Emerton–Gee stacks, Serre weights, and Breuil–Mézard conjectures for GSp_4

by

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Abstract

We construct a moduli stack of rank 4 symplectic projective étale (φ, Γ)-modules and prove its geometric properties for any prime $p > 2$ and finite extension $K/\mathbf{Q}_p$. When $K/\mathbf{Q}_p$ is unramified, we adapt the theory of local models recently developed by Le–Le Hung–Levin–Morra to study the geometry of potentially crystalline substacks in this stack. In particular, we prove the unibranch property at torus fixed points of local models and deduce that tamely potentially crystalline deformation rings are domain under genericity conditions. As applications, we prove, under appropriate genericity conditions, an GSp_4-analogue of the Breuil–Mézard conjecture for tamely potentially crystalline deformation rings, the weight part of Serre's conjecture formulated by Gee–Herzig–Savitt for global Galois representations valued in GSp_4 satisfying Taylor–Wiles conditions, and a modularity lifting result for tamely potentially crystalline representations.

박금희, 이철호, 나의 부모님께

To my parents, Park Geum Hee, Lee Cheol Ho

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I learned the word “Langlands Program” from the book Fermat’s Last Theorem when I was around ten. More than ten years later, when I was applying to graduate schools, I thought I might study the Langlands Program for my graduate studies. Yet, I was afraid of it rather than excited about it because of its notorious difficulty. Five years later, now, I feel incredibly grateful that I could have studied it, and I am very excited to learn more about it and make a small contribution to this grand subject. All these changes would not be possible without my advisor, Florian Herzig, who guided me to explore this complex and beautiful world, patiently answered my numerous questions, and encouraged me whenever I felt frustrated.

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Contents

Chapter 1

Introduction

Serre's modularity conjecture predicts that every continuous, irreducible, odd Galois representations $\overline{r}$: $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ arise from modular forms [Ser87]. Its refined version predicts the minimal weight of modular forms that give rise to $\overline{r}$ in terms of the restriction of $\overline{r}$ at p. The weight part of Serre's conjecture, saying that the original conjecture implies the refined conjecture, was crucial for the proof of the original conjecture due to Kisin and Khare–Winterberger [KW09a; KW09b; Kis09].

The weight part of Serre's conjecture has been generalized, most notably, to 2-dimensional Galois representations over a totally real field [BDJ10], to Galois representations over a totally real field valued in GL_n [Her09] and in more general groups G [GHS18]. (Both [Her09] and [GHS18] assume that p is unramified in the totally real field and the Galois representations are tamely ramified at places above p. See [Le+a, Conjecture 1.6.2] for a generalization to wildly ramified representations, and [Le+b, Definition 1.1.1] to a totally real field in which p ramifies.) In these generalizations, the notion of weight is replaced by *Serre weights*, irreducible $\overline{\mathbf{F}}_p$-representations of $G(\mathbf{F}_q)$.

Although stated in the global context, the weight part of Serre's conjecture is closely related to the mod p Langlands program. Let $K/\mathbf{Q}_p$ be a finite extension. The mod p Langlands program predicts a certain correspondence between admissible smooth $\overline{\mathbf{F}}_p$-representations of a p-adic reductive group $G(K)$ and continuous representations of $\mathrm{Gal}(\overline{K}/K)$ valued in the C-group ${}^C G(\overline{\mathbf{F}}_p)$ defined in [BG14]. (In this thesis, we only consider $G = \mathrm{GL}_n$ or $G = \mathrm{GSp}_4$, in which case ${}^C G$ is isomorphic to $G \times \mathrm{GL}_1$, and it is equivalent to work with G.) Such correspondence has been established only in the case $G = \mathrm{GL}_2$ and $K = \mathbf{Q}_p$ [Col10; Paš13] or $G = \mathrm{GL}_1$. Despite many recent advances (for example, see [Car+16; Bre+a]) in the program, its intricate nature still remains poorly understood, and a precise conjecture for $G = \mathrm{GL}_n$ was only made recently [EGH22] in a categorical language. However, using global methods, one can construct a candidate for a mod p Langlands correspondence, namely, an admissible smooth $\overline{\mathbf{F}}_p$-representation Π of $\mathrm{GL}_n(K)$ associated with an n-dimensional continuous $\overline{\mathbf{F}}_p$-representation $\overline{\rho}$ of $\mathrm{Gal}(\overline{K}/K)$. One major open problem is to prove that Π only depends on $\overline{\rho}$ and not on non-canonical choices made in the global argument. The weight part of Serre's conjecture describes the Serre weights appearing in the $\mathrm{GL}_n(\mathcal{O}_K)$-socle of Π explicitly in terms of $\overline{\rho}|_{I_K}$. Thus, the weight part of Serre's conjecture has been a guiding principle in the mod p Langlands program.

The theory of Galois deformations has played an important role in many advances in proving the weight part of Serre's conjecture ([Le+18; Le+20; Le+a]), in the mod p and p-adic Langlands program ([Col10; Paš13; Car+16; Bre+b]), and in proving modularity lifting results ([TW95; Wil95; CHT08; Bar+14; Box+21]). Recently, Emerton–Gee constructed an object which algebraically interpolates deformation spaces of n-

dimensional p-adic Galois representations (called the Emerton–Gee stack [EG23]). Thus, it allows us to employ more geometric methods to study p-adic Galois representations. Most importantly, it is an essential ingredient in formulating the categorical mod p and p-adic Langlands conjecture [EGH22]. When $K/\mathbf{Q}_p$ is unramified, [Le+a] developed a theory of local models to study the geometry of potentially crystalline substacks in Emerton–Gee stacks. In particular, it is applied to analyze singularities of tamely potentially crystalline deformation rings. This led to several applications, including the weight part of Serre's conjecture and the Breuil–Mézard conjecture. The objective of this thesis is to generalize Emerton–Gee stacks, local models, and their applications to Serre weight conjectures and the Breuil–Mézard conjecture to the group GSp_4.

Emerton–Gee stacks for GSp_4

Let p be a prime and $K/\mathbf{Q}_p$ be a finite extension with the ring of integers $\mathcal{O}_K$, a uniformizer ϖ, and the residue field k. We also fix a sufficiently large finite extension $E/\mathbf{Q}_p$ with the ring of integers $\mathcal{O}$, a uniformizer ϖ, and the residue field $\mathbf{F}$.

In [EG23], Emerton–Gee constructed $\mathcal{X}_{n,K}$ a moduli stack of rank n projective étale (φ,Γ)-modules. For a finite $\mathcal{O}$-algebra A, rank n projective étale (φ,Γ) modules with A-coefficients are equivalent to continuous representations of $G_K := \mathrm{Gal}(\overline{K}/K)$ on rank n projective A-modules. However, for general A, étale (φ,Γ)-modules behave better than G_K-representations in algebraic families. In particular, a family of reducible étale (φ,Γ)-modules can converge to an irreducible one, unlike G_K-representations. Thus, $\mathcal{X}_{n,K}$ is considered the correct notion of a moduli stack of "p-adic Langlands parameters".

To generalize Emerton–Gee stacks to GSp_4, we define a *symplectic* projective étale (φ,Γ)-module to be a triple (M,N,α) where M is a rank 4 projective étale (φ,Γ) module, N is a rank 1 projective étale (φ,Γ)-modules, and $\alpha : M \simeq M^\vee \otimes N$ is an essentially self-dual isomorphism satisfying a certain sign condition (Definition 4.1.1). These reflect the underlying 4-dimensional vector space of GSp_4, the similitude character of GSp_4, and the non-degenerate skew-symmetric bilinear form on the 4-dimensional vector space, respectively. Under Fontaine's equivalence, symplectic projective étale (φ,Γ)-modules with A-coefficients for finite local $\mathcal{O}$-algebra A are equivalent to continuous representations $\rho : G_K \to \mathrm{GSp}_4(A)$. We let $\mathcal{X}_{\mathrm{Sym},K}$ be a category fibered in groupoids over $\mathrm{Spf}\,\mathcal{O}$ whose groupoid of A-points, for a p-adically complete $\mathcal{O}$-algebra A, is the groupoid of symplectic projective étale (φ,Γ)-modules with A-coefficients. The following Theorem generalizes main properties of $\mathcal{X}_{n,K}$ to $\mathcal{X}_{\mathrm{Sym},K}$.

Theorem 1 (Theorem 4.1.5, Proposition 4.1.6, Theorem 4.1.10)**.** *Suppose that $p > 2$.*

1. *The category fibered in groupoids $\mathcal{X}_{\mathrm{Sym},K}$ is a Noetherian formal algebraic stack.*

2. *For each Hodge type λ and inertial type τ, there exist a closed substack $\mathcal{X}^{\lambda,\tau}_{\mathrm{Sym},K}$ (resp. $\mathcal{X}^{\mathrm{ss},\lambda,\tau}_{\mathrm{Sym},K}$) which is furthermore a p-adic formal algebraic stack and flat over $\mathcal{O}$. It is uniquely characterized as $\mathcal{O}$-flat closed substack such that for any finite flat $\mathcal{O}$-algebra A, $\mathcal{X}^{\lambda,\tau}_{\mathrm{Sym},K}(A)$ (resp. $\mathcal{X}^{\mathrm{ss},\lambda,\tau}_{\mathrm{Sym},K}(A)$) is equivalent to the groupoid of representations $\rho : G_K \to \mathrm{GSp}_4(A)$ such that $\rho \otimes_A E$ is potentially crystalline (resp. semistable) representations with Hodge type λ and inertial type τ.*

3. *The underlying reduced substack $\mathcal{X}_{\mathrm{Sym},K,\mathrm{red}} \subset \mathcal{X}_{\mathrm{Sym},K}$ is an algebraic stack over $\mathbf{F}$ and equidimensional of dimension $4[K:\mathbf{Q}_p]$. Moreover, its irreducible components are naturally labelled by Serre weights.*

The proofs of item (1) and (2) use the corresponding results for the Emerton–Gee stack for $\mathrm{GL}_4 \times \mathrm{GL}_1$ and can be easily generalized to GSp_{2n}. To prove item (3), we first construct irreducible components corresponding to Serre weights and prove that their union is equal to the stack $\mathcal{X}_{\mathrm{Sym},K,\mathrm{red}}$, following the strategy for GL_n. The construction of irreducible components requires a generalization of a result of Emerton–Gee constructing a family of extensions using abelian Galois cohomology. One difference between GSp_4 and GL_n is the presence of a non-abelian unipotent radical of a maximal parabolic subgroup Q called the Klingen parabolic subgroup. It is unclear how the construction of Emerton–Gee can be generalized to the non-abelian case. Instead, we construct a family of extensions valued in Q inside a family of extensions valued in the minimal parabolic of GL_4 containing Q by only using abelian Galois cohomology. To prove that their union is equal to the stack $\mathcal{X}_{\mathrm{Sym},K,\mathrm{red}}$, we need to prove the existence of crystalline lifts for any continuous representation $G_K \to \mathrm{GSp}_4(\mathbf{F})$. In this case, it seems inevitable to use non-abelian Galois cohomology. We prove this by using the main result in [Lina], which develops an obstruction theory for lifting mod p representations of G_K valued in reductive groups.

While we were writing this thesis, Zhongyipan Lin posted a preprint that constructs generalizations of Emerton–Gee stacks to tamely ramified reductive groups and proves their Noetherian formal algebraicity using a Tannakian formalism [Linb].

Local models for potentially crystalline stacks

From now on, we assume that $K/\mathbf{Q}_p$ is unramified of degree f.

In [Le+a], the authors constructed local models of potentially crystalline substacks of $\mathcal{X}_{n,K}$ using Breuil–Kisin modules. One can associate a Breuil–Kisin module with descent data to a lattice in a potentially crystalline representation. Using this, they identify the potentially crystalline stacks (with bounded p-adic Hodge type) with a certain closed substack of the moduli stack of Breuil–Kisin modules with descent data constructed in [CL18]. It is proved in *loc. cit.* that the moduli stack of Breuil–Kisin modules with descent data is smoothly equivalent to a Pappas–Zhu local model studied in [PZ13]. The innovation in [Le+a] is a construction of certain subvarieties in Pappas–Zhu local models whose open neighborhoods are, after p-adic completion, smoothly equivalent to open neighborhoods of potentially crystalline stacks.

The Pappas–Zhu local models exist for any connected reductive group which splits over a tamely ramified base change. Let $M(\leq \lambda)$ be the Pappas–Zhu local model over $\operatorname{Spec}\mathcal{O}$ associated to the group $\mathrm{Res}_{K/\mathbf{Q}_p}\mathrm{GSp}_4$, Iwahori subgroup of $\mathrm{GSp}_4(K)$, and a regular Hodge type λ. Following the idea of [Le+a], we construct a closed subvariety $M(\lambda, \nabla_{\mathbf{a}_\tau}) \subset M(\leq \lambda)$ for a tame inertial type τ. The following Theorem generalizes the main result on local models in [Le+a] to GSp_4.

