HIGHER EULER-KRONECKER CONSTANTS

BY

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ABSTRACT

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The coefficients that appear in the Laurent series of Dedekind zeta functions and their logarithmic derivatives are mysterious and seem to contain a lot of arithmetic information. Although the residue and the constant term have been widely studied, not much is known about the higher coefficients. In this thesis, we study these coefficients $\gamma_{K,n}$ that appear in the Laurent series expansion of $\frac{\zeta_K'(s)}{\zeta_K(s)}$ about $s = 1$, where K is a global field. For example, when K is a number field, we prove, under GRH,

$$\gamma_{K,n} \ll (\log(\log(|d_K|))^{n+1}$$

d_K being the absolute discriminant of K. Analogous bounds for the function field case are also shown. We prove (unconditionally) interesting arithmetic formulas satisfied by these constants.

We also study the distribution of values of higher derivatives of $\mathcal{L}(s,\chi) = L'(s,\chi)/L(s,\chi)$ at $s = 1$ and χ ranges over all non-trivial Dirichlet characters with a given large prime conductor m. In particular, we compute moments, i.e. the average of $P^{(a,b)}(\mathcal{L}^{(n)}(1,\chi))$, where $P^{(a,b)}(z) = z^a\overline{z}^b$ and study their asymptotic behaviour as $m \to \infty$. We then construct a density function $M_\sigma(z)$, for $\sigma = \text{Re}(s)$ and show that for $\text{Re}(s) > 1$

$$\text{Avg}_\chi \Phi(\mathcal{L}'(s,\chi)) = \int_{\mathbb{C}} M_\sigma(z)\Phi(z)|dz|$$

holds for any continuous function Φ on $\mathbb{C}$.

To my parents.

ACKNOWLEDGEMENTS

To be written later.

CONTENTS

1 INTRODUCTION

Since the advent of analytic number theory, the study of zeros of zeta and L-functions have engaged number theorists, perhaps more than any other theme. The famous Riemann hypothesis has been an open problem for more than 160 years now. A new approach to this problem was presented by Xian-Jin Li [Li97] in 1997, who showed that positivity of a sequence of coefficients coming from the Laurent series expansion, about $s = 1$, of the logarithmic derivative of these zeta functions is equivalent to the Riemann hypothesis. Later, Brown in [Br005] proved an effective version of Li's criterion relating positivity of the first finitely many terms in the sequence, to zero-free regions. This thesis is concerned with studying these higher coefficients.

1.1 ORGANIZATION OF CHAPTERS

In this introductory Chapter 1, we give a brief review of some well known results that will be useful for our later journey, as well as present the key motivation that led to this study. We then present a summary of our main results of this thesis.

Chapter 2 is about studying the higher *Euler-Kronecker constants* (to be defined in the next section) of a number field. In particular, after presenting some preliminary facts in section 2.1, we derive an arithmetic formula (unconditionally) for the first Euler-Kronecker constant in section 2.2. We then use this formula to derive upper bounds (under GRH) in section 2.3. In the subsequent section 2.4, we generalize these results, deduced in the previous sections, for higher constants.

Chapter 3 is a similar study in the case of a function field of a curve defined over a finite field. We prove analogous bounds.

In Chapter 4 we focus on Dirichlet L-functions. Again, section 4.1 - 4.4 is focused on deriving similar arithmetic formulas and bounds for these coefficients. The key new results in this chapter are on moments. After giving some historical background in section 4.5, we compute moments of $\mathcal{L}'(1,\chi)$ in section 4.6 under GRH, where $\mathcal{L}(s,\chi) = L'(s,\chi)/L(s,\chi)$. We then use zero sum estimates to prove an unconditional version of our result in section 4.7. Finally, we generalize these results to moments of higher derivatives in section 4.8.

Chapter 5 is on distribution of values of these higher derivatives of the logarithmic derivative of Dirichlet L-functions near $s = 1$. The main result is in section 5.3, and it is about showing the existence of a distribution function for the first derivative, for $\mathrm{Re}(s) > 1$. In section 5.4 we briefly discuss potential generalization to higher derivatives. Finally in the concluding section 5.5 we discuss future work and issues on extension of our result to parts of the critical strip : $\frac{1}{2} < \mathrm{Re}(s) \leq 1$.

1.2 SOME BACKGROUND AND MOTIVATION

Let K be an algebraic number field of finite degree n over $\mathbb{Q}$. The Dedekind zeta function of K is defined as

$$\zeta_K(s) = \sum_{\mathfrak{a}} (N\mathfrak{a})^{-s}$$

for $\mathrm{Re}(s) > 1$, where the sum is taken over all integral ideals $\mathfrak{a}$ of $\mathcal{O}_K$, the ring of integers of K. It also satisfies the Euler product formula

$$\zeta_K(s) = \prod_{\wp} \left(1 - \frac{1}{N\wp^s}\right)^{-1}$$

Hecke showed that $(s-1)\zeta_K(s)$ extends to an entire function. There is a simple pole of $\zeta_K(s)$ at $s = 1$ and the residue satisfies the famous *class number formula* :

$$\lim_{s\to 1} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} hR}{\omega\sqrt{|d_K|}}$$

where r_1 denotes the number of real embeddings of K, $2r_2$ is that of complex embeddings, h is the class number, R is the regulator, ω is the number of roots of unity, and d_K is the discriminant of K.

Moreover, we also have a functional equation of the form

$$\xi_K(s) = \xi_K(1-s)$$

where

$$\xi_K(s) = \left(\frac{\sqrt{|d_K|}}{2^{r_2}\pi^{n/2}}\right)^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_K(s)$$

where $[K:\mathbb{Q}] = n$. For the logarithmic derivative one can write :

$$-\frac{\zeta_K'(s)}{\zeta_K(s)} = \sum_{\mathfrak{a}} \frac{\Lambda(\mathfrak{a})}{N\mathfrak{a}^s} \tag{1.1}$$

$\Lambda(.)$ being the number field analogue of the von Mangoldt function given by :

$$\Lambda(\mathfrak{a}) = \begin{cases} \log \ N\mathfrak{p} & \text{if } \mathfrak{a} = \mathfrak{p}^k \text{ for some prime ideal } \mathfrak{p} \\ 0 & \text{otherwise.} \end{cases}$$

For the sake of completeness, we also recall that by applying the Tauberian theorem to the above (1.1), one can deduce the number field analogue of the prime number theorem, namely the prime ideal theorem :

Theorem 1.2.1. Let $\pi_K(x)$ be the number of prime ideals of $\mathcal{O}_K$ with norm less than or equal to x. Then

$$\pi_K(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty$$

For details on the above discussion, one may refer to any standard textbook on analytic number theory, for example [Dav00], [CF76] or [MM97].

The **generalized Riemann hypothesis** (GRH) states that all non-trivial zeros (i.e. those in the critical strip) of the Dedekind zeta function is on the $s = \frac{1}{2}$ line.

In [Li97], Li introduced the following sequence of numbers $\{\lambda_n\}$, now known as Li's coefficients :

$$\lambda_n = \frac{1}{(n-1)!} \frac{d^n}{ds^n} \left(s^{n-1} \log \xi_K(s)\right)\Big|_{s=1} \qquad \text{for } n \geq 1 \tag{1.2}$$

and showed the following theorem

Theorem 1.2.2. **(Li's Criterion)** The general Riemann hypothesis for $\zeta_K(s)$ holds iff λ_n is non-negative for all $n \geq 1$.

Later Bombieri and Lagarias also gave an alternative proof in [BL99]. Andrew Droll, in his PhD thesis formulated a much more generalized Li's Criterion for generalized quasi-Riemann hypothesis for functions in an extension of the Selberg class.

Brown in [Bro05] proved an effective version of Li's theorem, showing positivity of the first few λ_i's, give zero-free regions of a certain shape around $s = 1$. In particular he showed, just $\lambda_2 > 0$ implies non-existence of the exceptional Siegel zeroes.

Note that, if we write the Laurent series about $s = 1$ of the logarithmic derivative of $\zeta_K(s)$, then λ_2 involves the constant term and the first coefficient (that is the coefficient of $(s-1)$) together with some terms coming from the Γ- factors . This was our primary motivation to closely study this first coefficient. We later found that many of our results easily generalized to higher coefficients. A path was already led out by Ihara et. al. who, in a series of papers, systematically studied the constant term, which he called the *Euler-Kronecker constant*. We define

Definition 1.2.3. Let the Laurent series of the logarithmic derivative of $\zeta_K(s)$ about $s = 1$ be given by

$$\frac{\zeta_K'(s)}{\zeta_K(s)} = \frac{-1}{s-1} \; + \; \gamma_{K,0} \; + \; \sum_{n=1}^{\infty} \gamma_{K,n}(s-1)^n \tag{1.3}$$

$\gamma_{K,n}$ will be called the *n-th Euler-Kronecker constant*.

The first thing we showed were certain arithmetic formulas for $\gamma_{K,n}$. We later came to know that similar formulas for $\zeta(s)$ and $\zeta_K(s)$ exists in the literature and we end this section with a few of those.

Suppose we write

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{n=1}^{\infty} \mathfrak{s}_n (s-1)^n$$

In 1885, T. J. Stieltjes [HS05] showed that

$$\mathfrak{s}_n = \frac{(-1)^n}{n!} \lim_{x\to\infty} \left(\sum_{m=1}^{x} \frac{(\log m)^n}{m} - \frac{(\log x)^{n+1}}{n+1} \right)$$

These $\mathfrak{s}_n$ are called the Stieltjes constants, the generalized Euler constants or sometimes the Euler-Stieltjes constants. For the Dedekind zeta function, let us write

$$\zeta_K(s) = \sum_{n=-1}^{\infty} \mathfrak{s}_{K,n}(s-1)^n$$

The author found a similar formula in a much recent paper and does not know if similar formula has been written down in the past. The following is due to Eddin, see Theorem 2 of [Edd18].

$$\mathfrak{s}_{K,n} = \frac{(-1)^n}{n!} \lim_{x\to\infty} \left(\sum_{N\mathfrak{a}\leq x} \frac{(\log N\mathfrak{a})^n}{N\mathfrak{a}} - \mathfrak{s}_{K,-1}\frac{(\log x)^{n+1}}{n+1} \right) \qquad \text{for } n \geq 1$$

and

$$\mathfrak{s}_{K,0} = \lim_{x\to\infty} \left(\sum_{N\mathfrak{a}\leq x} \frac{1}{N\mathfrak{a}} - \mathfrak{s}_{K,-1}\log x \right) + \mathfrak{s}_{K,-1}$$

The following formula for the logarithmic derivative of the Riemann zeta function is also known. Let

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{-1}{(s-1)} + \sum_{n=0}^{\infty} \gamma_n (s-1)^n$$

then

$$\gamma_n = \frac{(-1)^{n-1}}{n!} \lim_{x\to\infty} \left(\sum_{m<x} \frac{\Lambda(m)(\log m)^n}{m} - \frac{(\log x)^{n+1}}{n+1} \right)$$

For a proof see [Tit58].

The author is not aware of existence of a similar formula for Dedekind zeta functions. The formulas we deduced are similar but a bit more involved.

1.3 STATEMENT OF MAIN RESULTS

Our first result is the following formula :

Theorem 1.3.1. *(Unconditionally)*

$$\gamma_{K,1} = \lim_{x\to\infty}\left[\,\Psi_K(x) - 1 - \frac{1+x}{1-x}\log x - \frac{1}{2}\log^2 x\,\right]$$

where, $\Psi_K(x) = \frac{1}{x-1}\sum_{k,\ N(P)^k < x}\left(\frac{x}{N(P)^k} - 1\right) k(\log N(P))^2 \qquad \text{for } x > 1$

Theorem 1.3.2. Under GRH, for $|d_K| \geq 8$, we have,

$$\gamma_{K,1} \ll (\log(\log(|d_K|)))^2$$

Similarly for the general case we write :

$$\Psi_K(n,x) = \frac{(-1)^n}{x-1}\sum_{k,\ N(P)^k < x}\left(\frac{x}{N(P)^k} - 1\right) k^n(\log N(P))^{n+1} \qquad \text{for} \quad x > 1$$

Theorem 1.3.3. *(Unconditionally)*

$$\gamma_{K,n} + (-1)^n n! = \lim_{x\to\infty}\left[-\Psi_K(n,x) + \frac{f(n,x)}{(x-1)}\right]$$

where $f(n,x)$ is recursively defined as :

$$f(n,x) = \frac{(-1)^n}{n+1}(x-1)(\log x)^{n+1} \ + \ (-1)^{n+1}(x+1)(\log x)^n \ + n(n-1)f(n-2)$$

$$f(1,x) = (1-x)\left[2 + \frac{1+x}{1-x}\log x + \frac{1}{2}\log^2 x\right]$$

$$f(0,x) = (x-1)\log(x)$$

Theorem 1.3.4. Under GRH, for $|d_K| \geq 8$, we have

$$\gamma_{K,n} \ll (\log(\log(|d_K|)))^{n+1}$$

We also prove analogous bounds in the function field case (unconditional, GRH being known). For example if K is the function field of a curve X of genus g, over $\mathbb{F}_q$, q being a prime power then

Theorem 1.3.5. For $g > 2$ or, $g = 2$ and $q > 2$, we have

$$\gamma_{K,n} + (-1)^n n! \ll n!(\log \alpha_K)^{n+1} \quad \text{where } \alpha_K = (g-1)\log q \tag{1.4}$$

Remark 1.3.6. All implicit constants in the above results are absolute.

Let K be a number field and χ be a primitive Dirichlet character on K. Let $L(s,\chi)$ be the L-function associated to it. For notational brevity we'll write

$$\mathcal{L}(s,\chi) := \frac{L'(s,\chi)}{L(s,\chi)}$$

Initially we proved similar formulae and bounds like that of $\gamma_{K,n}$, namely,

$$\mathcal{L}^{(n)}(1,\chi) = \lim_{x\to\infty} (-1)^{n+1}\, \Psi_K(\chi,n,x)$$

where

$$\Psi_K(\chi,n,x) = \frac{1}{x-1} \sum_{k,\, N(P)^k < x} k^n \left(\frac{x}{N(P)^k} - 1\right) \chi(P)^k (\log N(P))^{n+1}$$

But our main goal was to compute moments for the higher derivatives, motivated by the work of Ihara, Murty and Shimura, who in [IMS09], computed moments of $\mathcal{L}(1,\chi)$.

For this section we work with $K = \mathbb{Q}$. Let m be a prime and X_m denote the set of all non-principal multiplicative characters $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ and $L(s,\chi)$ denote the corresponding Dirichlet L-function. For any pair of non-negative integers (a,b) let $P^{(a,b)}(z) = z^a \bar{z}^b$. We showed :

Theorem 1.3.7. For any $\epsilon > 0$ we have, unconditionally,

$$\frac{1}{|X_m|} \sum_{\chi \in X_m} P^{(a,b)}(\mathcal{L}^{(r)}(1,\chi)) = (-1)^{(r+1)d} \mu^{(a,b)}(r) + O\left(m^{\epsilon-1}\right)$$

The implicit constant depends on a, b only. Under GRH, the error term is

$$O\left(\frac{(\log m)^{(r+1)d+2}}{m}\right)$$

with $d = a + b$. In particular, letting $m \to \infty$ we get

$$\lim_{m\to\infty} \frac{1}{|X_m|} \sum_{\chi\in X_m} P^{(a,b)}(\mathcal{L}^{(r)}(1,\chi)) \;=\; (-1)^{(r+1)d}\mu^{(a,b)}(r)$$

Here $\mu^{(a,b)}(r)$ is the following explicitly computable constant :

$$\mu^{(a,b)}(r) = \sum_{j=1}^{\infty} \frac{\ell^r\Lambda_a(j)\ \ell^r\Lambda_b(j)}{j^2}$$

where for $k > 0$, $r \geq 0$

$$\ell^r\Lambda_k(n) = \sum_{n_1 n_2 \cdots n_k = n} \left(\prod_{i=1}^{k} \Lambda(n_i)(\log n_i)^r \right)$$

whereas, for $k = 0$, $\ell^r\Lambda_0(n) = 1$ for $n = 1$ and 0 for $n > 1$.

Note : *$\ell^r\Lambda_a(.)$ is just notation. We are not actually multiplying by some ℓ^r or anything. The logarithm appears with exponent r in the above formula together with $\Lambda(.)$, this is just a way of book keeping.*

We then focus on the distribution of values of these higher derivatives of the logarithmic derivative of Dedekind zeta functions, in particular we show the existence of a density function for $\mathrm{Re}(s) > 1$.

Let K be either $\mathbb{Q}$ or an imaginary quadratic number field. Let χ run over all Dirichlet characters on K whose conductor (the non-archimedean part) is a prime divisor, such that $\chi(\wp_\infty) = 1$. We define the average of a complex valued function $\phi(\chi)$, over a family of χ as defined above, as follows :

$$\mathrm{Avg}_\chi \phi(\chi) = \lim_{m\to\infty} \mathrm{Avg}_{N(\mathbf{f})\leq m}\phi(\chi)$$

where

$$\mathrm{Avg}_{N(\mathbf{f})\leq m}\phi(\chi) = \frac{\sum_{N(\mathbf{f})\leq m} \left(\sum_{\mathbf{f}_\chi = \mathbf{f}} \phi(\chi) \right) \ / \ \sum_{\mathbf{f}_\chi=\mathbf{f}} 1}{\sum_{N(\mathbf{f})\leq m} 1}$$

Then the main result in this chapter states that, *Theorem* 1.3.8. For any $s \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s) > 1$ there exists a function $M_\sigma : \mathbb{C} \to \mathbb{R}$ satisfying, $M_\sigma(w) \geq 0$, and $\int_{\mathbb{C}} M_\sigma(w)\, |dw| = 1$, such that

$$\mathrm{Avg}_\chi\, \Phi(\mathcal{L}'(\chi, s)) \;=\; \int_{\mathbb{C}} M_\sigma(w)\; \Phi(w)\; |dw| \tag{1.5}$$

holds for any continuous function Φ of $\mathbb{C}$.

Note that M_σ is constructed as the limit of $M_{\sigma,P}$ functions, where P is a finite set of non-archimedean primes. The Fourier dual of $M_\sigma(z)$ given by

$$\tilde{M}_\sigma(z) = \int_{\mathbb{C}} M_\sigma(w) \psi_z(w) |dw|$$

where $\psi_z : \mathbb{C} \to \mathbb{C}^1$ is the additive character $\psi_z(w) = \exp(i \cdot \mathrm{Re}(\bar{z}w))$, satisfies the following :

$$\tilde{M}_\sigma(z) = \mathrm{Avg}_\chi\; \psi_z(\mathcal{L}'(\chi, s))$$

2

BOUNDS ON HIGHER EULER-KRONECKER CONSTANTS : NF CASE

2.1 PRELIMINARIES

Let K be an algebraic number field. Consider the Laurent series expansion of the Dedekind zeta function about $s = 1$:

$$\zeta_K(s) = \frac{c_{-1}}{s-1} + c_0 + c_1(s-1) + c_2(s-1)^2 + \cdots \qquad (c_{-1} \neq 0) \tag{2.1}$$

In [Iha06], Ihara studied the constant : $\gamma_K = c_0/c_{-1}$ attached to K and called it the *Euler-Kronecker constant*. For $K = \mathbb{Q}$, this is the famous Euler-Mascheroni constant

$$\gamma = \gamma_{\mathbb{Q},0} = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) = 0.57721566\cdots$$

This constant γ_K is also the constant term in the Laurent series expansion of the logarithmic derivative of $\zeta_K(s)$ about $s = 1$, i.e.

$$\gamma_K = \lim_{s\to 1}\left(\frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{1}{s-1}\right)$$

On the other hand using a lemma of Stark, (Lemma 3 in [Sta74]) we get :

$$-\frac{\zeta_K'(s)}{\zeta_K(s)} = \frac{1}{s} + \frac{1}{s-1} - \sum \frac{1}{s-\rho} + \alpha_K + \beta_K + \tilde{\Gamma}_K(s) \tag{2.2}$$

where the sum, ρ runs over all non-trivial zeros of $\zeta_K(s)$, counted with multiplicities. Here

$$\alpha_K = \frac{1}{2}\log|d_K|, \quad d_K \text{ being the absolute discriminant of } K\,;$$
$$\beta_K = -\{\frac{r_1}{2}(\gamma + \log 4\pi) + r_2(\gamma + \log 2\pi)\}$$
$$\tilde{\Gamma}_K(s) = \frac{r_1}{2}\left(g\left(\frac{s}{2}\right) - g\left(\frac{1}{2}\right)\right) + r_2\left(g(s) - g(1)\right) \quad \text{where } \; g(s) = \frac{\Gamma'(s)}{\Gamma(s)}$$

with r_1, r_2 : the number of real, imaginary places of K, respectively, $\gamma = \gamma_{\mathbb{Q}}$ as before. Thus we get

$$\gamma_K = \sum_\rho \frac{1}{1-\rho} - \alpha_K - \beta_K - 1$$
$$\Rightarrow \frac{1}{2}\sum_\rho \frac{1}{\rho(1-\rho)} = \gamma_{K,0} + \alpha_K + \beta_K + 1 \tag{2.3}$$

Equation (2.3) will be used in a later section while finding upper bounds of certain terms. Ihara proved the following upper bound for γ_K :

Theorem 2.1.1. **(Ihara)** For $n_K = [K:\mathbb{Q}] > 2$ or, $n_K = 2$ and $|d_K| > 8$, under GRH we have

$$\gamma_K \le \left(\frac{\alpha_K + 1}{\alpha_K - 1}\right)(2\log\alpha_K + 1)$$

Note that γ_K appears as the constant term in the Laurent series expansion of the logarithmic derivative $\zeta_K(s)$ about $s = 1$. Our work in this chapter is to analyze the coefficients of higher powers of $(s-1)$. We will refer to these as general or higher Euler-Kronecker coefficients.

