CERTAIN CASES OF HIKITA-NAKAJIMA CONJECTURE

by

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ABSTRACT

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Let $\mathfrak{M}_0$ be an affine Nakajima quiver variety, and $\mathcal{M}$ is the corresponding BFN Coulomb branch. Assume that $\mathfrak{M}_0$ can be resolved by the (smooth) Nakajima quiver variety $\mathfrak{M}$. The Hikita-Nakajima conjecture claims that there should be an isomorphism of (graded) algebras $H_S^*(\mathfrak{M}, \mathbb{C}) \simeq \mathbb{C}[\mathcal{M}_{\mathfrak{s}}^{\mathbb{C}^\times}]$, where $S \curvearrowright \mathfrak{M}_0$ is a torus acting on $\mathfrak{M}_0$ preserving the Poisson structure, $\mathcal{M}_{\mathfrak{s}}$ is the (Poisson) deformation of $\mathcal{M}$ over $\mathfrak{s} = \operatorname{Lie} S$, $\mathbb{C}^\times$ is a generic one-dimensional torus acting on $\mathcal{M}$, and $\mathbb{C}[\mathcal{M}_{\mathfrak{s}}^{\mathbb{C}^\times}]$ is the algebra of schematic $\mathbb{C}^\times$-fixed points of $\mathcal{M}_{\mathfrak{s}}$. In this thesis we prove the Hikita-Nakajima conjecture for $\mathfrak{M} = \widetilde{\mathbb{C}^2/\Gamma}$ (Kleinian singularities) and $\mathfrak{M} = \mathfrak{M}(n, r)$ Gieseker variety ($ADHM$ space). In the latter case we produce the isomorphism explicitly on generators. We also describe the Hikita-Nakajima isomorphism above using the realization of $\mathcal{M}_{\mathfrak{s}}$ as the spectrum of the center of rational Cherednik algebra corresponding to $S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ and identify all the algebras that appear in the isomorphism with the center of degenerate cyclotomic Hecke algebra.

To anyone,
who did something nice.

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I would like to thank everyone who I communicated with along the way. If you are reading this thinking "am I on this list?" the answer is "yes".

CONTENTS

1

INTRODUCTION

1.1 CONICAL SYMPLECTIC RESOLUTIONS AND SYMPLECTIC DUALITY

Let $\mathfrak{M}_0$ be an algebraic variety over $\mathbb{C}$. The following definition belongs to [Bea00].

Definition 1.1.1. We say that $\mathfrak{M}_0$ is *singular symplectic* (or has symplectic singularities) if

(1) $\mathfrak{M}_0$ is a normal Poisson variety,

(2) there exists a smooth, dense open subset $U \subset \mathfrak{M}_0$ on which the Poisson structure comes from the symplectic form that we denote by ω,

(3) there exists a resolution of singularities (birational and projective morphism) $\mathfrak{M} \to \mathfrak{M}_0$ such that the pullback of ω to $\mathfrak{M}$ has no poles.

We say that $\mathfrak{M}_0$ is a *conical symplectic singularity* if in addition to (1)-(3) one has a $\mathbb{C}^\times$-action on $\mathfrak{M}_0$ which acts on ω with some positive weight and contracts $\mathfrak{M}_0$ to the unique fixed point. We will denote contracting $\mathbb{C}^\times$ by $\mathbb{C}^\times_\hbar$.

Assume now that $\mathfrak{M}_0$ possesses a $\mathbb{C}^\times_\hbar$-equivariant symplectic resolution $\mathfrak{M} \to \mathfrak{M}_0$. We will call $\mathfrak{M} \to \mathfrak{M}_0$ a *conical symplectic resolution*. It is known (see [Nam08, Lemmas 12, 22, Proposition 13], [GK02, Theorem 1.13]) that there exist canonical symplectic (resp. Poisson) deformations of $\mathfrak{M}$ (resp. $\mathfrak{M}_0$) over the base $\mathfrak{t} = \mathfrak{t}_{\mathfrak{M}_0} := H^2(\mathfrak{M}, \mathbb{C})$ that we denote by

$$\mathfrak{M}_{\mathfrak{t}} \to \mathfrak{t},\ \mathfrak{M}_{0,\mathfrak{t}} \to \mathfrak{t}.$$

Remark 1.1.2. *One can show that the space $\mathfrak{t}$ does not depend on the choice of the resolution $\mathfrak{M}$ of $\mathfrak{M}_0$. This space can be defined even if $\mathfrak{M}_0$ does not have a symplectic resolution $\mathfrak{M}$. The deformation $\mathfrak{M}_{\mathfrak{t}}$ is the universal deformation of $\mathfrak{M}$, $\mathfrak{M}_{0,\mathfrak{t}}$ is a pullback of the universal deformation of $\mathfrak{M}_0$ along the morphism $\mathfrak{t} \to \mathfrak{t}/W$ for a certain group W acting on $\mathfrak{t}$ (called Namikawa-Weyl group).*

Example 1.1.3 (Cotangent bundle of $\mathbb{P}^1$)**.** Let $\mathcal{N}_{\mathfrak{sl}_2}$ be the nilpotent cone of Lie algebra $\mathfrak{sl}_2$. The most basic example of a symplectic resolution is

$$\pi : T^*\mathbb{P}^1 \to \mathcal{N}_{\mathfrak{sl}_2}$$

A more explicit description of these two varieties would be the following:

$$\mathcal{N}_{\mathfrak{sl}_2} = \left\{ A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in M_2(\mathbb{C}^2) : a^2 + bc = 0 \right\} \xrightarrow{\sim} \mathbb{C}^2/\mathbb{Z}_2$$

$$T^*\mathbb{P}^1 = \left\{ \left(\ell \subset \mathbb{C}^2, A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathcal{N} \right) \mid \ell \subset \ker A \right\}$$

and the map $\pi\colon T^*\mathbb{P}^1 \to \mathcal{N}$ simply forgets the line ℓ. The conical action comes from scaling the matrices and Hamiltonian action comes from $\mathbb{C}^2$.

This example can be generalized in several ways. One of them is the following resolution:

Example 1.1.4.

$$\pi : T^*\mathbb{P}^n \to \{A \in \mathfrak{sl}_n : A^2 = 0, \mathrm{rk}(A) \leq 1\}$$

The projection map forgets the line, like in the previous example. This can be generalized to resolutions of partial (and full) flag varieties.

A different set of examples would be a resolution of type A (or, more generally, ADE type resolutions). Namely, take $\Gamma \subset SL_2(\mathbb{C})$ a finite subgroup. Under the McKay correspondence, such subgroups are in bijection with simply laced Dynkin diagrams. The quotient $\mathbb{C}^2/\Gamma$ has a minimal symplectic resolution $\widetilde{\mathbb{C}^2/\Gamma}$ and we study its properties in more detail in Chapter 2.

Hilbert scheme of n points on $\mathbb{C}^2$ is another such example. We cover in in detail in Chapter 3 and then proceed to work with a more general object, called Gieseker variety.

Let $\mathrm{Aut}_{\mathbb{C}^\times_\hbar}(\mathfrak{M}_0)$ be the group of Poisson automorphisms of $\mathfrak{M}_0$ commuting with the contracting $\mathbb{C}^\times_\hbar$. This is a finite-dimensional algebraic group (possibly disconnected). We denote by $S = S_{\mathfrak{M}_0} \subset \mathrm{Aut}_{\mathbb{C}^\times_\hbar}(\mathfrak{M}_0)$ a maximal torus and set $\mathfrak{s}_{\mathfrak{M}_0} := \mathrm{Lie}\, S_{\mathfrak{M}_0}$. One can show that the action of S on $\mathfrak{M}_0$ and the contracting action of $\mathbb{C}^\times_\hbar$ extend naturally to the action of $S \times \mathbb{C}^\times_\hbar$ on $\mathfrak{M}_{0,\mathfrak{t}}$ (the torus S acts fiberwise). It can also be shown that the $S \times \mathbb{C}^\times_\hbar$-action above lifts to the action on $\mathfrak{M}_\mathfrak{t}$.

Often conical singularities come in "dual" pairs

$$\mathfrak{M}_0, \ \mathfrak{M}_0^!.$$

We refer the reader to [BLPW14] for details on symplectic duality.

Remark 1.1.5. It seems that there is no functorial way to define $\mathfrak{M}_0^!$ from $\mathfrak{M}_0$. Apparently one needs some richer mathematical structure to produce a pair of symplectically dual varieties. For example it is expected that every 3d $\mathcal{N} = 4$ super-symmetric quantum field theory should produce such a pair. In the case of gauge theories such a pair was produced mathematically in [BFN18].

Let us recall that for symplectically dual varieties $\mathfrak{M}_0$, $\mathfrak{M}_0^!$ one should have the natural identifications

$$\mathfrak{t}_{\mathfrak{M}_0} \simeq \mathfrak{s}_{\mathfrak{M}_0^!},\ \mathfrak{s}_{\mathfrak{M}_0} \simeq \mathfrak{t}_{\mathfrak{M}_0^!}.$$

Also it is expected that the choice of $\mathfrak{M}$ (choice of the symplectic resolution of $\mathfrak{M}_0$) should correspond on the dual side to a cocharacter $\nu_{\mathfrak{M}} \colon \mathbb{C}^\times \to S_{\mathfrak{M}_0^!}$ (actually to a choice of a certain chamber in $\mathrm{Hom}(\mathbb{C}^\times, T) \otimes_{\mathbb{Z}} \mathbb{R}$) such that the set $(\mathfrak{M}_0^!)^{\nu_{\mathfrak{M}}(\mathbb{C}^\times)}$ consists of one point.

1.2 SCHEMATIC FIXED POINTS AND HIKITA-NAKAJIMA CONJECTURES

1.2.1 *Schematic fixed points*

Given a variety Y with an action of some algebraic group G we can define the functor

$$Y^G \colon \mathbf{Schemes}_{\mathbb{C}} \to \mathbf{Sets}$$

from the category $\mathbf{Schemes}_{\mathbb{C}}$ of schemes over $\mathbb{C}$ to the category **Sets** of sets as follows (see [Fog], [Dri13, Section 1.2]):

$$Y^G(S) := \mathrm{Maps}^G(S, Y),\ S \in \mathbf{Schemes}_{\mathbb{C}},$$

where the action of G on S is trivial and $\mathrm{Maps}^G(S, Y)$ is the set of G-equivariant morphisms from S to Y.

It turns out that in some cases, functor Y^G is represented by a scheme that we call *schematic fixed points of* Y (for more details, see [Fog, Theorem 2.3]).

Consider the case $Y = \mathrm{Spec}\, B$ for some $\mathbb{C}$-algebra B and $G = \mathbb{C}^\times$. The action $\mathbb{C}^\times \curvearrowright Y$ here corresponds to the $\mathbb{Z}$-grading $B = \bigoplus_{i \in \mathbb{Z}} B_i$. Compare the following Proposition with [Dri13, Example 1.2.3].

Proposition 1.2.1. *If $Y = \mathrm{Spec}\, B$ is an affine variety and $G = \mathbb{C}^\times$, then $Y^{\mathbb{C}^\times}$ is represented by an affine scheme whose ring of functions can be described in two equivalent ways:*

$$\mathbb{C}[Y^{\mathbb{C}^\times}] = B_0 / \sum_{i>0} B_{-i} B_i = B/(b_i \in B_i,\ i \neq 0). \tag{1.1}$$

Proof. It is enough to show that the functor $Y^{\mathbb{C}^\times}$ restricted to the category of affine schemes over $\mathbb{C}$ is represented by the affine scheme with the algebra of functions as in (1.1).

Let $S = \operatorname{Spec} C$ be an affine scheme with trivial $\mathbb{C}^\times$-action. The set $\operatorname{Maps}^{\mathbb{C}^\times}(S, Y)$ can be identified with the set of graded homomorphisms $B \to C$, where the grading on C is the trivial one ($C = C_0$). Since $C = C_0$, we conclude that every such homomorphism $f\colon B \to C$ factors through $B/(b_i \in B_i,\, i \neq 0)$. Note now that every homomorphism $\overline{f}\colon B/(b_i \in B_i,\, i \neq 0) \to C$ induces the *graded* homomorphism $B \twoheadrightarrow B/(b_i \in B_i,\, i \neq 0) \to C$, so we must have $Y^{\mathbb{C}^\times} = \operatorname{Spec}(B/(b_i \in B_i,\, i \neq 0))$.

It remains to note that the natural morphism

$$B_0/\sum_{i>0} B_{-i}B_i \to B/(b_i \in B_i,\, i \neq 0),$$

given by

$$B_0/\sum_{i>0} B_{-i}B_i \ni [b] \mapsto [b] \in B/(b_i \in B_i,\, i \neq 0)$$

is an isomorphism. □

Remark 1.2.2. *Proposition 1.2.1 can be easily generalized to the case when G is a torus of arbitrary rank (or even an arbitrary reductive group).*

The following Proposition holds (see, for example, [Ive72]).

Proposition 1.2.3. *If Y is a smooth algebraic variety over $\mathbb{C}$ and G is reductive, then Y^G is smooth.*

1.2.2 *Hikita-Nakajima conjecture*

Recall that $\mathfrak{M}_0$ is a Nakajima quiver variety and $\mathcal{M} = \mathfrak{M}_0^!$ is the corresponding Coulomb branch. We assume that $\mathfrak{M}_0$ is resolved by the corresponding (smooth) Nakajima quiver variety $\mathfrak{M} \to \mathfrak{M}_0$. Let $H^*_{S_{\mathfrak{M}_0}}(\mathfrak{M}, \mathbb{C})$ be the algebra of $S_{\mathfrak{M}_0}$-equivariant cohomology of $\mathfrak{M}$. This is an algebra over $H^*_{S_{\mathfrak{M}_0}}(\mathrm{pt}) = \mathbb{C}[\mathfrak{s}_{\mathfrak{M}_0}]$.

Recall also that the choice of $\mathfrak{M}$ (resolution of $\mathfrak{M}_0$) corresponds to a (generic) cocharacter $\nu_{\mathfrak{M}}\colon \mathbb{C}^\times \to S_{\mathcal{M}}$. We can consider the algebra of functions of schematic fixed points $\mathbb{C}[(\mathcal{M}_{\mathfrak{t}_{\mathcal{M}}})^{\nu_{\mathfrak{M}}(\mathbb{C}^\times)}]$ that is an algebra over $\mathbb{C}[\mathfrak{t}_{\mathcal{M}}] = \mathbb{C}[\mathfrak{s}_{\mathfrak{M}_0}]$. Note also that the algebra $H^*_{S_{\mathfrak{M}_0}}(\mathfrak{M}, \mathbb{C})$ has a natural cohomological grading and the algebra $\mathbb{C}[(\mathcal{M}_{\mathfrak{t}_{\mathcal{M}}})^{\nu_{\mathfrak{M}}(\mathbb{C}^\times)}]$ is graded via the contracting $\mathbb{C}^\times_\hbar$-action. The algebra $H^*_{S_{\mathfrak{M}_0}}(\mathfrak{M}, \mathbb{C})$ is finitely generated over $\mathbb{C}[\mathfrak{s}_{\mathfrak{M}_0}]$. The algebra $\mathbb{C}[(\mathcal{M}_{\mathfrak{t}_{\mathcal{M}}})^{\nu_{\mathfrak{M}}(\mathbb{C}^\times)}]$ is finitely generated over $\mathbb{C}[\mathfrak{s}_{\mathfrak{M}_0}]$ iff $\mathcal{M}^{\nu_{\mathfrak{M}}(\mathbb{C}^\times)} = \mathcal{M}^{S_{\mathcal{M}}}$ (considered as a set) consists of one point. Since we want to identify the algebras above we must make the following assumption.

Assumption 1.2.4. The set $\mathcal{M}^{S_{\mathcal{M}}}$ consists of one point.

Remark 1.2.5. *Let us recall that symplectic duality predicts that $\mathfrak{M}_0$ has a symplectic resolution iff $(\mathfrak{M}_0^!)^{S_{\mathfrak{M}_0^!}}$ consists of one point.*

The following conjecture belongs to Hikita and Nakajima. We will call it *Hikita-Nakajima conjecture*.

Conjecture 1.2.6. There is an isomorphism of $\mathbb{Z}$-graded algebras over $\mathbb{C}[\mathfrak{s}_{\mathfrak{M}_0}]$:

$$H^*_{S_{\mathfrak{M}_0}}(\mathfrak{M},\mathbb{C}) \simeq \mathbb{C}[(\mathcal{M}_{\mathfrak{t}_{\mathcal{M}}})^{\nu_{\mathfrak{M}}(\mathbb{C}^\times)}].$$

Example 1.2.7. Returning to the setting of Example 1.1.3, we have the resolution

$$\pi : T^*\mathbb{P}^1 \to \mathcal{N}_{\mathfrak{sl}_2}.$$

This symplectic resolution is self-dual. The nilpotent cone can be described as $\{(a,b,c) \in \mathbb{C}^3, a^2+bc=0\}$. The corresponding ring of functions is given by $\mathbb{C}[a,b,c]/(a^2+bc)$. We have a conical action of $\mathbb{C}^\times$ on $\mathcal{N}$ given by

$$t\cdot(a,b,c) = (a,tb,t^{-1}c).$$

The only fixed point of such an action is $(0,0,0)$. The scheme structure, however, is non-trivial :

$$\mathbb{C}[\mathcal{N}^{\mathbb{C}^\times}] = \mathbb{C}[a,b,c]/(bc+a^2,b,c) = \mathbb{C}[a]/(a^2)$$

From this we obtain the non-equivariant Hikita conjecture:

$$H^*(T^*\mathbb{P}^1,\mathbb{C}) = H^*(\mathbb{P}^1,\mathbb{C}) \simeq H^*(S^2,\mathbb{C}) \simeq \mathbb{C}[a]/a^2$$

Both $\mathfrak{s}$ and $\mathfrak{t}$ are one - dimensional, so we have a one - dimensional deformation of the nilpotent cone. Its fibers can be realized as the spaces of traceless matrices with a non-zero determinant w^2. In terms of the ring of functions this looks like

$$\mathbb{C}[a]/(a^2-w^2) \simeq \mathbb{C}[a]/(a+w)(a-w) \simeq \mathbb{C}\oplus\mathbb{C}$$

Notice that if we take a generic parameter in the torus acting on $T^*\mathbb{P}^1$ then the corresponding equivariant cohomology will be the cohomology of the fixed points (with the coefficients in the cohomology of a point), in particular, the cohomology will be equal to $\mathbb{C}\oplus\mathbb{C}$ This shows the equivariant Hikita-Nakajima statement.