Theorem 2 (Theorem 4.4.3)**.** *Let $\mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym},K,\mathrm{reg}}$ (resp. $M_{\mathrm{reg}}(\leq \lambda, \nabla_{\mathbf{a}_\tau})$) be the union of $\mathcal{X}^{\lambda',\tau}_{\mathrm{Sym},K}$ (resp. $M(\lambda', \nabla_{\mathbf{a}_\tau})$) for all regular Hodge types $\lambda' \leq \lambda$. Suppose that λ is regular and τ is sufficiently generic (depending on λ). There exist Zariski open covers $\{\mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym},K,\mathrm{reg}}(\widetilde{z})\}_{\widetilde{z}}$ of $\mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym},K,\mathrm{reg}}$ and $\{U_{\mathrm{reg}}(\widetilde{z}, \leq \lambda, \nabla_{\mathbf{a}_\tau})\}_{\widetilde{z}}$ of $M_{\mathrm{reg}}(\leq \lambda, \nabla_{\mathbf{a}_\tau})$, and for each $\widetilde{z}$, a local model diagram*

$$\begin{tikzcd} & \widetilde{U}_{\mathrm{reg}}(\widetilde{z}, \leq \lambda, \nabla_{\mathbf{a}_\tau})^{\wedge_p} \arrow[dl] \arrow[dr] & \\ \mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym},K,\mathrm{reg}}(\widetilde{z}) & & U_{\mathrm{reg}}(\widetilde{z}, \leq \lambda, \nabla_{\mathbf{a}_\tau})^{\wedge_p}. \end{tikzcd}$$

Here, $\widetilde{U}_{\mathrm{reg}}(\widetilde{z}, \leq \lambda, \nabla_{\mathbf{a}_\tau}) := T^f \times U_{\mathrm{reg}}(\widetilde{z}, \leq \lambda, \nabla_{\mathbf{a}_\tau}) \to U_{\mathrm{reg}}(\widetilde{z}, \leq \lambda, \nabla_{\mathbf{a}_\tau})$ is the trivial T^f-torsor, and the left diagonal arrow is a T^f-torsor under a different T^f-action, where T is the diagonal maximal torus of

GSp_4, *and* $\wedge_p$ *denotes the* p*-adic completion.*

Note that if λ is equal to a cocharacter η lifting the half sum of positive coroots, then $\mathcal{X}^{\leq\eta,\tau}_{\mathrm{Sym},K,\mathrm{reg}} = \mathcal{X}^{\eta,\tau}_{\mathrm{Sym},K}$ and $M_{\mathrm{reg}}(\leq\eta,\nabla_{\mathbf{a}_\tau}) = M(\eta,\nabla_{\mathbf{a}_\tau})$.

Similar to symplectic étale (φ,Γ)-modules, we define *symplectic* Breuil–Kisin modules and denote by $Y^{\leq\lambda,\tau}_{\mathrm{Sym}}$ the moduli stack of symplectic Breuil–Kisin modules of height $\leq\lambda$ and inertial type τ. We prove that the stack $Y^{\leq\lambda,\tau}_{\mathrm{Sym}}$ is p-adic formal algebraic stack over $\mathrm{Spf}\,\mathcal{O}$ (Proposition 4.2.5) and locally smoothly equivalent to $M(\leq\lambda)^{\wedge_p}$ (Theorem 4.2.16). There exists a closed substack $Y^{\leq\lambda,\tau,\nabla_\infty}_{\mathrm{Sym}} \subset Y^{\leq\lambda,\tau}_{\mathrm{Sym}}$ which is isomorphic to $\mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym},K,\mathrm{reg}}$ (Proposition 4.4.2). We note that the locus $Y^{\leq\lambda,\tau,\nabla_\infty}_{\mathrm{Sym}}$ is *analytic*, i.e. it is given by formal power series, while in contrast, the subvariety $M(\lambda,\nabla_{\mathbf{a}_\tau}) \subset M(\leq\lambda)$ is algebraic. However, these conditions defining substack/subvariety are "congruent modulo power of p" (in a suitable sense). Then the left arrow in the above diagram is induced by applying Elkik's approximation theorem [Elk73]. In particular, it is highly non-canonical but canonical modulo p.

The local model $M(\lambda,\nabla_{\mathbf{a}_\tau})$ is a projective variety equipped with a natural T^f-action. Its T^f-fixed points are given by certain elements $\widetilde{z}$ in the extended affine Weyl group of $(\mathrm{GSp}_4)^f$. Under the above local model diagram, they correspond to tame $\overline{\rho} \in \mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym},K}(\mathbf{F})$. The main result on the geometry of local models is the following.

Theorem 3 (Theorem 3.4.1)**.** *Suppose that* λ *is regular and* τ *is sufficiently generic (depending on* λ*). Then* $M(\lambda,\nabla_{\mathbf{a}_\tau})$ *is unibranch at any* T^f*-fixed point* $\widetilde{z} \in M(\lambda,\nabla_{\mathbf{a}_\tau})(\mathbf{F})$.

Our proof of Theorem 3 follows [Le+a] closely. The novel idea in *loc. cit.* is comparing the (mixed characteristic) local model $M(\lambda,\nabla_{\mathbf{a}})$ (defined over $\mathrm{Spec}\,\mathcal{O}$) with an equal characteristic local model (defined over $\mathrm{Spec}\,\mathbf{F}[v]$) inside an universal local model (defined over $\mathrm{Spec}\,\mathbf{Z}[v]$). In general, the mixed and equal characteristic local models are *not* a base change of the universal local model. However, this is true under a genericity hypothesis, and they naturally have the same special fiber. We prove the unibranch property in the equal characteristic case and use it to prove the mixed characteristic case. Here, it is also crucial to compare the base change of the normalization of universal local models and normalizations of mixed and equal characteristic local models. This requires the base change of the normalization of universal local models to be normal, which also holds under a genericity hypothesis. Using these ideas, we first prove a preliminary unibranch property (Theorem 3.3.4). Then using this, we prove the main unibranch property for products of local models (Theorem 3.4.1).

Let $\overline{\rho} : G_K \to \mathrm{GSp}_4(\mathbf{F})$ be a continuous representation. We denote by $R^{\lambda,\tau}_{\overline{\rho}}$ the unique $\mathcal{O}$-flat quotient of a framed deformation ring $R^{\square}_{\overline{\rho}}$ of $\overline{\rho}$ whose $\overline{\mathbf{Q}}_p$-points parameterize potentially crystalline lifts of type (λ,τ). As a direct consequence of Theorem 2 and 3, we obtain the following result.

Corollary 4. *Suppose that* λ *is regular and* τ *is sufficiently generic (depending on* λ*). If* $\overline{\rho} : G_K \to \mathrm{GSp}_4(\mathbf{F})$ *is tame, then* $R^{\lambda,\tau}_{\overline{\rho}}$ *is a domain (if is nonzero).*

The importance of this Corollary is related to patching argument. Roughly speaking, patching argument constructs a module $M_\infty(\lambda-\eta,\tau)$ over (a certain modification of) $R^{\lambda,\tau}_{\overline{\rho}}$. Standard commutative algebra argument shows that the support of $M_\infty(\lambda-\eta,\tau)$ is a union of irreducible components in $\mathrm{Spec}\,R^{\lambda,\tau}_{\overline{\rho}}$. It is a folklore conjecture that the support is indeed equal to $\mathrm{Spec}\,R^{\lambda,\tau}_{\overline{\rho}}$, which almost immediately implies a modularity lifting result in a relevant setup. Moreover, it has applications to the Serre weight conjectures and the Breuil–Mézard conjecture (see [GHS18, §4.1] for its relationship to the Breuil–Mézard conjecture). Unfortunately, this is hard to prove in general. However, if $M_\infty(\lambda-\eta,\tau) \neq 0$ and $R^{\lambda,\tau}_{\overline{\rho}}$ is domain, then the support of $M_\infty(\lambda-\eta,\tau)$ has to be equal to $\mathrm{Spec}\,R^{\lambda,\tau}_{\overline{\rho}}$.

Remark 5. As in [Le+a], our main results on local models, as well as results in the remaining introduction, hold under suitable genericity conditions. In particular, genericity conditions imply that the Hodge type λ is small relative to p. We refer readers to §1.3 in *loc. cit.* for discussions on genericity conditions.

The geometric Breuil–Mézard conjecture

The original Breuil–Mézard conjecture [BM02] measures the complexity of the special fiber of potentially crystalline (or semistable) deformation rings in terms of the mod p representation theory of $\mathrm{GL}_n(k)$. It's geometric formulations are stated in [EG14] (for deformation rings) and [EG23] (for Emerton–Gee stacks) and proven in [Le+a] in the tamely potentially crystalline case under genericity assumptions. Also, Dotto formulated the Breuil–Mézard conjecture for central division algebras and proved that it follows from the conjecture for GL_n [Dot].

We formulate the geometric Breuil–Mézard conjecture for GSp_4 in the tamely potentially crystalline case and prove it under genericity assumptions. We first explain the conjecture for deformation rings. Let $R^{\square}_{\overline{\rho}}$ be a framed deformation ring of $\overline{\rho}$. Given a regular Hodge type λ and a mildly generic tame inertial type τ, there exists a $(4f+11)$-dimensional cycle $Z(R^{\lambda,\tau}_{\overline{\rho}}/\varpi)$ in $\operatorname{Spec} R^{\square}_{\overline{\rho}}/\varpi$ which counts the irreducible components in $\operatorname{Spec} R^{\lambda,\tau}_{\overline{\rho}}/\varpi$ with appropriate multiplicities. On the other hand, one can associate to the pair (λ,τ) a locally algebraic representation $\sigma(\tau)\otimes_{\mathcal{O}} V(\lambda-\eta)$ of $\mathrm{GSp}_4(\mathcal{O}_K)$. Here, $\sigma(\tau)$ is an irreducible representation of $\mathrm{GSp}_4(k)$ corresponding to τ under the inertial local Langlands correspondence, which we discuss below, and $V(\lambda-\eta)$ is the irreducible algebraic representation of $\mathrm{GSp}_4(\mathcal{O}_K)$ of highest weight $\lambda-\eta$.

Conjecture 6. *Let $\overline{\rho}: G_K \to \mathrm{GSp}_4(\mathbf{F})$ be a continuous representation. For each Serre weight σ, there exists a $(4f+11)$-dimensional cycle Z_σ in $\operatorname{Spec} R^{\square}_{\overline{\rho}}/\varpi$ such that for any regular Hodge type λ and any mildly generic tame inertial type τ,*

$$Z(R^{\lambda,\tau}_{\overline{\rho}}/\varpi) = \sum_{\sigma} [\overline{\sigma(\tau)\otimes V(\lambda-\eta)} : \sigma] Z_\sigma. \tag{1.0.1}$$

Similar to the case of deformation rings, one can construct a $4f$-dimensional cycle $\mathcal{Z}_{\lambda,\tau}$ in $\mathcal{X}_{\mathrm{Sym},K,\mathrm{red}}$ attached to $\mathcal{X}^{\lambda,\tau}_{\mathrm{Sym},K} \times_{\operatorname{Spec}\mathcal{O}} \operatorname{Spec}\mathbf{F}$.

Conjecture 7. *For each Serre weight σ, there exists a $4f$-dimensional cycle $\mathcal{Z}_\sigma$ in $\mathcal{X}_{\mathrm{Sym},K,\mathrm{red}}$ such that for any regular Hodge type λ and any mildly generic tame inertial type τ,*

$$\mathcal{Z}_{\lambda,\tau} = \sum_{\sigma} [\overline{\sigma(\tau)\otimes V(\lambda-\eta)} : \sigma] \mathcal{Z}_\sigma. \tag{1.0.2}$$

Remark 8. Both Conjecture 6 and 7 should extend to any inertial type τ. We only state them for mildly generic (more precisely, 1-generic in the sense of §2.1.7) tame inertial types because we prove the inertial local Langlands under this condition. Also, they are expected to generalize to potentially semistable deformation cases.

Our main result on the Breuil–Mézard conjecture is the following.

Theorem 9 (Corollary 6.1.17 and 6.1.18)**.** *Let Λ be a finite set of regular Hodge types λ.*

1. *For each Serre weight σ, there exists a $4f$-dimensional cycle $\mathcal{Z}_\sigma$ in $\mathcal{X}_{\mathrm{Sym},K,\mathrm{red}}$ such that (1.0.2) holds for any regular Hodge type $\lambda\in\Lambda$ and any sufficiently generic (depending on Λ) regular tame inertial type τ.*

2. *Let $\overline{\rho} : G_K \to \mathrm{GSp}_4(\mathbf{F})$ be sufficiently generic (depending on Λ). For each Serre weight σ, there exists a $(4f+11)$-dimensional cycle Z_σ in $\mathrm{Spec}\, R^{\square}_{\overline{\rho}}/\varpi$ such that (1.0.1) holds for any regular Hodge type $\lambda \in \Lambda$ and any regular tame inertial type τ.*

By using patching functors and Corollary 4, we first construct cycles Z_σ satisfying (1.0.1) for regular λ, sufficiently generic τ, and tame $\overline{\rho}$ (Theorem 6.1.3). Then we algebraically interpolate Z_σ for various $\overline{\rho}$ to construct $\mathcal{Z}_\sigma$ in item (1) following the axiomatic argument in [Le+a, §8.3]. Item (2) (where we do *not* assume that $\overline{\rho}$ is tame) essentially follows from item (1) by pulling back the cycles $\mathcal{Z}_\sigma$ to a versal ring for $\mathcal{X}_{\mathrm{Sym},K,\mathrm{red}}$ at $\overline{\rho}$.

The weight part of Serre's conjecture

Let F be a totally real field of even degree over $\mathbf{Q}$ in which p is unramified. Let $\mathcal{G}$ be an inner form of GSp_4 over F which is compact modulo center at infinity and splits at all finite places. Fix an isomorphism $\iota : \overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$. Given a Hecke character $\chi : \mathbb{A}_F^\times / F^\times \to \mathbf{C}^\times$, a level $U \subset \mathcal{G}(\mathbb{A}_F^\infty)$, and a $\mathcal{O}[\mathcal{G}(\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p)]$-module W, we define a space of algebraic automorphic forms $S_\chi(U, W)$ to be the $\mathcal{O}$-module of continuous functions $f : \mathcal{G}(F)\backslash \mathcal{G}(\mathbb{A}_F^\infty) \to W$ such that $f(zg) = (\iota^{-1} \circ \chi(z)) f(g)$ and $f(gu) = u_p^{-1} \cdot f(g)$ for all $z \in Z(\mathbb{A}_F^\infty), g \in \mathcal{G}(\mathbb{A}_F^\infty)$, and $u \in U$.

Let $\overline{r} : G_F \to \mathrm{GSp}_4(\mathbf{F})$ be a continuous and absolutely irreducible representation which is the mod p reduction of the Galois representation attached to a regular algebraic cuspidal automorphic representation of $\mathcal{G}(\mathbb{A}_F)$ (or equivalently, of $\mathrm{GSp}_4(\mathbb{A}_F)$). Then $\overline{r}$ determines a maximal ideal $\mathfrak{m}_{\overline{r}}$ of an appropriate Hecke algebra, and $S_\chi(U, \sigma)_{\mathfrak{m}_{\overline{r}}} \neq 0$ for some χ, U, and Serre weight σ. In this case, we say σ is a *modular* Serre weight for $\overline{r}$ and write $W(\overline{r})$ for the set of modular Serre weights for $\overline{r}$. When $\overline{r}$ is tame at places above p, [GHS18] defines a set $W^?(\otimes_{v|p} \overline{r}|_{I_{F_v}})$ (see Definition 2.4.7) using combinatorial recipes, where I_{F_v} denotes the inertia subgroup at v, and conjectures that $W(\overline{r}) = W^?(\otimes_{v|p} \overline{r}|_{I_{F_v}})$. We verify this conjecture under technical genericity assumptions.

