2(\mathcal{E}_m)$ for $m \in M$. In each fiber $\mathcal{E}_m$ the wavelets in $\mathcal{G}(\Psi_m, \Lambda_m)$ can be used to analyze the restriction $\mathcal{I}_{|\mathcal{E}_m}$ of a signal $\mathcal{I} : \mathcal{E} \to \mathbb{R}$.

### 3.3.3 Boundary detection property

The main reason why the construction of Gabor filters described in Theorem 3.3.15 uses the bundle $\mathcal{E}$ over the manifold $M = S(T^*B)$ rather than the tangent bundle $TB$ over the manifold $B$ is in order to obtain filters that are especially suitable to detect $(n-1)$-dimensional boundaries in a signal $f : B \to \mathbb{R}$ (lifted to a signal $\mathcal{I}(f) : \mathcal{E} \to \mathbb{R}$ as in Lemma 3.3.6).

To see this property, it suffices to focus on a single fiber $\mathcal{E}_m \simeq \mathbb{R}^n$. The restriction of the window function $\Psi$ to this fiber is a rapid decay function of the form (3.3.10), for $m = (b, p)$, where the unit cotangent vector $\eta_p$ parameterizes a choice of an oriented hyperplane in $\mathcal{E}_m \simeq \mathbb{R}^n$.

Consider a signal $f : B \to \mathbb{R}$ that is a characteristic function $f = \chi_U$ of a bounded open set $U \subset B$ with smooth boundary $\Sigma = \partial U$ given by an $(n-1)$-dimensional smooth hypersurface $\Sigma$ in $B$. Let $\mathcal{I}(f)_{|\mathcal{E}_m} : \mathcal{E}_m \to \mathbb{R}$ denote the lifted signals on the fibers of the bundle $\mathcal{E}$ of signal spaces. As in [SCP92], define the output function

$$O_b(f, \eta_p) := \int_{\mathcal{E}_{(b,p)}} \mathcal{I}(f)_{|\mathcal{E}_{(b,p)}}(V) : \Psi_{(b,p)}(V) \, dV. \quad (3.3.12)$$

Theorem 3.3.16. For a given signal $f : B \to \mathbb{R}$ of the form $f = \chi_U$, with corresponding lift $\mathcal{I}(f) : \mathcal{E} \to \mathbb{R}$, and for a fixed $b \in \Sigma \subset B$, the output function $O_b(\eta_p)$ has a local maximum for $p \in S^{n-1}$ the normal vector $\nu_b(\Sigma)$ at $b$ to the boundary hypersurface $\Sigma = \partial U$,

$$\arg\max_{p \in S^{n-1}} O_b(\chi_U, \eta_p) = \nu_b(\Sigma).$$

Proof. Let $g^*_{(b,p)}$ be the Riemannian metric on the fibers of $\mathcal{E}^{\vee}$, $\mathcal{E}_{(b,p)}^{\vee}$, induced by the Riemannian metric on $B$. Then, $O(\eta_p)$ has a local maximum when the gradient with respect
to $g^{n}_{(b,p)} \nabla^{g^{n}_{(b,p)}} \mathcal{O}(\eta_{p})$, is zero. Thus, we have the following

$$
\nabla^{g^{n}_{(b,p)}} \mathcal{O}_{b}(I(f), \eta_{p}) = \int_{\mathcal{E}(b,p)} I(f) |_{\mathcal{E}(b,p)} (V) \cdot \nabla^{g^{n}_{(b,p)}} \mathcal{V}_{(b,p)}(V) \ dV \\
= \int_{\mathcal{E}(b,p)} \chi \cdot f \exp h(V) \cdot i(\nabla^{g^{n}_{(b,p)}} \eta_{p}, V > b) \mathcal{V}_{(b,p)}(V) \ dV
$$

(3.3.13)

At the same time, it holds that

$$
I(f)(V) = \begin{cases} 
1, & \exp h(V) \in U \\
0, & \text{else}
\end{cases}
$$

and therefore equation (3.3.13) becomes

$$
\nabla^{g^{n}_{(b,p)}} \mathcal{O}_{b}(I(f), \eta_{p}) = \int_{h^{-1}(T_{b}U)} I(f) |_{h^{-1}(T_{b}U)} (V)i(\nabla^{g^{n}_{(b,p)}} \eta_{p}, V > b) \mathcal{V}_{(b,p)}(V) \ dV.
$$

(3.3.14)

It follows, from equation (3.3.14), that $\nabla^{g^{n}_{(b,p)}} \mathcal{O}_{b}(I(f), \eta_{p})$ is equal to zero exactly when

$$
\nabla^{g^{n}_{(b,p)}} \eta_{p}, V > b = 0, \text{ for every } V \in h^{-1}(T_{b}U).
$$

The latter condition holds when $\eta_{p}, V > b$ has a local maximum for every $V \in h^{-1}(T_{b}U)$, hence it holds when $\eta_{p}$ is normal to $\Sigma$, since $V \in T_{b}B$ at $b$.

\[ \square \]

### 3.3.4 Geometric Bargmann transforms and Gabor frames

In order to ensure that a Gabor system $\mathcal{G}(\Psi, \Lambda)$ on an $n$-dimensional manifold $B$, in the sense of Definition 3.3.14 has good signal analysis properties, we need a method to detect whether it satisfies the frame condition. The local frames $\{V_{1}, ..., V_{n}\}$ and $\{a, F_{1}^{*}a, ..., F_{n-1}^{*}a\}$ of $\mathcal{E}$ and $\mathcal{E}^{\vee}$ determine a local isomorphism between $\mathcal{E}$ and $\mathcal{E}^{\vee}$. For $(V, \eta)$ in the fiber $(\mathcal{E} \oplus \mathcal{E}^{\vee})_{(b)}$, we define the vector bundle morphism

$$
\mathcal{I} : \mathcal{E} \oplus \mathcal{E}^{\vee} \rightarrow \mathcal{E} \oplus \mathcal{E}^{\vee}, (V, \eta) \mapsto (\eta, -V) := \sum_{i=1}^{n} \eta_{i}V_{i} - v_{i}F_{i-1}^{*}a
$$

(3.3.15)

which satisfies the condition $\mathcal{I}^{2} = -1$ and gives a $\mathbb{C}$-linear isomorphism

$$
\mathcal{J}_{x,g} : \mathcal{E} \oplus \mathcal{E}^{\vee}_{x} \rightarrow \mathbb{C}^{n} \\
(V, \eta) \mapsto (v_{1} + i\eta_{1}, ..., v_{n} + i\eta_{n}).
$$
with scalar multiplication by \( \lambda \in \mathbb{C} \), \( \lambda = x + iy \) with \( x, y \in \mathbb{R} \) given by \( \lambda \cdot (V + i\eta) = (x + yI)(V, \eta) \).

**Definition 3.3.17.** The Bargmann Transform of a function \( f \in L^2(\mathcal{E}, \mathbb{C}) \) is the function \( Bf : \mathcal{E} \oplus \mathcal{E}^\vee \to \mathbb{C} \) defined at each fiber as

\[
Bf \big|_{(V, \eta)} (W) := \int_{\mathcal{E}_x} f \big|_{(V, \eta)} (W) e^{2\pi i (\Re W \cdot V - V^t A_x V - \frac{\pi}{2} \mathcal{P}(V, \eta))} d\text{vol}_{x,y}(W),
\]

where

\[
W * (V, \eta) := W^t A_x V + i\langle \eta, V \rangle
\]

and \( \mathcal{P} : \mathcal{E} \oplus \mathcal{E}^\vee \to \mathbb{C} \) the quadratic form associated to \( A \) defined as

\[
\mathcal{P}(V, \eta) := V^t A_x V + 2i\langle \eta, V \rangle - \eta^t \eta.
\]

The volume form \( d\text{vol}_{x,y}(W) \) is the volume form on the fibers of \( \mathcal{E} \) determined by the Riemannian metric on \( M \).

**Definition 3.3.18.** Let \( z \) denote \( \mathcal{J}(V, \eta + \frac{\eta y}{2\pi}) \) for some \( (V, \eta) \) in \( \mathcal{E} \oplus \mathcal{E}^\vee \), then

- The Bargmann-Fock space \( \mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee) \) is the space of functions \( f : \mathcal{E} \oplus \mathcal{E}^\vee \to \mathbb{C} \) such that the \( f \big|_{(V, \eta)} \circ \mathcal{J}^{-1} : \mathbb{C}^n \to \mathbb{C} \) is entire and they are bounded with respect to the norm

\[
\|f\|_{\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)} := \int_M \left( \int_{\mathbb{C}^n} |f \big|_{(V, \eta)} \circ \mathcal{J}^{-1}(z)|^2 e^{-\frac{\pi}{2}(\Re z^t A_x \Re z + \Im z^t \Im z)} dz \right)^{\frac{1}{2}} d\text{vol}_M
\]

- The fiberwise Bargmann-Fock space \( \mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y} \) is the space of all functions \( F : (\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y} \to \mathbb{C} \) such that \( F \circ \mathcal{J}^{-1} : \mathbb{C}^n \to \mathbb{C} \) is entire and

\[
\|F\|_{\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y}} := \left( \int_{\mathbb{C}^n} |F \big|_{(V, \eta)} \circ \mathcal{J}^{-1}(z)|^2 e^{-\frac{\pi}{2}(\Re z^t A_x \Re z + \Im z^t \Im z)} dz \right)^{\frac{1}{2}} < \infty
\]

- A subset \( \Lambda \) of \( \mathbb{C}^n \) is a set of sampling for \( \mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y} \) if there exist \( A \) and \( B \) smooth, positive functions on the local charts of \( M \) such that

\[
A_{x,y} \|F\|_{\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y}} \leq \sum_{\lambda \in \Lambda} |F \big|_{\mathcal{E} \oplus \mathcal{E}^\vee} \circ \mathcal{J}^{-1}(\lambda) \|e^{-\frac{\pi}{2}(\Re \lambda^t A_x \Re \lambda + \Im \lambda^t \Im \lambda)} \| \leq B_{x,y} \|F\|_{\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y}}
\]
Proposition 3.3.19. The Bargmann-Fock space $\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)$ with inner product

$$\langle F, G \rangle_{\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)} := \int_M \left( \int_{\mathbb{C}^n} F \circ J^{-1}(z) \overline{G \circ J^{-1}(z)} e^{-\frac{t}{2} \left( \Re(z)^t \frac{\Lambda_x}{2} \Re(z) + \Im(z)^t \Im(z) \right)} \, dz \right) \, \text{dvol}_M$$

and the fiberwise Bargmann-Fock space $\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,g}$ with inner product

$$\langle F, G \rangle_{\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,g}} := \int_{\mathbb{C}^n} F \circ J^{-1}(z) \overline{G \circ J^{-1}(z)} e^{-\frac{t}{2} \left( \Re(z)^t \frac{\Lambda_x}{2} \Re(z) + \Im(z)^t \Im(z) \right)} \, dz \quad (3.3.19)$$

are Hilbert spaces. Additionally, the mapping

$$\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee) \to \mathcal{F}^2(\mathbb{C}^n), \; F \mapsto F \circ J^{-1} \quad (3.3.20)$$

is an embedding if and only if

$$\rho(Q_x) \leq 1$$

where $Q_x$ is the invertible matrix such that $A_x = Q_x^t Q_x$ and $\rho(Q_x)$ the spectral radius of $A_x$.

Proof. To prove that $\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,g}$ is a Hilbert space, it suffices to prove that $\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,g}$ is complete with respect to the norm

$$\|F\|_{\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,g}} = \sqrt{\langle F, F \rangle_{\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,g}}}.$$ 

Let $\{F_n\}_{n \in \mathbb{N}} \subset \mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,g}$ be a Cauchy sequence, then $\{F_n \circ J^{-1} \circ P_{x}^{-1}\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{F}^2(\mathbb{C}^n)$, where $P_x = \begin{pmatrix} Q_x & 0 \\ 0 & 1 \end{pmatrix}$. Let, also, $F \in \mathcal{F}^2(\mathbb{C}^n)$ such that $\|F_n \circ J^{-1} \circ P_{x}^{-1} - F\|_{\mathcal{F}^2(\mathbb{C}^n)} \to 0$. Then,

$$\|F_n \circ J^{-1} \circ P_{x}^{-1} - F\|_{\mathcal{F}^2(\mathbb{C}^n)} = \int_{\mathbb{C}^n} | F_n \circ J^{-1} \circ P_{x}^{-1} - F \|^2 e^{-\pi|z|^2} \, dz$$

$$= \det(P_x) \int_{\mathbb{C}^n} | F_n \circ J^{-1} - F \circ P_{x} | e^{-\pi|z|^2} \, dz$$

$$= \det(P_x) \|F_n - F \circ J\|_{\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,g}}$$

Thus, $\{F_n \circ J^{-1} \circ P_{x}^{-1}\}_{n \in \mathbb{N}}$ converges to the function $F_{P_x} \circ J \in \mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,g}$, where $F_{P_x} = F \circ P_x$, with respect to the $\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,g}$ norm.

Moreover, the space $\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)$ is the direct integral of the Hilbert spaces $\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,g}$. The inner product

$$\langle F, G \rangle = \int_M \langle F, G \rangle_{\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,g}} \, \text{dvol}_M$$
Thus, the following relation is satisfied
turns $\mathcal{F}^2(E \oplus \mathcal{E}^\vee)$ into a Hilbert space.

Finally, to prove that the map $F \mapsto F \circ J^{-1}$ is an embedding of $\mathcal{F}^2((E \oplus \mathcal{E}^\vee)_{x,y})$ into $\mathcal{F}^2(C^n)$, let $F \in \mathcal{F}^2(E \oplus \mathcal{E}^\vee_{x,y})$. To show that $F \circ J^{-1}$ is in $\mathcal{F}^2(C^n)$, it suffices to show that $\mathcal{F}^2(C^n,e^{-\frac{1}{2}|z|^2}dz)$ is embedded in $\mathcal{F}^2(C^n)$. The latter holds exactly when $|Pz| \leq |z|$, which, in turn, is satisfied exactly when $\rho(Q \xi) \leq 1$. Since

$$|Pz|^2 \leq \rho(Q \xi)^2 |Re(z)|^2 + |Im(z)|^2 \leq |z|^2 \text{ and } \rho(Q \xi) \leq \|Q \xi\|.$$

□

**Remark 3.3.20.** For $(W, \xi) \in \mathcal{F}^2(E \oplus \mathcal{E}^\vee)_{x,y}$, we introduce the notation

$$z := J(W, \xi) \text{ and } \Psi(z) := Re(z)^t \frac{A_x}{\pi} Re(z) + Im(z)^t Im(z)$$

**Lemma 1.** The elementary Gabor functions $M_{\xi}^T_W \Psi_{x,y} : \mathcal{E}_{x,y} \to \mathbb{C}$ satisfy the condition

$$|\langle f, M_{\xi}^T_W \Psi_{x,y} \rangle| = e^{-\frac{\pi}{2} |\Psi(z)|} |Bf(z)|$$

for every $f$ in $L^2(\mathcal{E}_{x,y})$ and every $(\xi, W)$ in $(E \oplus \mathcal{E}^\vee)_{x,y}$.

