An arithmetic-geometric reciprocity between theta functions attached to real and imaginary quadratic fields

by

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A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy
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Abstract

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2020

Abstract: We use the theta correspondence to construct classical holomorphic modular forms associated to ideal classes in quadratic number fields. These modular forms are theta functions that were originally introduced by Hecke in the 1920s and have been investigated by several authors since. Our framework allows us to prove old and new results concerning the periods of these modular forms over certain geometric cycles defined by arithmetic data. In particular, we establish a reciprocity relationship between the periods of theta functions attached to ideal classes in real and imaginary quadratic fields. This provides an analogue of (and context for) Hecke's discovery that certain periods of his imaginary quadratic theta functions are special values of classical Eisenstein series at CM points.

Acknowledgements

A significant debt of gratitude is owed to my advisor, Professor Stephen Kudla, for his generosity with his time, ideas and compassion. His insight and influence can be easily identified all throughout this manuscript. Thanks are also due to my committee members, Professors Florian Herzig, Fiona Murnaghan and Kumar Murty for their support. Finally, an enormous thank you to Jemima Merisca for her tireless assistance (and apologies for all the extra work and headaches that I undoubtedly created).

Contents

Introduction

This thesis is concerned with a certain interplay between the arithmetic and geometry of quadratic number fields. On the arithmetic side, we have classical modular forms $\vartheta_{\mathfrak{a}}$ attached to integral ideals $\mathfrak{a}$ (and additional auxiliary data). These modular forms were first defined and studied by Hecke in the 1920s [11, 12, 14] and have since been studied by various authors (see, e.g., [19, 20, 22, 27, 29, 30, 36]). They occur in two types according as to whether the quadratic field is real or imaginary. In the former case, they are indefinite theta functions of weight 1; in the latter case, they are definite theta functions of weight 2.

On the geometric side, we associate to each ideal class $[\mathfrak{a}]$ an equivalence class of CM points $[z_{\mathfrak{a}}]$ or geodesic arcs $[C_{\mathfrak{a}}]$ in the upper half-plane $\mathcal{H}$, depending on whether our quadratic field is imaginary or real, respectively. The main result of the thesis examines the period of one type of theta function over the geometric object of the "opposite type." We show that these periods satisfy a certain reciprocity relation. A little more specifically, we prove, under certain conditions, that the period integral of an imaginary quadratic theta function $\vartheta_{\mathfrak{a}}$ over a real quadratic geodesic arc $C_{\mathfrak{b}}$ is equal to the period (or, rather, the special value) of the real quadratic theta function $\vartheta_{\mathfrak{b}}$ at the corresponding CM point $z_{\mathfrak{a}}$. That is, up to multiplicative constants, we have

$$\int_{C_{\mathfrak{b}}} \vartheta_{\mathfrak{a}} = \vartheta_{\mathfrak{b}}(z_{\mathfrak{a}});$$

or, more suggestively,

$$\int_{\text{RM cycle}} \vartheta_{\text{CM}} = \vartheta_{\text{RM}}(\text{CM point}).$$

We expound on this below after giving some orienting historical remarks.

Hecke's ϑ functions

In a series of papers [11, 12, 14], Hecke introduced and examined theta functions attached to quadratic fields. Their definitions are as follows.

Let K be an imaginary quadratic field of discriminant $D_K < 0$, and fix an integral ideal $\mathfrak{a}$ in $\mathcal{O}_K$ with norm $A = [\mathcal{O}_K : \mathfrak{a}]$, an element $r \in \mathfrak{a}$, and a positive integer Q. Associated to this data is the theta function

$$\vartheta_2(\tau; r, \mathfrak{a}, Q\sqrt{D_K}) = \sum_{\mu \equiv r \pmod{\mathfrak{a}Q\sqrt{D_K}}} \mu\, e(\tfrac{\mu\mu'}{AQ|D|}\tau) \qquad (\tau \in \mathcal{H}).$$

Hecke proved this this is a holomorphic cusp form of weight 2 for the congruence subgroup $\Gamma_0(Q|D_K|)$.

Now let F be a real quadratic field of discriminant $D_F > 0$, let $\mathfrak{b}$ be an integral ideal in $\mathcal{O}_F$ with norm $B = [\mathcal{O}_F : \mathfrak{b}]$, and let r be an element of $\mathfrak{b}$. Fix a positive integer Q and let $U_F^+(Q\sqrt{D}_F)$ be the group of totally positive units ε in F such that $\varepsilon \equiv 1 \pmod{Q\sqrt{D_F}}$. Associated to all this data is the theta function

$$\vartheta(\tau; r, \mathfrak{b}, Q\sqrt{D_F}) = \sum_{\substack{\mu \equiv r \pmod{\mathfrak{b}Q\sqrt{D_F}} \\ \mu\mu' > 0 \\ \text{mod } U_F^+(Q\sqrt{D_F})}} \operatorname{sgn}(\mu)\, e\left(\frac{\mu\mu'}{BQD_F}\tau\right) \qquad (\tau \in \mathcal{H}),$$

where the summation is over the nonzero $U_F^+(Q\sqrt{D_F})$ equivalence classes of all μ in F satisfying the indicated conditions. Hecke showed that this is a holomorphic cusp form of weight 1 for the congruence subgroup $\Gamma_0(QD_F)$.

Simultaneously with all this, Hecke also considered the (real-analytic) Eisenstein series

$$G_k(\tau,s;r_1,r_2,N)=\sum_{(m,n)\equiv(r_1,r_2)\ (N)}\frac{1}{(m\tau+n)^k|m\tau+n|^{2s}},\ (\tau\in\mathcal{H},\mathrm{Re}(s)>1-\frac{k}{2})$$

where k,N,r_1 and r_2 are non-negative integers, and where the sum runs over all $(m,n)\neq(0,0)$ in $\mathbf{Z}^2$ such that $(m,n)\equiv(r_1,r_2)$ (mod $N\mathbf{Z}^2$). He showed that G_1 admits an analytic continuation to the entire s-plane; and that, furthermore, the value at $s=0$ of this analytic continuation is a *holomorphic* Eisenstein series of weight 1 for the congruence subgroup $\Gamma(N)$. In certain respects, this Eisenstein series $G_1(\tau,0;r,N)$ is a counterpart to the real quadratic theta series ϑ introduced above. (See below.)

Hecke's interest in these functions appears to be rooted in a few different directions. Ultimately, he would use his constructions in these papers to obtain the decomposition of $V=H^0(X(p),\Omega^1)$, the vector space of regular differentials on the modular curve $X(p)=\Gamma(p)\backslash\mathcal{H}^*$ of prime level p, under the action of the finite modular group $PSL(2,p)=PSL(2,\mathbf{Z})/\Gamma(p)$. In particular, Hecke showed that when $p\equiv 3$ (mod 4) and $p>3$, V contains an invariant subspace V_0 spanned by his weight 2 theta functions associated to $\mathbf{Q}(\sqrt{-p})$; and, moreover, he showed that V_0 is isomorphic to $h(-p)$ (= class number of $\mathbf{Q}(\sqrt{-p})$) copies of one of the two irreducible representations of $PSL(2,p)$ of dimension $(p-1)/2$. Towards this end, Hecke examined certain periods of his theta functions and (essentially) showed that they coincide with periods of elliptic curves with CM by $\mathbf{Q}(\sqrt{-p})$ (compare [32]).

We only mention this work for its intrinsic appeal. (It appears to be one of the first uses of cohomology to construct representations of groups.) For more on this, see [2] and [8].

Our interest in this thesis lies in Hecke's work on the aforementioned periods. What Hecke did was evaluate the periods of his ϑ_2 functions attached to an imaginary quadratic field K along a vertical geodesic in the upper half-plane $\mathcal{H}$. He showed that such a period can be expressed as the special value of his weight 1 Eisenstein series G_1 at a CM point lying in K. (See Section 3.7.) Thus, Hecke concluded, "[...] *der Zusammenhang mit der komplexen Multiplikation gegeben.*"[1] [12, p.714].

Our main result in this thesis is a re-framing of Hecke's results which places them in a broader context. On the one hand, we show that his functions ϑ_2, ϑ and G_1 can all be realized as special instances of a general theta series afforded to us by the theta correspondence. (See the next section.) In this light, Hecke's calculation of the periods of ϑ_2 becomes a special case of Kudla's general seesaw reciprocity philosophy [25]. On the other hand, our construction realizes Hecke's real quadratic functions ϑ and G_1 as being two sides of the same coin: both arise from the orthogonal group of a two-dimensional quadratic space, namely, the given quadratic number field with its norm form (appropriately scaled) in the former case, and the hyperbolic plane $\mathbf{Q}^2$ in the latter case. As such, our new perspective allows us to obtain the complementary result concerning the periods of ϑ_2 along geodesic arcs in $\mathcal{H}$ that are naturally (and classically) associated to the quadratic field F. We find, under a mild compatibility condition, that these periods are special values

[1] " [...] the connection with complex multiplication is given."

of Hecke's ϑ at CM points in K. This new result allows us to establish a mysterious reciprocity between ideal classes and the associated geometric cycles coming from a pair of compatible real and imaginary quadratic fields.

The theta correspondence

It is worth noting explicitly that Hecke's theta functions contain, by construction, one of the divisive features of the arithmetic of imaginary *vs.* real quadratic fields. Namely, underyling each is the norm form of the relevant quadratic field.

In the imaginary quadratic case, this quadratic form is *positive-definite*, which classically has allowed for easier constructions (beginning even with Jacobi's theta function $\sum_n q^{n^2}$ associated to the positive-definite quadratic form $q(x) = x^2$). On the other hand, indefinite quadratic forms, such as the norm form of a real quadratic field, have proven to be more formidable. For instance, the naive theta series $\sum_{n,m} q^{n^2-m^2}$ associated to the indefinite quadratic form $q(x,y) = x^2 - y^2$ does not even converge. One here has the option of restricting the domain of summation (generally at the price of automorphy) or modifying the summand (generally at the price of holomorphicity). Thus, it is interesting that Hecke was able to perform *both* in his construction of ϑ and G_1 at the expense of neither automorphy nor holomorphicity.

Classically, the construction of indefinite theta series (due to Siegel [36]) proceeds by the introduction of so-called *majorants* for the indefinite quadratic form (cf., for instance, our discussion preceding Theorem 3.1.6). Aspects of this construction were incorporated into Weil's Acta paper [39] as part of a robust, representation-thereotic framework for theta functions. In Weil's theory, theta functions live in the representation space of the "oscillator" representation of (a 2-fold cover of) the symplectic group.

More generally, given a pair (G, H) of reductive subgroups in Sp that are each other's centralizer, one is (in principle) able to use Weil's representation to lift automorphic representations attached to G to automorphic representations attached to H, and vice versa. This is the so-called theta correspondence. For a nice introduction to the general philosophy here, we recommend the two papers [21] and [25] by Kudla. See also Chapter 1 for an overview geared towards our specific goals.

In retrospect, Hecke's constructions are all seen to take place within the framework of the theta correspondence—and, in particular, make use of Kudla's seesaw reciprocity relationship (§1.3).

Summary of results and outline of thesis

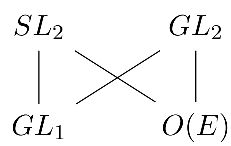
In Chapter 1 we give an overview of the aspects of the theta correspondence that our relevant to our end goal. Then in Chapter 2 we carry out, in detail, the construction of the seesaw of dual pairs that underlies all of our constructions.

In brief, let F and K be real and imaginary quadratic extensions of $\mathbf{Q}$, respectively. (We allow F to be the "split" algebra $\mathbf{Q} \oplus \mathbf{Q}$.) The indefinite orthogonal group $O(1,1)$ and the definite unitary group $U(1)$ associated to the norm forms of F and K sit together in a seesaw

$$\begin{array}{ccc} Sp(1) & & U(1,1) \\ | & \times & | \\ U(1) & & O(1,1) \end{array}$$

of dual pairs in $Sp(2)$ (§2.1). Then—at least, in principle—one takes the trivial representation of $U(1)$ and lifts it to an automorphic representation (form) on $SU(1,1)$ which essentially yields Hecke's imaginary quadratic theta function ϑ_2. Likewise, one lifts the trivial representation of $O(1,1)$ to $Sp(1)$, obtaining Hecke's ϑ in the case where F is anisotropic (a quadratic field), and Hecke's Eisenstein series G_1 in the isotropic case (the split algebra $\mathbf{Q}^2$). Moreover, the period integral of ϑ_2 over the embedded $O(1,1) \subset U(1,1)$ coincides, via Kudla's seesaw reciprocity, with the "period integral" of the real quadratic theta series over the embedded $U(1) \subset Sp(1)$. (Here $U(1)$ is compact at infinity, so the (archimedean component of the) latter period integral is really just evaluation at a point.)

Remark. It is of interest to compare our setup above to the (very similar) seesaw



where $O(E)$ is the orthogonal group associated to the norm form of an imaginary (resp., real) quadratic field $E/\mathbf{Q}$ ([25, §3, Example 1]). In this case, the theta lift of the trivial representation of $O(E)$ to SL_2 is essentially a classical theta function θ on SL_2 (compare Example 1.2.2). Likewise, the theta lift of a character ξ_s ($s \in \mathbf{C}$) is a classical Eisenstein series $E(\cdot, s)$ for GL_2 (an Epstein zeta function).

In this setup, Kudla's seesaw reciprocity gives the Petersson inner product identity

$$\langle \theta, \xi_s \rangle = \langle \chi_{\mathrm{triv}}, E(\cdot, s) \rangle.$$

Here the left-side is essentially the Mellin transform of θ, i.e., the zeta function of E (up to Γ-factors), while the right-side can be interpreted as: (i) the sum $\sum_{i=1}^{h} E(z_i, s)$ of E evaluated at the Heegner points z_i associated to the ideal classes of E, if E is imaginary; or (ii) the sum $\sum_{i=1}^{h} \int_{C_i} E(z,s)\, dz$ of integrals of E over geodesic arcs associated to the ideal classes of E. This is Hecke's famous "integral formula" [10] for the zeta fuction of a quadratic number field (cf. [40]).

It is a curious coincidence (?) that the seesaw principle captures two strikingly similar results of Hecke. ▲

Of course, all of this is just a formal sketch. In practice, as with all applications of the theta correspondence, there are many technical difficulties that must be surmounted. In particular, we will work *classically and not adelically* (§1.4). Thus, we extract from the seesaw of dual pairs above, and more precisely from their various embeddings

$$\begin{array}{ccc} U(1) \times O(1,1) & \xrightarrow{\mathrm{id}\times\iota} & U(1) \times U(1,1) \\ \downarrow{\scriptstyle \iota\times\mathrm{id}} & & \downarrow{\scriptstyle \sigma} \\ SL_2 \times O(1,1) & \xrightarrow{\sigma'} & Sp(2), \end{array}$$

embeddings of (models of) the associated real symmetric domains:

$$\begin{array}{ccc} \mathbf{R}_{>0} & \xrightarrow{\alpha} & \mathcal{H} \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \epsilon} \\ \mathcal{H} \times \mathbf{R}_{>0} & \xrightarrow{\epsilon'} & \mathcal{H}_2 \end{array}$$

(Theorem 2.3.1). Then, following Kudla [24], we consider theta functions ϑ_K on $\mathcal{H}$ and Θ_F on $\mathcal{H} \times \mathbf{R}_{>0}$ that restrict to the same function on $\mathbf{R}_{>0}$ (§3.1).

We prove that ϑ_K is a generalizations of Hecke's imaginary quadratic theta series ϑ_2 (3.6). We also show that the integral $\vartheta_F(\tau) = \int_{\Gamma_F \backslash \mathbf{R}_{>0}} \Theta_F(z,t)\, \frac{dt}{t}$, for an appropriate discrete subgroup Γ_F acting on $\mathbf{R}_{>0}$ (which trivial if $F = \mathbf{Q}^2$; else, it is a group of units in the quadratic field F), is a weight 1 holomorphic modular form that coincides with Hecke's theta series ϑ_F or his Eisenstein series G_1 depending on whether F is a real quadratic field or $F = \mathbf{Q}^2$, respectively (§3.4–3.5).

Our main theorem is a general period relation (Theorem 3.2.3) that results from our seesaw of dual reductive pairs, given above. Specifically, we show that the integrals of ϑ_K over the image $\alpha\colon \mathbf{R}_{>0} \hookrightarrow \mathcal{H}$ modulo appropriate discrete subgroups coincide with the special values of ϑ_F at CM point in K. The image of α is a vertical geodesic ray in the isotropic case and a geodesic arc associated to the real quadratic field F in the anisotropic case (§3.3).

Hecke's period relation results once we specialize our construction to the case $F = \mathbf{Q}^2$ and identify our theta functions ϑ_K and ϑ_F with his imaginary quadratic theta function ϑ_2 and Eisenstein series G_1, respectively. However, our framework suggests and easily proves the complementary result concerning the periods of ϑ_K along hyperbolic geodesic arcs associated to F—which Hecke apparently did not notice. We show, under a condition of compatibility between F and K (Lemma 3.6.1), that these periods of ϑ_K are special values of the anisotropic ϑ_F (hence Hecke's ϑ) at CM points in K (§3.7).

To be more explicit, let $\mathfrak{a}$ be an integral ideal in the imaginary quadratic field K with a fixed choice of integral basis (and "orientation"; cf. Lemma 2.4.1), and let $\delta_{\mathfrak{a}}$ be the CM point corresponding to $\mathfrak{a}$. (If $[a,b,c] = ax^2+bxy+cy^2$ is the binary quadratic form corresponding to $\mathfrak{a}$, then $\delta_{\mathfrak{a}} = \frac{-b+\sqrt{D_K}}{2a}$ is the root of $[a,b,c]$ in the upper half-plane $\mathcal{H}$.) Let Q be a positive integer and fix an $r \in \mathfrak{a}$. Then Hecke's theta function $\vartheta_2(\tau; \mathfrak{a}, r, Q\sqrt{D_K})$, which is a weight 2 cusp form with nebentypus, defines a holomorphic 1-form on a certain cover $\widetilde{Y}$ cover of the modular curve $Y(Q|D_K|) = \Gamma(Q|D_K|)\backslash\mathcal{H}$ for the principal congruence subgroup $\Gamma(Q|D_K|)$.

Now let $\mathfrak{b}$ be an integral ideal in the real quadratic field F, fix an oriented integral basis for $\mathfrak{b}$, and let $C_{\mathfrak{b}}$ be the associated geodesic in $\mathcal{H}$. (If $[a,b,c]$ is the binary quadratic form corresponding to $\mathfrak{b}$, then $C_{\mathfrak{b}}$ is the semicircle in $\mathcal{H}$ connecting the two real roots $\rho = \frac{-b+\sqrt{D_K}}{2a}$ and $\rho' = \frac{-b-\sqrt{D_K}}{2a}$ of $[a,b,c]$.) The projection of $C_{\mathfrak{b}}$ into $\tilde{Y}$ defines a closed geodesic $C_{\mathfrak{b}}(Q|D_K|)$.

Our general period relation relates the period integral of $\vartheta_2(\tau;\mathfrak{a},r,Q\sqrt{D_K})$ over $C_{\mathfrak{b}}(Q|D_K|)$ to the special value of $\vartheta(\tau;\mathfrak{b},r',Q'\sqrt{D_F})$ (for appropriate r' and Q') at the CM point $\tau_0 = |D_K|^{-1}|\delta_{\mathfrak{a}}|^{-2}\,i$. Namely, if we assume the "compatibility condition" $D_F \in N_{K/\mathbf{Q}}K$, we have

$$\int_{C_{\mathfrak{b}}(Q|D_K|)} \vartheta_2(\tau;r,\mathfrak{a},Q\sqrt{D_K})\,d\tau = C\ \vartheta(\tau_0;r',\mathfrak{b},Q|D_K|\sqrt{D_F}),$$

where C is a fairly explicit multiplicative constant. See Theorem 3.7.5.

If the compatibility condition is omitted, we still obtain a period relation. However, in this case the theta function ϑ_K is no longer one of Hecke's ϑ_2 series. Instead, it is a weight 2 cusp form for a cocompact arithmetic Fuchsian group Γ_K—i.e., $\vartheta_K(z)\,dz$ gives a holomorphic 1-form on the compact Shimura curve $\Gamma_K\backslash\mathcal{H}$ (§3.8). Our general period relation expresses the periods of these 1-forms over the aforementioned geodesics as special values of weight-1 elliptic modular forms (Theorem 3.8.3).

Chapter 1

The Theta Correspondence

The basic problem we wish to consider is that of the construction of automorphic forms. Classically this had been achieved with the aid of theta series attached to quadratic forms. This approach, pioneered by Riemann and Jacobi and later developed by Hecke, Maass, Siegel and many others, was provided with a systematic, representation-theoretic framework by Weil. For example, in Weil's theory, the classical theta series

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z}$$

attached to the quadratic form $q(x) = x^2$ arises as a theta kernel attached to the *dual pair* $(O(q), SL_2)$ by means of the *Weil representation*. (See Example 1.2.2.)

In this chapter we outline some basic facts of this theory that will be needed later. The presentation is economical and focuses on the special cases of interest rather than the most general results. References for the latter are given where appropriate.

1.1 The Weil representation

Fix a number field k and let $\mathbb{W}$ be a finite-dimensional symplectic space over k. Weil [38] constructed a representation of (a certain cover of) the symplectic group $Sp(\mathbb{W})$ and used it as a basis for his theory of theta series. (See also [39].)

Here we view $Sp(\mathbb{W}) = Sp(\mathbb{W}_k)$ as an algebraic group over k. For each place v of k, except when $k_v = \mathbf{C}$, the group $Sp(\mathbb{W}_v)$ admits a 2-fold central extension $\widetilde{Sp}(\mathbb{W}_v)$. This is the so-called **metaplectic group** of $\mathbb{W}_v$. These various groups piece together to give a 2-fold central extension $\widetilde{Sp}(\mathbb{W}_{\mathbf{A}})$ of the adelization $Sp(\mathbb{W}_{\mathbf{A}})$. (Here $\mathbf{A} = \mathbf{A}_k =$ adeles of k.) More precisely, we have commuting diagrams

$$\begin{array}{ccc} \widetilde{Sp}(\mathbb{W}_v) & \longrightarrow & \widetilde{Sp}(\mathbb{W}_{\mathbf{A}}) \\ \downarrow & & \downarrow \\ Sp(\mathbb{W}_v) & \longrightarrow & Sp(\mathbb{W}_{\mathbf{A}}) \end{array}$$

for each v. (When $k_v = \mathbf{C}$ we take $\widetilde{Sp}(\mathbb{W}_v) = Sp(\mathbb{W}_v)$.)

The group $Sp(\mathbb{W}_k)$ embeds into $Sp(\mathbb{W}_{\mathbf{A}})$ as a discrete subgroup. Moreover, there is a unique lifting

$$\begin{array}{ccc} & & \widetilde{Sp}(\mathbb{W}_{\mathbf{A}}) \\ & \nearrow & \downarrow \\ Sp(\mathbb{W}_k) & \longrightarrow & Sp(\mathbb{W}_{\mathbf{A}}) \end{array}$$

through which we view $Sp(\mathbb{W}_k)$ as a subgroup of $\widetilde{Sp}(\mathbb{W}_{\mathbf{A}})$. See, e.g., [5, 18].