Theorem 10 (Theorem 6.2.5)**.** *Let $\overline{r} : G_F \to \mathrm{GSp}_4(\mathbf{F})$ be as above. Moreover, we assume that $\overline{r}|_{I_{F_v}}$ is tame and sufficiently generic for $v|p$, and $\overline{r}$ satisfies* Taylor–Wiles conditions *(Definition 5.4.1). Then $W(\overline{r}) = W^?(\otimes_{v|p} \overline{r}|_{I_{F_v}})$.*

Previously, a similar conjecture was made by Herzig–Tilouine [HT13, Conjecture 1] when $F = \mathbf{Q}$ using étale cohomologies of Siegel modular varieties instead of algebraic automorphic forms on $\mathcal{G}$. We expect that our method can be used to prove the conjecture of Herzig–Tilouine (or its generalizations to totally real fields) if the conjecture on vanishing of mod p étale cohomologies of Hilbert–Siegel modular varieties localized at non-Eisenstein maximal ideal outside middle degree is known (c.f. [Car, Conjecture 4.3]).

Analogous conjectures for a rank n unitary group $U(n)$ over totally real field were proven under technical assumptions (when $U(n)$ splits at places above p by [GLS15] for $n = 2$, by [Le+18; Le+20] for $n = 3$, and by [Le+a] for general n; when $U(n)$ ramifies at places above p by [KM] for $n = 2$). For the group GSp_4, [GG12] proved modularity of *obvious weights* for $\overline{r}$ ordinary at places above p assuming modularity of a single obvious weight. Also, see [Yam, Theorem 1.3] for a result obtained by a different approach. Our result is independent of [GG12; Yam].

Congruent Kisin–Taylor–Wiles patching functors

The proofs of Theorem 9 and 10 use patching functors. Patching functors are first introduced in [EGS15] and have been a central object in the mod p and p-adic Langlands program (see [Car+16] and [EGH22]).

Our innovation in the global argument is introducing a *congruent family of fixed similitude patching functors* (Definition 5.2.4). It is a collection of fixed similitude patching functors, labelled by an appropriate set of similitude characters, that are congruent modulo p, i.e. they are identified when restricted to p-torsion objects. A single fixed similitude patching functor generalizes a (weak) patching functor in [Car+16; Le+a] to GSp_4 except the presence of the fixed similitude. The fixed similitude prevents important applications such as the comparison of the mod p reduction of patched modules over a crystalline deformation ring and a potentially crystalline deformation ring. However, a congruent family of fixed similitude patching functors makes such applications available through the congruence. We remark that our notion of *congruent* patching functors can describe Taylor's "Ihara avoidance" argument; see §5.4.8.

We construct patching functors for local and global applications. In the local case, it is a functor from a category of finite $\mathcal{O}$-modules with continuous $\mathrm{GSp}_4(\mathcal{O}_K)$-action with a fixed central character to a category of finitely generated modules over a certain deformation ring. In the global case, we replace the group $\mathrm{GSp}_4(\mathcal{O}_K)$ by $\mathrm{GSp}_4(\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p)$ where F is a totally real field. Although the construction of patching functors depends on non-canonical choices (e.g. Taylor–Wiles primes), these choices can be made independent of the fixed similitude character. Then the congruence property of fixed similitude patching functors in the global case follows naturally. We construct patching functors in the local case using the global ones, after realizing a given local Galois representation as a restriction of an automorphic global Galois representation with a minimality assumption, following [EG14, Appendix A] for GL_n. The congruent property is less obvious in the local case and requires the weight elimination result (Theorem 5.4.4) and certain potentially crystalline deformation rings with fixed similitude characters that are formally smooth and congruent modulo p (Theorem 5.1.4). In particular, our congruent patching functors improve a congruent *pair* of fixed similitude patching functors introduced in our previous work with John Enns [EL], where the patching functors in the local case was not available because the weight elimination result was not available.

We also incorporate the "Ihara avoidance" argument in our patching functors to prove the following modularity lifting result.

Theorem 11 (Theorem 6.3.1)**.** *Let $r, r' : G_F \to \mathrm{GSp}_4(\mathcal{O})$ be continuous representations that are isomorphic modulo ϖ, unramified at all but finitely many places, and potentially crystalline with regular Hodge type λ and sufficiently generic (depending on λ) tame inertial type τ at places above p. We further assume that $r \bmod \varpi$ is tame at places above p and satisfies Taylor–Wiles assumptions. If r' is automorphic, then r is also automorphic.*

The inertial local Langlands

For a mildly generic tame inertial type τ, we construct an irreducible representation $\sigma(\tau)$ of $\mathrm{GSp}_4(k)$ in characteristic zero such that if a smooth representation π of $\mathrm{GSp}_4(K)$ contains $\sigma(\tau)$ (viewed as a $\mathrm{GSp}_4(\mathcal{O}_K)$ via inflation) as a $\mathrm{GSp}_4(\mathcal{O}_K)$-subrepresentation, then τ is isomorphic to the restriction to inertia of the L-parameter of π (Theorem 2.4.1). This result is essential to our global arguments. It is proven by using the explicit theta correspondence in [GT11b] (in the non-supercuspidal case) and the depth-zero regular supercuspidal local Langlands correspondence in [DR09] (in the supercuspidal case).

Transfer from GSp_4 to GL_4

We define natural maps sending Deligne–Lusztig representations and Serre weights of $\mathrm{GSp}_4(k)$ to those of $\mathrm{GL}_4(k)$ (Proposition 2.4.10), which might be of independent interests. They are compatible with mod p

reduction under a very mild assumption. In the case of Deligne–Lusztig representations, it is also compatible with inertial local Langlands in a suitable sense. Moreover, this also shows that a Serre weight σ of $\mathrm{GSp}_4(k)$ is in the set $W^?(\overline{\rho}|_{I_K})$ of conjectural Serre weights associated with tame $\overline{\rho} : G_K \to \mathrm{GSp}_4(\mathbf{F})$ if and only if its transfer to $\mathrm{GL}_4(k)$ is in the set of conjectural Serre weights of $\mathrm{GL}_4(k)$ associated with $\overline{\rho}$ viewed as $\mathrm{GL}_4(\mathbf{F})$-valued representation (Corollary 2.4.12).

The work of Arthur

In this thesis, we use Arthur's multiplicity formula for the discrete spectrum of GSp_4 announced in [Art04] and later proven in [GT19] using the main results in [Art13]. We also use results from [Mok14; Box+21] which rely on Arthur's formula for GSp_4. At the time of writing, [Art13] is conditional on the twisted weighted fundamental lemma announced in [CL10], but whose proofs have not appeared, as well as on the references [A25], [A26], and [A27] in [Art13], which have not appeared publicly.

Specifically, we use [Mok14, Theorem 3.5] and [GT19, Theorem 7.4.1] for the construction of a Galois representation attached to a regular algebraic cuspidal automorphic representation of an inner form of GSp_4, and our construction of patching functors relies on it. Among theorems and corollaries stated above, proofs of Theorem 1 and 3 do not require patching functors, and all others rely on patching functors (and thus on [Art13]).

Organization

In §2, we first recall basic notions related to the representation theory of $\mathrm{GSp}_4(k)$. Then we prove the inertial local Langlands for GSp_4 in §2.4. In §2.5, we establish several results on mod p reduction of certain representations of $\mathrm{GSp}_4(k)$ which will be used throughout the thesis.

In §3, we introduce local models and prove its properties. Our presentation closely follows [Le+a]. When it is possible, we tried to simplify our argument by deducing results from analogous results for GL_4. After introducing the global affine Grassmannian and universal local models in §3.1–3.2, we introduce mixed characteristic local models and prove the unibranch property in §3.3–3.4. In §3.5–3.7, we discuss irreducible components in the special fiber of mixed characteristic local models. In §3.7, we classify torus fixed points in each irreducible components under genericity conditions.

We introduce moduli stacks of symplectic étale (φ, Γ)-modules in §4.1, of Breuil–Kisin modules in §4.2, and of étale φ-modules in §4.3. Then we bring them together to prove our main result on local model diagrams for potentially crystalline stacks (Theorem 4.4.3) and their for mod p reduction (Theorem 4.5.4). Both results rely on the existence of certain local lifts whose proofs are postponed to §5.4.

In §5, we discuss our setup for global arguments and construct a congruent family of fixed similitude patching functors. As applications, we prove the weight elimination result and the existence of certain local lifts in §5.4.

Finally, we formulate the geometric Breuil–Mézard conjecture and prove its restricted versions in §6.1. We prove our result on the weight part of a Serre's conjecture in §6.2 and a modularity lifting result in §6.3.

Notation

Let $p > 2$ be a prime. We let $K/\mathbf{Q}_p$ denote a finite field extension with the ring of integers $\mathcal{O}_K$ and the residue field k. Throughout this thesis except §4.1, we assume K to be unramified of degree f over $\mathbf{Q}_p$. We take E to be a finite extension of $\mathbf{Q}_p$ with ramification degree e, the ring of integers $\mathcal{O}$, and the residue field $\mathbf{F}$. We assume that E is sufficiently large unless mentioned otherwise. We let $[\cdot] : \mathbf{F}^\times \to W(\mathbf{F})^\times$ denote the multiplicative lift.

Let F be a field. We write $G_F := \mathrm{Gal}(\overline{F}/F)$ where $\overline{F}$ is a separable closure of F. If F is a non-archimedean local field, we write $I_F \subset G_F$ for the inertia subgroup and W_F for the Weil group.

Fix a separable closure $\overline{K}$ of K and suppose that $K/\mathbf{Q}_p$ is unramified of degree f. We choose $\pi \in \overline{K}$ such that $\pi^{p^f-1} = -p$. We denote by $\omega_K : G_K \to \mathcal{O}_K^\times$ the character defined by $g(\pi) = \omega_K(g)\pi$ for $g \in G_K$. For an embedding $\sigma : K \hookrightarrow E$, we write $\omega_{K,\sigma} = \sigma \circ \omega_K$.

Let ϵ denote the p-adic cyclotomic character. If V is a de Rham representation of G_K over E, then for each $\sigma \in \mathrm{Hom}_{\mathbf{Q}_p}(K, E)$, we let $\mathrm{HT}_\sigma(V)$ denote the multiset of Hodge–Tate weights labelled by σ normalized so that $\mathrm{HT}_\sigma(\epsilon) = \{1\}$.

Let F be a number field. For a place v of F, we let F_v be the completion of F at v with the ring of integers $\mathcal{O}_{F_v}$, a uniformizer ϖ_v, and the residue field k_v of size q_v. We write Frob_{F_v} for a geometric Frobenius element in G_{F_v}. We normalize the Artin map $\mathrm{Art}_{F_v} : F_v^\times \xrightarrow{\sim} W_{F_v}^{\mathrm{ab}}$ so that uniformizers are mapped to geometric Frobenius elements.

Let G be a split connected reductive group over $\mathbf{Z}$. In the body of the thesis, we take either $G = \mathrm{GSp}_4$ or $G = \mathrm{GL}_n$. We write $B \subset G$ for a choice of Borel subgroup, $T \subset B$ for a maximal torus, and $U \subset B$ for the unipotent radical of B. Let $\Phi^+ \subset \Phi$ be the subset of positive roots in the set of roots for (G, B, T). We denote by Δ the set of simple roots. We write $X^*(T)$ for the group of characters of T and $X_*(T)$ for the group of cocharacters of T. We write $\Lambda_R \subset X^*(T)$ and $\Lambda_R^\vee \subset X_*(T)$ for the root lattice and coroot lattice. We let W denote the Weyl group, W_a denote the affine Weyl group, and $\widetilde{W}$ denote the extended affine Weyl group for G.

We write $G^\vee$ for the split reductive group over $\mathbf{Z}$ defined by the root datum $(X_*(T), X^*(T), \Phi^\vee, \Phi)$. We write $T^\vee \subset G^\vee$ for the induced maximal split torus. We have isomorphisms $X^*(T^\vee) \simeq X_*(T)$ and $X_*(T^\vee) \simeq X^*(T)$. We let $W^\vee$, $W_a^\vee$, and $\widetilde{W}^\vee$ denote the Weyl group, affine Weyl group, and extended affine Weyl group for $G^\vee$.

We denote by $\mathcal{O}_p$ a finite étale $\mathbf{Z}_p$-algebra. In the body of this thesis, we take $\mathcal{O}_p$ to be either $\mathcal{O}_K$ or $\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p$ for some totally real field F in which p is unramified. Let $F_p = \mathcal{O}_p[1/p]$. Then F_p is isomorphic to a product $\prod_{v \in S_p} F_v$ for a finite set S_p and finite unramified extensions $F_v/\mathbf{Q}_p$. Also there is an isomorphism $\mathcal{O}_p \simeq \prod_{v \in S_p} \mathcal{O}_{F_v}$ where $\mathcal{O}_{F_v}$ is the ring of integer of F_v.

