Twisted eleven-dimensional supergravity and exceptional simple infinite dimensional super-Lie algebras

by

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Abstract

We study a class of formal moduli problems associated to eleven-manifolds with a rank 6 transversely holomorphic foliation and a transverse Calabi-Yau structure. On $\mathbb{R} \times \mathbb{C}^5$, the (-1)-shifted tangent complex of this formal moduli problem is L_∞-equivalent to a Lie-2 extension of an infinite dimensional exceptional simple super-Lie algebra called $E(5|10)$. In the first part of this thesis, we equip this formal moduli problem with the structure of a perturbative classical field theory in the Batalin-Vilkovisky formalism. Conjecturally, this theory describes the minimal twist of eleven-dimensional supergravity. We present strands of evidence for this conjecture by computing dimensional reductions and comparing with expected descriptions of twists of supergravity in lower dimensions, and by identifying the residual symmetries of the putative twist of eleven-dimensional supergravity within the symmetries of our theory.

In the second half of the thesis, we construct particular backgrounds for our theory which we conjecture are twisted avatars of the $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds of eleven-dimensional supergravity. To justify this conjecture, we study spaces of supergravity states on these backgrounds. We find that their characters match with prior expressions enumerating multi-gravitons in $AdS_4 \times S^7$ and $AdS_7 \times S^4$ respectively, and also admit specializations recovering generating functions of representation theoretic and enumerative significance, such as the MacMahon function. We study a decomposition of the state spaces we construct that exhibits them as direct sums of modules for two other exceptional linearly compact super-Lie algebras, $E(1|6)$ and $E(3|6)$ respectively. We conclude with some speculations about how our results can be used as input for holographic techniques.

(TODO: DEDICATE)

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(TODO: MORE)

Contents

Chapter 1

Introduction

1.1 Synopsis

In his historical article *the Unreasonable Effectiveness of Mathematics in the Natural Sciences*, Wigner comments on the utility of mathematical theories in elucidating the underlying structure of physical theories. To appreciate the surprising nature of this relationship, it is perhaps useful to disambiguate the two uses of the word *theory*. In physics, the word theory refers to an a priori non-rigorous procedure for computing a nominally measurable quantity about a collection of physical processes, from the knowledge of their participants and the geometry of the space where said processes occur. For instance, quantum field theories aim to model the dynamics of fundamental particles and how they interact by describing these particles as excitations of fields that are localized in space and time. Specifying the theory involves making sense of a putative integrand defined on a space of field configurations.

For a particular class of physical theories, supersymmetric field theories, the nature of their relationship with mathematical theories is even more surprising. Indeed, such physical theories can often be so tightly codified by mathematical theories that physical insights can be used to make novel predictions about the mathematical objects used to describe them. In particular, supersymmetric field theories admit a vast web of dualities, whereby two a priori distinct theories make the same predictions. Such dualities have frequently suggested the existence of nontrivial mathematical equivalences, partially reversing the flow of information in the phenomenon Wigner observed.

Several instances of this paradigm have emerged in the past several decades. A horribly non-exhaustive list of examples includes:

- 2d $\mathcal{N} = (2,2)$ sigma-models and their behavior under T-duality as the impetus for mirror symmetry
- Integrable sectors of families of 2d $\mathcal{N} = (2,2)$ gauge theories as a home for geometric constructions of quantum groups and their representations
- 3d $\mathcal{N} = 4$ theories and their behavior under 3d mirror symmetry as responsible for symplectic duality
- 4d $\mathcal{N} = 2$ gauge theory as a home for Donaldson's invariants of smooth 4-manifold

- 4d $\mathcal{N} = 4$ gauge theory and its behavior under S-duality as an organizing framework for the geometric Langlands program
- ...

These examples and many others are all unified in that the principal mediator between physical theories and mathematical theories is a construction known as *supersymmetric twisting*.

Twisting, which we expand on below, takes as input a supersymmetric field theory, and produces a simpler theory that only retains those processes and participants preserved by certain symmetries. Less impressionistically, supersymmetric field theories depend on data such as a fixed metric on the spacetime on which they are formulated. The twist however, only depends on much simpler structure on spacetime such as topology or a complex structure. Modern mathematical physics has triumphantly furnished many tools for investigating theories with such dependence. The measurements one can perform together with rules for combining them organize into structures such as vertex algebras, $\mathbb{E}_n$-algebras, and higher categories. Moreover, the construction of these kinds of objects from equations of motion describing twists often follow the paradigms of geometric representation theory. As such, twisting forms a basic conduit for mathematical codifications of the expectations of supersymmetric physics.

Until recently, theories of supergravity, had largely abstained from this dialogue. Though Einstein's theory of gravitation was perhaps one of the first illustrations of the phenomenon hi-lighted by Wigner, theories of supergravity had arguably inspired modern representation theory and geometry far less than their non-gravitational counterparts. This was in part due to the lack of a suitable analog of the twisting construction. Indeed, in the language introduced in the first paragraph of this introduction, gravity is subtle in part because the participants of the physical processes at hand are the underlying space itself - defining what it means for such processes to be unchanged by the relevant symmetries required new ideas. In [CL16], Costello and Li provided the missing ideas, which we describe below.

1.2 Supsersymmetric field theories and their twists

Twists of supersymmetric field theories have been a subject of active and fruitful study for many years. The notion of twisting was introduced by Witten and dates back to the first example [Wit88], in which a topological field theory related to Donaldson invariants of four-manifolds was constructed as a twist of 4d $\mathcal{N} = 2$ supersymmetric gauge theories. Shortly afterwards (thirty years ago), the field had expanded to include a diverse set of examples related to topological field theory [BBRT91].

For many years, twisting constructions were motivated by the goal of producing topological field theories, and correspondingly emphasized the role of particular modifications of Lorentz symmetry via "twisting homomorphisms" designed to produce scalar supercharges. From a more modern perspective, though, twisting a supersymmetric field theory is a fixed point construction, whereby one considers the derived fixed points with respect to the action of a one-parameter subgroup generated by a square zero supercharge. The data of a square zero supercharge determines a transversely holomorphic foliation (THF) on spacetime, and morally, the twist of the supersymmetric field theory picks out those field configurations that are flat with respect to a partial leaf-wise connection on this foliation.

The variety of appropriate square-zero elements was studied systematically in [ESW21], and has deep relations not only to the classification of twists, but to the construction of supersymmetric field theories via the pure spinor formalism [Ced14, EHSW21]. The name "pure spinor" originates because the variety of square-zero elements always contains a minimal stratum that is related to the space of Cartan pure spinors; in turn, these correspond to choices of complex structure on the spacetime, at least when the dimension is even. As such, any d-dimensional supersymmetric theory that admits a twist admits a *minimal* or *holomorphic* twist, in which $n = \lfloor d/2 \rfloor$ of the translations act nontrivially, and the THF is as minimal possible rank. Other twists can be understood as further twists of this holomorphic theory.

The importance and interest of holomorphic twists of supersymmetric gauge theories was emphasized in the work of Costello [Cos13], who developed the notion mathematically, building on earlier work on holomorphic field theories in the physics literature [LMNS96, BCOV94, for example]. Si Li pushed applications of holomorphic QFT further in his work on the higher genus B-model (see [Li11], and later [CL15] together with Costello). In the physics literature, further constructions related to holomorphic twists had appeared in the context of supersymmetric indices; see [Röm06], for example.

1.3 Twisted supergravity

In a supergravity theory, local supersymmetry is already gauged, so that the standard notion of twisting is *a priori* not a sensible operation. Twisted supergravity thus needs to be understood differently. In their seminal paper [**?**], Costello and Li gave a definition of twisted supergravity. As supersymmetries are gauged, the BV complex for a theory of supergravity (which iis $\mathbb{Z} \times \mathbb{Z}/2$-graded) includes a field of bidegree $(-1, 1)$ referred to as the *bosonic ghost.* The field is a section of a spinor bundle on spacetime, and its equations of motion stipulate that it be covariantly constant and square-zero. Costello and Li defined twisted supergravity to be perturbative supergravity in a background where the bosonic ghost takes a nonzero VEV. A feature of this definition is that it implies that twisting a supersymmetric field theory in the usual sense amounts to coupling it to such a supergravity background. Qualitatively, while a twist of a supersymmetric field theory describes deformations of partially flat connections or partially flat sections of bundles on THF manifolds, twists of supergravity describe in-part, deformations of the underlying THF itself.

For examples of supergravity in ten dimensions that arise as low energy limits of the closed string sector of a type II superstring theory, Costello-Li were able to give concrete conjectural descriptions of the twists of supergravity using ideas from topological string theory. A topological string theory is determined entirely by categorical data, and as such its closed string sector can be described in terms of Hochschild invariants of categories and its variants. Moreover, most examples of topological string theories related to twists are variants of the topological A and B-models. As such their closed string sector involves a description of the closed string B-model known as Kodaira-Spencer gravity or BCOV theory. This theory was introduced in [BCOV94] as the closed-string field theory of the topological B-model on Calabi–Yau three-folds. It was further extended to all Calabi–Yau manifolds in [CL15]. It is a sort of holomorphic version of gravity which at the genus zero describes fluctuations of the complex structure of the Calabi–Yau manifold. We recall relevant aspects of Kodaira–Spencer gravity in §2.5.1. The free limits of the conjectural descriptions of Costello-Li were recovered using the pure spinor superfield formalism in [SW21].

In this thesis, we study twists of eleven-dimensional supergravity. Eleven-dimensional supergravity is supposed to describe the low energy limit of M-theory, and at present, it is not known how to associate to M-theory some sort of categorical data. Nevertheless, we find that eleven-dimensional supergravity and its BPS solitons enjoy a hidden piece of structure - they seem to be controlled by three of the five exceptional simple infinite dimensional super-Lie algebras [Kac98].

1.4 Outline

This thesis is organized as follows:

The chapters of this thesis are drawn from original works done in collaboration with I. Saberi and B. Williams [RSW21], [RW22] and concern features of a certain holomorphic-topological field theory in eleven-dimensions with curious representation theoretic properties and of potentially central physical relevance.

We begin in chapter 2, which follows [RSW21], where we propose an interacting BV theory defined on manifolds locally of the form $S \times X$ where S is a smooth oriented 1-manifold and X is a Calabi-Yau 5-fold. The phase space of the theory describes a certain deformation of the cotangent bundle of the moduli of Calabi-Yau structures on X. On $\mathbb{R} \times \mathbb{C}^5$, the field contents matches with that of the minimal twist of the eleven-dimensional supergravity multiplet that was computed in [SW21]. Accordingly, we conjecture that the our theory is the minimal twist of eleven dimensional supergravity as an interacting theory.

We test this conjecture across multiple strands. We compute a local character of our theory iand find a refinement of the point-class contribution to the DT series of $\mathbb{C}^3$, which was previously recovered as a specialization of the graviton contribution to the index of M-theory on flat space [Nek09]. We also compare dimensional reductions of our theory down to dimensions 10 and 5 with conjectural descriptions of twists of supergravity in those dimensions. Most pertinently, we show that the linearized BRST cohomology of our model is L_∞-equivalent to a Lie-2 extension of an exceptional simple infinite dimensional super-Lie algebra called $E(5|10)$ in Kac's classification [Kac98]. This suggests an avenue by which the representation theory of $E(5|10)$ can play a role in the study of BPS invariants associated to eleven dimensional supergravity.

In chapter 3 which mostly follows [RW22], we take some first steps down this avenue and in doing so, find that infinite dimensional exceptional simple super-Lie algebras are even more intimately related to our model than previously though. We introduce conjectural twisted versions of the $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds of eleven-dimensional supergravity, via mimicry of their construction in the physical theory via backreaction of M2 and M5 branes respectively. We once again test this conjecture on several fronts. First, we construct spaces of supergravity states localized at the asymptotic boundary of AdS as costalks of certain factorization algebras associated to the compactifications of our theories along the transverse spheres. Characters of these spaces of supergravity states recover known counts of gravitons in $AdS_4 \times S^7$ and $AdS_4 \times S^7$ in the physics literature (see for example [BBMR08, **?**]), and admit specializations that recover familiar formulae in representation theory such as vacuum character of the $\mathcal{W}_{1+\infty}$ algebra.

Next we turn to symmetry considerations. We argue that our spaces of supergravity states admit actions of the minimally twisted 3d $\mathcal{N} = 8$ and 6d $\mathcal{N} = (2, 0)$ superconformal algebras respectively, in keeping with the appearance of the untwisted versions of these algebras as isometries of the physical $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds. Curiously, the states spaces constructed are naturally acted upon by two other infinite dimensional exceptional simple super-Lie algebras dubbed $E(1|6)$ and $E(3|6)$ in Kac's classification [Kac98]. These super-Lie algebras contain the relevant twisted superconformal algebras as finite dimensional subalgebras, and their presence can be thought of as analogous to the appearance of the Witt algebra in 2d chiral conformal field theory. We study decompositions of the spaces of supergravity states as modules for these super-Lie algebras and compute characters of the steps of the decomposition. Once again, the characters recover formulae of both physical and mathematical significance, such as superconformal indices for the 3d $\mathcal{N} = 8$ BLG theory and the 6d $\mathcal{N} = (2,0)$ abelian tensor multiplet, and vacuum characters of $\mathcal{W}_N$ algebras.

We conclude with some speculations on how the results in this thesis might be used as input for holographic considerations.

1.5 Cavents, conventions, and context

We conclude this introduction with some conventions and commentary on context. This is a mathematical thesis about objects in physics, and engages with ideas from physics on a slightly more direct level than many physically-inspired mathematical questions. As such, some remarks are in order regarding the formalism we use for discussing field theories. In particular, with the exception of the final, speculative, subsection, though this thesis features physics words, nearly all statements made are ones either in derived deformation theory or in the representation theory of super-Lie algebras.

Throughout this thesis, the language we use for describing field theories is a verison of the Batalin-Vilkovisky (BV) formalism, as articulated in the work of Costello [Cos11] and Costello-Gwilliam [CG17], [CG21b]. Here, a perturbative Lagrangian classical field theory is described by a sheaf of odd-shifted symplectic formal moduli problems, thought of as describing local solutions to Euler-Lagrange equations, imposed in a homotopically correct way. Using standard results in derived deformation theory [Lur11], [Pri19], such an object can be described using Lie-theoretic data, and we use this dictionary abundantly without comment. For a pithy reference, almost all of the field-theoretic package we use is exposited in the first few sections of [ESW20]

On occasion, we will discuss classical field theories that are accommodated by variants of the BV formalism intended for possibly non-Lagrangian theories, or theories with constrained fields. Here, the characteristic odd-shifted symplectic structure of the BV formalism bearing witness to the variational origin of the Euler-Lagrange equations is relaxed to be only shifted Poisson or shifted presymplectic. Though we do not rely on these structures in an essential way, we do make reference to examples; standard references for such variants of the BV formalism are [BY16], [SW20], [Rab].

Lastly, we on occasion use the language of factorization algebras as developed by Costello-Gwilliam [CG17], [CG21b], which provide a model for local observables in perturbative quantum field theories along with rules for combining them. We utilize only a very small piece of the structure of a factorization algebra; a factorization algebra is roughly a cosheaf of algebras, and in this thesis, we more or less only care about the costalk at a point as a vector space. Nevertheless, we state certain constructions using (classical) factorization algebras, partly as a means to keep an eye towards future work.

Chapter 2

Twisted Eleven-dimensional supergravity

In 1978, Cremmer, Julia, and Scherk constructed a theory of supergravity in eleven dimensions. Its fields include a metric, a gravitino, and a 3-form, and the action includes a Chern-Simons term for the 3-form []. In this chapter, we propose a perturbative description of the minimal twist of eleven-dimensional supergravity. Thanks to a straightforward classification [] which we recall below, there are essentially two types of twists which deform the flat background $\mathbb{R}^{11}$:

- The minimal (or holomorphic) twist. The resulting twisted theory is defined on $\mathbb{R} \times \mathbb{C}^5$ and is holomorphic along $\mathbb{C}^5$ and topological along the 'time' direction $\mathbb{R}$. The twist is $SU(5)$ invariant and involves a fixed choice of a Calabi–Yau form on $\mathbb{C}^5$.
- The non-minimal twist. The resulting twisted theory is defined on $\mathbb{R}^7 \times \mathbb{C}^2$ and is holomorphic along $\mathbb{C}^2$ and topological along $\mathbb{R}^7$. The twist is $SU(2)$ invariant and involves a fixed choice of a hyperkähler structure on $\mathbb{C}^2$.

Twists of eleven-dimensional supergravity were first studied by Costello in [Cos16, Cos17], where he introduced a five-dimensional holomorphic-topological gauge theory with a noncommutative deformation and argued that it describes eleven-dimensional supergravity in a particular Ω-background. In separate work [EH21], the free-field limit of the non-minimal twist was derived from the component-field multiplet as used in the physics literature.

Most relevant to the theory we study is the work of [SW21], where the authors used the pure spinor formalism to describe the *free field limit* of the minimal twist of eleven-dimensional supergravity on flat space, building on work of Cederwall [Ced10b, Ced10a] that constructed the eleven-dimensional theory in the pure spinor formalism.

The resulting theory is $\mathbb{Z}/2$ graded; the grading is inherited from the totalization of the $\mathbb{Z} \times \mathbb{Z}/2$-grading by ghost number and parity in the original, untwisted, eleven-dimensional theory. In §2.1 we introduce a full interacting theory in the Batalin–Vilkovisky (BV) formalism, the free limit of which is the twist computed in [SW21].

Our main conjecture is that this interacting BV theory is equivalent to the minimal twist of eleven-dimensional supergravity on flat space.

Conjecture 2.0.1 ([]). The eleven-dimensional holomorphic-topological theory on $\mathbb{R} \times \mathbb{C}^5$ that we will define in §2.1 is equivalent to twisted supergravity on $\mathbb{R}^{11}$, where the bosonic ghost takes a value corresponding to a holomorphic supercharge.

In this chapter we will provide evidence for this conjecture on a variety of fronts:

- In §2.1.5, we show that the character of the local operators in our 11-dimensional theory agrees with the index of 11-dimensional supergravity.
- In §2.3, we will show that this theory has all of the expected residual supersymmetries present after performing the holomorphic twist.
- In §2.4, we describe the non-minimal twist of eleven-dimensional supergravity as a background of our theory on $\mathbb{R} \times \mathbb{C}^5$. This non-minimal twist is invariant for the group $G_2 \times SU(2)$. We find a match with a conjectural description of this $G_2 \times SU(2)$ invariant twist formulated by Costello in [Cos16] and further developed in [RY19].
- In §2.5, we compute dimensional reductions and show that our model is compatible with descriptions of twists of theories of supergravity in lower dimensions. For instance, reducing along a circle in a complex plane agrees with the conjectural description of the $SU(4)$ twist of type IIA supergravity in [**?**].

2.0.1 A geometric description of the model

Our eleven-dimensional theory is defined for instance on manifolds of the form $S \times X$ where S is a smooth oriented real one-dimensional manifold, X is a Calabi–Yau five-fold. More generally, it can be placed on eleven-manifolds with a rank 6 complexified foliation and a transverse Calabi-Yau structure. A description of the formal moduli problem underlying our model as describing deformations of some geometric structure attached to $S \times X$ is desirable, yet unfortunately nebulous. However, there are particular loci which do have a clear deformation-theoretic meaning, as they describe for instance deformations of a chosen transversely holomorphic foliation of rank 9. This articulates a sense in which the theory can be viewed as having a "holomorphic gravitational" flavor.

We describe this locus. In our theory there is an even field

$$\mu^{1,1} \in C^\infty(S) \otimes \Omega^{0,1}(X, \mathrm{T}_X)$$

where T_X denotes the holomorphic tangent bundle on X. Locally, $\mu^{1,1}$ can be decomposed as Beltrami-like differential

$$\mu^{1,1} = \mu^i_j(z, \bar{z}, t) \mathrm{d}\bar{z}_i \frac{\partial}{\partial z_j}.$$

This field $\mu^{1,1}$ is a component of the metric which survives in the twisted theory; see §2.1.4. We will see that it participates in an equation of motion, which after a suitable choice of gauge, encodes the aforementioned partial deformation of complex structure on X.

Next, there are two fields

$$\gamma^{1,0} \in C^\infty(S) \otimes \Omega^{1,0}(X), \quad \gamma^{1,2} \in C^\infty(S) \otimes \Omega^{1,2}(X).$$

The field $\gamma^{1,2}$ is a component of the supergravity 3-form C-field that survives in the twisted theory, and he field $\gamma^{1,0}$ can be interpreted as a component of the one-form which is a ghost-for-a-ghost of the C-field.[1]

There is a background where the equation of motion involving the fields $\mu^{1,1}$, $\gamma^{1,0}$, and $\gamma^{1,2}$ reads

$$\bar{\partial}\mu^{1,1} + \frac{1}{2}[\mu^{1,1}, \mu^{1,1}] + \Omega^{-1} \vee \left(\partial\gamma^{1,0} \wedge \partial\gamma^{1,2}\right) = 0$$

Additionally, there are the conditions that $\mu^{1,1}$ preserve the holomorphic volume form on X and that all fields are locally constant along the topological direction S.

If we work in a background where one of $\gamma^{1,0}$ or $\gamma^{1,2}$ is zero, then we see that $\mu^{1,1}$ is exactly a deformation of complex structure along X. In terms of the eleven-dimensional geometry, these field configurations describe deformations of the natural transverse holomorphic foliation (THF structure) on $S \times X$ which further preserve the holomorphic volume form along the leaves. We further unpack the equations of motion in more general backgrounds in §2.1.4, but leave a complete study for future work.

2.0.2 Appearance of exceptional super-Lie algebras

The gauge symmetries of a field configuration in any theory appear as the zero-th cohomology of the (-1)-tangent complex at the specified point, and as such form a Lie algebra. The simplest field configuration in the twisted theory is the flat background; this corresponds to considering to our eleven-dimensional theory on $\mathbb{R} \times \mathbb{C}^5$, where we equip $\mathbb{C}^5$ with the flat holomorphic volume form. In this case, we find a striking relationship to a certain infinite-dimensional simple super Lie algebra studied by Kac and collaborators [Kac98, KR02].[2]

Theorem 2.0.2. The global symmetry algebra of the eleven-dimensional theory on $\mathbb{R} \times \mathbb{C}^5$ is equivalent to a central extension of the exceptional simple super Lie algebra $E(5|10)$.

In particular, correlation functions involving observables of the eleven-dimensional theory will be constrained by the infinite-dimensional symmetry algebra $E(5|10)$. Given our main conjecture that the interacting eleven-dimensional BV theory on $\mathbb{R} \times \mathbb{C}^5$ is the twist of supergravity, we obtain the following.

Conjecture 2.0.3. A central extension of the super Lie algebra $E(5|10)$ is a symmetry of supergravity on $\mathbb{R}^{11}$ which preserves the background where the bosonic ghost takes value equal to a holomorphic supercharge Q.

2.0.3 Relationship to other twists of supergravity

Through the dimensional reduction of our eleven-dimensional theory we will find a match with Costello and Li's descriptions of twists of 10-dimensional supergravity in terms of BCOV theory. In

[1] The C-field has a gauge symmetry of the form $\delta C = \mathrm{d}B$ where B is a two-form. This ghost B field has an additional gauge symmetry $\delta B = \mathrm{d}A$ for A a one-form. The field $\gamma^{1,0}$ is a component of A.

[2] The twist of the eleven-dimensional gravity multiplet was computed in [SW21] by reducing it to a particular pure spinor model based on $\mathrm{Gr}(2,5)$; a possible relationship between this pure spinor model and $E(5|10)$ was first pointed out by Martin Cederwall in [Ced21].

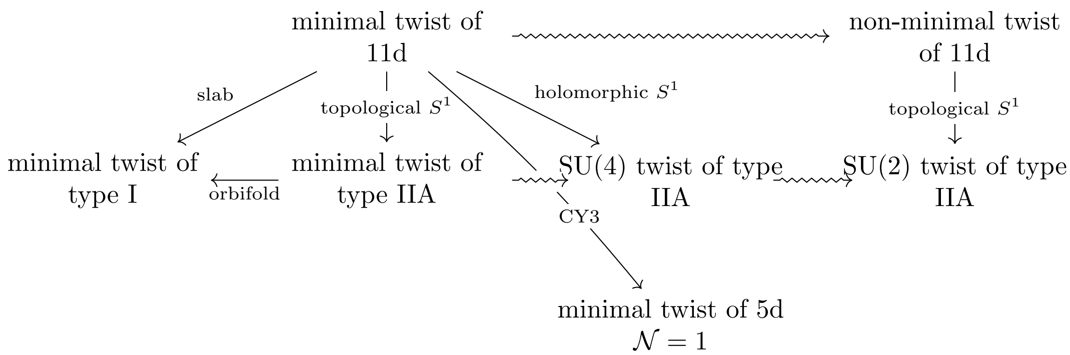


Table 2.1: Relations between various twists

Table 2.1, we provide a summary of the comparisons with twists of supergravity in lower dimensions we perform. The squiggly arrows denote further twists, and the solid arrows denote various kinds of reductions.

In §2.5.3 we will show that the reduction along $\{0\} \times \mathbb{R} \times \{0\} \subset \mathbb{C}^4 \times \mathbb{C} \times \mathbb{R}$ is equivalent to the $SU(4)$ twist of type IIA supergravity. The topological string approach does not lead to a description of the minimal, or holomorphic, twist of type IIA supergravity. In §2.5.6, we describe the reduction along the line $\{0\} \times \mathbb{R} \subset \mathbb{C}^5 \times \mathbb{R}$ to obtain a conjectural description of the holomorphic $SU(5)$ twist of type IIA supergravity. In §2.5.5, we show that the reduction along the interval $\{0\} \times [0,1] \subset \mathbb{C}^5 \times [0,1]$ with certain boundary conditions on the endpoints of the interval is equivalent to the twist of type I supergravity. We further observe in §2.5.6.1 that the twist of type I arises as the fixed points of a natural $\mathbb{Z}/2$ action on the minimal twist of IIA. Next, in §2.5.7 we study the reduction along a Calabi-Yau three-fold to obtain a conjectural description of the minimal twist of 5d $\mathcal{N} = 1$ supergravity. Finally, we show in §2.4 how the non-minimal twist of eleven-dimensional supergravity arises as a further twist of our holomorphic twist.

2.1 The minimal twist of eleven-dimensional supergravity

In this section we define the central theory of study within the Batalin–Vilkovisky (BV) formalism. The theory will be defined on any eleven-dimensional manifold of the form $X \times L$, where X is a Calabi–Yau five-fold and L is a smooth oriented one-manifold.

In [SW21], we showed that the underlying free limit of the theory we consider here is the free limit of the minimal twist of 11-dimensional supergravity on $\mathbb{C}^5 \times \mathbb{R}$. The goal of this section is to introduce interactions at the level of the minimal twist. The main result is Theorem 2.1.1 where we show that the theory is consistent within the BV formalism.

In the remainder of the paper we discuss further consistency with supersymmetry and string theory, but in this section we focus mostly on the theory as a partially holomorphic, partially topological, theory of gravity. Nevertheless, we do provide some preliminary justification towards the relationship to physical supergravity including a matching of indices in §2.1.5.

2.1.1 Divergence-free vector fields

We set up some notations and conventions in the context of complex geometry. Let V be a holomorphic vector bundle on a d dimensional complex manifold X. If j is an integer, we let $\Omega^{0,j}(X,V)$ denote the space of anti-holomorphic Dolbeault forms of type j on X with values in V. The $\bar\partial$ operator $\bar\partial\colon \Omega^{0,j}(X,V) \to \Omega^{0,j+1}(X,V)$ defines the Dolbeault complex of V:

$$\Omega^{0,\bullet}(X,V) = \left(\Omega^{0,j}(X,V)[-j],\ \bar\partial\right).$$

This is a free (smooth) resolution for the sheaf of holomorphic sections of V.

Suppose X is a Calabi–Yau manifold with holomorphic volume form Ω. The divergence $\partial_\Omega(\mu)$ of a holomorphic vector field μ is defined by the formula

$$\partial_\Omega(\mu) \wedge \Omega = L_\mu(\Omega)$$

where, on the right hand side, we mean the Lie derivative of Ω with respect to μ.

Let T_X denote the holomorphic tangent bundle and consider its Dolbeault complex $\Omega^{0,\bullet}(X,\mathrm{T}_X)$ resolving the sheaf of holomorphic vector fields. The divergence operator extends to the Dolbeault complex to yield a map of cochain complexes

$$\partial_\Omega\colon \Omega^{0,\bullet}(X,\mathrm{T}_X) \to \Omega^{0,\bullet}(X).$$

The resulting complex of sheaves

$$\begin{array}{ccc} \underline{0} & & \underline{1} \\ & & \\ \Omega^{0,\bullet}(X,\mathrm{T}_X) & \xrightarrow{\partial_\Omega} & \Omega^{0,\bullet}(X), \end{array} \tag{2.1}$$

resolves the sheaf of holomorphic divergence-free vector fields $\mathrm{Vect}_0(X)$. The anti-holomorphic Dolbeault degrees and the $\bar\partial$ operator are left implicit.

There is a direct way to extend the Lie bracket of vector fields to the complex (2.1). Denote by μ an element of $\Omega^{0,\bullet}(X,\mathrm{T}_X)$ and ν an element of $\Omega^{0,\bullet}(X)$ (for simplicity in notation, we will not expand the anti-holomorphic dependence). The Lie bracket defined by the formulas

$$[\mu,\mu'] = L_\mu \mu'$$
$$[\mu,\nu] = L_\mu \nu$$

is compatible with ∂_Ω and endows (2.1) with the structure of a sheaf of dg Lie algebras. We will refer to this sheaf by the symbol $\mathcal{L}_0(X)$, or just $\mathcal{L}_0$ if X is understood.

The sheaf $\mathcal{L}_0$ has the structure of a *local* dg Lie algebra [CG21b, §3.1.3]. This means that, as a graded sheaf, $\mathcal{L}_0$ is the smooth sections of a graded vector bundle, and its differential and Lie bracket are given by differential and bidifferential operators respectively.

2.1.1.1 Recall that an L_∞ algebra is a $\mathbb{Z}$-graded vector space $\mathcal{L}$ together with the datum of a square-zero, degree $+1$ derivation $\delta_\mathcal{L}$ of the free commutative graded algebra $\mathrm{Sym}\left(\mathcal{L}^\vee[-1]\right)$. The

Chevalley–Eilenberg cochain complex is

$$(\mathrm{Sym}\,(\mathcal{L}^\vee[-1])\,,\delta_{\mathcal{L}})\,.$$

The Taylor components of $\delta_{\mathcal{L}}$ define higher brackets $\{[-]_k\}_{k=1,2,\dots}$ where $[-]_k\colon \mathcal{L}^{\otimes k}\to\mathcal{L}[2-k]$. The condition that the differential $\delta_{\mathcal{L}}$ is square-zero is equivalent to the higher Jacobi relations.

An L_∞ morphism $\Phi:\mathcal{L}\rightsquigarrow\mathcal{L}'$ is the same datum as a map of commutative dg algebras

$$\Phi^*:\mathrm{C}^\bullet(\mathcal{L}')\to\mathrm{C}^\bullet(\mathcal{L}) \tag{2.2}$$

between their respective Lie algebra cochains. It follows from this that *any* automorphism Φ of the free commutative algebra on $\mathcal{L}^\vee[-1]$ defines a new model of the L_∞ algebra $\mathcal{L}$, for which the Chevalley–Eilenberg differential is obtained by conjugating $\delta_{\mathcal{L}}$ by Φ, and where Φ itself defines the L_∞ isomorphism.

$\mathcal{L}_0$ is the sheaf (2.1) resolving divergence-free vector fields equipped with the dg Lie algebra structure constructed in the previous section. We consider the following automorphism of $\mathrm{Sym}(\mathcal{L}_0^\vee[-1])$, defined by its action on generators:

$$\Psi_\infty:\nu\mapsto 1-e^{-\nu},\quad \mu\mapsto e^{-\nu}\mu. \tag{2.3}$$

This map defines a new model of the L_∞ algebra with underlying graded vector space the same as (2.1), which we will call $\mathcal{L}_\infty$.[3] The formulas for the automorphism above clearly arise from maps of vector bundles and hence endow $\mathcal{L}_\infty$ with the structure of a local L_∞ algebra, meaning all operations are given by polydifferential operators.

The notation refers to the fact that this new model has nonvanishing L_∞ brackets of every order. It is this new model that we will use to define the eleven-dimensional theory of twisted supergravity.

We can describe the L_∞ structure on our new model $\mathcal{L}_\infty$ explicitly. Recall that we have two types of elements: $\mu\in\mathrm{PV}^{1,\bullet}$ and $\nu\in\mathrm{PV}^{0,\bullet}[-1]$. (Here, and in what follows, we will use the symbol $\mathrm{PV}^{i,\bullet}$ for the Dolbeault resolution of *holomorphic polyvector fields;* by definition, this is the complex $\Omega^{0,\bullet}(X,\wedge^i\mathrm{T}_X)$.) The first few nonzero brackets are

$$\begin{aligned}[]
[\mu]_1 &= \bar\partial\mu+\partial_\Omega\mu\\
[\mu_1,\mu_2]_2 &= \partial_\Omega(\mu_1\wedge\mu_2)\\
[\nu,\mu_1,\mu_2]_3 &= \partial_\Omega(\nu\mu_1\wedge\mu_2)
\end{aligned}$$

For $k\geq 2$ the general formula for the k-ary brackets are

$$\begin{aligned}[]
[\nu_1,\dots,\nu_{k-2},\mu_1,\mu_2]_k &= \partial_\Omega(\nu_1\cdots\nu_k\mu_1\wedge\mu_2)\\
[\nu_1,\dots,\nu_{k-3},\mu_1,\mu_2,\gamma]_k &= \nu_1\cdots\nu_{k-3}(\mu\wedge\mu')\vee\partial\gamma.\\
[\nu_1,\dots,\nu_{k-2},\mu,\gamma]_k &= \nu_1\cdots\nu_{k-2}\mu\vee\partial\gamma.
\end{aligned}$$

[3]We are being slightly abusive and using the symbols ν,μ dually as coordinates, or operators, on the graded linear space $\mathcal{L}[1]$.

2.1.2 Theories of BF type

2.1.2.1 Suppose that $\mathcal{L}$ is an L_∞ algebra with L_∞ operations $\{[-]_k^{\mathcal{L}}\}_{k=1,2,\ldots}$ and that $(\mathcal{A}, \mathrm{d}_{\mathcal{A}})$ is a commutative dg algebra. The graded vector space $\mathcal{L} \otimes \mathcal{A}$ is equipped with the natural structure of an L_∞ algebra with operations $\{[-]_k\}_{k=1,2,\ldots}$ defined by

$$[x \otimes a]_1 = [x]_1^{\mathcal{L}} \otimes a + (-1)^{|x|} x \otimes \mathrm{d}_{\mathcal{A}} a$$
$$[x_1 \otimes a_1, \ldots, x_k \otimes a_k]_k = [x_1, \ldots, x_k]_k^{\mathcal{L}} \otimes (a_1 \cdots a_k), \qquad k \geq 2.$$

We apply this construction, taking $\mathcal{L}$ to be the sheaf resolving divergence-free holomorphic vector fields on a Calabi–Yau manifold X equipped with either the strict dg Lie algebra structure $\mathcal{L}_0(X)$ or its non-strict L_∞ structure $\mathcal{L}_\infty(X)$. The algebra $\mathcal{A}$ will be the smooth de Rham complex $(\Omega^\bullet(S), \mathrm{d}_S)$ where S is a smooth manifold.

We thus obtain the structure of an dg Lie algebra on $\mathcal{L}_0(X) \otimes \Omega^\bullet(S)$ or an L_∞ algebra $\mathcal{L}_\infty(X) \otimes \Omega^\bullet(S)$. These define equivalent local L_∞ algebras on the product manifold $X \times S$.

2.1.2.2 Associated to any local L_∞ algebra is a classical field theory in the BV formalism. Let $\mathcal{L}$ be a local L_∞ algebra on some manifold M, it is the sheaf of sections of some graded vector bundle L. For a section $A \in \mathcal{L}$, introduce the 'higher curvature map' defined by the formula

$$\mathsf{F}_A = [A]_1 + \frac{1}{2}[A, A]_2 + \frac{1}{3!}[A, A, A]_3 + \cdots .$$

The fields of the associated BV theory are pairs

$$(A, B) \in \mathcal{L}[1] \oplus \mathcal{L}^![-2].$$

Here $\mathcal{L}^!$ denotes the sheaf of sections of the bundle $L^* \otimes \mathrm{D}ens$, where $\mathrm{D}ens$ is the bundle of densities. The shifted symplectic BV pairing is the obvious integration pairing between A and B.

The action functional reads $S_{\mathrm{BF}} = \int_M B\, \mathsf{F}_A$ which leads to the equations of motion $\mathsf{F}_A = 0$ and $\mathsf{D}_A B = 0$ where D_A is the higher covariant derivative along A. We refer to this as the "BF theory" associated to $\mathcal{L}$.

We thus obtain a theory in the BV formalism on the product manifold $X \times S$ associated to both local L_∞ algebras $\mathcal{L}_0(X) \otimes \Omega^\bullet(S)$ and $\mathcal{L}_\infty(X) \otimes \Omega^\bullet(S)$.

2.1.2.3 For concreteness, we spell out the fields of the theories we have constructed on $X \times S$. In both cases, the space of fields equipped with the linear BRST operator is

$$\begin{array}{cccc} -n & -n+1 & -1 & 0 \\ \hline \Omega^0(X;S) \xrightarrow{\partial} & \Omega^1(X;S) & \mathrm{PV}^1(X;S) \xrightarrow{\partial_\Omega} & \mathrm{PV}^0(X;S). \end{array} \tag{2.4}$$

We denote the fields $(\beta, \gamma, \mu, \nu)$ respectively. We are using the shorthand notation

$$\begin{aligned} \Omega^i(X;S) &= \Omega^{i,\bullet;\bullet}(X;S) \\ &= \oplus_{j,k} \mathrm{PV}^{i,j}(X) \otimes \Omega^k(S)[-j-k]. \end{aligned}$$

which is equipped with the $\bar\partial + \mathrm{d}_S$ operator and similarly for $\mathrm{PV}^i(X;S)$.

The natural pairing between $\mathrm{PV}^i(X;S)$ and $\Omega^i(X;S)$ is of degree $-\dim_{\mathbb{C}}(X) - \dim_{\mathbb{R}}(S)$. As such, the $\mathbb{Z}$-grading indicated in (2.9) equips the sheaf of fields with a degree (-1) pairing, provided that we choose the shift to be given by

$$n = \dim_{\mathbb{C}}(X) + \dim_{\mathbb{R}}(S) - 1. \tag{2.5}$$

The pairing is defined by the formula

$$\int_{X\times S}^{\Omega} \mu \vee \gamma + \int_{X\times S}^{\Omega} \nu\beta$$

where $\int_{X\times S}^{\Omega} \alpha \overset{def}{=} \int_{X\times S} \alpha \wedge \Omega$.

We have constructed two equivalent descriptions of the BF theory which share the linear BRST complex (2.9). Explicitly, the action functional for BF theory associated to the local dg Lie algebra $\mathcal{L}_0(X) \otimes \Omega^\bullet(S)$ is

$$S_{BF,0} = \int^{\Omega} \left[\beta \wedge (\bar\partial + \mathrm{d}_S)\nu + \gamma \wedge (\bar\partial + \mathrm{d}_S)\mu + \beta \wedge \partial_\Omega \mu + \frac{1}{2}[\mu,\mu] \vee \gamma + [\mu,\nu]\beta\right]. \tag{2.6}$$

As in the Lie algebra structure of this strict model, notice that the Schouten–Nijenhuis bracket appears explicitly.

The action functional of BF theory associated to $\mathcal{L}_\infty(X) \otimes \Omega^\bullet(S)$ is non polynomial. In fact, it is related to the BCOV action functional via dimensional reduction (see §2.5). Explicitly, this action functional is

$$S_{BF,\infty} = \int^{\Omega} \left[\beta \wedge (\bar\partial + \mathrm{d}_S)\nu + \gamma \wedge (\bar\partial + \mathrm{d}_S)\mu + \beta \wedge \partial_\Omega \mu + \frac{1}{2}\frac{1}{1-\nu}\mu^2 \vee \partial\gamma\right]. \tag{2.7}$$

We demonstrated above that the two local L_∞ algebras on which these BF theories are based are equivalent. As such, the BF theories are also equivalent; the map (2.3) extends uniquely to an automorphism of BV theories. Explicitly, the automorphism is

$$\mu \mapsto e^{-\nu}\mu, \qquad \nu \mapsto 1 - e^{-\nu}, \qquad \beta \mapsto (\beta - \mu \vee \gamma)e^{\nu}, \qquad \gamma \mapsto e^{\nu}\gamma. \tag{2.8}$$

2.1.2.4 In what follows, we specialize to the case that X is a Calabi–Yau five-fold and that S is a one-dimensional smooth orientable manifold. In this case, with $n = 5 + 1 - 1 = 5$ the theories described in this section are $\mathbb{Z}$-graded in the BV formalism. Momentarily, we consider a new term in the action which will break this grading; as such, this integer shift will not play an essential role.

2.1.3 A deformation of BF theory

Let X be a Calabi–Yau five-fold and S be a smooth oriented one-dimensional real manifold. We will break the $\mathbb{Z}$-grading present in BF theory discussed in the previous section to a $\mathbb{Z}/2$ grading. For reference, this means that the linear cochain complex of fields of the model now takes the following form.

$$\begin{array}{cccc} \text{odd} & \text{even} & \text{odd} & \text{even} \\ \hline \Omega^0(X;S)_\beta & \xrightarrow{\partial} \Omega^1(X;S)_\gamma & \mathrm{PV}^1(X;S)_\mu & \xrightarrow{\partial_\Omega} \mathrm{PV}^0(X;S)_\nu . \end{array} \tag{2.9}$$

2.1.3.1 To define our classical field theory on $X \times S$, we consider a deformation of BF theory S_{BF} (this refers to either the presentation as $S_{BF,0}$ or $S_{BF,\infty}$). Such deformations are governed by the classical master equation: the parameterized family of actions

$$S_{BF} + gJ \tag{2.10}$$

defines a consistent theory in the BV formalism if and only if

$$\{S_{BF} + gJ, S_{BF} + gJ\} = 0. \tag{2.11}$$

Since this must hold for all g, and since the undeformed action S is already a solution to the classical master equation, this reduces to the pair of conditions

$$\{S_{BF}, J\} = \{J, J\} = 0. \tag{2.12}$$

The form of J depends on which presentation we use for BF theory. To begin, we will use the presentation of BF theory $S_{BF,\infty}$ which uses the the non-strict L_∞ structure on divergence-free holomorphic vector fields. The deformation J does not make reference to the Calabi–Yau structure explicitly, but it does involve the holomorphic de Rham operator ∂ on X.

The main result of this section is the following.

Theorem 2.1.1. Let X be a Calabi–Yau five-fold and S a smooth one-dimensional manifold, and consider the BV theory $(\mathcal{E}, S_{BF,\infty})$ on $X \times S$ defined above. The local functional

$$J = \frac{1}{6} \int_{X \times S} \gamma \wedge \partial\gamma \wedge \partial\gamma, \tag{2.13}$$

where $\gamma \in \Omega^{1,\bullet}(X;S)$, defines a deformation of $(\mathcal{E}, S_{BF,\infty})$ as a $\mathbb{Z}/2$-graded BV theory.

2.1.3.2 Before proceeding to the proof, we remark on grading issues. In the original $\mathbb{Z}$-grading on the BF theory given in (2.9) with $n = 5$, the component

$$\gamma^{1,i;j} \in \Omega^{1,i}(X) \otimes \Omega^j(S)$$

sits in degree $-4 + i + j$. Thus, we see that in the original $\mathbb{Z}$-grading on BF theory one has

$$\deg(J) = 6. \tag{2.14}$$

Thus $S_{BF} + J$ is not of homogenous $\mathbb{Z}$ grading (although it is even).

This is completely reasonable from the point of view of twisting supersymmetry in eleven dimensions. Indeed, the R-symmetry group is trivial, and there is not a way to regrade the fields of the twisted theory using twisting data . Nevertheless, if we break to the obvious $\mathbb{Z}/2$ grading, the functional $S_{BF} + gJ$ defines an even action functional. Unless otherwise stated, we will work with this $\mathbb{Z}/2$ grading for the remainder of this section.

2.1.3.3 We proceed to show that $S_{BF,\infty} + gJ$ solves the classical master equation. For notational simplicity we will omit the integral symbol $\int^{\Omega}$.

Proof. It is immediate from the form of the BV bracket that $\{J, J\} = 0$, since J depends only on the γ field. It remains to check that $\{S_{BF,\infty}, J\} = 0$. For the quadratic term in the BF action, we note that

$$\{\beta \wedge \partial_{\Omega}\mu, J\} = \frac{1}{2}\partial\beta \wedge \partial\gamma \wedge \partial\gamma = 0, \tag{2.15}$$

because total derivatives are equivalent to zero as local functionals.

The contribution from the remaining BF action takes the form

$$\left\{\frac{1}{2}\frac{1}{1-\nu}\partial\gamma \vee \mu^2, \frac{1}{6}\gamma \wedge \partial\gamma \wedge \partial\gamma\right\} = \frac{1}{2}(\mu \vee \partial\gamma) \wedge \partial\gamma \wedge \partial\gamma.$$

This expression is zero for symmetry reasons. Recall that $\partial\gamma$ is a two-form, and that the expression must be a totally symmetric local functional which is cubic in this two-form. We can ask whether such a contraction exists just at the level of $\mathfrak{sl}(5)$ representation theory. Let $\square$ denote the fundamental representation of $\mathfrak{sl}(5)$, which we identify with constant one-forms. Since the term must be a scalar, the contraction $(\partial\gamma^3$ must sit in the fundamental representation again, since it is dual to a vector field. Computing the decomposition of the tensor cube of the two-form, we find

$$\mathrm{Sym}^3\left(\ydiagram{1,1}\right) \cong \ydiagram{3,3} \oplus \ydiagram{2,2,1,1}. \tag{2.16}$$

(In fact, the absence of the relevant irreducible representation does not even depend on the parity of the field γ, since

$$\wedge^3\left(\ydiagram{1,1}\right) \cong \ydiagram{3,1,1,1} \oplus \ydiagram{2,2,2}; \tag{2.17}$$

the fundamental representation has symmetry type $\ydiagram{2,1}$.) $\square$

2.1.3.4 We make note of the dependence on the coupling constant g in the definition of the deformed action $S_{BF,\infty} + gJ$.

When $g = 0$ we recover BF theory for the L_∞ algebra $\mathcal{L}_\infty(\mathbb{C}^5) \otimes \Omega^\bullet(\mathbb{R})$. For any $g \neq 0$ the theories are essentially equivalent in perturbation theory. Indeed, if $g \neq 0$ we can make the following field redefinition

$$\gamma \mapsto \sqrt{g}\gamma, \quad \beta \mapsto \sqrt{g}\beta$$

to write the action as

$$\frac{1}{\sqrt{g}}\left(S_{BF,\infty} + J\right).$$

In perturbation theory, this has the affect of modifying the quantization parameter $\hbar$ to $\hbar/\sqrt{g}$. Thus, after modifying $\hbar$ and making the above field redefinition, the perturbative expansion of any theory is equivalent to the one with $g = 1$.

2.1.3.5 We remark on an alternative, equivalent, description of the deformed theory which involves the strict dg Lie algebra structure on divergence-free holomorphic vector fields.