Now, given a nontrivial additive character ψ of $\mathbf{A}$ that is trivial on k, Weil defines a unitary representation $\omega = \omega_\psi$ of $\widetilde{Sp}(\mathbb{W}_v)$. A model S of the smooth vectors in this representation is obtained as follows. Let

$$\mathbb{W} = \mathbb{U} \oplus \mathbb{U}'$$

be a complete polarization of $\mathbb{W}$, i.e. a decomposition into maximal totally isotropic subspaces. Then we may take S to be space of Schwarz–Bruhat functions on $\mathbb{U}_{\mathbf{A}}$. We omit the explicit description of ω on S since we will not have need for it (but see Example 1.2.2).

We define a linear functional Θ on S by

$$\Theta(\phi) = \sum_{x \in \mathbb{W}_k} \phi(x), \quad (\phi \in S).$$

The key fact about Θ is that it is $Sp(\mathbb{W}_k)$-invariant:

$$\Theta(\omega(g)\phi) = \Theta(\phi), \quad (g \in \mathrm{Sp}(\mathbb{W}_k), \phi \in S).$$

Thus, for each fixed ϕ, the assignment $g \mapsto \Theta(\omega(g)\phi)$ defines an automorphic form on $\widetilde{Sp}(\mathbb{W}_{\mathbf{A}})$—at least in the weak sense where by an automorphic form we merely mean a function on $\widetilde{Sp}(\mathbb{W}_{\mathbf{A}})$ that is $Sp(\mathbb{W})$-invariant. (Being built out of Schwarz–Bruhat functions, it is also of "moderate growth.") We write

$$\Theta_\phi(g) = \Theta(\omega(g)\phi) \quad (g \in \mathrm{Sp}(\mathbb{W}_{\mathbf{A}}))$$

for this automorphic form and call it the **theta function** attached to ϕ. A basic problem is the determination of the exact nature of these Θ_ϕ.

Example 1.1.1. The classical theta series attached to positive definite quadratic forms arise as special cases of this general construction. See Example 1.2.2. ▲

Now suppose that G is a reductive subgroup of $Sp(\mathbb{W}_k)$ and write $\widetilde{G(\mathbf{A})}$ for the inverse image of $G(\mathbf{A})$ in $\widetilde{Sp}(\mathbb{W}_{\mathbf{A}})$. Then we may consider the restriction $\Theta_\phi|_{\widetilde{G}}$ and study the resulting automorphic form. We will take up a more specific example of this type in the next section.

1.2 Dual reductive pairs and theta lifts

By a **dual reductive pair** (or **dual pair** for short) in $Sp(\mathbb{W})$ we mean a pair (G, H) of reductive k-subgroups of $Sp(\mathbb{W})$ that are mutual centralizers in $Sp(\mathbb{W})$:

$$Z_{Sp(\mathbb{W})}(G) = H \quad \text{and} \quad Z_{Sp(\mathbb{W})}(H) = G.$$

See [7].

Example 1.2.1. (i) $(O(V), Sp(W))$. Let V be a quadratic space over k, i.e. a finite-dimensional vector space endowed with a nondegenerate quadratic form q. The associated bilinear form $(\cdot, \cdot)$ is given by $(x, y) = q(x + y) - q(x) - q(y)$ for $x, y \in V$.

Let W by a symplectic space over k, i.e. a finite-dimensional vector space endowed with a symplectic form $\langle \cdot, \cdot \rangle_0$.

Then the space $\mathbb{W} = V \otimes W$ may be given the symplectic form $\langle \cdot, \cdot \rangle = (\cdot, \cdot) \otimes \langle \cdot, \cdot \rangle_0$.

The groups $O(V)$ and $Sp(W)$ may be viewed as a dual pair in $Sp(\mathbb{W})$ via the natural embedding $O(V) \times Sp(W) \hookrightarrow Sp(\mathbb{W})$.

(ii) $(U(V), U(W))$. Let $K/\mathbf{Q}$ be an imaginary quadratic field, viewed as a subfield of $\mathbf{C}$. Let V and W be Hermitian spaces over K, i.e. finite-dimensional vector spaces spaces endowed with nondegenerate Hermitian forms $(,)_V$ and $(,)_W$, respectively. Our convention for a Hermitian form $(\cdot, \cdot)$ is that it be linear in the first component and anti-linear in the second: $(\alpha_1 v, \alpha_2 v_2) = \alpha_1 \alpha_2^\sigma (v_1, v_2)$ where σ is the nontrivial Galois automorphism of K.

Let $\mathbb{W} = R_{K/\mathbf{Q}}(V \otimes_K W)$, where $R_{K/\mathbf{Q}}$ denotes restriction of scalars. Then

$\mathbb{W}$ may be given the symplectic form $\langle \cdot, \cdot \rangle = \mathrm{Im}[(\cdot, \cdot)_V \otimes (\cdot, \cdot)_W]$. The groups $U(W)$ and $U(V)$ sit inside $Sp(\mathbb{W})$ as a dual pair.

More generally, one can construct a (U, U) pair by beginning with an arbitrary quadratic extension and working with Hermitian and anti-Hermitian spaces defined over that extension (cf. [25]). The construction above, however, will suffice for our purposes. ▲

There is an intricate relationship (*Howe duality*, cf. [15, 16]) between the representation theories of two members in a dual pair. What is relevant to us is the following. The inverse images $\widetilde{G(\mathbf{A})}$ and $\widetilde{H(\mathbf{A})}$ of $G(\mathbf{A})$ and $H(\mathbf{A})$ in $\widetilde{Sp}(\mathbb{W}_{\mathbf{A}})$ commute, and so we obtain a map

$$\widetilde{G(\mathbf{A})} \times \widetilde{H(\mathbf{A})} \to \widetilde{Sp}(\mathbb{W}_{\mathbf{A}}).$$

Using this, and given a fixed Schwarz–Bruhat function $\phi \in S$, we define a function θ_ϕ by

$$\theta_\phi(g, h) = \Theta_\phi(\omega(g)\omega(h)) = \Theta(\omega(g)\omega(h)\phi) \quad ((g, h) \in \widetilde{G(\mathbf{A})} \times \widetilde{H(\mathbf{A})}).$$

This function is left-invariant under $G(k) \times H(k)$ and so defines a function on $G(k)\backslash\widetilde{G(\mathbf{A})} \times H(k)\backslash\widetilde{H(\mathbf{A})}$. We call the resulting function the **theta kernel** for (G, H) attached to ϕ.

The theta kernel $\theta_\phi(g, h)$ is in some sense simultaneously automorphic in G and H and therefore allows us to lift automorphic forms from G to H (and vice-versa) by integrating over one of the variables. More precisely, if f is a cusp form on $\widetilde{G(\mathbf{A})}$, then we may form the integral

$$\theta_\phi(f)(h) = \int_{G(k)\backslash\widetilde{G(\mathbf{A})}} f(g)\theta_\phi(g,h)\,dg.$$

(This is with respect to a suitably normalized Haar measure—e.g. Tamagawa measure.) We call this the **theta lift** of f (by θ_ϕ). As a function on $\widetilde{H(\mathbf{A})}$ it is left-invariant under $H(k)$. Moreover, if ϕ satisfies a certain finiteness property (cf. [18, Proposition 2.2]), then $\theta_\phi(f)$ will be of moderate growth and hence an automorphic form on $\widetilde{H(\mathbf{A})}$.

Likewise, starting with a cusp form on $\widetilde{H(\mathbf{A})}$ and a suitably finite ϕ, we may form its theta lift to an automorphic form on $\widetilde{G(\mathbf{A})}$.

Example 1.2.2. We will show how the classical theta series

$$\theta_Q(\tau) = \sum_{x\in\mathbf{Z}^m} e^{2\pi i Q(x)\tau} \qquad (\tau \in \mathcal{H}) \tag{1.1}$$

attached to a positive-definite quadratic form $Q = Q(x_1,\dots,x_m)$ over $\mathbf{Q}$ can be realized in terms of this general construction. The relevant dual pair will be $(O(V), Sp(W))$ where V is $\mathbf{Q}^m$ with the quadratic form Q and W is the standard two-dimensional symplectic space over $\mathbf{Q}$ so that $Sp(W) = SL_2$. If $\dim V = m$ is even, then the metaplectic cover $\widetilde{Sp}(\mathbb{W}_\mathbf{A})$ splits over

$$O(V_\mathbf{A}) \times SL_2(\mathbf{A}) \longrightarrow Sp(\mathbb{W}_\mathbf{A})$$

(cf. [26]) and so the Weil representation ω gives a representation of the groups $O(V_\mathbf{A})$ and $SL_2(\mathbf{A})$ (and not just of their inverse images in the metaplectic cover) on $S = \mathcal{S}(\mathbb{U}_\mathbf{A}) = \mathcal{S}(V_\mathbf{A})$ (Schwarz–Bruhat functions on $V_\mathbf{A}$). Moreover, in this case the action of $g \in O(V_{\mathbf{A}})$ is given simply by

$$(\omega(g)f)(x) = f(g^{-1}x) \tag{1.2}$$

while the action of $SL_2(\mathbf{A})$ can be described on generators as follows:

$$\left(\omega \begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} \phi\right)(x) = \psi(bQ(x))\phi(x) \tag{1.3}$$

$$\left(\omega \begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} \phi\right)(x) = \chi_V(a)|a|_{\mathbf{A}}^{\frac{m}{2}}\phi(x) \tag{1.4}$$

$$\left(\omega \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \phi\right)(x) = \gamma_V\, \widehat{\phi}(-x). \tag{1.5}$$

Here ψ is the fixed additive character used in defining ω, γ_V is a certain 8th root of unity, $\chi_V(\cdot) = (\cdot, (-1)^{m/2}\det Q)$ is the quadratic character of V, $\widehat{\phi}(x) = \int_{y \in V_{\mathbf{A}}} \phi(y)\psi(B(x,y))\, dy$ is the adelic Fourier transform and $B(x,y) = Q(x+y) - Q(x) - Q(y)$ is the bilinear form associated to Q. See [6, Theorem 2.2]. A convenient choice of ψ is

$$\psi(x) = e^{2\pi i x_\infty} \prod_p e^{-2\pi i \mathrm{Frac}_p(x_p)} \qquad (x \in \mathbf{A}).$$

If m is odd, the above can be used to define a *projective* representation of SL_2 and it will be necessary to go up to the metaplectic cover to get a proper representation. This behaviour is a manifestation of the fact that the theta function (1.1) in this case is a modular form of *half*-integral weight $\frac{m}{2}$. To keep matters simple, we shall restrict to the case where m is even.

To construct the appropriate theta kernel via the Weil machinery, we require a choice of Schwarz–Bruhat function $\phi \in S$. At the archimedean place we shall take the Gaussian

$$\phi_\infty(x) = e^{-2\pi Q(x)}$$

and at the non-archimedean place we shall take the characteristic function of the lattice $L_{\mathbf{A}}$ where L is the lattice $\mathbf{Z}^m$ in our quadratic space $V = \mathbf{Q}^m$. This gives us our kernel $\theta_\phi(g,h)$ ($g \in O(V), h \in SL_2$). Consider now the theta lift

$$\theta_\phi(\mathbb{1})(h) = \int_{O(V)\backslash O(V_{\mathbf{A}})} \theta_\phi(g,h)\, dg$$

of the constant function $\mathbb{1}$ on $O(V)\backslash O(V_{\mathbf{A}})$. Because (V,Q) is positive-definite, one knows that $O(V)\backslash O(V_{\mathbf{A}})$ is compact, so $\mathbb{1}$ is a cusp form and the above integral converges absolutely.

The key point is the following. For $\tau = u + iv \in H$, define an element $h_\tau = h_{\tau,\infty} \cdot 1 \in SL_2(\mathbf{A})$ by

$$h_{\tau,\infty} = \begin{bmatrix} 1 & u \\ & 1 \end{bmatrix} \begin{bmatrix} v^{1/2} & \\ & v^{-1/2} \end{bmatrix}$$

(so that $h_{\tau,\infty} \cdot i = \tau$). Then a straightforward computation using (1.2)—(1.4) shows that the archimedean component of the summand in $\theta_\phi(g,h)$ is

$$\omega(h_{\tau,\infty})\omega(g_\infty)\phi_\infty(x) = \omega(h_{\tau,\infty})\phi_\infty(g_\infty^{-1}x) = v^{\frac{m}{4}} e^{2\pi i Q(x)\tau}.$$

This is (up to the factor of $v^{m/4}$) the summand in the classical theta function (1.1)! Thus if we set

$$\theta_\phi(\tau) = v^{-\frac{m}{4}} \theta_\phi(\mathbb{1})(h_\tau) = v^{-\frac{m}{4}} \int_{O(V)\backslash O(V_{\mathbf{A}})} \theta_\phi(g,h)\, dg$$

then, upon "unfolding" the integral (cf. [9, §4.6] for the details), we find that

$$\theta_\phi(\tau) = \frac{1}{|\mathrm{Aut}(L)|} \theta_Q(\tau)$$

where $\mathrm{Aut}(L)$ is the stabilizer of $L = \mathbf{Z}^n$ in $O(Q)$. Thus we've exhibited the classical theta function $\theta_Q(\tau)$ as a theta lift. ▲

1.3 Kudla's seesaw reciprocity

The **theta correspondence** is the set of relations formed between theta lifts as described in the previous section. For instance, consider the formal pairing of automorphic forms on $\widetilde{G(\mathbf{A})}$ defined by

$$\langle f_1, f_2 \rangle_G = \int_{G(k)\backslash \widetilde{G(\mathbf{A})}} f_1(g)\overline{f_2(g)}\, dg.$$

If f_1 and f_2 are cusps forms on $\widetilde{G(\mathbf{A})}$ and $\widetilde{H(\mathbf{A})}$, respectively, and if we form their theta lifts $\theta_\phi(f_1)$ and $\theta_\phi(f_2)$, then by interchanging the order of integration, we obtain the formal identity

$$\langle \theta_\phi(f_1), f_2 \rangle_H = \langle f_1, \overline{\theta_\phi}(f_2) \rangle_G. \tag{1.6}$$

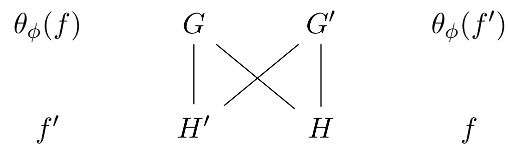
In many applications, the above integrals actually converge (or may be suitably regularized) and the order-interchange can be justified. This identity is then seen to convey interesting arithmetic information.

The scenario of major interest to us arises when one has *a pair* of dual pairs (G, H) and (G', H') in the same symplectic group $Sp(\mathbb{W})$ with $H \subset G'$ and $H' \subset G$. Following Kudla [25], we express this graphically with a **seesaw**:

$$\begin{array}{ccc} G & & G' \\ | & \times & | \\ H' & & H \end{array}$$

The vertical arrows denote inclusion while the diagonal arrows match up members of the same dual pair.

In such a situation, one can start with a cusp form f on H then lift it to G, and a cusp form f' on H' then lift it to G':



The formal identity (1.6) then reads:

$$\langle \mathrm{Res}_{H}^{G'} \theta_\phi(f), f' \rangle_{H'} = \langle f, \mathrm{Res}_{H'}^{G} \overline{\theta}_\phi(f') \rangle_{H}. \tag{1.7}$$

We call this relation **Kudla's seesaw reciprocity**.

Example 1.3.1. (U, U)–(O, Sp) seesaw. Let $K/\mathbf{Q}$ be an imaginary quadratic field and fix a purely imaginary element $\delta \in K$ (i.e. $\delta^\sigma = -\delta$, where σ is the nontrivial Galois automorphism of K). Fix also an embedding $K \hookrightarrow \mathbf{C}$ for which $\mathrm{Im}\,\delta > 0$.

Let V_1 and V_2 be Hermitian spaces over K and denote their respective Hermitian forms by $(\cdot,\cdot)_i$ for $i = 1, 2$. Then $(U(V_1), U(V_2))$ is a dual pair in $Sp(\mathbb{W})$ where $\mathbb{W} = R_{K/\mathbf{Q}}(V_1 \otimes_K V_2)$ as in Example 1.2.1 (ii).

Let $W_i = R_{K/\mathbf{Q}} V_i$ and give it the symplectic form $\langle \cdot,\cdot \rangle_i = \mathrm{Im}_\delta\,(\cdot,\cdot)_i$. Note that $U(V_i) \subset Sp(W_i)$. Choose maximal isotropic subspace $U_i \subset W_i$ for which $W_i = U_i \oplus \delta U_i$ is a complete polarization of W_i. (Any maximal isotropic U_i such that $U_i \cap \delta U_i = 0$ will work.) Then it is easily checked that $(\cdot,\cdot)_i$ restricts to a nondegenerate symmetric $\mathbf{Q}$-valued bilinear form on U_i and therefore that $O(U_i) = O(U_i, (\cdot,\cdot)_i|_{U_i}) \subset U(V_i)$.

Furthermore, we have $K \otimes_{\mathbf{Q}} U_i = V_i$ hence $V_1 \otimes_{\mathbf{Q}} U_2 = V_1 \otimes_K V_2$. Consequently,

$$W_1 \otimes_{\mathbf{Q}} U_2 = \mathbb{W}$$

and so $Sp(W_1)$ and $O(U_2)$ are a dual pair in $Sp(\mathbb{W})$. We have thus a constructed a seesaw

$$\begin{array}{ccc} Sp(W_1) & & U(V_2) \\ \Big| & \times & \Big| \\ U(V_1) & & O(U_2) \end{array}$$

in $Sp(\mathbb{W})$. ▲

The main result of this thesis will make fundamental use of Kudla's reciprocity relation applied to a certain $(U(1), U(1,1))$–$(O(1,1), Sp(1))$ seesaw (Section 2.1).

1.4 Reciprocity in classical terms

In what follows we shall work classically and not adelically. Thus we will ignore the non-archimedean aspects of the theta correspondence. What this amount to, for our purposes, is the study of period integrals over the symmetric spaces attached to the archimedean components of the various groups under consideration. We briefly summarize the relevant facts in this section. For simplicity, we assume that the metaplectic cover of $\widetilde{Sp}$ splits over the dual pairs under consideration, so that we can work with the groups themselves and not their inverse images in $\widetilde{Sp}$. This will be the case for our specific collection of groups.

A seesaw of dual pairs

$$\begin{array}{ccc} G & & G' \\ \Big| & \times & \Big| \\ H' & & H \end{array}$$

in $Sp(\mathbb{W})$ gives rise to a commuting diagram

$$\begin{array}{ccc} H \times H' & \xrightarrow{\mathrm{id}\times\iota} & H \times G \\ \Big\downarrow{\scriptstyle \iota\times\mathrm{id}} & & \Big\downarrow{\scriptstyle \sigma} \\ G' \times H' & \xrightarrow{\sigma'} & Sp(\mathbb{W}) \end{array}$$

where the top corner maps are the inclusions and the bottom corner maps are the ones afforded to us by the dual pairs. Fix a Schwarz–Bruhat function ϕ and let $\Theta = \Theta_\phi$ be the corresponding theta function on $Sp(\mathbb{W})$. By pulling Θ back to $H \times G$ and $G' \times H'$ we obtain two theta kernels

$$\theta(h, g) = \Theta(\sigma(h, g)) \quad \text{and} \quad \theta'(g', h') = \Theta(\sigma'(g', h'))$$

on the respective groups. We will use these to construct theta lifts of the identity functions $\mathbb{1}$ on H and H', respectively. Of course, $\mathbb{1}$ is generally not a cusp form, so some care is needed. For instance, if H is

(i) the orthogonal group of a k-anisotropic quadratic form, or

(ii) a unitary group that is definite at the archimedean places

then $H(k)\backslash H(\mathbf{A})$ is compact, and so there are no issues here. In other situations some regularization of the relevant integrals might be necessary. In any case, let us assume that these issues can be dealt with. We then obtain theta lifts

$$\theta(\mathbb{1}_H)(g) = \int_{H(k)\backslash H(\mathbf{A})} \theta(h, g)\, dh$$

and

$$\theta'(\mathbb{1}_{H'})(g') = \int_{H'(k)\backslash H'(\mathbf{A})} \theta'(g', h')\, dh'.$$

Write $\theta(g) = \theta(\mathbb{1}_H)(g)$ and $\theta'(g') = \overline{\theta'}(\mathbb{1}_{H'})(g')$ and view these theta functions as automorphic forms on $G(\mathbb{A})$ and $G'(\mathbb{A})$, respectively. Applying Kudla's seesaw reciprocity relation (1.7), we obtain the identity

$$\int_{H'(k)\backslash H'(\mathbf{A})} \theta(h')\, dh' = \int_{H(k)\backslash H(\mathbf{A})} \theta'(h)\, dh.$$

Supposing now that our choice of Schwarz–Bruhat function ϕ has trivial non-archimedean component, this identity reduces to essentially take the form

$$\int_{\Gamma'\backslash D'} \theta = \int_{\Gamma\backslash D} \theta' \tag{1.8}$$

(up to perhaps some scaling constants) where D' (resp. D) is the symmetric space attached to the real group $H'_\infty(\mathbb{R})$ (resp. $H_\infty(\mathbb{R})$) and Γ' and Γ are certain discrete subgroups of $H'_\infty(\mathbb{R})$ and $H_\infty(\mathbb{R})$. Thus what we have is a type of reciprocity between certain periods of two theta functions.

We hasten to point out that this is a high-level, formal discussion. In practice, several difficulties may arise. To obtain a meaningful interpretation of either side of the reciprocity relation (1.8), one needs to pick an appropriate Schwarz–Bruhat function ϕ in order to realize the theta lift as a familiar object (cf. Example 1.2.2). However, there are *two* theta lifts under consideration here—one coming from each dual pair in the seesaw. The choice of ϕ —and, as a consequence, the resulting associated theta kernel $\theta_\phi(\cdot,\cdot)$—might not be a favorable one for both pairs simultaneously.

Another, more subtle, difficulty is that the construction of the Weil representation that we have given (and shall use) relies on a *choice* of polarization $\mathbb{W} = \mathbb{U} \oplus \mathbb{U}'$ of the ambient symplectic space $\mathbb{W}$. Each dual pair in the seesaw might suggest a certain natural polarization.

Example 1.4.1. In an $(O(V), Sp(W))$ dual pair, where $\mathbb{W} = V \otimes_{\mathbf{Q}} W$, a natural choice is $\mathbb{U} = V \otimes_{\mathbf{Q}} U$, where U is a maximal isotropic subspace of W.

In a $(U(V_1), U(V_2))$ dual pair, where $\mathbb{W} = R_{K/\mathbf{Q}}(V_1 \otimes_K V_2)$, one may proceed as follows. Choose a maximal isotropic subspace U_i in the symplectic space $W_i = R_{K/\mathbf{Q}} V_i$ such that $U_i \cap \delta U_i = 0$ (cf. Example 1.3.1). Then $\mathbb{W} = W_1 \otimes_{\mathbf{Q}} U_2$ and so we may take $\mathbb{U} = U_1 \otimes_{\mathbf{Q}} U_2$.

In our construction of the $(U(V_1), U(V_2))$–$(O(U), Sp(W))$ seesaw from Example 1.3.1, where W is the symplectic space $R_{K/\mathbf{Q}}V_i$ associated to one of the V_i, we see that we will have an amicable polarization provided our U is a maximal isotropic subspace of the symplectic space $R_{K/\mathbf{Q}}V_j$ associated to the *other* Hermitian space. So, in some sense, once a "polarization" $O(U) \subset U(V_j)$ is chosen, it provides us with a natural choice of polarization for the ambient symplectic space $\mathbb{W}$.

