Descent techniques in algebraic K-theory

by

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Abstract

We investigate two different approaches to describing algebraic K-theory of schemes through descent techniques, one of global nature and the other of local nature.

The first half of the thesis is devoted to the study of an adelic descent statement for algebraic K-theory of Noetherian schemes, or more generally for any localizing invariants in place of algebraic K-theory. Given a Noetherian scheme X of finite Krull dimension, Beilinson's cosemisimplicial ring $A^{\bullet}_{\mathrm{red}}(X)$ of reduced adeles on X provides a resolution of the structure sheaf of X. We prove that for any localizing invariant E of small stable ∞-categories, e.g., nonconnective algebraic K-theory of Bass-Thomason, there is a natural equivalence $E(X) \simeq \lim_{\Delta_s} E(A^{\bullet}_{\mathrm{red}}(X))$. This can be viewed as a variant of the formal glueing problem for algebraic K-theory which concerns all irreducible closed subsets at once. We prove the descent statement by first converting the question to a cubical descent statement, and then constructing exact sequences of perfect module categories over adele rings.

In the second half of the thesis, we turn our attention to the study of p-adic K-theory of characteristic p rings. Specifically, we provide an alternative proof of Kelly-Morrow's generalization [KM21, Th. 2.1] of Geisser-Levine theorem to the Cartier smooth case. Our approach puts emphasis on utilizing motivic filtration and descent spectral sequence. Using the homological smoothness of Cartier smooth rings, we first compute their prismatic cohomology and syntomic cohomology complexes. Through motivic filtration, this computation gives a description of topological cyclic homology for Cartier smooth rings. Then, we use the pro-étale descent spectral sequence for topological cyclic homology and rigidity properties of the cyclotomic trace and syntomic cohomology complexes to deduce the result, computing algebraic K-theory of local Cartier smooth rings in terms of their logarithmic de Rham-Witt groups. We also collect some direct consequences of our arguments to prismatic cohomology complexes of Cartier smooth rings and their p-torsion free liftings to mixed characteristic.

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Finally, I would like to express my gratitude towards my parents.

μῆτερ ἐμή, χρειώ με κατήγαγεν εἰς Ἀίδαο
ψυχῇ χρησόμενον Θηβαίου Τειρεσίαο
(Hom. *Od.* XI.164-5).

Mother, it is need that has brought me down to the house of Hades,
the need to consult the spirit of Theban Teiresias.[1]

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[1] Homer, *The Odyssey*, Translated by Martin Hammond, Bloomsbury, 2000.

Contents

Chapter 1

Introduction

This manuscript is about studying algebraic K-theory of schemes in two different contexts, global and local respectively, through descent methods.

Let us start with a brief recollection of Segal's description of connective algebraic K-theory spectra. Let R be a connective $\mathbb{E}_\infty$-ring, e.g., an ordinary commutative ring, and consider the full subcategory $\mathrm{Proj}^{\mathrm{fg}}(R)$ of Mod_R consisting of direct summands of finite free R-modules, i.e., vector bundles on $\operatorname{Spec} R$. It admits finite coproducts, namely direct sums of modules, which endows the category with a coCartesian symmetric monoidal structure. By discarding non-invertible morphisms and taking group completion, one obtains an $\mathbb{E}_\infty$-group whose corresponding connective spectrum $\mathrm{K}_{\geq 0}(R)$ is a connective algebraic K-theory spectrum of R, i.e., $\Omega^\infty \mathrm{K}_{\geq 0}(R) \simeq \left(\mathrm{Proj}^{\mathrm{fg}}(R)^{\oplus,\simeq}\right)^{\mathrm{gp}}$. In particular, the 0-th K-group $\mathrm{K}_0(R) := \pi_0 \mathrm{K}_{\geq 0}(R)$ is nothing but the group completion of the monoid of isomorphism classes of finite locally free modules over $\pi_0 R$. This construction of connective algebraic K-theory in fact defines a Zariski sheaf of connective spectra, and hence via right Kan extension one can define connective K-theory spectra of qcqs[1] schemes. On the other hand, the Thomason-Trobaugh construction [TT90] (further clarified in [BGT13]) allows one to define a (nonconnective) algebraic K-theory spectrum $\mathrm{K}(X)$ for each qcqs schemes X whose connective cover recovers $\mathrm{K}_{\geq 0}(X)$. This construction also satisfies Zariski descent, i.e., the functor $\mathrm{K} : \mathrm{Sch}_{\mathrm{qcqs}}^{\mathrm{op}} \to \mathrm{Sp}$ is a Zariski sheaf of spectra.

An equivalent way of expressing the fact that algebraic K-theory satisfies Zariski descent is that ($\mathrm{K}(\emptyset) \simeq 0$, and that) K satisfies Zariski excision, i.e., for each qcqs scheme X and its quasicompact open subschemes U, V which together form a Zariski open cover of X, the natural diagram

$$\begin{tikzcd}
\mathrm{K}(X) \arrow[r] \arrow[d] & \mathrm{K}(V) \arrow[d] \\
\mathrm{K}(U) \arrow[r] & \mathrm{K}(U \cap V)
\end{tikzcd}$$

is a pullback square of spectra. In particular, we have a long exact Mayer-Vietoris sequence

$$\cdots \to \mathrm{K}_i(X) \to \mathrm{K}_i(U) \oplus \mathrm{K}_i(V) \to \mathrm{K}_i(U \cap V) \to \mathrm{K}_{i-1}(X) \to \cdots \quad (i \in \mathbb{Z})$$

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[1] Quasicompact and quasiseparated.

of abelian groups, where $\mathrm{K}_i(X) := \pi_i \mathrm{K}(X)$, etc. are the higher algebraic K-groups. This is one indication that algebraic K-theory behaves as a cohomology theory of schemes. This property is conveniently captured by the theory of localizing invariants [BGT13]; cf. see also section 2.1 for brief recollections. Another indication that a contravariant functor E on schemes behaves as a cohomology theory might be its $\mathbb{A}^1$-invariance, i.e., the requirement that the natural map $E(X) \to E(\mathbb{A}^1_X)$ is an equivalence; we will not explicitly address this property as well as surrounding theory and problems in this manuscript.

Turning our attention back to descent properties, one might naturally ask if algebraic K-theory satisfies the more useful étale descent. This would provide an analogue of Atiyah-Hirzebruch spectral sequence computing topological K-theory, through descent spectral sequence starting from étale cohomology groups and converging to algebraic K-theory groups. Unfortunately, this is not true; the problem is that algebraic K-theory does not satisfy Galois descent. Namely, given a Galois extension $R \to S$ of $\mathbb{E}_\infty$-rings for a finite group G, the natural map $\mathrm{K}(R) \to \mathrm{K}(S)^{hG}$ need not be an equivalence (even for the case of field extensions). In fact, this is the only obstacle, as algebraic K-theory satisfies Nisnevich descent [TT90]. Likewise, localizing invariants satisfy Nisnevich descent (cf. the final paragraph of section 1.1 and references therein) yet in general fail to satisfy étale (or equivalently, finite Galois) descent. Nevertheless, Thomason showed that the issue disappears after rationalization (i.e., $T(0)$-localization) or after $T(1)$-localization (under some technical conditions on the base scheme) applied to algebraic K-theory [Tho85; TT90]. As described below, it turned out later that Thomason's technical conditions in fact could be removed.

Thomason's result has undergone further clarification and generalization through recent remarkable advances in algebraic K-theory, notably through [CMNN20; CM21]. For instance, for any $n \geq 0$ and an implicit choice of prime p for periodic localizations, $L_{T(n)}\mathrm{K}$ satisfies finite flat descent (e.g., fppf descent in the case of ordinary commutative rings) [CMNN20, Th. 5.1, Prop. 5.4]. Moreover, in fact any localizing invariants valued in L_n^f-local spectra satisfy étale descent on qcqs ($\mathbb{E}_2$-)spectral algebraic spaces [CM21, Theorem 5.39], and under additional finiteness conditions on the base spectral algebraic space they satisfy étale hyperdescent [CM21, Theorem 7.14], generalizing Thomason's result for $L_{T(0)}\mathrm{K} \simeq \mathrm{K}(-)_{\mathbb{Q}}$ and $L_{T(1)}\mathrm{K}$.

Noticing that the failure of algebraic K-theory being an étale sheaf is a nuisance, one might try to remedy this by studying an étale sheafification $\mathrm{K}^{\text{ét}}$ of the algebraic K-theory functor which is viewed as a Zariski sheaf of spectra (on a qcqs base scheme or algebraic space). At least under some finiteness conditions on the base scheme X, the étale sheaf $\mathrm{K}^{\text{ét}}$ is a hypersheaf; in fact it is Postnikov complete, and hence gives a descent spectral sequence as in the situation of topological K-theory. For this approach to be meaningful, one should know that étale K-theory agrees with algebraic K-theory at least for sufficiently high enough degrees. This is essentially the subject of (now-proven) Quillen-Lichtenbaum conjecture; the natural map $\mathrm{K}(X) \to \mathrm{K}^{\text{ét}}(X)$ is sufficiently truncated.

A problem in this approach, as well as one of main difficulties in proving above Quillen-Lichtenbaum statement, is that $\mathrm{K}^{\text{ét}}$ in principle, although endowed with a better descent property, is not easy to comprehend. In most situations, sheafification $L_{\text{ét}} : \mathrm{Fun}^{\pi}(X_{\text{ét}}^{\text{op}}, \mathrm{Sp}) \to \mathrm{Shv}_{\mathrm{Sp}}(X_{\text{ét}})$ does not admit explicit descriptions, and it also does not help that the resulting functor $\mathrm{K}^{\text{ét}} = L_{\text{ét}}\mathrm{K}$ is not a localizing invariant. Thus, one might hope to find localizing invariants which simultaneously well-approximate algebraic K-theory and satisfy étale descent. We already have seen one good candidate for such purpose, which is $L_{T(1)}\mathrm{K}$. In fact, for X on which p is invertible, $L_{T(1)}\mathrm{K}$ approximates K well; under some finiteness conditions on X the map $\mathrm{K}(X) \to L_{T(1)}\mathrm{K}(X)$ is sufficiently truncated, cf. [RØ06], which relies on the theorem of Voevodsky-Rost [Ros02; Voe11].

To investigate what is happening at the residue characteristic p, we need help from a different localizing invariant, namely topological cyclic homology TC, and the cyclotomic trace map $\mathrm{tr} : \mathrm{K} \to \mathrm{TC}$ relating TC to the algebraic K-theory. See section 2.1 for a very brief review of relevant definitions, or better, see [DGM13]. First, TC has a better descent property; it satisfies étale descent (in fact even fpqc descent on connective $\mathbb{E}_\infty$-rings). Second, TC is a close enough approximation of K, at least much better than its predecessors HH or THH in that role. The difference between K and TC measured by tr is 'locally constant' [DGM13], which reduces comparison questions on connective $\mathbb{E}_1$-rings to those of ordinary rings. Using Nisnevich (hyper-)descent, one can often further reduce comparison questions to those of Henselian local rings. Then, for connective K-theory, rigidity [CMM21] enjoyed by the difference can further reduce quetions to the case of residue fields. In the residue characteristic p case, Geisser-Levine theorem [GL00], cf. chapter 4, can be used to analyze this situation. This in fact shows that the conclusion of the previous paragraph analogously holds (p-adically) for the cyclotomic trace map when X is entirely of residue characteristic p [CM21]. In *loc.cit.*, these two cases were simultaneously analyzed through the use of Selmer K-theory $\mathrm{K}^{\mathrm{Sel}} = L_{KU}\mathrm{K} \times_{L_{KU}\mathrm{TC}} \mathrm{TC}$, which by construction allows one to glue situations both at and away from residue characteristic p up to p-completion for each prime p. The Quillen-Lichtenbaum for $\mathrm{K}^{\text{ét}}$ as stated above then follows from the fact that the localizing invariant $\mathrm{K}^{\mathrm{Sel}}$ indeed computes $\mathrm{K}^{\text{ét}}$ in all high enough degrees [CM21, Th. 7.12 and Th. 7.13].

Despite the usefulness of localizing invariants, their inherent noncommutative nature still makes computations difficult to carry out in general. In order to remedy this, one might attempt to further linearize these invariants through filtrations in the hope of accessing the original invariant through its associated graded pieces which are arguably simpler and more linear in nature. As localizing invariants behave as noncommutative cohomology theories, one might expect that the associated graded pieces would act as cohomology complexes, which in principle can be better understood. For algebraic K-theory of a scheme X, Bloch's motivic cohomology complexes $\mathbb{Z}(n)^{\mathrm{mot}}(X)$ are the ones which appear as associated graded pieces of $\mathrm{K}(X)$ equipped with the motivic filtration, at least when X is smooth over a field [BS95; FS02; Lev08]. Due to the prevailing technical difficulties in studying motivic filtrations and algebraic cycles in general, one might instead study analogous filtrations on other localizing invariants closely related to algebraic K-theory in order to bypass such difficulties. For $L_{T(1)}\mathrm{K}$, it is Thomason's étale descent spectral sequence [Tho85; TT90] which gives such an analogue of motivic spectral sequence. For TC and pertinent Hochschild homology type invariants, [BMS19] constructed motivic filtrations and studied cohomology complexes appearing as associated graded pieces of such filtrations in p-adic setting. These BMS-style motivic filtrations were subsequently generalized to certain global and derived (or $\mathbb{E}_\infty$-) settings through [Mor20; BL22; HRW22] in recent years. In [Kim], we provided a construction of Thomason's motivic filtration through prismatic cohomology computations, which is closer to the spirit of [BMS19; HRW22]. In these approaches, the rough idea is to first study the case of big rings where filtrations can be easily handled, and then to right Kan extend to more general cases through descent properties enjoyed by localizing invariants.

In this thesis, we study two distinct results about algebraic K-theory of schemes using descent techniques. The first one is an adelic descent property of algebraic K-theory, or more generally of localizing invariants, as stated in Theorem 3.3.1. The second one is an alternative proof of Kelly-Morrow's extension [KM21] of Geisser-Levine theorem for Cartier smooth rings as stated in Corollary 4.4.4. The former result originally appeared in [Kim23], and our exposition of the adelic descent result which comprises the first part of this manuscript is largely commandeered from that paper. In the remaining parts of this chapter, we briefly explain these two results whose details are further presented in later chapters.

1.1 Adelic descent

To motivate us, let us consider the following arguably non-adelic situation first. Let R be an ordinary commutative ring and let I be its finitely generated ideal. One can ask, given a functorial construction of certain objects defined for qcqs schemes, whether it is possible or not to reconstruct each object on the entire $\operatorname{Spec} R$ from its restrictions on the schematic formal neighborhood $\operatorname{Spec} R^{\wedge_I}$ of the finitely presented closed embedding $V(I) \subseteq \operatorname{Spec} R$ and on the open complement $\operatorname{Spec} R \backslash V(I) \subseteq \operatorname{Spec} R$. The Beauville-Laszlo theorem, perhaps the most classical example of such *formal glueing* situation, concerns the case of finite projective modules over rings under suitable conditions [BL95] (cf. [Bha16, Prop. 5.6] for derived modules). Of course, an underlying intuition is that $\operatorname{Spec} R^{\wedge_I}$ acts as a tubular neighborhood of $V(I)$ in $\operatorname{Spec} R$ and that the square

$$\begin{tikzcd}
\operatorname{Spec} R^{\wedge_I} \backslash V(IR^{\wedge_I}) \arrow[r] \arrow[d] & \operatorname{Spec} R^{\wedge_I} \arrow[d] \\
\operatorname{Spec} R \backslash V(I) \arrow[r] & \operatorname{Spec} R
\end{tikzcd}$$

interprets $\operatorname{Spec} R$ as obtained by glueing its open subscheme $\operatorname{Spec} R \backslash V(I)$ with the tubular neighborhood $\operatorname{Spec} R^{\wedge_I}$ along the punctured tubular neighborhood $\operatorname{Spec} R^{\wedge_I} \backslash V(IR^{\wedge_I})$ of $V(I)$.

By using arc-topology [BM21], which was also independently considered by David Rydh, one can give the following interpretation of the idea above. Noting that the arc-topology on qcqs schemes is not subcanonical, consider the arc-sheafified Yoneda functor $h^\sharp$ from $\mathrm{Sch}_{\mathrm{qcqs}}$ into the ∞-topos[2] $\mathrm{Shv}((\mathrm{Sch}_{\mathrm{qcqs}})_{\mathrm{arc}})$. Then, the image of the above formal glueing square by $h^\sharp$ is a pushout diagram in $\mathrm{Shv}((\mathrm{Sch}_{\mathrm{qcqs}})_{\mathrm{arc}})$; in short, the above square of qcqs schemes is a pushout square up to arc-sheafification [BM21, Th. 6.6]. In particular, for any arc sheaf $F \in \mathrm{Shv}_{\mathcal{D}}((\mathrm{Sch}_{\mathrm{qcqs}})_{\mathrm{arc}})$ valued in a complete ∞-category $\mathcal{D}$, the natural square

$$\begin{tikzcd}
F(\operatorname{Spec} R) \arrow[r] \arrow[d] & F(\operatorname{Spec} R \backslash V(I)) \arrow[d] \\
F(\operatorname{Spec} R^{\wedge_I}) \arrow[r] & F(\operatorname{Spec} R^{\wedge_I} \backslash V(IR^{\wedge_I}))
\end{tikzcd}$$

is a pullback diagram in $\mathcal{D}$. It is also noted in *loc.cit.* that the use of arc-topology is in a sense essential; v-topolgy [Ryd10] (already stronger than the fpqc topology for instance) is still insufficient to guarantee an analogous claim.

Nevertheless, often formal glueing can be achieved by functors which do not satisfy arc descent. In fact, algebraic K-theory and more generally localizing invariants are examples of functors which sat-

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[2] Up to usual set-theoretic size considerations.

isfy formal glueing in Noetherian situation, yet in general fail to be arc sheaves. Let E be a localizing invariant valued in a stable ∞-category $\mathcal{D}$, and view it as a $\mathcal{D}$-valued presheaf on $\mathrm{Sch}_{\mathrm{qcqs}}$ by taking $E(X) := E(\mathrm{Perf}(X))$ for each qcqs scheme X. As remarked earlier, all we can say in general is that E defines a Nisnevich sheaf when restricted to X_{Nis} for each $X \in \mathrm{Sch}_{\mathrm{qcqs}}$, which is far from being an arc sheaf. However, it is precisely the fact that E is a localizing invariant *a priori* defined for all small stable ∞-categories (not necessarily of the form of perfect modules over qcqs schemes) which enables us to access the formal glueing. In fact, the equivalence $\mathrm{Perf}_{V(I)}(\mathrm{Spec}\,R) \simeq \mathrm{Perf}_{V(IR^{\wedge_I})}(\mathrm{Spec}\,R^{\wedge_I})$ which holds under the Noetherian assumption on R [TT90, Th. 2.6.3] together with the Thomason-Trobaugh argument as reviewed in the last paragraph of this section implies that the natural square

$$\begin{tikzcd}
E(\mathrm{Spec}\,R) \arrow[r] \arrow[d] & E(\mathrm{Spec}\,R\backslash V(I)) \arrow[d] \\
E(\mathrm{Spec}\,R^{\wedge_I}) \arrow[r] & E(\mathrm{Spec}\,R^{\wedge_I}\backslash V(IR^{\wedge_I}))
\end{tikzcd}$$

is a pullback diagram in $\mathcal{D}$.

We would like to study a variant of the formal glueing situation in Noetherian setting, where all the irreducible closed subsets of a scheme are considered at once. This is coherently captured by the cosimplicial ring $A^\bullet(X)$ of adeles and its variants attached to X. This object provides a resolution of X of global nature in a way generalizing the construction of classical (finite) adele rings for the ring of integers of global fields. To disregard repetitions in flags of irreducible closed subsets, we in particular consider the cosemisimplicial variant $A^\bullet_{\mathrm{red}}(X)$ of reduced adeles for our adelic version of formal glueing problem; see section 3.1 for a review of relevant definitions and details of constructions.

Now, let us state our *adelic* descent theorem for localizing invariants. Let X be a Noetherian scheme of finite Krull dimension n. Beilinson's construction [Bei80] of higher adeles on Noetherian schemes produces the cosemisimplicial ring $A^\bullet_{\mathrm{red}}(X)$ of reduced adeles on X, whose associated complex of abelian groups computes the cohomology of $\mathscr{O}_X$. Each ring $A^r_{\mathrm{red}}(X)$ of reduced adeles decomposes into a product of adele rings $A(i_0, ..., i_r)$ indexed by $0 \le i_0 < \cdots < i_r \le n$, i.e., subsets of $[n]$ of cardinality $r+1$. Moreover, the association $\{i_0, ..., i_r\} \mapsto A(i_0, ..., i_r)$ defines a functor A on $\mathcal{P}([n])\backslash\emptyset$, i.e, an n-cubical diagram A of rings without the initial vertex (see Remark 3.1.5 (1)). Our first main goal in this manuscript is to explain the following adelic descent result for nonconnective algebraic K-theory spectra (or more generally for any localizing invariant of stable ∞-categories):

Theorem 1.1.1. (Theorem 3.3.1, [Kim23, Th. 3.17]) Let X be a Noetherian scheme of finite Krull dimension n. Also, let $A^\bullet_{\mathrm{red}}(X)$ and $A(-)$ be the cosemisimplicial and cubical (without the initial vertex) diagram of adele rings on X respectively (see Remarks 3.1.2 and 3.1.5). Then, we have equivalences of (nonconnective) algebraic K-theory spectra

$$\mathrm{K}(X) \simeq \lim_{[r]\in(\Delta_s)_{\le n}} \mathrm{K}(A^r_{\mathrm{red}}(X)) \quad \text{and} \tag{1.1}$$

$$\mathrm{K}(X) \simeq \lim_{0\le i_0<\cdots<i_r\le n} \mathrm{K}(A(i_0, ..., i_r)). \tag{1.2}$$

Remark 1.1.2. As noted above, Theorem 1.1.1 remains valid if we replace the algebraic K-theory functor K by any localizing invariant $E : \mathrm{Cat}^{\mathrm{ex}} \to \mathcal{T}$ valued in a stable ∞-category $\mathcal{T}$. In fact, the proof of Theorem 3.3.1 does not use any properties specific to K except that K is a localizing invariant of small stable ∞-categories. In this manuscript, we do not require that localizing invariants commute with filtered colimits[3].

For the case of curves, Theorem 1.1.1 is closely related to Weil's description of vector bundles on X. Suppose $X = \operatorname{Spec} R$ for a Dedekind ring R which is not a field. Then, $A(0) = F$ is the field of fractions of R, $A(1) = O$ is the ring of integral adeles $\prod_{\mathfrak{p} \in (\operatorname{Spec} R)_0} R_{\mathfrak{p}}^{\wedge}$ (where the product is taken over maximal ideals of R), and $A(01) = A = F \otimes_R O$ is the ring of finite adeles. Moreover, $A_{\mathrm{red}}^{\bullet}(X)$ takes the form of $F \times O \rightrightarrows A$. Weil's adelic uniformization theorem implies we have an equivalence between the (1-)groupoid $\mathrm{BGL}_r(R)$ of rank r vector bundles on $\operatorname{Spec} R$ and the double quotient groupoid $[\mathrm{GL}_r(F) \backslash \mathrm{GL}_r(A) / \mathrm{GL}_r(O)]$ (see [Gro17, Corollary 3.38 and 3.39] for details and generalizations to Noetherian schemes). On objects, the equivalence sends each isomorphism class of finite projective R-module M of rank r to the double coset represented by $(\phi_\eta|_{F_p^\wedge} \circ \phi_p|_{F_p^\wedge}^{-1})_{p \in X_0} \in \mathrm{GL}_r(A)$, where ϕ_η is a trivialization of M on a nonempty open subset of X (hence gives a trivialization of $F \otimes_R M$) and each ϕ_p is a trivialization of $M_p^\wedge$ at the closed point p of X. In particular, each finite projective R-module is obtained by gluing finite projective (in fact, finite free) modules over F and O which are isomorphic to each other over A after base change. From this, we know there is an equalizer diagram $\pi_0\mathrm{Proj}^{\mathrm{fg}}(R) \to \pi_0\mathrm{Proj}^{\mathrm{fg}}(F) \times \pi_0\mathrm{Proj}^{\mathrm{fg}}(O) \rightrightarrows \pi_0\mathrm{Proj}^{\mathrm{fg}}(A)$ of commutative monoids[4]. After group-completion, we obtain a sequence $\mathrm{K}_0(R) \to \mathrm{K}_0(F) \oplus \mathrm{K}_0(O) \to \mathrm{K}_0(A)$ of abelian groups. Note that in general, this sequence does not realize $\mathrm{K}_0(R)$ as a kernel of the second map, as the group-completion functor (which is a left adjoint functor) does not preserve limits in general. Nonetheless, we can realize this sequence as a part of a long exact sequence through our descent result. By Theorem 1.1.1, we have a pullback square

$$\begin{tikzcd}
\mathrm{K}(R) \arrow[r] \arrow[d] & \mathrm{K}(F) \arrow[d] \\
\mathrm{K}(O) \arrow[r] & \mathrm{K}(A)
\end{tikzcd} \tag{1.3}$$

of spectra, and hence we have a long exact Mayer-Vietoris sequence

$$\cdots \to \mathrm{K}_i(R) \to \mathrm{K}_i(F) \oplus \mathrm{K}_i(O) \to \mathrm{K}_i(A) \to \mathrm{K}_{i-1}(R) \to \cdots \quad (i \in \mathbb{Z})$$

of abelian groups. Around degree $i = 0$, we have an exact sequence $\cdots \to \mathrm{K}_1(A) \to \mathrm{K}_0(R) \to \mathrm{K}_0(F) \oplus \mathrm{K}_0(O) \to \mathrm{K}_0(A) \to 0$ which extends the previous sequence of abelian groups obtained from the Weil uniformization theorem.

Another motivation for our result is the following adelic descent theorem of [Gro17] for perfect modules. Recall that for a Noetherian scheme X, Beilinson's construction indeed provides us the cosimplicial ring $A^{\bullet}(X)$ of adeles on X whose dual normalization is $A_{\mathrm{red}}^{\bullet}(X)$ [Hub91, Proposition 5.1.3].

Theorem 1.1.3 ([Gro17], Theorem 3.1)**.** Let X be a Noetherian scheme. Then, there is an equivalence of symmetric monoidal stable ∞-categories $\mathrm{Perf}(X) \simeq \lim_{[r] \in \Delta} \mathrm{Perf}(A^r(X))$ in $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{perf}})$.

As localizing invariants (and in particular the algebraic K-theory functor K) do not preserve

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[3] See the paragraph below Remark 2.1.5 for relevant definitions.

[4] Again, for each ordinary ring R, $\mathrm{Proj}^{\mathrm{fg}}(R)$ stands for the (nerve of the) category of finite projective R-modules equipped with the monoidal structure given by direct sums. Thus, the set $\pi_0\mathrm{Proj}^{\mathrm{fg}}(R)$ of isomorphism classes of finite projective R-modules admits a commutative monoid structure.

limits, in fact even pullbacks in general, we cannot deduce our descent result for K-theory spectra directly from Theorem 1.1.3. Instead, we follow a strategy which is more suited to investigate descent results for localizing invariants, and independent of the proof of Theorem 1.1.3 given in [Gro17].

We approach Theorem 1.1.1 as follows. Through a comparison between cubical and cosemisimplicial limits (Corollary 2.2.6), we deduce the cosemisimplicial descent (1.1) from the cubical descent (1.2). In order to prove (1.2), we introduce auxiliary stable subcategories $\mathrm{Perf}_{\leq i}(A(T))$ of the ∞-category $\mathrm{Perf}(A(T))$ of perfect modules[5] over the adele ring $A(T)$ for each $0 \leq i \leq n$ and $T \subseteq [n]$ (Definition 3.2.1), and prove that we have exact sequences

$$\mathrm{Perf}_{\leq i-1}(A(T)) \to \mathrm{Perf}_{\leq i}(A(T)) \to \mathrm{Perf}_{\leq i}(A(T \sqcup \{i\}))$$

of small stable ∞-categories for each $T \subseteq [i-1]$ (Proposition 3.2.16). When $i = n$, the image of the second map by K in this exact sequence recovers the n-cubical diagram $T \mapsto \mathrm{K}(A(T))$ of (1.2). For $n = 2$ (i.e., the case of surfaces), this 2-cube is obtained as an image of the right side 2-cube by K in the following diagram of small stable ∞-categories:

$$\begin{tikzcd}
\mathrm{Perf}_{\leq 1}(X) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathrm{Perf}(X) = \mathrm{Perf}_{\leq 2}(X) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathrm{Perf}_{\leq 2}(A(2)) \arrow[dr] \arrow[dd] & \\
& \mathrm{Perf}_{\leq 1}(A(1)) \arrow[rr] \arrow[dd] & & \mathrm{Perf}_{\leq 2}(A(1)) \arrow[rr] \arrow[dd] & & \mathrm{Perf}_{\leq 2}(A(12)) \arrow[dd] \\
\mathrm{Perf}_{\leq 1}(A(0)) \arrow[rr] \arrow[dr] & & \mathrm{Perf}_{\leq 2}(A(0)) \arrow[rr] \arrow[dr] & & \mathrm{Perf}_{\leq 2}(A(02)) \arrow[dr] & \\
& \mathrm{Perf}_{\leq 1}(A(01)) \arrow[rr] & & \mathrm{Perf}_{\leq 2}(A(01)) \arrow[rr] & & \mathrm{Perf}_{\leq 2}(A(012)).
\end{tikzcd} \tag{1.4}$$

After applying K, the four horizontal sequences in the diagram (1.4) become fiber sequences of spectra. Thus, by [Lura, 1.2.4.15] (see Proposition 2.2.2) the n-cube $T \mapsto \mathrm{K}(A(T))$ is a limit diagram precisely when the $(n-1)$-cube $T \mapsto \mathrm{K}(\mathrm{Perf}_{\leq n-1}(A(T)))$ (where $T \subseteq [n-1]$) is a limit diagram. For $n = 2$, this 1-cube is an image of the leftmost square of the diagram (1.4) by K. Using the exact sequences of Proposition 3.2.16, we can repeat this procedure on each i-cube $T \mapsto \mathrm{K}(\mathrm{Perf}_{\leq i}(A(T)))$ for all $0 \leq i \leq n$ until we reach $i = 0$. For our $n = 2$ case, the leftmost square of the diagram (1.4) fits into the following new diagram (as the right side 1-cube) whose rows are exact sequences of small stable ∞-categories:

$$\begin{tikzcd}
\mathrm{Perf}_{\leq 0}(X) \arrow[r] \arrow[d, "i=0"', "\simeq"] & \mathrm{Perf}_{\leq 1}(X) \arrow[r] \arrow[d] & \mathrm{Perf}_{\leq 1}(A(1)) \arrow[d] \\
\mathrm{Perf}_{\leq 0}(A(0)) \arrow[r] & \mathrm{Perf}_{\leq 1}(A(0)) \arrow[r] & \mathrm{Perf}_{\leq 1}(A(01)).
\end{tikzcd} \tag{1.5}$$

Note that after applying the functor K, the right side 1-cube of the diagram (1.5) takes the form of the square (1.3) for curves. By Proposition 3.2.16 and $\mathrm{Perf}_{\leq -1}(X) = 0$ (or by [TT90, Theorem 2.6.3]), the left vertical arrow $\mathrm{Perf}_{\leq 0}(X) \to \mathrm{Perf}_{\leq 0}(A(0))$ of the diagram (1.5) is an equivalence. Thus, the 0-cube $\mathrm{K}(\mathrm{Perf}_{\leq 0}(X)) \to \mathrm{K}(\mathrm{Perf}_{\leq 0}(A(0)))$ is a limit diagram, and we know the 1-cube $T \mapsto \mathrm{K}(\mathrm{Perf}_{\leq 1}(A(T)))$ and the original 2-cube $T \mapsto \mathrm{K}(A(T))$ are limit diagrams.

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[5] Here, we set $\mathrm{Perf}(A(\emptyset)) = \mathrm{Perf}(X)$.