Remark 1.2.8. *Actually Nakajima's version of this conjecture is even more general: on the LHS, we can consider the algebra $H^*_{S_{\mathfrak{M}_0}\times\mathbb{C}^\times_\hbar}(\mathfrak{M},\mathbb{C})$ and on the RHS we can consider the Rees algebra of "quantized schematic fixed points" (so-called B-algebra or Cartan subquotient) of the quantization of $\mathbb{C}[\mathcal{M}_{\mathfrak{t}_{\mathcal{M}}}]$ (see, for example, [Kam22, Section 5.6]). In [KMP21], another generalization of this conjecture is proposed (where one replaces cohomology on the LHS with quantum cohomology).*

Remark 1.2.9. *Let us note that if $\mathfrak{M}$, $\mathfrak{M}'$ are two symplectic resolutions of $\mathfrak{M}_0$, then the algebras $H^*_{S_{\mathfrak{M}_0}}(\mathfrak{M},\mathbb{C})$, $H^*_{S_{\mathfrak{M}_0}}(\mathfrak{M}',\mathbb{C})$ are isomorphic. This follows from the fact that the universal deformations $\mathfrak{M}_{\mathfrak{t}_{\mathfrak{M}_0}} \to \mathfrak{t}_{\mathfrak{M}_0}$, $\mathfrak{M}'_{\mathfrak{t}_{\mathfrak{M}_0}} \to \mathfrak{t}_{\mathfrak{M}_0}$ are locally trivial in C^∞-topology (see [Nam10, Section 1.2 and references therein]), so $\mathfrak{M}$, $\mathfrak{M}'$ are both diffeomorphic to a generic fiber of $\mathfrak{M}_{0,\mathfrak{t}_{\mathfrak{M}_0}} \to \mathfrak{t}_{\mathfrak{M}_0}$. Similarly, one can see that for any generic cocharacter $\nu\colon \mathbb{C}^\times \to S_{\mathcal{M}}$ the schematic fixed points $\mathcal{M}^{\nu(\mathbb{C}^\times)}_{\mathfrak{t}_{\mathcal{M}}}$ are the same and are isomorphic to the schematic fixed points $\mathcal{M}^{S_{\mathcal{M}}}_{\mathfrak{t}_{\mathcal{M}}}$. Indeed, note that $\mathcal{M}_{\mathfrak{t}_{\mathcal{M}}}$ can be $S_{\mathcal{M}}$-equivariantly embedded in some vector space O with a linear action of $S_{\mathcal{M}}$. Then $\mathcal{M}^{\nu(\mathbb{C}^\times)}_{\mathfrak{t}_{\mathcal{M}}}$, $\mathcal{M}^{S_{\mathcal{M}}}_{\mathfrak{t}_{\mathcal{M}}}$ are (schematic) intersections*

$$\mathcal{M}^{\nu(\mathbb{C}^\times)}_{\mathfrak{t}_{\mathcal{M}}} = \mathcal{M}_{\mathfrak{t}_{\mathcal{M}}} \cap O^{\nu(\mathbb{C}^\times)},\ \mathcal{M}^{S_{\mathcal{M}}}_{\mathfrak{t}_{\mathcal{M}}} = \mathcal{M}_{\mathfrak{t}_{\mathcal{M}}} \cap O^{S_{\mathcal{M}}}.$$

This reduces the claim to showing that $O^{\nu(\mathbb{C}^\times)} = O^{S_{\mathcal{M}}}$ for a generic $\nu\colon \mathbb{C}^\times \to S_{\mathcal{M}}$ that directly follows from Proposition 1.2.3 since O is smooth.

Remark 1.2.10. *Conjecture 1.2.6 has the following explicit "combinatorial" corollary. Pick a point $s \in \mathfrak{s}_{\mathfrak{M}_0}$. We have an isomorphism of algebras $H^*_{S_{\mathfrak{M}_0}}(\mathfrak{M})|_s \simeq H^*(\mathfrak{M}^s)$. Note that the semisimple quotient of $H^*(\mathfrak{M}^s)$ is the algebra $H^0(\mathfrak{M}^s)$ that is isomorphic to the direct sum of copies of $\mathbb{C}$, the number of copies is equal to $|\mathrm{Comp}(\mathfrak{M}^s)|$, where $\mathrm{Comp}(\mathfrak{M}^s)$ is the set of connected components of $\mathfrak{M}^s$. Conjecture 1.2.6 implies that the algebra $H^*(\mathfrak{M}^s)$ is isomorphic to the algebra $\mathbb{C}[\mathcal{M}_s^{\nu_{\mathfrak{M}}(\mathbb{C}^\times)}]$, where $\mathcal{M}_s$ is the fiber over $s \in \mathfrak{s}_{\mathfrak{M}_0}$ of the deformation $\mathcal{M}_{\mathfrak{s}_{\mathfrak{M}_0}} \to \mathfrak{s}_{\mathfrak{M}_0}$. The semisimple quotient of $\mathbb{C}[\mathcal{M}_s^{\nu_{\mathfrak{M}}(\mathbb{C}^\times)}]$ is isomorphic to the direct sum of copies of $\mathbb{C}$ labeled by the set $(\mathcal{M}_s)^{\nu_{\mathfrak{M}}(\mathbb{C}^\times)}$. So we obtain the bijection of sets*

$$\mathrm{Comp}(\mathfrak{M}^s) \simeq \mathcal{M}_s^{\nu_{\mathfrak{M}}(\mathbb{C}^\times)}.$$

Warning 1.2.11. *Hikita-Nakajima conjecture is not true as stated for arbitrary pairs of symplectically dual varieties. A counterexample is a part of a work in progress by Hoang, first author and Matvieievskyi.*

Let us briefly recall the current state of the Hikita-Nakajima conjecture. The Hikita conjecture was proven for the case of Hilbert scheme of points on $\mathbb{A}^2$, type A Slodowy slices and hypertoric varieties in [Hik, Theorem 1.1, Theorem A.1, Theorem B.1]. In [Shl19], the author has proven the case of $Y = \mathbb{C}^2/\Gamma$ where Γ is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. In the paper [Hat21, Theorem 1.0.5], Hatano proved that $H^*(\mathfrak{M}(n,r),\mathbb{C})$ and $\mathbb{C}[(\mathfrak{M}_0(n,r)^!)^{\mathbb{C}^\times}]$ are isomorphic as graded vector spaces. In [KTW+19a, Theorem 1.5], Kamnitzer, Tingley, Webster, Weekes and Yacobi have proven Hikita conjecture for the ADE slices in the affine Grassmanian. They have also proven the equivariant (see the Remark 1.2.8) version of the Conjecture 1.2.6 for type A quivers and some weaker form of this conjecture for DE quivers (see [KTW+19a, Section 8.3], [KTW+19b, Section 1.3 and Theorem 1.5], see also [Kam22, Section 6.6]). The results of [Shl19] were recently generalized to the equivariant case in [CWS23].

The main result of this thesis is the following Theorem.

Theorem 1.2.12. *Hikita-Nakajima conjecture holds for $\mathfrak{M} = \mathfrak{M}(n,r)$, the Gieseker variety.*

We will describe the isomorphism explicitly using certain generators of the algebras mentioned above (see Theorem 1.7.1 for more details).

Remark 1.2.13. *Let us point out that our approach gives new proof of the Hikita conjecture for the Hilbert scheme case (when $r = 1$) even in the non-equivariant setting (which was proved by Hikita himself). It also generalizes the results of [Hat21], where the author proves that $H^*(\mathfrak{M}(n,r), \mathbb{C})$, $\mathbb{C}[(\mathfrak{M}_0(n,r)^!)^{\mathbb{C}^\times}]$ are isomorphic as graded vector spaces (Appendix A repeats some of the arguments of the paper [Hat21]). The main idea of our thesis is that using deformations simplifies the picture.*

1.3 GIESEKER VARIETY ($ADHM$ SPACE)

The Gieseker variety $\mathfrak{M}(n,r)$ depends on a pair $n, r \in \mathbb{Z}_{\geqslant 1}$ of positive integer numbers. It can be realized as a Nakajima quiver variety corresponding to the Jordan quiver (see Definition 3.2.2). Variety $\mathfrak{M}(n,r)$ also has a realization as the moduli space of torsion-free sheaves on $\mathbb{P}^2$ of rank r with the second Chern class being n and with a fixed trivialization at the line at infinity (see [Nak99, Chapter 2] for details). Variety $\mathfrak{M}(n,r)$ is an important object that originally came from physics. Gieseker variety $\mathfrak{M}(n,r)$ is a resolution of singularities of the variety $\mathfrak{M}_0(n,r) = \operatorname{Spec} \mathbb{C}[\mathfrak{M}(n,r)]$. Variety $\mathfrak{M}_0(n,r)$ has a realization as an (affine) Nakajima quiver variety corresponding to the Jordan quiver (see Definition 3.2.2).

So we are taking $\mathfrak{M} = \mathfrak{M}(n,r)$, $\mathfrak{M}_0 = \mathfrak{M}_0(n,r)$. Then the torus $S_{\mathfrak{M}_0}$ can be described as follows. We have a natural symplectic action of SL_r on $\mathfrak{M}(n,r)$ (via changing the trivialization at infinity) and also the action of $\mathbb{T} = \mathbb{C}^\times$ via the action on $\mathbb{P}^2$ that multiplies one coordinate by t and another by t^{-1} (so-called "hyperbolic action"). Let $T_r \subset \mathrm{SL}_r$ be a maximal torus. The torus $S_{\mathfrak{M}_0}$ is the image of $A := \mathbb{T} \times T_r$ in $\mathrm{Aut}_{\mathbb{C}^\times_\hbar}(\mathfrak{M}(n,r))$ and $\mathfrak{s}_{\mathfrak{M}_0}$ naturally identifies with $\mathfrak{a} := \operatorname{Lie} A$. The space $\mathfrak{t}_{\mathfrak{M}_0(n,r)} = H^2(\mathfrak{M}(n,r), \mathbb{C})$ is known to be one-dimensional (follows, for example, from [MN18]).

1.4 SYMPLECTIC DUAL TO $\mathfrak{M}(n,r)$

Let us now give a rough description of the variety $\mathcal{M}(n,r) = \mathfrak{M}_0(n,r)^!$ and its deformation $\mathcal{M}(n,r)_\mathfrak{a} \to \mathfrak{a}$.

One way to construct a dual to $\mathfrak{M}_0(n,r)$ is via the Coulomb branches introduced in [BFN18]. In this approach, $\mathcal{M}(n,r)$ is equal to the spectrum of the algebra $H_*^{(\mathrm{GL}_n)_\mathcal{O}}(\mathcal{R}_{n,r})$ of $(\mathrm{GL}_n)_\mathcal{O}$-equivariant Borel-Moore homology of the variety of triples $\mathcal{R}_{n,r}$ corresponding to the Jordan quiver (see Section 4.1 for details). The variety $\mathcal{M}(n,r)_\mathfrak{a}$ is then the spectrum of $A \times (\mathrm{GL}_n)_\mathcal{O}$ - equivariant homology of the same variety of triples $\mathcal{R}_{n,r}$ (see Chapter 4 for details).

The Coulomb branch $\mathcal{M}(n,r)_{\mathfrak{a}}$ above can be realized (see [KN18], [BEF20], [Web]) as the spectrum of the center $Z(H_{n,r})$ of the rational Cherednik algebra $H_{n,r}$ corresponding to the group $\Gamma_n := S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ (see Section 4.2 for details).

Thus we have

$$\mathfrak{M}_0(n,r)^!_{\mathfrak{a}} = \mathcal{M}(n,r)_{\mathfrak{a}} = \operatorname{Spec} Z(H_{n,r}).$$

The algebra $H_{n,r}$ has a natural $\mathbb{Z}$-grading (see (4.4)) that induces the action of $\mathbb{T} = \mathbb{C}^\times$ on $\operatorname{Spec} Z(H_{n,r}) = \mathcal{M}(n,r)_{\mathfrak{a}}$.

Remark 1.4.1. *The action $\mathbb{T} \curvearrowright \mathcal{M}(n,r)_{\mathfrak{a}}$ in the Coulomb terms is described in [BFN18, Section 3 (v)].*

Remark 1.4.2. *One can also construct $\mathfrak{M}_0(n,r)^!$ as the (affine) Nakajima quiver variety $\mathcal{X}_0(n,r)$ corresponding to the cyclic quiver with r vertices labeled by $\mathbb{Z}/r\mathbb{Z}$ and having n-dimensional vector spaces placed at these vertices and one-dimensional framing at the vertex corresponding to zero. The deformation $\mathcal{X}_0(n,r)_{\mathfrak{a}}$ can be constructed similarly (c.f. Definition 3.1.4 below). The identification $\mathcal{X}_0(n,r)_{\mathfrak{a}} \simeq \operatorname{Spec} Z(H_{n,r})$ is given by the Etingof-Ginzburg isomorphism (that goes back to the paper [EG]). More detailed, in [Prz16, Section 3.3] it is explained (following the proof of [EG, Theorem 11.16]) how to construct isomorphisms between smooth fibers of the families $\mathcal{X}_0(n,r)_{\mathfrak{a}}$, $\operatorname{Spec} Z(H_{n,r})$ over $\mathfrak{a}$. One can show that these isomorphisms extend to the desired isomorphism $\mathcal{X}_0(n,r)_{\mathfrak{a}} \simeq \operatorname{Spec} Z(H_{n,r})$. The idea is to first extend these isomorphisms to the smooth locus over fiber over* every *$c \in \mathfrak{a}$ and then use the normality of our varieties to extend in codimension two.*

1.5 CYCLOTOMIC HECKE ALGEBRA AND ITS CENTER

Let $Q_{n,r}$ be the algebra of functions on schematic $\mathbb{T}$-fixed points of $\operatorname{Spec} Z(H_{n,r})$ (that can also be considered as an algebra of functions on $\mathcal{M}(n,r)^{\mathbb{T}}_{\mathfrak{a}}$). Recall that by Proposition 1.2.1 we have

$$Q_{n,r} = Z(H_{n,r})_0 / \sum_{i>0} Z(H_{n,r})_{-i} Z(H_{n,r})_i = Z(H_{n,r})/(b \in Z(H_{n,r})_i,\ i \neq 0),$$

where the grading on $H_{n,r}$ is as in (4.4):

$$\deg x_j = 1,\ \deg y_j = -1,\ \deg \Gamma_n = \deg \mathbf{h} = 0.$$

Our goal is to identify the algebra $Q_{n,r}$ with the algebra $H^*_A(\mathfrak{M}(n,r),\mathbb{C})$. It turns out that there is another algebra, isomorphic to both of the algebras above. This algebra is the center $Z(R^r(n))^{JM}$ of the cyclotomic degenerate Hecke algebra $R^r(n)$ (see Section 6.0.1 for the definition). It was observed in [SVV17] that algebras $Z(R^r(n))^{JM}$, $H^*_A(\mathfrak{M}(n,r),\mathbb{C})$ are isomorphic. We give an independent (but certainly similar) proof of this fact.

1.6 MAIN IDEA OF THE PROOF

It turns out that there exists one "universal" approach that allows us to identify algebras

$$H_A^*(\mathfrak{M}(n,r),\mathbb{C}),\ Z(R^r(n))^{JM},\ Q_{n,r} \simeq \mathbb{C}[\mathcal{M}(n,r)_{\mathfrak{a}}^{\mathbb{T}}] \tag{1.2}$$

with each other simultaneously. The idea is simple: we embed all of the algebras above inside the algebra

$$E := \bigoplus_{\lambda\in\mathcal{P}(r,n)} \mathbb{C}[\mathfrak{a}] = \mathbb{C}[\mathfrak{a}]^{\oplus|\mathcal{P}(r,n)|}$$

and show that their images coincide. Here $\mathcal{P}(r,n)$ is the set of r-multipartitions of n (see Definition 5.0.3). To show that images are the same we consider natural generators of these algebras and show that their images in E are the same. In particular, we obtain explicit descriptions for isomorphisms between the algebras in (1.2).

Remark 1.6.1. *Let F be the function field of the parameter space $\mathfrak{a} = \mathbb{A}^r$. We will see that the embeddings above become isomorphisms after tensoring by F or, more precisely, after localizing at a certain finite set of elements of $\mathbf{k} := \mathbb{C}[\mathfrak{a}]$ (certain "walls"). Compare with Remark 5.0.6 below.*

Remark 1.6.2. *In [BLP+11, Definition 2.1] the authors introduce a notion of* localization algebra*. The proof of our main Theorem can be described as follows: we observe that algebras that appear in (1.2) have natural structures of (strong, free) localization algebras and all of them are isomorphic (as algebras with this additional structure).*

1.6.1 *Embedding $Z(R^r(n))^{JM} \subset E$ and generators of $Z(R^r(n))^{JM}$*

For every $\lambda \in \mathcal{P}(r,n)$ one can consider the corresponding "universal" Specht modules over $R^r(n)$ that we denote by $\widetilde{S}_{\mathbf{k}}(\lambda)$ (see Chapter 6 for details). Acting by the center $Z(R^r(n))^{JM}$ of $R^r(n)$ on these modules we obtain the desired embedding

$$\psi\colon Z(R^r(n))^{JM} \subset \bigoplus_{\lambda\in\mathcal{P}(r,n)} \mathrm{End}_{R^r(n)}(\widetilde{S}_{\mathbf{k}}(\lambda)) = E.$$

It follows from [Bru08, Theorem 1] that natural generators of $Z(R^r(n))^{JM}$ are classes of elements

$$e_k(z_1,\dots,z_n),\ k = 1,\dots,n,$$

where $e_k \in \mathbb{C}[z_1,\dots,z_n]^{S_n}$ are elementary symmetric polynomials.

1.6.2 *Embedding $H_A^*(\mathfrak{M}(n,r),\mathbb{C}) \subset E$ and generators of $H_A^*(\mathfrak{M}(n,r),\mathbb{C})$*

The set of A-fixed points $\mathfrak{M}(n,r)$ can be parametrized by $\mathcal{P}(r,n)$ (see Chapter 5 for details).

We have the natural embedding

$$\iota\colon \mathfrak{M}(n,r)^A \subset \mathfrak{M}(n,r)$$

that induces the desired embedding

$$\iota^*\colon H_A^*(\mathfrak{M}(n,r),\mathbb{C}) \subset H_A^*(\mathfrak{M}(n,r)^A,\mathbb{C}) = E.$$

It remains to note that the algebra $H_A^*(\mathfrak{M}(n,r),\mathbb{C})$ is generated by the elements

$$c_k(\mathcal{V}),\ k = 1,\ldots,n,$$

where $c_k(\bullet)$ is the kth A-equivariant Chern class and $\mathcal{V}$ is the tautological n-dimensional vector bundle on $\mathfrak{M}(n,r)$. This, for example, follows from [MN18, Corollary 1.5].

1.6.3 *Embedding $Q_{n,r} \subset E$ and generators of $Q_{n,r}$*

Recall that $Q_{n,r}$ is a quotient of $Z(H_{n,r}) \subset H_{n,r}$. To every $\lambda \in \mathcal{P}(r,n)$ one can associate the corresponding induced (graded) $H_{n,r}$-module $\Delta(\lambda)$ on which $Z(H_{n,r})$ will act by some character and this action factors through the action of $Q_{n,r}$ (see Section 7.1). This gives us the desired embedding

$$\phi\colon Q_{n,r} \subset \bigoplus_{\lambda\in\mathcal{P}(r,n)} \operatorname{End}^{\mathrm{gr}}_{H_{n,r}}(\Delta(\lambda)) = E.$$

Generators of $Q_{n,r}$ are classes of

$$e_k(u_1,\ldots,u_n),\ k = 1,\ldots,n,$$

where $u_i \in H_{n,r}$ are Dunkl-Opdam elements (see Lemma 7.1.6).

Remark 1.6.3. *The identification $Z(H_{n,r}) \simeq H_*^{A\times(\mathrm{GL}_n)_{\mathcal{O}}}(\mathcal{R}_{n,r})$ sends $e_k(u_1,\ldots,u_n)$ to the function $m_k := c_k(V) * 1$, here $c_k(V) \in H^*_{A\times\mathrm{GL}_n}(\mathrm{pt})$ is the Chern class of the tautological $A\times\mathrm{GL}_n$-bundle $V = \mathbb{C}^n$ on pt and $*$ is the convolution product on $H_*^{A\times(\mathrm{GL}_n)_{\mathcal{O}}}(\mathcal{R}_{n,r})$ (see [Web, Section 4] for details). This should be compared with the fact that generators on the dual side (i.e., generators of $H_A^*(\mathfrak{M}(n,r),\mathbb{C})$) are the Chern classes $c_i(\mathcal{V})$ of the tautological bundle $\mathcal{V}$.*

Remark 1.6.4. *The embedding ϕ can be interpreted geometrically as a pullback homomorphism from schematic fixed points on $\mathcal{M}(n,r)_{\mathfrak{a}}$ to schematic fixed points on a resolution of $\mathcal{M}(n,r)_{\mathfrak{a}}$ (see Chapter 3 for $r = 1$ example and Section 9.1 for general conjectures in this direction).*

1.7 MAIN RESULTS AND STRUCTURE OF THE THESIS

1.7.1 *Identification of the parameters and the main Theorem*

We have the parameter spaces

$$\mathbf{h} = \mathbb{C}[\kappa, c_1, \dots, c_{l-1}],\ \mathbf{k} = \mathbb{C}[\kappa, a_1, \dots, a_r]/(a_1 + \dots + a_r) = \mathbb{C}[\mathfrak{a}], \quad (1.3)$$

used in definitions of algebras (1.2) that we want to identify. We identify the parameters as follows ($i = 1, \dots, r$, $k = 1, \dots, r-1$):

$$a_i = p(\eta^{i-1}), \quad (1.4)$$

where $p(q) = \frac{1}{r}\sum_{l=1}^{r-1} \frac{c_l}{\eta^{-l}-1} q^l \in \mathbb{C}[q]$, $\eta = e^{\frac{2\pi\sqrt{-1}}{r}}$. We will denote by F the field of fractions of the algebras (1.3).