We can replace $S_{BF,\infty}$ by $S_{BF,0}$ via applying the field automorphism (2.8). Doing this we see that J becomes

$$\tilde{J} = \frac{1}{6} e^{\nu}\gamma \wedge \partial(e^{\nu}\gamma) \wedge \partial(e^{\nu}\gamma).$$

Since this automorphism preserves the odd BV bracket, the actions $S_{BF,\infty} + gJ$ and $S_{BF,0} + g\tilde{J}$ are both solutions to the classical master equation, and are equivalent as $\mathbb{Z}/2$ graded BV theories.

2.1.4 Equations of motion of the component fields

Soon, we will provide a series of justifications for the assertion that the deformed theory $S_{BF,\infty} + gJ$ is the minimal twist of eleven-dimensional supergravity on flat space $X \times S = \mathbb{C}^5 \times \mathbb{R}$ where $\mathbb{C}^5$ is equipped with its flat Calabi–Yau form. For the moment, we briefly read off the equations of motion of the general theory on $X \times S$. Let Ω denote the Calabi–Yau form on X.

We consider the action $S_{BF,\infty} + gJ$. The equation of motion obtained by varying β is especially simple—in fact linear—since β only appears in the action via a quadratic term. It is

$$\bar{\partial}\nu + \mathrm{d}_S \nu + \partial_\Omega \mu = 0. \tag{2.18}$$

Varying γ we obtain the equation of motion

$$\bar{\partial}\mu + \mathrm{d}_S\mu + \frac{1}{2}\frac{1}{1-\nu}\partial_\Omega(\mu^2) + \frac{1}{2}(\partial\gamma \wedge \partial\gamma) \vee (g\Omega^{-1}) = 0. \tag{2.19}$$

The last term represents the contraction of an element of $\Omega^{4,\bullet}(X;S)$ with the nonvanishing section $\Omega^{-1} \in \mathrm{PV}^{5,\bullet}(X;S)$ to yield an element of $\mathrm{PV}^{1,\bullet}(X;S)$. If we vary the μ we obtain

$$(\bar{\partial} + \mathrm{d}_S)\gamma + \partial\beta + \frac{1}{1-\nu}(\mu \vee \partial\gamma) = 0. \tag{2.20}$$

Finally, if we vary ν we obtain

$$(\bar{\partial} + \mathrm{d}_S)\beta + \frac{1}{2}\frac{1}{(1-\nu)^2}\mu^2 \vee \partial\gamma = 0. \tag{2.21}$$

The equation of motion must hold for any inhomogenous superfields. We can get a better sense of the equations if we expand in components of these fields. The component fields of the eleven-dimensional theory on $X \times S$ have the following form:

- $\mu = \sum_{i,j} \mu^{i;j}$ is a superfield where

$$\mu^{i;j} \in \mathrm{PV}^{1,i}(X) \otimes \Omega^j(\mathbb{R}), \quad i = 0, \ldots, 5, \quad j = 0, 1.$$

  The component $\mu^{i;j}$ has parity $i + j + 1 \pmod 2$.

- $\nu = \sum_{i,j} \nu^{i;j}$ is a superfield where

$$\nu^{i;j} \in \mathrm{PV}^{0,i}(X, \mathrm{T}_X) \otimes \Omega^j(\mathbb{R}), \quad i = 0, \ldots, 5, \quad j = 0, 1.$$

The component $\nu^{i;j}$ has parity $i+j \pmod 2$.

- $\gamma = \sum_{i,j} \gamma^{i;j}$ is a superfield where

$$\gamma^{i;j} \in \Omega^{1,i}(X) \otimes \Omega^j(\mathbb{R}), \quad i = 0, \dots, 5, \quad j = 0, 1.$$

The component $\gamma^{i;j}$ has parity $i+j \pmod 2$.

-

- $\beta = \sum_{i,j} \beta^{i;j}$ is a superfield where

$$\beta^{i;j} \in \Omega^{0,i}(X) \otimes \Omega^j(\mathbb{R}), \quad i = 0, \dots, 5, \quad j = 0, 1.$$

The component $\beta^{i;j}$ has parity $i+j+1 \pmod 2$.

We look closely at the geometric meaning of (2.18). Let's make the simplifying assumption that all components of μ are divergence-free, and further that all fields are locally constant along S: that is, $\partial_\Omega \mu = 0$ and $\mathrm{d}_S \mu = \mathrm{d}_S \gamma = 0$. Then $\nu = 0$ is a solution to (2.18) and we can assume that all fields are functions, or zero-forms, along S. Then, there is a component of (2.19) which can be written as

$$\bar\partial \mu^{1;0} + \frac{1}{2}[\mu^{1;0}, \mu^{1;0}] + \left(\frac{1}{2}\partial\gamma^{1;0} \wedge \partial\gamma^{1;0} + \partial\gamma^{2;0} \wedge \partial\gamma^{0;0}\right) \vee (g\Omega^{-1}) = 0 \tag{2.22}$$

where now $[-,-]$ stands for the Schouten bracket.

To further simplify (2.22), we can look for a solutions where $\gamma^{1;0}$, the $(0,1)$ Dolbeault part of γ, is zero. Then, up to the term involving

$$\alpha \overset{def}{=} \partial\gamma^{0;0},$$

we find precisely the integrability equation for the complex structure determined by Beltrami differential $\mu^{1;0} \in \mathrm{PV}^{1,1} \otimes \Omega^0$. If $\bar\partial\alpha = 0$, the holomorphic two-form $\alpha \in \Omega^{2,hol}(X)$ defines a map of sheaves

$$\Omega^{2,hol}_X \xrightarrow{\wedge\alpha} \Omega^{4,hol}_X \cong_\Omega \mathcal{T}^{hol}_X$$

where $\mathcal{T}^{hol}_X$ denotes the sheaf of holomorphic vector fields and the last isomorphism uses the Calabi–Yau form Ω on X. The image of $\Omega^{2,hol}_X$ defines a subsheaf $\mathcal{F}_\alpha \subset \mathcal{T}^{hol}_X$. Since $\partial\alpha = 0$, this subsheaf is automatically integrable and hence determines a foliation.

Summarizing, see that there is a field configuration where the Beltrami diffrential $\xi = \mu^{1;0} \in \Omega^{0,1}(X, \mathrm{T}_X)$ satisfies the modified integrability condition

$$\bar\partial\xi + \frac{1}{2}[\xi, \xi] = \alpha \vee \rho$$

for some $\rho \in \mathrm{PV}^{2,2}(X)$. In other words, ξ defines an integrable complex structure deformation along the leaf space associated to the foliation $\mathcal{F}_\alpha$. We leave a more complete exploration of the moduli space of solutions of the equations of motion for future work.

In [SW21], the second two authors showed that the free limit of the minimal twist of eleven-dimensional supergravity agrees with the free limit of the eleven-dimensional theory that we have introduced here. Given this result, we can recognize many fields in the twisted theory as components of the physical fields of supergravity which remain after we twist.

- The components

$$\mu^{1;0} = \mu_i^j(z,\bar{z},t)\mathrm{d}\bar{z}_j\partial_{z_i}$$
$$\mu^{0;1} = \mu_i^t(z,\bar{z},t)\mathrm{d}t\partial_{z_i}$$

  of μ comprise components of the metric which remain after the twist. The components

$$\mu^{0;0} = \mu_i(z,\bar{z},t)\partial_{z_i}$$

  comprise the ghosts for infinitesimal (holomorphic) diffeomorphisms.

- The three-form fields

$$\beta^{3;0} = \beta^{ijk}(z,\bar{z},t)\mathrm{d}\bar{z}_i\mathrm{d}\bar{z}_j\mathrm{d}\bar{z}_k, \quad \beta^{2;1} = \beta_t^{ij}(z,\bar{z},t)\mathrm{d}\bar{z}_i\mathrm{d}\bar{z}_j\mathrm{d}t$$
$$\gamma^{2;0} = \gamma^{ijk}(z,\bar{z},t)\mathrm{d}z_i\mathrm{d}\bar{z}_j\mathrm{d}\bar{z}_k, \quad \gamma^{1;1} = \gamma_t^{ij}(z,\bar{z},t)\mathrm{d}z_i\mathrm{d}\bar{z}_j\mathrm{d}t. \quad (2.23)$$

  comprise components of the supergravity C-field which remain after the twist. The two-form fields $\beta^{2;0}, \beta^{1;1}, \gamma^{1;0}, \gamma^{0;1}$, the one-form fields $\beta^{1;0}, \beta^{0;1}$, and the zero-form field $\beta^{0;0}$ is what remains of the ghost system (ghosts, ghosts for ghosts, etc.) for the supergravity C-field.

2.1.5 Local character

We consider the eleven-dimensional theory on the manifold $\mathbb{C}^5 \times \mathbb{R}$, where $\mathbb{C}^5$ is equipped with its standard Calabi–Yau structure. On this background, the theory is manifestly $SU(5)$ invariant. In this section, we compute the corresponding character of the local operators at the origin.

The local character is only sensitive to the free limit of the theory. Furthermore, the linear BRST operator is an $SU(5)$-invariant deformation of the $(\bar{\partial} + \mathrm{d}_{\mathbb{R}})$ operator. Therefore, to compute the character it suffices to compute the $SU(5)$-equivariant character of the $\bar{\partial}$ cohomology.

The solutions to the $(\bar{\partial}+\mathrm{d}_{\mathbb{R}})$-equations of motion simply say that all fields are holomorphic along $\mathbb{C}^5$ and constant along $\mathbb{R}$. Thus, the solutions can be identified with

$$\mu^i\partial_{z_i} \in \mathrm{Vect}(\mathbb{C}^5) \cong \mathcal{O}(\mathbb{C}^5)\partial_{z_i}, \quad \nu \in \mathcal{O}(\mathbb{C}^5)$$
$$\beta \in \mathcal{O}(\mathbb{C}^5), \quad \gamma^i\mathrm{d}z_i \in \Omega^1(\mathbb{C}^5) \cong \mathcal{O}(\mathbb{C}^5)\mathrm{d}z_i$$

where z_i is a holomorphic coordinate on $\mathbb{C}^5$.

Corresponding to each of the above, we have a tower of linear local operators labeled by $(m_j) = (m_1, m_2, m_3, m_4, m_5) \in \mathbb{Z}^5_{\geq 0}$; these are given by

$$\boldsymbol{\mu}^i_{(m_j)} : \mu^i \mapsto \partial_{z_1}^{m_1}\partial_{z_2}^{m_2}\partial_{z_3}^{m_3}\partial_{z_4}^{m_4}\partial_{z_5}^{m_5}\mu^i(0)$$
$$\boldsymbol{\nu}_{(m_j)} : \nu \mapsto \partial_{z_1}^{m_1}\partial_{z_2}^{m_2}\partial_{z_3}^{m_3}\partial_{z_4}^{m_4}\partial_{z_5}^{m_5}\nu(0)$$
$$\boldsymbol{\gamma}^i_{(m_j)} : \gamma^i \mapsto \partial_{z_1}^{m_1}\partial_{z_2}^{m_2}\partial_{z_3}^{m_3}\partial_{z_4}^{m_4}\partial_{z_5}^{m_5}\gamma^i(0)$$
$$\boldsymbol{\beta}_{(m_j)} : \beta \mapsto \partial_{z_1}^{m_1}\partial_{z_2}^{m_2}\partial_{z_3}^{m_3}\partial_{z_4}^{m_4}\partial_{z_5}^{m_5}\beta(0)$$

It is easiest to label the Cartan subgroup of $SU(5)$ by fugacities $q_1, \dots, q_5$ subject to the constraint that $\prod_{i=1}^5 q_i = 1$. We first compute the single particle index. This is the $SU(5)$ character of the space of linear local operators.

Lemma 2.1.2. The single particle index is

$$i(q_1, \dots, q_5) = \frac{\sum_{i=1}^5 q_i}{\prod_{i=1}^5 (1-q_i)} + \frac{\sum_{i=1}^5 q_i^{-1}}{\prod_{i=1}^5 (1-q_i^{-1})}$$

where the fugacities satisfy the constraint $\prod_{i=1} q_i = 1$.

Proof. The linear local operators $\boldsymbol{\nu}_{(m_j)}$ and $\boldsymbol{\beta}_{(m_j)}$ are of the same q-weight but opposite parity. Thus, they do not contribute to the single particle index.

The q-weight of the odd local operator $\boldsymbol{\mu}^i_{(m_j)}$ is

$$q_1^{m_1+1} \cdots q_i^{m_i} \cdots q_5^{m_5+1}.$$

The q-weight of the even local operator $\boldsymbol{\gamma}^i_{(m_j)}$ is

$$q_1^{m_1} \cdots q_i^{m_i+1} \cdots q_5^{m_5}.$$

Thus we find that the single particle index is given by the infinite series

$$\sum_{i=1}^5 \left(\sum_{(m_i)\in\mathbb{Z}^5_{\geq 0}} q_1^{m_1} \cdots q_i^{m_i+1} \cdots q_5^{m_5} - \sum_{(m_i)\in\mathbb{Z}^5_{\geq 0}} q_1^{m_1+1} \cdots q_i^{m_i} \cdots q_5^{m_5+1} \right) \tag{2.24}$$

which sums to the expression

$$-\frac{\sum_{i=1}^5 q_1 \cdots \hat{q}_i \cdots q_5}{\prod_{i=1}^5 (1-q_i)} + \frac{\sum_{i=1}^5 q_i}{\prod_{i=1}^5 (1-q_i)}. \tag{2.25}$$

This simplifies to the stated expression. □

This single particle index for our space of local operators agrees with the one computed in [Nek09]. To obtain the full index of local operators we apply the plethystic exponential $\mathrm{P}E[f(x)] = \exp\left(\sum_n \frac{1}{n} f(x^n)\right)$.

Proposition 2.1.3. The character of local operators of the eleven-dimensional theory on $\mathbb{C}^5 \times \mathbb{R}$ is

$$\prod_{i=1}^5 \prod_{(m_i)\in\mathbb{Z}^5_{\geq 0}} \frac{1 - q_1^{m_1+1} \cdots q_i^{m_i} \cdots q_5^{m_5+1}}{1 - q_1^{m_1} \cdots q_i^{m_i+1} \cdots q_5^{m_5}}$$

Proof. Recall that the plethystic exponential takes sums to products and monomials to geometric series. Apply this to the infinite series (2.24). □

2.1.6 One-loop quantization

In [GRW21] an existence result for one-loop quantizations of mixed topological-holomorphic theories was established. The crucial thing is that these results apply to more than just topological-holomorphic theories arising from twists of supersymmetric gauge theories. They also apply to twists of supergravity theories. We apply this to the eleven-dimensional model at hand.

The eleven-dimensional theory is a mixed topological-holomorphic theory. On flat space $\mathbb{C}^5_z \times \mathbb{R}_t$, this means that the theory is translation invariant and that the following act homotopically trivially:

- the vector fields $\partial_{\bar{z}_1}, \ldots, \partial_{\bar{z}_5}$ corresponding to infinitesimal anti-holomorphic translations,
- the vector field ∂_t corresponding to infinitesimal translations in the $\mathbb{R}_t$ direction.

Recall that the action functional of the eleven-dimensional theory is $S_{BF,\infty} + gJ$. Since the cubic and higher interactions only involve holomorphic derivatives, we obtain the following directly from the main result of [GRW21].

Theorem 2.1.4. There exists a gauge fixing condition for the eleven-dimensional theory on $\mathbb{C}^5 \times \mathbb{R}$ which renders its one-loop quantization finite and anomaly-free.

When $g = 0$, this result is actually exact, since there are no Feynman diagrams present past one-loop order in this case. When $g \neq 0$, on the other hand, this result does not immediately imply the existence of a gauge-invariant perturbative quantization to higher orders in $\hbar$. The presence of the functional $J = \frac{1}{6}\int \gamma\partial\gamma\partial\gamma$ allows one to construct Feynman graphs at arbitrary loop order.

In [Cos16], Costello argues that, upon performing the Ω-background, the theory localizes to a five-dimensional theory on $\mathbb{C}^2 \times \mathbb{R}$. Via a cohomological argument, it is shown that this effective five-dimensional theory exhibits an essentially unique quantization in perturbation theory. We will return to the existence and uniqueness of a higher order quantization of the eleven-dimensional theory (before turning on the Ω-background) in future work.

2.2 Infinite-dimensional symmetry in flat backgrounds

2.2.1 Global symmetry algebra

In any field theory, the cohomology classes of states of odd ghost number have the structure of a Lie algebra. More generally, after shifting the cohomological degree by one, the full cohomology of states with respect to the linear BRST operator is naturally a graded Lie algebra. If we forget the grading to a $\mathbb{Z}/2$ grading, then this global symmetry algebra has the structure of a super Lie algebra.

In general, taking cohomology loses information. If the dg Lie (or L_∞) algebra we start with is not formal, then there exist higher-order operations on the linearized BRST cohomology. We will refer to this L_∞ algebra as the global symmetry algebra of the theory.

Before taking cohomology with respect to the linear BRST operator, we described the super L_∞ structure on the parity shift of the eleven-dimensional fields in the previous section. This is encoded by the full BV action of the eleven-dimensional theory. The cubic component of the full BV action induces the super Lie algebra structure present in the linearized BRST cohomology.

Our main result is to relate the global symmetry algebra of the minimal twist of eleven-dimensional supergravity on $\mathbb{C}^5 \times \mathbb{R}$ to a certain infinite-dimensional exceptional super Lie algebra studied by Kac [Kac98, KR02] called $E(5|10)$. We recall the definition below.

Theorem 2.2.1. Let $\Pi\mathcal{E}(\mathbb{C}^5 \times \mathbb{R})$ be the parity shift of the fields of eleven-dimensional supergravity on $\mathbb{C}^5 \times \mathbb{R}$ and denote by $\delta^{(1)}$ the linearized BRST operator.

1. As a super Lie algebra, the $\delta^{(1)}$-cohomology of $\Pi\mathcal{E}(\mathbb{C}^5 \times \mathbb{R})$ is isomorphic to the trivial one-dimensional central extension of the super Lie algebra $E(5|10)$.
2. The global symmetry algebra is equivalent, as a super L_∞ algebra, to the non-trivial central extension of $E(5|10)$ determined by the even cocycle defined in (2.33).

This result implies that the action functional $S_{BF,\infty} + J$ of the eleven-dimensional theory is invariant for the infinite-dimensional Lie algebra $E(5|10)$.

2.2.2 Linearized BRST cohomology

We compute the linearized BRST cohomology of eleven-dimensional supergravity. Then we will describe the induced structure of a super Lie algebra present in the parity shift of the cohomology, proving part (1) of Theorem 2.2.1.

2.2.2.1 First we recall the definition of the exceptional simple super Lie algebra $E(5|10)$. Recall that $\mathrm{Vect}_0(\mathbb{C}^5)$ is the Lie algebra of divergence-free holomorphic vector fields on $\mathbb{C}^5$. Let $\Omega^2_{cl}(\mathbb{C}^5)$ be the module of holomorphic 2-forms that are closed for the holomorphic de Rham operator ∂.

The even part of the super Lie algebra $E(5|10)$ is the Lie algebra

$$E(5|10)_+ = \mathrm{Vect}_0(\mathbb{C}^5)$$

of divergence-free vector fields on $\mathbb{C}^5$, whose elements we continue to denote by μ. The odd piece is the module

$$E(5|10)_- = \Omega^2_{cl}(\mathbb{C}^5),$$

whose elements we denote by α. Besides the natural module structure, there is odd bracket $E(5|10)_- \otimes E(5|10)_- \to E(5|10)_+$ The bracket uses the isomorphism $\Omega^{-1} \vee (-) : \Omega^4 \cong \mathrm{Vect}(\mathbb{C}^5)$ induced by the standard Calabi–Yau form $\mathrm{d}^5 z$, and is defined by

$$[\alpha, \alpha'] = \Omega^{-1} \vee (\alpha \wedge \alpha'). \tag{2.26}$$

Since both α, α' are closed two-forms, the resulting vector field on the right hand side is divergence free. In coordinates, if $f^{ij}\mathrm{d}z_i \wedge \mathrm{d}z_j$ and $g^{kl}\mathrm{d}z_k \wedge \mathrm{d}z_l$ are two closed two-forms, their bracket is the vector field $\epsilon_{ijklm} f^{ij} g^{kl} \partial_{z_m}$.

To be precise, Kac studied a more algebraic version of the algebra we have just introduced, where holomorphic functions are replaced by holomorphic polynomials. As such, the simple super Lie algebra that appears in the classification in [Kac98] is a dense sub Lie algebra of what we call $E(5|10)$, consisting of those vector fields and two-forms that have polynomial coefficients.

2.2.2.2 If $\mathcal{E}$ is the space of fields of any theory in the BV or BRST formalism, the shift $\mathcal{L} = \mathcal{E}[-1]$ has the structure of a Lie, possibly L_∞ algebra. In the $\mathbb{Z}/2$ graded world, the parity shifted object $\mathcal{L} = \Pi\mathcal{E}$ has the structure of a super L_∞ algebra.

In this section, we use the description of the eleven-dimensional theory as the deformation of the BF action $S_{BF,\infty}$ by the functional J of Theorem 2.1.1. We set the coupling $g = 1$. For any other nonzero value of g, we will obtain an isomorphic super L_∞ algebra as explained above. We would also obtain equivalent results if we used the other model of the eleven-dimensional theory explained in §2.1.3.5.

The full differential on the cochain complex of observables of the theory is given by the BV bracket with the BV action. For us, this is

$$\delta = \{S_{BF,\infty} + J, -\}.$$

The linear BRST operator (dual to the differential on the cochain complex of fields) comes only from the quadratic summands in $S_{BF,\infty}$, and is of the form

$$\delta^{(1)} = \bar\partial + \mathrm{d}_{\mathbb{R}} + \partial_\Omega|_{\mu\to\nu} + \partial|_{\beta\to\gamma}. \tag{2.27}$$

To compute the cohomology with respect to $\delta^{(1)}$ we can use a spectral sequence, first taking the cohomology with respect to $\bar\partial + \mathrm{d}_{\mathbb{R}}$ and then with respect to ∂_Ω. By the $\bar\partial$ and de Rham Poincaré lemmas, the cohomology of the space of fields of the eleven-dimensional theory on $\mathbb{C}^5 \times \mathbb{R}$ with respect $\bar\partial + \mathrm{d}_{\mathbb{R}}$ results in the cochain complex

$$\begin{array}{ccc} - & & + \\ \hline \mathrm{Vect}(\mathbb{C}^5) & \xrightarrow{\partial_\Omega} & \mathcal{O}(\mathbb{C}^5) \\ \mathcal{O}(\mathbb{C}^5) & \xrightarrow{\partial} & \Omega^1(\mathbb{C}^5). \end{array} \tag{2.28}$$

Recall that $\mathrm{Vect}(\mathbb{C}^5), \mathcal{O}(\mathbb{C}^5)$, and $\Omega^1(\mathbb{C}^5)$ denote the space of holomorphic vector fields, functions, and one-forms, respectively.

The cohomology with respect to the remaining linearized BRST operator consists of the space of triples $(\mu, [\gamma], b)$ where:

- μ is a divergence-free holomorphic vector field on $\mathbb{C}^5$, which is constant along $\mathbb{R}$

$$\mu = \mu \otimes 1 \in \Pi\mathrm{Vect}_0(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R}).$$

  Note that μ is a ghost in the $\mathbb{Z}/2$ graded theory.

- $[\gamma]$ is an equivalence class of a holomorphic one-form modulo exact holomorphic one-forms along $\mathbb{C}^5$, which are also constant along $\mathbb{R}$

$$[\gamma] = [\gamma] \otimes 1 \in \left(\Omega^1(\mathbb{C}^5)/\mathrm{d}\mathcal{O}(\mathbb{C}^5)\right) \otimes \Omega^0(\mathbb{R}).$$

- A constant function $b \in \Pi\mathbb{C}$ on $\mathbb{C}^5 \times \mathbb{R}$. This is a β-type field in the eleven-dimensional theory, any constant function is closed for the de Rham differential. This element is also a ghost in the $\mathbb{Z}/2$-graded theory.

2.2.2.3 After parity shifting, we've identified the solutions to the linear equations of motion with triples

$$(\mu, [\gamma], b) \in \mathrm{Vect}_0(\mathbb{C}^5) \oplus \Pi\Omega^1(\mathbb{C}^5)/\partial\mathcal{O}(\mathbb{C}^5) \oplus \mathbb{C}.$$

The bracket induced by the cubic component of $S_{BF,\infty}$ in the classical BV action is the usual bracket on divergence-free vector fields together with the module structure on holomorphic one-forms by Lie derivative. Notice that the Lie derivative commutes with the ∂ operator, so this action descends to equivalence classes as above. The elements b are central.

The final term in the BV action $J = \frac{1}{6}\int \gamma \wedge \partial\gamma \wedge \partial\gamma$ induces the following Lie bracket on the solutions to the linearized equations of motion

$$[[\gamma], [\gamma']] = \Omega^{-1} \vee (\partial\gamma \wedge \partial\gamma') \in \mathrm{Vect}_0(\mathbb{C}^5). \tag{2.29}$$

where Ω^{-1} denotes the section of $\mathrm{PV}^{5,hol}(\mathbb{C}^5)$ which is inverse to the Calabi–Yau form Ω on $\mathbb{C}^5$. Notice that this bracket is well-defined as it does not depend on the particular equivalence classes and that the resulting vector field is automatically divergence-free.

2.2.2.4 Having described the linearized BRST cohomology as a super vector space, we turn to the proof of Theorem 2.2.1.

Proof of Theorem 2.2.1. For the first part, we write down an explicit map between the cohomology computed above and the algebra $E(5|10)$.

The relationship of the μ-elements in $E(5|10)$ and the eleven-dimensional theory is apparent.

Next, we need to relate the equivalence classes $[\gamma]$ with the closed two-forms α in $E(5|10)$. On flat space, any closed differential form is exact (this is a holomorphic version of the Poincaré lemma). In other words, there is an isomorphism

$$\partial : \Omega^1(\mathbb{C}^5)/\mathrm{d}\mathcal{O}(\mathbb{C}^5) \xrightarrow{\cong} \Omega^2_{cl}(\mathbb{C}^5)$$

induced by the holomorphic de Rham differential. This gives the relationship between the equivalence class $[\gamma]$ in the eleven-dimensional theory and a closed two-form in $E(5|10)$ by $\alpha = \partial\gamma$. It is clear from Equations (2.26) and (2.29) that this assignment intertwines the Lie brackets in $E(5|10)$ and the twist of eleven-dimensional supergravity. This completes the proof of part (1).

For part (2), we first produce the following homotopy data:

$$K \circlearrowright (\Pi\mathcal{E}, \delta^{(1)}) \underset{i}{\overset{q}{\rightleftarrows}} (E(5|10) \oplus \mathbb{C}_b\,, 0)\,, \tag{2.30}$$

- On the ν's we take K to be any operator $K : \mathcal{O} \to \mathrm{Vect}$ such that $\partial_\Omega K\nu = \nu$. On the γ's we take K to be any operator $K : \Omega^1 \to \Omega^0$ which satisfies the homotopy relation

$$\tilde{K}\partial\gamma + \partial K\gamma = \gamma \tag{2.31}$$

  for some auxiliary operator $\tilde{K} : \Omega^2_{cl} \to \Omega^1$.

  The precise form of each of these operators will not be needed. The existence of such operators is guaranteed by the holomorphic Poincaré lemma. The operator K annihilates fields β and μ.

- The map q is described as follows. First $q(\mu) = \mu - K\partial_\Omega(\mu)$. Notice that $q(\mu)$ is automatically divergence-free. Next, $q(\gamma) = \partial\gamma$. If β is a holomorphic function, then $q(\beta) = \beta(z = 0)$.
- The map i embeds μ and b in the obvious way. On a closed two form α, we have that $i(\alpha) = \tilde{K}\gamma$.

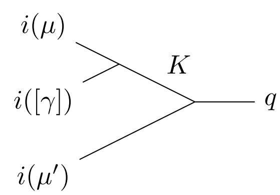
It is straightforward to check that this comprises well-defined homotopy data, the only nontrivial thing to check is the relation $\mathrm{Id} - i \circ q = \delta^{(1)}K - K\delta^{(1)}$. Plugging in the field γ we see that we must check that

$$\gamma - \tilde{K}\partial\gamma = \partial K\gamma$$

which is precisely (2.31).

Given this homotopy data, we can compute the homotopy transferred L_∞ structure on the linearized BRST cohomology. Since ν does not survive to cohomology and the fact that there are no nontrivial Lie brackets involving the field β, this transferred structure is easy to compute.

There is a single diagram which contributes to the transferred structure, it is given by



(2.32)

together with a similar diagram with the μ and μ' flipped.

This diagram leads to a new 3-ary bracket on $E(5|10) \oplus \mathbb{C}_b$

$$\left[\mu, \mu', [\gamma]\right]_3 = \varphi(\mu, \mu', [\gamma])$$

where $\varphi \in \mathrm{C}^{\mathrm{even}}(E(5|10))$ is the even Lie algebra cocycle defined by the formula

$$\begin{aligned} \varphi : E(5|10) \times E(5|10) \times E(5|10) &\to \mathbb{C}_b \\ \varphi(\mu, \mu', \alpha) &= \langle \mu \wedge \mu', \alpha \rangle|_{z=0}. \end{aligned} \tag{2.33}$$

Since b is central, this cocycle defines a central extension of $E(5|10)$. □

2.2.2.5 We briefly remark on Lie algebra cohomology for super Lie algebras. The Lie algebra cohomology $\mathrm{C}^{\bullet,\bullet}(\mathcal{L})$ of any super Lie algebra $\mathcal{L}$ is graded by $\mathbb{Z} \times \mathbb{Z}/2$. The first grading is by the symmetric degree in the Chevalley–Eilenberg complex. The second grading is the internal parity of the super Lie algebra $\mathcal{L}$. The Chevalley–Eilenberg differential is degree $(1, +)$.

The cocycle φ has homogenous bigrading $(3, -)$. In the above discussion we forgot the bigrading to a totalized $\mathbb{Z}/2$ grading where

$$\mathrm{C}^{\mathrm{even}}(\mathcal{L}) = \mathrm{C}^{2\bullet,+}(\mathcal{L}) \oplus \mathrm{C}^{2\bullet+1,-}(\mathcal{L})$$
$$\mathrm{C}^{\mathrm{odd}}(\mathcal{L}) = \mathrm{C}^{2\bullet,-}(\mathcal{L}) \oplus \mathrm{C}^{2\bullet+1,+}(\mathcal{L}).$$

With this totalization, φ is an even cocycle and hence determines a super L_∞ central extension by the one-dimensional even vector space $\mathbb{C}$.

2.3 Residual supersymmetry

In this section we consider the minimal twist of eleven-dimensional supersymmetry explicitly. We compute the residual supersymmetry algebra given by taking the cohomology of the eleven-dimensional supersymmetry algebra with respect to the minimal twisting supercharge. In order for this to map to the gauge symmetries of the eleven-dimensional theory, it is necessary to consider an extension of the eleven-dimensional supersymmetry algebra corresponding to the M2 brane. We will see how this extension is compatible, upon twisting by the minimal supercharge, with the central extension of $E(5|10)$ we found as the global symmetry algebra in the previous section.

2.3.1 Supersymmetry in eleven dimensions

The (complexified) eleven-dimensional supertranslation algebra is a complex super Lie algebra of the form

$$\mathfrak{t}_{11d} = V \oplus \Pi S$$

where S is the (unique) spin representation and $V \cong \mathbb{C}^{11}$ the complex vector representation, of $\mathfrak{so}(11,\mathbb{C})$. The bracket is the unique surjective $\mathfrak{so}(11,\mathbb{C})$-equivariant map from the symmetric square of S to V; this decomposes into three irreducibles,

$$\mathrm{Sym}^2(S) \cong V \oplus \wedge^2 V \oplus \wedge^5 V. \tag{2.34}$$

Denote by $\Gamma_{\wedge^1}, \Gamma_{\wedge^2}, \Gamma_{\wedge^5}$ the projections onto each of the summands above. The bracket in $\mathfrak{t}_{11d}$ is defined using the first projection

$$[\psi, \psi'] = \Gamma_{\wedge^1}(\psi, \psi').$$

The super Poincaré algebra is

$$\mathfrak{siso}_{11d} = \mathfrak{so}(11,\mathbb{C}) \ltimes \mathfrak{t}_{11d}.$$

The R-symmetry is trivial in eleven-dimensional supersymmetry.

2.3.2 Extensions of the supersymmetry algebra

Extensions of the supersymmetry algebra correspond to the existence of extended objects, such as branes, in the supergravity theory. In eleven-dimensional supersymmetry, there are two such extensions corresponding to the M2 brane and the M5 brane. We begin by describing a less standard dg Lie algebra model for the M2 brane algebra. In the next section we will explain the relationship to other descriptions in terms of L_∞ algebras [BH05, BL07, FSS15].

Our model for the M2 brane algebra is a dg Lie algebra extension of the super Poincaré algebra $\mathfrak{siso}_{11d}$.

Introduce the cochain complex $\Omega^\bullet(\mathbb{R}^{11})$ of (complex valued) differential forms on $\mathbb{R}^{11}$ equipped with the de Rham differential d. The M2 brane algebra arises as an extension of $\mathfrak{siso}_{11d}$ by the cochain complex $\Omega^\bullet(\mathbb{R}^{11})[2]$ and is defined by a cocycle

$$c_{M2} \in \mathrm{C}^{2,+}\left(\mathfrak{siso}_{11d}\ ;\ \Omega^\bullet(\mathbb{R}^{11})[2]\right).$$

The formula is

$$c_{M2}(\psi,\psi') = \Gamma_{\wedge^2}(\psi,\psi') \in \Omega^2(\mathbb{R}^{11})$$

where $\Gamma_{\wedge^2}$ is the projection onto $\wedge^2 V$, thought of as the space of constant coefficient two-forms, as in the decomposition (2.34).

Definition 2.3.1. The algebra $\mathfrak{m2brane}$ is the $\mathbb{Z}\times\mathbb{Z}/2$-graded dg Lie algebra defined by the extension of $\mathfrak{siso}_{11d}$ by the cocycle c_{M2}.

Here, we are using a bigrading by $\mathbb{Z}\times\mathbb{Z}/2$. The super Poincaré algebra is concentrated in zero integer grading and carries its natural $\mathbb{Z}/2$ grading as a super Lie algebra. The complex $\Omega^\bullet(\mathbb{R}^{11})[2]$ is concentrated in integer degrees $[-2,9]$ and has even parity. The bracket in $\mathfrak{m2brane}$ is bidegree $(0,+)$ and the differential is bidegree $(1,+)$.

2.3.3 The minimal twist

Fix a supercharge $Q \in S$ satisfying $Q^2 = 0$ that is in the lowest stratum of the nilpotence variety. Such a supercharge has a six-dimensional image in the space of (complexified) translations on $\mathbb{R}^{11}$ and defines the minimal twist of eleven-dimensional supersymmetry [SW21]. We characterize the cohomology of the algebra $\mathfrak{m2brane}$ with respect to this supercharge.

Q defines a maximal isotropic subspace $L \subset V$. In turn, we will decompose the super Poincaré algebra into $\mathfrak{sl}(L) = \mathfrak{sl}(5)$ representations. First, the defining and spinor representations decompose as

$$V = L \oplus L^\vee \oplus \mathbb{C}_t, \qquad S = \wedge^\bullet L. \tag{2.35}$$

In the expression for S, we are omitting factors of $\det(L)^{\frac{1}{2}}$ for simplicity. Also, $\mathfrak{so}(11,\mathbb{C})$ decomposes as

$$\mathfrak{sl}(L) \oplus \wedge^2 L \oplus \wedge^2 L^\vee \oplus L \oplus L^\vee \oplus \mathbb{C}.$$

Furthermore, the spin representation can be identified with

$$S = \wedge^\bullet(L) = \mathbb{C} \oplus L \oplus \wedge^2 L \oplus \wedge^3 L \oplus \wedge^4 L \oplus \wedge^5 L.$$

The element Q lives in the first summand. Let

$$\mathrm{Stab}(Q) = \mathfrak{sl}(L) \oplus \wedge^2 L^\vee \oplus L^\vee \subset \mathfrak{so}(11,\mathbb{C})$$

be the stabilizer of Q. This is a parabolic subalgebra whose Levi factor is $\mathfrak{sl}(5)$.

2.3.4 Q-cohomology of $\mathfrak{m2brane}$

Any element $Q \in S$ satisfying $Q^2 = 0$ determines a deformation of the dg Lie algebra $\mathfrak{m2brane}$. To deform d by Q we must break the $\mathbb{Z}\times\mathbb{Z}/2$ bigrading. The supercharge Q is odd and of cohomological degree zero. Recall, the original differential on $\mathfrak{m2brane}$ is the de Rham differential d which just acts on the summand $\Omega^\bullet(\mathbb{R}^{11})[2]$ and is even of cohomological degree $+1$. Thus, only the totalized $\mathbb{Z}/2$ grading makes the differential $\mathrm{d} + [Q, -]$ homogenous.

Definition 2.3.2. The Q-twist $\mathfrak{m2brane}^Q$ of $\mathfrak{m2brane}$ is the super dg Lie algebra whose differential is $\mathrm{d} + [Q, -]$. The bracket is unchanged.

Let Q be a minimal supercharge satisfying $Q^2 = 0$. We first determine $H^\bullet(\mathfrak{m2brane}^q)$ as a super vector space.

Lemma 2.3.3. As a $\mathbb{Z}/2$ graded space, the cohomology of the Q-twist $\mathfrak{m2brane}^Q$ is

$$L \oplus \mathrm{Stab}(Q) \oplus \Pi\left(\wedge^2 L^\vee\right) \oplus \mathbb{C} \tag{2.36}$$

whose elements we denote by (v, m, ψ, c).

Proof. The cohomology of the non-centrally extended algebra was computed in [SW21], we briefly recall the result. The element Q only acts nontrivially on the summands $\wedge^4 L$ and $\wedge^5 L$ in S. The image of $\wedge^4 L \cong L^\vee$ trivializes the antiholomorphic translations while the image of $\wedge^5 L$ trivializes the time translation. So, of the translations, only the holomorphic ones, which live in L, survive. The map

$$[Q, -] \colon \mathfrak{so}(11, \mathbb{C}) \to S$$

is the projection onto $\wedge^0 L \oplus \wedge^1 L \oplus \wedge^2 L$. The kernel of $[Q, -]$ is the stabilizer $\mathrm{Stab}(Q)$.

In summary, the space of odd translations which survive cohomology is $\wedge^3 L \cong \wedge^2 L^\vee$; two such elements bracket to a holomorphic translation by taking the wedge product to get an element of $\wedge^4 L^\vee \cong L$. This completes the calculation of the cohomology. □

The main result of this subsection is the following:

Proposition 2.3.4. The cohomology of the Q-twist $H^\bullet(\mathfrak{m2brane}^Q)$ has the following structures:

1. As a super Lie algebra, $H^\bullet(\mathfrak{m2brane}^Q)$ is the natural extension of $\mathrm{Stab}(Q)$ together with the bracket
$$[\psi, \psi']_2 = \psi \wedge \psi' \in \wedge^4 L^\vee \cong L_v \tag{2.37}$$
2. $\mathfrak{m2brane}^Q$ is not formal as a super dg Lie algebra. As a super L_∞ algebra, the Q-twist is equivalent to (2.36) with 2-brackets described in (1) where we additionally introduce the 3-ary bracket
$$[v, v', \psi]_3 = 4\langle v \wedge v', \psi\rangle \in \mathbb{C}_b. \tag{2.38}$$

It will be useful to list the formulas for the brackets in terms of coordinates. Let $\{z_i\}$ denote a basis for L, which we will also think of as a linear coordinate on $\mathbb{C}^5$. Let $\{\partial_{z_i}\}$ be a dual basis for $L^\vee$, which we will also think of as translation invariant vector fields. The 2-ary bracket above is

$$[z_i \wedge z_j, z_k \wedge z_l]_2 = \epsilon_{ijklm}\partial_{z_m}$$

and the 3-ary bracket is

$$[\partial_{z_i}, \partial_{z_j}, z_k \wedge z_\ell]_3 = 4(\delta^i_k \delta^j_\ell - \delta^i_\ell \delta^j_k).$$

2.3.4.1 One way to prove the proposition above is to use homotopy transfer directly to $\mathfrak{m2brane}^Q$, just as we did in the proof of Theorem 2.2.1 to deduce the form of the 3-ary bracket. Instead, we will use the following minimal model for $\mathfrak{m2brane}^Q$ to prove Proposition 2.3.4. This minimal model also has the advantage of being more directly related to the eleven-dimensional supergravity theory.

Lemma 2.3.5. Let $\mathfrak{g}$ denote the following $\mathbb{Z}/2$ graded dg Lie algebra which as a cochain complex is

$$H^\bullet(\mathfrak{m2brane}^Q) \oplus (L^\vee \xrightarrow{\mathrm{Id}} \Pi L^\vee).$$

Denote the elements of the second summand by $(\lambda, \tilde{\lambda})$. The Lie structure extends the one on $H^\bullet(\mathfrak{m2brane}^Q)$ described in (1) of Proposition 2.3.4 together with the brackets

$$[v, \lambda] = \langle v, \lambda \rangle \in \mathbb{C}_b$$
$$[v, \psi] = \langle v, \psi \rangle \in \Pi L^\vee_{\tilde{\lambda}}.$$

There is an L_∞ map

$$\mathfrak{g} \rightsquigarrow \mathfrak{m2brane}^Q$$

which is a quasi-isomorphism of cochain complexes.

Proof. We embed $\mathfrak{g}$ into $\mathfrak{m2brane}^Q$ in the following way: $\mathrm{Stab}(Q)$ and L sit inside in the evident way. The central element maps to $c \mapsto -1 \in \Omega^0(\mathbb{R}^{11})$. The summand L_λ is mapped to the linear functions in $\Omega^0(\mathbb{R}^{11})$ and $\Pi L_{\tilde{\lambda}}$ is sent to the constant coefficient one-forms in $\Pi\Omega^1(\mathbb{R}^{11})$. It remains to define where $\psi \in \wedge^2 L$ is mapped.

Notice that, at least naively, $\psi \in \wedge^2 L$ is not Q-closed due to the presence of the central extension. To embed $\wedge^2 L$ we introduce the operator

$$H \colon \Omega^2(\mathbb{R}^{11}) \to \Omega^1(\mathbb{R}^{11}),$$

which sends a two-form α to the one-form $H\alpha$ defined by the formula $(H\alpha)(x) = \int_0^x \alpha$ where we integrate over a straight line path from 0 to x.

Notice that if α is d-closed then $\mathrm{d}(H\alpha) = \alpha$. It follows that any element $\psi \in \wedge^2 L \subset S$ can be lifted to a closed element at the cochain level in $\mathfrak{m2brane}^Q$ by the formula

$$\tilde{\psi} = \psi - H\Gamma_{\wedge^2}(Q, \psi) \in \Pi S \oplus \Pi\Omega^1.$$

Thus, sending $\psi \mapsto \tilde{\psi}$ defines a cochain map $\mathfrak{g} \to \mathfrak{m2brane}^Q$.

The Lie bracket $[\tilde{\psi}, \tilde{\psi}']$ agrees with $[\psi, \psi']$. On the other hand, in $\mathfrak{m2brane}^Q$ there is the Lie bracket

$$[v, \tilde{\psi}] = -L_v(H\Gamma_{\wedge^2}(Q, \psi)) = -\langle v, \Gamma_{\wedge^2}(Q, \psi)\rangle - \mathrm{d}\langle v, H\Gamma_{\wedge^2}(Q, \psi)\rangle.$$

The first term agrees with the bracket $[v, \psi]_{\mathfrak{g}}$ in $\mathfrak{g}$. The other term is exact in $\mathfrak{m2brane}^Q$ and can hence be corrected by the following bilinear

$$v \otimes \psi \mapsto \langle v, H\Gamma_{\wedge^2}(Q, \psi)\rangle \in L_\lambda.$$

Together with the cochain map described above, this bilinear term prescribes the desired L_∞ map.

□

2.3.4.2 We now proceed to the proof of proposition 2.3.4.

Proof of Proposition 2.3.4. Using the model $\mathfrak{g}$, the first part of Proposition 2.3.4 follows immediately. We deduce the second part using homotopy transfer.

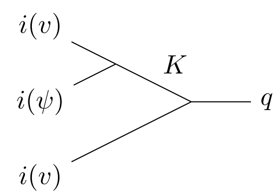
Recall that we described the cohomology of $\mathfrak{m2brane}^Q$ in (2.36). Let δ denote the differential on $\mathfrak{g}$ which simply maps ΠL to L by the identity map. We produce the homotopy data

$$K \circlearrowright (\mathfrak{g}, \delta) \underset{i}{\overset{q}{\rightleftarrows}} \left(H^\bullet(\mathfrak{m2brane}^Q)\, , \, 0\right) , \tag{2.39}$$

as follows.

- The operator K annihilates $H^\bullet(\mathfrak{m2brane}^Q)$ and is the identity map $K \colon \Pi L_{\tilde{\lambda}} \to L_\lambda$.
- The map q is the identity on $H^\bullet(\mathfrak{m2brane}^Q)$ and annihilates the summand $L \to \Pi L$.
- The map i embeds $H^\bullet(\mathfrak{m2brane}^Q)$ in the obvious way.

It is immediate to verify this data prescribes valid homotopy data. There is only a single term in the L_∞ structure generated by homotopy transfer. It is determined by the following tree diagram



(2.40)

together with a similar diagram with the v and v' reversed. It is an immediate calculation to show that these trees recover the formula in (2) of Proposition 2.3.4. □

2.3.5 Embedding supersymmetry into the eleven-dimensional theory

Consider now the super L_∞ algebra $\mathcal{L}$ underlying the eleven-dimensional theory on $\mathbb{C}^5 \times \mathbb{R}$.

Proposition 2.3.6. Endow the cohomology of $\mathfrak{m2brane}^Q$ with the L_∞ structure of Proposition 2.3.4 and let $\mathcal{L}(\mathbb{C}^5 \times \mathbb{R})$ be the super L_∞ algebra underlying eleven-dimensional supergravity on $\mathbb{C}^5 \times \mathbb{R}$. There is a map of super L_∞ algebras

$$H^\bullet(\mathfrak{m2brane}^Q) \rightsquigarrow \mathcal{L}(\mathbb{C}^5 \times \mathbb{R})$$

In particular, the Q-twisted algebra $\mathfrak{m2brane}^Q$ is a symmetry of eleven-dimensional theory on $\mathbb{C}^5 \times \mathbb{R}$.

Proof. Recall that the cohomology of $\mathfrak{m2brane}^Q$ takes the following form

$$\begin{array}{ccc} \underline{\textit{even}} & \underline{\textit{odd}} & \underline{\textit{even}} \\ & & \\ L^\vee & (\wedge^2 L^\vee)_2 & L \\ & & \\ (\wedge^2 L^\vee)_1 & & \\ & & \\ \mathfrak{sl}(5) & & \mathbb{C}_b \end{array} \tag{2.41}$$

The lefthand column is Stab(Q). The subscripts are used to distinguish between the two copies of $\wedge^2 L^\vee$.