In [TT90], Thomason showed that algebraic K-theory satisfies Zariski descent for qcqs (i.e., quasicompact quasiseparated) schemes through Zariski excision [TT90, Theorem 8.1]. There, he first established exact sequences

$$\mathrm{Perf}_Z(X) \to \mathrm{Perf}(X) \xrightarrow{\jmath^*} \mathrm{Perf}(U)$$

of perfect modules for each quasicompact open embedding $\jmath : U \hookrightarrow X$ with complement $Z = X \backslash U$. Then, he used the equivalence

$$\pi^* : \mathrm{Perf}_Z(X) \simeq \mathrm{Perf}_{\pi^{-1}Z}(Y)$$

for each flat morphism $Y \xrightarrow{\pi} X$ of qcqs schemes which is an isomorphism over Z in order to prove the equivalence $\mathrm{K}(X) \simeq \mathrm{K}(Y) \times_{\mathrm{K}(\pi^{-1}U)} \mathrm{K}(U)$ as in the situation of diagram (1.5). Our proof of Theorem 1.1.1 applies this Thomason-Trobaugh argument to cubical diagrams of perfect modules at each induction step to deduce the descent result. In a sense, we are solving a version of formal glueing problem at each step until we cover all flags and reach the full adelic descent. As already noted, Thomason's approach proves that any localizing invariants satisfy Nisnevich excision, and hence satisfy Nisnevich descent for qcqs schemes ([Lurs, 3.7.5.1], see also [CM21, Proposition 5.15] for qcqs spectral algebraic spaces). Theorem 1.1.1, although not a descent result for a particular Grothendieck topology, provides a descent result for localizing invariants for an adelic resolution $A^\bullet_{\mathrm{red}}(X)$ of each Noetherian scheme X of finite Krull dimension, allowing one to understand $\mathrm{K}(X)$ via K-theory of adele rings and maps between them. In particular, one can view algebraic K-theory (or localizing invariant) of X as an object right Kan extended from those of rings of the form of adele rings appearing in the adelic resolution.

1.2 Algebraic K-theory of Cartier smooth rings

Fix a prime p. Thanks to Quillen's classical computation [Qui72] of algebraic K-theory of finite fields, we have the following concrete description of algebraic K-theory groups in all degrees:

$$\mathrm{K}(\mathbb{F}_p) : \begin{cases} \mathrm{K}_{<0}(\mathbb{F}_p) = 0, \\ \mathrm{K}_0(\mathbb{F}_p) = \mathbb{Z}, \\ \mathrm{K}_{2i}(\mathbb{F}_p) = 0, \\ \mathrm{K}_{2i-1}(\mathbb{F}_p) = \mathbb{Z}/(p^i - 1). \end{cases}$$

Thus, we have vanishing of even (positive) degrees and p-torson freeness in all degrees. In particular, $\mathrm{K}(\mathbb{F}_p, \mathbb{Z}_p) := \mathrm{K}(\mathbb{F}_p)^{\wedge_p} \simeq H\mathbb{Z}_p$ as an $\mathbb{E}_\infty$-ring. Note that while the input $\mathbb{F}_p$ is in characteristic p (in other words $H\mathbb{F}_p$ is of chromatic height -1, as $H\mathbb{F}_p \otimes \mathbb{S}[1/p] \simeq 0$), after p-completion the output algebraic K-theory looks like a lifting of a characteristic p object to the mixed characteristic $(0, p)$ (and $\mathrm{K}(\mathbb{F}_p)^{\wedge_p}$ is of chromatic height 0[6]). One might hope that, although computing (integral) algebraic K-groups would generally be difficult, describing p-adic K-theory of characteristic p rings akin to the consequence $\mathrm{K}(\mathbb{F}_p)^{\wedge_p} \simeq H\mathbb{Z}_p$ of Quillen's theorem might be more approachable. In fact,

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[6] More precisely, $H\mathbb{Z}_p \otimes \mathbb{S}[1/p] \not\simeq 0$ while $H\mathbb{Z}_p \otimes T(1) \simeq 0$, where the last equivalence holds for any Eilenberg-MacLane spectra. Hence, the equivalence $K(\mathbb{F}_p, \mathbb{Z}_p) \simeq H\mathbb{Z}_p$ can be seen as a baby example of chromatic redshift phenomenon for algebraic K-theory.

Geisser and Levine showed the following generalization of (the aforementioned p-adic consequence of) Quillen's computation in the smooth case.

Theorem 1.2.1 ([GL00])**.** Let R be a local smooth algebra over a perfect field k of characteristic p. Then, for each $i \geq 0$ and $r \geq 1$, the followings hold:
(1) $\mathrm{K}_i(R)$ is p-torsion free.
(2) $\mathrm{K}_i(R)/p^r \simeq W_r\Omega^i_{R,\log}$.

Here, $W_r\Omega^i_{R,\log}$ stands for the logarithmic part of the i-th de Rham-Witt group of length r [Ill79]. Under the p-torsion freeness condition of (1), the condition (2) amounts to the equivalence $\mathrm{K}_i(R, \mathbb{Z}_p) \simeq W\Omega^i_{R,\log}$ (where $\mathrm{K}_i(R, \mathbb{Z}_p) = \pi_i(\mathrm{K}(R)^{\wedge_p})$ as usual). As the de Rham-Witt complex $W\Omega^*_R$ provides a lifting of the de Rham complex Ω^*_R of R to mixed characteristic $(0, p)$, one observes that the pattern observed in the case of $R = \mathbb{F}_p = k$ remains valid.

Example 1.2.2. Let $R = \mathbb{F}_p$. In this case, the de Rham-Witt complex is given by $W\Omega^*_{\mathbb{F}_p} \simeq \widehat{\Omega}^*_{\mathbb{Z}_p}$, the p-completed de Rham complex of $W(R) = \mathbb{Z}_p$. Moreover, since $L^{\wedge_p}_{\mathbb{Z}_p/\mathbb{Z}} \simeq 0$, we know $W\Omega^*_{\mathbb{F}_p} \simeq \widehat{\Omega}^*_{\mathbb{Z}_p} \simeq \mathbb{Z}_p$. Thus, $W\Omega^0_{\mathbb{F}_p,\log} = \mathbb{Z}_p$ and $W\Omega^{i>0}_{\mathbb{F}_p,\log} = 0$, and Geisser-Levine theorem recovers $\mathrm{K}(\mathbb{F}_p, \mathbb{Z}_p) \simeq H\mathbb{Z}_p$. Cf. Lemma 4.4.7 for relevant details and generalizations.

Remark 1.2.3. Note that in the Geisser-Levine description of algebraic K-groups, both sides of the equivalence commute with filtered colimits of rings. Thus, Theorem 1.2.1 remains valid for local ind-smooth k-algebras. In particular, Theorem 1.2.1 applies to regular local rings in characteristic p, since these rings are ind-smooth over $\mathbb{F}_p$ by Néron-Popescu desingularization.

Another class of rings which contain characteristic p finite fields is that of perfect rings. As arguments of Example 1.2.2 look compatible with perfect rings (more or less trivially, up to verifying that the de Rham-Witt complex should be the ring of p-typical Witt vectors), or more interestingly with smooth algebras over perfect rings (again once de Rham-Witt complex of such rings are properly defined), it is tempting to speculate that Geisser-Levine description should remain true for smooth algebras over perfect rings. In [KM21], Kelly and Morrow proved that that is indeed the case. In fact, they introduced the notion of Cartier smoothness for characteristic p rings, and proved that Geisser-Levine description holds for local Cartier smooth rings. Our second main goal in this manuscript is to explain an alternative proof of the following theorem of Kelly and Morrow:

Theorem 1.2.4 ([KM21])**.** Let S be a local Cartier smooth algebra over $\mathbb{F}_p$. Then, for each $i \geq 0$ and $r \geq 1$, the followings hold:
(1) $\mathrm{K}_i(S)$ is p-torsion free.
(2) $\mathrm{K}_i(S)/p^r \simeq W_r\Omega^i_{S,\log}$.

Contemporaneous with [KM21], Cartier smoothness was also studied in [KST21]. We will review the notion of Cartier smoothness and some of its characterizations in Chapter 4. Here, let us be contented with some archetypal examples of such rings.

Example 1.2.5. The following class of rings are Cartier smooth:
(1) (ind-)smooth algebras over a perfect field k of characteristic p.
(2) perfect rings.
(3) (ind-)smooth algebras over a perfect ring.
(4) filtered colimits and localizations of Cartier smooth rings.

(5) (Néron-Popescu) regular Noetherian rings over $\mathbb{F}_p$.

(6) (Gabber [KST21, App. A]) valuation rings over $\mathbb{F}_p$.

Remark 1.2.6. (1) A Noetherian $\mathbb{F}_p$-algebra is Cartier smooth if and only if it is regular, cf. [BM23, Th. 4.15] (in the Cartier smooth case, the main technical ingredient is in [BM23, Prop. 4.19]). Thus, Cartier smoothness can be regarded as a non-Noetherian generalization of regularity in characteristic p.

(2) Despite (1), note that Cartier smooth rings in general have non-vanishing negative K-groups, e.g., perfect rings might have nonzero negative K-groups. In the setting of Geisser-Levine Theorem 1.2.1, ind-smooth k-algebras of course have vanishing negative K-groups, so describing the connective part was already enough. Theorem 1.2.4 gives a description of the connective p-adic K-theory for Cartier smooth rings, but says nothing about the negative K-group.

(3) Both [KM21] and our approach to Theorem 1.2.4 in Chapter 4 relies on the trace method in the p-adic setting, which by its very nature can only control the connective part of algebraic K-theory. This might be the main reason for the limitation described in (2).

Remark 1.2.7. For general (p-adic) situation, [BM23] introduced the notion of F-smoothness for any p-quasisyntomic rings in order to capture an absolute version of p-complete smoothness[7]. This notion precisely generalizes the notion of Cartier smoothness in characteristic p; a quasisyntomic $\mathbb{F}_p$-algebra is Cartier smooth if and only if it is F-smooth [BM23, Prop. 4.14]. Moreover, they computed (mod-p^r) syntomic complexes of p-torsion free F-smooth rings in terms of truncated nearby cycles (which, as usual, is the étale Tate twist on the locus where p acts invertibly) modified by the image of the symbol map (which is the only visible part on the vanishing locus of p, exactly as in Corollary 4.2.1) at the top cohomological degree [BM23, Th. 1.8]; this is a vast p-adic generalization of our computation of syntomic complexes in Corollary 4.2.1 for the characteristic p case, albeit exact methods of computations are different.

In [KM21], Theorem 1.2.4 was proved by reduction to Theorem 1.2.1 of Geisser-Levine through trace methods. More precisely, they first handle homotopy groups $\pi_*(\mathrm{TC}(S)/p^r)$ via computations of topological restriction theory groups $\pi_*(\mathrm{TR}(S)/p^r)$ and Frobenius on them, using the classical description of TC/p^r as a Frobenius fixed point of TR/p^r. Then, they use CMM rigidity [CMM21] in trace methods to ensure that the difference between algebraic K-theory groups and logarithmic de Rham-Witt groups for local Cartier smooth rings, where the latter is related to TC/p^r by the previous computation, is exactly the same as the difference between them in the case of ind-smooth algebras, which is nothing but zero due to the Geisser-Levine theorem.

Our alternative approach to Theorem 1.2.4 in chapter 4 follows the same strategy of reducing the problem to the case of ind-smooth algebras through trace methods. However, our way of describing TC and treating the entire situation p-completely requires us to employ different techniques and offers us additional useful information complementary to [KM21]. First, we approach topological cyclic homology of Cartier smooth rings $\mathrm{TC}(S, \mathbb{Z}_p)$ through its motivic filtration [BMS19][8]. More precisely, we compute syntomic cohomology complexes $\mathbb{Z}_p(n)(S)$ of Cartier smooth rings S, which are associated graded pieces of the motivic filtration on $\mathrm{TC}(S, \mathbb{Z}_p)$ and hence act as p-adic (pro-

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[7] Roughly speaking, the condition of F-smoothness is designed to enforce the Segal conjecture true under the presense of finiteness of conjugate filtration.

[8] We note that it was already indicated in [KM21] that it would be possible to use the motivic filtration to reformulate some of their computations.

)étale motivic cohomology complexes of $\operatorname{Spec} S$. It turns out syntomic cohomology complexes are precisely given by logarithmic de Rham-Witt sheaves up to shift, and this gives a description of $\mathrm{TC}(-, \mathbb{Z}_p)$ in terms of a pro-étale sheaf on $\operatorname{Spec} S$, cf. Corollary 4.2.1 and Corollary 4.3.1. Then, we use pro-étale descent spectral sequence to obtain natural exact sequences relating homotopy groups of p-adic TC and syntomic cohomology complexes. These exact sequences, for mod-p^r coefficients, also appeared in [KM21] due to a different, purely computational reason; here, we realize them in a more conceptual context of descent spectral sequences. Finally, we use CMM rigidity as well as AMMN rigidity property [AMMN22] for syntomic cohomology complexes to compute $\pi_i \mathrm{K}_{\geq 0}(S, \mathbb{Z}_p)$ in Theorem 4.4.3, which implies Theorem 1.2.4. Note that the key point of the computation of syntomic complexes of Cartier smooth rings given here is that the prismatic cohomology complexes $\mathbb{\Delta}_S$ of those rings are precisely given by their de Rham-Witt complexes together with the data of Nygaard filtrations, cf. Proposition 4.1.7 and Proposition 4.5.4. We will briefly remark on this and relevant consequences involving p-torsion free lifts of Cartier smooth rings (which are instances of mixed characteristic F-smooth rings) in section 4.5.

Chapter 2

Categorical preliminaries

In this chapter we review and explain necessary preliminaries on ∞-categories. In 2.1 we review the notion of exact sequences of stable ∞-categories and localizing invariants following [BGT13]. In 2.2 we study cubical and cosemisimplicial diagrams and their limits through Cartesian fibrations. As we use the language of sheaves of module spectra following [Lurs], we briefly recall some of their theory in 2.3. Finally, we fix our conventions on filtered objects and explain basic properties of associated graded functor and spectral sequence constructions in 2.4.

2.1 Stable ∞-categories and localizing invariants

Let $\mathrm{Cat}^{\mathrm{ex}}$ be the ∞-category of small stable ∞-categories and exact functors, and let $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be the ∞-category of presentable stable ∞-categories and left adjoint (i.e., colimit preserving) functors. The ind-completion construction $\mathrm{Ind} : \mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ (as in [Lur09, 5.3.5]) relates these two categories, and factors through the full subcategory of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ spanned by compactly generated stable ∞-categories. Let $\mathrm{Cat}^{\mathrm{perf}}$ be the full subcategory of $\mathrm{Cat}^{\mathrm{ex}}$ consisting of idempotent complete small stable ∞-categories. Then, the construction $(\mathrm{Ind}(-))^{\omega} : \mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Cat}^{\mathrm{perf}}$ is well-defined and behaves as indempotent-completion (i.e., provides a left adjoint to the inclusion functor $\mathrm{Cat}^{\mathrm{perf}} \subseteq \mathrm{Cat}^{\mathrm{ex}}$). In fact, Ind induces an equivalence $\mathrm{Ind} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$ from $\mathrm{Cat}^{\mathrm{perf}}$ onto the ∞-category $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$ of compactly generated (presentable) stable ∞-categories and compact left adjoint functors (i.e., those preserving compact objects, or equivalently those with filtered-colimit preserving right adjoints [Lur09, 5.5.7.2]), whose inverse is given by the functor $(-)^{\omega}$ taking (ω-)compact objects of each ∞-categories. Recall that both $\mathrm{Cat}^{\mathrm{ex}}$ and $\mathrm{Cat}^{\mathrm{perf}}$ admit all (small) limits and colimits, and the inclusion $\mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Cat}_{\infty}$ preserves limits and filtered colimits ([Lura, 1.1.4] and [BGT13, 4.25]). Likewise, recall that $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ has all (small) limits and colimits, and the inclusion $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}} \to \widehat{\mathrm{Cat}}_{\infty}$ preserves limits and (all) colimits ([Mat16, Proposition 2.4 and its proof], see also [Lur09, 5.5.3.13, 5.5.3.18]). Note that the inclusion $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega} \to \widehat{\mathrm{Cat}}_{\infty}$, although preserving colimits [Lur09, 5.5.7.6, 5.5.7.7], does not preserve limits in general; as noted in [Kel94, Section 2], fiber products of compactly generated presentable stable ∞-categories may not be compactly generated.

Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be a sequence in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Recall that the sequence is called *exact* if the composite functor is zero, $\mathcal{A} \to \mathcal{B}$ is fully faithful, and the induced functor $\mathcal{B}/\mathcal{A} \to \mathcal{C}$ is an equivalence. Here, $\mathcal{B}/\mathcal{A}$ is a cofiber of the functor $\mathcal{A} \to \mathcal{B}$ (i.e., a pushout of functors $\mathcal{A} \to \mathcal{B}$ and $\mathcal{A} \to 0$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$), which can be described via Bousfield localization in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. In fact, the homotopy category of $\mathcal{B}/\mathcal{A}$ is equivalent to the Verdier quotient of the inclusion $\mathrm{h}\mathcal{A} \to \mathrm{h}\mathcal{B}$, and the sequence is exact precisely if the corresponding sequence of homotopy categories is an exact sequence of triangulated categories [BGT13, 5.9–5.11]. Now, we call a sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in $\mathrm{Cat}^{\mathrm{ex}}$ *exact* if the resulting sequence $\mathrm{Ind}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{B}) \to \mathrm{Ind}(\mathcal{C})$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is exact in the previous sense. This is equivalent to the condition that the composite functor is zero, $\mathcal{A} \to \mathcal{B}$ is fully faithful, and the induced functor $\mathcal{B}/\mathcal{A} \to \mathcal{C}$ is an equivalence *after* idempotent completion [BGT13, 5.13]. One can describe the cofiber $\mathcal{B}/\mathcal{A}$ in $\mathrm{Cat}^{\mathrm{ex}}$ intrinsically (i.e., without embedding into $\mathrm{Ind}(\mathcal{B})$) through Dwyer-Kan localization in a way compatible with Bousfield localization in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, and still $\mathrm{h}(\mathcal{B}/\mathcal{A}) \simeq \mathrm{h}\mathcal{B}/\mathrm{h}\mathcal{A}$ holds. In fact, one has a description of the mapping space as a filtered colimit $\mathrm{Map}_{\mathcal{B}/\mathcal{A}}(\overline{b}, \overline{c}) \simeq \mathrm{colim}_{a \in \mathcal{A}_{/c}} \mathrm{Map}_{\mathcal{B}}(b, \mathrm{cof}(a \to c))$, where $\overline{b}$ and $\overline{c}$ denote images of $b, c \in \mathcal{B}$ respectively [NS18, I.3.3].

Let us briefly explain the notion of fiber (or kernel) categories of exact functors between stable ∞-categories, which will be useful in the description of split-exact sequences.

Proposition 2.1.1. Let $\mathcal{C}$ be a pointed ∞-category, and let $\mathcal{B} \xrightarrow{q} \mathcal{C}$ be a functor from an ∞-category $\mathcal{B}$. Fix any zero object $0 : \Delta^0 \to \mathcal{C}$ of $\mathcal{C}$. Then, the ∞-category $\mathrm{fib}(q) = \Delta^0 \times_{\mathcal{C}} \mathcal{B}$ is equivalent to the full subcategory of $\mathcal{B}$ generated by the objects $\{b \in \mathcal{B} \mid q(b) \simeq 0\}$.

Proof. First, let us describe $\mathrm{fib}(q)$ explicitly in terms of quasicategories. Let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ generated by zero objects.

Lemma 2.1.2. The inverse image simplicial set $q^{-1}(\mathcal{C}') \hookrightarrow \mathcal{B}$ is a quasicategory equivalent to a pullback $\mathrm{fib}(q) = \Delta^0 \times_{\mathcal{C}} \mathcal{B}$ of ∞-categories.

Proof. As $\mathcal{C}'$ is a contractible Kan complex [Lur09, 1.2.12.9], the canonical map $\mathcal{C}' \to \Delta^0$ is a categorical equivalence, and hence its section $\Delta^0 \to \mathcal{C}'$ given by 0 is an equivalence. This induces an equivalence $\mathrm{fib}(q) \simeq \mathcal{C}' \times_{\mathcal{C}} \mathcal{B}$, and hence to prove the claim, it suffices to check the latter can be computed by a simplicial set $q^{-1}(\mathcal{C}')$. The inclusion $\mathcal{C}' \overset{\imath}{\hookrightarrow} \mathcal{C}$ of a subcategory is by definition an inner fibration, and it is also an isofibration, since any isomorphism $\imath(0') \to 0''$ in $\mathcal{C}$ from a zero object comes from an isomorphism $0' \to 0''$ in $\mathcal{C}'$. Hence, $\mathcal{C}' \overset{\imath}{\hookrightarrow} \mathcal{C}$ is a categorical fibration, and the pullback ∞-category $\mathcal{C}' \times_{\mathcal{C}} \mathcal{B}$ is computed by the inverse image quasicategory $q^{-1}(\mathcal{C}') \hookrightarrow \mathcal{B}$. □

By Lemma 2.1.2, it suffices to check that the quasicategory $q^{-1}(\mathcal{C}')$ is the full subcategory of $\mathcal{B}$ determined by the set of objects $\{b \in \mathcal{B} \mid q(b) \simeq 0\}$. The inverse image quasicategory $q^{-1}(\mathcal{C}')$ is the pullback of simplicial sets viewed as set-valued presheaves on Δ. As $\mathcal{C}' \hookrightarrow \mathcal{C}$ is a full subcategory, it is a full simplicial subset [Ker23, tag 01CU], and hence its inverse image $q^{-1}(\mathcal{C}') \hookrightarrow \mathcal{B}$ is a full simplicial subset. As the inclusion is an inner fibration, it is an embedding of a full subcategory. The vertex of $q^{-1}(\mathcal{C}')$ is precisely $\{b \in \mathcal{B} \mid q(b) \simeq 0\}$, and since there should be a unique full simplicial subset (in this case, automatically a full subcategory) of $\mathcal{B}$ with the given vertex set [Ker23, tag 01CV], this concludes the proof. □

In particular, for the case of exact functors $\mathcal{B} \xrightarrow{q} \mathcal{C}$ between stable ∞-categories, we call $\mathrm{fib}(q)$ (for any choice of a zero object) a *fiber*, or even a *kernel* of q. Following the description of Proposition 2.1.1, we identify $\mathrm{fib}(q)$ with the stable subcategory of $\mathcal{B}$ generated by $\{b \in \mathcal{B} \mid q(b) \simeq 0\}$. Note that this description is independent of the choice of zero objects or choice of isomorphisms between zero objects, which is not immediate from the definition of $\mathrm{fib}(q)$ as a pullback ∞-category.

An important class of exact sequences is provided by semiorthogonal decompositions of stable ∞-categories. Given a stable ∞-category $\mathcal{B}$ and its stable subcategory $\mathcal{C}$, we denote $\mathcal{C}^{\perp}$ as the full (stable) subcategory of $\mathcal{B}$ generated by $b \in \mathcal{B}$ with $\mathrm{Map}_{\mathcal{B}}(c, b) \simeq *$ for all $c \in \mathcal{C}$, and similarly denote ${}^{\perp}\mathcal{C}$ as the stable subcategory of $\mathcal{B}$ generated by $b \in \mathcal{B}$ with $\mathrm{Map}_{\mathcal{B}}(b, c) \simeq *$ for all $c \in \mathcal{C}$.

Proposition 2.1.3. Let $\mathcal{A} \to \mathcal{B} \xrightarrow{q} \mathcal{C}$ be a sequence in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. If q admits a fully faithful right adjoint and $\mathcal{A} \to \mathcal{B}$ induces an equivalence between $\mathcal{A}$ and $\mathrm{fib}(q)$, then the sequence is exact in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

Proof. The only nontrivial part to check is that q induces $\mathcal{B}/\mathcal{A} \simeq \mathcal{C}$. By assumption, we can identify the fully faithful embedding $\mathcal{A} \to \mathcal{B}$ as $\mathcal{A} = \mathrm{fib}(q) \hookrightarrow \mathcal{B}$. By [BGT13, 5.6], cofiber $\mathcal{B}/\mathcal{A}$ is equivalent to the Bousfield localization of $\mathcal{B}$ at morphisms whose cofibers are in the essential image of $\mathcal{A}$, i.e., $\mathcal{B}/\mathcal{A}$ is equivalent to the stable subcategory of $\mathcal{B}$ generated by objects $b \in \mathcal{B}$ such that $\mathrm{Map}_{\mathcal{B}}(a, b) \simeq *$ for all $a \in \mathcal{A}$. If we denote a right adjoint of $\mathcal{A} \hookrightarrow \mathcal{B}$ by g', then (from $\mathrm{Map}_{\mathcal{B}}(a, b) \simeq \mathrm{Map}_{\mathcal{A}}(a, g'(b))$ for all $a \in \mathcal{A}$) we have $\mathcal{B}/\mathcal{A} \simeq \mathrm{fib}(g')$. Now, via the fully faithful right adjoint $\mathcal{C} \hookrightarrow \mathcal{B}$ of q, let us identify $\mathcal{C}$ as a stable subcategory of $\mathcal{B}$. Then by adjunction $\mathcal{A} = \mathrm{fib}(q) = {}^{\perp}\mathcal{C}$. On the other hand, again using the right adjoint g' one immediately computes $({}^{\perp}\mathcal{C})^{\perp} = \mathrm{fib}(g')$. From $\mathcal{C} = ({}^{\perp}\mathcal{C})^{\perp}$ [Lurs, 7.2.1.8], one has $\mathcal{C} = \mathrm{fib}(g') \simeq \mathcal{B}/\mathcal{A}$. □

We call exact sequences of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ satisfying the conditions of Propostion 2.1.3 *split-exact*. The point of Proposition 2.1.3 is that the conditions of [BGT13, 5.18] (in $\mathrm{Cat}^{\mathrm{ex}(\kappa)}$, with *a priori* given exactness assumption) automatically ensure the sequence is exact in the case of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ (for instance, see [Tam18, Recollection 9] for the statement). By our discussions on fiber and cofiber ∞-categories, a split-exact sequence $\mathcal{A} \to \mathcal{B} \xrightarrow{q} \mathcal{C}$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is simultaneously a fiber and a cofiber sequence in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Also, note that given that the stable subcategory $\mathcal{C} \hookrightarrow \mathcal{B}$ (via right adjoint) is closed under equivalences, the condition of Proposition 2.1.3 is precisely saying we have a semiorthogonal decomposition of $\mathcal{B}$ of the form $(\mathcal{A}, \mathcal{C})$ [Lurs, 7.2.1.7].

Example 2.1.4. Let $\mathcal{B}$ be a small stable ∞-category and let $\mathcal{A}$ be its stable subcategory. By [NS18, I.3.5], the resulting sequence $\mathrm{Ind}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{B}) \to \mathrm{Ind}(\mathcal{B}/\mathcal{A})$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ exhibits $\mathrm{Ind}(\mathcal{A})$ as a fiber of the second compact functor $\mathrm{Ind}(\mathcal{B}) \to \mathrm{Ind}(\mathcal{B}/\mathcal{A})$, and this functor admits a fully faithful right adjoint. Hence by Proposition 2.1.3, the sequence is a split-exact sequence in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Moreover, the right adjoint $\mathrm{Ind}(\mathcal{B}/\mathcal{C}) \to \mathrm{Ind}(\mathcal{B})$ is also in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ (i.e., preserves all small colimits), and corresponds to the Yoneda functor $\mathcal{B}/\mathcal{A} \to \mathrm{Ind}(\mathcal{B})$ sending the image of $b \in \mathcal{B}$ in $\mathcal{B}/\mathcal{A}$ to the filtered colimit $\mathrm{colim}_{a \in \mathcal{A}_{/b}} \mathrm{cof}(a \to b)$ in $\mathrm{Ind}(\mathcal{B})$ (i.e., $\mathrm{colim}_{a \in \mathcal{A}_{/b}} \mathrm{Map}_{\mathcal{B}}(-, \mathrm{cof}(a \to b)) \in \mathcal{P}(\mathcal{B})$). This immediately follows from the description of the mapping space of $\mathcal{B}/\mathcal{A}$, as well as from the fact that $\mathrm{Ind}(\mathcal{B}/\mathcal{A}) \to \mathrm{Ind}(\mathcal{B})$ is already exact, so it suffices to consider the case of filtered colimits when verifying that the functor commutes with all small colimits. In particular, the unit map for the adjunction associated with the second compact functor on $b \in \mathcal{B}$ takes the form $b \to \mathrm{colim}_{a \in \mathcal{A}_{/b}} \mathrm{cof}(a \to b)$.

Remark 2.1.5. Suppose we are given a split exact sequence $\mathcal{K} \to \mathcal{B} \xrightarrow{\jmath^*} \mathcal{C}$ of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, with $\mathcal{K}$ given as a stable subcategory of $\mathcal{B}$ and $\jmath^* : \mathcal{B} \to \mathcal{C}$ be in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$ (i.e., a compact functor). Let $\imath^! : \mathcal{B} \to \mathcal{K}$ and $\jmath_* : \mathcal{C} \to \mathcal{B}$ be right adjoints of the functors $\mathcal{K} \hookrightarrow \mathcal{B}$ and $\jmath^* : \mathcal{B} \to \mathcal{C}$ respectively.
(1) We have fiber sequences $\imath^! b \to b \to \jmath_* \jmath^* b$ in $\mathcal{B}$ fuctorial on $b \in \mathcal{B}$ [Lurs, 7.2.0.2]. Also note that $\imath^!$ commutes with filtered colimits, as it is equivalent to a fiber of the unit map $id \to \jmath_* \jmath^*$, whose source and target functors commute with filtered colimits due to the compactness assumption on $\jmath^*$.
(2) Suppose we have a stable subcategory $\mathcal{A} \subseteq \mathcal{K}$ closed under filtered colimits and suppose $\imath^!$ maps

compact objects $\mathcal{B}^\omega$ into $\mathcal{A}$. Then, $\imath^!$ induces an equivalence $\mathcal{K} \to \mathcal{A}$, with an inverse given by inclusion $\mathcal{A} \subseteq \mathcal{K}$. In fact, by assumption $\imath^!$ induces $\mathcal{B} \simeq \mathrm{Ind}(\mathcal{B}^\omega) \to \mathcal{A}$, and satisfies $a \simeq \imath^! a$ for $a \in \mathcal{A}$ by restriction of the unit map (which is an equivalence) on $\mathcal{A}$. For $k \in \mathcal{K} = \mathrm{fib}(\jmath^*)$, the canonical fiber sequence $\imath^! k \to k \to \jmath_* \jmath^* k$ has zero cofiber part, and hence the counit also induces an equivalence $\imath^! k \simeq k$ for $k \in \mathcal{K}$.

We finally recall the notion of localizing invariants in stable setting.

Definition 2.1.6. A functor $E : \mathrm{Cat}^{\mathrm{ex}} \to \mathcal{T}$ defined on small stable ∞-categories and valued in a stable ∞-category $\mathcal{T}$ is called *localizing* if it factors through the idempotent-completion $\mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Cat}^{\mathrm{perf}}$ and sends exact sequences of $\mathrm{Cat}^{\mathrm{ex}}$ to fiber sequences of $\mathcal{T}$.

Archetypal examples are the nonconnective K-theory functor $\mathrm{K} : \mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Sp}$ of Bass-Thomason and various functors related to it via trace maps, e.g., topological Hochschild homology THH and topological cyclic homology TC. These are all valued in the ∞-category Sp of spectra, although $\mathcal{T}$ might be any stable ∞-category in theory—for instance THH is canonically valued in $\mathcal{T} = \mathrm{CycSp}$ and is a localizing invariant valued in $\mathcal{T}$.

A localizing invariant E is called *finitary* if it commutes with filtered colimits—algebraic K-theory functor K and THH are standard examples of finitary localizing invariants, while TC is not finitary. Unless specified as finitary, localizing invariants in this manuscript are *not* assumed to be compatible with filtered colimits.