Theorem 1.7.1. *After the identification of parameters (1.4) $\mathbb{Z}$-graded algebras*

$$H_A^*(\mathfrak{M}(n,r), \mathbb{C}),\ Z(R^r(n))^{JM},\ Q_{n,r},\ H_*^{A\times(\mathrm{GL}_n)_{\mathcal{O}}}(\mathcal{R}_{n,r})$$

are isomorphic. The isomorphisms above identify generators as follows

$$c_k(\mathcal{V}) = [e_k(z_1, \dots, z_n)] = [e_k(u_1, \dots, u_n)] = [m_k], \quad (1.5)$$

where $\mathcal{V}$ is the tautological rank n vector bundle on $\mathfrak{M}(n,r)$, $m_k = c_k(V) * 1$, $u_i \in H_{n,r}$ are Dunkl-Opdam elements (Definition (7.1.3)).

Remark 1.7.2. *Note that the isomorphisms above are automatically graded since they preserve the degree of the generators (1.5). Note that the isomorphism $H_A^*(\mathfrak{M}(n,r), \mathbb{C}) \simeq H_*^{A\times(\mathrm{GL}_n)_{\mathcal{O}}}(\mathcal{R}_{n,r})$ is already an isomorphism of $\mathbb{C}[\kappa, a_1, \dots, a_n]/(a_1 + \dots + a_n) = \mathbb{C}[\mathfrak{a}]$-algebras and there is no need in any identification of parameters.*

1.7.2 *Structure of the thesis*

The thesis is organized as follows. In Chapter 2 we prove the non-equivariant version of Conjecture 1.2.6 for the Slodowy slice to the sub-regular orbit in a simply laced Lie algebra $\mathfrak{g}$. In Chapter 3, we prove the Hikita-Nakajima conjecture for $r = 1$, i.e., for the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{A}^2)$ using results of [Vas01]. In Chapter 4, we consider the case of arbitrary r and describe symplectically dual variety to $\mathfrak{M}_0(n,r)$ and its deformation using Coulomb branches and rational Cherednik algebras.

In Chapters 5, 6 and 7 we implement the idea of the proof of Hikita-Nakajima conjecture that we briefly explain in Section 1.6. More detailed, in Chapter 5 we describe the embedding $H_A^*(\mathfrak{M}(n,r), \mathbb{C}) \subset E$ and determine the image of generators $c_i(\mathcal{V}) \in H_A^*(\mathfrak{M}(n,r),\mathbb{C})$ under this embedding. In Chapter 6 we define the cyclotomic degenerate Hecke algebra $R^r(n)$ and recall its representation theory. We then describe the embedding $Z(R^r(n))^{JM} \subset E$ (using the representation theory of $R^r(n)$) and determine its image. In Chapter 7, we recall the representation theory of the rational Cherednik algebra $H_{n,r}$ and then describe the embedding $Q_{n,r} \subset E$ (using the representation theory of $H_{n,r}$). In Lemma 7.1.6, we describe generators of $Q_{n,r}$ and then determine their images under the embedding $Q_{n,r} \subset E$. As a corollary of results of Chapters 5, 6, 7, we obtain Theorem 1.7.1 (see Theorem 8.1.1 in Chapter 8). In Chapter 9, we discuss a possible approach to proving the Hikita-Nakajima conjecture for more general quivers. Appendix A contains a (short) proof of the fact that $Q_{n,r} \simeq \mathbb{C}[\mathcal{M}(n,r)^{\mathbb{T}}_{\mathfrak{a}}]$ is flat over the space of parameters.

2 HIKITA CONJECTURE FOR DU VAL SINGULARITIES

Let $\mathfrak{g}$ be a simply laced simple Lie algebra. The closure of its minimal nilpotent orbit is expected to be dual to the Slodowy slice to the subregular orbit.

The Slodowy slice to the subregular orbit in a Lie algebra $\mathfrak{g}$ is the same as $\mathbb{C}^2/\Gamma$ (see [Slo80], for example), where Γ is a finite subgroup of $SL(2,\mathbb{C})$ (corresponding to $\mathfrak{g}$). It is a symplectic variety with rational double points. It admits a unique symplectic resolution $\widetilde{\mathbb{C}^2/\Gamma}$, given by the minimal resolution. The cohomology algebra of this resolution is known (we will prove it) to be $Sym[\mathfrak{h}]/Sym^{\geq 2}[\mathfrak{h}]$, where $\mathfrak{h}$ is the abstract Cartan algebra of a Lie algebra $\mathfrak{g}$, corresponding to Γ.

The "dual" symplectic variety to it is given by the closure of the minimal nilpotent orbit in $\mathfrak{g}$, or, equivalently (via the isomorphism), the closure of the minimal orbit $\mathcal{O}_{min}$ in $\mathfrak{g}^*$. We will work with the latter.

If we choose a generic action of $\mathbb{C}^*$, Hikita conjecture (non-equivariant) for this pair of singular symplectic varieties states that

$$H^*(\widetilde{\mathbb{C}^2/\Gamma}) = \mathbb{C}[\overline{\mathcal{O}_{min}}]^{\mathbb{C}^*}. \tag{2.1}$$

Let us first examine the algebra structure of the LHS of 2.1:

Theorem 2.0.1. *The cohomology ring of the resolution $H^*(\widetilde{\mathbb{C}^2/\Gamma})$ is equal to $Sym[\mathfrak{h}]/Sym^{\geq 2}[\mathfrak{h}]$, where $\mathfrak{h}$ is the abstract Cartan algebra of $\mathfrak{g}$.*

Proof. On both $\mathbb{C}^2/\Gamma$ and $\widetilde{\mathbb{C}^2/\Gamma}$ there is an action of $\mathbb{C}^*$, that contracts the lower to the point zero. Thus, due to homotopy equivalence, we can restrict the computation of cohomology ring of $\widetilde{\mathbb{C}^2/\Gamma}$ to the fiber of the resolution over zero point of $\mathbb{C}^2/\Gamma$. This is known to be a tree of $\mathbb{P}^1$'s that is a Dynkin diagram of the types ADE. Let us show that its cohomology ring is given by $Sym[\mathfrak{h}]/Sym^{\geq 2}[\mathfrak{h}]$. π_1 of this variety is 0, since we can retract every loop to a tree, formed by the points of intersection and connecting lines between them, and tree is contractible. We are left to deal with H^2. To find it, one can observe that our tree has a decomposition into affine cells and dots, therefore we obtain the generators for H^2 from each affine line. Since the number of P^1's (and, thus, $\mathbb{A}^1$'s) equals the rk of the corresponding Lie algebra $\mathfrak{g}$ of type ADE, we thus obtain the cohomology ring, isomorphic to $Sym[\mathfrak{h}]/Sym^{\geq 2}[\mathfrak{h}]$, where $\mathfrak{h}$ is the abstract Cartan algebra of $\mathfrak{g}$.

□

After the left part of 2.1 is found we have to show that the right hand side is the same. To do this it will be useful to simplify the problem in the following way.

Notice that taking $\mathbb{C}^*$-invariants means the same as intersecting with the Cartan subalgebra: $\overline{\mathcal{O}_{min}}^{\mathbb{C}^*} = \mathfrak{h} \cap \overline{\mathcal{O}_{min}}$ as a scheme. So, instead of working with $\mathbb{C}^*$-invariant functions we can simply take the ideal of $\overline{\mathcal{O}_{min}}$ in $Sym[\mathfrak{g}]$, look at the image of its projection in $Sym[\mathfrak{h}]$ and factorize by it. The result of the factorization will be the ring we seek.

Now, let $\mathfrak{g}$ be a simple Lie algebra, fix $\mathfrak{h}$ a Cartan subalgebra and let $\overline{\mathcal{O}_{min}}$ denote the closure of the minimal nilpotent orbit in $\mathfrak{g}^*$. The statement about the equality of algebras (with the reasoning above) will follow from the following theorem:

Theorem 2.0.2. *Let I be the defining ideal of $\overline{\mathcal{O}_{min}}$ in $Sym[\mathfrak{g}]$. Then its image under the projection*

$$Sym[\mathfrak{g}] \twoheadrightarrow Sym[\mathfrak{h}],$$

induced by the inclusion $\mathfrak{h}^* \hookrightarrow \mathfrak{g}^*$ is given by $Sym[\mathfrak{h}]$ in degree ≥ 2.

Before moving to the proof let's take a closer look at some special cases of this statement.

Example 2.0.3. Consider the easiest example possible: the case of $\mathfrak{sl}(2)$. Since we have chosen h, we have both e and f and the nilpotent orbit (in this case there is only one nilpotent orbit apart from 0) is given by the equation $h^2 + ef = 0$ (or, equivalently, by the ideal, generated by $h^2 + ef$). This, after the projection to $Sym[\mathfrak{h}]$ will give us h^2, which clearly generates the whole algebra in degree ≥ 2.

Example 2.0.4. One more example would be the Lie algebra $\mathfrak{sl}(n)$ case. If a matrix belongs to the minimal nilpotent orbit its square is zero and its rank is 1. In terms of matrix equations such a matrix A is given by the $A^2 = 0, rk(A) = 1$ – every 2×2 minor should be zero and the matrix squared should be zero. If we restrict those to the Cartan subalgebra we will get $a_{ii}^2 = 0$ (from the $A^2 = 0$) and $a_{ii}a_{jj} = 0$ (from $det_{ij} = 0$) which gives us all the functions in degree 2 of $Sym[\mathfrak{h}]$ for $\mathfrak{gl}(n)$, thus in $\mathfrak{sl}(n)$ too.

To understand the theorem better we are going to use the following knowledge about the adjoint representation of $\mathfrak{g}$. One should note that

$\mathfrak{g}^*$ is a representation of the type V^*_θ there θ is the highest weight for the adjoint representation. Moreover, $Sym^n[\mathfrak{g}]$ can be decomposed as a sum $Sym^n[\mathfrak{g}] = V(n\theta) \oplus L_n$, where L_n stands for the sum of representations of lower weight. The ideal we are interested in is given by the $L := \bigoplus_n L_n$. Indeed, every function from a representation of lower weight kills the highest weight vector v_θ from $\mathfrak{g}^* = V^*(\theta)$. To find the generators of this ideal we can observe that it is actually given by the kernels of maps like

$$V(n\theta) \otimes V(m\theta) \twoheadrightarrow V((n+m)\theta).$$

The objects $V(n\theta)$ form a subring in the ring of all highest weight representations of our algebra $\mathfrak{g}$. Moreover, we are interested of the projection of this subring to $\mathrm{Sym}[\mathfrak{h}]$, in particular to $\mathrm{Sym}^2[\mathfrak{h}]$

The structure of generators of such a ring is given by the following theorem, in spirit of Kostant (see [FH], [LT]).

Theorem 2.0.5. *Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathfrak{g}^* = V(\theta)$ be the representation with the highest weight θ. Let*

$$A = \mathrm{Sym}(V\theta)).$$

This is a commutative graded algebra, and we can split it into pieces

$$A^a = \mathrm{Sym}^a(V(\theta))$$

where a is a nonnegative integer. A^a then is the direct sum of the representation $V(n\theta)$, and the sum L^a of the representations with strictly smaller highest weights. In particular, $L = \bigoplus_a L^a$ is an ideal in A, and it is generated by the elements of the form

$$\left(\left[\sum_{i=1}^{n}(x_i \otimes y_i + y_i \otimes x_i)\right] - 2(\theta,\theta)\,\mathrm{Id} \otimes \mathrm{Id}\right) \cdot\ (v_1 \otimes v_2),$$

where v_1 and v_2 stand for vectors in $V(\theta)$, and the sum $[\sum_{i=1}^n (x_i \otimes y_i + y_i \otimes x_i)]$ denotes the Casimir element of the Lie algebra $\mathfrak{g}$.

Proof. We will explain the theorem in case of Sym^2, as this is the case we will require. We are interested in the kernel of the map $\mathfrak{g}^* \otimes \mathfrak{g}^* = V(\theta) \otimes V(\theta) \to V(2\theta)$. Notice that $V(\theta) \otimes V(\theta) \simeq V(2\theta) \oplus J$, where J is an ideal that consists of representations of lower weights. In particular, Casimir element action separates these two spaces.

Casimir element $\mathcal{C}$ acts on the representation of weight θ by $\langle\theta,\theta\rangle + 2\langle\rho,\theta\rangle$. Thus, on representation $V(2\theta)$, $\mathcal{C}$ acts by $4\langle\theta,\theta\rangle + 4\langle\rho,\theta\rangle$. Then an operator $\mathcal{C} - (4\langle\theta,\theta\rangle + 4\langle\rho,\theta\rangle)\,\mathrm{Id}$ will be a projection onto the kernel that we want to describe.

Let us look more closely at the action of $\mathcal{C}$ on our space. Suppose x_i, y_i is a dual basis with respect to Killing form. Then the action of $\mathcal{C}$ on $\mathfrak{g}^* \otimes \mathfrak{g}^*$ is given by

$$\mathcal{C}(v_1 \otimes v_2) = \mathcal{C} \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathcal{C} + \left[\sum_{i=1}^{n}(x_i \otimes y_i + y_i \otimes x_i)\right]$$

Now, $\mathcal{C}$ acts on v_1 and v_2 via $\langle\theta,\theta\rangle + 2\langle\rho,\theta\rangle$, because $v_1, v_2 \in V(\theta)$. Thus the action of $\mathcal{C} - (4\langle\theta,\theta\rangle + 4\langle\rho,\theta\rangle)$ takes the form

$$\left(\left[\sum_{i=1}^{n}(x_i \otimes y_i + y_i \otimes x_i)\right] - 2(\theta,\theta)Id \otimes Id\right) \cdot \ (v_1 \otimes v_2).$$

□

For our case the theorem says that the kernel of the morphism $\mathfrak{g}^* \otimes \mathfrak{g}^* = V(\theta) \otimes V(\theta) \to V(2\theta)$, is generated by the elements of the form $C(v \cdot w) - 2(\theta,\theta)v \cdot w$, where C is the Casimir element. Since we are interested in the image of the projection of this subspace on the $\mathrm{Sym}^2[\mathfrak{h}]$ we can take vector $h \in \mathfrak{h}$ for both v, w and see what happens to it. If we choose a basis e_i, f_i, h_j in $\mathfrak{g}$, Casimir element can be written as a sum

$$C = \sum_i (e_i \otimes f_i + f_i \otimes e_i) + \sum_j (h_i \otimes h_i).$$

The first part of this sum sends the element $h \otimes h$ to a kernel of the projection $\mathrm{Sym}[\mathfrak{g}] \to \mathrm{Sym}[\mathfrak{h}]$, whereas the second part acts by zero. So, after projection we will be left with $2(\theta,\theta)h \otimes h$ and elements of this form generate all the $\mathrm{Sym}^2[\mathfrak{h}]$. The latter generates all the algebra $\mathrm{Sym}[\mathfrak{h}]$ in degree ≥ 2. This proves the Theorem 2.0.2, from which 2.1 follows.

3
HIKITA-NAKAJIMA CONJECTURE FOR HILBERT SCHEMES

In this Chapter, we give an “elementary” proof of the Hikita-Nakajima conjecture for the Hilbert scheme of points on $\mathbb{A}^2$. As a corollary (setting the equivariant parameter to be zero), we obtain a theorem that was originally proved by Hikita (see [Hik]) using different methods.

The idea is the following: we identify the algebra of schematic fixed points with the Rees algebra of the center $Z(\mathbb{C}S_n)$ of $\mathbb{C}S_n$ and then use the identification

$$\mathrm{Rees}(Z(\mathbb{C}S_n)) \simeq H^*_{\mathbb{T}}(\mathrm{Hilb}_n(\mathbb{A}^2)) \quad \text{(see [Vas01])} \tag{3.1}$$

to obtain the Hikita-Nakajima conjecture for $\mathrm{Hilb}_n(\mathbb{A}^2)$. An alternative approach to the identification (3.1) appears in Chapters 5, 6.

3.1 NAKAJIMA QUIVER VARIETIES

Let $I = (I_0, I_1)$ be a finite quiver, where I_0 is the set of vertices, and I_1 is the set of oriented edges. Let $\mathbf{v} = (v_i)_{i \in I_0}$, $\mathbf{w} = (w_i)_{i \in I_0}$ be I_0-tuples of nonnegative integer numbers. Let $V = \bigoplus_{i \in I_0} V_i$, $W = \bigoplus_{i \in I_0} W_i$ be I_0-graded vector spaces with $\dim V_i = v_i$, $\dim W_i = w_i$.

Consider the representation space

$$\mathbf{N} = \mathbf{N}(\mathbf{v}, \mathbf{w}) = \mathbf{N}_I(\mathbf{v}, \mathbf{w}) := \bigoplus_{(i \to j) \in I_1} \mathrm{Hom}(V_i, V_j) \oplus \bigoplus_{i \in I_0} \mathrm{Hom}(W_i, V_i).$$

We also consider the cotangent space $\mathbf{M}_I(\mathbf{v}, \mathbf{w}) = \mathbf{M}(\mathbf{v}, \mathbf{w}) = \mathbf{M} := T^*\mathbf{N}$ that can be identified with

$$\bigoplus_{(i \to j) \in I_1} \mathrm{Hom}(V_i, V_j) \oplus \bigoplus_{(i \to j) \in I_1} \mathrm{Hom}(V_j, V_i) \oplus \bigoplus_{i \in I_0} \mathrm{Hom}(W_i, V_i) \oplus \bigoplus_{i \in I_0} \mathrm{Hom}(V_i, W_i).$$

We can represent elements of $\mathbf{M}(\mathbf{v},\mathbf{w})$ as quadruples (X,Y,γ,δ), where

$$X \in \bigoplus_{(i\to j)\in I_1} \operatorname{Hom}(V_i,V_j),\ Y \in \bigoplus_{(i\to j)\in I_1} \operatorname{Hom}(V_j,V_i),$$

$$\gamma \in \bigoplus_{i\in I_0} \operatorname{Hom}(W_i,V_i),\ \delta \in \bigoplus_{i\in I_0} \operatorname{Hom}(V_i,W_i)$$

The space $\mathbf{M}(\mathbf{v},\mathbf{w}) = T^*\mathbf{N}$ carries a natural symplectic form. We set

$$G_{\mathbf{v}} := \prod_{i\in I_0} \mathrm{GL}(V_i),\ \mathfrak{g}_{\mathbf{v}} := \bigoplus_{i\in I_0} \mathfrak{gl}(V_i).$$

The group $G_{\mathbf{v}}$ acts naturally on the vector space $\mathbf{M}(\mathbf{v},\mathbf{w})$. This action is symplectic with the moment map:

$$\mu\colon \mathbf{M}(\mathbf{v},\mathbf{w}) \to \mathfrak{g}_{\mathbf{v}},\ (X,Y,\gamma,\delta) \mapsto [X,Y] + \gamma\delta.$$

Definition 3.1.1. A quadruple $(X,Y,\gamma,\delta) \in \mathbf{M}(\mathbf{v},\mathbf{w})$ is called *stable* if for every X,Y-invariant graded subspace $S \subset V$ such that S contains $\operatorname{im}\gamma$ we have $S = V$. We denote by $\mathbf{M}(\mathbf{v},\mathbf{w})^{\mathrm{st}} \subset \mathbf{M}(\mathbf{v},\mathbf{w})$ the (open) subset of stable quadruples.

Definition 3.1.2. The Nakajima quiver varieties $\mathfrak{M}(\mathbf{v},\mathbf{w})$, $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$ are defined as the following quotients:

$$\mathfrak{M}(\mathbf{v},\mathbf{w}) := \mu^{-1}(0)^{\mathrm{st}}/G_{\mathbf{v}},\ \mathfrak{M}_0(\mathbf{v},\mathbf{w}) := \mu^{-1}(0)/\!/G_{\mathbf{v}}.$$

We have the natural (projective) morphism $\pi\colon \mathfrak{M}(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_0(\mathbf{v},\mathbf{w})$.

Assumption 3.1.3. We assume that π is a resolution of singularities.