Definition 2.1.2. Let the Laurent series of the logarithmic derivative of $\zeta_K(s)$ about $s = 1$ be given by

$$\frac{\zeta_K'(s)}{\zeta_K(s)} = \frac{-1}{s-1} \; + \; \gamma_{K,0} \; + \; \gamma_{K,1}(s-1) \; + \; \cdots \tag{2.4}$$

$\gamma_{K,n}$ will be called the *n-th Euler-Kronecker constant*.

So with the above notation, $\gamma_K = \gamma_{K,0}$. We will first investigate the next coefficient $\gamma_{K,1}$ and try to see whether the methods used by Ihara in [Iha06] to deduce upper bounds of $\gamma_{K,0}$ generalizes. We will then consider the most general case. Note that

$$\lim_{s\to 1}\left[\frac{d}{ds}\frac{\zeta_K'(s)}{\zeta_K(s)}-\frac{1}{(s-1)^2}\right] = \gamma_{K,1} \tag{2.5}$$

2.2 AN 'EXACT FORMULA' FOR $\gamma_{K,1}$

Now, following Ihara's notation, we also denote :

$$Z_K(s)=-\frac{\zeta_K'(s)}{\zeta_K(s)}=\sum_{P,k\geq 1}\frac{\log N(P)}{N(P)^{ks}}$$

Hence, differentiating (2.2) we get,

$$Z_K'(s)=-\frac{1}{s^2}-\frac{1}{(s-1)^2}+\sum\frac{1}{(s-\rho)^2}+\tilde{\Gamma}_K'(s) \tag{2.6}$$

with $\tilde{\Gamma}_K'(s)=\frac{r_1}{4}g'\left(\frac{s}{2}\right)+r_2g'(s)$. Substituting this in equation (2.5) we get

$$\gamma_{K,1}=1-\sum\frac{1}{(1-\rho)^2}-\tilde{\Gamma}_K'(1) \tag{2.7}$$

On the other hand, taking term by term derivative of the Dirichlet series, we obtain :

$$Z_K'(s)=\sum_{P,k\geq 1}\frac{-k(\log N(P))^2}{N(P)^{ks}} \tag{2.8}$$

We wish to find a similar 'exact formula' as in the case of $\gamma_{K,0}$ by Ihara. To do so, we evaluate the integral :

$$\Psi_K^{(\mu)}(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s-\mu}}{s-\mu}Z_K'(s)\,ds \qquad \text{for } c\gg 0$$

For $\mu=0$ and 1 in two different ways using equation (2.8) and equation (2.6) and the classical formulas:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s}\,ds=\begin{cases}0 & 0<y<1\\ \frac{1}{2} & y=1\\ 1 & y>1\end{cases} \tag{2.9}$$

And for $n \geq 1$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s^{n+1}}\,ds = \begin{cases} 0 & 0 < y \leq 1 \\ \frac{1}{n!}(\log y)^n & y > 1 \end{cases} \tag{2.10}$$

Thus from (2.8) we get :

$$\begin{aligned}
x\Psi_K^{(1)}(x) - \Psi_K^{(0)}(x) &= x \cdot \sum_{P,k\geq 1} \frac{-k(\log N(P))^2}{N(P)^k}\left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty} \frac{1}{s-1}\left(\frac{x}{N(P)^k}\right)^{s-1} ds\right] \\
&\quad - \sum_{P,k\geq 1} -k(\log N(P))^2 \left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty} \frac{1}{s}\left(\frac{x}{N(P)^k}\right)^{s} ds\right] \\
&= x \cdot \sum_{k,\ N(P)^k < x} \frac{-k(\log N(P))^2}{N(P)^k} + \sum_{k,\ N(P)^k = x} \frac{-k(\log N(P))^2}{2} \\
&\quad - \sum_{k,\ N(P)^k < x} -k(\log N(P))^2 - \sum_{k,\ N(P)^k = x} \frac{-k(\log N(P))^2}{2} \\
&= \sum_{k,\ N(P)^k < x} \left(\frac{x}{N(P)^k} - 1\right)\left(-k(\log N(P))^2\right)
\end{aligned}$$

Looking at the above computation, we define :

$$\boxed{\Psi_K(x) = \frac{1}{x-1}\sum_{k,\ N(P)^k < x}\left(\frac{x}{N(P)^k} - 1\right) k(\log N(P))^2 \qquad \text{for } x > 1} \tag{2.11}$$

Remark 2.2.1. The reason for dividing by $(x-1)$ will become apparent while computing $x\Psi_K^{(1)}(x) - \Psi_K^{(0)}(x)$ the other way.

From (2.6) we get :

$$\begin{aligned}
x\Psi_K^{(1)}(x) - \Psi_K^{(0)}(x) &= \frac{x}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s-1}\left[-\frac{1}{s^2} - \frac{1}{(s-1)^2} + \sum \frac{1}{(s-\rho)^2} + \tilde{\Gamma}_K'(s)\right] ds \\
&\quad - \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s}\left[-\frac{1}{s^2} - \frac{1}{(s-1)^2} + \sum \frac{1}{(s-\rho)^2} + \tilde{\Gamma}_K'(s)\right] ds
\end{aligned}$$

Let us first look at the contribution from the term : $-\frac{1}{s^2} - \frac{1}{(s-1)^2}$

$$\int x^s \left[-\frac{1}{s^2(s-1)} - \frac{1}{(s-1)^3} + \frac{1}{s^3} + \frac{1}{s(s-1)^2} \right] ds$$

$$= \int x^s \left[\frac{1}{s^2} + \frac{1}{s} - \frac{1}{(s-1)} - \frac{1}{(s-1)^3} + \frac{1}{s^3} + \frac{1}{(s-1)^2} - \frac{1}{s-1} + \frac{1}{s} \right] ds$$

$$= 2\int \frac{x^s}{s}\, ds \;-\; 2x \int \frac{x^{s-1}}{s-1} ds \;+ \int x^s \left[\frac{1}{s^2} + \frac{1}{(s-1)^2} \right] ds \;+ \int x^s \left[\frac{1}{s^3} - \frac{1}{(s-1)^3} \right] ds$$

Thus using (2.9) and (2.10) we have :

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+\infty} x^s \left[-\frac{1}{s^2(s-1)} - \frac{1}{(s-1)^3} + \frac{1}{s^3} + \frac{1}{s(s-1)^2} \right] ds$$

$$= 2 - 2x + \log x(1+x) + \frac{1}{2}(\log x)^2(1-x)$$

$$= (1-x) \left[2 + \frac{1+x}{1-x} \log x + \frac{1}{2} \log^2 x \right] \tag{2.12}$$

Now for contribution from the $\sum_\rho$ term, we do some contour manipulation. As in Figure 2.1 (below), for large R and T, take the contour $C_{R,T}$ to be the rectangle : $c - iT \to c + iT \to -R + iT \to -R - iT \to c - iT$.

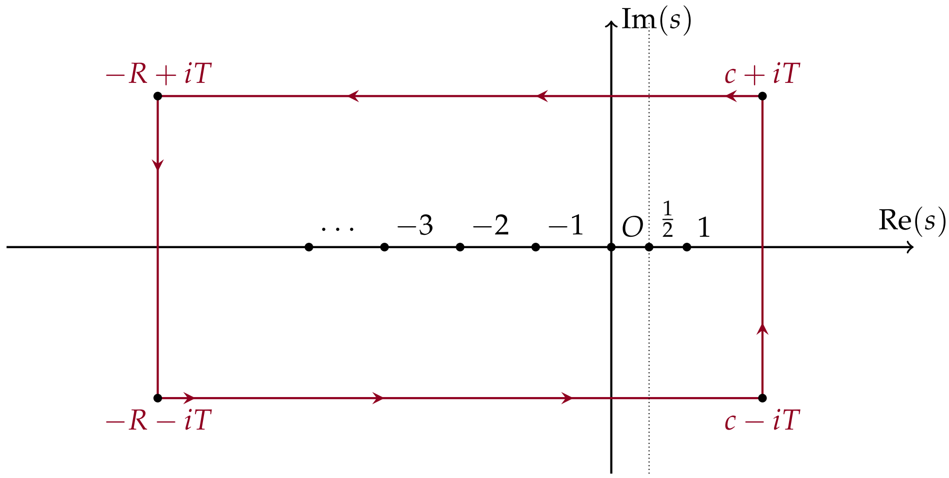

Figure 2.1

$$\left|\int_{-R-iT}^{-R+iT} \frac{x^s}{s}\cdot\frac{1}{(s-\rho)^2}\,ds\right| \le \int_{-T}^{T}\frac{x^{-R}}{(t^2+aRt+b)^{3/2}}\,dt \ll x^{-R}T^{-2}$$

And thus,

$$\left|\int_{-R-iT}^{-R+iT} \frac{x^s}{s}\cdot\frac{1}{(s-\rho)^2}\,ds\right| \xrightarrow{R,\,T\to\infty} 0.$$

Similarly,

$$\left|\int_{-R-iT}^{c-iT} \frac{x^s}{s}\cdot\frac{1}{(s-\rho)^2}\,ds\right| \ll T^{-2}\int_{-R}^{c} x^{\sigma}\,d\sigma \xrightarrow{R,\,T\to\infty} 0.$$

Now let us compute the poles and the residues.
The pole at $s=0$ has residue : $\dfrac{1}{\rho^2}$.
The pole at ρ (double pole) has residue : $\lim_{s\to\rho}\dfrac{d}{ds}(\dfrac{x^s}{s}) = \dfrac{\rho x^{\rho}\log x - x^{\rho}}{\rho^2}$
(Computations for $s=1$ are exactly similar).

Therefore by Residue theorem, net contribution from the $\sum_\rho$ term is given by :

$$x\left[\sum \frac{1}{(1-\rho)^2} + \frac{(\rho-1)x^{\rho-1}\log x - x^{\rho-1}}{(1-\rho)^2}\right] - \sum \frac{1}{\rho^2} + \frac{\rho x^\rho \log x - x^\rho}{\rho^2}$$

$$= (x-1)\sum \frac{1}{(1-\rho)^2} + \sum \frac{\rho^2(\rho-1)x^\rho\log x - \rho^2 x^\rho - \rho(\rho-1)^2 x^\rho \log x + (1-\rho)^2 x^\rho}{\rho^2(1-\rho)^2}$$

$$= (x-1)\sum \frac{1}{(1-\rho)^2} + \sum \frac{\rho(\rho-1)x^\rho\log x(\rho-\rho+1) - \rho^2 x^\rho + (1-\rho)^2 x^\rho}{\rho^2(1-\rho)^2}$$

$$= (x-1)\left[\sum \frac{\rho(\rho-1)\log x - \rho^2 + (1-\rho)^2}{\rho^2(1-\rho)^2}\cdot\frac{x^\rho}{x-1} + \sum \frac{1}{(1-\rho)^2}\right]$$

$$= (x-1)\left[r(x) + \sum \frac{1}{(1-\rho)^2}\right] \qquad \text{(say)} \tag{2.13}$$

For the Gamma term, we look at it's series expansion :

$$\tilde{\Gamma}(s) = -r_1\left(\frac{1-s}{s} + \sum_{n=1}^{\infty}\left(\frac{1}{s+2n} - \frac{1}{1+2n}\right)\right) - r_2\left(\frac{1-s}{s} + \sum_{n=1}^{\infty}\left(\frac{1}{s+n} - \frac{1}{1+n}\right)\right)$$

Since we are only interested in integrating along a line which is far to the right, the sums are absolutely convergent and hence taking term by term derivatives, we get :

$$\tilde{\Gamma}'(s) = \frac{r_1+r_2}{s^2} + r_1\left(\sum_{n=1}^{\infty}\frac{1}{(s+2n)^2}\right) + r_2\left(\sum_{n=1}^{\infty}\frac{1}{(s+n)^2}\right)$$

We want to compute :

$$\frac{x}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s-1}}{s-1}\tilde{\Gamma}'(s)\,ds \;-\; \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s}\tilde{\Gamma}'(s)\,ds$$

We compute the three terms individually :

$$
\begin{aligned}
&(r_1+r_2)\left[\frac{x}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s-1}}{s^2(s-1)}\,ds \;-\; \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s^3}\,ds\right]\\
&=(r_1+r_2)\left[\frac{x}{2\pi i}\int_{c-i\infty}^{c+\infty}x^{s-1}\left(\frac{1}{s-1}-\frac{1}{s}-\frac{1}{s^2}\right)\,ds \;-\; \frac{1}{2}(\log x)^2\right]\\
&=(r_1+r_2)\left[x-1-\log x-\frac{1}{2}(\log x)^2\right]
\end{aligned}
$$

Note that for the two series we can again refer to Figure 2.1 and use similar residue computations as $\frac{1}{(s-\rho)^2}$ with $\rho=-2n$ and $\rho=-n$. Thus we get :

$$
\begin{aligned}
&\frac{x}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s-1}}{s-1}\tilde{\Gamma}'(s)\,ds \;-\; \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s}\tilde{\Gamma}'(s)\,ds\\
&=(r_1+r_2)(x-1)\left[1-\frac{1}{x-1}\log x-\frac{1}{2(x-1)}(\log x)^2\right]\\
&+r_1(x-1)\left[\sum_{n=1}^{\infty}\frac{2n(2n+1)\log x-4n^2+(1+2n)^2}{4n^2(1+2n)^2}\cdot\frac{x^{-2n}}{x-1}\;+\;\sum_{n=1}^{\infty}\frac{1}{(1+2n)^2}\right]\\
&+r_2(x-1)\left[\sum_{n=1}^{\infty}\frac{n(n+1)\log x-n^2+(1+n)^2}{n^2(1+n)^2}\cdot\frac{x^{-n}}{x-1}\;+\;\sum_{n=1}^{\infty}\frac{1}{(1+n)^2}\right]\\
&+r_1\left[\sum_{n=1}^{\infty}\frac{1}{(1+2n)^2}-\sum_{n=1}^{\infty}\frac{1}{4n^2}\right]-r_2\\
&=(x-1)\left[\ell(x)+\tilde{\Gamma}'(1)\right]\qquad\text{(say)}
\end{aligned}
\tag{2.14}
$$

Here we're denoting the rest of the expression by $\ell(x)$. The full expression is available below in the proof of Theorem 2.2.2.

Putting all these together along with equation (2.7) we get

$$
\begin{aligned}
\Psi_K(x)&=2+\frac{1+x}{1-x}\log x+\frac{1}{2}\log^2 x-r(x)-\sum\frac{1}{(1-\rho)^2}-\ell(x)-\tilde{\Gamma}'(1)\\
&=\gamma_{K,1}\;+1+\frac{1+x}{1-x}\log x+\frac{1}{2}\log^2 x\;-r(x)\;-\ell(x)
\end{aligned}
$$

$$\boxed{\gamma_{K,1} = \Psi_K(x) - 1 - \frac{1+x}{1-x}\log x - \frac{1}{2}\log^2 x \ + r(x) \ + \ell(x)} \tag{2.15}$$

Theorem 2.2.2. *(Unconditionally)*

$$\gamma_{K,1} = \lim_{x\to\infty}\left[\Psi_K(x) - 1 - \frac{1+x}{1-x}\log x - \frac{1}{2}\log^2 x\right]$$

Proof. Looking at equation (2.14), $\ell(x)$ looks like the following :

$$\begin{aligned}\ell(x) = &-\frac{r_1+r_2}{x-1}\left[\log x + \frac{1}{2}(\log x)^2\right] \\ &+ r_1\left[\sum_{n=1}^{\infty}\frac{2n(2n+1)\log x - 4n^2 + (1+2n)^2}{4n^2(1+2n)^2}\cdot\frac{x^{-2n}}{x-1}\right] \\ &+ r_2\left[\sum_{n=1}^{\infty}\frac{n(n+1)\log x - n^2 + (1+n)^2}{n^2(1+n)^2}\cdot\frac{x^{-n}}{x-1}\right] \\ &+ \frac{r_1}{x-1}\left[\sum_{n=1}^{\infty}\frac{1}{(1+2n)^2} - \sum_{n=1}^{\infty}\frac{1}{4n^2}\right] - \frac{r_2}{x-1}\end{aligned} \tag{2.16}$$

Since all the series are absolutely convergent, we can take limit term-by-term and see that $\lim_{x\to\infty}\ \ell(x) = 0$.

To show, same is true for $r(x)$ we make use of standard zero-free region of the Dedekind zeta function. In particular, we will use the following Lemma 8.1 of [LO77] which states that :

Theorem 2.2.3. There is an absolute, effectively computable positive constant c such that $\zeta_K(s)$ has no zeros $\rho = \beta + i\gamma$ in the region :

$$|\gamma| \geq \frac{1}{1+4\log d_K}\ , \quad \beta \geq 1 - \frac{c}{\log d_K + n_k\log(|\gamma|+2)}$$

We have (writing $\rho = \beta + i\gamma$)

$$r(x) = \sum \frac{\rho(\rho-1)\log x - \rho^2 + (1-\rho)^2}{\rho^2(1-\rho)^2} \cdot \frac{x^\rho}{x-1}$$
$$\ll \sum \frac{(\log x)\ x^{\beta-1}}{\gamma^2}$$

Since $\beta < 1$, we can assume the condition on β in the above theorem holds, by excluding finitely many zeros. Thus we get :

$$\sum \frac{(\log x)\ x^{\beta-1}}{\gamma^2} < \sum \frac{(\log x)\ x^{-c(\log d_K + n_K \log(|\gamma|+2))^{-1}}}{\gamma^2}$$
$$= \sum_{\log d_K + n_K \log(|\gamma|+2)\ <\ T} + \sum_{\log d_K + n_K \log(|\gamma|+2)\ \geq\ T} \qquad (2.17)$$

where we choose $T = \sqrt{\log x}$. Thus for the first sum :

$$\sum_{\log d_K + n_K \log(|\gamma|+2)\ <\ T} \frac{(\log x)\ x^{-c(\log d_K + n_K \log(|\gamma|+2))^{-1}}}{\gamma^2}$$
$$< \sum \frac{(\log x) x^{-cT^{-1}}}{\gamma^2} = \left(\sum \frac{1}{\gamma^2}\right)(\log x)\exp(-c\sqrt{\log x})$$

Note that the last equality follows from :

$$\exp(-c\sqrt{\log x}) = \exp(-c\log x(\sqrt{\log x})^{-1}) = \exp(\log x^{-cT^{-1}}) = x^{-cT^{-1}}$$

Now as $x \to \infty$, clearly $\exp(-c\sqrt{\log x}) \to 0$. We also have

$$\lim_{x\to\infty} \frac{\log x}{e^{c\sqrt{\log x}}} = \lim_{y\to\infty} \frac{y^2}{e^{cy}} \ \to\ 0$$

Now let us consider the second sum in 2.17. Note that

$$\log d_K + n_K \log(|\gamma|+2)\ \geq\ \sqrt{\log x}\ \Rightarrow\ |\gamma| \geq -2 + \exp\left(\frac{\sqrt{\log x} - \log d_K}{n_K}\right)$$

We will write the expression on the right as u, i.e. $|\gamma| \geq u$. Note that as $x \to \infty$, so does u.

We will also use the following result on counting the number of zeros in a rectangle in the critical strip (it can be deduced from Jensen's theorem) :

If $N_K(T)$ denote the number of zeros of $\zeta_K(s)$ in the region $0 < \Re(s) < 1$ and $|\Im(s)| < T$, then we have

$$|N_K(T+1) - N_K(T)| \ll n_K \log T \tag{2.18}$$

where the implied constant is absolute.

Since $x > 1$ we have

$$\begin{aligned}
&\sum_{|\gamma|\geq u} \frac{(\log x)\; x^{-c(\log d_K + n_K \log(|\gamma|+2))^{-1}}}{\gamma^2} \\
&< \; (\log x)\left(\sum_{|\gamma|\geq u} \frac{1}{\gamma^2}\right) \\
&\leq (\log x)\left(\sum_{j>u}\; \sum_{j<|\gamma|<j+1} \frac{1}{j^2}\right) \\
&\leq \; (\log x)\sum_{j>u} \frac{c_1 n_K \log j}{j^2} \\
&\leq c_1 n_K (\log x) \frac{\log u + 1}{u} \\
&\leq c_1 \frac{(\log x)(\sqrt{\log x} - \log d_K + n_K)}{-2 + \exp(\sqrt{\log x})} \to 0 \;\text{ as }\; x \to \infty
\end{aligned}$$

Therefore we have, $\lim_{x\to\infty} r(x) = 0$ and thus letting $x \to \infty$ in (3.8), completes the proof. □

In the next section, we will deduce conditional bounds for $\gamma_{K,1}$.

2.3 BOUNDS FOR $\gamma_{K,1}$ UNDER GRH

Bound for $\ell(x)$

$$\begin{aligned}
\ell(x) = &-\frac{r_1+r_2}{x-1}\left[\log x + \frac{1}{2}(\log x)^2\right] \\
&+ r_1\left[\sum_{n=1}^{\infty}\frac{2n(2n+1)\log x - 4n^2 + (1+2n)^2}{4n^2(1+2n)^2}\cdot\frac{x^{-2n}}{x-1}\right] \\
&+ r_2\left[\sum_{n=1}^{\infty}\frac{n(n+1)\log x - n^2 + (1+n)^2}{n^2(1+n)^2}\cdot\frac{x^{-n}}{x-1}\right] \\
&+ \frac{r_1}{x-1}\left[\sum_{n=1}^{\infty}\frac{1}{(1+2n)^2} - \sum_{n=1}^{\infty}\frac{1}{4n^2}\right] - \frac{r_2}{x-1} \\
&\ll \frac{r_1+r_2}{x}(\log x)^2 + \frac{r_1}{x^3}(\log x) + \frac{r_2}{x^2}(\log x) + \frac{n_k}{x} \\
&\ll \frac{n_k(\log x)^2}{x} + \frac{n_k \log x}{x^2} \\
&\ll \frac{n_k(\log x)^2}{x}
\end{aligned} \tag{2.19}$$

Remark 2.3.1. Note that the above bound for $\ell(x)$ is unconditional.