We will see in the subsequent Chapters that the nature of this embedded $O \subset U$ can give qualitatively different interpretations of the seesaw reciprocity relation. ▲

Chapter 2

The $(U(1), U(1,1))$–$(O(1,1), Sp(1))$ Seesaw

In this chapter we describe the seesaw of dual pairs that will play a key part in our main result. It will be a special instance of the general construction given in Example 1.3.1.

2.1 Basic setup

Let $K/\mathbf{Q}$ be an imaginary quadratic field of discriminant $D_K < 0$. Fix an embedding $K \hookrightarrow \mathbf{C}$ so that $\delta := \sqrt{D_K}$ lies the upper half-plane $\mathcal{H}$. Define a Hermitian form on K by $\langle z, w\rangle_K = z\overline{w}$ where $w \mapsto \overline{w}$ is the nontrivial Galois automorphism of K. Note that $\overline{\delta} = -\delta$ and $\mathrm{trace}_{K/\mathbf{Q}}(\delta) = 0$. We will write $U(K)$ for the unitary group associated to the Hermitian space $(K, \langle \cdot, \cdot \rangle_K)$, which is definite of signature $(1, 0)$.

Next, let $F/\mathbf{Q}$ be a real quadratic field of discriminant $D_F > 0$. Set $\rho := \sqrt{D_F}$. Endowed with the norm form $\mathrm{N}_{\mathrm{F}/\mathbf{Q}}(a) = aa'$ ($a \mapsto a'$ is Galois conjugation), F may be viewed as a quadratic space over $\mathbf{Q}$. Note that $\rho' = -\rho$ and $\mathrm{trace}_{F/\mathbf{Q}}(\rho) = 0$. The associated bilinear form is given by $(a, b)_F = \frac{1}{2}\mathrm{trace}_{F/\mathbf{Q}}(ab')$ and has signature $(1, 1)$. We will denote its orthogonal group by $O(F)$.

The natural extension of $(\cdot,\cdot)_F$ to a Hermitian form $(\cdot,\cdot)$ on $K \otimes_{\mathbf{Q}} F$, i.e. via

$$(z \otimes a, w \otimes b) = z\overline{w}(a, b)_F = (z, w)_K (a, b)_F,$$

has signature $(1, 1)$ and thus gives us an indefinite unitary group $U(K \otimes_{\mathbf{Q}} F)$.

The unitary groups $U(K)$ and $U(K \otimes_{\mathbf{Q}} F)$ sit together as a dual pair in the symplectic group $Sp(\mathbb{W})$ of the symplectic space

$$\mathbb{W} = \mathrm{R}_{K/\mathbf{Q}}(K \otimes_K (K \otimes_{\mathbf{Q}} F)) \cong \mathrm{R}_{K/\mathbf{Q}}(K \otimes_{\mathbf{Q}} F)$$

endowed with the symplectic form

$$\langle \cdot,\cdot \rangle = \mathrm{Im}_\delta\ [(\cdot,\cdot)_K \otimes (\cdot,\cdot)] = \mathrm{Im}_\delta\,(\cdot,\cdot),$$

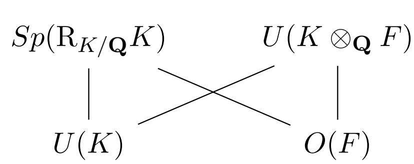
where $\mathrm{Im}_\delta(z) = \dfrac{z - \overline{z}}{2\delta}$.

If we let $W = \mathrm{R}_{K/\mathbf{Q}}K$ and give it the symplectic form

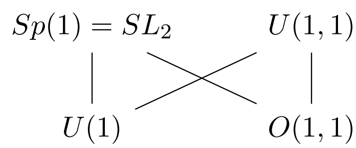
$$\langle \cdot,\cdot \rangle_W = \mathrm{Im}_\delta\,(\cdot,\cdot)_K$$

then the symplectic group $Sp(W)$ sits together with the orthogonal group $O(F)$ as a dual pair in the symplectic group $Sp(W \otimes_{\mathbf{Q}} F) = Sp(\mathbb{W})$.

Thus we have produced a seesaw



of dual pairs in $Sp(\mathbb{W})$. Or, in more descriptive if less precise terms:



in $Sp(2)$.

We close this section with a minor observation for later reference. The special orthogonal group $SO(F) = \{g \in O(F) \mid \det g = 1\}$ can be identified with the group F^1 of units ε in F of norm 1.

Proposition 2.1.1. *The map $F^1 \to SO(F)$ sending $a \in F^1$ to the multiplication-by-a map m_a is an isomorphism.*

Proof. Since $\det m_a = \mathrm{N}_{\mathrm{F}/\mathbf{Q}}\, a$, this map is well-defined. It is also clearly injective. To show that it is surjective, take $g \in SO(F)$ and set $a = g1$. Then $\mathrm{N}_{\mathrm{F}/\mathbf{Q}}\, a = \mathrm{N}_{\mathrm{F}/\mathbf{Q}}\, 1 = 1$ so $a \in F^1$. The basis $\{1, \rho\}$ for F is orthogonal. Thus so is $\{g1, g\rho\} = \{a, g\rho\}$. We claim that $a \perp a\rho$. Indeed:

$$(a, a\rho)_F = \frac{1}{2}\mathrm{trace}_{\,F/\mathbf{Q}}(aa'\rho') = \frac{1}{2}\mathrm{trace}_{\,F/\mathbf{Q}}(\rho) = 0.$$

Since $\dim_{\mathbf{Q}} F = 2$, it follows that $g\rho = ra\rho$ for some $r \in \mathbf{Q}$. Writing $a = u + v\rho$ ($u, v \in \mathbf{Q}$), we find that the matrix of g with respect to the basis $\{1, \rho\}$ is given by

$$\begin{bmatrix} u & rv\rho^2 \\ v & ru \end{bmatrix}$$

whence $\det g = r\mathrm{N}_{\mathrm{F}/\mathbf{Q}}\, a = r$. So $r = 1$ and therefore $g = m_a$. ■

In fact, since Galois conjugation on F also preserves $\mathrm{N}_{\mathrm{F}/\mathbf{Q}}$ and has determinant -1, we have $O(F) = F^1 \rtimes \langle \text{Gal conj} \rangle = SO(F) \rtimes \mu_2$.

2.2 Symmetric spaces

The seesaw of dual pairs constructed in the Section 2.1 gives us a commuting diagram

$$\begin{array}{ccc} U(K) \times O(F) & \xrightarrow{\mathrm{id} \times \iota} & U(K) \times U(K \otimes_{\mathbf{Q}} F) \\ \big\downarrow{\scriptstyle \iota \times \mathrm{id}} & & \big\downarrow{\scriptstyle \sigma} \\ Sp(\mathrm{R}_{K/\mathbf{Q}} K) \times O(F) & \xrightarrow{\sigma'} & Sp(\mathbb{W}) \end{array}$$

where the top corner maps are the obvious inclusions, and the bottom corner maps σ_i are the ones afforded to us by the dual pairs. In particular, note that the definite unitary group $U(K)(\mathbf{R}) = U(1)$ embeds into $Sp(\mathrm{R}_{K/\mathbf{Q}} K)(\mathbf{R}) = Sp(1, \mathbf{R})$ as a maximal compact subgroup.

At the level of symmetric spaces, we have

$$\begin{array}{ccc} B & \xrightarrow{\alpha} & D \\ \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \epsilon} \\ H \times B & \xrightarrow{\epsilon'} & H_2 \end{array}$$

where B, D, H and H_2 are the symmetric spaces associated to the real groups $O(F)(\mathbf{R})$, $U(K \otimes_{\mathbf{Q}} F)(\mathbf{R})$, $Sp(\mathrm{R}_{K/\mathbf{Q}} K)(\mathbf{R})$ and $Sp(\mathbb{W})(\mathbf{R})$, respectively. The symmetric space associated to the compact group $U(K)(\mathbf{R})$ is a point; it appears in the above diagram only implicitly (but importantly) as follows: we have $\beta(x) = (\tau_0, x)$, where $\tau_0 \in H$ is the fixed point given by the inclusion of the maximal compact subgroup $U(K)(\mathbf{R}) \hookrightarrow Sp(\mathrm{R}_{K/\mathbf{Q}} K)(\mathbf{R})$.

Following [24], we shall give explicit descriptions of the above maps. First, some identifications are in order. We will give Grassmannian models of the various symmetric spaces and then, once appropriate coordinates are chosen, we will be able to identify each model with a familiar space. The end result of all this is summarized in Section 2.3.

2.2.1 The symmetric space B

The symmetric space B associated to the group $O(F)(\mathbf{R}) = O(1,1)$ is the Grassmannian of negative lines in $F \otimes_{\mathbf{Q}} \mathbf{R}$:

$$B = \{\ell \in \mathrm{Gr}_1(F \otimes_{\mathbf{Q}} \mathbf{R}) \mid (\cdot,\cdot)_F \text{ is negative-definite on } \ell\}.$$

We identify $F \otimes_{\mathbf{Q}} \mathbf{R} = \mathbf{R} \times \mathbf{R}$ via the two real embeddings $F \hookrightarrow \mathbf{R}$: $a \mapsto (a, a')$. The norm form $\mathrm{N_{F/\mathbf{Q}}}(a) = aa'$ on F then transports to give the quadratic form $q(x,y) = xy$ on $\mathbf{R}^2$. To be precise, these identifications amount to the following. Fix a $\mathbf{Q}$-basis $\{\omega_1, \omega_2\}$ for F and let $Q_F = [(\omega_i, \omega_j)_F] \in GL_2(\mathbf{Q})$ be the corresponding matrix of the bilinear form $(\cdot,\cdot)_F$. Let T_F be a matrix in $GL_2^+(\mathbf{R})$ such that

$$T_F^t Q_F T = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

Then T_F^{-1} defines an isomorphism

$$F \otimes_{\mathbf{Q}} \mathbf{R} = \mathbf{R} \times \mathbf{R} \xrightarrow{\sim} \mathbf{R}^2$$

under which our orthogonal group $O(F)(\mathbf{R})$ is conjugated onto the "standard" orthogonal group $O(1,1)$ of the the quadratic form $q(x,y) = xy$ with matrix $\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$.

Under these identifications, a line $\ell \in \mathrm{Gr}_1(F \otimes_{\mathbf{Q}} \mathbf{R})$ is negative if and only if it is spanned by a vector of the form

$$T_F \begin{bmatrix} x \\ y \end{bmatrix}$$

with $xy < 0$, or equivalently, by a vector of the form

$$T_F \begin{bmatrix} -t \\ 1 \end{bmatrix}$$

with $t > 0$. Thus the map

$$t \mapsto \operatorname{span} T_F \begin{bmatrix} -t \\ 1 \end{bmatrix}$$

identifies $\mathbf{R}_+$ with B.

2.2.2 The symmetric space D

The symmetric space D associated to the group $U(K \otimes_{\mathbf{Q}} F)(\mathbf{R}) = U(1,1)$ is also a negative Grassmannian:

$$D = \{\ell \in \mathrm{Gr}_1((K \otimes_{\mathbf{Q}} F) \otimes_{\mathbf{Q}} \mathbf{R}) \mid (\cdot,\cdot)_K \text{ is negative-definite on } \ell\}.$$

Here, as above, we identify $(K \otimes_{\mathbf{Q}} F) \otimes_{\mathbf{Q}} \mathbf{R} = \mathbf{C}^2$. The corresponding Hermitian form h on $\mathbf{C}^2$ is the Hermitian extension of q from the preceding section: $h((z_1, w_1), (z_2, w_2)) = z_1\overline{w_2} + \overline{z_2}w_1$. Furthermore, the matrix T_F defined above induces an isomorphism

$$T_F^{-1} \colon (K \otimes_{\mathbf{Q}} F) \otimes_{\mathbf{Q}} \mathbf{R} \xrightarrow{\sim} \mathbf{C}^2$$

under which our unitary group $U(K \otimes_{\mathbf{Q}} F)(\mathbf{R})$ is conjugated onto the "standard" unitary group $U(1,1)$ of the Hermitian form with matrix $\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$.

So a negative line ℓ will be spanned by the vector

$$T_F \begin{bmatrix} z \\ w \end{bmatrix}$$

if and only if $z\overline{w} + \overline{z}w < 0$, that is, if and only if $\mathrm{Re}(z\overline{w}) < 0$. We may arrange for the basis vector of ℓ to have $w = \delta^{-1}$. In this case, the condition for ℓ to belong to D becomes that $\mathrm{Re}(-z\delta^{-1}) < 0$ or equivalently that $\mathrm{Im}(z) > 0$. We may thus identify the upper half-plane $\mathcal{H}$ with D via the map

$$z \mapsto \operatorname{span} T_F \begin{bmatrix} z \\ \delta^{-1} \end{bmatrix}.$$

2.2.3 The symmetric spaces $\mathcal{H}$ and $\mathcal{H}_2$

Now we come to the issue of determining models for the symmetric spaces associated to the two symplectic groups $Sp(\mathrm{R}_{K/\mathbf{Q}}K)(\mathbf{R})$ and $Sp(\mathbb{W})(\mathbf{R})$.

We begin with some generalities. Let $(W, \langle , \rangle)$ be a real symplectic space of dimension $2n$. Then one model of the symmetric space associated to the symplectic group $Sp(W)$ is Siegel's generalized upper half-space:

$$\mathcal{H}_n = \{Z \in M_n(\mathbf{C}) \mid Z^t = Z \text{ and } \mathrm{Im}(Z) > 0\}.$$

Since ultimately we wish to construct maps $B \times \mathcal{H} \to \mathcal{H}_2$ and $D \to \mathcal{H}_2$, and since the spaces B and D were just given models as subspaces of Grassmannians, it would be useful to have a similar description of $\mathcal{H}_n$. This can be obtained as follows. View $W(\mathbf{C})$ as the complexification of W and denote complex conjugation by $w \mapsto \overline{w}$. If we extend $\langle \cdot, \cdot \rangle$ to a $\mathbf{C}$-bilinear (skew-symmetric) form on $W(\mathbf{C})$, then

$$h(w, z) = \delta \langle w, \overline{z} \rangle$$

defines a Hermitian form on $W(\mathbf{C})$. Let

$$\mathfrak{h}_n = \{\ell \in \mathrm{Gr}_n(W(\mathbf{C})) \mid \langle , \rangle_\ell = 0,\ h|_\ell < 0\}.$$

Fix a symplectic basis for W and suppose that the subspace $\ell \in \mathfrak{h}_n$ is spanned by the columns of the $2n \times n$ matrix

$$\begin{bmatrix} Z \\ I_n \end{bmatrix}$$

with respect to this basis. Then the map $\mathfrak{h}_n \xrightarrow{\sim} \mathcal{H}_n$ given by $\ell \mapsto Z$ is well-defined and gives us our desired identification (which is $Sp(n)$-equivariant, with respect to the natural actions on both spaces). See [31, §7].

For our symplectic spaces $R_{K/\mathbf{Q}}K$ and $\mathbb{W}$, we may now obtain Grassmannian models of the form $\mathfrak{h}_1$ and $\mathfrak{h}_2$ and then map them into Siegel spaces $\mathcal{H}_1 = \mathcal{H}$ and $\mathcal{H}_2$, respectively, by the process outlined above. For this, we need to choose symplectic bases. A convenient choice will be bases coming from complete polarizations of the form

$$R_{K/\mathbf{Q}}K = U \oplus \delta U \quad \text{and} \quad \mathbb{W} = U' \oplus \delta U',$$

where U (resp. U') is a maximal isotropic subspace of $R_{K/\mathbf{Q}}K$ (resp. $\mathbb{W}$).

For instance, we may take $\{1, \delta\}$ and $\{\omega_1, \omega_2, \delta e_1, \delta e_2\}$, respectively, where $\{\omega_1, \omega_2\}$ is any $\mathbf{Q}$-basis for F (e.g. the basis used to obtain the identifications $B = \mathbf{R}_+$ and $D = \mathcal{H}$ in the previous two sections). We will describe some useful choices in Section 2.4.

The point, however, of choosing bases that come from polarizations as above is that the restrictions of the Hermitian forms $(\cdot,\cdot)_K$ and $(\cdot,\cdot)$ to U and U', respectively, are non-degenerate symmetric bilinear forms. (If we use our choice of bases in the previous paragraph, then in fact $U' = F$ and $(\cdot,\cdot)|_{U'} = (\cdot,\cdot)_F$.)

Consequently, if we let $Q_K \in GL_1(\mathbf{Q})$ and $Q_F \in GL_2(\mathbf{Q})$ be the matrices of these aforementioned bilinear forms with respect to our choice of bases (in particular, $Q_F = [(\omega_i, \omega_j)_F]$ will coincide with the Q_F introduced in Section 2.2.1 if we choose a **Q**-basis as above), then the matrices of the symplectic forms will be given by

$$\langle\cdot,\cdot\rangle \sim \begin{bmatrix} & -Q_K \\ Q_K & \end{bmatrix}$$

and

$$\langle\cdot,\cdot\rangle_W \sim \begin{bmatrix} & -Q_F \\ Q_F & \end{bmatrix}.$$

Thus, as in the previous sections, our symplectic groups $Sp(R_{K/\mathbf{Q}}K)(\mathbf{Q})$ and $Sp(\mathbb{W})(\mathbf{Q})$ are *conjugates* of the standard $Sp(1,\mathbf{Q})$ and $Sp(2,\mathbf{Q})$, namely

$$Sp(R_{K/\mathbf{Q}}K)(\mathbf{Q}) = \begin{bmatrix} 1 & \\ & Q_K \end{bmatrix} Sp(1,\mathbf{Q}) \begin{bmatrix} 1 & \\ & Q_K \end{bmatrix}^{-1}.$$

and similarly for $Sp(\mathbb{W})(\mathbf{Q})$. Therefore, at the level of symmetric spaces, a point in our $\mathfrak{h}_1$ will be a subspace spanned by the columns of

$$\begin{bmatrix} 1 & \\ & Q_K \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} z \\ Q_K \end{bmatrix}$$

or, equivalently, by the columns of

$$\begin{bmatrix} zQ_K^{-1} \\ 1 \end{bmatrix}$$

and so our map $\mathfrak{h}_1 \to \mathcal{H}_1$ sends it to $zQ_K^{-1} \in \mathcal{H}_1$ (the upper-half plane). A similar comment applies to the map $\mathfrak{h}_2 \to \mathcal{H}_2$.

At any rate, in what follows we will identify $Sp(R_{K/\mathbf{Q}}K)(\mathbf{Q})$ and $Sp(\mathbb{W})(\mathbf{Q})$ with $Sp(1,\mathbf{Q})$ and $Sp(2,\mathbf{Q})$ with the bases and conjugation given above. The same identification passes through to the **R**-points.

2.2.4 The maps α, β, ϵ and ϵ'

We now turn to the issue of describing the maps

$$\begin{array}{ccc} B & \xrightarrow{\alpha} & D \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\epsilon} \\ H \times B & \xrightarrow{\epsilon'} & H_2 \end{array}$$

between our models of the symmetric spaces obtained in the preceding subsections.

The map $\alpha\colon B \to D$ is obtained from the natural map $O(F) \to U(K \otimes_{\mathbf{Q}} F)$. In terms of the Grassmannian models of B and D described above, it is given by $\ell \mapsto \ell \otimes \mathbf{C}$.

Next, recall that we've already observed that $\beta(t) = (\tau_0, t)$ where τ_0 is the unique fixed point of $U(K)(\mathbf{R}) \hookrightarrow Sp(\mathrm{R}_{K/\mathbf{Q}}K)(\mathbf{R})$ in H.

Finally, we briefly describe the maps ϵ and ϵ'. For the details, refer to §2 of [24]. We shall first define them as maps into $\mathfrak{h}_2$, and then apply the map $\mathfrak{h}_2 \to \mathcal{H}_2$ from the previous section. Carrying forward the notation we introduced there, we consider the complexification $\mathbb{W}(\mathbf{C})$ of $\mathbb{W}(\mathbf{R})$. The underlying $\mathbf{Q}$-vector space $\mathbb{W} = R_{K/\mathbf{Q}}(K \otimes_{\mathbf{Q}} F)$ has an endomorphism Δ induced by multiplication by δ on K. Extend Δ to $\mathbb{W}(\mathbf{C})$ and note that, because $\Delta^2 = \pm\delta^2$, we obtain a decomposition $\mathbb{W}(\mathbf{C}) = \mathbb{W}(\mathbf{C})^+ \oplus \mathbb{W}(\mathbf{C})^-$ into $\pm\delta$-eigenspaces. The maps $\varphi^{\pm}(v) = \pm\delta v + \Delta v$ give an isomorphism (resp. anti-isomorphism) of $(K \otimes_{\mathbf{Q}} F)(\mathbf{R}) = \mathbb{W}(\mathbf{R})$ onto $\mathbb{W}(\mathbf{C})^{\pm}$. Since Δ is real, we have that $\overline{\mathbb{W}(\mathbf{C})^+} = \mathbb{W}(\mathbf{C})^-$. One can check that the map φ^+ (resp. φ^-) takes negative (resp. positive) vectors in $(K \otimes_{\mathbf{Q}} F)(\mathbf{R})$ with respect to $(\cdot,\cdot)_K$ to negative vectors in $\mathbb{W}(\mathbf{C})$ with respect to h. Our desired equivariant map $D \to \mathfrak{h}_2$ is then given by

$$\ell \mapsto \varphi^+(\ell) \oplus \varphi^-(\ell^\perp)$$

where $\ell^\perp$ is the orthogonal complement to $\ell \in D$ in $(K \otimes_{\mathbf{Q}} F)(\mathbf{R})$. From this we get ϵ.