An important characterization of the K-theory functor is given by the corepresentability result of [BGT13, 9.8]. There is a stable presentable ∞-category $\mathcal{M}_{\mathrm{loc}} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, the presentable stable ∞-category of *noncommutative motives*, and a finitary localizing invariant $[-]_{\mathrm{loc}} : \mathrm{Cat}^{\mathrm{ex}} \to \mathcal{M}_{\mathrm{loc}}$ into it which is universal in the sense that any finitary localizing invariants into any $\mathcal{T} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ uniquely (up to homotopy) factor through $[-]_{\mathrm{loc}}$, i.e., there is an equivalence $\mathrm{Fun}_{\mathrm{loc}}(\mathrm{Cat}^{\mathrm{ex}}, \mathcal{T}) \xleftarrow{-\circ[-]_{\mathrm{loc}}} \mathrm{Fun}^{\mathrm{L}}(\mathcal{M}_{\mathrm{loc}}, \mathcal{T})$. Via this equivalence, K is described as $\mathrm{K}(-) \simeq \underline{\mathrm{Map}}_{\mathcal{M}_{\mathrm{loc}}}([\mathrm{Perf}(\mathbb{S})]_{\mathrm{loc}}, [-]_{\mathrm{loc}})$. More generally, mapping spectra $\underline{\mathrm{Map}}_{\mathcal{M}_{\mathrm{loc}}}([\mathcal{C}]_{\mathrm{loc}}, [\mathcal{D}]_{\mathrm{loc}})$ in $\mathcal{M}_{\mathrm{loc}}$ from a smooth proper $\mathcal{C} \in \mathrm{Cat}^{\mathrm{ex}}$ (e.g., $\mathcal{C} = \mathrm{Perf}(\mathbb{S})$) can be expressed as a K-theory spectrum (see [BGT13, 9.36] and for the additive version see [BGT13, 9.9]), and this often enables one to extend results about K-theory to results for localizing invariants in general.

Construction 2.1.7. Let us briefly review how the cyclotomic trace map $\mathrm{tr} : \mathrm{K} \to \mathrm{TC}$ can be constructed out of THH. Recall that the topological Hochschild homology construction defines a localizing invariant $\mathrm{THH} : \mathrm{Cat}^{\mathrm{ex}} \to \mathrm{CycSp}$ valued in the ∞-category of cyclotomic spectra CycSp [NS18]. For instance, on the level of $\mathbb{E}_1$-rings, $\mathrm{THH}(A)$ is realized as a geometric realization of the cyclic spectrum $(\cdots A \otimes A \otimes A \rightrightarrows\!\!\!\!\to A \otimes A \rightrightarrows A)$, where each of the term $A^{\otimes n}$ is equipped with the natural action of the finite cyclic group C_n. Let $\mathrm{tr} : \mathcal{M}_{\mathrm{loc}} \to \mathrm{CycSp}$ be a functor in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ corresponding to the localizing invariant THH. It induces a map of spectra $\mathrm{K}(\mathcal{C}) \simeq \underline{\mathrm{Map}}_{\mathcal{M}_{\mathrm{loc}}}([\mathrm{Perf}(\mathbb{S})]_{\mathrm{loc}}, [\mathcal{C}]_{\mathrm{loc}}) \xrightarrow{\mathrm{tr}} \underline{\mathrm{Map}}_{\mathrm{CycSp}}(\mathrm{THH}(\mathbb{S}), \mathrm{THH}(\mathcal{C}))$ natural on $\mathcal{C} \in \mathrm{Cat}^{\mathrm{ex}}$; this precisely defines the cyclotomic trace map $\mathrm{tr} : \mathrm{K}(\mathcal{C}) \to \mathrm{TC}(\mathcal{C})$ as a morphism of localizing invariants. Note that postcomposing $\underline{\mathrm{Map}}_{\mathrm{CycSp}}(\mathrm{THH}(\mathbb{S}), \mathrm{THH}(\mathcal{C})) \xrightarrow{\mathrm{forget}} \underline{\mathrm{Map}}_{\mathrm{Sp}}(\mathbb{S}, \mathrm{THH}(\mathcal{C}))$ recovers the Dennis trace map $\mathrm{K}(\mathcal{C}) \to \mathrm{THH}(\mathcal{C})$.

Remark 2.1.8. Let us briefly remark on the question of commuting pullbacks with localizing invariants. Generally, localizing invariants (in particular, algebraic K-theory) do not commute with limits as a functor from $\mathbb{E}_1$-rings, which of course is partly responsible for difficulties and importance of descent questions. Often such questions can be reduced to the case of finite limits or even pullbacks of $\mathbb{E}_1$-rings, which again localizing invariants do not preserve in general. Consider the square of ordinary commutative rings

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ B \arrow[r] & B'. \end{tikzcd} \tag{2.1}$$

We have already seen two cases in which any localizing invariants map the square (2.1) to a pullback square in the target stable ∞-category. First is the case of affine Nisnevich square; take $A \to B$ to be an étale map which induces an isomorphism $A/(t) \to B/(t)$ for some $t \in A$, take $A \to A'$ to be the localization map $A \to A[1/t]$, and take $B \to B'$ to be the cobase change $B \to B[1/t]$ making the diagram a pushout square of $\mathbb{E}_\infty$-rings. From the exactness of the sequence $0 \to A \to B \oplus A[1/t] \to B[1/t] \to 0$, we know (2.1) is a pullback square of $\mathbb{E}_1$-rings. By Thomason-Trobaugh argument as reviewed in 1.1, localizing invariants map the square (2.1) to a pullback square; by Morel-Voevodsky this affine Nisnevich excision implies that localizing invariants satisfy Nisnevich descent. The other case is that of Noetherian formal glueing square; take $A \to A'$ to be the localization map $A \to A[1/t]$ for some $t \in A$ from a Noetherian ring A, take $A \to B$ to be the completion map $A \to A^{\wedge f}$, and take $B \to B'$ to be the cobase change $A^{\wedge f} \to A^{\wedge f}[1/f]$. The arguments we used in the previous case applies *ad verbum*, and we again see the image of (2.1) by any localizing invariant is a pullback diagram. Note that one can use that $A \to A[1/f]$ is Tor-unital and appeal to [Tam18, Th. 28] (or [LT19, Cor. 1.4]) to deduce the same conclusion.

On the other hand, consider the case of Milnor square; take $A \to A'$ and $B \to B'$ to be surjections, and take $A \to B$ to be a map sending the kernel of $A \to A'$ isomorphically onto the kernel of $B \to B'$. As before, this gives a pullback square of $\mathbb{E}_1$-rings. Compared to the previous two cases, this case is however significantly more difficult to analyze; in fact localizing invariants generally do not send Milnor squares to pullback squares, unless *e.g.* they are truncating (which is not the case for algebraic K-theory). To remedy this in the case of algebraic K-theory, one has to take all infinitesimal thickenings of the ideals simultaneously into consideration and naturally study the resulting pro-excision statements, cf. [KST18; Mor18; LT19]. Note that the notion of (weak) equivalences in Pro(Sp) used in these pro-excison (or pro-cdh descent) results are weaker than that of genuine equivalences.

2.2 Limits of diagrams

This section is specifically for the discussion of adelic descent for localizing invariants in chapter 3. First, recall the following basic behaviours of cubical limits.

Proposition 2.2.1. [Lura, 1.2.4.13] Let $\mathcal{C}$ be a stable ∞-category, and let $n \geq 0$. Suppose we are given a diagram $F \in \operatorname{Fun}((\Delta^1)^n, \mathcal{C})$. Then, F is a limit diagram if and only if F is a colimit diagram.

Proposition 2.2.2. Let $\mathcal{C}$ be a stable ∞-category, and let $n \geq 1$. Suppose we are given a diagram $F \in \operatorname{Fun}((\Delta^1)^n, \mathcal{C})$, which can be identified with an object F' of $\operatorname{Fun}\left(\Delta^1, \mathcal{C}^{(\Delta^1)^{n-1}}\right)$ by choosing a component Δ^1 of $(\Delta^1)^n$. Take any choice of a fiber functor $\operatorname{Fun}\left(\Delta^1, \mathcal{C}^{(\Delta^1)^{n-1}}\right) \xrightarrow{\mathrm{fib}_{n-1}} \operatorname{Fun}((\Delta^1)^{n-1}, \mathcal{C})$. Then, F is a limit diagram if and only if $\mathrm{fib}_{n-1}(F')$ is a limit diagram.

Proof. This follows immediately from [Lura, 1.2.4.15]. More precisely, it treats the (more general) case of colimits over any simplicial set K whose shapes in $\mathcal{C}$ admit colimits. In our case, K can be taken as the finite simplicial set satisfying $K^{\triangleleft} = (\Delta^1)^{n-1}$. Combined with Proposition 2.2.1 above, we have the result. $\square$

Let us investigate a relationship between cubical and cosemisimplicial limits. View $\mathcal{P}(\mathbb{N})$ as a small category via its poset structure determined by inclusions of subsets. Also, recall that the standard semi-simplicial category Δ_s is the subcategory of the standard simplicial category Δ with the same objects but only with injective order-preserving maps as morphisms (in other words, degeneracy maps are dropped), cf. [Lur09, 6.5.3.6]. Let $\mathcal{P}(\mathbb{N})\backslash\emptyset \xrightarrow{c} \Delta_s$ be the functor determined by sending $T = (0 \leq i_0 < \cdots < i_r) \subseteq \mathbb{N}$ to $[r]$, and $T\backslash i_k \hookrightarrow T$ to the k-th face map $[r-1] \xrightarrow{d^k} [r]$. For each $n \geq 0$, it restricts to functors $\mathcal{P}([n])\backslash\emptyset \xrightarrow{c_n} (\Delta_s)_{\leq n}$.

Lemma 2.2.3. The functor $\mathcal{P}(\mathbb{N})\backslash\emptyset \xrightarrow{c} \Delta_s$, and hence $\mathcal{P}([n])\backslash\emptyset \xrightarrow{c_n} (\Delta_s)_{\leq n}$ for each $n \geq 0$, is a Cartesian fibration of ordinary categories.

Proof. Suppose we are given $\emptyset \neq T = (0 \leq i_0 < \cdots < i_{r'}) \subseteq \mathbb{N}$ and $[r] \xrightarrow{\alpha} [r']$ in Δ_s. We have to check that there is a c-Cartesian lifting of α in $\mathcal{P}(\mathbb{N})\backslash\emptyset$ whose target is T [Ker23, tag 01RN]. As α is injective, we have a well-defined $S := (0 \leq i_{\alpha(0)} < \cdots < i_{\alpha(r)}) \subseteq T$ such that $c(S) = [r]$. Note that any inclusion $S' = (0 \leq j_0 < \cdots < j_s) \subseteq T$ with $c(S') = [s]$ maps to $[s] \xrightarrow{\alpha'} [r']$ in Δ_s in a way that $j_k = i_{\alpha'(k)}$. Thus, the morphism $S \subseteq T$ lifts α. To check $S \subseteq T$ is c-Cartesian, suppose we are given $S' \subseteq T$ as before and a morphism $[s] \xrightarrow{\beta} [r]$ in Δ_s such that $S' \subseteq T$ maps to $\alpha \circ \beta$ by c. Then, $j_k = i_{\alpha(\beta(k))} \in S$ for each $k \in [s]$, and hence we know $S' \subseteq S$. The image $[s] \xrightarrow{\gamma} [r]$ of the morphism $S' \subseteq S$ satisfies $j_k = i_{\alpha(\gamma(k))}$ for each $k \in [s]$. As α is injective, we know $\gamma = \beta$, i.e., $S' \subseteq S$ lifts β. $\square$

We explain a slight generalization of [CS02, Proposition 40.2] in an ∞-categorical setting for our purpose. Loosely speaking, this interprets an intergration along the fibers formula for (co)limits over Grothendieck constructions. Given an ∞-category $\mathcal{C}$, let us denote the Grothendieck construction by $\mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty) \xrightarrow{\mathrm{Gr}} \mathrm{coCFib}(\mathcal{C})$, and the dual construction by $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty) \xrightarrow{\mathrm{Gr}^-} \mathrm{CFib}(\mathcal{C})$. (Here, $\mathrm{coCFib}(\mathcal{C})$ and $\mathrm{CFib}(\mathcal{C})$ denotes the ∞-category of coCartesian fibrations and Cartesian fibrations over $\mathcal{C}$ respectively.)

Proposition 2.2.4. Let $H \in \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty)$ be a functor from an ∞-category $\mathcal{C}$, and let $F \in \mathrm{Fun}(\mathrm{Gr}(H), \mathcal{E})$ be a functor from $\mathrm{Gr}(H)$ into an ∞-category $\mathcal{E}$. Suppose $\mathcal{E}$ admits colimits indexed over $H(c)$ for each $c \in \mathcal{C}$, as well as over $\mathcal{C}$. Then, a colimit of F exists[1]. Moreover, there exists a functor $p_!F \in \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ such that each $p_!F(c)$ is equivalent to a colimit of $F|_{H(c)}$, and there is an equivalence between colimits $\mathrm{colim}_{\mathcal{C}}\, p_!F \simeq \mathrm{colim}_{\mathrm{Gr}(H)}\, F$ in $\mathcal{E}$ canonical on F.

Proof. Let $\mathrm{Gr}(H) \xrightarrow{p} \mathcal{C}$ be a coCartesian fibration corresponding to H, and let $\mathrm{Fun}(\mathcal{C}, \mathcal{E}) \xrightarrow{p^*} \mathrm{Fun}(\mathrm{Gr}(H), \mathcal{E})$ be the induced functor. By assumption its left adjoint, the functor of left Kan extensions $\mathrm{Fun}(\mathrm{Gr}(H), \mathcal{E}) \xrightarrow{p_!} \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ exists, and is computed pointwisely. More precisely, by

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[1] Note that by the proof below, we can weaken the existence of colimits condition slightly: assume $\mathcal{E}$ admits $H(c)$-indexed colimits for all c. Then, it suffices to require the LHS colimit $\mathrm{colim}_{\mathcal{C}}\, p_!F$ of the formula exists, rather than all colimits over $\mathcal{C}$.

[Maz19, 1.16], each $H(c)$ is canonically equivalent to $\mathrm{Gr}(H) \times_{\mathcal{C}} c$, and this fiber product as ∞-categories is equivalent to the fiber product computed as quasicategories (simplicial sets). Thus [Lur09, 4.3.3.10] (with $q = id_S$ and $\delta = p$) applies to ensure $p_!$ exists, and satisfies $(p_!F)(c) \simeq \mathrm{colim}(H(c) \hookrightarrow \mathrm{Gr}(H) \xrightarrow{F} \mathcal{E})$. Again by assumption a left Kan extension $s_!p_!F \simeq \mathrm{colim}_{\mathcal{C}}\, p_!F$ of $p_!F$ along $\mathcal{C} \xrightarrow{s} \Delta^0$ exists, and it gives a left Kan extension of F along $s \circ p : \mathrm{Gr}(H) \to \Delta^0$. □

Remark 2.2.5. (1) Informally speaking, Proposition 2.2.4 says that there is a canonical equivalence

$$\mathrm{colim}_{c\in\mathcal{C}} \left(\mathrm{colim}\left(H(c) \hookrightarrow \mathrm{Gr}(H) \xrightarrow{F} \mathcal{E}\right)\right) \simeq \mathrm{colim}_{\mathrm{Gr}(H)} F.$$

(2) Dually, given $H \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty)$ and $F \in \mathrm{Fun}(\mathrm{Gr}^-(H), \mathcal{E})$ such that $\mathcal{E}$ admits limits indexed over $H(c)$ (for all $c \in \mathcal{C}$) and $\mathcal{C}$, we know a limit of F exists, and have a canonical equivalence

$$\lim_{\mathrm{Gr}^-(H)} F \simeq \lim_{c\in\mathcal{C}} \left(\lim\left(H(c) \hookrightarrow \mathrm{Gr}^-(H) \xrightarrow{F} \mathcal{E}\right)\right).$$

More precisely, we have an equivalence between limits $\lim_{\mathrm{Gr}^-(H)} F \simeq \lim_{\mathcal{C}} p_*F$ in $\mathcal{E}$, and the functor p_*F is given as a right Kan extension of F along the Cartesian fibration $\mathrm{Gr}^-(H) \xrightarrow{p} \mathcal{C}$ corresponding to H.

Corollary 2.2.6. Let $F : (\Delta^1)^{n+1}\backslash\emptyset \simeq \mathrm{N}\mathcal{P}([n])\backslash\emptyset \to \mathcal{T}$ be a n-cubical diagram (without the initial vertex) valued in a finitely complete ∞-category $\mathcal{T}$. Then, its limit $\lim_{(\Delta^1)^{n+1}\backslash\emptyset} F$ exists, and is equivalent to

$$\lim_{[r]\in(\Delta_s)_{\leq n}} \left(\prod_{0\leq i_0<\cdots<i_r\leq n} F(i_0, ..., i_r)\right).$$

Proof. Consider the Cartesian fibration $\mathcal{P}([n])\backslash\emptyset \xrightarrow{c_n} (\Delta_s)_{\leq n}$ of Lemma 2.2.3. Note that each fiber category $c_n\mathrm{inv}([r])$ is finite discrete. Also, note that the nerve $\mathrm{N}(\Delta_s)_{\leq n}$ viewed as a simplicial set is finite. Indeed if $i > n$, then each i-simplex in $\mathrm{Fun}([i], (\Delta_s)_{\leq n})$ viewed as a composition of i-number of morphisms must contain an identity morphism. For any ordinary category $\mathcal{C}$, an i-simplex of $\mathrm{N}\mathcal{C}$ is nondegenerate precisely if it can be represented by a sequence $x_0 \to \cdots \to x_i$ of morphisms which does not include identities, so nondegenerate simplices of $\mathrm{N}(\Delta_s)_{\leq n}$ are concentrated in degrees $\leq n$, and hence their number is finite. By Proposition 2.2.4, a limit $\lim_{(\Delta^1)^{n+1}\backslash\emptyset} F$ exists, and is equivalent to $\lim((c_n)_*F) \simeq \lim_{[r]\in(\Delta_s)_{\leq n}} \left(\prod_{0\leq i_0<\cdots<i_r\leq n} F(i_0, ..., i_r)\right)$. □

2.3 Sheaves of modules

We briefly recall some conventions and results of [Lurs] about sheaves of modules which we will use here. Informally speaking, we will mostly consider modules over ordinary schemes in a derived sense. In particular, functors between module categories should be read as derived ones of their classical counterparts unless otherwise specified (e.g., in construction of adele rings). More precisely, given a qcqc (ordinary) scheme X, we consider the symmetric monoidal ∞-category $\mathrm{Mod}(\mathscr{O}_X) \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ of sheaves of $\mathscr{O}_X$-module spectra, and likewise $\mathrm{Mod}(\mathscr{O})$ for any Zariski sheaf $\mathscr{O}$ of discrete commutative rings on X [Lurs, 2.1.0.1]. The ∞-category $\mathrm{Mod}(\mathscr{O})$ has a canonical t-structure specified via connective $\mathscr{O}$-modules, and its heart $\mathrm{Mod}(\mathscr{O})^\heartsuit$ recovers the abelian category of discrete $\mathscr{O}$-modules.

Remark 2.3.1. The canonical functor $\mathcal{D}(\mathrm{Mod}(\mathscr{O})^\heartsuit) \to \mathrm{Mod}(\mathscr{O})$ from the derived ∞-category induced from the canonical embedding of the heart is a fully faithful embedding, and identifies $\mathcal{D}(\mathscr{O}) = \mathcal{D}(\mathrm{Mod}(\mathscr{O})^\heartsuit)$ with the stable subcategory spanned by $\mathscr{O}$-modules whose underlying sheaves of spectra are hypercomplete [Lurs, 2.1.2.3]. In particular if the underlying ∞-topos $\mathrm{Shv}_\mathcal{S}(X_{\mathrm{Zar}})$ associated with X is hypercomplete, then it is an equivalence. For example, the assumption holds if X is Noetherian of finite Krull dimension.

The theory of quasicoherent sheaves [Lurs, 2.2] applied to $X = (X, \mathscr{O}_X)$ gives $\mathrm{QCoh}(X) \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ as a stable subcategory of $\mathrm{Mod}(\mathscr{O}_X)$, which inherits a symmetric monoidal structure and a t-structure recovering the abelian category of discrete quasicoherent sheaves on X as $\mathrm{QCoh}(X)^\heartsuit \subseteq \mathrm{Mod}(\mathscr{O}_X)^\heartsuit$. For $X = \operatorname{Spec} R$ affine we recover $\mathrm{QCoh}(\operatorname{Spec} R) \simeq \mathrm{Mod}(HR)$, which we simply write as $\mathrm{Mod}(R)$ [Lurs, 2.2.3.3].

Remark 2.3.2. For a qcqs scheme X, let $\mathcal{D}_{\mathrm{QCoh}}(X) = \mathcal{D}_{\mathrm{QCoh}}(\mathrm{Mod}(\mathscr{O}_X)^\heartsuit)$ be the stable subcategory of $\mathcal{D}(\mathscr{O}_X) = \mathcal{D}(\mathrm{Mod}(\mathscr{O}_X)^\heartsuit)$ spanned by $\mathscr{O}_X$-modules whose homologies are discrete quasicoherent sheaves. Then, the embedding of Remark 2.3.1 induces an equivalence $\mathcal{D}_{\mathrm{QCoh}}(X) \simeq \mathrm{QCoh}(X)$ [Lurs, 2.2.6.2].

Let R be a discrete commutative ring[2]. Recall that the ∞-category of perfect R-modules is the smallest stable subcategory $\mathrm{Perf}(R)$ of $\mathrm{Mod}(R)$ containing R and closed under retractions. In fact, $\mathrm{Perf}(R) = \mathrm{Mod}(R)^{\mathrm{dual}} = \mathrm{Mod}(R)^\omega$, i.e., perfect modules are precisely dualizable modules, and are again precisely compact modules [Lura, 7.2.4]. For qcqs schemes such identification remains true. Let X be a qcqs scheme, and let $\mathrm{Perf}(X)$ be the stable subcategory of $\mathrm{QCoh}(X)$ spanned by quasicoherent modules affine-locally perfect. Equivalently, $\mathrm{Perf}(X) \simeq \lim_{\operatorname{Spec} R \to X} \mathrm{Perf}(R)$; see also [Sta23, tag 08CM] for a classical approach. Then $\mathrm{Perf}(X) = \mathrm{QCoh}(X)^{\mathrm{dual}} = \mathrm{QCoh}(X)^\omega$ [Lurs, 6.2.6.2, 9.1.5.5], where $\mathrm{QCoh}(X)^{\mathrm{dual}}$ is the full subcategory spanned by dualizable objects in the symmetric monoidal ∞-category $\mathrm{QCoh}(X)$. Note that the compactness assumption (qcqs property) is needed precisely for the identification of compactness and dualizability. We in particular know the functor $X \mapsto \mathrm{QCoh}(X)$ on qcqs schemes is valued in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$.

Let $E : \mathrm{Cat}^{\mathrm{ex}} \to \mathcal{T}$ be a localizing invariant valued in a stable ∞-category $\mathcal{T}$. Composing with $\mathrm{Perf}(-) : \mathrm{Sch}^{\mathrm{op}}_{\mathrm{qcqs}} \to \mathrm{Cat}^{\mathrm{perf}} \subseteq \mathrm{Cat}^{\mathrm{ex}}$ defined via pullbacks, we can view E as a functor defined on (the nerve of) the category of qcqs schemes, and we set $E(X) := E(\mathrm{Perf}(X))$. After restriction, it is in particular defined on the category of discrete commutative rings, and we denote $E(R) := E(\mathrm{Perf}(R))$. As E is localizing, it is an additive invariant [BGT13, 6.1] and in particular E commutes with finite products of rings. Note that localizing (or additive) invariants however do not commute with finite limits, even pullbacks of rings in general as remarked in section 2.1.

Remark 2.3.3. (1) For product $\prod_{i \in I} R_i$ of commutative rings (indexed by a small set I which might not be finite), Bhatt's theorem [Bha16] guarantees the map $\mathrm{Perf}(\prod_{i \in I} R_i) \to \prod_{i \in I} \mathrm{Perf}(R_i)$ induced by projections of rings is a fully faithful embedding, cf. [Gro17, Theorem 3.15].
(2) On (almost-)perfect modules over Noetherian rings, extension of scalars by completions realize derived completions, see [Lurs, 7.2 and 7.3]. Let R be a Noetherian commutative ring and I be its ideal. For $C \in \mathrm{Perf}(R)$ the canonical map $R^\wedge_I \otimes_R C \to C^\wedge_I$ is an equivalence in $\mathrm{Mod}(R)$ [Lurs, 7.3.5.7]. In particular $R^\wedge_I \otimes_R C \in \mathrm{Mod}^{\mathrm{Cpl}(I)}(R)$ is an I-complete object.

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[2] Or in fact any connective $\mathbb{E}_\infty$-ring.

(3) There is a version of derived Nakayama lemma for I-complete modules. Let R be a (discrete) Noetherian commutative ring and let I be an ideal of R. Let $C \in \mathrm{Mod}^{\mathrm{Cpl}(I)}(R)$ be an I-complete module. If C in $\mathrm{Mod}(R)$ satisfies $R/I \otimes_R C \simeq 0$, then $C \simeq 0$ [Sta23, 0G1U].

2.4 Filtrations

This section is specfically for chapter 4. We review the notion of filtered objects and study the standard spectral sequence associated with filtered objects in Proposition 2.4.8.

Definition 2.4.1. Let $\mathcal{D}$ be a stable presentable ∞-category.
(1) A *filtered object* in $\mathcal{D}$ is a $\mathbb{Z}^{\mathrm{op}}$-shaped diagram $F^{\geq \bullet} : \mathrm{N}\mathbb{Z}^{\mathrm{op}} \to \mathcal{D}$ in $\mathcal{D}$, where $\mathbb{Z}$ as a poset is equipped with the usual partial order. We denote $\mathcal{D}^{\mathrm{fil}} := \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{D})$ for the ∞-category of filtered objects in $\mathcal{D}$.
(2) A *graded object* in $\mathcal{D}$ is a $\mathbb{Z}^{\mathrm{disc}}$-shaped diagram $X : \mathrm{N}\mathbb{Z}^{\mathrm{disc}} \to \mathcal{D}$ in $\mathcal{D}$, where $\mathbb{Z}^{\mathrm{disc}}$ is the discrete category whose underlying set of objects is $\mathbb{Z}$. We denote $\mathcal{D}^{\mathrm{gr}} := \mathrm{Fun}(\mathbb{Z}^{\mathrm{disc}}, \mathcal{D})$ for the ∞-category of graded objects in $\mathcal{D}$.

In particular, we will only consider $\mathbb{Z}$-graded objects and (descending or increasing) $\mathbb{Z}$-filtered objects in this manuscript. Consider the unique map $p : \mathrm{N}\mathbb{Z}^{\mathrm{op}} \to \Delta^0$ of simplicial sets and the induced functor $p^* : \mathcal{D} \to \mathcal{D}^{\mathrm{fil}}$ which assigns each object of $\mathcal{D}$ to a constant filtration. As p is a Cartesian and coCartesian fibration, we have adjunctions $p_! \dashv p^* \dashv p_*$ given by the functors of left and right Kan extensions along p, which are computed as $p_!(F) \simeq \mathrm{colim}_{n \to -\infty} F^{\geq n}$ and $p_*(F) \simeq \lim_{n \to \infty} F^{\geq n}$ respectively.

Definition 2.4.2. Let $F^{\geq \bullet}$ be a filtered object in a stable presentable ∞-category $\mathcal{D}$.
(1) $F^{\geq \bullet}$ is *complete* if $p_*(F) \simeq 0$, i.e., $\lim_{n \to \infty} F^{\geq n} \simeq 0$.
(2) An object $F \in \mathcal{D}$ is an *underlying object* of $F^{\geq \bullet}$ if there is an equivalence $F \simeq p_!(F)$, i.e., $\mathrm{colim}_{n \to -\infty} F^{\geq n} \simeq F$. In this case, $F^{\geq \bullet}$ is called an *exhausive filtration* of F.

Above definitions formalize the idea of linear algebraic filtrations prevalent in algebra and geometry. An important operation useful in studying filtrations is taking the associated graded object of a given filtered object, which we will review below shortly; this often makes the study of the underlying object more accessible by providing further linear algebraic data or a more commutative object. Conceptually, this operation also provides a deformation picture, that each $\mathbb{Z}$-filtered object gives a one-parameter deformation of its associated graded object (i.e., the value at the special fiber) by its underlying object (i.e., the value at the generic fiber).

Example 2.4.3. An archetypal example is the sheaf of differential operators $\mathcal{D}_X$ on a smooth variety X over a characteristic 0 field k, equipped with the natural exhaustive increasing $\mathbb{Z}$-filtration $\mathrm{Fil}_{\leq \bullet}\mathcal{D}_X$ by order. The associated graded object is naturally isomorphic to $\mathrm{Sym}^{\bullet} T_{X/k}$ as a (graded Poisson) $\mathscr{O}_X$-algebra, and hence $\mathcal{D}_X$ is a one-parameter deformation or a *quantization* of the cotangent bundle of X; this is important in algebraic geometry and representation theory. Analogous statements can be made for their modules, e.g., for modules whose underlying sheaves are vector bundles on X, flat connections equipped with a filtration compatible with the filtration on $\mathcal{D}_X$ deform graded Higgs bundles.

Definition 2.4.4. Let $F^{\geq \bullet} \in \mathcal{D}^{\mathrm{fil}}$. Its *associated graded* object $\mathrm{gr}(F) \in \mathcal{D}^{\mathrm{gr}}$ is defined as $\mathrm{gr}(F)^n = \mathrm{gr}^n(F) := \mathrm{cof}(F^{\geq n+1} \to F^{\geq n})$ for each $n \in \mathbb{Z}$. This construction naturally defines a functor $\mathrm{gr} : \mathcal{D}^{\mathrm{fil}} \to \mathcal{D}^{\mathrm{gr}}$.

By construction gr commutes with small colimits, and hence is a left adjoint functor under our assumption on $\mathcal{D}$. Its right adjoint can be described as follows:

Proposition 2.4.5. Let $\mathcal{D}$ be a stable presentable ∞-category. Consider the functor $t : \mathcal{D}^{\mathrm{gr}} \to \mathcal{D}^{\mathrm{fil}}$ naturally described as $X \mapsto (\cdots \xrightarrow{0} X^{n+1} \xrightarrow{0} X^n \xrightarrow{0} X^{n-1} \xrightarrow{0} \cdots)$, i.e., $t(X)^{\geq n} \simeq X^n$ and $(t(X)^{\geq n+1} \to t(X)^{\geq n}) \simeq 0$ for all $n \in \mathbb{Z}$. Then, t is a right adjoint functor, and it fits into an adjoint situation $\mathrm{gr} \dashv t : \mathcal{D}^{\mathrm{gr}} \to \mathcal{D}^{\mathrm{fil}}$.

Proof. First, for each $n \in \mathbb{Z}$, consider the functor $c_n : \mathcal{D} \to \mathcal{D}^{\mathrm{fil}}$ naturally described by sending $Y \in \mathcal{D}$ to $(\cdots \to 0 \to Y \to 0 \to \cdots)$, i.e., $c_n(Y)^{\geq n} \simeq Y$ and $c_n(Y)^{\geq i} \simeq 0$ for $i \neq n$.

Lemma 2.4.6. There is an adjoint situation $\mathrm{gr}^n \dashv c_n : \mathcal{D} \to \mathcal{D}^{\mathrm{fil}}$.