Bound for $r(x)$ under GRH

Under GRH, $\rho = \frac{1}{2} + i\gamma$, thus $1-\rho = \frac{1}{2} - i\gamma = \bar{\rho}$. We have

$$\begin{aligned}
r(x) &= \sum \frac{\rho(\rho-1)\log x - \rho^2 + (1-\rho)^2}{\rho^2(1-\rho)^2}\cdot\frac{x^\rho}{x-1} \\
&= \frac{\log x}{x-1}\sum\frac{x^\rho}{\rho(\rho-1)} + \frac{1}{x-1}\sum\left(\frac{x^\rho}{\rho^2} - \frac{x^\rho}{(1-\rho)^2}\right) \\
&= \frac{\log x}{x-1}\sum\frac{x^\rho}{\rho(\rho-1)} + \frac{1}{x-1}\sum\frac{x^{1-\rho} - x^\rho}{(1-\rho)^2} \\
&= \frac{\log x}{x-1}\sum\frac{x^\rho}{\rho(\rho-1)} + \frac{1}{x-1}\sum\frac{-2i\sqrt{x}\sin(\gamma\log x)}{(1-\rho)^2}
\end{aligned}$$

where the last equality follows from $x^{1-\rho} - x^{\rho} = x^{\bar{\rho}} - x^{\rho} = \sqrt{x}e^{-i\gamma\log x} - \sqrt{x}e^{i\gamma\log x}$. We also note that $|(1-\rho)^2| = \frac{1}{4} + \gamma^2 = |\rho(\rho-1)| = \rho(1-\rho)$. Thus,

$$\begin{aligned}
|r(x)| &\leq \frac{\sqrt{x}\log x}{x-1}\sum\frac{1}{|\rho(\rho-1)|} \; + \; \frac{2\sqrt{x}}{x-1}\sum\frac{1}{|(1-\rho)^2|} \\
&= \frac{\sqrt{x}(\log x + 2)}{x-1}\sum\frac{1}{\rho(1-\rho)} \\
&= \frac{2\sqrt{x}(\log x+2)}{x-1}(\gamma_{K,0} + \alpha_K + \beta_K + 1) \qquad \text{(using equation 2.3)}
\end{aligned}$$

Now by Ihara's Theorem 2.1.1, $\gamma_{K,0} \ll \log\alpha_K$. Thus we get,

$$r(x) \ll \frac{2\log x}{\sqrt{x}}(\alpha_k + \beta_K) \tag{2.20}$$

Remark 2.3.2. We are keeping the 2 in the numerator as it slightly helps our later computations while minimizing.

Estimates for $\Psi_K(x)$

Recall,

$$\begin{aligned}
\Psi_K(x) &= \frac{1}{x-1}\sum_{k,\, N(P)^k < x}\left(\frac{x}{N(P)^k} - 1\right)k(\log N(P))^2 \\
&= \frac{1}{x-1}\sum_{k,\, N(P)^k < x}\left(\frac{x}{N(P)^k} - 1\right)(\log N(P))(\log N(P)^k) \\
&\leq (\log x)\Phi_K(x)
\end{aligned} \tag{2.21}$$

Where $\Phi_K(x)$ is the counterpart of our $\Psi_K(x)$ used by Ihara in [Iha06] to compute $\gamma_{K,0}$. Ihara showed (e.g. proof of Theorem 3, 1.6.35 and 1.6.36)

$$\Phi_K(x) \leq n_K \cdot \Phi_{\mathbb{Q}}(x) < n_k \log x \tag{2.22}$$

And hence $\Psi_K(x) < n_k(\log x)^2$. In the computation below we show a better bound and replace the n_k by a small absolute constant.

$$\Psi'_K(t) = \frac{1}{t-1}\sum_{N(P)^k<t}\frac{k(\log N(P))^2}{N(P)^k} - \frac{1}{(t-1)^2}\sum_{k,\, N(P)^k<t}\left(\frac{t}{N(P)^k}-1\right)k(\log N(P))^2$$
$$= \frac{1}{(t-1)^2}\sum_{N(P)^k<t}\left(\frac{(t-1)}{N(P)^k}-\frac{t}{N(P)^k}+1\right)k(\log N(P))^2$$
$$= \frac{1}{(t-1)^2}\sum_{N(P)^k<t}\left(1-\frac{1}{N(P)^k}\right)k(\log N(P))^2$$
$$\leq \frac{1}{(t-1)^2}\sum_{N(P)^k<t}k(\log N(P))^2 \ll \frac{1}{(t-1)^2}(\log t)^2\cdot\frac{t}{\log t} \ll \frac{\log t}{t}$$

Note that, $\Psi_K(x)$ remains zero till the first real number a (>1) such that $N(P) = a$ for some prime ideal P. Choose small $\epsilon > 0$ with $1+\epsilon < a$ and write

$$\Psi_K(x) = \int_{1+\epsilon}^{x}\Psi'_K(t)dt \ll (\log x)^2 \tag{2.23}$$

We now have all the estimates to prove our theorem :

***Theorem* 2.3.3.** Under GRH, for $|d_K| \geq 8$

$$\gamma_{K,1} \ll \left(\log\ \log\sqrt{|d_K|}\right)^2 \tag{2.24}$$

Proof. Plugging in the bounds obtained for $\ell(x)$ in (3.10), $r(x)$ in (3.11), $\Psi_K(x)$ in (3.12) into the equation (3.8) we get :

$$\gamma_{K,1} \ll (\log x)^2\ + \frac{n_K(\log x)^2}{x} + \frac{2\log x}{\sqrt{x}}(\alpha_k+\beta_K)$$
$$= \left[(\log x)^2+\frac{2\log x}{\sqrt{x}}(\alpha_K)\right] + \left[\frac{n_K(\log x)^2}{x}+\frac{2\log x}{\sqrt{x}}\beta_K\right]$$

We first focus on the second sum. Recall,

$$\beta_K = -\{\frac{r_1}{2}(\gamma+\log 4\pi)+r_2(\gamma+\log 2\pi)\} \leq\ -\frac{(\gamma+\log 2\pi)}{2}n_K\ < -\ n_k$$

Note that the last inequality follows from $\gamma + \log 2\pi = 2.4150927\ldots$ Thus the second term

$$\left[\frac{n_K(\log x)^2}{x} + \frac{2\log x}{\sqrt{x}}\beta_K\right] = n_k\left[\frac{(\log x)^2}{x} - \frac{2\log x}{\sqrt{x}}\right] < 0$$

Choosing $x = \alpha_K^2$ we get,

$$\gamma_{K,1} \ll (\log \alpha_K)^2 + 4\log \alpha_K \tag{2.25}$$

and hence we have our result. □

Remark 2.3.4. Note that our bound in Theorem 2.3.3 is not necessarily optimal. The main term $\left[(\log x)^2 + \frac{2\log x}{\sqrt{x}}(\alpha_K)\right] = (\log x)\left[\log x + \frac{2\alpha_K}{\sqrt{x}}\right]$. The reason for choosing $x = \alpha_K^2$ is because it minimizes the term in square brackets.

2.4 GENERALIZATIONS TO HIGHER CONSTANTS $\gamma_{K,n}$

We first note that :

$$\lim_{s\to 1}\left[\frac{d^n}{ds^n}\frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{d^n}{ds^n}\frac{1}{(s-1)}\right] = \gamma_{K,n} \tag{2.26}$$

And thus differentiating equation (2.2) n times gives :

$$Z_K^{(n)}(s) = (-1)^n\frac{n!}{s^{n+1}} + (-1)^n\frac{n!}{(s-1)^{n+1}} + (-1)^{n+1}\sum\frac{n!}{(s-\rho)^{n+1}} + \tilde{\Gamma}_K^{(n)}(s) \tag{2.27}$$

and therefore, letting $\lim s \to 1$

$$-\gamma_{K,n} = (-1)^n n! + (-1)^{n+1}\sum\frac{n!}{(1-\rho)^{n+1}} + \tilde{\Gamma}_K^{(n)}(1)$$

$$\Rightarrow \gamma_{K,n} = (-1)^{n+1}n! + (-1)^n\sum\frac{n!}{(1-\rho)^{n+1}} - \tilde{\Gamma}_K^{(n)}(1) \tag{2.28}$$

Note : Recall we were writing $Z_K(s) = -\zeta_K'/\zeta_K(s)$.

On the other hand, taking term by term derivative of the Dirichlet series, we obtain :

$$Z_K^{(n)}(s) = \sum_{P,k\geq 1} \frac{(-1)^n k^n (\log N(P))^{n+1}}{N(P)^{ks}} \tag{2.29}$$

Like the $\gamma_{K,1}$ case we evaluate the integral :

$$\Psi_K^{(\mu)}(n,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-\mu}}{s-\mu} Z_K^{(n)}(s)\, ds \qquad \text{for } c \gg 0$$

For $\mu = 0$ and 1 in two different ways using equation (2.29) and (2.27).

Thus from (2.29) we get :

$$x\Psi_K^{(1)}(n,x) - \Psi_K^{(0)}(n,x) = \sum_{k,\ N(P)^k < x} \left(\frac{x}{N(P)^k} - 1\right)((-1)^n k^n (\log N(P))^{n+1}) \tag{2.30}$$

And hence we define :

$$\boxed{\Psi_K(n,x) = \frac{(-1)^n}{x-1} \sum_{k,\ N(P)^k < x} \left(\frac{x}{N(P)^k} - 1\right) k^n (\log N(P))^{n+1} \qquad \text{for } \ x > 1} \tag{2.31}$$

Lemma 2.4.1.

$$\Psi_K(n,x) \ll (\log x)^{n+1} \tag{2.32}$$

Proof.

$$\begin{aligned}
|\Psi_K(n,x)| &= \frac{1}{x-1} \sum_{k,\ N(P)^k < x} \left(\frac{x}{N(P)^k} - 1\right) k^n (\log N(P))^{n+1} \\
&= \frac{1}{x-1} \sum_{k,\ N(P)^k < x} \left(\frac{x}{N(P)^k} - 1\right) k (\log N(P))^2 (\log N(P)^k)^{n-1} \\
&\leq (\log x)^{n-1} \Psi_K(x) \ll (\log x)^{n+1}
\end{aligned}$$

where the last inequality follows from equation (3.12). □

We now focus on equation (2.27) and first look at the contribution from the term : $(-1)^n \frac{n!}{s^{n+1}} + (-1)^n \frac{n!}{(s-1)^{n+1}}$. To evaluate this, let

$$
\begin{aligned}
f(n,x) &= \frac{1}{2\pi i}\int_{(c)} (-1)^n n!\; x^s \left[\frac{1}{s^{n+1}(s-1)} + \frac{1}{(s-1)^{n+2}} - \frac{1}{s^{n+2}} - \frac{1}{s(s-1)^{n+1}}\right] ds \\
&= (-1)^n n!(x-1)\cdot\frac{1}{(n+1)!}(\log x)^{n+1} \;+ \\
&\qquad \frac{1}{2\pi i}\int_{(c)} (-1)^n n!\; x^s \left[\frac{1}{s^{n+1}(s-1)} - \frac{1}{s(s-1)^{n+1}}\right] ds \\
&= \frac{(-1)^n}{n+1}(x-1)(\log x)^{n+1} \;+ \\
&\qquad \frac{1}{2\pi i}\int_{(c)} (-1)^n n!\; x^s \left[\frac{1}{s^{n}(s-1)} - \frac{1}{s^{n+1}} + \frac{1}{s(s-1)^{n}} - \frac{1}{(s-1)^{n+1}}\right] ds
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(-1)^n}{n+1}(x-1)(\log x)^{n+1} \;+\; (-1)^{n+1}(x+1)(\log x)^n \;+ \\
&\qquad \frac{1}{2\pi i}\int_{(c)} (-1)^n n!\; x^s \left[\frac{1}{s^{n}(s-1)} + \frac{1}{s(s-1)^{n}}\right] ds \\
&= \frac{(-1)^n}{n+1}(x-1)(\log x)^{n+1} \;+\; (-1)^{n+1}(x+1)(\log x)^n \;+ \\
&n(n-1)\cdot\frac{1}{2\pi i}\int_{(c)} (-1)^{n-2}(n-2)!\; x^s \left[\frac{1}{s^{n-1}(s-1)} - \frac{1}{s^{n}} - \frac{1}{s(s-1)^{n-1}} + \frac{1}{(s-1)^{n}}\right] ds \\
&= \frac{(-1)^n}{n+1}(x-1)(\log x)^{n+1} \;+\; (-1)^{n+1}(x+1)(\log x)^n \;+ n(n-1)f(n-2,x)
\end{aligned}
$$

Note that from our previous computations, e.g. see equation (2.12) we get the $n = 1$ case and from [Iha06], 1.3.15 for the $n = 0$ case :

$$
\begin{aligned}
f(1) &= (1-x)\left[2 + \frac{1+x}{1-x}\log x + \frac{1}{2}\log^2 x\right] \\
f(0) &= (x-1)\log(x)
\end{aligned}
$$

Now let us compute the contribution from the sum of non-trivial zeros. Similar to the $n = 1$ case we do this by contour manipulation.

The pole at $s = 0$ has residue : $\dfrac{n!}{\rho^{n+1}}$.
The pole at ρ (order = $n+1$) has residue :

$$(-1)^{n+1} n! \lim_{s\to\rho} \frac{d^n}{ds^n}(\frac{x^s}{s}) = (-1)^{n+1} n! \frac{x^\rho (\log x)^n}{\rho} + \cdots$$

(Computations for $s = 1$ are exactly similar). Thus we see that the net contribution is :

$$n!(-1)^{n+1} x \sum \frac{1}{(1-\rho)^{n+1}} - n! \sum \frac{1}{\rho^{n+1}} + r(n,x)$$
$$= n!(-1)^{n+1}(x-1) \sum \frac{1}{(1-\rho)^{n+1}} + r(n,x) \qquad \text{for } n \text{ odd, and}$$
$$n!(-1)^{n+1}(x+1) \sum \frac{1}{(1-\rho)^{n+1}} + r(n,x) \qquad \text{for } n \text{ even.}$$

where

$$\begin{aligned} r(n,x) &= (-1)^{n+1} n! (\log x)^n \left[\sum \frac{x^\rho}{\rho - 1} - \sum \frac{x^\rho}{\rho} \right] + \cdots \\ &= (-1)^{n+1} n! (\log x)^n \sum \frac{x^\rho}{\rho(1-\rho)} + \cdots \\ &\ll (n!)(\log x)^n \sqrt{x}\, (\alpha_K + \beta_K) \end{aligned} \tag{2.33}$$

The last inequality is under GRH. The $(\alpha_K + \beta_K)$ is coming from equation (2.3) and Ihara's theorem 2.1.1. Also recall, as we showed in the proof of Theorem 2.3.3, $\beta_K < -n_K$. Looking at this error term, we can ditch the even-odd distinction and will just use the expression for n odd.

Now for the Gamma factors,

$$\tilde{\Gamma}^{(n)}(s) = (-1)^{n+1} n! \left[\frac{(r_1 + r_2)}{s^{n+1}} + r_1 \sum_{k=1}^{\infty} \frac{1}{(s+2k)^{n+1}} + r_2 \sum_{k=1}^{\infty} \frac{1}{(s+k)^{n+1}} \right]$$

we do similar residue computations at $s = 0$, $s = 1$, $s = -k$ and $s = -2k$. Note that

$$\frac{1}{s^{n+1}(s-1)} = \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2} - \cdots - \frac{1}{s^{n+1}}$$

So contribution from the first part looks like :

$$(-1)^{n+1} n!(r_1 + r_2) \left[x - 1 - \frac{1}{(n+1)!} (\log x)^{n+1} + p(\log x) \right]$$

where $p(.)$ is a polynomial of degree n and no constant term. Contribution from the second term will look like :

$$(-1)^{n+1}(n!) r_1 \left[x \sum_{k=1}^{\infty} \frac{1}{(1+2k)^{n+1}} - \sum_{k=1}^{\infty} \frac{1}{(2k)^{n+1}} + (\log x)^n \sum_{k=1}^{\infty} \frac{x^{-2k}}{2k(2k-1)} + \cdots \right]$$

We write this as :

$$(-1)^{n+1}(n!) r_1 (x-1) \left[\sum_{k=1}^{\infty} \frac{1}{(1+2k)^{n+1}} + \frac{(\log x)^n}{x-1} \sum_{k=1}^{\infty} \frac{x^{-2k}}{2k(2k-1)} + \cdots \right]$$

Similarly, contribution from the second term is given by :

$$(-1)^{n+1}(n!) r_2 (x-1) \left[\sum_{k=1}^{\infty} \frac{1}{(1+k)^{n+1}} + \frac{(\log x)^n}{x-1} \sum_{k=1}^{\infty} \frac{x^{-k}}{k(k-1)} + \cdots \right]$$

Putting these together, we see that the net contribution from the Gamma factors can be written as :

$$(x-1) \left[\tilde{\Gamma}^{(n)}(1) + \ell(n, x) \right] \tag{2.34}$$

where

$$\begin{aligned} \ell(n,x) &= \frac{(-1)^n}{n+1} \cdot \frac{r_1 + r_2}{(x-1)} (\log x)^{n+1} + O\left(\frac{n_K (\log x)^n}{x^2} \right) \\ &= O\left(\frac{n_K}{n+1} \frac{(\log x)^{n+1}}{x} \right) \end{aligned} \tag{2.35}$$

Where as before, $n_K = [K : \mathbb{Q}]$. In particular $\ell(n, x) \to 0$ as $x \to \infty$.

We are now ready to state and prove our results for the general n-th Euler-Kronecker constants :

Theorem 2.4.2. Under GRH, for $|d_K| \geq 8$,

$$\gamma_{K,n} + (-1)^n n! \ll \left(\log\ \log \sqrt{|d_K|}\right)^{n+1}$$

Proof. Putting our above computations together we get :

$$\begin{aligned}(x-1)\Psi_K(n,x) &= f(n,x) + n!(-1)^{n+1}(x-1)\sum \frac{1}{(1-\rho)^{n+1}} + r(n,x) \\ &\quad + (x-1)\left[\tilde{\Gamma}^{(n)}(1) + \ell(n,x)\right] \\ \Rightarrow \Psi_K(n,x) &= \frac{f(n,x)}{(x-1)} - \left[(-1)^n \sum \frac{n!}{(1-\rho)^{n+1}} - \tilde{\Gamma}^{(n)}(1)\right] \\ &\quad + \frac{r(n,x)}{(x-1)} + \ell(n,x)\end{aligned}$$

Now from equation (2.28) we get :

$$\Rightarrow \gamma_{K,n} - (-1)^{n+1} n! = -\Psi_K(n,x) + \frac{f(n,x)}{(x-1)} + \frac{r(n,x)}{(x-1)} + \ell(n,x) \qquad (2.36)$$

Now by Lemma 2.4.1 and the expression for $f(n,x)$ we see that,

$$-\Psi_K(n,x) + \frac{f(n,x)}{(x-1)} = O\left(\log x\right)^{n+1}$$

Thus,

$$\begin{aligned}\gamma_{K,n} + (-1)^n n! &\ll (\log x)^{n+1} + \frac{n_K}{n+1}\frac{(\log x)^{n+1}}{x} + (n!)\frac{(\log x)^n}{\sqrt{x}}\left(\alpha_K + \beta_K\right) \\ &< (\log x)^n \left(\log x + \frac{n!\alpha_K}{\sqrt{x}}\right) + n_K (\log x)^n \left(\frac{1}{n+1}\frac{\log x}{x} - \frac{n!}{\sqrt{x}}\right)\end{aligned}$$

Note that the second sum is negative. Choosing $x = \left(\frac{n!\alpha_K}{2}\right)^2$ minimizes the term in the brackets of the first term and gives us our bound. □

We can also prove the following general arithmetic formula like that in Theorem 2.2.2, unconditionally.

Theorem 2.4.3.

$$\gamma_{K,n} + (-1)^n n! = \lim_{x\to\infty}\left[-\Psi_K(n,x) + \frac{f(n,x)}{(x-1)}\right] \tag{2.37}$$

where $f(n,x)$ is recursively defined as :

$$f(n,x) = \frac{(-1)^n}{n+1}(x-1)(\log x)^{n+1} \; + \; (-1)^{n+1}(x+1)(\log x)^n \; + n(n-1)f(n-2)$$

$$f(1,x) = (1-x)\left[2 + \frac{1+x}{1-x}\log x + \frac{1}{2}\log^2 x\right]$$

$$f(0,x) = (x-1)\log(x)$$

Proof. As we saw in equation (2.36)

$$\Rightarrow \gamma_{K,n} - (-1)^{n+1}n! = -\Psi_K(n,x) + \frac{f(n,x)}{(x-1)} + \frac{r(n,x)}{(x-1)} + \ell(n,x)$$

From (2.35) we get $\lim_{x\to\infty}\ell(n,x) = 0$.

It was the $r(n,x)$ term which we estimated using GRH. To prove an unconditional result, we will use zero free regions, as we did in Theorem 2.2.2. We first look at the $r(n,x)$ term more carefully. Recall, we wrote the residue at the pole at ρ (order = $n+1$) as :

$$(-1)^{n+1}n! \lim_{s\to\rho}\frac{d^n}{ds^n}(\frac{x^s}{s}) = (-1)^{n+1}n!\frac{x^\rho(\log x)^n}{\rho} + \cdots$$

To make it more precise, we use some cute calculus, namely Leibniz rule, the one that generalizes the product rule of differentiation :

$$\frac{d^n}{dx^n}(f(x)\cdot g(x)) = \sum_{k=0}^{n}\binom{n}{k}f^{(n-k)}(x)g^{(k)}(x)$$

Therefore,

$$\frac{d^n}{ds^n}\frac{x^s}{s} = \sum_{k=0}^{n}\binom{n}{k}x^s(\log x)^{n-k}\cdot\frac{(-1)^k k!}{s^{k+1}}$$

which makes the residue

$$(-1)^n(n!)\sum_{k=0}^{n}\binom{n}{k}x^{\rho}(\log x)^{n-k}\cdot\frac{(-1)^k k!}{\rho^{k+1}} \tag{2.38}$$

Thus

$$\begin{aligned} r(n,x) &= \sum_{\rho}\Bigg(x(-1)^n(n!)\sum_{k=0}^{n}\binom{n}{k}x^{\rho-1}(\log x)^{n-k}\cdot\frac{(-1)^k k!}{(\rho-1)^{k+1}} \\ &\quad -(-1)^n(n!)\sum_{k=0}^{n}\binom{n}{k}x^{\rho}(\log x)^{n-k}\cdot\frac{(-1)^k k!}{\rho^{k+1}}\Bigg) \\ &\ll n!(\log x)^n\sum\frac{x^{\beta}}{\gamma^2} \end{aligned}$$

where $\rho=\beta+i\gamma$. Thus, $\frac{r(n,x)}{(x-1)} \ll n!(\log x)^n\sum\frac{x^{\beta-1}}{\gamma^2}$.