To get an explicit expression for $\epsilon\colon \mathcal{H} \to \mathcal{H}_2$, we can proceed as follows. With respect to the bases given in the previous section, the map Δ is given by

$$\begin{bmatrix} & I_2 \\ \delta^2 I_2 & \end{bmatrix}$$

and so the maps $\varphi^{\pm}\colon (K \otimes_{\mathbf{Q}} F) \otimes_{\mathbf{Q}} \mathbf{R} \to \mathbb{W}(\mathbf{C})^{\pm}$ are given by

$$\varphi^{+}(v) = \begin{bmatrix} v \\ \delta v \end{bmatrix} \quad \text{and} \quad \varphi^{-}(v) = \overline{\varphi^{+}(v)} = \begin{bmatrix} \overline{v} \\ -\delta\overline{v} \end{bmatrix}.$$

The point $z \in \mathcal{H}$ corresponds to the line $\ell = \operatorname{span} T_F \begin{bmatrix} z \\ \delta^{-1} \end{bmatrix}$. And since $\ell^{\perp} = \operatorname{span} T_F \begin{bmatrix} \overline{z} \\ \delta^{-1} \end{bmatrix}$, we have that the point in $\mathfrak{h}_2$ corresponding to $\epsilon(z)$ is given by the span of the columns

$$\begin{bmatrix} \phi^{+}(T_K \otimes \ell), \, \phi^{-}(T_K \otimes \ell^{\perp}) \end{bmatrix} = \begin{bmatrix} T_K \otimes T_F \begin{bmatrix} z \\ \delta^{-1} \end{bmatrix} & T_K \otimes T_F \begin{bmatrix} z \\ -\delta^{-1} \end{bmatrix} \\ Q_K T_K \otimes Q_F T_F \begin{bmatrix} \delta z \\ 1 \end{bmatrix} & Q_K T_K \otimes Q_F T_F \begin{bmatrix} -\delta z \\ 1 \end{bmatrix} \end{bmatrix}$$

In view of our remarks at the end of the previous section, it follows that the desired point $\epsilon(z)$ in $\mathcal{H}_2$ is given by

$$T_K T_F \begin{bmatrix} z & z \\ \delta^{-1} & -\delta^{-1} \end{bmatrix} \left(Q_K T_K Q_F T_F \begin{bmatrix} \delta z & -\delta z \\ 1 & 1 \end{bmatrix} \right)^{-1}. \tag{2.1}$$

Next, to get a description for ϵ' note first that the above discussion for ϵ amounts to giving a map from $D = \mathfrak{h}_1 = \mathfrak{h}$ to $\mathfrak{h}_2$. This in turn can be used to define a map $\mathfrak{h} \times B \to \mathfrak{h}_2$ by

$$(\ell_0, \ell) \mapsto \varphi^{+}(\ell_0) \otimes \ell^{\perp} + \varphi^{-}(\ell_0^{\perp}) \otimes \ell.$$

This is well-defined because F is isotropic for $\langle \cdot, \cdot \rangle$ and the signs of $(\cdot, \cdot)_F$ and h agree by construction. From this we get our desired equivariant map $\epsilon'\colon \mathcal{H} \times B \to \mathcal{H}_2$. For an explicit expression, we can argue as above. We find that the point in $\mathfrak{h}_2$

corresponding to $\epsilon'(\tau, t)$ is given by the span of the columns of

$$\left[\begin{bmatrix}1 & \\ & Q_F\end{bmatrix}\begin{bmatrix}\overline{\tau}\\1\end{bmatrix}\otimes T_F\begin{bmatrix}-t\\1\end{bmatrix}, \begin{bmatrix}1 & \\ & Q_F\end{bmatrix}\begin{bmatrix}\tau\\1\end{bmatrix}\otimes T_F\begin{bmatrix}t\\1\end{bmatrix}\right] = \begin{bmatrix}T_F\begin{bmatrix}-\overline{\tau}t & \tau t\\ \overline{\tau} & \tau\end{bmatrix}\\ Q_F T_F\begin{bmatrix}-t & t\\ 1 & 1\end{bmatrix}\end{bmatrix}.$$

So $\epsilon'(\tau, t) \in \mathcal{H}_2$ is given by

$$T_F\begin{bmatrix}-\overline{\tau}t & \tau t\\ \overline{\tau} & \tau\end{bmatrix}\left(Q_F T_F\begin{bmatrix}-t & t\\ 1 & 1\end{bmatrix}\right)^{-1}. \tag{2.2}$$

Finally, note that all of our constructions above clearly commute, in the sense that $\epsilon \circ \alpha = \epsilon' \circ \beta$.

2.3 Summary

Our seesaw of dual pairs 2.1

$$\begin{array}{ccc} U(K) \times O(F) & \xrightarrow{\mathrm{id}\times\iota} & U(K) \times U(K \otimes_{\mathbf{Q}} F) \\ \big\downarrow{\scriptstyle \iota\times\mathrm{id}} & & \big\downarrow{\scriptstyle \sigma} \\ Sp(\mathrm{R}_{K/\mathbf{Q}}K) \times O(F) & \xrightarrow{\sigma'} & Sp(\mathbb{W}) \end{array}$$

induces a commuting diagram of maps of symmetric spaces

$$\begin{array}{ccc} B & \xrightarrow{\alpha} & D \\ \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \epsilon} \\ H \times B & \xrightarrow{\epsilon'} & H_2 \end{array}$$

with the symmetric space associated to the compact group $U(K)$ (a point) implicitly occurring in the vertical map $\beta(x) = (\tau_0, x)$ as the fixed point $\tau_0 \in H$ of $U(K)(\mathbf{R}) \hookrightarrow Sp(R_{K/\mathbf{Q}}K)(\mathbf{R})$.

Upon choosing compatible coordinates for the relevant vector spaces, we can obtain identifications

$$U(K)(\mathbf{R}) = U(1;\mathbf{R}), \quad O(F)(\mathbf{R}) = O(1,1;\mathbf{R}), \quad U(K\otimes_{\mathbf{Q}} F) = U(1,1;\mathbf{R}),$$

$$Sp(\mathrm{R}_{K/\mathbf{Q}}K) = Sp(1;\mathbf{R}) \quad \text{and} \quad Sp(\mathbb{W}) = Sp(2;\mathbf{R}),$$

where $U(1)$, $Sp(1)$ and $Sp(2)$ are the standard unitary and symplectic groups, while $O(1,1)$ and $U(1,1)$ are the subgroups of GL_2 preserving the bilinear (resp. Hermitian) form with matrix

$$\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

This amounts to making the following choices. First, choose complete polarizations

$$R_{K/\mathbf{Q}}K = U \oplus \delta U \quad \text{and} \quad \mathbb{W} := R_{K/\mathbf{Q}}(K\otimes_{\mathbf{Q}} F) = U' \oplus \delta U',$$

where U (resp. U') is a maximal isotropic subspace of the symplectic space $R_{K/\mathbf{Q}}K$ (resp. $\mathbb{W}$). Here $\dim_{\mathbf{Q}} U = 1$ and $\dim_{\mathbf{Q}} U' = 2$.

Next, choose $\mathbf{Q}$-bases $\{\xi\}$ and $\{\omega_1, \omega_2\}$ for U and U', and let

$$Q_K = [(\xi,\xi)_K] \quad \text{and} \quad Q_F = [(\omega_i,\omega_j)_F]$$

be the corresponding matrices of $(\cdot,\cdot)_K$ and $(\cdot,\cdot)_F$.

Finally, choose matrices $T_K \in GL_1^+(\mathbf{R})$ and $T_F \in GL_2^+(\mathbf{R})$ such that

$$T_K^t Q_K T_K = [1] \quad \text{and} \quad T_F^t Q_F T_F = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

By conjugating with T_K and T_F, we can isomorphically map our groups onto the "standard" groups mentioned above, obtaining the seesaw

$$\begin{array}{ccc} U(1;\mathbf{R}) \times O(1,1;\mathbf{R}) & \xrightarrow{\mathrm{id}\times\iota} & U(1;\mathbf{R}) \times U(1,1;\mathbf{R}) \\ \downarrow{\scriptstyle \iota\times\mathrm{id}} & & \downarrow{\scriptstyle \sigma} \\ Sp(1;\mathbf{R}) \times O(1,1;\mathbf{R}) & \xrightarrow{\sigma'} & Sp(2;\mathbf{R}). \end{array}$$

Under these isomorphisms, the symmetric spaces B, D, H and H_2 admit models as $\mathbf{R}_+, \mathcal{H}, \mathcal{H}$ and $\mathcal{H}_2$, respectively, where $\mathcal{H}$ is the upper-half plane and and $\mathcal{H}_2$ is Siegel space.

The upshot is (compare [24, Proposition 2.1]):

Theorem 2.3.1. *The seesaw of dual pairs*

$$\begin{array}{ccc} U(K) \times O(F) & \xrightarrow{id\times\iota} & U(K) \times U(K \otimes_{\mathbf{Q}} F) \\ \downarrow{\scriptstyle \iota\times id} & & \downarrow{\scriptstyle \sigma} \\ Sp(R_{K/\mathbf{Q}}K) \times O(F) & \xrightarrow{\sigma'} & Sp(\mathbb{W}) \end{array}$$

constructed in Section 2.1 gives rise to a commuting diagram

$$\begin{array}{ccc} \mathbf{R}_+ & \xrightarrow{\alpha} & \mathcal{H} \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \epsilon} \\ \mathcal{H} \times \mathbf{R}_+ & \xrightarrow{\epsilon'} & \mathcal{H}_2 \end{array}$$

of equivariant maps of the corresponding symmetric spaces, given as follows:

(i) $\alpha(t) = -\delta^{-1}t$ *for all* $t > 0$.

(ii) $\beta(t) = (-\delta^{-1}Q_K^{-1}, t)$ *for all* $t > 0$.

(iii) $\epsilon(z) = \omega_2(z)\omega_1(z)^{-1}$ *for all* $z \in \mathcal{H}$, *where*

$$\omega_2(z) = T_K T_F \begin{bmatrix} z & z \\ \delta^{-1} & -\delta^{-1} \end{bmatrix} \quad \text{and} \quad \omega_1(z) = Q_K T_K \delta Q_F T_F \begin{bmatrix} z & -z \\ \delta^{-1} & \delta^{-1} \end{bmatrix}.$$

In particular, the map $\epsilon\colon \mathcal{H} \to \mathcal{H}_2$ *is holomorphic.*

(iv) $\epsilon'(\tau, t) = \eta_2(\tau, t)\eta_1(t)^{-1}$ *for all* $(\tau, t) \in \mathcal{H} \times \mathbf{R}_+$, *where*

$$\eta_2(\tau, t) = T_F \begin{bmatrix} -\overline{\tau}t & \tau t \\ \overline{\tau} & \tau \end{bmatrix} \quad \text{and} \quad \eta_1(t) = Q_F T_F \begin{bmatrix} -t & t \\ 1 & 1 \end{bmatrix}.$$

In (iii) *and* (iv), Q_K, T_K, Q_F *and* T_F *are as defined in the beginning of this section.*

Proof. Everything has been established except for the expressions in (i)–(iv). The ones for ϵ and ϵ' follow immediately from (2.1) and (2.2). Thus all that remains is the determination of the maps α and β.

The map α is given by $\ell \mapsto \ell \otimes \mathbf{C}$. In terms of our description of $B = \mathbf{R}_+$ and $D = \mathcal{H}$, we have that if

$$\ell \longleftrightarrow \operatorname{span} T_F \begin{bmatrix} -t \\ 1 \end{bmatrix}$$

then

$$\ell \otimes \mathbf{C} \longleftrightarrow \operatorname{span} T_F \begin{bmatrix} -t \\ 1 \end{bmatrix} = \operatorname{span} T_F \begin{bmatrix} -\delta^{-1}t \\ \delta^{-1} \end{bmatrix}$$

which is precisely what it means to say that $\alpha(t) = -\delta^{-1}t$. This proves (i).

Next, we have $\beta(t) = (\tau_0, t)$ where τ_0 is the fixed point of the embedding $U(K)(\mathbf{R}) \hookrightarrow Sp(\mathrm{R}_{K/\mathbf{Q}}K)(\mathbf{R})$ on $\mathcal{H}$. Much like how we defined a map $D \to \mathfrak{h}_2$, we can define a map the one-point symmetric space $\{t_0\}$ corresponding to $U(1)$ to $\mathfrak{h}$ by sending the one point t_0 to the span of $\phi^+(t_0) + \phi^-(t_0^\perp) = \phi^+(t_0)$. Here, ϕ^+ is the map

$$\phi^+ \colon K \otimes \mathbf{R} \to W(\mathbf{C})^+,$$

where $W(\mathbf{C})^+$ is the δ-eigenspace of $W(\mathbf{C}) := R_{K/\mathbf{Q}}K$. With our choice of coordinates (i.e., in terms of the basis for the given polarization of $W = R_{K/\mathbf{Q}}K$), we have

$$\phi^+(z) = \begin{bmatrix} z \\ \delta z \end{bmatrix}.$$

Thus, in $\mathfrak{h}$, τ_0 is given by the span of

$$\begin{bmatrix} 1 & \\ & Q_K \end{bmatrix} \begin{bmatrix} T_K \\ -\delta T_K \end{bmatrix} = \begin{bmatrix} T_K \\ -\delta Q_K T_K \end{bmatrix},$$

and so, in the upper half-plane $\mathcal{H}$,

$$\tau = T_K(-\delta Q_K T_K)^{-1} = -\delta^{-1} Q_K^{-1},$$

as claimed. ■

Remark 2.3.2. Notice that, by (ii), our choices of coordinates allow us to identify the fixed point $\tau_0 \in \mathcal{H}$ of $U(1) \hookrightarrow Sp(1)$ as $\tau_0 = -\delta^{-1} Q_K^{-1}$. Here Q_K is a rational number that depends on our choice of polarization for $R_{K/\mathbf{Q}} K$. ▲

2.4 An arithmetic set-up

In this section we will explain how to choose coordinates (in effect, $\mathbf{Q}$-bases for our quadratic fields K and F) in the manner discussed preceding the statement of Theorem 2.3.1. Our choices will be integral bases for ideals in K and F, and will be key to our main results in Chapter 3.

Thus, let $\mathfrak{a}$ (resp. $\mathfrak{b}$) be an integral ideal in K (resp. F). We regard $\mathfrak{a}$ (resp. $\mathfrak{b}$) as a rank 2 Hermitian (resp. quadratic) $\mathbf{Z}$-lattice in K (resp. F) with respect to the Hermitian (resp. quadratic) form from Section 2.1. Note that $\mathfrak{b}$ is anisotropic and has signature $(1, 1)$. The next lemma shows that $\mathfrak{a}$ and $\mathfrak{b}$ admit appropriately "oriented" integral bases.

Lemma 2.4.1. *Let $E/\mathbf{Q}$ be a quadratic number field of discriminant D. Let $\mathfrak{c}$ be* *an integral ideal in E. Then $\mathfrak{c}$ admits an (ordered) integral basis $\{\omega_1, \omega_2\}$ such that*

$$\frac{\omega_2\omega_1' - \omega_1\omega_2'}{\sqrt{D}} > 0,$$

where $'$ denotes Galois conjugation and $\sqrt{D}$ is the square root of D with positive imaginary part if $D < 0$. In particular, this implies that the element $\omega := \omega_2/\omega_1 \in E$ is a quadratic irrational, which lies on the real axis if $D > 0$, or else lies in the upper half-plane $\mathcal{H}$ if $D > 0$.

Proof. Let $\{\omega_1, \omega_2\}$ be an integral basis for $\mathfrak{c}$. Note that

$$\omega_2\omega_1' - \omega_1\omega_2' = \det \begin{bmatrix} \omega_2 & \omega_2' \\ \omega_1 & \omega_1' \end{bmatrix} \neq 0.$$

Furthermore,

$$(\omega_2\omega_1' - \omega_1\omega_2')' = -(\omega_2\omega_1' - \omega_1\omega_2').$$

It follows that $\omega_2\omega_1' - \omega_1\omega_2'$ is real if $D > 0$ and purely imaginary if $D < 0$. On the other hand, since $\omega_2\omega_1' - \omega_1\omega_2'$ switches signs when ω_1 and ω_2 are interchanged, upon re-labeling if necessary we find that

$$\frac{\omega_2\omega_1' - \omega_1\omega_2'}{\sqrt{D}} > 0$$

as claimed.

In particular, if $\omega := \omega_2/\omega_1$, then

$$\frac{\omega - \omega'}{\sqrt{D}} = \frac{w_1 w_1'}{|w_1|^2} \cdot \frac{\omega - \omega'}{\sqrt{D}} = \frac{1}{|w_1|^2} \cdot \frac{\omega_2\omega_1' - \omega_1\omega_2'}{\sqrt{D}} > 0$$

The final assertion in follows from this. ■

Remarks 2.4.2. We can attach geometric meaning to the element ω given in the preceding lemma. Let

$$q_{\mathfrak{c}}(x,y) = \frac{N(x\omega_1 - y\omega_2)}{N\mathfrak{c}} = ax^2 + bxy + cy^2$$

be the binary quadratic form associated to the ideal $\mathfrak{c}$ with integral basis $\{\omega_1, \omega_2\}$ as above.

(i) Suppose that $D < 0$, so that E is imaginary and $q_{\mathfrak{c}}$ is positive-definite. The element ω is the zero in the upper half-plane $\mathcal{H}$ of $q_{\mathfrak{c}}(x,1)$:

$$\omega = \frac{-b+\sqrt{D}}{2a}.$$

Under the action of $\gamma \in \Gamma = SL_2(\mathbf{Z})$ on forms, ω is translated to $\gamma\omega$ (with γ acting on $\mathcal{H}$ as a linear fractional transformation), which will be the zero in $\mathcal{H}$ of the corresponding γ-translated form. Since Γ-orbits of forms correspond bijectively to ideal classes (cf. [3, §7B]), we can thus associate to each ideal class $[\mathfrak{c}]$ in E the point $[\omega]$ in the modular curve $\Gamma\backslash\mathcal{H}$. In this way we obtain h_E CM points in $\Gamma\backslash\mathcal{H}$ associated to the imaginary quadratic field E, where h_E is the class number of E.

(ii) In the real quadratic case, the situation is slightly more subtle. The form $q_{\mathfrak{c}}$ is indefinite and ω is one of the two real roots of $q_{\mathfrak{c}}(x,1)$. The other root is the Galois-conjugate ω' of ω and we have:

$$\omega = \frac{-b+\sqrt{D}}{2a} \quad \text{and} \quad \omega' = \frac{-b-\sqrt{D}}{2a}.$$

The relevant geometric object here is the geodesic arc $C_{\omega,\omega'}$ in $\mathcal{H}$ connecting these two roots. The action of $\gamma \in \Gamma$ on forms translates the roots to $\gamma\omega$ and $\gamma\omega'$, and translates the geodesic arc accordingly. Consequently, each Γ-orbit of forms gives rise to a geodesic cycle[1] $[C_{\omega,\omega'}]$ in $\Gamma\backslash\mathcal{H}$. (See Section 3.3 for an elaboration on this.) The corresponding field theoretic objects are not ideal classes like in the definite case, but rather *narrow* ideal classes (cf. [1, Chapter 2, §7]). Thus, to each narrow ideal class $[\mathfrak{c}]$ in E we can associate the geodesic cycle $[C_{\omega,\omega'}]$ in $\Gamma\backslash\mathcal{H}$. In this way we obtain h_E^+ cycles in $\Gamma\backslash\mathcal{H}$ associated to the real quadratic field E, where h_E^+ is the narrow class number of E. (One can think of these as "real multiplication" (RM) cycles.) ▲

Now, given integral ideals $\mathfrak{a}$ and $\mathfrak{b}$ in K and F, respectively, with oriented bases $\{\xi_1, \xi_2\}$ and $\{\omega_1, \omega_2\}$, we can obtain polarizations for the symplectic spaces $R_{K/\mathbf{Q}}(K)$ and $\mathbb{W} = R_{K/\mathbf{Q}}(K \otimes_{\mathbf{Q}} F)$ as follows. Let $\delta_{\mathfrak{a}} = \xi_2/\xi_1$ be the CM point associated to the basis for $\mathfrak{b}$, as described in the remark above. Let $U = \operatorname{span}_{\mathbf{Q}}\{\delta_{\mathfrak{a}}\}$. Then

$$R_{K/\mathbf{Q}}(K) = U \oplus \delta U$$

is a complete polarization. The corresponding matrix $Q_K \in M_{1\times 1}(\mathbf{Q})$ (cf. Section 2.3), which we will identify with a rational number, is given by

$$Q_K = (\delta_{\mathfrak{a}}, \delta_{\mathfrak{a}}) = N_{K/\mathbf{Q}}\delta_{\mathfrak{a}} = |\delta_{\mathfrak{a}}|^2. \tag{2.3}$$

Thus, if we let $T_K = |\delta_{\mathfrak{a}}|^{-1}$, we have $T_K^t Q_K T_K = 1$.

Similarly, let $U' = \operatorname{span}_{\mathbf{Q}}\{\omega_1, \omega_2\} = F$. Then

$$\mathbb{W} = U' \oplus \delta U'$$

is a complete polarization. We let $Q_F = [(\omega_i, \omega_j)_F]$ and we fix $T_F \in GL_2^+(\mathbf{R})$ such

[1] The image of $C_{\omega,\omega'}$ in $\Gamma\backslash\mathcal{H}$ is *closed*.

that

$$T_F^T Q_F T_F = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

Remark 2.4.3. If instead of the norm form of F we had used the quadratic form

$$q_{\mathfrak{b}}(x,y) = ax^2 + bxy + cy^2$$

associated to the ideal $\mathfrak{b}$ (and the oriented basis $\{\omega_1, \omega_2\}$), so that in particular $b^2 - 4ac = D_F$, then since

$$q_{\mathfrak{b}}(x,y) = \left(\sqrt{a}x + \frac{b}{2\sqrt{a}}y\right)^2 - \left(\sqrt{\frac{D_F}{4a}}y\right)^2$$

we would have

$$S^t Q_F S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{where } S = \begin{bmatrix} \sqrt{a} & \frac{b}{2\sqrt{a}} \\ 0 & \frac{1}{2}\sqrt{\frac{D_F}{a}} \end{bmatrix}.$$

On the other hand,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = R^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} R, \quad \text{where } R = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

So we find that, in this scenario, we can take

$$T_F = S^{-1}R = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 - \frac{b}{\sqrt{D_F}} & 1 - \frac{b}{\sqrt{D_F}} \\ \frac{2a}{\sqrt{D_F}} & \frac{2a}{\sqrt{D_F}} \end{bmatrix} = \frac{2a}{\sqrt{2D_F}} \begin{bmatrix} \rho_{\mathfrak{b}}' & \rho_{\mathfrak{b}} \\ 1 & 1 \end{bmatrix}, \tag{2.4}$$

where $\rho_{\mathfrak{b}} = \omega_2/\omega_1$. This will be used in Section 3.3. ▲

2.5 Coda: $F = \mathbf{Q} \oplus \mathbf{Q}$

It will be of interest to note that our construction in Section 2.1 carries over with only minor changes if F is the split quadratic $\mathbf{Q}$-algebra $\mathbf{Q} \oplus \mathbf{Q}$ instead of a quadratic field. The role of the field norm in this case will be taken by the multiplication map

$N_{F/\mathbf{Q}}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = ab$, and therefore the associated bilinear form $(\cdot,\cdot)_F$ will be given by

$$\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a' \\ b' \end{bmatrix}\right)_F = \frac{1}{2}(ab' + a'b).$$

Evidently, $(\cdot,\cdot)_F$ is *isotropic* in this case, unlike in the case where F is a real quadratic field. Its signature remains $(1,1)$, and its associated matrix (with respect to the standard basis $\{e_1, e_2\}$ of F) is now

$$Q_F = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

(So $T_F = I_2$.) The arithmetic setup in Section 2.4 is no longer interesting in this case (in place of an integral ideal $\mathfrak{b}$, we can work with an integral lattice, which for our purposes will always be of the form $\mathfrak{b} = \mathbf{Z}^2$ or $N\mathbf{Z}^2$ for some $N \in \mathbf{Z}_{>0}$) and Proposition 2.1.1 should be replaced with:

Proposition 2.5.1. *Let $F = \mathbf{Q}^2$, viewed as a quadratic space over $\mathbf{Q}$ as above. Then*

$$SO(F) = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a \in \mathbf{Q}^\times \right\}$$

and $O(F) = SO(F) \coprod SO(F)Q_F$, where $Q_F = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ as above.

Proof. We have $O(F) = \{g \in GL(F) \colon N(gx) = N(x) \text{ for all } x \in F\}$. By considering $x = e_1$, e_2 and $e_1 + e_2$, we quickly see that $g \in O(F)$ must be of the form

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \text{ or } \begin{bmatrix} 0 & b \\ b^{-1} & 0 \end{bmatrix}.$$

This shows that

$$O(F) = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \right\} \coprod \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \right\} Q$$

which simultaneously establishes all claims in the proposition. ∎

Remark 2.5.2. We note for future reference that if $g = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \in SO(F)$ preserves the lattice $\mathfrak{b} = \mathbf{Z}^2$ then both a and a^{-1} must be integers, and consequently $g = \pm I$. ▲

On the other hand, the material in Section 2.3 goes through verbatim, as does Theorem 2.3.1.