Let $\mathfrak{z}_{\mathbf{v}} \subset \mathfrak{g}_{\mathbf{v}}$ be the center of $\mathfrak{g}_{\mathbf{v}}$. Varieties $\mathfrak{M}(\mathbf{v},\mathbf{w})$, $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$ admit certain natural deformations over the space $\mathfrak{z}_{\mathbf{v}}$.

Definition 3.1.4. The "universal" quiver varieties $\mathfrak{M}(\mathbf{v},\mathbf{w})_{\mathfrak{z}_{\mathbf{v}}}$, $\mathfrak{M}_0(\mathbf{v},\mathbf{w})_{\mathfrak{z}_{\mathbf{v}}}$ are defined as follows:

$$\mathfrak{M}(\mathbf{v},\mathbf{w})_{\mathfrak{z}_{\mathbf{v}}} := \mu^{-1}(\mathfrak{z}_{\mathbf{v}})^{\mathrm{st}}/G_{\mathbf{v}},\ \mathfrak{M}_0(\mathbf{v},\mathbf{w})_{\mathfrak{z}_{\mathbf{v}}} := \mu^{-1}(\mathfrak{z}_{\mathbf{v}})/\!/G_{\mathbf{v}}.$$

For $\mathbf{a} \in \mathfrak{z}_{\mathbf{v}}$, we denote by $\mathfrak{M}(\mathbf{v},\mathbf{w})_{\mathbf{a}}$, $\mathfrak{M}_0(\mathbf{v},\mathbf{w})_{\mathbf{a}}$ the fibers of these families over $\mathbf{a}$.

3.2 HILBERT SCHEME $\mathrm{Hilb}_n(\mathbb{A}^2)$ AND $S^n(\mathbb{A}^2)$ AS NAKAJIMA QUIVER VARIETIES

Our main object of study in this section is $\mathrm{Hilb}_n(\mathbb{A}^2)$ the Hilbert scheme of n points on $\mathbb{A}^2$.

Definition 3.2.1. The variety $\mathrm{Hilb}_n(\mathbb{A}^2)$ is the variety whose $\mathbb{C}$-points are ideals $J \subset \mathbb{C}[x,y]$ of codimension n. The (affine) variety $S^n(\mathbb{A}^2)$ is the categorical quotient $(\mathbb{A}^2)^n/S_n$.

Recall that we have the Hilbert-Chow morphism

$$\mathrm{Hilb}_n(\mathbb{C}^2) \to S^n(\mathbb{A}^2),\ J \mapsto \mathrm{Supp}(\mathbb{C}[x,y]/J).$$

This morphism is a symplectic resolution of singularities.

Let us now recall the description of $\mathrm{Hilb}_n(\mathbb{C}^2)$ as a Nakajima quiver variety (corresponding to the Jordan quiver).

Definition 3.2.2. We denote by $\mathfrak{M}(n,r)$, $\mathfrak{M}_0(n,r)$ the Nakajima quiver varieties corresponding to the quiver I consisting of one vertex and one loop with $\dim V = n$, $\dim W = r$.

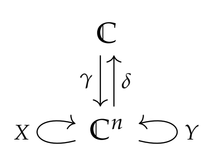
The following Proposition holds by [Nak99, Theorem 2.1 and Proposition 2.9].

Proposition 3.2.3. *There exist isomorphisms*

$$\mathfrak{M}(n,1) \xrightarrow{\sim} \mathrm{Hilb}_n(\mathbb{A}^2),\ \mathfrak{M}_0(n,1) \xrightarrow{\sim} S^n(\mathbb{A}^2) \tag{3.2}$$

compatible with natural morphisms $\mathfrak{M}(n,1) \to \mathfrak{M}_0(n,1)$, $\mathrm{Hilb}_n(\mathbb{A}^2) \to S^n(\mathbb{A}^2)$.

We conclude that the points of $\mathrm{Hilb}_n(\mathbb{A}^2)$, $S^n(\mathbb{A}^2)$ can be represented as certain quadruples (X,Y,γ,δ) that can be considered as representations of the following quiver (here we identify $V = \mathbb{C}^n$, $W = \mathbb{C}$):



3.3 CALOGERO-MOSER SPACE, DEFORMATIONS OF $\mathrm{Hilb}_n(\mathbb{A}^2)$ AND TORUS ACTIONS

We see that varieties $\mathfrak{M}(n,1)_{\mathfrak{z}_n}$, $\mathfrak{M}_0(n,1)_{\mathfrak{z}_n}$ (see Definition 3.1.4) are one-parameter deformations of $\mathrm{Hilb}_n(\mathbb{A}^2)$ and $S^n(\mathbb{A}^2)$, where the base of the deformation is the center $\mathfrak{z}_n \subset \mathfrak{gl}(V)$ that can be identified with $\mathbb{A}^1$ via the map $\mathbb{A}^1 \ni \mathbf{a} \mapsto \mathbf{a} \cdot \mathrm{Id}_V \in \mathfrak{gl}(V)$.

Let us now discuss torus actions. Let $\mathbb{T}$, $\mathbb{C}_\hbar^\times$ be copies of $\mathbb{C}^\times$. We have an action of $\mathbb{T} \times \mathbb{C}_\hbar^\times$ on $\mathrm{Hilb}_n(\mathbb{A}^2)$, $S^n(\mathbb{A}^2)$ that is induced by the action $\mathbb{T} \times \mathbb{C}_\hbar^\times \curvearrowright \mathbb{A}^2$ given by

$$(t, \hbar) \cdot (x, y) = (t\hbar^{-1}x, t^{-1}\hbar^{-1}y),\ t \in \mathbb{T},\ \hbar \in \mathbb{C}_\hbar^\times.$$

After the identifications (3.2) the action of $\mathbb{T} \times \mathbb{C}_\hbar^\times$ can be described as follows: it is induced from the following action on $\mathfrak{M}(n, 1)$:

$$(t, \hbar) \cdot (X, Y, \gamma, \delta) = (t\hbar^{-1}X, t^{-1}\hbar^{-1}Y, \hbar^{-1}\gamma, \hbar^{-1}\delta). \tag{3.3}$$

Remark 3.3.1. *Note that $\mathbb{T}$ acts symplectically, while $\mathbb{C}_\hbar^\times$ scales the symplectic form with the weight 2.*

Formula (3.3) induces actions

$$\mathbb{T} \times \mathbb{C}_\hbar^\times \curvearrowright \mathfrak{M}(n, 1)_{\mathfrak{z}_n},$$
$$\mathbb{T} \times \mathbb{C}_\hbar^\times \curvearrowright \mathfrak{M}_0(n, 1)_{\mathfrak{z}_n}.$$

Consider $\mathbf{a} \in \mathbb{C}^\times$. Let us describe the fibers $\mathfrak{M}(n, 1)_\mathbf{a}$, $\mathfrak{M}(n, 1)_\mathbf{a}$ of the families $\mathfrak{M}(n, 1)_{\mathfrak{z}_n}$, $\mathfrak{M}(n, 1)_{\mathfrak{z}_n}$ over $\mathbf{a}$. Note that the action of $\mathbb{C}_\hbar$ induces identifications

$$\mathfrak{M}(n, 1)_\mathbf{a} \simeq \mathfrak{M}(n, 1)_1,$$
$$\mathfrak{M}_0(n, 1)_\mathbf{a} \simeq \mathfrak{M}_0(n, 1)_1.$$

Definition 3.3.2. Recall that V is a vector space of dimension n. We define the Calogero-Moser variety $\mathcal{C}(n)$ as the following quotient:

$$\mathcal{C}(n) := \{(X, Y) \in \mathrm{End}(V)^{\oplus 2} \mid \mathrm{rk}\left([X, Y] - \mathrm{Id}_V\right) = 1\} / \mathrm{GL}(V).$$

The following proposition relates it to the fiber of our family (see, for example, [Wil, Section 1]).

Proposition 3.3.3. *Natural morphisms*

$$\mathfrak{M}(n, 1)_1 \to \mathfrak{M}_0(n, 1)_1 \to \mathcal{C}(n),$$

given by

$$[(X, Y, \gamma, \delta)] \mapsto [X, Y, \gamma, \delta] \mapsto [(X, Y)]$$

are isomorphisms.

So families $\mathfrak{M}(n, 1)_{\mathfrak{z}_n}$, $\mathfrak{M}_0(n, 1)_{\mathfrak{z}_n}$ are $\mathbb{C}_\hbar^\times$-equivariant deformations of $\mathrm{Hilb}_n(\mathbb{A}^2)$, $S^n(\mathbb{A}^2)$ over $\mathbb{A}^1$. Over a non-zero parameter their fibers are isomorphic to the Calogero-Moser variety $\mathcal{C}(n)$.

3.4 HIKITA-NAKAJIMA CONJECTURE FOR $\mathrm{Hilb}_n(\mathbb{A}^2)$

We denote by $H^*_{\mathbb{T}}(\mathrm{Hilb}_n(\mathbb{A}^2))$ the $\mathbb{T}$-equivariant cohomology of $\mathrm{Hilb}_n(\mathbb{A}^2)$. This is a $\mathbb{Z}$-graded algebra over $H^*_{\mathbb{T}}(\mathrm{pt}) = \mathbb{C}[\mathrm{Lie}\,\mathbb{T}]$.

We are now ready to state the "deformed" version of the Hikita conjecture for $\mathrm{Hilb}_n(\mathbb{A}^2)$ that we refer to as the Hikita-Nakajima conjecture (see Conjecture 1.2.6 above).

Theorem 3.4.1. *We have an isomorphism of $\mathbb{Z}$-graded algebras over $\mathbb{C}[\mathfrak{z}_n] \simeq \mathbb{C}[\mathrm{Lie}\,\mathbb{T}]$ (the identification induced by the isomorphism $\mathfrak{z}_n \simeq \mathbb{A}^1 \simeq \mathrm{Lie}\,\mathbb{T}$):*

$$\mathbb{C}\left[\mathfrak{M}_0(n,1)^{\mathbb{T}}_{\mathfrak{z}_n}\right] \simeq H^*_{\mathbb{T}}(\mathrm{Hilb}_n(\mathbb{A}^2)).$$

Our goal is to prove this theorem. We will do it by showing that both of algebras in the Theorem 3.4.1 are isomorphic to the Rees algebra of the center $Z(\mathbb{C}S_n)$ of the group algebra of S_n.

3.5 EQUIVARIANT COHOMOLOGY OF $\mathrm{Hilb}_n(\mathbb{A}^2)$ AND THE CENTER OF $\mathbb{C}S_n$

Let $Z(\mathbb{C}S_n)$ in $\mathbb{C}S_n$ be the center of the group algebra $\mathbb{C}S_n$. Consider the grading on the vector space $\mathbb{C}S_n$ defined in the following way: pick a permutation $\sigma \in S_n$ and let $\ell(\sigma)$ be the number of cycles in the decomposition of σ as a product of disjoint cycles. We then define

$$\deg \sigma := 2(n - \ell(\sigma)).$$

The grading above induces the increasing $\mathbb{Z}_{\geqslant 0}$-filtration on Z_n:

$$\mathbb{C} = F_0 Z_n = F_1 Z_n \subset F_2 Z_n = F_3 Z_n \subset \ldots \subset Z_n = F_{2n-2} Z_n = F_{2n-1} Z_n = \ldots \tag{3.4}$$

We denote by $\mathrm{Rees}(Z(\mathbb{C}S_n))$ the Rees algebra corresponding to the filtration (3.4). Recall that the algebra $\mathrm{Rees}(Z(\mathbb{C}S_n))$ is defined as follows:

$$\mathrm{Rees}(Z(\mathbb{C}S_n)) := \bigoplus_{m \geqslant 0} \kappa^m F_{2m} Z(\mathbb{C}S_n) \subset Z(\mathbb{C}S_n)[\kappa],$$

where κ is a formal parameter of degree 2. We consider $\mathrm{Rees}(Z(\mathbb{C}S_n))$ as an algebra over $\mathbb{C}[\mathbb{A}^1] = \mathbb{C}[\kappa]$.

The following result holds by [Vas01] or Chapters 5, 6 (see also [SVV17, Theorem 4.7 and Corollary 4.8]).

Proposition 3.5.1. *There is an isomorphism of $\mathbb{Z}$-graded algebras over $\mathbb{C}[\operatorname{Lie}\mathbb{T}] \simeq \mathbb{C}[\mathbb{A}^1]$ (the identification is induced by the isomorphism $\operatorname{Lie}\mathbb{T} \simeq \mathbb{A}^1$):*

$$H^*_{\mathbb{T}}(\operatorname{Hilb}_n(\mathbb{A}^2)) \simeq \operatorname{Rees}(Z(\mathbb{C}S_n)).$$

Remark 3.5.2. *Proposition 3.5.1 can also be proved using the same argument as we use in the proof of Theorem 3.6.1 below.*

Let us now recall the description of the center $Z(\mathbb{C}S_n)$. To every $k \in 1, \ldots, n$ we can associate the corresponding Jucys–Murphy element JM_k defined as follows:

$$JM_k := (1\,k) + (2\,k) + \ldots + (k-1\,k) \in \mathbb{C}S_n,$$

where $(i\,k) \in S_n$ is the transposition switching i, k.

Remark 3.5.3. *Note that $JM_1 = 0$.*

The following proposition provides us with a generating set in the center $Z(\mathbb{C}S_n)$ (see, [Mur83, Theorem 1.9]).

Proposition 3.5.4. *The center $Z(\mathbb{C}S_n)$ is generated (as a vector space) by the elements $f(JM_1, \ldots, JM_n)$, where f runs through the symmetric functions on n variables.*

3.6 CONSTRUCTION OF THE ISOMORPHISM BETWEEN SCHEMATIC FIXED POINTS OF $\mathfrak{M}_0(n,1)_{\mathfrak{z}_n}$ AND $\operatorname{Rees}(Z(\mathbb{C}S_n))$

We will prove the following theorem and, using Proposition 3.5.1, obtain Theorem 3.4.1 as a corollary.

Theorem 3.6.1. *There is an isomorphism of $\mathbb{Z}$-graded algebras over $\mathbb{C}[\mathbb{A}^1] = \mathbb{C}[\operatorname{Lie}\mathbb{T}]$*

$$\operatorname{Rees}(Z(\mathbb{C}S_n)) \xrightarrow{\sim} \mathbb{C}\left[\mathfrak{M}_0(n,1)^{\mathbb{T}}_{\mathfrak{z}_n}\right] \tag{3.5}$$

that sends $f(JM_1, \ldots, JM_n) \in Z(\mathbb{C}Z_n)$ to the restriction of the function

$$\mathfrak{M}_0(n,1)_{\mathfrak{z}_n} \ni [(X, Y, \gamma, \delta)] \mapsto f(\alpha_1, \ldots, \alpha_n)$$

to $\mathfrak{M}_0(n,1)^{\mathbb{T}}_{\mathfrak{z}_n}$. Here f is a symmetric function on n variables and $\alpha_1, \ldots, \alpha_n$ are roots of the characteristic polynomial of $YX \in \operatorname{End}(V)$, $\mathbb{Z}$-grading on the LHS of (3.5) is the natural grading on $\operatorname{Rees}(Z(\mathbb{C}S_n))$, $\mathbb{Z}$-grading on the RHS is the one induced by the action of $\mathbb{C}^\times_\hbar$.

The rest of the section is devoted to describing the idea of the proof of Theorem 3.6.1. We start with the following proposition, the proof of which is given in Appendix A.

Proposition 3.6.2. *The algebra* $\mathbb{C}\left[\mathfrak{M}_0(n,1)^{\mathbb{T}}_{\mathfrak{z}_n}\right]$ *is flat (hence, free) over* $\mathfrak{z}_n = \mathbb{A}^1$. *In particular, we have an isomorphism of* $\mathbb{Z}$*-graded algebras*

$$\mathbb{C}\left[\mathfrak{M}_0(n,1)^{\mathbb{T}}_{\mathfrak{z}_n}\right] \simeq \operatorname{Rees}(\mathbb{C}[\mathfrak{M}_0(n,1)^{\mathbb{T}}_1]) = \operatorname{Rees}(\mathbb{C}[\mathcal{C}(n)^{\mathbb{T}}]).$$

We conclude that to prove Theorem 3.6.1 it is enough to construct the isomorphism of filtered algebras $\mathbb{C}[\mathcal{C}(n)^{\mathbb{T}}] \simeq Z(\mathbb{C}S_n)$. To do so we need to describe the algebra $\mathbb{C}[\mathcal{C}(n)^{\mathbb{T}}]$ more explicitly. Let us note that the variety $\mathcal{C}(n)$ is smooth, hence, the scheme $\mathcal{C}(n)^{\mathbb{T}}$ is also smooth (see Proposition 1.2.3) and in particular reduced.

The description of the set of fixed points $\mathcal{C}(n)^{\mathbb{T}}$ was given by Wilson in [Wil, Proposition 6.11], we recall it in Section 3.7. The set $\mathcal{C}(n)^{\mathbb{T}}$ is finite and can be parametrized by the set $\mathcal{P}(n)$ of partitions of n, we denote by $[(X^{\lambda}, Y^{\lambda})]$ the fixed point corresponding to $\lambda \in \mathcal{P}(n)$ (see Definition 3.7.2). Since every finite reduced scheme over $\mathbb{C}$ is just the spectrum of the direct sum of copies of $\mathbb{C}$ then we must have

$$\mathbb{C}[\mathcal{C}(n)^{\mathbb{T}}] = \bigoplus_{\lambda \in \mathcal{P}(n)} \mathbb{C}\chi_{\lambda}, \tag{3.6}$$

where $\chi_{\lambda} \in \mathbb{C}[\mathcal{C}(n)^{\mathbb{T}}]$ is the characteristic function of the $\mathbb{T}$-fixed point $[(X^{\lambda}, Y^{\lambda})]$ corresponding to $\lambda \in \mathcal{P}(n)$.

Recall now that we have the natural identification

$$Z(\mathbb{C}S_n) = \bigoplus_{\mathcal{P}(n)} \mathbb{C}\mathbf{e}_{\lambda}, \tag{3.7}$$

here $\mathbf{e}_{\lambda} \in Z(\mathbb{C}S_n)$ is the idempotent corresponding to the Specht module $S(\lambda)$ (in other words, for $\nu \in \mathcal{P}(n)$ the element $\mathbf{e}_{\lambda} \in Z(\mathbb{C}S_n)$ acts on $S(\nu)$ via $\delta_{\lambda\nu} \cdot \operatorname{Id}_{S(\nu)}$).

Composing (3.6) and (3.7) we obtain the isomorphism of algebras

$$\Theta \colon Z(\mathbb{C}S_n) \xrightarrow{\sim} \mathbb{C}[\mathcal{C}(n)^{\mathbb{T}}],\ \mathbf{e}_{\lambda} \mapsto \chi_{\lambda}.$$

To prove Theorem 3.6.1 it remains to show that the isomorphism Θ is one that we need, i.e., it sends element $f(JM_1, \ldots, JM_n)$ to the restriction of the function $\mathcal{C}(n) \ni [(X,Y)] \mapsto f(\alpha_1, \ldots, \alpha_n)$ to $\mathcal{C}(n)^{\mathbb{T}}$. From this we will conclude that the isomorphism Θ is filtration-preserving.

To prove that Θ sends element $f(JM_1, \ldots, JM_n)$ to the function $\mathcal{C}(n)^{\mathbb{T}} \ni [(X,Y)] \mapsto f(\alpha_1, \ldots, \alpha_n)$ we just need to show that for every symmetric function f on n variables we have

$$f(JM_1, \ldots, JM_n)|_{S(\lambda)} = f(\alpha_1, \ldots, \alpha_n) \operatorname{Id}_{S(\lambda)},$$

where $\alpha_1, \ldots, \alpha_n$ is the multiset of eigenvalues of $Y^\lambda X^\lambda$. Recall that $f(JM_1, \ldots, JM_n)$ acts on $S(\lambda)$ via the multiplication by $f(c_1, \ldots, c_n)$, where $c_1, \ldots, c_n$ is the multiset of contents of boxes of the Young diagram $\mathbb{Y}(\lambda)$ corresponding to λ (see (3.8)). It remains to check that the multiset of eigenvalues of $Y^\lambda X^\lambda$ is the same as the multiset of contents of boxes of $\mathbb{Y}(\lambda)$.