We see in the computations done for the proof of Theorem 2.2.2, $\log x$ replaced by any power of $\log x$ also works! Hence $\lim_{x\to\infty}\frac{r(n,x)}{(x-1)}=0$ and we have

$$\gamma_{K,n}+(-1)^n n! = \lim_{x\to\infty}\left[-\Psi_K(n,x)+\frac{f(n,x)}{(x-1)}\right]$$

□

3

BOUNDS FOR THE FUNCTION FIELD CASE

In this chapter we deduce similar bounds as in Chapter 2 for the function field case. Let q be a power of a prime and $\mathbb{F}_q$ be the finite field with q elements. Let K be the function field of a curve X over $\mathbb{F}_q$ of genus g. A good reference for the basic facts about the zeta function $\zeta_K(s)$ is [Ros00].

We set $u = q^{-s}$, then the $\zeta_K(s)$ is a rational function of u of the form

$$\zeta_K(s) = \frac{\prod_{i=1}^{g}(1-\pi_i u)(1-\overline{\pi_i}u)}{(1-u)(1-qu)} \qquad \text{with} \quad \pi_i\overline{\pi_i} = q \quad \text{for all} \ \ 1 \le i \le g \tag{3.1}$$

Note that each zero $\frac{1}{\pi_i}$ or $\frac{1}{\overline{\pi_i}}$ of $\zeta_K(s)$ in u corresponds to infinitely many zeros in s and all of them are translations of a zero by $\frac{2\pi i n}{\log q}$, $n \in \mathbb{Z}$. Similarly, poles are translations of 0 and 1 by $\frac{2\pi i n}{\log q}$, $n \in \mathbb{Z}$.

Also, $\zeta_K(s)$ has a simple pole at $s = 1$, and thus like the number field case we can write the Laurent series of it's logarithmic derivative as

$$\frac{\zeta_K'(s)}{\zeta_K(s)} = \frac{-1}{s-1} + \gamma_{K,0} + \gamma_{K,1}(s-1) + \cdots \tag{3.2}$$

and define $\gamma_{K,n}$ as the general *Euler-Kronecker* constants.

Ihara in [Iha06], 1.3.10 derives the following Stark like lemma.

Lemma 3.0.1.

$$-\frac{\zeta_K'(s)}{\zeta_K(s)} = \frac{1}{s} + \frac{1}{s-1} - \sum_{\rho}\frac{1}{s-\rho} + (g-1)\log q + \sum_{\theta \neq 0,1}\frac{1}{s-\theta} \tag{3.3}$$

where ρ runs over the non-trivial zeros of and θ runs over all poles $\neq 0,1$ of $\zeta_K(s)$.

Proof. Write a simpler rational form $\zeta_K(s) = \prod_{\alpha \in A}(1-\alpha q^{-s})^{\lambda_\alpha}$, where $\lambda_\alpha = \pm 1$ and A is a finite subset of $\mathbb{C}^\times$. Taking the logarithmic derivative we get,

$$\frac{\zeta_K'(s)}{\zeta_K(s)} = \sum_{\alpha \in A} \lambda_\alpha \frac{-\alpha q^{-s}\log q}{1-\alpha q^{-s}} \quad \Rightarrow -\frac{\zeta_K'(s)}{\zeta_K(s)} = \sum_{\alpha \in A} \lambda_\alpha \frac{\log q}{1-\alpha^{-1}q^{s}} \tag{3.4}$$

Now consider the partial fraction formula :

$$\frac{1}{e^z-1} + \frac{1}{2} = \lim_{T\to\infty} \sum_{n=-T}^{T} \frac{1}{z-2\pi i n}$$

Substituting $e^z = \alpha^{-1}q^s \Rightarrow z = s\log q - \log\alpha$ we get,

$$\begin{aligned}
\frac{1}{\alpha^{-1}q^s-1} + \frac{1}{2} &= \lim_{T\to\infty} \sum_{n=-T}^{T} \frac{1}{s\log q - \log\alpha - 2\pi i n} \\
\frac{\log q}{\alpha^{-1}q^s-1} + \frac{\log q}{2} &= \lim_{T\to\infty} \sum_{n=-T}^{T} \frac{1}{s - \frac{\log\alpha + 2\pi i n}{\log q}} \\
&= \lim_{T\to\infty} \sum_{\substack{q^\beta=\alpha \\ |\beta|\le T}} \frac{1}{s-\beta} = \sum_{q^\beta=\alpha} \frac{1}{s-\beta}
\end{aligned} \tag{3.5}$$

Putting (3.5) in equation (3.4) we get,

$$\begin{aligned}
-\frac{\zeta_K'(s)}{\zeta_K(s)} &= \sum_{\alpha\in A} \lambda_\alpha \left(\frac{\log q}{2} - \sum_{q^\beta=\alpha} \frac{1}{s-\beta} \right) \\
&= \frac{\log q}{2} \sum_{\alpha\in A} \lambda_\alpha + \left(\frac{1}{s} + \frac{1}{s-1} + \sum_{\substack{\text{poles } \theta \\ \theta\neq 0,1}} \frac{1}{s-\theta} \right) - \sum_{\text{zeros}} \frac{1}{s-\rho} \\
&= \frac{1}{s} + \frac{1}{s-1} - \sum \frac{1}{s-\rho} + (g-1)\log q + \sum_{\theta \neq 0,1} \frac{1}{s-\theta}
\end{aligned}$$

□

We have

$$\gamma_{K,0} = \lim_{s\to 1} \left(\frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{1}{s-1} \right)$$

and using Lemma 3.0.1, we get

$$\gamma_{K,0} = \sum \frac{1}{1-\rho} - (g-1)\log q - \sum_{\theta \neq 0,1} \frac{1}{1-\theta} - 1$$

In [Iha06], Ihara deduces the following upper bound for $\gamma_{K,0}$:

Theorem 3.0.2. **(Ihara)** For $g > 2$ or, $g = 2$ and $q > 2$, and $\alpha_K = (g-1)\log q$ we have

$$\gamma_{K,0} \leq \left(\frac{\alpha_K + 1}{\alpha_K - 1}\right)(2\log \alpha_K + 1 + \log q)$$

3.1 BOUNDS FOR $\gamma_{K,1}$

Looking at the similarities of equation (3.3) to the NF case of (2.2), we see that we only need to tweak our computations a little bit, particularly, instead of the gamma factors, we need to do the computations for the sum related to the poles, rest is similar.

Hence, differentiating (3.0.1) we get,

$$Z_K'(s) = -\frac{1}{s^2} - \frac{1}{(s-1)^2} + \sum \frac{1}{(s-\rho)^2} - \sum_{\theta \neq 0,1} \frac{1}{(s-\theta)^2} \tag{3.6}$$

Taking limit $s \to 1$ we get,

$$\gamma_{K,1} = 1 - \sum \frac{1}{(1-\rho)^2} + \sum_{\theta \neq 0,1} \frac{1}{(1-\theta)^2} \tag{3.7}$$

We do the same process of computing $\Psi_K(x)$ and focus on the contribution from the $\sum_\theta$ term.

$$(x-1)\left[\sum_{\theta \neq 0,1} \frac{\theta(\theta-1)\log x - \theta^2 + (1-\theta)^2}{\theta^2(1-\theta)^2} \cdot \frac{x^\theta}{x-1} + \sum_{\theta \neq 0,1} \frac{1}{(1-\theta)^2}\right]$$

$$= (x-1)\left[\ell(x) + \sum_{\theta \neq 0,1} \frac{1}{(1-\theta)^2}\right] \qquad \text{(say)}$$

Thus we have the formula,

$$\boxed{\gamma_{K,1} = \Psi_K(x) - 1 - \frac{1+x}{1-x}\log x - \frac{1}{2}\log^2 x \ + r(x) \ - \ell(x)} \tag{3.8}$$

where as before,

$$\Psi_K(x) = \frac{1}{x-1}\sum_{k,\ N(P)^k < x}\left(\frac{x}{N(P)^k} - 1\right)k(\log N(P))^2 \qquad \text{for } \ x > 1 \tag{3.9}$$

and

$$r(x) = \sum \frac{\rho(\rho-1)\log x - \rho^2 + (1-\rho)^2}{\rho^2(1-\rho)^2}\cdot\frac{x^\rho}{x-1}$$

Upper bound for $\ell(x)$

$$\begin{aligned}
\ell(x) &= \sum_{\theta\neq 0,1}\frac{\theta(\theta-1)\log x - \theta^2 + (1-\theta)^2}{\theta^2(1-\theta)^2}\cdot\frac{x^\theta}{x-1} \\
&= \frac{\log x}{x-1}\sum_{\theta\neq 0,1}\frac{x^\theta}{\theta(\theta-1)} \ + \frac{1}{x-1}\sum_{\theta\neq 0,1}\left[\frac{x^\theta}{\theta^2} - \frac{x^\theta}{(1-\theta)^2}\right] \\
&\ll \log x \quad \text{(Note that } \mathrm{Re}(\theta) = 0 \text{ or 1, and the series are abs. convg.)}
\end{aligned} \tag{3.10}$$

Upper bound for $r(x)$

Since GRH holds in the Function field case, $\rho = \frac{1}{2} + i\gamma$, thus $1 - \rho = \frac{1}{2} - i\gamma = \bar{\rho}$.
We will have, like in te number field case, (and [Iha06], 1.3.11)

$$r(x) \ll \frac{\log x}{\sqrt{x}}(\gamma_{K,0} + \alpha_K + \frac{q+1}{2(q-1)}\log q) \tag{3.11}$$

Upper bound for $\Psi_K(x)$

As in the NF case :

$$\Psi_K(x) \ll (\log x)^2 \tag{3.12}$$

We now have all the estimates to prove our theorem :

Theorem 3.1.1. For $g > 2$ or, $g = 2$ and $q > 2$, we have

$$\gamma_{K,1} \ll (\log \alpha_K)^2 \quad \text{where } \alpha_K = (g-1)\log q \tag{3.13}$$

Proof. Plugging in the bounds obtained for $\ell(x)$ in (3.10), $r(x)$ in (3.11), $\Psi_K(x)$ in (3.12) into the equation (3.8) we get :

$$\begin{aligned}\gamma_{K,1} &\ll (\log x)^2 \;+\; \frac{\log x}{\sqrt{x}}\left(\gamma_{K,0} + \alpha_K + \frac{q+1}{2(q-1)}\log q\right) \\ &= \log x\left[\log x + \frac{2\delta_K}{\sqrt{x}}\right] \quad \text{(say)}\end{aligned}$$

where $2\delta_K = \gamma_{K,0} + \alpha_K + \frac{q+1}{2(q-1)}\log q$. Choosing $x = \delta_K^2$ minimizes the RHS.

$$\gamma_{K,1} \ll (\log \delta_K)^2 \ll (\log \alpha_K)^2$$

where the last inequality follows from the bound on $\gamma_{K,0}$ due to Ihara, as in Theorem 3.0.2. □

3.2 GENERAL CASE : BOUNDS FOR $\gamma_{K,n}$

Hence, differentiating (3.0.1) n times we get,

$$Z_K^{(n)}(s) = \frac{(-1)^n n!}{s^{n+1}} + \frac{(-1)^n n!}{(s-1)^{n+1}} + (-1)^{n+1}\sum \frac{n!}{(s-\rho)^{n+1}} + (-1)^n \sum_{\theta\neq 0,1} \frac{n!}{(s-\theta)^{n+1}} \tag{3.14}$$

and therefore, letting $\lim s \to 1$

$$\begin{aligned}-\gamma_{K,n} &= (-1)^n n! + (-1)^{n+1}\sum \frac{n!}{(1-\rho)^{n+1}} + (-1)^n \sum_{\theta\neq 0,1} \frac{n!}{(1-\theta)^{n+1}} \\ \Rightarrow \gamma_{K,n} &= (-1)^{n+1} n! + (-1)^n \sum \frac{n!}{(1-\rho)^{n+1}} + (-1)^{n+1} \sum_{\theta\neq 0,1} \frac{n!}{(1-\theta)^{n+1}}\end{aligned} \tag{3.15}$$

We do the same contour manipulation as in the NF case. We define,

$$\Psi_K(n,x) = \frac{(-1)^n}{x-1} \sum_{k,\, N(P)^k < x} \left(\frac{x}{N(P)^k} - 1 \right) k^n (\log N(P))^{n+1} \qquad \text{for } x > 1 \tag{3.16}$$

and we have

$$\Psi_K(n,x) \ll (\log x)^{n+1} \tag{3.17}$$

The contribution from the term : $\frac{(-1)^n n!}{s^{n+1}} + \frac{(-1)^n n!}{(s-1)^{n+1}}$ can be similarly computed to be the function $f(n,x)$ as defined in Theorem 2.4.3. Also, we had $f(n,x) \ll x(\log x)^{n+1}$. Similarly, since GRH is known in this case, contribution from the non-trivial zeros is

$$n!(-1)^{n+1}(x-1) \sum \frac{1}{(1-\rho)^{n+1}} + r(n,x)$$

where,

$$r(n,x) \ll (n!)(\log x)^n \sqrt{x}\, \alpha_K$$

The only new thing we need to compute is the contribution of the poles. Which again, looking at the similarity of the term to that of the zeros, looks like :

$$n!(-1)^n(x-1) \sum \frac{1}{(1-\theta)^{n+1}} + \ell(n,x)$$

where

$$\begin{aligned}
\ell(n,x) &= \sum_{\substack{\text{poles} \\ \theta \neq 0,1}} \Bigg(x(-1)^n (n!) \sum_{k=0}^{n} \binom{n}{k} x^{\theta-1} (\log x)^{n-k} \cdot \frac{(-1)^k k!}{(\theta-1)^{k+1}} \\
&\qquad - (-1)^n (n!) \sum_{k=0}^{n} \binom{n}{k} x^{\theta} (\log x)^{n-k} \cdot \frac{(-1)^k k!}{\theta^{k+1}} \Bigg) \\
&\ll n! x (\log x)^n \sum_{k=0}^{n} \binom{n}{k} \left| \sum_{\substack{\text{poles} \\ \theta \neq 0,1}} \frac{1}{(\theta-1)^{k+1}} - \frac{1}{\theta^{k+1}} \right| \\
&\ll n! x (\log x)^n
\end{aligned}$$

Note that the first inequality follows from the fact that $\text{Re}(\theta) = 0$ or 1 whereas, all the series in the second inequality is absolutely convergent.

Remark 3.2.1. To see the expression we wrote for $\ell(x)$ look back at the last part of the proof of Theorem 2.4.3.

We are now ready to generalize Theorem 3.1.1.

***Theorem* 3.2.2.** For $g > 2$ or, $g = 2$ and $q > 2$, we have

$$\gamma_{K,n} + (-1)^n n! \ll n!(\log \alpha_K)^{n+1} \quad \text{where } \alpha_K = (g-1)\log q \tag{3.18}$$

Proof. Putting our computations together,

$$\begin{aligned}(x-1)\Psi_K(n,x) &= f(n,x) + n!(-1)^{n+1}(x-1)\sum \frac{1}{(1-\rho)^{n+1}} + r(n,x)\\ &\quad + n!(-1)^n(x-1)\sum \frac{1}{(1-\theta)^{n+1}} + \ell(n,x)\end{aligned}$$

$$\begin{aligned}\Rightarrow \Psi_K(n,x) &= \frac{f(n,x)}{(x-1)} - \left[(-1)^n \sum \frac{n!}{(1-\rho)^{n+1}} + (-1)^{n+1} \sum_{\theta \neq 0,1} \frac{n!}{(1-\theta)^{n+1}}\right]\\ &\quad + \frac{r(n,x)}{(x-1)} + \frac{\ell(n,x)}{(x-1)}\end{aligned}$$

$$\begin{aligned}\gamma_{K,n} + (-1)^n n! &= -\Psi_K(n,x) + \frac{f(n,x)}{(x-1)} + \frac{r(n,x)}{(x-1)} + \frac{\ell(n,x)}{(x-1)}\\ &\ll (\log x)^{n+1} + \frac{(n!)(\log x)^n \alpha_K}{\sqrt{x}} + n!(\log x)^n\\ &\ll n!(\log x)^n \left(\log x + \frac{\alpha_K}{\sqrt{x}}\right)\end{aligned}$$

choosing $x = \alpha_K^2$ to minimize the sum in the brackets gives our result! $\square$

4

MOMENTS OF HIGHER DERIVATIVES OF $\mathcal{L}(s,\chi)|_{s=1}$

4.1 PRELIMINARIES

Let K be a number field and χ be a primitive Dirichlet character on K (i.e. a primitive Hecke character with finite order). Let $L(s,\chi)$ be the L-function associated to it. In particular, when $\chi = \chi_0$, the principal character, $L(s,\chi) = \zeta_K(s)$, the Dedekind zeta function of K. The completed L-function is of the form :

$$\xi(s,\chi) = AB^{\frac{s}{2}}\Gamma\left(\frac{s+1}{2}\right)^{a}\Gamma\left(\frac{s}{2}\right)^{a'}\Gamma(s)^{r_2}\,L(s,\chi) \tag{4.1}$$

and satisfies a functional equation : $\xi(s,\chi) = \varepsilon(\chi)\xi(1-s,\overline{\chi})$, where $\varepsilon(\chi)$ is a constant of absolute value 1. A, B are constants involving $2, \pi$, the discriminant of K and the conductor $\mathfrak{f}_\chi$. As we'll be concerned with higher derivatives, we haven't written them down explicitly, but interested reader can have a look at p.211 of [CF76] or for Hecke's original proof see [Hec83].

Also note that, here a (resp. a') is the number of real places of K where χ is ramified (resp. unramified), $r_1 = a + a'$ is the number of real places of K and r_2 is the number of complex places in K.

For $\chi \neq \chi_0$, taking the logarithmic derivative of (4.1) and using Hadamard product one can then deduce a Stark like lemma (e.g. see Lemma 2.1 of [Sta74] or p.83 of [Dav00]) :

$$\frac{L'(s,\chi)}{L(s,\chi)} = C - \frac{a}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) - \frac{a'}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+1}{2}\right) - r_2\frac{\Gamma'}{\Gamma}(s) + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) \tag{4.2}$$

C being a constant involving log of terms in B in (4.1) etc.

For the rest of the chapter, we will denote the LHS by $\mathcal{L}(s,\chi)$, i.e.

$$\mathcal{L}(s,\chi) = \frac{L'(s,\chi)}{L(s,\chi)} \qquad \text{(say)}$$

On the other hand, by taking the logarithmic derivative of the Euler product of $L(s,\chi)$ we get :

$$\mathcal{L}(s,\chi) = -\sum_{P,k} \left(\frac{\chi(P)}{N(P)^s} \right)^k \log N(P) \tag{4.3}$$

Ihara, Murty and Shimura proved the following theorem in [IMS09] :

Theorem 4.1.1. **(Ihara, Murty, Shimura)**
If $\chi \neq \chi_0$, then

$$\mathcal{L}(1,\chi) = -\lim_{x\to\infty} \Phi_{K,\chi}(x) \tag{4.4}$$

where

$$\Phi_{K,\chi}(x) = \frac{1}{x-1} \sum_{N(P)^k \leq x} \left(\frac{x}{N(P)^k} - 1 \right) \chi(P)^k \, \log N(P) \quad (\text{ for } x > 1)$$

Here, k is a positive integer and the sum is taken over non-archimedean primes. Under GRH, they have shown the following upper bound :

$$|\mathcal{L}(1,\chi)| < 2\, \log\log \sqrt{d_\chi} \; + 1 - \gamma_{K,0} + O\left(\frac{\log |d_K| + \log\log d_\chi}{\log d_\chi} \right)$$

Here, $d_\chi = |d_K| N(\mathfrak{f}_\chi)$ and $\gamma_{K,0}$ is the Euler-Kronecker constant of K.

The proof of the above theorem follows its counterpart for the Dedekind zeta function, due to Ihara in [Iha06]. It is based on computing the integral

$$\Phi^{(\mu)}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s-\mu}}{s-\mu} \mathcal{L}(s,\chi)\, ds \qquad \text{for } c \gg 0$$

for $\mu = 0$ and 1, in two different ways using the equations (4.2) and (4.3) and then estimating the terms.

Remark 4.1.2. Due to equation (4.2) and (4.3), using the same methods as in the case of $\gamma_{K,n}$ in Chapter 2, similar formulas and bounds for higher derivatives of $\mathcal{L}(s,\chi)$ at $s=1$ can be computed. We present some of those computations in the next sections.