We define $G_0 := \mathrm{Res}_{\mathcal{O}_p/\mathbf{Z}_p} G_{/\mathcal{O}_p}$ and $\underline{G} := (G_0)_{/\mathcal{O}}$. Let B be a choice of Borel subgroup and $T \subset B$ be a maximal split torus. We define $B_0, \underline{B}$ and $T_0, \underline{T}$ similarly to $G_0, \underline{G}$. Let $\mathcal{J} = \mathrm{Hom}_{\mathbf{Z}_p}(\mathcal{O}_p, \mathcal{O})$. Then $(\underline{G}, \underline{B}, \underline{T})$ is naturally identified with $(G^{\mathcal{J}}_{/\mathcal{O}}, B^{\mathcal{J}}_{/\mathcal{O}}, T^{\mathcal{J}}_{/\mathcal{O}})$. The root datum of $(\underline{G}, \underline{B}, \underline{T})$ is given by

$$(X^*(\underline{T}), X_*(\underline{T}), \underline{\Phi}, \underline{\Phi}^\vee) \simeq (X^*(T)^{\mathcal{J}}, X_*(T)^{\mathcal{J}}, \Phi^{\mathcal{J}}, \Phi^{\vee,\mathcal{J}}).$$

We have $\underline{\Lambda}_R \simeq \Lambda_R^{\mathcal{J}}$, $\underline{W} \simeq W^{\mathcal{J}}$, $\underline{W}_a \simeq W_a^{\mathcal{J}}$, $\underline{\widetilde{W}} \simeq \widetilde{W}^{\mathcal{J}}$, and similarly for $\underline{\Lambda}_R^\vee$, $\underline{W}^\vee$, $\underline{W}_a^\vee$, $\underline{\widetilde{W}}^\vee$.

Let φ be the absolute Frobenius on $\mathcal{O}_p/p$ and its lift to $\mathcal{O}_p$. If S is a set and $s = (s_\sigma)_{\sigma \in \mathcal{J}} \in S^{\mathcal{J}}$, then we define $\pi(s)$ by $\pi(s)_\sigma = s_{\sigma \circ \varphi^{-1}}$. When $\mathcal{O}_p = \mathcal{O}_K$, we fix an embedding $\sigma_0 : K \hookrightarrow E$ and define $\sigma_j := \sigma_0 \circ \varphi^{-j}$ for $j \in \mathbf{Z}/f\mathbf{Z}$. This identifies $\mathcal{J}$ with $\mathbf{Z}/f\mathbf{Z}$.

We write $\mathrm{Diag}(a_1, \ldots, a_n)$ for the diagonal matrix in GL_n with entries $a_1, \ldots, a_n$. If μ is a cocharacter and a is a scalar, we write a^μ to denote $\mu(a)$. We write Ind for the unnormalized parabolic induction and ind for the compact induction.

Chapter 2

Types and weights

In this chapter, we recall basic notions regarding the representation theory and the extended Weyl group of GSp_4 and prove the inertial local Langlands correspondence for GSp_4 (Theorem 2.4.1). We follow [Le+a, §2] closely throughout this chapter except §2.4.

2.1 Preliminaries

2.1.1 The group GSp_4

Let $G = \mathrm{GSp}_4$ be the split reductive group over **Z** defined by

$$\mathrm{GSp}_4(R) = \left\{ g \in \mathrm{GL}_4(R) \mid {}^t g J g = \mathrm{sim}(g) J \text{ for some } \mathrm{sim}(g) \in R^\times \right\}$$

for any commutative ring R, where

$$J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

Then $\mathrm{sim} : g \mapsto \mathrm{sim}(g)$ defines a character of GSp_4, called the *similitude character*. Let $W = N(T)/T$ be the Weyl group of G. We identify W with the subgroup of $N(T)$ generated by two simple reflections

$$s_1 := \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}, \; s_2 := \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix}.$$

We write $w_0 = s_1 s_2 s_1 s_2$ for the longest element in W. We use the same notation to denote $\mathcal{J}$-fold product of w_0 in $\underline{W}$.

Let B be the upper triangular Borel subgroup of GSp_4. For any subset $A \subset \{s_1, s_2\}$, we have a standard parabolic subgroup P_A generated by A and B. When $A = \{s_1\}$, we call $S := P_A$ the *Siegel parabolic subgroup*, and when $A = \{s_2\}$, we call $Q := P_A$ the *Klingen parabolic subgroup*. If P is a standard parabolic subgroup with unipotent radical N, we write $\overline{P}$ and $\overline{N}$ to denote the opposite parabolic and its unipotent radical.

We identify the character group $X^*(T)$ with $\mathbf{Z}^3$ by defining the character corresponding to $(a, b; c) \in \mathbf{Z}^3$ by

$$(a, b; c) : \mathrm{Diag}(x, y, zy^{-1}, zx^{-1}) \mapsto x^a y^b z^c.$$

Similarly, we identify the cocharacter group $X_*(T)$ with $\mathbf{Z}^3$ by

$$(a, b; c) : x \mapsto \mathrm{Diag}(x^a, x^b, x^{c-b}, x^{c-a}).$$

Sets of simple roots and coroots are given by

$$\begin{aligned}\Delta &= \{\alpha_1 = (1, -1; 0), \alpha_2 = (0, 2; -1))\} \\ \Delta^\vee &= \{\alpha_1^\vee = (1, -1; 0), \alpha_2^\vee = (0, 1; 0)\}\end{aligned}$$

Sets of positive roots and coroots are given by

$$\begin{aligned}\Phi^+ &= \{\alpha_1, \alpha_2, \alpha_3 = 2\alpha_1 + \alpha_2, \alpha_4 = \alpha_1 + \alpha_2\} \\ \Phi^{\vee,+} &= \{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee, \alpha_4^\vee\}\end{aligned}$$

where $\alpha_3 = \alpha_1 + \alpha_2$, $\alpha_4 = 2\alpha_1 + \alpha_2$, $\alpha_3^\vee = \alpha_1^\vee + 2\alpha_2^\vee$, and $\alpha_4^\vee = \alpha_1^\vee + \alpha_2^\vee$.

The dual group of GSp_4 is isomorphic to GSp_4 by an exceptional isomorphism. We often write $\mathrm{GSp}_4^\vee$ (and $T^\vee$ for its diagonal maximal torus) to emphasize that we are working on the dual side. We fix the duality isomorphism by

$$\begin{aligned}(X^*(T), X_*(T), \Phi, \Phi^\vee) &\xrightarrow{\sim} (X_*(T^\vee), X^*(T^\vee), \Phi^\vee, \Phi) \\ \phi : X^*(T) &\to X_*(T^\vee) \\ (a, b; c) &\mapsto (a + b + c, a + c; a + b + 2c).\end{aligned}$$

Then ϕ maps α_1 to $\alpha_2^\vee$ and α_2 to $\alpha_1^\vee$. We also write a map $\phi : W \xrightarrow{\sim} W^\vee$ sending $s_1 \mapsto s_2$, $s_2 \mapsto s_1$, so that

$$\phi(w(\lambda)) = \phi(w)(\phi(\lambda))$$

for any $w \in W$ and $\lambda \in X^*(T)$. We fix an element $\eta = (2, 1; 0) \in X^*(T)$. We often use the same notation to denote the $\mathcal{J}$-fold product of η in $X^*(\underline{T})$ and its image in $X_*(\underline{T}^\vee)$ under ϕ.

2.1.2 The group GL_n

If $G = \mathrm{GL}_n$, we add subscript n to the objects introduced above when we need to distinguish them from the case $G = \mathrm{GSp}_4$. For example, we write $T = T_n, B = B_n, U = U_n, W = W_n$. We identify $X^*(T_n) \simeq X_*(T_n) \simeq \mathbf{Z}^n$ in the standard way. We fix an element $\eta' = (n-1, n-2, \ldots, 0) \in X^*(T_n)$ and use the same notation for its $\mathcal{J}$-fold product in $X^*(\underline{T}_n)$.

2.1.3 Affine Weyl group

Let G be a split reductive group over $\mathbf{Z}$. Recall that the affine Weyl group $W_a \simeq \Lambda_R \rtimes W$ and the extended Weyl group $\widetilde{W} \simeq X^*(T) \rtimes W$ for G. Similarly, we write $W_a^\vee \simeq \Lambda_R^\vee \rtimes W^\vee$ and $\widetilde{W}^\vee \simeq X_*(T^\vee) \rtimes W^\vee$ for the affine Weyl group and extended Weyl group for $G^\vee$.

Let $\mathcal{A}$ be the set of alcoves of $X^*(T) \otimes_{\mathbf{Z}} \mathbf{R}$. We denote by A_0 the dominant base alcove with respect to our choice of the Borel subgroup B. The group W_a acts simply transitively on $\mathcal{A}$, and the choice of A_0 defines a Bruhat order on W_a which we denote by $\leq$. We also recall the upper arrow ordering on the set $\mathcal{A}$ (see, [Jan03, §II.6.5]). This induces an ordering on W_a which we denote by $\uparrow$.

Let $\Omega \leq \widetilde{W}$ be the subgroup stabilizing A_0. Then we have $\widetilde{W} \simeq W_a \rtimes \Omega$. We extend a Bruhat order and upper arrow order to $\widetilde{W}$ by: for $\widetilde{w}_1, \widetilde{w}_2 \in W_a$ and $\delta_1, \delta_2 \in \Omega$, when $\delta_1 = \delta_2$, $\widetilde{w}_1\delta_1 \leq \widetilde{w}_2\delta_2$ (resp. $\widetilde{w}_1\delta_1 \uparrow \widetilde{w}_2\delta_2$) if and only if $\widetilde{w}_1 \leq \widetilde{w}_2$ (resp. $\widetilde{w}_1 \uparrow \widetilde{w}_2$), and when $\delta_1 \neq \delta_2$, $\widetilde{w}_1\delta_1$, $\widetilde{w}_2\delta_2$ are incomparable.

We define a Bruhat order on $W_a^\vee$ induced by the antidominant base alcove and extend to $\widetilde{W}^\vee$ as in the previous paragraph. We define a bijection

$$\begin{aligned} (-)^* : \widetilde{W} &\to \widetilde{W}^\vee \\ \widetilde{w} = t_\nu w &\mapsto \widetilde{w}^* := \phi(w)^{-1} t_{\phi(\nu)}. \end{aligned}$$

This is an antihomomorphism of groups and preserves Bruhat ordering on both sides. We also write $(-)^*$ for its inverse, i.e. if $\widetilde{w} \in \widetilde{W}$ and $\widetilde{z} = \widetilde{w}^*$, then $\widetilde{z}^* = \widetilde{w}$.

Definition 2.1.4. We define the *admissible set* associated to $\lambda_0 \in X^*(T)$ as

$$\mathrm{Adm}(\lambda_0) := \left\{ \widetilde{w} \in \widetilde{W} \mid \widetilde{w} \leq t_{w\lambda_0} \text{ for some } w \in W \right\}.$$

If $\lambda \in X^*(\underline{T})$, we define $\mathrm{Adm}(\lambda) = \prod_{j \in \mathcal{J}} \mathrm{Adm}(\lambda_j)$

We similarly define $\mathrm{Adm}^\vee(\lambda)$ for $\lambda \in X_*(\underline{T}^\vee)$. The map $(-)^*$ induces a bijection between $\mathrm{Adm}(\phi^{-1}(\lambda))$ and $\mathrm{Adm}^\vee(\lambda)$.

The *p-dot action* of $t_\nu w \in \widetilde{W}$ on $\lambda \in X^*(T) \otimes \mathbf{R}$ is defined as

$$t_\nu w \cdot \lambda := w(\lambda + \eta) - \eta + p\nu$$

A *p-alcove* is a connected component of the complement

$$(X^*(T) \otimes \mathbf{R}) \setminus \cup_{\alpha \in \Phi, m \in \mathbf{Z}} \{x \mid \langle x + \eta, \alpha^\vee \rangle = pm\}.$$

We let C_0 denote the dominant base p-alcove, i.e.

$$C_0 := \left\{ x \in X^*(T) \otimes \mathbf{R} \mid 0 < \langle x + \eta, \alpha^\vee \rangle < p, \forall \alpha \in \Phi^+ \right\}.$$

Similarly, we define p-alcoves in $X^*(\underline{T}) \otimes \mathbf{R}$ and denote the dominant base p-alcove by $\underline{C}_0$.

We say that an alcove A (resp. a p-alcove C) is *restricted (resp. p-restricted)* if for all $x \in A$ (resp. $x \in C$)

and $\alpha \in \Phi^+$, $0 < \langle x, \alpha^\vee \rangle < 1$ (resp. $0 < \langle x+\eta, \alpha^\vee \rangle < p$). We define

$$\begin{aligned}\widetilde{W}^+ &:= \{\widetilde{w} \in \widetilde{W} \mid \widetilde{w}(A_0) \text{ is dominant}\}\\ \widetilde{W}_1^+ &:= \{\widetilde{w} \in \widetilde{W} \mid \widetilde{w}(A_0) \text{ is restricted}\}\end{aligned}$$

and $\underline{\widetilde{W}}^+ := \widetilde{W}^{+,\mathcal{J}}$ and $\underline{\widetilde{W}}_1^+ := \widetilde{W}_1^{+,\mathcal{J}}$.

We use above notations for $G = \mathrm{GSp}_4$. For $G = \mathrm{GL}_n$, we again add subscript n when we need to distinguish them from the case of GSp_4. For example, we write $\widetilde{W}_n$, $\widetilde{W}_n^+$, and $\widetilde{W}_{1,n}^+$ for $\widetilde{W}$, $\widetilde{W}^+$, and $\widetilde{W}_1^+$ respectively.

2.1.5 Transfer map

We write $\mathrm{std} : \mathrm{GSp}_4^\vee \to \mathrm{GL}_4^\vee$ for the standard representation. It induces a map between the tori $\underline{T}^\vee \to \underline{T}_4^\vee$. We also write the induced map between cocharacter groups by $\mathrm{std} : X_*(\underline{T}^\vee) \to X_*(\underline{T}_4^\vee)$. Explicitly, we have

$$(a, b; c) \overset{\mathrm{std}}{\longmapsto} (a, b, c-b, c-a).$$

We also define $\mathcal{T} : X^*(\underline{T}) \to X^*(\underline{T}_4)$ to be a unique map which makes the following diagram commute

$$\begin{array}{ccc} X^*(\underline{T}) & \xrightarrow{\mathcal{T}} & X^*(\underline{T}_4) \\ \big\downarrow{\scriptstyle\phi} & & \big\| \\ X_*(\underline{T}^\vee) & \xrightarrow{\mathrm{std}} & X_*(\underline{T}_4^\vee) \end{array}$$

Explicitly, $\mathcal{T}$ maps $(a, b; c) \in X^*(\underline{T}) \mapsto (a+b+c, a+c, b+c, c) \in X^*(\underline{T}_4)$.