**Proof.** Let $f \in L^2(\mathcal{E}_{x,y})$ and $(\xi, W) \in (E \oplus \mathcal{E}^\vee)_{x,y}$, then

$$\langle f, M_{\xi}^T_W \Psi_{x,y} \rangle = \int_{\mathcal{E}_{x,y}} f(V)e^{(V-W)^tA\xi(V-W)}e^{i\eta y(V-W)}-2\pi i(\xi, V)dvol_{\mathcal{E}_{x,y}}(V)$$

$$= e^{-i(\eta y, W)}e^{-\pi(W^tA\xi W)}$$

$$\int_{\mathcal{E}_{x,y}} f(V)e^{-\pi(V^tA\xi V)+2\pi i(\eta y, W)e^{i\eta y(V-W)}-2\pi i(\eta, V)dvol_{\mathcal{E}_{x,y}}(V)$$

$$= e^{-i(\eta y, W)}e^{-i\pi(\xi - \frac{\eta y}{2\pi} W^tA\xi W)}e^{-\frac{\pi}{2}(W^tA\xi W+(\xi - \frac{\eta y}{2\pi})(\xi - \frac{\eta y}{2\pi})))}$$

$$\int_{\mathcal{E}_{x,y}} f(V)e^{V^tA\xi V+2\pi V^*(W, -\xi + \frac{\eta y}{2\pi})}e^{-\frac{\pi}{2}P(V, -\xi + \frac{\eta y}{2\pi})dvol_{\mathcal{E}_{x,y}}(V).}$$

Thus, the following relation is satisfied

$$|\langle f, M_{\xi}^T_W \Psi_{x,y} \rangle| = e^{-\frac{\pi}{2} \left[ (W^tA\xi W+(\xi - \frac{\eta y}{2\pi})(\xi - \frac{\eta y}{2\pi}))) \right]} |Bf(W, -(\xi - \frac{\eta y}{2\pi}))| .$$

□

**Proposition 3.3.21.** Let $P : E \oplus \mathcal{E}^\vee \to \mathbb{C}$ be the quadratic form associated to $A$ defined as

$$P(V, \eta) := V^t \frac{A_x}{\pi} V + 2i\langle \eta, V \rangle - \eta^t \eta,$$
then the following hold:

1. The vectors $e_a(z) = \det(P) \left( \frac{|z|^a}{a!} \right)^{1/2} (J_{(x,y)}^{-1}(Pz))^a$ for $a = (a_1, ..., a_n)$ with $a_j \geq 0$, form an orthonormal basis for $F^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y}$.  

2. The Hilbert space $F^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y}$ is a reproducing kernel Hilbert space, that is, for every $z_0 \in C^n$ the evaluation functional $f \mapsto f(z)$ is bounded. The reproducing kernel is $F_{z_0}(z) = \det^2(P)e^{\pi(|Pz_0|^2)}$.

Proof. 1. The inner product of the vectors $e_a(z)$ in $F^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y}$ can be expressed in terms of the inner product of the vectors

$$E_a(z) = \left( \frac{|z|^a}{a!} \right)^{1/2} (z)^a, z \in C^n$$

in the Bargmann-Fock space $F^2(C^n)$ as

$$\langle e_a(z), e_\beta(z) \rangle_{F^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y}} = \det(P^{-1})\langle E_a(z), E_\beta(z) \rangle_{F(C^n)}.$$  

Since $\{E_a(z) : a \geq 0\}$ is an orthonormal basis of $F^2(C^n)$, [Bar61], it follows that the vectors $e_a(z)$ form an orthonormal system of $F^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y}$. To prove completeness, we take $f \in F^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y}$ such that $\langle f, e_a(z) \rangle_{F^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y}} = 0$ for all $a$. Since $\langle f, e_a(z) \rangle_{F^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y}} = \det(P^{-1})\langle f \circ J^{-1} \circ P^{-1}, E_\beta(z) \rangle_{F(C^n)}$ and $\{E_a(z) : a \geq 0\}$ is an orthonormal basis of $F^2(C^n)$, $f \equiv 0$.

2. Since $f(z) = \sum_{a \geq 0} \langle f, e_a \rangle_{F^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y}} e_a(z)$, by the Cauchy-Schwarz inequality we obtain

$$|f(z)| \leq (\sum_{a \geq 0} |\langle f, e_a \rangle_{F^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y}}|^2)^{1/2} \left( \sum_{a \geq 0} \det^2(P) \frac{|z|^a}{a!} |Pz|^2 \right)^{1/2} \leq \det^2(P)\|f\|_{F^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y}} \cdot e^{\pi|Pz|^2/2}.$$  

Therefore, the point evaluations for $z \in C^n$

$$F^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y} \to C$$

$$f \mapsto f(z)$$

are continuous linear functionals and for each $z_0 \in C^n$ there exists a reproducing kernel $F_{z_0} \in F^2(\mathcal{E} \oplus \mathcal{E}^\vee)_{x,y}$ such that

$$f(z_0) = \langle f, F_{z_0} \rangle.$$  

(3.3.24)

Expanding the reproducing kernel $F_{z_0}$ with respect to the orthonormal basis $\{e_a\}_{a \geq 0}$,
we obtain the following equation

\[ F_{z_0}(z) = \sum_a (F_{z_0}, e_a)e_a(z) \]
\[ = \sum_a e_\alpha(z_0)e_a(z) \]
\[ = \sum_a \det^2(\mathcal{P}) \frac{\pi^{na}}{a!}(\mathcal{P}z_0)^a(\mathcal{P}z)^a \]
\[ = \det^2(\mathcal{P})e^{\pi(\mathcal{P}z_0)\mathcal{P}z} \]

and that completes the proof.

\[ \square \]

**Proposition 3.3.22.** The Bargmann Transform of definition 3.3.17 is a linear bijection from \( L^2(\mathcal{E}(\xi,\eta)) \) onto \( \mathcal{F}^n(\mathcal{E}(\xi,\eta) \oplus \mathcal{E}'(\xi,\eta)) \), for any \((\xi,\eta) \in M, \) and

\[ \| Bf \|_{\mathcal{F}(\mathcal{E}(\xi,\eta) \oplus \mathcal{E}'(\xi,\eta))} = \sqrt{\frac{\pi^n}{\det A_\xi}} \| f \|_{L^2(\mathcal{E}(\xi))}. \]  \( (3.3-25) \)

**Proof.** Let \((\xi,\eta) \) in \( M, (\theta, \nu) \) in \((\mathcal{E} \oplus \mathcal{E}'')_\xi, \eta \) and \( f \) in \( L^2(\mathcal{E}(\xi,\eta)) \). From the orthogonality relations of the Short Time Fourier Transform it follows that

\[ \| \langle f, M_\eta T_\nu \Psi_{\xi,\eta} \rangle \|^2_{L^2(\mathcal{E}(\xi,\eta))} = \| f \|^2_{L^2(\mathcal{E}(\xi,\eta))} \| \Psi_{\xi,\eta} \|^2_{L^2(\mathcal{E}(\xi,\eta))}. \]

Additionally, for the norm \( \| \Psi_{\xi,\eta} \|^2_{L^2(\mathcal{E}(\xi,\eta))} \) of the window function holds that

\[ \| \Psi_{\xi,\eta} \|^2_{L^2(\mathcal{E}(\xi,\eta))} = \int_{\mathcal{E}(\xi,\eta)} e^{-WA_\xi W} d\text{vol}_{\mathcal{E}(\xi,\eta)}(W) = \sqrt{\frac{\pi^n}{\det A_\xi}}, \]

and by Lemma 1 it follows that the fiberwise Bargmann Transform is bounded, injective and

\[ \| Bf \|^2_{\mathcal{F}^2(\mathcal{E}(\xi,\eta) \oplus \mathcal{E}'(\xi,\eta))} = \sqrt{\frac{\pi^n}{\det A_\xi}} \| f \|^2_{L^2(\mathcal{E}(\xi))}. \]

To prove surjectivity of \( B \) onto \( \mathcal{F}^2(\mathcal{E}(\xi,\eta) \oplus \mathcal{E}'(\xi,\eta)) \), it suffices to prove that \( B(L^2(\mathcal{E}(\xi,\eta))) \) is dense in \( \mathcal{F}^2(\mathcal{E}(\xi,\eta) \oplus \mathcal{E}'(\xi,\eta)) \). We write \( z = J(W, \xi) \) and \( z_0 = J(V, \eta) \). After some bookkeeping and after applying equation \( (3.3-21) \), we obtain that

\[ B(T_\nu M_\eta \Psi_{\xi,\eta})(W, -\xi) = h(V, \eta, \frac{\eta_\xi}{2\pi} \det(\mathcal{P})e^{\pi(\mathcal{P}z_0)\mathcal{P}z}, \]

for some \( h \in \mathbb{C} \) depending on \( V, \eta \) and \( \frac{\eta_\xi}{2\pi} \). Thus, the reproducing kernel \( K_{z_0}(z) \) is in the range of \( B \). Suppose that there exists some \( F \in \mathcal{F}^2((\mathcal{E} \oplus \mathcal{E}'')_\xi, \eta) \) such that \( \langle F, Bf \rangle = 0, \) for
all \( f \in L^2(\mathcal{E}_{x,g}) \). Then, for every \( z_0 = \mathcal{J}(V, \eta) \)
\[
0 = \langle B(T_{V}M_{\Psi}^{\bar{x},\bar{y}}), F \rangle = h(V, \eta, \frac{\eta g}{2\pi}) \det(P) \langle K_{z_0}z, F \rangle = F(z_0)
\]
and therefore \( F \equiv 0 \). Thus, \( B(L^2(\mathcal{E}_{x,g})) = \mathcal{F}^2((\mathcal{E} \oplus \mathcal{E}^\vee)_{x,g}) \)
\[\square\]

Finally, by considering a complex lattice as in 2.1.14 we have the following statement.

**Theorem 3.3.23.** Let \( \mathcal{G}(\Psi_{(x,y)}, \Lambda_{x,y}) \) a Gabor system with window function as defined in (3.3.11). Suppose that there exist normalized lattices \( L_1, \ldots, L_n \) in \( \mathbb{C} \), \( M \) in \( GL(n, \mathbb{C}) \) and \( a \in \mathbb{C}^* \) such that

\[
\mathcal{J}(\Lambda_{x,g}) = aM \bigoplus_{i=1}^{n} L_i
\]

and for the characteristic indices \( \gamma_1, \ldots, \gamma_n \) of \( \left( \frac{Q_s}{\sqrt{\pi}}, 0 \right) M \) it holds that \( 0 < \gamma_i < 1 \) for \( i = 1, \ldots, n \), where \( Q_s \) is the invertible matrix such that \( A_s = Q_s^TQ_s \), then \( \mathcal{G}(\Psi_{(\bar{x},\bar{y})}, \Lambda_{x,y}) \) is a frame.

**Proof.** From Lemma 1 it follows that \( \mathcal{G}(\Psi_{(x,y)}, \Lambda_{x,g}) \) is a frame exactly when \( \overline{\Lambda_{x,g}} - i \frac{\eta y}{2\pi} \) is a set of sampling for the Bargmann-Fock space \( \mathcal{F}^2((\mathcal{E} \oplus \mathcal{E}^\vee)_{x,g}) \). Hence, it suffices to prove that there exist \( A : U \to \mathbb{R}^+ \) and \( B : U \to \mathbb{R}^+ \) smooth functions in the local chart \( U \subset M \) that contains \((\bar{x}, \bar{y})\) such that

\[
A_{x,g}\|F\|^2_{\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee_{x,y})} \leq \sum_{\lambda \in \overline{\Lambda_{x,g}} - i \frac{\eta y}{2\pi}} |F|_{\mathcal{E} \oplus \mathcal{E}_{x,y}} \circ \mathcal{J}^{-1}(\lambda) \left| e^{-\frac{1}{2}(\text{Re}(\lambda)^t A \text{Re}(\lambda) + \text{Im}(\lambda)^t \text{Im}(\lambda))} \right| \leq B_{x,y}\|F\|^2_{\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee_{x,y})}
\]

for every \( F \in \mathcal{F}^2((\mathcal{E} \oplus \mathcal{E}^\vee)_{x,g}) \) or equivalently

\[
A_{x,g}\|F\|^2_{\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee_{x,y})} \leq \sum_{\lambda \in \overline{\Lambda_{x,g}} - i \frac{\eta y}{2\pi}} |F|_{\mathcal{E} \oplus \mathcal{E}_{x,y}} \circ \mathcal{J}^{-1}(P^{-1}\lambda) \left| e^{-\frac{1}{2}(\text{Re}(\lambda)^t A \text{Re}(\lambda) + \text{Im}(\lambda)^t \text{Im}(\lambda))} \right| \leq B_{x,y}\|F\|^2_{\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee_{x,y})}
\]

where \( P = \left( \frac{Q_s}{\sqrt{\pi}}, 0 \right) \) and \( \overline{\Lambda_{x,g}} = PA_{x,g} \). By Theorem 9 of [Grö01] and Proposition 4.5 of [LM23], it follows that \( \overline{\Lambda_{x,g}} - i \frac{\eta y}{2\pi} \) is a sampling set of \( \mathcal{F}^2(\mathbb{C}^n) \) since for the characteristic indices \( \{\gamma_i, i = 1, \ldots, n\} \) of \( PM \) it holds that \( 0 < \gamma_i < 1 \). Additionally, \( F \mid_{\mathcal{E} \oplus \mathcal{E}_{x,g}^\vee} \circ \mathcal{J}^{-1} \) is
in $\mathcal{F}^2(C^n,e^{-\frac{2}{n}\|p\|^2}dz)$ since $F \in \mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee_{\xi,\bar{g}})$ and $\|F \circ J^{-1} \circ P^{-1}\|_{\mathcal{F}^2(C^n)}^2 = det(P)^2\|F \circ J^{-1}\|_{\mathcal{F}^2(C^n,e^{-\frac{2}{n}\|p\|^2}dz)}^2$. Thus the following inequality holds

$$A_{x,g}det(P)^2\|F \circ J^{-1}\|_{\mathcal{F}^2(C^n,e^{-\frac{2}{n}\|p\|^2}dz)}^2 \leq \sum_{\lambda \in A_{x,\bar{g}} - \frac{y_g}{2\pi}} |F|_{\mathcal{E} \oplus \mathcal{E}^\vee_{\xi,\bar{g}}} \circ J^{-1}(P^{-1}\lambda)^2e^{-\frac{2}{n}(Re(\lambda)^iRe(\lambda) + Im(\lambda)^iIm(\lambda))} \leq B_{x,g}det(P)^2\|F \circ J^{-1}\|_{\mathcal{F}^2(C^n,e^{-\frac{2}{n}\|p\|^2}dz)}^2,$$

The latter is equivalent to the inequality

$$A_{x,g}det(P)^2\|F\|_{\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee_{\xi,\bar{g}})}^2 \leq \sum_{\lambda \in A_{x,\bar{g}} - \frac{y_g}{2\pi}} |F|_{\mathcal{E} \oplus \mathcal{E}^\vee_{\xi,\bar{g}}} \circ J^{-1}(\lambda)^2e^{-\frac{2}{n}(Re(\lambda)^iRe(\lambda) + Im(\lambda)^iIm(\lambda))} \leq B_{x,g}det(P)^2\|F\|_{\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee_{\xi,\bar{g}})},$$

which proves that $\Lambda_{x,\bar{g}} - \frac{y_g}{2\pi}$ is a set of sampling for $\mathcal{F}^2(\mathcal{E} \oplus \mathcal{E}^\vee_{\xi,\bar{g}})$.