The L_∞ map from the dg Lie model $\mathfrak{g}$ to the fields of the twisted eleven-dimensional supergravity theory has a linear piece $\Phi^{(1)}$ and a quadratic piece $\Phi^{(2)}$. Define the linear map $\Phi^{(1)} \colon \mathfrak{g} \to \mathcal{L}$ as follows:

$$\begin{aligned} L^\vee &\mapsto 0 \\ \wedge^2 L_1^\vee &\mapsto 0 \\ z_i \wedge z_j \in \wedge^2 L_2^\vee &\mapsto \frac{1}{2}(z_i \mathrm{d} z_j - z_j \mathrm{d} z_i) \in \Omega^{1,0}(\mathbb{C}^5) \,\widehat{\otimes}\, \Omega^0(\mathbb{R}) \\ A_{ij} \in \mathfrak{sl}(5) &\mapsto \sum_{ij} A_{ij} z_i \partial_{z_j} \in \mathrm{PV}^{1,0}(\mathbb{C}^5) \,\widehat{\otimes}\, \Omega^0(\mathbb{R}) \\ \partial_{z_j} \in L &\mapsto \partial_{z_i} \in \mathrm{PV}^{1,0}(\mathbb{C}^5) \,\widehat{\otimes}\, \Omega^0(\mathbb{R}^5) \\ 1 \in \mathbb{C}_b &\mapsto 1 \in \Omega^{0,0}(\mathbb{C}^5) \,\widehat{\otimes}\, \Omega^0(\mathbb{R}). \end{aligned}$$

It is immediate to check that this is a map of cochain complexes, since all elements in the image of this map lie in the kernel of the linearized BRST operator (2.27).

This map also preserves the bracket between odd elements in $\wedge^2 L_2^\vee$. In the cohomology of $\mathfrak{m2brane}^Q$ we have the bracket

$$[z_i \wedge z_j, z_k \wedge z_l] = \epsilon_{ijklm} \partial_{z_m}$$

which is precisely the bracket induced by the cubic term in the action $J = \frac{1}{6} \in \gamma \partial \gamma \partial \gamma$.

This map does not preserve all of the brackets, however. Indeed, in the eleven-dimensional theory $\mathcal{L}(\mathbb{C}^5 \times \mathbb{R})$ there is the bracket

$$[\partial_{z_i}, z_j \mathrm{d} z_k - z_k \mathrm{d} z_j] = \delta^i_j \mathrm{d} z_k - \delta^i_k \mathrm{d} z_j$$

arising from the cubic term in $\frac{1}{2} \int \frac{1}{1-\nu} \mu^2 \partial \gamma$. To remedy the failure for $\Phi^{(1)}$ to preserve the brackets, we introduce the odd bilinear map $\Phi^{(2)} \colon \mathfrak{g} \otimes \mathfrak{g} \to \Pi \mathcal{L}$ defined by

$$\Phi^{(2)}\left(\partial_{z_i}, z_j \wedge z_k\right) = \frac{1}{2}(\delta^i_j z_k - \delta^i_k z_j). \tag{2.42}$$

Notice that the field on the right hand side is of type β.

The bilinear map $\Phi^{(2)}$ provides a homotopy trivialization for the failure for $\Phi^{(1)}$ to preserve the

2-ary bracket:

$$[\Phi^{(1)}(\partial_{z_i}), \Phi^{(1)}(z_j \wedge z_k)] = \partial\Phi^{(2)}\left(\partial_{z_i}, z_j \wedge z_k\right).$$

The lefthand side is $\frac{1}{2}(\delta^i_j \mathrm{d}z_k - \delta^i_k \mathrm{d}z_j)$ which is precisely the de Rham differential applied to (2.42).

To define an L_∞ morphism $\Phi^{(1)} + \Phi^{(2)}$ must satisfy additional higher relations. There is a single nontrivial cubic relation to verify:

$$\begin{aligned}\Phi^{(1)}\left[\partial_{z_i}, \partial_{z_j}, z_k \wedge z_l\right]_3 = [\Phi^{(1)}(\partial_{z_i}), \Phi^{(1)}(\partial_{z_i}), \Phi^{(1)}(z_k \wedge z_l)]_3 \\ + [\partial_{z_i}, \Phi^{(2)}(\partial_{z_j}, z_k \wedge z_l)] + [\partial_{z_j}, \Phi^{(2)}(\partial_{z_i}, z_k \wedge z_l)]\end{aligned} \quad (2.43)$$

where $[-]_3$ on the left hand side is the 3-ary bracket defined in Proposition 2.3.4 and $[-]_3$ on the right hand side is the 3-ary bracket defined by the quartic part of the action $\frac{1}{2}\int \frac{1}{1-\nu}\mu^2 \vee \partial\gamma$. The two terms in the second line of (2.43) cancel for symmetry reasons and the quartic term in the BV action induces precisely the correct 3-ary bracket.

□

2.3.5.1 From the previous proposition we can readily compare the super L_∞ algebra $H^\bullet(\mathfrak{m2brane}^Q)$ with the global symmetry algebra of our theory.

Corollary 2.3.7. There is a map of super L_∞ algebras

$$H^\bullet(\mathfrak{m2brane}^Q) \to E(\hat{5|}10),$$

where $E(\hat{5|}10)$ is a central extension of $E(5|10)$ by the cocycle (2.33).

Proof. Because this map preserves differentials, it descends to a map in cohomology. We have already computed the cohomology of $\mathcal{L}$ on $\mathbb{C}^5 \times \mathbb{R}$; it is the trivial one-dimensional central extension of $E(5,10)$. The Lie algebra structure present in the cohomology of $\mathfrak{m2brane}^Q$ is described in part (1) of Proposition 2.3.4. The map

$$L \oplus \mathrm{Stab}(Q) \oplus \Pi\left(\wedge^3 L\right) \oplus \mathbb{C}_b \to E(5,10) \oplus \mathbb{C}_{b'}$$

is defined by very similar formulas as above

$$\begin{aligned} L_1^\vee &\mapsto 0 \\ \wedge^2 L_1^\vee &\mapsto 0 \\ z_i \wedge z_j \in \wedge^2 L_2 &\mapsto \mathrm{d}z_i \wedge \mathrm{d}z_j \in \Omega^2_{cl}(\mathbb{C}^5) \\ A_{ij} \in \mathfrak{sl}(5) &\mapsto \sum_{ij} A_{ij} z_i \partial_{z_j} \in \mathrm{Vect}_0(\mathbb{C}^5) \\ \partial_{z_i} \in L &\mapsto \partial_{z_i} \in \mathrm{Vect}_0(\mathbb{C}^5) \\ b \in \mathbb{C}_b &\mapsto b \in \mathbb{C}_{b'}. \end{aligned}$$

The relationship between the transferred L_∞ structures can be described as follows. Recall that the linear BRST cohomology of the parity shift of the fields of the eleven-dimensional theory is equivalent to the super L_∞ algebra $E(\hat{5|}10)$. Also, we described the L_∞ structure present in the cohomology of $\mathfrak{m2brane}^Q$ in part (2) of Proposition 2.3.4. Each of these L_∞ structure involved introducing a single new 3-ary bracket, which are easily seen to be compatible. □

2.3.5.2 In this short section we compare to another description of the M2 brane algebra given as a one-dimensional L_∞ central extension of the super Poincaré algebra. Such central extensions were studied in [BH11, SSS09, FSS15], following [CDF91]. In these references, the algebra $\mathfrak{m2brane}$ is defined as an L_∞ central extension of $\mathfrak{siso}_{11d}$.

Recall that given two spinors $\psi, \psi' \in S$ we can form the constant coefficient two-form $\Gamma_{\wedge^2}(\psi, \psi')$. Using this two-form we can define the following four-linear expression

$$\mu_2(\psi, \psi', v, v') = \langle v \wedge v', \Gamma(\psi, \psi')\rangle.$$

This expression is symmetric on the spinors and antisymmetric on the vectors, therefore it defines an element in $\mathrm{C}^4(\mathfrak{siso}_{11d})$. This expression defines a nontrivial class in $H^4(\mathfrak{siso}_{11d})$ so defines a one dimensional central extension of $\mathfrak{siso}_{11d}$ as a Lie 3-algebra. Instead of working with a one-dimensional central extension by $\mathbb{C}[2]$, we work with a central extension by the resolution $\Omega^\bullet(\mathbb{R}^{11})[2]$ determined by a cocycle c_{M2}, see §2.3.2. There is a quasi-isomorphism $\mathrm{C}^\bullet(\mathfrak{siso}_{11d}) \to \mathrm{C}^\bullet(\mathfrak{siso}_{11d}, \Omega^\bullet(\mathbb{R}^{11}))$ induced by the embedding of constant functions into the full de Rham complex. The cocycles μ_2 and c_{M2} are cohomologous via a two-step zig-zag in the double complex $\mathrm{C}^\bullet(\mathfrak{siso}_{11d}, \Omega^\bullet(\mathbb{R}^{11}))$.

2.4 The non-minimal twist

We have provided numerous consistency checks that the eleven-dimensional theory defined on a manifold with $SU(5)$ holonomy is a twist of supergravity. We have referred to this theory as "minimal," since it renders the minimal number of translations homotopically trivial, or (slightly improperly) as "holomorphic." In this section we characterize the unique further twist of eleven-dimensional supergravity on flat space, as seen through the lens of the holomorphic theory. This further twist is invariant for the group $G_2 \times SU(2)$ and is fully topological along seven directions, as opposed to just a single direction as in the minimal twist. This is easiest to see by decomposing the eleven-dimensional spinor as a representation of $\mathrm{Spin}(4) \times \mathrm{Spin}(7)$; from this perspective, a square-zero element is a rank-one element in the tensor product of a chiral spin representation of Spin(4) and the spin representation of Spin(7). Elements of the latter fall into two distinct orbits under the Spin(7) action, the minimal orbit—"Cartan pure spinors"—and the generic orbit [Igu70]. The stabilizer of an element of the generic orbit is G_2, almost by definition.

We will show that the non-minimal twist is equivalent to an interacting theory on $\mathbb{C}^2 \times \mathbb{R}^7$ that we call "Poisson" Chern–Simons theory, using a direct description of the further twist together with an indirect cohomological argument. This completes the confirmation of a conjecture in the literature ([Cos16]; see also [RY19]); the result was checked at the level of the free theory in [EH21] by computing the nonminimal twist of the eleven-dimensional multiplet directly at the component-field level.

In the BV formalism, the theory is $\mathbb{Z}/2$ graded, with fields given by

$$A \in \Pi\Omega^{0,\bullet}(\mathbb{C}^2) \mathbin{\widehat{\otimes}} \Omega^\bullet(\mathbb{R}^7),$$

where Π, as always, denotes parity shift. The equations of motion are of the form

$$\bar{\partial}A + \mathrm{d}_{\mathbb{R}^7}A + \partial_{z_1}A \wedge \partial_{z_2}A = 0.$$

The action functional depends on the holomorphic symplectic structure on $\mathbb{C}^2$ through the Poisson bracket on the algebra of holomorphic functions. We give a precise definition below.

The main result of this section is the following.

Theorem 2.4.1. The non-minimal twist of the eleven-dimensional theory is equivalent to Poisson Chern–Simons theory on

$$\mathbb{C}^2 \times \mathbb{R}^7.$$

From the point of view of the untwisted theory, the non-minimal twist is defined by working in a background where the fermionic ghost in the physical theory is equal to a supertranslation of the form

$$Q + Q_{nm}$$

where Q is the supertranslation which defines the minimal twist, see §2.3.3. The minimal twist of supergravity is obtained by setting a fermionic ghost equal to Q.

In the language of the minimal twist, the supercharge Q_{nm} determines a square-zero element in the Q-cohomology of the original supersymmetry algebra (which we will denote by the same letter). The characterization of this cohomology in Proposition 2.3.4 implies that Q_{nm} is an element

$$Q_{nm} \in \wedge^2 (L^\vee)$$

where $L \cong \mathbb{C}^5$ is the defining $SU(5)$ representation. In other words, Q is a translation invariant holomorphic two-form on $\mathbb{C}^5$. The condition that $[Q_{nm}, Q_{nm}] = 0$ simply says that $Q_{nm} \wedge Q_{nm} = 0$ as a translation invariant four-form on $\mathbb{C}^5$. By a linear change of coordinates, all such two-forms Q are of the form $Q_{nm} = \mathrm{d}z_i \wedge \mathrm{d}z_j$ where $i, j = 1, \ldots, 5$.

From hereon in this section we will rename coordinates by

$$\mathbb{C}^5 \times \mathbb{R} = \mathbb{C}^2_{z_i} \times \mathbb{C}^3_{w_a} \times \mathbb{R}$$

which is most natural from the point of view of the non-minimal twist. We will fix the non-minimal supercharge

$$Q_{nm} = \mathrm{d}z_1 \wedge \mathrm{d}z_2.$$

Notice that this choice of supercharge breaks the holonomy of the eleven-dimensional theory from $SU(5)$ to $SU(2) \times SU(3)$.

2.4.1 Index matching

As a first consistency check, we can compare deformation invariants attached to the holomorphic twist and the nonminimal twist. We will find that the local character of the latter agrees with a specialization of the local character computed in §2.1.5

Proposition 2.4.2. The local character of the nonminimal twist of eleven-dimensional supergravity on flat space is given by

$$\prod_{(n_1,n_2)\in\mathbb{Z}^2_{\geq 0}} \frac{1}{1-q^{-n_1+n_2}}.$$

This agrees with the specializaiton of the local character computed in proposition 2.1.3.

Proof. The space of solutions to linearized equations of motion is parametrized by a holomorphic function A on $\mathbb{C}^2_{w_j}$. The corresponding linear local operators are labeled by $(n_1, n_2) \in \mathbb{Z}^2_{\geq 0}$ and are given by

$$\boldsymbol{A}_{(n_1,n_2)} : A \mapsto \partial^{n_1}_{w_1}\partial^{n_2}_{w_2} A(0).$$

The character of the linear span of these is given by the geometric series

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2_{\geq 0}} q^{-n_1+n_2} \tag{2.44}$$

with plethystic exponential given by

$$\prod_{(n_1,n_2)\in\mathbb{Z}^2_{\geq 0}} \frac{1}{1-q^{-n_1+n_2}}.$$

For the last part, it suffices to observe the specialization at the level of single particle indices. A natural choice of fugacities for $SU(2) \times SU(3)$ is given in terms of the fugacities q_i for $SU(5)$ chosen in §2.1.5 by requiring the additional constraints

$$q_1 q_2 = 1, \quad q_3 q_4 q_5 = 1.$$

After imposing the above constraints, the single particle index (2.25) is

$$i(q) = \frac{1}{(1-q)(1-q^{-1})}$$

where $q = q_1 = q_1^{-1}$. This is exactly the sum of the geometric series (2.44). □

2.4.2 The non-minimal global symmetry algebra

We constructed an embedding of the Q-cohomology of the supersymmetry algebra into the fields of our eleven-dimensional theory on $\mathbb{C}^5 \times \mathbb{R}$. The further twist is obtained by working in a background where a certain field on $\mathbb{C}^5 \times \mathbb{R}$ takes nonzero value Q_{nm}. Explicitly, the element $Q_{nm} \in \wedge^2 L$ corresponds to the image under ∂ of a γ-field of type $\Omega^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R})$. According to the embedding in §2.3.5 this is the γ-field

$$\gamma_{nm} = \frac{1}{2}(z_1 \mathrm{d}z_2 - z_2 \mathrm{d}z_1) \in \Omega^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R}) \tag{2.45}$$

Notice that $\partial\gamma_{nm} = \mathrm{d}z_1 \wedge \mathrm{d}z_2$ as desired.

2.4.2.1 Before proceeding to the proof of the theorem above, we perform a simple calculation of the global symmetry algebra present in the Q_{nm}-twisted theory.

Recall that up to a copy of constant functions, the global symmetry algebra of the holomorphic twist of the eleven-dimensional theory is the super Lie algebra $E(5,10)$. From this point of view, the global symmetry algebra of the Q_{nm}-twisted theory is given by deformation of this super Lie algebra by the Maurer–Cartan element

$$\mathrm{d}z_1 \wedge \mathrm{d}z_2 \in \Omega^2_{cl}(\mathbb{C}^5).$$

We recall that the space of closed two-forms on $\mathbb{C}^5$ is precisely the odd part of the super Lie algebra $E(5,10)$.

We compute the cohomology of $E(5,10)$ with respect to the differential which is bracketing with this Maurer–Cartan element. Recall that we are using the holomorphic coordinates $(z_1, z_2, w_1, w_2, w_3)$ on $\mathbb{C}^5$.

There are the following brackets in the super Lie algebra $E(5,10)$

$$\begin{aligned}
[f_l \partial_{z_l}, \mathrm{d}z_1 \wedge \mathrm{d}z_2] &= \partial f_i \wedge \mathrm{d}z_j - \partial f_j \wedge \mathrm{d}z_i \\
[g_a \partial_{w_a}, \mathrm{d}z_1 \wedge \mathrm{d}z_2] &= 0 \\
[h^{ab} \mathrm{d}w_a \wedge \mathrm{d}w_b, \mathrm{d}z_1 \wedge \mathrm{d}z_2] &= \epsilon_{abc} h^{ab} \partial_{w_c}.
\end{aligned}$$

where $f_l \partial_{z_l}$, $g_a \partial_{w_a}$ are divergence-free vector fields on $\mathbb{C}^5$ and $h^{ab} \mathrm{d}w_a \wedge \mathrm{d}w_b$ is a closed two-form.

From these relations, we see that the following elements are in the kernel of $[\mathrm{d}z_1 \wedge \mathrm{d}z_2, -]$:

- $f(z_i, w_a) dz_1 \wedge dz_2$ for f a holomorphic function on $\mathbb{C}^5$.
- $f(z_i)\partial_{z_1} + g(z_i)\partial_{z_2}$ for holomorphic functions f, g on $\mathbb{C}_{z_1} \times \mathbb{C}_{z_2}$ which satisfy

  $$\partial_{z_1} f + \partial_{z_2} g = 0.$$

  In other words, this is a divergence-free vector field on $\mathbb{C}_{z_1} \times \mathbb{C}_{z_2}$.
- $f_b(z_i, w_a)\partial_{w_b}$ for f_b a holomorphic function on $\mathbb{C}^5$ where $b = 1, 2, 3$.

It is immediate to check that these are the only nonzero elements in the kernel. Further, any element of the first type is clearly exact and any element of the last type is clearly exact by the closed two-form $\epsilon^{ijklm} f \mathrm{d}w_l \mathrm{d}w_m$.

Thus, the cohomology is the (purely bosonic) Lie algebra of divergence-free vector fields on $\mathbb{C}^2 = \mathbb{C}_i \times \mathbb{C}_j$

$$H^\bullet\big(E(5,10), [\mathrm{d}z_1 \wedge \mathrm{d}z_2, -]\big) \simeq \mathrm{Vect}_0(\mathbb{C}^2).$$

We proved in Theorem 2.2.1 that the global symmetry algebra of the eleven-dimensional theory on $\mathbb{C}^5 \times \mathbb{R}$ is equivalent to a central extension $\hat{E(5,10)}$ of the super Lie algebra $E(5,10)$.

The Lie algebra of divergence-free vector fields on $\mathbb{C}^2$ also admits a central extension:

$$0 \to \mathbb{C} \to \mathcal{O}(\mathbb{C}^2) \to \mathrm{Vect}_0(\mathbb{C}^2) \to 0 \tag{2.46}$$

where $\mathcal{O}(\mathbb{C}^2)$ is equipped with the Poisson bracket with respect to the symplectic form $\mathrm{d}z_1 \wedge \mathrm{d}z_2$. These two central extension are compatible.

Proposition 2.4.3. Let $E(\hat{5,10})$ be the central extension of $E(5,10)$ which is equivalent to the global symmetry algebra of the eleven-dimensional theory on $\mathbb{C}^5 \times \mathbb{R}$. Then there is an isomorphism of Lie algebras

$$H^\bullet\big(E(\hat{5,10}), [\mathrm{d}z_1 \wedge \mathrm{d}z_2, -]\big) \simeq \mathcal{O}(\mathbb{C}^2).$$

Proof. The only thing to check is that, in cohomology, the cocycle defining the central extension of $E(5,10)$ is the cocycle exhibiting $\mathcal{O}(\mathbb{C}^2)$ as the central extension of divergence-free vector fields. Recall that the formula (2.33) for the cocycle is

$$\varphi(\mu, \mu', \alpha) = \langle \mu \wedge \mu', \alpha \rangle|_{z=0}.$$

In cohomology, we obtain the cocycle for divergence-free vector fields by plugging in $\alpha = \mathrm{d}z_1 \wedge \mathrm{d}z_2$. This gives the cocycle on $\mathrm{Vect}_0(\mathbb{C}^2)$

$$(f_i \partial_{z_i}, g_j \partial_{z_j}) \mapsto (f_1 g_2 - f_2 g_1)(z_1 = z_2 = 0).$$

This is the cocycle defining (2.46), as desired. □

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This proposition implies that the global symmetry algebra of the non-minimal twist of eleven-dimensional supergravity is the Lie algebra $\mathcal{O}(\mathbb{C}^2)$. We will see that this is compatible with the calculation of the non-minimal twist of the full BV theory.

2.4.3 The non-minimal twist of the eleven-dimensional theory

Now, we turn to deducing the action functional of the non-minimal twist and hence the proof of Theorem 2.4.1. We will show that the eleven-dimensional theory on $\mathbb{C}^5 \times \mathbb{R}$ placed in the background where the $(1,0)$ component of γ takes the value γ_{nm} (2.45) is equivalent to a theory with a purely Chern–Simons-like action functional that we referred to in the introduction to this section.

Poisson Chern–Simons theory is defined on any manifold of the form

$$Z \times M$$

where Z is a hyper Kähler surface and M is a smooth manifold of real dimension seven. The fundamental field of the theory is

$$\alpha \in \Pi\Omega^{0,\bullet}(Z) \hat{\otimes} \Omega^\bullet(M).$$

Just in our original eleven-dimensional theory, this theory is also only $\mathbb{Z}/2$ graded.

The holomorphic symplectic form $\omega_Z^{2,0}$ on Z induces a Poisson bracket define on all Dolbeault forms $\Omega^{0,\bullet}(Z)$ which we denote by $\{-,-\}_{pb}$. In local Darboux coordinates (z_1, z_2), this bracket reads

$$\{\alpha^I(z,\bar{z})\mathrm{d}\bar{z}_I, \alpha'^J(z,\bar{z})\mathrm{d}\bar{z}_J\}_{pb} = (\partial_{z_1}\alpha^I \partial_{z_2}\alpha^J \pm \partial_{z_2}\alpha^I \partial_{z_1}\alpha^J)\mathrm{d}\bar{z}_I \wedge \mathrm{d}\bar{z}_J.$$

The action functional of Poisson Chern–Simons theory is

$$\frac{1}{2}\int_{Z\times M} (\alpha \wedge \mathrm{d}\alpha) \wedge \omega_Z^{2,0} + \frac{1}{6}\int_{Z\times M} \alpha \wedge \{\alpha, \alpha\}_{pb} \wedge \omega_Z^{2,0} \tag{2.47}$$

where $\{-,-\}$ is the Poisson bracket induced from the symplectic form ω_Z on Z.

For simplicity, we will work only on flat space $\mathbb{C}^5 \times \mathbb{R} = \mathbb{C}^2_z \times (\mathbb{C}^3_w \times \mathbb{R})$, where we view $Z = \mathbb{C}^2_z$ as a hyper Kähler manifold with its standard holomorphic symplectic form $\omega^{2,0} = \mathrm{d}^2 z$.

We will decompose the fields according to these coordinates.

Proof of Theorem 2.4.1. Decompose the μ-field as $\mu = \mu_z + \mu_w$ where

$$\mu_z \in \mathrm{PV}^{1,\bullet}(\mathbb{C}^2_z) \otimes \mathrm{PV}^{0,\bullet}(\mathbb{C}^3_w) \otimes \Omega^\bullet(\mathbb{R})$$
$$\mu_w \in \mathrm{PV}^{0,\bullet}(\mathbb{C}^2_z) \otimes \mathrm{PV}^{1,\bullet}(\mathbb{C}^3_w) \otimes \Omega^\bullet(\mathbb{R}).$$

and similarly $\gamma = \gamma_z + \gamma_w$. We will also use the notation ∂^z for the holomorphic de Rham differential along $\mathbb{C}^2_z$ and similarly ∂^w for the holomorphic de Rham differential along $\mathbb{C}^3_w$.

To twist, we expand near the background where the field γ_z takes value γ_{nm} as in (2.45). This will generate new kinetic and interacting terms.

There are two types of interactions in the original theory. The first is

$$\frac{1}{2}\int_{\mathbb{C}^2\times\mathbb{C}^3\times\mathbb{R}} \frac{1}{1-\nu}\left(\partial\gamma \vee \mu^2\right) \wedge (\mathrm{d}^2 z \wedge \mathrm{d}^3 w) \tag{2.48}$$

and the second is

$$\frac{1}{6}\int_{\mathbb{C}^2\times\mathbb{C}^3\times\mathbb{R}} \gamma\partial\gamma\partial\gamma. \tag{2.49}$$

Expanding (2.48) around the background where γ takes value γ_{nm}, we obtain,

$$\int \frac{1}{1-\nu}\left(\frac{1}{2}\partial^w\gamma_w \vee \mu_w^2 + \partial^z\gamma_w \vee \mu_w\mu_z + \partial^w\gamma_z \vee \mu_w\mu_z + \frac{1}{2}\partial^z\gamma_z \vee \mu_z^2\right) \wedge (\mathrm{d}^2 z \wedge \mathrm{d}^3 w)$$
$$+ \frac{1}{2}\int \frac{1}{1-\nu}\left(\mathrm{d}^2 z \vee \mu_z^2\right) \wedge (\mathrm{d}^2 z \wedge \mathrm{d}^3 w). \tag{2.50}$$

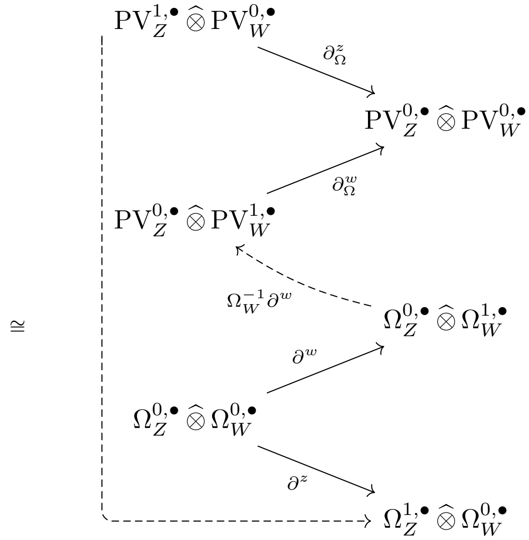
We similarly expand (2.49),

$$\frac{1}{6}\int (\gamma_w\partial^z\gamma_w\partial^z\gamma_w + \gamma_w\partial^w\gamma_w\partial^z\gamma_z + \gamma_w\partial^w\gamma_z\partial^w\gamma_z) + \frac{1}{2}\int (\gamma_w\partial^w\gamma_w) \wedge \mathrm{d}^2 z \tag{2.51}$$

The new terms in the non-minimally twisted linearized BRST differential arise from the quadratic terms in the action in Equations (2.50) and (2.51):

$$\frac{1}{2}\int (\mathrm{d}^2 z \vee \mu_z^2) \wedge (\mathrm{d}^2 z \wedge \mathrm{d}^3 w) + \frac{1}{2}\int (\gamma_w \wedge \partial^w\gamma_w) \wedge \mathrm{d}^2 z. \tag{2.52}$$

The non-minimally twisted linear BRST complex thus takes the form



(2.53)

Here, we write $Z = \mathbb{C}^2_z$ and $X = \mathbb{C}^3_w$ for notational simplicity.

Here, the dashed arrow along the outside of the diagram corresponds to the BV antibracket with the first term in (2.52). It is given by the isomorphism

$$\Omega^{1,\bullet}_Z \,\widehat{\otimes}\, \Omega^{0,\bullet}_W \xrightarrow{\omega^{2,0}_Z \otimes \mathrm{Id}} \mathrm{PV}^{1,\bullet}_Z \,\widehat{\otimes}\, \mathrm{PV}^{0,\bullet}_W$$

induced holomorphic symplectic form on Z. The other dashed arrow corresponds to the BV antibracket with the second term in (2.52). It is given by the composition

$$\Omega^{0,\bullet}_Z \,\widehat{\otimes}\, \Omega^{1,\bullet}_W \xrightarrow{\mathrm{Id}\otimes\partial^w} \Omega^{0,\bullet}_Z \,\widehat{\otimes}\, \Omega^{2,\bullet}_W \xrightarrow{\mathrm{Id}\otimes\Omega_W} \mathrm{PV}^{0,\bullet}_Z \,\widehat{\otimes}\, \mathrm{PV}^{1,\bullet}_W$$

given by applying the holomorphic de Rham operator along X followed by contracting with the inverse holomorphic volume form along X.

We replace this linear BRST complex, up to quasi-isomorphism, with a smaller BRST complex. Consider the complex

$$\Omega^{0,\bullet}_Z \,\widehat{\otimes}\, \Omega^{\bullet,\bullet}_W \,\widehat{\otimes}\, \Omega^{\bullet}_L = \oplus_{k=0}^3 \Omega^{0,\bullet}_Z \,\widehat{\otimes}\, \Omega^{k,\bullet}_W \,\widehat{\otimes}\, \Omega^{\bullet}_L \tag{2.54}$$

which is equipped with the differential $\bar\partial^z + \bar\partial^w + \partial^w + \mathrm{d}_{\mathbb{R}}$. Write $\alpha = \alpha^0 + \cdots + \alpha^3$ for a field in this complex, using the decomposition on the right hand side. The full Poisson Chern–Simons action S_{pCS} equips this complex with the structure of a dg Lie algebra.

Define the following non-linear map of BRST complexes from (2.54) to the twisted theory (2.53). It is defined by the equations

$$\mu_z = (1-\tilde\alpha^3)(\partial_{z_1}\wedge\partial_{z_2}) \vee \partial^z\alpha^0, \quad \mu_w = (\partial_{w_1}\wedge\partial_{w_2}\wedge\partial_{w_3})\vee\alpha^2, \quad \nu = \tilde\alpha^3$$
$$\beta = \alpha^0, \quad \gamma_w = \alpha^1, \quad \gamma_z = 0. \tag{2.55}$$

In the above equation we have introduced the notation $\tilde\alpha^3 = \Omega_X^{-1}\vee\alpha^3$. Notice the only non-linearity appears in the definition of μ_z.

The restriction of the kinetic terms $\int \gamma(\bar\partial + \mathrm{d}_{\mathbb{R}})\mu + \beta(\bar\partial + \mathrm{d}_{\mathbb{R}})\nu$ along (2.55) is

$$\int \sum_{k=0}^{3} \alpha^k(\bar\partial + \mathrm{d}_{\mathbb{R}})\alpha^{3-k} \tag{2.56}$$

The restriction of the kinetic term $\int \beta\partial_\Omega \mu$ along (2.55) is

$$\int \alpha^0\partial^w\alpha^2 - \int \alpha^0\partial^z\alpha^0\partial^z\alpha^3. \tag{2.57}$$

Finally, the restriction of the kinetic term $\frac{1}{2}\int \gamma\partial^w\gamma$ in (2.51) along (2.55) is

$$\int \frac{1}{2}\alpha^1\partial^w\alpha^1. \tag{2.58}$$

Together, (2.56)–(2.58) give the kinetic term in Poisson Chern–Simons theory.

The formulas above show that the linear terms in (2.55) define a map of linear BRST complexes. Applying the apparent contracting homotopy, we see that this map is a quasi-isomorphism. We will show that the full non-linear map intertwines the action functionals up to cohomologically exact terms, and hence defines an equivalence of BV theories.

We substitute the values for the fields in (2.55) into the original eleven-dimensional action. Notice that any terms involving γ_z can be discarded. Restricting (2.50) along this map, we obtain the action functional

$$\frac{1}{2}\int \frac{1}{1-\tilde\alpha^3}\partial^w\alpha^1(\tilde\alpha^2)^2\mathrm{d}^2z + \int \alpha^2\partial^z\alpha^0\partial^z\alpha^1 + \frac{1}{2}\int (1-\alpha^3)\partial^z\alpha^0\partial^z\alpha^0. \tag{2.59}$$

Here, $\tilde\alpha^2$ denotes the element of $\Omega^{0,\bullet}_Z \widehat\otimes \mathrm{PV}^{1,\bullet}_W \widehat\otimes \Omega^\bullet_L$ corresponding to α^2 determined by the Calabi–Yau form d^3w. Notice that the very last term is equivalent to the functional $-\frac{1}{2}\int \alpha^3\partial^z\alpha^0\partial^z\alpha^0$ since the quadratic part is a total derivative.

There is only one cubic term left in (2.51) when we substitute the fields according to (2.55). It is

$$\frac{1}{6}\int \alpha^1\partial^z\alpha^1\partial^z\alpha^1. \tag{2.60}$$

Combining all of these terms, we see that the total action restricted along the map (2.55) is

$$S_{pCS}(\alpha) + \int \frac{1}{2}\frac{1}{1-\tilde\alpha^3}\partial^w\alpha^1(\tilde\alpha^2)^2\mathrm{d}^2z \tag{2.61}$$

where S_{pCS} is the Poisson Chern–Simons action in (2.47).

We will show that the term not appearing in $S_{pCS}(\alpha)$ is cohomologically trivial. Consider the odd local functional

$$\frac{1}{6}\int \frac{1}{1-\tilde\alpha^3}\alpha^2(\tilde\alpha^2)^2. \tag{2.62}$$

Applying the linearized BRST operator (in the non-minimal twist) this becomes

$$\frac{1}{2}\int \frac{1}{1-\tilde\alpha^3}\partial^w\alpha^1(\tilde\alpha^2)^2 + \frac{1}{6}\int \frac{1}{1-\tilde\alpha^3}\partial^w(\alpha^2)\alpha^2(\tilde\alpha^2)^2.$$

The first term in this expression agrees the term in (2.61) which is not in $S_{pCS}(\alpha)$.

The latter term $(1-\tilde{\alpha}^3)^{-1}\partial^w(\alpha^2)\alpha^2(\tilde{\alpha}^2)^2$ is of polynomial degree ≥ 4. Since this term has trivial self BV bracket, it determines a cocycle for the dg Lie algebra underlying Poisson Chern–Simons theory. Since it is manifestly translation invariant, it arises via descent from a cocycle in the dg Lie algebra of ∞-jets of fields at $0 \in \mathbb{C}^2 \times \mathbb{R}^7$. This dg Lie algebra is quasi-isomorphic to the Lie algebra $\mathbb{C}[[z_1, z_2]]$ equipped with the holomorphic Poisson bracket. (This is the formal power series version of the Lie algebra from Proposition 2.4.3.)

There is a weight grading on this Lie algebra, given by declaring $|z_1^{n+1} z_2^{m+1}| = n + m$; in turn, this induces a grading on the Gelfand–Fuks Lie algebra cohomology. The weight of the Gelfand–Fuks class corresponding to our deformation is $+2$, since no derivatives of z_1 or z_2 appear. Results from [Fuk86] show that there is no cohomology in this weight. As such, the cocycle must define a trivial deformation of the dg Lie algebra up to equivalence. This completes the proof that the non-minimal twist is equivalent to Poisson Chern–Simons theory. □

2.5 Dimensional reduction and ten-dimensional supergravity

In this section we demonstrate that our proposal for the action of minimally twisted eleven-dimensional supergravity agrees with conjectural descriptions of twisted type IIA and type I supergravities due to Costello and Li.

The original motivation for M-theory was as the strong coupling limit for type IIA string theory. Roughly, the radius of the M-theory circle plays the role of this coupling constant. Additionally, at low energies M-theory is expected to be approximated by eleven-dimensional supergravity in the same way that the low energy limit of type IIA/IIB string theory is type IIA/IIB supergravity. Combining these two pictures, various checks have been made that the dimensional reduction of eleven-dimensional supergravity along the M-theory circle is type IIA supergravity.

Motivated by the topological string, Costello and Li have laid out a series of conjectures for twists of type IIA/IIB supergravity [**?**] and type I supergravity [CL20]. Their description was inspired by the model of the open and closed B-model topological string on a Calabi–Yau manifold. The open sector is holomorphic Chern–Simons theory [Wit95] and the closed sector is called Kodaira–Spencer theory [BCOV94]. There are a few different versions of Kodaira–Spencer theory, but the shared characteristic is that they are all 'gravitational' in nature; they describe fluctuations of the Calabi–Yau structure. From this point of view, Kodaira–Spencer theory is at the heart of the formulation of the various flavors of twisted ten-dimensional supergravity.

We begin by introducing certain variants of Kodaira–Spencer theory which will feature in the descriptions of twists of type IIA and type I supergravity.

2.5.1 Kodaira–Spencer theory

Let X be a Calabi–Yau manifold; for now it can be of arbitrary complex dimension d. Define

$$\mathrm{PV}^{i,j}(X) = \Omega^{0,j}(X, \wedge^i \mathrm{T}_X). \tag{2.63}$$

We will consider the graded space $\mathrm{PV}^{\bullet,\bullet}(X) = \oplus_{i,j}\mathrm{PV}^{i,j}(X)[-i-j]$ where the piece of type (i,j) sits in degree $i+j$.

For each fixed i, while we let j vary, the $\bar\partial$ operator defines a cochain complex $\mathrm{PV}^{i,\bullet}(X) = (\oplus_j \mathrm{PV}^{i,j}(X)[-j], \bar\partial)$ which provides a resolution for the sheaf of holomorphic polyvector fields of type i. The divergence operator extends to an operator of the form

$$\partial_\Omega \mathcal{O}lon \mathrm{PV}^{i,\bullet}(X) \to \mathrm{PV}^{i-1,\bullet}(X).$$

Motivated by the states of the topological B-model, one defines the fields of Kodaira–Spencer gravity on X to be the cochain complex

$$\left(\mathrm{PV}^{\bullet,\bullet}(X)[[u]][2]\,,\, \bar\partial + u\partial_\Omega\right). \tag{2.64}$$

Here, u is a parameter of cohomological degree $+2$, which turns $\delta^{(1)}_{KS} = \bar\partial + u\partial_\Omega$ into an operator of homogenous degree $+1$. We also have performed an overall cohomological shift by 2 so that $u^k \mathrm{PV}^{i,j}$ sits in degree $i + j + 2k - 2$. More precisely, this is a model for the S^1-equivariant cohomology of the states of the B-model on a closed disk. We refer to [CL20, **?**] for detailed justification for this ansatz.

2.5.1.1 The original action for Kodaira–Spencer theory posited by [BCOV94] has a nonlocal kinetic term. In the BV formalism, this is codified by stipulating that the BV pairing is a degenerate odd Poisson tensor rather than an odd symplectic form. The Poisson kernel is given by the expression

$$(\partial_\Omega \otimes 1)\delta_{\Delta \subset X\times X} \in [\mathrm{PV}^{\bullet,\bullet}(X)]^{\widehat{\otimes}\, 2},$$

see [CL15, §1.4]. Here, we view the δ-distribution as a polyvector field using the Calabi–Yau form. Notice that the shifted Poisson tensor does not involve the parameter u at all. For this reason, only the duals of a small number of fields pair nontrivially under the resulting odd BV bracket.

2.5.1.2 There is a natural local interaction which equips the complex (2.5.1.3) with the structure of $\mathbb{Z}/2$ graded Poisson BV theory. Explicitly, it is given by

$$I_{BCOV}(\Sigma) = \mathrm{Tr}_X \langle \exp \Sigma \rangle_0 = \sum_{n\geq 0} \mathrm{Tr}_X \langle \Sigma^{\otimes n}\rangle_0 \tag{2.65}$$

where $\mathrm{Tr}_X \Phi = \int_X (\Phi \vee \Omega) \wedge \Omega$ and where $\langle - \rangle_0$ denotes the genus zero Gromov-Witten invariant with marked points

$$\langle u^{k_1}\mu_1 \otimes \cdots \otimes u^{k_m}\mu_m \rangle_0 := \left(\int_{\overline{\mathcal{M}}_{0,m}} \psi_1^{k_1} \cdots \psi_m^{k_m} \right) \mu_1 \cdots \mu_m = \binom{m-3}{k_1, \cdots, k_m} \mu_1 \cdots \mu_m. \tag{2.66}$$

This interaction is extremely natural from the point of view of string field theory. Indeed, the B-model localizes to the space of constant maps into X, which factors as a product of $\overline{\mathcal{M}}_{0,m} \times X$. This is in keeping with finding an interaction that factors as an integral over X times an integral over $\overline{\mathcal{M}}_{0,m}$.

In [BCOV94] the authors show that the above interaction satisfies the classical master equation. Moreover, they show that the L_∞ structure determined by the above action is equivalent to a natural dgla structure on the complex of fields with Lie bracket given by the Schouten bracket. Explicitly, the equivalence is given by the transcendental automorphism

$$\Sigma \mapsto [u(\exp(\Sigma/u) - 1)]_+$$

where $[-]_+$ denotes projection onto positive powers of u.

2.5.1.3 We pointed out in §2.5.1.1 that the majority of fields pair to zero under the Poisson tensor. Physically these correspond to closed string fields that do not propogate. In the supergravity approximation, the fields that survive are those closed string fields that propogate. In terms of our description of closed string field theory in terms of Kodaira–Spencer theory, this motivates us to consider the smallest cochain complex containing those fields thathave nonzero pairing under the Poisson tensor. This is referred to as minimal Kodaira–Spencer theory.

The fields of minimal Kodaira–Spencer theory are given by the subcomplex of

$$\left(\bigoplus_{i+j \leq d-1} u^i \mathrm{PV}^{j,\bullet}(X)[2], \bar{\partial} + u\partial_\Omega \right). \tag{2.67}$$

We observe that the original odd Poisson tensor lives in this subcomplex. There is a natural action functional given by restricting I_{BCOV} to this space.

2.5.2 The $SU(4)$ twist of type IIA supergravity

We recall the description of the $SU(4)$ twist of type IIA supergravity conjectured in [**?**]. In principal, there is also a minimal, $SU(5)$ invariant, twist of type IIA supergravity but so far no description, even conjecturally, exists. We turn to this in §2.5.6.

Let X be a Calabi–Yau manifold of complex dimension four. The $\mathbb{Z}/2$ graded complex of fields of minimal Kodaira–Spencer theory on X takes the form

$$\begin{array}{ccccccc}
- & & + & & - & & + \\
\hline
 & & & & & & \mathrm{PV}^{0,\bullet} \\
 & & & & \mathrm{PV}^{1,\bullet} & \xrightarrow{u\partial_\Omega} & u\mathrm{PV}^{0,\bullet} \\
 & & \mathrm{PV}^{2,\bullet} & \xrightarrow{u\partial_\Omega} & u\mathrm{PV}^{1,\bullet} & \xrightarrow{u\partial_\Omega} & u^2\mathrm{PV}^{0,\bullet} \\
\mathrm{PV}^{3,\bullet} & \xrightarrow{u\partial_\Omega} & u\mathrm{PV}^{2,\bullet} & \xrightarrow{u\partial_\Omega} & u^2\mathrm{PV}^{1,\bullet} & \xrightarrow{u\partial_\Omega} & u^3\mathrm{PV}^{0,\bullet}
\end{array} . \tag{2.68}$$

Denote this complex by $\mathcal{E}_{mKS}(X)$. Here, $u^\ell \mathrm{PV}^{k,i}$ is placed in parity $k + i - 1 \mod 2$. The classical BCOV action I_{BCOV} follows from the general formula we gave above.

With this in hand the conjecture of [**?**] takes the following form.

Conjecture 2.5.1. The $SU(4)$-invariant twist of type IIA supergravity on $\mathbb{R}^2 \times \mathbb{C}^4$ is the $\mathbb{Z}/2$-graded

Poisson BV theory with fields

$$\alpha = \sum_n \alpha_n u^n \in \mathcal{E}_{mKS}(\mathbb{C}^4) \otimes \Omega^\bullet(\mathbb{R}^2). \tag{2.69}$$

The classical interaction takes the form

$$I_{IIA} = \int_{\mathbb{C}^4 \times \mathbb{R}^2} \alpha_0^3 + \cdots$$

We will need a more detailed description of the classical action. For the moment, let us introduce some notations for the fields of this IIA model. As always, we leave the internal Dolbeault degree implicit:

$$\begin{aligned} \eta \in \mathrm{PV}^{0,\bullet}(\mathbb{C}^4) \otimes \Omega^\bullet(\mathbb{R}^2), \quad \mu + u\nu \in \mathrm{PV}^{1,\bullet}(\mathbb{C}^4) \otimes \Omega^\bullet(\mathbb{R}^2) \oplus u\mathrm{PV}^{0,\bullet}(\mathbb{C}^4) \otimes \Omega^\bullet(\mathbb{R}^2) \\ \Pi \in \mathrm{PV}^{3,\bullet}(\mathbb{C}^4) \otimes \Omega^\bullet(\mathbb{R}^2), \quad \sigma \in \mathrm{PV}^{3,\bullet}(\mathbb{C}^4) \otimes \Omega^\bullet(\mathbb{R}^2). \end{aligned} \tag{2.70}$$

We will not need an explicit notation for the remaining descendant fields.

With this notation in hand, we have the more precise form of the action appearing in the conjecture:

$$I_{IIA} = \frac{1}{2}\mathrm{Tr}_{\mathbb{C}^4\times\mathbb{R}^2} \frac{1}{1-\nu}\mu^2 \wedge \Pi + \mathrm{Tr}_{\mathbb{C}^4\times\mathbb{R}^2} \frac{1}{1-\nu}\eta \wedge \mu \wedge \sigma + \frac{1}{2}\mathrm{Tr}_{\mathbb{C}^4\times\mathbb{R}^2} \frac{1}{1-\nu}\eta \wedge \Pi^2 + \cdots \tag{2.71}$$

where the $\cdots$ denotes terms involving higher-order descendants.

2.5.3 Reduction to IIA supergravity

We now turn back to our eleven-dimensional theory. The first goal is to compare the dimensional reduction of our eleven-dimensional theory on $\mathbb{C}^5 \times \mathbb{R}$ with the $SU(4)$ invariant twist of type IIA on $\mathbb{R}^2 \times \mathbb{C}^4$. Doing so will require a slight modification to the description of the $SU(4)$ twist of IIA supergravity recollected in §2.5.2.

2.5.3.1 Recall that in the physical theory, the components of the C-field in eleven dimensions that are not supported along the M-theory circle become the components of the Ramond–Ramond 2-form of type IIA. However, as noted in [**?**] components of Ramond–Ramond fields do not appear as fields in Kodaira–Spencer theory; rather it is components of their field strengths that appear. We recalled in §2.1.4 that components of the C-field become components of γ_{11d} in $\mathcal{E}$. This suggests that we must modify our description of the twist of type IIA to include potentials for certain fields.