3.7 DESCRIPTION OF $\mathcal{C}(n)^{\mathbb{T}}$ AND EIGENVALUES OF $Y^\lambda X^\lambda$

The parametrization of $\mathcal{C}(n)^{\mathbb{T}}$ by the elements of $\mathcal{P}(n)$ goes as follows (the description was obtained in [Wil], we follow [Prz16]). Pick $m \in \mathbb{Z}_{\geqslant 1}$ and $1 \leqslant k \leqslant m$.

Definition 3.7.1. By D_m we will denote the $m \times m$ matrix with 1's on the first diagonal and 0's elsewhere, i.e., $D_m = \sum_{i=1}^{m-1} E_{i\,i+1}$. Now, let $Y(m,k)$ be the $m \times m$ matrix such that its only non-zero entries are on the -1st diagonal, and it satisfies the relation $[Y(m,k), D_m] = mE_{k\,k}$. In other words, the numbers below the diagonal are $1, 2, \ldots, k-1, -m+k, \ldots, -2, -1$:

$$Y(m,k) = \begin{pmatrix} 0 & 0 & \cdots & & & & 0 \\ 1 & 0 & \cdots & & & & 0 \\ 0 & 2 & 0 & \cdots & & & 0 \\ \vdots & \cdots & \ddots & 0 & \cdots & & 0 \\ \cdots & & k-1 & 0 & \cdots & & \\ \cdots & & & -m+k & 0 & \cdots & \\ \cdots & & & & & & \\ \cdots & & & & & -1 & 0 \end{pmatrix}$$

Pick $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathcal{P}(n)$. Following [Prz16, Section 4.1] we denote by

$$\mathbb{Y}(\lambda) := \{(i,j) \,|\, 1 \leqslant i \leqslant l,\, 1 \leqslant j \leqslant \lambda_i\} \tag{3.8}$$

the corresponding Young diagram. If $\square = (i,j) \in \mathbb{Y}(\lambda)$ is a cell let $c(\square) := j - i$ be the *content* of $\square$. By a *hook* associated to the cell (i,j) we call the set

$$\mathbb{H}_{(i,j)} := \{(i,j)\} \cup \{(i',j) \in \mathbb{Y}(\lambda) \,|\, i' > i\} \cup \{(i,j') \in \mathbb{Y}(\lambda) \,|\, j' > j\}.$$

Box (i,j) is called the *root* of the hook $\mathbb{H}_{(i,j)}$. A hook of the form $\mathbb{H}_{(i,i)}$ is called a *Frobenius hook* of $\mathbb{Y}(\lambda)$. Diagram $\mathbb{Y}(\lambda)$ is the disjoint union of its Frobenius hooks. Suppose that $(1,1), (2,2), \ldots, (s,s)$ are cells of $\mathbb{Y}(\lambda)$ with zero content. Let $\mathbb{H}_i$ be the Frobenius hook with root (i,i). Let k_i be the height of $\mathbb{H}_i$ and n_i be the size of $\mathbb{H}_i$.

We are ready to describe the tuple $[(X^\lambda, Y^\lambda)] \in \mathcal{C}(n)^{\mathbb{T}}$ corresponding to λ.

Definition 3.7.2. Tuple (X^λ, Y^λ) is defined as follows. We have $X^\lambda = D_n$. The $n_i \times n_i$ diagonal blocks of Y^λ are given by the matrices $Y(n_i, k_i)$, and the off-diagonal blocks satisfy the following property: For $i \neq j$, $(Y^\lambda)_{ij}$ is the unique $n_i \times n_j$ matrix with non-zero entries on the diagonal $k_i - k_j$, satisfying the following property:

$$(Y^\lambda)_{ij} D_{n_j} - D_{n_i} (Y^\lambda)_{ij} = n_i E_{k_i k_j}.$$

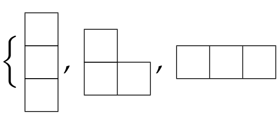
Remark 3.7.3. *If $\lambda \in \mathcal{P}(n)$ is a hook of height k, then we have $Y^\lambda = Y(n,k)$. Note that the diagonal matrix elements of $Y^\lambda X^\lambda = Y(n,k) D_n$ are precisely the contents of cells of the hook λ.*

Proposition 3.7.4. *The eigenvalues of $Y^\lambda X^\lambda = Y^\lambda D_n$ are the same as the eigenvalues of blocks $(Y^\lambda)_{ii} D_{n_i} = Y(n_i, k_i) D_{n_i}$ as if off-diagonal blocks in $Y^\lambda X^\lambda$ were not present. So the eigenvalues of $Y^\lambda X^\lambda$ are diagonal elements of $Y(n_i, k_i) D_{n_i}$ that are exactly the multiset of contents of boxes of λ.*

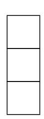
Proof. Follows from the proof of [Wil, Proposition 6.13]. □

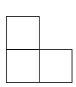
Corollary 3.7.5. *The isomorphism $\Theta \colon Z(\mathbb{C}S_n) \xrightarrow{\sim} \mathbb{C}[\mathcal{C}(n)]^{\mathbb{T}}$ sends $f(JM_1, \ldots, JM_n)$ to the restriction of the function $[(X,Y)] \mapsto f(\alpha_1, \ldots, \alpha_n)$ to $\mathcal{C}(n)^{\mathbb{T}}$, here f is a symmetric function on n variables.*

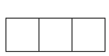
Example 3.7.6. Assume that $n = 3$. Let us describe the $\mathbb{C}^\times$-fixed points of $\mathcal{C}(3)$ in terms of X and Y. Recall that

$\mathcal{P}(3) = \{$  $\}$.

The corresponding $\mathbb{C}^\times$-fixed points of $\mathcal{C}(3)$ can be written as follows:

 $: Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow Y_1 X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$

 $: Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow Y_2 X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$

 $: Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, X_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow Y_3 X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$

We see that the diagonal terms of XY are precisely the contents of the corresponding Young diagrams.

3.8 PROOF OF THEOREM 3.6.1

We have already constructed (see Section 3.6) the isomorphism

$$\Theta\colon Z(\mathbb{C}S_n) \xrightarrow{\sim} \mathbb{C}[\mathcal{C}(n)^{\mathbb{T}}]$$

and have shown that this isomorphism sends generators $f(JM_1, \ldots, JM_n) \in Z(\mathbb{C}S_n)$ to functions $\mathcal{C}(n)^{\mathbb{T}} \ni (X, Y) \mapsto f(\alpha_1, \ldots, \alpha_n)$ (see Corollary 3.7.5). To finish the proof of Theorem 3.6.1 it remains to show that the isomorphism Θ is filtered. Let us note that if f has degree k then $\deg f(JM_1, \ldots, JM_n) = 2k$ and the degree of the function $(X, Y) \mapsto f(\alpha_1, \ldots, \alpha_n)$ is also equal to $2k$. Thus in order to show that Θ is filtration-preserving it is enough to check that $F_{2k}Z(\mathbb{C}S_n)$ is generated (as a vector space) by

$$\Big\{ f(JM_1, \ldots, JM_n) \mid f \text{ is a homogeneous symmetric polynomial of degree } \leqslant k \Big\}.$$

This is a direct corollary of the following (classical) proposition:

Proposition 3.8.1. *The algebra* $\operatorname{gr} Z(\mathbb{C}S_n)$ *is generated by the elements*

$$\Big\{ \operatorname{gr}\Big(f(JM_1, \ldots, JM_n)\Big) \mid f \textit{ is a homogeneous symmetric polynomial} \Big\}.$$

Proof. The claim follows from the proof of [Mur83, Theorem 1.9]. □

We finish this chapter with the following conjecture:

Conjecture 3.8.2. The isomorphism $\operatorname{gr}\Theta\colon \operatorname{gr} Z(\mathbb{C}S_n) \xrightarrow{\sim} \mathbb{C}[(S^n(\mathbb{A}^2))^{\mathbb{T}}]$ coincides with the isomorphism constructed in [Hik, Section 2].

4

DESCRIPTION OF SYMPLECTICALLY DUAL TO GIESEKER VARIETY

Recall that the Gieseker variety is the Nakajima quiver variety corresponding to the Jordan quiver (see Definition 3.2.2). It depends on the pair $n, r \in \mathbb{Z}_{\geqslant 1}$ and is denoted by $\mathfrak{M}(n,r)$. The corresponding affine Poisson variety is denoted by $\mathfrak{M}_0(n,r)$. In this chapter, we give two different ways to describe the symplectically dual variety $\mathfrak{M}_0(n,r)^!$ and its deformation $\mathfrak{M}_0(n,r)^!_{\mathfrak{a}}$.

In the paper [BFN18], the candidate for symplectically dual variety to every Nakajima quiver variety was constructed.

4.1 CONSTRUCTION OF COULOMB BRANCH

Let us revisit the construction, starting from the Jordan quiver with the dimension vector $n \in \mathbb{Z}_{\geqslant 1}$ and framing $r \in \mathbb{Z}_{\geqslant 1}$.

Recall the vector space $\mathbf{N} = \operatorname{Hom}(V,V) \oplus \operatorname{Hom}(V,W)$ and the group $G_n = \mathrm{GL}(V)$ acting on $\mathbf{N}$ (see Section 3.1).

Definition 4.1.1. We define $\mathrm{Gr}_{\mathrm{GL}(V)}$ as the moduli space of the data $(\mathcal{P}, \varphi)$, where

(a) $\mathcal{P}$ is a $\mathrm{GL}(V)$-bundle on $\mathbb{P}^1$;

(b) $\varphi\colon \mathcal{P}^{\mathrm{triv}}|_{\mathbb{P}^1\setminus\{0\}} \xrightarrow{\sim} \mathcal{P}|_{\mathbb{P}^1\setminus\{0\}}$ is a trivialization of $\mathcal{P}$ restricted to $\mathbb{P}^1 \setminus \{0\}$.

We then consider the moduli space of triples $\mathcal{R}_{n,r}$ (corresponding to the Jordan quiver, dimension vector n and framing r) defined as follows.

Definition 4.1.2. Let $\mathcal{R}_{n,r}$ be the moduli space of triples $\{(\mathcal{P}, \varphi, s)\}$, where $(\mathcal{P}, \varphi)$ is a point of $\mathrm{Gr}_{\mathrm{GL}(V)}$ and s is a section of the associated vector bundle $\mathcal{P}_{\mathbf{N}} = \mathcal{P} \times_{\mathrm{GL}(V)} \mathbf{N}$ such that it is sent to a regular section of a trivial bundle under φ.

Set $\mathrm{GL}(V)_{\mathcal{O}} := \mathrm{GL}(V)[[z]]$. We can consider the equivariant Borel-Moore homology $H_*^{\mathrm{GL}(V)_{\mathcal{O}}}(\mathcal{R}_{n,r})$ (see [BFN18, Section 2(ii)] for the definition and detailed discussion), this vector space is equipped with an algebra structure via convolution $*$ (see [BFN18, Section 3]). It follows from [BFN18, Proposition 5.15] that the algebra $(H_*^{\mathrm{GL}(V)_{\mathcal{O}}}(\mathcal{R}_{n,r}), *)$ is commutative.

Definition 4.1.3. Coulomb branch $\mathcal{M}(n,r)$ is defined as the spectrum of the algebra $H_*^{\mathrm{GL}(V)_{\mathcal{O}}}(\mathcal{R}_{n,r})$:

$$\mathcal{M}(n,r) := \mathrm{Spec}(H_*^{\mathrm{GL}(V)_{\mathcal{O}}}(\mathcal{R}_{n,r})).$$

The variety $\mathcal{M}(n,r)$ is conjectured to be symplectically dual to $\mathfrak{M}(n,r)$. The deformation $\mathfrak{M}(n,r)_{\mathfrak{a}} \to \mathfrak{a}$ of $\mathcal{M}(n,r)$ can be constructed as follows (compare with [BFN18, Section 3(viii)]). Let $\mathbb{T}$ be the copy of $\mathbb{C}^\times$. Let $T_r \subset \mathrm{SL}(W)$ be a maximal torus. We have the natural action of $A = \mathbb{T} \times T_r$ on $\mathbf{N}$ given by

$$(t,g) \cdot (X,\gamma) = (tX, \gamma \circ g^{-1}),\ (t,g) \in \mathbb{T} \times T_r.$$

Warning 4.1.4. *Note that the action of $\mathbb{T}$ on $\mathbf{N}$ is not the scaling action (as in [BFN18]). This action for example appears in [BEF20, Section 5.5].*

We can identify $\mathrm{Lie}\,\mathbb{T} \simeq \mathbb{A}^1$, so $\mathbb{C}[\mathrm{Lie}\,\mathbb{T}] = \mathbb{C}[\kappa]$ for some variable κ. We can also identify $\mathfrak{t}_r := \mathrm{Lie}\,T_r$ with the subspace of $\mathbb{C}^r$ consisting of points with the sum of coordinates being zero, i.e., $\mathbb{C}[\mathfrak{t}_w] = \mathbb{C}[a_1, \ldots, a_r]/(a_1 + \ldots + a_r)$.

Definition 4.1.5. We define

$$\mathcal{M}(n,r)_{\mathfrak{a}} := \mathrm{Spec}(H_*^{A \times \mathrm{GL}(V)_{\mathcal{O}}}(\mathcal{R}_{n,r})).$$

Let us now give a more explicit description of the algebra $\mathbb{C}[\mathcal{M}(n,r)_{\mathfrak{a}}]$. We will realize it as a spherical subalgebra of the double affine rational Cherednik algebra $H_{n,r}$ corresponding to the group $\Gamma_n = S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$.

4.2 DOUBLE AFFINE RATIONAL CHEREDNIK ALGEBRA CORRESPONDING TO $S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$

We start by recalling some definitions and notations. Consider the subgroup $\Gamma_n \subset \mathrm{GL}_n$ of monomial matrices with entries being rth roots of unity. Let $\eta \in \mathbb{C}^\times$ be a rth primitive root of unity. We set $\epsilon_j = \mathrm{diag}(1, \ldots, 1, \eta, 1, \ldots, 1)$. Note that we have the natural embedding $S_n \subset \Gamma_n$. We obtain the natural identification $S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n \xrightarrow{\sim} \Gamma_n$. Consider the standard representation $\Gamma_n \curvearrowright \mathfrak{h} := \mathbb{C}^n$ induced by the embedding $\Gamma_n \subset \mathrm{GL}_n$. Let $x_1, \ldots, x_n$ be the standard basis in $\mathfrak{h} = \mathbb{C}^n$ and denote by $y_1, \ldots, y_n \in \mathfrak{h}^*$ the dual basis.

Let $t, \kappa, c_1, \ldots, c_{r-1}$ be formal parameters. Define

$$\mathbf{h} := \mathbb{C}[\kappa, c_1, \ldots, c_{r-1}].$$

We set $c(q) := \sum_{i=1}^{r-1} c_i q^i$, here q is a formal variable. Following [Web] and [Prz16, Section 2.3] we define rational Cherednik algebra $\mathcal{H}_{\Gamma_n}$ as follows:

Definition 4.2.1. Algebra $\mathcal{H}_{\Gamma_n} = \mathcal{H}_{n,r}$ is a quotient of the semi-direct product

$$\left(\mathbb{C}\Gamma \ltimes T^{\bullet}(\mathfrak{h} \oplus \mathfrak{h}^*)\right) \otimes \mathbf{h}[\hbar]$$

subject to the relations

$$[x_i, x_j] = [y_i, y_j] = 0, \tag{4.1}$$

$$[x_i, y_i] = -\hbar - \kappa \sum_{j \neq i} \sum_{p=0}^{r-1} (ij) \epsilon_i^p \epsilon_j^{-p} - c(\epsilon_i), \tag{4.2}$$

$$[x_i, y_j] = \kappa \sum_{p=0}^{r-1} \eta^p (ij) \epsilon_i^p \epsilon_j^{-p} \qquad (i \neq j) \tag{4.3}$$

We also set $H_{n,r} := \mathcal{H}_{n,r}/(\hbar)$.

Remark 4.2.2. *Our parameters $(\hbar, \kappa, c_1, \ldots, c_{r-1})$ and the parameters $(\underline{t}, \underline{\kappa}, \underline{c}_1, \ldots, \underline{c}_{r-1})$ of [Mar14] are related as follows: $\underline{t} = -\hbar$, $\underline{\kappa} = \kappa$, $\underline{c}_i = \frac{c_i}{(1-\eta^{-i})}$ (note that $x_i^{Martino} = y_i$, $y_i^{Martino} = x_i$).*

Definition 4.2.3. Set $\mathbf{e} := \frac{1}{|\Gamma|} \sum_{g \in \Gamma} g$. We denote by $\mathcal{H}_{n,r}^{\text{sph}}$ the spherical subalgebra $\mathbf{e}\mathcal{H}_{n,r}\mathbf{e} \subset \mathcal{H}_{n,r}$. We denote by $H_{n,r}^{\text{sph}}$ the subalgebra $\mathbf{e}H_{n,r}\mathbf{e} \subset H_{n,r}$.

Recall that $Z(H_{n,r}) \subset H_{n,r}$ is the center. The following Proposition holds by [EG].

Proposition 4.2.4. *The composition $Z(H_{n,r}) \subset H_{n,r} \twoheadrightarrow \mathbf{e}H_{n,r}\mathbf{e}$ induces the identification $Z(H_{n,r}) \xrightarrow{\sim} \mathbf{e}H_{n,r}\mathbf{e}$.*

Algebra $H_{n,r}$ is $\mathbb{Z}$-graded as follows:

$$\deg x_j = 1, \deg y_j = -1, \deg \Gamma_n = \deg \mathbf{h} = 0. \tag{4.4}$$

Remark 4.2.5. *One can consider grading above as the $\mathbb{C}^{\times}$-action on $H_{n,r}$.*

4.3 COULOMB BRANCH FOR JORDAN QUIVER AS THE CENTER OF RATIONAL CHEREDNIK ALGEBRA FOR $S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$

Following [Web] set

$$p(q) := \frac{1}{r}\sum_{l=1}^{r-1}\frac{c_l}{\eta^{-l}-1}q^l = \frac{1}{r}\sum_{l=1}^{r-1}\frac{c_l\eta^l}{1-\eta^l}q^l.$$

Then we can write

$$c(q) = r(p(\eta^{-1}q) - p(q)). \tag{4.5}$$

The next proposition was proven in [KN18, Theorem 1.1], see also [Web, Theorem 4.1] and [BEF20, Theorem 2.19].

Proposition 4.3.1. *There exists the isomorphism of algebras*

$$H_{n,r}^{\mathrm{sph}} \simeq H_*^{A\times \mathrm{GL}(V)_{\mathcal{O}}}(\mathcal{R}_{n,r})$$

that identifies parameters in the following way $\kappa = \kappa$, $a_i = p(\eta^{i-1})$. *The isomorphism above sends* $e_i(u_1,\ldots,u_n)$ *to* m_i, *where* $m_i = c_i(V) * 1$.