**Corollary 3.3.24.** Let $\mathcal{G}(\Psi(x,y),\Lambda_{b_1,..,b_n} \oplus \Lambda^\vee_{c_1,..,c_n})$ be the Gabor system with window function as defined in (3.3.13) and lattice as defined in (3.3.9). If $0 < b_i < 1$ and $b_i = \pm c_i$, then $\mathcal{G}(\Psi(x,y),\Lambda_{b_1,..,b_n} \oplus \Lambda^\vee_{c_1,..,c_n})$ satisfies the frame condition.

**Proof.** Indeed, the lattice can be written as

$$J(\Lambda_{b_1,..,b_n} \oplus \Lambda^\vee_{c_1,..,c_n}) = \begin{pmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & 0 & \vdots \\ \vdots & \cdots & \vdots \\ 0 & \cdots & c_n \\ \end{pmatrix} \begin{pmatrix} z + iz \\ \vdots \\ z + iz \end{pmatrix}.$$

**3.3.5 Geometric examples and applications**

**Hypercomplex Manifolds**

Let $(b_1,p) = (b_1,..,b_n,p_1,..,p_n)$ be local coordinates on $T^*M$, and $\lambda$, the tautological 1-form on $T^*M$, locally expressed as $\lambda_{(b_1,p)} = \sum_{i=1}^n p_idb_i$. If $J$ is an almost complex structure on $M$, the twist of $\lambda$ by $J$ is the 1-form $\lambda_J$ with local expression $\lambda_J = \sum_{i,j} p_iJdb_j$. If the manifold is equipped with more than one almost complex structures, each one introduces
a new 1-form. Almost hypercomplex manifolds are an example of manifolds with more than one almost complex structure.

**Definition 3.3.25.** A manifold $B$ of dimension $4n$, is an **almost hypercomplex manifold** if it admits a triple $\{I, J, K\}$ of almost complex structures satisfying the quaternionic identities

$$I^2 = J^2 = K^2 = -\text{id} \quad \text{and} \quad IJ = -JI = K.$$

The three almost complex structures are locally expressed as

$$I = \sum_{k,l} I^k_l \, db_l \otimes \partial b_k, \quad J = \sum_{k,l} J^k_l \, db_l \otimes \partial b_k \quad \text{and} \quad K = \sum_{k,l} K^k_l \, db_l \otimes \partial b_k.$$

If the almost complex structures are integrable the manifold $B$ is a **hypercomplex manifold**.

If $(B, (I, J, K))$ is an (almost) hypercomplex manifold, each (almost) complex structure, introduces a “twisted” canonical 1-form on the cotangent bundle $T^*B$, locally expressed as

$$\lambda_I = \sum_{k,l} p_k I^k_l \, db_l, \quad \lambda_J = \sum_{k,l} p_k J^k_l \, db_l \quad \text{and} \quad \lambda_K = \sum_{k,l} p_k K^k_l \, db_l.$$

In the following proposition we prove that the 1-forms $\lambda, \lambda_I, \lambda_J$ and $\lambda_K$ restricted on the unit cotangent bundle of a hypercomplex manifold $B$, introduce a quadruple of contact 1-forms on $S(T^*B)$,

$$a := \lambda|_{S(T^*B)}, \quad a_I := \lambda_I|_{S(T^*B)}, \quad a_J := \lambda_J|_{S(T^*B)}, \quad a_K := \lambda_K|_{S(T^*B)},$$

which are appropriate for the construction of the Gabor system

$$\mathcal{G}(\Psi, \Lambda) = \{ M_{\lambda'} T_{\lambda} \Psi \mid (\lambda, \lambda') \in \Gamma(M, \Lambda) \},$$

from definition 3.3.14.

**Lemma 3.3.26.** Let $M^7 = S(T^*B)$ be the co-sphere bundle of a hypercomplex $(B, I, J, K)$ with Riemannian metric $g$.

1. The 1-forms $a_I, a_J$ and $a_K$, induced by the twisted 1-forms $\lambda_I, \lambda_J$ and $\lambda_K$ respectively, are contact 1-forms on $M$ and form a linearly independent set of $T^*_b(M)$ at each $(b, [p]) \in S(T^*B)$.

2. Additionally, the fibers $\pi^{-1}(b)$ for each $b \in B$ of the fiber bundle $\pi : S(T^*B) \to B$ are Legendrian submanifolds of the contact distributions induced by $a_I, a_J$ and $a_K$ and of the canonical contact distribution induced by $\lambda |_{S(T^*B)}$.

3. If $V \to M$ is the vertical tangent bundle of $\pi : M \to B$, $\Gamma(V)$ denotes the vertical vector
fields of $M$ and $\mathcal{R}_a, \mathcal{R}_{a_1}, \mathcal{R}_{a_j}, \mathcal{R}_{a_K}$ are the Reeb fields of $a, a_1, a_j, a_K$ respectively, then

$$\text{span}\{\mathcal{R}_a, \mathcal{R}_{a_1}, \mathcal{R}_{a_j}, \mathcal{R}_{a_K}\} \cap \Gamma(V) = \{0\}.$$  

Proof. First, we are going to prove that the one forms $a_1, a_j, a_K$ are contact. The almost complex structure $I$ on $B$ comes from a complex structure on $B$, since it is integrable. Hence we can consider a holomorphic local coordinate chart $(U, x_1 + iy_1, x_2 + iy_2)$ and the corresponding local trivialization chart $(T^*U, (x_1 + iy_1, x_2 + iy_2, p_1^x dx_1 + p_1^y dy_1 + p_2^x dx_2 + p_2^y dy_2))$ of $T^*B$. The 1-form $\lambda_1$ has a local expression $\lambda_1 = \sum_{i=1}^{n} (-p_i^y dx_i + p_i^x dy_i)$. Without loss of generality we assume that $p_1^x \neq 0$ to obtain the local expression of $\lambda_1$ restricted on $S(T^*B)$,

$$a_1 = \lambda_1 |_{S(T^*B)} = dy_1 - p_j^y dx_1 + \sum_{i=2}^{4} (-p_i^y dx_i + p_i^x dy_i).$$

It follows that $a_1 \land (da_1)^2 = \lambda_1^2 = (dp_1^x \land d p_j^y \land dx_i \land dy_i) \neq 0$. By expressing $\lambda_J$ and $\lambda_K$ in the holomorphic coordinates induced by $J$ and $K$ respectively, one can see that $a_J$ and $a_K$ are contact 1-forms.

Next, we are going to prove that $a, a_1, a_J$ and $a_K$ are linearly independent at each $T_1^{*}\{b, [p]\} M$. Let $(b, [p]) \in S(T^*B)$ and $\lambda, \lambda_1, \lambda_J$ and $\lambda_K$ real functions on $S(T^*B)$ such that $\lambda a(b, [p]) + \lambda_1 a_1(b, [p]) + \lambda_J a_J(b, [p]) + \lambda_K a_K(b, [p]) = 0$, namely

$$\sum_{i=1}^{4} (p_i \lambda Id + p_i \lambda_1 I^1_i + p_i \lambda_J I^J_i + p_i \lambda_K K^i_j) = 0, \; \forall j \in \{1, ..., 4\}, \tag{3.3.27}$$

where $p = (p_1, ..., p_4) \in T_b S(T^*B)$. Equation (3.3.27) is equivalent to $p$ being in the kernel of the linear transformation $(\lambda a(b, [p])Id + \lambda_1 a_1(b, [p])I + \lambda_J a_J(b, [p])J + \lambda_K a_K(b, [p])K)$.

Since $p \neq 0$, the latter holds if and only if $-\lambda$ is an eigenvalue for $\lambda a(b, [p])Id + \lambda_1 a_1(b, [p])I + \lambda_J a_J(b, [p])J + \lambda_K a_K(b, [p])K$ and $p$ a corresponding eigenvector, in which case we have the following

$$(\lambda a(b, [p])Id + \lambda_1 a_1(b, [p])I + \lambda_J a_J(b, [p])J + \lambda_K a_K(b, [p])K)p = -\lambda p$$

$$\iff (\lambda a(b, [p])Id + \lambda_1 a_1(b, [p])I + \lambda_J a_J(b, [p])J + \lambda_K a_K(b, [p])K)^2 p = \lambda^2 p$$

$$\iff (-\lambda_1^2 - \lambda_J^2 - \lambda_K^2) p = \lambda^2 p,$$

and therefore $\lambda = \lambda_1 = \lambda_J = \lambda_K = 0$.

For each $b \in B$, the fiber $\pi^{-1}(b)$ is Legendrian with respect to any of the contact 1-forms $a, a_1, a_J$ and $a_K$, since they do not depend on the local covector fields $dp_i$. \hfill \Box

Lemma 3.3.26 states that the contact 1-forms $a, a_1, a_J$ and $a_K$ share the same properties with the contact 1-forms $F_i a_{i=0}^3$ introduced in section 3.3.2. Hence we obtain the following
Corollary which is an adaptation of 3.3.9 for a hypercomplex manifold \((B, (I, J, K))\).

**Corollary 3.3.27.** The contact 1-forms \(a, a_I, a_J\) and \(a_K\) on the manifold \(M = \mathbb{S}(T^*B)\) of a hypercomplex manifold \((B, (I, J, K))\) determine a framed lattice bundle structure on the rank 8 vector bundle \(E \oplus E^\vee\) of (3.3.5).

The proof of Corollary 3.3.27 is the same as the proof of Lemma 3.3.9. Finally, the following statement follows directly from 3.3.26 and 3.3.27.

**Proposition 3.3.28.** Let \((B, (I, J, K))\) be a 4-dimensional compact hypercomplex manifold. The collection \(a, a_I, a_J, a_K\) of contact 1-forms on \(M = \mathbb{S}(T^*B)\) and the function \(\Phi\) of (3.3.9) determine a Gabor system \(\mathcal{G}(\Psi, \Lambda)\) on \(B\), with window function \(\Psi\) as in (3.3.10) and with lattice bundle structure \(\Lambda\) on \(E \oplus E^\vee\) as in Corollary 3.3.27 (with either the Reeb lattice bundle structure of (3.3.7) or the dual-basis lattice bundle structure of (3.3.8)).

**Configuration Spaces in Robotics**

In this paragraph we describe an application of the construction described in section 3.3.1 in configuration spaces of robotic motion with environmental constraints. Consider a mechanism whose possible movements in the ambient 3-dimensional space are parameterized by a configuration space \(\mathcal{M}(\mathbb{R})\), which is a manifold of some higher dimension \(N = \dim \mathcal{M}(\mathbb{R})\). The configuration space represents all the possible positions the mechanisms can take, dictated by its structure. However the motion of the mechanism can be limited by environmental constraints/ obstacles which can be described as subsets of the configuration manifold. More generally, we think of a constraint according to the following definition.

**Definition 3.3.29.** Let \(\mathcal{L}(\mathcal{M}(\mathbb{R}))\) be the Lebesgue \(\sigma\)-algebra of \(\mathcal{M}(\mathbb{R})\) and \(g\) a Riemannian metric on \(\mathcal{M}(\mathbb{R})\). A constraint on the configuration space is a probability measure

\[
\mu : \mathcal{L}(\mathcal{M}(\mathbb{R})) \to [0, 1]
\]

which is absolutely continuous with respect to the volume measure \(d\text{vol}_g\).

This definition allows us to consider the case of soft constraints as well. By soft constraints we mean giving a degree of preference for certain motions and configurations over others, for example for the purpose of motion planning, instead of realizing rigid constraints which exclude parts of the configuration space. A constraint on the configuration space can be identified with its density function. For simplicity we will use the letter \(\mu\) for the density function \(\mu : \mathcal{M}(\mathbb{R}) \to \mathbb{R}_+\) of a constraint \(\mu\).

**Proposition 3.3.30.** Let \(\mathcal{M}_c(\mathbb{R})\) be the configuration space of a robotic arm \(\mathcal{R}\) and \(U \subset \mathcal{M}_c(\mathbb{R})\) a relatively compact subset of configurations that are not attainable and its boundary \(\partial U = \Sigma\) is
smooth. For fixed $b \in \Sigma \subset B$ and for $\mu : \mathcal{M}_\ell(\mathcal{R}) \to \mathbb{R}_+$ the density function of the constraint indicating the constraints, the output function $\mathcal{O}_b(\mu, \eta_p)$ has a local maximum for $p \in S^{n-1}$ the normal vector $v_b(\Sigma)$ at $b$ to the boundary hypersurface $\Sigma = \partial \mathcal{U},$
\[
\arg\max_{p \in S^{n-1}} \mathcal{O}_b(\mu, \eta_p) = v_b(\Sigma).
\]

**Proof.** The output function $\mathcal{O}(\eta_p)$ has a local maximum when the gradient with respect to the metric $g^*_{F(b, p)}$ on the fibers of $\mathcal{E}$, $\nabla g^*_{F(b, p)} \mathcal{O}(\eta_p)$, is zero. Since $\mu$ is compactly supported, the lifted signal $\mathcal{I}(\mu) : \mathcal{E} \to \mathbb{R}_+$ is non-zero only on $h^{-1}(TU)$, then it follows that
\[
\nabla g^*_{F(b, p)} \mathcal{O}_b(\mathcal{I}(f), \eta_p) = \int_{h^{-1}(T_bU)} I(f)|_{h^{-1}(T_bU)}(V)i(\nabla g^*_{F(b, p)} \langle \eta_p, V \rangle_b) \Psi_{(b, p)}(V) \, dV.
\]
Thus, the gradient $\nabla g^*_{F(b, p)} \mathcal{O}_b(\mathcal{I}(f), \eta_p)$ is equal to zero exactly when
\[
\nabla g^*_{F(b, p)} \langle \eta_p, V \rangle_b = 0, \text{ for every } V \in h^{-1}(T_bU),
\]
which holds exactly when $\langle \eta_p, V \rangle_b = 0$ for every $V \in h^{-1}(T_bU)$, hence $\eta_p$ is normal to $\Sigma$, since $V \in T_bB$ at $b$. \qed

**Remark 3.3.31.** If the constraint is sharp, namely $\mu = \chi_U$ the previous proposition follows directly from 3.3.16.