The map std also induces a map between $\underline{W}^\vee$ and $\underline{W}_4^\vee$. We again denote by $\mathcal{T} : \underline{W} \to \underline{W}_4$ the composition $\mathrm{std} \circ \phi$ followed by the identification $\underline{W}_4^\vee = \underline{W}_4$. We have $\mathcal{T}(w\lambda) = \mathcal{T}(w)\mathcal{T}(\lambda)$ for all $\lambda \in X^*(\underline{T})$, $w \in \underline{W}$. As a result, $\mathcal{T}$ extends to a map between $\widetilde{W}$ and $\widetilde{W}_4$. Note that $\mathcal{T}$ is equal to the composition

$$\widetilde{W} \xrightarrow{(-)^*} \widetilde{W}^\vee \xrightarrow{\mathrm{std}} \widetilde{W}_4^\vee \xrightarrow{(-)^*} \widetilde{W}_4.$$

The image of $\mathcal{T}$ in $\underline{\widetilde{W}}_4$ can be interpreted as the invariant of a certain involution. Let $w_0' \in \underline{W}_4$ be the longest element. Let $\underline{\widetilde{W}}_{4,a,\mathbf{Q}} = (X^*(T_4) \otimes_{\mathbf{Z}} \mathbf{Q}) \rtimes W_4$. Define $\Theta : \underline{\widetilde{W}}_{4,a,\mathbf{Q}} \to \underline{\widetilde{W}}_{4,a,\mathbf{Q}}$ to be the unique group homomorphism satisfying

$$\Theta(s) = w_0' s w_0', \quad \Theta(a, b, c, d) = \frac{a+b+c+d}{2}(1,1,1,1) - w_0'(a, b, c, d)$$

for $s \in \underline{W}_4$ and $(a, b, c, d) \in X^*(\underline{T}_4)$. Then $\widetilde{w} \in \underline{\widetilde{W}}_4$ is in the image of $\mathcal{T}$ if and only if $\Theta(\widetilde{w}) = \widetilde{w}$.

Lemma 2.1.6. *The map $\mathcal{T} : \underline{\widetilde{W}} \to \underline{\widetilde{W}}_4$ respects Bruhat ordering, i.e. for all $\widetilde{w}_1, \widetilde{w}_2 \in \underline{\widetilde{W}}$,*

$$\widetilde{w}_1 \leq \widetilde{w}_2 \text{ in } \underline{\widetilde{W}} \text{ if and only if } \mathcal{T}(\widetilde{w}_1) \leq \mathcal{T}(\widetilde{w}_2) \text{ in } \underline{\widetilde{W}}_4.$$

Moreover, for $\lambda \in X^*(\underline{T})$, we have $\mathrm{Adm}(\lambda) \overset{\mathcal{T}}{\simeq} \mathrm{Adm}(\mathcal{T}(\lambda))^\Theta$.

Proof. The first claim is the result of Kottwitz–Rapoport [KR00, Proposition 2.3]. The second claim is the result of Haines–Châo [HC02, Proposition 5]. Note that in *loc. cit.*, the set $\mathrm{Perm}(\mu)$ is equal to $\mathrm{Adm}(\mu)$ (for the group GL_n) by Theorem 1 in *loc. cit.*. □

2.1.7 Genericity

For a character $\lambda \in X^*(T)$ (resp. a cocharacter $\lambda \in X_*(T^\vee)$), we define $h_\lambda = \max_{\alpha\in\Phi}\{\langle \lambda, \alpha^\vee\rangle\}$ (resp. $h_\lambda = \max_{\alpha\in\Phi}\{\langle \phi^{-1}(\lambda), \alpha^\vee\rangle\}$).

Definition 2.1.8 (cf. Definition 2.1.10 in [Le+a])**.** Let $\lambda \in X^*(T)$ be a character and $m \in Z_{\geq 0}$.

1. We say that λ is *m-deep in a p-alcove C* if
$$n_\alpha p + m < \langle \lambda + \eta, \alpha^\vee\rangle < (n_\alpha + 1)p - m$$
for all $\alpha \in \Phi^+$ where $C = \{\lambda \in X^*(T) \otimes \mathbf{R} \mid n_\alpha p < \langle \lambda + \eta, \alpha^\vee\rangle < (n_\alpha + 1)p, \forall \alpha \in \Phi^+\}$
2. We say λ is *m-deep* if it is m-deep in some p-alcove C.
3. For $\widetilde{w} = wt_\nu \in \widetilde{W}$, we say that $\widetilde{w}$ is *m-generic* if $\nu - \eta$ is m-deep.
4. For $\widetilde{w} = wt_\nu \in \widetilde{W}$, we say that $\widetilde{w}$ is m-small if $h_\nu \leq m$.
5. For $\widetilde{z} \in \widetilde{W}^\vee$, we say that $\widetilde{z}$ is *m-generic* (resp. *m-small*) if $\widetilde{z}^*$ is m-generic (resp. m-small).
6. For $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \in \mathbf{F}^3$, we say that $\mathbf{a}$ is m-generic if
$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_1 - \mathbf{a}_2\} \cap \{-m, -m+1, \dots, m-1, m\} = \emptyset$$
where $\{-m, -m+1, \dots, m-1, m\}$ is considered as a subset of $\mathbf{F}$ using $\mathbf{Z} \to \mathbf{F}$.
7. Let $P(X_1, X_2, X_3) \in \mathbf{Z}[X_1, X_2, X_3]$ be a polynomial and let R be a commutative ring. We say that $\mathbf{a} \in R^3$ (resp. $\mathbf{a} = (\mathbf{a}_j)_{j\in\mathcal{J}} \in (R^3)^{\mathcal{J}}$) is *$P$-generic* if $P(\mathbf{a}) \bmod p$ (resp. $P(\mathbf{a}_j) \bmod p$) is in $(R/p)^\times$ (resp. for all $j \in \mathcal{J}$).
8. We define $P_m(X_1, X_2, X_3) \in \mathbf{Z}[X_1, X_2, X_3]$ to be the polynomial
$$P_m(X_1, X_2, X_3) = \prod_{a=1}^{m}(X_1 - X_2 - a)(X_2 - a)(X_1 + X_2 + a).$$
Note that $\lambda - \eta \in C_0$ is m-deep if and only if λ (viewed as an element in $\mathbf{Z}^3$) is P_m-generic.

Lemma 2.1.9. *If $\lambda \in X^*(T)$ is m-deep, then $\mathcal{T}(\lambda) \in X^*(T_4)$ is m-deep in the sense of [Le+a, Definition 2.1.10]. Similarly, if $\widetilde{w} \in \widetilde{W}$ is m-generic (resp. m-small), then $\mathcal{T}(\widetilde{w}) \in \widetilde{W}_4$ is m-generic (resp. m-small).*

Proof. This follows from a direct computation. □

2.2 Serre weights and Deligne–Lusztig representations

Let G be either GSp_4 or GL_n. Recall that we have a finite étale $\mathbf{Z}_p$-algebra $\mathcal{O}_p$ and $G_0 = \mathrm{Res}_{\mathcal{O}_p/\mathbf{Z}_p} G_{/\mathcal{O}_p}$. We remark that $G_0(\mathbf{F}_p) = G(k)$ when $\mathcal{O}_p = \mathcal{O}_K$, and $G_0(\mathbf{F}_p) = G(\mathcal{O}_F/p)$ when $\mathcal{O}_p = \mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p$ with F a totally real field in which p is unramified.

2.2.1 Serre weights

A *Serre weight* of $G_0(\mathbf{F}_p)$ is an irreducible $\mathbf{F}$-representation of $G_0(\mathbf{F}_p)$. Recall the set of p-restricted dominant weights

$$X_1^*(\underline{T}) = \left\{\lambda \in X^*(\underline{T}) \mid \forall\alpha^\vee \in \underline{\Delta}^\vee, 0 \le \langle\lambda, \alpha^\vee\rangle \le p-1\right\}.$$

For $\lambda \in X_1^*(\underline{T})$, we write $L(\lambda)$ for the unique up to isomorphism irreducible $\underline{G}_{/\mathbf{F}}$-representation of highest weight λ. Then $F(\lambda) := L(\lambda)|_{\underline{G}(\mathbf{F}_p)}$ is a Serre weight. Moreover, we have the following bijection ([GHS18, Lemma 9.2.4])

$$\frac{X_1^*(\underline{T})}{(p-\pi)X^0(T)} \xrightarrow{\sim} \{\text{Serre weights of } G_0(\mathbf{F}_p)\} / \simeq$$
$$\lambda \mapsto F(\lambda)$$

where $X^0(\underline{T}) := \{\lambda \in X^*(\underline{T}) \mid \forall\alpha \in \underline{\Phi}, \langle\lambda, \alpha^\vee\rangle = 0\}$. For an integer $m \ge 0$, we say a Serre weight σ is m-*deep* if $\sigma \simeq F(\lambda)$ for some m-deep λ.

Let $X_{\mathrm{reg}}(\underline{T}) \subset X_1^*(\underline{T})$ be the subset of λ such that $0 \le \langle\lambda, \alpha^\vee\rangle < p-1$ for all $\alpha^\vee \in \underline{\Delta}^\vee$. We say a Serre weight σ is *regular* if $\sigma \simeq F(\lambda)$ for some $\lambda \in X_{\mathrm{reg}}(\underline{T})$.

Let $\widetilde{w}_h := w_0 t_{-\eta}$. We denote by $\mathcal{R}$ an endomorphism on $X_{\mathrm{reg}}(\underline{T})$ given by $\lambda \mapsto \widetilde{w}_h \cdot \lambda$. It preserves $X^0(T)$ and induces a map on the set of regular Serre weights by $\mathcal{R}(F(\lambda)) := F(\mathcal{R}(\lambda))$.

Definition 2.2.2. Let $\omega - \eta \in \underline{C}_0 \cap X^*(\underline{T})$ and $\widetilde{w}_1 \in \underline{\widetilde{W}}_1^+$. We define

$$F_{(\widetilde{w}_1,\omega)} := F(\pi^{-1}(\widetilde{w}_1) \cdot (\omega - \eta)).$$

Consider the equivalence relation $(\widetilde{w}_1, \omega) \sim (t_\nu \widetilde{w}_1, \omega - \nu)$ for all $\nu \in X^0(\underline{T})$. The map $(\widetilde{w}_1, \omega) \mapsto F_{(\widetilde{w}_1,\omega)}$ sends equivalent pairs to the same Serre weights. We call the equivalence class of $(\widetilde{w}_1, \omega)$ as a *lowest alcove presentation* of $F_{(\widetilde{w}_1,\omega)}$. Let $m \in \mathbf{Z}_{\ge 0}$. If $\omega - \eta$ is m-deep in $\underline{C}_0$, we say that $(\widetilde{w}_1, \omega)$ is m-*generic* lowest alcove presentation of $F_{(\widetilde{w}_1,\omega)}$.

2.2.3 Deligne–Lusztig representations

Let $(s, \mu) \in \underline{W} \times X^*(\underline{T})$. By [GHS18, Proposition 9.2.1 and 9.2.2], we can attach a Deligne–Lusztig representation $R_s(\mu)$ of $G_0(\mathbf{F}_p)$. By [DL76, Proposition 10.10], it is a genuine representation of $G_0(\mathbf{F}_p)$ if (s, μ) is a *good pair* (see [LLL19, §2.2]).

Definition 2.2.4. Let R be a Deligne–Luszitg representation and $m \in \mathbf{Z}_{\ge 0}$.

1. We say that $(s, \mu - \eta)$ is a m-*generic lowest alcove presentation* of R if $R \simeq R_s(\mu)$ and $\mu - \eta$ is m-deep in $\underline{C}_0$.

2. We say that R is m-*generic* if there exists a m-generic lowest alcove presentation of R.

Remark 2.2.5. If $\mu - \eta$ is 0-deep in $\underline{C}_0$, then (s, μ) is a good pair. Moreover if $\mu - \eta$ is 1-deep in $\underline{C}_0$, then $R_s(\mu)$ is irreducible by [DL76, Theorme 6.8].

The following Proposition gives the Jordan–Hölder factors of the reduction of Deligne–Lusztig representations in generic cases.

Proposition 2.2.6 (Proposition 2.3.6 in [Le+a]). *Let $(s, \mu - \eta)$ be a lowest alcove presentation of $R_s(\mu)$ that is 6-generic if $G = \mathrm{GSp}_4$ and $2(n-1)$-generic if $G = \mathrm{GL}_n$. For $\lambda \in X_1^*(\underline{T})$, $F(\lambda) \in \mathrm{JH}(\overline{R_s(\mu)})$ if and only if there exists $\widetilde{w} = wt_{-\nu} \in \underline{\widetilde{W}}$ such that*

$$\widetilde{w} \cdot (\mu - \eta + s\pi(\nu))) \uparrow \widetilde{w}_h \cdot \lambda$$

and $\widetilde{w} \cdot \underline{C}_0 \uparrow \widetilde{w}_h \cdot \underline{C}_0$.

Definition 2.2.7. Let $w \in \underline{W}$. Let $\nu \in X^*(\underline{T})$ be an element up-to-$X^0(\underline{T})$ uniquely determined by the condition $wt_{-\nu} \in \underline{\widetilde{W}}_1$. We define $\widehat{w} := wt_{-\nu} \mod X^0(\underline{T})$.

By abuse of notation, we also let $\widehat{w}$ denote a representative in its class.

Definition 2.2.8. Let $R \simeq R_s(\mu)$ be a Deligne–Lusztig representation that is 6-generic if $G = \mathrm{GSp}_4$ and $2(n-1)$-generic if $G = \mathrm{GL}_n$. By Proposition 2.2.6, we can define a function

$$\begin{aligned} F_R : \underline{W} &\to \mathrm{JH}(\overline{R_s(\mu)}) \\ w &\mapsto F(\widetilde{w}_h^{-1}\pi^{-1}(\widehat{w}) \cdot (\mu - \eta + s(\widehat{w}^{-1}(0)))). \end{aligned}$$

Note that this does not depend on the choice of $\widehat{w}$. The map F_R depends on the choice of the lowest alcove presentation, which will be clear in the context. So we suppress the dependence on it in the notation.

Serre weights in the image of F_R are called *outer weights* in [Le+20].