Example 4.3.2. Let us illustrate how (quantized version of) the isomorphism $H_{n,r}^{\mathrm{sph}} \simeq H_*^{A\times \mathrm{GL}(V)_{\mathcal{O}}}(\mathcal{R}_{n,r})$ works for $n = 1$. We have $A = \mathbb{C}^\times \times (\mathbb{C}^\times)^{r-1}$ and $\mathbf{N} = \mathbb{C}\oplus(\mathbb{C}^r)^*$. By [BFN18, Section 4(iii)] the algebra $H_*^{A\times(\mathrm{GL}(V)_{\mathcal{O}}\rtimes\mathbb{C}^\times)}(\mathcal{R}_{n,r})$ has the following description: it is generated over $\mathbb{C}[\kappa, a_1,\ldots,a_r]/(a_1+\ldots+a_r)$ by r_1, r_{-1}, b subject to relations

$$r_1r_{-1} = \prod_{i=1}^{r}(b-a_i),\ r_{-1}r_1 = \prod_{i=1}^{r}(b-a_i-\hbar),\ [r_1,b] = \hbar r_1,\ [r_{-1},b] = -\hbar r_{-1}.$$

The algebra $H_{1,r}$ is generated by $\mathbb{C}\mathbb{Z}/r\mathbb{Z}$ and x, y subject to the relations

$$[x,y] = -\hbar - c(\epsilon),\ \epsilon x = \eta x\epsilon,\ \epsilon y = \eta^{-1}y\epsilon.$$

Set $u := \frac{1}{r}(xy+\hbar) + p(\eta^{-1}\epsilon)$ (compare with Definition 7.1.3). The isomorphism

$$H_*^{A\times(\mathrm{GL}(V)_{\mathcal{O}}\rtimes\mathbb{C}_\hbar^\times)}(\mathcal{R}_{1,r}) \xrightarrow{\sim} \mathcal{H}_{1,r}^{\mathrm{sph}}$$

is then given by

$$r_{-1}\mapsto \mathbf{e}x^r\mathbf{e},\ r_1\mapsto \frac{1}{r^r}\mathbf{e}y^r\mathbf{e},\ b\mapsto \mathbf{e}u\mathbf{e},\ \kappa\mapsto\kappa,\ a_i\mapsto p(\eta^{i-1}) - \frac{(i-1)\hbar}{r}.$$

Using Proposition 4.2.4 we conclude that the deformation $\mathcal{M}(n,r)_{\mathfrak{a}}$ of the variety $\mathcal{M}(n,r)$ can be described as

$$\mathcal{M}(n,r)_{\mathfrak{a}} = \operatorname{Spec}(Z(H_{n,r})).$$

5 EQUIVARIANT COHOMOLOGY OF GIESEKER VARIETY

We start by describing the combinatorial parametrization of the $A = \mathbb{T} \times T_r$-fixed points of the Gieseker variety $\mathfrak{M}(n, r)$.

Consider the case $r = 1$. The variety $\mathfrak{M}(n, 1)$ coincides with the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{A}^2)$ (see Proposition 3.2.3 above or [Nak99, Section 2.2]). This Hilbert scheme parametrizes the codimension n ideals in $\mathbb{C}[x, y]$.

The torus T_1 is zero-dimensional, thus $(\mathrm{Hilb}_n(\mathbb{A}^2))^{\mathbb{T}\times T_1} = (\mathrm{Hilb}_n(\mathbb{A}^2))^{\mathbb{T}}$. The fixed point set $(\mathrm{Hilb}_n(\mathbb{A}^2))^{\mathbb{T}}$ is the set of monomial ideals $J \in \mathrm{Hilb}_n(\mathbb{A}^2)$. Such ideals are parametrized by the set $\mathcal{P}(n)$ of partitions of n as follows. Recall that (following notations of [Prz16, Section 4.1]) we associate to $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathcal{P}(n)$ the Young diagram

$$\mathbb{Y}(\lambda) = \{(i, j) \mid 1 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant \lambda_i\}.$$

We fill $\mathbb{Y}(\lambda)$ with monomials putting $x^{i-1}y^{j-1}$ into the box $(i, j) \in \mathbb{Y}(\lambda)$. The ideal J_λ that corresponds to λ is spanned by the monomials outside the diagram. We have the following result.

Proposition 5.0.1. *The fixed point set $(\mathrm{Hilb}_n(\mathbb{A}^2))^{\mathbb{T}}$ is identified with the set $\mathcal{P}(n)$ via the map $\lambda \mapsto J_\lambda$ described above.*

Now we proceed to the case of arbitrary r. Let us introduce some notation. Let $w_1, \ldots, w_r \in W$ be a basis of W consisting of eigenvectors of T_r.

The following lemma is classical. (see, for example, [NY03])

Lemma 5.0.2. *The variety $\mathfrak{M}(n, r)^{T_r}$ is isomorphic to the disjoint union*

$$\bigsqcup_{\sum_{l=1}^{r} n_l = n} \prod_{l=1}^{r} \mathfrak{M}(n_l, 1) \simeq \bigsqcup_{\sum_{l=1}^{r} n_l = n} \prod_{l=1}^{r} \mathrm{Hilb}_{n_l}(\mathbb{A}^2).$$

Definition 5.0.3. We say that an ordered r-tuple $\boldsymbol{\lambda} = (\lambda^0, \ldots, \lambda^{r-1})$ of partitions defines an r-multipartition of $n \in \mathbb{Z}_{\geqslant 0}$ if $\sum_{l=0}^{r-1} |\lambda^l| = n$. Let $\mathcal{P}(r,n)$ denote the set of r-multipartitions of n.

Proposition 5.0.4. *The set of fixed points $\mathfrak{M}(n,r)^A$ is identified with the set $\mathcal{P}(r,n)$. A multipartition $\boldsymbol{\lambda} = (\lambda^0, \lambda^1, \ldots, \lambda^{r-1})$, corresponds to the following quiver data V, W, $X^{\boldsymbol{\lambda}}, Y^{\boldsymbol{\lambda}} \in \mathrm{End}(V)$, $\gamma^{\boldsymbol{\lambda}} \in \mathrm{Hom}(W,V)$, $\delta^{\boldsymbol{\lambda}} \in \mathrm{Hom}(V,W)$:*

$$V := \bigoplus_{l=0}^{r-1} \mathbb{C}[x,y]/J_{\lambda^l},\ W = \bigoplus_{l=0}^{r-1} \mathbb{C}w_l;$$

$$A^{\boldsymbol{\lambda}} := \bigoplus_{i=0}^{r-1} L_x^l,\ B^{\boldsymbol{\lambda}} := \bigoplus_{l=0}^{r-1} L_y^l;$$

$$\gamma^{\boldsymbol{\lambda}} \text{ sends } w_i \in W \text{ to } [1] \text{ in } \mathbb{C}[x,y]/J_{\lambda^i},\ \delta^{\boldsymbol{\lambda}} = 0,$$

by L_x^l, L_y^l we denote operators of multiplications by x, y on $\mathbb{C}[x,y]/J_{\lambda^l}$.

Proof. This follows from Lemma 5.0.2 and the description of $\mathbb{T}$-fixed points of Hilbert schemes above. □

We have the A-equivariant embedding

$$\iota\colon \mathfrak{M}(n,r)^A \subset \mathfrak{M}(n,r).$$

This embedding induces the homomorphism

$$\iota^*\colon H_A^*(\mathfrak{M}(n,r)) \to H_A^*(\mathfrak{M}(n,r)^A) = \mathbb{C}[\mathfrak{a}]^{\oplus|\mathcal{P}(r,n)|} = E.$$

Lemma 5.0.5. *The homomorphism ι^* is an embedding that becomes isomorphism after tensoring by $F = \mathbb{C}(\mathfrak{a})$.*

Proof. Follows from the Atiyah-Bott localization theorem (see [AB84]) together with the fact that $H_A^*(\mathfrak{M}(n,r), \mathbb{C})$ is a free $\mathbb{C}[\mathfrak{a}]$-module (see [Nak01, Theorem 7.3.5]). □

Remark 5.0.6. *Actually we do not need to tensor by the whole $\mathbb{C}(\mathfrak{a})$ but only have to to localize at elements $h \in \mathbb{C}[\mathfrak{a}]$ corresponding to the cocharacters $\nu\colon \mathbb{C}^\times \to A$ such that $\mathfrak{M}(n,r)^{\nu(\mathbb{C}^\times)}$ is* infinite*. One can describe these elements h explicitly.*

Recall now that by [MN18] the algebra $H_A^*(\mathfrak{M}(n,r), \mathbb{C})$ is generated by the A-equivariant Chern classes $c_k(\mathcal{V})$, $k = 1, \ldots, n$, where $\mathcal{V}$ is the tautological rank n vector bundle on $\mathfrak{M}(n,r)$.

Lemma 5.0.7. *The image $\iota^*(c_k(\mathcal{V}))$ is equal to the collection*

$$(e_k(\kappa c_1^0 + a_1, \ldots, \kappa c_{|\lambda^0|}^0 + a_1, \ldots, \kappa c_1^{r-1} + a_r, \ldots, \kappa c_{|\lambda^{r-1}|}^{r-1} + a_r))_{\boldsymbol{\lambda} \in \mathcal{P}(r,n)},$$

where for $l = 0, 1, \ldots, r-1$ $c_1^l, c_2^l, \ldots, c_{|\lambda^l|}^l$ *is the multiset of contents of boxes of the diagram* $\mathbb{Y}(\lambda^l)$.

Proof. The homomorphism ι^* sends $c_k(\mathcal{V})$ to

$$c_k(\mathcal{V}|_{\mathfrak{M}(n,r)^A}) = (c_k(\mathcal{V}|_{(X^\lambda, Y^\lambda, \gamma^\lambda, 0)}))_{\lambda \in \mathcal{P}(r,n)}.$$

Note that $c_k(\mathcal{V}|_{(X^\lambda, Y^\lambda, \gamma^\lambda, 0)})$ is nothing but $e_k(\alpha_1, \ldots, \alpha_n)$, where $\alpha_1, \ldots, \alpha_n$ is the multiset of $\mathfrak{a}$-weights of $\mathcal{V}|_{(X^\lambda, Y^\lambda, \gamma^\lambda, 0)}$.

Recall that $\mathbb{C}[\operatorname{Lie}\mathbb{T}] = \mathbb{C}[\kappa]$, $\mathbb{C}[\mathfrak{t}_r] = \mathbb{C}[a_1, \ldots, a_r]/(a_1 + \ldots + a_r)$ and $\mathfrak{a} = (\operatorname{Lie}\mathbb{T}) \oplus \mathfrak{t}_r$. We claim that the $\mathfrak{a}$-weight of $[x^i y^j] \in \mathbb{C}[x,y]/J_{\lambda^l}$ is equal to $\kappa(j-i) + a_l$ and this will conclude the proof. Indeed taking $g = (t^\kappa, t^{a_1}, \ldots, t^{a_r}) \in \mathbb{T} \times T_r$ we see that this element acts on $(X^\lambda, Y^\lambda, \gamma^\lambda, 0)$ as follows

$$g \cdot X^\lambda = \bigoplus_{l=0}^{r-1} t^\kappa L_x^l,\ g \cdot Y^\lambda = \bigoplus_{l=0}^{r-1} t^{-\kappa} L_y^l,\ (g \cdot \gamma^\lambda)(w_l) = t^{-a_l} w_l.$$

Consider the element $g' \in \prod_{l=0}^{r-1} \operatorname{GL}(\mathbb{C}[x,y]/J_{\lambda^l})$ that acts on $x^i y^j \in \mathbb{C}[x,y]/J_{\lambda^l}$ via the multiplication by $t^{a_l + \kappa j - \kappa i}$. Directly from the definitions, we see that

$$g \cdot (A^\lambda, B^\lambda, \gamma^\lambda, 0) = g' \cdot (A^\lambda, B^\lambda, \gamma^\lambda, 0).$$

□

Combining the results of this chapter we obtain the following Proposition.

Proposition 5.0.8. *The homomorphism* $\iota^* \colon H_A^*(\mathfrak{M}(n,r)) \hookrightarrow H_A^*(\mathfrak{M}(n,r)^A) = E$ *is injective and becomes isomorphism after tensoring by F. On generators* $c_k(\mathcal{V})$, $k = 1, \ldots, n$ *it is given by*

$$c_k(\mathcal{V}) \mapsto (e_k(\kappa c_1^0 + a_1, \ldots, \kappa c_{|\lambda^0|}^0 + a_1, \ldots, \kappa c_1^{r-1} + a_r, \ldots, \kappa c_{|\lambda^{r-1}|}^{r-1} + a_r))_{\lambda \in \mathcal{P}(r,n)}.$$

6 CENTER OF CYCLOTOMIC DEGENERATE HECKE ALGEBRA

Recall the space $\mathbf{k} = \mathbb{C}[\kappa, a_1, \dots, a_r]/(a_1 + \dots + a_r)$.

Definition 6.0.1. The degenerate affine Hecke algebra $R(n) = R(S_n)$ is generated by $\mathbb{C}S_n$ and $\mathbf{k}[z_1, \dots, z_n]$ subject to relations

$$s_i z_j = z_{s_i(j)} s_i + \kappa(\delta_{i+1,j} - \delta_{i,j}).$$

The cyclotomic degenerate Hecke algebra $R^r(n)$ is the quotient of $R(n)$ by the ideal generated by $\prod_{i=1}^r (z_1 - a_i)$.

Definition 6.0.2. Let $Z(R^r(n))^{JM} \subset R^r(n)$ be the image of $\mathbf{k}[z_1, \dots, z_n]^{S_n} \subset R(n)$ in $R^r(n)$.

Remark 6.0.3. *By [Lus88, Theorem 6.5] subalgebra $\mathbf{k}[z_1, \dots, z_n]^{S_n} \subset R(n)$ is nothing else but the center of $R(n)$. It follows from [Bru08, Theorem 1] that the algebra $Z(R^r(n))^{JM}$ is the center of $R^r(n)$.*

To every $\boldsymbol{\lambda} \in \mathcal{P}(r, n)$ one can associate the "universal" Specht module $\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda})$ over $R^r(n)$ (compare with [Bru08, Section 4]) in the following way.

Recall that $\boldsymbol{\lambda} = (\lambda^0, \lambda^1, \dots, \lambda^{r-1})$. For $i = 0, 1, \dots, r-1$ set $n_i := |\lambda^i|$. We slightly modify the algebras $R(n)$, $R^r(n)$ first.

Definition 6.0.4. Let $R_{\mathbb{C}[\kappa, a_1, \dots, a_r]}(n)$ be the algebra generated by $\mathbb{C}S_n$ and $\mathbb{C}[z_1, \dots, z_n]$ over $\mathbb{C}[\kappa, a_1, \dots, a_r]$ subject to the relations

$$s_i z_j = z_{s_i(j)} s_i + \kappa(\delta_{i+1,j} - \delta_{i,j}).$$

We denote by $R^r_{\mathbb{C}[\kappa, a_1, \dots, a_r]}(n)$ the quotient of $R_{\mathbb{C}[\kappa, a_1, \dots, a_r]}(n)$ by the ideal generated by $\prod_{i=1}^r (z_1 - a_i)$.

For every $i = 0, 1, \dots, r-1$ consider the usual Specht module $// \ S_{n_i} \curvearrowright S((\lambda^i)^t)$, corresponding to $(\lambda^i)^t$. We can extend $S(\lambda^i) \otimes \mathbb{C}[\kappa, a_i]$ to the $R_{\mathbb{C}[\kappa, a_i]}(n_i)$-module by letting z_1 act via the multiplication by a_i. Note that we have the natural embedding

$$R_{\mathbb{C}[\kappa, a_1, \dots, a_r]}(n_0, \dots, n_{r-1}) := R_{\mathbb{C}[\kappa, a_1]}(n_0) \otimes_{\mathbb{C}[\kappa]} \dots \otimes_{\mathbb{C}[\kappa]} R_{\mathbb{C}[\kappa, a_r]}(n_{r-1}) \subset R_{\mathbb{C}[\kappa, a_1, \dots, a_r]}(n)$$

induced by the embedding $S_{n_0} \times \ldots \times S_{n_{r-1}} \subset S_n$. Then we define

$$\widetilde{S}_{\mathbb{C}[\kappa,a_1,\ldots,a_r]}(\boldsymbol{\lambda}) :=$$
$$R_{\mathbb{C}[\kappa,a_1,\ldots,a_r]}(n) \otimes_{R_{\mathbb{C}[\kappa,a_1,\ldots,a_r]}(n_0,\ldots,n_{r-1})} \Big((S(\lambda^0) \otimes \mathbb{C}[\kappa, a_1]) \otimes_{\mathbb{C}[\kappa]} \ldots \otimes_{\mathbb{C}[\kappa]} (S(\lambda^{r-1}) \otimes \mathbb{C}[\kappa, a_r]) \Big)$$

that is an $R^r_{\mathbb{C}[\kappa,a_1,\ldots,a_r]}(n)$-module. Modding out by $a_1 + \ldots + a_r = 0$ we obtain the desired $R^r(n)$-module $\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda})$.

We now compute the action of $Z(R^r(n))^{JM}$ on $\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda})$. Let μ be a partition of some $m \in \mathbb{Z}_{\geqslant 1}$ and consider the action $R_{\mathbb{C}[\kappa,a]}(m) \curvearrowright S(\mu) \otimes \mathbb{C}[\kappa, a]$.

Lemma 6.0.5. *Let B be a Young tableau on μ and recall that $p_B \in S(\mu)$ is the corresponding vector. The element $z_i \in R_{\mathbb{C}[\kappa,a]}(m)$ acts on p_B via the multiplication by $\kappa \operatorname{ct}(B(i)) + a$.*

Proof. The i-th Jucys-Murphy element is given by

$$JM_i = \sum_{j<i} (ij) \in \mathbb{C}S_m.$$

Recall now that $s_i z_{i+1} = z_i s_i + \kappa$ so we have

$$z_{i+1} = s_i z_i s_i + s_i \kappa.$$

It follows that

$$z_j = s_{j-1} s_{j-2} \ldots s_1 z_1 s_1 \ldots s_{j-1} + \kappa JM_i$$

so the action of z_j on $S(\mu) \otimes \mathbb{C}[\kappa, a]$ coincides with the action of $\kappa JM_i + a$ and this concludes the proof of the lemma. □

From Lemma 6.0.5 we obtain the following proposition (see also [Bru08, Section 4]):

Proposition 6.0.6. *The class $[f(z_1, \ldots, z_n)] \in Z(R^r(n))^{JM}$ of the element $f(z_1, \ldots, z_n)$ in $\mathbb{C}[z_1, \ldots, z_n]^{S_n} \subset R(n)$ acts on the representation $\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda})$ via the multiplication by*

$$f(\kappa c^0_1 + a_1, \ldots, \kappa c^0_{|\lambda^0|} + a_1, \ldots, \kappa c^{r-1}_1 + a_r, \ldots, \kappa c^{r-1}_{|\lambda^{r-1}|} + a_r),$$

where $c^l_1, c^l_2, \ldots, c^l_{|\lambda^l|}$ is the multiset of contents of boxes of the diagram $\mathbb{Y}(\lambda^l)$.

Since every element of $Z(R^r(n))^{JM}$ is central we can consider the homomorphism $\psi\colon Z(R^r(n))^{JM} \to \bigoplus_{\boldsymbol{\lambda}} \operatorname{End}_{R^r(n)}(\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda}))$ induced by the action of $Z(R^r(n))^{JM}$ on representations $\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda})$.

Proposition 6.0.7. *The homomorphism*

$$\psi\colon Z(R^r(n))^{JM} \to \bigoplus_{\boldsymbol{\lambda}} \operatorname{End}_{R^r(n)} \widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda}) = E$$

becomes isomorphism after tensoring by $F = \operatorname{Frac}(\mathbf{k})$. This homomorphism is injective. It sends generators $[e_k(z_1, \ldots, z_n)]$ to the collection

$$(e_k(\kappa c_1^0 + a_1, \ldots, \kappa c_{|\lambda^0|}^0 + a_1, \ldots, \kappa c_1^{r-1} + a_r, \ldots, \kappa c_{|\lambda^{r-1}|}^{r-1} + a_r))_{\boldsymbol{\lambda} \in \mathcal{P}(r,n)},$$

where $c_1^l, c_2^l, \ldots, c_{|\lambda^l|}^l$ is the multiset of contents of boxes of the diagram $\mathbb{Y}(\lambda^l)$.