**Example 3.3.32 (Planar Robotic Arm).** Consider $\mathcal{R}$ to be a robot arm, that is a collection of $n$-bars of given lengths $\ell_1, \ldots, \ell_n$ connected with each other with joints that allow full rotation and we will denote the space of configurations of $\mathcal{R}$ by $\mathcal{M}_\ell(\mathcal{R})$. The simplest restriction that one can impose to the motion of the robotic arm is to assume that the first joint is attached to the origin. If we allow the arm to intersect itself, then the configuration space is the $n$-torus,
\[
\mathcal{M}_\ell(\mathcal{R}) = S^1 \times \ldots \times S^1 \subset \mathbb{C}^n.
\]
with coordinate charts given by the angle functions $\theta_i \mapsto e^{i \theta_i}, 0 < \theta_i < 2\pi$ for $i = 1, \ldots, n$. The workspace of $\mathcal{R}$ is the variety of positions on the end-point of the arm denoted as $W$ and the robot arm workspace map is the map
\[
f_\mathcal{R} : \mathcal{M}_\ell(\mathcal{R}) \to W
\]
\[
(\theta_1, \ldots, \theta_n) \mapsto \ell_1 \theta_1 + \ldots + \ell_n \theta_n.
\]
Suppose that the planar arm has two bars of the same length $\ell_1 = \ell_2$, then the preimage $f^{-1}_\mathcal{R}(0)$ is the anti-diagonal $\Delta^* = \{(\theta_1, \theta_2) \in \mathcal{M}_\ell(\mathcal{R}) : \theta_1 = -\theta_2\}$. If we consider $U$ to be $\mathcal{M}_\ell(\mathcal{R})/\Delta^*$
and \( \mu = \chi_U \), then it follows by Theorem 3.3.16 that the filters

\[
\Psi_{(b,p)}(V) = \exp(-V^t A_b V - i\langle \eta_p, V \rangle_b), \quad b \in \mathcal{M}_\ell(\mathbb{R}) \text{ and } p \in S(T_b^* \mathcal{M}_\ell(\mathbb{R})),
\]

detect the configurations that make the robotic arm to fold in half.
3.4 Contact Manifolds and Gabor Bundles

In this section, we introduce a framework for Gabor frames on contact manifolds, which provides a more convenient approach for studying the stability of Gabor frames under contact transformations. The functional architecture of $V_1$ serves as the archetypal framework for Gabor frames on certain classes of contact manifolds, as presented in 3.1 and generalized in 3.3. However, it is important to note that this framework imposes various constraints and assumptions dictated by the unique structure of $V_1$, for instance the existence of two contact forms. In this section, we present a framework that is tailored to the existing literature on deformations of Gabor frames where the lattice is transformed symplectically. The objective is to address the following problem:

Let $(K, \tau)$ be a contact manifold with coorientable contact distribution $\tau$. If $G(\phi, \Lambda)$ is a Gabor system with an appropriate choice of window function $\phi : \tau \to \mathbb{R}$ and lattice $\Lambda \subset \tau$ which satisfies the frame condition, for which contact transformations $F : (K, \tau) \to (K, \tau)$ does the Gabor system $G(F^*\phi, dF\Lambda)$ satisfy the frame condition?

This project is currently underway in collaboration with B. Khesin and M. Marcolli, who have kindly given me permission to include in this thesis.

3.4.1 Contact structure and conformal symplectic structure on the contact planes

We will review some necessary background information that is relevant to this section.

**Definition 3.4.1.** A symplectic vector bundle over a smooth manifold $B$ is a pair $(E, w)$ where $E \xrightarrow{\pi} B$ is a smooth vector bundle and $w$ is a smooth section of the bundle $\wedge^2 E^* \to B$ such that $w_b$ is a symplectic linear form on each fiber $E_b = \pi^{-1}(b)$. Two symplectic vector bundles $(E_1, w_1)$ and $(E_2, w_2)$ are isomorphic if there exists a bundle isomorphism $\Phi : E_1 \to E_2$ such that $\Phi^* w_2 = w_1$.

**Example 3.4.2.** Consider the bundle of contact planes $\tau \to M$ of a cooriented contact manifold $(M, \tau)$. Given a contact form $a$, the restriction of the 2-form $da|_\tau$ on the contact planes $\tau$, is a smooth section $\sigma : K \to \wedge^2 \tau^*$ such that $\sigma(q) = (da)|_{\tau_q}$ is non-degenerate on $\tau_q$. The pair $(\tau, da|_\tau)$ is a symplectic vector bundle.

We consider a compact manifold $K$ equipped with a co-orientable contact structure, i.e. a distribution $\tau$ of hyperplanes. Let $a_i$, $i = 1, 2$ two contact 1-forms for the same hyperplane distribution $\tau$

$$\tau = \ker(a_i), \ i = 1, 2,$$

then there exists a function $f : K_1 \to \mathbb{R}^*$ such that $a_1 = f(q)a_2$. 
**Definition 3.4.3.** Given a symplectic vector bundle \((E, w)\) over a manifold \(B\), the **conformal symplectic structure on** \(E_b\) is the equivalence class
\[
[w_b] = \{ \sigma \in \wedge^2 E_b^* : \sigma = \lambda(b)w_b, \text{ for } \lambda(b) \neq 0 \}.
\]
Moreover, the conformal class of the symplectic bundle structure \((E, w)\) is the equivalence class
\[
[w] = \{ \sigma \in \mathcal{A}(E) : \sigma_b = f(b)w_b \text{ for some } f \in C^\infty(B) \text{ non-vanishing } \}.
\]

**Lemma 3.4.4.** Given a contact structure \(\tau\), the corresponding conformal symplectic structure is well-defined.

*Proof.* Let \(\alpha\) be a contact form defining the contact structure \(\tau\), then any other contact form defining the same contact structure \(\tau\) has the form \(f\alpha\) for a function \(f\) on the contact manifold \(K\). Thus, the restriction of \(d(f\alpha)\) to hyperplanes \(\tau = \ker \alpha\) is
\[
d(f\alpha) \mid_\tau = (df \wedge \alpha) \mid_\tau + (f \, d\alpha) \mid_\tau = f \, d\alpha \mid_\tau
\]
i.e. at a point \(q \in K\) the symplectic structure is multiplied by \(f(q)\), and hence its conformal class is well-defined. \(\square\)

**Lemma 3.4.5.** Contact transformations of \((K, \tau)\) preserve the associated conformal symplectic structure.

*Proof.* Let \(\alpha\) a 1-form on \(K\) such that \(\tau_q = ker(\alpha_q)\) for every \(q\) in some open \(U \subseteq K\) and \(F : K \rightarrow K\) a contact transformation of \((K, \tau)\). Then, \(ker(F^*\alpha_q) = \tau_q\) for every \(q \in U\) and therefore there exists some function \(f : U \rightarrow \mathbb{R}^+\) such that \(F^*\alpha_q = f(q)\alpha_q\). Since it holds that \(dF^*\alpha_q \mid_q = f(q)d\alpha_q \mid_q\) for every \(q \in U\), the conformal class of \(\tau\) is preserved,
\[
[d\alpha_q] = [dF^*\alpha_q] \text{ for every } q \in U.
\]
\(\square\)

Let us revisit some fundamental concepts and principles in sub-Riemannian geometry which can be found in detail in [ABB19]. A sub-Riemannian structure on a smooth connected manifold \(M\) of dimension \(n \geq 3\) is a pair \((\mathcal{D}, \langle \cdot, \cdot \rangle)\), where \(\mathcal{D}\) is a tangent distribution \(\mathcal{D} \subset TM\) of fixed rank \(k < n\) which satisfies the Hörmander condition
\[
\text{Lie}_q(\mathcal{D}) := \text{span}\{[X_1, [X_2, ..., [X_{n-1}, X_n]]] : X_i \in \mathcal{D}_q, n \in \mathbb{N}\} = T_qM
\]
and \(\langle \cdot, \cdot \rangle\) is a smooth bundle metric on \(\mathcal{D}\), namely \(\langle \cdot, \cdot \rangle_q\) is an inner product on \(\mathcal{D}_q\). Contact manifolds \((M, \tau)\) satisfy the Hörmander condition since \(\tau\) is a nonintegrable codimension
1 distribution and therefore we have that

\[ [X_q, Y_q] \notin \tau_q, \text{ where } X_q, Y_q \in \tau_q \text{ for any } q \in K. \]

**Definition 3.4.6.** A contact distribution \( \tau \) on a connected contact manifold \((M, \tau)\) with a bundle metric \(\langle \cdot, \cdot \rangle\) on \(\tau\) is a codimension 1 sub-Riemannian structure on \(M\). The triple \((M, \tau, \langle \cdot, \cdot \rangle)\) is a contact sub-Riemannian manifold.

Finally, an almost CR structure compatible to the conformal symplectic bundle structure of a contact manifold \((K, \tau)\) is a pair \((\tau, J)\) where \(J\) is an endomorphism of \(\tau\) with \(J^2 = -Id\) such that the assignment

\[ q \mapsto \{G_q : \tau \times \tau \to \mathbb{R}\} \]

\[ G_q(X, Y) = \sigma(X, JY) \]

is a sub-Riemannian structure on \((K, \tau)\) for any \(\sigma\) in the conformal symplectic bundle structure of \(\tau\).

When a contact sub-Riemannian manifold \((K, \tau, \langle \cdot, \cdot \rangle)\) is coorientable, for each contact form \(\alpha\) there exists a section of the bundle \(End(\tau) \to K\)

\[ J^\alpha : K \to End(\tau) \]

defined by the equation

\[ d\alpha_q(X_q, J^\alpha_q Y_q) = \langle X_q, Y_q \rangle_q \]

and satisfying the condition \((J^\alpha)^2 = -I\) for every \(q \in K\). Each pair \((\tau, J^\alpha)\) is an almost CR-structure on \((K, \tau)\). The superscript of \(J^\alpha\) indicates the dependence of the almost CR-structure on the contact 1-form \(\alpha\) and will be omitted when the choice of \(\alpha\) is easily implied.

For simplicity, when referring to a contact manifold \(K\) we will assume that it is connected and coorientable, unless stated otherwise.

**Remark 3.4.7.** For every contact 1-form \(\alpha\) with a Legendre fibration \(K \xrightarrow{p} B\) and an almost CR-structure \(J : \tau \to \tau\), the vector bundle \(J(L) \subset TK\), where \(L = \bigsqcup_{q \in K} T_q p^{-1}(b)\), is a Lagrangian subbundle of the symplectic vector bundle \((\tau, da|_\tau)\)

In this section, instead of using the standard notion of Gabor systems and Gabor frames as presented in section 2.1, we will work with a more general system of functions obtained by the action of the Heisenberg-Weyl operator on \(L^2(\mathbb{R}^n)\). Let \((x, w)\) denote a point on the standard symplectic vector space \((\mathbb{R}^{2n}, \sigma)\).

**Definition 3.4.8.** Let \(\phi\) be a non-zero square integrable function (hereafter called window) on
$\mathbb{R}^n$, and let $\Lambda$ be a regular (full rank) lattice in a symplectic space $\mathbb{R}^{2n}$ with a symplectic structure $\sigma$ (and canonical coordinates $(x, w)$). The associated $h$-Gabor system is the set of square-integrable functions

$$G^h(\phi, \Lambda) = \{ \hat{T}^h(x, w)\phi \mid (x, w) \in \Lambda \},$$

where $\hat{T}^h((x, w)) = e^{-i\sigma(x, -w, (x, w))/h}$ is the Heisenberg-Weyl operator.

The action of this operator on functions $\phi : \mathbb{R}^n \to \mathbb{R}$ is explicitly given by the formula

$$\hat{T}^h((x, w))\phi(t) = e^{i\langle w, t \rangle - \langle w, x \rangle / 2} \phi(t - x)$$

where $\langle , \rangle$ denotes the standard inner product on $\mathbb{R}^n$.

The Weyl-Heisenberg operators $\hat{T}^h((x, w))$ are the quantum-mechanical analogues of the phase space translation operators $T(x_0, w_0) : (x, w) \to (x, w) + (x_0, w_0)$. The choice $h = 1/2\pi$ associates the Weyl-Heisenberg operators with the time-frequency shifts

$$M_w T_x \phi(t) = e^{2\pi i \langle w, x \rangle} \phi(t - x)$$

in the following way

$$\hat{T}^{1/2\pi}((x, w))\phi = e^{-i\pi \langle w, x \rangle} M_w T_x \phi. \quad (3.4.1)$$

Consequently, we have that

$$| \langle f, \hat{T}^{1/2\pi}((x, w))\phi \rangle | = | \langle f, M_w T_x \phi \rangle |$$

and therefore the notion of a standard Gabor frame and a $h$-Gabor frame for $h = \frac{1}{2\pi}$ coincide. For a more detailed discussion about the Heisenberg-Weyl operators see [De 06].

### 3.4.2 Gabor $h$-Bundles on contact manifolds

Having recalled the necessary background and definitions in the previous section, we are ready to introduce the notion of Gabor System Bundles and Gabor frame bundles. Moreover, in this section we present a construction of a Gabor system bundle for a contact manifold $(K, \tau)$ with a Legendrian fibration and an almost $CR$ structure.

**Definition 3.4.9.** Consider a symplectic vector bundle $(E \to B, \sigma)$ together with a Lagrangian subbundle $L \subset E$. Given a local frame $\{V_i\}_{i=1}^{2n}$ of $E$ such that $\{V_i\}_{i=1}^n \subset L$, one can consider a
family of lattices (a section of a lattice bundle over B)

\[ \Lambda = \sum_{i=1}^{n} \mathbb{Z} V_i + \sum_{i=n+1}^{2n} \mathbb{Z} V_i \subset E. \]  

If E is equipped with a family of functions \( \phi_b : E_b \to \mathbb{R} \) varying smoothly in \( b \), then we define a Gabor \( h^- \) System bundle over E to be the set of functions

\[ G_E(\phi, \Lambda) := \bigcup_{b \in B} \{ T^{h(i)}(\lambda) \phi_{|_{L_b}} : \lambda \in \Lambda_b \}, \]

where the Gabor functions \( T^{h(i)}(\lambda) \phi_{|_{L_b}} \) are given by

\[ T^{h(i)}(\lambda) \phi_{|_{L_b}} = e^{i(c_q(w,t) - c_q(w,x)/2)/h} \phi_{|_{L_b}}(t - x). \]

In the upcoming paragraphs, our objective is to build a Gabor system \( G_K(\phi, \Lambda) \) on a contact manifold \((K, \tau)\), with the following properties:

- \( K \) is compact and connected and the contact structure \( \tau \) is coorientable
- \( K \) is equipped with a Legendrian fibration \( K \to B \) over a base manifold \( B \)
- \( K \) is equipped with and an almost \( CR \)-structure \( J : \tau \to \tau \) compatible to the conformal symplectic structure of \( \tau \).

**Remark 3.4.10.** Any contact manifold admits many Legendrian foliations. Actually, by Darboux theorem, a neighbourhood of a Legendrian submanifold of any contact manifold is contactomorphic to the jet space over that submanifold and locally admits a fibration, where this submanifold is the one of the fibers.