Proof. It suffices to check that there is a natural equivalence $\mathrm{Map}_{\mathcal{D}^{\mathrm{fil}}}(F, c_n(Y)) \simeq \mathrm{Map}_{\mathcal{D}}(\mathrm{gr}^n(F), Y)$ for $F \in \mathcal{D}^{\mathrm{fil}}$ and $Y \in \mathcal{D}$. Recall that mapping spaces of $\mathcal{D}^{\mathrm{fil}} = \mathrm{Fun}(\mathrm{N}\mathbb{Z}^{\mathrm{op}}, \mathcal{D})$ can be described through an end construction [Gla16, Prop. 2.3]. In our case, the twisted arrow category $\mathrm{Tw}(\mathbb{Z}^{\mathrm{op}})$ is given by (the nerve of) $\{(p,q) \mid p \geq q\} \subseteq \mathbb{Z} \times \mathbb{Z}^{\mathrm{op}}$. Hence,

$$\begin{aligned}\mathrm{Map}_{\mathcal{D}^{\mathrm{fil}}}(F, c_n(Y)) \simeq \int_{\mathbb{Z}^{\mathrm{op}}} \mathrm{Map}_{\mathcal{D}}(F^{\geq \bullet}, c_n(Y)^{\geq \bullet}) &\simeq \lim_{\{(p,q) \in \mathbb{Z} \times \mathbb{Z}^{\mathrm{op}} \ \mid \ p \geq q\}} \mathrm{Map}_{\mathcal{D}}(F^{\geq p}, c_n(Y)^{\geq q}) \\ &\simeq \mathrm{fib}\left(\mathrm{Map}_{\mathcal{D}}(F^{\geq n}, c_n(Y)^{\geq n}) \to \mathrm{Map}_{\mathcal{D}}(F^{\geq n+1}, c_n(Y)^{\geq n})\right) \\ &\simeq \mathrm{Map}_{\mathcal{D}}(\mathrm{cof}(F^{\geq n+1} \to F^{\geq n}), c_n(Y)^{\geq n}) \simeq \mathrm{Map}_{\mathcal{D}}(\mathrm{gr}^n(F), Y).\end{aligned}$$

While transitioning to the second line, we used that $c_n(Y)^{\geq q} \simeq 0$ for $q \neq n$. ☐

Now, we can check that there is a natural equivalence $\mathrm{Map}_{\mathcal{D}^{\mathrm{gr}}}(\mathrm{gr}\, F, X) \simeq \mathrm{Map}_{\mathcal{D}^{\mathrm{fil}}}(F, t(X))$ for $F = F^{\geq \bullet} \in \mathcal{D}^{\mathrm{fil}}$ and $X \in \mathcal{D}^{\mathrm{gr}}$. In fact, from $\mathcal{D}^{\mathrm{gr}} \simeq \prod_{\mathbb{Z}} \mathcal{D}$, we have

$$\begin{aligned}\mathrm{Map}_{\mathcal{D}^{\mathrm{gr}}}(\mathrm{gr}\, F, X) &\simeq \prod_{n \in \mathbb{Z}} \mathrm{Map}_{\mathcal{D}}(\mathrm{gr}^n(F), X^n) \\ &\simeq \prod_{n \in \mathbb{Z}} \mathrm{Map}_{\mathcal{D}^{\mathrm{fil}}}(F, c_n(X^n)) \simeq \mathrm{Map}_{\mathcal{D}^{\mathrm{fil}}}(F, \prod_{n \in \mathbb{Z}} c_n(X^n)) \simeq \mathrm{Map}_{\mathcal{D}^{\mathrm{fil}}}(F, t(X)).\end{aligned}$$

Here, for the second equivalence we used the adjunction of Lemma 2.4.6, and for the final equivalence we used the natural equivalence $t(X) \simeq \prod_{n \in \mathbb{Z}} c_n(X^n)$ in $\mathcal{D}^{\mathrm{fil}}$. ☐

Remark 2.4.7. Suppose $\mathcal{D}$ is a symmetric monoidal stable presentable ∞-category. Then, $\mathcal{D}^{\mathrm{fil}}$ and $\mathcal{D}^{\mathrm{gr}}$ admit symmetric monoidal structures via Day convolution operations[3]. Moreover, arguments of [Lurw, Sec. 3.2], applied *mutatis mutandis*, show that gr is symmetric monoidal, and hence the adjunction of Proposition 2.4.5 enhances to a symmetric monoidal adjunction.

Proposition 2.4.8. Let $\mathcal{D}$ be a stable presentable ∞-category equipped with a t-structure compatible with filtered colimits (i.e., $\mathcal{D}_{\geq 0}$ is Grothendieck prestable). Suppose $F^{\geq \bullet}$ is a filtered object of $\mathcal{D}$ which is complete, and suppose $F \in \mathcal{D}$ is its underlying object (i.e., $F^{\geq \bullet}$ is an exhaustive filtration on

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[3] Here, both $\mathrm{N}\mathbb{Z}^{\mathrm{op}}$ and $\mathrm{N}\mathbb{Z}^{\mathrm{disc}}$ are regarded as symmetric monoidal ∞-categories via addtion $+$ on $\mathbb{Z}$.

F). Then, there is a conditionally convergent spectral sequence $E_2^{i,j} = \pi_{-i-j}(\operatorname{gr}^{-j} F^{\geq \bullet}) \Rightarrow \pi_{-i-j}F$ in $\mathcal{D}^{\heartsuit}$ (of cohomological grading).

Proof. This is a very standard result used in a vast number of literatures in stable homotopy theory, often with slightly different indexings depending on the context. Nevertheless, let us state and prove the proposition here to clarify our convention. The spectral sequence can be constructed by considering the object $F(\bullet) := F^{\geq -\bullet} \in \operatorname{Fun}(\mathbb{Z}, \mathcal{D})$, and applying the construction of [Lura, 1.2.2.6] on $F(\bullet)$ (by viewing it as a gapped object on $\mathbb{Z} \cup -\infty$ and then restricting to $\mathbb{Z}$). Equivalently, one can appeal to [Boa99]. In fact, fiber sequences $F^{\geq i+1} \to F^{\geq i} \to \operatorname{gr}^i F^{\geq \bullet}$ induce an exact couple

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{-*}F^{\geq i+1} \arrow[rr] & & \pi_{-*}F^{\geq i} \arrow[dl] \arrow[r] & \cdots \\
& & \pi_{-*}\operatorname{gr}^i F^{\geq *} \arrow[ul, "+1"] & &
\end{tikzcd}$$

in $\mathcal{D}^{\heartsuit}$, and the first page of the associated spectral sequence takes the form $E_1^{i,j} = \pi_{-i-j}\operatorname{gr}^i F^{\geq \bullet} \xrightarrow{d_1} E_1^{i+1,j} = \pi_{-i-j-1}\operatorname{gr}^{i+1} F^{\geq \bullet}$. To get the desired form, set $\ell = -i$ and $k = i + j - \ell = 2i + j$ (hence $i = -\ell$ and $j = k + 2\ell$). As $(i, j) \mapsto (i + r, j - r + 1)$ sends (k, ℓ) to $(k + r + 1, \ell - r)$, we can define the new spectral sequence $((E')_r^{k,\ell}, d_r)_{r\geq 2}$ by $(E_{r-1}^{i,j} = E_{r-1}^{-\ell,k+2\ell}, d_{r-1})_{r\geq 2}$. In particular, the initial second page consists of objects $(E')_2^{k,\ell} = \pi_{-k-\ell}\operatorname{gr}^{-\ell} F^{\geq \bullet}$. Since $\lim_{i\to\infty} F^{\geq i} \simeq 0$, we know $\lim^p_{i\to\infty} \pi_{-*}F^{\geq i} = 0$ for $p = 0, 1$. Thus, our spectral sequence converges conditionally to the colimit $\pi_{-*}F$. $\square$

Chapter 3

Adelic descent for localizing invariants

In this chapter, we explain the proof of our first main result, Theorem 3.3.1. First, we review cosemisimplicial and cubical sheaves of adele rings, as well as modules over adele rings on Noetherian schemes in 3.1. In 3.2 we explain the construction of certain exact sequences in $\mathrm{Cat}^{\mathrm{ex}}$ involving categories of perfect modules over sheaves of adeles (Proposition 3.2.16). Using these exact sequences and properties of cubical and semisimplicial diagrams explained in previous sections, we derive the descent result in 3.3.

3.1 Adeles on Noetherian schemes

Let X be a Noetherian scheme. Its underlying set of points admits a canonical partial order given by specializations of points, i.e., for points p and q of X, we say $p \leq q$ if $p \in \overline{q}$ (i.e., if p is a specialization of q). We write the simplicial set obtained as the nerve of the poset structure on X as $S_\cdot(X) = \mathrm{N}(X)$. By definition for each $r \geq 0$, one has $S_r(X) = \{(p_0, ..., p_r) \in X^{r+1} \mid p_0 \leq p_1 \leq \cdots \leq p_r\}$. We also consider the semi-simplicial set $S_\cdot^{\mathrm{red}}(X)$ consisting of $S_r^{\mathrm{red}}(X) = \{(p_0, ..., p_r) \in X^{r+1} \mid p_0 < p_1 < \cdots < p_r\}$ for each $r \geq 0$. After restriction to Δ_s^{op} we can view $S_\cdot(X)$ as a semi-simplicial set, and $S_\cdot^{\mathrm{red}}(X)$ is defined to be its semi-simplicial subset with vertices specified as above.

For each $r \geq 0$, a subset $T \subseteq S_r(X)$, and a quasicoherent sheaf $F \in \mathrm{QCoh}(X)^\heartsuit$, we can define the sheaf of adeles $A_T(F) = A(T, F)$ as an object of $\mathrm{Mod}(\mathscr{O}_X)^\heartsuit$. Below, $T_q = \{(p_0, ..., p_{r-1}) \in S_{r-1}(X) \mid (p_0, ..., p_{r-1}, q) \in T\}$ and $h_{sq} : \mathrm{Spec}\, \mathscr{O}_q/\mathfrak{m}_q^s \to X$ is the canonical map for each $q \in X$ and $s \geq 0$. Functors $(h_{sq})_*$ and h_{sq}^* in this subsection are underived pushforwards and pullbacks respectively.

Definition 3.1.1. For each $r \geq 0$ and $T \subseteq S_r(X)$, we let $A_T = A(T, -) : \mathrm{QCoh}(X)^\heartsuit \to \mathrm{Mod}(\mathscr{O}_X)^\heartsuit$ be the exact functor uniquely characterized by the following three conditons [Hub91, Proposition 2.1.1], [Gro17, Definition 1.4]:

(1) A_T commutes with filtered colimits[1].

(2) If $r = 0$, then $A_T(F) = \prod_{q \in T} \lim_{s \geq 0} (h_{sq})_* h_{sq}^* F$ for each coherent sheaf F.

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[1] Hence it suffices to determine values of A_T on each coherent sheaves on X.

(3) If $r > 0$, then $A_T(F) = \prod_{q\in X} \lim_{s\geq 0} A_{T_q}((h_{sq})_* h_{sq}^* F)$ for each coherent sheaf F.

By taking local sections, we recover abelian groups of adeles associated with T and F restricted on each opens. Also, by construction each functor A_T is lax symmetric monoidal, so each $A_T(\mathscr{O}_X)$ is canonically a sheaf of commutative $\mathscr{O}_X$-algebras. We will often omit $\mathscr{O} = \mathscr{O}_X$ in the notation, and simply write as $A_T = A_T(\mathscr{O}) = A(T, \mathscr{O})$. Note that over an affine $X = \operatorname{Spec} R$, global sections rings $\Gamma(A_T)$ are flat over R due to exactness of A_T [Gro17, Lemma 1.10].

It turns out that the construction of $A_T(F)$ is also sufficiently functorial on T. In fact, for each $F \in \operatorname{QCoh}(X)^\heartsuit$ the association $[r] \mapsto A^r(X, F) := A(S_r(X), F)$ assembles to a cosimplicial object $A^\bullet(X, F)$ of $\operatorname{Mod}(\mathscr{O}_X)^\heartsuit$, and likewise the association $[r] \mapsto A^r_{\mathrm{red}}(X, F) := A(S^{\mathrm{red}}_r(X), F)$ assembles to a cosemisimplicial object $A^\bullet_{\mathrm{red}}(X, F)$ of $\operatorname{Mod}(\mathscr{O}_X)^\heartsuit$ [Hub91, Theorem 2.4.1], [Mor12, Theorem 8.12]. See also [Gro17, Proposition 1.7].

Remark 3.1.2. Let us briefly review the functoriality of $A_T(F)$ on T, and in particular explain how the cosemisimplicial object $A^\bullet_{\mathrm{red}}(X, F)$ is defined. It suffices to describe maps between local sections, and after restriction we are reduced to the case of global sections. So let us abuse notations slightly and understand $A_T(F)$ as the module of global sections. By [Hub91, Proposition 2.1.4] for each $T \subseteq S^{\mathrm{red}}_r(X)$, there is an embedding $A_T(F) \hookrightarrow \prod_{\xi\in T} A_\xi(F)$ into the product of local factors $A_\xi(F) = A(\{\xi\}, F)$ canonical on F. Suppose we are given a map $\alpha : [r] \to [r']$ of Δ_s, such that the induced $\alpha^* : S^{\mathrm{red}}_{r'}(X) \to S^{\mathrm{red}}_r(X) = (p_0, ..., p_{r'}) \mapsto (p_{\alpha(0)}, ..., p_{\alpha(r)})$ maps $S \subseteq S^{\mathrm{red}}_{r'}(X)$ into $T \subseteq S^{\mathrm{red}}_r(X)$. For each $\eta \in S^{\mathrm{red}}_{r'}(X)$, there is a canonical map $\alpha_\eta : A_{\alpha^*(\eta)}(F) \to A_\eta(F)$ of local factors [Mor12, 8.3, p. 59], [Hub91, Definition 2.2.3]. Now, one defines the map

$$\alpha_{*,F} : \prod_{\xi\in T} A_\xi(F) \to \prod_{\eta\in S} A_\eta(F)$$

as the composition

$$\prod_{\xi\in T} A_\xi(F) \to \prod_{\eta\in S} A_{\alpha^*(\eta)}(F) \xrightarrow{\prod_{\eta\in S}\alpha_\eta} \prod_{\eta\in S} A_\eta(F),$$

where the first map is induced from canonical projections. By [Hub91, Proof of Theorem 2.4.1 and Proposition 2.2.4] applied to a decomposition of α into composition of face maps, we know $\alpha_{*,F}$ induces the map $\alpha_{*,F} : A_T(F) \to A_S(F)$, and satisfies transitivity $\alpha_{*,F} \circ \beta_{*,F} = (\alpha \circ \beta)_{*,F}$ for β satisfying analogous conditions as α. In particular, $\prod_{\xi\in S^{\mathrm{red}}_\cdot(X)} A_\xi(F)$ and $A^\bullet_{\mathrm{red}}(X, F) := A(S^{\mathrm{red}}_\bullet(X), F)$ are well-defined as cosemisimplicial objects. Moreover, for a quasicoherent sheaf B of (discrete) commutative $\mathscr{O}_X$-algebras induced maps between local factors and adeles are maps of algebras, and both $\prod_{\xi\in S^{\mathrm{red}}_\cdot(X)} A_\xi(B)$ and $A^\bullet_{\mathrm{red}}(X, B)$ are cosemisimplicial objects in commutative $\mathscr{O}_X$-algebras.

We will consider ∞-category of modules over sheaves A_T of adele rings on X. Let A_T be the sheaf of adele rings associated with $T \subseteq S^{\mathrm{red}}_r(X)$ and $\mathscr{O}_X$, and let $\operatorname{Perf}(A_T)$ be the stable subcategory of $\mathcal{D}(A_T) \hookrightarrow \operatorname{Mod}(A_T)$ spanned by perfect complexes over A_T [Sta23, tag 08CM], which we simply call as perfect A_T-modules. These are objects of $\mathcal{D}(A_T)$ which are Zariski-locally on X equivalent to objects represented by bounded complexes of direct summands of finite free A_T-modules.

Remark 3.1.3. In fact, we can identify this category with the ∞-category of perfect modules over the global section ring. Let A_T be the sheaf of adele rings as above, and let $\Gamma(A_T)$ be its ring of global sections. By [Gro17, Corollary 2.23], the global sections functor induces an equivalence $\mathrm{Perf}(A_T) \simeq \mathrm{Perf}(\Gamma(A_T))$ of (symmetric monoidal) stable ∞-categories. In particular, their values $E(A_T) := E(\mathrm{Perf}(A_T))$ and $E(\Gamma(A_T)) = E(\mathrm{Perf}(\Gamma(A_T)))$ for each localizing invariant E are equivalent.

Let X be a Noetherian scheme of finite Krull dimension n. Given an increasing sequence $0 \leq i_0 < \cdots < i_r \leq n$ of integers, let $\underline{i_0, ..., i_r} := \{(p_0, ..., p_r) \in S_r^{\mathrm{red}}(X) \mid \dim \overline{p_k} = i_k \text{ for all } 0 \leq k \leq r\}$. Note that $S_r^{\mathrm{red}}(X) = \coprod_{0 \leq i_0 < \cdots < i_r \leq n} \underline{i_0, ..., i_r}$. Hence by [Hub91, Proposition 2.1.5] the sheaf of reduced adele ring $A_{\mathrm{red}}^r(X) := A(S_r^{\mathrm{red}}(X), \mathscr{O}_X)$ decomposes into

$$A_{\mathrm{red}}^r(X) \cong \prod_{0 \leq i_0 < \cdots < i_r \leq n} A(i_0, ..., i_r),$$

where $A(i_0, ..., i_r) := A(\underline{i_0, ..., i_r}, \mathscr{O}_X)$. As each subset $S \subseteq [n]$ defines a unique increasing sequence $0 \leq i_0 < \cdots < i_r \leq n$ consisting of its elements, we use the notation $A(S) = A(\underline{i_0, ..., i_r}, \mathscr{O}_X)$ for any such S[2].

Example 3.1.4. Let X be a Noetherian scheme of dimension 1. Then $A^{\cdot}(X)$ takes the form of $F \times O \rightrightarrows A \times (F \times O) \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \cdots$, and similarly $A^{\cdot}_{\mathrm{red}}(X)$ is of the form $F \times O \rightrightarrows A$ [Hub91, Proposition. 3.3.3]. Here $F = A(1)$ is the sheaf of rings of fractions of X whose global section ring is $\prod_{\eta \in X_1} \mathscr{O}_\eta$ (here the product is taken over generic points of X), $O = A(0)$ is the sheaf of integral adele rings of X whose global section ring is $\prod_{p \in X_0} \mathscr{O}_p^\wedge$ (where the product is taken over closed points), and $A = A(01) = F \otimes_{\mathscr{O}} O$ is the sheaf of finite adele rings of X. Hence the classical notion of finite adeles for global fields fits into the framework of higher adeles.

Remark 3.1.5. Let X be a Noetherian scheme of finite Krull dimension n.
(1) The association $A := T \mapsto A(T) : \mathcal{P}([n]) \to \mathrm{CAlg}(\mathscr{O}_X)^\heartsuit$ gives a cubical object. For $\emptyset \neq S = (0 \leq i_0 < \cdots < i_r \leq n) \subseteq [n]$, denote $\underline{S} = \underline{i_0, ..., i_r} \subseteq S_{c(S)}^{\mathrm{red}}(X)$ and $A(S) = A(\underline{S}, \mathscr{O}_X)$ as above, where $c(S) = c_n(S) = [r] = [|S| - 1]$. Each $\emptyset \neq S \subseteq T \subseteq [n]$ induces a map $c_{S \subseteq T} = c(\imath_{S \subseteq T}) : c(S) \to c(T)$ such that $c_{S \subseteq T}^* : S_{c(T)}^{\mathrm{red}}(X) \to S_{c(S)}^{\mathrm{red}}(X)$ satisfies $c_{S \subseteq T}^*(\underline{T}) \subseteq \underline{S}$, so from the transitivity of Remark 3.1.2 we know the association $(S \subseteq T) \mapsto (c_{S \subseteq T})_{*, \mathscr{O}_X} : A(S) \to A(T)$ defines a functor $\mathcal{P}([n]) \backslash \emptyset \to \mathrm{CAlg}(\mathscr{O}_X)^\heartsuit$. By defining $A(\emptyset) = \mathscr{O}_X$, we have an extension of the functor to the cube $\mathcal{P}([n])$.
(2) Let $\alpha : [r] \to [r']$ be a map in $(\Delta_s)_{\leq n}$. For each $T \in \mathcal{P}([n]) \backslash \emptyset$ of $c(T) = [r']$, let $\alpha^* T$ be the element of $\mathcal{P}([n]) \backslash \emptyset$ with $c(\alpha^* T) = [r]$ obtained by the Cartesian fibration structure of c_n. Then the induced map $((c_n)_* A)[r] \to ((c_n)_* A)[r']$ is described as a composition

$$\prod_{S \in c_n^{-1}([r])} A(S) \to \prod_{T \in c_n^{-1}([r'])} A(\alpha^* T) \to \prod_{T \in c_n^{-1}([r'])} A(T).$$

Here, the first map is induced from the projections $\prod_{S \in c_n^{-1}([r])} A(S) \to A(\alpha^* T)$ for each $T \in c_n^{-1}([r'])$, and the second map is the product of the maps $A(\alpha^* T) \to A(T)$ induced from $\alpha^* T \subseteq T$ over $T \in c_n^{-1}([r'])$. Let $E : \mathrm{Cat}^{\mathrm{ex}} \to \mathcal{T}$ be a localizing invariant. As the functor E on rings commutes with finite products and as E does not distinguish sheaves of adeles and their global section rings, [Lur09, 4.3.3.10] implies $E((c_n)_* A)$ is equivalent to a right Kan extension of $E(A) = E \circ A$ along c_n. On the other hand, on each local sections rings the induced map $\prod_{S \in c_n^{-1}([r])} A(S) \to \prod_{T \in c_n^{-1}([r'])} A(T)$ for each $\alpha : [r] \to [r']$ is compatible with the map $\alpha_{*, \mathscr{O}_X} : \prod_{\xi \in S_r^{\mathrm{red}}} A_\xi(\mathscr{O}_X) \to \prod_{\eta \in S_{r'}^{\mathrm{red}}} A_\eta(\mathscr{O}_X)$

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[2] Hence for example $A(ji) = A(ij) = A(\underline{ij}, \mathscr{O}_X)$ for $j > i$.

by construction (through embedding into products of local factors). As the cosemisimplicial object $\prod_{\xi \in S_{\cdot}^{\mathrm{red}}} A_\xi(\mathscr{O}_X)$ induces $A^\bullet_{\mathrm{red}}(X)$ by restriction of each structure maps, we know $(c_n)_* A \cong A^\bullet_{\mathrm{red}}(X)$.

3.2 Exact sequences of categories of modules over adele rings

Let X be a Noetherian scheme of finite Krull dimension n.

Definition 3.2.1. For each $0 \leq i \leq n$, denote the stable subcategory of $\mathrm{Perf}(X)$ generated by $C \in \mathrm{Perf}(X)$ with $\mathrm{Supp}(C) \subseteq X \setminus (X_n \cup \cdots \cup X_{i+1})$ by $\mathrm{Perf}_{\leq i}(X)$. Here, $X_j := \{p \in X \mid \dim \overline{p} = j\}$ and $\mathrm{Perf}_{\leq n}(X) = \mathrm{Perf}(X)$. Hence, $\mathrm{Perf}_{\leq i}(X)$ consists of $C \in \mathrm{Perf}(X)$ with $\dim \mathrm{Supp}(C) \leq i$. Similarly, let $0 \leq i_0 < \cdots < i_r \leq n$, and denote the stable subcategory of $\mathrm{Perf}(A(i_0, ..., i_r))$ generated by the essential image of the exact functor

$$\mathrm{Perf}_{\leq i}(X) \hookrightarrow \mathrm{Perf}(X) \xrightarrow{A(i_0,\ldots,i_r)\otimes_{\mathscr{O}} -} \mathrm{Perf}(A(i_0, ..., i_r))$$

by $\mathrm{Perf}_{\leq i}(A(i_0, ..., i_r)) \in \mathrm{Cat}^{\mathrm{ex}}$.

Remark 3.2.2. Let $A = A(i_0, ..., i_r)$ as in the definition above. The stable subcategory $\mathrm{Perf}_{\leq n}(A)$ of $\mathrm{Perf}(A)$ may not be closed under retracts, but is closed under finite colimits and contains A. Thus, after idempotent completion

$$\mathrm{Idem}(\mathrm{Perf}_{\leq n}(A)) \simeq \mathrm{Ind}(\mathrm{Perf}_{\leq n}(A))^\omega \simeq \mathrm{Mod}(A)^\omega \simeq \mathrm{Perf}(A)$$

in $\mathrm{Cat}^{\mathrm{perf}}$, and localizing invariants do not distinguish between $\mathrm{Perf}_{\leq n}(A)$ and $\mathrm{Perf}(A)$.

On the other hand, by decreasing induction on $i \leq n$ one observes $\mathrm{Perf}_{\leq i}(X) \in \mathrm{Cat}^{\mathrm{perf}}$. For $i = n$ one has $\mathrm{Perf}_{\leq n}(X) = \mathrm{Perf}(X) \in \mathrm{Cat}^{\mathrm{perf}}$. In general one has a fiber sequence $\mathrm{Perf}_{\leq i}(X) \to \mathrm{Perf}_{\leq i+1}(X) \to \mathrm{Perf}(A(i+1))$, since for $C \in \mathrm{Perf}_{\leq i+1}(X)$, one has $A(i+1) \otimes_{\mathscr{O}} C \simeq 0$ iff $C_q^\wedge \simeq 0$ for all $q \in X_{i+1}$, which is equivalent to the condition $\mathrm{Supp}(C) \subseteq X \setminus X_{i+1}$ due to the faithful flatness of $\mathscr{O}_q \to \mathscr{O}_q^\wedge$ under Noetherian assumption [Sta23, tag 00MC]. As $\mathrm{Supp}(C)$ does not contain $X_n \cup \cdots \cup X_{i+2}$ already, the condition is equivalent to $C \in \mathrm{Perf}_{\leq i}(X)$. In particular, $\mathrm{Perf}_{\leq i}(X) \in \mathrm{Cat}^{\mathrm{perf}}$ as a fiber of exact functors in $\mathrm{Cat}^{\mathrm{perf}}$, since fiber products of idempotent-complete ∞-categories over any ∞-category is idempotent complete [Tam18, Lemma 8-(ii)].

We will consider exact sequences in $\mathrm{Cat}^{\mathrm{ex}}$ of the form

$$\mathcal{A} \to \mathrm{Perf}_{\leq i}(A(T)) \to \mathrm{Perf}_{\leq i}(A(T \sqcup \{i\})),$$

where $T \subseteq [i-1]$. The following proposition treats the case of $T = \emptyset$:

Proposition 3.2.3. Let X be a Noetherian scheme of finite Krull dimension n, and let $0 \leq i \leq n$. Then, we have an exact sequence

$$\mathrm{Perf}_{\leq i-1}(X) \to \mathrm{Perf}_{\leq i}(X) \xrightarrow{A(i)\otimes_{\mathscr{O}} -} \mathrm{Perf}_{\leq i}(A(i)) \quad \text{in } \mathrm{Cat}^{\mathrm{ex}}.$$

Here, we set $\mathrm{Perf}_{\leq -1}(X) = 0$. Before giving a proof of Proposition 3.2.3, let us consider the following two lemmas as preparation. Our goal is to describe a functor $\jmath_* : \mathrm{IndPerf}_{\leq i}(A(i)) \to$

$\mathrm{IndPerf}_{\leq i}(X)$ which is right adjoint to the functor $\jmath^* = A(i) \otimes_{\mathscr{O}} -$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ through Lemma 3.2.5. We will use Lemma 3.2.4 for the proof of Lemma 3.2.5 as well as the proof of Proposition 3.2.3 below.

Lemma 3.2.4. Let X be a Noetherian scheme of finite Krull dimension n, and let $0 \leq i \leq n$. Fix $C \in \mathrm{Perf}_{\leq i}(X)$. For each $q \in X$, let $h_q : \operatorname{Spec} \mathscr{O}_q \to X$ be the canonical flat morphism, and let $C_q = h_q^* C \in \mathrm{Perf}(\mathscr{O}_q)$. Then the followings hold:
(1) $A(i) \otimes_{\mathscr{O}} C$ viewed as an object of $\mathrm{Mod}(\mathscr{O}_X)$ is equivalent to $\prod_{q \in S} (h_q)_* C_q$ for some finite subset $S \subseteq X_i$ of dimension-i points.
(2) $A(i) \otimes_{\mathscr{O}} C$ viewed as an object of $\mathrm{Mod}(\mathscr{O}_X)$ is in $\mathrm{IndPerf}_{\leq i}(X)$.

Proof. Let $S \subseteq X_i$ be the set of dimension-i points with $C_q \not\simeq 0$, which is finite due to our assumption, and consider the $\mathscr{O}_X$-module object $\prod_{q \in S} (h_q)_* C_q$. Note that by Noetherian assumption on X, the canonical map $h_q : \operatorname{Spec} \mathscr{O}_q \to X$ is quasicompact quasiseparated, and hence $\prod_{q \in S} (h_q)_* C_q$ is in $\mathrm{QCoh}(X)$ [Sta23, tag 08D5].

We first consider the affine case $X = \operatorname{Spec} R$ for (1). As before let S be the finite set consisting of points $\mathfrak{q} \in X_i$ satisfying $C_{\mathfrak{q}}^{\wedge} \not\simeq 0$ (equivalently $C_{\mathfrak{q}} \not\simeq 0$ due to fully faithfulness of $R_{\mathfrak{q}} \to R_{\mathfrak{q}}^{\wedge}$). Then, the perfect complex $A(i) \otimes_R C = \prod_{\mathfrak{q} \in X_i} R_{\mathfrak{q}}^{\wedge} \otimes_R C$ is equivalent to $\prod_{\mathfrak{q} \in S} C_{\mathfrak{q}}^{\wedge}$ over $A(i)$ via projection $A(i) \to \prod_{\mathfrak{q} \in S} R_{\mathfrak{q}}^{\wedge}$ (as both give equivalent data in $\prod_{\mathfrak{q} \in X_i} \mathrm{Perf}(R_{\mathfrak{q}}^{\wedge})$). Now, note that each $C_{\mathfrak{q}} = R_{\mathfrak{q}} \otimes_R C \in \mathrm{Perf}(R_{\mathfrak{q}})$ satisfies $(C_{\mathfrak{q}})_{\mathfrak{p}} \simeq C_{\mathfrak{p}} \simeq 0$ for all $\mathfrak{p} \in \operatorname{Spec} R_{\mathfrak{q}} \backslash \{\mathfrak{q}\}$, due to assumption $C \in \mathrm{Perf}_{\leq i}(\operatorname{Spec} R)$. Hence, $C_{\mathfrak{q}}$ is canonically a perfect complex over $R_{\mathfrak{q}}^{\wedge}$, with $C_{\mathfrak{q}} \simeq C_{\mathfrak{q}}^{\wedge}$.

Now suppose X is a Noetherian scheme, possibly non-affine. For each affine open $\operatorname{Spec} R \subseteq X$, the restriction of $\prod_{q \in S} (h_q)_* C_q$ on $\operatorname{Spec} R$ viewed as an R-module is $\prod_{q \in S \cap \operatorname{Spec} R} C_q$, and we know that there is an equivalence $(A(i) \otimes_{\mathscr{O}} C)|_{\operatorname{Spec} R} \simeq \prod_{q \in S \cap \operatorname{Spec} R} C_q$ of $A(i)|_{\operatorname{Spec} R}$-modules obtained from the canonical projection $A(i)|_{\operatorname{Spec} R} \to \prod_{q \in S \cap \operatorname{Spec} R} \mathscr{O}_q^{\wedge}$ and $C_q^{\wedge} \simeq C_q$ over $\mathscr{O}_q$, compatible with restrictions. (Here, we are implicitly using the fact that $\mathrm{Perf}(A(i)|_{\operatorname{Spec} R}) \simeq \mathrm{Perf}(A(i)(\operatorname{Spec} R))$.) Thus $\prod_{q \in S} (h_q)_* C_q$ admits an $A(i)$-module structure and is equivalent to $A(i) \otimes_{\mathscr{O}} C$ over $A(i)$ via canonical projections, hence in particular equivalent over $\mathscr{O}_X$ when viewed as objects in $\mathrm{Mod}(\mathscr{O}_X)$.

Finally, (2) follows from (1). Note that $(h_q)_* C_q \simeq ((h_q)_* \mathscr{O}_{\operatorname{Spec} \mathscr{O}_q}) \otimes_{\mathscr{O}} C$ by derived projection formula [Sta23, tag 0B54], and $(h_q)_* \mathscr{O}_{\operatorname{Spec} \mathscr{O}_q} \in \mathrm{QCoh}(X)$ due to Noetherian assumption. Thus $(h_q)_* C_q$ is equivalent to a filtered colimit of the form $\operatorname{colim}_k E_k \otimes_{\mathscr{O}} C$, where each E_k is in $\mathrm{Perf}(X)$. Thus each of $E_k \otimes_{\mathscr{O}} C$ is in $\mathrm{Perf}_{\leq i}(X)$, and we know $(h_q)_* C_q \in \mathrm{IndPerf}_{\leq i}(X)$. Since $A(i) \otimes_{\mathscr{O}} C$ is a finite product of such objects by (1), it is also in $\mathrm{IndPerf}_{\leq i}(X)$. □

Lemma 3.2.5. Let X be a Noetherian scheme of dimension n, and fix any $0 \leq i \leq n$. Consider the essentially surjective functor $\jmath^* = A(i) \otimes_{\mathscr{O}} - : \mathrm{Perf}_{\leq i}(X) \to \mathrm{Perf}_{\leq i}(A(i))$ which induces a compact functor

$$\jmath^* = A(i) \otimes_{\mathscr{O}} - : \mathrm{IndPerf}_{\leq i}(X) \to \mathrm{IndPerf}_{\leq i}(A(i))$$

in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ still denoted by $\jmath^*$. Also, consider the restriction of scalars functor $\rho : \mathrm{Mod}(A(i)) \to \mathrm{Mod}(\mathscr{O}_X)$ induced by $\mathscr{O}_X \to A(i)$.