2.6 Strategy

The basic machinery is now all set up. What remains is the input from the theta correspondence. This will take the following form. (Cf. Section 1.4.)

In principle, we will attempt to construct a theta function Θ on $Sp(\mathbb{W})$ whose restrictions to $U(K) \times U(K \otimes_{\mathbf{Q}} F)$ and $Sp(\mathrm{R}_{K/\mathbf{Q}} K) \times O(F)$ will essentially coincide with certain classical theta series ϑ_K and ϑ_F originally considered by Hecke. However, this will not quite work, because we are not able to choose an appropriate Schwarz–Bruhat function: the natural choices yield theta kernels whose restrictions are non-holomorphic. We will therefore have to tweak our theta kernels to mend the situation.

This notwithstanding, Kudla's reciprocity relationship (1.8) will take the form

$$\int_{\Gamma_F^+ \backslash B} \vartheta_K = \int_{\Gamma' \backslash D'} \vartheta_F$$

where D' is the Hermitian space associated to the compact real group $U(K)(\mathbb{R})$—i.e. it is *a point!*—and Γ_F^+ is a certain discrete subgroup of $SO(F)(\mathbb{R})$. (If F is a real quadratic field, then Γ_F^+ will be a subgroup of the group F^1 of norm-1 units in F. If $F = \mathbf{Q}^2$, then Γ_F^+ will be trivial.) Here we are viewing $\Gamma_F^+ \backslash B$ as a cycle in $\Gamma_K \backslash D$ (for some certain Γ_K) by the embedding described in the previous section. Thus the above identity is of the form

$$\text{period integral of } \vartheta_K = \text{special value of } \vartheta_F.$$

We will elucidate this in the next chapter, where we will see in particular that the period integral is is over a cycle obtained from a geodesic arc in $\mathcal{H}$ that is naturally associated to (an ideal class in) F, while the special value is a CM point in K (cf. Remarks 2.4.2).

Chapter 3

Theta Functions

We continue with the notation from the preceding chapter. Thus F (resp. K) is a real (resp. imaginary) quadratic extension of $\mathbf{Q}$, and we have a seesaw of dual pairs

$$\begin{array}{ccc} Sp(\mathrm{R}_{K/\mathbf{Q}}K) & & U(K\otimes_{\mathbf{Q}}F) \\ | & \times & | \\ U(K) & & O(F) \end{array}$$

$$\iff \quad \begin{array}{ccc} U(K)\times O(F) & \xrightarrow{\mathrm{id}\times\iota} & U(K)\times U(K\otimes_{\mathbf{Q}}F) \\ \downarrow{\scriptstyle \iota\times\mathrm{id}} & & \downarrow{\scriptstyle \sigma} \\ Sp(\mathrm{R}_{K/\mathbf{Q}}K)\times O(F) & \xrightarrow{\sigma'} & Sp(\mathbb{W}) \end{array}$$

in the symplectic group of $\mathbb{W} = R_{K/\mathbf{Q}}(K\otimes_{\mathbf{Q}}F)$. The maps in the bottom diagram induce equivariant maps between the associated symmetric spaces

$$\begin{array}{ccc} \mathbf{R}_+ & \xrightarrow{\alpha} & \mathcal{H} \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \epsilon} \\ \mathcal{H}\times\mathbf{R}_+ & \xrightarrow{\epsilon'} & \mathcal{H}_2 \end{array} \tag{3.1}$$

in the manner described in Section 2.3.

We will construct a theta function Θ on $\mathcal{H}_2$ whose (suitably modified) pullbacks

ϑ_K and Θ_F to $\mathcal{H}$ and $\mathcal{H}\times\mathbf{R}_+$ transform like weight 2 (resp. weight 1) modular forms in the $\mathcal{H}$ variable. Functions of the form ϑ_K and Θ_F were originally considered by Hecke in the context of theta series attached to quadratic number fields. In view of our commuting diagram above, they will have a common specialization to $\mathbf{R}_+$, and we will find that in fact

$$\vartheta_K(\alpha(t)) = \tau_0^{-1}\,\Theta_F(\tau_0, t),$$

for all t. Here τ_0 is the fixed point of $U(K)(\mathbf{R}) \hookrightarrow Sp(\mathrm{R}_{K/\mathbf{Q}}K)(\mathbf{R})$ acting on $\mathcal{H}$ (cf. Theorem 2.3.1 and Section 2.4).

In the case where $F/\mathbf{Q}$ is a field, we will then have, for a certain unit $\varepsilon > 1$ of F,

$$\int_1^{\varepsilon} \vartheta_K(\alpha(t))\frac{dt}{t} = i\int_1^{\varepsilon} \Theta_F(\tau_0, t)\,\frac{dt}{t}.$$

The right-hand integral is the value at $\tau = \tau_0$ of

$$\vartheta_F(\tau) = \int_1^{\varepsilon} \Theta_F(\tau, t)\,\frac{dt}{t}.$$

This function is actually a holomorphic modular form of weight 1. Thus what we have done above essentially amounts to an identity of the form

$$\int_{\langle\varepsilon\rangle\backslash\mathbf{R}_+} \vartheta_K(z)\,dz = \vartheta_F(\tau_0)$$

which expresses the period of ϑ_K over an arc in $\mathcal{H}$ associated to a certain unit ε of F as a special value of ϑ_F at a certain CM point of K; cf. Theorem 3.2.3 and the discussion that follows it.

In the case where $F = \mathbf{Q}^2$, instead of integrating over (what amounts to) a hyperbolic geodesic arc in $\mathcal{H}$ as above, we integrate over the geodesic ray from 0 to $i\infty$. The resulting integral is then essentially the Mellin transform evaluated at $s = 0$ of $\Theta_F(\tau, t)$, viewed as a function of $t > 0$. In this case, the Mellin transform $\vartheta_F(\tau)$ is seen to be a holomorphic Eisenstein series of weight 1, and our relation above exhibits the period of $\vartheta_K(z)$ again as the special value of this Eisenstein series at $\tau = \tau_0$. This result was discovered by Hecke [12, Satz 1].

3.1 A theta function Θ on Siegel space $\mathcal{H}_2$

We begin by introducing an auxiliary "reduced theta function" in a general setting. For a full-rank $\mathbf{Z}$-lattice $L \subset \mathbf{Q}^2$ and points $r \in \mathbf{Q}^2$, $\tau \in \mathcal{H}_2$ and $v \in \mathbf{C}^2$, put

$$\theta(v, \tau; r, L) = \sum_{y \in r+L} e\left(\frac{1}{2}v^t(\tau - \overline{\tau})^{-1}v + \frac{1}{2}y^t\tau y + y^t v\right),$$

where $e(\cdot) = \exp(2\pi i \cdot)$; and for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp(2, \mathbf{R})$, let

$$\theta|_g(v, \tau; r, L) = \det J(g, \tau)^{-1/2}\theta(J(g, \tau)^{-t}v, g\tau; r, L),$$

where $J(g, \tau) = c\tau + d$ and we've chosen a continuous branch of $\det J(g, \tau)^{1/2}$ on $\mathcal{H}_2$. This theta function satisfies the following automorphy relation: there is an $M \in \mathbf{Z}_{>0}$ such that

$$\theta|_\gamma(v, \tau; r, L) = \lambda(\gamma)\theta(v, \tau; r, L) \tag{3.2}$$

for all $\gamma \in \Gamma(M) := \{g \in Sp(2, \mathbf{Z}) \mid g \equiv I_2 \pmod{M}\}$. Here $\lambda(\gamma)$ is a fourth root of unity that depends on γ and the chosen branch of $\det J(g, \tau)^{1/2}$; and we may take $M = 2m^2$ where m is any positive integer such that $mr \in \mathbf{Z}^2$. See [33, Proposition 1.4].

Kudla [24] discovered that certain modifications to the derivative

$$\Theta(\tau; r, L) = \frac{1}{2\pi i} \frac{\partial}{\partial v_1}\bigg|_{v=0} \theta(v, \tau; r, L) = \sum_{y \in r+L} y_1 e(\tfrac{1}{2} y^t \tau y) \qquad (\tau \in \mathcal{H}_2)$$

of θ will produce functions that have desirable properties once pulled back to $\mathcal{H}$ and $\mathcal{H} \times \mathbf{R}_+$. In fact, [24] gives much more general results. In what follows we will describe the relevant constructions for our specific set-up.

Thus, we now assume that we have a pair quadratic extensions K and F of $\mathbf{Q}$ as in the previous Chapter. We will also fix integral ideals $\mathfrak{a}$ and $\mathfrak{b}$ in K and F, respectively, with oriented integral bases $\{\xi_1, \xi_2\}$ and $\{\omega_1, \omega_2\}$ as in Section 2.4. Furthermore, let $\delta_{\mathfrak{a}} = \xi_2/\xi_1$ be the CM point corresponding to $\mathfrak{a}$ and its chosen integral basis (cf. Remark 2.4.2). (If $F = \mathbf{Q}^2$ is the split $\mathbf{Q}$-algebra, we let $\mathfrak{b} = \mathbf{Z}$, and take $\{\omega_1, \omega_2\}$ to be the standard basis.) Finally, let Q_K, T_K, Q_F and T_F be as they were in Sections 2.4 and 2.5.

With this data, and in the manner discussed Section 2.3, we can identify K and F with $\mathbf{Q}^2$. In our examples of interest, the lattice $L \subset \mathbf{Q}^2$ used in the definition of $\Theta(\tau; r, L)$ will always be a fractional ideal (in fact, an explicit multiple of an integral ideal) in one of these quadratic extensions.

We begin with the definition of the modified pullback of Θ to $\mathcal{H}$. Let

$$\vartheta_K(z; r, L) = Q_K^{\frac{1}{2}} |\det Q_F|^{\frac{1}{4}} \det \omega_1(z)^{-\frac{1}{2}} \frac{1}{2\pi i} \frac{\partial}{\partial v_1}\bigg|_{v=0} \theta(\check{\omega}_1(z) v, \epsilon(z); r, L)$$

for $z \in \mathcal{H}$, where $\check{\omega}_1(z) = \omega_1(z)^{-t}$ and $\epsilon(z)$ and $\omega_1(z)$ are as in Theorem 2.3.1 (iii). The normalization constant of $Q_K^{\frac{1}{2}} |\det Q_F|^{\frac{1}{4}}$ here matches that of [24] and allows for cleaner formulas later.

Now, for M as above, let

$$\Gamma_K^*(M) = \{g \in SU(K \otimes_{\mathbf{Q}} F)(\mathbf{R}) \mid \sigma(g) \in \Gamma(M)\}$$

where we've identified $Sp(\mathbb{W})(\mathbf{R})$ with $Sp(2, \mathbf{C})$ as described in Section 2.2.3.

Remark 3.1.1. To be a bit more precise here, we observe that our identification of $Sp(\mathbb{W})$ with $Sp(2, \mathbf{Q})$ involved the identifications $K \cong \mathbf{Q}^2$ and $F \cong \mathbf{Q}^2$, hence $K \otimes_{\mathbf{Q}} F \cong \mathbf{Q}^4$, via the choice of integral bases given above. This can be done in a manner compatible with the integral structure. Namely, under our usual identifications, we see that the copy of $\mathbf{Z}^4 \subset \mathbf{Q}^4$ needed to make sense of $Sp(2, \mathbf{Z})$ (hence of $\Gamma(M)$ and $\Gamma_K^*(M)$) is the $\mathbf{Z}$-module spanned by $\{\omega_1, \omega_2, \omega_1\delta_{\mathfrak{a}}, \omega_2\delta_{\mathfrak{a}}\}$. ▲

In view of the preceding remark, $\Gamma_K^*(M)$ should really be denoted by $\Gamma_K^*(M; \delta_{\mathfrak{a}}, \{\omega_1, \omega_2\})$ to highlight the dependence on the underlying integral structure. However, this notation is rather cumbersome and shall be suppressed in the sequel. Notice that $\Gamma_K^*(M)$ is a discrete subgroup of $SU(K \otimes_{\mathbf{Q}} F)(\mathbf{R}) \cong SU(1,1) \cong SL_2(\mathbf{R})$.

Theorem 3.1.2. *The function ϑ_K is holomorphic on $\mathcal{H}$. Furthermore, there is a finite-index, torsion-free subgroup $\Gamma_K(M) \subset \Gamma_K^*(M)$ such that for all $g \in \Gamma_K(M)$ we have $\vartheta_K|_g = \vartheta_K$ and so $\vartheta_K(z; r, L)\, dz$ defines a holomorphic $(1,0)$-form on $\Gamma_K(M)\backslash\mathcal{H}$.*

Proof. This follows from [24, Corollary 4.3]. In our set-up, we take $w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in the differential operator $\nabla(w)$ instead of Kudla's $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This difference has no effect because of symmetry. Finally, we can guarantee that our Γ_K be torsion-free by replacing it, if necessary, with a finite-index subgroup. ■

Thus ϑ_K is a weight 2 holomorphic modular form on $\mathcal{H}$ of level $\Gamma_K(M)$. In fact it is a cusp form: if $(K \otimes_{\mathbf{Q}} F, (\cdot,\cdot))$ is anisotropic then it will turn out that $\Gamma_K(M)\backslash\mathcal{H}$ is *compact*; see Section 3.8. If, on the other hand, $(K \otimes_{\mathbf{Q}} F, (\cdot,\cdot))$ is isotropic then this follows from [24, Proposition 7.4]. Moreover we can extract from [24, Proposition 7.4] a more explicit expression for ϑ_K in this case:

Theorem 3.1.3. *If $(K \otimes_{\mathbf{Q}} F, (\cdot,\cdot))$ is isotropic then*

$$\vartheta_K(z; r, L) = |Q_K|^{\frac{1}{2}}(-\delta)^{\frac{1}{2}} \sum_{y \in r+L} (y_1 + \delta y_2)\, e(|\delta|^2(|y_1|^2 - \delta^2|y_2|^2)z).$$

Proof. Our $\vartheta_K(z; r, L)$ is an instance of (7.15) in [24, Proposition 7.4] and in our case, using the notation of [loc. cit.], the summand reduces to

$$(-\delta)^{1/2}\, y^t\overline{w}\, e(\frac{1}{2}y^t e^{*0}(z)y)$$

where w is as it is there except with $\begin{bmatrix}1\\0\end{bmatrix}$ instead of $\begin{bmatrix}0\\1\end{bmatrix}$. Thus

$$y^t\overline{w} = \delta^{-1}y_1 + y_2 = \delta^{-1}(y_1 + \delta y_2).$$

Finally, $\frac{1}{2}y^t e^{*0}(z)y$ is evaluated at the bottom of page 55 [loc. cit.] which for us reduces simply to

$$|\delta|^2|y|^2 z = |\delta|^2(|y_1|^2 - \delta^2|y_2|^2)z.$$

This completes the proof. ∎

Next we consider the pullback to $\mathcal{H} \times \mathbf{R}_+$. Let

$$\Theta_F(\tau, t; r, L) = (\mathrm{Im}_\delta \tau)^{\frac{1}{2}} |\det Q_F|^{\frac{1}{4}} \det \eta_1(t)^{-\frac{1}{2}}\ \frac{1}{2\pi i}\ \frac{\partial}{\partial v_1}\bigg|_{v=0} \theta(\check{\eta}_1(t)v, \epsilon'(\tau, t); r, L) \tag{3.3}$$

for $(\tau, t) \in \mathcal{H} \times \mathbf{R}_+$, where η_1 and ϵ' are as in Theorem 2.3.1. And similarly to what was done above, set

$$\Gamma_F^*(M) = \Gamma_F^*(M; \delta_{\mathfrak{a}}, \{\omega_1, \omega_2\}) = \{g \in Sp(\mathrm{R}_{K/\mathbf{Q}}K) \mid \sigma'(g) \in \Gamma(M)\}.$$

Notice that in this case, if we identify $Sp(\mathrm{R}_{K/\mathbf{Q}}K)$ with $Sp(1, \mathbf{Q})$, then we can identify $\Gamma_F^*(M)$ with a subgroup of $Sp(1, \mathbf{Z}) \subset Sp(1, \mathbf{Q})$, where $\mathbf{Z}^2 \subset \mathbf{Q}^2$ corresponds to $\mathbf{Z} \oplus \delta_{\mathfrak{a}}\mathbf{Z} \subset K$; cf. Remark 3.1.1.

Theorem 3.1.4. *We have*

$$\Theta_F(g\tau, t; r, L) = \lambda(\sigma'(g))j(g, \tau)\Theta_F(\tau, t; r, L)$$

for all $g \in \Gamma_F^*(M)$, where $\lambda(\sigma'(g))$ is a fourth of unity as before, and $j(g, \tau) = c\tau + d$ is the standard automorphy factor for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp(R_{K/\mathbf{Q}}K) = Sp(1)$.

Proof. This follows from Kudla [24, Corollary 4.5]. ■

Remark 3.1.5. The theta series Θ_F above is of the type considered in [22]. Using the results there (cf. [loc. cit., Proposition 1.1] in particular), we can pinpoint its level a little more explicitly. Indeed, the automorphy relation in the preceding theorem holds for all $g \in \Gamma_0(M)$, where we can take for M any positive integer satisfying:

$$ML^\vee \subset L \quad \text{and} \quad M(x, y) \subset 2\mathbf{Z} \text{ for all } x, y \in L^\vee$$

where $L^\vee = \{x \in \mathbf{Q}^2 \mid (x, L) \subset \mathbf{Z}\}$ is the dual lattice of L in F with respect to the bilinear form induced by the quadratic from on F. ▲

In fact, Proposition 5.2 of [24] gives a more explicit description of Θ_F that will be useful to us. In order to utilize it in our set-up, a preliminary computation is necessary. We will need to make use of the majorant associated to each $t \in \mathbf{R}_+$, where here we revert to viewing this space as B, the symmetric domain of the group $O(F)(\mathbf{R}) = O(1,1)$ (cf. Section 2.2.1).

Recall that each point $t \in B$ corresponds to a negative line ℓ in $V = F \otimes_{\mathbf{Q}} \mathbf{R} = \mathbf{R}^2$. This gives a decomposition of V into negative and positive definite subspaces: $V = \ell \oplus \ell^\perp$. Each such decomposition determines a positive definite form $(,)_t$—a *majorant* of the form $(,)$—on V obtained by changing the sign of $(,)$ on ℓ:

$$(y, y)_t = -(\mathrm{proj}_\ell y, \mathrm{proj}_\ell y) + (\mathrm{proj}_{\ell^\perp} y, \mathrm{proj}_{\ell^\perp} y).$$

We had previously associated to each $t > 0$ the negative line $\ell = \operatorname{span} T_F \begin{bmatrix} -t \\ 1 \end{bmatrix}$. Recall that under the identification $V = F \otimes_{\mathbf{Q}} \mathbf{R} = \mathbf{R}^2$ the norm form is transported to the quadratic form $q(y) = y_1 y_2$ on $\mathbf{R}^2$. The corresponding bilinear form is $(x, y) = \frac{1}{2}(x_1 y_2 + x_2 y_1)$. Then $\ell^\perp = \operatorname{span} T_F \begin{bmatrix} t \\ 1 \end{bmatrix}$ and we compute

$$\mathrm{proj}_\ell y = \frac{-y_1 t^{-1} + y_2}{2} \begin{bmatrix} -t \\ 1 \end{bmatrix} \quad \text{and} \quad \mathrm{proj}_{\ell^\perp} y = \frac{y_1 t^{-1} + y_2}{2} \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

Consequently,

$$(\mathrm{proj}_\ell y, \mathrm{proj}_\ell y) = -\frac{1}{4}(y_1 t^{-1/2} - y_2 t^{1/2})^2$$

and

$$(\mathrm{proj}_{\ell^\perp} y, \mathrm{proj}_{\ell^\perp} y) = \frac{1}{4}(y_1 t^{-1/2} + y_2 t^{1/2})^2.$$

With this in hand, we are ready for the following key result.

Theorem 3.1.6. *We have*

$$\Theta_F(\tau, t; r, L) = (\mathrm{Im}_\delta \tau)^{\frac{1}{2}} \sum_{y \in r+L} (y_1 t^{-\frac{1}{2}} + y_2 t^{\frac{1}{2}}) \times$$
$$\times\, e(\tfrac{1}{4}\tau (y_1 t^{-\frac{1}{2}} + y_2 t^{\frac{1}{2}})^2 - \tfrac{1}{4}\overline{\tau}(y_1 t^{-\frac{1}{2}} - y_2 t^{\frac{1}{2}})^2).$$

Proof. Proposition 5.2 in [24] reduces, in our case, to

$$\begin{aligned}\Theta_F(\tau, t; r, L) &= (\mathrm{Im}_\delta \tau)^{1/2} \sum_{y \in r+L} (y, \begin{bmatrix} t \\ 1 \end{bmatrix}) e(\tfrac{1}{2}(y,y)_{\tau,t}) \\ &= (\mathrm{Im}_\delta \tau)^{1/2} \sum_{y \in r+L} \frac{y_1 + y_2 t}{2} e(\tfrac{1}{2}(y,y)_{\tau,t}),\end{aligned}$$

where

$$\begin{aligned}(y,y)_{\tau,t} &= \frac{1}{2}(\tau + \overline{\tau})(y,y) + \frac{1}{2}(\tau - \overline{\tau})(y,y)_t \\ &= \frac{1}{2}\tau((y,y) + (y,y)_t) + \frac{1}{2}\overline{\tau}((y,y) - (y,y)_t) \\ &= \tau(\mathrm{proj}_{\ell^\perp} y, \mathrm{proj}_{\ell^\perp} y) - \overline{\tau}(\mathrm{proj}_\ell y, \mathrm{proj}_\ell y) \\ &= \frac{1}{4}\tau(y_1 t^{-\frac{1}{2}} + y_2 t^{\frac{1}{2}})^2 + \frac{1}{4}\overline{\tau}(y_1 t^{-\frac{1}{2}} - y_2 t^{\frac{1}{2}})^2.\end{aligned}$$

This completes the proof. ■

In this form, Θ_F resembles one of the theta series introduced by Hecke in [11, 14]. In Section 3.4 we pursue this point further.

3.2 The period relation

Given data as in the previous section (namely, K, F, $\mathfrak{a}$, $\mathfrak{b}$, and their integral bases), fix the subgroup $\Gamma_K(M)$ of $SU(K \otimes_{\mathbf{Q}} F)(\mathbf{R})$ from Theorem 3.1.2, and put

$$\Gamma_F(M) = SO(F)(\mathbf{R}) \cap \Gamma_K(M)$$

where here we are viewing $SO(F)$ as a subgroup of $SU(K \otimes F)$ via our embedding ι (the natural inclusion). It is important to not confuse $\Gamma_F(M)$ with the subgroup $\Gamma_F^*(M)$ from Theorem 3.1.4, the latter being a subgroup of $Sp(R_{K/\mathbf{Q}} K)$.

If F is a real quadratic field, then $SO(F) = F^1$ is the group of norm-1 units in F (Proposition 2.1.1). So if we set $\Gamma_F^+(M) = SO(F)(\mathbf{R})^0 \cap \Gamma_F(M)$, where SO^0 denotes the connected component of the identity in SO, then we may view $\Gamma_F^+(M)$ as a cyclic discrete subgroup of $SO(F)(\mathbf{R})^0 \cong \mathbf{R}_+$, say with generator a totally positive unit $\varepsilon_M > 1$ of F.