The lattice

**Lemma 3.4.11.** Suppose \((K, \tau)\) is a contact manifold with a coorientable contact structure. Additionally, suppose that \( K \to B \) is a Legendre fibration of \( K \) over a manifold \( B \) and \( J : \tau \to \tau \) an almost \( CR \)-structure compatible to the conformal symplectic structure of \( \tau \). Then, for each local frame \( \ell = \{ \ell_1, ..., \ell_n \} \) of the vertical bundle \( L = \ker(dp) \) there exists a bundle of framed lattices \( \Lambda \subset \tau \) associated to \( J \) of the form

\[ \Lambda_{\ell} = \Lambda_L \oplus \Lambda_{L}^\perp \]

where \( \Lambda_L \) is a subset of the vertical bundle \( L \) and \( \Lambda_{L}^\perp \) is the orthogonal of \( \Lambda_L \) with respect to the metric compatible to \( J \) and the conformal symplectic structure of \( \tau \). Moreover, for each
representative \( w \) of the conformal symplectic class there exists a unique bundle of normalized lattices, namely there exists a smooth function \( f : K \to \mathbb{R}^+ \)

\[
\Lambda_{w,\ell}^J = \frac{1}{\sqrt{f}} \Lambda_L \oplus \frac{1}{\sqrt{f}} \Lambda_L^\perp
\]  

(3.4.6)

Proof. For a point \( q \in K \), there exist a chart \( U \subset K \) around \( q \) such that \( \ell_1, ..., \ell_n : U \to L \) are smooth Legendrian vector fields and the vectors \( \ell_1(k), ..., \ell_n(k) \) span \( L_k \) for each \( k \in U \). Then, the smooth vector fields \( J\ell_1, ..., J\ell_n : U \to TK \) are Legendrian and the vectors \( \ell_1(k), ..., \ell_n(k), J\ell_1(k), ..., J\ell_n(k) \) form a symplectic basis with respect to the conformal symplectic structure of \( \tau \), namely for each \( w \) in the (positive) conformal symplectic structure of \( \tau \) there exists a function \( f : U \to \mathbb{R}^+ \) such that

\[
w(\ell_i(k), J\ell_i(k)) = f(k) \quad \text{and} \quad w(\ell_i, \ell_j) = w(J\ell_i, J\ell_j) = 0 \quad \text{for} \quad i, j \in \{1, ..., n\}
\]

By employing a partition of unity, we obtain the lattice bundle structure on \( \tau \) given by

\[
\Lambda^J \ell = \text{span}_Z \{ \ell_i \} \oplus \text{span}_Z \{ J\ell_i \}.
\]

Moreover, the normalized vector fields \( \frac{\ell_1}{\sqrt{f}}, ..., \frac{\ell_n}{\sqrt{f}}, \frac{J\ell_1}{\sqrt{f}}, ..., \frac{J\ell_n}{\sqrt{f}} \) give a unique lattice bundle structure corresponding to \( w \)

\[
\Lambda_{w,\ell}^J = \text{span}_Z \left\{ \frac{\ell_i}{\sqrt{f}} \right\} \oplus \text{span}_Z \left\{ \frac{J\ell_i}{\sqrt{f}} \right\}.
\]

\( \square \)

Proposition 3.4.12. Let \( \Phi : B \to B \) a diffeomorphism of \( B \) and \( F : (K, \tau) \to (K, \tau) \) a contactomorphism of \( K \) such that \( F \) is also a bundle map of \( K \to B \) with respect to \( \Phi \). Then, the differential \( dF : TK \to TK \) maps a bundle of framed lattices associated to the almost \( CR- \) structure \( J \)

\[
\Lambda^J = \Lambda_V \oplus \Lambda_V^\perp
\]

(3.4.7)

to a bundle of framed lattices \( dF\Lambda^J \ell \) with the following decomposition

\[
dF\Lambda^J \ell = dF\Lambda_V \oplus dF\Lambda_V^\perp,
\]

(3.4.8)

where \( dF\Lambda_V^\perp \) is the orthogonal lattice with respect to the metric compatible to \( J \) and the conformal symplectic structure of \( \tau \).

Proof. Let \( \ell = \{ \ell_1, ..., \ell_n \} \) be a smooth Legendrian vectors fields \( \ell_i : U \to TK \) on a local
chart $U$ of $K$ such that the vectors $\{\ell_1(k), ..., \ell_n(k), J\ell_1(k), ..., J\ell_n(k)\}$ form a symplectic basis on the contact plane $\tau_q$, for each $k \in U$. The differential $dF$ of $F$ maps Legendrian vector fields $\ell_1, ..., \ell_n : U \to TK$ to Legendrian vector fields $dF\ell_1, ..., dF\ell_n : F(U) \to TK$ such that $\{dF\ell_1, ..., dFJ\ell_1, ..., dFJ\ell_n\}$ is a symplectic basis of $\tau_q$, for each $k \in F(U)$, with respect to the conformal symplectic structure of $\tau$. Indeed, since $F$ is a contactomorphism for each $w$ in the conformal class of $\tau$, $F^*w$ is in the conformal class of $\tau$ and there exists a function $\Phi : F(U) \to \mathbb{R}_+$ such that

$$w(dF\ell_i(k), dFJ\ell_i(k)) = F^*w(\ell_i(k), J\ell_i(k)) = f(k)$$

and $w(dF\ell_i(k), dF\ell_i(k)) = F^*w(\ell, \ell) = 0$ for each $k \in F(U)$.

Thus, the bundle of framed lattices $dFA^\mathcal{J}_l$ is given by

$$dFA^\mathcal{J}_l = \text{span}_\mathbb{Z}\{dF\ell_i\} \oplus \text{span}_\mathbb{Z}\{dFJ\ell_i\} = dFA^\mathcal{V} \oplus dFA^\mathcal{V}_l.$$

The window function The next step is to define a window function on each contact plane $\tau_q$, $q \in K$. Here is a summary of the subsequent construction.

Consider the tangent bundle $TB$ and take a sufficiently nice window function $\phi_b$ at the tangent spaces $T_bB$ at each point $b \in B$ of the base (e.g. Gaussian). Next, every contact space at a point $q \in K$ has a natural projection (in the Legendre fibration) to the codimension 1 subspace $p_*(\tau_q)$ of the tangent space $T_{p(q)}B$. Thus one can lift to this contact space $\tau_q$ the restriction to $p_*(\tau_q)$ of the window function $\phi_b$ for $b = p(q)$ and define $p^*(\phi|_{p_*(\tau_q)})$. Now the function $p^*(\phi|_{p_*(\tau_q)})$ is a function on the $(2k$-dimensional) contact plane $\tau_q$ depending on $k$ variables, and can be regarded as a window function. The latter requires the $L^2$-integrability condition to be satisfied and this can be assured by the corresponding choice of the window function $\phi$ (e.g. taking it to be Gaussian). Thus, it is necessary to take into account window functions $\phi : TB \to \mathbb{R}$ that belong to a specific function space. For this purpose we consider the following topological vector space associated to a vector bundle.

Consider $E \to K$ to be a vector bundle of rank $k$ and let $E_q$ denote the fiber over $q \in K$. Then, the vector space $S(E)$ of functions that are rapidly decaying along the fibers, is the locally convex topological vector space

$$S(E) := \{\phi \in \mathcal{E}(E; E \times \mathbb{R}) : \sup_{q \in K} \left( \sup_{\xi \in E_q} |\xi^\beta \partial^\alpha \phi(\xi)| \right) < \infty \text{ for all } \alpha, \beta \in \mathbb{N}^p \text{ and } p \in \mathbb{N} \}$$

(3.4.9)
Lemma 3.4.13. Suppose \( \phi \) is a smooth real-valued function on \( TB \), rapidly decaying along the fibers. Then, the pullback \( p^*(\phi_{|p_*(q)}) \) is a smooth real-valued function on the vector subbundle \( JL \), rapidly decaying along the fibers \( JL_{q^*} \), where \( L \) is the vertical bundle of \( K \overset{p}{\rightarrow} B \).

Proof. First we need to prove that \( p^*(\phi_{|p_*(q)}) \) is in \( C^\infty(JL_p) \). Indeed, since \( \phi \in C^\infty(TB) \), the restriction \( \phi_{|p_*(q)} \) on \( p_*(\tau) \subset TB \) is a smooth function \( \phi_{|p_*(\tau)} : p_*(\tau) \rightarrow \mathbb{R} \). Let \( q \) some arbitrary point in \( K \), \( (p^*\phi_{|p_*(\tau)}) \) is well-defined on tangent vectors \( W \in T_qK \) such that \( p_*(W) \in p_*(\tau_q) \), that is when \( W \in \tau_q \). Writing \( W \) as \( W = projLW + projJLW \) where \( projLW \in L_q \) and \( projJLW \in JL_q \), it follows that

\[
p^*\phi_{|p_*(\tau)}(W) = \phi_{|p_*(\tau)}(p_*(projLW + projJLW)) = \\
\phi_{|p_*(\tau)}(p_*(projLW)) = p^*\phi_{|p_*(\tau)(projJLW)}
\]

and therefore \( p^*\phi_{|p_*(\tau)} \) is a function in \( C^\infty(JL) \). Now it is left to prove that \( p^*\phi_{|p_*(\tau)} \) is in \( S(JL) \). Let \( \xi \in JL \), there exists some \( \zeta \in p_*(JL) \) such that \( p_*(\xi) = \zeta \). Consider two auxiliary metrics \( g_K \) on \( K \) and \( g_B \) on \( B \), consequently we have

\[
|\xi^\beta \partial^a \phi(\xi)| = |\xi^\beta \partial^a \phi(\xi)| = |(p_*)^{-1}(\xi)^\beta \partial^a \phi(\xi)| \leq \|(p_*)^{-1}\|_{JL} \| (\xi)^\beta \partial^a \phi(\xi)| < \infty
\]

and therefore \( p^*\phi_{|p_*(\tau)} \) is rapidly decaying along the fibers of \( JL \).

\[\square\]

Proposition 3.4.14. Let \( f : B \rightarrow B \) a diffeomorphism of \( B \) and \( F : (K, \tau) \rightarrow (K, \tau) \) a contactomorphism of \( K \) such that \( F \) is also a bundle map of \( K \overset{p}{\rightarrow} B \) with respect to \( f \). Then, the pullback of the window function \( p^*\phi_{|p_*(\tau)} \in S(JL) \) via \( F \) is the window function \( p^*(f^*\phi_{|p_*(\tau)})) \in S(JL) \). In particular, if \( f \) is the identity map, then the pullback \( F^*(p^*\phi_{|p_*(\tau)}) \) is equal to \( p^*\phi_{|p_*(\tau)} \).

Proof. Firstly, since \( F \) is a contactomorphism and maps the vertical bundle \( L \) to itself, the symplectic orthogonal \( JL \) of \( L \) is mapped onto itself,

\[dF(JL) = L.\]

Let \( X \) be a vector in \((JL)_q\) for some point \( q \in K \) such that \( p(q) = b \). Then, it follows that

\[
(F^*p^*\phi_{|p_*(\tau)})_q(X) = (p^*\phi_{|p_*})_{F(q)}(F_*(X))
\]
\[
= (\phi_{|p_*(\tau)})_{f(b)}(p_*F_*X)
\]
\[
= (\phi_{|p_*(\tau)})_b(f_*p_*X)
\]
\[
= (f^*(\phi_{|p_*(\tau)}))_b(p_*X)
\]
\[
= p^*(f^*(\phi_{|p_*(\tau)}))_q(X).
\]
Additionally, if \( f \) is the identity map, then \( F^*((p^*\phi|_{p_*(\tau)})) = p^*\phi|_{p_*(\tau)}. \)

**Remark 3.4.15.** If \( TB \) can be trivialized, it suffices to define a window function in one tangent space and extend it to all.

### 3.4.3 Invariance with respect to contact diffeomorphisms

In the preceding paragraphs we saw that a contact transformation \( F : K \to K \) which is a bundle map of the fiber bundle \( K \to B \) transforms a bundle of lattices on the contact distribution \( \tau \) and a collection of window functions on \( JL \) simultaneously. In particular, the lattice \( \Lambda^q_{\ell} \) at a point \( q \in K \) is transformed symplectically by \( F_\tau \). This observation directs our attention towards the symplectic transformation properties of Gabor frames.

The properties of a Gabor frames \( G(\phi, \Lambda) \) on \( \mathbb{R}^n \) after a symplectic transformation of the lattice, have been studied by M. De Gosson and K. Gröchenig, among others. More specifically, Gröchenig employs the metaplectic representation of the symplectic group to investigate different symmetries in time-frequency analysis, in [Grö01].

Let \( \text{Aut}(\mathbb{H}_n) \) denote the group of automorphisms of the Heisenberg group \( \mathbb{H}_n = \mathbb{R}^{2n+1} \) with product rule

\[
(\tau, x, w) \cdot (\tau', x', w') = (\tau + \tau' + \frac{1}{2}\sigma((x, w), (x', w')), x + x', w + w')
\]

where \( \sigma : \mathbb{R}^{2n} \to \mathbb{R} \) is the standard symplectic form on \( \mathbb{R}^{2n} \). For any matrix \( S \in Sp(n) \) the map

\[
T_S(\tau, x, w) = (\tau, S(x, w))
\]

is an automorphism of \( \mathbb{H}_n \). Consider the Schrödinger representation of the Heisenberg group on \( L^2(\mathbb{R}^n) \),

\[
\rho : \mathbb{H}_n \to \mathcal{U}(L^2(\mathbb{R}^n))
\]

\[
(\tau, x, w) \mapsto (f(t) \mapsto e^{2\pi i t \tau} e^{2\pi i wt - \pi i xw} f(t + x))
\]

where \( \mathcal{U}(L^2(\mathbb{R}^n)) \) denotes the unitary operators on \( L^2(\mathbb{R}^n) \). According to the Stone-Von Neuman theorem, and since \( \rho \circ T_S \) is irreducible there exists a unitary operator \( \mu(T_S) \) on \( L^2(\mathbb{R}^n) \) such that

\[
\rho \circ T_S(\tau, x, w) = \mu(T_S)\rho(\tau, x, w)\mu(T_S)^{-1}.
\]

**Remark 3.4.16.** For an arbitrary automorphism \( T \in \text{Aut}(\mathbb{H}) \), the unitary operator \( \mu(T) \in \mathcal{U}(L^2(\mathbb{R}^n)) \) such that

\[
\rho \circ T(x, w, \tau) = \mu(T)\rho(x, w, \tau)\mu(T)^{-1}
\]
can only be determined up to a sign $\pm I$. Thus,
\begin{equation}
\mu : \text{Aut}(\mathbb{H}) \to \mathcal{U}(L^2(\mathbb{R}^n))/\{\pm I\} \\
T \mapsto [\mu(T)]
\end{equation}

is a homomorphism and $\mu$ defines a projective unitary representation of $T$.

**Definition 3.4.17.** The metaplectic representation of $Sp(n)$ on $\mathcal{U}(L^2(\mathbb{R}^n))$ is the continuous homomorphism
\begin{equation}
\mu : Sp(n) \to \mathcal{U}(L^2(\mathbb{R}^n))/\{\pm I\} \\
S \mapsto \mu(T_S)
\end{equation}

where $\mathcal{U}(L^2(\mathbb{R}^n))$ is equipped with the strong operator topology and $\mathcal{U}(L^2(\mathbb{R}^n))/\{\pm I\}$ with the quotient topology.

At the same time, the Heisenberg-Weyl operators $\hat{T}^{1/2\pi}(x,w)$ for $\hbar = \frac{1}{2\pi}$ can be written in terms of the Schrödinger representation $\rho : \mathbb{H} \to \mathcal{U}(L^2(\mathbb{R}^n))$ as
\begin{equation}
\rho(\tau,x,w)\phi(t) = e^{-2\pi it} \hat{T}^{1/2\pi}(x,w)\phi(t).
\end{equation}

and therefore it follows that the operators $\hat{T}^{1/2\pi}(x,w)$ satisfy a symplectic covariance relation
\begin{equation}
\mu(T_S)\hat{T}^{1/2\pi}(S(x,w))\mu(T_S)^{-1} = \hat{T}^{1/2\pi}(x,w)
\end{equation}

where $S \in Sp(n)$ and $\mu(T_S)$ is the corresponding to $S$ unitary operator as in **Definition 3.4.17.** More generally, one can obtain the same relation for the operators $\hat{T}^{\hbar}(x,w)$ for arbitrary $\hbar$,
\begin{equation}
\mu(T_S)\hat{T}^{\hbar}(S(x,w))\mu(T_S)^{-1} = \hat{T}^{\hbar}(x,w).
\end{equation}

The symplectic covariance relation leads to the following proposition, according to which if the lattice $\Lambda$ of a Gabor Frame $\mathcal{G}(\phi,\Lambda)$ is transformed by some symplectic transformation $S \in Sp(n)$ and at the same time the window $\phi$ is trasformed by the associated unitary operator $\mu(T_S)$, then one obtains a new Gabor Frame $\mathcal{G}(\mu(T_S)\phi, S\Lambda)$.