Then, the restriction $\mathrm{IndPerf}_{\leq i}(A(i)) \to \mathrm{Mod}(\mathscr{O}_X)$ of the functor ρ to $\mathrm{IndPerf}_{\leq i}(A(i))$ factors through $\mathrm{IndPerf}_{\leq i}(X)$, and the resulting functor

$$\rho' : \mathrm{IndPerf}_{\leq i}(A(i)) \to \mathrm{IndPerf}_{\leq i}(X)$$

is a right adjoint of $\jmath^*$.

Proof. Since restriction of scalars functor ρ commutes with filtered colimits, it suffices to check that for each $C \in \mathrm{Perf}_{\leq i}(X)$ the object $A(i) \otimes_{\mathscr{O}} C$ of $\mathrm{IndPerf}_{\leq i}(A(i))$, now viewed as an object of $\mathrm{Mod}(\mathscr{O}_X)$ via ρ, in fact sits inside a stable subcategory $\mathrm{IndPerf}_{\leq i}(X)$. This follows from Lemma 3.2.4 (2). From the already-existing adjunction $A(i) \otimes_{\mathscr{O}} - \dashv \rho : \mathrm{Mod}(A(i)) \to \mathrm{Mod}(\mathscr{O}_X)$, one knows $\rho' : \mathrm{IndPerf}_{\leq i}(A(i)) \to \mathrm{IndPerf}_{\leq i}(X)$ is right adjoint to $\jmath^*$. □

Proof of Proposition 3.2.3. We check that Ind-completion of the sequence of $\mathrm{Cat}^{\mathrm{ex}}$ in question is a split exact sequence of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ by following the criterion provided by Proposition 2.1.3. Let

$$\jmath^* = A(i) \otimes_{\mathscr{O}} - : \mathrm{IndPerf}_{\leq i}(X) \to \mathrm{IndPerf}_{\leq i}(A(i))$$

be an Ind-completion of the functor $A(i) \otimes_{\mathscr{O}} - : \mathrm{Perf}_{\leq i}(X) \to \mathrm{Perf}_{\leq i}(A(i))$, and let

$$\jmath_* : \mathrm{IndPerf}_{\leq i}(A(i)) \to \mathrm{IndPerf}_{\leq i}(X)$$

be its right adjoint. As $\jmath^*$ is compact, $\jmath_*$ commutes with filtered colimits. We verify that $\jmath_*$ is fully faithful, i.e., the counit map $\jmath^*\jmath_*(\jmath^*C) \to \jmath^*C$ is an equivalence for all $C \in \mathrm{Perf}_{\leq i}(X)$. By our description of $\jmath_*$ as ρ' in Lemma 3.2.5, this means we have to check the canonical map $A(i) \otimes_{\mathscr{O}} A(i) \otimes_{\mathscr{O}} C \to A(i) \otimes_{\mathscr{O}} C$ is an equivalence for $C \in \mathrm{Perf}_{\leq i}(X)$. As the statement is Zariski-local on X, we may assume that $X = \mathrm{Spec}\, R$ is affine. By Lemma 3.2.4 (1), we know $A(i) \otimes_R C \simeq \prod_{\mathfrak{q} \in S} C^{\wedge}_{\mathfrak{q}} \simeq \prod_{\mathfrak{q} \in S} C_{\mathfrak{q}}$ as R-modules. We have a canonical equivalence $A(i) \otimes_R \prod_{\mathfrak{q} \in S} C_{\mathfrak{q}} \simeq A(i) \otimes_R C$ obtained as a base change of the equivalence $\prod_{\mathfrak{q} \in S} R_{\mathfrak{q}} \otimes_R \prod_{\mathfrak{q} \in S} C_{\mathfrak{q}} \simeq \prod_{\mathfrak{q} \in S} R_{\mathfrak{q}} \otimes_R C$. In fact, as products are taken over finite sets, it suffices to check that we have canonical equivalences $R_{\mathfrak{p}} \otimes_R \prod_{\mathfrak{q} \in S} C_{\mathfrak{q}} \simeq R_{\mathfrak{p}} \otimes_R C$ for each $\mathfrak{p} \in S$. As the involved base changes are flat, we can assume $C \simeq M[0]$ for some discrete finitely generated R-module M. As $M_{\mathfrak{q}}$ is supported on $\{\mathfrak{q}\}$, each $x \in M_{\mathfrak{q}}$ admits an $r > 0$ with $(\mathfrak{q}R_{\mathfrak{q}})^r \cdot x = 0$. If $\mathfrak{q} \neq \mathfrak{p} \in S$, then one can find $f \in (R \backslash \mathfrak{p}) \cap \mathfrak{q}$, and $x = f^r x / f^r = 0 \in R_{\mathfrak{p}} \otimes_R M_{\mathfrak{q}}$. Thus $R_{\mathfrak{p}} \otimes_R M_{\mathfrak{q}} \simeq 0$ for $\mathfrak{p} \neq \mathfrak{q}$, and the claim follows.

As the composition of the sequence is zero, it remains to compute the fiber of $\jmath^*$. To show the fiber is equivalent to $\mathrm{IndPerf}_{\leq i-1}(X)$, we use the description of $\mathrm{hPerf}_{\leq i}(X)/\mathrm{hPerf}_{\leq i-1}(X)$ given as a consequence of [Bal07, 3.24] (see also (7) in the proof of [Bal07, Theorem 2]), that the canonical triangulated functor $\mathrm{hPerf}_{\leq i}(X)/\mathrm{hPerf}_{\leq i-1}(X) \to \oplus_{q \in X_i} \mathrm{hPerf}_{\{q\}}(\mathscr{O}_q)$ exhibits the target as an idempotent completion of triangulated categories. Since the functor is precisely the image of $\mathrm{Perf}_{\leq i}(X)/\mathrm{Perf}_{\leq i-1}(X) \to \oplus_{q \in X_i} \mathrm{Perf}_{\{q\}}(\mathscr{O}_q)$ in $\mathrm{Cat}^{\mathrm{ex}}$ by taking homotopy categories, [BGT13, 5.15] implies we have an exact sequence

$$\mathrm{IndPerf}_{\leq i-1}(X) \to \mathrm{IndPerf}_{\leq i}(X) \to \mathrm{Ind}(\oplus_{q \in X_i} \mathrm{Perf}_{\{q\}}(\mathscr{O}_q))$$

in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. In particular, note that for each $C \in \mathrm{Perf}_{\leq i}(X)$, the unit map of the adjunction $C \to \oplus_{q \in X_i} (h_q)_* C_q \simeq \prod_{q \in S} (h_q)_* C_q$ for the second left adjoint functor agrees with the unit map $C \to \jmath_*\jmath^*C$ by Lemma 3.2.4. Hence, by description of the unit map in Example 2.1.4 associated with the functor $\mathrm{Ind}(\mathrm{Perf}_{\leq i}(X)) \to \mathrm{Ind}(\mathrm{Perf}_{\leq i}(X)/\mathrm{Perf}_{\leq i-1}(X)) \simeq \mathrm{Ind}(\oplus_{q \in X_i} \mathrm{Perf}_{\{q\}}(\mathscr{O}_q))$, we know

$$\jmath_*\jmath^*C \simeq \oplus_{q \in X_i} (h_q)_* C_q \simeq \mathrm{colim}_{F \in \mathrm{Perf}_{\leq i-1}(X)_{/C}} \mathrm{cof}(F \to C)$$

in $\mathrm{IndPerf}_{\leq i}(X)$. Now, consider the fiber sequence $\mathrm{colim}_{F \in \mathrm{Perf}_{\leq i-1}(X)_{/C}} F \to C \to \jmath_*\jmath^*C$ obtained from taking a filtered colimt of the fiber sequences $F \to C \to \mathrm{cof}(F \to C)$ indexed by the filtered ∞-category $\mathrm{Perf}_{\leq i-1}(X)_{/C}$. From this, we know that the right adjoint

$$\imath^! : \mathrm{IndPerf}_{\leq i}(X) \to \mathrm{fib}(\jmath^*)$$

of the inclusion $\mathrm{fib}(\jmath^*) \hookrightarrow \mathrm{IndPerf}_{\leq i}(X)$ maps compact objects $\mathrm{Perf}_{\leq i}(X)$ to $\mathrm{IndPerf}_{\leq i-1}(X)$. By Remark 2.1.5 (2), we know $\mathrm{fib}(\jmath^*) \simeq \mathrm{IndPerf}_{\leq i-1}(X)$. ☐

Example 3.2.6. For $i = n = \dim X$, one in particular has the exact sequence

$$\mathrm{Perf}_{\leq n-1}(X) \to \mathrm{Perf}(X) \to \mathrm{Perf}(A(n))$$

in $\mathrm{Cat}^{\mathrm{perf}}$ by Proposition 3.2.3. Note that full faithfulness of $\jmath_*$ in the proof of Proposition 3.2.3 for this case can also be explained through the second formula of Lemma 3.2.7 below.

We note the following lemma, which is useful in the case of $i = n$ and motivates our approach to the problem:

Lemma 3.2.7. Let X be a Noetherian scheme of finite Krull dimension n, and let $0 \leq i_0 < \cdots < i_r < n$. Then, the following canonical maps of sheaves of rings

$$A(n) \otimes_{\mathscr{O}} A(i_0, ..., i_r) \to A(n, i_0, ..., i_r) \quad \text{and} \quad A(n) \otimes_{\mathscr{O}} A(n) \to A(n)$$

are isomorphisms.

Proof. As the statement is Zariski-local on X, we can assume $X = \operatorname{Spec} R$ is affine. Note that the set of generic points X^0 of X is finite, and in particular the set of dimension n-points $X_n \subseteq X^0$ is finite. By the characterizing properties of sheaves of adeles, we compute

$$\begin{aligned}
A(n, i_0, ..., i_r) &\cong \prod_{\eta \in X_n} A((i_0, ..., i_r, n)_\eta, (h_\eta)_* h_\eta^* \mathscr{O}) \\
&\cong \prod_{\eta \in X_n} \mathrm{colim}_{\eta \in D(f)} A(i_0, ..., i_r, \widetilde{R_f}) \\
&\cong \prod_{\eta \in X_n} \mathrm{colim}_{\eta \in D(f)} \mathrm{colim}(A(i_0, ..., i_r, \mathscr{O}_{\operatorname{Spec} R}) \xrightarrow{f\cdot} A(i_0, ..., i_r, \mathscr{O}_{\operatorname{Spec} R}) \xrightarrow{f\cdot} \cdots) \\
&\cong \prod_{\eta \in X_n} \mathrm{colim}_{\eta \in D(f)} A(i_0, ..., i_r, \mathscr{O})_f \cong A(n) \otimes_{\mathscr{O}} A(i_0, ..., i_r).
\end{aligned}$$

For the second map, it suffices to check that $R_{\mathfrak{p}} \otimes_R R_{\mathfrak{q}} \cong 0$ for minimal prime ideals $\mathfrak{p} \neq \mathfrak{q}$. By assumption $\mathfrak{q}R_{\mathfrak{q}}$ is the unique prime ideal of $R_{\mathfrak{q}}$, hence is the nilradical of $R_{\mathfrak{q}}$. We can take $f \in (R \backslash \mathfrak{p}) \cap \mathfrak{q}$, and $f^r = 0$ in $\mathfrak{q}R_{\mathfrak{q}}$ for some $r > 0$. Hence $1 = f^r/f^r = 0$ in the localization $R_{\mathfrak{p}} \otimes_R R_{\mathfrak{q}}$ of $R_{\mathfrak{q}}$, and we have $R_{\mathfrak{p}} \otimes_R R_{\mathfrak{q}} \cong 0$. ☐

The following proposition describes remaining exact sequences of the form

$$\mathcal{A} \to \mathrm{Perf}_{\leq i}(A(T)) \to \mathrm{Perf}_{\leq i}(A(T \sqcup \{i\}))$$

for $T \neq \emptyset$ (see Definition 3.2.1 for notations):

Proposition 3.2.8. Let X be a Noetherian scheme of finite Krull dimension n, and let $0 \leq i_0 < \cdots < i_r < i \leq n$. We have an exact sequence

$$\mathrm{Perf}_{\leq i-1}(A(i_0, ..., i_r)) \to \mathrm{Perf}_{\leq i}(A(i_0, ..., i_r)) \xrightarrow{A(i,i_0,...,i_r) \underset{A(i_0,...,i_r)}{\otimes} -} \mathrm{Perf}_{\leq i}(A(i, i_0, ..., i_r)) \qquad \text{in } \mathrm{Cat}^{\mathrm{ex}}.$$

Proof. For convenience, let us denote $\underline{j} := (i_0, ..., i_r)$. Let

$$\jmath^* = A(i, \underline{j}) \otimes_{A(\underline{j})} - : \mathrm{IndPerf}_{\leq i}(A(\underline{j})) \to \mathrm{IndPerf}_{\leq i}(A(i, \underline{j}))$$

be an Ind-completion of the functor $A(i,\underline{j}) \otimes_{A(\underline{j})} : \mathrm{Perf}_{\leq i}(A(\underline{j})) \to \mathrm{Perf}_{\leq i}(A(i, \underline{j}))$. It is a restriction of the functor

$$\jmath^* = A(i, \underline{j}) \otimes_{A(\underline{j})} - : \mathrm{Mod}(A(\underline{j})) \to \mathrm{Mod}(A(i, \underline{j})),$$

which we still denote by $\jmath^*$. We would like to check an Ind-completion of the given sequence in $\mathrm{Cat}^{\mathrm{ex}}$ is a split-exact sequence of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ by applying Proposition 2.1.3. First, consider the following decomposition property:

Lemma 3.2.9. The canonical map $A(\underline{j}) \otimes_{\mathscr{O}} A(i) \to A(i, \underline{j})$ of $A(\underline{j})$-algebras induces an equivalence

$$A(\underline{j}) \otimes_{\mathscr{O}} A(i) \otimes_{\mathscr{O}} C \simeq A(i, \underline{j}) \otimes_{\mathscr{O}} C$$

in $\mathrm{Mod}(A(\underline{j}))$ for all $C \in \mathrm{Perf}_{\leq i}(X)$.

The proof will be given below. The right adjoint $\jmath_* : \mathrm{Mod}(A(i, \underline{j})) \to \mathrm{Mod}(A(\underline{j}))$ of the functor $\jmath^*$ is given as the restriction of scalars functor induced by $A(\underline{j}) \to A(i, \underline{j})$, and hence commutes with filtered colimits. By restriction to $\mathrm{IndPerf}_{\leq i}(A(i, \underline{j}))$ it induces $\jmath_* : \mathrm{IndPerf}_{\leq i}(A(i, \underline{j})) \to \mathrm{IndPerf}_{\leq i}(A(\underline{j}))$, since for $C \in \mathrm{Perf}_{\leq i}(X)$, one has $\jmath_*(A(i, \underline{j}) \otimes_{\mathscr{O}} C) \simeq A(\underline{j}) \otimes_{\mathscr{O}} (A(i) \otimes_{\mathscr{O}} C)$ by Lemma 3.2.9, with $A(i) \otimes_{\mathscr{O}} C \in \mathrm{IndPerf}_{\leq i}(X)$ by Lemma 3.2.5. Thus, it is still a right adjoint of $\jmath^* = A(i, \underline{j}) \otimes_{A(\underline{j})} - : \mathrm{IndPerf}_{\leq i}(A(\underline{j})) \to \mathrm{IndPerf}_{\leq i}(A(i, \underline{j}))$.

Using this description of the right adjoint

$$\jmath_* : \mathrm{IndPerf}_{\leq i}(A(i, \underline{j})) \to \mathrm{IndPerf}_{\leq i}(A(\underline{j})),$$

we check that this functor $\jmath_*$ is fully faithful, i.e., the counit map for the associated adjunction is an equivalence. It suffices to verify the canonical equivalence

$$A(i, \underline{j}) \otimes_{A(\underline{j})} A(i, \underline{j}) \otimes_{\mathscr{O}} C \simeq A(i, \underline{j}) \otimes_{\mathscr{O}} C$$

in $\mathrm{Mod}(A(i, \underline{j}))$ for $C \in \mathrm{Perf}_{\leq i}(X)$. From Proposition 3.2.3, we know $A(i) \otimes_{\mathscr{O}} A(i) \otimes_{\mathscr{O}} C \xrightarrow{\sim} A(i) \otimes_{\mathscr{O}} C$. Base change to $A(i, \underline{j})$ over $A(i)$ gives the canonical equivalence $A(i, \underline{j}) \otimes_{\mathscr{O}} A(i) \otimes_{\mathscr{O}} C \xrightarrow{\sim} A(i, \underline{j}) \otimes_{\mathscr{O}} C$. As the source is equivalent to $A(i, \underline{j}) \otimes_{A(\underline{j})} A(\underline{j}) \otimes_{\mathscr{O}} A(i) \otimes_{\mathscr{O}} C$, again Lemma 3.2.9 gives a desired equivalence. Before computing the fiber of $\jmath^*$, let us give a proof of the Lemma:

Proof of lemma 3.2.9. For $i = n$, we have an isomorphism $A(\underline{j}) \otimes_{\mathscr{O}} A(n) \simeq A(n, \underline{j})$ by Lemma 3.2.7, hence by tensoring with C the result follows. Now we give a proof which works for the general case. As the statement is Zariski-local on X, we can further assume $X = \operatorname{Spec} R$ is affine. We have to check $A(\underline{j}) \otimes_R A(i) \otimes_R C \simeq A(i, \underline{j}) \otimes_{A(i)} A(i) \otimes_R C$. By Lemma 3.2.4 (1), $A(i) \otimes_R C \simeq \prod_{\mathfrak{q} \in S} C^{\wedge}_{\mathfrak{q}}$

for some finite set $S \subseteq X_i$, and each $C_{\mathfrak{q}}^{\wedge}$ is in $\mathrm{Perf}_{\{\mathfrak{q}\}}(R_{\mathfrak{q}}^{\wedge})$. In particular, $C_{\mathfrak{q}}$ is canonically a perfect module over $R_{\mathfrak{q}}^{\wedge}$, and $C_{\mathfrak{q}} \simeq C_{\mathfrak{q}}^{\wedge}$. Thus, we have to prove $A(\underline{j}) \otimes_R \prod_{\mathfrak{q} \in S} C_{\mathfrak{q}} \simeq A(i, \underline{j}) \otimes_{A(i)} \prod_{\mathfrak{q}} C_{\mathfrak{q}}$, and since the product is over a finite set, we are reduced to proving that $A(\underline{j}) \otimes_R C_{\mathfrak{q}} \simeq A(i, \underline{j}) \otimes_{A(i)} C_{\mathfrak{q}}$, i.e.,

$$(A(i) \otimes_R R_{\mathfrak{q}}) \otimes_{R_{\mathfrak{q}}} C_{\mathfrak{q}} \simeq \left(A(i, \underline{j}) \otimes_{A(i)} R_{\mathfrak{q}}^{\wedge}\right) \otimes_{R_{\mathfrak{q}}^{\wedge}} C_{\mathfrak{q}} \qquad \text{over } A(\underline{j}),$$

for $\mathfrak{q} \in X_i$ and $C_{\mathfrak{q}} \simeq C_{\mathfrak{q}}^{\wedge} \in \mathrm{Perf}_{\{\mathfrak{q}\}}(R_{\mathfrak{q}}^{\wedge})$. As $A(i, \underline{j}) \cong \prod_{q' \in X_i} \lim_s A_{s\overline{q'}}(i, \underline{j})$ by construction [Mor12, p. 65], the perfect module $A(i, \underline{j}) \otimes_{A(i)} C_{\mathfrak{q}} \in \mathrm{Perf}(A(i, \underline{j}))$ is equivalent to $\lim_s A_{s\overline{\mathfrak{q}}}(i, \underline{j}) \otimes_{R_{\mathfrak{q}}^{\wedge}} C_{\mathfrak{q}}$, and we have to prove that

$$\left(A(\underline{j}) \otimes_R R_{\mathfrak{q}}\right) \otimes_{R_{\mathfrak{q}}} C_{\mathfrak{q}} \simeq (\lim_s A_{s\overline{\mathfrak{q}}}(i, \underline{j})) \otimes_{R_{\mathfrak{q}}^{\wedge}} C_{\mathfrak{q}} \qquad \text{over } A(\underline{j}). \tag{3.1}$$

Lemma 3.2.10. The natural map

$$(A(\underline{j}) \otimes_R R_{\mathfrak{q}})_{\mathfrak{q}}^{\wedge} \xrightarrow{\sim} \lim_s A_{s\overline{\mathfrak{q}}}(i, \underline{j})$$

is an isomorphism. Here, the completion in the source is taken at the ideal $\mathfrak{q}(A(\underline{j}) \otimes_R R_{\mathfrak{q}})$.

Proof. For each s, one has

$$(A(\underline{j}) \otimes_R R_{\mathfrak{q}})/\mathfrak{q}^s(A(\underline{j}) \otimes_R R_{\mathfrak{q}}) \cong (A(\underline{j})/\mathfrak{q}^s A(\underline{j})) \otimes_R R_{\mathfrak{q}} \cong (A(\underline{j}) \otimes_R R/\mathfrak{q}^s) \otimes_R R_{\mathfrak{q}}.$$

Now, observe that by viewing $R/\mathfrak{q}^s$ as a coherent R-module, one has

$$A(\underline{j}) \otimes_R R/\mathfrak{q}^s \cong A(\underline{j}, R/\mathfrak{q}^s) = A(\underline{j}, \imath_* \mathscr{O}_{s\overline{\mathfrak{q}}}) \cong A_{s\overline{\mathfrak{q}}}(\underline{j}).$$

For the last isomorphism, note that

$$\underline{j} = \left(\underline{j} \cap S_{r-1}^{\mathrm{red}}(V(\mathfrak{q}^s))\right) \coprod \{(p_0, ..., p_r) \in \underline{j} \mid p_r \in \operatorname{Spec} R \backslash V(\mathfrak{q}^s)\},$$

so $A(\underline{j}, \imath_* \mathscr{O}_{s\overline{\mathfrak{q}}}) \cong A\left(\underline{j} \cap S_{r-1}^{\mathrm{red}}(V(\mathfrak{q}^s)), \imath_* \mathscr{O}_{s\overline{\mathfrak{q}}}\right) \times A\left(\{(p_0, ..., p_r) \in \underline{j} \mid p_r \in \operatorname{Spec} R \backslash V(\mathfrak{q}^s)\}, \imath_* \mathscr{O}_{s\overline{\mathfrak{q}}}\right) \cong A_{\operatorname{Spec} R/\mathfrak{q}^s}(\underline{j}) \times 0$, using [Hub91, Proposition 2.1.5]. Hence, we can continue the chain of canonical isomorphisms as

$$\begin{aligned}(A(\underline{j}) \otimes_R R_{\mathfrak{q}})/\mathfrak{q}^s(A(\underline{j}) \otimes_R R_{\mathfrak{q}}) &\cong A_{s\overline{\mathfrak{q}}}(\underline{j}) \otimes_{R/\mathfrak{q}^s} R/\mathfrak{q}^s \otimes_R R_{\mathfrak{q}} \\ &\cong A_{s\overline{\mathfrak{q}}}(\underline{j}) \otimes_{R/\mathfrak{q}^s} R_{\mathfrak{q}}/\mathfrak{q}^s R_{\mathfrak{q}} \cong A_{s\overline{\mathfrak{q}}}(\underline{j}) \otimes_{R/\mathfrak{q}^s} \mathrm{Frac}(R/\mathfrak{q}^s) \cong A_{s\overline{\mathfrak{q}}}(i, \underline{j}).\end{aligned}$$

These isomorphisms (for each s) are compatible with each other, and hence induce an isomorphism between limits. □

Hence combined with the lemma below (applied to $A = A(\underline{j}) \otimes_R R_{\mathfrak{q}}$), we have the canonical equivalence (3.1), finishing the proof. Note that $A(\underline{j})$ is flat over R [Gro17, Lemma 1.10].

Lemma 3.2.11. Let $R = (R, \mathfrak{m}, \kappa)$ be a Noetherian local ring, $R^{\wedge}$ be its completion at $\mathfrak{m}$, and A be a flat R-algebra. Also, let $A^{\wedge} = \lim_s A/\mathfrak{m}^s A$, which is canonically an algebra over A and $R^{\wedge}$.

Then the canonical map of exact functors

$$A \otimes_R (-) \to A^\wedge \otimes_{R^\wedge} (-) (\simeq A^\wedge \otimes_R -)$$

from $\mathrm{Perf}_{\{\mathfrak{m}\}}(R) \simeq \mathrm{Perf}_{\{\mathfrak{m}\}}(R^\wedge)$ to $\mathrm{Mod}(A)$ is an equivalence.

Proof. By induction on the number of nonzero homotopy modules, it suffices to check the equivalence $A \otimes_R M \simeq A^\wedge \otimes_R M$ for discrete finitely generated R-modules M supported on the point $\{\mathfrak{m}\}$. More precisely, for each $C \in \mathrm{Perf}_{\{\mathfrak{m}\}}(R)$ one can apply exact functors on a truncation fiber sequence of the form $\pi_k(C)[-k] \to C \to \tau_{<k}C$, with $\tau_{<k}C$ having strictly less number of nonvanishing homotopy modules. By assumption on M, there is an $r > 0$ with $\mathfrak{m}^r M \simeq 0$. Thus, by applying exact functors on fiber sequences $\mathfrak{m}^i M \to M \to M/\mathfrak{m}^i M$ $(0 \leq i \leq r)$, one knows it suffices to verify the equivalence for R-modules $\mathfrak{m}^{i-1}M/\mathfrak{m}^i M$, or more generally for finite $R/\mathfrak{m}$-modules viewed as R-modules. Hence, it suffices to verify the equivalence $A \otimes_R \kappa \simeq A^\wedge \otimes_R \kappa$. Note that by [Sta23, tag 0AGW] or [Yek18, Theorem 0.1], $A^\wedge$ is still flat over R, and the involved base changes are underived. Thus, both sides are canonically equivalent to $A/\mathfrak{m}A$. ☐

This finishes the proof of Lemma 3.2.9. ☐

It remains to describe the fiber of $\jmath^*$. Let $A(\underline{j}) \otimes_{\mathscr{O}} C \in \mathrm{Perf}_{\leq i}(A(\underline{j}))$, where $C \in \mathrm{Perf}_{\leq i}(X)$. By Lemma 3.2.9, its unit map $A(\underline{j}) \otimes_{\mathscr{O}} C \to \jmath_* \jmath^*(A(\underline{j}) \otimes_{\mathscr{O}} C) \simeq A(i, \underline{j}) \otimes_{\mathscr{O}} C$ is equivalent to the image of the unit map $C \to A(i) \otimes_{\mathscr{O}} C$ of Proposition 3.2.3 by $A(\underline{j}) \otimes_{\mathscr{O}} -$. Again by Proposition 3.2.3, Remark 2.1.5 (1), and exactness of $A(\underline{j}) \otimes_{\mathscr{O}} -$, we have a fiber sequence $A(\underline{j}) \otimes_{\mathscr{O}} F \to A(\underline{j}) \otimes_{\mathscr{O}} C \to \jmath_* \jmath^*(A(\underline{j}) \otimes_{\mathscr{O}} C)$ in $\mathrm{IndPerf}_{\leq i}(A(\underline{j}))$, where $F \in \mathrm{IndPerf}_{\leq i-1}(X)$. Thus, a right adjoint

$$\imath^! : \mathrm{IndPerf}_{\leq i}(A(\underline{j})) \to \mathrm{fib}(\jmath^*)$$

of the inclusion $\mathrm{fib}(\jmath^*) \hookrightarrow \mathrm{IndPerf}_{\leq i}(A(\underline{j}))$ maps $\mathrm{Perf}_{\leq i}(A(\underline{j}))$ to $\mathrm{IndPerf}_{\leq i-1}(A(\underline{j}))$, and by Remark 2.1.5 (2), we know $\mathrm{fib}(\jmath^*) \simeq \mathrm{IndPerf}_{\leq i-1}(A(\underline{j}))$. ☐

Example 3.2.12. For $i = n = \dim X$ and for $0 \leq i_1 \leq \cdots \leq i_r < n$, we in particular have an exact sequence

$$\mathrm{Perf}_{\leq n-1}(A(i_0, ..., i_r)) \to \mathrm{Perf}_{\leq n}(A(i_0, ..., i_r)) \xrightarrow{A(n,i_0,...,i_r) \underset{A(i_0,...,i_r)}{\otimes} -} \mathrm{Perf}_{\leq n}(A(n, i_0, ..., i_r)) \quad \text{in } \mathrm{Cat}^{\mathrm{ex}}.$$

Note that fully faithfulness of a right adjoint of (an Ind-completion of) the functor $A(n, i_0, ..., i_r) \underset{A(i_0,...,i_r)}{\otimes} -$ can be explained by the second isomorphism in Lemma 3.2.7. By Remark 3.2.2, applying any localizing invariant $E : \mathrm{Cat}^{\mathrm{ex}} \to \mathcal{T}$ to above exact sequence yields the fiber sequence

$$E(\mathrm{Perf}_{\leq n-1}(A(i_0, ..., i_r))) \to E(A(i_0, ..., i_r)) \to E(A(n, i_0, ..., i_r))$$

in a stable ∞-category $\mathcal{T}$.

Remark 3.2.13. In fact, we can verify that the fiber of the functor $\jmath^* = A(i, \underline{j}) \otimes_{A(\underline{j})} -$ in Proposition 3.2.8 on compact objects $\mathrm{Perf}_{\leq i}(A(\underline{j}))$ is $\mathrm{Perf}_{\leq i-1}(A(\underline{j}))$ via direct computation. We have a canonical morphism (i.e., a square) from $\mathrm{Perf}_{\leq i}(X) \to \mathrm{Perf}_{\leq i}(A(i))$ of Proposition 3.2.3 to $\mathrm{Perf}_{\leq i}(A(\underline{j})) \to$

$\mathrm{Perf}_{\leq i}(A(i,\underline{j}))$, whose component functors are essentially surjective. Thus, it suffices to prove the following:

Proposition 3.2.14. Let X be a Noetherian scheme of finite Krull dimension n, and take i and $\underline{j}$ as in Proposition 3.2.8. For $C \in \mathrm{Perf}_{\leq i}(X)$, the vanishing $A(i,\underline{j}) \otimes_{\mathscr{O}} C \simeq 0$ implies $A(i) \otimes_{\mathscr{O}} C \simeq 0$.

Lemma 3.2.15. Proposition 3.2.14 holds for $i = n$. In other words, $A(n,\underline{j}) \otimes_{\mathscr{O}} C \simeq 0$ implies $A(n) \otimes_{\mathscr{O}} C \simeq 0$ for $C \in \mathrm{Perf}(X)$.

Proof. The question is Zariski-local on X, so we can assume $X = \operatorname{Spec} R$. We proceed by induction on $\dim X$. The case of $\dim X = 0$ is tautological, as the only possible choice of the sheaf is $A(0)$. Suppose $\dim X > 0$, and let $\underline{j} = (i_0, ..., i_r)$. By Lemma 3.2.7, the assumption equivalently says $A(n) \otimes_R \big(A(\underline{j}) \otimes_R C\big) \simeq 0$, so $A(\underline{j}) \otimes_R C$ vanishes at each points of X_n. Fix any $\eta \in X_n$. We can take an affine open subset where $A(\underline{j}) \otimes_R C$ vanishes, since it is concentrated in finitely many degrees and $A(\underline{j})$ is flat over R. Thus we can assume our $C \in \mathrm{Perf}(R)$ satisfies $A(\underline{j}) \otimes_R C \simeq 0$ (where $i_r < n$). By base change to $A(n-1,\underline{j})$ over $A(\underline{j})$ (if $i_r < n-1$), we can further assume $i_r = n-1$. We are reduced to checking that this condition, together with the induction hypothesis, imply $C_\eta \simeq 0$.