4.2 AN "EXACT FORMULA" FOR $\mathcal{L}'(1,\chi)$

Differentiating equation (4.2) we get,

$$\mathcal{L}'(s,\chi) = -\sum_{\rho} \frac{1}{(s-\rho)^2} + \tilde{\Gamma}'_{\chi}(s) \tag{4.5}$$

$$\begin{aligned}
\tilde{\Gamma}_{\chi}(s) &= -\frac{a}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+1}{2}\right) - \frac{a'}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) - r_2\frac{\Gamma'}{\Gamma}(s) \\
&= \frac{n}{2}\gamma + \frac{a}{2}\sum_{k=0}^{\infty}\left(\frac{2}{s+1+2k} - \frac{2}{2k+2}\right) + \frac{a'}{2}\sum_{k=0}^{\infty}\left(\frac{2}{s+2k} - \frac{2}{1+2k}\right) \\
&\quad + r_2\sum_{k=0}^{\infty}\left(\frac{1}{s+k} - \frac{1}{1+k}\right)
\end{aligned}$$

Here $\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \ln n\right] \sim 0.5772\ldots$ is the Euler–Mascheroni constant and $n=[K:\mathbb{Q}]$. Differentiating we get,

$$\tilde{\Gamma}'_{\chi}(s) = -\,a\sum_{k=0}^{\infty}\frac{1}{(s+1+2k)^2} - a'\sum_{k=0}^{\infty}\frac{1}{(s+2k)^2} - r_2\sum_{k=0}^{\infty}\frac{1}{(s+k)^2}$$

Differentiating the Euler product in equation (4.3) we get,

$$\mathcal{L}'(s,\chi) = \sum_{P,k} k\left(\frac{\chi(P)}{N(P)^s}\right)^k (\log N(P))^2 \tag{4.6}$$

To find a similar 'exact formula' as in the case of $\gamma_{K,1}$, we evaluate the integral :

$$\Psi_{\chi}^{(\mu)}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s-\mu}}{s-\mu}\mathcal{L}'(\chi,s)\,ds \qquad \text{for } c \gg 0$$

For $\mu = 0$ and 1 in two different ways using equation (4.6) and equation (4.2) and the classical formulas:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s}\,ds = \begin{cases} 0 & 0<y<1 \\ \frac{1}{2} & y=1 \\ 1 & y>1 \end{cases} \tag{4.7}$$

From the Euler product we get :

$$x\Psi_\chi^{(1)}(x) - \Psi_\chi^{(0)}(x) = \sum_{k,\,N(P)^k<\,x} k\left(\frac{x}{N(P)^k}-1\right)\chi(P)^k(\log N(P))^2 \tag{4.8}$$

Looking at the above computation, we define :

$$\boxed{\Psi_\chi(x) = \frac{1}{x-1}\sum_{k,\,N(P)^k<\,x} k\left(\frac{x}{N(P)^k}-1\right)\chi(P)^k(\log N(P))^2 \qquad \text{for } \ x>1} \tag{4.9}$$

On the other hand, from equation (4.5) we get,

$$\begin{aligned} x\Psi_\chi^{(1)}(x) - \Psi_\chi^{(0)}(x) &= \frac{x}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s-1}}{s-1}\left[-\sum\frac{1}{(s-\rho)^2}+\tilde{\Gamma}'_\chi(s)\right]\,ds \\ &\quad -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s}\left[-\sum\frac{1}{(s-\rho)^2}+\tilde{\Gamma}'_\chi(s)\right]\,ds \end{aligned}$$

Computing similar contour integrals as $\gamma_{K,1}$ we see that, contribution from the $\sum_\rho$ term is :

$$\begin{aligned} &(1-x)\left[\sum\frac{\rho(\rho-1)\log x-\rho^2+(1-\rho)^2}{\rho^2(1-\rho)^2}\cdot\frac{x^\rho}{x-1} \;+\; \sum\frac{1}{(1-\rho)^2}\right] \\ &= (1-x)\left[r_\chi(x)+\sum\frac{1}{(1-\rho)^2}\right] \qquad \text{(say)} \end{aligned} \tag{4.10}$$

For contribution from the $\tilde{\Gamma}'_\chi(s)$, we first re-write it as follows :

$$\tilde{\Gamma}'_\chi(s) = -\frac{a' + r_2}{s^2} - \ a \sum_{k=0}^{\infty} \frac{1}{(s+1+2k)^2} \ - a' \sum_{k=1}^{\infty} \frac{1}{(s+2k)^2} \ - r_2 \sum_{k=1}^{\infty} \frac{1}{(s+k)^2}$$

For the first term in the above equation, we have

$$\begin{aligned}
&- (a' + r_2) \left[\frac{x}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s^2(s-1)} \, ds \ - \ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s^3} \, ds \right] \\
&= -(a' + r_2) \left[\frac{x}{2\pi i} \int_{c-i\infty}^{c+\infty} x^{s-1} \left(\frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2} \right) \, ds \ - \ \frac{1}{2} (\log x)^2 \right] \\
&= -(a' + r_2) \left[x - 1 - \log x - \frac{1}{2} (\log x)^2 \right] \\
&= (a' + r_2)(1 - x) \left[1 - \frac{\log x}{(x-1)} - \frac{(\log x)^2}{2(x-1)} \right]
\end{aligned}$$

Note that, as before, here we're using the classical formula : (for $n \geq 1$)

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^s}{s^{n+1}} \, ds = \begin{cases} 0 & 0 < y \leq 1 \\ \frac{1}{n!} (\log y)^n & y > 1 \end{cases}$$

For the rest of the terms involving series, the computations are similar to that of $\frac{1}{(s-\rho)^2}$ and we get the total contribution of $\tilde{\Gamma}'_\chi(s)$ terms to be :

$$(a'+r_2)(1-x)\left[1-\frac{\log x}{(x-1)}-\frac{(\log x)^2}{2(x-1)}\right]+$$
$$a(1-x)\left[\sum_{k=0}^{\infty}\frac{(2k+1)(2k+2)\log x-(1+2k)^2+(2+2k)^2}{(2+2k)^2(1+2k)^2}\cdot\frac{x^{-2k-1}}{x-1}+\sum_{k=0}^{\infty}\frac{1}{(2+2k)^2}\right]+$$
$$a'(1-x)\left[\sum_{k=1}^{\infty}\frac{2k(2k+1)\log x-4k^2+(1+2k)^2}{4k^2(1+2k)^2}\cdot\frac{x^{-2k}}{x-1}+\sum_{k=1}^{\infty}\frac{1}{(1+2k)^2}\right]+$$
$$r_2(1-x)\left[\sum_{k=1}^{\infty}\frac{k(k+1)\log x-k^2+(1+k)^2}{k^2(1+k)^2}\cdot\frac{x^{-k}}{x-1}+\sum_{k=1}^{\infty}\frac{1}{(1+k)^2}\right]$$
$$=(1-x)\left[\ell_\chi(x)-\tilde{\Gamma}'_\chi(1)\right]\qquad\text{(say)}\tag{4.11}$$

Putting together equation (4.8), (4.10) and (4.11) we get

$$(x-1)\Psi_\chi(x)=(1-x)\left[r_\chi(x)+\sum\frac{1}{(1-\rho)^2}\right]+(1-x)\left[\ell_\chi(x)-\tilde{\Gamma}'_\chi(1)\right]$$
$$-\Psi_\chi(x)=\ell_\chi(x)+r_\chi(x)-\tilde{\Gamma}'_\chi(1)+\sum\frac{1}{(1-\rho)^2}$$
$$\boxed{\mathcal{L}'(1,\chi)\;=\;\Psi_\chi(x)+r_\chi(x)+\ell_\chi(x)}\tag{4.12}$$

Lemma 4.2.1. For $\chi\neq\chi_0$, we have (unconditionally)

$$\ell_\chi(x)=O\left(\frac{n_K(\log x)^2}{x}\right)$$

Here the implied constant is absolute.

Proof. Note that the series are absolutely convergent and thus, contribution from the series terms are : $O(\frac{(a+r_2)\log x}{x^2}+\frac{a'\log x}{x^3})=O(\frac{n_K\log x}{x^2})$. Whereas, the first term is $O(\frac{n_K(\log x)^2}{x})$. □

We are now ready to state and prove our exact formula :

***Theorem* 4.2.2.** For $\chi \neq \chi_0$, we have, unconditionally,

$$\mathcal{L}'(\chi, 1) = \lim_{x\to\infty} \Psi_\chi(x)$$

Proof. From the above lemma, $\lim_{x\to\infty} \ell_\chi(x) = 0$. For the r_χ term, note that following the exact same steps as in the computation of $\lim_{x\to\infty} r(x)$, for $\gamma_{K,1}$ we can show that $\lim_{x\to\infty} r_\chi(x) = 0$. □

4.3 A GRH BOUND FOR $\mathcal{L}'(1,\chi)$

Since $r_\chi(x)$ has the same expression as that of $r(x)$ in the computaion of $\gamma_{K,1}$, we get, under GRH, (writing $\rho = \frac{1}{2} + i\gamma$)

$$r_\chi(x) = \frac{\log x}{x-1} \sum \frac{x^\rho}{\rho(\rho-1)} + \frac{1}{x-1} \sum \frac{-2i\sqrt{x}\sin(\gamma \log x)}{(1-\rho)^2} \tag{4.13}$$

Thus, (for $\chi \neq \chi_0$)

$$\begin{aligned} |r(x)| &\leq \frac{\sqrt{x}\log x}{x-1} \sum \frac{1}{|\rho(\rho-1)|} + \frac{2\sqrt{x}}{x-1} \sum \frac{1}{|(1-\rho)^2|} \\ &= \frac{\sqrt{x}(\log x + 2)}{x-1} \sum \frac{1}{\rho(1-\rho)} \\ &= \frac{2\sqrt{x}(\log x + 2)}{x-1} (\mathcal{L}(\chi, 1) + \alpha_{K,\chi} + \beta_{K,\chi}) \end{aligned}$$

The last equality follows from, Theorem 2 of [IMS09]. Here,

$$\begin{cases} \alpha_{K,\chi} = \frac{1}{2}\log d_\chi & \text{where } d_\chi = |d_K| N(\mathfrak{f}_\chi) \\ \beta_{K,\chi} = -\frac{a+r_2}{2}(\gamma + \log 4\pi) - \frac{a'+r_2}{2}(\gamma + \log \pi) \end{cases}$$

$\mathfrak{f}_\chi$ being the conductor of χ and $\gamma = \gamma_{\mathbb{Q},0}$ being the Euler-Mascheroni constant. Note that, from Theorem 3 of [IMS09] we have $\mathcal{L}(\chi, 1) \ll \log \alpha_{K,\chi}$. Thus we can write,

$$r_\chi(x) \ll \frac{\log x}{\sqrt{x}} (\alpha_{K,\chi}) \tag{4.14}$$

Theorem 4.3.1. For $\chi \neq \chi_0$, and $|d_K| \geq 8$, we have, under GRH

$$\mathcal{L}'(\chi, 1) \ll \left(\log\log\sqrt{d_\chi}\right)^2$$

Proof. Note that $|\Psi_\chi(x)| \leq \Psi_K(x) \ll (\log x)^2$. Thus,

$$\mathcal{L}'(\chi, 1) \ll (\log x)^2 + \frac{\log x}{\sqrt{x}}(\alpha_{K,\chi}) + \frac{n_K(\log x)^2}{x}$$

Taking $x = \alpha_{K,\chi}^2$ we get,

$$\begin{aligned}\mathcal{L}'(\chi, 1) &\ll (2\log\alpha_{K,\chi})^2 + 2\log\alpha_{K,\chi} + 4n_k\left(\frac{\log\alpha_{K,\chi}}{\alpha_{K,\chi}}\right)^2 \\ &\ll \left(\log\log\sqrt{d_\chi}\right)^2\end{aligned}$$

□

4.4 GENERALIZATION TO HIGHER DERIVATIVES

In this section we present a generalization of the limit formula as in Theorem 4.2.2. For $n \geq 1$, we look at the n-th derivative of the Euler product in (4.3) :

$$\mathcal{L}^{(n)}(s, \chi) = (-1)^{n+1}\sum_{P,k} k^n \left(\frac{\chi(P)}{N(P)^s}\right)^k (\log N(P))^{n+1} \tag{4.15}$$

Similarly differentiating (4.2) n-times,

$$\mathcal{L}^{(n)}(s, \chi) = (-1)^n n! \sum_\rho \frac{1}{(s-\rho)^{n+1}} + \tilde{\Gamma}_\chi^{(n)}(s) \tag{4.16}$$

We similarly evaluate the integral :

$$\Psi_\chi(\mu, n, x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-\mu}}{s-\mu}\mathcal{L}^{(n)}(\chi, s)\, ds \qquad \text{for } c \gg 0$$

For $\mu = 0$ and 1 in two different ways using equation (4.15) and (4.16). Thus on one hand we have,

$$x\Psi_\chi(1,n,x) - \Psi_\chi(0,n,x) = (-1)^{n+1} \sum_{k,\, N(P)^k < x} k^n \left(\frac{x}{N(P)^k} - 1\right) \chi(P)^k (\log N(P))^{n+1} \tag{4.17}$$

On the other hand we can similarly compute the contribution from the $\sum_\rho$ term and Γ-factor. For the non-trivial zeros we do similar contour computations. The pole at $s = 0$ (resp. $s = 1$) has residue $-\frac{n!}{\rho^{n+1}}$ (resp. $\frac{(-1)^n n!}{(1-\rho)^{n+1}}$) where as residue at $s = \rho$ (pole of order $n+1$) is $(-1)^n \lim_{s\to\rho} \frac{d^n}{ds^n}\left(\frac{x^s}{s}\right)$ (resp. s replaced by $s-1$ while computing $\Psi_K(1,n,x)$) so that the total contribution will be of the form $(x-1)[r(\chi,n,x) - (-1)^n n! \sum_\rho \frac{1}{(1-\rho)^{n+1}}]$ where

$$r(\chi,n,x) = \frac{(-1)^n}{x-1}\left[x \sum_\rho \lim_{s\to\rho} \frac{d^n}{ds^n}\left(\frac{x^{s-1}}{s-1}\right) - \sum_\rho \lim_{s\to\rho} \frac{d^n}{ds^n}\left(\frac{x^s}{s}\right)\right]$$
$$\ll \frac{(\log x)^n}{\sqrt{x}} \quad \text{(under GRH)}$$

Also, following along the same lines as Theorem 2.4.3 of Chapter 2, unconditionally we have $r(\chi,n,x) \to 0$ as $x \to \infty$.

Now for the Gamma factors, first note

$$\tilde{\Gamma}_\chi^{(n)}(s) = (-1)^n n! \left[\frac{(a'+r_2)}{s^{n+1}} + a\sum_{k=0}^{\infty} \frac{1}{(s+1+2k)^{n+1}} + a'\sum_{k=1}^{\infty} \frac{1}{(s+2k)^{n+1}} + r_2 \sum_{k=1}^{\infty} \frac{1}{(s+k)^{n+1}}\right]$$

from the residue at $s = 0$ the $\tilde{\Gamma}_\chi^{(n)}(1)$ term will come and we will similarly be able to write, the total contribution as $(x-1)[\ell(\chi,n,x) + \tilde{\Gamma}_\chi^{(n)}(1)]$. The main term of $\ell(\chi,n,x)$ comes from the $\frac{a'+r_2}{s^{n+1}}$ term as before and thus is $\ll \frac{(\log x)^n}{x}$, in particular $\ell(\chi,n,x) \to 0$ as $x \to \infty$. Therefore we have the following theorem.

Theorem 4.4.1. For $\chi \neq \chi_0$, we have, unconditionally

$$\mathcal{L}^{(n)}(1,\chi) = \lim_{x\to\infty} (-1)^{n+1}\, \Psi_K(\chi, n, x)$$

and under GRH, for $x > 1$,

$$\mathcal{L}^{(n)}(1,\chi) = (-1)^{n+1}\, \Psi_K(\chi, n, x) + O\left(\frac{(\log x)^n}{\sqrt{x}}\right)$$

and so

$$\mathcal{L}^{(n)}(1,\chi) \ll \left(\log\log\sqrt{d_\chi}\right)^{n+1}$$

where

$$\Psi_K(\chi, n, x) = \frac{1}{x-1} \sum_{k,\, N(P)^k < x} k^n \left(\frac{x}{N(P)^k} - 1\right) \chi(P)^k (\log N(P))^{n+1}$$

Remark 4.4.2. Note the difference in the limit formula in comparison to Theorem 2.4.3, in particular the the absence fo the $f(n,x)$ term.

We will now focus our attention to the case when $K = \mathbb{Q}$ and in the following sections, study the moments of higher derivatives of $\mathcal{L}(s,\chi)$ at $s = 1$, where χ runs over all non-principal multiplicative characters of large prime conductors. Before that, let us first have a brief look at some of the rich history of the study of moments of L-functions.

4.5 MOMENTS : A BRIEF HISTORY

The distribution of values of Dirichlet L-functions $L(1,\chi)$, for variable χ has been studied extensively and has a vast literature. However the study of the same for logarithmic derivatives $L'(1,\chi)/L(1,\chi)$ is more recent. Let m be a prime and X_m denote the set of all non-principal multiplicative characters $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ and $L(s,\chi)$ denote the corresponding Dirichlet L-function.

For any pair of non-negative integers (a, b) let $P^{(a,b)}(z) = z^a \overline{z}^b$. A result of Paley and Selberg states that (e.g. see [Pal31])

$$\frac{1}{|X_m|} \sum_{\chi \in X_m} P^{(1,1)}(L(1,\chi)) = \zeta(2) + O((\log m)^2/m)$$

This was later improved and by many authors. W. Zhang [Zha90] generalized to the case of $P^{(k,k)}$. In [IMS09], Ihara, Murty and Shimura studied the moments of the logarithmic derivative and proved the following theorem :

Theorem 4.5.1. **(Ihara, Murty, Shimura)**
Let m be a large prime number, and let X_m be the collection of all non-principal primitive Dirichlet characters $\chi : (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$. Then

$$\frac{1}{|X_m|} \sum_{\chi \in X_m} P^{(a,b)}(L'(1,\chi)/L(1,\chi)) = (-1)^{a+b}\mu^{a,b} + O(m^{\varepsilon-1}) \tag{4.18}$$

for any $\varepsilon > 0$. In particular,

$$\lim_{m\to\infty} \frac{1}{|X_m|} \sum_{\chi \in X_m} P^{(a,b)}(L'(1,\chi)/L(1,\chi)) = (-1)^{a+b}\mu^{a,b}$$

Here $\mu^{a,b}$ is a non-negative real number defined as follows :

$$\mu^{(a,b)} = \sum_{n=1}^{\infty} \frac{\Lambda_a(n)\Lambda_b(n)}{n^2} \quad \text{where} \quad \Lambda_k(n) = \sum_{n=n_1\cdots n_k} \Lambda(n_1)\cdots\Lambda(n_k)$$

$k > 0$ and $\Lambda(n) = \log p$, when n is a prime power and 0 otherwise (the von Mangoldt function).

In the subsequent section the author wishes to derive similar theorems on moments of the higher derivatives of $\mathcal{L}(s,\chi) = L'(s,\chi)/L(s,\chi)$ at $s = 1$. Note that, the author was not able to find a good reference that studies moments of higher derivatives of $L(s,\chi)$ at $s = 1$ but the case of $s = \frac{1}{2}$ (and fractional moments) has been studied by Conrey [CON88], Milinovich [Mil11], Heath-Brown [HB10], Soundararajan [Sou09], Sono etc. For example, Sono [Son14] recently showed :

Under GRH, for $1/2 < k < 2$ and $m \in \mathbb{Z}_{\geq 0}$ we have,

$$\frac{1}{\phi(q)} \sum_{\substack{\chi (\text{mod } q) \\ \chi \neq \chi_0}} P^{(k,k)}\left(L^{(m)}\left(\frac{1}{2}, \chi\right)\right) \ll (\log q)^{k^2+2km}$$

whereas for $k \geq 2$, for any $\epsilon > 0$, under GRH,

$$\frac{1}{\phi(q)} \sum_{\chi (\text{mod } q)}^{*} P^{(k,k)}\left(L^{(m)}\left(\frac{1}{2}, \chi\right)\right) \ll (\log q)^{k^2+2km+\epsilon}$$

where $\sum^*$ is over all primitive Dirichlet characters modulo q.

However note that, methods used in the above do not seem to apply to our case. Ours is more of an extension of the work done in [IMS09].

4.6 MOMENTS OF $\mathcal{L}'(1,\chi)$ (CONDITIONAL : UNDER GRH)

Before we dive right into our theorems, let us look at the following neat connection due to the orthogonality relations of characters.

Let $\alpha : \mathbb{N} \to \mathbb{C}$ be such that, for any $\epsilon > 0$, $\alpha(n) = O(n^{\epsilon})$. Consider the Dirichlet series (absolutely convergent for $\text{Re}(s) > 1$)

$$f(s) = \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s}$$

Let $X_m^{\star} = X_m \cup \{\chi_0\}$. For each $\chi \in X_m^{\star}$, consider the associated series

$$f_{\chi}(s) = \sum_{n=1}^{\infty} \frac{\chi(n)\alpha(n)}{n^s}$$

Let $\alpha_k(n)$ denote the Dirichlet coefficient of n^{-s} in $f(s)^k$ for $k \geq 0$. Then from the orthogonality relation for characters lead to the asymptotic formula : (writing $\sigma = \text{Re}(s)$)

$$\frac{1}{|X_m^{\star}|} \sum_{\chi \in X_m^{\star}} P^{(a,b)}(f_{\chi}(s)) = \sum_{n=1}^{m-1} \frac{\alpha_a(n)\overline{\alpha_b(n)}}{n^{2\sigma}} + O_{a,b}(m^{1+\epsilon-\sigma}) \tag{4.19}$$

For any s with $\sigma > 1+\epsilon$. In particular, taking the limit $m \to \infty$

$$\lim_{m\to\infty} \frac{1}{|X_m^\star|} \sum_{\chi \in X_m^\star} P^{(a,b)}(f_\chi(s)) = \sum_{n=1}^{\infty} \frac{\alpha_a(n)\overline{\alpha_b(n)}}{n^{2\sigma}}$$

Now one can ask whether this holds for $s=1$ (or even for $\mathrm{Re}(s) \le 1$) when $X_m^\star$ is replaced by X_m. It turns out it depends on the analytic properties of $f(s)$ to the left of 1. For the case $f(s) = \mathcal{L}(s,\chi)$, Ihara, Murty and Shimura in [IMS09], first showed, under GRH, the error term for each χ is small. This together with bounds obtained for $\mathcal{L}(1,\chi)$ gives a formula similar to (4.18), where the main term is same and the error term is $O(\frac{(\log m)^{a+b+2}}{m})$.

To obtain the unconditional result, as stated in Theorem 4.5.1, they used Montgomery's result in [Mon71] on estimating the number of zeros in a rectangular region for $\sigma \ge 4/5$ and showed that the average value, of the absolute value of the error terms, is sufficiently small. Following in their footsteps we also first prove a conditional result.