2.3 Tame inertial L-parameters

Recall that we take $\mathcal{O}_p$ to be a finite étale $\mathbf{Z}_p$-algebra. It is isomorphism to $\prod_{v \in S_p} \mathcal{O}_v$ where S_p is a finite set and $\mathcal{O}_v$ is the ring of intergers in some finite unramified extension $F_v/\mathbf{Q}_p$. Following [Le+a, §1.8.2], we have the following definitions.

Definition 2.3.1. Let A be a topolocigal $\mathcal{O}$-algebra.

1. An *L-homomorphism over A* is a continuous homomorphism $W_{\mathbf{Q}_p} \to {}^L\underline{G}(A)$. An *$L$-parameter over A* is a $\underline{G}^\vee(A)$-conjugacy class of L-homomorphisms. When A is finite, any L-homomorphism extends to $G_{\mathbf{Q}_p}$.

2. An *inertial L-homomorphism over A* is a continuous homomorphism $I_{\mathbf{Q}_p} \to \underline{G}^\vee(A)$ which has open kernel and extends to an L-homomorphism over A. An *inertial L-parameter over A* is a $\underline{G}^\vee(A)$-conjugacy class of inertial L-homomorphism.

3. An *inertial type for K over A* is a $G^\vee(A)$-conjugacy class of homomorphisms $I_K \to G^\vee(A)$ which has open kernel and extends to homomorphisms $W_K \to G^\vee(A)$.

Note that a choice of an isomorphism $\overline{F}_v \simeq \overline{\mathbf{Q}}_p$ for each $v \in S_p$ induces an embedding $G_{F_v} \hookrightarrow G_{\mathbf{Q}_p}$. This induces a bijection between L-homomorphisms over A and collections of continuous homomorphisms $W_{F_v} \to G^\vee(A)$ indexed by S_p. Once we take $\underline{G}^\vee(A)$-conjugacy classes and $G^\vee(A)$-conjugacy classes respectively, the bijection is independent of the choice of isomorphisms. The same bijection holds for inertial L-parameters and collections of inertial types for F_v indexed by S_p. In particular, when $\mathcal{O}_p = \mathcal{O}_K$, the inertial L-parameter over A is equivalent to an inertial type for K over A. When A is finite, one can replace Weil groups in the definition by Galois groups.

Let $(s, \mu) \in \underline{W} \times X^*(\underline{T})$ and $s' = \mathcal{T}(s)$, $\mu' = \mathcal{T}(\mu)$. Then [LLL19, Definition 2.2.1] defines a tame inertial L-parameter

$$\tau(s', \mu' + \eta') : I_{\mathbf{Q}_p} \to \underline{T}_4^\vee(\mathcal{O}).$$

By our choice of (s', μ'), this factors through $\underline{T}^\vee(\mathcal{O}) \subset \underline{T}_4^\vee(\mathcal{O})$, and we let $\tau(s, \mu+\eta)$ denote the induced tame inertial L-parameter valued in $\underline{T}^\vee(\mathcal{O})$.

Let F^* denote the endomorphism $p\pi^{-1}$ on $X_*(\underline{T}^\vee)$, and $d \geq 1$ be an integer such that $(F^* \circ s^{-1})^d = p^d$. By [Le+a, §2.4], we have the following explicit description:

$$\tau(s, \mu+\eta) := \left(\sum_{i=0}^{d-1} (F^* \circ \phi(s)^{-1})^i (\phi(\mu+\eta))\right)(\omega_d) : I_{\mathbf{Q}_p} \to \underline{T}^\vee(\mathcal{O}).$$

When $\mathcal{O}_p = \mathcal{O}_K$, we also write $\tau(s, \mu+\eta)$ to denote the corresponding tame inertial type for K. By following Example 2.4.1 of *loc. cit.*, we write $s_\tau := s_0 s_1 \dots s_{f-1} \in W$, $r := |s_\tau|$, and $\mathbf{a}^{(0)} := (\sum_{j=0}^{f-1} (F^* \circ s^{-1})^j (\mu+\eta))_0 \in X^*(T)$. Then we have

$$\tau(s, \mu+\eta) = \left(\sum_{k=0}^{r-1} p^{fk} \phi(s_\tau)^{-k} \left(\phi\left(\mathbf{a}^0\right)\right)\right)(\omega_{fr}) : I_K \to T^\vee(\mathcal{O}).$$

Note that base change $- \otimes_{\mathcal{O}} E$ and $- \otimes_{\mathcal{O}} \mathbf{F}$ induce bijections between tame inertial L-parameters over $\mathcal{O}$, E, and $\mathbf{F}$. For a tame inertial L-parameter τ over $\mathcal{O}$ or E, we write $\overline{\tau}$ for the corresponding tame inertial L-parameter over $\mathbf{F}$.

Definition 2.3.2. Let τ (resp. $\overline{\tau}$) be a tame inertial L-parameter over E (resp. $\mathbf{F}$). Let $m \in \mathbf{Z}_{\geq 0}$.

1. A pair $(s, \mu) \in \underline{W} \times X^*(\underline{T})$ is a *m-generic lowest alcove presentation* of τ (resp $\overline{\tau}$) if μ is m-deep in $\underline{C}_0$ and $\tau \simeq \tau(s, \mu+\eta)$ (resp $\overline{\tau} \simeq \overline{\tau}(s, \mu+\eta)$).
2. We say τ (resp. $\overline{\tau}$) is *m-generic* if there is a m-generic lowest alcove presentation (s, μ) of τ (resp. $\overline{\tau}$).

2.4 Inertial local Langlands for GSp_4

In this section, we take $G = \mathrm{GSp}_4$ and $\mathcal{O}_p = \mathcal{O}_K$.

In [GT11a], Gan and Takeda established the local Langlands correspondence for GSp_4, which we denote by $\mathrm{rec}_{\mathrm{GT}}$. It is a surjective finite-to-one map that takes equivalence classes of smooth irreducible representation of $\mathrm{GSp}_4(K)$ to GSp_4-conjugacy classes of Weil–Deligne representations of W_K valued in $\mathrm{GSp}_4(\mathbf{C})$. Fix once and for all an isomorphism $\iota : \mathbf{C} \simeq \overline{\mathbf{Q}}_p$. This induces a correspondence $\mathrm{rec}_{\mathrm{GT},\iota}$ over $\overline{\mathbf{Q}}_p$. We define a normalized local Langlands correspondence by

$$\mathrm{rec}_{\mathrm{GT},p}(\pi) := \mathrm{rec}_{\mathrm{GT},\iota}(\pi \otimes |\mathrm{sim}|^{-3/2})$$

for any smooth irreducible $\overline{\mathbf{Q}}_p$-representation π of $\mathrm{GSp}_4(K)$.

Let τ be a tame inertial L-parameter with 1-generic lowest alcove presentation (s, μ). We attach a tame type $\sigma(\tau) := R_s(\mu+\eta)$ to τ. We often view $\sigma(\tau)$ as $\mathrm{GSp}_4(\mathcal{O}_K)$-representation by inflation.

Theorem 2.4.1. *Let π be a smooth irreducible $\overline{\mathbf{Q}}_p$-representation of $\mathrm{GSp}_4(K)$ and τ be a tame inertial type for K with 1-generic lowest alcove presentation (s,μ). If $\sigma(\tau)\subset\pi|_{\mathrm{GSp}_4(\mathcal{O}_K)}$, then $\mathrm{rec}_{\mathrm{GT}}(\pi)|_{I_K}\simeq\tau$. Moreover, $\mathrm{Hom}_{\overline{\mathbf{Q}}_p[\mathrm{GSp}_4(\mathcal{O}_K)]}(\sigma(\tau),\pi|_{\mathrm{GSp}_4(\mathcal{O}_K)})$ is 1-dimensional.*

Recall that given a pair $(s,\mu)\in\underline{W}\times X^*(\underline{T})$, we defined $(s_\tau,\mathbf{a}^{(0)})\in W\times X^*(T)$ by

$$s_\tau=s_0s_1\dots s_{f-1},\ \mathbf{a}^{(0)}=\sum_{j=0}^{f-1}((F^*\circ s^{-1})^j(\mu+\eta))_0.$$

For such $(s_\tau,\mathbf{a}^{(0)})$, we can associate a pair $(\mathbb{T},\theta):=(T_{s_\tau},\theta_{s_\tau,\mathbf{a}^{(0)}})$ and the Deligne–Lusztig representation $\epsilon_{\mathrm{GSp}_4}\epsilon_{\mathbb{T}}R_{\mathbb{T}}^{\theta}$ following [Her09, Lemma 4.2]. Using [DM20, Corollary 10.5 and Proposition 12.2] and [CGP15, Proposition A.5.15(1)], one can see that $\epsilon_{\mathrm{GSp}_4}\epsilon_{\mathbb{T}}R_{\mathbb{T}}^{\theta}$ is isomorphic to $R_s(\mu+\eta)$ as $\mathrm{GSp}_4(k)\simeq G_0(\mathbf{F}_p)$-representation. We say that $R_s(\mu+\eta)$ is *cuspidal* if the torus $\mathbb{T}$ is *not* contained in any proper Levi subgroup of GSp_4. Otherwise, it is called *non-cuspidal*.

Let P be the minimal standard parabolic subgroup of GSp_4 containing $\mathbb{T}$ with Levi factor M and the unipotent radical N. By [DL76, Proposition 8.2] we have

$$R_{\mathbb{T}}^{\theta}=\mathrm{Ind}_{P(\mathbf{F}_q)}^{G(\mathbf{F}_q)}(R_{\mathbb{T},P}^{\theta}).$$

Note that M decomposes into a product $\prod_{i=1}^r M_i$ (see [RS07, §2.1]). We also write $\mathbb{T}=\prod_{i=1}^r\mathbb{T}_i$ where $\mathbb{T}_i\subset M_i$ is a maximal torus and $\theta_i=\theta|_{\mathbb{T}_i}$. By Künneth Theorem, we have an isomorphism between $M(k)$-representations

$$R_{\mathbb{T},P}^{\theta}=\bigotimes_{i=1}^r R_{\mathbb{T}_i,M_i}^{\theta_i}.$$

We give explicit descriptions of $\mathbb{T}_i$ and θ_i. Let us write $\mathbf{a}^{(0)}=(a_1,a_2;a_3)$. Recall that we write T_2 for the diagonal torus of GL_2 and W_2 for the Weyl group for GL_2. Only in this section, we write $W_2=\{1,s\}$.

1. When $s_\tau=1$, we have $P=B$, $r=3$, $M_i=\mathbb{T}_i=\mathbf{G}_m$ for $1\le i\le 3$. Then θ_i is given by $a_i\in X^*(\mathbf{G}_m)$.
2. When $s_\tau\in\{s_1,s_2s_1s_2\}$, we have $P=S$, $r=2$, $M_1\simeq\mathrm{GL}_2$ and $M_2\simeq\mathbf{G}_m$. Then $(\mathbb{T}_1,\theta_1)=(T_s,\theta_{s,\mu})$ where $\mu=(a_1,a_2)$ (resp. $(a_1,-a_2)$) in $X^*(T_2)$ when $s_\tau=s_1$ (resp. $s_2s_1s_2$), and θ_2 is given by $a_3\in X^*(\mathbf{G}_m)$.
3. When $s_\tau\in\{s_2,s_1s_2s_1\}$, we have $P=Q$, $r=2$, $M_1\simeq\mathbf{G}_m$, and $M_2\simeq\mathrm{GL}_2$. Then θ_1 is given by a_1 (resp. a_2) in $X^*(\mathbf{G}_m)$, and $(\mathbb{T}_2,\theta_2)=(T_s,\theta_{s,\mu})$ where $\mu=(a_2+a_3,a_3)$ (resp. (a_1+a_3,a_3)) in $X^*(T_2)$ when $s_\tau=s_2$ (resp. $s_1s_2s_1$).
4. When $s_\tau\in\{s_1s_2,s_2s_1,w_0\}$, we have $P=\mathrm{GSp}_4$. This is the only case that $R_{\mathbb{T}}^{\theta}$ is cuspidal.

Let π and $\sigma(\tau)$ be as in Theorem 2.4.1. When $\sigma(\tau)$ is non-cuspidal, we can use the above data to describe π as a parabolically induced representation.

Proposition 2.4.2. *Let τ be a 1-generic tame inertial L-parameter such that $\sigma(\tau)$ is non-cuspidal. Let π be a smooth irreducible $\overline{\mathbf{Q}}_p$-representation of $G(K)$ such that $\sigma(\tau)\subset\pi|_{\mathrm{GSp}_4(\mathcal{O}_K)}$. Following the above* *notations, there exists a* $M_i(\mathcal{O}_K)K^\times$*-representation* $\widetilde{R}^{\theta_i}_{\mathbb{T}_i}$ *extending* $\epsilon_{M_i}\epsilon_{\mathbb{T}_i}R^{\theta_i}_{\mathbb{T}_i}$ *such that*

$$\pi \simeq \operatorname{Ind}^{\mathrm{GSp}_4(K)}_{P(K)}\left(\bigotimes_{i=1}^{r}\operatorname{ind}^{M_i(K)}_{M_i(\mathcal{O}_K)K^\times}\widetilde{R}^{\theta_i}_{\mathbb{T}_i}\right).$$

Moreover, π *contains* $\sigma(\tau)$ *with multiplicity one.*

Proof. Let $\mathfrak{P} \subset \mathrm{GSp}_4(\mathcal{O}_K)$ be the parahoric subgroup defined as the inverse image of $P(k) \subset \mathrm{GSp}_4(k)$. We let $\sigma_{\mathfrak{P}}$ and σ_M denote the inflation of $\epsilon_P\epsilon_{\mathbb{T}}R^\theta_{\mathbb{T},P}$ to $\mathfrak{P}$ and $M(\mathcal{O}_K)$ respectively. By [Mor99, Lemma 3.6], we have an isomorphism $\pi^{\sigma_{\mathfrak{P}}} \simeq (\pi_N)^{\sigma_M}$ where π_N is the unnormalized Jacquet module. Since $\sigma(\tau) \subset \pi|_{\mathrm{GSp}_4(\mathcal{O}_K)}$, we get $\sigma_M \hookrightarrow \pi_N|_{M(\mathcal{O}_K)}$. The pair $(M(\mathcal{O}_K), \sigma_M)$ is $M(K)$-type. This implies that there is an isomorphism of $M(K)$-representations $\pi_N \xrightarrow{\sim} \tau_{\sigma_M}$ for a supercuspidal representation of $M(K)$

$$\tau_{\sigma_M} = \bigotimes_{i=1}^{r}\operatorname{ind}^{M_i(K)}_{M_i(\mathcal{O}_K)K^\times}\widetilde{R}^{\theta_i}_{\mathbb{T}_i,M_i}$$

where $\widetilde{R}^{\theta_i}_{\mathbb{T}_i}$ is a $M_i(\mathcal{O}_K)K^\times$-representation extending $\epsilon_{M_i}\epsilon_{\mathbb{T}_i}R^{\theta_i}_{\mathbb{T}_i}$ (see [Mor99, Proposition 4.1]). By Frobenius reciprocity, we get a non-zero map

$$\pi \to \operatorname{Ind}^{\mathrm{GSp}_4(K)}_{P(K)}\tau_{\sigma_M}.$$

This is an isomorphism by [GT11b, Lemma 5.1.(a), 5.2.(b)] (for $P = S$ or Q) and [Box+21, Proposition 2.4.6] (for $P = B$).