From $A(\underline{j}) \simeq \prod_{\mathfrak{q} \in (\operatorname{Spec} R)_{i_r}} \lim_s A_{s\overline{\mathfrak{q}}}(\underline{j})$ (e.g., [Mor12, p. 65]), we know $0 \simeq \lim_s A_{s\overline{\mathfrak{q}}}(\underline{j}) \otimes_R C$ for all $\mathfrak{q} \in (\operatorname{Spec} R)_{i_r}$. In particular, $0 \simeq A_{\overline{\mathfrak{q}}}(\underline{j}) \otimes_R C \simeq A_{R/\mathfrak{q}}(\underline{j}) \otimes_{R/\mathfrak{q}} (R/\mathfrak{q} \otimes_R C)$ holds. By the induction hypothesis applied to $\operatorname{Spec} R/\mathfrak{q}$, we know $R/\mathfrak{q} \otimes_R C$ satisfies $0 \simeq \mathrm{Frac}(R/\mathfrak{q}) \otimes_{R/\mathfrak{q}} (R/\mathfrak{q} \otimes_R C) \simeq \kappa(R^{\wedge}_{\mathfrak{q}}) \otimes_{R^{\wedge}_{\mathfrak{q}}} R^{\wedge}_{\mathfrak{q}} \otimes_R C \simeq \kappa(R^{\wedge}_{\mathfrak{q}}) \otimes_{R^{\wedge}_{\mathfrak{q}}} C^{\wedge}_{\mathfrak{q}}$. (Here, $\kappa(R^{\wedge}_{\mathfrak{q}})$ stands for the residue field of $R^{\wedge}_{\mathfrak{q}}$.) By the derived Nakayama lemma (Remark 2.3.3), we have $C^{\wedge}_{\mathfrak{q}} \simeq 0 \in \mathrm{Perf}(R^{\wedge}_{\mathfrak{q}})$ for all $\mathfrak{q} \in (\operatorname{Spec} R)_{i_r}$. By the Noetherian hypothesis $R_{\mathfrak{q}} \to R^{\wedge}_{\mathfrak{q}}$ is faithfully flat [Sta23, tag 00MC], and we in particular know ($C_{\mathfrak{q}} \simeq 0$, and) $\mathrm{Supp}(C) \subseteq \operatorname{Spec} R \backslash (\operatorname{Spec} R)_{i_r}$. Thus, the closed subset $\mathrm{Supp}(C) \subseteq \operatorname{Spec} R$ (C is perfect) should not contain codimension 1 points of $\overline{\eta}$, and we know $\eta \notin \mathrm{Supp}(C)$, i.e., $C_\eta \simeq 0$. ☐

Proof of Proposition 3.2.14. As the statement is Zariski-local on X, we can assume $X = \operatorname{Spec} R$. From the assumption $0 \simeq A(i,\underline{j}) \otimes_{A(i)} (A(i) \otimes_R C)$, we have $0 \simeq \lim_s A_{s\overline{\mathfrak{q}}}(i,\underline{j}) \otimes_{R^{\wedge}_{\mathfrak{q}}} C^{\wedge}_{\mathfrak{q}} \simeq \lim_s A_{s\overline{\mathfrak{q}}}(i,\underline{j}) \otimes_R C$ for all $\mathfrak{q} \in X_i$. In particular, $A_{R/\mathfrak{q}}(i,\underline{j}) \otimes_R C \simeq 0$, or equivalently $A_{R/\mathfrak{q}}(i,\underline{j}) \otimes_{R/\mathfrak{q}} (R/\mathfrak{q} \otimes_R C) \simeq 0$ for all $\mathfrak{q} \in (\operatorname{Spec} R)_i$. By Lemma 3.2.15 applied to $\operatorname{Spec} R/\mathfrak{q}$, we have $0 \simeq A_{R/\mathfrak{q}}(i) \otimes_{R/\mathfrak{q}} (R/\mathfrak{q} \otimes_R C) \simeq \mathrm{Frac}(R/\mathfrak{q}) \otimes_R C \simeq R^{\wedge}_{\mathfrak{q}}/\mathfrak{q}R^{\wedge}_{\mathfrak{q}} \otimes_{R^{\wedge}_{\mathfrak{q}}} R^{\wedge}_{\mathfrak{q}} \otimes_R C \simeq \kappa(R^{\wedge}_{\mathfrak{q}}) \otimes_{R^{\wedge}_{\mathfrak{q}}} C^{\wedge}_{\mathfrak{q}}$. By the derived Nakayama lemma (Remark 2.3.3), we know $C^{\wedge}_{q} \simeq 0$ for all $q \in X_i$, i.e., $A(i) \otimes_{\mathscr{O}} C \simeq 0$. ☐

By combining Proposition 3.2.3 and Proposition 3.2.8, we have the following:

Proposition 3.2.16. Let X be a Noetherian scheme of finite Krull dimension n, and let $0 \leq i \leq n$. Then for each $T \in \mathcal{P}([i-1])$, we have an exact sequence

$$\mathrm{Perf}_{\leq i-1}(A(T)) \to \mathrm{Perf}_{\leq i}(A(T)) \xrightarrow{A(T \sqcup \{i\}) \underset{A(T)}{\otimes} -} \mathrm{Perf}_{\leq i}(A(T \sqcup \{i\})) \qquad \text{in } \mathrm{Cat}^{\mathrm{ex}}.$$

Proof. The case of $T = \emptyset$ follows from Proposition 3.2.3, and the remaining case of $T \neq \emptyset$ follows from Proposition 3.2.8. ☐

3.3 Adelic descent for localizing invariants

Let X be a Noetherian scheme of finite Krull dimension n. Recall that we have cosemisimplicial and cubical diagrams

$$A^\bullet_{\mathrm{red}}(X) := A^\bullet_{\mathrm{red}}(X, \mathscr{O}_X) : (\Delta_s)_{\leq n} \to \mathrm{CAlg}(\mathscr{O}_X)^\heartsuit$$

and

$$A(-) : \mathcal{P}([n]) \to \mathrm{CAlg}(\mathscr{O}_X)^\heartsuit$$

of $\mathscr{O}_X$-algebras (Remarks 3.1.2 and 3.1.5). By setting $A^{-\infty}_{\mathrm{red}}(X) := \mathscr{O}_X$, we can view $A^\bullet_{\mathrm{red}}(X)$ as an augmented cosemisimplicial diagram, and after composition with $\mathrm{Perf}(-)$ we have an augmented cosemisimplicial diagram $\mathrm{Perf}(A^\bullet_{\mathrm{red}}(X))$ in $\mathrm{Cat}^{\mathrm{perf}}$. Likewise, we have an n-cubical diagram $\mathrm{Perf}(A(-))$ in $\mathrm{Cat}^{\mathrm{perf}}$. For each $0 \leq i \leq n$, we also consider the n-cube $\mathrm{Perf}_{\leq i}(A(-))$ in $\mathrm{Cat}^{\mathrm{ex}}$ induced as a subfunctor of $\mathrm{Perf}(A(-))$. Note that $\mathrm{Perf}_{\leq i}(A(-))$ can be regarded as an i-cubical diagram after restriction to $\mathcal{P}([i])$, since $\mathrm{Perf}_{\leq i}(X) \to \mathrm{Perf}(A(i_0, ..., i_r))$ factors through $\mathrm{Perf}(A(i_r))$ and hence $\mathrm{Perf}_{\leq i}(A(i_0, ..., i_r)) \simeq 0$ for $i_r > i$. By further composing these diagrams with a localizing invariant $E : \mathrm{Cat}^{\mathrm{ex}} \to \mathcal{T}$, we obtain (augmented) cosemisimplicial and cubical diagrams $E(A^\bullet_{\mathrm{red}}(X))$, $E(A(-))$, and $E(\mathrm{Perf}_{\leq i}(A(-)))$ in a stable ∞-category $\mathcal{T}$.

Theorem 3.3.1. ([Kim23, Th. 3.17]) Let X be a Noetherian scheme of finite Krull dimension n, and let $E : \mathrm{Cat}^{\mathrm{ex}} \to \mathcal{T}$ be a localizing invariant valued in a stable ∞-category $\mathcal{T}$.
(1) For each $0 \leq i \leq n$, the n-cubical diagram $E(\mathrm{Perf}_{\leq i}(A(-))) : \mathrm{N}\mathcal{P}([n]) \to \mathcal{T}$ is a limit diagram. In particular, the n-cubical diagram $E(A(-)) : \mathrm{N}\mathcal{P}([n]) \to \mathcal{T}$ is a limit diagram, and we have an equivalence

$$E(X) \simeq \lim_{0 \leq i_0 < \cdots < i_r \leq n} E(A(i_0, ..., i_r)) \ \text{ in } \mathcal{T}.$$

(2) The (truncated) augmented cosemisimplicial diagram $E(A^\bullet_{\mathrm{red}}(X)) : \mathrm{N}((\Delta_s)_+)_{\leq n} \to \mathcal{T}$ is a limit diagram, and we have an equivalence

$$E(X) \simeq \lim_{[r] \in (\Delta_s)_{\leq n}} E(A^r_{\mathrm{red}}(X)) \ \text{ in } \mathcal{T}.$$

Proof. We prove (1) through induction on i. By Proposition 3.2.3 the underlying functor of the 0-cubical diagram $\mathrm{Perf}_{\leq 0}(X) \to \mathrm{Perf}_{\leq 0}(A(0))$ is an equivalence, and we in particular have $i = 0$ case by applying E. Suppose $0 < i \leq n$, and consider the n-cubical diagrams $\mathrm{Perf}_{\leq i}(A(-))$ and $E(\mathrm{Perf}_{\leq i}(A(-)))$. In order to check $E(\mathrm{Perf}_{\leq i}(A(-)))$ is a limit diagram, it suffices to check the i-cubical diagram obtained by a restriction to $\mathcal{P}([i])$ is a limit diagram, as images of the other vertices are zero. Now, consider the decomposition $\mathcal{P}([i]) = \mathcal{P}([i-1]) \coprod (\mathcal{P}([i-1]) \sqcup \{i\})$ and view the i-cube $\mathrm{Perf}_{\leq i}(A(-))|_{\mathcal{P}([i])}$ as a morphism

$$\mathrm{Perf}_{\leq i}(A(-))|_{\mathcal{P}([i-1])} \to \mathrm{Perf}_{\leq i}(A(i,-))|_{\mathcal{P}([i-1])}$$

of $(i-1)$-cubical diagrams, and similarly for $E(\mathrm{Perf}_{\leq i}(A(-)))$. By applying E to the exact sequences of Proposition 3.2.16, we have an equivalence

$$\mathrm{fib}\left(E(\mathrm{Perf}_{\leq i}(A(-)))|_{\mathcal{P}([i-1])} \to E(\mathrm{Perf}_{\leq i}(A(i,-)))|_{\mathcal{P}([i-1])}\right) \simeq E(\mathrm{Perf}_{\leq i-1}(A(-)))|_{\mathcal{P}([i-1])}.$$

By induction hypothesis this $(i-1)$-cubical diagram is a limit diagram, and hence by Proposition 2.2.2 we know the original i-cubical diagram $E(\mathrm{Perf}_{\leq i}(A(-)))|_{\mathcal{P}([i])}$ is a limit diagram, i.e, $E(\mathrm{Perf}_{\leq i}(A(-)))$ is a limit diagram. This establishes (1), and in particular for $i = n$, we have

$$E(X) \simeq \lim_{T \in \mathcal{P}([n]) \setminus \emptyset} E(A(T))$$

by Remark 3.2.2. By Corollary 2.2.6 and Remark 3.1.5, we know $E(A^{\bullet}_{\mathrm{red}}(X))$ is also a limit diagram, and have

$$E(X) \simeq \lim_{T \in \mathcal{P}([n]) \setminus \emptyset} E(A(T)) \simeq \lim_{[r] \in (\Delta_s)_{\leq n}} E(A^{r}_{\mathrm{red}}(X))$$

by (1). □

Chapter 4

The Geisser-Levine theorem for Cartier smooth rings

In this chapter, we study topological cyclic homology and its motivic filtration [BMS19] for Cartier smooth rings. In particular, we compute pertinent cohomology complexes, especially prismatic and syntomic cohomology complexes of Cartier smooth rings. Using these computations carried out in 4.1-4.3 and 4.5, we explain an alternative proof of Kelly-Morrow's generalization [KM21] of Geisser-Levine theorem for Cartier smooth rings in 4.4.

Conventions. Throughout this chapter, we fix a prime p. For each ordinary commutative ring R, we denote $\mathcal{D}(R) \simeq \mathrm{Mod}_R$ for its derived ∞-category. When R is p-complete, we denote $\widehat{\mathcal{D}}(R) \simeq \mathrm{Mod}_R^{\mathrm{Cpl}(p)}$ for its p-complete derived ∞-category, and similarly $\widehat{\mathcal{DF}}(R) \simeq \widehat{\mathcal{D}}(R)^{\mathrm{fil}}$ for the filtered p-complete derived ∞-category of R.

4.1 De Rham-Witt complex of Cartier smooth rings

In this section, we recall the notion of Cartier smoothness and study de Rham-Witt complex of Cartier smooth rings. The slogan is that every de Rham-type invariant is non-derived, i.e., derived objects agree with their underived analogues, for Cartier smooth rings. De Rham-Witt complexes [Ill79] of Deligne-Illusie are objects $W\Omega_X^*$ naturally constructed for each $\mathbb{F}_p$-scheme X which, in the smooth case, provide an explicit complex of sheaves whose (hyper-)cohomology computes crystalline cohomology of Berthelot-Grothendieck for each smooth $\mathbb{F}_p$-scheme. Similarly, one can consider the derived de Rham-Witt complex $LW\Omega_X$ which computes derived crystalline cohomology of a (derived) scheme X over $\mathbb{F}_p$. In order to remedy some computational and conceptual difficulties involved in [Ill79], an alternative construction of de Rham-Witt complexes was proposed by [BLM21]; for each $\mathbb{F}_p$-scheme X, they attach the saturated de Rham-Witt complex $\mathcal{W}\Omega_X^*$ which agrees with Illusie's $W\Omega_X^*$ whenever X is smooth. By construction saturated de Rham-Witt complexes have arguably simpler universal property and enable one to detour technical complications from the use of pro-complexes (which [Ill79] cannot avoid) at the cost of being difficult to carry out explicit computations outside of the smooth case and to describe (co)limits in general due to saturation. Nevertheless, we will show that for Cartier smooth rings (and hence for Cartier smooth schemes), all three de Rham-

Witt complexes $LW\Omega_S$, $W\Omega^*_S$, and $\mathcal{W}\Omega^*_S$ agree with each other. We refer the reader to [Ill79] and [BLM21] for detailed constructions of $W\Omega^*_X$ and $\mathcal{W}\Omega^*_X$ as well as relevant objects, e.g., Dieudonné algebras [BLM21, Ch. 3].

Let us briefly recall the notion of (nonabelian) derived functors from [Lur09]. Objects in the category $\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}$ of ordinary commutative $\mathbb{F}_p$-algebras can be described as filtered colimits of finitely presented $\mathbb{F}_p$-algebras, each of which is a (reflexive) coequalizer of finitely generated polynomial algebras. Reversing the process, one can consider the ∞-category $\mathrm{CAlg}^{\mathrm{an}}_{\mathbb{F}_p} = \mathcal{P}_\Sigma(\mathrm{CAlg}^{\mathrm{poly}}_{\mathbb{F}_p})$ of derived commutative $\mathbb{F}_p$-algebras freely generated by the category $\mathrm{CAlg}^{\mathrm{poly}}_{\mathbb{F}_p}$ of finitely generated polynomial $\mathbb{F}_p$-algebras under sifted colimits. The category $\mathrm{CAlg}^{\mathrm{an}}_{\mathbb{F}_p}$ is equivalent to the underlying ∞-category of the simplicial model category of simplicial commutative $\mathbb{F}_p$-algebras [Lur09, Cor. 5.5.9.3]. By definition, for any ∞-category $\mathcal{D}$ admitting sifted colimits, composition with the Yoneda embedding $\mathrm{CAlg}^{\mathrm{poly}}_{\mathbb{F}_p} \hookrightarrow \mathrm{CAlg}^{\mathrm{an}}_{\mathbb{F}_p}$ induces an equivalence $\mathrm{Fun}_\Sigma(\mathrm{CAlg}^{\mathrm{an}}_{\mathbb{F}_p}, \mathcal{D}) \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{poly}}_{\mathbb{F}_p}, \mathcal{D})$ with inverse given by left Kan extension; here the source consists of sifted colimit preserving functors. Given a functor $F : \mathrm{CAlg}^{\mathrm{poly}}_{\mathbb{F}_p} \to \mathcal{D}$, we call its left Kan extension $LF \in \mathrm{Fun}_\Sigma(\mathrm{CAlg}^{\mathrm{an}}_{\mathbb{F}_p}, \mathcal{D})$ along the Yoneda embedding a (nonabelian) *derived* functor of F. In particular, given a functor $G : \mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p} \to \mathcal{D}$, there is a natural map $L(G|_{\mathrm{CAlg}^{\mathrm{poly}}_{\mathbb{F}_p}})|_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}} \to G$ in $\mathrm{Fun}(\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}, \mathcal{D})$ which is up to equivalence a unique one extending the identity map of $G|_{\mathrm{CAlg}^{\mathrm{poly}}_{\mathbb{F}_p}}$.

Lemma 4.1.1. For each $r \geq 1$, there is an equivalence of functors $LW\Omega_{(-)} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/p^r \simeq LW_r\Omega_{(-)}$ in $\mathrm{Fun}_\Sigma(\mathrm{CAlg}^{\mathrm{an}}_{\mathbb{F}_p}, \mathcal{D}(\mathbb{Z}))$.

Proof. Consider the functor $R \mapsto W\Omega^*_R \otimes_{\mathbb{Z}} \mathbb{Z}/p^r$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{poly}}_{\mathbb{F}_p}, \mathcal{D}(\mathbb{Z}))$. For a smooth $\mathbb{F}_p$-algebra R (e.g., $R \in \mathrm{CAlg}^{\mathrm{poly}}_{\mathbb{F}_p}$), $W\Omega^*_R \otimes_{\mathbb{Z}} \mathbb{Z}/p^r \simeq W_r\Omega^*_R$ by [Ill79, Cor. I.3.17] and p-torsion freeness of each $W\Omega^i_R$ [Ill79, Cor. I.3.6]. On the other hand, the functor $R \mapsto LW\Omega_R \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/p^r$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{an}}_{\mathbb{F}_p}, \mathcal{D}(\mathbb{Z}))$ commutes with sifted colimits and for each $R \in \mathrm{CAlg}^{\mathrm{poly}}_{\mathbb{F}_p}$, takes the value $W\Omega^*_R \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/p^r \simeq W\Omega^*_R \otimes_{\mathbb{Z}} \mathbb{Z}/p^r$ (using [Ill79, Cor. I.3.6]). Thus, both of the functors in question correspond to $W\Omega^*_{(-)} \otimes_{\mathbb{Z}} \mathbb{Z}/p^r$ (being its left Kan extensions), and hence are equivalent to each other. □

Now, let us recall the notion of Cartier smoothness for characteristic p rings as introduced in [KM21] (and also studied in [KST21]).

Definition 4.1.2. ([KM21]) An $\mathbb{F}_p$-algebra S is called *Cartier smooth* if its cotangent complex $L_{S/\mathbb{F}_p} = L\Omega^1_{S/\mathbb{F}_p}$ is a flat (ordinary) S-module and the inverse Cartier map $\Omega^i_{S^{(1)}/\mathbb{F}_p} \xrightarrow{C^{-1}} H^i(\Omega^*_{S/\mathbb{F}_p})$ is an isomorphism for all $i \geq 0$. An $\mathbb{F}_p$-scheme X is *Cartier smooth* if for every affine open subset of X, its ring of functions is Cartier smooth.

In particular, for Cartier smooth S, the fiber sequence for compositions shows that the relative cotangent complex $L_{S/\mathbb{Z}_p}$ has Tor-amplitude in cohomological degrees $[-1, 0]$, i.e., S is p-quasisyntomic.

Lemma 4.1.3. Let S be a Cartier smooth $\mathbb{F}_p$-algebra. Then, the natural map $L\Omega_S \to \Omega^*_{S/\mathbb{F}_p}$ in $\mathcal{D}(\mathbb{F}_p)$ is an equivalence. This equivalence respects conjugate filtrations, i.e., $\mathrm{Fil}^{\mathrm{conj}}_{\leq \bullet} L\Omega_S \simeq \tau^{\leq \bullet}\Omega^*_{S/\mathbb{F}_p}$.

Proof. Here, $L\Omega_S$ is the derived de Rham complex of S, and the conjugate filtration $\mathrm{Fil}^{\mathrm{conj}}_{\leq \bullet} L\Omega_S$ is defined as the value of the derived functor of the Postnikov filtration construction $R \mapsto \tau^{\leq \bullet}\Omega^*_R$ at S. By construction, the natural map $\mathrm{Fil}^{\mathrm{conj}}_{\leq \bullet} L\Omega_S \to \tau^{\leq \bullet}\Omega^*_S$ enhances the natural map $L\Omega_S \to \Omega^*_S$

as a map of $\mathbb{Z}_{\geq 0}$-indexed increasing filtered objects, as Postnikov filtration and hence conjugate filtration admit Ω^*_S and $L\Omega_S$ respectively as their underlying objects. Thus, in order to show the claim, we need to check that the natural map $\mathrm{Fil}^{\mathrm{conj}}_{\leq i} L\Omega_S \to \tau^{\leq i}\Omega^*_S$ is an equivalence for each $i \geq 0$. By induction, it suffices to check that the induced natural map of i-th associated graded pieces is an equivalence for each $i \geq 0$. By construction, $\mathrm{gr}_i \mathrm{Fil}^{\mathrm{conj}}_{\leq \bullet} L\Omega_S \simeq L(R \mapsto H^i(\Omega^*_{R/\mathbb{F}_p})[-i])(S) \simeq L(R \mapsto \Omega^i_{R^{(1)}/\mathbb{F}_p}[-i])(S) \simeq L\Omega^i_{S^{(1)}}[-i]$ via Cartier isomorphism for polynomial algebras, and hence the natural map $\mathrm{gr}_i \mathrm{Fil}^{\mathrm{conj}}_{\leq \bullet} L\Omega_S \to \mathrm{gr}_i(\tau^{\leq \bullet}\Omega^*_{S/\mathbb{F}_p})$ between i-th associated graded pieces is given by the inverse Cartier map $L\Omega^i_{S^{(1)}}[-i] \simeq \left(\bigwedge^i L_{S^{(1)}/\mathbb{F}_p}\right)[-i] \to H^i(\Omega^*_{S/\mathbb{F}_p})[-i]$. The conditions required for Cartier smoothness of S ensures this map is an equivalence for all $i \geq 0$. □

Proposition 4.1.4. ([BLM21, Th. 9.4.1]) Let S be a Cartier smooth $\mathbb{F}_p$-algebra. Then, the classical de Rham-Witt complex of [Ill79] agrees with the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_S$ of [BLM21]. More precisely, the natural map $\gamma : W\Omega^*_S \to \mathcal{W}\Omega^*_S$ of commutative differential graded algebras in [BLM21, Cor. 4.4.11] is an isomorphism of strict Dieudonné algebras.

Proof. By [BLM21, Th. 9.4.1], the map $\gamma_1 : W_1\Omega^*_S = \Omega^*_{S/\mathbb{F}_p} \to \mathcal{W}_1\Omega^*_S$ is an isomorphism. Thus, it suffices to check the proof of [BLM21, Th. 4.4.12] which reduces the statement to the aforementioned proposition remains valid for S. It suffices to check $W\Omega^*_S$ is a saturated Dieudonné algebra (which is in fact also strict). However, by p-torsion freeness [KM21, Th. 2.8 (i)] and $FW\Omega^i_S = d^{-1}(pW\Omega^{i+1}_S)$ [KM21, Th. 2.8 (ii)] together with the injectivity of F, the claim follows. □

By Proposition 4.1.4 (which is in a sense a reinterpretation of [KM21, Th. 2.8]), we can freely use nice properties of $\mathcal{W}\Omega^*_{(-)}$ explained in [BLM21] for $W\Omega^*_S$. For instance, the functor $R \mapsto \mathcal{W}\Omega^*_R : \mathrm{CAlg}_{\mathbb{F}_p} \to \mathrm{DA}_{\mathrm{str}}$ is a left adjoint functor [BLM21, Cor. 4.1.5]. Also, note that by [BLM21, Rem. 2.7.3] there are quasi-isomorphisms $W\Omega^*_S/p^r W\Omega^*_S \to W_r\Omega^*_S$ for each $r \geq 1$. We no longer distinguish between the two constructions for Cartier smooth algebras.

Lemma 4.1.5. Let S be a Cartier smooth $\mathbb{F}_p$-algebra. Then, the natural map $LW_r\Omega_S \to W_r\Omega^*_S$ in $\mathcal{D}(\mathbb{Z})$ is an equivalence for all $r \geq 1$.

Proof. Let $R \in \mathrm{CAlg}^{\mathrm{poly}}_{\mathbb{F}_p}$. By p-torsion freeness of each $W\Omega^i_R$ [Ill79, Cor. I.3.6], we have an exact sequence of complexes $0 \to W\Omega^*_R/p \xrightarrow{p^r} W\Omega^*_R/p^{r+1} \to W\Omega^*_R/p^r \to 0$. This sequence induces a fiber sequence $W_1\Omega^*_R \to W_{r+1}\Omega^*_R \to W_r\Omega^*_R$ in $\mathcal{D}(\mathbb{Z})$ by [Ill79, Cor. I.3.17], which is natural in R. On the other hand, by Proposition 4.1.4, the argument remains valid for Cartier smooth algebras replacing R. Thus, there is a natural map of fiber sequences in $\mathcal{D}(\mathbb{Z})$ from $LW_1\Omega_S \to LW_{r+1}\Omega_S \to LW_r\Omega_S$ to $W_1\Omega^*_S \to W_{r+1}\Omega^*_S \to W_r\Omega^*_S$. This reduces our problem to the case of $r = 1$, i.e., to check the natural map $L\Omega_S \to \Omega^*_{S/\mathbb{F}_p}$ is an equivalence. This follows from Lemma 4.1.3. □

Recall that each de Rham-Witt complex admits a descending filtration $\mathcal{N}^{\geq \bullet}W\Omega_X$ called *Nygaard filtration*, which behaves as the p-adic filtration on the de Rham complex of the flat p-adic formal scheme with smooth special fiber. For Cartier smooth rings, the Nygaard filtration on $W\Omega^*_S$ takes the form $\mathcal{N}^{\geq i}W\Omega^*_S = \cdots \xrightarrow{d} pVW\Omega^{i-2}_S \xrightarrow{d} VW\Omega^{i-1}_S \xrightarrow{d} W\Omega^i_S \xrightarrow{d} W\Omega^{i+1}_S \xrightarrow{d} \cdots$, where V is the Verschiebung map of $W\Omega^*_S$.

Lemma 4.1.6. ([BMS19, Lem. 8.2] for smooth algebras) Let S be a Cartier smooth $\mathbb{F}_p$-algebra. Then, the composition $\mathcal{N}^{\geq i}W\Omega^*_S \xrightarrow{\varphi/p^i} W\Omega^*_S \xrightarrow{\mathrm{mod}\ p} \Omega^*_{S/\mathbb{F}_p}$ factors through $\tau^{\leq i}\Omega^*_{S/\mathbb{F}_p}$ and maps

$\mathcal{N}^{\geq i+1}W\Omega_S^*$ to zero. Moreover, the induced map $\operatorname{gr}^i_{\mathcal{N}} W\Omega_S^* = \mathcal{N}^{\geq i}W\Omega_S^*/\mathcal{N}^{\geq i+1}W\Omega_S^* \to \tau^{\leq i}\Omega^*_{S/\mathbb{F}_p}$ is a quasi-isomorphism.

Proof. In fact, proof *loc. cit.* works due to [KM21, Th. 2.8]. Alternatively, by Proposition 4.1.4, $W\Omega_S^*$ is a saturated Dieudonné complex, and hence [BLM21, Prop. 8.2.1] applies (where, notationally, $M^* \xrightarrow{\alpha_F} \eta_p M^* \to M^*$ equals the Frobenius φ used here). Note that $\tau^{\leq i}(W\Omega_S^*/p)$ is quasi-isomorphic to $\tau^{\leq i}\Omega^*_{S/\mathbb{F}_p}$. □

Proposition 4.1.7. Let S be a Cartier smooth $\mathbb{F}_p$-algebra. Then, the natural map $LW\Omega_S \to W\Omega_S^*$ in $\operatorname{CAlg}(\mathcal{D}(\mathbb{Z}))$ is an equivalence. This equivalence respects Nygaard filtrations and divided Frobenii.

Proof. The functor $LW\Omega_{(-)}$ takes values in the p-complete derived category $\widehat{\mathcal{D}}(\mathbb{Z}_p)$ in $\mathcal{D}(\mathbb{Z})$, and hence a left adjoint p-completion together with Lemma 4.1.1 and Lemma 4.1.5 provide natural equivalences $LW\Omega_S \simeq \lim_r LW\Omega_S \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/p^r \simeq \lim_r LW_r\Omega_S \simeq \lim_r W_r\Omega_S^*$ in $\mathcal{D}(\mathbb{Z})$. As each of the pro-objects $\{W_r\Omega^i_S\}_r$ has surjective transition maps, the last object is equivalent to $W\Omega_S^*$. On smooth $\mathbb{F}_p$-algebras, this construction reduces to the natural identification $LW\Omega_R = W\Omega_R^*$, and hence the composed natural equivalence is equivalent to the natural map $LW\Omega_S \to W\Omega_S^*$ of question.

Consider the diagram of $\mathbb{E}_\infty$-rings in $\mathcal{D}(\mathbb{Z})$

$$\begin{tikzcd}
\mathcal{N}^{\geq i}LW\Omega_S \arrow[r] \arrow[d, "\varphi_i"'] & \mathcal{N}^{\geq i}W\Omega_S^* \arrow[d, "\varphi/p^i"] \\
LW\Omega_S \arrow[r, "\simeq"'] & W\Omega_S^*
\end{tikzcd}$$

induced from left Kan extensions (and similarly for the canonical inclusion maps in place of divided Frobenii φ/p^i). The top horizontal arrow for $i = 0$ is the same as the bottom horizontal arrow by definition. To prove the top horizontal arrow is an equivalence for all $i \geq 0$, we consider the diagram

$$\begin{tikzcd}
\mathcal{N}^{\geq i+1}LW\Omega_S \arrow[r] \arrow[d] & \mathcal{N}^{\geq i}LW\Omega_S \arrow[r] \arrow[d] & \operatorname{Fil}^{\mathrm{conj}}_{\leq i} L\Omega_S \arrow[d, "\simeq"] \\
\mathcal{N}^{\geq i+1}W\Omega_S^* \arrow[r] & \mathcal{N}^{\geq i}W\Omega_S^* \arrow[r] & \tau^{\leq i}\Omega^*_{S/\mathbb{F}_p}
\end{tikzcd}$$

for each $i \geq 0$, where the horizontal lines are fiber sequences [BMS19, Lem. 8.2], Lemma 4.1.6. By Lemma 4.1.3, we know the right vertical arrow is an equivalence. Now, by induction on i we have the claimed result. □

Remark 4.1.8. Combined with Lemma 4.1.3 and Lemma 4.1.6, Proposition 4.1.7 in particular gives an equivalence $\operatorname{gr}^i_{\mathcal{N}} LW\Omega_S \simeq \operatorname{Fil}^{\mathrm{conj}}_{\leq i} L\Omega_S$ for Cartier smooth S. Note that in the language of prismatic complexes this takes the form $\operatorname{gr}^i_{\mathrm{Nyg}} F^*\mathbb{\Delta}_{S/\mathbb{Z}_p} \simeq \operatorname{Fil}^{\mathrm{conj}}_{\leq i}\overline{\mathbb{\Delta}}_S$; this equivalence itself does not require S to be Cartier smooth, and the base (in this case, $\mathbb{F}_p$) being perfectoid is sufficient to guarantee this.