As mentioned before, for the rest of the chapter, we are considering the case $K = \mathbb{Q}$, unless otherwise specified. Let m run over all odd prime numbers and for each m, let X_m be the collection of all non-principal primitive Dirichlet characters $\chi : (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$. As ususal by $\Lambda(n)$ we wil denote the von Mangoldt function :

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

Similar to [IMS09], we define

$$\Lambda_0(n) = \begin{cases} 1 & n = 1, \\ 0 & n > 1 \end{cases}$$

$$\Lambda_k(n) = \sum_{n=n_1\cdots n_k} \Lambda(n_1)\cdots\Lambda(n_k) \quad \text{for } k > 0. \tag{4.20}$$

Note that $\Lambda_k(n) = 0$ unless the sum of exponents in the prime factorization of n is at least k. Also we have, for $1 \le k \le r$

$$\Lambda_k(p^r) = \sum_{i_1+\cdots+i_k=r} \Lambda(p^{i_1})\cdots\Lambda(p^{i_k}) = \binom{r-1}{k-1}(\log p)^k \tag{4.21}$$

Following Section 3.8 of [Iha08], we see that if n has the prime factorization $n = \prod_{i=1}^r p_i^{\alpha_i}$ then, $\Lambda_k(n)$ is the coefficient of the monomial $x_1^{\alpha_1}\cdots x_r^{\alpha_r}$ in the polynomial

$$\left(\sum_{i=1}^r (\log p_i)(x_i + x_i^2 + \cdots + x_i^{\alpha_i})\right)^k$$

Letting $x_i = 1$ for all $i = 1, \cdots r$ we see that

$$\Lambda_k(n) \le \left(\sum_{i=1}^r \alpha_i(\log p_i)\right)^k = (\log n)^k \tag{4.22}$$

For our purposes, we also define :

$$\ell^1\Lambda_k(n) = \sum_{n=n_1\cdots n_k} \Lambda(n_1)\cdots\Lambda(n_k)(\log n_1)\cdots(\log n_k) \quad \text{for } k > 0. \tag{4.23}$$

and for $k = 0$ it is equal to $\Lambda_0(n)$. Note that applying arithmetic mean is greater than or equal to geometric mean inequality we see that :

$$\prod_{i=1}^k \log n_i \le \frac{(\log n)^k}{k^k} \quad \text{and so,}$$

$$\ell^1\Lambda_k(n) \le \frac{(\log n)^k}{k^k}\Lambda_k(n) \le \frac{(\log n)^{2k}}{k^k} \tag{4.24}$$

We now have a look again on $\Psi_\chi(x)$ as in equation (4.9) or $\Psi(\chi, 1, x)$ of Theorem 4.4.1. In particular, for $K = \mathbb{Q}$, it takes the form :

$$\begin{aligned}\Psi_\chi(x) &= \frac{1}{x-1}\sum_{k,p^k<x} k\left(\frac{x}{p^k}-1\right)\chi(p)^k(\log p)^2\\ &= \frac{1}{x-1}\sum_{k,p^k<x}\left(\frac{x}{p^k}-1\right)\chi(p^k)(\log p)(\log p^k)\\ &= \frac{1}{x-1}\sum_{n\le x}\left(\frac{x}{n}-1\right)\chi(n)\Lambda(n)(\log n) \end{aligned}\tag{4.25}$$

For each pair (a,b) of non-negative integers, we define

$$\tilde{\mu}^{(a,b)} = \tilde{\mu}^{(b,a)} = \sum_{n=1}^{\infty}\frac{\ell^1\Lambda_a(n)\ \ell^1\Lambda_b(n)}{n^2} \tag{4.26}$$

Note that $\tilde{\mu}^{(0,0)} = 1$, $\tilde{\mu}^{(a,0)} = 0$ for all $a > 0$, in all other cases $\tilde{\mu} > 0$. In particular,

$$\tilde{\mu}^{(1,1)} = \sum_{n=1}^{\infty}\left(\frac{\Lambda(n)\log(n)}{n}\right)^2$$

***Theorem* 4.6.1.** For each pair (a,b) of non-negative integers and for $x \ge m$, we have

$$\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}\ (\Psi_\chi(x)) = \tilde{\mu}^{(a,b)} + O_{a,b}\left(\frac{(\log x)^{2d}}{m}\right) \tag{4.27}$$

Here $\Psi_\chi(x)$ is as in equation(4.25) and $d = a+b+1$.

Proof. Note that for $\chi = \chi_0$, $\Psi_{\mathbb{Q}}(x) = O((\log x)^2)$. Thus if we include the principal character in proving the theorem, it will effect the results by $O\left(\frac{(\log x)^{2a+2b}}{m}\right)$ which is less than the error term. As before, we write $X_m^\star = X_m \cup \{\chi_0\}$,

$$\tilde{\mu}^{(a,b)}(x) = \frac{1}{|X_m^\star|}\sum_{\chi\in X_m^\star} P^{(a,b)}\ (\Psi_\chi(x)) = \frac{1}{|X_m^\star|}\sum_{\chi\in X_m^\star}\Psi_\chi(x)^a\Psi_{\overline{\chi}}(x)^b \tag{4.28}$$

For our purposes, we present in the following lemma, a general version of 4.2.2 and 4.2.3 of [IMS09].

Lemma 4.6.2. For some $x > 1$, and $\chi \in X_m^\star$ if $g_\chi(x) = \sum_{n\leq x} g(x,n)\chi(n)$ then,

$$\frac{1}{|X_m^\star|}\sum_{\chi\in X_m^\star} g_\chi(x)^a g_{\overline{\chi}}(x)^b = \sum_{j=1}^{m-1}\lambda^{(a)}(j,x)\lambda^{(b)}(j,x) \tag{4.29}$$

where

$$\lambda^{(k)}(j,x) = \sum_{\substack{n_1,\cdots,n_k<x\\ n_1\cdots n_k\equiv j\ (\mathbf{mod} m)}}\prod_{i=1}^{k} g(x,n_i)$$

for $k\geq 1$, and for $k=0$ define $\lambda^{(0)}(j,x) = 1$ for $j=1$ and 0 for $j>1$. (Recall m here is a prime number and a,b non-negative integers.)

Proof. This is a direct consequence of orthogonality relations of characters. In particular, a typical term in the sum in the LHS of (4.29) looks like

$$\left(\prod_{i=1}^{a} g(x,n_i)\prod_{j=1}^{b} g(x,m_j)\right)\chi(n_1\cdots n_a)\overline{\chi}(m_1\cdots m_b)$$

When summed over all χ, it has a nonzero contribution only when $(n_1\cdots n_a)\equiv(m_1\cdots m_b)\pmod m$ and hence we have our result. □

Thus applying Lemma 4.6.2, in our case with $g(x,n) = \frac{1}{x-1}\left(\frac{x}{n}-1\right)\Lambda(n)\log n$, we get, $g_\chi(x) = \Psi_\chi(x)$ and hence from equation (4.28)

$$\tilde{\mu}^{(a,b)}(x) = \sum_{j=1}^{m-1}\lambda^{(a)}(j,x)\lambda^{(b)}(j,x) \tag{4.30}$$

where

$$\begin{aligned}\lambda^{(k)}(j,x) &= \frac{1}{(x-1)^k}\sum_{\substack{n_1,\cdots,n_k<x\\ n_1\cdots n_k\equiv j\ (\mathbf{mod}\ m)}}\prod_{i=1}^{k}\left(\frac{x}{n_i}-1\right)\Lambda(n_i)\log n_i\\ &= \frac{1}{(x-1)^k}\sum_{l=0}^{[(x^k-j)/m]}\sum_{\substack{n_1,\cdots,n_k<x\\ n_1\cdots n_k=j+lm}}\prod_{i=1}^{k}\left(\frac{x}{n_i}-1\right)\Lambda(n_i)\log n_i\\ &= \sum_{l=0}^{[(x^k-j)/m]} L^{(k)}(j+lm,x)\quad\text{(say)}\end{aligned} \tag{4.31}$$

here [.] in the upper limit of the sum, is the greatest integer function. Note that $L^{(k)}(N,x) \neq 0$ only when $N < x^k$ and in this case,

$$
\begin{aligned}
L^{(k)}(N,x) &= \frac{1}{(x-1)^k} \sum_{\substack{n_1,\cdots,n_k<x \\ n_1\cdots n_k=N}} \prod_{i=1}^{k} \left(\frac{x}{n_i}-1\right) \Lambda(n_i)\log n_i \\
&\leq \frac{1}{N} \sum_{\substack{n_1,\cdots,n_k<x \\ n_1\cdots n_k=N}} \prod_{i=1}^{k} \Lambda(n_i)\log n_i \\
&\leq \frac{1}{N}\,\ell^1\Lambda_k(N) \leq \frac{(\log N)^{2k}}{k^k\,N} < k^k\frac{(\log x)^{2k}}{N}
\end{aligned} \tag{4.32}
$$

Thus the net contribution of the terms $l > 0$ in (4.31) is given by :

$$
\begin{aligned}
\sum_{l=1}^{[(x^k-j)/m]} L^{(k)}(j+lm,x) &< k^k\frac{(\log x)^{2k}}{m}\left(1+\frac{1}{2}\cdots+\frac{1}{[x^k/m]}\right) \\
&= O\left(\frac{(\log x)^{2k+1}}{m}\right)
\end{aligned}
$$

Therefore we have,

$$
\lambda^{(k)}(j,x) = L^{(k)}(j,x) + O\left(\frac{(\log x)^{2k+1}}{m}\right) \tag{4.33}
$$

For the main term, we also use the inequality as in (4.2.9) of [IMS09] :

For $x > 0$ and $i,j \geq 1$ we have $(x-i)(x-j) \geq (x-1)(x-ij)$

Generalizing,

$$
(x-1)^k \geq (x-n_1)\cdots(x-n_k) \geq (x-1)^{k-1}(x-n_1\cdots n_k)
$$

Thus for $n_i \geq 1$ and $n_1\cdots n_k = j$,

$$
\frac{1}{(x-1)^k}\prod_{i=1}^{k}\left(\frac{x}{n_i}-1\right) = \frac{1}{(x-1)^k}\frac{\prod_{i=1}^{k}(x-n_i)}{j} \leq \frac{1}{j}
$$

On the other hand,

$$\frac{1}{(x-1)^k}\prod_{i=1}^{k}\left(\frac{x}{n_i}-1\right) \geq \frac{1}{(x-1)}\frac{x-j}{j}$$

$$\Rightarrow 0 \leq \frac{1}{j} - \frac{1}{(x-1)^k}\prod_{i=1}^{k}\left(\frac{x}{n_i}-1\right) \leq \frac{j-1}{j(x-1)}$$

That is,

$$\frac{1}{(x-1)^k}\prod_{i=1}^{k}\left(\frac{x}{n_i}-1\right) = \frac{1}{j} + O\left(\frac{1}{x}\right) \tag{4.34}$$

Note that, in the sum of $L^{(k)}(j,x)$, since $j < m$, if we choose $x \geq m$, the condition $n_1, \cdots, n_k < x$ is automatic. Thus,

$$\begin{aligned} L^{(k)}(j,x) &= \frac{1}{(x-1)^k}\sum_{n_1\cdots n_k=j}\prod_{i=1}^{k}\left(\frac{x}{n_i}-1\right)\Lambda(n_i)\log n_i \\ &= \frac{\ell^1\Lambda_k(j)}{j} + O\left(\frac{(\log m)^{2k}}{m}\right) \end{aligned} \tag{4.35}$$

and so,

$$\lambda^{(k)}(j,x) = \frac{\ell^1\Lambda_k(j)}{j} + O\left(\frac{(\log x)^{2k+1}}{m}\right) \tag{4.36}$$

Plugging this in equation (4.30)

$$\tilde{\mu}^{(a,b)}(x) = \sum_{j=1}^{m-1}\frac{\ell^1\Lambda_a(j)\ \ell^1\Lambda_b(j)}{j^2} + O\left(\frac{(\log x)^{2(a+b+1)}}{m}\right) \tag{4.37}$$

Note that for $j \geq m$,

$$\begin{aligned} \sum_{j\geq m}\frac{\ell^1\Lambda_a(j)\ \ell^1\Lambda_b(j)}{j^2} &\leq \frac{1}{a^a b^b}\sum_{j\geq m}\frac{(\log j)^{2a+2b}}{j^2} \\ &= \left(\frac{(\log m)^{2a+2b}}{m}\right) \end{aligned}$$

Therefor for $x \geq m$,

$$\tilde{\mu}^{(a,b)}(x) = \tilde{\mu}^{(a,b)} + O_{a,b}\left(\frac{(\log x)^{2(a+b+1)}}{m}\right) \tag{4.38}$$

and that completes the proof.

□

We are now ready to prove the main theorem of this section, which is essentially a corollary of Theorem 4.6.1 and Theorem 4.2.2.

***Theorem* 4.6.3.** Under GRH,

$$\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}(\mathcal{L}'(1,\chi)) = \tilde{\mu}^{(a,b)} + O\left(\frac{(\log m)^{2(a+b+1)}}{m}\right)$$

the implicit constant depends on a, b. In particular,

$$\lim_{m\to\infty}\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}(\mathcal{L}'(1,\chi)) \;=\; (-1)^{a+b}\tilde{\mu}^{(a,b)}$$

Proof. Note that for $K = \mathbb{Q}$, lemma 4.2 takes the form :

$$\frac{L'(s,\chi)}{L(s,\chi)} = -\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right) + B(\chi) + \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) \tag{4.39}$$

For example see p.83 of [Davoo] . Here $a = 0$ (respectively, $a = 1$) if χ is even (resp. odd) and $B(\chi) = \xi'(0,\chi)/\xi(0,\chi)$. The sum is over all non-trivial zeros ρ of $L(s,\chi)$, i.e. zeros in the critical strip.

Writing $\mathcal{L}(s,\chi) = L'(s,\chi)/L(s,\chi)$ and differentiating we get

$$\mathcal{L}'(s,\chi) = -\sum_{k=0}^{\infty}\frac{1}{(s+a+2k)^2} - \sum_{\rho}\frac{1}{(s-\rho)^2} \tag{4.40}$$

From the exact formula in (4.12), Lemma 4.2.1 and equation(4.14) we get, under GRH

$$\mathcal{L}'(1,\chi) = \Psi_\chi(x) + O\left(\frac{\log m\ \log x}{\sqrt{x}} + \frac{(\log x)^2}{x}\right) \tag{4.41}$$

the implicit constant being absolute. Putting $x = m^2$ in both equation (4.41) and Theorem 4.6.1 completes the proof. □

Remark 4.6.4. The proof Theorem 4.6.1 and the Lemma 4.6.2 suggests that we should be able to generalize these ideas for the moments of higher derivatives, $\mathcal{L}^{(n)}(1,\chi)$. We will explore more on this in a later section.

Remark 4.6.5. Note that Theorem 4.6.1 is unconditional, we are only using GRH in Theorem 4.6.3 essentially to estimate the (error) difference between $\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}(\mathcal{L}'(1,\chi))$ and $\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}(\Psi_\chi(x))$. Without GRH, it's a little more work to manage this error term, but it can be done. This is what we explore in the next section.

4.7 MOMENTS OF $\mathcal{L}'(1,\chi)$ (UNCONDITIONAL)

In this section we prove an unconditional version of Theorem 4.6.3 :

***Theorem* 4.7.1.** For any $\epsilon > 0$, we have, unconditionally,

$$\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}(\mathcal{L}'(1,\chi)) = \tilde{\mu}^{(a,b)} + O\left(m^{\epsilon-1}\right)$$

the implicit constant depends on a, b. In particular,

$$\lim_{m\to\infty}\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}(\mathcal{L}'(1,\chi)) \;=\; (-1)^{a+b}\tilde{\mu}^{(a,b)}$$

Remark 4.7.2. The key difference here, is that under GRH, the individual terms for each χ in the error were small, whereas unconditionally we will show that the average of the error terms is small. In particular, we will show that for large x,

$$\frac{1}{|X_m|}\sum_{\chi\in X_m}\left|P^{(a,b)}(\mathcal{L}'(1,\chi)) - P^{(a,b)}(\Psi_\chi(x))\right| \ll m^{\epsilon-1}$$

To do this, much like Section 5.4 of [IMS09], we will employ zero-sum estimates of $L(s,\chi)$. Note that the above result together with Theorem 4.6.1, will give our unconditional Theorem 4.7.1.

We start with an easy inequality (This was used in 6.8 of [Iha08] and 5.3 of [IMS09]), we include a short proof as well.

Proposition 4.7.3. For any $w, z \in \mathbb{C}$ we have

$$|P^{(a,b)}(z+w) - P^{(a,b)}(z)| \le (a+b)|w|(|z|+|w|)^{a+b-1}$$

Proof. First note that for any $n \ge 1$,

$$\begin{aligned}|(z+w)^n - z^n| &= \left|\binom{n}{1}z^{n-1}w + \cdots + \binom{n}{n}w^n\right| \\ &\le n|w|\left(\sum_{i=1}^{n}\binom{n-1}{i-1}|z|^{n-i}|w|^{i-1}\right) \\ &= n|w|(|z|+|w|)^{n-1}\end{aligned}$$

where the last inequality follows from $\binom{n}{i} \le n\binom{n-1}{i-1}$ for $1 \le i \le n$. Thus,

$$\begin{aligned}&|P^{(a,b)}(z+w) - P^{(a,b)}(z)| = |(z+w)^a(\overline{z+w})^b - z^a\overline{z}^b| \\ &= |(z+w)^a\overline{(z+w)}^b - z^a(\overline{z+w})^b + z^a(\overline{z+w})^b - z^a\overline{z}^b| \\ &\le |z+w|^b|(z+w)^a - z^a| + |z|^a|(\overline{z+w})^b - \overline{z}^b| \\ &\le a|w|(|z|+|w|)^{a+b-1} + b(|z|+|w|)^a|\overline{w}|(|\overline{z}|+|\overline{w}|)^{b-1} \\ &\le (a+b)|w|(|z|+|w|)^{a+b-1}\end{aligned}$$

□

Choosing, $z = \mathcal{L}'(1,\chi)$ and $w = \Psi_\chi(x) - \mathcal{L}'(1,\chi)$ gives,

$$\begin{aligned}\left|P^{(a,b)}(\mathcal{L}'(1,\chi)) - P^{(a,b)}(\Psi_\chi(x))\right| &\le (a+b)\left|\Psi_\chi(x) - \mathcal{L}'(1,\chi)\right| \cdot \\ &\quad \left(\left|\Psi_\chi(x) - \mathcal{L}'(1,\chi)\right| + \left|\mathcal{L}'(1,\chi)\right|\right)^{a+b-1}\end{aligned} \tag{4.42}$$

Let us denote the unique real quadratic character in X_m by χ_1.

We will show the following bounds :

Proposition 4.7.4.

1. For $\chi \in X_m$, $x \geq m$ and $\epsilon > 0$, we have

$$\left|P^{(a,b)}(\mathcal{L}'(1,\chi)) - P^{(a,b)}(\Psi_\chi(x))\right| \ll \begin{cases} (\log x)^{2a+2b-1} m^{\epsilon(a+b)} & \text{for } \chi = \chi_1 \\ \left((\log x)^2(\log m)\right)^{(a+b-1)} |\Psi_\chi(x) - \mathcal{L}'(1,\chi)| & \text{for } \chi \neq \chi_1 \end{cases}$$

2. For $x \geq m^{12}$, we have

$$\sum_{\substack{\chi \in X_m \\ \chi \neq \chi_1}} \left|\Psi_\chi(x) - \mathcal{L}'(1,\chi)\right| \ll (\log x)^{16} \tag{4.43}$$

We postpone the proof of this proposition to the end of this section as we will need several Lemmas to prove it.

Recall from the exact formula (4.12), and Lemma 4.2.1, for $\chi \in X_m$,

$$\left|\Psi_\chi(x) - \mathcal{L}'(1,\chi)\right| = |r_\chi(x)| + O\left(\frac{(\log x)^2}{x}\right) \tag{4.44}$$

where, (writing $\rho = \beta + i\gamma$)

$$\begin{aligned} |r_\chi(x)| &= \left|\sum \frac{\rho(\rho-1)\log x - \rho^2 + (1-\rho)^2}{\rho^2(1-\rho)^2} \cdot \frac{x^\rho}{x-1}\right| \\ &\leq \frac{1}{(x-1)} \sum \left(\frac{\log x}{|\rho(\rho-1)|} + \frac{1}{|\rho|^2} + \frac{1}{|(1-\rho)|^2}\right) x^\beta \end{aligned} \tag{4.45}$$

Zero-sum estimates

We now write down several lemmas to essentially estimate (4.45) and prove Prop 4.7.4. These lemmas depends on the behavior and estimates of zeros of $L(s,\chi)$ in the critical strip. To begin with, we will use the following two well-known results : (due to Gronwall, Titchmarsh, Siegel etc, e.g. see [Davoo], §14, 16 and 21)

Theorem. (A) There exists an absolute and effective positive constant c such that if $\rho = \beta + i\gamma$ is a non-trivial zero of $L(s,\chi)$ with $|\gamma| \leq T$, $T \geq 1$, then either,

$$\text{Min}(1-\beta,\beta) \; > \; \frac{c}{\log(mT)}$$

or, $\chi = \chi_1$ and $\rho = \beta_1$ or $1-\beta_1$ is a real simple zero satisfying $\beta_1 > \frac{1}{2}$ and $1-\beta_1 \gg m^{-\varepsilon}$.

Theorem. (B) Let Z_χ be the set of non-trivial zeros of $L(s,\chi)$. Then

$$\#\{\beta + i\gamma \in Z_\chi \; : \; |\gamma - T| < 1\} \ll \log(m(T+2))$$

Lemma 4.7.5.

$$\sum_{|\gamma|\leq 1}{}' \left(\frac{\log x}{|\rho(\rho-1)|} + \frac{1}{|\rho|^2} + \frac{1}{|(1-\rho)|^2} \right) x^\beta \ll x(\log mx)(\log m)^2 \qquad (4.46)$$

where $\sum'$ is the sum over all ρ excluding the possible exceptional zero.