On the other hand, if $F = \mathbf{Q}^2$, then $\Gamma_F \subseteq \{\pm I\}$ by Remarks 2.5.2 and 3.1.1. We set $\Gamma_F^+(M) = \{I\}$ in this case.

Now, for $\tau \in \mathcal{H}$, define

$$\vartheta_F(\tau; r, L, M) = \int_{\Gamma_F^+(M)\backslash\mathbf{R}_+} \Theta_F(\tau, t; r, L)\,\frac{dt}{t}.$$

Theorem 3.2.1. *The function ϑ_F is a holomorphic modular form of weight 1.*

Proof. This follows from Kudla [24, Theorem 5.3] and the results in [22] (see Theorem 2.6 in particular; cf. also [23]). ■

Remark 3.2.2. The relationship between ϑ_F and Hecke's weight 1 modular forms associated to real quadratic fields will be given in Sections 3.4 and 3.5. ▲

We are now ready to state our main result which relates the special value of ϑ_F at $\tau = \tau_0$ to a certain period integral of ϑ_K. Here

$$\tau_0 = -\delta^{-1} Q_K^{-1} = -\delta^{-1}|\delta_{\mathfrak{a}}|^{-2}$$

is the fixed point of $U(K) \hookrightarrow Sp(R_{K/\mathbf{Q}}K)$ acting on $\mathcal{H}$ that was obtained in Theorem 2.3.1 (see also (2.3)).

Let $C_F(M)$ denote the image of the map $\alpha\colon \Gamma_F^+(M)\backslash\mathbf{R}_+ \hookrightarrow \Gamma_K(M)\backslash\mathcal{H}$.

Theorem 3.2.3 (Abstract Period Relation)**.**

$$\int_{C_F(M)} \vartheta_K(z; r, L)\,dz = i\,\vartheta_F(\tau_0; r, L, M).$$

The proof of this relation will require some preliminary computations, which we collect in the following lemma.

Lemma 3.2.4. *In the notation established so far, we have:*

(i) $T_K = T_K^{-1} Q_K^{-1}$.

(ii) $(\mathrm{Im}_\delta \tau_0)^{\frac{1}{2}} = |\delta|^{-1} Q_K^{-\frac{1}{2}}$.

(iii) $\omega_1(\alpha(t)) = Q_K T_K\, \eta_1(t)$ *for all* $t \in \mathbf{R}_+$*, where* ω_1 *is as in Theorem 2.3.1(iii).*

Proof. The identity in (i) follows immediately from the definition of T_K:

$$T_K Q_K T_K = 1.$$

For (ii), we observe that

$$\mathrm{Im}_\delta(\tau_0) = \mathrm{Im}_\delta(-\delta^{-1} Q_K^{-1}) = Q_K^{-1} \mathrm{Im}_\delta(-\delta^{-1})$$

from which the identity follows, since $\delta^{-1} = |\delta|^{-2}\overline{\delta} = -|\delta|^{-2}\delta$. Finally, using the expressions in Theorem 2.3.1 (i), (iii) and (iv), we have

$$\begin{aligned}
\omega_1(\alpha(t)) &= \omega_1(-\delta^{-1} t) \\
&= Q_K T_K \delta Q_F T_F \begin{bmatrix} -\delta^{-1} t & \delta^{-1} t \\ \delta^{-1} & \delta^{-1} \end{bmatrix} \\
&= Q_K T_K Q_F T_F \begin{bmatrix} -t & t \\ 1 & 1 \end{bmatrix} \\
&= Q_K T_K \eta_1(t),
\end{aligned}$$

as claimed. ■

Proof of Theorem 3.2.3. We are integrating the 1-form $\vartheta_K(z; r, L)\, dz$ along a path in $\Gamma_K \backslash \mathcal{H}$. As we've already remarked, ϑ_K is a cusp form, so this integral is finite.

To ease notation, we will write $\theta_K(z)$ for $\theta_K(z; r, L)$. We have

$$\begin{aligned}\int_{C_F(M)} \vartheta_K(z)\,dz &= \int_{\Gamma_F^+(M)\backslash\mathbf{R}_+} \alpha^*(\vartheta_K(z)\,dz)\\ &= \int_{\Gamma_F^+(M)\backslash\mathbf{R}_+} \vartheta_K(\alpha(t))\,\frac{d(\alpha(t))}{t}\\ &= -\delta^{-1}\int_{\Gamma_F^+(M)\backslash\mathbf{R}_+} \vartheta_K(\alpha(t))\,\frac{dt}{t}\end{aligned}$$

since $\alpha(t) = -\delta^{-1}t$ (Theorem 2.3.1 (i)). Now consider the common specializations of ϑ_K and Θ_F to $\mathbf{R}_+$ in (3.1). Using the fact that $\epsilon(\alpha(t)) = \epsilon'(\beta(t)) = \epsilon'(\tau_0, t)$, together with the identities established in Lemma 3.2.4, we have

$$\begin{aligned}\vartheta_K(\alpha(t)) &= Q_K^{\frac{1}{2}}|\det Q_F|^{\frac{1}{4}}\det\omega_1(\alpha(t))^{-\frac{1}{2}}\,\frac{1}{2\pi i}\,\frac{\partial}{\partial v_1}\bigg|_{v=0}\theta(\check{\omega}_1(\alpha(t))v, \epsilon(\alpha(t)))\\ &= Q_K^{-\frac{1}{2}}|\det Q_F|^{\frac{1}{4}}T_K^{-1}\det\eta(t)^{-\frac{1}{2}}T_K^{-1}Q_K^{-1}\,\frac{1}{2\pi i}\,\frac{\partial}{\partial v_1}\bigg|_{v=0}\theta(\check{\eta}_1(t))v, \epsilon'(\tau_0, t)).\\ &= |\delta|(\mathrm{Im}_\delta\tau_0)^{\frac{1}{2}}|\det Q_F|^{\frac{1}{4}}\det\eta(t)^{-\frac{1}{2}}\,\frac{1}{2\pi i}\,\frac{\partial}{\partial v_1}\bigg|_{v=0}\theta(\check{\eta}_1(t)v, \epsilon'(\tau_0, t))\\ &= |\delta|\Theta_F(\tau_0, t).\end{aligned}$$

Consequently, our integral becomes

$$\begin{aligned}\int_{C_F(M)} \vartheta_K(z)\,dz &= -\delta^{-1}|\delta|\int_{\Gamma_F^+(M)\backslash\mathbf{R}_+} \Theta_F(\tau_0, t)\,\frac{dt}{t}\\ &= i\,\vartheta_F(\tau_0; r, L, M),\end{aligned}$$

as claimed. ■

Remark 3.2.5. From the vantage point of the theta correspondence, our functions ϑ_K and Θ_F are in principle theta kernels. Thus, in the preceding proof, the integral

$$\int_{\Gamma_F^+(M)\backslash\mathbf{R}_+} \vartheta_K(\alpha(t); r, L)\,\frac{dt}{t}$$

is in effect the archimedean component of the Petersson inner product

$$\left\langle \mathrm{Res}^{U(1,1)}_{O(1,1)} \theta(\mathbb{1}_{U(1,1)}), \mathbb{1}_{O(1,1)} \right\rangle_{O(1,1)}$$

on $O(1,1)(\mathbf{Q})\backslash O(1,1)(\mathbf{A})$. Likewise, the integral

$$\int_{\Gamma_F^+(M)\backslash \mathbf{R}_+} \Theta_F(\tau_0, t; r, L)\, \frac{dt}{t}$$

is effectively the archimedean component of

$$\left\langle \mathrm{Res}^{Sp(1)}_{U(1)} \theta(\mathbb{1}_{O(1,1)}), \mathbb{1}_{U(1)} \right\rangle_{U(1)}$$

on $U(1)(\mathbf{Q})\backslash U(1)(\mathbf{A})$. In these terms, the proof above can be interpreted to be an application of Kudla's seesaw reciprocity. ▲

3.3 A geometric interpretation of $C_F(M)$

According to Theorem 2.3.1 (i), α maps the symmetric space $\mathbf{R}_+$ attached to $O(F)$ bijectively onto the line $Y = \{\mathrm{Re}\,\tau = 0\}$ in $\mathcal{H}$. This line is the geodesic ray in $\mathcal{H}$ from 0 to $i\infty$.

If $F = \mathbf{Q}^2$, then $\Gamma_F^+(M)$ is trivial, and therefore $C_F(M)$ is simply this geodesic ray. The case where F is a field is more interesting. Here $\Gamma_F^+(M)$ is an (infinite cyclic) group of totally positive units in F, and the image $C_F(M)$ of $\alpha\colon \Gamma_F^+(M)\backslash \mathbf{R}_+ \hookrightarrow \Gamma_K(M)\backslash\mathcal{H}$ can be interpreted as the projection of one of the geodesic arcs associated to F as described in Remark 2.4.2 (ii). We explain this below.

If F is given fixed integral structure $(\mathfrak{b}, \rho_{\mathfrak{b}})$ as described in Section 2.4, then we can associate to this data the geodesic semicircle $C_{\rho_{\mathfrak{b}}, \rho_{\mathfrak{b}}'}$ in $\mathcal{H}$ connecting the points $\rho_{\mathfrak{b}}$ and $\rho_{\mathfrak{b}}'$ on the real axis. Here we view $C_{\rho_{\mathfrak{b}}, \rho_{\mathfrak{b}}'}$ as living in the symmetric space $\mathcal{H}$

associated to the real group $SU(K \otimes_{\mathbf{Q}} F)(\mathbf{R}) \cong SU(1,1)$. Our map $\alpha\colon \mathbf{R}_+ \to \mathcal{H}$, which is given by

$$\alpha(t) = -\delta_{\mathfrak{a}}^{-1} t$$

parameterizes an arc in $\mathcal{H}$. In the form given above, we are implicitly using an identification of the symmetric space $\mathbf{R}_+$ associated to $SO(F)(\mathbf{R})$ which hides the integral structure. If we keep track of the integral structure on the domain alone, then by what we had discussed in Section 2.4 (see in particular (2.4)), the map is more efficaciously expressed as

$$t \mapsto \frac{2a}{\sqrt{2D_F}} \begin{bmatrix} \rho_{\mathfrak{b}}' & \rho_{\mathfrak{b}} \\ 1 & 1 \end{bmatrix} \cdot (-\delta^{-1} t) = \frac{\rho_{\mathfrak{b}}' + \rho_b(-\delta^{-1}t)}{1 + (-\delta^{-1}t)}, \qquad (0 < t < \infty).$$

This is a parameterization of $C_{\rho_{\mathfrak{b}},\rho_{\mathfrak{b}}'}$! Indeed, the right-side is a linear fractional transformation that maps $\{0, |D_K|^{-1}i, \infty\}$ to $\{\rho_{\mathfrak{b}}'$, the apex of $C_{\rho_{\mathfrak{b}},\rho_{\mathfrak{b}}'}$, $\rho_{\mathfrak{b}}\}$, hence must map the vertical geodesic through $|D_K|^{-i}$ to the hyperbolic geodesic $C_{\rho_{\mathfrak{b}},\rho_{\mathfrak{b}}'}$. (Notice, in particular, that α provides a clockwise orientation of $C_{\rho_{\mathfrak{b}},\rho_{\mathfrak{b}}'}$: from $\rho_{\mathfrak{b}}'$ to $\rho_{\mathfrak{b}}$.)

Our embedded subgroup $SO(F)(\mathbf{R}) \cong SO(1,1)$ acts on $\mathcal{H}$ and the arithmetic subgroup

$$\Gamma_F^+(M) = \Gamma_K(M) \cap SO^+(F) \subset F^1$$

maps $C_{\rho_{\mathfrak{b}},\rho_{\mathfrak{b}}'}$ to itself. Indeed, the full group of norm-1 units F^1 maps $C_{\rho_{\mathfrak{b}},\rho_{\mathfrak{b}}'}$ to itself: if ε_F is the smallest unit $\varepsilon_F > 1$ in F^1, and if we write

$$\varepsilon_F = d + c\rho_{\mathfrak{b}}$$

$$\varepsilon_F \rho = a + b\rho_{\mathfrak{b}}$$

with $a, b, c, d \in \mathbf{Q}$, then the matrix

$$\gamma_F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is in $SL_2(\mathbf{Q})$ and has fixed points $\rho_{\mathfrak{b}}$ and $\rho_{\mathfrak{b}}'$ on the real axis, i.e. it is *hyperbolic*. Any hyperbolic element in $SL_2(\mathbf{R})$ stabilizes the geodesic arc connecting its fixed points. Since γ_F generates F^1, it follows that F^1 stabilizes $C_{\rho_{\mathfrak{b}},\rho_{\mathfrak{b}}'}$.

Consequently, the image $C_F(M)$ of the induced map $\alpha\colon \Gamma_F^+(M)\backslash\mathbf{R}_+ \hookrightarrow \Gamma_K(M)\backslash\mathcal{H}$ may be interpreted as being the projection of this arc into the curve $\Gamma_K(M)\backslash\mathcal{H}$.

3.4 Hecke's real quadratic theta function and ϑ_F

In this section we consider only the case where F is a field. (The case where $F = \mathbf{Q}^2$ will be taken up in Section 3.5.) Let $D_F = \operatorname{disc} F$, let $\mathfrak{b}$ be an integral ideal in $\mathcal{O}_F$ with norm $B = [\mathcal{O}_F\colon \mathfrak{b}]$, and let r be an element of $\mathfrak{b}$. Fix a positive integer M and let $U_F^+(Q\sqrt{D_F})$ denote the group of totally positive units ε in F such that $\varepsilon \equiv 1 \pmod{M\sqrt{D_F}}$. Hecke [11, 14] introduced the theta function

$$\vartheta(\tau; r, \mathfrak{b}, M\sqrt{D_F}) = \sum_{\substack{\mu \equiv r \pmod{\mathfrak{b}M\sqrt{D_F}} \\ \mu\mu' > 0 \\ \operatorname{mod} U_F^+(M\sqrt{D_F})}} \operatorname{sgn}(\mu)\, e\left(\frac{\mu\mu'}{BMD_F}\tau\right) \qquad (\tau \in \mathcal{H}).$$

Here the summation is over the nonzero $U_F^+(M\sqrt{D_F})$ equivalence classes of all μ in F satisfying the above conditions. Hecke showed that this function is (if nonzero[1]) a holomorphic cusp form of weight 1. More specifically, he proved that

$$\vartheta\left(\frac{a\tau+b}{c\tau+d}; r, \mathfrak{b}, M\sqrt{D_F}\right) = \left(\frac{|D_F|}{d}\right) e\left(\frac{abrr'}{BMD_F}\right)(c\tau+d)\,\vartheta(\tau; ar, \mathfrak{b}, M\sqrt{D_F})$$

[1] If F contains a negative unit ε such that $\varepsilon \equiv 1 \pmod{M\sqrt{D_F}}$, then $\vartheta_F \equiv 0$. Conversely, if there is no such unit, then at least $\vartheta_F(\tau; 1, (1), M\sqrt{D_F})$ does not vanish identically [11, Satz 1].

for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(MD_F)$.

Example 3.4.1. Let $\vartheta(\tau) = \vartheta(\tau; 1, (1), 1\sqrt{12})$ (i.e. $D_F = 12$ so $F = \mathbf{Q}(\sqrt{3})$). Then

$$\vartheta(\tau + 1) = e(1/12)\,\vartheta(\tau) \quad \text{and} \quad \vartheta\left(\frac{-1}{\tau}\right) = -i\tau\vartheta(\tau).$$

Thus ϑ is a cusp form of weight 1 and level 12 and must therefore be a scalar multiple of $\Delta^{1/12}$, where

$$\Delta(\tau) = e(\tau) \prod_{n=1}^{\infty} (1 - e(n\tau))^{24}$$

is the discriminant function. The coefficient of $q = e(\tau)$ in ϑ is obviously 1, which coincides with the coefficient of q in $\Delta^{1/12}$. Thus it must be the case that

$$\vartheta(\tau; 1, (1), \sqrt{12}) = \Delta^{1/12}(\tau). \qquad \blacktriangle$$

We now show how our ϑ_F can give Hecke's ϑ. This amounts to picking the appropriate lattice L in the definition of Θ_F.

Theorem 3.4.2. *There is a choice of lattice $L = L(\mathfrak{b}, M, D_F)$ in $F \cong \mathbf{Q}^2$ such that for all $\tau \in \mathcal{H}$ and $r \in \mathfrak{b}$,*

$$\vartheta_F(\tau; r, L, M) = 2|\delta|^{-1/2} \frac{[U_F^+(M\sqrt{D_F}) \colon \Gamma_F^+(M) \cap U_F^+(M\sqrt{D_F})]}{[\Gamma_F^+(M) \colon \Gamma_F^+(M) \cap U_F^+(M\sqrt{D_F})]}\ \vartheta(\tau; r, \mathfrak{b}, M\sqrt{D_F}),$$

where $U_F^+(M\sqrt{D_F})$ is as defined in the beginning of this section.

Proof. Specifically, we will take L to be $\mathfrak{b}M\sqrt{D_F}$, where we are viewing this as a lattice in $F \cong \mathbf{Q}^2$ under our usual identification—namely, with respect to the fixed integral basis $\{\omega_1, \omega_2\}$ for $\mathfrak{b}$. However, instead of using our usual norm form $N_{F/\mathbf{Q}}$ on F, we re-scale it to $\dfrac{1}{BMD_F} N_{F/\mathbf{Q}}$. (This has no effect on any of the theory we have developed thus far.) We use this to define an isometric embedding $L \hookrightarrow F \otimes \mathbf{R} \cong \mathbf{R}^2$

via the map

$$\mu \mapsto \frac{1}{\sqrt{BMD_F}}(\mu, \mu').$$

We identify L with its image in $\mathbf{R}^2$ under this mapping. Then, according to Theorem 3.1.6, we have

$$\Theta_F(\tau, t; r, L) = (\mathrm{Im}_\delta \tau)^{\frac{1}{2}} \sum_{\mu \in r+L} \frac{\mu t^{-\frac{1}{2}} + \mu' t^{\frac{1}{2}}}{\sqrt{BMD_F}} \times$$
$$\times\, e\left(\tfrac{1}{4BMD_F}(\tau(\mu t^{-\frac{1}{2}} + \mu' t^{\frac{1}{2}})^2 - \overline{\tau}(\mu t^{-\frac{1}{2}} - \mu' t^{\frac{1}{2}})^2)\right).$$

We want to compute

$$\vartheta_F(\tau; r, L, M) = \int_{\Gamma_F^+(M)\backslash \mathbf{R}_+} \Theta_F(\tau, t; r, L)\, \frac{dt}{t}.$$

We shall suppress the (r, L, M) in the rest of the proof. Recall that $\Gamma_F^+(M)$ (whose existence depends on the construction of Θ and therefore on the choice of lattice L) is generated by a totally positive unit $\varepsilon_M > 1$. The transformation $\mu \mapsto \varepsilon_M \mu$ has the effect of changing t to $\varepsilon_M^{-1} t$ in the above summands. Consequently, we see that

$$\int_{\Gamma_F^+(M)\backslash \mathbf{R}_+} \sum_{\mu \in r+L} = \int_1^{\varepsilon_M} \sum_{\mu \in r+L} = \sum_{\mu \in r+L} \int_1^{\varepsilon_M} = \sum_{\substack{\mu \in r+L \\ \bmod \Gamma_F^+(M)}} \int_0^\infty .$$

Therefore, writing $\tau = u + iv$, we have

$$\vartheta_F(\tau) = \sqrt{\tfrac{v}{|\delta| BMD_F}} \sum_{\substack{\mu \in r+L \\ \bmod \Gamma_F^+(M)}} \int_0^\infty (\mu t^{-1/2} + \mu' t^{1/2}) \times$$
$$\times\, e\left(\tfrac{1}{4BMD_F}(\tau(\mu t^{-1/2} + \mu' t^{1/2})^2 - \overline{\tau}(\mu t^{-1/2} - \mu' t^{1/2})^2)\right) \frac{dt}{t}$$

$$= \sqrt{\tfrac{v}{|\delta| BMD_F}} \sum_{\substack{\mu \in r+L \\ \bmod \Gamma_F^+(M)}} \int_0^\infty (\mu t^{-1/2} + \mu' t^{1/2}) e\left(\tfrac{\mu\mu' u}{BMD_F}\right) e\left(\tfrac{iv}{2BMD_F}(\mu^2 t^{-1} + \mu'^2 t)\right) \frac{dt}{t}$$

$$= \sqrt{\tfrac{v}{|\delta|BMD_F}} \sum_{\substack{\mu\in r+L \\ \bmod \Gamma_F^+(M)}} e(\tfrac{\mu\mu' u}{BMD_F}) \int_0^\infty (\mu t^{-1/2} + \mu' t^{1/2}) e(\tfrac{iv}{2BMD_F}(\mu^2 t^{-1} + \mu'^2 t)) \, \frac{dt}{t}.$$

A change of variables $t = e^x$ leaves us with the integral

$$\int_{-\infty}^{\infty} (\mu e^{-x/2} + \mu' e^{x/2}) e(\tfrac{iv}{2BMD_F}(\mu^2 e^{-x} + \mu'^2 e^x)) \, dx. \tag{3.4}$$

To evaluate this, we need a lemma.

Lemma 3.4.3. (i) $e^{-2\pi a} = \sqrt{a} \int_{-\infty}^{\infty} e^{v/2} e^{-\pi a(e^{-v}+e^v)}$ *for all* $a > 0$.

(ii) $$\frac{e^{-2\pi t|\mu\mu'|}}{|\mu'|\sqrt{t}} = \int_{-\infty}^{\infty} e^{x/2} e^{-\pi t(\mu^2 e^{-x} + \mu'^2 e^x)} \, dx \text{ for all } t > 0.$$

Proof. (i) The substitution $u = e^{v/2}$ transforms the integral into

$$2\sqrt{a} \int_0^\infty e^{-\pi a(u^{-2}+u^2)} \, du.$$

Introduce a parameter s and consider

$$I(s) = \int_0^\infty e^{-\pi a(su^{-2}+u^2)} \, du.$$

We want $2\sqrt{a}\, I(1)$. We know from the familiar Gaussian integral that $I(0) = 1/(2\sqrt{a})$. To obtain an expression for general s, we differentiate under the integral sign to get

$$I'(s) = \frac{-\pi a}{\sqrt{s}} \int_0^\infty \frac{1}{u^2} e^{-\pi a(su^{-2}+u^2)} \, du.$$

The integral on the right is $I(s)$ as can be seen by making the change of variables $u \leftrightarrow 1/u$. Thus we end up with the differential equation $I'(s) = \frac{-\pi a}{\sqrt{s}} I(s)$, $I(0) = 1/(2\sqrt{a})$, whose solution is $I(s) = \frac{1}{2\sqrt{a}} e^{-2\pi a\sqrt{s}}$. In particular, our desired integral is $2\sqrt{a}\, I(1) = e^{-2\pi a}$, as claimed.