Let X be an $\mathbb{F}_p$-scheme, and view $W\Omega_X^*$ as a complex in $\operatorname{Shv}_{\mathrm{Ab}}(X_{\text{proét}})$ (with an additional structure of a Dieudonné complex). More precisely, for an $\mathbb{F}_p$-scheme X, each $W_r\Omega^i_X$ defined as the presheaf $(\operatorname{Spec} A)_X \mapsto W_r\Omega^*_A$ on $X^{\mathrm{aff}}_{\text{ét}}$ is a sheaf [Ill79, Prop. I.1.14] (cf. [BLM21, Th. 5.3.7]), and by taking the left adjoint geometric morphism $\nu^* : \mathrm{Shv}_{\mathrm{Ab}}(X_{\text{ét}}) \to \mathrm{Shv}_{\mathrm{Ab}}(X_{\text{proét}})$ we view (each $\nu^* W_r\Omega^i_X$ and) the limit $W\Omega^i_X$ naturally as a sheaf on $X_{\text{proét}}$, cf. [BS15, Prop. 5.6.2]. As in [BMS19], define $W_r\Omega^i_{X,\log}$ to be the image of the map $d\log : \mathbb{G}^{\otimes i}_{m/X} \to W_r\Omega^i_X = a_1 \otimes \cdots \otimes a_i \mapsto d[a_1]/[a_1] \wedge \cdots \wedge d[a_i]/[a_i]$ of pro-étale sheaves on X, and let $W\Omega^i_{X,\log}$ be the (underived) limit of them.

Proposition 4.1.9. ([BMS19, Prop. 8.4] for smooth schemes over perfect base) Let X be a Cartier smooth $\mathbb{F}_p$-scheme. Then, the sequence of complexes of pro-étale sheaves $0 \to W\Omega^i_{X,\log}[-i] \to \mathcal{N}^{\geq i} W\Omega^*_X \xrightarrow{\varphi/p^i - 1} W\Omega^*_X \to 0$ is exact, and $W\Omega^i_{X,\log} \simeq \lim_r W_r\Omega^i_{X,\log}$, where the limit is taken in $\mathcal{D}(\mathbb{Z})$.

Proof. Since each $W\Omega^n_A$ for affine open $\operatorname{Spec} A$ of X is p-torsion free [KM21, Th. 2.8 (i)], the divided Frobenius $\varphi/p^i : \mathcal{N}^{\geq i} W\Omega^*_X \to W\Omega^*_X$ is defined as in the smooth case. Now, the proof in [BMS19, Prop. 8.4] for smooth schemes over perfect fields works *ad verbum*. Let us reproduce the proof *loc. cit.* here for convenience. Let $\operatorname{Spec} A$ be an affine open of X. For degrees $n \neq i$, the map $(\mathcal{N}^{\geq i} W\Omega^*_A)^i \xrightarrow{\varphi/p^i - 1} W\Omega^i_A$ is an isomorphism. Indeed, for $n > i$, $\varphi/p^i - 1 = p^{n-i}F - 1 : W\Omega^n_A \to W\Omega^n_A$, and since $p^{n-i}F$ is p-adically contracting and $W\Omega^n_A$ is p-complete, this map is invertible. For $n < i$, consider the commutative diagram

$$\begin{tikzcd}
W\Omega^n_A \arrow[r, "p^{i-1-n}V"] \arrow[rr, bend right, "1-p^{i-1-n}V"'] & \mathcal{N}^{\geq i} W\Omega^n_A \arrow[r, "\varphi/p^i - 1"] & W\Omega^n_A.
\end{tikzcd}$$

The first map $p^{i-1-n}V$ is an isomorphism by definition. The lower curved arrow is also an isomorphism, since for $n < i-1$, $p^{i-1-n}V$ is p-adically contracting, and for $n = i-1$, $W\Omega^n_A$ is V-complete (which follows from [KM21, Prop. 2.6 (ii)]). Thus, the map $\varphi/p^i - 1$ is an isomorphism.

It remains to study the case of $n = i$, i.e., the map $W\Omega^i_X \xrightarrow{F-1} W\Omega^i_X$. By [Mor19, Cor. 4.1] (extending [Ill79, Th. I.5.7.2] in the smooth case), the sequence $0 \to \{W_r\Omega^i_{X,\log}\}_r \to \{W_r\Omega^i_X\}_r \xrightarrow{F-1} \{W_r\Omega^i_X\}_r \to 0$ is exact in $\mathrm{Pro}(\mathrm{Shv}_{\mathrm{Ab}}(X_{\text{ét}}))$, and by taking the limit, we obtain the exact sequence $0 \to W\Omega^i_{X,\log} \to W\Omega^i_X \xrightarrow{F-1} W\Omega^i_X \to 0$ in $\mathrm{Shv}_{\mathrm{Ab}}(X_{\text{proét}})$ and $\lim^q_r W\Omega^i_{X,\log} \simeq 0$ for $q > 0$. ☐

4.2 Syntomic cohomology of Cartier smooth rings

Using the description of derived de Rham-Witt complexes of Cartier smooth rings from the previous section, we can immediately compute syntomic cohomology complexes $\mathbb{Z}_p(i)$ of Cartier smooth rings. Let us give a minimalistic exposition here sufficient for stating the results (especially in characteristic p), and refer the reader to [BMS19] for more details. Let S be a p-quasisyntomic ring of characteristic p (e.g., a Cartier smooth ring), and let $\mathrm{CAlg}^{\mathrm{QSyn}}_S$ be the full subcategory of the category of ordinary S-algebras consisting of p-quasisyntomic S-algebras T, i.e., whose structure map $S \to T$ is flat and $L_{T/S}$ has Tor amplitude in cohomological degrees $[-1, 0]$. The opposite of this category admits a site structure with respect to the quasisyntomic topology via faithfully flat p-quasisyntomic maps, called the quasisyntomic site of S [BMS19, Var. 4.35]. Similarly, the (opposite of the) category $\mathrm{CAlg}^{\text{proét}}_S$ of weakly étale S-algebras form the pro-étale site of S [BS15]. The derived de Rham-Witt complex construction $LW\Omega_{(-)}$ and its Nygaard filtration satisfy quasisyntomic descent on $\operatorname{Spec} S$, and hence the fiber $\mathbb{Z}_p(i)$ of $\varphi_i - \mathrm{can} : \mathcal{N}^{\geq i} LW\Omega_{(-)} \to LW\Omega_{(-)}$ for each $i \geq 0$ is a quasisyntomic sheaf on $\operatorname{Spec} S$. This fiber $\mathbb{Z}_p(i)$ precisely computes the i-th syntomic cohomology complex of $\operatorname{Spec} S$.

Corollary 4.2.1. ([BMS19, Cor. 8.21] for smooth algebras over perfect fields) Let S be a Cartier smooth $\mathbb{F}_p$-algebra. Then, for each $i \geq 0$, the syntomic cohomology $\mathbb{Z}_p(i)$ in the pro-étale topology of $\operatorname{Spec} S$ is naturally equivalent to $W\Omega^i_{\operatorname{Spec} S, \log}[-i]$. More precisely, the pushforward of the sheaf $\mathbb{Z}_p(i) \in \mathrm{Shv}_{\mathrm{Sp}}((\mathrm{CAlg}_S^{\mathrm{QSyn}})^{\mathrm{op}})$ along the canonical map $(\mathrm{CAlg}_S^{\mathrm{QSyn}})^{\mathrm{op}} \to (\mathrm{CAlg}_S^{\mathrm{proét}})^{\mathrm{op}}$ of sites (induced from $\mathrm{CAlg}_S^{\mathrm{proét}} \subseteq \mathrm{CAlg}_S^{\mathrm{QSyn}}$) is naturally equivalent to $W\Omega^i_{\operatorname{Spec} S, \log}[-i]$.

Proof. Note that S is quasisyntomic. By definition $\mathbb{Z}_p(i)$ fits into the fiber sequence $\mathbb{Z}_p(i) \to \mathcal{N}^{\geq i} LW\Omega_- \xrightarrow{\varphi_i - 1} LW\Omega_-$ of sheaves on $\mathrm{CAlg}_S^{\mathrm{QSyn}}$, which after pushforward can be viewed as a fiber sequence in $\mathrm{Shv}_{\mathrm{Sp}}((\mathrm{CAlg}_S^{\mathrm{proét}})^{\mathrm{op}}) \simeq \mathrm{Shv}_{\mathrm{Sp}}((\operatorname{Spec} S)_{\mathrm{proét}})$. By Lemma 4.2.2 below, objects of $\mathrm{CAlg}_S^{\mathrm{proét}}$ are Cartier smooth. Hence, the map $\mathcal{N}^{\geq i} LW\Omega_- \xrightarrow{\varphi_i - 1} LW\Omega_-$ is equivalent to $\mathcal{N}^{\geq i} W\Omega^*_{\operatorname{Spec} S} \xrightarrow{\varphi/p^i - 1} W\Omega^*_{\operatorname{Spec} S}$ by Proposition 4.1.7. Now, Proposition 4.1.9 implies that there is a natural equivalence $\mathbb{Z}_p(i)_{(\operatorname{Spec} S)_{\mathrm{proét}}} \simeq W\Omega^i_{\operatorname{Spec} S, \log}[-i]$. ☐

Lemma 4.2.2. Let $R \to R'$ be a weakly étale map of $\mathbb{F}_p$-algebras. Following the convention of [BLM21, Not. 9.5.8], set $R^{(1)} = R$ and write R when we view it as an $R^{(1)}$-algebra via Frobenius $R^{(1)} \to R$. Then,

(1) The commutative diagram

$$\begin{tikzcd} R^{(1)} \arrow[r, "F_R"] \arrow[d] & R \arrow[d] \\ R'^{(1)} \arrow[r, "F_{R'}"'] & R' \end{tikzcd}$$

is a pushout square of $\mathbb{F}_p$-algebras.

(2) If R is Cartier smooth, then so is R'.

Proof. (1) The relative Frobenius map $\operatorname{Spec} R' \to \operatorname{Spec} R'^{(1)} \times_{\operatorname{Spec} R^{(1)}} \operatorname{Spec} R$ is weakly étale by assumption [BS15, Prop. 2.3.3 (4)]. On the other hand, it is a universal homeomorphism [Sta23, tag 0CCB], and hence must be an isomorphism [Sta23, tag 0F6V].

(2) From $L_{R'^{(1)}/R^{(1)}} \simeq 0$, one observes $L_{R'^{(1)}/\mathbb{F}_p}$ is equivalent to a flat $R'^{(1)}$-module. As in the proof of [BLM21, Prop. 9.5.11], (1) implies that the inverse Cartier operator $\Omega^\bullet_{R'^{(1)}} \to H^\bullet(\Omega^*_{R'})$ is an isomorphism of graded $R'^{(1)}$-algebras by base change. ☐

Corollary 4.2.3. Let S be a Cartier smooth $\mathbb{F}_p$-algebra. Then, for each $i \geq 0$ and $r \geq 1$, the mod-p^r syntomic cohomology $\mathbb{Z}/p^r(i)$ in the étale topology of $\operatorname{Spec} S$ is naturally equivalent to $W_r\Omega^i_{\operatorname{Spec} S, \log}[-i]$.

Proof. The equivalence is obtained by smashing the equivalence $\mathbb{Z}_p(i) \simeq W\Omega^i_{\operatorname{Spec} S, \log}[-i]$ of Corollary 4.2.1 (pushed forward to $\mathrm{Shv}_{\mathrm{Sp}}((\mathrm{CAlg}_S^{\mathrm{ét}})^{\mathrm{op}})$) with the mod-$p^r$ Moore spectrum $\mathbb{S}/p^r$ pointwisely. By definition $\mathbb{Z}_p(i)/p^r = \mathbb{Z}/p^r(i)$. On the other hand, [KM21, Th. 2.11] implies that $W\Omega^i_{R,\log}/p^r \simeq W_r\Omega^i_{R,\log}$ naturally for any local Cartier smooth $\mathbb{F}_p$-algebra R. As Cartier smoothness is closed under filtered colimits, we know $W\Omega^i_{\operatorname{Spec} S, \log}/p^r \simeq W_r\Omega^i_{\operatorname{Spec} S, \log}$ from étale-local stalk computations. ☐

4.3 $\mathrm{TC}(-, \mathbb{Z}_p)$ of Cartier smooth rings

Recall that for any ring R of characteristic p, its topological Hochschild homology $\mathrm{THH}(R)$ is a connective and p-complete cyclotomic spectrum, and hence $\mathrm{TC}(R) = \mathrm{TC}(\mathrm{THH}(R))$ is p-complete. Let S be a p-quasisyntomic ring of characteristic p. We know that p-complete TC satisfies p-complete faithfully flat descent [BMS19, Cor. 3.4], and in particular have $\mathrm{TC}(-) \simeq \mathrm{TC}(-, \mathbb{Z}_p) \in \mathrm{Shv}_{\mathrm{Sp}}((\mathrm{CAlg}_S^{\mathrm{QSyn}})^{\mathrm{op}})$ and hence $\mathrm{TC}/p^r \in \mathrm{Shv}_{\mathrm{Sp}}((\mathrm{CAlg}_S^{\mathrm{QSyn}})^{\mathrm{op}})$ for each $r \geq 1$ as well. Through the construction of [BMS19, Sec. 7.4], we know that $\mathrm{TC}(-, \mathbb{Z}_p)$ admits a $\mathbb{Z}_{\geq 0}$-indexed descending complete and exhaustive filtration, called the *motivic* filtration, whose i-th associated graded piece is given by the (shift of the) syntomic complex $\mathbb{Z}_p(i)_{\mathrm{Spec}\, S}[2i]$. Of course, the construction fundamentally relies on the quasisyntomic descent and the novel interpretation of p-adic TC by [NS18] in terms of the equalizer of the Frobenius and the canonical map between p-adic TC^- and TP.

For a quasisyntomic $\mathbb{F}_p$-algebra S, let $\mathrm{Shv}_{\mathrm{Sp}}((\mathrm{CAlg}_S^{\mathrm{QSyn}})^{\mathrm{op}}) \xrightarrow{\lambda_*} \mathrm{Shv}_{\mathrm{Sp}}((\mathrm{CAlg}_S^{\mathrm{proét}})^{\mathrm{op}})$ be the natural map induced from the geometric morphism comparing underlying ∞-topoi. Similarly, let $\mathrm{Shv}_{\mathrm{Sp}}((\mathrm{CAlg}_S^{\mathrm{QSyn}})^{\mathrm{op}}) \xrightarrow{\lambda'_*} \mathrm{Shv}_{\mathrm{Sp}}((\mathrm{CAlg}_S^{\mathrm{ét}})^{\mathrm{op}})$ be the analogous natural map.

Corollary 4.3.1. Let S be a Cartier smooth $\mathbb{F}_p$-algebra. Then, there is an isomorphism

$$\pi_i \lambda_* \mathrm{TC}(-, \mathbb{Z}_p) \xrightarrow{\sim} W\Omega^i_{\mathrm{Spec}\, S, \log}$$

in $\mathrm{Shv}_{\mathrm{Ab}}((\mathrm{CAlg}_S^{\mathrm{proét}})^{\mathrm{op}})$ for all $i \geq 0$, and the sheaf $\lambda_* \mathrm{TC}(-, \mathbb{Z}_p)$ is connective. Here, $\pi_i = \pi_i^{\mathrm{proét}}$ is taken with respect to the canonical t-structure on $\mathrm{Shv}_{\mathrm{Sp}}((\mathrm{CAlg}_S^{\mathrm{proét}})^{\mathrm{op}})$ [Lurs, 1.3.2.3].

Proof. By Proposition 2.4.8 applied to the image of the motivic filtration on $\mathrm{TC}(-, \mathbb{Z}_p)$ by λ_*, there is a spectral sequence $E_2^{i,j} = \pi_{-i-j}(\lambda_* \mathbb{Z}_p(-j)[-2j])$ in $\mathrm{Shv}_{\mathrm{Ab}}((\mathrm{CAlg}_S^{\mathrm{proét}})^{\mathrm{op}})$ which converges (conditionally) to $\pi_{-i-j} \lambda_* \mathrm{TC}(-, \mathbb{Z}_p)$. Corollary 4.2.1 tells us that the spectral sequence degenerates at the second page, and that $\pi_i \lambda_* \mathrm{TC}(-, \mathbb{Z}_p) \simeq H^i(\lambda_* \mathbb{Z}_p(i))$, which is isomorphic to $W\Omega^i_{\mathrm{Spec}\, S, \log}$ for $i \geq 0$ and is zero for $i < 0$. ☐

Corollary 4.3.2. Let S be a Cartier smooth $\mathbb{F}_p$-algebra. Then, there is an isomorphism

$$\pi_i \lambda'_* \mathrm{TC}/p^r \xrightarrow{\sim} W_r\Omega^i_{\mathrm{Spec}\, S, \log}$$

in $\mathrm{Shv}_{\mathrm{Ab}}((\mathrm{CAlg}_S^{\mathrm{ét}})^{\mathrm{op}})$ for all $i \geq 0$ and $r \geq 1$, and the sheaf $\lambda'_* \mathrm{TC}/p^r$ is connective. Here, $\pi_i = \pi_i^{\mathrm{ét}}$ is taken with respect to the canonical t-structure on $\mathrm{Shv}_{\mathrm{Sp}}((\mathrm{CAlg}_S^{\mathrm{ét}})^{\mathrm{op}})$ [Lurs, 1.3.2.3].

Proof. From the motivic filtration $\cdots \to \mathrm{Fil}^{\geq 1}\mathrm{TC}(-, \mathbb{Z}_p) \to \mathrm{Fil}^{\geq 0}\mathrm{TC}(-, \mathbb{Z}_p) = \mathrm{TC}(-, \mathbb{Z}_p)$, we get the exhaustive complete $\mathbb{Z}_{\geq 0}$-indexed filtration $\cdots \to (\mathrm{Fil}^{\geq 1}\mathrm{TC}(-, \mathbb{Z}_p))/p^r \to (\mathrm{Fil}^{\geq 0}\mathrm{TC}(-, \mathbb{Z}_p))/p^r \simeq \mathrm{TC}/p^r$ on TC/p^r, i.e., we set $\mathrm{Fil}^{\geq i}\mathrm{TC}/p^r = (\mathrm{Fil}^{\geq i}\mathrm{TC}(-, \mathbb{Z}_p))/p^r$. In particular, its associated graded object is $\mathrm{gr}^i\, \mathrm{TC}/p^r = \mathbb{Z}/p^r(i)$. Applying the right adjoint functor λ'_*, we obtain the exhaustive complete $\mathbb{Z}_{\geq 0}$-indexed filtration $\lambda'_* \mathrm{Fil}^{\geq i}\mathrm{TC}/p^r$. By applying Proposition 2.4.8 to this filtration, we have a spectral sequence $E_2^{i,j} = \pi_{-i-j}(\lambda'_* \mathbb{Z}/p^r(-j)[-2j])$ which (conditionally) converges to $\pi_{-i-j}\lambda'_* \mathrm{TC}/p^r$ in $\mathrm{Shv}_{\mathrm{Ab}}((\mathrm{CAlg}_S^{\mathrm{ét}})^{\mathrm{op}})$. Corollary 4.2.3 tells us that the spectral sequence degenerates at the second page, and that $\pi_i \lambda'_* \mathrm{TC}/p^r \simeq H^i(\lambda'_* \mathbb{Z}/p^r(i))$, which is isomorphic to $W_r\Omega^i_{\mathrm{Spec}\, S, \log}$ for $i \geq 0$ and is zero for $i < 0$. ☐

From now on, let us drop λ_* (resp. λ'_*) and view $\mathrm{TC}(-,\mathbb{Z}_p)$ (resp. TC/p^r) as a pro-étale sheaf (resp. an étale sheaf) of spectra on $\operatorname{Spec} S$.

Proposition 4.3.3. Let S be a Cartier smooth $\mathbb{F}_p$-algebra. Also, let $\pi_i^{\text{ét}}(\mathrm{K}_{\geq 0}/p^r)$ be the étale sheafification of the presheaf $\pi_i(\mathrm{K}_{\geq 0}(-)/p^r)$ on $(\operatorname{Spec} S)_{\text{ét}}$. Then, there is an isomorphism

$$\pi_i^{\text{ét}}(\mathrm{K}_{\geq 0}/p^r) \xrightarrow{\sim} W_r\Omega^i_{\operatorname{Spec} S,\log}$$

for each $i \geq 0$ and $r \geq 1$.

Proof. Consider the composition $\pi_i^{\text{ét}}(\mathrm{K}_{\geq 0}/p^r) \xrightarrow{\mathrm{tr}} \pi_i(\mathrm{TC}/p^r) \to W_r\Omega^i_{\operatorname{Spec} S,\log}$. The second map is the isomorphism of Corollary 4.3.2. To prove that the first map is also an isomorphism, it suffices to check the map induces isomorphisms on stalks. As $\mathrm{K}_{\geq 0}$ and TC/p^r commutes with filtered colimits of rings (see [CMM21, Th. G] for the latter), we are reduced to the statement that the map $\pi_i(\mathrm{K}_{\geq 0}(R)/p^r) \xrightarrow{\mathrm{tr}} \pi_i(\mathrm{TC}(R)/p^r)$ is an isomorphism for R strictly Henselian local of characteristic p, and this holds by [CMM21, Th. C]. ☐

In the next section, we will refine this to Zariski topology in the form of Corollary 4.4.4. For the case of smooth rings over characteristic p perfect fields, this was proved by Geisser-Levine [GL00].

4.4 Geisser-Levine theorem for Cartier smooth rings via syntomic cohomology

In this section, we complete our proof of Theorem 4.4.3 and deduce Corollary 4.4.4 as stated in [KM21, Th. 2.1]. As noted in Remark 4.4.5, these two are equivalent statements. We also remark a quick consequence of the theorem to the higher algebraic K-groups of perfect rings.

We first compute the descent spectral sequence of $\mathrm{TC}(-,\mathbb{Z}_p)$ in pro-étale topology in Proposition 4.4.2. Note that in pro-étale ∞-topos of schemes, hypercomplete objects are Postnikov complete [BS15, Prop. 3.2.3]. Nevertheless, we provide a different proof verifying the Postnikov completeness of $\mathrm{TC}(-,\mathbb{Z}_p)$ and computing the resulting descent spectral sequence.

Lemma 4.4.1. Let $f : \operatorname{Spec} S' \to \operatorname{Spec} S$ be the map given by an ind-étale map $S \to S'$ of commutative $\mathbb{F}_p$-algebras. Then, for each $r \geq 1$ and $j \geq 0$, there is a natural isomorphism $f^*\mathcal{W}_r\Omega^*_{\operatorname{Spec} S} \simeq \mathcal{W}_r\Omega^*_{\operatorname{Spec} S'}$ of étale sheaves of commutative differential graded algebras.

Proof. Consider the square

$$\begin{tikzcd}
\mathrm{Shv}_{\mathrm{Ab}}((\operatorname{Spec} W_r(S))_{\text{ét}}) \arrow[r] \arrow[d, "f_r^*"'] & \mathrm{Shv}_{\mathrm{Ab}}((\operatorname{Spec} S)_{\text{ét}}) \arrow[d, "f^*"] \\
\mathrm{Shv}_{\mathrm{Ab}}((\operatorname{Spec} W_r(S'))_{\text{ét}}) \arrow[r] & \mathrm{Shv}_{\mathrm{Ab}}((\operatorname{Spec} S')_{\text{ét}})
\end{tikzcd}$$

induced from the commutative square

$$\begin{tikzcd}
W_r(S) \arrow[r] \arrow[d] & S \arrow[d] \\
W_r(S') \arrow[r] & S'.
\end{tikzcd}$$

By topological invariance of the étale site, the horizontal arrows induced by pullbacks are equivalences. Moreover, under this equivalence, each $\mathcal{W}_r\Omega^j_{\operatorname{Spec} S}$ corresponds to the quasicoherent sheaf on $(\operatorname{Spec} W_r(S))_{\text{ét}}$ given by the $W_r(S)$-module $\mathcal{W}_r\Omega^j_S$, and similarly for $\mathcal{W}_r\Omega^j_{\operatorname{Spec} S'}$ [BLM21, Th. 5.3.7]. Thus, it suffices to check that the natural map $W_r(S') \otimes_{W_r(S)} \mathcal{W}_r\Omega^*_S \to \mathcal{W}_r\Omega^*_{S'}$ is an isomorphism of graded $W_r(S')$-algebras. Since $W_r(-)$ commutes with filtered colimits, by writing $S \to S' \simeq \operatorname{colim}_\alpha S'_\alpha$ as a filtered colimit of étale S-algebras, we have $W_r(S') \simeq \operatorname{colim}_\alpha W_r(S'_\alpha)$ compatible with maps to $S' \simeq \operatorname{colim}_\alpha S'_\alpha$. Hence, $W_r(S') \otimes_{W_r(S)} \mathcal{W}_r\Omega^*_S \simeq \operatorname{colim}_\alpha W_r(S'_\alpha) \otimes_{W_r(S)} \mathcal{W}_r\Omega^*_S \simeq \operatorname{colim}_\alpha \mathcal{W}_r\Omega^*_{S'_\alpha} \simeq \mathcal{W}_r\Omega^*_{S'}$, where the second isomorphism uses [BLM21, Cor. 5.3.5], while the final isomorphism follows from [BLM21, Cor. 4.3.5]. □

Proposition 4.4.2. Let S be a Cartier smooth $\mathbb{F}_p$-algebra. Then, the followings hold:
(1) The prestable ∞-category of connective pro-étale sheaves $\operatorname{Shv}_{\mathrm{Sp}}((\operatorname{Spec} S)_{\text{proét}})_{\geq 0}$ has enough objects of cohomological dimension ≤ 1 with coefficients in $\{\pi_j \mathrm{TC}(-, \mathbb{Z}_p)\}_{j \in \mathbb{Z}}$ (cf. [CM21, Def. 2.8]).
(2) The pro-étale sheaf $\mathrm{TC}(-, \mathbb{Z}_p)$ is Postnikov complete, and the resulting descent spectral sequence gives functorial exact sequences $0 \to H^1((\operatorname{Spec} S)_{\text{proét}}, W\Omega^{i+1}_{\operatorname{Spec} S, \log}) \to \pi_i \mathrm{TC}(S, \mathbb{Z}_p) \to W\Omega^i_{S, \log} \to 0$ for each $i \geq 0$.

Remark. Similarly, for each $r \geq 1$, the étale descent spectral sequence for $\pi_j(\mathrm{TC}/p^r)$ gives functorial exact sequences $0 \to H^1((\operatorname{Spec} S)_{\text{ét}}, W_r\Omega^{i+1}_{\operatorname{Spec} S, \log}) \to \pi_i(\mathrm{TC}(S)/p^r) \to W_r\Omega^i_{S, \log} \to 0$ for each $i \geq 0$ using Corollary 4.3.2 (cf. [CM21, Cor. 5.20]).

Proof. (1) Note that the category is equivalent to $\operatorname{Shv}_{\mathrm{Sp}^{\mathrm{cn}}}((\operatorname{Spec} S)_{\text{proét}})$ [Lurs, 1.3.5.7], and hence is generated under colimits by representable sheaves. It suffices to check that for each $j \geq 0$ and weakly étale map $\operatorname{Spec} S' \to \operatorname{Spec} S$, there exists a faithfully flat weakly étale map $\operatorname{Spec} S'' \to \operatorname{Spec} S'$ such that $H^i((\operatorname{Spec} S'')_{\text{proét}}, W\Omega^j_{\operatorname{Spec} S, \log}) = 0$ for $i > 1$ by Corollary 4.3.1. Since weakly étale maps can be refined by ind-étale maps [BS15, Th. 2.3.4], we are reduced to check that for each $j \geq 0$ and a map $\operatorname{Spec} S' \to \operatorname{Spec} S$ given by an ind-étale ring map $S \to S'$, we have $H^i((\operatorname{Spec} S')_{\text{proét}}, W\Omega^j_{\operatorname{Spec} S, \log}) = 0$ for $i > 1$. The exact sequence $0 \to W\Omega^j_{\operatorname{Spec} S, \log} \to W\Omega^j_{\operatorname{Spec} S} \xrightarrow{F-1} W\Omega^j_{\operatorname{Spec} S} \to 0$ of Proposition 4.1.9 further reduces this to the vanishing $H^i((\operatorname{Spec} S')_{\text{proét}}, W\Omega^j_{\operatorname{Spec} S}|_{\operatorname{Spec} S'}) = 0$ for $i > 0$. Note that $W\Omega^j_{\operatorname{Spec} S}|_{\operatorname{Spec} S'} \simeq (\mathrm{R}\lim_r \nu^* W_r\Omega^j_{\operatorname{Spec} S})|_{\operatorname{Spec} S'} \simeq \mathrm{R}\lim_r \nu^*(W_r\Omega^j_{\operatorname{Spec} S}|_{\operatorname{Spec} S'})$ using [BS15, Lem. 5.4.1] and that $\operatorname{Spec} S' \to \operatorname{Spec} S$ is weakly étale. We then have

$$\begin{aligned}
&\mathrm{R}\Gamma((\operatorname{Spec} S')_{\text{proét}}, W\Omega^j_{\operatorname{Spec} S}|_{\operatorname{Spec} S'}) \simeq \mathrm{R}\Gamma((\operatorname{Spec} S')_{\text{proét}}, \mathrm{R}\lim_r \nu^*(W_r\Omega^j_{\operatorname{Spec} S}|_{\operatorname{Spec} S'})) \\
&\simeq \mathrm{R}\lim_r \mathrm{R}\Gamma((\operatorname{Spec} S')_{\text{proét}}, \nu^*(W_r\Omega^j_{\operatorname{Spec} S}|_{\operatorname{Spec} S'})) \simeq \mathrm{R}\lim_r \mathrm{R}\Gamma((\operatorname{Spec} S')_{\text{ét}}, W_r\Omega^j_{\operatorname{Spec} S}|_{\operatorname{Spec} S'}) \\
&\simeq \mathrm{R}\lim_r \mathrm{R}\Gamma((\operatorname{Spec} S')_{\text{ét}}, W_r\Omega^j_{\operatorname{Spec} S'}) \simeq \mathrm{R}\lim_r W_r\Omega^j_{S'} \simeq W\Omega^j_{S'},
\end{aligned}$$

where the second equivalence uses the fact that $\mathrm{R}\Gamma$ and $\mathrm{R}\lim$ commutes, the third equivalence uses [BS15, Cor. 5.1.6], the 4th equivalence follows from Lemma 4.4.1, and the 5th equivalence follows from [BLM21, Th. 5.3.7] and its proof. This in particular implies $H^0((\operatorname{Spec} S')_{\text{proét}}, W\Omega^j_{\operatorname{Spec} S}|_{\operatorname{Spec} S'}) = W\Omega^j_{S'}$ and the claimed vanishing for nonzero degrees.
(2) Recall that the quasisyntomic sheaf $\mathrm{TC}(-, \mathbb{Z}_p)$ is hypercomplete. Since the left adjoint geometric morphism $\operatorname{Shv}_{\mathrm{Sp}}((\operatorname{Spec} S)_{\text{proét}}) \to \operatorname{Shv}_{\mathrm{Sp}}((\mathrm{CAlg}^{\mathrm{QSyn}}_S)^{\mathrm{op}})$ is t-exact (in particular preserves ∞-connective objects), the corresponding right adjoint geometric morphism (i.e., the pushforward)

preserves hypercompleteness, and in particular $\mathrm{TC}(-,\mathbb{Z}_p)$ understood as a pro-étale sheaf is hypercomplete. Then, by (1) above and [CM21, Prop. 2.10], we know $\mathrm{TC}(-,\mathbb{Z}_p)$ is in fact a Postnikov complete pro-étale sheaf. In particular, we have a descent spectral sequence $E_2^{i,j} = H^i((\operatorname{Spec} S)_{\text{proét}}, W\Omega^j_{\operatorname{Spec} S,\log})$ which converges conditionally to $\pi_{-i+j}\mathrm{TC}(S,\mathbb{Z}_p)$. By

$$\mathrm{R}\Gamma((\operatorname{Spec} S)_{\text{proét}}, W\Omega^j_{\operatorname{Spec} S}) \simeq W\Omega^j_S$$

computed above in the process of proving (1), we know $E_2^{i,j} = 0$ for $i \neq 0, 1$. Thus, the spectral sequence degenerates at the second page, and we obtain the claimed exact sequence. ☐

Now, we are ready to prove the main theorem of this section:

Theorem 4.4.3. Let S be a local Cartier smooth $\mathbb{F}_p$-algebra. Then, the map $\pi_i\mathrm{K}_{\geq 0}(S,\mathbb{Z}_p) \xrightarrow{\mathrm{tr}} \pi_i\mathrm{TC}(S,\mathbb{Z}_p) \to W\Omega^i_{S,\log}$ given as a composition of the cyclotomic trace and the natural map of Propostion 4.4.2 is an isomorphism for all $i \geq 0$.