Proof. For $|\gamma| \leq 1$, by Theorem (A) with $T = 1$, and ρ not being exceptional, we see that $|\rho| \geq |\beta| \gg \frac{1}{\log m}$, similarly $|1-\rho| \gg \frac{1}{\log m}$. Hence,

$$\left| \frac{1}{\rho(\rho-1)} \right| \ll \log m, \quad \frac{1}{|\rho|^2} \ll (\log m)^2, \quad \frac{1}{|1-\rho|^2} \ll (\log m)^2$$

Therefore,

$$\begin{aligned} &\sum_{|\gamma|\leq 1}{}' \left(\frac{\log x}{|\rho(\rho-1)|} + \frac{1}{|\rho|^2} + \frac{1}{|(1-\rho)|^2} \right) x^\beta \\ &\ll x(\log x \log m + (\log m)^2) \sum_{|\gamma|\leq 1}{}' 1 \\ &\ll x(\log mx)(\log m)^2 \end{aligned}$$

□

Lemma 4.7.6. For $T \geq 1$

$$\sum_{|\gamma|>T}\left(\frac{\log x}{|\rho(\rho-1)|}+\frac{1}{|\rho|^2}+\frac{1}{|(1-\rho)|^2}\right)x^{\beta} \ll \frac{x(\log x)(\log mT)}{T} \tag{4.47}$$

Proof.

$$\begin{aligned}
&\sum_{|\gamma|>T}\left(\frac{\log x}{|\rho(\rho-1)|}+\frac{1}{|\rho|^2}+\frac{1}{|(1-\rho)|^2}\right)x^{\beta}\\
&\ll (x\log x)\sum_{|\gamma|>T}\frac{1}{\gamma^2}\\
&\ll (x\log x)\sum_{j=[T]}^{\infty}\frac{1}{j^2}\sum_{|\gamma-(j+1)|<1} 1\\
&\ll (x\log x)\sum_{j=[T]}^{\infty}\frac{\log(m(j+3))}{j^2}\\
&\ll \frac{x(\log x)(\log mT)}{T}
\end{aligned}$$

□

The following result is part of the proof of a sublemma (5.4.4) of [IMS09]. We record it here as a lemma and for the sake of completion also include the proof.

Lemma 4.7.7. **(Ihara, Murty and Shimura)** For $T \geq 2$ and $x \geq (mT)^6$

$$\sum_{\chi\in X_m}\sideset{}{'}\sum_{\substack{\rho\in Z_\chi\\|\gamma|\leq T}} x^{\beta} \ll x(\log x)^{14} \tag{4.48}$$

Proof. Let us denote

$$\tilde{S}(x,m,T)=\sum_{\chi\in X_m}\sideset{}{'}\sum_{\substack{\rho\in Z_\chi\\|\gamma|\leq T}} x^{\beta}$$

The lemma is a consequence of well-known bounds for the number $N(\sigma, T, m)$ related to the number of zeros of $L(s,\chi)$ in a rectangle. In particular, for $0 \leq \sigma \leq 1$ and $T \geq 2$, define

$$\begin{cases} N(\sigma, T, \chi) = \#\{\rho = \beta + i\gamma \in Z_\chi \ : \ \beta \geq \sigma, \ |\gamma| \leq T\} \\ N(\sigma, T, m) = \sum_{\chi \in X_m} N(\sigma, T, \chi) \end{cases}$$

It is well known that $N(0, T, \chi) \ll T \log(mT)$ (e.g. see §16 of [Dav00]) and thus $N(0, T, m) \ll mT \log(mT)$. We will also use the following result by Montgomery (Theorem 12.1) of [Mon71], also [Mon69] :

For $\sigma \geq 4/5$ and $T \geq 2$,

$$N(\sigma, T, m) \ll (mT)^{\frac{2(1-\sigma)}{\sigma}} (\log mT)^{14} \ll (mT)^{\frac{5}{2}(1-\sigma)} (\log mT)^{14} \tag{4.49}$$

Similar result can also be found in [HJ77]. We rewrite $\tilde{S}(x, m, T)$ as

$$\tilde{S}(x, m, T) = \sum_{\chi \in X_m} \sideset{}{'}\sum_{\substack{\rho \in Z_\chi \\ |\gamma| \leq T \\ \beta < 4/5}} x^\beta + \sum_{\chi \in X_m} \sideset{}{'}\sum_{\substack{\rho \in Z_\chi \\ |\gamma| \leq T \\ 4/5 \leq \beta < 1}} x^\beta$$

The first summand is

$$\ll x^{4/5} N(0, T, m) \ll x^{4/5} (mT)(\log mT) \ll x^{4/5 + 1/6} \log x \ll x$$

where the last inequality is due to the imposed condition $x \geq (mT)^6$. The second summand is

$$\leq \left| \int_{4/5}^{1} x^\sigma d_\sigma N(\sigma, T, m) \right| \leq x^{4/5} N(4/5, T, m) + \left| \int_{4/5}^{1} (x^\sigma \log x) N(\sigma, T, m) \, d\sigma \right|$$

$$\ll x^{4/5} (mT)^{1/2} (\log mT)^{14} + (\log x)(mT)^{5/2} (\log mT)^{14} \int_{4/5}^{1} \left(\frac{x}{(mT)^{5/2}} \right)^\sigma \, d\sigma$$

Note that the first term is $\ll x$. Whereas the integral

$$\int_{4/5}^{1}\left(\frac{x}{(mT)^{5/2}}\right)^{\sigma} d\sigma = \left[\frac{\left(\frac{x}{(mT)^{5/2}}\right)^{\sigma}}{\log\left(\frac{x}{(mT)^{5/2}}\right)}\right]_{4/5}^{1}$$
$$\ll \frac{x}{(mT)^{5/2}(\log x)}$$

and so the second term is $\ll x(\log mT)^{14} \ll x(\log x)^{14}$. Hence the lemma is proved. □

Lemma 4.7.8. For $T > 1$ and $x \geq (mT)^6$ we have

$$\sum_{\chi\in X_m}\sideset{}{'}\sum_{\substack{\rho\in Z_\chi\\|\gamma|\leq T}}\left(\frac{\log x}{|\rho(\rho-1)|}+\frac{1}{|\rho|^2}+\frac{1}{|(1-\rho)|^2}\right)x^{\beta} \ll x(\log x)^{16} \tag{4.50}$$

Proof. Keeping similar notation as in [IMS09], 5.6, let us denote,

$$S(x,m,T) = \sum_{\chi\in X_m}\sideset{}{'}\sum_{\substack{\rho\in Z_\chi\\|\gamma|\leq T}}\left(\frac{\log x}{|\rho(\rho-1)|}+\frac{1}{|\rho|^2}+\frac{1}{|(1-\rho)|^2}\right)x^{\beta} \tag{4.51}$$

Note that for all ρ with $\beta \leq \frac{4}{5}$, $S(x,m,T) \ll x^{4/5}(\log mx)(\log m)^2 \ll x$. This is essentially from Lemma 4.7.5 and 4.7.6. So let us focus on the zeros ρ with $\beta \geq \frac{4}{5}$. In this case, like before we divide the sum for $|\gamma| \leq 2$ and $2 < |\gamma| \leq T$. Note that, Since, $\beta \geq \frac{4}{5} > \frac{1}{3}$, we have

$$\text{Min}(\beta, 1-\beta) = 1-\beta > \frac{c}{\log(mT)}$$

Thus,

$$|\rho(\rho-1)| \geq \text{Re}\,\rho(1-\rho) = \beta(1-\beta)+\gamma^2 > \frac{c_1}{\log(mT)}$$
$$|\rho|^2 = \beta^2+\gamma^2 > \frac{16}{25} \quad \text{and}$$
$$|1-\rho|^2 \geq (1-\beta)^2 > \frac{c^2}{(\log mT)^2}$$

Thus we have,

$$S(x,m,T) \ll \left((\log mT)^2 + (\log mT)(\log x)\right) \tilde{S}(x,m,2) + S_1(x,m,T) \tag{4.52}$$

where,

$$\begin{aligned} S_1(x,m,T) &= \sum_{\chi \in X_m} \sideset{}{'}\sum_{2<|\gamma|\le T} \frac{x^\beta}{\gamma^2} \\ &\le \sum_{\substack{j\ge 0\\ 2^{j+1}\le T}} \frac{1}{4^j} \sum_{\chi\in X_m} \sideset{}{'}\sum_{2^j<|\gamma|\le 2^{j+1}} x^\beta \\ &\le \sum_{\substack{j\ge 0\\ 2^{j+1}\le T}} \frac{\tilde{S}(x,m,2^{j+1})}{4^j} \ll x(\log x)^{14} \end{aligned} \tag{4.53}$$

Since, $x \ge (mT)^6$, we thus get, putting equation (4.52) and (4.53) together,

$$S(x,m,T) \ll (\log mT)(\log mTx)x(\log x)^{14} \ll x(\log x)^{16}$$

□

We are now ready to prove the proposition.

***Proof of Proposition* 4.7.4**

1. By Lemma 4.7.5 and 4.7.6 with $T=1$ we see that, for $\chi \ne \chi_1$,

   $$r_\chi(x) \ll (\log x)(\log m)^2$$

   and for $\chi = \chi_1$,

   $$r_\chi(x) \ll (\log x)(\log m)^2 + (\log x)m^\epsilon \ll (\log x)m^\epsilon$$

   This inequality is given by the Theorem (A), stated before Lemma 4.7.5. Putting these in (4.44) we get

   $$|\Psi_\chi(x) - \mathcal{L}'(1,\chi)| \ll \begin{cases} (\log x)(\log m)^2 & \text{for } \chi \ne \chi_1 \\ (\log x)m^\epsilon & \textbf{for } \chi = \chi_1 \end{cases} \tag{4.54}$$

Recall that $\Psi_\chi(x) \ll (\log x)^2$, and so with the above bound for $r_\chi(x)$, we get $\mathcal{L}'(1,\chi) \ll (\log x)^2(\log m)$ for $x \geq m$ and $\chi \neq \chi_1$, whereas, $\mathcal{L}'(1,\chi_1) \ll (\log x)(\log x + m^\epsilon)$. Substituting these bounds in equation (4.42) we get, For $\chi \neq \chi_1$

$$\left|P^{(a,b)}(\mathcal{L}'(1,\chi)) - P^{(a,b)}(\Psi_\chi(x))\right| \ll \left((\log x)^2(\log m)\right)^{(a+b-1)} |\Psi_\chi(x) - \mathcal{L}'(1,\chi)|$$

and for $\chi = \chi_1$ we get,

$$\begin{aligned}\left|P^{(a,b)}(\mathcal{L}'(1,\chi)) - P^{(a,b)}(\Psi_\chi(x))\right| &\ll (\log x)m^\epsilon \left((\log x)^2 m^\epsilon\right)^{(a+b-1)} \\ &\ll (\log x)^{2a+2b-1} m^{\epsilon(a+b)}\end{aligned}$$

2. Putting $T = m$ in Lemma 4.7.6 we get that,

$$\sum_{\chi \in X_m} \frac{1}{x-1} \sum_{|\gamma|>m} \left(\frac{\log x}{|\rho(\rho-1)|} + \frac{1}{|\rho|^2} + \frac{1}{|(1-\rho)|^2}\right) x^\beta \ll (\log x)(\log m)$$

whereas for $T = m$, Lemma 4.7.8 gives, for $x \geq m^{12}$

$$\sum_{\chi \in X_m} \frac{1}{x-1} \sideset{}{'}\sum_{|\gamma| \leq m} \left(\frac{\log x}{|\rho(\rho-1)|} + \frac{1}{|\rho|^2} + \frac{1}{|(1-\rho)|^2}\right) x^\beta \ll (\log x)^{16}$$

Therefore, $\sum_{\substack{\chi \in X_m \\ \chi \neq \chi_1}} |\Psi_\chi(x) - \mathcal{L}'(1,\chi)| \ll (\log x)^{16}$.

□

***Proof of Theorem** 4.7.1*

Putting $x = m^{12}$ in Proposition 4.7.4, we get

$$\begin{aligned}&\sum_{\chi \in X_m} \left|P^{(a,b)}(\mathcal{L}'(1,\chi)) - P^{(a,b)}(\Psi_\chi(x))\right| \\ &\ll (\log m)^{2a+2b-1} m^{\epsilon(a+b)} + (\log m)^{3(a+b-1)} \sum_{\substack{\chi \in X_m \\ \chi \neq \chi_1}} |\Psi_\chi(m^{12}) - \mathcal{L}'(1-\chi)| \\ &\ll (\log m)^{2a+2b-1} m^{\epsilon(a+b)} + (\log m)^{3(a+b-1)+16} \ll m^{\epsilon'}\end{aligned}$$

Hence we have,

$$\begin{aligned}\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}(\mathcal{L}'(1,\chi)) &= \frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}(\Psi_\chi(m^{12})) + O(m^{\epsilon'-1}) \\ &= \tilde{\mu}^{(a,b)} + O(m^{\epsilon'-1})\end{aligned}$$

Note that the last equality follows from Theorem 4.6.1 with $x = m^{12}$. □

4.8 MOMENTS OF HIGHER DERIVATIVES $\mathcal{L}^{(n)}(1,\chi)$

We will now generalize the results in section 4.6 to higher derivatives. For this we look back at Theorem 4.4.1. Recall we defined,

$$\Psi_K(\chi,r,x) = \frac{1}{x-1}\sum_{k,\ N(P)^k< x} k^r\left(\frac{x}{N(P)^k}-1\right)\chi(P)^k(\log N(P))^{r+1}$$

In particular, for $K=\mathbb{Q}$, it takes the form

$$\begin{aligned}\Psi(\chi,r,x) &= \Psi_{\mathbb{Q}}(\chi,r,x) \\ &= \frac{1}{x-1}\sum_{k,\ p^k< x} k^r\left(\frac{x}{p^k}-1\right)\chi(p)^k(\log p)^{r+1} \\ &= \frac{1}{x-1}\sum_{k,\ p^k< x} \left(\frac{x}{p^k}-1\right)\chi(p^k)(\log p)(\log p^k)^r \\ &= \frac{1}{x-1}\sum_{n< x}\left(\frac{x}{n}-1\right)\chi(n)\Lambda(n)(\log n)^r\end{aligned}$$

Therefore we define, for $k>0$, $r\geq 0$

$$\ell^r\Lambda_k(n) = \sum_{n_1n_2\cdots n_k=n}\left(\prod_{i=1}^{k}\Lambda(n_i)(\log n_i)^r\right) \tag{4.55}$$

whereas, for $k=0$, $\ell^r\Lambda_0(n)=\Lambda_0(n)$. With this, define, for $r\geq 0$,

$$\mu^{(a,b)}(r) = \sum_{j=1}^{\infty}\frac{\ell^r\Lambda_a(j)\ \ell^r\Lambda_b(j)}{j^2} \tag{4.56}$$

In particular, $\mu^{(a,b)}(0) = \mu^{(a,b)}$ as in 4.1.5 of [IMS09] or Theorem 4.5.1, whereas $\mu^{(a,b)}(1) = \tilde{\mu}^{(a,b)}$ as defined in equation (4.26) in the previous sections. We are now ready to state a generalization of Theorem 4.6.1.

***Theorem* 4.8.1.** For each pair (a,b) of non-negative integers, $r \geq 0$ and for $x \geq m$, we have

$$\frac{1}{|X_m|} \sum_{\chi \in X_m} P^{(a,b)} \ (\Psi(\chi,r,x)) = \ \mu^{(a,b)}(r) + O_{a,b}\left(\frac{(\log x)^{(r+1)d+2}}{m}\right) \quad (4.57)$$

Here $d = a+b$.

Proof. The proof follows the $r = 1$ case in Theorem 4.6.1 very closely. Applying Lemma 4.6.2 with $g(x,n) = \frac{1}{(x-1)}\left(\frac{x}{n}-1\right)\Lambda(n)(\log n)^r$ we get,

$$\frac{1}{|X_m^\star|} \sum_{\chi \in X_m^\star} P^{(a,b)}(\Psi(\chi,r,x)) = \sum_{j=1}^{m-1} \lambda^{(a)}(j,x)\ \lambda^{(b)}(j,x) \quad (4.58)$$

where

$$\begin{aligned} \lambda^{(k)}(j,x) &= \sum_{\substack{n_1,\cdots,n_k<x \\ n_1\cdots n_k \equiv j \pmod m}} \prod_{i=1}^{k} \frac{1}{(x-1)}\left(\frac{x}{n_i}-1\right)\Lambda(n_i)(\log n_i)^r \\ &= \frac{1}{(x-1)^k} \sum_{l=0}^{[(x^k-j)/m]} \sum_{\substack{n_1,\cdots,n_k<x \\ n_1\cdots n_k = j+lm}} \prod_{i=1}^{k}\left(\frac{x}{n_i}-1\right)\Lambda(n_i)(\log n_i)^r \end{aligned} \quad (4.59)$$

As before we write it as

$$\lambda^{(k)}(j,x) = \sum_{l=0}^{[(x^k-j)/m]} L^{(k)}(j+lm,x)$$

and show that the total contribution from $l > 0$ terms is small. For this we note that, $L^{(k)}(N,x) \neq 0$ only when $N < x^k$ and in this case,

$$L^{(k)}(N,x) \leq \frac{1}{N} \sum_{\substack{n_1,\cdots,n_k<x \\ n_1\cdots n_k=N}} \left(\prod_{i=1}^{k} \Lambda(n_i)(\log n_i)^r \right)$$
$$\leq \frac{1}{N} \ell^r \Lambda_k(N) \leq k^k \frac{(\log x)^{(r+1)k}}{N}$$

where the last inequality follows from : (similar to (4.24))

$$\ell^r \Lambda_k(n) \leq \frac{(\log n)^{rk}}{k^{rk}} \Lambda_k(n) \leq \frac{(\log n)^{(r+1)k}}{k^{rk}}$$

and therefore,

$$\sum_{l=1}^{[(x^k-j)/m]} L^{(k)}(j+lm,x) < k^k \frac{(\log x)^{(r+1)k}}{m} \left(1 + \frac{1}{2} + \cdots + \frac{1}{[x^k/m]}\right)$$
$$= O\left(\frac{(\log x)^{(r+1)k+1}}{m}\right)$$

For the $l = 0$ term using (4.34) and for $x \geq m$ we have

$$L^{(k)}(j,x) = \frac{\ell^r \Lambda_k(j)}{j} + O\left(\frac{(\log m)^{(r+1)k}}{m}\right) \tag{4.60}$$

and hence,

$$\lambda^{(k)}(j,x) = \frac{\ell^r \Lambda_k(j)}{j} + O\left(\frac{(\log x)^{(r+1)k+1}}{m}\right)$$

and so, (writing $d = a + b$)

$$\frac{1}{|X_m^\star|} \sum_{\chi \in X_m^\star} P^{(a,b)}(\Psi(\chi,r,x)) = \sum_{j=1}^{m-1} \frac{\ell^r \Lambda_a(j)\ \ell^r \Lambda_b(j)}{j^2} + O\left(\frac{(\log x)^{(r+1)d+2}}{m}\right)$$

which, together with the inequality :

$$\sum_{j\geq m} \frac{\ell^r\Lambda_a(j)\ \ell^r\Lambda_b(j)}{j^2} \leq \frac{1}{(a^a b^b)^r} \sum_{j\geq m} \frac{(\log j)^{(r+1)(a+b)}}{j^2}$$
$$= O\left(\frac{(\log m)^{(r+1)(a+b)}}{m}\right)$$

proves the theorem. □

Now applying Theorem 4.4.1 directly, with $x = m^2$ gives,

$$\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}(\mathcal{L}^{(r)}(1,\chi))$$
$$= \frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}\left((-1)^{r+1}\Psi(\chi,r,m^2)\right) + O\left(\frac{(\log m)^{rd}}{m^d}\right)$$
$$= (-1)^{(r+1)d}\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}\left(\Psi(\chi,r,m^2)\right) + O\left(\frac{(\log m)^{rd}}{m^d}\right)$$
$$= (-1)^{(r+1)d}\mu^{(a,b)}(r) + O\left(\frac{(\log m)^{(r+1)d+2}}{m}\right)$$

Here the last line follows from Theorem 4.8.1 with $x = m^2$. Therefore we have the following general version of Theorem 4.6.3 :

Theorem 4.8.2. Under GRH,

$$\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}(\mathcal{L}^{(r)}(1,\chi)) = (-1)^{(r+1)d}\mu^{(a,b)}(r) + O\left(\frac{(\log m)^{(r+1)d+2}}{m}\right)$$

Here $d = a + b$ and the implicit constant depends on a, b. In particular,

$$\lim_{m\to\infty}\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}(\mathcal{L}'(1,\chi)) = ((-1)^{(r+1)d}\mu^{(a,b)}(r)$$

Remark 4.8.3. Note that for an unconditional version of the above theorem, we need to have a closer look at the general $r_\chi(n,x)$ term. We have,

$$r_\chi(n,x) = \frac{(-1)^n n!}{x-1}\left[x\sum_\rho \lim_{s\to\rho}\frac{d^n}{ds^n}\left(\frac{x^{s-1}}{s-1}\right) - \sum_\rho \lim_{s\to\rho}\frac{d^n}{ds^n}\left(\frac{x^s}{s}\right)\right]$$

$$= \frac{(-1)^n n!}{x-1}\sum_\rho\left[x\sum_{k=0}^{n}\binom{n}{k}x^{\rho-1}(\log x)^{n-k}\cdot\frac{(-1)^k k!}{(\rho-1)^{k+1}} - \sum_{k=0}^{n}\binom{n}{k}x^{\rho}(\log x)^{n-k}\cdot\frac{(-1)^k k!}{\rho^{k+1}}\right]$$

$$\ll \frac{(\log x)^n}{x}\sum_\rho\left|\frac{x^\rho}{\rho(\rho-1)}\right| \quad \text{(\textit{The implicit constant depends on } n.)}$$

This reduces the case to that shown by Ihara, Murty and Shimura, see Sublemma 5.4.4 of [IMS09] with the difference that the implicit constant depends on n as well, apart from a and b. Therefore we have

***Theorem* 4.8.4.** For any $\epsilon > 0$, we have, unconditionally,

$$\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}(\mathcal{L}^{(r)}(1,\chi)) = (-1)^{(r+1)d}\mu^{(a,b)}(r) + O\left(m^{\epsilon-1}\right)$$

The implicit constant depends on a, b and r. In particular,

$$\lim_{m\to\infty}\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}(\mathcal{L}'(1,\chi)) \;=\; ((-1)^{(r+1)d}\mu^{(a,b)}(r)$$

5 DISTRIBUTION

5.1 PRELIMINARIES

For most of this chapter K is either $\mathbb{Q}$ or an imaginary quadratic number field. In particular K has exactly one Archimedean prime denoted by $\wp_\infty$. Let χ run over all Dirichlet characters on K whose conductor (the non-archimedean part) is a prime divisor, such that $\chi(\wp_\infty) = 1$.