The fundamental fields of the $SU(4)$ twist of IIA supergravity were given in (2.69). We modify the space of fields by introducing potentials for both the Π and σ fields. First, we introduce a field $\gamma \in \Omega^{1,\bullet}(\mathbb{C}^4) \otimes \Omega^\bullet(\mathbb{R}^2)$ (not to be confused, yet, with the γ field in our eleven-dimensional theory) which satisfies $\Pi \vee \Omega = \partial\gamma$ where Ω is the Calabi–Yau form on $\mathbb{C}^4$. This condition does not uniquely fix γ. There is a new linear gauge symmetry determined by $\gamma \to \gamma + \partial_\Omega \beta$ where β is a ghost that we must also introduce. Similarly, we introduce a field $\theta \in \Omega^{0,\bullet}(\mathbb{C}^4) \otimes \Omega^\bullet(\mathbb{R}^2)$ which satisfies $\sigma \vee \Omega = \partial\theta$, there is no extra gauge symmetry present in this condition.[4]

[4]Using the Calabi–Yau form we have normalized the potential fields γ, β, θ to be written as differential forms

In diagrammatic detail, the potential theory we are considering has underlying cochain complex of fields

$$
\begin{array}{ccc}
- & & + \\
\hline
 & & \mathrm{PV}^{0,\bullet}(\mathbb{C}^4) \otimes \Omega^\bullet(\mathbb{R}^2)_\eta \\
 & & \\
\mathrm{PV}^{1,\bullet}(\mathbb{C}^4) \otimes \Omega^\bullet(\mathbb{R}^2)_\mu & \xrightarrow{u\partial_\Omega} & u\mathrm{PV}^{0,\bullet}(\mathbb{C}^4) \otimes \Omega^\bullet(\mathbb{R}^2)_\nu \\
 & & \\
u^{-1}\Omega^{0,\bullet}(\mathbb{C}^4) \otimes \Omega^\bullet(\mathbb{R}^2)_\beta & \xrightarrow{u\partial} & \Omega^{1,\bullet}(\mathbb{C}^4) \otimes \Omega^\bullet(\mathbb{R}^2)_\gamma \\
 & & \\
\Omega^{0,\bullet}(\mathbb{C}^4) \otimes \Omega^\bullet(\mathbb{R}^2)_\theta & &
\end{array}
\tag{2.72}
$$

.

The original space of fields of the twist of IIA supergravity on $\mathbb{C}^4 \times \mathbb{R}^2$ was equipped with an odd Poisson bivector which was degenerate. In other words, it did not define a theory in the conventional BV formalism. One of the key features of this new complex of fields, after we have taken these potentials, is that it is equipped with an odd nondegenerate pairing, thus equipping it with the structure of a theory in the conventional BV formalism.

The pairing is $\mathrm{R}es_u \frac{\mathrm{d}u}{u} \int^\Omega_{\mathbb{C}^4\times\mathbb{R}^2} \alpha \vee \alpha'$ where α, α' are two general fields in this potential theory on $\mathbb{C}^4 \times \mathbb{R}^2$. Explicitly, in the description of the fields in (2.78) the pairing is

$$
\int^\Omega_{\mathbb{C}^4\times\mathbb{R}^2} \eta\theta + \int^\Omega_{\mathbb{C}^4\times\mathbb{R}^2} \mu \vee \gamma + \int^\Omega_{\mathbb{C}^4\times\mathbb{R}^2} \nu\beta .
$$

This pairing is compatible with the odd Poisson bracket present in the original theory on $\mathbb{C}^4 \times \mathbb{R}^2$.

The type IIA action completely determines the action of this theory with potentials. One simply takes the (2.71) and replaces all appearances of Π with $\partial_\Omega \gamma$ and all appearances of σ with $\partial_\Omega \theta$. This yields the interaction of the potential theory

$$
\tilde{I}_{IIA} = \frac{1}{2}\int^\Omega_{\mathbb{C}^4\times\mathbb{R}^2} \frac{1}{1-\nu}\mu^2 \vee \partial\gamma + \int^\Omega_{\mathbb{C}^4\times\mathbb{R}^2} \frac{1}{1-\nu}(\eta\wedge\mu)\vee\partial\theta + \frac{1}{2}\int_{\mathbb{C}^4\times\mathbb{R}^2} \frac{1}{1-\nu}\eta\wedge\partial\gamma\wedge\partial\gamma \tag{2.73}
$$

Notice that the terms involving higher descendants vanishes since these fields are set to zero in the potential theory.

2.5.3.2 We turn to the proof of the main result of this section that the dimensional reduction of our eleven-dimensional theory agrees with the twist of IIA supergravity just introduced.

We recall the notion of dimensional along a holomorphic direction following [ESW20]. Suppose that $V_\mathbb{R}$ is a real vector space and denote by V its complexification. We consider a field theory defined on $M \times V$, which is holomorphic along V (in particular, this means that the theory is translation invariant along V). We consider the dimensional reduction along the projection

$$
M \times V \to M \times V_\mathbb{R} \tag{2.74}
$$

instead of polyvector fields.

induced by $\mathrm{Re}\mathcal{O}lonV \to V_{\mathbb{R}}$. Most relevant for us is the case when $V = \mathbb{C}$ and M is $\mathbb{C}^4 \times \mathbb{R}$, but the explicit form of the theory along M is not important at the moment.

For illustrative purposes, let us first assume that M is a point and that the space of fields is of the form $\Omega^{0,\bullet}(V) \otimes W$ for W some graded vector space. As properly formulated in [ESW20], it is shown that the dimensional reduction along $V \to V_{\mathbb{R}}$ is equivalent to the theory whose fields are $\Omega^{\bullet}(V_{\mathbb{R}}) \otimes W$. In other words, the dimensional reduction of the holomorphic theory on V is a topological theory on $V_{\mathbb{R}}$.

If we put M back in, the result is similar. Suppose the original theory is of the form $\mathcal{E}(M) \otimes \Omega^{0,\bullet}(V) \otimes W$. Then, the dimensional reduction along (2.74) is the theory whose space of fields is $\mathcal{E}(M) \otimes \Omega^{\bullet}(V_{\mathbb{R}}) \otimes W$.

An explicit model for this reduction can be described as follows. Suppose $V\mathcal{O}ng\mathbb{C}^n$ and place the theory on $(\mathbb{C}^{\times})^{\times n} \subset \mathbb{C}^n$. The dimensional reduction along $\mathbb{C}^n \to \mathbb{R}^n$ agrees with the compactification of the theory along $S^1 \times \cdots \times S^1$ where one throws away all nonzero winding modes around each circle.

Proposition 2.5.2. The $SU(4)$ invariant twist of type IIA on $\mathbb{C}^4 \times \mathbb{R}^2$ is the dimensional reduction of the eleven-dimensional theory along

$$\mathbb{C}^4 \times \mathbb{C} \times \mathbb{R}_t \to \mathbb{C}^4 \times \mathbb{R}_x \times \mathbb{R}_t \mathcal{O}ng \mathbb{C}^4 \times \mathbb{R}^2.$$

Proof. Let us denote the holomorphic coordinate we are reducing along by $z_5 = x + iy$. We first read off the dimensional reduction of each component field of the eleven-dimensional theory. Per the above discussion, this is obtained by taking all fields to be independent of y and replacing $\mathrm{d}\bar{z}_5$ by $\mathrm{d}x$. To not confuse the notations of fields in ten and eleven dimensions, we use the notation α_{11d} to denote an eleven-dimensional field.

The reductions of the eleven-dimensional fields ν_{11d}, β_{11d} are easy to describe. Recall that

$$\nu_{11d} \in \mathrm{PV}^{0,\bullet}(\mathbb{C}^5) \otimes \Omega^{\bullet}(\mathbb{R}).$$

The reduction of this field is a ten-dimensional ν field

$$\nu(z_i, x, t) = \nu_{11d}(z_i, x, y = 0, t)|_{\mathrm{d}\bar{z}_5 = \mathrm{d}x}.$$

Similarly, the reduction of β_{11d} is a ten-dimensional β field

$$\beta(z_i, x, t) = \beta_{11d}(z_i, x, y = 0, t)|_{\mathrm{d}\bar{z}_5 = \mathrm{d}x}.$$

The reduction of the eleven-dimensional fields μ_{11d} and γ_{11d} require a bit of massaging. We break the $SU(5)$ symmetry to $SU(4)$ to write

$$\mu_{11d} = \mu^0_{11d} + \theta_{11d}\partial_{z_5}$$

where

$$\mu^0_{11d} \in \mathrm{PV}^{1,\bullet}(\mathbb{C}^4) \otimes \Omega^{0,\bullet}(\mathbb{C}_{z_5}) \otimes \Omega^\bullet(\mathbb{R}_t)$$
$$\theta_{11d} \in \Omega^{0,\bullet}(\mathbb{C}^4) \otimes \Omega^{0,\bullet}(\mathbb{C}_{z_5}) \otimes \Omega^\bullet(\mathbb{R}_t).$$

The dimensional reduction of μ^0_{11d} is a ten-dimensional μ field

$$\mu(z_i, x, t) = \mu^0_{11d}(z_i, x, y = 0, t)|_{\mathrm{d}\bar{z}_5 = \mathrm{d}x}.$$

The dimensional reduction of θ_{11d} is a θ field

$$\theta(z_i, x, t) = \theta_{11d}(z_i, x, y = 0, t)|_{\mathrm{d}\bar{z}_5 = \mathrm{d}x}.$$

Finally, write the eleven-dimensional field γ_{11d} as

$$\gamma_{11d} = \gamma^0_{11d} + \eta_{11d}\mathrm{d}z_5$$

where

$$\gamma^0_{11d} \in \Omega^{1,\bullet}(\mathbb{C}^4) \otimes \Omega^{0,\bullet}(\mathbb{C}_{z_5}) \otimes \Omega^\bullet(\mathbb{R}_t)$$
$$\eta_{11d} \in \mathrm{PV}^{0,\bullet}(\mathbb{C}^4) \otimes \Omega^{0,\bullet}(\mathbb{C}_{z_5}) \otimes \Omega^\bullet(\mathbb{R}_t).$$

The dimensional reduction of γ^0_{11d} is a ten-dimensional γ field

$$\gamma(z_i, x, t) = \gamma^0_{11d}(z_i, x, y = 0, t)|_{\mathrm{d}\bar{z}_5 = \mathrm{d}x}.$$

The dimensional reduction of η_{11d} is an η field

$$\eta(z_i, x, t) = \eta_{11d}(z_i, x, y = 0, t)|_{\mathrm{d}\bar{z}_5 = \mathrm{d}x}.$$

Next, we read off the dimensional reduction of the eleven-dimensional action. Let us first focus on the term present in BF theory which is $\int^\Omega \frac{1}{1-\nu_{11d}}\mu^2_{11d} \vee \partial\gamma_{11d}$. Upon reduction, this becomes

$$\int^{\Omega_{\mathbb{C}^4}}_{\mathbb{C}^4\times\mathbb{R}^2} \frac{1}{1-\nu}\mu^2 \vee \partial\gamma + \int^{\Omega_{\mathbb{C}^4}}_{\mathbb{C}^4\times\mathbb{R}^2} \frac{1}{1-\nu}(\theta \wedge \mu) \vee \partial\eta \tag{2.75}$$

Next, consider the cubic term in the eleven-dimensional action $J = \frac{1}{6}\int \gamma_{11d} \wedge \partial\gamma_{11d} \wedge \partial\gamma_{11d}$. Upon reduction, this becomes

$$\int_{\mathbb{C}^4\times\mathbb{R}^2} \eta \wedge \partial\gamma \wedge \partial\gamma. \tag{2.76}$$

The sum of the action functionals (2.75) and (2.76) does not precisely agree with the IIA action $\tilde{I}_{IIA}$. To relate the two actions we must make the following field redefinition:

$$\tilde{\theta} = \frac{1}{1-\nu}\theta, \quad \tilde{\eta} = (1-\nu)\eta, \quad \tilde{\beta} = \beta + \frac{1}{1-\nu}\eta \wedge \theta.$$

Notice that this change of coordinates is compatible with the odd symplectic pairing on the fields.

Under this field redefinition the total dimensionally reduced action can be written as

$$\int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega_{\mathbb{C}^4}} \frac{1}{1-\nu}\mu^2 \vee \partial\gamma + \int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega_{\mathbb{C}^4}} \frac{1}{1-\nu}\tilde{\eta}\wedge\partial\gamma\wedge\partial\gamma + \int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega_{\mathbb{C}^4}} (\tilde{\theta}\wedge\mu)\vee\partial\left(\frac{1}{1-\nu}\tilde{\eta}\right) + \int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega_{\mathbb{C}^4}} \frac{1}{1-\nu}(\tilde{\eta}\wedge\tilde{\theta})\partial_\Omega\mu. \quad (2.77)$$

The first line comes from plugging in the new fields into the interactions (2.75) and (2.76). The second line comes from plugging in the new fields into the kinetic term $\int \beta\partial_\Omega\mu$, which because of the non-linear change of coordinates now contributes to the interaction. We observe that the first two terms agree with the first and third terms in (2.73).

After integrating by parts, the remaining terms can be written as

$$-\int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega_{\mathbb{C}^4}} \left(\frac{1}{1-\nu}\tilde{\eta}\right)\partial_\Omega(\tilde{\theta}\mu) + \int_{\mathbb{C}^4\times\mathbb{R}^2}^{\Omega_{\mathbb{C}^4}} \left(\frac{1}{1-\nu}\tilde{\eta}\right)\tilde{\theta}\partial_\Omega\mu.$$

Applying the identity $\partial_\Omega(\tilde{\theta}\mu) = \tilde{\theta}\partial_\Omega\mu + \partial(\tilde{\theta})\vee\mu$, we see that this agrees exactly with the second term in (2.73). □

2.5.4 The twist of type I supergravity

We now turn to a different type of redution of the eleven-dimensional theory, this time involving type I supergravity. We begin by briefly recalling the description of type I supergravity following [CL20] which was motivated by the unoriented B-model. In [SW21], the second two authors verified the conjectural description of the space of fields recalled below using the pure spinor formalism. Unlike type IIA supergravity, there only exists an $SU(5)$ invariant twist of type I supergravity and it is holomorphic in the maximal number of dimensions.

Concretely, the space of fields of the $SU(5)$ twist of type I supergravity is a subspace of minimal Kodaira–Spencer theory on $\mathbb{C}^5$. The $\mathbb{Z}/2$ graded space of fields equipped with its linear BRST operator is

$$\begin{array}{cccc} - & + & - & + \\ \hline \mathrm{PV}^{1,\bullet}(\mathbb{C}^5) \xrightarrow{u\partial_\Omega} u\mathrm{PV}^{0,\bullet}(\mathbb{C}^5) & & & \\ & & & \\ \mathrm{PV}^{3,\bullet}(\mathbb{C}^5) \xrightarrow{u\partial_\Omega} u\mathrm{PV}^{2,\bullet}(\mathbb{C}^5) & \xrightarrow{u\partial_\Omega} u^2\mathrm{PV}^{1,\bullet}(\mathbb{C}^5) & \xrightarrow{u\partial_\Omega} u^3\mathrm{PV}^{0,\bullet}(\mathbb{C}^5) & \end{array} . \quad (2.78)$$

Let us give a description of the classical action. Introduce notations for the fields of this type I model:

$$\mu + u\nu \in \mathrm{PV}^{1,\bullet}(\mathbb{C}^5) \oplus u\mathrm{PV}^{0,\bullet}(\mathbb{C}^5), \quad \sigma \in \mathrm{PV}^{3,\bullet}(\mathbb{C}^5). \quad (2.79)$$

We will not need an explicit notation for the remaining descendant fields.

Conjecture 2.5.3. The twist of type I supergravity on $\mathbb{C}^5$ is the $\mathbb{Z}/2$-graded theory with fields $\mu + u\nu, \sigma$ as above and with classical action

$$I_{\text{type I}} = \mathrm{Tr}_{\mathbb{C}^5}\frac{1}{1-\nu}\mu^2\vee\sigma + \cdots \quad (2.80)$$

where the $\cdots$ stands for terms involving the higher descendant fields.

2.5.4.1 Like in the type IIA discussion, there is a slight modification of the type I model above which is most directly related to eleven-dimensional supergravity.

This modification involves replacing the field σ above by a potential $\tilde{\gamma} \in \Omega^{1,\bullet}(\mathbb{C}^5)$ which satisfies $\Omega \vee \sigma = \partial \tilde{\gamma}$. This condition does not fix $\tilde{\gamma}$ uniquely, there is a gauge symmetry of the form $\tilde{\gamma} \to \tilde{\gamma} + \partial \tilde{\beta}$.

In detail, this potential theory we are considering has underlying cochain complex of fields

$$\begin{array}{ccc} - & + & - \\ \hline \mathrm{PV}^{1,\bullet}(\mathbb{C}^5)_\mu \xrightarrow{\partial_\Omega} & \mathrm{PV}^{0,\bullet}(\mathbb{C}^5)_\nu & \\ & \Omega^{0,\bullet}(\mathbb{C}^5)_{\tilde{\beta}} \xrightarrow{\partial} & \Omega^{1,\bullet}(\mathbb{C}^5)_{\tilde{\gamma}}. \end{array} \tag{2.81}$$

This space of fields is equipped with an odd nondegenerate pairing. Like the eleven-dimensional theory, it is a classical BV theory in the $\mathbb{Z}/2$-graded sense.

The type I action (2.80) completely determines the action of this theory with potentials. One simply takes the action and replaces all appearances of σ with $\Omega^{-1} \vee \partial \tilde{\gamma}$. This yields the interaction of the potential theory

$$\tilde{I}_{\text{type I}} = \frac{1}{2} \int_{\mathbb{C}^5}^{\Omega} \frac{1}{1-\nu} \mu^2 \vee \partial \tilde{\gamma}. \tag{2.82}$$

Notice that the terms involving higher descendants vanishes since these fields are set to zero in the potential theory.

2.5.5 Slab compactification

We consider placing twisted eleven-dimensional supergravity on the manifold $\mathbb{C}^5 \times [0,1]$. In order to do this, we must choose appropriate boundary conditions at $t=0$ and $t=1$. Our eleven-dimensional theory on such manifolds fits nicely into the formalism of [BY16, Rab] in that it is topological in the direction transverse to the boundary.

The phase space of the theory at $t=0$ or $t=1$ is

$$\begin{array}{cc} - & + \\ \hline \mathrm{PV}^{1,\bullet}(\mathbb{C}^5)_\mu \xrightarrow{\partial_\Omega} & \mathrm{PV}^{0,\bullet}(\mathbb{C}^5)_\nu \\ \Omega^{0,\bullet}(\mathbb{C}^5)_\beta \xrightarrow{\partial} & \Omega^{1,\bullet}(\mathbb{C}^5)_\gamma. \end{array} \tag{2.83}$$

The wedge an integrate pairing between the top and bottom lines induces an *even* symplectic structure on the phase space. Denote this phase space by $\mathcal{E}_\partial$ for the moment.

The phase space is equipped with the restriction of the linear BRST operator of the full eleven-dimensional theory. There is also a non linear BRST operator, just like in the bulk theory. The BV action induces a L_∞ structure on the parity shift $\Pi\mathcal{E}_\partial$ whose cohomology is still a trivial central extension of $E(5,10)$.

A boundary condition is given by a Lagrangian subspace of $\mathcal{E}_\partial$ with respect to this even symplectic structure. To make sense of the theory on $\mathbb{C}^5 \times [0,1]$ we must make the choice of two separate boundary conditions

$$\mathcal{M}_{t=0}, \mathcal{M}_{t=1} \subset \mathcal{E}_\partial.$$

Moreover, these boundary conditions carry non linear BRST operators endowing their parity shifts $\Pi\mathcal{M}_{t=0}, \Pi\mathcal{M}_{t=1}$ with the structures of L_∞ algebras. These L_∞ structures must be compatible with the one on the phase space. In fact, in our context these boundary conditions are abstractly isomorphic. We will explain the explicit boundary conditions momentarily.

An important thing to note is that the fields of the theory compactified on the slab is computed by the *derived* intersection of the two Lagrangians:

$$\mathcal{M}_{t=0} \overset{\mathbb{L}}{\underset{\mathcal{E}_\partial}{\times}} \mathcal{M}_{t=1}.$$

To compute this derived intersection we must suitably resolve the boundary conditions.

2.5.5.1 At $t = 0$, the boundary condition of the eleven-dimensional theory is determined by declaring

$$\mathcal{M}_{t=0} : \quad \gamma|_{t=0} = \beta|_{t=0} = 0.$$

We will place the theory on $\mathbb{C}^5 \times [0,1]$ by imposing the same boundary condition at $t = 1$:

$$\mathcal{M}_{t=1} : \quad \gamma|_{t=1} = \beta|_{t=1} = 0.$$

Proposition 2.5.4. With these boundary conditions for the classical eleven-dimensional theory on $\mathbb{C}^5 \times [0,1]$, the dimensional reduction along

$$\mathbb{C}^5 \times [0,1] \to \mathbb{C}^5$$

is equivalent to the twist of type I supergravity on $\mathbb{C}^5$.

Proof. Notice that both $\mathcal{M}_{t=0}$ and $\mathcal{M}_{t=0}$ are abstractly isomorphic to the complex resolving divergence-free vector fields

$$\begin{array}{cc} - & + \\ \hline \mathrm{PV}^{1,\bullet}(\mathbb{C}^5)_\mu & \xrightarrow{\partial_\Omega} \mathrm{PV}^{0,\bullet}(\mathbb{C}^5)_\nu . \end{array} \tag{2.84}$$

To compute the derived intersection between the two Lagrangians at $t = 0$ and $t = 1$ we replace the Lagrangian morphism $\mathcal{M}_{t=0} \hookrightarrow \mathcal{E}_\partial$. Consider the cochain complex $\tilde{\mathcal{M}}_{t=0}$ defined by

$$\begin{array}{ccc} - & + & - \\ \hline \mathrm{PV}^{1,\bullet}(\mathbb{C}^5)_\mu \xrightarrow{\partial_\Omega} & \mathrm{PV}^{0,\bullet}(\mathbb{C}^5)_\nu & \\ \Omega^{0,\bullet}(\mathbb{C}^5)_\beta \xrightarrow{\partial} & \Omega^{1,\bullet}(\mathbb{C}^5)_\gamma & \\ \quad\searrow^{\mathrm{Id}} & \quad\searrow^{\mathrm{Id}} & \\ & \Omega^{0,\bullet}(\mathbb{C}^5)_{\tilde\beta} \xrightarrow{\partial} & \Omega^{1,\bullet}(\mathbb{C}^5)_{\tilde\gamma} . \end{array} \tag{2.85}$$

Notice that as a graded vector space, this complex is of the form $\mathcal{E}_\partial \oplus (\Omega^{0,\bullet} \oplus \Pi\Omega^{1,\bullet})$. The L_∞ structure on $\Pi\tilde{\mathcal{M}}_{t=0}$ extends the one on $\mathcal{E}_\partial$ coming from the bulk BV action. Notice that the obvious embedding $\mathcal{M}_{t=0} \hookrightarrow \tilde{\mathcal{M}}_{t=0}$ is a quasi-isomorphism.

The projection map $\tilde{\mathcal{M}}_{t=0} \twoheadrightarrow \mathcal{E}_\partial$ factors the original Lagrangian inclusion as

$$\mathcal{M}_{t=0} \hookrightarrow \tilde{\mathcal{M}}_{t=0} \twoheadrightarrow \mathcal{E}_\partial .$$

To compute the derived intersection of $\mathcal{M}_{t=0}$ and $\mathcal{M}_{t=1}$ we can compute the ordinary intersection of $\tilde{\mathcal{M}}_{t=0}$ and $\mathcal{M}_{t=1}$.

Let $\mu_{t=1}$ and $\nu_{t=1}$ denote the fields present in the other boundary condition $\mathcal{M}_{t=1}$. The intersection $\tilde{\mathcal{M}}_{t=0} \times_{\mathcal{E}_\partial} \mathcal{M}_{t=1}$ is computed by setting the fields β, γ to zero and $\mu = \mu_{t=1}$, $\nu = \nu_{t=1}$. Thus, we are left with

$$\begin{array}{ccc} - & + & - \\ \hline \mathrm{PV}^{1,\bullet}(\mathbb{C}^5)_\mu \xrightarrow{\partial_\Omega} & \mathrm{PV}^{0,\bullet}(\mathbb{C}^5)_\nu & \\ & \Omega^{0,\bullet}(\mathbb{C}^5)_{\tilde\beta} \xrightarrow{\partial} & \Omega^{1,\bullet}(\mathbb{C}^5)_{\tilde\gamma} \end{array} \tag{2.86}$$

This is precisely the underlying cochain complex of fields for the type I model with potentials. The odd nondegenerate pairing on this complex agrees with the one on this particular potential theory for the twist of type I supergravity. The L_∞ structure on the parity shift of this complex is compatible with the one induced from the BV action in (2.82). □

2.5.6 The $SU(5)$ twist of type IIA supergravity

Thus, given that our eleven-dimensional theory correctly describes the $SU(5)$-invariant twist of supergravity on $\mathbb{C}^5 \times \mathbb{R}$, to obtain the $SU(5)$ twist of type IIA supergravity we should reduce along the topological $\mathbb{R}$ direction. This results in a $SU(5)$ invariant, holomorphic, theory on $\mathbb{C}^5$.

Let us briefly spell out the fields present in this dimensional reduction. The reduction is obtained by replacing $\Omega^\bullet(\mathbb{R})$ with its translation invariant subalgebra $\mathbb{C}[\epsilon] = \mathbb{C}[\mathrm{d}t]$. Here, ϵ is an odd parameter playing the role of the translation invariant one-form $\mathrm{d}t \in \Omega^1(\mathbb{R})$. Equivalently, we are compactifying the theory along

$$\mathbb{C}^5 \times S^1 \to \mathbb{C}^5 .$$

The field μ_{11d} is replaced by the field

$$\mu + \epsilon\mu' \in \Pi\mathrm{PV}^{1,\bullet}(\mathbb{C}^5)[\epsilon].$$

Notice that the lowest component of μ is odd (just like μ_{11d}), but the lowest component of μ' is now even. Completely similarly, the remaining fields reduce as $\nu + \epsilon\nu'$, $\gamma + \epsilon\gamma'$, and $\beta + \epsilon\beta'$.

In summary, the linear complex of fields of the dimensionally reduced theory on $\mathbb{C}^5$ is

$$\begin{array}{cccc} \text{odd} & \text{even} & \text{even} & \text{odd} \\ \hline \mathrm{PV}^{1,\bullet}(\mathbb{C}^5)_\mu \xrightarrow{\partial} & \mathrm{PV}^{0,\bullet}(\mathbb{C}^5)_\nu & & \\ & \epsilon\Omega^{0,\bullet}(\mathbb{C}^5)_{\beta'} \xrightarrow{\partial_\Omega} & \epsilon\Omega^{1,\bullet}(\mathbb{C}^5)_{\gamma'} . & \\ & \epsilon\mathrm{PV}^{1,\bullet}(\mathbb{C}^5)_{\mu'} \xrightarrow{\partial_\Omega} & \epsilon\mathrm{PV}^{0,\bullet}(\mathbb{C}^5)_{\nu'} & \\ & & \Omega^{0,\bullet}(\mathbb{C}^5)_\beta \xrightarrow{\partial_\Omega} & \Omega^{1,\bullet}(\mathbb{C}^5)_\gamma . \end{array} \tag{2.87}$$

We can compute the dimensional reduction of the eleven-dimensional action $S_{BF,\infty} + J$ in a similar way to how we have done in the past few sections. We arrive at the action functional described below.

Conjecture 2.5.5. The $SU(5)$ twist of type IIA supergravity on $\mathbb{C}^5$ is equivalent to the theory whose linear BRST complex of fields is displayed in (2.87). The full action functional is

$$\int_{\mathbb{C}^5}^{\Omega} \left(\beta' \wedge \bar\partial\nu + \beta \wedge \bar\partial\nu' + \gamma' \wedge \bar\partial\mu + \gamma \wedge \bar\partial\mu' + \beta' \wedge \partial_\Omega \mu + \beta \wedge \partial_\Omega \mu'\right) \\ + \int_{\mathbb{C}^5}^{\Omega} \left(\frac{1}{2}\frac{1}{1-\nu}\mu^2 \vee \partial\gamma' + \frac{1}{1-\nu}(\mu \wedge \mu') \vee \partial\gamma' + \frac{1}{2}\frac{\nu'}{(1-\nu)^2}\mu^2 \vee \partial\gamma\right) \\ + \frac{1}{2}\int_{\mathbb{C}^5} \gamma' \wedge \partial\gamma \wedge \partial\gamma. \quad (2.88)$$

The first two lines in (2.88) arise from the reduction of the BF action $S_{BF,\infty}$. The final line arises from the reduction of $J = \frac{1}{6}\int \gamma_{11d}\partial\gamma_{11d}\partial\gamma_{11d}$.

2.5.6.1 The slab compactification of the previous section was one way to implement the $S^1/\mathbb{Z}/2$ reduction of the eleven-dimensional theory. We offer another point of view of this $S^1/\mathbb{Z}/2$ reduction.

First off, there is the following $\mathbb{Z}/2$ action on the eleven-dimensional theory on $\mathbb{C}^5 \times S^1$ before compactifying. We obtain it by the following tensor product of $\mathbb{Z}/2$ actions. First, $\mathbb{Z}/2$ acts on $\Omega^\bullet(S^1)$ by orientation reversing diffeomorphisms. Second, we declare that the eigenvalue of the $\mathbb{Z}/2$ action on $\mathrm{PV}^{k,\bullet}(\mathbb{C}^5)$, for $k = 0, 1$ is $+1$ and the eigenvalue of the $\mathbb{Z}/2$ action on $\Omega^{k,\bullet}(\mathbb{C}^5)$ for $k = 0, 1$ is -1. This determines a $\mathbb{Z}/2$ action on the full space of fields of the eleven-dimensional theory.

Upon S^1 compactification the $\mathbb{Z}/2$ action is easy to describe: μ, ν both have eigenvalue $+1$, μ', ν' both have eigenvalue -1, γ, β both have eigenvalue -1, and γ', β' both have eigenvalue $+1$. In particular, we see that the $\mathbb{Z}/2$ fixed points simply pick out the $\mu, \nu, \gamma', \beta'$ fields; this comprises the first two lines of (2.87).

The fields match precisely with the fields in the twist of type I supergravity that we recalled in §2.5.4.1 (Under the relabeling $\gamma' \leftrightarrow \tilde\gamma, \beta' \leftrightarrow \tilde\beta$). Furthermore, restricting the action in the above conjecture agrees precisely with the action of this twisted type I model.

2.5.7 Compactification along a CY3

In the first section we saw that the eleven-dimensional theory can be defined on any manifold that is a product of a Calabi–Yau five-fold with a smooth oriented one-manifold. In this section, we investigate an important compactification of the eleven-dimensional theory which involves the Calabi–Yau manifold $X \times \mathbb{C}^2$ where X is a simply connected compact Calabi–Yau three-fold.

The compactification of the theory along the three-fold X

$$X \times \mathbb{C}^2 \times \mathbb{R} \to \mathbb{C}^2 \times \mathbb{R}$$

yields an effective five-dimensional theory which is holomorphic along $\mathbb{C}^2$ and topological along $\mathbb{R}$. Upon compactification, we will find a match with a description of the twist of five-dimensional minimally supersymmetric supergravity.

Proposition 2.5.6. The compactification of the eleven-dimensional theory along a Calabi–Yau three-fold X is equivalent to the twist of five-dimensional $\mathcal{N} = 1$ supergravity with $h^{1,1}(X) - 1$ vector multiplets and $h^{1,2}(X) + 1$ hypermultiplets.

2.5.7.1 We give a conjectural capitulation of the twist of five-dimensional $\mathcal{N} = 1$ supergravity. Before twisting, a general five-dimensional $\mathcal{N} = 1$ supergravity contains a gravity multiplet coupled to some number of vector and hypermultiplets. The twist of the vector and hypermultiplet has been computed in [ESW20], and we recall it below. The twist of the gravity multiplet is less clear. A thorough computation of the twist has yet to appear, though some checks have been established by Elliott and the last author in [EW21]. We give a description of the twist now, but leave a detailed computation from first principles to future work.

The gravity multiplet, see [CCDF95] for instance, consists of a graviton e, a gravitino ψ, and a one-form gauge field $\mathcal{A}_{grav}$. After twisting, the graviton and components of the gravitino decompose into two Dolbeault-de Rham valued fields

$$\alpha, \eta \in \Pi\Omega^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}),$$

whose lowest components both carry odd parity. The one-form gauge field $\mathcal{A}_{grav}$ and the remaining components of the gravitino decompose into two more Dolbeault-de Rham valued fields

$$A_{grav}, B_{grav} \in \Pi\Omega^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}),$$

whose lowest components also both carry odd parity.

Conjecture 2.5.7. The twist of five-dimensional supergravity (with nonzero Chern–Simons term) with vector multiplets valued in a Lie algebra $\mathfrak{g}$ and hypermultiplets valued in a representation V has BV fields

- $\alpha, A_{grav} \in \Pi\Omega^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R})$ with conjugate BV fields η, B_{grav},
- $A \in \Pi\Omega^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}) \otimes \mathfrak{g}$ with conjugate BV field B,
- $\chi \in \Omega^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}) \otimes V$ with conjugate BV field ψ.

The action is

$$\int^{\Omega}_{\mathbb{C}^2\times\mathbb{R}} \left(\eta\bar{\partial}\alpha + B_{grav}\bar{\partial}A_{grav} + B\bar{\partial}A + \psi\bar{\partial}\chi\right) + \int^{\Omega}_{\mathbb{C}^2\times\mathbb{R}} \left(\frac{1}{2}\eta\{\alpha,\alpha\} + B_{grav}\{\alpha, A_{grav}\} + B\{\alpha, A\} + \psi\{\alpha,\chi\}\right) + \frac{1}{6}\int_{\mathbb{C}^2\times\mathbb{R}} B_{grav}\partial B_{grav}\partial B_{grav}. \quad (2.89)$$

2.5.7.2 With this description of the twist of five-dimensional supergravity, we turn to the proof of Proposition 2.5.6.

First, we set up some notation. Let Ω_X be the holomorphic volume form on X. To define the eleven-dimensional theory on $X \times \mathbb{C}^2 \times \mathbb{R}$ we use the Calabi–Yau form $\Omega_X \wedge \mathrm{d}z_1 \wedge \mathrm{d}z_2$, where $\{z_i\}$

is a holomorphic coordinate on $\mathbb{C}^2$. Let $\omega \in \Omega^{1,1}(X)$ be a fixed Kähler form on X. For any k, let $H^k(X, \Omega^k_X)_\perp$ denote the cohomology of the primitive elements.

Proof. Consider the eleven-dimensional field ν_{11d}. Under the equivalence

$$\begin{aligned} \mathrm{PV}^{0,\bullet}(X \times \mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}) &\simeq H^\bullet(X, \mathcal{O}) \otimes \mathrm{PV}^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}) \\ &= \mathrm{PV}^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}) \oplus \Pi\bar{\Omega}_X \mathrm{PV}^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}) \end{aligned}$$

the ν_{11d} field decomposes as

$$\nu_{11d} = \nu + \bar{\Omega}_X \tilde{\nu}.$$

Here $\bar{\Omega}_X$ is the complex conjugate to the holomorphic volume form on X. Notice that the zero form component of $\tilde{\nu}$ is a field with even parity.

Next, consider the eleven-dimensional field μ_{11d}. Under the equivalence

$$\begin{aligned} \Pi\mathrm{PV}^{1,\bullet}(X \times \mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}) &\simeq \Pi H^\bullet(X, \mathcal{O}) \otimes \mathrm{PV}^{1,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}) \\ &\oplus \Pi H^\bullet(X, \mathrm{T}_X) \otimes \mathrm{PV}^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}) \\ &= \Pi\mathrm{PV}^{1,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}) \oplus \bar{\Omega}_X \mathrm{PV}^{1,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}) \\ &\oplus H^1(X, \mathrm{T}_X) \otimes \mathrm{PV}^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}) \oplus \Pi H^2(X, \mathrm{T}_X) \otimes \mathrm{PV}^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}) \end{aligned}$$

the field μ_{11d} decomposes as

$$\begin{aligned} \mu_{11d} &= \mu + \bar{\Omega}_X \tilde{\mu} \\ &\quad + e^i \chi_i + f^a A_a + (\Omega_X^{-1} \vee \omega^2) A_{grav}. \end{aligned}$$

Here, $\{e^i\}_{i=1,\ldots,h^{2,1}}$ is a basis for $H^1(X, \mathrm{T}_X)$ and $\{f^a\}_{a=1,\ldots,h^{1,1}-1}$ is a basis for

$$H^2(X, \Omega^2_X)_\perp \subset H^2(X, \Omega^2_X) \mathcal{O}ng H^2(X, \mathrm{T}_X).$$

Notice that the zero form part of $\tilde{\mu}$ is an even field, the zero form part of χ_i is an even field, the zero form part of A_a is an odd field, and the zero form part of μ_ω is an odd field.

The decomposition for the eleven-dimensional fields γ_{11d} and β_{11d} is similar. We record it here:

$$\begin{aligned} \beta_{11d} &= \beta + \bar{\Omega}_X \tilde{\beta} \\ \gamma_{11d} &= \gamma + \bar{\Omega}_X \tilde{\gamma} + e_i \psi^i + f_a B^a + \omega \wedge B_{grav}. \end{aligned}$$

Here, $\{e_i\}_{i=1,\ldots,h^{2,1}}$ is a basis for $H^2(X, \Omega^1_X)$ dual to the basis $\{e^i\}$ under the Serre pairing. Also, $\{f_a\}_{a=1,\ldots,h^{1,1}-1}$ is a basis for $H^1(X, \Omega^1_X)_\perp$ dual to the basis $\{f^a\}$.

To compare most directly to the description of the twist of five-dimensional $\mathcal{N} = 1$ supergravity we modestly modify the fields. Let ∂ be the holomorphic de Rham differential along $\mathbb{C}^2$. First, we introduce a potential for the fields μ and $\tilde{\mu}$. Let

$$\alpha, \chi \in \Omega^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R})$$

be differential forms satisfying $\partial\alpha = \mu \vee \Omega_{\mathbb{C}^2}$ and $\partial\chi = \tilde{\mu} \vee \Omega_{\mathbb{C}^2}$. The fields $\nu, \tilde{\nu}$ are set to zero.

Dually, we replace the fields $\gamma, \tilde{\gamma}$ their 'field strengths', suitably renormalized with respect to the volume form

$$\eta = (\mathrm{d}^2 z)^{-1} \vee \partial \tilde{\gamma}, \quad \psi = (\mathrm{d}^2 z)^{-1} \vee \partial \gamma \in \Omega^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}).$$

The roles of $\beta, \tilde{\beta}$ were as gauge symmetries implementing $\gamma \to \gamma + \partial\beta$ and $\tilde{\gamma} \to \tilde{\gamma} + \partial\tilde{\beta}$. Since we are replacing $\gamma, \tilde{\gamma}$ by their images under the operator ∂, these gauge symmetries are set to zero.

In summary, we are left with the following fields

$$\begin{array}{rclrcll} \alpha, A_{grav} & \in & \Pi\Omega^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}), & \eta, B_{grav} & \in & \Pi\Omega^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}) & \\ \chi, \chi_i & \in & \Omega^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}), & \psi, \psi^i & \in & \Omega^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}), & i = 1, \ldots, h^{2,1} \\ A_a & \in & \Pi\Omega^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}), & B^a & \in & \Pi\Omega^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}), & a = 1, \ldots, h^{1,1} - 1. \end{array}$$

Let us plug these fields in to the eleven-dimensional action. First, consider the BF term $\frac{1}{2}\int^\Omega \frac{1}{1-\nu_{11d}} \mu_{11d}^2 \gamma_{11d}$. With the field redefinitions above, this decomposes as

$$\int_{\mathbb{C}^2 \times \mathbb{R}}^\Omega \left(\frac{1}{2} \partial\alpha \wedge \partial\alpha \wedge \eta + \partial A_{grav} \wedge \partial A_{grav} \wedge B_{grav}\right) \\ + \int_{\mathbb{C}^2 \times \mathbb{R}}^\Omega \left(\partial\alpha \wedge \partial\chi \wedge \psi + \partial\alpha \wedge \partial\chi_i \wedge \psi^i + \partial\alpha \wedge \partial A_a \wedge B^a\right). \quad (2.90)$$

This term agrees with the second line in the five-dimensional action (2.89).

Finally, consider the term in the eleven-dimensional action $J(\gamma_{11d}) = \frac{1}{6}\int \gamma_{11d} \wedge \partial\gamma_{11d} \wedge \partial\gamma_{11d}$. This induces the five-dimensional Chern–Simons term

$$\frac{1}{6} \int_{\mathbb{C}^2 \times \mathbb{R}} B_{grav} \partial B_{grav} \partial B_{grav}. \quad (2.91)$$

This completes the proof. □

2.5.7.3 In §2.2.1 we computed the global symmetry algebra of the eleven-dimensional theory on $\mathbb{C}^5 \times \mathbb{R}$ and found a close relationship to the exceptional super Lie algebra $E(5, 10)$. In this section we deduce the form of the global symmetry algebra of the five-dimensional compactified theory on $\mathbb{C}^2 \times \mathbb{R}$.

Consider the full de Rham cohomology of X by

$$H^\bullet(X, \Omega^\bullet) = \oplus_{i,j} H^i(X, \Omega_X^j).$$

This is a graded commutative algebra using the wedge product of differential forms. Next, consider the space of holomorphic functions $\mathcal{O}(\mathbb{C}^2)$ on $\mathbb{C}^2$. The Poisson bracket $\{-,-,\}$ associated to the standard holomorphic symplectic structure on $\mathbb{C}^2$ endows $\mathcal{O}(\mathbb{C}^2)$ with the structure of a Lie algebra. In particular, we can tensor $\mathcal{O}(\mathbb{C}^2)$ with $H^\bullet(X, \Omega^\bullet)$ to obtain the structure of a graded Lie algebra on

$$H^\bullet(X, \Omega^\bullet) \otimes \mathcal{O}(\mathbb{C}^2).$$

Let $[\omega] \in H^1(X, \Omega_X^1)$ be the class of the Kähler form on X.

The global symmetry algebra of the compactified theory along the Calabi–Yau three-fold X is equivalent to a deformation of this graded Lie algebra. The deformation introduces the following

Lie bracket

$$\big[[\omega] \otimes f, [\omega] \otimes g\big] = [\omega^2] \otimes \{f, g\} \in H^{2,2}(X, \Omega^2) \otimes \mathcal{O}(\mathbb{C}^2).$$

Chapter 3

Twisted Spectrometry on $AdS_4 \times S^7$ and $AdS_7 \times S^4$

Having described our eleven-dimensional model on flat spacetimes, we now pursue descriptions on maximally symmetric spacetimes. We begin by describing twisted versions of the $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds. In eleven-dimensional supergravity, these backgrounds arise as near-horizon limits of the backgrounds sourced by some number of M2 and M5 branes in flat space respectively. In the first section of this chapter 3.3, we describe an analogous procedure natively in our twisted context.

To do so, we motivate ansatzae for the leading order couplings of our eleven-dimensional model to M2 and M5 branes. These couplings determine certain curved deformations of the L_∞-algebra underlying our model - we conjecture that deforming the theory in the complement of the brane by a solution to the resulting curved Maurer-Cartan equation is perturbatively equivalent to the twist of the theory on $AdS_4 \times S^7$ and $AdS_7 \times S^4$.

The next two sections provide evidence for this conjecture. We begin in section 3.2 with numerical checks. We give definitions of supergravity states in our twisted $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds, which can be thought of as particular field configurations that are localized at points on the conformal boundary of AdS. We compute characters of the proposed state spaces and find exact matches with counts of gravitons on $AdS_4 \times S^7$ and $AdS_7 \times S^4$ respectively.

The next strand of evidence we pursue is by matching symmetries. In the physical theory, the $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds carry actions of the 3d $\mathcal{N} = 8$ and 6d $\mathcal{N} = (2,0)$ superconformal algebras. We show that our conjectural descriptions of twists of these backgrounds carry actions of the minimal twists of the corresponding superconformal algebras.

With these pieces of evidence in hand, in sections 3.4, 3.5, we then turn to study some representation theoretic aspects of the state spaces constructed in section 3.2. We identify certain $\mathbb{C}^\times$ actions on our eleven-dimensional model that combine rescalings in directions normal to branes with a certain rescaling of the space of fields - this induces a decomposition of the space of fields that we dub the *graviton decomposition.* The weight 0 parts of these decompositions are certain local L_∞-algebras whose costalks recover the linearly compact super-Lie algebras $E(1|6)$ and $E(3|6)$. We thus see that these linearly compact super-Lie algebras act on the spaces of supergravity states constructed in section 3.2. We explicate their action on nonzero weight spaces of the graviton decomposition.

In the final section of the chapter, we motivate some current work in progress that leverages the uncovered appearance of exceptional linearly compact super-Lie algebras for holographic means. Eleven-dimensional supergravity on $AdS_4 \times S^7$ and $AdS_7 \times S^4$ is expected to be equivalent to the large N limit of the worldvolume theories of N M2 branes and N M5 branes respectively. We offer some conjectures that apply the twisted holography proposal of [] to [] (TODO: FINISH)

3.1 Twisted Backreactions

As remarked above, in eleven-dimensional supergravity, the $AdS_7 \times S^4$ and $AdS_4 \times S^7$ backgrounds are obtained by backreacting a number of M5 branes and M2 branes in flat space [Mal98, Wit98]. In this section, we wish to give an account of this procedure at the level of our twisted theory in eleven-dimensions. Before describing the specific examples of interest, we begin with some generalities.

Suppose we have a theory of gravity on the total space of a vector bundle. In this thesis, we are interested in holomorphic-topological field theories, and in this context, the bundle projection is a map of THF manifolds, and the gravitational theory is a local moduli problem that describes, in part, deformations of the THF structure on the total space. Operationally, producing the theory in the backreacted geometry is the output of the following two-step procedure.

- Place the theory on the complement of the zero section.
- Deform the theory on the complement of the zero section by a certain Maurer–Cartan element, thought of as the flux sourced by branes wrapping the zero section. More rigorously, the zero section determines a certain curved Maurer-Cartan equation, and the desired Maurer-Cartan element is a solution to this equation.

This procedure is implemented at the level of the Ω-deformed nonminimal twist on flat space in the appendix of [Cos16], and in [RW22] the procedure is carried about for M5 branes in our eleven-dimensional model in some global generality. For the purposes of this thesis however, we will content ourselves with examples on flat space.

3.1.1 The $\mathrm{A}dS_4 \times S^7$ background

In this section we introduce the analog of the $\mathrm{A}dS_4 \times S^7$ background in our conjectural description of the minimal twist of eleven-dimensional supergravity.

3.1.1.1 We begin by viewing the eleven-dimensional manifold $\mathbb{R} \times \mathbb{C}^5$ as

$$\mathrm{Tot}(K_{\mathbb{C}}^{1/4} \otimes \mathbb{C}^4 \to \mathbb{R} \times \mathbb{C}_z)$$

where we have abusively used $K_{\mathbb{C}}^{1/4} \otimes \mathbb{C}^4$ to denote its pullback along the natural projection $\mathbb{R} \times \mathbb{C} \to \mathbb{C}$. Thinking of flat space in this way is simply a way to record weights under natural scaling actions. We will use w_a to denote holomorphic fiber coordinates on $K_{\mathbb{C}}^{1/4} \otimes \mathbb{C}^4$.

We carry out the above procedure. Consider a stack of N M2 branes wrapping the zero section $\mathbb{R} \times \mathbb{C}_z$. A natural interaction to consider is

$$I_{M2}(\gamma) = N \int_{\mathbb{C}_z} \gamma + \cdots$$

which is nonzero only on the component of γ in $\Omega^1(\mathbb{R}) \otimes \Omega^{1,1}(\mathbb{C}^5)$. We have only indicated the lowest order coupling, the $\cdots$ indicate higher-order couplings which will be higher order in the fields of the eleven-dimensional theory and explicitly involve the fields in the worldvolume theory.

This coupling is justified by comparison with the physical theory and by dimensional reduction. Indeed, as discussed in §2.1.4, the component of γ which participates in the above coupling is a component of the C-field of eleven dimensional supergravity. Thus, the proposal mirrors electric couplings of M2 branes in the physical theory, which simply involves integrating the C-field over the worldvolume of the brane.