Proof. Consider the map

$$\underline{\mathrm{Map}}_{\mathrm{Shv}_{\mathrm{Sp}}((\operatorname{Spec} S)_{\text{proét}})}(h_{(\operatorname{Spec} S)_{\text{proét}}}, \tau_{\leq *}\mathrm{TC}(-,\mathbb{Z}_p)) \to \underline{\mathrm{Map}}_{\mathrm{Shv}_{\mathrm{Sp}}((\operatorname{Spec} S)_{\text{ét}})}(h_{(\operatorname{Spec} S)_{\text{ét}}}, \tau_{\leq *}(\mathrm{TC}/p^r))$$

in $\mathrm{Fun}(\mathbb{Z},\mathrm{Sp})$ induced from the right adjoint geometric morphism ν_*(together with the canonical map $h_{(\operatorname{Spec} S)_{\text{ét}}} \to \nu_* h_{(\operatorname{Spec} S)_{\text{proét}}}$ and with taking mod-p^r cofiber). This induces a map between the associated descent spectral sequences, and in particular by Proposition 4.4.2 induces a commutative square

$$\begin{tikzcd}
\pi_i(\mathrm{TC}(S,\mathbb{Z}_p)) \arrow[r] \arrow[d] & \pi_i(\mathrm{TC}(S)/p^r) \arrow[d] \\
W\Omega^i_{S,\log} \arrow[r, "\mathrm{can}"'] & W_r\Omega^i_{S,\log}.
\end{tikzcd}$$

Thus, the composition $\pi_i\mathrm{TC}(S,\mathbb{Z}_p) \to \lim_r \pi_i(\mathrm{TC}(S)/p^r) \to \lim_r W_r\Omega^i_{S,\log}$ agrees with the map $\pi_i\mathrm{TC}(S,\mathbb{Z}_p) \to W\Omega^i_{S,\log}$ from the pro-étale descent spectral sequence. Taking cyclotomic trace maps into account, we have a commutative diagram

$$\begin{tikzcd}
\pi_i(\mathrm{K}_{\geq 0}(S,\mathbb{Z}_p)) \arrow[r] \arrow[d, "\mathrm{tr}"'] & \lim_r \pi_i(\mathrm{K}_{\geq 0}(S)/p^r) \arrow[d, "\mathrm{tr}"] \arrow[dd, bend left=60] \\
\pi_i(\mathrm{TC}(S,\mathbb{Z}_p)) \arrow[r] \arrow[dr] & \lim_r \pi_i(\mathrm{TC}(S)/p^r) \arrow[d] \\
 & W\Omega^i_{S,\log}.
\end{tikzcd}$$

Let R be a local ind-smooth $\mathbb{F}_p$-algebra, and set $S = R$ in the diagram above. By Geisser-Levine [GL00, Th. 8.3], we know each $\pi_i(\mathrm{K}_{\geq 0}(R))$ is p-torsion free and that the cyclotomic trace induces isomorphisms $\pi_i(\mathrm{K}_{\geq 0}(R)/p^r) \xrightarrow{\mathrm{tr}} \pi_i(\mathrm{TC}(R)/p^r) \to W_r\Omega^i_{R,\log}$ compatible with each other for all $r \geq 1$. Here, the second map is obtained from the étale descent spectral sequence for TC/p^r. Hence, the top horizontal map (due to p-torsion freeness) and the rightmost bent arrow are isomorphisms, i.e., we have the claimed result for $S = R$ ind-smooth over $\mathbb{F}_p$. In particular, $\pi_i(\mathrm{K}_{\geq 0}(R,\mathbb{Z}_p)) \xrightarrow{\mathrm{tr}} \pi_i(\mathrm{TC}(R,\mathbb{Z}_p))$ is split injective.

Now, we consider S as stated above. Let R be an ind-smooth local $\mathbb{F}_p$-algebra which admits a surjection onto S with Henselian kernel [Hoo06]. Consider the commutative diagram

$$\begin{tikzcd}
\pi_i(\mathrm{K}_{\geq 0}(R, \mathbb{Z}_p)) \arrow[r, "\mathrm{tr}"] \arrow[d] & \pi_i(\mathrm{TC}(R, \mathbb{Z}_p)) \arrow[r] \arrow[d] & W\Omega^i_{R,\log} \arrow[d] \\
\pi_i(\mathrm{K}_{\geq 0}(S, \mathbb{Z}_p)) \arrow[r, "\mathrm{tr}"] & \pi_i(\mathrm{TC}(S, \mathbb{Z}_p)) \arrow[r] & W\Omega^i_{S,\log}.
\end{tikzcd}$$

Here, the right horizontal maps are obtained by Proposition 4.4.2. By rigidity [CMM21, Th. A] and the (split) injectivity of tr for R, we have short exact Mayer-Vietoris type sequences $0 \to \pi_i(\mathrm{K}_{\geq 0}(R, \mathbb{Z}_p)) \to \pi_i(\mathrm{TC}(R, \mathbb{Z}_p)) \oplus \pi_i(\mathrm{K}_{\geq 0}(S, \mathbb{Z}_p)) \to \pi_i(\mathrm{TC}(S, \mathbb{Z}_p)) \to 0$, i.e., the left square in the diagram is biCartesian (in particular a pushout square). To check that the right square in the diagram is biCartesian, it suffices to check the map between kernels of horizontal maps of the right square is an isomorphism. By Proposition 4.4.2, this map equals $H^1((\operatorname{Spec} R)_{\text{proét}}, W\Omega^{i+1}_{\operatorname{Spec} R,\log}) \to H^1((\operatorname{Spec} S)_{\text{proét}}, W\Omega^{i+1}_{\operatorname{Spec} S,\log})$, which can be rewritten as the natural map $H^{i+2}(\mathbb{Z}_p(i+1)(R)) \to H^{i+2}(\mathbb{Z}_p(i+1)(S))$ using Corollary 4.3.1. By rigidity [AMMN22, Th. 5.2] of syntomic cohomology, we know this map is an isomorphism. ☐

Corollary 4.4.4. ([KM21, Th. 2.1]) Let S be a local Cartier smooth $\mathbb{F}_p$-algebra. Then, each $\pi_i \mathrm{K}_{\geq 0}(S)$ is p-torsion free and the map $\pi_i(\mathrm{K}_{\geq 0}(S)/p^r) \xrightarrow{\mathrm{tr}} \pi_i(\mathrm{TC}(S)/p^r) \to W_r\Omega^i_{S,\log}$ induced by the cyclotomic trace and the natural map is an isomorphism for all $i \geq 0$ and $r \geq 1$.

Proof. Consider the following diagram

$$\begin{tikzcd}
 & \mathrm{K}^M_i(S)/p^r \arrow[d] \arrow[ddd, bend right=60, two heads, "d\log"'] & & & \\
0 \arrow[r] & \pi_i(\mathrm{K}_{\geq 0}(S))/p^r \arrow[r] \arrow[d, hook] & \pi_i(\mathrm{K}_{\geq 0}(S)/p^r) \arrow[r] \arrow[d, "\simeq"] & \pi_{i-1}(\mathrm{K}_{\geq 0}(S))[p^r] \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \pi_i(\mathrm{K}_{\geq 0}(S, \mathbb{Z}_p))/p^r \arrow[r] \arrow[d, "\simeq"] & \pi_i(\mathrm{K}_{\geq 0}(S, \mathbb{Z}_p)/p^r) \arrow[r] & \pi_{i-1}(\mathrm{K}_{\geq 0}(S, \mathbb{Z}_p))[p^r] \arrow[r] & 0. \\
 & W_r\Omega^i_{S,\log} & & &
\end{tikzcd}$$

The two exact rows are derived from exact sequences induced from mod p^r fiber sequences for $\mathrm{K}_{\geq 0}(S)$ and $\mathrm{K}_{\geq 0}(S, \mathbb{Z}_p)$ respectively, and in particular the two squares in the diagram commute. By Theorem 4.4.3 and Proposition 4.1.4, the bottom right object $\pi_{i-1}(\mathrm{K}_{\geq 0}(S, \mathbb{Z}_p))[p^r]$ is zero. By the usual diagram chasing, we know that among those squares the left vertical arrow is injective and has cokernel isomorphic to $\pi_{i-1}(\mathrm{K}_{\geq 0}(S))[p^r]$.

The left bottom vertical isomorphism is the map induced by the isomorphism of Theorem 4.4.3, and hence by the commutative diagram

$$\begin{tikzcd}
\pi_i(\mathrm{K}_{\geq 0}(S))/p^r \arrow[r] \arrow[d, "\mathrm{tr}"'] & \pi_i(\mathrm{K}_{\geq 0}(S, \mathbb{Z}_p))/p^r \arrow[d, "\mathrm{tr}"] \\
\pi_i(\mathrm{TC}(S))/p^r \arrow[r] \arrow[d] & \pi_i(\mathrm{TC}(S, \mathbb{Z}_p))/p^r \arrow[d] \arrow[dl] \\
\pi_i(\mathrm{TC}(S)/p^r) \arrow[r] & W_r\Omega^i_{S,\log}
\end{tikzcd}$$

the map $\pi_i(\mathrm{K}_{\geq 0}(S))/p^r \to \pi_i(\mathrm{K}_{\geq 0}(S, \mathbb{Z}_p))/p^r \to W_r\Omega^i_{S,\log}$ in the first diagram equals the map obtained as a composition of the cyclotomic trace and the natural map obtained from the étale descent spectral sequence for TC/p^r. Since cyclotomic trace lifts the $d\log$ map on symbols [GH99, Lem. 4.2.3 and Cor. 6.4.1], we know the first diagram is commutative.

Now, since $d\log$ is surjective by definition, the map $\pi_i(\mathrm{K}_{\geq 0}(S))/p^r \to \pi_i(\mathrm{K}_{\geq 0}(S, \mathbb{Z}_p))/p^r$ is surjective and hence an isomorphism. In particular, each $\pi_{i-1}(\mathrm{K}_{\geq 0}(S))$ is p-torsion free as argued in the first paragraph, and the map $\pi_i(\mathrm{K}_{\geq 0}(S))/p^r \simeq \pi_i(\mathrm{K}_{\geq 0}(S)/p^r) \to W_r\Omega^i_{S,\log}$ of question is an isomorphism. □

Remark 4.4.5. Let S be a local Cartier smooth ring.
(1) The p-torsion freeness part of Corollary 4.4.4 for each $\mathrm{K}_i(S)$ $(i \geq 0)$ can also be shown directly; the following argument was provided to the author by Matthew Morrow. Consider the commutative diagram

$$\begin{tikzcd}
 & \mathrm{K}_i^M(S)/p \arrow[dl] \arrow[d] \arrow[dd, twoheadrightarrow, bend left=60, "d\log"] \\
\mathrm{K}_i(S)/p \arrow[r, hook] & \mathrm{K}_i(S, \mathbb{Z}/p) \arrow[d, "\simeq"] \\
 & \Omega^i_{S,\log}.
\end{tikzcd}$$

Since $d\log$ is surjective by definition, the upper vertical arrow is surjective, and hence the horizontal map is surjective (equivalently an isomorphism).
(2) Hence, by (1), mod p^r-versions imply the p-adic version. Since the pro-object $\{\pi_{i+1}(\mathrm{K}_{\geq 0}(S)/p^r)\}_r$ is isomorphic to $\{\mathrm{K}_{i+1}(S)/p^r\}_r$ by (1), we in particular know its transition maps are surjective. Hence, the Milnor exact sequence gives an isomorphism $\pi_i\mathrm{K}_{\geq 0}(S, \mathbb{Z}_p) \to \lim_r \pi_i(\mathrm{K}_{\geq 0}(S)/p^r)$. Composing with the cyclotomic trace induced map, we obtain an isomorphism $\pi_i\mathrm{K}(S, \mathbb{Z}_p) \to W\Omega^i_{S,\log}$.

Remark 4.4.6. Although [KM21] proves Corollary 4.4.4 through a different method, it was indicated in *loc. cit.* that an approach through the motivic filtration would be possible.

Let us record one application of Kelly-Morrow's generalization of Geisser-Levine theorem for perfect rings.

Lemma 4.4.7. Let S be a perfect ring over $\mathbb{F}_p$. Then, there is a natural isomorphism $W\Omega_S^* \simeq W(S)$ of (strict) Dieudonné algebras.

Proof. The natural map $W(S) \otimes^{\mathrm{L}}_{\mathbb{Z}_p} L_{\mathbb{Z}_p/\mathbb{Z}} \to L_{W(S)/\mathbb{Z}}$ is an equivalence, since its cofiber $L_{W(S)/\mathbb{Z}_p} \simeq W(S) \otimes^{\mathrm{L}}_S L_{S/\mathbb{F}_p}$ vanishes. On the other hand, from the fiber sequence $\mathbb{F}_p \otimes^{\mathrm{L}}_{\mathbb{Z}_p} L_{\mathbb{Z}_p/\mathbb{Z}} \to L_{\mathbb{F}_p/\mathbb{Z}} \to L_{\mathbb{F}_p/\mathbb{Z}_p}$

whose second map is an equivalence, we have $(L_{\mathbb{Z}_p/\mathbb{Z}})/p \simeq 0$. Combining these, we know $\widehat{L_{W(S)/\mathbb{Z}}} \simeq 0$ (i.e., the p-completed cotangent complex vanishes). In particular, the p-completed de Rham complex $\widehat{\Omega}^*_{W(S)}$ is isomorphic to $W(S)$ as a Dieudonné algebra. Note that $W(S)$ is a strict Dieudonné algebra [BLM21, Ex. 2.5.6]. Now, for any strict Dieudonné algebra A^*, we have natural isomorphisms $\mathrm{Mor}_{\mathrm{CAlg}_{\mathbb{F}_p}}(S, A^0/VA^0) \simeq \mathrm{Mor}_F(W(S), A^0) \simeq \mathrm{Mor}_{\mathrm{DA}}(\widehat{\Omega}^*_{W(S)}, A^*) \simeq \mathrm{Mor}_{\mathrm{DA}_{\mathrm{str}}}(W(S), A^*)$ by [BLM21, Prop. 3.6.3 and Var. 3.3.1]. Combined with Proposition 4.1.4, we know $\mathcal{W}\Omega^*_S \simeq W\Omega^*_S \simeq W(S)$ as strict Dieudonné algebras. ☐

Proposition 4.4.8. (Hiller) Let S be a perfect ring over $\mathbb{F}_p$. Then, $\mathrm{K}_i(S)$ is a $\mathbb{Z}[1/p]$-module for all $i > 0$.

Proof. Since S is a filtered colimit of colimit perfections of finitely presented perfect $\mathbb{F}_p$-algebras (as the colimit perfection is a left adjoint functor), we can assume that S itself is of the form S'_{perf} for some S' finite type over $\mathbb{F}_p$. In particular, the underlying topological space of $\mathrm{Spec}\, S$ is the same as that of $\mathrm{Spec}\, S'$. By Zariski descent spectral sequence (note that K is a Postnikov complete sheaf by [CM21, Th. 3.2]), we are further reduced to the case of S being local. By Corollary 4.4.4, we know that each $\mathrm{K}_i(S)$ is p-torsion free and that $\mathrm{K}_i(S)/p \simeq \mathrm{K}_i(S, \mathbb{Z}_p)/p \simeq W\Omega^i_{S,\log}/p$, which is zero by Lemma 4.4.7. Thus, p acts invertibly on $\mathrm{K}_i(S)$. ☐

Remark. After the argument above was written down, the author was informed later that the idea of explaining Proposition 4.4.8 through Geisser-Levine theorem appeared about two months earlier in [AMM22]. It might be interesting to see if the approach using de Rham-Witt complexes can be generalized to cover certain mixed characteristic cases.

4.5 More on prismatic complexes of Cartier smooth rings

In this section, we collect and verify some consequences of 4.1 relevant to prismatic cohomology complexes of Cartier smooth rings and their p-torsion free liftings following [BL22]. In the stated results, reference to [BLM21] or [BL22] in their titles indicate that the corresponding results stated in there for regular Noetherian $\mathbb{F}_p$-algebras or ind-smooth maps remain valid for Cartier smooth rings or Cartier smooth maps; we explain how necessary modifications can be made.

Lemma 4.5.1. ([BLM21, Th. 10.1.1] for smooth algebras over perfect fields) Let S be a Cartier smooth $\mathbb{F}_p$-algebra. Then, there is a natural equivalence $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}\, S/\mathbb{Z}_p) \simeq W\Omega_S$ in $\mathrm{CAlg}(\widehat{\mathcal{D}}(\mathbb{Z}_p))$.

Remark 4.5.2. In Lemma 4.5.1, note that the right hand side fits into equivalences $LW\Omega_S \simeq \mathcal{W}\Omega_S \simeq W\Omega_S$ in $\mathrm{CAlg}(\widehat{\mathcal{D}}(\mathbb{Z}_p))$ by Proposition 4.1.7 and Proposition 4.1.4. Also, one can replace $\mathrm{Spec}\, S$ by any Cartier smooth $\mathbb{F}_p$-scheme X, as both sides satisfy Zariski descent.

Proof of Lemma 4.5.1. It suffices to check that there is a natural equivalence $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}\, S/\mathbb{Z}_p) \simeq LW\Omega_S$ due to Proposition 4.1.7, and this follows from [BLM21, Prop. 10.2.16 and Prop. 10.3.1]. For convenience, let us briefly explain their contents here. Write the restriction of $LW\Omega_{(-)}$ on quasisyntomic $\mathbb{F}_p$-algebras as $A(-) \in \mathrm{CAlg}(\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{QSyn}}_{\mathbb{F}_p}, \widehat{\mathcal{D}}(\mathbb{Z}_p)))$. After the base change along $\widehat{\mathcal{D}}(\mathbb{Z}_p) \to \mathcal{D}(\mathbb{F}_p)$, it admits an equivalence $L\Omega_{(-)} \to \mathbb{F}_p \otimes^{\mathrm{L}}_{\mathbb{Z}_p} A(-)$ as commutative algebra objects of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{QSyn}}_{\mathbb{F}_p}, \mathcal{D}(\mathbb{F}_p))$. By [BLM21, Prop. 10.2.16], there is a morphism $\mathrm{R}\Gamma_{\mathrm{crys}}(-/\mathbb{Z}_p) \to A(-)$ in $\mathrm{CAlg}(\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{QSyn}}_{\mathbb{F}_p}, \widehat{\mathcal{D}}(\mathbb{Z}_p)))$ which intertwines augmentation maps $\mathrm{R}\Gamma_{\mathrm{crys}}(-/\mathbb{Z}_p) \xrightarrow{\epsilon^{\mathrm{crys}}} id$

and $A(-) \xrightarrow{\epsilon} id$. To show the morphism is an equivalence, it suffices to check its image in $\mathrm{CAlg}(\mathrm{Fun}(\mathrm{CAlg}_{\mathbb{F}_p}^{\mathrm{QSyn}}, \mathcal{D}(\mathbb{F}_p)))$ is an equivalence. This reduction mod p gives a morphism $L\Omega_{(-)} \to L\Omega_{(-)}$ in $\mathrm{CAlg}(\mathrm{Fun}(\mathrm{CAlg}_{\mathbb{F}_p}^{\mathrm{QSyn}}, \mathcal{D}(\mathbb{F}_p)))$ compatible with the augmentation map $L\Omega_{(-)} \xrightarrow{\epsilon^{\mathrm{dR}}} id$. Now, [BLM21, Proof of Prop. 10.3.1] assures the morphism is an equivalence. ☐

Corollary 4.5.3. Let X be a Cartier smooth $\mathbb{F}_p$-scheme. Then, there is a natural equivalence $\mathrm{R}\Gamma_{\mathbbl{\Delta}}(X) \simeq W\Omega_X$ in $\mathrm{CAlg}(\widehat{\mathcal{D}}(\mathbb{Z}_p))$. In particular, for $X = \operatorname{Spec} S$, we have $\mathbbl{\Delta}_S \simeq W\Omega_S$.

Proof. The equivalence is given by the composition of the crystalline comparison equivalence

$$\gamma_{\mathbbl{\Delta}}^{\mathrm{crys}} : \mathrm{R}\Gamma_{\mathbbl{\Delta}}(X) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(X/\mathbb{Z}_p)$$

of [BL22, Th. 4.6.1] and the equivalence of Lemma 4.5.1. ☐

We can refine the above equivalence as a natural equivalence between filtered objects.

Proposition 4.5.4. (cf. [BL22, Prop. 5.3.8]) Let S be a Cartier smooth $\mathbb{F}_p$-algebra. Then, there is an equivalence $\mathrm{Fil}_{\mathrm{N}}^{\bullet} F^*\mathbbl{\Delta}_{S/\mathbb{Z}_p} \simeq \mathcal{N}^{\geq \bullet} W\Omega_S$ in $\widehat{\mathcal{DF}}(\mathbb{Z}_p)$ refining the equivalence of Corollary 4.5.3.

Remark 4.5.5. In other words, we have an equivalence $\mathrm{Fil}_{\mathrm{N}}^{\bullet} \mathbbl{\Delta}_S \simeq \mathcal{N}^{\geq \bullet} W\Omega_S$ in $\widehat{\mathcal{DF}}(\mathbb{Z}_p)$ through the equivalence of [BL22, Th. 5.6.2].

Proof of Proposition 4.5.4. In [BL22, Prop. 5.3.8], S is stated to be a regular Noetherian $\mathbb{F}_p$-algebra. In fact, the proof *loc. cit.* works in our case as well. More precisely, the natural map $\mathrm{Fil}^{\bullet}(\varphi) : \mathcal{N}^{\geq \bullet} LW\Omega_S \to p^{\bullet}\mathbbl{\Delta}_{S/\mathbb{Z}_p}$ (left Kan extended from the Frobenius map) realizes the source as the Nygaard filtration on $F^*\mathbbl{\Delta}_{S/\mathbb{Z}_p}$ for all $S \in \mathrm{CAlg}_{\mathbb{F}_p}^{\mathrm{an}}$, and Proposition 4.1.7 identifies it with $\mathcal{N}^{\geq \bullet} W\Omega_S$ in Cartier smooth case. ☐

Corollary 4.5.6. (cf. [BL22, Rem. 5.4.3]) Let R be a p-torsion-free commutative ring whose mod-p quotient $\overline{R} := R/pR$ is a Cartier smooth $\mathbb{F}_p$-algebra. Then, there are natural equivalences $\mathrm{R}\Gamma_{\mathrm{crys}}(\overline{R}/\mathbb{Z}_p) \simeq \mathbbl{\Delta}_{\overline{R}} \simeq \widehat{\mathrm{dR}}_R \simeq \widehat{\Omega}_R$ in $\mathrm{CAlg}(\widehat{\mathcal{D}}(\mathbb{Z}_p))$. In particular, for a p-adic formal scheme $\mathfrak{X}$ flat over $\operatorname{Spf} \mathbb{Z}_p$ whose special fiber $X = \mathfrak{X} \times_{\operatorname{Spf} \mathbb{Z}_p} \operatorname{Spec} \mathbb{F}_p$ is Cartier smooth, there is a natural equivalence $\mathrm{R}\Gamma_{\mathrm{crys}}(X/\mathbb{Z}_p) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(\mathfrak{X})$.

Proof. Combine Corollary 4.5.3, [BL22, Th. 5.4.2] (absolute de Rham comparison), and [BL22, Prop. E.12]. ☐

Remark 4.5.7. (cf. [BL22, Rem. 4.7.4]) Let R be a p-torsion-free commutative ring whose quotient R/pR is Cartier smooth. Then, by [BL22, Prop. E.12], $L\widehat{\Omega}_R^n \simeq \widehat{\Omega}_R^n$ for all $n \geq 0$. In particular, one can compute the conjugate filtration $\mathrm{Fil}_{\leq \bullet}^{\mathrm{conj}} \widehat{\Omega}_R^{\not{D}}$ of the p-complete diffracted Hodge cohomology R; by the previous computation $\mathrm{gr}_n^{\mathrm{conj}} \widehat{\Omega}_R^{\not{D}} \simeq L\widehat{\Omega}_R^n[-n]$ is concentrated in degree n for all $n \in \mathbb{Z}_{\geq 0}$, and hence $\mathrm{Fil}_{\leq \bullet}^{\mathrm{conj}} \widehat{\Omega}_R^{\not{D}} \simeq \tau^{\leq \bullet} \widehat{\Omega}_R^{\not{D}}$.

Proposition 4.5.8. (cf. [BL22, Prop. 5.8.2]) Let R be a p-torsion-free commutative ring whose quotient R/pR is Cartier smooth. Then, for each $n \in \mathbb{Z}$, the prismatic complex $\mathbbl{\Delta}_R\{n\}$ is Nygaard-complete.

Proof. Let us explain how the proof *loc. cit.* can be adjusted to prove the claimed case. The result follows from [BL22, Lem. 5.8.3], which in turn relies on [BL22, Cor. 5.2.11]:

Lemma 4.5.9. (cf. [BL22, Cor. 5.2.11]) Let (A, I) be a bounded prism and let R be a p-completely flat $\overline{A}(=A/I)$-algebra for which $\overline{A}/p\overline{A} \to R/pR$ is a Cartier smooth map (see Lemma 4.5.11 and above). Then, $F^*\Delta_{R/A}$ is Nygaard-complete.

Although the statement *loc. cit.* is made for ind-smooth $\overline{A}/p\overline{A}$-algebras, the proof works without any change. As explained in the proof of [BL22, Prop. 5.8.2 and Lem. 5.8.3], the question is reduced to the statement that the relative Nygaard filtration on $F^*\Delta_{(\overline{A}^m \otimes R)/A^m}$ is complete. By Lemma above, it suffices to check that for each $m \geq 0$, the natural map $\overline{A}^m/p\overline{A}^m \to (\overline{A}^m \otimes R)/p(\overline{A}^m \otimes R)$ is Cartier smooth.[1] From the p-torsion-freeness of R and the fact that (A^m, I^m) is transverse, we know $\overline{A}^m \otimes (R/pR) \simeq (\overline{A}^m \otimes R)/p(\overline{A}^m \otimes R)$, and hence $\overline{A}^m/p\overline{A}^m \otimes_{\mathbb{F}_p} (R/pR) \simeq (\overline{A}^m \otimes R)/p(\overline{A}^m \otimes R)$. In particular, by Lemma 4.5.11, the natural map $\overline{A}^m/p\overline{A}^m \to (\overline{A}^m \otimes R)/p(\overline{A}^m \otimes R)$ is Cartier smooth. ☐

Proposition 4.5.10. (cf. [BL22, Prop. 5.7.9]) Let R be a p-torsion-free commutative ring whose quotient R/pR is Cartier smooth. Then, for each $n \in \mathbb{Z}$, the map $\mathrm{Fil}^\bullet(\varphi\{n\}) : \mathrm{Fil}^\bullet_{\mathrm{N}}\Delta_R\{n\} \to \Delta_R^{[\bullet+n]}$ exhibits the source as the connective cover of the target with respect to the Beilinson t-structure.

Proof. The proof *loc. cit.* works *ad verbum*, using Remark 4.5.7 and Propsition 4.5.8 above. ☐

Recall that a map $A \to B$ of commutative $\mathbb{F}_p$-algebras is Cartier smooth [BL22, Def. E.10] if it is flat, $L_{B/A}$ is a flat discrete B-module, and the inverse Cartier map $\Omega^i_{B^{(1)}/A} \to H^i(\Omega^*_{B/A})$ is an isomorphism of $B^{(1)} = A \otimes_{\varphi, A} B$-modules for all $i \geq 0$.

Lemma 4.5.11. Let $A \to B$ a map of commutative $\mathbb{F}_p$-algebras and let $A' \to B'$ be its base change along a flat map $A \to A'$. If $A \to B$ is Cartier smooth, then so is $A' \to B'$. If $A \to A'$ is faithfully flat, then the converse holds.

Proof. Since properties of being flat and of being an isomorphism are preserved under base change and satisfy faithfully flat descent, it suffices to check that $B'^{(1)} \otimes_{B^{(1)}} C^{-1}_{B/A} : B'^{(1)} \otimes_{B^{(1)}} \Omega^i_{B^{(1)}/A} \to B'^{(1)} \otimes_{B^{(1)}} H^i(\Omega^*_{B/A})$ agrees with $C^{-1}_{B'/A'} : \Omega^i_{B'^{(1)}/A'} \to H^i(\Omega^*_{B'/A'})$ as graded $B'^{(1)}$-algebra maps (where C^{-1} denotes the inverse Cartier map). Consider the commutative diagram

$$\begin{tikzcd}
A \arrow[rrr, "\varphi"] \arrow[dr] \arrow[dd] & & & A \arrow[dl] \arrow[dr] \arrow[dd] & \\
& B \arrow[r] \arrow[dd] & B^{(1)} \arrow[rr] \arrow[dd] & & B \arrow[dd] \\
A' \arrow[rrr, "\varphi"] \arrow[dr] & & & A' \arrow[dl] \arrow[dr] & \\
& B' \arrow[r] & B'^{(1)} \arrow[rr] & & B',
\end{tikzcd}$$

where the four vertical edges of the outer cube are Frobenius. By definition the left, upper, and bottom sides of the left hexahedron are pushout squares, and hence the right side of it is a pushout square. In particular, we have $B'^{(1)} \otimes_{B^{(1)}} \Omega^i_{B^{(1)}/A} \simeq \Omega^i_{B'^{(1)}/A'}$. As the right side of the outer cube is a pushout square, we know the front square of the right prism is a pushout square. Combined with

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[1] Recall that (A^0, I^0) is the transverse prism $(\mathbb{Z}[a_0, a_1^{\pm 1}, a_2, ...]^\wedge_{(p,a_0)}, (a_0))$, $(A^\bullet, I^\bullet)$ is the associated cosimplicial (transverse) prism given by coproducts of copies of (A^0, I^0), and $\overline{A}^m = A^m/I^m$.

$\Omega^*_{B/A} \simeq \Omega^*_{B/B^{(1)}}$ and $\Omega^*_{B'/A'} \simeq \Omega^*_{B'/B'^{(1)}}$ [Sta23, tag 0CCC], this gives $B'^{(1)} \otimes_{B^{(1)}} \Omega^*_{B/A} \simeq \Omega^*_{B'/A'}$. These identify source and target of both of the maps (using the flatness assumption). To prove the claim, it suffices to check that under these identifications, $B'^{(1)} \otimes_{B^{(1)}} C^{-1}_{B/A}$ agrees with $C^{-1}_{B'/A'}$ on degree 1 elements of the form $d(1\otimes g')$ for each $g' \in B'$. Let $g' = \sum_j f'_j \otimes g_j \in B' = A' \otimes_A B$. Its image in $B'^{(1)}$ is $1\otimes g' = \sum_j 1\otimes(f'_j\otimes g_j) = \sum_j \varphi(f'_j)\cdot 1\otimes(1\otimes g_j)$, and hence $d(1\otimes g') = \sum_j \varphi(f'_j) d(1\otimes(1\otimes g_j))$. The image of the latter element by $C^{-1}_{B'/A'}$ is $\sum_j \varphi(f'_j) \cdot [(1\otimes g_j)^{p-1} d(1\otimes g_j)]$. On the other hand, $d(1\otimes g')$ corresponds to $\sum_j \varphi(f'_j) \otimes d(1\otimes g_j)$ in $B'^{(1)} \otimes_{B^{(1)}} \Omega^1_{B^{(1)}/A}$, and its image by $B'^{(1)} \otimes_{B^{(1)}} C^{-1}_{B/A}$ is $\sum_j \varphi(f'_j) \otimes [g_j^{p-1} dg_j]$, which in turn corresponds to $C^{-1}_{B'/A'}(d(1\otimes g'))$ as computed above. ☐

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