**Theorem 3.4.18.** [Gröb1] Let $\phi$ in $L^2(\mathbb{R}^n)$ or in $S(\mathbb{R}^n)$. A Gabor system $\mathcal{G}(\phi, \Lambda)$ is a $\hbar$-Frame if and only if the Gabor system $\mathcal{G}(\mu(T_S)\phi, S\Lambda)$ is an $\hbar$-Frame.

Calculating the operator $\mu(T_S)$ for an arbitrary symplectic matrix $S$ can be tedious, however we will only need to refer to two specific examples, for now.
Example 3.4.19. Let \( g \) be the standard 1-dimensional Gaussian \( g(t) = 2^{1/4} e^{-\pi t^2} \). Suppose we have the Gabor System \( \mathcal{G}(g, \Lambda) \), and we rotate the lattice using the symplectic transformation induced by
\[
R_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.
\]
The corresponding transformation of the window is \( \mu(T_{R_\theta})g = cg \) for some \( |c| = 1 \) and therefore the Gabor system \( \mathcal{G}(g, \Lambda) \) is a frame exactly when the ”rotated” Gabor system \( \mathcal{G}(g, R_\theta \Lambda) \) is a frame.

Example 3.4.20. Let \( g \) be a window function in \( L^2(\mathbb{R}^n) \). Suppose we have the Gabor System \( \mathcal{G}(g, \Lambda) \). The Gabor system \( \mathcal{G}(g, \Lambda) \) is a frame if and only if the Gabor system \( \mathcal{G}(\mu(T_f)g, J\Lambda) \) is a frame, where \( f = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and the operator \( \mu(T_f) \) is the Fourier transform
\[
\mu(T_f)f(t) = F^{-1}f(t) = \left( \frac{1}{2\pi \hbar} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{2\pi i < t, y>} f(y) dy.
\] (3.4.16)

In a similar manner as in the preceding sections, we will say that a Gabor System Bundle over the vector bundle \( E \rightarrow B \) if a bundle of Gabor \( h \)-frames if it satisfies the frame condition fiberwise.

Definition 3.4.21. A Gabor System Bundle over the vector bundle \( E \rightarrow B \) is a Gabor-\( h \)-frame bundle if there exist smooth \( \mathbb{R}^* \)-valued functions on the local charts of \( K \) and smooth \( h : K \rightarrow \mathbb{R}^* \) such that the following inequality holds
\[
a(q)||f||^2 \leq \sum_{z_0 \in \Lambda(q)} |\langle f, T^h(q)(z_0)\phi(q) \rangle|^2 \leq b(q)||f||^2 \text{ for every } f \in L^2(L_q) \quad (3.4.17)
\]

We will consider a particular class of window functions \( \psi \in \mathcal{S}(\mathcal{J}L) \). In particular, to each \( \sigma \) in the conformal symplectic class of the symplectic bundle \( \tau \rightarrow B \) we associate the smooth function
\[
\psi_{\sigma} : \mathcal{J}L \rightarrow \mathbb{R} \\
V' \mapsto \exp(-\pi \sigma(V', \mathcal{J}V')),
\]
which is rapidly decaying along the fibers of \( \mathcal{J}L \). These window functions can be regarded as the analogues of Gaussian windows.

Theorem 3.4.22. Let \( (K, \tau) \) be a contact manifold with a Legendrian fibration \( K \rightarrow B \) over a smooth manifold \( B \) and an almost \( CR \)-structure \( \mathcal{J} : \tau \rightarrow \tau \) compatible with the conformal symplectic class of \( \tau \). The Gabor System Bundle \( \mathcal{G}(\psi_{\sigma}, \Lambda^\mathcal{J}_{\ell}) \) is a Gabor-\( h \)-frame bundle where \( \Lambda^\mathcal{J}_{\ell} \) is a bundle of framed lattices as in proposition 3.4.12 if and only if the Gabor System Bundle
\( \mathcal{G}(\mathcal{J}_{|\mathcal{J}L}^*, \psi_\sigma, \mathcal{J}(\Lambda^q_\ell)) \) is a frame bundle, where \( \mathcal{J}_{|\mathcal{J}L}^* \psi_\sigma \) is pull-back of \( \psi_\sigma \) under the bundle map

\[
\mathcal{J}_{|\mathcal{J}L} : \mathcal{J}L \to L.
\]

**Proof.** We start by calculating \( \mathcal{J}_{|\mathcal{J}L}^* \psi_\sigma \). Assume \( q \) is a point in \( K \), then

\[
\mathcal{J}_{|\mathcal{J}L}^* (\psi_\sigma)_q (V') = (\psi_\sigma)_q (\mathcal{J}_{|\mathcal{J}L} V')
\]

\[
= \exp(-\pi \sigma_q (\mathcal{J}V' - V')) = \exp(-\pi \sigma_q (V', \mathcal{J}V'))
\]

\[
= (\psi_\sigma)_q (V'), \text{ for } V' \in L_q.
\]

Additionally, let \( z_0 := (X, W) \in \Lambda^q_\ell \) and \( z'_0 := (\mathcal{J}X, \mathcal{J}W) \in \mathcal{J} \Lambda^q_\ell \), we are going to show that

\[
\{ \mathcal{J}_{|\mathcal{J}L}^* (T^{h(q)}(-z_0)\psi_\sigma) : z_0 \in \Lambda^q_\ell \} = \{ T^{h(q)}(z'_0)\mathcal{J}_{|\mathcal{J}L}^* \psi_\sigma : z'_0 \in \mathcal{J} \Lambda^q_\ell \} \subset L^2(\mathcal{J}L).
\]

as discrete subsets of \( L^2(\mathcal{J}L) \)

Indeed, we have that

\[
T^{h(q)}(z'_0)(\mathcal{J}_{|\mathcal{J}L}^* \psi_\sigma)(V') = e^{(\sigma_q(\mathcal{J}W, V') - \sigma_q(\mathcal{J}W, \mathcal{J}X)/2)/h} \exp(-\pi \sigma(\mathcal{J}V' - \mathcal{J}X, \mathcal{J}(V' - \mathcal{J}X)))
\]

\[
= e^{(\sigma_q(-W, V') - \sigma_q(W, X)/2)/h} \exp(-\pi \sigma(\mathcal{J}V' + X, \mathcal{J}(\mathcal{J}V' + X)))
\]

\[
= T^{h(q)}(-z_0)(\psi_\sigma)(\mathcal{J}V'), \text{ for every } V' \in \mathcal{J}L.
\]

Moreover,

\[
\mathcal{J}_{|\mathcal{J}L}^* (T^{h(q)}(-z_0)\psi_\sigma)(V') = e^{(\sigma_q(-W, V') - \sigma_q(W, X)/2)/h} \exp(-\pi \sigma(\mathcal{J}V' + X, \mathcal{J}(\mathcal{J}V' + X)))
\]

\[
= T^{h(q)}(-z_0)(\psi_\sigma)(\mathcal{J}V'), \text{ for every } V' \in \mathcal{J}L
\]

and therefore \( T^{h(q)}(z'_0)(\mathcal{J}_{|\mathcal{J}L}^* \psi_\sigma) = \mathcal{J}_{|\mathcal{J}L}^* (T^{h(q)}(-z_0)\psi_\sigma)(V') \) which implies (3.4.18).

Lastly, to prove that \( \mathcal{G}(\psi_\sigma, \Lambda^q_\ell) \) is a frame if and only if \( \mathcal{G}(\mathcal{J}_{|\mathcal{J}L}^* \psi_\sigma, \mathcal{J}(\Lambda^q_\ell)) \) is a frame, assume \( a \) and \( b \) are smooth functions strictly positive in a chart \( U \subset K \) of \( q \). Let \( g \) be a function in \( L^2(\mathcal{J}L_q) \), then there exists some \( f \in L^2(L_q) \) such that \( g = \mathcal{J}_{|\mathcal{J}L}^* f \). It follows that

\[
\langle g, T^{h(q)}(z'_0)(\mathcal{J}_{|\mathcal{J}L}^* \psi_\sigma) \rangle_{\mathcal{J}L_q} = \langle g, \mathcal{J}_{|\mathcal{J}L}^* (T^{h(q)}(-z_0)\psi_\sigma) \rangle_{\mathcal{J}L_q} = \langle \mathcal{J}_{|\mathcal{J}L}^* f, \mathcal{J}_{|\mathcal{J}L}^* (T^{h(q)}(-z_0)\psi_\sigma) \rangle_{\mathcal{J}L_q} = \langle f, T^{h(q)}(-z_0)(\psi_\sigma) \rangle_{L_q}
\]

where \( \langle \cdot, \cdot \rangle_{L_q} \) denotes the inner product of \( L^2(L_q) \) and \( \langle \cdot, \cdot \rangle_{\mathcal{J}L_q} \) denotes the inner product of \( L^2(\mathcal{J}L_q) \).
Finally, it follows that

$$\sum_{z_0^\prime \in J \Lambda(q)} |\langle g, T^{h(q)}(z_0^\prime)(\mathcal{J}_L^\ast \psi_{\sigma}) \rangle_{\mathcal{J}_L^q}|^2 = \sum_{z_0 \in \Lambda(q)} |\langle f, T^{h(q)}(z_0)(\psi_{\sigma}) \rangle|_{\mathcal{J}_L^q}^2$$

and therefore $a(q)$ and $b(q)$ are frame bounds for

$$\{T^{h(q)}(z_0^\prime)(\mathcal{J}_L^\ast \psi_{\sigma}) : z_0^\prime \in J \Lambda^\ast(q) \} \subset L^2(\mathcal{J}_L^q)$$

if and only if they are frame bounds for $$\{T^{h(q)}(z_0)(\psi_{\sigma}) : z_0 \in \Lambda^\ast(q) \} \subset L^2(L_q)$$.

**Remark 3.4.23.** The previous theorem can be interpreted in two distinct manners: either as the analogue of example 3.4.19 where the transformation of the lattice is a $90^\circ$ rotations and the window is the Gaussian $g(t) = e^{-\pi t^2}$, or as analogue of example 3.4.20, since the image of a Gaussian via the Fourier transform is a Gaussian.

### 3.4.4 Compatibility with models of the visual cortex

While the set $\mathcal{G}(\phi, \Lambda)$ constructed as above provides a Gabor system, this is not yet the one that we need for signal analysis, generalizing the case of the visual cortex to higher dimensions. In fact, there are two reasons why this construction needs to be modified.

1. The dimension of the Gabor frames obtained is not the desired one, as the filters should be functions on $TB$ (or on the pullback of $TB$ to $M$) rather than on the codimension-one $p_*^\ast(\tau) \subset TB$.

2. The window $\phi$ in this construction depends only on the coordinates on $TB$ (and in fact on $p_*^\ast(\tau) \subset TB$), while one would like to obtain a family of filters, depending on the coordinates on $M$, where the Legendrian directions along the foliation $L \subset M$ parameterize the dependence of filters on additional geometric parameters (such as the angle dependence in the case of the visual cortex) that make the filters “adapted” to certain classes of signals (such as detecting one-dimensional boundaries in a two-dimensional image in the case of the visual cortex).

We address how to modify the construction above to accommodate these two properties.

In comparison with the model of the contact geometry of the visual cortex, this construction misses one dimension, in the sense that in such models, one wants to work directly with a window function on $TB$, not on a bundle of hyperplanes in $TB$ like $p_*^\ast(\tau)$. One wants to obtain $2n$-dimensional Gabor systems (with an $n$-dimensional space
of translations and an \(n\)-dimensional space of modulations), rather than a \(2(n-1)\)-dimensional Gabor system with \((n-1)\)-dimensional spaces of translations and modulation.

Thus, we need to modify this construction so that, instead of working with the \(2(n-1)\)-symplectic spaces \(\tau\) given by the contact hyperplanes distribution of \(K\), we work with an associated contact bundle with \(2n\)-dimensional symplectic spaces. This shift of dimension by 2 can be achieved by passing to the symplectization of \(K\).

We recall the symplectization \(S(K, \tau)\) of the contact manifold \((M, \alpha)\), described in section 2.2, which we can describe as the complement of the zero section in the real line bundle \(TK/\tau\),

\[
S(K, \tau) = (TK/\tau)^0 = (TK/\tau) \setminus \{0 \text{ section}\}. \tag{3.4.19}
\]

with symplectic form given by \(\omega = d(\alpha)\). We will be working with one of the two connected components of \(S(K, \tau)\), which is an \(\mathbb{R}^*\)-bundle over the contact manifold \(K\). With the change of variables \(r = e^{\theta}\) we can identify the connected component \(S(K, \tau)\) with \(S(K, \tau) = K \times \mathbb{R}\), with \(\omega = d(e^{\theta}\alpha)\). \(\tag{3.4.20}\)

A first general observation is that, given a basis \(\{V_i\}_{i=1}^{2(n-1)}\) of \(\tau\), with vector fields \(V_i\) on \(K\), it is possible to complete it to a basis of the tangent bundle \(TS(K, \tau)\). Indeed, by choosing an auxiliary metric the tangent space \(TK\) is isomorphic to \(\tau \oplus \tau^\perp\). The line bundle \(\tau^\perp \cong TK/\tau\) is trivial since \(\tau\) is coorientable and can be identified with the line subbundle \(\langle R_\alpha \rangle \subset TK\) spanned by the Reeb field \(R_\alpha\) for the contact form \(\alpha\). Under the identification (3.4.20), \(S(K, \tau)\) is an \(\mathbb{R}\)-bundle over \(K\) with projection \(\pi_S : S(K, \tau) \to K\) and therefore we can decompose \(TS(K, \tau)\) into a sum of the pullback to \(K \times \mathbb{R}\) of \(\tau\), the pullback to \(K \times \mathbb{R}\) of the trivial Reeb line bundle \(\mathcal{R}_\alpha\), the real line bundle spanned by the Reeb vector field \(R_\alpha\) on \(K\), and the pullback of the trivial line bundle \(T\mathbb{R}\) spanned by \(\partial_s\) in the \(\mathbb{R}\)-direction of \(S(K, \tau)\),

\[
TS = \pi_S^\perp(\tau) + \pi_S^\perp(\langle R_\alpha \rangle) + \pi_S^\perp(T\mathbb{R}). \tag{3.4.21}
\]

Given a family of lattices \(\Lambda \subset \tau\) as before, we can then obtain a family of lattices

\[
\tilde{\Lambda} \subset TS(K, \tau), \quad \tilde{\Lambda} = \Lambda \oplus \mathbb{Z}R_\alpha \oplus \mathbb{Z}\partial_s, \tag{3.4.22}
\]

with the sum decomposition as in the decomposition of \(TS(K, \tau)\) discussed above. This is now a rank \(2n\) lattice in the rank \(2n\) vector bundle \(TS(K, \tau)\).