Moreover, reducing on a circle transverse to the M2 brane yields the $SU(4)$ twist of type IIA supergravity on $\mathbb{R}^2 \times \mathbb{C}_z \times \mathbb{C}^3$ with N $D2$ branes wrapping $\mathbb{R} \times \mathbb{C}_z$. As is shown in [**?**], an electric coupling of D2 branes to the $SU(4)$ twist of type IIA supergravity is given by

$$I_{D2}(\gamma) = N \int_{\mathbb{R}\times\mathbb{C}_z} \gamma + \cdots$$

where γ now denotes the 1-form field of the $SU(4)$ twist of type IIA supergravity. It is immediate that the pullback of I_{M2} along the map in the proof of proposition 2.5.2 recovers I_{D2}.

3.1.1.2 The backreacted geometry will be given by a solution to the equations of motion upon deforming the eleven-dimensional action by the interaction $I_{M2}(\gamma)$. Varying the deformed action with respect to γ, we obtain the equation of motion

$$\bar{\partial}\mu + \frac{1}{2}[\mu, \mu] + \partial\gamma\partial\gamma = N\Omega^{-1}\delta_{w=0}. \tag{3.1}$$

Here $[-, -]$ is the Schouten bracket. Varying β, we obtain the equation of motion

$$\partial_\Omega \mu = 0. \tag{3.2}$$

Lemma 3.1.1. Let

$$F_{M2} = \frac{6}{(2\pi i)^4} \frac{\sum_{a=1}^4 \bar{w}_a \mathrm{d}\bar{w}_1 \cdots \widehat{\mathrm{d}\bar{w}_a} \cdots \mathrm{d}\bar{w}_4}{\|w\|^8} \partial_z.$$

Then the background where $\mu = NF_{M2}$ and $\gamma = 0$ satisfies the above equations of motion in the presence of a stack of N M2 branes:

$$\bar{\partial}(NF_{M2}) + \frac{1}{2}[NF_{M2}, NF_{M2}] = N\Omega^{-1}\delta_{w=0}$$
$$\partial_\Omega(NF_{M2}) = 0.$$

Here we set all components of the field γ equal to zero (as well as the fields ν, β).

Proof. Upon specializing $\gamma = 0$, the last term in the first equation above vanishes. The equation $\bar{\partial}F_{M2} = \Omega^{-1}\delta_{w=0}$ characterizes the Bochner–Martinelli kernel representing the residue class on $\mathbb{C}^4 \setminus 0$. It is clear that $\partial_\Omega F_{M2} = 0$ and

$$[F_{M2}, F_{M2}] = 0$$

by simple type reasons. □

We summarize the output of our computation with a definition.

Definition 3.1.2. Let $\mathcal{E}^N_{AdS_4\times S^7}$ denote the classical BV theory on $\mathrm{Tot}(K_{\mathbb{C}}^{1/4} \otimes \mathbb{C}^4 \to \mathbb{R}\times\mathbb{C}_z) \setminus 0(\mathbb{R}\times\mathbb{C})$ given by the sheaf of cochain complexes $\mathcal{E}|_{(\mathbb{R}\times\mathbb{C})\times(\mathbb{C}^4\setminus\{0\})}$, with BV pairing induced from $\mathcal{E}$, deformed by the interaction

$$S_{BF,\infty}(\mu + NF_{M2}, \nu, \beta, \gamma) + J(\gamma).$$

Remark 3.1.3. Note that upon expanding the interaction around NF_{M2}, the cubic term in $S_{BF,\infty}$ will contribute a differential which acts on γ and μ by bracketing with NF_{M2}. We accordingly denote this differential $[NF_{M2}, -]$, and we see that the sheaf of cochain complexes underlying $\mathcal{E}_{AdS_4\times S^7}$ is in fact

$$\left(\mathcal{E}|_{(\mathbb{R}\times\mathbb{C})\times(\mathbb{C}^4\setminus\{0\})}, \delta^{(1)} + [NF_{M2}, -]\right)$$

where $\delta^{(1)}$ denotes the original linearized BRST differential.

Conjecture 3.1.4. The minimal twist of eleven-dimensional supergravity on the $AdS_4 \times S^7$ background with N units of M2 brane flux supported on S^7 is perturbatively equivalent to $\mathcal{E}^N_{AdS_4\times S^7}$.

To verify this conjecture, we should directly twist eleven-dimensional supergravity on the $AdS_4 \times S^7$ spacetime. Doing so seems difficult - while it is likely not hard to identify the covariantly constant nilpotent spinors which define the twist, it seems more difficult to establish a perturbative equivalence with our description above. A modification of the pure spinor superfield formalism to symmetric spaces such as cosets for the superconformal group might make such checks more feasible. In lieu of such, we will instead pursue other consistency checks in the following two sections.

3.1.2 The $\mathrm{A}dS_7 \times S^4$ background

We similarly introduce an analog of the $\mathrm{A}dS_7 \times S^4$ background in our description of the minimal twist of eleven-dimensional supergravity. As before, we begin by viewing our eleven-dimensional manifold $\mathbb{R}\times\mathbb{C}^5$ as

$$\mathrm{Tot}(\mathbb{R} \oplus K_{\mathbb{C}^3}^{1/2} \otimes \mathbb{C}^2 \to \mathbb{C}^3_z)$$

to record weights under natural scaling actions. We once again will use w_a to denote holomorphic fiber coordinates on $K_{\mathbb{C}^3}^{1/2} \otimes \mathbb{C}^2$, and we use t to denote a fiber coordinate on the trivial bundle $\mathbb{R} \to \mathbb{C}^3_z$.

3.1.2.1 To repeat the procedure in the previous subsection, we begin by discussing how the eleven-dimensional theory couples to M5 branes. Consider a stack of N M5 branes wrapping the zero section $\mathbb{C}^3_z$.

It is natural to consider the nonlocal interaction

$$I_{M5} = N \int_{\mathbb{C}^3_z} \partial_\Omega^{-1} \mu \vee \Omega + \cdots$$

Note that this expression is only nonzero on the component of μ in $\mathrm{PV}^{1,3}$. We argue that this coupling is consistent with expectations from the physical theory and from dimensional reduction.

The twisted field $\mu^{1,3}$ is a component of the Hodge star of the G-flux in the physical theory (§2.1.4). In the physical theory, M5 branes magnetically couple to the C-field; the coupling involves choosing a primitive for the Hodge star of the G-flux and integrating it over the M5 worldvolume. Our twist contains no fields corresponding to components of such a primitive; hence such a magnetic coupling is reflected in the appearance of ∂_Ω^{-1}.

We may once again justify this coupling by dimensional reduction to IIA supergravity. Reducing on the circle along the directions the M5 branes wrap yields the $SU(4)$ invariant twist of type IIA supergravity on $\mathbb{C}^4 \times \mathbb{R}^2$ with N $D4$ branes wrapping $\mathbb{C}^2 \times \mathbb{R}$.

In [?], it is shown that the magnetic coupling of $D4$ branes to the $SU(4)$ twist of IIA is of the form

$$N \int_{\mathbb{C}^2\times\mathbb{R}} \partial_\Omega^{-1}\mu \vee \Omega_{\mathbb{C}^4} + \cdots .$$

Again, we have only explicitly indicated the first-order piece of the coupling.

3.1.2.2 The backreacted geometry will be given by a solution to the equations of motion upon deforming the eleven-dimensional action by the interaction $I_{M5}(\mu)$.

Varying the potential $\partial_\Omega^{-1}\mu$, we obtain the following equation of motion involving the field γ:

$$\bar\partial\partial\gamma + \partial_\Omega\left(\frac{1}{1-\nu}\mu\right) \wedge \partial\gamma = N\delta_{w_1=w_2=t=0}. \tag{3.3}$$

Notice that there is an extra derivative compared to the equation of motion arising from varying the field μ. This equation only depends on γ through its field strength $\partial\gamma$.

Varying γ we obtain the equation of motion

$$(\bar\partial + \mathrm{d}_{\mathbb{R}})\mu + \partial\gamma\partial\gamma = 0. \tag{3.4}$$

Again, this only depends on γ through its field strength $\partial\gamma$.

Lemma 3.1.5. Let

$$F_{M5} = \frac{1}{(2\pi i)^3}\frac{\bar w_1 \mathrm{d}\bar w_2 \wedge \mathrm{d}t - \bar w_2 \mathrm{d}\bar w_1 \wedge \mathrm{d}t + t\mathrm{d}\bar w_1 \wedge \mathrm{d}\bar w_2}{(\|w\|^2+t^2)^{5/2}} \wedge \mathrm{d}w_1 \wedge \mathrm{d}w_2$$

Then, $\partial\gamma = NF_{M5}$, $\mu = 0$, and $\nu = 0$ satisfies the equations of motion in the presence of a stack of N M5 branes sourced by the term $N\delta_{w_1=w_2=t=0}$:

$$\bar\partial(NF_{M5}) + \mathrm{d}_{\mathbb{R}}(NF_{M5}) = N\delta_{w_1=w_2=t=0}$$
$$(NF_{M5}) \wedge (NF_{M5}) = 0.$$

Here, we set all components of the field μ equal to zero (as well as the fields ν, β).

Proof. The first equation,

$$\bar\partial F + \mathrm{d}_{\mathbb{R}} F = N\delta_{w_1=w_2=t=0},$$

characterizes the kernel representing N times the residue class for a four-sphere in

$$(\mathbb{C}^2 \times \mathbb{R}) \setminus 0 \simeq S^4 \times \mathbb{R}.$$

That is

$$\oint_{S^4} NF = N$$

for any four-sphere centered at $0 \in \mathbb{C}^2 \times \mathbb{R}$.

The second equation $F \wedge F = 0$ follows by simple type reasons. □

Once again, we summarize our findings in a definition.

Definition 3.1.6. Let $\mathcal{E}^N_{AdS_7 \times S^4}$ denote the classical BV theory on $\mathrm{Tot}(\mathbb{R} \oplus K^{1/2}_{\mathbb{C}^3} \otimes \mathbb{C}^2 \to \mathbb{C}^3_z) \setminus 0(\mathbb{C}^3_z)$ given by the sheaf of cochain complexes $\mathcal{E}|_{\mathbb{C}^3 \times (\mathbb{R} \times \mathbb{C}^2 \setminus \{0\})}$, with BV pairing induced from that on $\mathcal{E}$, deformed by the interaction

$$S_{BF,\infty}(\mu, \nu, \beta, \gamma + N\partial^{-1}F_{M5}) + J(\gamma + N\partial^{-1}F_{M5}).$$

Remark 3.1.7. Note that both terms in the action only depend on γ through its holomorphic derivatives so the above expression for the action is indeed well-defined.

As before, upon expanding the interactions around NF_{M5}, the cubic terms in both $S_{BF,\infty}$ and J will contribute differentials. From $S_{BF,\infty}$, we get a differential which takes a μ type field to the Schouten bracket $N[F_{M5}, \mu]$ and from J, we get a differential which acts as $\gamma \mapsto NF_{M5} \wedge \partial\gamma$. We accordingly denote this differential $[NF_{M5}, -]$, and the sheaf of cochain complexes underlying $\mathcal{E}_{AdS_7 \times S^4}$ is in fact

$$\left(\mathcal{E}|_{\mathbb{C}^3 \times (\mathbb{R} \times \mathbb{C}^2 \setminus \{0\})}, \delta^{(1)} + [NF_{M5}, -]\right)$$

where $\delta^{(1)}$ denotes the original linearized BRST differential.

Conjecture 3.1.8. The minimal twist of eleven-dimensional supergravity on the $AdS_7 \times S^4$ background with N units of M5 brane flux supported on S^4 is perturbatively equivalent to $\mathcal{E}_{AdS_7 \times S^4}$.

3.2 Twisted supergravity states

In this section, we pursue a check of conjectures 3.1.4 and 3.1.8. We enumerate supergravity states in the twists of the $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds and compare them with expressions in the literature enumerating gravitons in these geometries.

We begin by giving a definition of supergravity states suited to our context. The definition is meant to codify the following situation. Suppose we have a theory of gravity defined on a background of the form $AdS_{d+1} \times S^{d'}$ where the conformal boundary of AdS_{d+1} is a d manifold M. We can compactify the theory, retaining all the Kaluza-Klein harmonics, to get a theory on AdS_{d+1}. A supergravity state is traditionally defined to be a solution to linearized equations of motion in this compactified theory with a given boundary value on M^d [Wit98]. Often, this definition is made in situations where the relevant boundary value problem has a unique solution, in which case one may label states by the corresponding boundary values. Moreover, one may think of such boundary values as arising from modifications of a vacuum boundary condition at a point.

3.2.0.1 For twists of supergravity, we procedurally implement this as follows. First, we wish to describe a model for sphere compactification on twisted AdS backgrounds. As we saw in the last section, our proposal for twists of backgrounds of the form $AdS_{d+1} \times S^{d'}$ involve placing certain local L_∞-algebras on certain manifolds of the form $X = \mathrm{Tot}(V \to Z) \setminus 0(Z)$. Note that Z can be written as a sphere bundle over $\mathbb{R}_{>0} \times Z$ - the sphere compactification of our theory on X is given by the pushforward to $\mathbb{R}_{>0} \times Z$. Such sphere compactifications can be described using a method for computing pushforwards of modules for Lie algebroids associated to foliations [Kor14, Sec. 4.2], [KT75].

The compactified theory will admit a natural boundary condition at $\{\infty\} \times Z \subset \mathbb{R}_{>0} \times Z$, given by certain local L_∞-algebras $\mathcal{L}$. In fact, the presence of extra differentials involving bracketing with the flux sourced by branes wrapping Z will induce an interesting shifted-Poisosn structure on $\mathcal{L}$. Given a local L_∞-algebra $\mathcal{L}$ underlying a perturbative classical field theory, the Chevalley-Eilenberg cochains $\mathrm{C}^\bullet(\mathcal{L})$ carry the structure of a $\mathbb{P}_0$-factorization algebra [CG17], [CG21b]. Associated to a factorization algebra, we can define a space of local operators, which is the costalk at a point of the underlying cosheaf. The sought-after spaces of supergravity states will be a space of local operators associated to $\mathrm{C}^\bullet(\mathcal{L})$.

Remark 3.2.1. Crucially, we will only implement this procedure at the level of the free limits of the theories defined in 3.1.2, 3.1.6 - this will suffice for the purposes of extracting spaces of states and will allow us to forgo discussion involving homotopy transfer of L_∞ structure. As such, the boundary conditions we specify at $\{\infty\} \times Z$ will in fact be abelian local Lie algebras. On the other hand, in subsection 3.2.3 we will identify boundary conditions for the pushforward of the *interacting* theories that exist after turning off the flux, which yield the same spaces of supergravity states.

Let us now carry out this construction for the theories defined in 3.1.2, 3.1.6.

3.2.1 States on twisted $AdS_4 \times S^7$

Following the above prescription, we consider the S^7 bundle

$$\begin{array}{ccc} S^7 & \longrightarrow & \mathrm{Tot}(K_{\mathbb{C}}^{1/4} \otimes \mathbb{C}^4 \to \mathbb{R} \times \mathbb{C}) \setminus 0(\mathbb{R} \times \mathbb{C}) \\ & & \big\downarrow p \\ & & \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{C} \end{array}$$

We wish to describe the free limit of the pushforward $p_* \mathcal{E}^N_{AdS_4 \times S^7}$ as a sheaf of cochain complexes on $\mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{C}$.

Proposition 3.2.2. The pushforward is given by the sheaf of complexes $\Omega^\bullet_{\mathbb{R}_{>0}\times\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{V}^N_{\mathbb{C}})$ on $\mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{C}$, where $\mathcal{V}^N_{\mathbb{C}}$ is the following dg-vector bundle on $\mathbb{C}$:

$$\begin{array}{ccc} - & & + \\ H^\bullet(\mathbb{C}^4 \setminus 0, \mathrm{T}) \otimes \mathcal{O} & \xrightarrow{\partial_\Omega^W} & H^\bullet(\mathbb{C}^4 \setminus 0) \otimes \mathcal{O} \\ H^\bullet(\mathbb{C}^4 \setminus 0) \otimes \mathrm{T} & \nearrow_{\partial_\Omega^Z} & \\ H^\bullet(\mathbb{C}^4 \setminus 0) \otimes \mathcal{O} & \xrightarrow{\partial_Z} & H^\bullet(\mathbb{C}^4 \setminus 0) \otimes \Omega^1 \\ & \searrow_{\partial_W} & H^\bullet(\mathbb{C}^4 \setminus 0, \Omega^1) \otimes \mathcal{O} \end{array} \tag{3.5}$$

where the differentials are as follows:

- The differentials ∂_Ω^Z and ∂_Ω^W are the divergence operators along the base and fiber respectively.
- The differentials ∂_Z and ∂_W are components of the holomorphic deRham differentials along the base and fiber respectively.
- Internal to each summand is a differential given by bracketing with the flux NF_{M2}.

Before proceeding with the proof, we explicate the internal differential in the third item above. Recall that for $\mathcal{F} = \mathcal{O}, \mathrm{T}$, or Ω^1, the cohomology $H^\bullet(\mathbb{C}^4 \setminus 0, \mathcal{F})$ is concentrated in degrees 0 and 3. We will make use of the following dense embeddings.

$$\begin{aligned}
\mathbb{C}[w_1, \dots, w_4] &\hookrightarrow H^0(\mathbb{C}^4 \setminus 0, \mathcal{O}) \\
\mathbb{C}[w_1, \dots, w_4]\{\partial_{w_a}\} &\hookrightarrow H^0(\mathbb{C}^4 \setminus 0, \mathrm{T}) \\
\mathbb{C}[w_1, \dots, w_4]\{\mathrm{d}w_a\} &\hookrightarrow H^0(\mathbb{C}^4 \setminus 0, \Omega^1)
\end{aligned}$$

and

$$\begin{aligned}
(w_1 \cdots w_4)^{-1}\mathbb{C}[w_1^{-1}, \dots, w_4^{-1}] &\hookrightarrow H^3(\mathbb{C}^4 \setminus 0, \mathcal{O}) \\
(w_1 \cdots w_4)^{-1}\mathbb{C}[w_1^{-1}, \dots, w_4^{-1}]\{\partial_{w_a}\} &\hookrightarrow H^3(\mathbb{C}^4 \setminus 0, \mathrm{T}) \\
(w_1 \cdots w_4)^{-1}\mathbb{C}[w_1^{-1}, \dots, w_4^{-1}]\{\mathrm{d}w_a\} &\hookrightarrow H^3(\mathbb{C}^4 \setminus 0, \Omega^1).
\end{aligned}$$

The flux NF_{M2} of lemma 3.1.1 is represented by the section $N(w_1 \cdots w_4)^{-1}\partial_z \in H^3(\mathbb{C}^4 \setminus 0) \otimes \mathrm{T}$ and acts on each summand by Lie derivative along z and multiplying by $(w_1 \cdots w_4)^{-1}$

Proof. To compute the pushforward of the kind of local L_∞-algebra associated to a holomorphic-topological field theory along a map of THF manifolds, we can use a result of [Kor14, Sec. 4.2], [KT75] for describing direct images of Lie algebroid modules along maps of Lie algebroids. Schematically, if we have a proper submersion $p : X \to Z$ of THF manifolds, and a sheaf of complexes $\mathcal{E}$ on X which resolves sections of some bundle flat along the leaves of the foliation on X, then the pushforward $p_*\mathcal{E}$ has a model as a partially flat bundle on Z. The fiber of this partially flat bundle on Z is the THF cohomology of the fiber of p, with respect to the induced foliation on the fiber, with coefficients in the pullback of the bundle to the fiber. In the case of a holomorphic submersion, this recovers the usual construction of the Gauss-Manin connection for instance.

In our case, the pushforward $p_*\mathcal{E}_{AdS_4 \times S^7}$ is a complex of $\Omega^\bullet_{\mathbb{R}_{>0}} \otimes \Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}$- modules given by

$$\begin{array}{ccc}
\underline{\quad - \quad} & & \underline{\quad \pm \quad} \\
H^\bullet_{\mathrm{THF}}(S^7, \mathrm{T}) \otimes \mathcal{O} & \xrightarrow{\partial_\Omega^W} & H^\bullet_{\mathrm{THF}}(S^7) \otimes \mathcal{O} \\
& \nearrow_{\partial_\Omega^Z} & \\
H^\bullet_{\mathrm{THF}}(S^7) \otimes \mathrm{T} & & \\
H^\bullet_{\mathrm{THF}}(S^7) \otimes \mathcal{O} & \xrightarrow{\partial_Z} & H^\bullet_{\mathrm{THF}}(S^7) \otimes \Omega^1 \\
& \searrow^{\partial_W} & \\
& & H^\bullet_{\mathrm{THF}}(S^7, \Omega^1) \otimes \mathcal{O}
\end{array} \tag{3.6}$$

Then the proposition follows from the fact that the map $S^7 \to \mathbb{C}^4 \setminus \{0\}$ induces an isomorphism in THF cohomology. $\square$

3.2.1.1 Continuing, we wish to understand a natural boundary condition we can place on the fields at $\{\infty\} \times \mathbb{R} \times \mathbb{C} \subset \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{C}$. Such a boundary condition is specified by a Lagrangian in the phase space. The restriction $(p_* \mathcal{E}^N_{AdS_4 \times S^7})|_{\{\infty\}\times\mathbb{C}}$ describes the phase space and is easily seen to be $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{V}^N_{\mathbb{C}})$. We wish to describe a shifted Lagrangian therein.

We begin by rewriting the phase space in the following form. Note that there is a higher residue pairing

$$H^0(\mathbb{C}^4 \setminus 0) \otimes H^3(\mathbb{C}^4 \setminus 0) \to \mathbb{C};$$

together with the natural pairings between T, Ω^1 and the integration pairing along $\mathbb{R} \times \mathbb{C}_z$, this equips $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{V}^N_{\mathbb{C}})$ with a local even-shifted symplectic structure.

Together with this shifted symplectic structure, we can identify the phase space with a twisted cotangent bundle

$$\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{V}^N_{\mathbb{C}}) = T^*_\Pi \left(\begin{array}{ccc} \underline{\quad} & & \pm \\ \mathbb{C}[w_1,\dots,w_4]\{\partial_{w_a}\} \otimes \Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}} & \xrightarrow{\partial^W_\Omega} & \mathbb{C}[w_1,\dots,w_4] \otimes \Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}} \\ \mathbb{C}[w_1,\dots,w_4] \otimes \Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathrm{T}) & \xrightarrow{\partial^Z_\Omega} & \\ \mathbb{C}[w_1,\dots,w_4] \otimes \Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}} & \xrightarrow{\partial_Z} & \mathbb{C}[w_1,\dots,w_4] \otimes \Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\Omega^1) \\ & \xrightarrow{\partial_W} & \mathbb{C}[w_1,\dots,w_4]\{\mathrm{d}w_a\} \otimes \Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}} \end{array} \right)$$

For now, the subscript of Π is just meant to indicate that the extra differential given by bracketing with the flux constitutes a deformation of the cotangent bundle.

Thus we see that a natural Lagrangian in the phase space is given by the sheaf of cochain complexes $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^\bullet_{\mathbb{C}}(\mathcal{L}^N_{AdS_4 \times S^7})$ where $\mathcal{L}^N_{AdS_4 \times S^7}$ is the following dg-vector bundle

$$\begin{array}{ccc} \pm & & \underline{\quad} \\ (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]\{\partial_{w_a}\} \otimes \mathcal{O} & \xrightarrow{\partial^W_\Omega} & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}] \otimes \mathcal{O} \\ (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}] \otimes \mathrm{T} & \xrightarrow{\partial^Z_\Omega} & \\ (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}] \otimes \mathcal{O} & \xrightarrow{\partial_Z} & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}] \otimes \Omega^1 \\ & \xrightarrow{\partial_W} & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]\{\mathrm{d}w_a\} \otimes \mathcal{O} \end{array} \tag{3.7}$$

Remark 3.2.3. The extra term in the differential in fact equips $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^\bullet_{\mathbb{C}}(\mathcal{L}^N_{AdS_4 \times S^7})$ with an interesting N dependent odd-shifted Poisson structure. This will not play a significant role in the narrative of this thesis, but we flag it for later commentary nonetheless.

Definition 3.2.4. The *space of supergravity states* on twisted $AdS_4 \times S^7$ is given by the costalk at zero of the factorization algebra on $\mathbb{R} \times \mathbb{C}$ given by $\mathrm{C}^\bullet\left(\Pi\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^N_{AdS_4 \times S^7})\right)$

3.2.1.2 Let us explicate definition 3.2.4 a bit. We wish to first understand the space of supergravity states as a cochain complex.

Lemma 3.2.5. The space of supergravity states on twisted $AdS_4 \times S^7$ is the symmetric algebra $\mathrm{Sym}(\mathcal{H}_{AdS_4\times S^7})$ where $\mathcal{H}_{AdS_4\times S^7}$ is the cochain complex

$$\begin{array}{ccc}
\underline{-} & & \underline{+} \\
\mathbb{C}[w_1,\dots,w_4]\{\partial_{w_a}\} \otimes \mathbb{C}[\partial_z]\delta_{z=0} & \xrightarrow{\partial_\Omega^W} & \mathbb{C}[w_1,\dots,w_4] \otimes \mathbb{C}[\partial_z]\delta_{z=0} \\
\mathbb{C}[w_1,\dots,w_4]\partial_z \otimes \mathbb{C}[\partial_z]\delta_{z=0} & \xrightarrow{\partial_\Omega^Z} & \\
\mathbb{C}[w_1,\dots,w_4] \otimes \mathbb{C}[\partial_z]\delta_{z=0} & \xrightarrow{\partial_z} & \mathbb{C}[w_1,\dots,w_4]\mathrm{d}z \otimes \mathbb{C}[\partial_z]\delta_{z=0} \\
& \xrightarrow{\partial_W} & \mathbb{C}[w_1,\dots,w_4]\{\mathrm{d}w_a\} \otimes \mathbb{C}[\partial_z]\delta_{z=0}
\end{array} \tag{3.8}$$

Proof. To compute the costalk at 0 of a factorization algebra, we consider a nested sequence of open sets containing the origin and compute the limit of the value of the factorization algebra over this sequence. Consider open sets in $\mathbb{R}\times\mathbb{C}$ of the form $I \times D$ where $I \subset \mathbb{R}$ is an interval and $D \subset \mathbb{C}$ is a disc. The sections of the sheaf of cochain complexes $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^N_{AdS_4\times S^7})$ over this open set is given by

$$\begin{array}{ccc}
\underline{+} & & \underline{-} \\
(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]\{\partial_{w_a}\} \otimes \mathcal{O}(D) & \xrightarrow{\partial_\Omega^W} & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}] \otimes \mathcal{O}(D) \\
(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}] \otimes \Gamma(D,\mathrm{T}) & \xrightarrow{\partial_\Omega^Z} & \\
(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}] \otimes \mathcal{O}(D) & \xrightarrow{\partial_z} & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}] \otimes \Gamma(D,\Omega^1) \\
& \xrightarrow{\partial_W} & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]\{\mathrm{d}w_a\} \otimes \mathcal{O}(D)
\end{array} \tag{3.9}$$

Now note that there is a canonical map $\mathcal{O}(D) \to \mathbb{C}[\![z]\!]$ given by taking the Taylor expansion at the origin. Given a functional on the fields that only depends on the value of their derivatives at the origin, then the functional must factor through the Taylor expansion. Therefore, we have that the costalk of our factorization algebra is given by

$$\mathrm{C}^\bullet\left(\begin{array}{ccc}
\underline{-} & & \underline{+} \\
(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]\{\partial_{w_a}\} \otimes \mathbb{C}[\![z]\!] & \xrightarrow{\partial_\Omega^W} & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}] \otimes \mathbb{C}[\![z]\!] \\
(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}] \otimes \mathbb{C}[\![z]\!]\partial_z & \xrightarrow{\partial_\Omega^Z} & \\
(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}] \otimes \mathbb{C}[\![z]\!] & \xrightarrow{\partial_z} & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}] \otimes \mathbb{C}[\![z]\!]\mathrm{d}z \\
& \xrightarrow{\partial_W} & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]\{\mathrm{d}w_a\} \otimes \mathbb{C}[\![z]\!]
\end{array}\right) \tag{3.10}$$

The definition of the Chevalley-Eilenberg complex above involves the continuous linear dual of a chain complex of topological vector spaces. The duals of each of the tensor factors are as follows:

- There is an isomorphism between the continuous linear dual of $\mathbb{C}[\![z]\!]$ and $\mathbb{C}[\partial_z]\delta_{z=0}$: every continuous linear functional on $\mathbb{C}[\![z]\!]$ is given by a derivative of the δ-function at zero.

	z	w_1	w_2	w_3	w_4
t_1	0	1	0	0	-1
t_2	0	0	1	0	-1
t_3	0	0	0	1	-1
q	-1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Table 3.1: Fugacities for the fields of the holomorphic twist of eleven-dimensional supergravity for the geometry $\mathbb{R} \times \mathbb{C}^5 \setminus (\mathbb{R} \times \mathbb{C}^2)$.

- The higher residue pairing lets us identify the continuous linear dual of $(w_1 \cdots w_4)^{-1}\mathbb{C}[w_1^{-1}, \ldots, w_4^{-1}]$ with $\mathbb{C}[w_1, \cdots, w_4]$.
- The tensor factors involving one-forms and vector fields are dual to each other in the obvious way.

Thus, we see that $\mathcal{H}_{AdS_4 \times S^7}$ is indeed as claimed. □

3.2.1.3 We proceed to computing a local character for the factorization algebra defined in 3.2.4; thanks to lemma 3.2.5 we can compute this as a character of $\mathrm{Sym}(\mathcal{H}_{AdS_4 \times S^7})$.

We first observe the following action of $\mathfrak{sl}(4) \oplus \mathfrak{sl}(2)$ on $\mathcal{H}_{AdS_4 \times S^7}$. The $\mathfrak{sl}(4)$ summand acts on the tensor factor $\mathbb{C}[w_1, w_2, w_3, w_4]$ in the obvious way - it's the symmetric algebra on the fundamental representation. The $\mathfrak{sl}(2)$ summand acts by bracketing with the vector fields

$$\frac{\partial}{\partial z}, \quad z\frac{\partial}{\partial z} - \frac{1}{4}\sum_{a=1}^{4} w_a \frac{\partial}{\partial w_a}, \quad z\left(z\frac{\partial}{\partial z} - \frac{1}{2}\sum_{a=1}^{4} w_a \frac{\partial}{\partial w_a}\right).$$

We choose the following explicit generators for a choice of Cartan as follows:

- t_1, t_2, t_3 denote generators for the Cartan of $\mathfrak{sl}_4$ which is spanned by the vector fields

$$h_1 = w_1 \frac{\partial}{\partial w_1} - w_4 \frac{\partial}{\partial w_4}, \qquad h_2 = w_2 \frac{\partial}{\partial w_2} - w_4 \frac{\partial}{\partial w_4}, \qquad h_3 = w_3 \frac{\partial}{\partial w_3} - w_4 \frac{\partial}{\partial w_4}$$

- q denotes a generator for the Cartan of $\mathfrak{sl}_2$ which is spanned by the vector field

$$\Delta = \frac{1}{4}\sum_{a=1}^{4} w_a \frac{\partial}{\partial w_a} - z\frac{\partial}{\partial z}.$$

The weights of $\mathcal{H}_{AdS_4 \times S^7}$ with respect to the generators of this Cartan subalgebra are entirely determined by the weights of the holomorphic coordinates $z, w_a, a = 1, \cdots, 4$, which we summarize in table 3.1

With this in hand, we wish to compute the character of the space of supergravity states $\mathrm{Sym}(\mathcal{H}_{AdS_4 \times S^7})$. Note that the space of supergravity states was defined to be a symmetric algebra - therefore its character can be computed using plethystic exponentiation of the character of $\mathcal{H}_{AdS_4 \times S^7}$ - the latter may be referred to as a *single particle index* and is defined by

$$f_{AdS_4 \times S^7}(t_1, t_2, t_3, q) = \mathrm{Tr}_{\mathcal{H}_{AdS_4 \times S^7}} (-1)^F q^{\Delta} t_1^{h_1} t_2^{h_2} t_3^{h_3}.$$

Proposition 3.2.6. The single particle index of the space of supergravity states $\mathcal{H}_{AdS_4 \times S^7}$ is given by

$$f_{AdS_4\times S^7}(t_1,t_2,t_3,q) = \frac{q\begin{pmatrix} q^{1/4}(t_1+t_2+t_3+t_1^{-1}t_2^{-1}t_3^{-1}) + q^{-1} \\ -q^{-1/4}(t_1^{-1}+t_2^{-1}+t_3^{-1}+t_1t_2t_3) - q \end{pmatrix}}{(1-q)(1-q^{1/4}t_1)(1-q^{1/4}t_2)(1-q^{1/4}t_3)(1-q^{1/4}t_1^{-1}t_2^{-1}t_3^{-1})}$$

Proof. The two summands not involving holomorphic vector fields or forms appear with opposite parity, so their contributions to the character cancel. For the remaining summands, It is straightforward to compute the character of each tensor factor:

- The factor $\mathbb{C}[\partial_z]\delta_{z=0}$ contributes a factor of
$$\frac{q}{1-q}.$$
- The tensor factor $\mathbb{C}[w_1,\cdots,w_4]$ contributes a factor of
$$\frac{1}{(1-q^{1/4}t_1)(1-q^{1/4}t_2)(1-q^{1/4}t_3)(1-q^{1/4}t_1^{-1}t_2^{-1}t_3^{-1})}.$$
- The tensor factors involving vector fields and forms contribute a factor of
$$-q^{-1/4}(t_1^{-1}+t_2^{-1}+t_3^{-1}+t_1t_2t_3) + q^{1/4}(t_1+t_2+t_3+t_1^{-1}t_2^{-1}t_3^{-1}) - q + q^{-1}.$$

□

3.2.1.4 Upon subtracting one and making the substitution

$$q = x^2, \qquad t_1 = (y_2y_3)^{1/2}/y_1^{1/2}, \qquad t_2 = (y_1y_3)^{1/2}/y_2^{1/2}, \qquad t_3 = (y_1y_2)^{1/2}/y_3^{1/2}$$

this character matches the expression in [BBMR08, Eq. 2.17]. The discrepancy of one is accounted for by a zero mode that we have introduced in writing our theory in such a way that M2 branes couple electrically. Indeed, this is an avatar of the central element in the central extension $\widehat{E(5|10)}$ of section 2.2.1.

3.2.2 States on twisted $AdS_7 \times S^4$

Next, we consider the sphere reduction of $\mathcal{E}^N_{AdS_7\times S^4}$ As before, we consider the S^4 bundle

$$\begin{array}{ccc} S^4 & \longrightarrow & \mathrm{Tot}(\mathbb{R}\oplus K^{1/2}\otimes\mathbb{C}^2 \to \mathbb{C}^3)\setminus 0(\mathbb{C}^3) \\ & & \downarrow p \\ & & \mathbb{R}_{>0}\times\mathbb{C}^3 \end{array}$$

We wish to describe the free limit of the pushforward $p_*\mathcal{E}_{AdS_7\times S^4}$ as a sheaf of cochain complexes on $\mathbb{R}_{>0}\times\mathbb{C}^3$.

Proposition 3.2.7. The pushforward $p_*\mathcal{E}^N_{AdS_7\times S^4}$ is given by the sheaf of cochain complexes $\Omega^\bullet_{\mathbb{R}_{>0}} \otimes \Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{V}^N_{\mathbb{C}^3})$ where $\mathcal{V}^N_{\mathbb{C}^3}$ is the following dg-vector bundle on $\mathbb{C}^3$:

$$
\begin{array}{ccc}
\underline{-} & & \underline{+} \\
H^\bullet_{\text{THF}}\left((\mathbb{R}\times\mathbb{C}^2)\setminus 0, \mathrm{T}\right)\otimes\mathcal{O} & \xrightarrow{\partial^W_\Omega} & H^\bullet_{\text{THF}}\left((\mathbb{R}\times\mathbb{C}^2)\setminus 0\right)\otimes\mathcal{O} \\
 & \xrightarrow{\partial^Z_\Omega} & \\
H^\bullet_{\text{THF}}\left((\mathbb{R}\times\mathbb{C}^2)\setminus 0\right)\otimes\mathrm{T} & & \\
H^\bullet_{\text{THF}}\left((\mathbb{R}\times\mathbb{C}^2)\setminus 0\right)\otimes\mathcal{O} & \xrightarrow{\partial_Z} & H^\bullet_{\text{THF}}\left((\mathbb{R}\times\mathbb{C}^2)\setminus 0\right)\otimes\Omega^1 \\
 & \xrightarrow{\partial_W} & H^\bullet_{\text{THF}}\left((\mathbb{R}\times\mathbb{C}^2)\setminus 0, \Omega^1\right)\otimes\mathcal{O}
\end{array}
\tag{3.11}
$$

where the differentials are as follows:

- The differentials ∂^Z_Ω and ∂^W_Ω are the divergence operators along the base and fiber respectively.
- The differentials ∂_Z and ∂_W are components of the holomorphic deRham differentials along the base and fiber respectively.
- The dotted arrows are N dependent differentials roughly given by bracketing with the flux, and are explicated below.

Before proceeding with the proof, it will again be useful to explicate the internal differentials above. The THF cohomology of $(\mathbb{R}\times\mathbb{C}^2)\setminus 0$ possibly with coefficients in a sheaf $\mathcal{F}$ equipped with a partial flat connection along the leaves of the THF can be described as the cohomology of the following quotient of the deRham complex

$$\Omega^\bullet\left((\mathbb{R}\times\mathbb{C}^2)\setminus 0\right)/(\mathrm{d}w_1, \mathrm{d}w_2).$$

The cohomology is accordingly concentrated in degrees zero and two. We will make use of the dense embeddings

$$
\begin{aligned}
\mathbb{C}[w_1,w_2] &\hookrightarrow H^0\left(\Omega^\bullet\left(\mathbb{C}^2_w\times\mathbb{R}\setminus 0\right)/(\mathrm{d}w_1,\mathrm{d}w_2)\right)\\
w_1^{-1}w_2^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}] &\hookrightarrow H^2\left(\Omega^\bullet\left(\mathbb{C}^2_w\times\mathbb{R}\setminus 0\right)/(\mathrm{d}w_1,\mathrm{d}w_2)\right)
\end{aligned}
$$

along with the analogous versions with coefficients in the sheaf $\mathcal{F}=\Omega^1, T$.

The flux NF_{M5} is then represented by a class of the form $N(w_1w_2)^{-1}\mathrm{d}w_1\mathrm{d}w_2 \in (w_1w_2)^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}]\otimes\Omega^2$. The dotted differentials in equation 3.11 are then explicitly given by maps

$$
\begin{aligned}
\mathbb{C}[w_1,w_2]\{\partial_{w_a}\}\otimes\mathcal{O} &\to (w_1w_2)^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}]\{\mathrm{d}w_a\}\otimes\mathcal{O}\\
\mathbb{C}[w_1,w_2]\otimes\Omega^1 &\to (w_1w_2)^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}]\otimes\mathrm{T}
\end{aligned}
$$

where the first map is given by contracting with $\mathrm{d}w_1\mathrm{d}w_2$ and multiplying by $(w_1w_2)^{-1}$, while the second map is given by applying ∂_Z, wedging with the $\mathrm{d}w_1\mathrm{d}w_2$ and contracting with the inverse of the holomorphic volume form on $\mathbb{C}^5$ to get a vector field along $\mathbb{C}^3$.

Proof. The proof is exactly analogous to that of proposition 3.2.2 . Using results from [Kor14, Sec. 4.2], [KT75] to compute the pushforward, we find a sheaf of $\Omega^\bullet_{\mathbb{R}_{>0}} \otimes \Omega^{0,\bullet}_{\mathbb{C}^3}$-modules whose sections have a tensor factor given by the THF cohomology of S^4. Next, we use the isomorphism in THF cohomology afforded by the deformation retraction of $(\mathbb{R} \times \mathbb{C}^2) \setminus 0$ onto S^4. □

3.2.2.1 As in the previous subsection, we ask for a boundary condition we can place on the fields at $\{\infty\} \times \mathbb{C}^3 \subset \mathbb{R}_{>0} \times \mathbb{C}^3$. The phase space, given by $(p_* \mathcal{E}^N_{AdS_7 \times S^4})|_{\{\infty\} \times \mathbb{C}^3}$, is seen to be $\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{V}_{\mathbb{C}^3})$, and we search for a shifted Lagrangian therein.

Completely analogously to before, we may rewrite the phase space as a shifted cotangent bundle. The higher residue pairing, the natural pairing between T, Ω^1, and the integration pairing along $\mathbb{C}^3$, all conspire to give the phase space an even-shifted symplectic structure. Together with this, we may ionce again identify the phase space with a twisted cotangent bundle

$$\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{V}_{\mathbb{C}^3}) = T^*_\Pi \left(\begin{array}{ccc} \underline{-} & & \underline{+} \\ \mathbb{C}[w_1, w_2]\{\partial_{w_a}\} \otimes \Omega^{0,\bullet}_{\mathbb{C}^3} & \xrightarrow{\partial^W_\Omega} & \mathbb{C}[w_1, w_2] \otimes \Omega^{0,\bullet}_{\mathbb{C}^3} \\ \mathbb{C}[w_1, w_2] \otimes \Omega^{0,\bullet}_{\mathbb{C}^3}(\mathrm{T}) & \nearrow_{\partial^Z_\Omega} & \\ \mathbb{C}[w_1, w_2] \otimes \Omega^{0,\bullet}_{\mathbb{C}^3} & \xrightarrow{\partial_Z} & \mathbb{C}[w_1, w_2] \otimes \Omega^{0,\bullet}_{\mathbb{C}^3}(\Omega^1) \\ & \searrow_{\partial_W} & \mathbb{C}[w_1, w_2]\{\mathrm{d}w_a\} \otimes \Omega^{0,\bullet}_{\mathbb{C}^3} \end{array} \right)$$

Exactly analogously to the case of $AdS_4 \times S^7$, Π here denotes the extra N-dependent differential induced by bracketing with the flux, which deforms the cotangent bundle.

A natural Lagrangian in the phase space is thus given by $\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}_{AdS_7 \times S^4})$ where $\mathcal{L}_{AdS_7 \times S^4}$ is the dg-vector bundle given by

$$\begin{array}{ccc} \underline{-} & & \underline{+} \\ (w_1 w_2)^{-1}\mathbb{C}[w_1^{-1}, w_2^{-1}]\{\partial_{w_a}\} \otimes \mathcal{O} & \xrightarrow{\partial^W_\Omega} & (w_1 w_2)^{-1}\mathbb{C}[w_1^{-1}, w_2^{-1}] \otimes \mathcal{O} \\ (w_1 w_2)^{-1}\mathbb{C}[w_1^{-1}, w_2^{-1}] \otimes \mathrm{T} & \nearrow_{\partial^Z_\Omega} & \\ (w_1 w_2)^{-1}\mathbb{C}[w_1^{-1}, w_2^{-1}] \otimes \mathcal{O} & \xrightarrow{\partial_Z} & (w_1 w_2)^{-1}\mathbb{C}[w_1^{-1}, w_2^{-1}] \otimes \Omega^1 \\ & \searrow_{\partial_W} & (w_1 w_2)^{-1}\mathbb{C}[w_1^{-1}, w_2^{-1}]\{\mathrm{d}w_a\} \otimes \mathcal{O} \end{array} \tag{3.12}$$

Definition 3.2.8. The space of supergravity states on twisted $AdS_7 \times S^4$ is given by the costalk at zero of the factorization algebra on $\mathbb{C}^3$ given by $\mathrm{C}^\bullet\left(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^N_{AdS_7 \times S^4})\right)$.

3.2.2.2 An argument exactly analogous to the proof of lemma 3.2.5 gives us the following

Lemma 3.2.9. The space of supergravity states on twisted $AdS_7 \times S^4$ is the symmetric algebra $\mathrm{Sym}(\mathcal{H}_{AdS_7 \times S^4})$ where $\mathcal{H}_{AdS_7 \times S^4}$ is given by the cochain complex

$$\begin{array}{ccc} + & & - \\ \hline \mathbb{C}[w_1,w_2]\{\partial_{w_a}\} \otimes \mathbb{C}[\partial_{z_1},\partial_{z_2},\partial_{z_3}]\delta_{z_i=0} & \xrightarrow{\partial^W_\Omega} & \mathbb{C}[w_1,w_2] \otimes \mathbb{C}[\partial_{z_1},\partial_{z_2},\partial_{z_3}]\delta_{z_i=0} \\ & \nearrow{\scriptstyle \partial^Z_\Omega} & \\ \mathbb{C}[w_1,w_2]\{\partial_{z_i}\} \otimes \mathbb{C}[\partial_{z_1},\partial_{z_2},\partial_{z_3}]\delta_{z_i=0} & & \\ & & \\ \mathbb{C}[w_1,w_2] \otimes \mathbb{C}[\partial_{z_1},\partial_{z_2},\partial_{z_3}]\delta_{z_i=0} & \xrightarrow{\partial_Z} & \mathbb{C}[w_1,w_2]\{\mathrm{d}z_a\} \otimes \mathbb{C}[\partial_{z_1},\partial_{z_2},\partial_{z_3}]\delta_{z_i=0} \\ & \searrow{\scriptstyle \partial_W} & \\ & & \mathbb{C}[w_1,w_2]\{\mathrm{d}w_a\} \otimes \mathbb{C}[\partial_{z_1},\partial_{z_2},\partial_{z_3}]\delta_{z_i=0} \end{array} \tag{3.13}$$

3.2.2.3 As before, we may compute the local character of the factorization algebra defined in 3.2.8 as a character of $\mathrm{Sym}(\mathcal{H}_{AdS_7\times S^4})$.

We will use an action of $\mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{gl}(1)$ on $\mathcal{H}_{AdS_7\times S^4}$ which we may explicitly realize as follows:

- The subalgebra $\mathfrak{sl}(3)$ acts as vector fields rotating the plane $\mathbb{C}^3_z$

$$\sum_{ij} A_{ij} z_i \frac{\partial}{\partial z_j} \quad (A_{ij}) \in \mathfrak{sl}(3). \tag{3.14}$$

- The subalgebra $\mathfrak{sl}(2)$ acts by the triple of vector fields

$$w_1 \frac{\partial}{\partial w_2}, \quad w_2 \frac{\partial}{\partial w_1}, \quad \frac{1}{2}\left(w_1 \frac{\partial}{\partial w_1} - w_2 \frac{\partial}{\partial w_2}\right). \tag{3.15}$$

- The subalgebra $\mathfrak{gl}(1)$ acts as the vector field

$$\Delta = \sum_{i=1}^{3} z_i \frac{\partial}{\partial z_i} - \frac{3}{2} \sum_{a=1}^{2} w_a \frac{\partial}{\partial w_a}. \tag{3.16}$$

The character will be a function on a Cartan in $\mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{gl}(1)$; we choose one whose generators are given as follows.

- t_1, t_2 denote generators for the Cartan of $\mathfrak{sl}(3)$ which is spanned by the vector fields

$$h_1 = z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}, \quad h_2 = z_2 \frac{\partial}{\partial z_2} - z_3 \frac{\partial}{\partial z_3}. \tag{3.17}$$

- r denotes a generator for the Cartan of a $\mathfrak{sl}(2)$ which is generated by the element

$$h = \frac{1}{2}\left(w_1 \frac{\partial}{\partial w_1} - w_2 \frac{\partial}{\partial w_2}\right). \tag{3.18}$$

- q denotes a generator for the Cartan of the $\mathfrak{gl}(1)$ which is generated by the element Δ from equation (3.16).