Let $\mathrm{GSp}_4(\mathcal{O}_K)_1 := \ker(\mathrm{GSp}_4(\mathcal{O}_K) \twoheadrightarrow \mathrm{GSp}_4(k))$. Also we let $\widetilde{\sigma}_M$ be the $M(\mathcal{O}_K)K^\times$-representation extending σ_M by letting $K^\times$ act by the central character of π. Then the multiplicity one assertion follows from

$$\pi^{\sigma(\tau)} \simeq \operatorname{Hom}_{\mathrm{GSp}_4(k)}(\sigma(\tau), \pi^{\mathrm{GSp}_4(\mathcal{O}_K)_1}) \simeq \pi^{\sigma_{\mathfrak{P}}} \simeq (\pi_N)^{\sigma_M} \simeq (\tau_{\sigma_M})^{\widetilde{\sigma}_M} \simeq \operatorname{Hom}_{M(K)}(\tau_{\sigma_M}, \tau_{\sigma_M})$$

where the second isomorphism is induced by Frobenius reciprocity for finite groups, the fourth isomorphism exists because $\pi_N \simeq \tau_{\sigma_M}$ admits a central character, and the last isomorphism is induced by the Frobenius reciprocity for compact inductions. □

Now suppose that $\sigma(\tau)$ is cuspidal. We write $\widetilde{\sigma}(\tau)$ for the $G(\mathcal{O}_K)K^\times$-representation extending $\sigma(\tau)$ by letting $K^\times$ act by central character of π. By [DR09, Lemma 4.5.1], we have an isomorphism of $G(K)$-representations

$$\pi \simeq \operatorname{ind}^{G(K)}_{G(\mathcal{O}_K)K^\times}(\widetilde{\sigma}(\tau)).$$

We prove Theorem 2.4.1 using an explicit description of the local Langlands correspondence. In the non-cuspidal case, we use the explicit theta correspondence in [GT11b]. In the cuspidal case, we use the explicit construction of the local Langlands correspondence for tame regular semisimple elliptic Langlands parameters in [DR09]. Note that the compatibility between the local Langlands correspondence for GSp_4 of DeBacker–Reeder and Gan–Takeda is proven by Lust ([Lus13, Theorem 1.1]).

2.4.3 Explicit theta correspondences

In [GT11b], Gan–Takeda established theta correspondences between the group GSp_4 and various similitude orthogonal groups $\mathrm{GSO}_{2,2}$, $\mathrm{GSO}_{3,3}$, and $\mathrm{GSO}_{4,0}$. This was used to prove the local Langlands conjecture for GSp_4 ([GT11a]). The parabolically induced representations of GSp_4 has a non-zero theta lift to $\mathrm{GSO}_{3,3}$. The group $\mathrm{GSO}_{3,3}$ admits an accidental isomorphism

$$\mathrm{GSO}_{3,3} \simeq (\mathrm{GL}_4 \times \mathrm{GL}_1)/\left\{(z, z^{-2}) \mid z \in \mathrm{GL}_1)\right\}.$$

Using this isomorphism, we view a representation of $\mathrm{GSO}_{3,3}$ as a representation of $\mathrm{GL}_4 \times \mathrm{GL}_1$.

Let π be a irreducible smooth $\mathbf{C}$-representation of $\mathrm{GSp}_4(K)$. If the theta lifting of π to $\mathrm{GSO}_{3,3}$ is given by (non-zero) $\Pi \boxtimes \chi$, the L-parameter of $\Pi \boxtimes \chi$ (which is valued in $\mathrm{GL}_4(\mathbf{C}) \times \mathrm{GL}_1(\mathbf{C})$) factors through the map

$$\mathrm{GSp}_4(\mathbf{C}) \xrightarrow{\mathrm{std} \times \mathrm{sim}} \mathrm{GL}_4(\mathbf{C}) \times \mathrm{GL}_1(\mathbf{C})$$

which provides the L-parameter $\mathrm{rec}_{\mathrm{GT}}(\pi)$ of π.

In the following Theorem, we write ϕ_τ for the L-parameter attached to a smooth irreducible $\overline{\mathbf{Q}}_p$-representation τ of $\mathrm{GL}_2(K)$ by [HT01] conjugated by ι. We write ω_τ for the central character of τ.

Proposition 2.4.4. *Let π be a smooth irreducible $\overline{\mathbf{Q}}_p$-representation of $\mathrm{GSp}_4(K)$.*

1. *($P = B$ the Borel subgroup) If $\pi \simeq \mathrm{Ind}_{B(K)}^{\mathrm{GSp}_4(K)}(\chi_1, \chi_2; \chi)$, then*

$$\mathrm{rec}_{\mathrm{GT},p}(\pi) = \chi_1\chi_2\chi|\cdot|^{-3} \oplus \chi_1\chi|\cdot|^{-2} \oplus \chi_2\chi|\cdot|^{-1} \oplus \chi : W_K \to T(E) \subset \mathrm{GSp}_4(E).$$

2. *($P = Q$ the Klingen parabolic) If $\pi \simeq \mathrm{Ind}_{Q(K)}^{\mathrm{GSp}_4(K)}(\chi \otimes \tau)$, then*

$$\mathrm{rec}_{\mathrm{GT},p}(\pi) = \phi_\tau|\cdot|^{-1/2} \oplus \phi_\tau\chi|\cdot|^{-5/2} : W_K \to M_Q(E) \subset \mathrm{GSp}_4(E).$$

3. *($P = S$ the Siegel parabolic) If $\pi \simeq \mathrm{Ind}_{P(K)}^{\mathrm{GSp}_4(K)}(\tau \otimes \chi)$, then*

$$\mathrm{rec}_{\mathrm{GT},p}(\pi) = \chi \oplus \phi_\tau\chi|\cdot|^{-3/2} \oplus \chi\omega_\tau|\cdot|^{-3} : W_K \to M_S(E) \subset \mathrm{GSp}_4(E).$$

Proof. This is a special case of [GT11b, Proposition 13.1] (vi), (iv), (v) (for (1), (2), (3) respectively). Note that the induction in *loc. cit.* is normalized. □

Proof of Theorem 2.4.1 in the non-cuspidal case. This follows from Proposition 2.4.2 and 2.4.4, and the inertial local Langlands correspondence for GL_2 and GL_1 (e.g. [Le+a, Proposition 2.5.5]). □

2.4.5 Depth zero regular supercuspidal local Langlands

Let $\tau \simeq \tau(s, \mu+\eta)$ be a 1-generic tame inertial type for K over E and $\psi : G_K \to \mathrm{GSp}_4(E)$ be a continuous representation extending τ. We also assume $\sigma(\tau)$ cuspidal. After taking conjugation, we can assume that the image of I_K is contained in $T(E)$ and $\psi(\mathrm{Frob}_K) \in N_G(T)$. We can write $\psi(\mathrm{Frob}_K) = wt$ for a unique $w \in W$ and some $t \in T(E)$. Note that t gives a well-defined class in $T/(1-w)T$, and thus $\mathrm{sim}(t)$ is independent of the choice of t. By the construction of τ, we must have $w = \phi(s_\tau)$. The cuspidality of $\sigma(\tau)$ implies $w \in \{s_1 s_2, s_2 s_1, w_0\}$. Then ψ is TRSELP (tame regular semisimple elliptic L-parameter) in the sense of [DR09, §4.1].

For any TRSELP ψ, DeBacker–Reeder constructed the L-packet of depth-zero supercuspidal representations associated with it. These representations are distributed among the pure inner forms of GSp_4. The pure inner forms are parameterized by Galois cohomology $H^1(K, \mathrm{GSp}_4)$. Since $H^1(K, \mathrm{Sp}_4) = 1$ (because Sp_4 is simply-connected) and $H^1(K, \mathbf{G}_m) = 1$, we have $H^1(K, \mathrm{GSp}_4) = 1$. Thus, all these representations are of GSp_4.

Now we explain the construction in [DR09, §4] in a special case. The L-packet of ψ, denoted by $\Pi(\psi)$, is parameterized by $\mathrm{Irr}(C_\psi)$ the (finite) set of irreducible representations of $C_\psi = \pi_0(Z_G(\mathrm{Im}\,\psi))$. One can check that C_ψ is trivial when $s_\tau = s_1 s_2$ or $s_2 s_1$ and is isomorphic to $\mathbf{Z}/2\mathbf{Z}$ when $s_\tau = w_0$. In both cases, we simply take the trivial representation of C_ψ. Then the corresponding element in $\Pi(\psi)$ is given by

$$\mathrm{ind}_{G(\mathcal{O}_K)K^\times}^{G(K)} \widetilde{R}_{\mathbb{T}}^{\theta}$$

where $(\mathbb{T}, \theta) = (T_{s_\tau}, \theta_{s_\tau, \mathbf{a}^{(0)}})$, $\widetilde{R}_{\mathbb{T}}^{\theta}|_{G(\mathcal{O}_K)} \simeq \epsilon_G \epsilon_{\mathbb{T}} R_{\mathbb{T}}^{\theta}$, and $\widetilde{R}_{\mathbb{T}}^{\theta}|_{K^\times}$ sends p to $\mathrm{sim}(t)$.

Proof of Theorem 2.4.1 in the cuspidal case. We know that $\sigma(\tau) \subset \pi|_{G(\mathcal{O}_K)}$ implies that $\pi \simeq \mathrm{ind}_{G(\mathcal{O}_K)K^\times}^{G(K)}(\widetilde{\sigma}(\tau))$ for some $\widetilde{\sigma}(\tau)$ extending $\sigma(\tau)$. Let ω_π be the central character of π. By the above construction, π is contained in the L-packet of ψ such that $\psi|_{I_K} \simeq \tau$ and $\mathrm{sim}(\psi)(\mathrm{Frob}_K) = \omega_\pi(p)$. The multiplicity one assertion follows from

$$\mathrm{Hom}_{G(\mathcal{O}_K)}(\sigma(\tau), \pi|_{G(\mathcal{O}_K)}) \simeq \mathrm{Hom}_{G(\mathcal{O}_K)K^\times}(\widetilde{\sigma}(\tau), \pi|_{G(\mathcal{O}_K)K^\times}) \simeq \mathrm{Hom}_{G(K)}(\pi, \pi)$$

where the first isomorphism exists because π admits a central character, and the second isomorphism follows from the Frobenius reciprocity for compact inductions. □

2.4.6 Serre weights of a tame inertial L-parameter

We define the conjectural set of Serre weights associated to a tame inertial L-parameter following [GHS18, Definition 9.2.5].

Definition 2.4.7. Let $\overline{\rho}$ be a tame inertial L-parameter over $\mathbf{F}$. We define $W^?(\overline{\rho})$ to be the set $\mathcal{R}(\mathrm{JH}(\overline{\sigma}([\overline{\rho}])))$.

Definition 2.4.8. Let $\overline{\rho} \simeq \overline{\tau}(s, \mu)$ be a 6-generic tame inertial L-parameter over $\mathbf{F}$. We define a function

$$\begin{aligned} F_{\overline{\rho}} : \underline{W} &\to W^?(\overline{\rho}) \\ w &\mapsto \mathcal{R}(F_{\sigma([\overline{\rho}])}(w)). \end{aligned}$$

We define $W_{\mathrm{obv}}(\overline{\rho})$ to be the image of $F_{\overline{\rho}}$ and call its elements as *obvious weights* of $\overline{\rho}$.

Note that $W_{\mathrm{obv}}(\overline{\rho})$ does not depend on the choice of the lowest alcove presentation of $\overline{\rho}$ and coincides with $W_{\mathrm{obv}}(\overline{\rho})$ defined in [Le+a, Definition 2.6.3]. In *loc. cit.*, $F_{\overline{\rho}}(w)$ is called as the obvious weight of $\overline{\rho}$ corresponding to w.

2.4.9 Transfer to GL_4

Recall that we have a map $\mathcal{T}$ defined in §2.1 which maps $\underline{W}$ to $\underline{W}_4$ and $X^*(\underline{T})$ to $X^*(\underline{T}_4)$. Using this, we define the *transfer* of Deligne–Lusztig representations and Serre weights of $G_0(\mathbf{F}_p)$ to $(\mathrm{GL}_4)_0(\mathbf{F}_p)$.

Proposition 2.4.10. *The map $\mathcal{T}$ induces a well-defined assignment from Deligne–Lusztig representations (resp. Serre weights) of $G_0(\mathbf{F}_p)$ to the Deligne–Lusztig representations (resp. Serre weights) of $(\mathrm{GL}_4)_0(\mathbf{F}_p)$ given by*

$$
\begin{aligned}
R_s(\mu) &\mapsto \mathcal{T}(R_s(\mu)) := R_{\mathcal{T}(s)}(\mathcal{T}(\mu)) \\
F(\lambda) &\mapsto \mathcal{T}(F(\lambda)) := F(\mathcal{T}(\lambda)).
\end{aligned}
$$

Suppose that $\mu - \eta \in \underline{C}_0$ is 6-deep. Then $F(\lambda) \in \mathrm{JH}(\overline{R_s(\mu)})$ implies $\mathcal{T}(F(\lambda)) \in \mathrm{JH}(\overline{\mathcal{T}(R_s(\mu))})$. The converse is true if furthermore $\mu - \eta = (a_j, b_j; c_j)_{j\in\mathcal{J}}$ and $|a_j - p/2| > 3/2$ for each $j \in \mathcal{J}$.