Proof. The last part of the claim is Proposition 6.0.6. Recall now that by [Bru08, Therorem 1] $Z(R^r(n))^{JM}$ is a free $\mathbf{k}$-module of rank $|\mathcal{P}(n,r)|$. Thus injectivity of ψ would follow if we show that ψ becomes isomorphism after tensoring by F. In order to do that it is enough to check that ψ becomes surjective after tensoring by F (using the equality of dimensions of $Z(R^r(n))^{JM}, \bigoplus_{\boldsymbol{\lambda}} \operatorname{End}_{R^r(n)}(\widetilde{S}_{\mathbf{k}}(\boldsymbol{\lambda}))$ over $\mathbf{k}$). Surjectivity is a corollary of Proposition 6.0.6 and Proposition 5.0.8. □

7

SCHEMATIC FIXED POINT OF THE CENTRE OF RATIONAL CHEREDNIK ALGEBRA

7.1 STANDARD AND SIMPLE REPRESENTATIONS OF $H_{n,r}$, GRADING ON THEM

We set $\zeta_{i,j} := \frac{1}{r}\sum_{p=0}^{r-1} \epsilon_i^p \epsilon_j^{-p}$ (projector to the invariants under $\epsilon_i\epsilon_j^{-1}$). The Jucys-Murphy elements are

$$JM_{\Gamma_n,i} := \sum_{j<i} \zeta_{i,j}(ij) \in \mathbb{C}\Gamma_n. \tag{7.1}$$

Recall that $\mathcal{P}(r,n)$ is the set of r-multipartitions of n (Definition 5.0.3). Pick $\boldsymbol{\lambda} \in \mathcal{P}(r,n)$ and consider the corresponding r-tuple of Young diagrams

$$\mathbb{Y}(\boldsymbol{\lambda}) = (\mathbb{Y}(\lambda^0), \dots, \mathbb{Y}(\lambda^{r-1})). \tag{7.2}$$

Given a cell $b \in \mathbb{Y}(\boldsymbol{\lambda})$, define $\beta(b) = k$ if $b \in \mathbb{Y}(\boldsymbol{\lambda}^{k-1})$ and $\mathrm{ct}(b) = j - i$ if b is in the ith row and jth column of $\mathbb{Y}(\lambda^k)$. There is a bijection $\boldsymbol{\lambda} \mapsto S(\boldsymbol{\lambda})$ from the set of r-partitions of n to the set of irreducible Γ_n-modules such that $S(\boldsymbol{\lambda})$ has a basis p_B indexed by standard Young tableaux B on $\boldsymbol{\lambda}$, and p_B is determined up to scalars by the equations (see, for example, [Gri18, Equation (2.16)])

$$JM_{\Gamma_n,i} \cdot p_B = \mathrm{ct}(B(i))p_B,\ \epsilon_i \cdot p_B = \eta^{\beta(B(i))} p_B. \tag{7.3}$$

We can consider $S(\boldsymbol{\lambda}) \otimes \mathbf{h}$ as a module over $(\mathbf{h} \otimes \mathbb{C}\Gamma_n) \ltimes S^\bullet\mathfrak{h}^*$ via the trivial action of $S^\bullet\mathfrak{h}^*$. Let $\Delta(\boldsymbol{\lambda}) := \mathrm{Ind}_{(\mathbf{h}\otimes\mathbb{C}\Gamma)\ltimes S^\bullet\mathfrak{h}^*}^{H_{n,r}} (S(\boldsymbol{\lambda}) \otimes \mathbf{h})$ be the induced module (sometimes called the standard module corresponding to $\boldsymbol{\lambda}$). Recall that the algebra $H_{n,r}$ is graded via

$$\deg x_j = 1,\ \deg y_j = -1,\ \deg \Gamma_n = \deg \mathbf{h} = 0.$$

This grading induces a grading on $\Delta(\boldsymbol{\lambda})$. The following lemma describes all the *graded* endomorphisms of our modules $\Delta(\boldsymbol{\lambda})$.

Lemma 7.1.1. *We have* $\mathrm{End}^{\mathrm{gr}}_{H_{n,r}}(\Delta(\boldsymbol{\lambda})) = \mathbf{h}$.

Proof. Note that $\Delta(\boldsymbol{\lambda})_0 = S(\boldsymbol{\lambda}) \otimes \mathbf{h}$ and $\Delta(\boldsymbol{\lambda})_0$ generates $\Delta(\boldsymbol{\lambda})$ over $H_{n,r}$. Now the claim follows from the equality $\mathrm{End}_{\mathbf{h}\otimes\mathbb{C}\Gamma_n}(S(\boldsymbol{\lambda}) \otimes \mathbf{h}) = \mathbf{h}$. □

Recall the algebra

$$Q_{n,r} := Z(H_{n,r})_0 / \sum_{i>0} Z(H_{n,r})_{-i} Z(H_{n,r})_i = Z(H_{n,r})/(b \in Z(H_{n,r})_i,\ i \neq 0)$$

of functions on schematic fixed points $(\operatorname{Spec} Z(H_{n,r}))^{\mathbb{T}}$.

Note that the action of $Z(H_{n,r})_0$ on $\Delta(\boldsymbol{\lambda})_0$ factors through $Q_{n,r}$ so we obtain a homomorphism

$$\phi\colon Q_{n,r} \to \bigoplus_{\boldsymbol{\lambda}} \mathrm{End}^{\mathrm{gr}}_{H_{n,r}}(\Delta(\boldsymbol{\lambda})) = \bigoplus_{\boldsymbol{\lambda}} \mathbf{h} = E.$$

Remark 7.1.2. *In [Gor02] Gordon defines so-called baby Verma modules over certain quotients of $H_{n,r}$ (called restricted Cherednik algebras of Γ_n). Homomorphism ϕ can be also defined by replacing $\Delta(\boldsymbol{\lambda})$ by the (universal analogs) of baby Verma modules and acting by $Q_{n,r}$ on them (compare with [Mar14]).*

We have constructed the homomorphism ϕ, the next step is to describe generators of the algebra $Q_{n,r}$.

Definition 7.1.3. Dunkl-Opdam operators u_i, $i = 1, \dots, n$ are the following elements of $H_{n,r}$:

$$u_i := \frac{1}{r} y_i x_i - \kappa \sum_{j>i} (ij)\zeta_{i,j} + p(\epsilon_i) = \frac{1}{r} x_i y_i + \kappa J M_{\Gamma_n, i} + p(\eta^{-1}\epsilon_i), \qquad (7.4)$$

where the last equality comes from the fact that

$$[x_i, y_i] = -\kappa \sum_{j\neq i} \sum_{p=0}^{r-1} (ij)\epsilon_i^p \epsilon_j^{-p} - c(\epsilon_i)$$

together with the equality $c(\epsilon_i) = rp(\eta^{-1}\epsilon_i) - rp(\epsilon_i)$. We will denote by the same symbol u_i the class of u_i in $H_{n,r}$.

Remark 7.1.4. *Note that the element u_i identifies with the element $\frac{z_i}{r}$ defined in [Mar14, Section 3.2].*

Lemma 7.1.5. *The subalgebra $\mathbf{h}[u_1, \dots, u_n]^{S_n} \subset H_{n,r}$ is central.*

Proof. This follows from [Mar14, Theorem 3.4], the statement can be also deduced from the presentation of $H_{n,r}$ given in [Web]. □

The Lemma 7.1.6 describes generators of the algebra $Q_{n,r}$, see Proposition 9.1.6 for alternative argument that covers a more general situation.

Lemma 7.1.6. *Classes of elements $e_k(u_1,\dots,u_n)$, $k=1,\dots,n$ generate the algebra $Q_{n,r}$.*

Proof. The claim follows from the proof of [Mar14, Theorem 5.5]. Let us repeat the argument. Recall that $Q_{n,r}$ is the quotient of the algebra $Z(H_{n,r}) = \mathbf{e}H_{n,l}\mathbf{e}$. Consider the following filtration on $H_{n,r}$:

$$\deg x_i = \deg y_i = 1,\ \deg \kappa = \deg c_j = \deg \Gamma_n = 0.$$

We have

$$\operatorname{gr} H_{n,r} = \left(\mathbb{C}\Gamma_n \ltimes S^\bullet(\mathfrak{h}\oplus\mathfrak{h}^*)\right)\otimes \mathbf{h}.$$

We need to prove that classes of the elements $\sum_{i=1}^n u_i^{ra+c}$, $a,c\in\mathbb{Z}_{\geqslant 0}$ generate the algebra $Q_{n,r}$. To see that it is enough to show that the elements $\operatorname{gr}\left(\sum_{i=1}^n u_i^{ra+c}\right) = \operatorname{gr}\left(\sum_i (x_iy_i)^{ra+c}\right)$ do generate $\operatorname{gr} Q_{n,r}$. Note that $\operatorname{gr} Q_{n,r}$ is the quotient of

$$\begin{aligned}\operatorname{gr}\mathbf{e}H_{n,r}\mathbf{e} = \mathbf{h}[x_1,\dots,x_n,y_1,\dots,y_n]^{S_n\ltimes(\mathbb{Z}/r\mathbb{Z})^n} =\\ = \mathbf{h}[x_1y_1,\dots,x_ny_n,x_1^r,\dots,x_n^r,y_1^r,\dots,y_n^r]\end{aligned}$$

that is generated by $\{\sum_{i=1}^n (x_i^r)^a(y_i^r)^b(x_iy_i)^c,\ a,b,c\in\mathbb{Z}_{\geqslant 0}\}$ ([EG, Lemma 11.17], [Wanoo, Lemma 1] see also Appendix A for more detailed discussion of the generators of $\operatorname{gr} Q_{n,r}$). It remains to note that the class of $\sum_{i=1}^n (x_i^r)^a(y_i^r)^b(x_iy_i)^c$ in $\operatorname{gr} Q_{n,r}$ is zero if $a\neq b$ and for $a=b$ we have $\sum_{i=1}^n (x_i^r)^a(y_i^r)^a(x_iy_i)^c = \sum_{i=1}^n (x_iy_i)^{ra+c}$. □

Lemma 7.1.7. *Let B be a Young tableau on $\boldsymbol{\lambda}$ and recall that $p_B\in S(\boldsymbol{\lambda})$ is the corresponding vector. The element u_i acts on p_B via the multiplication by*

$$\kappa\,\mathrm{ct}(B(i)) + p(\eta^{\beta(B(i))-1}).$$

Proof. Follows from the definition of u_i (see 7.4) together with (7.3). □

Proposition 7.1.8. *The homomorphism $\phi\colon Q_{n,r}\to\bigoplus_{\boldsymbol{\lambda}}\operatorname{End}^{\mathrm{gr}}_{H_{n,r}}(\Delta(\boldsymbol{\lambda}))$ becomes isomorphism after tensoring by $F=\mathrm{Frac}(\mathbf{h})$. This homomorphism is injective. It sends generators $[e_k(u_1,\dots,u_n)]$ to the collection*

$$(e_k(\kappa c_1^0+p(1),\dots,\kappa c^0_{|\lambda^0|}+p(1),\dots,\kappa c_1^{r-1}+p(\eta^{r-1}),\dots,\kappa c^{r-1}_{|\lambda^{r-1}|}+p(\eta^{r-1})))_{\boldsymbol{\lambda}\in\mathcal{P}(r,n)},$$

where $c_1^l, c_2^l,\dots,c^l_{|\lambda^l|}$ is the multiset of contents of boxes of the diagram $\mathbb{Y}(\lambda^l)$.

Proof. The proof repeats the structure from Proposition 6.0.7. The only difference is that we use Appendix A (flatness of $Q_{n,r}$ over $\mathbf{h}$) and the fact that the fiber of $Q_{n,r}$ over a generic point is $\mathbb{C}^{\oplus|\mathcal{P}(r,n)|}$ (follows from [Gor02]) instead of [Bru08, Theorem 1] and Lemma 7.1.7 instead of Proposition 6.0.6. □

8

PROOF OF THE THEOREM

8.1 PROOF OF THEOREM 1.7.1

Let us now repeat the statement of Theorem 1.7.1.

Theorem 8.1.1. *After the identification of parameters (1.4) graded algebras*

$$H_A^*(\mathfrak{M}(n,r)),\ Z(R^r(n))^{JM},\ Q_{n,r},\ H_*^{A\times(\mathrm{GL}_n)_{\mathcal{O}}}(\mathcal{R}_{n,r})$$

are isomorphic. Isomorphisms above identify generators as follows

$$c_k(\mathcal{V}) = [e_k(z_1,\dots,z_n)] = [e_k(u_1,\dots,u_n)] = [m_k]. \tag{8.1}$$

Proof. The claim follows from Propositions 5.0.8, 6.0.7, 7.1.8. The equality $e_k(u_1,\dots,u_n) = m_k$ follows from [Web, Section 4]. □

Remark 8.1.2. $\mathfrak{M}_0(n,r)^!$ *and its deformation have realization as certain quiver varieties (see Remark 1.4.2). It is a natural question to describe functions m_k in "quiver terms". For $r = 1$ these functions appear in Chapter 3 and are given by $(X,Y,\gamma,\delta) \mapsto e_k(\alpha_1,\dots,\alpha_n)$, where $\alpha_1,\dots,\alpha_n$ are roots of the characteristic polynomial of $YX \in \mathrm{End}(V)$. Functions m_k can be similarly described for arbitrary r, see [Prz16, Section 6] for details (for $r = l > 1$ there is a minor computational error in this paper that should be fixed, that's why we decided not to go into details here). Let us finally mention that the algebra generated by $\{m_k,\ k = 1,\dots,n\}$ defines an integrable system on the Coulomb branch $\mathcal{M}(n,r)$. This Coulomb branch can be identified with a certain Cherkis bow variety and this integrable system can be naturally described in these terms (see [NY03] for details, integrable systems are denoted by ϖ_C, Ψ in loc. cit.).*

9

POSSIBLE GENERALIZATIONS

9.1 CHANGING THE QUIVER

Let $I = (I_0, I_1)$ be a finite quiver and let $\mathbf{v} = (v_i)_{i \in I_0}$ be a dimension vector for I and $\mathbf{w} = (w_i)_{i \in I_0}$ be a framing. We can consider the corresponding (smooth) Nakajima quiver variety that we denote $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ (see Section 3.1). There is a natural torus A acting on $\mathbf{N}(\mathbf{v}, \mathbf{w})$. We can consider the symplectically dual variety $\mathcal{M}(\mathbf{v}, \mathbf{w})_{\mathfrak{a}}$ that can be described as the spectrum of the algebra $H_*^{A \times (G_{\mathbf{v}})_{\mathcal{O}}}(\mathcal{R}_{\mathbf{v},\mathbf{w}})$ of equivariant homology of the variety of triples $\mathcal{R}_{\mathbf{v},\mathbf{w}}$ corresponding to I, $\mathbf{v}$, $\mathbf{w}$ (see [BFN18]). Let $\mathbb{T}$ be $(\mathbb{C}^\times)^{|I_0|}$. We have a natural action of $\mathbb{T}$ on $H_*^{A \times (G_{\mathbf{v}})_{\mathcal{O}}}(\mathcal{R}_{\mathbf{v},\mathbf{w}})$ (see [BFN18, Section 3(v)]).

Recall that $\mathfrak{M}(\mathbf{v}, \mathbf{w}) = \mu^{-1}(0)^{\mathrm{st}}/G_{\mathbf{v}}$ so $H_A^*(\mathfrak{M}(\mathbf{v}, \mathbf{w}), \mathbb{C}) = H_{A \times G_{\mathbf{v}}}^*(\mu^{-1}(0)^{\mathrm{st}}, \mathbb{C})$ and we have the natural *surjective* (see [MN18]) homomorphism

$$H_{A \times G_{\mathbf{v}}}^*(\mathrm{pt}, \mathbb{C}) \twoheadrightarrow H_{A \times G_{\mathbf{v}}}^*(\mu^{-1}(0)^{\mathrm{st}}, \mathbb{C}) = H_A^*(\mathfrak{M}(\mathbf{v}, \mathbf{w}), \mathbb{C}). \tag{9.1}$$

Similarly we can consider the natural composition

$$H_{A \times G_{\mathbf{v}}}^*(\mathrm{pt}, \mathbb{C}) \subset H_*^{A \times (G_{\mathbf{v}})_{\mathcal{O}}}(\mathcal{R}_{\mathbf{v},\mathbf{w}}) \twoheadrightarrow \mathbb{C}[\mathrm{Spec}(H_*^{A \times (G_{\mathbf{v}})_{\mathcal{O}}}(\mathcal{R}_{\mathbf{v},\mathbf{w}}))^{\mathbb{T}}] \tag{9.2}$$

that is also surjective (see Proposition 9.1.6 below).

The equivariant version of the following conjecture was formulated in [HKW20, Conjecture 5.26].

Conjecture 9.1.1. The kernel of (9.1) is equal to the kernel of (9.2) i.e. they define the same closed subschemes of $\mathrm{Spec}\, H_{A \times G_{\mathbf{v}}}^*(\mathrm{pt}, \mathbb{C})$.

Remark 9.1.2. *If I is a Jordan quiver then Conjecture 9.1.1 follows from Theorem 1.7.1.*

The approach used in this thesis has a chance to be generalized to (some) other quivers. We start from the following Conjecture.

Conjecture 9.1.3. The algebra $\mathbb{C}[\mathrm{Spec}(H_*^{A \times (G_{\mathbf{v}})_{\mathcal{O}}}(\mathcal{R}_{\mathbf{v},\mathbf{w}}))^{\mathbb{T}}]$ is flat over $\mathbb{C}[\mathfrak{a}]$.

Remark 9.1.4. *Note that when I is the Jordan quiver then Conjecture 9.1.3 follows from Appendix A, note also that the "dual" statement to the Conjecture 9.1.3 is indeed true and holds by [Nak01, Theorem 7.3.5].*

Remark 9.1.5. *For our purposes, it is enough to prove that* $\mathbb{C}[\mathrm{Spec}(H_*^{A\times(G_{\mathbf{v}})_\mathcal{O}}(\mathcal{R}_{\mathbf{v},\mathbf{w}}))^{\mathbb{T}}]$ *is torsion free over* $\mathbb{C}[\mathfrak{a}]$*.*

Recall now that by [MN18] the algebra $H_A^*(\mathfrak{M}(\mathbf{v},\mathbf{w}))$ that appears at the LHS of Conjecture 9.1.1 is generated over $H_A^*(\mathrm{pt})$ by the Chern classes of tautological bundles $c_k(\mathcal{V}_i)$. It turns out that the "dual" statement can also be proven (without any restrictions on the quiver I) i.e., that the algebra $\mathbb{C}[\mathrm{Spec}(H_*^{A\times(G_{\mathbf{v}})_\mathcal{O}}(\mathcal{R}_I(\mathbf{v},\mathbf{w})))^{\mathbb{T}}]$ of schematic fixed points is generated over $H_A^*(\mathrm{pt})$ by the classes of $m_{k,i} := c_k(V_i) * 1$. This is equivalent to the following Proposition, which proof was explained to us by Ben Webster and Alex Weekes and which detailed proof will appear in their joint work with Joel Kamnitzer and Oded Yacobi.

Proposition 9.1.6. *The natural embedding* $H_{A\times G_{\mathbf{v}}}^*(\mathrm{pt}) \subset H_*^{A\times(G_{\mathbf{v}})_\mathcal{O}}(\mathcal{R}_{\mathbf{v},\mathbf{w}})$ *induces surjection* $H_{A\times G_{\mathbf{v}}}^*(\mathrm{pt}) \twoheadrightarrow \mathbb{C}[(\mathrm{Spec}\, H_*^{A\times(G_{\mathbf{v}})_\mathcal{O}}(\mathcal{R}_{\mathbf{v},\mathbf{w}}))^{\mathbb{T}}]$*.*

Proof. The claim follows from [Wee19, Proposition 3.1] (see also [BFN18, Remark 6.7]) using that the dressed minuscule monopole operators have a nonzero degree with respect to $\mathbb{T}$ so their images in $\mathbb{C}[(\mathrm{Spec}\, H_*^{A\times(G_{\mathbf{v}})_\mathcal{O}}(\mathcal{R}_{\mathbf{v},\mathbf{w}}))^{\mathbb{T}}]$ are zero. □

Assume now that Conjecture 9.1.3 holds. Assume also that there exists a resolution of singularities $\widetilde{\mathcal{M}}(\mathbf{v},\mathbf{w})_\mathfrak{a} \to \mathcal{M}(\mathbf{v},\mathbf{w})_\mathfrak{a}$ (see [Wee22] for the discussion). It induces the morphism $\widetilde{\mathcal{M}}(\mathbf{v},\mathbf{w})_\mathfrak{a}^{\mathbb{T}} \to \mathcal{M}(\mathbf{v},\mathbf{w})_\mathfrak{a}^{\mathbb{T}}$ that gives us the embedding $\mathbb{C}[\mathcal{M}(\mathbf{v},\mathbf{w})_\mathfrak{a}^{\mathbb{T}}] \subset \widetilde{\mathcal{M}}(\mathbf{v},\mathbf{w})_\mathfrak{a}^{\mathbb{T}}$, the fact that this is indeed an embedding follows from Conjecture 9.1.3. We can also consider the embedding $H_A^*(\mathfrak{M}(\mathbf{v},\mathbf{w})) \subset H_A^*(\mathfrak{M}(\mathbf{v},\mathbf{w})^A)$ that corresponds to the restriction $\mathfrak{M}(\mathbf{v},\mathbf{w})^A \subset \mathfrak{M}(\mathbf{v},\mathbf{w})$. Using that $c_k(\mathcal{V}_i)$, $m_{k,i}$ are generators of our algebras it then remains to show that the images of $c_k(\mathcal{V}_i)$, $m_{k,i}$ under these embeddings coincide.