In particular, suppose \(K\) is endowed with a Legendrian fibration \(K \overset{p}{\to} B\) with fiber
$L := p^{-1}(b)$, and let $\mathcal{F}_L$ denote the foliation of $K$ induced by the fibration

$$L \xrightarrow{\iota_L} K \xrightarrow{p} B,$$

we obtain a Lagrangian foliation $\pi_S^*(\mathcal{F}_L)$ of the symplectization $S(K, \tau)$, where the leaves $\mathcal{L}$ are the inverse images under $\pi_S$ of the leaves of $\mathcal{F}_L$. A basis of $TL$ is determined by $\partial_s$ and a basis for $TL \subset \tau$. Additionally, we obtain the Lagrangian fibration

$$\mathcal{L} \rightarrow S(K, \tau) \xrightarrow{\Pi} B$$

where $\Pi$ is the composition of $\pi_S : S \rightarrow K$ and $p : K \rightarrow B$, namely

$$\Pi = p \circ \pi_S.$$

The fiber $\mathcal{L} \simeq \Pi^{-1}(b)$ is the pullback $\iota_L^*(S)$ of $S$ to $L$ under the inclusion $\iota_L : L \rightarrow K$, where we view $S = S(M, \xi)$ as an $\mathbb{R}$-bundle over $M$. The foliation induced by the fibration $\Pi : S \rightarrow B$ is the Lagrangian foliation $\pi_S(\mathcal{F}_L)$.

For the upcoming lemma, we write $TL^\vee$ for the dual, with respect to the pairing given by the sub-Riemannian structure $da(-, J-)$ on $\tau$, of the Lagrangian subbundle $TL \subset \tau$.

**Lemma 3.4.24.** There exists an almost complex structure on $\pi_S^*((\langle R_a \rangle)) \oplus \pi_S^*(\mathcal{T}\mathcal{R})$,

$$\mathcal{J}_{\pi_S^*((\langle R_a \rangle)) \oplus \pi_S^*(\mathcal{T}\mathcal{R})} : \pi_S^*((\langle R_a \rangle)) \oplus \pi_S^*(\mathcal{T}\mathcal{R}) \rightarrow \pi_S^*((\langle R_a \rangle)) \oplus \pi_S^*(\mathcal{T}\mathcal{R})$$

with the following properties:

1. The bundle map

$$\mathcal{J}_S : TS \rightarrow TS$$

$$\mathcal{J}_S := \pi_S^*J \oplus \mathcal{J}_{\pi_S^*((\langle R_a \rangle)) \oplus \pi_S^*(\mathcal{T}\mathcal{R})}$$

is an almost complex structure on $S(K, \tau)$ compatible with $\omega = d(e^a)$.  

2. There is an identification of the dual of $T\mathcal{L}$ with respect to the Riemannian metric $\omega(-, J_S-)$ with $\pi_S^*TL^\vee \oplus \pi_S^*\mathcal{R}_a$, where $\mathcal{R}_a$ is the Reeb line bundle, and the summands on the right-hand-side stand for the pullbacks from $K$ to $S(K, \tau)$.

**Proof.** The sections $R_a$ and $\partial_s$ form a global frame for $\pi_S^*((\langle R_a \rangle)) \oplus \pi_S^*(\mathcal{T}\mathcal{R})$ so we can define the vector bundle morphism

$$\mathcal{J}_{\pi_S^*((\langle R_a \rangle)) \oplus \pi_S^*(\mathcal{T}\mathcal{R})} : \pi_S^*((\langle R_a \rangle)) \oplus \pi_S^*(\mathcal{T}\mathcal{R}) \rightarrow \pi_S^*((\langle R_a \rangle)) \oplus \pi_S^*(\mathcal{T}\mathcal{R})$$

$$\omega_{R_a} \rightarrow \omega_{R_a} R_a - \omega_{R_a} \partial_s.$$
Using $\mathcal{J}_{\pi^S_{3}((R_{\alpha})) \oplus \pi^S_{3}(\mathbb{T} \mathbb{R})}$ we can extend the almost CR-structure $\mathcal{J}$ on $\tau$ to an almost complex structure $\mathcal{J}_{S}$ on $TS(K, \tau)$,

$$\mathcal{J}_{S} : TS \to TS \quad (3.4.25)$$

$$\mathcal{J}_{S} := \pi^S_{3} J \oplus J_{\pi^S_{3}((R_{\alpha})) \oplus \pi^S_{3}(\mathbb{T} \mathbb{R})}$$

1. To prove that $\mathcal{J}_{S}$ is compatible to $\omega$, it suffices to prove that for $s = (q, \eta)$ being a point in the symplectization $S(K, \tau)$ such that $\pi_S(s) = q$, if $\{\ell_1, ..., \ell_n, J\ell_1, ..., J\ell_n\}$ is a symplectic basis of $(\tau_q, da)$, then

$$\{\ell_1, ..., \ell_n, \partial_s, J\ell_1, ..., J\ell_n, R_a\}$$

is (up to scaling) a symplectic basis of $(T_sS, \omega)$ of the form

$$e_1, ..., e_{n+1}, f_1 = J_{S}e_1, ..., f_{n+1} = J_{S}e_{n+1}.$$ 

Indeed, if we take $e_i = \ell_i$ for $i = 1, ..., n$, $e_{n+1} = \partial_s$, $f_i = J\ell_i$ for $i = 1, ..., n$ and $f_{n+1} = R_a$, then it follows that $J_{S}e_i = f_i$ for $i = 1, ..., n + 1$ and

$$\omega(\ell_i, J_{S}\ell_j) = e^s da((\ell_j, J\ell_j) = e^s \delta_{ij}$$

$$\omega(\partial_s, R_a) = e^s ds \wedge da(\partial_s, R_a) = e^s$$

$$\omega(\ell_i, R_a) = \omega(\ell_i, \partial_s) = \omega(J\ell_i, R_a) = \omega(J\ell_i, \partial_s) = 0.$$ 

2. The subbundle $T\mathcal{L} \subset TS$ is the vertical bundle with respect to the fibration $\Pi : S(K, \tau) \to B$ and therefore

$$T\mathcal{L} = \ker(\Pi) = \ker(p_+ \pi_S) \simeq \pi^S_{3}(\mathbb{T} \mathbb{R}) \oplus \pi^S_{3}(T\mathcal{L}).$$

Let $X \in TS$ such that

$$\omega(X, J_{S}Y) = 0 \quad \text{for all} \quad Y \in T\mathcal{L},$$

where $Y$ can be written as $Y = Y_{\mathbb{R}} + Y_{\mathbb{T}}$ with $Y_{\mathbb{T}} \in \pi^S_{3}(\mathbb{T} \mathbb{R})$ and $Y_{\mathbb{R}} \in \pi^S_{3}(\mathbb{T} \mathbb{L})$. Then, $\omega(X, J_{S}Y) = 0$ if and only if

$$\omega(X, J_{\pi^S_{3}((R_{\alpha})) \oplus \pi^S_{3}(\mathbb{T} \mathbb{R})}Y_{\mathbb{R}}) = \omega(X, \pi^S_{3}JY_{\mathbb{T}}) = 0,$$

for every $J_{\pi^S_{3}((R_{\alpha})) \oplus \pi^S_{3}(\mathbb{T} \mathbb{R})}Y_{\mathbb{R}} \in \pi^S_{3}(R_{\alpha})$ and for every $\pi^S_{3}JY_{\mathbb{T}} \in \pi^S_{3}(T\mathcal{L}^\vee)$. Equivalently, at every point $s \in S$ $X_s$ lies in the intersection of the symplectic orthogonal vector spaces of $\pi^S_{3}(T\mathcal{L}^\vee)_s$ and $\pi^S_{3}(R_{\alpha})_s$ with respect to $\omega_s$ and therefore

$$X \in \pi^S_{3}(T\mathcal{L}^\vee) \oplus \pi^S_{3}(R_{\alpha}).$$
Similarly, we write $\mathcal{L}^\vee \mathcal{L}$ for the dual, with respect to the Riemannian metric $\omega(-, \mathcal{J}_S -)$ on $TS$ of the Lagrangian subbundle $\mathcal{L}$ in $TS(K, \tau)$.

**Proposition 3.4.25.** Given a smooth function $\phi : TB \to \mathbb{R}$, there exists a family of window functions $\psi(s) : T\mathcal{L}^\vee \to \mathbb{R}$ parametrized by $s \in S$ and together with the lattice

$$\tilde{\Lambda} \subset TS(K, \tau) \, , \, \tilde{\Lambda} = \Lambda \oplus \mathbb{Z} \alpha \oplus \mathbb{Z} \partial_s$$

they form a Gabor system

$$\mathcal{G}(\psi, \tilde{\Lambda}) = \{ T^n(\lambda) \psi | \lambda \in \tilde{\Lambda} \}$$

(3.4.26)
on $L^2(T\mathcal{L}^\vee)$.

**Proof.** Consider then the pullback $\mathcal{E} = p^*TB$ to $K$ of the tangent bundle of $B$ under the projection $p : K \to B$ of the Legendrian fibration. Using the projection $\pi_S : S(K, \tau) \to B$, we can further pull back $\mathcal{E}$ to $S(K, \tau)$. We denote the resulting bundle on $S(K, \tau)$ by $\tilde{\mathcal{E}} := \pi_S(\mathcal{E})$. As discussed above, we can identify $T\mathcal{L}^\vee \simeq T\mathcal{L}^\vee \oplus \mathcal{R}_\alpha$, where $\mathcal{R}_\alpha$ is the Reeb line bundle, and the summands on the right-hand-side stand for the pullbacks from $K$ to $S(K, \tau)$. The Lagrangian fibration $L \hookrightarrow S(K, \tau) \to B$ gives an identification of the pullback $\tilde{\mathcal{E}}$ of $TB$ with the subbundle $T\mathcal{L}^\vee$ of $TS(K, \tau)$ spanned by the pullbacks of $T\mathcal{L}^\vee$ and $\mathcal{R}_\alpha$.

Consider a smooth function $\phi : TB \to \mathbb{R}$, of rapid decay in the fiber directions given by

$$\phi_b(V) = \exp(-V^t A_b V) \quad \text{with} \quad V \in T_b B$$

with $A$ a smooth section of $T^*B \otimes T^*B$ that is symmetric and positive definite as a quadratic form on the fibers of $TB$. Then, we can consider a window function $\psi$ on $T\mathcal{L}^\vee$ over $S(K, \tau)$, obtained by taking

$$\psi_b(V) = \phi_b(V) \exp(i \langle \eta, V \rangle_p) ,$$

(3.4.27)

where we write a point $s \in S(K, \tau)$ as $s = (q, \eta) = (b, \ell, \eta)$ with $q \in K$ and $\eta \in T^*_q K$ with $\text{Ker}(\eta) = \tau_q$, and $q = (\ell, b)$ with $b = p(q)$ with respect to the Legendrian foliation $L \hookrightarrow K \to B$ and $\langle \eta, V \rangle$ is the duality pairing.

Finally, using the lattice $\tilde{\Lambda}$ as described in section 3.4.22, we can then construct the Gabor system associated to the choice of the window function $\psi$

$$\mathcal{G}(\psi, \tilde{\Lambda}) = \{ T^n(\lambda) \psi | \lambda \in \tilde{\Lambda} \}$$

(3.4.28)
with $T^h(\lambda)\psi$ defined as in (3.4.4) with $\lambda = (x, w)$ with respect to the Darboux coordinates on $T\mathcal{S}(K, \tau)$ and $t \in T\mathcal{L}^\vee$.

\[\text{\(\square\)}\]

**Remark 3.4.26.** The Gabor system constructed in the previous proposition is an anlogue of the Gabor systems $G(\Psi_0, \Lambda_{\alpha, J} \oplus \Lambda_{\alpha, J}^\vee)$, constructed in 3.1.7 for the models of the visual cortex, as well as the Gabor systems $G(\Psi, \Lambda)$ from definition 3.3.14. In particular, the window function $\psi_s : T\mathcal{L}^\vee \rightarrow \mathbb{R}$ depends on the point $b \in B$ of the base of the fibration and the pair $(\ell, \eta) \in \mathcal{L}_b$ of the Lagrangian fiber over $b$ as parameters. Thus, one can regard the pair $(\ell, \eta)$ as a generalization of the angle coordinate $\theta$ of the window function

$$
\Psi_{0, (x, y, \theta)}(V) = \exp \left( -V^t A_{(x, y)} V - i \langle \eta, V \rangle_{(x, y)} \right).
$$

which models the receptive profiles in the models of $V_1$. The dependence of the window function $\psi_s$ on $s$ can be used to maximize the response to certain classes of signals (for example to the presence of higher dimensional boundaries in the case where $M = S(T^*B)$ and the Legendrian fibration is given by the bundle projection $S^{n-1} \hookrightarrow S(T^*B) \rightarrow B$ in section 3.3).
Chapter 4

Appendix

4.1 The Fourier Transform of functions on $T^*M$ to functions on $TM$

In 1993, Landsman, in [Lan93], introduced a Fourier transform that maps functions on the co-tangent bundle $T^*M$ of a Riemannian manifold $(M, g)$ to those on the tangent bundle $TM$. Let $\{U_a\}_{a \in I}$ coordinate charts of $M$ with coordinate functions $m^i_a : U_a \to \mathbb{R}$. This leads to canonical coordinates $(m^i, p_i)$ on $T^*M$ and $(m^i, v^i)$ on $TM$. The metric $g$ induces volume forms $d\text{vol}_{T^*M}$ on $T^*_mM$ and $d\text{vol}_{TM}$ on $T^*_mM$ for each $m \in M$. In coordinates one has

$$d\text{vol}_{T^*_mM} = \sqrt{g_m} dv$$
$$d\text{vol}_{T^*_mM} = \frac{dp}{\sqrt{g_m}}.$$

The fiberwise Fourier transform of a function $\phi : T^*M \to \mathbb{R}$ measurable in $T^*M$ and integrable on each fiber is

$$\mathcal{F}(\phi)_m(v) = \frac{1}{(2\pi)^n} \int_{T^*_mM} \phi(m, p) e^{2\pi i p_i v^i} \frac{dp^n}{\sqrt{g_m}}. \quad (4.1.1)$$

where $\langle -, - \rangle$ is the duality pairing between vectors and covectors. In local coordinates this leads to

$$\mathcal{F}(\phi)(m, v) = \frac{1}{(2\pi)^n} \int_{T^*_M} \phi(p) e^{2\pi i (p_i v^i)} d\text{vol}_{T^*_M}.$$ 

In particular, the Fourier transform maps functions on $T^*M$ which are rapidly decaying along the fibers $T^*_mM$ to function on $TM$ which are rapidly decaying along the fibers $T_mM$ and therefore one obtains the map 3.1.6.
4.1.1 Metric on the Cotangent Bundle of a Riemannian manifold

Let \((M, g)\) be a Riemannian manifold where \(g_{ij}\) are the components of \(g\) on a coordinate chart \((U, m^i)\) of \(M\). We consider the lift \(\tilde{g}\) on the cotangent bundle \(\pi : T^*M \to M\) introduced in [Sat68]. In a local chart \((\pi^{-1}(U), (m^i, p_i))\) of \(T^*M\), we consider the line element

\[
d\sigma^2 = g_{ij} dm^i dm^j + \tilde{g}^{ij} Dp_i Dp_j,
\]

where \(Dp_i\) is the covariant differential of \(p_i\),

\[
Dp_i = dm^i + \Gamma^i_{jk} p_j dm^k.
\]

The components of \(\tilde{g}\) for \(i\) and \(j\) from 1 to \(n = \text{dim}(M)\) are

\[
\tilde{g}_{ij} = g_{ij} + \gamma_{ik} \gamma_{jl} \tilde{g}^{kl}, \quad \tilde{g}_{i(n+j)} = -g^{ia} \gamma_{aj}, \quad \tilde{g}_{(n+i)(n+j)} = g^{ij} \quad \text{where} \quad \gamma_{ij} := \Gamma^a_{ij} p_a.
\]

or as a matrix

\[
\begin{pmatrix}
 g_{ij} + \gamma_{ik} \gamma_{jl} \tilde{g}^{kl} & -g^{ia} \gamma_{aj} \\
 -g_{ia} \gamma_{aj} & g^{ij}
\end{pmatrix}.
\]

4.2 Fiber Bundles

In this paragraph we present some of the definitions and theorems regarding fiber bundles that have been used in the thesis. For a more extended discussion see [Coh98]. Let \(M\) be a connected space with a base point \(m_0 \in M\) and a continuous map \(\pi : E \to M\).