The weights of twisted supergravity states with respect to the generators of the Cartan subalgebra above are completely determined by the weights of the holomorphic coordinates w_1, w_2, z_1, z_2, z_3, which we summarize in table 3.2.

	z_1	z_2	z_3	w_1	w_2
t_1	1	0	−1	0	0
t_2	0	1	−1	0	0
r	0	0	0	1	−1
q	−1	−1	−1	$\frac{3}{2}$	$\frac{3}{2}$

Table 3.2: Fugacities for the fields of the holomorphic twist of eleven-dimensional supergravity for the geometry $\mathbb{R} \times \mathbb{C}^5 \setminus \mathbb{C}^3$.

We enumerate single particle supergravity states via computing the super trace of the operator $q^{\Delta} t_1^{h_1} t_2^{h_2} r^h$ acting on $\mathcal{H}_{AdS_7 \times S^4}$:

$$f_{AdS_7 \times S^4}(t_1, t_2, r, q) = \mathrm{Tr}_{\mathcal{H}_{AdS_7 \times S^4}} (-1)^F q^{\Delta} t_1^{h_1} t_2^{h_2} r^h. \tag{3.19}$$

Proposition 3.2.10. The single particle index of the space of twisted supergravity states $\mathcal{H}_{AdS_7 \times S^4}$ is given by the following expression

$$f_{AdS_7 \times S^4}(t_1, t_2, r, q) = \frac{q^4(t_1^{-1} + t_1 t_2^{-1} + t_2) - q^2(t_1 + t_1^{-1} t_2 + t_2^{-1}) + (q^{3/2} - q^{9/2})(r + r^{-1})}{(1 - t_1^{-1} q)(1 - t_2 q)(1 - t_1 t_2^{-1} q)(1 - r q^{3/2})(1 - r^{-1} q^{3/2})}. \tag{3.20}$$

The full (multiparticle) index is defined to be the plethystic exponential

$$\mathrm{PExp}\left[f_{AdS_7 \times S^4}(t_1, t_2, r, q)\right]. \tag{3.21}$$

Proof. As before, the two summands not involving holomorphic vector fields or forms appear with opposite parity, so their contributions to the character will cancel. For the remaining summands, it is again straightforward to compute the character of each tensor factor.

- The factor $\mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0}$ contributes a factor of

$$\frac{q^3}{(1 - t_1^{-1} q)(1 - t_2 q)(1 - t_1 t_2^{-1} q)}.$$

- The factor of $\mathbb{C}[w_1, w_2]$ contributes a factor of

$$\frac{1}{(1 - r q^{3/2})(1 - r^{-1} q^{3/2})}.$$

- The tensor factors involving vector fields and forms contribute a factor of

$$q(t_1^{-1} + t_1 t_2^{-1} + t_2) - q^{-1}(t_1 + t_1^{-1} t_2 + t_2^{-1}) + (q^{-3/2} - q^{3/2})(r + r^{-1})$$

□

3.2.2.4 To simplify the form of this index we can introduce a different parametrization of the Cartan of $\mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{gl}(1)$. First, we can parameterize the Cartan of $\mathfrak{sl}(3)$ by the vector fields

$$-(\log y_1) z_1 \frac{\partial}{\partial z_1} - (\log y_2) z_2 \frac{\partial}{\partial z_2} - (\log y_3) z_3 \frac{\partial}{\partial z_3}. \tag{3.22}$$

where y_1, y_2, y_3 are parameters which satisfy the single constraint

$$y_1 y_2 y_3 = 1. \tag{3.23}$$

In terms of the variables t_1, t_2 used above we have

$$y_1 = t_1^{-1}, \quad y_2 = t_1 t_2^{-1}, \quad y_3 = t_2. \tag{3.24}$$

Second, we can parametrize the Cartan of the remaining subalgebra $\mathfrak{sl}(2) \oplus \mathfrak{gl}(1)$ by the two vector fields

$$\tilde{h} = h + \frac{1}{2}\Delta \quad \text{and} \quad \Delta \tag{3.25}$$

where Δ is as in equation (3.16) and h is as in (3.18). We denote by y the generator of the Cartan corresponding to the vector field $\tilde{h}$ and by q (as above) the generator corresponding to Δ. In terms of the variable r used above we have

$$y = q^{1/2} r. \tag{3.26}$$

Using the parametrization of the Cartan given by the variables y_i, y, Δ we obtain the equivalent expression for the index (3.20) as

$$f_{AdS_7 \times S^4}(y_i, y, q) = \frac{q^4(y_1 + y_2 + y_3) - q^2(y_1^{-1} + y_2^{-1} + y_3^{-1}) + (1 - q^3)(yq + y^{-1}q^2)}{(1 - y_1 q)(1 - y_2 q)(1 - y_3 q)(1 - yq)(1 - y^{-1}q^2)}, \tag{3.27}$$

We note that this matches exactly with the index computed in [KKKL13, Eq. (3.23)] with the change of variables.

Our formula (3.20) also matches with [BBMR08, Eq. (3.24)] where we use the change of variables

$$q = x^4, \quad t_1 = y_2, \quad t_2 = y_1, \quad r^2 = z. \tag{3.28}$$

(Notice the variables y_1, y_2 used in [BBMR08] differ from the variables we introduced in (3.22).)

3.2.2.5 We consider the specialization of this index

$$q = r^2, t_2 = 1 \tag{3.29}$$

which is known as the Schur limit. Applying this limit to (3.20) yields the plethystic exponential of the following single particle index

$$f_{AdS_7 \times S^4}(q, t_1, t_2 = 1, r = q^{1/2}) = \frac{q}{(1-q)^2}$$

This plethystic exponential yields the MacMahon function, which is the character of the vacuum module of the $W_{1+\infty}$-algebra. We will revisit this observation in section 3.6.

3.2.3 Transverse boundary conditions

In the previous subsections, we discussed boundary conditions in the phase spaces of the sphere compactifications $p_* \mathcal{E}^N_{AdS_4 \times S^7}$ and $p_* \mathcal{E}^N_{AdS_7 \times S^4}$ viewed as free theories, that exist for generic values of N. However, there are distinguished boundary conditions that exist for $N = 0$ that we will use in the sequel. Moreover, these distinguished boundary conditions are in fact boundary conditions for the interacting theory.

Indeed, recall that in equations (3.6.4.4), (3.6.4.4) we wrote the phase spaces as twisted cotangent bundles, where the nontrivial Poisson tensor was induced by the terms in the differential coming from bracketing with the flux. When we specialize $N = 0$, this Poisson tensor vanishes, and there is an additional Lagrangian given by the zero section.

Explicitly these Lagrangians in the phase space are described as follows.

- Let $\mathcal{L}^{r=0}_{AdS_4 \times S^7}$ denote the following dg-vector bundle on $\mathbb{C}$

$$
\begin{array}{ccc}
\underline{-} & & \pm \\
\mathbb{C}[w_1, \ldots, w_4]\{\partial_{w_a}\} \otimes \mathcal{O} & \xrightarrow{\partial^W_\Omega} & \mathbb{C}[w_1, \ldots, w_4] \otimes \mathcal{O} \\
 & \nearrow_{\partial^Z_\Omega} & \\
\mathbb{C}[w_1, \ldots, w_4] \otimes \mathrm{T} & & \\
\mathbb{C}[w_1, \ldots, w_4] \otimes \mathcal{O} & \xrightarrow{\partial_Z} & \mathbb{C}[w_1, \ldots, w_4] \otimes \Omega^1 \\
 & \searrow_{\partial_W} & \\
 & & \mathbb{C}[w_1, \ldots, w_4]\{\mathrm{d}w_a\} \otimes \mathcal{O}
\end{array}
\tag{3.30}
$$

  The desired Lagrangian in $(p_* \mathcal{E}^0_{AdS_4 \times S^7})|_{\{\infty\} \times \mathbb{R} \times \mathbb{C}}$ is given by $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4 \times S^7})$.

- Let $\mathcal{L}^{r=0}_{AdS_7 \times S^4}$ denote the following dg-vector bundle on $\mathbb{C}^3$

$$
\begin{array}{ccc}
\pm & & \underline{-} \\
\mathbb{C}[w_1, w_2]\{\partial_{w_a}\} \otimes \mathcal{O} & \xrightarrow{\partial^W_\Omega} & \mathbb{C}[w_1, w_2] \otimes \mathcal{O} \\
 & \nearrow_{\partial^Z_\Omega} & \\
\mathbb{C}[w_1, w_2] \otimes \mathrm{T} & & \\
\mathbb{C}[w_1, w_2] \otimes \mathcal{O} & \xrightarrow{\partial_Z} & \mathbb{C}[w_1, w_2] \otimes \Omega^1 \\
 & \searrow_{\partial_W} & \\
 & & \mathbb{C}[w_1, w_2]\{\mathrm{d}w_a\} \otimes \mathcal{O}
\end{array}
\tag{3.31}
$$

  The desired Lagrangian in $(p_* \mathcal{E}^0_{AdS_7 \times S^4})|_{\{\infty\} \times \mathbb{C}^3}$ is given by $\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7 \times S^4})$.

Note that the same formulae defining the L_∞ structure on the parity shift of the eleven-dimensional theory $\mathcal{E}$ 2.1.1 equip these Lagrangians in the $N = 0$ phase spaces with L_∞ structures. Moreover, the canonical maps

$$
\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4 \times S^7}) \to (p_* \mathcal{E}^0_{AdS_4 \times S^7})|_{\{\infty\} \times \mathbb{R} \times \mathbb{C}}, \quad \Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7 \times S^4}) \to (p_* \mathcal{E}^0_{AdS_7 \times S^4})|_{\{\infty\} \times \mathbb{C}^3}
$$

preserve the L_∞ brackets even taking into account the potential for additional higher brackets on the target coming from homotopy transfer. Indeed, such brackets must necessarily involve classes in $H^2_{\mathrm{THF}}\big((\mathbb{R} \times \mathbb{C}^2) \setminus 0\big)$

Remark 3.2.11. These boundary conditions have a very natural physical interpretation. Recall that we constructed our avatars of the $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds by codifying their appearance as backreactions of M2 and M5 branes respectively. In the absence of the fluxes sourced by these branes, we may ask that the supergravity fields extend over the former locations of these branes. As such, we can think of the boundary conditions defined above as finite-type models for the restriction of the fields of the eleven-dimensional theory to the location of branes.

3.2.3.1 The following lemma illustrates that the state space of definitions 3.2.4 3.2.8 can also be computed using these alternate boundary conditions.

These Lagrangians afford an alternative description of twisted supergravity states, which will be used to investigate their representation theoretic properties.

Proposition 3.2.12. There are isomorphisms

$$\mathcal{U}\left(\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4\times S^7})\right)(0) \cong \mathrm{Sym}(\mathcal{H}_{AdS_4\times S^7})$$
$$\mathcal{U}\left(\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^S})\right)(0) \cong \mathrm{Sym}(\mathcal{H}_{AdS_7\times S^4})$$

Proof. For a local L_∞ algebra $\mathcal{L}$, its factorization envelope $\mathcal{U}(L)$ is defined to be $\mathrm{C}_\bullet(\mathcal{L}_c)$ the Lie algebra chains on the cosheaf of compactly supported sections. Therefore, it suffices to show that in each case, the costalk of the cosheaf of comapctly supported sections is quasi-isomorphic to $\mathcal{H}_{AdS_4\times S^7}$ and $\mathcal{H}_{AdS_7\times S^4}$ respectively. This is a consequence of the following observations.

Note that by ellipticity, there are quasi-isomorphisms

$$\overline{\Omega}^{0,\bullet}_{\mathbb{C},c}(D) \to \Omega^{0,\bullet}_{\mathbb{C},c}(D), \quad \overline{\Omega}^{0,\bullet}_{\mathbb{C}^3,c} \to \Omega^{0,\bullet}_{\mathbb{C}^3,c}(D^3).$$

coming from the inclusion of compactly supported distributional sections into compactly supported smooth sections. Now contracting the Dolbeault resolution, there are quasi-isomorphisms

$$\mathbb{C}[\partial_z]\delta_{z=0} \to \overline{\Omega}^{0,\bullet}_{\mathbb{C},c}(D), \quad \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z=0} \to \overline{\Omega}^{0,\bullet}_{\mathbb{C}^3,c}(D^3).$$

These results apply equally as well for sections of holomorphic bundles. Now, computing the limit in the definition of the costalk over a collection of open sets containing the origin gives the result. □

3.3 Twisted global symmetries

As we indicated in the beginning of this chapter, a feature of the physical $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds is that they have as isometries, the 3d $\mathcal{N} = 8$ and 6d $\mathcal{N} = (2,0)$ superconformal algebras respectively. In fact, the complex forms of these two super-Lie algebras are the same.

In this section we provide evidence for conjectures 3.1.43.1.8 by arguing that the global sections of the local moduli problems $p_*\mathcal{E}_{AdS_4\times S^7}$ and $p_*\mathcal{E}_{AdS_7\times S^4}$ carry actions by the minimal twists of the relevant superconformal algebras. We will find that the twist of the superconformal algebra is the same in each case, but the actions are slightly different.

3.3.1 Superconformal algebras

The complex form of the algebra of isometries for supergravity in both the $\mathrm{A}dS_4$ and $\mathrm{A}dS_7$ backgrounds is $\mathfrak{osp}(8|4)$ (though, their real forms differ). This agrees with the complex form of the 6d $\mathcal{N} = (2,0)$ superconformal algebra and the 3d $\mathcal{N} = 8$ superconformal algebra. The bosonic part of this algebra is isomorphic to $\mathfrak{so}(8) \oplus \mathfrak{sp}(2) \cong \mathfrak{so}(8) \oplus \mathfrak{so}(5)$.

The following is a mild rephrasing of a result in [SW22] where twisted superconformal symmetry in six dimensions is studied in some detail.

Theorem 3.3.1 (Saberi-Williams)**.** There is a map of super-Lie algebras $\phi : \mathfrak{siso}_{11d} \to \mathfrak{osp}(8|4)$. Letting $Q \in \mathfrak{siso}_{11d}$ be the odd square-zero element used to define the minimal twist of eleven-dimensional supergravity, there is an equivalence of dg super-Lie algebras

$$(\mathfrak{osp}(8|4), [\phi(Q), -]) \cong \mathfrak{osp}(6|2).$$

The super-Lie algebra $\mathfrak{osp}(6|2)$ will therefore play the role of the residual isometries of the twisted AdS background. The bosonic part of $\mathfrak{osp}(6|2)$ is the direct sum Lie algebra $\mathfrak{sl}(4) \oplus \mathfrak{sl}(2)$. The odd part of the algebra $\mathfrak{osp}(6|2)$ is $\wedge^4 W \otimes R$ where W is the fundamental $\mathfrak{sl}(4)$ representation and R is the fundamental $\mathfrak{sl}(2)$ representation.

3.3.2 Global symmetries of twisted $\mathrm{A}dS_4 \times S^7$

To provide further evidence for conjecture 3.1.4 we wish to articulate a sense in which $\mathfrak{osp}(6|2)$ is witnessed as a symmetry of $\mathcal{E}^N_{AdS_4\times S^7}$. To this end, we will provide evidence for the claim that there is a Lie map

$$\mathfrak{osp}(6|2) \to H^\bullet\left(\Pi\mathcal{E}^N_{AdS_4\times S^7}\left((\mathbb{R}\times\mathbb{C})\times\mathbb{C}^4\setminus\{0\}\right)\right).$$

We first focus on the case where the flux $N = 0$. In this case, we will show that the embedding factors through the natural restriction map from the theory on flat space

$$\begin{array}{ccc} & & H^\bullet\left(\Pi\mathcal{E}(R\times\mathbb{C}^5)\right)\cong\widehat{E(5|10)} \\ & \overset{i_{M2}}{\nearrow} & \Big\downarrow \mathrm{res} \\ \mathfrak{osp}(6|2) & \longrightarrow & H^\bullet\left(\Pi\mathcal{E}^0_{AdS_4\times S^7}\left((\mathbb{R}\times\mathbb{C})\times\mathbb{C}^4\setminus\{0\}\right)\right) \end{array}$$

3.3.2.1 We begin by describing the map i_{M2}. As recalled above, the bosonic part of $\mathfrak{osp}(6|2)$ is $\mathfrak{sl}(4) \oplus \mathfrak{sl}(2)$. In its incarnation as a twist of 3d $\mathcal{N} = 8$ superconformal symmetry, it is useful to think of the $\mathfrak{sl}(2)$-summand as describing conformal transformations on $\mathbb{C}_z$, while the Lie algebra $\mathfrak{sl}(4)$ is a residual R-symmetry describing rotations on $\mathbb{C}^4_w$.

The restriction of i_{M2} to the bosonic summand will actually realize $\mathfrak{sl}(4) \oplus \mathfrak{sl}(2)$ as the global symmetries corresponding to the vector fields in equation 3.6.4.4.

- The image of the bosonic summand $\mathfrak{sl}(2)$ under i_{M2} is spanned by the vector fields

$$\frac{\partial}{\partial z}, \quad z\frac{\partial}{\partial z} - \frac{1}{4}\sum_{a=1}^4 w_a\frac{\partial}{\partial w_a}, \quad z\left(z\frac{\partial}{\partial z} - \frac{1}{2}\sum_{a=1}^4 w_a\frac{\partial}{\partial w_a}\right) \in \mathrm{PV}^{1,0}(\mathbb{C}^5)\otimes\Omega^0(\mathbb{R}).$$

  These vector fields are divergence free and reduce to the usual holomorphic conformal transformations along $w = 0$.

- The image of $B_{ab} \in \mathfrak{sl}(4)$ under i_{M2} is given by the vector field

$$B_{ab} w_a \frac{\partial}{\partial w_b} \in \mathrm{PV}^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R}).$$

To describe the image of the fermionic part of $\mathfrak{osp}(6|2)$ under the map i_{M2} It is natural to split $R = \mathbb{C}_{+1} \oplus \mathbb{C}_{-1}$, so that the odd part decomposes as

$$(\wedge^2 \mathbb{C}^4)_{+1} \oplus (\wedge^2 \mathbb{C}^4)_{-1}.$$

In terms of residual 3d $\mathcal{N} = 8$ superconformal symmetries, the fermionic summand $(\wedge^2\mathbb{C}^4)_{+1}$ consists of residual supertranslations, while the fermionic summand $(\wedge^2\mathbb{C}^4)_{-1}$ consists of the remaining superconformal transformations.

- For $e_a \wedge e_b \in (\wedge^2\mathbb{C}^4)_{+1}$ we have that

$$i_{M2}(e_a \wedge e_b) = \frac{1}{2}(w_a \mathrm{d}w_b - w_b \mathrm{d}w_a) \in \Omega^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R}).$$

- For $e_a \wedge e_b \in (\wedge^2\mathbb{C}^4)_{-1}$ we have that

$$i_{M2}(e_a \wedge e_b) = \frac{1}{2} z(w_a \mathrm{d}w_b - w_b \mathrm{d}w_a) \in \Omega^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R}).$$

The following is a straightforward check.

Lemma 3.3.2. The map i_{M2} is a Lie map.

It is clear that the image of the chain-level map i_{M2} defined above is closed for the linearized BRST differential $\delta^{(1)}$ so descends to a map $i_{M2} : \mathfrak{osp}(6|2) \to H^\bullet\left(\Pi\mathcal{E}(R \times \mathbb{C}^5)\right) \cong \widehat{E(5|10)}$ as claimed. As such, the composition $\mathrm{res} \circ i_{M2}$ defines an inner action of $\mathfrak{osp}(6|2)$ on the cohomology of global sections $H^\bullet\left(\Pi\mathcal{E}^0_{AdS_4\times S^7}\left((\mathbb{R}\times\mathbb{C})\times\mathbb{C}^4 \setminus \{0\}\right)\right)$.

3.3.2.2 Next, we turn on $N \neq 0$ units of nontrivial flux. Note that not all fields in the image of the map $\mathrm{res} \circ i_{M2}$ commute with bracketing with the flux NF_{M2}, and as such are not compatible with the total differential $\delta^{(1)} + [NF_{M2}, -]$ on $\Pi\mathcal{E}^N_{AdS_4\times S^7}\left((\mathbb{R}\times\mathbb{C})\times\mathbb{C}^4\setminus\{0\}\right)$. Nevertheless, we have the following:

Proposition 3.3.3. There exist N-dependent corrections to the fields defining the embedding of $\mathfrak{osp}(6|2)$ summarized above which are closed for the modified BRST differential $\delta^{(1)} + [NF_{M2}, -]$. Furthermore, these order N corrections define an embedding of

$$\mathfrak{osp}(6|2) \to H^\bullet\left(\Pi\mathcal{E}^N_{AdS_4\times S^7}\left((\mathbb{R}\times\mathbb{C})\times\mathbb{C}^4\setminus\{0\}\right)\right).$$

Proof. For notational convenience, we will let $\mathcal{L}$ denote the local L_∞ algebra on $(\mathbb{R}\times\mathbb{C})\times(\mathbb{C}^4\setminus\{0\})$ given by $\Pi\mathcal{E}^N_{AdS_4\times S^7}$ and set $F = F_{M2}$. We will show that the image of the map $\mathrm{res}\circ i_{M2}$ survives to the last page of a spectral sequence that abuts to the target of the above map. The spectral sequence is the one associated to the bicomplex whose differentials are the linearized BRST differential $\delta^{(1)}$

and the operator $[F,-]$ given by bracketing with the flux. Recall from 3.6.4.4 that F is an element of $\mathrm{PV}^{1,3}(\mathbb{C}^4_w \setminus 0) \otimes \Omega^{0,0}(\mathbb{C}_z) \otimes \Omega^0(\mathbb{R})$ and $[F,-]$ acts on the fields according to two types of maps:

$$[F,-] : \mathrm{PV}^{i,\bullet}(\mathbb{C}^4_w \setminus 0) \otimes \mathrm{PV}^{j,\bullet}(\mathbb{C}_z) \otimes \Omega^\bullet(\mathbb{R}) \to \mathrm{PV}^{i,\bullet+3}(\mathbb{C}^4_w \setminus 0) \otimes \mathrm{PV}^{j,\bullet}(\mathbb{C}_z) \otimes \Omega^\bullet(\mathbb{R})$$
$$[F,-] : \Omega^{i,\bullet}(\mathbb{C}^4_w \setminus 0) \otimes \Omega^{j,\bullet}(\mathbb{C}_z) \otimes \Omega^\bullet(\mathbb{R}) \to \Omega^{i,\bullet+3}(\mathbb{C}^4_w \setminus 0) \otimes \Omega^{j,\bullet}(\mathbb{C}_z) \otimes \Omega^\bullet(\mathbb{R}).$$

The first page of the spectral sequence is the cohomology with respect to the original linearized BRST differential $\delta^{(1)}$; this is exactly $H^\bullet\left(\Pi\mathcal{E}^0_{AdS_4\times S^7}\left((\mathbb{R}\times\mathbb{C})\times\mathbb{C}^4\setminus\{0\}\right)\right)$. It will be useful to compute this page explicitly.

Recall that the linearized BRST differential decomposes as

$$\delta^{(1)} = \bar\partial + \mathrm{d}_{\mathbb{R}} + \partial_\Omega|_{\mu\to\nu} + \partial|_{\beta\to\gamma}.$$

To compute this page, we use an auxiliary spectral sequence which simply filters by the holomorphic form and polyvector field type. This first page of this auxiliary spectral sequence is simply given by the cohomology with respect to $\bar\partial + \mathrm{d}_{\mathbb{R}}$. This cohomology is given by

$$\begin{array}{cc} \underline{+} & \underline{-} \\ H^\bullet(\mathbb{C}^4\setminus 0,\mathrm{T})\otimes H^\bullet(\mathbb{C},\mathcal{O}) & H^\bullet(\mathbb{C}^4\setminus 0,\mathcal{O})\otimes H^\bullet(\mathbb{C},\mathcal{O}) \\ H^\bullet(\mathbb{C}^4\setminus 0,\mathcal{O})\otimes H^\bullet(\mathbb{C},\mathrm{T}) & \\ H^\bullet(\mathbb{C}^4\setminus 0,\mathcal{O})\otimes H^\bullet(\mathbb{C},\mathcal{O}) & H^\bullet(\mathbb{C}^4\setminus 0,\mathcal{O})\otimes H^\bullet(\mathbb{C},\Omega^1) \\ & H^\bullet(\mathbb{C}^4\setminus 0,\Omega^1)\otimes H^\bullet(\mathbb{C},\mathcal{O}) \end{array} \tag{3.32}$$

The cohomology of $\mathbb{C}$ is of course concentrated in degree zero and there is a dense embedding $\mathbb{C}[z] \hookrightarrow H^\bullet(\mathbb{C},\mathcal{F})$ for $\mathcal{F} = \mathcal{O},\mathrm{T}$, or Ω^1. It follows that up to completion, the cohomology $H^\bullet\left(\mathcal{L}\left((\mathbb{R}\times\mathbb{C})\times\mathbb{C}^4\setminus\{0\}\right); d+\bar\partial\right)$ is given by the direct sum of $H^\bullet\left(\Pi\mathcal{E}(\mathbb{R}\times\mathbb{C}^5); d+\bar\partial\right)$ with

$$\begin{array}{ccc} \underline{-} & & \underline{+} \\ (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\cdots,w_4^{-1}][z]\{\partial_{w_i}\} & \xrightarrow{\partial_\Omega^W} & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\cdots,w_4^{-1}][z] \\ (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\cdots,w_4^{-1}][z]\partial_z & \nearrow_{\partial_\Omega^Z} & \\ (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\cdots,w_4^{-1}][z] & \xrightarrow{\partial_Z} & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\cdots,w_4^{-1}][z]\mathrm{d}z \\ & \searrow_{\partial_W} & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\cdots,w_4^{-1}][z]\{\mathrm{d}w_i\}. \end{array} \tag{3.33}$$

The remaining piece of the original BRST operator is drawn in dotted lines. The first page of the spectral sequence converging to the cohomology with respect to $\delta^{(1)} + [NF,-]$ is thus given by the cohomology of the global symmetry algebra on $\mathbb{C}^5\times\mathbb{R}$, which we computed in §2.2.1, plus the cohomology of the above complex with respect to the dotted-line operators. Indeed, this is exactly $H^\bullet\left(\Pi\mathcal{E}^0_{AdS_4\times S^7}\left((\mathbb{R}\times\mathbb{C})\times\mathbb{C}^4\setminus\{0\}\right)\right)$

Recall that the image of the flux F at this page in the spectral sequence corresponds to the class

$$[F] = (w_1 \cdots w_4)^{-1}\partial_z \in (w_1 \cdots w_4)^{-1}\mathbb{C}[w_1^{-1}, \cdots, w_4^{-1}][z]\partial_z$$

The next page of the spectral sequence is given by computing the cohomology with respect to the operator $[NF, -]$. As observed above, this operator maps Dolbeault degree zero elements to Dolbeault degree three elements. For degree reasons, there are no further differentials and the spectral sequence collapses after the second page.

We now wish to argue that the image of the map $res \circ i_{M2}$ is annihilated by $\big[N[F], -\big]$. This is a direct calculation. For instance, recall that an element in the image of the odd summand $(\wedge^2\mathbb{C}^2)_{-1}$ (which corresponds to a superconformal transformation) is of the form $zw_a \wedge \mathrm{d}w_b = z(w_a\mathrm{d}w_b - w_b\mathrm{d}w_a)$. We have

$$\big[[F], z(w_a\mathrm{d}w_b - w_b\mathrm{d}w_a)\big] = (w_1 \cdots w_4)^{-1}(w_a\mathrm{d}w_b - w_b\mathrm{d}w_a) = 0$$

since the class $(w_1 \cdots w_4)^{-1}$ is in the kernel of the operator given by multiplication by w_a for any $a = 1, \ldots 4$. □

Remark 3.3.4. We comment on an alternate method to compute the first page of the auxiliary spectral sequence we used to compute the first page of the spectral sequence converging to the cohomology with respect to $\delta^{(1)} + [NF, -]$. We could have used a Serre-type spectral sequence for certain kinds of sheaves on THF manifolds [KT75], [Kor14], applied to the pushforward $p_*\Pi\mathcal{E}^N_{AdS_4\times S^7}$ from section 3.2.7. In this case, this Serre-type spectral sequence degenerates at the E_2-page.

3.3.3 Global symmetries of twisted $AdS_7 \times S^4$

We now wish to repeat the analysis of the previous section for the twisted $AdS_7 \times S^4$ background so as to provide evidence for conjecture 3.1.8. As before, we wish to provide evidence for the claim that there is a Lie map

$$\mathfrak{osp}(6|2) \to H^\bullet\left(\Pi\mathcal{E}^N_{AdS_7\times S^4}\left(\mathbb{C}^3 \times (\mathbb{R}\times\mathbb{C}^2)\setminus 0\right)\right).$$

We first focus on the case $N = 0$ where once again the embedding factors through the natural restriction map from the theory on flat space

$$\begin{array}{ccc} & & H^\bullet\left(\Pi\mathcal{E}(R\times\mathbb{C}^5)\right) \cong \widehat{E(5|10)} \\ & \overset{i_{M5}}{\nearrow} & \big\downarrow \mathrm{res} \\ \mathfrak{osp}(6|2) & \longrightarrow & H^\bullet\left(\Pi\mathcal{E}^0_{AdS_7\times S^4}\left(\mathbb{C}^3 \times (\mathbb{R}\times\mathbb{C}^2)\setminus 0\right)\right) \end{array}$$

3.3.3.1 We begin by describing the map i_{M5}. Recall that the bosonic part of $\mathfrak{osp}(6|2)$ is the direct sum Lie algebra $\mathfrak{sl}(4) \oplus \mathfrak{sl}(2)$. In its incarnation as the minimal twist of the 6d $\mathcal{N} = (2, 0)$ superconformal algebra, the roles of the $\mathfrak{sl}(4)$ and $\mathfrak{sl}(2)$ summands are interchanged compared to the case of the M2 brane. Indeed, the Lie algebra $\mathfrak{sl}(4)$ represents conformal transformations along $\mathbb{C}^3_z$, while $\mathfrak{sl}(2)$ is a residual R-symmetry describing rotations on $\mathbb{C}^2_w$.

Moreover, the restriction of i_{M5} to a copy of $\mathfrak{sl}(3) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \subset \mathfrak{sl}(4) \oplus \mathfrak{sl}(2)$ will realize this subalgebra as the global symmetries corresponding to the vector fields in equation 3.2.2.3.

- The bosonic abelian subalgebra $\mathbb{C}^3 \subset \mathfrak{sl}(4)$ of translations is mapped to the obvious vector fields
$$\frac{\partial}{\partial z_i} \in \mathrm{PV}^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R}), \quad i = 1, 2, 3.$$
- The image of $A_{ij} \in \mathfrak{sl}(3) \subset \mathfrak{sl}(4)$ under i_{M5} is given by the vector field
$$A_{ij} z_i \frac{\partial}{\partial z_j} \in \mathrm{PV}^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R}), \quad (A_{ij}) \in \mathfrak{sl}(3).$$
- The image of $\mathfrak{gl}(1) \subset \mathfrak{sl}(4)$ corresponding to rescaling $\mathbb{C}^3$ under i_{M5} is the element
$$\sum_{i=1}^{3} z_i \frac{\partial}{\partial z_i} - \frac{3}{2} \sum_{a=1}^{2} w_a \frac{\partial}{\partial w_a} \in \mathrm{PV}^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R}).$$
- The image of the remaining subalgebra of $\mathfrak{sl}(4)$, which describes special conformal transformations on $\mathbb{C}^3$, is spanned by the elements
$$z_j \left(\sum_{i=1}^{3} z_i \frac{\partial}{\partial z_i} - 2 \sum_{a=1}^{2} w_a \frac{\partial}{\partial w_a} \right) \in \mathrm{PV}^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R}).$$
Notice that these vector fields are divergence-free and restrict to the ordinary special conformal transformations along $w = 0$.
- The image of the bosonic summand $\mathfrak{sl}(2)$ corresponding to residual R-symmetry is spanned by the vector fields
$$w_1 \frac{\partial}{\partial w_2}, w_2 \frac{\partial}{\partial w_1}, \frac{1}{2} \left(w_1 \frac{\partial}{\partial w_1} - w_2 \frac{\partial}{\partial w_2} \right) \in \mathrm{PV}^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R}).$$

To describe the image of the fermionic part of $\mathfrak{osp}(6|2)$, which is given by $\wedge^2 W \oplus R$ with W the fundamental $\mathfrak{sl}(4)$ representation and R the fundamental $\mathfrak{sl}(2)$ representation, it is natural to split $W = L \oplus \mathbb{C}$ with $L = \mathbb{C}^3$ the fundamental $\mathfrak{sl}(3) \subset \mathfrak{sl}(4)$ representation. The odd part then decomposes as
$$L \otimes R \oplus \wedge^2 L \otimes R \cong \mathbb{C}^3 \otimes \mathbb{C}^2 \oplus \wedge^2 \mathbb{C}^3 \otimes \mathbb{C}.$$

- The summand $L \otimes R$ consists of the remaining 6d supertranslations. Its image under i_{M5} is spanned by the fields
$$z_i \mathrm{d} w_a \in \Omega^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R}), \quad a = 1, 2, \quad i = 1, 2, 3.$$
- The summand $\wedge^2 L \otimes R$ consists of the remaining 6d superconformal transformations. Its image under i_{M5} is spanned by the fields
$$\frac{1}{2} w_a (z_i \mathrm{d} z_j - z_j \mathrm{d} z_i) \in \Omega^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R}), \quad a = 1, 2, \quad k = 1, 2, 3.$$

The following is a straightforward check.

Lemma 3.3.5. The map i_{M5} is a Lie map.

It is again clear that the image of the chain-level map i_{M5} defined above is closed for the linearized BRST differential $\delta^{(1)}$ on $\Pi\mathcal{E}$ so descends to a map $i_{M5} : \mathfrak{osp}(6|2) \to H^\bullet(\Pi\mathcal{E}(\mathbb{R} \times \mathbb{C}^5)) \cong \widehat{E(5|10)}$ as claimed. As such, the composition $\mathrm{res} \circ i_{M5}$ will define an inner action of $\mathfrak{osp}(6|2)$ on the cohomology of the global sections $H^\bullet\left(\Pi\mathcal{E}^0_{AdS_7 \times S^4}\left(\mathbb{C}^3 \times (\mathbb{R} \times \mathbb{C}^2) \setminus 0\right)\right)$.

3.3.3.2 Next, we turn on $N \neq 0$ units of nontrivial flux. Again, not all fields in the image of the map $\mathrm{res} \circ i_{M5}$ are compatible with the total differential $\delta^{(1)} + [NF, -]$ on $\Pi\mathcal{E}^N_{AdS_7 \times S^4}\left(\mathbb{C}^3 \times (\mathbb{R} \times \mathbb{C}^2) \setminus 0\right)$. Nevertheless, we have the following version of proposition 3.3.3

Proposition 3.3.6. There exist N-dependent corrections to the fields defining the embedding of $\mathfrak{osp}(6|2)$ summarized above which are closed for the modified BRST differential $\delta^{(1)} + [NF_{M5}, -]$. Furthermore, these N-dependent corrections define an embedding

$$H^\bullet\left(\Pi\mathcal{E}^N_{AdS_7 \times S^4}\left(\mathbb{C}^3 \times (\mathbb{R} \times \mathbb{C}^2) \setminus 0\right)\right).$$

Proof. We proceed exactly analogously to the proof of proposition 3.3.3. For notational convenience, we will let $\mathcal{L}$ denote the local L_∞ algebra on $\mathbb{C}^3 \times (\mathbb{R} \times \mathbb{C}^2) \setminus 0$ given by $\Pi\mathcal{E}^N_{AdS_7 \times S^4}$ and set $F = F_{M5}$. We will show that the image of the map $\mathrm{res} \circ i_{M5}$ survives to the last page of a spectral sequence that abuts to the target of the above map. The spectral sequence is the one associated to the bicomplex whose differentials are the linearized BRST differential $\delta^{(1)}$ and the operator $[F, -]$ given by bracketing with the flux.

The first page of this spectral sequence is the cohomology with respect to the original linearized BRST differential $\delta^{(1)}$; this is exactly $H^\bullet\left(\Pi\mathcal{E}^0_{AdS_7 \times S^4}\left(\mathbb{C}^3 \times (\mathbb{R} \times \mathbb{C}^2) \setminus 0\right)\right)$. It will be useful to compute this page explicitly.

We once again do so by way of an auxiliary spectral sequence which simply filters by the holomorphic form and polyvector field type. This first page of this auxiliary spectral sequence is simply given by the cohomology with respect to $\mathrm{d}_{\mathbb{R}} + \bar\partial$.

It follows that up to completions, the cohomology $H^\bullet\left(\mathcal{L}\left(\right); \mathrm{d}_{\mathbb{R}} + \bar\partial\right)$ is the direct sum of the cohomology on flat space $H^\bullet(\Pi\mathcal{E}(\mathbb{C}^5 \times \mathbb{R}), \mathrm{d}_{\mathbb{R}} + \bar\partial)$ with

$$\begin{array}{ccc}
+ & & - \\
w_1^{-1}w_2^{-1}\mathbb{C}[w_1^{-1}, w_2^{-1}][z_1, z_2, z_3]\{\partial_{w_i}\} & \xrightarrow{\partial_\Omega^W} & w_1^{-1}w_2^{-1}\mathbb{C}[w_1^{-1}, w_2^{-1}][z_1, z_2, z_3] \\
 & \nearrow_{\partial_\Omega^Z} & \\
w_1^{-1}w_2^{-1}\mathbb{C}[w_1^{-1}, w_2^{-1}][z_1, z_2, z_3]\{\partial_{z_i}\} & & \\
w_1^{-1}w_2^{-1}\mathbb{C}[w_1^{-1}, w_2^{-1}][z_1, z_2, z_3] & \xrightarrow{\partial_W} & w_1^{-1}w_2^{-1}\mathbb{C}[w_1^{-1}, w_2^{-1}][z_1, z_2, z_3]\{\mathrm{d}z_i\} \\
 & \searrow^{\partial_Z} & w_1^{-1}w_2^{-1}\mathbb{C}[w_1^{-1}, w_2^{-1}][z_1, z_2, z_3]\{\mathrm{d}w_i\}.
\end{array} \tag{3.34}$$

The first page of the spectral sequence converging to the cohomology with respect to $\delta^{(1)} + [NF, -]$ is given by the cohomology of the global symmetry algebra on $\mathbb{C}^5 \times \mathbb{R}$, which we computed in §2.2.1, plus the cohomology with respect to the dotted-line operators in (3.34). This is indeed the cohomology of global sections $H^\bullet\left(\Pi\mathcal{E}^0_{AdS_7 \times S^4}\left(\mathbb{C}^3 \times (\mathbb{R} \times \mathbb{C}^2) \setminus 0\right)\right)$.

Recall that the flux F was defined as the image under ∂ of some γ-type field. Therefore, the class $[F]$ does not live inside this page of the spectral sequence, but the operator $[[F], -]$ does act on this page nevertheless. For instance, if $f^i(z,w)\mathrm{d}z_i$ is a one-form living in $H^0(\mathbb{C}^5, \Omega^1) \otimes H^0(\mathbb{R})$, then

$$[[F], f^i(z,w)\mathrm{d}z_i] = \epsilon_{ijk} w_1^{-1} w_2^{-1} \partial_{z_j} f^i(z,w) \partial_{z_k}$$

which is an element in

$$\mathbb{C}[w_1^{-1}, w_2^{-1}][z_1, z_2, z_3]\{\partial_{z_i}\} \subset H^0(\mathbb{C}^3, \mathrm{T}) \otimes H^2\big(\Omega^\bullet(\mathbb{C}^2 \times \mathbb{R} \setminus 0)/(\mathrm{d}w_1, \mathrm{d}w_2)\big).$$

The next page of the spectral sequence is given by computing the cohomology with respect to the operator $[NF, -]$. This operator maps Dolbeault-de Rham degree zero elements to Dolbeault-de Rham degree two elements. For degree reasons, there are no further differentials and the spectral sequence collapses after the second page.

We now wish to argue that the image of the map $\mathrm{res} \circ i_{M5}$ is annihilated by $\big[N[F], -\big]$. This is a direct calculation. For instance, recall that an element in the image of the odd summand $\wedge^2 L \otimes R = \wedge^2 \mathbb{C}^3 \otimes \mathbb{C}^2$ (which corresponds to a superconformal transformation) is of the form $w_a(z_i \mathrm{d}z_j - z_j \mathrm{d}z_i)$, $a = 1, 2, i, j = 1, 2, 3$. We have

$$\big[[F], w_a(z_i \mathrm{d}z_j - z_j \mathrm{d}z_i)\big] = 2\epsilon_{ijk}(w_1^{-1} w_2^{-1}) \cdot w_a \partial_{z_k} = 0$$

since the class $w_1^{-1} w_2^{-1}$ is in the kernel of the operator given by multiplication by w_a for $a = 1, 2$. Verifying that the remaining elements in the image of i_{M5} are in the kernel of $\big[[F], -\big]$ is similar. This completes the proof. □

Remark 3.3.7. As in remark 3.3.4 comment on an alternate method to compute the first page of the auxiliary spectral sequence we used to compute the first page of the spectral sequence converging to the cohomology with respect to $\delta^{(1)} + [NF, -]$. We could have used a Serre-type spectral sequence for certain kinds of sheaves on THF manifolds [KT75], [Kor14], applied to the pushforward $p_* \Pi \mathcal{E}^N_{AdS_7 \times S^4}$ from section 3.2.7. In this case, this Serre-type spectral sequence degenerates at the E_2-page. We will return to similarly flavored constructions in later work.

3.4 $E(1|6)$ modules from gravitons on $AdS_4 \times S^7$

Having justified that the spaces of supergravity states constructed in the previous subsection are in fact counting gravitons on $AdS_4 \times S^7$ and $AdS_7 \times S^4$ respectively, we turn to studying representation theoretic properties of these state spaces. In this section, we focus on the case of gravitons on $AdS_4 \times S^7$, using the description of the state space afforded by proposition 3.2.12 which describes it as the costalk of a factorization envelopes of the boundary conditions $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4 \times S^7})$.

We construct a certain $\mathbb{C}^\times$-action on the boundary fields $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4 \times S^7})$ equipped with the L_∞ structure from remark with the feature that the zeroth weight spaces are a local version of another exceptional linearly compact super-Lie algebras, $E(1|6)$. This in particular readily gives a decomposition of the state space $\mathcal{H}_{AdS_4 \times S^7}$ into $E(1|6)$-modules. We explicitly characterize the summands of this decomposition with their module structures and give closed form expressions for their characters.

3.4.1 The graviton decomposition of twisted $AdS_4 \times S^7$

3.4.1.1 We consider a particular decomposition of the space of states $\mathcal{H}_{AdS_4 \times S^7}$. It is induced by a decomposition of the boundary fields $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4 \times S^7})$ introduced in section 3.2.3 The decomposition is induced by a $\mathbb{C}^\times$ action on the boundary fieldsboundary fields $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4 \times S^7})$ that mixes fiberwise rescalings on spacetime with a fiberwise rescaling of the space of fields.

Explicitly, the action is given as follows

- On the fields

  $$\mu(t; w_a, z) \in \mathbb{C}[w_1, \cdots, w_4]\{\partial_{w_a}\} \otimes \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}(U) \oplus \mathbb{C}[w_1, \cdots, w_4] \otimes \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}(U, T)$$

  the action is
  $$\lambda \cdot \mu(t; w_a, z) = \mu(t; \lambda w_a, z).$$

- On the fields $\nu(t; w_a, z) \in \mathbb{C}[w_1, \cdots, w_4] \otimes \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}(U)$ the action is

  $$\lambda \cdot \nu(t; w_a, z) = \nu(t; \lambda w, z).$$

- On the fields $\beta(t; w_a, z) \in \mathbb{C}[w_1, \cdots, w_4] \otimes \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}(U)$ the action is

  $$\lambda \cdot \beta(t; w_a, z) = \lambda^{-2} \beta(t; \lambda w_a, z).$$

- On the fields

  $$\gamma(t; w_a, z) \in \mathbb{C}[w_1, \cdots, w_4]\{\mathrm{d}w_a\} \otimes \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}(U) \oplus \mathbb{C}[w_1, \cdots, w_4] \otimes \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}(U, \Omega^1)$$

  the action is
  $$\lambda \cdot \gamma(t; w_a, z) = \lambda^{-2} \gamma(t; \lambda w_a, z).$$

The following result is a straightforward if lengthy computation. We state it without proof.

Proposition 3.4.1. The L_∞ structure on $\Pi\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4 \times S^7})$ identified in section 3.2.3 is equivariant for this $\mathbb{C}^\times$ action.

This result induces a product decomposition

$$\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4 \times S^7}) = \prod_{n \geq 2} \mathcal{F}^{(n)}_{\mathbb{R} \times \mathbb{C}}$$

where for each open set $I \times U \subset \mathbb{R} \times \mathbb{C}$, we have that

$$\mathcal{F}^{(n)}_{\mathbb{R} \times \mathbb{C}}(I \times U) \subset \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}(U, \mathcal{L}^{r=0}_{AdS_4 \times S^7})$$

is the weight n eigenspace with respect to the above $\mathbb{C}^\times$ action. In particular, we see that $\mathcal{F}^{(0)}_{\mathbb{R} \times \mathbb{C}}$ is itself a local dg-Lie algebra, for which every $\mathcal{F}^{(n)}_{\mathbb{R} \times \mathbb{C}}$ is a module.

3.4.2 The lowest piece: the holomorphic-topological twist of the 3d $\mathcal{N} = 8$ BLG theory

3.4.2.1 The first nontrivial case is the weight (-2) piece. We have the following

Lemma 3.4.2. There is a quasi-isomorphism

$$\mathcal{F}^{(-2)}_{\mathbb{R}\times\mathbb{C}} \cong \Omega^\bullet_{\mathbb{R}\times\mathbb{C}}.$$

Proof. The only sections which contribute are those of type β or γ with no form components along the fiber directions. Therefore, we see directly that

$$\mathcal{F}^{(-2)}_{\mathbb{R}\times\mathbb{C}} \cong \Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{O} \xrightarrow{\partial} \Omega^1).$$

□

3.4.2.2 The next nontrivial case is the weight (-1) piece.

Lemma 3.4.3. There is a quasi-isomorphism

$$\mathcal{F}^{(-1)}_{\mathbb{R}\times\mathbb{C}} \cong \Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}} \left(K^{1/4} \otimes (\mathbb{C}^4)^* \oplus \Pi K^{3/4} \otimes \mathbb{C}^4\right)$$

Proof. On an open set of the form $I \times U$, the sections of the specified weight are:

- fields of type μ of the form $\mu_a(t;z)\partial_{w_a}$. As the w_a are fiber coordinates on $K^{1/4}_{\mathbb{C}}$, these fields transform as sections of $K^{1/4}_{\mathbb{C}}$.
- fields of type γ of the form $\gamma_a(t;z)\mathrm{d}w_a$. These fields transform as sections of $K^{3/4}_{\mathbb{C}}$.

□

3.4.2.3 We wish to flag an appearance of $\mathcal{F}^{(-1)}_{\mathbb{R}\times\mathbb{C}}$ in supersymmetric physics in three-dimensions. There is a highly supersymmetric Chern-Simons-matter theory discovered independently by Bagger-Lambert [BL07], [BL08] and Gustavsson [Gus09]. The aptly named BLG theory has $\mathcal{N} = 8$ superconformal symmetry, and admits a holomorphic-topological twist that was computed by Garner in [?].