(ii) The change of variables $a = |\mu\mu'|t$ and $v = x + \log|\frac{\mu'}{\mu}|$ applied to (i) gives us

$$e^{-2\pi t|\mu\mu'|} = \sqrt{t|\mu\mu'|}\int_{-\infty}^{\infty}\sqrt{\left|\frac{\mu}{\mu'}\right|}e^{x/2}e^{-\pi t(\mu^2e^{-x}+\mu'^2e^x)}\,dx$$

and the result follows from this. ■

From (ii), we see that

$$\int_{-\infty}^{\infty}\mu' e^{x/2}e(\tfrac{iv}{2BMD_F}(\mu^2e^{-x}+{\mu'}^2e^x))\,dx = \sqrt{\frac{BMD_F}{v}}\,\mathrm{sgn}(\mu')e^{-2\pi\frac{v}{BMD_F}|\mu\mu'|} \qquad (3.5)$$

By symmetry,

$$\int_{-\infty}^{\infty}\mu e^{-x/2}e(\tfrac{iv}{2BMD_F}(\mu^2e^{-x}+{\mu'}^2e^x))\,dx = \sqrt{\frac{BMD_F}{v}}\,\mathrm{sgn}(\mu)e^{-2\pi\frac{v}{BMD_F}|\mu\mu'|}.$$

Thus our integral in (3.4) will be equal to 0 if μ and μ' have different signs, i.e., if $\mu\mu' < 0$, and otherwise it will be equal to

$$2\sqrt{\frac{BMD_F}{v}}\,\mathrm{sgn}(\mu)e^{-2\pi\frac{v}{BMD_F}|\mu\mu'|} = 2\sqrt{\frac{BMD_F}{v}}\,\mathrm{sgn}(\mu)e(\tfrac{iv}{BMD_F}\mu\mu').$$

Hence

$$\begin{aligned}\vartheta_F(\tau) &= 2|\delta|^{-1/2}\sum_{\substack{\mu\in r+L\\ \mu\mu'>0\\ \mathrm{mod}\ \Gamma_F^+(M)}} e(\tfrac{\mu\mu' u}{BMD_F})\mathrm{sgn}(\mu)e(\tfrac{iv}{BMD_F}\mu\mu')\\ &= 2|\delta|^{-1/2}\sum_{\substack{\mu\equiv r\ (\mathrm{mod}\ \mathfrak{b}\sqrt{D_F})\\ \mu\mu'>0\\ \mathrm{mod}\ \Gamma_F^+(M)}}\mathrm{sgn}(\mu)e(\tfrac{\mu\mu'}{BMD_F}\tau).\end{aligned}$$

Finally, it remains to match up the groups $\Gamma_F^+(M)$ and $U_F^+(M\sqrt{D_F})$. Note that if $\varepsilon = a + b\rho$ $(a, b \in \mathbf{Q})$ is any unit in $\Gamma_F^+(M)$ then $\varepsilon \equiv 1 \pmod M$. Indeed, the matrix representation of ε with respect to the basis $\{1, \rho\}$ is

$$\begin{bmatrix} a & b\rho^2 \\ b & a \end{bmatrix}$$

Therefore, $a, b \in \mathbf{Z}$ by Remark 3.1.1, and $a \equiv 1 \pmod{M}$ and $b \equiv 0 \pmod{M}$ because $\sigma'(\varepsilon) \equiv 1 \pmod{M}$.

Since $U_F^+(M)$ is infinite cyclic and contains $\Gamma_F^+(M)$ and $U_F^+(M\sqrt{D_F})$ as non-trivial subgroups, we have that $[\Gamma_F^+(M) \colon \Gamma_F^+(M) \cap U_F^+(M\sqrt{D_F})] < \infty$. So if we specialize the sum in ϑ_F and ϑ to units modulo $\Gamma_F^+(M) \cap U_F^+(M\sqrt{D_F})$, we obtain

$$\begin{aligned}
&[\Gamma_F^+(M) \colon \Gamma_F^+(M) \cap U_F^+(M\sqrt{D_F})]\ \vartheta_F(\tau; r, L, M) \\
&\qquad = 2|\delta|^{-1/2}\, [U_F^+(M\sqrt{D_F}) \colon \Gamma_F^+(M) \cap U_F^+(M\sqrt{D_F})]\, \vartheta(\tau; r, \mathfrak{b}, M\sqrt{D_F}).
\end{aligned}$$

So, up to a constant, our ϑ_F and Hecke's ϑ are one and the same. ■

Remark 3.4.4. We can perform a consistency check at this point. By the preceding theorem, and Hecke's work in [14], we can conclude *a posteriori* that $\vartheta_F(\tau; r, L, M)$ is automorphic of level $\Gamma_0(MD_F)$. On the other hand, we know that ϑ_F is automorphic of level $\Gamma_0(M)$ for M as specified in Remark 3.1.5. Now, the dual lattice of the ideal $\mathfrak{b}$ with respect to the trace pairing on F is $\mathfrak{b}^\vee = \mathfrak{d}^{-1}\mathfrak{b}^{-1} = B^{-1}\mathfrak{d}^{-1}\mathfrak{b}$ where $\mathfrak{d} = (\sqrt{D_F})$ is the different of F. Thus the dual of $L = \mathfrak{b}M\sqrt{D_F}$ with respect to the pairing $\frac{1}{BMD_F}\mathrm{tr}_{F/\mathbf{Q}}$ consists of all $x \in F$ such that

$$\begin{aligned}
\frac{1}{BMD_F}\,\mathrm{tr}(x, L) \subset \mathbf{Z} &\iff \frac{1}{BMD_F}\,\mathrm{tr}(x, \mathfrak{b}M\sqrt{D_F}) \subset \mathbf{Z} \\
&\iff \mathrm{tr}(\frac{1}{B\sqrt{D_F}}x, \mathfrak{b}) \subset \mathbf{Z} \\
&\iff x \in B\sqrt{D_F}\mathfrak{b}^\vee \\
&\iff x \in \mathfrak{b}.
\end{aligned}$$

That is, $L^\vee = \mathfrak{b}$. Hence $MD_F L^\vee \subset L$ (as $D_F\mathfrak{b} = \sqrt{D_F}\sqrt{D_F}\mathfrak{b} \subseteq \sqrt{D_F}\mathfrak{b}$) and, since the trace pairing is even, we may take $M = MD_F$ to satisfy the requirements in Remark 3.1.5. This matches Hecke's result. ▲

Remark 3.4.5. The preceding remark contains the useful observation that the ideal $\mathfrak{b}$ may be recovered as the dual lattice of the quadratic lattice $L = L(\mathfrak{b}, M, D_F)$ given in Theorem 3.4.2. For instance, the condition $r \in \mathfrak{b}$ may be recast as $r \in L^\vee$. ▲

3.5 Hecke's Eisenstein series and $\vartheta_{\mathbf{Q}^2}$

In this section we consider the case where $F = \mathbf{Q}^2$. We will show that in this case our ϑ_F reduces to an Eisenstein series of weight 1 that was introduced by Hecke (cf. [11, §6, eq.(23)] and [12, §1, eq.(10)]). Namely, for $k, N \in \mathbf{Z}_{>0}$, put

$$G_k(\tau, s; r_1, r_2, N) = \sum_{(m,n)\equiv(r_1,r_2)\ (N)} \frac{1}{(m\tau+n)^k |m\tau+n|^{2s}}, \qquad (\tau \in \mathcal{H}, s \in \mathbf{C})$$

where the sum runs over all $(m,n) \neq (0,0)$ in $\mathbf{Z}^2$ satisfying the congruence condition $(m,n) \equiv (r_1, r_2) \pmod{N\mathbf{Z}^2}$. Although the above series only converges for $k + \mathrm{Re}(2s) > 2$, the function G_k admits an analytic continuation to the entire s-plane ([35, Theorem 9.7]). Let $G_k(\tau; r, N) = G_k(\tau, 0; r_1, r_2, N)$ denote the value of this continued function at $s = 0$. Then, for $k \neq 2$, $G_k(\tau; r, N)$ is a holomorphic modular form of level $\Gamma(N)$. For $k = 1$, its Fourier expansion is given by

$$-\frac{N}{2\pi i} G_1(\tau; r, N) = A + \sum_{\substack{(m_1,m_2)\equiv(r_1,r_2)\ (N) \\ m_1 m_2 > 0}} \mathrm{sgn}(m_1) e\left(\frac{m_1 m_2}{N}\tau\right), \qquad (3.6)$$

where

$$A = \begin{cases} \frac{1}{2} - \left\{\frac{r_2}{N}\right\} & \text{if } r_1 \equiv 0 \pmod N \text{ and } r_2 \not\equiv 0 \pmod N \\ \frac{1}{2} - \left\{\frac{r_1}{N}\right\} & \text{if } r_1 \not\equiv 0 \pmod N \text{ and } r_2 \equiv 0 \pmod N \\ 0 & \text{otherwise.} \end{cases}$$

Here $\{x\}$ denotes the fractional part of x. See the discussion following Satz 9 in [11] (or [35, §9 and §3] for a different formulation).

Theorem 3.5.1. *Let $r = (r_1, r_2) \in \mathbf{Z}^2$ satisfy $r_1 r_2 \not\equiv 0 \pmod N$. Then there is a lattice $L = L(N)$ in F such that*

$$\vartheta_F(\tau; r, L) = -|\delta|^{-1} \frac{N}{2\pi i} G_1(\tau; r, N).$$

Proof. As in the previous section, we must pick an appropriate lattice in F. We shall take $L = N\mathbf{Z}$, where $N \in \mathbf{Z}_{>0}$ is as above.

To get everything to match appropriately, we must perform some renormalizations. First, we re-scale the quadratic form $N_{F/\mathbf{Q}}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = ab$ to $\frac{1}{N} N_{F/\mathbf{Q}}$ so that $(y, y) = \frac{y_1 y_2}{N}$. Then, for $r = (r_1, r_2) \in \mathbf{Z}^2$, Theorem 3.1.6 gives us

$$\Theta_F(\tau, t; r, L) = (\mathrm{Im}_\delta \tau)^{\frac{1}{2}} \sum_{y \in r + L} \frac{y_1 t^{-\frac{1}{2}} + y_2 t^{\frac{1}{2}}}{\sqrt{N}} \times$$
$$\times\, e\left(\tfrac{1}{4N}(\tau(y_1 t^{-\frac{1}{2}} + y_2 t^{\frac{1}{2}})^2 - \overline{\tau}(y_1 t^{-\frac{1}{2}} - y_2 t^{\frac{1}{2}})^2)\right).$$

Letting $\tau = u + iv$, we can re-write this as

$$\Theta_F(\tau, t; r, L) = \sqrt{\frac{v}{|\delta| N}} \sum_{y \in r + L} (y_1 t^{-\frac{1}{2}} + y_2 t^{\frac{1}{2}}) \times$$
$$\times\, e\left(\tfrac{1}{N}(u y_1 y_2 + \tfrac{1}{2} i v (y_2^2 t^{-1} + y_2^2 t))\right).$$

Now,

$$\vartheta_F(\tau; r, L) = \int_{\mathbf{R}_+} \Theta_F(\tau, t; r, L)\, \frac{dt}{t},$$

and by [22, Theorem 2.6] our congruence condition on r allows us to integrate the series Θ_F termwise. So we are reduced to having to evaluate the integral

$$\int_0^\infty (y_1 t^{-\frac{1}{2}} + y_2 t^{\frac{1}{2}})\, e(\tfrac{1}{2N} iv(y_2^2 t^{-1} + y_2^2 t))\, \frac{dt}{t}. \tag{3.7}$$

Proceeding as we did in the proof of Theorem 3.4.2 (cf. (3.5)), we find that

$$\begin{aligned}\int_0^\infty (y_i t^{-\frac{1}{2}})\, e(\tfrac{1}{2N} iv(y_2^2 t^{-1} + y_2^2 t))\, \frac{dt}{t} &= \sqrt{\frac{N}{v}} \mathrm{sgn}(y_i) e^{-2\pi \frac{v}{N} |y_1 y_2|} \\ &= \sqrt{\frac{N}{v}} \mathrm{sgn}(y_i) e\left(\frac{v}{N} |y_1 y_2|\right)\end{aligned}$$

for $i = 1, 2$. Therefore, our integral in (3.7) vanishes if y_1 and y_2 have different signs, and is otherwise equal to

$$\sqrt{\frac{N}{v}} \mathrm{sgn}(y_1) e\left(\frac{v}{N} y_1 y_2\right).$$

We conclude that

$$\vartheta_F(\tau; r, L) = |\delta|^{-1} \sum_{\substack{(m,n) \equiv (y_1, y_2)\ (N) \\ r_1 r_2 > 0}} \mathrm{sgn}(y_1) e\left(\frac{v}{N} y_1 y_2\right).$$

This completes the proof. ■

Remark 3.5.2. If one of r_1 or r_2 is congruent to 0 mod N, then it can be shown that $\vartheta_F(\tau; r, L)$, for L as in the preceding proof, is also a constant multiple of Hecke's

$G_1(\tau; r, N)$. See [22, Theorem 3.2 (ii)]. ▲

3.6 Hecke's imaginary quadratic theta function and ϑ_K

We proceed as we did in the previous two sections. Thus let D_K be the discriminant of our imaginary quadratic field K, and fix an integral ideal $\mathfrak{a}$ in $\mathcal{O}_K$ with norm $A = [\mathcal{O}_K \colon \mathfrak{a}]$, an element $r \in \mathfrak{a}$, and a positive integer M. In [11, 12] Hecke studied the theta function

$$\vartheta_2(\tau; r, \mathfrak{a}, M\sqrt{D_K}) = \sum_{\mu \equiv r \pmod{\mathfrak{a}M\sqrt{D_K}}} \mu\, e(\tfrac{\mu\mu'}{AM|D_K|}\tau) \qquad (\tau \in \mathcal{H}),$$

and proved that

$$\vartheta_2\left(\frac{a\tau+b}{c\tau+d}; r, \mathfrak{a}, M\sqrt{D_K}\right) = \left(\frac{|D_K|}{d}\right) e^{2\pi i \frac{abrr'}{AM|D_K|}} (c\tau+d)^2\, \vartheta_2(\tau; ar, \mathfrak{a}, M\sqrt{D_K})$$

for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(M|D_K|)$. Thus ϑ_2 is a weight 2 cusp form (or identically zero).

As we did with ϑ_F, we now show how ϑ_2 arises as a special instance of our ϑ_K. In order to do this we will need $(K \otimes_{\mathbf{Q}} F, (\cdot,\cdot))$ to be isotropic (as a Hermitian space over K). The following lemma gives a criterion for this to be the case. (See also (3.8.1) in the next section.)

Lemma 3.6.1. (i) *If F is a field, then the Hermitian space $(K \otimes_{\mathbf{Q}} F, (\cdot,\cdot))$ is isotropic if and only if $D_F \in N_{K/\mathbf{Q}}K$.*

(i) *IF $F = \mathbf{Q}^2$, then the Hermitian space $(K \otimes_{\mathbf{Q}} F, (\cdot,\cdot))$ is isotropic.*

Proof. For (i), note that, with respect to the K-basis $\{1, \sqrt{D_F}\}$ of $K \otimes_{\mathbf{Q}} F$, the matrix of $(\cdot,\cdot)$ is given by

$$H = \begin{bmatrix} 1 & \\ & -D_F \end{bmatrix}.$$

Thus $v = a + b\sqrt{D_F}$ $(a, b \in K)$ is a nonzero isotropic vector if and only if

$$[a \;\; b]\, H \begin{bmatrix} \overline{a} \\ \overline{b} \end{bmatrix} = 0 \iff a\overline{a} - D_F b\overline{b} = 0 \iff D_F = N_{K/\mathbf{Q}}\left(\frac{a}{b}\right).$$

This completes the proof of (i). The split case $F = \mathbf{Q}^2$ is trivial since F is already isotropic. ∎

Remarks 3.6.2. (i) We will reformulate the condition $D_F \in N_{K/\mathbf{Q}}K$ in the language of quaternion algebras in Section 3.8.

(ii) The condition is, of course, not automatic. For example, if $K = \mathbf{Q}(i)$ and $F = \mathbf{Q}(\sqrt{p})$ where p is an odd integer prime, then $D_F \in N_{K/\mathbf{Q}}K$ if and only if $p \equiv 1 \pmod 4$. ▲

Using the lemma, we have:

Theorem 3.6.3. *If $D_F \in N_{K/\mathbf{Q}}K$ then there is a choice of lattice $L = L(\mathfrak{a}, M, D_K)$ in $K \cong \mathbf{Q}^2$ such that for all $\tau \in \mathcal{H}$ and $r \in \mathfrak{a}$,*

$$\vartheta_K(\tau; r, L) = \frac{1}{\sqrt{-\delta AM|D_K|}}\,\vartheta_2(\tau; r, \mathfrak{a}, M\sqrt{D_K}).$$

Proof. For our lattice we take $L = \mathfrak{a}M\sqrt{D_K}$ in $K \cong \mathbf{Q}^2$ (the isomorphism being $a + b\delta_{\mathfrak{a}} \leftrightarrow \begin{bmatrix} a \\ b \end{bmatrix}$). We scale the the Hermitian form on K to $h(z,w) = \frac{1}{AM|D_K||\delta_{\mathfrak{a}}|^{1/2}}(z,w)_K$ and define an isometric embedding $L \hookrightarrow K \otimes \mathbf{R} = \mathbf{C} \cong \mathbf{R}^2$ via the map

$$\mu \mapsto \tilde{\mu} := \frac{1}{|\delta_{\mathfrak{a}}|\sqrt{AM|D_K|}}\mu$$

so that if $\mu = \mu_1 + \mu_2\delta_{\mathfrak{a}} \in L$ then its image in $\mathbf{R}^2$ is $\tilde{\mu} = \frac{1}{|\delta_{\mathfrak{a}}|\sqrt{AM|D_K|}}\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$. Then upon identifying L with its image as such, Theorem 3.1.3 gives

$$\begin{aligned}\vartheta_K(z;r,L) &= |Q_K|^{\frac{1}{2}}(-\delta)^{-\frac{1}{2}}\sum_{\mu\in r+L}(\tfrac{1}{|\delta_{\mathfrak{a}}|\sqrt{AM|D_K|}}\mu_1+\delta_{\mathfrak{a}}\tfrac{1}{|\delta_{\mathfrak{a}}|\sqrt{AM|D_K|}}\mu_2)\times\\ &\qquad\times e(|\delta|^2(\tfrac{1}{|\delta_{\mathfrak{a}}|^2AM|D_K|}\mu_1^2-\delta_{\mathfrak{a}}^2\tfrac{1}{|\delta_{\mathfrak{a}}|^2AM|D_K|}\mu_2^2)z)\\ &= |\delta_{\mathfrak{a}}|(-\delta)^{-\frac{1}{2}}\frac{1}{|\delta_{\mathfrak{a}}|\sqrt{AM|D_K|}}\sum_{\mu\in r+L}\mu\, e(\tfrac{1}{AM|D_K|}\mu\mu' z)\\ &= \frac{1}{\sqrt{-\delta AM|D_K|}}\,\vartheta_2(z;r,\mathfrak{a},M\sqrt{D_K}).\end{aligned}$$

Thus our ϑ_K and Hecke's ϑ_2 are the same, up to a constant. ■

3.7 The period relation revisited

In [11], Hecke considered certain periods of his imaginary quadratic theta functions. To be more precise, let's fix K, D_K, $\mathfrak{a}$, r and M as in Section 3.6, and let $\vartheta_2(\tau;r,\mathfrak{a},M\sqrt{D_F})$ be the corresponding theta function. Now consider

$$j(\tau;r,\mathfrak{a},M)=\frac{2\pi i}{AM|D_K|}\int_{i\infty}^{\tau}\vartheta_2(\tau;r,\mathfrak{a},M)\,d\tau$$

where the integral is along the vertical ray in $\mathcal{H}$ from $\tau+i\infty$ to τ. Hecke showed that this function $j(\tau)$ satisfies the transformation formula

$$j\left(-\frac{1}{\tau};r,\mathfrak{a},M\sqrt{D_K}\right)=\frac{1}{M\sqrt{D_K}}\sum_{\alpha}e\left(-\frac{1}{AM|D_K|}\mathrm{trace}\,(\alpha' r)\right)j(\tau;\alpha,\mathfrak{a},M\sqrt{D_K})+P,$$

where the summation is over equivalence classes of $\alpha\in\mathfrak{a}$ modulo $\mathfrak{a}M\sqrt{D_K}$, and where the *period* $P=P(r,\mathfrak{a},M\sqrt{D_K})$ is a constant independent of τ.

Letting τ tend to $i\infty$, we find on the one hand that $j(\tau)\to 0$, and on the other hand from the above transformation formula we obtain

$$\begin{aligned}
P &= j(0; r, \mathfrak{a}, M\sqrt{D_K}) \\
&= \frac{2\pi i}{AM|D_K|}\int_{\infty}^{0} \vartheta_2(\tau; r, \mathfrak{a}, M\sqrt{D_K})\, d\tau \\
&= \sum_{\mu \equiv r \ (\mathrm{mod}\ \mathfrak{a}M\sqrt{D_K})} \frac{1}{\mu'} e(\tfrac{\mu\mu'}{AM|D_K|}\tau) \\
&= \lim_{x\to 1} \sum_{\mu \equiv r \ (\mathrm{mod}\ \mathfrak{a}M\sqrt{D_K})} \frac{x^{\mu\mu'}}{\mu'}.
\end{aligned}$$

Hecke noticed that the limit above is the value at $s = 0$ of the analytic continuation of the function

$$G(s) = \sum_{\mu \equiv r \ (\mathrm{mod}\ \mathfrak{a}M\sqrt{D_K})} \frac{1}{\mu'|\mu|^s},$$

(which a priori is only defined for $\mathrm{Re}\, s > 1$). Now, upon choosing an oriented integral basis $\{\omega_1, \omega_2\}$ for $\mathfrak{a}$ as in Lemma 2.4.1 (so that $\delta = \omega_2/\omega_1$), we can further rewrite this sum (for $\mathrm{Re} s > 1$) as

$$\sum_{(m_1,m_2)\equiv(r_1,r_2)\ (M|D_K|)} \frac{-\sqrt{D_K}|D_K|^{\frac{s}{2}}}{(m_1\omega_1' + m_2\omega_2')|m_1\omega_1 + m_2\omega_2|^s},$$

where (r_1, r_2) are the coefficients of $r\sqrt{D_K}$ with respect to the chosen integral basis. In this way Hecke was able to relate the period of ϑ_2 along the geodesic ray from 0 to $i\infty$ to a special value at $\tau_0 = (\delta^{-1})' = -\delta^{-1}$ of the Eisenstein series

$$G_1(\tau; r, M|D_K|) = \sum_{(m_1,m_2)\equiv(r_1,r_2)\ (M|D_K|)} \frac{1}{(m_1\tau + m_2)|m_1\tau + m_2|^{2s}}\Bigg|_{s=0},$$

where here we are viewing $r \in \mathfrak{a}$ as giving the pair $(r_1, r_2) \in \mathbf{Z}^2$ in the manner described above.

This result (up to harmless multiplicative constants) is an immediate consequence of our period relation (Theorem 3.2.3) together with the identifications

$\vartheta_K \sim \vartheta_2$ and $\vartheta_{\mathbf{Q}^2} \sim G_1$ obtained in the previous two sections. (Notice that if $F = \mathbf{Q}^2$ then $V = K \otimes_{\mathbf{Q}} F$ is automatically isotropic by Lemma 3.6.1.)