**Definition 4.2.1.** The map \(\pi : E \to M\) is a fiber bundle with fiber \(F\) if it satisfies the following properties

1. \(\pi^{-1}(b_0) = F\)
2. \(\pi : E \to M\) is surjective
3. For every point \(m \in M\) there is an open neighbourhood \(U_m \subset M\) and a homeomorphism \(\Phi : \pi^{-1}(U_m) \to U_m \times F\) which preserves the fiber.

4.2.1 The pullback bundle

**Definition 4.2.2.** Let \(E_1, E_2\) and \(M\) smooth manifolds and \(f : E_1 \to M, g : E_1 \to M\) smooth maps. The fiber product, denoted as \(E_1 \times^f g E_2\), is the set

\[
E_1 \times^f g E_2 = \{ (x, y) \in E_1 \times E_2 : f(x) = g(y) \}.
\]
Proposition 1. If the maps $f: E_1 \to M$ and $g: E_2 \to M$ are transversal, i.e. $\text{Im}(df_x) + \text{Im}(dg_y) = T_mM$ for all $x \in E_1$, $y \in E_2$ and $m \in M$ such that $f(x) = g(y) = m$, the fiber product $E_1 \times_g E_2$ is an embedded submanifold of $E_1 \times M$.

When the maps $f: E_1 \to M$ and $g: E_2 \to M$ are easily implied, the fiber product will be denoted as $E_1 \times_M E_2$.

Definition 4.2.3. Let $\pi: E \to M$ be a fiber bundle over $M$ and $f: N \to M$ a map. The fiber product

$$E \times_M N = \{ (x, y) \in E \times N : \pi(x) = f(y) \}$$

is a fiber bundle over $M$ with fiber $E_m$, called the pull-back bundle of $E$ to $N$ with respect to the map $f$ and denoted as $f^*E$. If $(U, \phi)$ is a local trivialization of $E$, $(f^{-1}(U), \text{proj}_2 \circ \phi)$ is a local trivialization of $f^*E$, where $\text{proj}_2$ is the projection on the second coordinate.

Principal $G$-bundles and Associated Bundles

Definition 4.2.4. Let $G$ be a Lie group. A principal $G$-bundle over a manifold $B$ is a fiber bundle $\pi: E \to B$ with fiber $G$ satisfying the following properties:

1. the total space $E$ has a free, right action $\mu: E \times G \to E$ such that the following diagram commutes

$$
\begin{array}{ccc}
E \times G & \xrightarrow{\mu} & E \\
\pi \times e & \downarrow & \pi \\
B \times \{e\} & \xrightarrow{=} & B
\end{array}
$$

2. the induced action on the fibers

$$\mu: \pi^{-1}(b) \times G \to \pi^{-1}(b)$$

is free and transitive

3. there exist local trivializations $\Phi: \pi^{-1}(U) \to U \times G$ which are $G$-equivariant with respect to action of $G$ on $U \times G$ by multiplication on the right. That is the following diagram commutes

$$
\begin{array}{ccc}
\pi^{-1}(U) \times G & \xrightarrow{\Phi \times e} & U \times G \times G \\
\mu & \downarrow & id \times m \\
\pi^{-1}(U) & \xrightarrow{\Phi} & U \times G
\end{array}
$$

where $m$ is the multiplication on $G$, $(g_1, g_2) \to g_1 g_2$.

Remark 4.2.5. Properties 1. and 2. equivalently state that the orbits of the free, right action of $G$ on $E$ are exactly the fibers $\pi^{-1}(b)$.
The above definition of a principal $G$–bundle implies that the structure group of $\pi : E \to B$ is $G$. Indeed, if $p$ is in a fiber $\pi^{-1}(b)$ such that $b$ lies on the intersection of two local trivialization domains $(U_i, \Phi_i)$ and $(U_j, \Phi_j)$, and $p = \Phi_i^{-1}(b, f)$. Then, there exists some $g \in G$ such that $\Phi_j^{-1}(b, f) = \mu(p, g)$ since the right action of $G$ on $E$ is transitive on the fibers. Finally, $\Phi_i \circ \Phi_j^{-1}(b, f) = \Phi_i(\mu(p, g)) = (b, fg)$. Since the action is free, $g \in G$ is unique and therefore the structure group of $E \to B$ is $G$.

Given a real vector bundle $E \xrightarrow{\pi} B$ of dimension $n$, a frame at a point $b \in B$ is a linear isomorphism $T_b : \mathbb{R}^n \to E_b$. The bundle of frames, denoted as $F_{GL(n)}(E)$ is the disjoint union of the set of frames at each point $b$

$$F_{GL(n)}(E) = \bigsqcup_{b \in B} \{ T_b : T_b \text{ linear isomorphism from } \mathbb{R}^n \text{ onto } E_b \}.$$ 

The frame bundle has a natural projection $\pi_F : F_{GL(n)}(E) \to B$ that maps $(b, T_b)$ to $b$, as well as a topology and bundle structure induced by the bundle structure of $E$. Namely, a local trivialization $\phi$ of $E$ over an neighbourhood $U$ of $b$,

$$\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$$

induces a local trivialization of $F_{GL(n)}(E)$

$$\Phi : \pi_{F_{GL(n)}}^{-1}(U) \to U \times GL_n(\mathbb{R})$$

$$(b, T_b) \mapsto (\phi(b), \phi \circ T_b).$$

The group $GL_n(\mathbb{R})$ acts on $F_{GL(n)}(E)$ on the right via composition

$$T_b \circ A : \mathbb{R}^n \to F_{GL(n)}(E_b), \ A \in GL_n(\mathbb{R})$$

and the fibers $\pi_{F_{GL(n)}}^{-1}(b)$ are exactly the orbits of the action. Thus, the fiber bundle $F_{GL(n)}(E) \to B$ is a principal $GL_n(\mathbb{R})$–bundle.

When the vector bundle $E \xrightarrow{\pi} B$ is equipped with a bundle metric, namely a smooth map

$$g : E \otimes E \to \mathbb{R}$$

such that the restriction on $E_b \otimes E_b$ is an inner product on $E_b$, one can consider the bundle of orthonormal frames

$$F_{O(n)}(E) := \bigsqcup_{b \in B} \{ T_b : T_b \text{ linear isometry from } \mathbb{R}^n \text{ onto } E_b \}$$

where $\mathbb{R}^n$ is equipped with the Euclidean metric. The orthonormal frame bundle is a principal $O(n)$–bundle with topology and bundle structure induced by the bundle
structure of $E$. Additionally, if $E$ is oriented, the bundle of oriented orthonormal frames $F_{SO(n)}(E)$ is a principal $SO(n)-$bundle.

Given a manifold $F$ with a left $G$ action, one can construct a unique fiber bundle with fiber $F$, associated to a given principal $G-$bundle.

**Definition 4.2.6.** Let $E \xrightarrow{\pi} B$ be a principal $G$ bundle and $F$ a manifold with a left $G-$action. The associated fiber bundle is $E \times_G F := (E \times F)/G$

where the action of $G$ on $E$ by which we quotient is $g : (e,f) \mapsto (eg^{-1}, gf)$.

The space $E \times_G F$ is a fiber bundle over $B = E/G$ with abstract fiber $F$ and projection $\pi_G : [(e,f)] \mapsto \pi(e)$.

**Proposition 4.2.7.** Let $(B, g)$ be a Riemannian manifold. The unit cotangent bundle $S(T^*B)$ is isomorphic to the sphere bundle $F_{SO(n)}(T^*B) \times_{SO(n)} S^{n-1}$ associated to $F_{SO(n)}(T^*B)$.

**Proof.** Let $T$ be an oriented orthonormal co-frame $T : \mathbb{R}^n \to T^*_b B$ at a point $b$ in $B$ and $p$ a unit vector of $\mathbb{R}^n$. The map

$$\sigma : F_{SO(n)}(T^*B) \times S^{n-1} \to S(T^*B)$$

$$(T, p) \mapsto T(p)$$

descents to a map on the quotient

$$F_{SO(n)}(T^*B) \times S^{n-1}/SO(n) \to S(T^*B)$$

since it holds that $\sigma(T \circ A^{-1}, Ap) = T \circ A^{-1}(Ap) = T(p) = \sigma(T, p)$ for any $A$ in $SO(n)$. The induced map $F_{SO(n)}(T^*B) \times_{SO(n)} S^{n-1} \to S(T^*B)$ is a fiber bundle morphism since $\pi_{SO(n)}([T, p]) = \pi_{F_{SO(n)}(T^*B)}(T) = \pi_S(T(p))$. Lastly, to prove that the bundle morphism is an isomorphism it suffices to find a bundle morphism

$$\rho : S(T^*B) \to F_{SO(n)}(T^*B) \times_{SO(n)} S^{n-1}$$

such that $\rho \circ \sigma = \sigma \circ \rho = id$. Let $(U, \Phi)$ be a local trivialization of $S(T^*B)$ and $a \in \pi_S^{-1}(U)$. Then, one can consider the map

$$\rho : S(T^*B) \to F_{SO(n)}(T^*B) \times_{SO(n)} S^{n-1}$$

$$a \mapsto [(\Phi^{-1}_{\mid_{\pi^{-1}(b)}}, \Phi_{\mid_{\pi^{-1}(b)}}(a))].$$
If there exists a trivialization \((V, \Psi)\) such that \(b \in V \cap U\) then \(\Psi|_{\pi^{-1}(b)} = A \circ \Phi|_{\pi^{-1}(b)} \) for some \(A \in SO(n)\) and

\[
([\Psi^{-1}|_{\pi^{-1}(b)}, \Psi|_{\pi^{-1}(b)}(a)]) = ([\Phi|_{\pi^{-1}(b)} \circ A^{-1}, A \circ \Phi|_{\pi^{-1}(b)}(a)]) = ([\Phi|_{\pi^{-1}(b)}, \Phi|_{\pi^{-1}(b)}(a)])
\]

which implies that the map is coordinate independent. Finally, for an oriented orthonormal frame \(f\) over \(b \in B\) there exists some \(A \in SO(n)\) such that \(\Phi^{-1}|_{\pi^{-1}(b)} \circ T = A\), therefore we have the following

\[
\rho \circ \sigma([T, p]) = \rho(T(p)) = [(\Phi^{-1}|_{\pi^{-1}(b)}, \Phi|_{\pi^{-1}(b)}(T(p)))] = [(TA^{-1}, Ap)]
\]

and \(\sigma \circ \rho(a) = \sigma([\Phi^{-1}|_{\pi^{-1}(b)}, \Phi|_{\pi^{-1}(b)}(a)]) = a\),

and therefore \(\rho \circ \sigma = \sigma \circ \rho = id\). □

A principal \(G\)-bundle \(E\) has also a left action of the group \(G\), \(\nu : E \times G \rightarrow E\), which commutes with the right action \(\mu : G \times E \rightarrow E\) and where each action map

\[
\nu_g : P \rightarrow P
\]

\[
p \mapsto \nu(p, g)
\]

is a bundle automorphism of \(E\) such that the induced map on the base \(B\) is the identity.

**Proposition 4.2.8.** The left action of \(G\) on \(E\) induces an action of \(G\) on the associated bundle \(E \times_G F\) such that the orbits \(G\) are exactly the fibers of \(\pi^{-1}_G(b)\) for every \(b \in B\), where \(\pi_G : E \times_G F \rightarrow B\) is the projection of the associated bundle as defined in 4.2.6. The

**Proof.** For each \(g \in G\) the bundle automorphism \(\nu_g : E \rightarrow E\) induces a map on the product \(E \times F\),

\[
\nu_g \times id : E \times F \rightarrow E \times F.
\]

The maps \(\nu_g \times id\) descent on the quotient \(E \times F / G\), since for \((e, f) \in E \times F\) and for every \(g\) and \(h\) in \(G\) we have that

\[
[(\nu_g \mu_h(e), hf)] = [(\mu_h(\nu_g(e)), hf)] = [(\nu_g(e), f)].
\]

Therefore, one can consider the action map

\[
G \times E \times_G F \rightarrow E \times_G F, \ (g, [e, f]) \mapsto ([\nu_g(e), f]).
\]

Finally, since \(\pi_G[\nu_g e, f] = \pi(e)\) for all \(g \in G\), it follows that the orbits of the action of \(G\) on \(E \times_G F\) are exactly the fibers \(\pi^{-1}(b)\), for all \(b \in B\).
Remark 4.2.9. The lemma 3.3.1 follows as a corollary of the previous proposition.

4.3 Direct Integrals

We present the definition of the direct integral of Hilbert spaces which has been used in the proof of 3.3.19. For more details see [Hal13].

Definition 4.3.1. Let $X$ be a measure space with measure $\mu$ and $H(x)$ a Hilbert space assigned to every point $x \in X$. Then a direct integral of the family of Hilbert spaces $\{H(x)\}_{x \in X}$ denoted by

$$
\int_X H(x)d\mu
$$

is the set of all functions (up to a set of measure 0) $f : x \to \bigcup_{x \in X} H(x)$ such that

1. $f(x) \in H(x)$
2. $\|f(x)\|_{H(x)} \in L^2(X, \mu)$

A direct integral of separable Hilbert spaces is a Hilbert space with inner product given by the formula

$$
\langle f_1, f_2 \rangle = \int_X \langle f_1(x), f_2(x) \rangle_x d\mu.
$$

Indeed, since $f_1$ and $f_2$ are square integrable and $| \langle f_1(x), f_2(x) \rangle |_x \leq \|f_1(x)\|_x \|f_2(x)\|_x$ the inner product is well-defined. One can prove that a direct integral of Hilbert spaces is a Hilbert space using an argument similar to the proof of completeness of $L^2$ spaces.
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