Remark 9.1.7. *Let us point out that even without assuming Conjecture 9.1.3 one can consider the image of* $\mathbb{C}[\mathcal{M}(\mathbf{v},\mathbf{w})_\mathfrak{a}^{\mathbb{T}}] \to \mathbb{C}[\widetilde{\mathcal{M}}(\mathbf{v},\mathbf{w})_\mathfrak{a}^{\mathbb{T}}]$ *and if the images of* $c_k(\mathcal{V}_i)$*,* $m_{k,i}$ *coincide then we obtain a surjective homomorphism of algebras* $\mathbb{C}[\mathcal{M}(\mathbf{v},\mathbf{w})_\mathfrak{a}^{\mathbb{T}}] \twoheadrightarrow H_A^*(\mathfrak{M}(\mathbf{v},\mathbf{w}),\mathbb{C})$ *that is an isomorphism generically. Conjecture 9.1.3 implies that this surjection is an isomorphism.*

Let us finally note that to show that the images of $c_k(\mathcal{V}_i)$, $m_{k,i}$ coincide, it is enough to do the following. For a generic $\mathbf{a} \in \mathfrak{a}$ we need to construct a bijection

$$\mathfrak{M}(\mathbf{v},\mathbf{w})^A \xrightarrow{\sim} \mathcal{M}(\mathbf{v},\mathbf{w})_\mathbf{a}^{\mathbb{T}},\ p \mapsto p'$$

such that

$$e_k(\alpha_1,\ldots,\alpha_{v_i}) = m_{k,i}(p'),$$

where $\alpha_1,\ldots,\alpha_{v_i}$ are eigenvalues of $\mathbf{a} \in \mathfrak{a}$ acting on $\mathcal{V}_i|_{p'}$ and $m_{k,i}$ is considered as a function on $\mathcal{M}(\mathbf{v},\mathbf{w})$.

A

APPENDIX

The goal of this Appendix is to give a self-contained proof of the fact that the algebra $Q_{n,r}$ of functions on the schematic fixed points

$$(\operatorname{Spec} Z(H_{n,r}))^{\mathbb{T}} = \mathcal{M}(n,r)^{\mathbb{T}}_{\mathfrak{a}}$$

is a flat (hence, free) $\mathbf{h}$-module of rank $|\mathcal{P}(r,n)|$. Let us first of all note that by graded Nakayama lemma together with the fact that $\dim_F(Q_{n,r} \otimes_{\mathbf{h}} F) = |\mathcal{P}(r,n)|$ (this follows from [Gor02], see also [Prz16, Section 5]) in order to prove this fact it is enough to show that

$$\dim_{\mathbb{C}} Q_{n,r}/(\kappa, c_1, \ldots, c_{r-1}) \leqslant |\mathcal{P}(r,n)|$$

i.e. that

$$\dim_{\mathbb{C}} \mathbb{C}[(\mathbb{A}^{2n}/\Gamma_n)^{\mathbb{T}}] \leqslant |\mathcal{P}(r,n)|. \tag{A.1}$$

The goal of this chapter is to prove the inequality (A.1). Our argument simply follows papers [Hik] (for $r=1$ case) and [Hat21] (in general) but is much shorter since we do not need any explicit formulas for the multiplication rule of the elements of $\mathbb{C}[(\mathbb{A}^{2n}/\Gamma_n)^{\mathbb{T}}]$ and only need to estimate the dimension of this algebra from above since the estimate from below follows from the deformation argument (so we only need [Hik, Lemma 2.5] for $r=1$ case and [Hat21, Lemma 2.1.4] for general r). We start from the case $r=1$ i.e. from the case when $\Gamma_n = S_n$.

A.1 Hilbert scheme case ($r=1$)

Let us recall some notation (we follow [Hik]).

Definition A.1.1. An unordered sequence $\Lambda = (a_1,b_1)\ldots(a_l,b_l)$ with $(a_i,b_i) \in \mathbb{Z}^2_{\geqslant 0} \setminus \{(0,0)\}$ is called bipartite partition of $(a,b) \in \mathbb{Z}^2_{\geqslant 0} \setminus \{(0,0)\}$ if $\sum_{i=1}^{l} a_i = a$, $\sum_{i=1}^{l} b_i = b$. We set $\ell(\Lambda) = l$, $|\Lambda| = (a,b)$.

We have a natural surjection

$$\mathbb{C}[S^{n+1}(\mathbb{A}^2)] = \mathbb{C}[x_1,\ldots,x_{n+1},y_1,\ldots,y_{n+1}]^{S_{n+1}} \twoheadrightarrow \twoheadrightarrow \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]^{S_n} = \mathbb{C}[S^n(\mathbb{A}^2)]$$

and denote by S the inverse limit

$$S := \varprojlim \mathbb{C}[S^n(\mathbb{A}^2)].$$

We denote by $m_\Lambda \in S$ the symmetrization of the monomial $x_1^{a_1} y_1^{b_1} \dots x_l^{a_l} y_l^{b_l}$. We have

$$S = \mathbb{C}[m_{(a,b)} \mid (a,b) \in \mathbb{Z}^2_{\geqslant 0} \setminus \{(0,0)\}].$$

For $(a,b) \in \mathbb{Z}^2_{\geqslant 0} \setminus \{(0,0)\}$ we set $(a,b)\Lambda := (a,b)(a_1,b_1)\dots(a_l,b_l)$. If $(a,b) = (a_i,b_i)$ for some $i \in \{1,2,\dots,l\}$ we set $\Lambda \setminus (a,b) := (a_1,b_1)\dots(a_{i-1},b_{i-1})(a_{i+1},b_{i+1})\dots(a_l,b_l)$ We set $\overline{S} := S/(m_{(a,b)},\ a \neq b)$ and denote by $\overline{m}_\Lambda$ the image of m_Λ in $\overline{S}$. Note that directly from the definitions for every $n \in \mathbb{Z}_{\geqslant 1}$ we have a surjective homomorphism $\overline{S} \twoheadrightarrow \mathbb{C}[(S^n(\mathbb{A}^2))^{\mathbb{T}}]$ which sends every $\overline{m}_\Lambda$ with $\ell(\Lambda) > n$ to zero.

The following Lemma is clear.

Lemma A.1.2. *Let Λ be a bipartite partition. We have*

$$m_{(a,b)} m_\Lambda = k m_{(a,b)\Lambda} + \sum_{(i,j)\in\Lambda} k_{(i,j)} m_{(a+i,b+j)\Lambda\setminus(i,j)}$$

for some $k, k_{(i,j)} \in \mathbb{Z}_{>0}$.

For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ we denote by $(\lambda, 0)$ the bipartite partition $(\lambda_1, 0), \dots, (\lambda_l, 0)$.

This Lemma is [Hik, Lemma 2.5].

Lemma A.1.3. *$\{\overline{m}_{(\lambda,0)(0,1)^{|\lambda|}} \mid \lambda$ is partition$\}$ spans $\overline{S}$.*

Proof. Let us first of all note that the functions $\overline{m}_{(a,a)(b,b)(c,c)\dots}$ span $\overline{S}$. Indeed to prove this it is enough to show that every $\overline{m}_\Lambda$ can be obtained as a linear combination of $\overline{m}_{(a,a)(b,b)(c,c)\dots}$. This can be proved by the induction on $\ell(\Lambda)$ using Lemma A.1.2 together with the fact that $\overline{m}_{(a,b)} = 0$ for $a \neq b$.

It remains to show that every $\overline{m}_{(a,a)(b,b)(c,c)\dots}$ can be expanded in terms of $\overline{m}_{(\lambda,0)(0,1)^{|\lambda|}}$. To see that it is enough to show that every $\overline{m}_{(a_1,b_1)\dots(a_l,b_l)(0,1)^m}$ with $a_i \geqslant b_i$ can be obtained as a linear combination of $\overline{m}_{(\lambda,0)(0,1)^{|\lambda|}}$. We prove this by the induction on $d = \sum_{i=1}^l b_i$. For $d = 0$ the claim is clear.

For the induction step without losing the generality, we can assume that $b_1 > 0$. Using Lemma A.1.2 we obtain:

$$m_{(a_1,b_1-1)} m_{(a_2,b_2)\dots(a_l,b_l)(0,1)^{m+1}} = k_1 m_{(a_1,b_1)\dots(a_l,b_l)(0,1)^m} +$$

$$+ k_0 m_{(a_1,b_1-1)(a_2,b_2)\dots(a_l,b_l)(0,1)^{m+1}} + \sum_{i=2}^{l} k_i m_{(a_2,b_2)\dots(a_1+b_i,b_1+b_i-1)\dots(a_l,b_l)(0,1)^{m+1}}$$

for some $k_0, k_1, \dots, k_l \in \mathbb{Z}$ with $k_1 \neq 0$. Induction hypothesis together with the fact that $\overline{m}_{(a_1,b_1-1)}$ finish the proof. □

Corollary A.1.4. *The image of the set* $\{\overline{m}_{(\lambda,0)(0,1)^{|\lambda|}}, |\, \ell(\lambda)+|\lambda| \leqslant n\}$ *spans* $\mathbb{C}[S^n(\mathbb{A}^2)] = \mathbb{C}[\mathbb{A}^{2n}/S_n]$. *In particular, we have*

$$\dim \mathbb{C}[\mathbb{A}^{2n}/S_n] \leqslant |\mathcal{P}(n)|.$$

Proof. Clearly the elements $\overline{m}_{(\lambda,0)(0,1)^{|\lambda|}}$ with $\ell(\lambda)+|\lambda| > n$ lie in the kernel of $\overline{S} \twoheadrightarrow \mathbb{C}[S^n(\mathbb{A}^2)]$. Now the first claim follows from Lemma A.1.3. It remains to note that we have a bijection

$$\{\lambda \,|\, \ell(\lambda)+|\lambda| \leqslant n\} \xrightarrow{\sim} \mathcal{P}(n)$$

that sends a partition $\lambda = (1^{\alpha_1}2^{\alpha_2}\dots)$ to the partition $\hat{\lambda} \in \mathcal{P}(n)$ given by

$$\hat{\lambda} = 1^{n-\ell(\lambda)-|\lambda|}2^{\alpha_1}3^{\alpha_2}\dots i^{\alpha_{i-1}}\dots.$$

□

A.2 GENERAL CASE (r IS ARBITRARY)

Let us now generalize the arguments of Section A.1 to the arbitrary $r \in \mathbb{Z}_{\geqslant 1}$. We follow [Hat21].

We start with some notation. Recall that $\mathbb{C}[\mathbb{A}^{2n}/\Gamma_n]$ is nothing else but

$$\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]^{S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n} \simeq$$
$$\simeq \Big(\mathbb{C}[x_1',\dots,x_n',y_1',\dots,y_n',z_1',\dots,z_n']/(x_1'y_1'-(z_1')^r,\dots,x_n'y_n'-(z_n')^r)\Big)^{S_n}$$

where the isomorphism is given by

$$x_i' \mapsto x_i^r,\ y_i' \mapsto y_i^r,\ z_i' \mapsto x_iy_i.$$

Remark A.2.1. *Geometrically isomorphism above corresponds to the identification* $\mathbb{C}[\mathbb{A}^{2n}/\Gamma_n] \simeq S^n(\mathbb{A}^2/(\mathbb{Z}/r\mathbb{Z}))$.

Set $I_n := (x_1'y_1'-(z_1')^r,\dots,x_n'y_n'-(z_n')^r) \subset \mathbb{C}[x_1',\dots,x_n',y_1',\dots,y_n',z_1',\dots,z_n']$. Set

$$S' := \varprojlim \mathbb{C}[x_1',\dots,x_n',y_1',\dots,y_n',z_1',\dots,z_n']^{S_n},\ I := \varprojlim I_n,\ S := S/I.$$

For every tripartition $\Lambda = (a_1,b_1,c_1)(a_2,b_2,c_2),\dots,(a_l,b_l,c_l)$ let $m_\Lambda' \in S'$ be the symmetrization of the monomial $(x_1')^{a_1}(y_1')^{b_1}(z_1')^{c_1}\dots(x_l')^{a_l}(y_l')^{b_l}(z_l')^{c_l}$, we set $\ell(\Lambda) := l$. We denote by $m_\Lambda \in S$ the image of m'_Λ. The following Lemma is clear.

Lemma A.2.2. *The set* $\{m'_\Lambda \mid \Lambda\text{-tripartition}\}$ *spans* S'. *So*

$$\{m_\Lambda \mid \Lambda = (a_1, b_1, c_1)\dots, c_i \leqslant r-1\}$$

spans S.

Lemma A.2.3. *Let* Λ *be a tripartition and* $(a,b,c) \in \mathbb{Z}^3_{\geqslant 0} \setminus \{(0,0,0)\}$. *Then we have*

$$m'_{(a,b,c)} m'_\Lambda = ? \cdot m'_{(a,b,c)\Lambda} + \sum_{(i,j,k)\in\Lambda} ? \cdot m'_{(a+i,b+j,c+k)\Lambda\setminus(i,j,k)}.$$

with ? *being some positive numbers. As a corollary we have*

$$\begin{aligned} m_{(a,b,c)} m_\Lambda = ? \cdot m_{(a,b,c)\Lambda} + \sum_{(i,j,k)\in\Lambda,\, c+k\leqslant r-1} ? \cdot m_{(a+i,b+j,c+k)\Lambda\setminus(i,j,k)} + \\ + \sum_{(i,j,k)\in\Lambda,\, c+k\geqslant r} ? \cdot m_{(a+i+1,b+j+1,c+k-r)\Lambda\setminus(i,j,k)}. \end{aligned}$$

with ? *being some positive numbers.*

For $\Lambda = (a_1, b_1, c_1)\dots(a_l, b_l, c_l)$ we set $\deg\Lambda := \sum_{i=1}^l a_i - \sum_{i=1}^l b_i$. Let $J \subset S$ be the ideal generated by $\{m_\Lambda \mid \deg\Lambda \neq 0\}$. We set $\overline{S} := S/J$. We denote by $\overline{m}_\Lambda \in \overline{S}$ the image of m_Λ.

For an r-tuple of partitions $\boldsymbol{\lambda} = (\lambda^0, \lambda^1, \dots, \lambda^{r-1})$ we define tripartition to be denoted by the same symbol

$$\boldsymbol{\lambda} := (\lambda^0_1, 0, 0)\dots(\lambda^0_{\ell(\lambda^0)}, 0, 0), \dots, (\lambda^{r-1}_1, 0, r-1), \dots, (\lambda^{r-1}_{\ell(\lambda^{r-1})}, 0, r-1).$$

Recall that $\ell(\boldsymbol{\lambda}) = \sum_{i=0}^{r-1} \ell(\lambda^i)$, $|\boldsymbol{\lambda}| = \sum_{i=0}^{r-1} |\lambda^i|$.

This Lemma is [Hat21, Lemma 2.1.4].

Lemma A.2.4. *The set* $\{\overline{m}_{\boldsymbol{\lambda}(0,1,0)^{|\boldsymbol{\lambda}|}}\}$ *spans* $\overline{S}$.

Proof. We start from proving that elements $\overline{m}_{(a_1,a_1,c_1)(a_2,a_2,c_2)\dots}$ span $\overline{S}$. It is enough to show that every m_Λ can be presented as a linear combination of $\overline{m}_{(a_1,a_1,c_1)(a_2,a_2,c_2)\dots}$. We can assume that there exists $(a,b,c) \in \Lambda$ such that $a \neq b$ (otherwise there is nothing to prove). From Lemma A.2.3 it follows that

$$0 = \overline{m}_{(a,b,c)} \overline{m}_{\Lambda\setminus(a,b,c)} = k\overline{m}_\Lambda + \sum_{\Lambda', \ell(\Lambda')=\ell(\Lambda)-1} ? \cdot \overline{m}_{\Lambda'}$$

and the claim follows by the induction on the length of Λ.

It remains to show that every element $\overline{m}_{(a_1,a_1,c_1)\dots(a_l,a_l,c_l)}$ can be written as a linear combination of $\overline{m}_{\lambda(0,1,0)^{|\lambda|}}$. We prove more general statement: that every element $\overline{m}_{(a_1,b_1,c_1)\dots(a_l,b_l,c_l)(0,1,0)^k}$ with $a_i \geqslant b_i$ can be written as a linear combination of $\overline{m}_{\lambda(0,1,0)^{|\lambda|}}$. We prove this claim by the induction on $b+l$, where $b := \sum_{i=1}^{l} b_i$. Let us, first of all, note that we can assume that $\sum_{i=1}^{l} a_i = b+k$. For $b=0$ we must have $b_i = 0$ for every i and then there is nothing to prove. Suppose now that $b > 0$. Without losing the generality we can assume that $b_1 > 0$. By Lemma A.2.3 we have

$$\begin{aligned} 0 = \overline{m}_{(a_1,b_1-1,c_1)}\overline{m}_{(a_2,b_2,c_2)\dots(a_l,b_l,c_l)(0,1,0)^{k+1}} = \\ =? \cdot \overline{m}_{(a_1,b_1-1,c_1)(a_2,b_2,c_2)\dots(a_l,b_l,c_l)(0,1,0)^{k+1}} + u\overline{m}_{(a_1,b_1,c_1)\dots(a_l,b_l,c_l)(0,1,0)^k} + \\ + \sum_{c_1+c_i\leqslant r-1} ? \cdot \overline{m}_{(a_2,b_2,c_2)\dots(a_i+a_1,b_i+b_1-1,c_i+c_1)\dots(a_l,b_l,c_l)(0,1,0)^{k+1}} + \\ \sum_{c_i+c_1\geqslant r} ? \cdot \overline{m}_{(a_2,b_2,c_2)\dots(a_i+a_1+1,b_i+b_1,c_i+c_1-r)\dots(a_l,b_l,c_l)(0,1,0)^{k+1}} \end{aligned}$$

with $u \in \mathbb{Z}_{>0}$. □

Now the claim follows from the induction hypothesis.

Corollary A.2.5. *The image of the set* $\{\overline{m}_{\lambda(0,1,0)^{|\lambda|}} \mid \ell(\lambda) + |\lambda| \leqslant n\}$ *spans* $\mathbb{C}[\mathbb{A}^{2n}/\Gamma_n]$. *In particular*

$$\dim \mathbb{C}[\mathbb{A}^{2n}/\Gamma_n] \leqslant |\mathcal{P}(r,n)|.$$

Proof. Note that $\ell(\lambda(0,1,0)^{|\lambda|}) = \ell(\lambda) + |\lambda|$ so the elements $\overline{m}_{\lambda(0,1,0)^{|\lambda|}}$ with $\ell(\lambda) + |\lambda| > n$ lie in the kernel of $\overline{S} \twoheadrightarrow \mathbb{C}[\mathbb{A}^{2n}/\Gamma_n]$. Now the first claim follows from Lemma A.2.4. It remains to note that we have a bijection

$$\{\lambda \mid \ell(\lambda) + |\lambda| \leqslant n\} \xrightarrow{\sim} \mathcal{P}(r,n)$$

that sends an r-partition λ with $\lambda^0 = 1^{\alpha_1}2^{\alpha_2}\dots$ to the r-partition $\hat{\lambda} \in \mathcal{P}(r,n)$ given by

$$\hat{\lambda}^0 = 1^{n-\ell(\lambda)-|\lambda|}2^{\alpha_1}3^{\alpha_2}\dots k^{\alpha_{k-1}}\dots,\ \hat{\lambda}^i = \lambda^i \text{ for } i = 1,\dots,r-1.$$

The inverse map sends $\mu \in \mathcal{P}(r,n)$ with $\mu^0 = 1^{\beta_1}2^{\beta_2}\dots$ to the partition λ given by

$$\lambda^0 = 1^{\beta_2}2^{\beta_3}\dots k^{\beta_{k+1}}\dots,\ \lambda^i = \mu^i \text{ for } i = 1,\dots,r-1.$$

□

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