The average of a complex valued function $\phi(\chi)$, over a family of χ as defined above, is taken as follows :

$$\mathrm{Avg}_\chi \phi(\chi) = \lim_{m\to\infty} \mathrm{Avg}_{N(\mathbf{f})\le m}\phi(\chi)$$

where

$$\mathrm{Avg}_{N(\mathbf{f})\le m}\phi(\chi) = \frac{\sum_{N(\mathbf{f})\le m} \left(\sum_{\mathbf{f}_\chi=\mathbf{f}} \phi(\chi)\right) \ / \ \sum_{\mathbf{f}_\chi=\mathbf{f}} 1}{\sum_{N(\mathbf{f})\le m} 1}$$

For the above setting, the following distribution theorem was proved by Ihara in [Iha08] :

Theorem 5.1.1. **(Ihara)** For K as above and for $\sigma = \mathrm{Re}(s) > 1$, there exists a real valued function $M_\sigma : \mathbb{C} \to \mathbb{R}$ satisfying, $M_\sigma(w) \ge 0$, is C^∞ in w and $\int_{\mathbb{C}} M_\sigma(w)\ |dw| = 1$, such that

$$\mathrm{Avg}_\chi\ \Phi\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) \ = \ \int_{\mathbb{C}} M_\sigma(w)\ \Phi(w)\ |dw| \qquad (5.1)$$

holds for any continuous function Φ of $\mathbb{C}$. Moreover,

$$\mathrm{Avg}_\chi\ \psi_z\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) \ = \tilde{M}_\sigma(z)$$

where $\tilde{M}_\sigma(z)$ comes from the Fourier transform of $M_\sigma(z)$ in the sense that

$$\tilde{M}_\sigma(z) = \int_{\mathbb{C}} M_\sigma(w)\ \psi_z(w)\ |dw|$$

here $\psi_z : \mathbb{C} \to \mathbb{C}^1$ is the additive character $\psi_z(w) = \exp(i \cdot \mathrm{Re}(\bar{z}w))$

Remark 5.1.2. Note that Ihara shows this more generally, in the sense that he considers certain function fields of one variable over a finite field (the theorem is true in this case for $\sigma > 3/4$), $K = \mathbb{Q}$ and χ runs over characters of the form $N(\wp)^{-\tau i}$ and when K is a number field having more that one archimedean prime and χ runs over all "normalized unramified Grössencharacters" of K modifying the definition of average accordingly.

Remark 5.1.3. In a later paper [IM11], Ihara together with Matsumoto showed the above theorem for $\sigma \geq \frac{1}{2} + \epsilon$, under GRH and with "mild" conditions on the test function namely, $\Phi(w) \ll e^{a|w|}$ holds for some $a > 0$, or Φ is the characteristic function of either a compact subset or the compliment of such a subset.

Similar distribution result for real characters was also proved by Mourtada, my academic sister, in her thesis, see [Mou13], [Mou15]. She showed, for a fundamental discriminant D and a real character χ_D attached to D, let

$$N(y) := \{|D| \leq y\ :\ D \text{ is a fundamental discriminant}\}$$

then the following theorem holds.

Theorem 5.1.4. **(Mourtada)** Let $\sigma > \frac{1}{2}$ and assume GRH. Then there exists a density function $\mathcal{Q}_\sigma(x)$ such that

$$\lim_{y\to\infty} \frac{1}{N(y)} \sum_{\substack{|D|\leq y \\ D \text{ fund. disc.}}} \Phi\left(\frac{L'(\sigma.\chi_D)}{L(\sigma,\chi_D)}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{Q}_\sigma(x)\Phi(x)dx$$

holds for any bounded continuous function Φ on $\mathbb{R}$. It also holds when Φ is the characteristic function of either a compact subset of $\mathbb{R}$ or the complement of such a subset.

Our goal in this chapter is to deduce similar distribution theorems for higher derivatives of the logarithmic derivative of Dirichlet L-functions. We have been able to prove results similar to that of Theorem 5.1.1, for

$\sigma > 1$. Whereas generalization of the later developments are still a work in progress. We'll show in a concluding section where these later developed techniques fail if we consider higher derivatives.

5.2 DISTRIBUTION FUNCTIONS : SOME BACKGROUND

In this section we present some background related to distribution. The results presented are based on the paper [JW35] of Jessen and Wintner.

Let $\mathbb{R}^K$ be a k-dimensional Euclidean space and $\mathbf{x} = (x_1, \cdots, x_k)$ be a variable point.

Definition 5.2.1. A completely additive, non-negative set function $\phi(E)$ defined for all Borel sets E in $\mathbb{R}^k$ and having the value 1 for $E = \mathbb{R}^k$ will be called a *distribution function* in $\mathbb{R}^k$.

Notation. An integral with respect to ϕ will be denoted by

$$\int_E f(\mathbf{x})\phi(d\mathbf{x})$$

and is to be understood in the Lebesgue-Radon (or Lebesgue-Stieltjes) sense.

Definition 5.2.2. A set E is called a *continuity set* of ϕ if $\phi(E^\circ) = \phi(\overline{E})$ where E° denotes the set formed by all interior points of E and $\overline{E}$ is the closure of E.

Definition 5.2.3. A sequence of distribution functions ϕ_n is said to be *convergent* if there exists a distribution function ϕ such that $\phi_n(E) \to \phi(E)$ for all continuity sets E of the limit function ϕ, which is then unique. We will use the notation $\phi_n \to \phi$.

Proposition 5.2.4. A sequence of distribution functions $\{\phi_n\}$ converges to a distribution function ϕ if and only if

$$\int_{\mathbb{R}^k} f(\mathbf{x})\phi_n(d\mathbf{x}) \to \int_{\mathbb{R}^k} f(\mathbf{x})\phi(d\mathbf{x})$$

holds for all continuous and bounded functions f. Moreover, if $\phi_n \to \phi$ then,

$$\int_{\mathbb{R}^k} f(\mathbf{x})\phi(d\mathbf{x}) \leq \liminf \int_{\mathbb{R}^k} f(\mathbf{x})\phi_n(d\mathbf{x})$$

holds for every non-negative, continuous function f.

Definition 5.2.5. If ϕ_1 and ϕ_2 are two distribution functions, then we define a new distribution function as their convolution, as follows :

$$\phi_1 * \phi_2(E) \;:=\; \int_{\mathbb{R}^k} \phi_1(E - \mathbf{x})\phi_2(d\mathbf{x})$$

for every Borel set E. Here $E - \mathbf{x}$ denotes the set obtained from E by the translation $-\mathbf{x}$.

Note that one can show, $\phi_1 * \phi_2 = \phi_2 * \phi_1$

Definition 5.2.6. The spectrum $S = S(\phi)$ of a distribution function ϕ is the set of points $\mathbf{x} \in \mathbb{R}^k$ for which $\phi(E) > 0$ for any set E containing $\mathbf{x}$ as an interior point. We note that S is always a non-empty closed set.

5.3 construction of $M_{\sigma,P}$ functions

Let P be any finite set of non-archimedean primes of K and set
$T_P := \prod_{\wp} \mathbb{C}^1$, where $\mathbb{C}^1$ denotes, $\{z : |z| = 1\}$.
The following lemma was proved in [Iha08], (Lemma 4.3.1)

Lemma 5.3.1. Let K be as above and χ run over a family as above, excluding those characters such that $\mathbf{f}_\chi \in P$. For each such χ, let $\chi_P = (\chi(\wp))_\wp \in T_P$. Then $(\chi_P)_\chi$ is uniformly distributed on T_P. i.e. for any continuous function $\Psi : T_P \to \mathbb{C}$ we have,

$$\mathrm{Avg}_\chi\ \Psi(\chi_P) = \int_{T_P} \Psi(t_P)\ d^*t_P$$

where $d^*t_\wp = (2\pi i t_\wp)^{-1} dt_\wp$ is the normalized Haar measure on the $t_\wp$-unit circle.

Remark 5.3.2. The above lemma is the key ingredient of our results. The idea is to make suitable change of variables in the above lemma, so that from the Jacobian a density function can be extracted.

Now define, $g_{\sigma,P} : T_P \to \mathbb{C}$ by

$$g_{\sigma,P}(t_P) = \sum_{\wp \in P} g_{\sigma,\wp}(t_\wp) \quad \text{where} \quad g_{\sigma,\wp}(t_\wp) = \frac{t_\wp N\wp^\sigma (\log N\wp)^2}{(t_\wp - N\wp^\sigma)^2}$$

where $t_P = (t_\wp)_{\wp \in P}$, in particular, $|t_\wp| = 1$.

For any character χ of K which is unramified over P, let

$$L_P(\chi, s) = \prod_{\wp \in P} (1 - \chi(\wp)\ N\wp^{-s})^{-1}$$

Then we have,

$$\mathcal{L}_P(\chi, s) := \frac{L_P'(\chi, s)}{L_P(\chi, s)} = \sum_{\wp \in P} -\frac{\chi(\wp) N\wp^{-s} \log N\wp}{(1 - \chi(\wp)\ N\wp^{-s})}$$

and so,

$$\mathcal{L}'_P(\chi, s) = \frac{d}{ds}\frac{L'_P(\chi, s)}{L_P(\chi, s)} = \sum_{\wp \in P} \frac{\chi(\wp) N\wp^{-s} (\log N\wp)^2}{(1 - \chi(\wp) N\wp^{-s})^2} = g_{\sigma,P}(\chi_P NP^{-it})$$

where $t = \mathrm{Im}(s)$ and $\chi_P NP^{-it} = (\chi(\wp) N\wp^{-it})_{\wp \in P}$.

For $(\mathbf{f}_\chi, P) = 1$, since, $\{\chi_P\}_\chi$ is uniformly distributed on T_P, so is its translate $\{\chi_P NP^{-it}\}_\chi$. Thus for any continuous function Φ on $\mathbb{C}$, by the above lemma 5.3.1, applied to $\Psi = \Phi \circ g_{\sigma,P}$, we get

$$\mathrm{Avg}_\chi \left(\Phi\left(\mathcal{L}'_P(\chi, s)\right)\right) = \int_{T_p} \Phi(g_{\sigma,P}(t_P))\, d^*t_P \tag{5.2}$$

We first note the following.

Lemma 5.3.3. For fixed s, with $\sigma = \mathrm{Re}(s) > 1$, and for $P = P_y = \{\wp : N\wp \le y\}$, as $y \to \infty$, $\mathcal{L}'_P(\chi, s)$ tends uniformly to $\mathcal{L}'(\chi, s)$.

Proof. For any χ we have,

$$|\mathcal{L}'(\chi, s) - \mathcal{L}'_P(\chi, s)| \le \sum_{\wp \notin P} \frac{N\wp^\sigma \log N\wp^2}{(N\wp^\sigma - 1)^2}$$

Thus letting $y \to \infty$, RHS tends to 0. □

***Theorem* 5.3.4.** Let $\sigma > 0$. Then there exists a function $M_{\sigma,P} : \mathbb{C} \to \mathbb{R}$ such that, for any continuous function $\Phi(w)$ on $\mathbb{C}$,

$$\int_{\mathbb{C}} M_{\sigma,P}(w)\Phi(w)|dw| = \int_{T_P} \Phi(g_{\sigma,P}(t_P))\, d^*t_P$$

where $w = x + iy$ and $|dw| = (2\pi)^{-1} dxdy$, and d^*t_P is the normalized Haar measure on T_P. This $M_{\sigma,P}$ function is compactly supported and satisfies the following properties :

1. $M_{\sigma,P}(w) \ge 0$,
2. $M_{\sigma,P}(\overline{w}) = M_{\sigma,P}(w)$,

3. $\int_{\mathbb{C}} M_{\sigma,P}(w)\ |dw| = 1.$

Proof. We first consider the case when $|P| = 1$, say $P = \{\wp\}$. Let $T_\wp = \mathbb{C}^1$ and write $t_\wp = e^{i\theta}$ and so $d^*t_\wp = \frac{1}{2\pi}d\theta$.

We consider the open unit disc, $z = re^{i\theta}$ for $0 \leq r < 1$ and $0 \leq \theta < 2\pi$. Consider the map

$$w = w(z) = \frac{(\log N\wp)^2\ re^{i\theta}}{(1 - re^{i\theta})^2} = \frac{A\ re^{i\theta}}{(1 - re^{i\theta})^2}$$

For computational brevity we'll write $A = (\log N\wp)^2$ as this is just a constant. Let ρ be a real number such that, $N\wp^{-\sigma} < \rho < 1$ and let B_ρ be the region surrounded by the curve :

$$w = \frac{A\ \rho e^{i\theta}}{(1 - \rho e^{i\theta})^2}$$

Thus $w = w(z)$ gives a one-to-one correspondence between the region $B_{\sigma,\wp}$ and the disc $C_\rho := \{z\ :\ |z| < \rho\}$.

Let us now compute the Jacobian of this mapping. We see that,

$$w(z) = A\ \frac{r\cos\theta - 2r^2 + r^3\cos\theta}{|1 - re^{i\theta}|^4} + i\ A\ \frac{r\sin\theta - r^3\sin\theta}{|1 - re^{i\theta}|^4} = U + iV \quad \text{(say)} \tag{5.3}$$

Thus the Jacobian is given by :

$$J\ =\ \begin{vmatrix} \frac{\partial U}{\partial r} & \frac{\partial U}{\partial \theta} \\ \frac{\partial V}{\partial r} & \frac{\partial V}{\partial \theta} \end{vmatrix}\ =\ \frac{A^2\ r\ |1 + re^{i\theta}|^2}{|1 - re^{i\theta}|^6}$$

Note : This Jacobian was computed using a computer algebra system.

And so we have

$$\begin{aligned}\int_{T_\wp} \Phi(g_{\sigma,\wp}(t_\wp))\, d^*t_\wp &= \frac{1}{2\pi}\int_0^{2\pi} \Phi\left(\frac{e^{i\theta}N\wp^{\sigma}(\log N\wp)^2}{(e^{i\theta}-N\wp^{\sigma})^2}\right)\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \Phi\left(\frac{e^{i\theta}N\wp^{-\sigma}(\log N\wp)^2}{(1-N\wp^{-\sigma}e^{i\theta})^2}\right)\, d\theta \\ &= \frac{1}{2\pi}\int\int_{B_{\sigma,\wp}} \Phi(w)\delta(r-N\wp^{-\sigma})\, J^{-1}\, dU\, dV\end{aligned}$$

where $\delta(.)$ denotes the Dirac delta distribution and $w = U + iV$. Therefore we define $M_{\sigma,\wp}(w)$ in the following way :

$$M_{\sigma,\wp}(w) = J^{-1}\delta(r-N\wp^{-\sigma}) = \frac{|1-re^{i\theta}|^6}{(\log N\wp)^2\,|1+re^{i\theta}|^2}\,\frac{\delta(r-N\wp^{-\sigma})}{r} \tag{5.4}$$

for $w \in B_{\sigma,\wp}$ and $M_{\sigma,\wp}(w) = 0$ otherwise. Plugging this in, we get

$$\int_{T_\wp} \Phi(g_{\sigma,\wp}(t_\wp))\, d^*t_\wp \;=\; \int_{\mathbb{C}} M_{\sigma,\wp}(\omega)\, \Phi(\omega)\, |d\omega|$$

This proves the case $P = \{\wp\}$. For the general case, we define the function using convolution product. That is,

$$M_{\sigma,P}(w) = *_{\wp\in P}\, M_{\sigma,\wp}(w)$$

in other words, for $P = P' \cup \{\wp\}$ define

$$M_{\sigma,P}(w) = \int_{\mathbb{C}} M_{\sigma,P'}(w')\, M_{\sigma,\wp}(w-w')\, |dw'| \tag{5.5}$$

Note that, for any open set $U \subseteq \mathbb{C}$ we get,

$$\int_U M_{\sigma,P}(w)\, |dw| = \operatorname{Vol}(g_{\sigma,P}^{-1}(U)) \tag{5.6}$$

where the volume is with respect to d^*t_P and thus $\int_{\mathbb{C}} M_{\sigma,P}(w)\, |dw| = 1$. The Haar measure is normalized, i.e the total volume of T_P is 1. □

Our next Goal is to show, for $P = P_y = \{\wp \;:\; N\wp \leq y\}$ as before, as $y \to \infty$, $M_{\sigma,P_y}(w)$ converges to a function $M_\sigma(w)$ uniformly in w.

Proposition 5.3.5. If $P = P_y$ and $y \to \infty$, for $\sigma > 1/2$, $M_{\sigma,P}(w)$ converges to $M_\sigma(w)$ uniformly in w. The limit, $M_\sigma(w)$ is therefore continuous in w and non-negative.

Proof. For $\wp \notin P$ we have, (writing $N\wp^{-\sigma} = q$)

$$|M_{\sigma,P\cup\{\wp\}}(w) - M_{\sigma,P}(w)| = \left| \frac{1}{2\pi} \frac{1}{(\log N\wp)^2} \int_0^{2\pi} \frac{|1-qe^{i\theta}|^6}{|1+qe^{i\theta}|^2} M_{\sigma,P}(z - qe^{i\theta})\, d\theta \right|$$

$$\ll \frac{q^4}{(\log N\wp)^2} \ll \left(\frac{1}{N\wp^\sigma} \right)^4$$

Note that, by (1) and (3) of Theorem 5.3.4, $M_{\sigma,P}$ is bounded. Thus we see that $M_{\sigma,P}(w)$ converges uniformly to a function, say $M_\sigma(w)$ for $\sigma > 1/2$ (in fact, 1/4). □

Remark 5.3.6. We also have, $\int_{\mathbb{C}} M_\sigma(w)|dw| = 1$. But we will show this after showing the next theorem. Note that since, $\int_{\mathbb{C}} M_{\sigma,P}(w)|dw| = 1$ for all P, the uniform convergence already gives,

$$\int_{\mathbb{C}} M_\sigma(w)|dw| \le 1$$

Theorem 5.3.7. For any $s \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s) > 1$

$$\mathrm{Avg}_\chi\, \Phi(\mathcal{L}'(\chi, s)) \;=\; \int_{\mathbb{C}} M_\sigma(w)\, \Phi(w)\, |dw| \tag{5.7}$$

holds for any continuous function Φ of $\mathbb{C}$.

Proof. From equation 5.2 and Theorem 5.3.4 we have,

$$\mathrm{Avg}_\chi \left(\Phi\left(\mathcal{L}'_P(\chi, s) \right) \right) = \int_{T_p} \Phi(g_{\sigma,P}(t_P))\, d^* t_P$$

$$= \int_{\mathbb{C}} M_{\sigma,P}(w) \Phi(w) |dw|$$

The theorem is proved by taking the limit and from Lemma 5.3.3 and Proposition 5.3.5. □

Note that if we take the particular case of $\Phi(w) = P^{(a,b)}(w) = w^a\overline{w}^b$, Then our results from moments give : (Theorem 4.7.1)

$$(-1)^{a+b}\tilde{\mu}^{(a,b)} = \int_{\mathbb{C}} M_\sigma(w)P^{(a,b)}(w)|dw|$$

In particular, taking $a = b = 0$ gives,

$$\int_{\mathbb{C}} M_\sigma(w)|dw| = \tilde{\mu}^{(0,0)} = 1$$

Also note that if we consider the Fourier dual of $M_\sigma(z)$ given by

$$\tilde{M}_\sigma(z) = \int_{\mathbb{C}} M_\sigma(w)\psi_z(w)|dw|$$

Then from the above Theorem 5.3.7, we have

$$\tilde{M}_\sigma(z) = \mathrm{Avg}_\chi\ \psi_z(\mathcal{L}'(\chi, s))$$

5.4 A NOTE ON HIGHER DERIVATIVES

We note that the above technique theoretically generalizes to higher derivatives. We just need to appropriately choose the $g_{\sigma,\wp}(t_\wp)$ function such that for a local factor, we get

$$\mathcal{L}_\wp^{(n)}(\chi, s) = g_{\sigma,\wp}(\chi(\wp)N\wp^{-it})$$

But note that for higher derivatives, computing these $M_{\sigma,P}$ functions explicitly becomes very involved. Even in our case we used a computer to simplify the Jacobian.

5.5 POSSIBLE EXTENSION OF OUR RESULT TO $\frac{1}{2} < \sigma \leq 1$

For $\sigma > 1$, the image of $g_{\sigma,P}$ remains bounded as $|P| \to \infty$. Since the support of $M_{\sigma,P}$ is the image of the mapping, $g_{\sigma,P}$, so the support of M_σ is also bounded. Therefore, in the proof of the above theorem we can just assume Φ to be continuous.

This is no longer true for $\sigma > \frac{1}{2}$, i.e. image of $g_{\sigma,P}$ need not be bounded. As remarked earlier, Ihara and Matsumoto, in a later paper [IM11] extends it to $\sigma > 1/2$ under GRH and some conditions on the test function. They introduced the idea of admissible functions and developed a more general notion of $g_{\sigma,\wp}$ which they called g-functions. However their approach does not seem to apply for higher derivatives. It fails at essential steps in section 3.1 and 3.3 of their paper. We are yet to discover a way of doing this and this is currently a work in progress.

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