The sheaf of complexes $\mathcal{F}^{(-1)}_{\mathbb{R}\times\mathbb{C}}$ matches the field contents of the holmorphic-topological twist of the BLG theory, and as such, it can be equipped with an L_∞ structure under which it is perturbatively equivalent to the twisted BLG theory. In work-in-progress with Garner and Williams, we show that the action of $\mathcal{F}^{(0)}_{\mathbb{R}\times\mathbb{C}}$ on $\mathcal{F}^{(-1)}_{\mathbb{R}\times\mathbb{C}}$ in fact preserves this L_∞-structure.

3.4.3 The zero-th piece: A local version of $E(1|6)$

3.4.3.1 The next nontrivial case is the weight (0) piece. This factor is special because it carries the induced structure of a local L_∞ algebra on $\mathbb{R} \times \mathbb{C}$. We will prove that it is equivalent to a local Lie algebra version of the exceptional super-Lie algebra $E(1|6)$.

We first recall the definition of this super-Lie algebra [Kac98]

Definition 3.4.4. Let $E(1|6)$ be the following super-Lie algebra.

- The even part of $E(1|6)_0$ given by the semidirect product Lie algebra $\Gamma(\hat{D}, T) \ltimes \left(\Gamma(\hat{D}, \mathcal{O}) \otimes \mathfrak{sl}(4)\right)$
- The odd part $E(1|6)_1$ is given by the (unique) nontrivial extension of $E(1|6)_0$-modules

$$0 \to \mathrm{Sym}^2(\mathbb{C}^4) \otimes K_{\mathbb{C}}^{1/2} \to E(1|6)_1 \to \wedge^2(\mathbb{C}^4) \otimes K_{\mathbb{C}}^{-1/2} \to 0.$$

The only remaining bracket to be specified, the odd bracket, is given as follows.

- Given sections $A \otimes f\mathrm{d}z^{1/2} \in \mathrm{Sym}^2(\mathbb{C}^4) \otimes K_{\mathbb{C}}^{1/2}$ and $B \otimes g\partial_z^{1/2} \in \wedge^2(\mathbb{C}^4) \otimes K_{\mathbb{C}}^{-1/2}$, we have that

$$[A \otimes f\mathrm{d}z^{1/2}, B \otimes g\partial_z^{1/2}] = A * B \otimes fg \in \mathfrak{sl}(4) \otimes \mathcal{O}.$$

  Here, $*$ refers to the hodge star of B and we are viewing A and $*B$ as symmetric and skew-symmetric 4×4 matrices respectively; their product is traceless.

- Given sections $A \otimes f\partial_z^{1/2}, B \otimes g\partial_z^{1/2} \in \Gamma(\hat{D}, \wedge^2(\mathbb{C}^4) \otimes K_{\mathbb{C}}^{1/2})$, we have that

$$\begin{aligned}[A \otimes f\mathrm{d}z^{-1/2}, B \otimes g\mathrm{d}z^{-1/2}] &= \mathrm{Tr}(A * B) \otimes fg\partial_z + \frac{1}{2}(A * B)_0 \otimes \left(\partial(f\mathrm{d}z^{-1/2})g\mathrm{d}z^{-1/2} + f\mathrm{d}z^{-1/2}\partial(g\mathrm{d}z^{-1/2})\right) \\ &\in \Gamma(\hat{D}, T) \ltimes \left(\mathfrak{sl}(4) \otimes \Gamma(\hat{D}, \mathcal{O})\right).\end{aligned}$$

  where again $*$ denotes the Hodge star and the subscript of zero denotes projection to the traceless part.

The relationship between this super-Lie algebra and our decomposition is established through the following result.

Proposition 3.4.5. There is an equivalence of super-Lie algebras

$$\mathcal{F}^{(0)}_{\mathbb{R}\times\mathbb{C},c}(0) \cong E(1|6).$$

Proof. We will begin by trying to characterize the local L_∞-algebra $\mathcal{F}^{(0)}_{\mathbb{R}\times\mathbb{C}}$. We claim that it is quasi-isomorphic to a local version of $E(1|6)$.

Indeed, it is easy to see that the weight zero sections consists of the following cochain complex

$$\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}} \left(\begin{array}{ccc} \underline{\textit{even}} & & \underline{\textit{odd}} \\ \mathbb{C}[w_a\partial_{w_b}] \otimes \mathcal{O} & \xrightarrow{\partial^W_\Omega} & \mathcal{O} \\ \mathrm{T} & \nearrow & \\ \mathrm{Sym}^2(\mathbb{C}^4) & \xrightarrow{\partial_W} & \mathbb{C}[w_a dw_b] \otimes K^{-1/2} \\ & \searrow & \mathrm{Sym}^2(\mathbb{C}^4) \otimes \Omega^1 \otimes K^{-1/2} \end{array} \right) \tag{3.35}$$

Of course, the differentials are just appropriate components of the divergence operator and holomorphic deRham operator. We can compute cohomology by way of a spectral sequence whose first page is the cohomology with respect to $\partial^W_\Omega + \partial_W$. We see that the differential ∂^W maps surjectively onto functions and its kernel is isomorphic to $\mathfrak{sl}(4) \otimes \mathcal{O}$. Likewise, the differential ∂_W is the canonical inclusion of $\mathfrak{sl}(4)$ representations $\mathrm{Sym}^2(\mathbb{C}^4) \to \mathbb{C}^4 \otimes \mathbb{C}^4$. Its cokernel is a copy of $\wedge^2\mathbb{C}^4$.

Thus, we see that this page of the spectral sequence is given by

$$\mathcal{E}(1|6) \overset{def}{=} \Omega^\bullet_\mathbb{R} \otimes \Omega^{0,\bullet}_\mathbb{C} \begin{pmatrix} \underline{even} & \underline{odd} \\ & \\ \mathrm{T} & \wedge^2(\mathbb{C}^4) \otimes K^{-1/2} \\ & \\ \mathfrak{sl}(4) \otimes \mathcal{O} & \mathrm{Sym}^2(\mathbb{C}^4) \otimes K^{1/2} \end{pmatrix} \tag{3.36}$$

and there are no non-zero differentials so the spectral sequence degenerates.

To see that the Lie structure induced from the L_∞-structure on $\Omega^\bullet_\mathbb{R} \otimes \Omega^{0,\bullet}_\mathbb{C}(\mathcal{L}^{r=0}_{AdS_4 \times S^7})$ is in fact given by the same formulae as the brackets on $E(1|6)$ in equation 3.4.4, it will be useful to provide an explicit quasi-isomorphism $\Psi^{(0)} : \mathcal{E}(1|6) \to \mathcal{F}^{(0)}_{\mathbb{R}\times\mathbb{C}}$. On an open set $I \times D \subset \mathbb{R} \times \mathbb{C}$, this is defined as follows

- Given a section $g(t;z)\partial_z \in \Omega^\bullet_\mathbb{R}(I) \otimes \Omega^{0,\bullet}_\mathbb{C}(D,T)$ where $g(t;z)$ is a mixed deRham-Dolbeault form on $I \times D$, we define

$$\begin{aligned} \Psi^{(0)}(g(t;z)\partial_z) &= g(t;z)\partial_z - \frac{1}{4}(\partial_z g(t;z)) w_a \partial_{w_a} \\ &\in \Omega^\bullet_\mathbb{R}(I) \otimes \Omega^{0,\bullet}_\mathbb{C}(D, T \oplus \mathbb{C}\{w_a \partial_{w_b}\}) \end{aligned}$$

- Given a section $A_{ab} \otimes g(t;z) \in \Omega^\bullet_\mathbb{R}(I) \otimes \Omega^{0,\bullet}_\mathbb{C}(D, \mathfrak{sl}(4) \otimes \mathcal{O})$ where $g(t;z)$ is a mixed deRham-Dolbeault form on $I \times D$ and $A_{ab} \in \mathfrak{sl}(4)$

$$\begin{aligned} \Psi^{(0)}(A_{ab} \otimes g(t;z)) &= g(t;z) A_{ab} w_a \partial_{w_b} \\ &\in \Omega^\bullet_\mathbb{R}(I) \otimes \Omega^{0,\bullet}_\mathbb{C}(D, \mathbb{C}\{w_a \partial_{w_b}\}) \end{aligned}$$

- Given a section $A_{ab} \otimes g(t;z)\mathrm{d}z^{-1/2} \in \Omega^\bullet_\mathbb{R}(I) \otimes \Omega^{0,\bullet}_\mathbb{C}\left(D, \wedge^2(\mathbb{C}^4) \otimes K^{-1/2}\right)$ where $g(t;z)$ is a mixed deRham-Dolbeault form on $I \times D$ and $A_{ab} \in \wedge^2(\mathbb{C}^4)$ we define

$$\begin{aligned} \Psi^{(0)}(A_{ab} \otimes g(t;z)\mathrm{d}z^{-1/2}) &= g(t;z) A_{ab} w_a \partial_{w_b} \\ &\in \Omega^\bullet_\mathbb{R}(I) \otimes \Omega^{0,\bullet}_\mathbb{C}\left(D, \mathbb{C}\{w_a \partial_{w_b}\} \otimes K^{-1/2}\right) \end{aligned}$$

- Given a section $A_{ab} \otimes g(t;z)\mathrm{d}z^{1/2} \in \Omega^\bullet_\mathbb{R}(I) \otimes \Omega^{0,\bullet}_\mathbb{C}\left(D, \mathrm{Sym}^2(\mathbb{C}^4) \otimes K^{1/2}\right)$ where $g(t;z)$ is a mixed deRham-Dolbeault form on $I \times D$ and $A_{ab} \in \mathrm{Sym}^2(\mathbb{C}^4)$ we define

$$\begin{aligned} \Psi^{(0)}(A_{ab} \otimes g(t;z)\mathrm{d}z^{1/2}) &= g(t;z) A_{ab} w_a w_b \mathrm{d}z \\ &\in \Omega^\bullet_\mathbb{R}(I) \otimes \Omega^{0,\bullet}_\mathbb{C}\left(D, \mathrm{Sym}^2(\mathbb{C}^4) \otimes \Omega^1 \otimes K^{-1/2}\right) \end{aligned}$$

It is easy to see that $\Psi^{(0)}$ is a quasi-isomorphism and a straightforward if lengthy check confirms that it preserves Lie brackets. The result then follows from computing the limit of $\mathcal{E}(1|6)_c(I \times D)$ over open sets containing the origin. □

Remark 3.4.6. We note that the map i_{M2} from lemma 3.3.2 in fact defines a Lie map from $\mathfrak{osp}(6|2)$ to the sections of the boundary condition $\Pi\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4 \times S^7})$ over every open set containing the origin. The image of the map lands exactly in the step $\mathcal{F}^{(0)}_{\mathbb{R}\times\mathbb{C}}$ of the decomposition from proposition 3.4.1. Therefore we see that $E(1|6)$ contains $\mathfrak{osp}(6|2)$ as a finite dimensional subalgebra.

3.4.4 General summands and $E(1|6)$-modules

We now move on to giving an explicit description of the general summand $\mathcal{F}^{(j)}$ for $j \geq 1$.

We first fix some notation for irreducible highest weight representations of $\mathfrak{sl}(4)$. Let $\mathfrak{h} \subset \mathfrak{sl}$ be the Cartan given by diagonal matrices and let $L_i \in \mathfrak{h}^*$ be the linear functional that picks out the i-th diagonal entry. We may accordingly write $\mathfrak{h}^* = \mathbb{C}\{L_1, L_2, L_3, L_4\}/(L_1 + \cdots + L_4)$. We will write Γ_{a_1,a_2,a_3} for the irreducible representation of $\mathfrak{sl}(4)$ of highest weight $(a_1+a_2+a_3)L_1 + (a_2+a_3)L_2 + a_3L_3$.

Proposition 3.4.7. Let $j \geq 1$. The complex of vector bundles $\mathcal{F}^{(j)}_{\mathbb{R}\times\mathbb{C}}$ is quasi-isomorphic to

$$\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}} \begin{pmatrix} \underline{\textit{even}} & \underline{\textit{odd}} \\ \Gamma_{j,1,0} \otimes K^{-j/4} & \mathrm{Sym}^{j+2}(\mathbb{C}^4) \otimes \Omega^1 \otimes K^{-(j+2)/4} \\ \mathrm{Sym}^j(\mathbb{C}^4) \otimes \mathrm{T} \otimes K^{-j/4} & \Gamma_{j+1,0,1} \otimes K^{-(j+2)/4} \end{pmatrix} \tag{3.37}$$

Proof. We begin by noting that we can explicitly describe $\mathcal{F}^{(j)}_{\mathbb{R}\times\mathbb{C}}$ as $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(F^{(j)})$ where $F^{(j)}$ denotes the following dg-vector bundle:

$$\begin{array}{ccc} \underline{\textit{even}} & & \underline{\textit{odd}} \\ \mathrm{Sym}^{j+1}(\mathbb{C}^4) \otimes (\mathbb{C}^4)^* \otimes K^{-j/4} & \xrightarrow{\partial^W_\Omega} & \\ & & \mathrm{Sym}^j(\mathbb{C}^4) \otimes K^{-j/4} \\ \mathrm{Sym}^j(\mathbb{C}^4) \otimes \mathrm{T} \otimes K^{-j/4} & \xrightarrow{\partial^V_\Omega} & \\ & & \mathrm{Sym}^{j+2}(\mathbb{C}^4) \otimes \mathrm{T}^* \otimes K^{-(j+2)/4} \\ \mathrm{Sym}^{j+2}(\mathbb{C}^4) \otimes K^{-(j+2)/4} & \begin{array}{c}\xrightarrow{\partial_V} \\ \xrightarrow{\partial_W}\end{array} & \\ & & K^{-(j+2)/4} \otimes \mathrm{Sym}^{j+1}(\mathbb{C}^4) \otimes \mathbb{C}^4. \end{array} \tag{3.38}$$

Note that the differentials here are all $\mathfrak{sl}(4)$ equivariant maps, tensored with a differential operator acting on sections of a bundle on $\mathbb{C}$. In particular

- The differential ∂_Ω^W involves the canonical projection

$$\mathrm{Sym}^{j+1}(\mathbb{C}^4) \otimes (\mathbb{C}^4)^* \to \mathrm{Sym}^j(\mathbb{C}^4).$$

  Its kernel is precisely the irreducible highest weight representation $\Gamma_{j+1,0,1}$.

- The differential ∂_W is the canonical inclusion

$$\mathrm{Sym}^{j+2}(\mathbb{C}^4) \to \mathrm{Sym}^{j+1}(\mathbb{C}^4) \otimes \mathbb{C}^4.$$

  Its cokernel is the irreducible highest weight representation $\Gamma_{j,1,0}$.

We can compute the cohomology using a spectral sequence whose first page is given by the cohomology with respect to $\partial_\Omega^W + \partial_W$. There are no further differentials on this page so the result follows. □

3.4.5 Characters of $E(1|6)$-modules

Note that the decomposition of the state space $\mathrm{Sym}(\mathcal{H}_{AdS_4 \times S^7}) = \prod_{j \geq -2} \mathcal{U}(\mathcal{F}^{(j)}_{\mathbb{R}\times\mathbb{C}})(0)$ gives a product formula for the characters computed in proposition 3.6.4.4

$$\chi\left(\mathrm{Sym}(\mathcal{H}_{AdS_4 \times S^7})\right) = \prod_{j \geq -2} \chi\left(\mathcal{U}(\mathcal{F}^{(j)}_{\mathbb{R}\times\mathbb{C}})(0)\right).$$

We end this section by computing each of the characters $\chi\left(\mathcal{U}(\mathcal{F}^{(j)}_{\mathbb{R}\times\mathbb{C}})(0)\right)$. We will express our characters in terms of characters of highest weight respresentations of $\mathfrak{sl}(4)$ which we denote $\chi^{\mathfrak{sl}(4)}(\Gamma_{a_1,a_2,a_3})$.

3.4.5.1 From the characterization in 3.6.4.4, the lowest step of the decomposition $\mathcal{F}^{(-2)}$ is just given by the deRham complex on $\mathbb{R} \times \mathbb{C}$, and accordingly the character of $\mathcal{F}^{(-2)}_c(0)$ is the constant function 1.

3.4.5.2 We proceed to the next step of the decomposition, using the characterization in 3.6.4.4.

Proposition 3.4.8. The character $\chi\left(\mathcal{U}(\mathcal{F}^{(-1)}_{\mathbb{R}\times\mathbb{C}})(0)\right)$ is given by the plethystic exponential of the following expression:

$$f_{-1}(t_1,t_2,t_3,q) = \frac{q\left(q^{-3/4}(t_1+t_2+t_3+t_1^{-1}t_2^{-1}t_3^{-1}) - q^{-1/4}(t_1^{-1}+t_2^{-1}+t_3^{-1}+t_1t_2t_3)\right)}{(1-q)} \tag{3.39}$$

Proof. The proof proceeds by the same trick as in the proof of proposition 3.2.12. To describe the costalk, we wish to compute a limit of sections of $\mathcal{F}^{(-1)}_{\mathbb{R}\times\mathbb{C},c}$ on open sets of the form $I \times D$ containing the origin in $\mathbb{R} \times \mathbb{C}$. Using ellipticity, we can describe such sections as a module over the ring generated by holomorphic derivatives of the delta function.

Accordingly, we have contributions from the following summands:

- An even copy of $\mathbb{C}^4 \otimes \mathbb{C}\{\mathrm{d}z^{3/4}\} \otimes \mathbb{C}[\partial_z]\delta_{z=0}$. The character of this summand is

$$\frac{q\left(q^{-3/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{1,0,0})\right)}{(1-q)} = \frac{q\left(q^{-3/4}(t_1+t_2+t_3+t_1^{-1}t_2^{-1}t_3^{-1})\right)}{(1-q)}$$

- An odd copy of $\mathbb{C}^4 \otimes \mathbb{C}\{\mathrm{d}z^{1/4}\} \otimes \mathbb{C}[\partial_z]\delta_{z=0}$. The character of this summand is

$$\frac{-q\left(q^{-1/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{0,0,1})\right)}{(1-q)} = \frac{-q\left(q^{-3/4}(t_1^{-1}+t_2^{-1}+t_3^{-1}+t_1t_2t_3)\right)}{(1-q)}$$

□

Note that under the change of fugacities in 3.6.4.4, this matches exactly with the single particle index for the theory on a single M2 brane [BBMR08, Eq. (2.32)].

3.4.5.3 We continue to the next step of the decomposition given by $\mathcal{F}^{(0)}_{\mathbb{R}\times\mathbb{C}}$.

Arguing similarly as in the proof of the previous proposition, we have the following.

Proposition 3.4.9. The character $\chi\left(\mathcal{U}(\mathcal{F}^{(0)}_{\mathbb{R}\times\mathbb{C}})(0)\right)$ is given by the plethystic exponential of the following expression:

$$f_0(t_1,t_2,t_3,q) = \frac{q}{(1-q)}\left(q^{1/2}\chi^{\mathfrak{sl}(4)}(\Gamma_{0,1,0}) + q^{-1/2}\chi^{\mathfrak{sl}(4)}(\Gamma_{2,0,0}) - q - \chi^{\mathfrak{sl}(4)}(\Gamma_{1,0,1})\right) \tag{3.40}$$

3.4.5.4 Finally, we continue to the general step of the decomposition.

Proposition 3.4.10. Let $j \geq 1$. The character $\chi\left(\mathcal{U}(\mathcal{F}^{(j)}_{\mathbb{R}\times\mathbb{C}})(0)\right)$ is the plethystic exponential of the following expression:

$$f_j(t_1,t_2,t_3,q) = \frac{q}{(1-q)}\begin{pmatrix} q^{(j-2)/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{j+2,0,0}) + q^{(j+2)/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{j+1,0,1}) \\ -q^{j/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{j,1,0}) - q^{(j+1)/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{j,0,0}) \end{pmatrix} \tag{3.41}$$

3.4.5.5 As a consequence, of the above we have that $f_{AdS_4\times S^7}(t_i,q) = \sum_{j\geq -2} f_j(t_i,q)$, or explicitly:

$$\begin{aligned} &\frac{q\begin{pmatrix} q^{1/4}(t_1+t_2+t_3+t_1^{-1}t_2^{-1}t_3^{-1}) + q^{-1} \\ -q^{-1/4}(t_1^{-1}+t_2^{-1}+t_3^{-1}+t_1t_2t_3) - q \end{pmatrix}}{(1-q)(1-q^{1/4}t_1)(1-q^{1/4}t_2)(1-q^{1/4}t_3)(1-q^{1/4}t_1^{-1}t_2^{-1}t_3^{-1})} \\ &= 1 + \frac{q}{1-q}\begin{pmatrix} q^{-3/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{1,0,0}) - q^{-1/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{0,0,1}) \\ +q^{1/2}\chi^{\mathfrak{sl}(4)}(\Gamma_{0,1,0}) + q^{-1/2}\chi^{\mathfrak{sl}(4)}(\Gamma_{2,0,0}) - q - \chi^{\mathfrak{sl}(4)}(\Gamma_{1,0,1}) \end{pmatrix} \\ &+ \frac{q}{1-q}\sum_{j\geq 1}\begin{pmatrix} q^{(j-2)/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{j+2,0,0}) + q^{(j+2)/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{j+1,0,1}) \\ -q^{j/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{j,1,0}) - q^{(j+1)/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{j,0,0}) \end{pmatrix} \end{aligned}$$

In [BBMR08, Eq. (2.15, 2.16)], the index counting gravitons on $f_{AdS_4\times S^7}$ is expressed as a sum of characters of irreducible representations of the 3d $\mathcal{N}=8$ superconformal algebra that the authors call *graviton representations*. Comparison with the above expansion suggests the following conjecture

Conjecture 3.4.11. For $j \geq -1$, the minimal twist of the $j+2$nd graviton representation in [BBMR08, Eq. (2.15, 2.16)] is exactly $\mathcal{F}^{(j)}_{\mathbb{R}\times\mathbb{C},c}(0)$.

Remark 3.4.12. This conjecture implies that the minimal twist of these graviton representations, which is a priori a module for the minimally twisted 3d $\mathcal{N}=8$ superconformal algebra $\mathfrak{osp}(6|2)$, is in fact a module for the larger infinite dimensional super-Lie algebra $E(1|6)$. This can be thought of as analogous to the enhancement of conformal symmetries to the action of the Witt algebra of vector fields in 2d chiral conformal field theory. Such symmetry enhancements in 3 dimensions is the topic of joint work in progress with Garner and Williams.

3.5 $E(3|6)$-modules from gravitons on $AdS_7 \times S^4$

We now repeat the analysis of the previous subsection for gravitons on $AdS_7 \times S^4$ respectively, making use of the description of supergravity states on $AdS_7 \times S^4$ as the costalk of the factorization envelopes of the boundary condition and $\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$

As before, we construct certain $\mathbb{C}^\times$ actions on the boundary fields $\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$; we find that the zeroth weight space is a local version of another exceptional linearly compact super-Lie algebra $E(3|6)$. The decomposition of $\mathcal{H}_{AdS_7\times S^4}$ as a direct sum of $E(3|6)$ modules incidentally turns out to be very closely related to a decomposition of $E(5|10)$ into $E(3|6)$ modules studied by Cheng-Kac [].

3.5.1 The graviton decomposition of twisted $AdS_7 \times S^4$

We wish to consider a particular decomposition of the space of states $\mathcal{H}_{AdS_7\times S^4}$. It is induced by a decomposition of the boundary fields $\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$ introduced in section 3.2.3.

Let $U \subset \mathbb{C}^3$ be an open; explicitly, the $\mathbb{C}^\times$ action on $\Omega^{0,\bullet}_{\mathbb{C}^3}(U, \mathcal{L}^{r=0}_{AdS_7\times S^4})$ is given as follows.

- On the fields $\mu(w_a, z_i) \in \mathbb{C}[w_1, w_2]\{\partial_{w_a}\} \otimes \Omega^{0,\bullet}_{\mathbb{C}^3}(U) \oplus \mathbb{C}[w_1,w_2] \otimes \Omega^{0,\bullet}_{\mathbb{C}^3}(U, T)$ the action is

$$\lambda \cdot \mu(w_a, z_i) = \mu(\lambda w_a, z_i).$$

- On the fields $\nu(w_a, z_i) \in \mathbb{C}[w_1, w_2] \otimes \Omega^{0,\bullet}_{\mathbb{C}^3}(U)$ the action is

$$\lambda \cdot \nu(w_a, z_i) = \nu(\lambda w, z).$$

- On the fields $\beta(w_a, z_i) \in \mathbb{C}[w_1, w_2] \otimes \Omega^{0,\bullet}_{\mathbb{C}^3}(U)$ the action is

$$\lambda \cdot \beta(w_a, z_i) = \lambda^{-1}\beta(\lambda w_a, z_i).$$

- On the fields $\gamma(w_a, z_i) \in \mathbb{C}[w_1, w_2]\{\mathrm{d}w_a\} \otimes \Omega^{0,\bullet}_{\mathbb{C}^3}(U) \oplus \mathbb{C}[w_1,w_2] \otimes \Omega^{0,\bullet}_{\mathbb{C}^3}(U, \Omega^1)$ the action is

$$\lambda \cdot \gamma(w_a, z_i) = \lambda^{-1}\gamma(\lambda w_a, z_i).$$

The following proposition is a straightforward if lengthy computation - we state it without proof.

Proposition 3.5.1. The L_∞ structure on $\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$ identified in remark 3.6.4.4 is equivariant for this $\mathbb{C}^\times$ action.

This result induces a product decomposition

$$\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4}) = \prod_{n\geq -1} \mathcal{G}^{(n)}_{\mathbb{C}^3} \tag{3.42}$$

where for each open set $U \subset \mathbb{C}^3$

$$\mathcal{G}^3_{\mathbb{C}}(U)^{(n)} \subset \Omega^{0,\bullet}_{\mathbb{C}^3}(U, \mathcal{L}^{r=0}_{AdS_7\times S^4})$$

is the weight n eigenspace with respect to the above $\mathbb{C}^\times$ action. In particular, we see that $\mathcal{G}^{(0)}_{\mathbb{C}^3}$ is itself a local dg-Lie algebra (that we will soon describe). Moreover, every $\mathcal{G}^{(n)}_{\mathbb{C}^3}$, $n \geq -1$ is a (local) module for this local dg-Lie algebra.

3.5.2 The lowest piece: the holomorphic twist of the abelian 6d $\mathcal{N} = (2,0)$ tensor multiplet

The first non trivial case is the weight (-1) piece.

Lemma 3.5.2. There is an equivalence of abelian local Lie algebras

$$\mathcal{G}^{(-1)}_{\mathbb{C}^3} \cong \Omega^{0,\bullet}_{\mathbb{C}^3}\left(\begin{array}{cc} + & - \\ \\ \mathbb{C}^2 \otimes K^{1/2}_{\mathbb{C}^3} & \\ \\ \mathcal{O}_{\mathbb{C}^3} & \xrightarrow{\partial} \Omega^1_{\mathbb{C}^3} \end{array}\right)$$

Proof. We readily see that the fields of weight -1 include

- fields of type μ of the form $\mu_a(z_i)\partial_{w_a}$. As w_a are fiber coordinates on $K^{1/2}_{\mathbb{C}^3}$, these fields transform as sections of $K^{1/2}_{\mathbb{C}^3}$.
- fields of type β with no w_a-dependence. These fields constitute a copy of $\mathcal{O}_{\mathbb{C}^3}$.
- fields of type γ of the form $\gamma_i(z_i)\mathrm{d}z_i$. These fields constitute a copy of $\Pi\Omega^1_{\mathbb{C}^3}$.

Since ∂ is weight zero for this $\mathbb{C}^\times$ action, the fields of the last two type combine to give the complex of sheaves

$$\mathcal{O}_{\mathbb{C}^3} \xrightarrow{\partial} \Pi\Omega^1_{\mathbb{C}^3}.$$

□

3.5.2.1 In [SW20] Saberi and Williams, the authors studied the minimal twist of the 6d $\mathcal{N} = (2,0)$ abelian tensor multiplet. The twist is a free theory and can be defined on any complex three-fold admitting a square root of its canonical bundle. On $\mathbb{C}^3$, the $\mathbb{Z}\times\mathbb{Z}/2$ graded sheaf of complexes $\mathcal{E}_{tens}$ encoding its field content is given by

$$\underline{-1} \qquad\qquad \underline{0}$$

$$\Pi\mathbb{C}^2 \otimes \Omega^{0,\bullet}_{\mathbb{C}^3} \otimes K^{1/2}_{\mathbb{C}^3} \tag{3.43}$$

$$\Omega^{2,\bullet}_{\mathbb{C}^3} \xrightarrow{\partial} \Omega^{3,\bullet}_{\mathbb{C}^3}$$

Here we recall in the $\mathbb{Z} \times \mathbb{Z}/2$ bigrading the differential has bidegree $(1, 0)$.

We observe the following:

Proposition 3.5.3. There is a quasi-isomorphism of factorization algebras valued in $\mathbb{Z}/2$ graded commutative dg algebras on $\mathbb{C}^3$

$$\mathcal{U}(\mathcal{G}^{(-1)}_{\mathbb{C}^3}) \xrightarrow{\simeq} \mathrm{C}^\bullet(\Pi\mathcal{E}_{tens})$$

Proof. Recall that the factorization algebra $\mathcal{U}(\mathcal{G}^{(-1)}_{\mathbb{C}^3})$ assigns to an open set $U \subset \mathbb{C}^3$ the graded symmetric algebra on the complex

$$\underline{-} \qquad\qquad \underline{\pm}$$

$$\Omega^{0,\bullet}_{\mathbb{C}^3,c}(U) \xrightarrow{\partial} \Omega^{1,\bullet}_{\mathbb{C}^3,c}(U) \tag{3.44}$$

$$\Omega^{0,\bullet}_{\mathbb{C}^3,c}(U, \mathbb{C}^2 \otimes K^{1/2})$$

On the other hand, if we totalize the $\mathbb{Z} \times \mathbb{Z}/2$-grading on $\mathcal{E}_{tens}$ to a $\mathbb{Z}/2$-grading, the factorization algebra $\mathrm{C}^\bullet(\Pi\mathcal{E}_{tens})$ assigns to an open set $U \subset \mathbb{C}^3$ the symmetric algebra on the complex

$$\underline{-} \qquad\qquad \underline{\pm}$$

$$\Omega^{2,\bullet}_{\mathbb{C}^3}(U)^\vee \xrightarrow{\partial} \Omega^{3,\bullet}_{\mathbb{C}^3}(U)^\vee \tag{3.45}$$

$$\Omega^{0,\bullet}_{\mathbb{C}^3}(U, \mathbb{C}^2 \otimes K^{1/2})^\vee$$

Here the superscript refers to the topological dual, which is described in terms of compactly supported distributional sections of the Serre dual vector bundle. Thus, we see that the above complex is the same as

$$\underline{-} \qquad\qquad \underline{\pm}$$

$$\overline{\Omega}^{0,\bullet}_{\mathbb{C}^3,c}(U) \xrightarrow{\partial} \overline{\Omega}^{1,\bullet}_{\mathbb{C}^3,c}(U) \tag{3.46}$$

$$\overline{\Omega}^{0,\bullet}_{\mathbb{C}^3,c}(U, \mathbb{C}^2 \otimes K^{1/2})$$

where the degree shift is coming from Serre duality. The result then follows from the fact that by ellipticity, the natural inclusion $\Omega^{0,\bullet}_{\mathbb{C}^3,c} \to \overline{\Omega}^{0,\bullet}_{\mathbb{C}^3,c}$ is a quasi-isomorphism. □

3.5.3 The zero-th piece: a local version of $E(3|6)$

As before, the weight zero summand $\mathcal{G}^{(0)}_{\mathbb{C}^3}$ is special because it carries the induced structure of a local L_∞-algebra on $\mathbb{C}^3$ inherited from the L_∞ structure on $\Pi\Omega^{0,\bullet}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$ identified in section 3.2.3. We will prove that it is equivalent to a local Lie algebra version of the exceptional super-Lie algebra $E(3|6)$ [Kac98].

We first recall the definition of this super-Lie algebra.

Definition 3.5.4. Let $E(3|6)$ be the following super-Lie algebra.

- The even part, $E(3|6)_0$ is given by the semidirect product Lie algebra $\Gamma(\widehat{D},T)\ltimes\Big(\mathfrak{sl}(2)\otimes\Gamma(\widehat{D},\mathcal{O})\Big)$.
- The odd part, $E(3|6)_1$ is given by $\mathbb{C}^2\otimes\Gamma(\widehat{D},\Omega^1(K^{-1/2}))$.

The remaining brackets to be specified, are as follows:

- The action of $E(3|6)_0$ on $E(3|6)_1$ is given by the Lie derivative, along with the fundamental action of $\mathfrak{sl}(2)$.
- The bracket between two odd elements is given by

$$\begin{aligned}[v_1\otimes f_i\mathrm{d}z_i\otimes(\partial_{z_1}\partial_{z_2}\partial_{z_3})^{1/2}, v_2\otimes g_j\mathrm{d}z_j\otimes(\partial_{z_1}\partial_{z_2}\partial_{z_3})^{1/2}]&\\ =\omega(v_1,v_2)\varepsilon^{ijk}f_ig_j\partial_{z_k}&\\ +(v_1\odot v_2)\left(\partial(f_i\mathrm{d}z_i)g_j\mathrm{d}z_j - f_i\mathrm{d}z_i\partial(g_j\mathrm{d}z_j)\right)\vee(\partial_{z_1}\partial_{z_2}\partial_{z_3})&\end{aligned}$$

  where ω denotes a symplectic form on $\mathbb{C}^2$ and $\odot:\mathbb{C}^2\otimes\mathbb{C}^2\to\mathfrak{sl}(2)$ is the canonical $\mathfrak{sl}(2)$-equivariant projection.

The relationship between this super-Lie algebra and our decomposition is established through the following result.

Proposition 3.5.5. There is an equivalence of super-Lie algebras

$$\mathcal{G}^{(0)}_{\mathbb{C}^3,c}(0)\cong E(3|6)$$

Proof. We begin by characterizing the local L_∞-algebra $\mathcal{G}^{(0)}_{\mathbb{C}^3}$. We claim that it is quasi-isomorphic to a local version of $E(3|6)$.

Indeed, we readily see that the weight zero sections consist of the following cochain complex

$$\Omega^{0,\bullet}_{\mathbb{C}^3}\left(\begin{array}{ccc} \underline{\mathit{even}} & & \underline{\mathit{odd}} \\ \mathbb{C}\{w_a\partial_{w_b}\}\otimes\mathcal{O} & \xrightarrow{\partial^W_\Omega} & \mathcal{O} \\ & \nearrow_{\partial^Z_\Omega} & \\ \mathrm{T} & & \\ \mathbb{C}^2\otimes\mathcal{O} & \xrightarrow{\partial_W} & \mathbb{C}\{\mathrm{d}w_a\}\otimes K^{-1/2} \\ & \searrow^{\partial_Z} & \\ & & \mathbb{C}^2\otimes\Omega^1\otimes K^{-1/2} \end{array}\right) \tag{3.47}$$

The differentials are again components of the divergence operator and holomorphic deRham operator. We can compute cohomology by way of a spectral sequence whose first page is the cohomology with respect to $\partial^W_\Omega + \partial_W$. We see that the differential ∂^W_Ω maps surjectively onto functions and its kernel is isomorphic to $\mathfrak{sl}(2)\otimes\mathcal{O}$. Likewise, the differential ∂_W is just the identity map between $\mathbb{C}^2$ and $\mathbb{C}\{\mathrm{d}w_a\}$.

Thus we see that this page of the spectral sequence is given by

$$\mathcal{E}(3|6) \stackrel{def}{=} \Omega^{0,\bullet}_{\mathbb{C}^3}\left(\begin{array}{cc} \underline{\mathit{even}} & \underline{\mathit{odd}} \\ \mathrm{T} & \mathbb{C}^2\otimes\Omega^1_{\mathbb{C}^3}(K^{-1/2}) \\ \mathfrak{sl}(2)\otimes\mathcal{O} & \end{array}\right) \tag{3.48}$$

and there are no non-zero differentials so the spectral sequence degenerates.

To see that the Lie structure induced from the L_∞-structure on $\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$ is in fact given by the same formulae as the brackets on $E(3|6)$ given in 3.5.4, it will be useful to provide an explicit quasi-isomorphism $\Phi^{(0)} : \mathcal{E}(3|6) \to \mathcal{G}^{(0)}_{\mathbb{C}^3}$. On an open set $U \subset \mathbb{C}^3$, this is defined as follows.

- Given a section $g_i(z)\partial_{z_i} \in \Omega^{0,\bullet}_{\mathbb{C}^3}(U,T)$ where $g_i(z)$ is a Dolbeault form on U, we define

$$\begin{aligned}\Phi^{(0)}(g_i(z)\partial_{z_i}) &= g_i(z)\partial_{z_i} - \frac{1}{2}(\partial_{z_i}g_i(z))w_a\partial_{w_a} \\ &\in \Omega^{0,\bullet}_{\mathbb{C}^3}\left(U, T\oplus\mathbb{C}\{w_a\partial_{w_b}\}\otimes\mathcal{O}\right).\end{aligned}$$

- Given a section $A\otimes g(z) \in \Omega^{0,\bullet}_{\mathbb{C}^3}(U,\mathfrak{sl}(2)\otimes\mathcal{O})$ where $g(z)$ is a Dolbeault form on U, and $A_{ab}\in\mathfrak{sl}(2)$ we define

$$\begin{aligned}\Phi^{(0)}(A_{ab}\otimes g(z)) &= g(z)A_{ab}w_a\partial_{w_b} \\ &\in \Omega^{0,\bullet}_{\mathbb{C}^3}\left(\mathbb{C}\{w_a\partial_{w_b}\}\otimes\mathcal{O}\right).\end{aligned}$$

- Given a section $v\otimes g_i(z)\mathrm{d}z_i(\partial_{z_1}\partial_{z_2}\partial_{z_3}) \in \Omega^{0,\bullet}_{\mathbb{C}^3}(U,\mathbb{C}^2\otimes\Omega^1\otimes K^{-1/2})$ where $g_i(z)$ is a Dolbeault form on U and $v \in \mathbb{C}^2$, we define

$$\Phi^{(0)}(v \otimes g_i(z) \mathrm{d}z_i(\partial_{z_1}\partial_{z_2}\partial_{z_3})) = (w_1(v)w_1 + w_2(v)w_2) \otimes g_i(z) \\ \in \Omega^{0,\bullet}_{\mathbb{C}^3}\left(\mathbb{C}^2 \otimes \Omega^1 \otimes K^{-1/2}\right).$$

The result then follows from computing the limit of $\mathcal{E}(3|6)_c(D^3)$ over open sets containing the origin. □

Remark 3.5.6. We note that the map i_{M5} from lemma 3.3.5 in fact defines a Lie map from $\mathfrak{osp}(6|2)$ to the sections of the boundary condition $\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$ over every open set containing the origin. The image of the map lands exactly in the step $\mathcal{G}^{(0)}_{\mathbb{C}^3}$ of the decomposition from proposition 3.5.1. Therefore we see that $E(3|6)$ contains $\mathfrak{osp}(6|2)$ as a finite dimensional subalgebra.

3.5.4 General summands and $E(3|6)$-modules

We move on to give the following general description of the weight j component $\mathcal{G}^{(j)}_{\mathbb{C}^3}$. Since we have already described $j = -1, 0$ we focus on $j \geq 1$.

Proposition 3.5.7. Let $j \geq 1$. The complex of vector bundles $\mathcal{G}^{(j)}$ is quasi-isomorphic to

$$\Omega^{0,\bullet}_{\mathbb{C}^3}\begin{pmatrix} \underline{\mathit{even}} & \underline{\mathit{odd}} \\ \mathrm{Sym}^j(\mathbb{C}^2)\otimes \mathrm{T}\otimes K^{-j/2} & \mathrm{Sym}^{j-1}(\mathbb{C}^2)\otimes K^{-(j+1)/2} \\ \mathrm{Sym}^{j+2}(\mathbb{C}^2)\otimes K^{-j/2} & \mathrm{Sym}^{j+1}(\mathbb{C}^2)\otimes \mathrm{T}^*\otimes K^{-(j+1)/2}. \end{pmatrix} \tag{3.49}$$

Proof. We begin by noting that we can explicitly describe the weight j component $\mathcal{G}^{(j)}_{\mathbb{C}^3}$ as $\Omega^{0,\bullet}_{\mathbb{C}^3}(G^{(j)})$

where $G^{(j)}$ is the following dg-vector bundle

$$
\begin{array}{ccc}
\underline{\textit{even}} & & \underline{\textit{odd}} \\
\mathrm{Sym}^{j+1}(\mathbb{C}^2)\otimes\mathbb{C}^2\otimes K^{-j/2} & & \\
 & \xrightarrow{\partial_\Omega^W} & \mathrm{Sym}^{j}(\mathbb{C}^2)\otimes K^{-j/2} \\
\mathrm{Sym}^{j}(\mathbb{C}^2)\otimes \mathrm{T}\otimes K^{-j/2} & \xrightarrow{\partial_\Omega^Z} & \\
 & & \\
 & & \mathrm{Sym}^{j+1}(\mathbb{C}^2)\otimes\Omega^1\otimes K^{-(j+1)/2} \\
 & \xrightarrow{\partial_Z} & \\
\mathrm{Sym}^{j+1}(\mathbb{C}^2)\otimes K^{-(j+1)/2} & & \\
 & \xrightarrow{\partial_W} & \\
 & & \mathrm{Sym}^{j}(\mathbb{C}^2)\otimes\mathbb{C}^2\otimes K^{-(j+1)/2}
\end{array}
\tag{3.50}
$$

Note that the differentials here are $\mathfrak{sl}(2)$-equivariant maps, tensored with a differential operator acting on sections of a bundle on $\mathbb{C}^3$. In particular

- The differential ∂_Ω^W is the canonical projection

$$\mathrm{Sym}^{j+1}(\mathbb{C}^2)\otimes\mathbb{C}^2 \cong \mathrm{Sym}^{j+2}(\mathbb{C}^2)\oplus\mathrm{Sym}^{j}(\mathbb{C}^2) \twoheadrightarrow \mathrm{Sym}^{j}(\mathbb{C}^2)$$

  tensored with the identity acting on $K^{-j/2}$.

- The differential ∂_W is the canonical inclusion

$$\mathrm{Sym}^{j+1}(\mathbb{C}^2) \hookrightarrow \mathrm{Sym}^{j-1}(\mathbb{C}^2)\oplus\mathrm{Sym}^{j+1}(\mathbb{C}^2) \cong \mathrm{Sym}^{j}(\mathbb{C}^2)\otimes\mathbb{C}^2.$$

  tensored with the identity acting on $K^{-(j+1)/2}$.

There is a spectral sequence whose first term is computed by the $\partial_\Omega^W + \partial_W$-cohomology. The result is the complex of sheaves in equation 3.49. There are no further differentials so the spectral sequence collapses at this page and the result follows. $\square$

3.5.4.1 The decomposition of $\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$ in equation (3.42) is closely related to a decomposition of the exceptional simple super Lie algebra $E(5|10)$ studied in [KR01]. In 2.2.1, we showed that the global sections of the parity shifted fields of our eleven-dimensional theory on flat space is quasi-isomorphic to a Lie 2-extension of $E(5|10)$, which we denoted $\widehat{E(5|10)}$. More precisely, we found a Lie 2-extension of a version of $E(5|10)$ built out of polynomials rather than Taylor series. Given that the L_∞ structure on $\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$ is given by the same formulas as that on $\Pi\mathcal{E}$, it is easy to see that the space of ∞-jets of $\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$ at the origin, which following lemma 3.6.4.4 has underlying vector space $\mathcal{H}_{AdS_7\times S^4}$, is quasi-isomorphic to $\widehat{E(5|10)}$.

In [KR01] the following weight decomposition of $E(5|10)$ is constructed. Splitting $\mathbb{C}^5 = \mathbb{C}^2_{w_a} \times \mathbb{C}^3_{z_i}$ as we have been doing, we stipulate that

- the coordinate z_i has weight zero, $\widetilde{(}z_i) = 0$.
- the coordinate w_a has weight $+1$, $\widetilde{(}w_a) = +1$.
- the parity of an element carries an additional weight of -1. Thus, for example, the odd element $[\mathrm{d}w_1 \mathrm{d}z_1] \in \Omega^{2,cl}(\widehat{D}^5)$ carries weight $+1 - 1 = 0$. Viewing the odd part as the space of closed two-forms, then equivalently this grading translates to the one-form symbol $\mathrm{d}(-)$ as carrying weight $-1/2$.

It is straightforward to verify that this weight grading is compatible with the super Lie algebra structure on $E(5|10)$. Moreover, we see that similarly to the decomposition of $\mathcal{H}_{AdS_7 \times S^4}$ induced by our $\mathbb{C}^\times$ action in 3.5.1, the weight grading is concentrated in degrees ≥ -1. In particular, there is a decomposition of super vector spaces

$$E(5|10) = \tilde{U}_{-1} \times \prod_{j \geq 0} U_j \tag{3.51}$$

Further, this decomposition also has the property that the 0-th piece U_0 is isomorphic to $E(3|6)$. As such, each U_j is an $E(3|6)$-module; Kac characterizes these modules explicitly and identifies them as certain irreducible $E(3|6)$-modules. In the notation of [KR01] we have that $U_{-1} = I(0,0;1;-1)^*$ and for $j \geq 1$, $U_j = I(0,0;j-1;j+1)^*$.

The decomposition of $\mathcal{H}_{AdS_7 \times S^4}$ afforded by proposition 3.5.1 induces a weight grading of $\widehat{E(5|10)}$ which extends the one on $E(5|10)$ that we have just described by declaring that the central term have weight -1. In this way, we get a related decomposition of super L_∞ algebras

$$\widehat{E(5|10)} = \prod_{j \geq -1} U_j \tag{3.52}$$

were U_{-1} is a $\mathbb{C}$-extension of $\tilde{U}_{-1}$ defined in the decomposition (3.51) and for $j \geq 0$ the U_j's are the same as in the non centrally extended case. As a corollary, we see that each of the $E(3|6)$ modules which we have identified as the costalk at 0 $\mathcal{G}^{(j)}_{\mathbb{C}^3,c}(0)$ is in fact irreducible.

3.5.5 Characters of $E(3|6)$-modules

The decomposition of the state space $\mathrm{Sym}\,(\mathcal{H}_{AdS_7 \times S^4}) = \prod_{j \geq -1} \mathcal{U}(\mathcal{G}^{(j)})(0)$ gives a product formula for the characters computed in proposition 3.2.10

$$\chi\left(\mathrm{Sym}\mathcal{H}_{AdS_7 \times S^4}\right) = \prod_{j \geq -1} \chi\left(\mathcal{U}(\mathcal{G}^{(j)})(0)\right)$$

We end this section by computing each of the characters $\chi\left(\mathcal{U}(\mathcal{G}^{(j)})(0)\right)$. We will express our characters in terms of characters of highest weight representations of $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$, which we denote by $\chi^{\mathfrak{sl}(2)}_k$ and $\chi^{\mathfrak{sl}(3)}_{[k,l]}$.