We would like to apply our period relation to Hecke's theta functions ϑ and ϑ_2. In view of Sections 3.4 and 3.6, we can do this provided we can find a lattice in $\mathbf{R}^2$ on which we can simultaneously realize our constructions of the ϑ_F and ϑ_K that match Hecke's theta functions. In particular we will assume that $D_F \in N_{K/\mathbf{Q}}K$ so that we are in the isotropic case, as in the previous section. In Theorems 3.4.2 and 3.6.3 the necessary lattice was the image in $\mathbf{R}^2$ of a lattice in F (resp. K) of the form $\mathfrak{c}M\sqrt{D}$ where $\mathfrak{c}$ was an ideal in the relevant quadratic field and D was the field discriminant.

Thus let us examine the following situation. Consider the lattice $\mathfrak{b}M\sqrt{D_F}$ in the real quadratic field F. Embed it into $\mathbf{R}^2$ via the map $\mu \mapsto \frac{1}{\sqrt{AMD_F}}(\mu, \mu')$ and call the image L. We had given the lattice $\mathfrak{b}M\sqrt{D_F}$ the quadratic form $\frac{1}{AMD_F}N_{F/\mathbf{Q}}$ and noted that on L the corresponding bilinear form is $q(x, y) = \frac{1}{2}(x_1 y_2 + x_2 y_1)$. We would like to realize L as the isometric image of a similar lattice in K, say $L' = \mathfrak{a}M'\sqrt{D_K}$ where $\mathfrak{a}$ is an integral ideal in K and D_K is the discriminant of K. This latter lattice is endowed with a Hermitian form h, and we require its underlying bilinear form to match q. That is, we require

(i) a $\mathbf{Z}$-linear isomorphism $L' \xrightarrow{\sim} L$ together with

(ii) a 'complex multiplication', i.e a $\mathbf{Q}$-linear embedding $\varphi\colon K = L' \otimes_{\mathbf{Z}} \mathbf{Q} \hookrightarrow \mathrm{End}_{\mathbf{Q}}(L \otimes_{\mathbf{Z}} \mathbf{Q})$ such that

(iii) the underlying bilinear form of h (see (3.8) below) agrees with the bilinear form q on $L \otimes_{\mathbf{Z}} \mathbf{Q}$.

Note that (i) simply amounts to a choice of bases for L and L'. It induces an isomorphism

$$L \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\sim} L' \otimes_{\mathbf{Z}} \mathbf{Q} = R_{K/\mathbf{Q}} K$$

and it's in this context that (ii) is to be interpreted as a 'complex multiplication' on $L \otimes_{\mathbf{Z}} \mathbf{Q}$. (This terminology is not entirely standard.)

We turn to some generalities. Let V be a K-vector space equipped with a Hermitian form h. (In the situation above, $V = K$ with $h = (\cdot, \cdot)_K$.) Let $W = R_{K/\mathbf{Q}} V$ and let

$$q_h(x, y) = h(x + y, x + y) - h(x, x) - h(y, y), \quad x, y \in W, \tag{3.8}$$

be the underlying bilinear form of h. Then (W, q_h) is a quadratic space over $\mathbf{Q}$ and the action of K on V gives an embedding $\varphi \colon K \hookrightarrow \mathrm{End}_{\mathbf{Q}}(W)$. The key point here is the following.

Lemma 3.7.1. *$q_h(\varphi(z)x, y) = q_h(x, \varphi(\overline{z})y)$ for $x, y \in W$ and $z \in K$. Here $\overline{z}$ is the Galois conjugate of $z \in K$.*

Proof. First note that $q_h(x, y) = h(x, y) + \overline{h(x, y)}$ from (3.8). (Here the bar denotes complex conjugation.) So

$$\begin{aligned} q_h(\varphi(z)x, y) &= h(\varphi(z)x, y) + \overline{h(\varphi(z)x, y)} \\ &= h(zx, y) + \overline{h(zx, y)} \\ &= h(x, \overline{z}y) + \overline{h(x, \overline{z}y)} \\ &= q_h(x, \varphi(\overline{z})y). \end{aligned}$$

as required. ∎

Conversely, suppose that W is a $\mathbf{Q}$-vector space and that there is given an embedding $\varphi\colon K \hookrightarrow \mathrm{End}_{\mathbf{Q}}(W)$. This turns W into a K-vector space V. Suppose q is a bilinear form on W that satisfies the condition in the above lemma, i.e.

$$q(\varphi(z)x, y) = q(x, \varphi(\overline{z})y) \text{ for all } x, y \in W \text{ and z } \in K. \tag{3.9}$$

Then q is q_h for a Hermitian form h on V:

Lemma 3.7.2. *Assuming the preceding notation, let*

$$h(x,y) = \frac{1}{2}q(x,y) - \frac{1}{2}\delta^{-1}q(\varphi(\delta)x, y)$$

for $x, y \in V$. Then h is a Hermitian form on V whose underlying bilinear form q_h is q.

Proof. Linearity in the first variable is clear. Consider then

$$\begin{aligned}
\overline{h(x,y)} &= \frac{1}{2}q(x,y) + \frac{1}{2}\delta^{-1}q(\varphi(\delta)x, y) \\
&= \frac{1}{2}q(x,y) + \frac{1}{2}\delta^{-1}q(x, \varphi(-\delta)y) \qquad (\text{as } \overline{\delta} = -\delta) \\
&= \frac{1}{2}q(y,x) - \frac{1}{2}\delta^{-1}q(\varphi(\delta)y, x) \\
&= h(y,x)
\end{aligned}$$

Thus h is Hermitian.

Next, as noted in the proof of Lemma 3.7.1,

$$q_h(x,y) = h(x,y) + \overline{h(x,y)} = q(x,y).$$

This completes the proof. ■

Now let us return to the situation at hand, with our lattice L in $\mathbf{R}^2$. Fix a $\mathbf{Z}$-basis

$\{e_1, e_2\}$ for L. Consider the lattice $L' = \mathfrak{a}M\sqrt{D_K}$ in K. We may write $L' = \mathbf{Z} + \mathbf{Z}\omega$ for some $\omega \in K$ (cf. Lemma 2.4.1) and then obtain a $\mathbf{Z}$-isomorphism $L' \xrightarrow{\sim} L$ by sending $\{1, \omega\}$ to $\{e_1, e_2\}$. We are in search of an embedding $\varphi\colon K \hookrightarrow \mathrm{End}_{\mathbf{Q}}(L \otimes_{\mathbf{Z}} \mathbf{Q})$ that satisfies (3.9) where $L \otimes_{\mathbf{Z}} \mathbf{Q}$ is given the bilinear form $q(x, y) = \frac{1}{2}(x_1 y_2 + x_2 y_1)$. Given such a φ, we can then transport the Hermitian structure from $\mathbf{Z} + \mathbf{Z}\omega$ onto L in such a way that the bilinear structures match. Note that φ is completely determined by its values $\varphi(1) = 1$ and $\varphi(\delta)$ on the $\mathbf{Q}$-basis $\{1, \delta\}$ for K.

To get a more explicit understanding of (3.9) we can re-write it in terms of matrices as follows. We identify $\mathrm{End}_{\mathbf{Q}}(L \otimes \mathbf{Q}) \cong M_2(\mathbf{Q})$ via the given basis for L. The matrix of q with respect to this basis is

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and so (3.9) amounts to the assertion that $\varphi(z)^t J = J\varphi(\overline{z})$ or, equivalently, that

$$\varphi(\overline{z}) = J^{-1}\varphi(z)^t J \quad \text{for all } z \in K. \tag{3.10}$$

Thus if

$$\varphi(\delta) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then (3.10) becomes

$$\varphi(\overline{\delta}) = \begin{bmatrix} d & c \\ b & a \end{bmatrix}.$$

Now in view of the $\mathbf{Q}$-linearity of φ, the matrices $\varphi(\delta)$ and $\varphi(\overline{\delta}) = -\varphi(\delta)$ must be annihilated by the minimal polynomial $m_\delta(x) = x^2 - \delta^2$ of δ. This gives us explicit equations in a, b, c, d. A simple computation shows these to be:

$$a + d = 0 \quad \text{and} \quad ad - bc = -\delta^2,$$

or in other words

$$\operatorname{trace} \varphi(\delta) = \operatorname{trace}_{K/\mathbf{Q}} \delta \quad \text{and} \quad \det \varphi(\delta) = \mathrm{N}_{K/\mathbf{Q}} \delta.$$

So in some sense the restriction imposed by (3.9) is no restriction at all. In any case, we have a solution $(a, b, c, d) = (0, \delta^2, 1, 0) \in \mathbf{Q}^4$. Hence, by defining

$$\varphi(\delta) = \begin{bmatrix} 0 & \delta^2 \\ 1 & 0 \end{bmatrix}$$

we obtain our desired embedding φ. In summary:

Theorem 3.7.3. *Let F and K be real and imaginary quadratic fields of discriminants D_F and D_K, respectively, and let $(\mathfrak{a}, \{\xi_1, \xi_2\})$ and $(\mathfrak{b}, \{\omega_1, \omega_2\})$ be integral ideals in F and K, respectively, with oriented integral bases as described in Section 2.4, which provide us with isomorphisms $K \cong \mathbf{Q}^2$ and $F \cong \mathbf{Q}^2$.*

If $D_F \in N_{K/\mathbf{Q}}K$ then there is a choice of lattice $L \subset \mathbf{Q}^2$ for which the corresponding ϑ_F and ϑ_K simultaneously agree with Hecke's ϑ and ϑ_2 in the manner described by Theorems 3.4.2 and 3.6.3.

Remarks 3.7.4. (i) If we drop the condition $D_F \in N_{K/\mathbf{Q}}K$ then all that is lost is our ability to identify ϑ_K with Hecke's ϑ_2.

(ii) Given lattices $\mathfrak{a}M\sqrt{D_K}$ and $\mathfrak{b}M'\sqrt{D_F}$ which have been identified with L as above, the dual lattices $\mathfrak{a}$ and $\mathfrak{b}$ may be identified with $L^\vee \subset \mathbf{R}^2$. In this way we obtain a $\mathbf{Z}$-linear bijection that associates each $r \in \mathfrak{a}$ with a corresponding element $\mathfrak{b}$ that we will denote, for brevity, by $\check{r}$. ▲

Using this, we now obtain the following reformulation of our period relation in Theorem 3.2.3.

Theorem 3.7.5. *Let F and K be real and imaginary quadratic fields of discriminants D_F and D_K, respectively, and set $\delta = \sqrt{D_K} \in \mathcal{H}$.*

Let $\mathfrak{a}$ be an integral ideal K of norm A. Fix an oriented integral basis $\{\xi_1, \xi_2\}$ for $\mathfrak{a}$ and denote by $\delta_{\mathfrak{a}} = \delta_{\mathfrak{a};\{\xi_1,\xi_2\}}$ its associated CM point in $\mathcal{H}$ (cf. Remark 2.4.2 (i)). Let $M \in \mathbf{Z}_{>0}$ and $r \in \mathfrak{a}$. Hecke's weight-2 theta function $\vartheta_2(\tau; r, \mathfrak{a}, M\sqrt{D_K})$ defines a holomorphic 1-form on the modular curve $Y(r, A, M, |D_K|) := \Gamma(r, A, M, |D_K|)\backslash\mathcal{H}$, where $\Gamma(r, A, M, |D_K|)$ is the intersection of the kernel of the nebentypus

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \left(\frac{|D_K|}{d}\right) e^{2\pi i \frac{abrr'}{AM|D_K|}}$$

and the congruence subgrouup $\Gamma_0(M|D_K|)$ (cf. the definition of ϑ_2 in Section 3.6).

Let $\mathfrak{b}$ be an integral ideal in F. Fix an oriented integral basis $\{\omega_1, \omega_2\}$ and denote by $C_{\mathfrak{b}} = C_{\mathfrak{b};\{\omega_1,\omega_2\}}$ its associated geodesic arc in $\mathcal{H}$ (cf. Remark 2.4.2 (ii)). Let $C_{\mathfrak{b}}(r, A, M, |D_K|)$ be the projection of $C_{\mathfrak{b}}$ in $Y(r, A, M, |D_K|)$.

Assume that $D_F \in N_{K/\mathbf{Q}}K$. Then

$$\int_{C_{\mathfrak{b}}(r,A,M,|D_K|)} \vartheta_2(\tau; r, \mathfrak{a}, M\sqrt{D_K})\, d\tau = C\ \vartheta(-\delta^{-1}|\delta_{\mathfrak{a}}|^{-2}; \check{r}, \mathfrak{b}, M|D_K|\sqrt{D_F}),$$

where $\check{r}$ is as defined in Remark 3.7.4 (ii), and where the multiplicative constant C is given by

$$C = -2\sqrt{iAM|D_K|}\ r$$

with r a positive rational number (expressible in terms of indices of subgroups of unit groups with congruence conditions).

Proof. The fields K and F, together with their given integral structure, give us our usual seesaw of dual groups. In particular, we have our map $\alpha\colon \mathbf{R}_+ \hookrightarrow \mathcal{H}$ which we may interpret as parameterizing $C_{\mathfrak{b}}$, as described in Section 3.3. We also have, for each lattice $L \subset \mathbf{Q}^2$ and $r \in \mathbf{Q}^2$, our theta function $\vartheta_K(\tau; r, L)$, which we may view as being a holomorphic 1-form on the curve $Y(\Gamma_K(M_0)) = \Gamma_K(M_0)\backslash\mathcal{H}$ for an appropriate M_0, where $\Gamma_K(M_0)$ is Theorem 3.1.2. We can then create our theta function $\vartheta_F(\tau; r, L, M)$ which is a weight-1 cusp form for the arithmetic group $\Gamma_F^*(M)$.

Assuming now that $D_F \in N_{K/\mathbf{Q}}K$, we may realize $\vartheta_2(\tau; r, \mathfrak{a}, Q)$ as an instance of our $\vartheta_K(\tau; r, L)$ with $L = \mathfrak{a}M\sqrt{D_K}$ by Theorem 3.6.3. Therefore, *a posteriori*, we may take $M_0 = M|D_k|$—more specifically, we can identify $\Gamma_K(M_0)$ with $\Gamma(r, A, M, |D_K|)$ as defined in the statement of the theorem. The group $\Gamma(r, A, M, |D_K|)$ is a subgroup of the principle congruence subgroup $\Gamma(M|D_K|)$. Consequently, the group $\Gamma_F^+(M_0)$ in the definition of ϑ_F, which is given by

$$\Gamma_K(M_0) \cap SO(F)(\mathbf{R})^0 = \Gamma(r, A, M, |D_K|) \cap SO(F)(\mathbf{R})^0,$$

is seen to be, under our identification $SO(F) = F^1$, a subgroup of the group of totally positive units in F that are congruent to 1 modulo $M|D_K|$, i.e., it is a (nontrivial) subgroup of $U_F^+(M|D_K|)$.

Finally, by tracing through our abstract period relation in Theorem 3.2.3 (with $\tau_0 = -\delta Q_K^{-1} = -\delta|\delta_{\mathfrak{a}}|^{-2}$), noting that we are essentially integrating over $C_F(M|D_K|)$, and our identifications in Theorem 3.4.2 and 3.6.3, we immediately get the claimed period relation with multiplicative constant

$$C = -2\sqrt{\delta|\delta|^{-1}AM|D_K|}\,\frac{[U_F^+(M|D_K|\sqrt{D_F})\colon \Gamma_F^+(M|D_K|) \cap U_F^+(M|D_K|\sqrt{D_F})]}{[\Gamma_F^+(M|D_K|)\colon \Gamma_F^+(M|D_K|) \cap U_F^+(M|D_K|\sqrt{D_F})]}. \quad \blacksquare$$

Thus, just as in the case of the split algebra $F = \mathbf{Q}^2$, the period of ϑ_2 over a geodesic cycle is equal to the special value of a weight-1 elliptic modular (cusp) form. We can also reverse this line of reasoning to conclude that the values of the weight-1 theta and Eisenstein series ϑ and G_1 at CM points may be realized as periods of ϑ_2 over suitable geodesic arcs. This can be used to obtain rationality statements about these special values.

3.8 The anisotropic case

We close with some comments about the case where $(K \otimes_{\mathbf{Q}} F, (\cdot,\cdot))$ is anisotropic and where we do not have as convenient an expression for ϑ_K as given in Theorem 3.1.6. Nonetheless we can still conclude that ϑ_K is cuspidal. This is because $\Gamma_K \backslash \mathcal{H}$ in this case is compact.

To explain why, we quickly review the necessary general facts. To this end, consider an arbitratry two-dimensional vector space V over K equipped with a nondegenerate Hermitian form h. Let H be the matrix of h with respect to some fixed K-basis for V. Then since

$$\det(SHS^*) = \det H \ \det S \overline{\det S} = \det H \, N_{K/\mathbf{Q}}(\det S)$$

we find that the assignment $h \mapsto \det H$ associates to h a well-defined equivalence class in $K^\times / N_{K/\mathbf{Q}} K^\times$. In fact, since $\overline{H} = H^t$ we see that $\det H \in \mathbf{Q}^\times$ and so $\det H$ is really a class in $\mathbf{Q}^\times / N_{K/\mathbf{Q}} K^\times$. We denote this class by $\det h$ and call it the **determinant** of the Hermitian form h. Note that the matrix of h can always be diagonalized and put into the form $\mathrm{diag}\{r, rs\}$ $(a, r \in \mathbf{Q})$.

Lemma 3.8.1. $\det h = [-1] \iff h$ *is isotropic.*

Proof. Similar to the proof of Lemma 3.6.1. ∎

Now, one can associate to any such V—isotropic or not—a quaternion algebra

$B(V,h)$ over $\mathbf{Q}$ as follows. Fix a K-basis for V with respect to which $H = \operatorname{diag}\{r, rs\}$. Then let

$$B(V,h) = \left(\frac{D_K, -s}{\mathbf{Q}}\right) = \left\{ \begin{bmatrix} a & b \\ -s\bar{b} & \bar{a} \end{bmatrix} \mid a, b \in K = \mathbf{Q}(\sqrt{D_K}) \right\}. \tag{3.11}$$

We have the following easy facts (cf. [34, Lemmas 4.3 and 4.4]).

Proposition 3.8.2. (i) *$B(V,h)$ is determined up to isomorphism over $\mathbf{Q}$ by $\det h$. In particular, $B(V,h)$ is split over $\mathbf{Q}$ if and only if $\det h = [-1]$, i.e., if and only if h is isotropic.*

(ii) *$B(V,h) \cong V$ as $\mathbf{Q}$-vector spaces.*

(iii) *$SU(V) \cong B^1(V,h)$ where $B^1(V,h) = \{b \in B(V,h) \mid \det b = 1\}$ is the group of norm 1 units in $B(V,h)$.*

Proof. (i) If $H = \operatorname{diag}\{r', r's'\}$ is another diagonal representation of H then one has a corresponding algebra $B' = \left(\frac{D_K, -s'}{\mathbf{Q}}\right)$ defined as above and one knows that

$$B \cong B' \iff \frac{s}{s'} \in N_{K/\mathbf{Q}} K^\times.$$

Thus B is completely determined by the class of $\det h$. In particular, if h is isotropic, then we can take $s = -1$ in the above and we conclude that B is split over $\mathbf{Q}$ if and only if h is isotropic, by Lemma 3.8.1.

(ii) There is an evident $\mathbf{Q}$-linear isomorphism $V \xrightarrow{\sim} B$ given by

$$\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ -s\bar{b} & \bar{a} \end{bmatrix}$$

where we've used the given K-basis to identify elements of V with column vectors.

(iii) An easy computation shows that B consists of the matrices $g \in M_2(K)$

satisfying $g^\iota H = H\overline{g^t}$, where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\iota = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

Consequently, $B^\times = \{g \in GL_2(K) \mid gH\overline{g^t} = \det gH\}$. Indeed, if $g \in B^\times$ then $gH\overline{g^t} = gg^\iota H = \det g\, H$. Conversely, if $gH\overline{g^t} = \det g\, H$ then $gH\overline{g^t}H^{-1} = \det g$ so $H\overline{g^t}H^{-1} = g^\iota$ whence g is in B and is in $B^\times$ if $g \in GL_2$. The result follows from this. ∎

Now we return to the case where $V = K \otimes_{\mathbf{Q}} F$ and consider the corresponding quaternion algebra B given by (3.11), namely $\left(\frac{D_K, D_F}{\mathbf{Q}}\right)$.

By part (i) of the previous proposition, this quaternion algebra is split over $\mathbf{Q}$ if and only if $D_F \in N_{K/\mathbf{Q}}K$. This is precisely our isotropy condition from Lemma 3.6.1. In the case where the quaternion algebra is *nonsplit*, part (iii) of the proposition tells us that $SU(V) = B^1$. The symmetric space for $SU(V)$ is, of course, still $\mathcal{H}$, but now the subgroup $\Gamma_K(M)$ of Theorem 3.1.2 is a cocompact arithmetic Fuchsian group. The compact quotient $\Gamma_K(M)\backslash\mathcal{H}$ is the set of complex points of a Shimura curve. Note that the period relation (Theorem 3.2.3) still applies to the cusp form ϑ_K in this situation. We obtain, in analogous fashion to Theorem 3.7.5:

Theorem 3.8.3. *Let F and K be real and imaginary quadratic fields of discriminants D_F and D_K, respectively, and assume that $D_F \notin N_{K/\mathbf{Q}}K$.*

Let $\mathfrak{a}$ be an integral ideal K with oriented integral basis $\{\omega_1, \omega_2\}$ and associated CM point $\delta_{\mathfrak{a}}$ in $\mathcal{H}$. Let $M \in \mathbf{Z}_{>0}$ and $r \in \mathfrak{a}$. The weight-2 theta function $\vartheta_2(\tau; r, \mathfrak{a}M\sqrt{D_K})$ associated to the Hermitian lattice $\mathfrak{a}M\sqrt{D_K}$ in $K \cong \mathbf{Q}^2$ defines a holomorphic 1-form on the compact Shimura curve $X(\Gamma_K(M)) = \Gamma_K(M)\backslash\mathcal{H}$, where M is as in Theorem 3.1.2.

(i) Let $\mathfrak{b}$ be an integral ideal in F with oriented basis $\{\omega_1, \omega_2\}$ and let $C_{\mathfrak{b}}(M)$ denote the projection in $X(\Gamma_K(M))$ of the geodesic semicircle in $\mathcal{H}$ corresponding to $\mathfrak{b}$ and this integral basis. Then

$$\int_{C_{\mathfrak{b}}(M)} \vartheta_2(\tau; r, \mathfrak{a}M\sqrt{D_K})\, d\tau = -2\sqrt{iAM|D_K|}\,\vartheta(-\delta^{-1}|\delta_{\mathfrak{a}}|^{-2}; \check{r}, \mathfrak{b}, M|D_K|\sqrt{D_F}).$$

(ii) Let $C_\infty(M)$ denote the projection in $X(\Gamma_K(M))$ of the vertical geodesic ray from 0 to $i\infty$ in $\mathcal{H}$. Then

$$\int_{C_\infty(M)} \vartheta_2(\tau; r, \mathfrak{a}Q\sqrt{D_K})\, d\tau = \delta^{-1}\frac{M|D_K|}{2\pi i}\; G_1(-\delta^{-1}|\delta_{\mathfrak{a}}|^{-2}; \check{r}, M|D_K|),$$

where $\check{r} = (r_1, r_2) \in \mathbf{Z}^2$ are the coordinates of $r \in \mathfrak{a}$ with respect to the fixed oriented integral basis $\{\omega_1, \omega_2\}$: $r = r_1\omega_1 + r_2\omega_2$. ■

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