3.5.5.1 We begin with the lowest step of the decomposition, using the characterization given in 3.5.2.

Proposition 3.5.8. The character $\chi\left(\mathcal{U}(\mathcal{G}^{(-1)}_{\mathbb{C}^3})(0)\right)$ is given by the plethystic exponential of the following expression:

$$g_{-1}(t_1, t_2, r, q) = \frac{q^{3/2}(r + r^{-1}) - q^2(t_1 + t_1^{-1}t_2 + t_2^{-1}) + q^3}{(1 - t_1^{-1}q)(1 - t_1t_2^{-1}q)(1 - t_2q)}. \tag{3.53}$$

Proof. Note that in light of proposition 3.5.3 we can equivalently compute the character of the costalk at the origin of $\mathrm{C}^\bullet(\Pi\mathcal{E}_{tens})$. We first give a more explicit description of the costalk as a cochain complex. Proceeding exactly analogously to the proof of lemma 3.2.5, we see that the costalk is given by the symmetric algebra on the following cochain complex

$$\begin{array}{ccc} - & & + \\ \mathbb{C}\{\mathrm{d}z_i\mathrm{d}z_j\} \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0} & \xrightarrow{\partial} & \mathbb{C}\{\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\} \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0} \\ & & \mathbb{C}^2 \otimes \mathbb{C}\{\mathrm{d}z_i^{1/2}\} \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0} \end{array}$$

Computing summand-by-summand, we see:

- the odd summand $\mathbb{C}\{\mathrm{d}z_i\mathrm{d}z_j\} \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0}$ contributes

$$-q^2\frac{\chi^{\mathfrak{sl}(3)}_{[0,1]}(t_1, t_2)}{(1 - t_1^{-1}q)(1 - t_1t_2^{-1}q)(1 - t_2q)} = -q^2\frac{t_1 + t_1^{-1}t_2 + t_2^{-1}}{(1 - t_1^{-1}q)(1 - t_1t_2^{-1}q)(1 - t_2q)}.$$

  where $\chi^{\mathfrak{sl}(3)}_{[1,0]}(t_1, t_2)$ is the $\mathfrak{sl}(3)$ character of highest weight $[1, 0]$.

- the even summand $\mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0}$ contributes

$$q^3\frac{1}{(1 - t_1^{-1}q)(1 - t_1t_2^{-1}q)(1 - t_2q)}.$$

- the even summand $\mathbb{C}^2 \otimes \mathbb{C}\{\mathrm{d}z_i^{1/2}\} \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0}$ contributes

$$q^{3/2}\frac{\chi^{\mathfrak{sl}(2)}_1(r)}{(1 - t_1^{-1}q)(1 - t_1t_2^{-1}q)(1 - t_2q)} = q^{3/2}\frac{(r + r^{-1})}{(1 - t_1^{-1}q)(1 - t_1t_2^{-1}q)(1 - t_2q)}.$$

  where $\chi^{\mathfrak{sl}(2)}_1(r)$ is the $\mathfrak{sl}(2)$ character of highest weight one.

□

In terms of the parameters y_1, y_2, y_3, y, q introduced in 3.6.4.4 this single particle character reads

$$g_{-1}(y_i, y, q) = \frac{qy + q^2y^{-1} - q^2(y_1^{-1} + y_2^{-1} + y_3^{-1}) + q^3}{(1 - y_1q)(1 - y_2q)(1 - y_3q)}. \tag{3.54}$$

The expression matches exactly with the index of the abelian six-dimensional superconformal theory. For example, compare with [KKKL13, Eq. (3.1)] or [BBMR08, Eq. (3.35)].

From now on, we will give all formulas for the index in terms of the parameters y_1, y_2, y_3, y, q.

3.5.5.2 We continue to the next step of the decomposition, which is given by $\mathcal{G}^{(0)}_{\mathbb{C}^3}$.

Proposition 3.5.9. The character $\chi\left(\mathcal{U}(\mathcal{G}^{(0)}_{\mathbb{C}^3})(0)\right)$ is the plethystic exponential of following expression:

$$g_0(y_i, y, q) = \frac{q^4(y_1+y_2+y_3) + q^2(y^2+q+q^2y^{-2}) - q^3(y+qy^{-1})(y_1^{-1}+y_2^{-1}+y_3^{-1})}{(1-y_1q)(1-y_2q)(1-y_3q)}. \tag{3.55}$$

Proof. As usual, we wish to describe the costalk $\mathcal{G}^{(0)}_{\mathbb{C}^3,c}(0)$ more explicitly. By the same argument as in the proofs of propositions 3.6.4.4, 3.6.4.4, we may use elliptic regularity to describe the compactly supported smooth sections on a disc in terms of derivatives of the delta function at the origin in $\mathbb{C}^3$.

Accordingly, we have contributions from the following summands.

- An even copy of $\mathbb{C}\{\partial_{z_i}\} \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0}$. The character of this summand is

$$q^4 \frac{\chi^{\mathfrak{sl}(3)}_{[1,0]}(y_i)}{(1-y_1q)(1-y_2q)(1-y_3q)} = q^4 \frac{y_1+y_2+y_3}{(1-y_1q)(1-y_2q)(1-y_3q)}.$$

- An even copy of $\mathfrak{sl}(2) \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0}$. The character of this summand is

$$q^3 \frac{\chi^{\mathfrak{sl}(2)}_2(q^{-1/2}y)}{(1-y_1q)(1-y_2q)(1-y_3q)} = \frac{q^2y^2+q^3+q^4y^{-2}}{(1-y_1q)(1-y_2q)(1-y_3q)}.$$

- An odd copy of $\mathbb{C}^2 \otimes \mathbb{C}\{\mathrm{d}z_i\} \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0}$. The character of this summand is

$$q^{7/2} \frac{\chi^{\mathfrak{sl}(2)}_1(q^{-1/2}y)\,\chi^{\mathfrak{sl}(3)}_{[0,1]}(y_i)}{(1-y_1q)(1-y_2q)(1-y_3q)} = q^3 \frac{(y+qy^{-1})(y_1^{-1}+y_2^{-1}+y_3^{-1})}{(1-y_1q)(1-y_2q)(1-y_3q)}.$$

□

3.5.5.3 Finally, we continue to the general step of the decomposition.

Proposition 3.5.10. Let $j \geq 1$. The character $\chi\left(\mathcal{U}(\mathcal{G}^{(j)}_{\mathbb{C}^3})(0)\right)$ is the plethystic exponential of following expression:

$$g_j(y_i, y, q) = \frac{q^3\begin{pmatrix} q^{1+3j/2}\chi^{\mathfrak{sl}(2)}_j(q^{-1/2}y)\chi^{\mathfrak{sl}(3)}_{[1,0]}(y_i) + q^{3j/2}\chi^{\mathfrak{sl}(2)}_{j+2}(q^{-1/2}y) \\ -q^{3(j+1)/2}\chi^{\mathfrak{sl}(2)}_{j-1}(q^{-1/2}y) - q^{-1+3(j+1)/2}\chi^{\mathfrak{sl}(2)}_{j+1}(q^{-1/2}y)\chi^{\mathfrak{sl}(3)}_{[0,1]}(y_i)\end{pmatrix}}{(1-y_1q)(1-y_2q)(1-y_3q)}. \tag{3.56}$$

Proof. We proceed exactly analogously to all the previous cases. Using elliptic regularity on sections of $\mathcal{G}^{(j)}_{\mathbb{C}^3}$ over a disc containing the origin, we see that $\mathcal{U}(\mathcal{G}^{(j)}_{\mathbb{C}^3})(0)$ is a symmetric algebra on a cochain complex with the following summands

- An even copy of $\mathrm{Sym}^j(\mathbb{C}^2) \otimes \mathbb{C}\{\partial_{z_i}\} \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0} \otimes (\partial_{z_1}\partial_{z_2}\partial_{z_3})^{-j/2}$ which contributes

$$\frac{q^3\left(q^{1+3j/2}\chi^{\mathfrak{sl}(2)}_j(q^{-1/2}y)\chi^{\mathfrak{sl}(3)}_{[1,0]}(y_i)\right)}{(1-y_1q)(1-y_2q)(1-y_3q)} \tag{3.57}$$

- An even copy of $\mathrm{Sym}^{j+2}(\mathbb{C}^2) \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0} \otimes (\partial_{z_1}\partial_{z_2}\partial_{z_3})^{-j/2}$ which contributes

$$\frac{q^3 \left(q^{3j/2}\chi^{\mathfrak{sl}(2)}_{j+2}(q^{-1/2}y)\right)}{(1-y_1q)(1-y_2q)(1-y_3q)} \tag{3.58}$$

- An odd copy of $\mathrm{Sym}^{j-1}(\mathbb{C}^2) \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0} \otimes (\partial_{z_1}\partial_{z_2}\partial_{z_3})^{-(j+1)/2}$ which contributes

$$\frac{-q^3 \left(q^{3(j+1)/2}\chi^{\mathfrak{sl}(2)}_{j-1}(q^{-1/2}y)\right)}{(1-y_1q)(1-y_2q)(1-y_3q)} \tag{3.59}$$

- An odd copy of $\mathrm{Sym}^{j+1}(\mathbb{C}^2) \otimes \mathbb{C}\{\mathrm{d}z_i\} \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0} \otimes (\partial_{z_1}\partial_{z_2}\partial_{z_3})^{-(j+1)/2}$ which contributes

$$\frac{-q^3 \left(q^{-1+3(j+1)/2}\chi^{\mathfrak{sl}(2)}_{j+1}(q^{-1/2}y)\chi^{\mathfrak{sl}(3)}_{[0,1]}(y_i)\right)}{(1-y_1q)(1-y_2q)(1-y_3q)} \tag{3.60}$$

□

3.5.5.4 As a consequence, we have that $f_{AdS_7\times S^4}(y_i, y, q) = \sum_{j\geq -1} g_j(y_i, y, q)$, or explicitly:

$$\begin{aligned}
&\frac{q^4(y_1+y_2+y_3) - q^2(y_1^{-1}+y_2^{-1}+y_3^{-1}) + (1-q^3)(yq+y^{-1}q^2)}{(1-y_1q)(1-y_2q)(1-y_3q)(1-yq)(1-y^{-1}q^2)} \\
&= \frac{qy + q^2y^{-1} - q^2(y_1^{-1}+y_2^{-1}+y_3^{-1}) + q^3}{(1-y_1q)(1-y_2q)(1-y_3q)} \\
&+ \frac{q^4(y_1+y_2+y_3) + q^2(y^2+q+q^2y^{-2}) - q^3(y+qy^{-1})(y_1^{-1}+y_2^{-1}+y_3^{-1})}{(1-y_1q)(1-y_2q)(1-y_3q)} \\
&+ \sum_{j\geq 1} \frac{q^3 \begin{pmatrix} q^{1+3j/2}\chi^{\mathfrak{sl}(2)}_{j}(q^{-1/2}y)\chi^{\mathfrak{sl}(3)}_{[1,0]}(y_i) + q^{3j/2}\chi^{\mathfrak{sl}(2)}_{j+2}(q^{-1/2}y) \\ -q^{3(j+1)/2}\chi^{\mathfrak{sl}(2)}_{j-1}(q^{-1/2}y) - q^{-1+3(j+1)/2}\chi^{\mathfrak{sl}(2)}_{j+1}(q^{-1/2}y)\chi^{\mathfrak{sl}(3)}_{[0,1]}(y_i) \end{pmatrix}}{(1-y_1q)(1-y_2q)(1-y_3q)}
\end{aligned}$$

In [BBMR08, Eq. (3.22, 3.23)], the index counting gravitons on $f_{AdS_7\times S^4}$ is expressed as a sum of characters of irreducible representations of the 6d $\mathcal{N} = (2,0)$ superconformal algebra. In [CDI16, Table 24] these representations are labeled as $\mathcal{D}_1[0,0,0]^{(0,m)}_{2m}$ where $m \geq 1$. The characters of these modules have been computed (see for example [AFI+20, Eq. (166)] and match exactly with $g_{m-2}(y_i, y, q)$ after a suitable change of variables. Thus, we conjecture the following

Conjecture 3.5.11. For $j \geq -1$, the minimal twist of $\mathcal{D}_1[0,0,0]^{(0,j+2)}_{2(j+2)}$ is exactly $\mathcal{G}^{(j)}_{\mathbb{C}^3,c}(0)$.

Remark 3.5.12. As we remarked in 3.4.12, this conjecture implies that the minimal twist of $\mathcal{D}_1[0,0,0]^{(0,j+2)}_{2(j+2)}$ which is a priori a module for the minimally twisted 6d $\mathcal{N} = (2,0)$ superconformal algebra $\mathfrak{osp}(6|2)$, is in fact a module for the larger infinite dimensional super-Lie algebra $E(3|6)$. This can be thought of as analogous to the enhancement of conformal symmetries to the action of the Witt algebra of vector fields in 2d chiral conformal field theory.

3.6 Holographic Speculations

We began this thesis with some remarks on how dualities between physical theories can often be used to uncover novel equivalences between the mathematical objects that describe them. In this final section of the thesis, we offer some speculations to this effect. We caution the reader that a large portion of this section involves recalling constructions from physics without any attention to rigor for motivational purposes.

In sections 3.4.2, 3.5.2, we commented on how minimal twists of the 3d $\mathcal{N} = 8$ BLG theory and 6d $\mathcal{N} = (2, 0)$ tensor multiplets are are visible as pieces of the graviton decompositions of twisted $AdS_4 \times S^7$ and $AdS_7 \times S^4$ respectively. Famous instances of the AdS/CFT correspondence posit equivalences between the higher rank 3d $\mathcal{N} = 8$ theories studied by ABJM and the higher rank 6d $\mathcal{N} = (2, 0)$ theories of type A_N with M-theory on $AdS_4 \times S^7$ and $AdS_7 \times S^4$ respectively. It is natural to wonder whether the twisted holograpahy proposal mentioned in the introduction can be applied to our descriptions of the twisted $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds to study the minimal twists of the higher rank 3d $\mathcal{N} = 8$ and 6d $\mathcal{N} = (2, 0)$ superconformal field theories respectively.

Our goal in this section is to posit some expectations regarding the minimal twist of the 6d $\mathcal{N} = (2, 0)$ theory of type A_N. This theory is notorious for being both ubiquitous and nebulous. On the one hand, almost every superconformal field theory that has had interesting applications to geometry, topology, or representation theory occurs as one of its dimensional reductions, so it has long been expected to contain very rich mathematics. On the other hand, it does not admit a Lagrangian description. Its only free parameter is the rank of an ADE Lie algebra, and outside of the abelian case, a field realization is not even known.

We begin by recalling some features of the AdS/CFT correspondence. We will begin with a more physical language, and work towards some concrete mathematical expectations.

3.6.1 The AdS/CFT correspondence

Traditional formulations of the AdS/CFT correspondence relate two theories, schematically denoted T_{CFT} and T_{grav} on manifolds M_1, M_2 respectively, together with a conformal diffeomorphism $\partial M_2 \cong M_1$. The theories have the feature that boundary values of fields of T_{grav} denoted $\phi|_\partial$, may be identified with sources for T_{CFT} denoted J. The two theories are considered to be holographically dual when their partition functions are equivalent $Z_{CFT}[J] = Z_{grav}[\phi|_\partial]$.

In examples of stringy origin, T_{CFT} describes the low energy dynamics of a stack of N branes in supergravity, in the large N limit, and T_{grav} describes gravitational dynamics in the background the branes source.

3.6.1.1 Let's identify some salient features of the primordial example of such a duality so as to inform our desiderata in the sequel.

Conjecture 3.6.1 (Maldacena [Mal98], [Wit98]). The following are equivalent:

- $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$. In addition to the rank of the gauge group, the theory has a parameter the Yang-Mills coupling constant g_{YM}.
- type IIB superstring theory on $AdS_5 \times S^5$ with N units of five-form flux on S^5. The theory has two free parameters, the string coupling g_s and a parameter L/ℓ_s which describes the scale

of AdS relative to the length of the string.

Under this equivalence, the parameters of the two sides are identified as follows $g_{YM}^2 = 2\pi g_s$ and $2g_{YM}^2 N = (L/\ell_s)^4$.

It is convenient to introduce a parameter $\lambda = g_{YM}^2 N$, the so-called *'t Hooft coupling*; in the perturbative regime where the number of colors is also large (a limit that we will introduce momentarily), the β-function keeps λ of the same order.

It is very difficult to perform explicit calculations of most observables associated to either theory at generic values of the parameters on either side. However, there are certain limits which afford more tractability.

- The first limit we can take involves sending the string coupling g_s to zero and keeping the parameter L/ℓ_s fixed. In this limit, contributions from higher genus worldsheets in string perturbation theory are suppressed. Under the above identification of parameters, we see that this limit should involve taking $g_{YM} \to 0$ while keeping the 't Hooft coupling finite; that is, we must take the large N limit of the gauge theory. This limit is traditionally referred to as the *'t Hooft limit*. Corrections in $\frac{1}{N}$ then correspond to turning on quantum effects in string theory.

- After taking the 't Hooft limit, we may further consider the limit where L/ℓ_s is large. In this limit, strings are small and particle like compared to the scale of AdS and the theory looks like classical type IIB supergravity on $AdS_5 \times S^5$. On the gauge theory side, this corresponds to the limit where the 't Hooft coupling is large. As such, we see that even this simplified form of the AdS/CFT correspondence is extremely powerful as it relates strongly coupled gauge theory to classical perturbative supergravity!

3.6.2 BPS observables in AdS/CFT

Many checks of the AdS/CFT correspondence involve computing quantities on either side that are independent of the coupling and comparing them. Such quantities are typically BPS, and as such can be studied at the level of twists. We introduce two such quantities which we will further expand on in our relevant example below.

3.6.2.1 Suppose that T_{CFT} is superconformal, such as in the above example. In such examples, one expects that the superconformal algebra in fact acts on M_2 as isometries, at least asymptotically.

Superconformal field theories admit a plethora of protected quantities that can be computed exactly at weak coupling. One such quantity is the superconformal index, which in a Hamiltonian formulation of the theory can be thought of as a Witten-index in radial quantization. Schematically, such a quantity takes the form

$$\mathrm{Tr}_{\mathcal{H}}\left((-1)^F \exp(-\beta\{Q,\overline{Q}\}) x_1^{J_1}\cdots x_n^{J_n} y_1^{H_1}\cdots y_n^{H_n}\right)$$

where $(-1)^F$ is the fermion number operator, β is an inverse temperature, Q is a supercharge and the x_i are fugacities keeping track of charges under angular momenta, and y_i are fugacities keeping track of charges under R-symmetries. The superconformal index gives a generating function for the difference between bosonic and fermionic states annihilated by a particular supercharge. Under an operator-state correspondence, the superconformal index can also be thought of as a signed count of local operators preserved under a single supercharge.

In terms of partition functions, the superconformal index is gotten by a partition function on a twisted product $M_1 = S^1 \times_\omega S^{d-1}$ where the twisting ω is determined by a background connection for the global symmetries of the problem. The expectation that the AdS/CFT correspondence can be expressed as an equality of partition functions therefore suggests a recapitulation of the superconformal index in gravitational terms. An exciting body of work aims to make this gravitiational incarnation precise, see for example [Mur20] and references therein.

Note that by definition, the superconformal index provides a lower bound on the number of fractionally BPS states of T_{CFT}. It is often the case, however, that T_{grav} includes in its spectrum, black holes, which are expected to have a thermodynamic entropy proportional to the event-horizon-area at leading order, as given by the Beckenstein-Hawking formula. As such, the growth of states in T_{CFT}, and hopefully the superconformal index, should reflect this.

3.6.2.2 Another such quantity is the algebra of BPS local operators in T_{CFT}. This vector space underlying this algebra is precisely a costalk of the factorization algebra of observables of a twist of T_{CFT}. In light of the aforementioned operator-state correspondence, this can be thought of as categorifying the superconformal index. Under the AdS/CFT dictionary, local operators of T_{CFT} are supposed to match with certain kinds of states in T_{grav}.

Moreover, both kinds of objects transform in representations of a superconformal algebra and the map between them preserves the actions. Local operators in the CFT are equipped with an interesting algebraic structure given by operator-product-expansion, and the AdS/CFT correspondence intertwines this algebraic structure with scattering of supergravity states. Indeed, the equality of partition functions along with the matching of sources for CFT local operators with boundary values of gravitational fields gives a prescription for computing correlation functions between CFT local operators by varying the gravitational action evaluated on field configurations subject to certain boundary values with respect to the boundary value. This recipe can be recast as a tree-level computation in the gravitational theory, involving computation of so-called Witten diagrams [**?**].

3.6.3 Twisted holography

Introduced by Costello and Li in [**?**], the twisted holography proposal posits an avatar of the AdS/CFT correspondence that holds at the level of factorization algebras associated to supersymmetric twists of T_{CFT} and T_{grav}. There is an exciting body of work being developed around this program including tests of this proposal from both the gravitational and gauge theory sides.

3.6.3.1 Concretely, the twisted holography proposal suggests that the type of duality between the factorization algebras associated to a gravitational theory and to the worldvolume theory of a number of branes is a general version of *Koszul duality*.

Ordinary Koszul duality for associative algebras (so quantum mechanical systems) associates to an (augmented) algebra A a dual algebra $A^!$ whose appropriate derived category of modules is the same as that of A. Following the work of [**?**, **?**] (see also the review in [PW21]) there is a simple physical interpretation of Koszul duality. If A is the algebra of operators of some bulk quantum field theory (perturbatively we can even consider a theory of gravity) then $A^!$ is the algebra of operators on the universal topological line defect. Universal here means that algebra of operators on any other line defect which couples to the bulk system admits a unique map of algebras from $A^!$.

The general theory of Koszul duality for factorization algebras has not been developed, and we do not do so here, but see [Lur17] for the case of $\mathbb{E}_n$-algebras and [GLZ22], [Tam03] for the case of particular kinds of vertex algebras. This sort of duality would allow one to make sense of universality statements as above for higher dimensional, possibly non-topological, defects in an arbitrary bulk quantum field theory. Roughly, one expects the Koszul dual of a factorization algebra to be the factorization algebra corepresenting the functor of looking at solutions to a Maurer-Cartan equation in a tensor product.

3.6.3.2 Let us now make a more concrete, yet slightly informal, statement of twisted holography which fits into the approach of this thesis. Let X be a smooth manifold, and let Obs_{grav} denote a factorization algebra on X that we view as the observables of a bulk gravitational theory. Suppose we have, in addition, a stack of N branes, wrapping a closed submanifold $Y \hookrightarrow X$ whose worldvolume theory has a factorization algebra of observables Obs^N_{CFT}.

Note that Obs_{grav} is a factorization algebra on X, while Obs^N_{CFT} is a factorization algebra on the closed submanifold Y so we cannot yet compare them. We can, however, attempt to restrict Obs_{grav} to a factorization algebra just on Y, which we denote by $\mathrm{Obs}_{grav}|_Y$.

Expectation 3.6.2 (Twisted holographic principle following [**?**])**.** There is a map of factorization algebras

$$(\mathrm{Obs}_{grav}|_Y)^! \to \mathrm{Obs}^N_{CFT}$$

that becomes an equivalence in the large N limit.

As we recalled in the previous subsection, traditional formulations of the AdS/CFT correspondence relate local operators of the gauge theory to states of the gravitational theory on AdS. Therefore, a natural desideratum in relating the above to more traditional statements is a precise relation between the source of the above map and gravitational states in AdS. Moreover, there is an operational definition of the operator-product-expansion on the costalk of a Koszul dual factorization algebra which realizes the expectation that Koszul duality corepresents the functor taking Maurer-Cartan elements in the tensor product. Another desideratum is to relate the output of this procedure with the scattering product on gravitational states computed by Witten diagrams.

Remark 3.6.3. In this context, the definition of Koszul duality involves another ingredient, namely the backreaction of branes wrapping Y. This is meant to capture the fact that $(\mathrm{Obs}_{grav}|_Y)$ may not be canonically augmented, but we may try to deform it in a way that kills off the obstruction to being augmented. More precisely, one expects that the correct version of Koszul duality for application in holographic contexts is a version of *curved* Koszul duality for factorization algebras.

Remark 3.6.4. For finite N, this map will in general be neither injective nor surjective. The kernel and cokernel of this map for finite N correspond to interesting nonperturbative effects in the gravitational theory. For instance, in gauge theories:

- This map has a kernel given by trace relations. Syzygies between trace relations are conjecturally related to the worldvolume theories of certain other branes in the gravitational theory, so-called *giant gravitons* [GL21], [CKL+23], [Ima22] [1]

- This map also has a cokernel. By fiat, these are classes that exist in the finite N cohomology of the observables of a gauge theory that are not in the image of the natural map from the large N theory. Recent developments in cohomological approaches to counting quantum microstates of $\frac{1}{16}$-BPS black holes in $AdS_5 \times S^5$ [CKL+23] [CL23] [CY13] can be cast as trying to characterize the cokernel of a specific example of this map.

3.6.3.3 The above expectation can be tested in instances where both sides of the duality admit explicit descriptions. This has been carried out in many examples including:

- A stack of $D3$ branes in twisted Ω-deformed type IIB supergravity on flat space. The theory on the stack of $D3$ branes is dual to the closed string B-model on the deformed conifold [CG21a]. This can be understood as a twisted Ω-deformed version of the physical AdS/CFT duality between 4d $\mathcal{N} = 4$ super Yang-Mills and type IIB string theory on $AdS_5 \times S^5$. Here the duality can be formulated in terms of vertex algebras.

- M2 branes and M5 branes in twisted Ω-deformed M-theory on Taub-NUT space [Cos16, Cos17]. In the particular Ω-background, M2 branes are localized to a topological quantum mechanical system where the duality can be phrased in terms of associative algebras and ordinary Koszul duality. The koszul dual algebra bears close relations to the spherical Cherednik algebra. The Ω-background localizes M5 to a complex plane and the observables of the localized theory are an affine W_N vertex algebra. Many celebrated features of the representation theory of these algebras and their relations with geometry have found natural explanations from the perspective of this twist of M-theory [GR20], [OZ21].

The example we consider is closely related to the second of these. Indeed, there is an odd nilpotent element in $\mathfrak{osp}(6|2)$, which we refer to as S in the sequel. Using the inner action of $\mathfrak{osp}(6|2)$ on our eleven-dimensional model on twisted $AdS_7 \times S^4$ as identified in proposition 3.3.6, S affords a deformation of our model. This is the deformation considered in [BRvR15], and it induces a specialization of characters called the Schur limit.

3.6.3.4

3.6.4 M5 branes, holomorphy, and holography

The results in the second half of this thesis can be viewed as baby steps in investigating twisted holography for the minimal twist of the 6d $\mathcal{N} = (2,0)$ theory. Let us begin by spelling out the objects in expectation 3.6.2 adapted to our setting.

- The 11d spacetime manifold X is $\mathbb{R} \times \mathbb{C}^5$ and Y is a copy of $\mathbb{C}^3$.

- The factorization algebra $\mathrm{Obs}_{grav}|_{\mathbb{C}^3}$ has the feature that its semiclassical free limit is the factorization algebra denoted $\mathrm{C}^\bullet\left(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^N_{AdS_7\times S^4})\right)$ in definition 3.2.8.

- The factorization algebra Obs^N_{CFT} describes local observables in the minimal twist of the theory on a stack of N M5 branes wrapping $\mathbb{C}^3$.

[1] We thank Ji-Hoon Lee for conversations related to this topic

Our goal is to try and use this map, and expectations about its kernel and cokernel, to give a concrete description of the target. There have been various approaches to try and characterize the spectrum of $\frac{1}{16}$-BPS states in the 6d $\mathcal{N} = (2,0)$ theories of type A_{N-1}, which furnish consistency checks to test our proposal against. Some of these involve applications of instanton counting techniques in 5d $\mathcal{N} = 2$ gauge theory [?] and some of them involve holographic techniques [Ima22].

As we remarked in subsection 3.6.2, the first ingredient is a map of representations of the superconformal algebra between local operators of the CFT and supergravity states. In order to codify such a matching in terms of the kinds of data in the statement of expectation 3.6.2, we require a matching between supergravity states and the costalk at the origin of the factorization algebra $(\mathrm{Obs}_{grav}|_{\mathbb{C}^3})^!$. This is precisely the content of proposition 3.2.12.

3.6.4.1 We have the following conjectural large N statements

Conjecture 3.6.5 (R-Saberi-Williams)**.** There is an equivalence of holomorphic $\mathbb{P}_0$-factorization algebras

$$\left(\mathrm{C}^\bullet(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^N_{AdS_7\times S^4}))\right)^! \cong \mathcal{U}_\omega\left(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})\right).$$

where the right hand side denotes a twisted factorization envelope of the local L_∞-algebra $\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$. Moreover, upon deforming by the Maurer-Cartan element $S \in \mathfrak{osp}(6|2)$, the factorization algebra $\mathcal{U}_\omega\left(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})\right)$ has no sections away from a copy of $\mathbb{C} \subset \mathbb{C}^3$, and its restriction to this copy of $\mathbb{C}$ is equivalent to a twisted factorization envelope of the local Lie algebra $\mathrm{Diff}_{\mathbb{C}}$ of holomorphic differential operators on $\mathbb{C}$.

Here, the twisting cocycle ω comes from the shifted Poisson tensor that was induced by the flux in section 3.3. The content in verifying this conjecture is to explicitly compute the twisting coming from the flux sourced by the brane, and check that upon deforming by the element S, it induces the correct cocycle on $\mathrm{Diff}(\mathbb{C}^\times)$

The comment in 3.2.2.5 constitutes a very meager consistency check for the second part of this conjecture, where we observe that the Schur limit of the character of the costalk of $\mathcal{U}_\omega\left(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})\right)$ recovers the vacuum character of the $W_{1+\infty}$ vertex algebra.

There is a deformation of the twisted factorization envelope of $\mathrm{Diff}_{\mathbb{C}}$ which yields the $\mathcal{W}_{1+\infty}$ vertex algebra, also referred to as the affine Yangian of $\mathfrak{gl}(1)$. In [Cos16], Costello finds this deformation from a loop level computation in his 5d noncommutative gauge theory. We also expect to be able to lift this to the minimal twist. We summarize this expectation in a conjecture.

Conjecture 3.6.6 (R-Saberi-Williams)**.** There is an equivalence of holomorphic factorization algebras

$$\left(\mathrm{C}^\bullet_\hbar(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^N_{AdS_7\times S^4}))\right)^! \cong \mathcal{U}_{\hbar,\omega}\left(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})\right).$$

where the right hand side denotes a deformation of the factorization algebra in the previous conjecture induced by loop-level effects in our eleven-dimensional model. Moreover, upon deforming by the Maurer-Cartan element $S \in \mathfrak{osp}(6|2)$, the factorization algebra $\mathcal{U}_{\hbar,\omega}\left(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})\right)$ has no sections away from a copy of $\mathbb{C} \subset \mathbb{C}^3$, and its restriction to this copy of $\mathbb{C}$ is equivalent to the $\mathcal{W}_{1+\infty}$ vertex algebra.

3.6.4.2 We now move on to finite N statements. For the lowest steps of the filtration, we can make some very concrete statements.

Conjecture 3.6.7 (R-Saberi-Williams)**.** Upon deforming by $S \in \mathfrak{osp}(6|2)$, the factorization algebra $\mathcal{U}_\omega(\mathcal{G}^{(-1)}_{\mathbb{C}^3})$ has no sections away from a copy of $\mathbb{C} \subset \mathbb{C}^3$ and its restriction to this copy of $\mathbb{C}$ is equivalent to the Heisenberg vertex algebra.

To check this conjecture, it remains to compute the pullback of the twisting cocycle ω under the inclusion of $\mathcal{G}^{(0)}$ and see that it deforms to the Heisenberg cocycle.

3.6.4.3 There is a distinguished Lie sub-algebra of the algebra of differential operators on $\mathbb{C}^\times$ which is given by the Witt-algebra of vector fields. The central extension of $\mathrm{Diff}(\mathbb{C}^\times)$ induced by ω above restricts to the Virasoro central extension. Similarly, in proposition 3.5.5 we have identified a distinguished local super-Lie algebra inside $\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$ given by $\mathcal{E}(3|6)$.

Accordingly, we conjecture the following

Conjecture 3.6.8 (R-Saberi-Williams)**.** Upon deforming by $S \in \mathfrak{osp}(6|2)$, the factorization algebra $\mathcal{U}_\omega(\mathcal{E}(3|6))$ has no sections away from a copy of $\mathbb{C} \subset \mathbb{C}^3$ and its restriction to this copy of $\mathbb{C}$ is equivalent to the Virasoro vertex algebra.

Again, to check this conjecture it remains to compute the pullback of the twisting cocycle ω along the inclusion $\mathcal{E}(3|6) \to \Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$ and compare its deformation with the cocycle giving the Virasoro central extension.

We can once again perform a consistency check at the level of characters of costalks. Indeed, we see that the plethystic exponential of the specialized character $g_0(y=1, y_3=1, q) = \frac{q^2}{1-q}$ is exactly the vacuum character of the Virasoro algebra.

Moreover, note that combining with conjecture 3.6.5, we expect maps

$$\mathcal{U}_\omega(\mathcal{E}(3|6)) \to \Big(\mathrm{C}^\bullet(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^N_{AdS_7\times S^4}))\Big)^! \to \mathrm{Obs}^N_{CFT}$$

for every N. This map can be thought of as a Noether-type map associated to an $\mathcal{E}(3|6)$-symmetry of the minimal twist of any finite rank 6d $\mathcal{N}=(2,0)$ theory [CG21b].

3.6.4.4 More generally, the $\mathcal{W}_{1+\infty}$ algebra has as quotients, the $\mathcal{W}_N$ algebras when the central charge is set equal to N. Accordingly, we dream of the following:

Speculation 3.6.9. Under an integrality condition on the central charge, the map

$$(\mathrm{Obs}_{grav}|_{\mathbb{C}^3})^! \to \mathrm{Obs}^N_{CFT}$$

factors as

$$\begin{array}{ccc} (\mathrm{Obs}_{grav}|_{\mathbb{C}^3})^! & \longrightarrow & \mathrm{Obs}^N_{CFT} \\ \downarrow & \nearrow & \\ \mathcal{U}_{\hbar,\omega}(\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^N_{AdS_7\times S^4}))/\mathcal{U}_{\hbar,\omega}(\prod_{j\geq N-1}\mathcal{G}^{(j)}_{\mathbb{C}^3}) & & \end{array}$$

We can perform a consistency check of the above speculation at the level of characters of costalks. It is expected that the superconformal deformation deforms Obs^N_{CFT} to the $\mathcal{W}_N$ vertex algebra. On the other hand, we have that

Proposition 3.6.10. Upon specializing $y = 1, y_3 = 1$ (so that $y_1 y_2 = 1$), one has

$$\chi\left(\Omega^{0,\bullet}_{\mathbb{C}^3,c}(\mathcal{L}^N_{AdS_7\times S^4})(0)/\left(\mathcal{G}^{(-1)}_{\mathbb{C}^3,c}\times\prod_{j\geq N-1}\mathcal{G}^{(j)}_{\mathbb{C}^3,c}\right)(0)\right) = \sum_{j\geq 0}^{N-2} g_j(y_1,y_2,y_3=1,y=1,q) = \frac{q^2+q^3+\cdots+q^N}{1-q}$$

The plethystic exponential of the right hand side agrees with the vacuum character of the W_N vertex algebra.

Proof. By induction it suffices to show that the specialization of the single particle local character g_j of the factorization algebra $\mathcal{U}(\mathcal{G}^{(j)})$ is $q^{j+2}/(1-q)$. We have already seen this in the case $j = -1, 0$, so it suffices to show this when $k \geq 1$.

First observe that the denominator becomes

$$(1-y_1 q)(1-y_2 q)(1-q). \tag{3.61}$$

Next, we observe that the numerator of $g_j(y_1, y_2, y_3 = 1, y = 1, q)$ can be factored as

$$q^{3+3j/2}\left(q^{-(j+2)/2}+q^{-(j-2)/2}-q^{-j/2}(y_1+y_2)\right) = q^{j+2}(1+q^2-q(y_1+y_2)) = q^{j+2}(1-y_1q)(1-y_2q)$$

where in the last line we have used $y_1 y_2 = 1$. The result follows. □

3.6.4.5 In [RW22] we try to explicitly characterize the discrepancy between the characters of $\mathcal{U}(\mathcal{G}^{(j)}_{\mathbb{C}^3})$ and expectations about the superconformal index of the finite rank 6d $\mathcal{N} = (2, 0)$ theories computed via instanton counting techniques [KKKL13] and the "giant graviton expansion" [AFI+20, **?**], [Ima22]. It would be interesting to try and categorify the discrepancies and identify them in terms of modules for $E(3|6)$.

Bibliography

[AFI$^+$20] Reona Arai, Shota Fujiwara, Yosuke Imamura, Tatsuya Mori, and Daisuke Yokoyama, *Finite-n corrections to the m-brane indices*, Journal of High Energy Physics **2020** (2020), no. 11.

[BBMR08] Jyotirmoy Bhattacharya, Sayantani Bhattacharyya, Shiraz Minwalla, and Suvrat Raju, *Indices for Superconformal Field Theories in 3,5 and 6 Dimensions*, JHEP **02** (2008), 064.

[BBRT91] Danny Birmingham, Matthias Blau, Mark Rakowski, and George Thompson, *Topological field theory*, Physics Reports **209** (1991), no. 4-5, 129–340.

[BCOV94] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Kodaira–Spencer theory of gravity and exact results for quantum string amplitudes*, Commun. Math. Phys. **165** (1994), 311–428.

[BH05] Anirban Basu and Jeffrey A. Harvey, *The m2-m5 brane system and a generalized nahm's equation*, Nuclear Physics B **713** (2005), no. 1-3, 136–150.

[BH11] John C. Baez and John Huerta, *Division algebras and supersymmetry II*, Adv. Theor. Math. Phys. **15** (2011), no. 5, 1373–1410. MR 2989835

[BL07] Jonathan Bagger and Neil Lambert, *Modeling multiple m2-branes*, Physical Review D **75** (2007), no. 4.

[BL08] ———, *Gauge symmetry and supersymmetry of multiple M2-branes*, Phys. Rev. D **77** (2008), 065008.

[BRvR15] Christopher Beem, Leonardo Rastelli, and Balt C. van Rees, *W symmetry in six dimensions*, J. High Energy Phys. (2015), no. 5, 017, front matter+37. MR 3359380

[BY16] Dylan Butson and Philsang Yoo, *Degenerate classical field theories and boundary theories.*

[CCDF95] A. C. Cadavid, A. Ceresole, R. D'Auria, and S. Ferrara, *Eleven-dimensional supergravity compactified on Calabi–Yau threefolds*, Phys. Lett. B **357** (1995), no. 1-2, 76–80. MR 1352855

[CDF91] Leonardo Castellani, Riccardo D'Auria, and Pietro Fré, *Supergravity and superstrings. A geometric perspective. Vol. 1*, World Scientific Publishing Co., Inc., Teaneck, NJ, 1991, Mathematical foundations. MR 1120022

[CDI16] Clay Cordova, Thomas T. Dumitrescu, and Kenneth Intriligator, *Multiplets of superconformal symmetry in diverse dimensions*, 2016.

[Ced10a] Martin Cederwall, *$D = 11$ supergravity with manifest supersymmetry*, Mod. Phys. Lett. A **25** (2010), 3201–3212.

[Ced10b] ______, *Towards a manifestly supersymmetric action for eleven-dimensional supergravity*, Journal of High Energy Physics **2010** (2010), no. 1, 1–19.

[Ced14] ______, *Pure spinor superfields: an overview*, Breaking of Supersymmetry and Ultraviolet Divergences in Extended Supergravity (2014), 61–93.

[Ced21] ______, *$SL(5)$ supersymmetry.*

[CG17] Kevin Costello and Owen Gwilliam, *Factorization algebras in quantum field theory. Vol. 1*, New Mathematical Monographs, vol. 31, Cambridge University Press, Cambridge, 2017. MR 3586504

[CG21a] Kevin Costello and Davide Gaiotto, *Twisted holography*, 2021.

[CG21b] Kevin Costello and Owen Gwilliam, *Factorization algebras in quantum field theory. Vol. 2*, New Mathematical Monographs, vol. 41, Cambridge University Press, Cambridge, 2021.

[CKL+23] Sunjin Choi, Seok Kim, Eunwoo Lee, Siyul Lee, and Jaemo Park, *Towards quantum black hole microstates*, 2023.

[CL15] Kevin Costello and Si Li, *Quantization of open-closed BCOV theory, I.*

[CL16] ______, *Twisted supergravity and its quantization.*

[CL20] ______, *Anomaly cancellation in the topological string*, Adv. Theor. Math. Phys. **24** (2020), no. 7, 1723–1771.

[CL23] Chi-Ming Chang and Ying-Hsuan Lin, *Words to describe a black hole*, Journal of High Energy Physics **2023** (2023), no. 2.

[Cos11] K. Costello, *Renormalization and effective field theory*, Mathematical surveys and monographs, American Mathematical Society, 2011.

[Cos13] Kevin Costello, *Notes on supersymmetric and holomorphic field theories in dimensions two and four*, Pure Appl. Math. Q. **9** (2013), no. 1, 73–165. MR 3126501

[Cos16] ______, *M-theory in the Omega-background and five-dimensional noncommutative gauge theory.*

[Cos17] ______, *Holography and Koszul duality: the example of the M2 brane.*

[CY13] Chi-Ming Chang and Xi Yin, *1/16 bps states in n=4 sym*, Physical Review D **88** (2013), no. 10.

[EH21] Richard Eager and Fabian Hahner, *Maximally twisted eleven-dimensional supergravity.*

[EHSW21] Richard Eager, Fabian Hahner, Ingmar Saberi, and Brian R. Williams, *Perspectives on the pure spinor formalism.*

[ESW20] Chris Elliott, Pavel Safronov, and Brian R. Williams, *A taxonomy of twists of supersymmetric Yang–Mills theory.*

[ESW21] Richard Eager, Ingmar Saberi, and Johannes Walcher, *Nilpotence varieties*, Annales Henri Poincaré **22** (2021), no. 4, 1319–1376.

[EW21] Chris Elliott and Brian R. Williams, *Holomorphic Poisson field theories*, High. Struct. **5** (2021), no. 1, 1–28.

[FSS15] Domenico Fiorenza, Hisham Sati, and Urs Schreiber, *Super-Lie n-algebra extensions, higher WZW models and super-p-branes with tensor multiplet fields*, Int. J. Geom. Methods Mod. Phys. **12** (2015), no. 2, 1550018, 35.

[Fuk86] D. B. Fuks, *Cohomology of infinite-dimensional Lie algebras*, Contemporary Soviet Mathematics, Consultants Bureau, New York, 1986, Translated from the Russian by A. B. Sosinski. MR 874337

[GL21] Davide Gaiotto and Ji Hoon Lee, *The Giant Graviton Expansion.*

[GLZ22] Zhengping Gui, Si Li, and Keyou Zeng, *Quadratic duality for chiral algebras*, 2022.

[GR20] Davide Gaiotto and Miroslav Rapcak, *Miura operators, degenerate fields and the m2-m5 intersection*, 2020.

[GRW21] Owen Gwilliam, Eugene Rabinovich, and Brian R. Williams, *Quantization of topological-holomorphic field theories: local aspects.*

[Gus09] Andreas Gustavsson, *Algebraic structures on parallel M2-branes*, Nucl. Phys. B **811** (2009), 66–76.

[Igu70] Jun-ichi Igusa, *A classification of spinors up to dimension twelve*, American Journal of Mathematics **92** (1970), no. 4, 997–1028.

[Ima22] Yosuke Imamura, *Analytic continuation for giant gravitons.*

[Kac98] Victor G. Kac, *Classification of infinite-dimensional simple linearly compact Lie superalgebras*, Adv. Math. **139** (1998), no. 1, 1–55. MR 1652530

[KKKL13] Hee-Cheol Kim, Seok Kim, Sung-Soo Kim, and Kimyeong Lee, *The general M5-brane superconformal index.*

[Kor14] Eric O. Korman, *Elliptic involutive structures and generalized Higgs algebroids*, ProQuest LLC, Ann Arbor, MI, 2014, Thesis (Ph.D.)–University of Pennsylvania. MR 3251109

[KR01] Victor G. Kac and Alexei Rudakov, *Representations of the exceptional Lie superalgebra $E(3,6)$. II. Four series of degenerate modules*, Comm. Math. Phys. **222** (2001), no. 3, 611–661. MR 1888090

[KR02] ———, *Complexes of modules over exceptional Lie superalgebras $E(3,8)$ and $E(5,10)$*, Int. Math. Res. Not. (2002), no. 19, 1007–1025. MR 1903328

[KT75] Franz W. Kamber and Philippe Tondeur, *Foliated bundles and characteristic classes*, Lecture Notes in Mathematics, Vol. 493, Springer-Verlag, Berlin-New York, 1975. MR 0402773

[Li11] Si Li, *Calabi–Yau geometry and higher-genus mirror symmetry*, ProQuest LLC, Ann Arbor, MI, 2011, Thesis (Ph.D.)–Harvard University. MR 2898602

[LMNS96] Andrei Losev, Gregory W. Moore, Nikita Nekrasov, and Samson Shatashvili, *Four-dimensional avatars of two-dimensional RCFT*, Nucl. Phys. B Proc. Suppl. **46** (1996), 130–145.

[Lur11] Jacob Lurie, *Derived algebraic geometry x: Formal moduli problems*, 2011.

[Lur17] ———, *Higher algebra*, 2017.

[Mal98] Juan Martin Maldacena, *The Large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998), 231–252.

[Mur20] Sameer Murthy, *The growth of the $\frac{1}{16}$-bps index in 4d $\mathcal{N}=4$ sym*, 2020.

[Nek09] Nikita A. Nekrasov, *Instanton partition functions and M-theory*, Jpn. J. Math. **4** (2009), no. 1, 63–93. MR 2491283

[OZ21] Jihwan Oh and Yehao Zhou, *Twisted holography of defect fusions*, SciPost Phys. **10** (2021), no. 5, 105.

[Pri19] J. P. Pridham, *Unifying derived deformation theories*, 2019.

[PW21] Natalie M. Paquette and Brian R. Williams, *Koszul duality in quantum field theory.*

[Rab] Eugene Rabinovich, *Factorization algebras for bulk-boundary systems*, Thesis (Ph.D.), University of California, Berkeley.

[Röm06] Christian Römelsberger, *Counting chiral primaries in $\mathcal{N}=1$, $d=4$ superconformal field theories*, Nuclear Physics B **747** (2006), no. 3, 329–353.

[RSW21] Surya Raghavendran, Ingmar Saberi, and Brian R. Williams, *Twisted eleven-dimensional supergravity*, 2021.

[RW22] Surya Raghavendran and Brian R. Williams, *A holographic approach to the six-dimensional superconformal index*, 2022.

[RY19] Surya Raghavendran and Philsang Yoo, *Twisted S-duality*, 2019.

[SSS09] Hisham Sati, Urs Schreiber, and Jim Stasheff, *Fivebrane structures*, Rev. Math. Phys. **21** (2009), no. 10, 1197–1240. MR 2588823

[SW20] Ingmar Saberi and Brian R. Williams, *Constraints in the BV formalism: six-dimensional supersymmetry and its twists.*

[SW21] ———, *Twisting pure spinor superfields, with applications to supergravity.*

[SW22] ———, *Derived enhancements of superconformal symmetry in six dimensions*, To appear, 2022.

[Tam03] Dimitri Tamarkin, *Deformations of chiral algebras*, 2003.

[Wit88] Edward Witten, *Topological quantum field theory*, Communications in Mathematical Physics **117** (1988), no. 3, 353–386.

[Wit95] E. Witten, *Chern–Simons gauge theory as a string theory*, The Floer memorial volume, Progr. Math., vol. 133, Birkhäuser, Basel, 1995, pp. 637–678. MR 1362846

[Wit98] Edward Witten, *Anti de sitter space and holography.*