Proof. The first claim for Deligne–Lusztig representations follows from the fact that the map $\mathcal{T}$ respects the Weyl group action on the character lattice and p-dot actions. For Serre weights, it follows from that $\mathcal{T}$ maps $X_1^*(\underline{T})$ and $X^0(\underline{T})$ into $X_1^*(\underline{T}_4)$ and $X^0(\underline{T}_4)$ respectively.

Suppose that $F(\lambda) \in \mathrm{JH}(\overline{R_s(\mu)})$. By Proposition 2.2.6, there exists $w \in \underline{W}$ and $\widetilde{w} \in \underline{\widetilde{W}}_1$ such that $\widehat{w} \uparrow \widetilde{w}$ and

$$
F(\lambda) \simeq F\left(\widetilde{w}_h^{-1}\widetilde{w} \cdot (\mu - \eta + s\pi(\widehat{w}^{-1}(0)))\right) \in \mathrm{JH}(\overline{R_s(\mu)}).
$$

Let $\mathcal{T}(\widetilde{w}_h) = \widetilde{w}'_h$. By applying $\mathcal{T}$ to the above equation, we have

$$
\mathcal{T}(F(\lambda)) \simeq F\left(\widetilde{w}_h'^{-1}\mathcal{T}(\widetilde{w}) \cdot (\mathcal{T}(\mu) - \eta' + \mathcal{T}(s\pi(\widehat{w}^{-1}(0))))\right).
$$

Since two ordering $\leq$ and $\uparrow$ coincide on $\underline{\widetilde{W}}^+$, we have $\mathcal{T}(\widehat{w}) \uparrow \mathcal{T}(\widetilde{w})$ by Lemma 2.1.6. Then the above equality implies $\mathcal{T}(F(\lambda)) \in \mathrm{JH}(\overline{\mathcal{T}(R_s(\mu))})$ by Proposition 2.2.6.

For the converse, suppose $\mathcal{T}(F(\lambda)) \in \mathrm{JH}(\overline{\mathcal{T}(R_s(\mu))})$. Then there exists $w' \in \underline{W}_4$ and $\widetilde{w}' \in \underline{\widetilde{W}}_1$ such that $\widehat{w}' \uparrow \widetilde{w}'$ and

$$
\mathcal{T}(F(\lambda)) \simeq F\left(\widetilde{w}_h'^{-1}\widetilde{w}' \cdot (\mathcal{T}(\mu) - \eta' + \mathcal{T}(s)\pi(\widehat{w}'^{-1}(0)))\right).
$$

This shows that $\widetilde{w}_h'^{-1}\widetilde{w}' \cdot (\mathcal{T}(\mu) - \eta' + \mathcal{T}(s)\pi(\widehat{w}'^{-1}(0)))$ is fixed by Θ (defined in §2.1). Suppose that $\widetilde{w}'$ is fixed by Θ. Then $\widehat{w}'^{-1}(0)$ is fixed by Θ, and simple computation shows that $\widehat{w}'$ is fixed by Θ as well. Thus $\widetilde{w}' = \mathcal{T}(\widetilde{w})$ and $\widehat{w}' = \mathcal{T}(\widehat{w})$ for some $\widetilde{w} \in \underline{\widetilde{W}}_1$ and $w \in \underline{W}$. As in the previous paragraph, this shows that $F(\lambda) \in \mathrm{JH}(\overline{R_s(\mu)})$.

We finish the proof by showing that $\widetilde{w}'$ is fixed by Θ. Let $\widetilde{w}_0 \in \underline{\widetilde{W}}$ be an element such that $\lambda \in \widetilde{w}_0 \cdot \underline{C}_0$. Then we can write $\widetilde{w}' = \mathcal{T}(\widetilde{w}_0)\delta'$ for some $\delta' \in \underline{\Omega}_4$. Suppose that δ' is not fixed by Θ. Since $\Omega_4/\mathcal{T}(\Omega)$ is cyclic group of order 2, we can assume that δ'_j is the generator of Ω_4 sending (a, b, c, d) to $(b, c, d, a - p)$ for at least one $j \in \mathcal{J}$. Let us write $\mu - \eta = (a_j, b_j; c_j)_{j\in\mathcal{J}}$ and $\mathcal{T}(s)\pi(\widehat{w}'^{-1}) = (x_j, y_j, z_j, w_j)_{j\in\mathcal{J}}$. Then

$$
\begin{aligned}
\left(\delta' \cdot (\mathcal{T}(\mu - \eta) + \mathcal{T}(s)\pi(\widehat{w}'^{-1}))\right)_j &= \delta'_j \cdot (a_j + b_j + c_j + x_j, a_j + c_j + y_j, b_j + c_j + z_j, c_j + w_j) \\
&= (a_j + c_j + y_j, b_j + c_j + z_j, c_j + w_j, a_j + b_j + c_j + x_j - p).
\end{aligned}
$$

Since $\delta' \cdot (\mathcal{T}(\mu-\eta)+\mathcal{T}(s)\pi(\widehat{w}'^{-1}))$ is fixed by Θ, we have $2a_j - p = z_j + w_j - y_j - x_j$. However, $|z_j + w_j - y_j - x_j| \leq 3$ by [LLL19, Remark 4.1.4]. This leads to a contradiction and δ' is fixed by Θ. $\square$

Example 2.4.11. The condition $|a_j - p/2| > 3/2$ in the above Lemma may seem dubious but it is necessary. For example, let $K = \mathbf{Q}_p$ and $(s, \mu-\eta) \in W \times X^*(T)$ be a lowest alcove presentation of $R_s(\mu)$ where $s = e$, $\mu - \eta = (a, b; c)$. We take $a = (p-1)/2$. Let $w' \in W_4$ be the element sending (x, y, z, w) to (y, z, w, x). Then $\widehat{w}' = w't_{-\nu}$ where $\nu = (1, 0, 0, 0)$. Let

$$\begin{aligned}\lambda' &:= \widetilde{w}_h'^{-1}\widehat{w} \cdot (\mathcal{T}(\mu) - \eta' + \nu) \\ &= \widetilde{w}_h'^{-1} \cdot \mathcal{T}(a, a-b; b+c-a)\end{aligned}$$

so that $F(\lambda') \in \mathrm{JH}(\overline{\mathcal{T}(R_e(\mu))})$. However, one can easily check that

$$F(\widetilde{w}_h^{-1} \cdot (a, a-b; b+c-a)) \notin \mathrm{JH}(\overline{R_e(\mu)}).$$

Corollary 2.4.12. *Let $\overline{\rho} \simeq \tau(s, \mu)$ be a 6-generic tame inertial L-parameter. Recall the set $W^?(\mathrm{std}(\overline{\rho}))$ defined in [Le+a, Definition 2.6.1] Suppose that $\mu - \eta = (a_j, b_j; c_j)_{j\in\mathcal{J}}$ and $|a_j - p/2| > 3/2$ for each $j \in \mathcal{J}$. Then*

$$\mathcal{T}(W^?(\overline{\rho})) = W^?(\mathrm{std}(\overline{\rho})) \cap \{\mathcal{T}(F(\lambda)) \mid \lambda \in X_1^*(\underline{T})\}.$$

The following Corollary shows the compatibility between the inertial local Langlands correspondence and the transfer of Deligne–Lusztig representations.

Corollary 2.4.13. *Let π be a smooth irreducible $\overline{\mathbf{Q}}_p$-representation of $\mathrm{GSp}_4(K)$ and $\psi = \mathrm{rec}_{\mathrm{GT},p}(\pi)$. Let Π be a smooth irreducible $\overline{\mathbf{Q}}_p$-representation of $\mathrm{GL}_4(K)$ corresponding to $\mathrm{std}(\psi)$ under the local Langlands correspondence of [HT01] (conjugated by ι). Let τ be a 1-generic tame inertial L-parameter. If $\sigma(\tau) \subset \pi|_{\mathrm{GSp}_4(\mathcal{O}_K)}$, then $\mathcal{T}(\sigma(\tau)) \subset \Pi|_{\mathrm{GL}_4(\mathcal{O}_K)}$.*

Proof. By Theorem 2.4.1, we have $\psi|_{I_K} \simeq \tau$. Then the claim follows from the construction of $\mathcal{T}(\sigma(\tau))$ and [Sho18, Theorem 3.7 (2)]. (Note that although the local Langlands correspondence used in *loc. cit.* is normalized, it only differs by unramified twist.) $\square$

Remark 2.4.14. The converse of the above Corollary is not true, as the L-packet of ψ can have two elements (e.g. if $\psi|_{I_K} \simeq \tau(s, \mu)$ and $s_\tau = w_0$).

2.5 Combinatorics between types and weights

Let G be either GSp_4 or GL_n. We first introduce some notations and definitions.

Notation 2.5.1. Let (s, μ) be a lowest alcove presentation of a Deligne–Lusztig representation R (resp. a tame inerital L-parameter τ). We let $\widetilde{w}(R)$ (resp. $\widetilde{w}(\tau)$) denote $t_{\mu+\eta}s \in \underline{\widetilde{W}}$. If $\overline{\rho}$ is another tame inertial L-parameter with a lowest alcove presentation (s', μ'), we write

$$\widetilde{w}(\overline{\rho}, \tau) := \widetilde{w}(\tau)^{-1}\widetilde{w}(\overline{\rho}) \in \underline{\widetilde{W}}.$$

Definition 2.5.2. We call an element of $X^*(\underline{Z})$ as an *algebraic central character* and an element of $X^*(\underline{Z})/(p-\pi)X^*(\underline{Z})$ as a *central character*.

Note that $X^*(\underline{Z})/(p-\pi)X^*(\underline{Z}) \simeq \mathrm{Hom}(Z(\mathcal{O}_p/p), \mathbf{F}^\times)$ which justifies the term central character.

Recall that we have an isomorphism

$$\underline{\Omega} \simeq \underline{\widetilde{W}}/\underline{W}_a \simeq X^*(\underline{Z}). \tag{2.5.3}$$

If σ is a Serre weight with a lowest alcove presentation $(\widetilde{w}_1, \omega)$, then the central character of σ can be described as the image of $t_{\omega-\eta}\widetilde{w}_1$ under (2.5.3) composed with $X^*(\underline{Z}) \twoheadrightarrow X^*(\underline{Z})/(p-\pi)X^*(\underline{Z})$. Similarly, if R is a Deligne–Lusztig representation with a lowest alcove presentation (s,μ), then the central character of R is given by the lift of the image of $\widetilde{w}(R)$ under (2.5.3) composed with $X^*(\underline{Z}) \twoheadrightarrow X^*(\underline{Z})/(p-\pi)X^*(\underline{Z})$.

For a dominant character $\lambda \in X^*(\underline{T})$, let $W(\lambda)_{/\mathcal{O}}$ be the unique up to isomorphism irreducible algebraic $\underline{G}_{/\mathcal{O}}$-representation of highest weight λ. Let $V(\lambda)$ be the restriction of $W(\lambda)_{/\mathcal{O}}$ to $G_0(\mathbf{Z}_p)$. We define a *type* (of $G_0(\mathbf{Z}_p)$) to be a pair $(\lambda+\eta, \tau)$ where $\lambda \in X_*(\underline{T}^\vee)$ is a dominant cocharacter and τ is a 1-generic tame inertial L-parameter. To a type $(\lambda+\eta,\tau)$, we associate a locally algebraic representations of $G_0(\mathbf{Z}_p)$

$$\sigma(\lambda,\tau) := \sigma(\tau) \otimes_{\mathcal{O}} V(\phi^{-1}(\lambda)).$$

Let $W(\lambda)_{/\mathbf{F}}$ be the dual Weyl module of highest weight λ for the algebraic group $\underline{G}_{/\mathbf{F}}$ and $W(\lambda)$ be the restriction of $W(\lambda)_{/\mathbf{F}}$ to $G_0(\mathbf{F}_p)$. Note that $V(\lambda)\otimes_{\mathcal{O}} \mathbf{F} \simeq W(\lambda)$.

Definition 2.5.4.

1. Let $(\widetilde{w}_1, \omega)$ be a lowest alcove presentation of a Serre weight σ. We say $(\widetilde{w}_1,\omega)$ is *compatible with* $\zeta \in X^*(\underline{Z})$ if the image of $t_{\omega-\eta}\widetilde{w}_1$ in $X^*(\underline{Z})$ under (2.5.3) is equal to ζ. We also say that σ *has algebraic central character* ζ *(with respect to* $(\widetilde{w}_1,\omega)$*)*.

2. Let (s,μ) be a lowest alcove presentation of a Deligne–Lusztig representation R (resp. a tame inertial L-parameter τ over E). Let $\lambda \in X^*(\underline{T})$. We say (s,μ) is λ*-compatible with* $\zeta \in X^*(\underline{Z})$ if the image of $t_\lambda t_{\mu+\eta}s$ in $X^*(\underline{Z})$ under (2.5.3) is equal to ζ. When $\lambda = 0$, we also say that R (resp. τ) *has algebraic central character* ζ *(with respect to* (s,μ)*)*.

3. Let (s,μ) be a lowest alcove presentation of a tame inertial L-parameter $\overline{\tau}$ over $\mathbf{F}$. Let $\lambda \in X^*(\underline{T})$. We say (s,μ) is λ*-compatible with* $\zeta \in X^*(\underline{Z})$ if the image of $t_\lambda t_\mu s$ in $X^*(\underline{Z})$ under (2.5.3) is equal to ζ. (Note that this differs by t_η from item (2); see [Le+a, Remark 2.4.2].) When $\lambda = 0$, we also say that τ *has algebraic central character* ζ *(with respect to* (s,μ)*)*.

4. We say that a lowest alcove presentation of tame inertial type τ over E (or $\overline{\tau}$ over $\mathbf{F}$) is *(*λ*-)compatible with* a lowest alcove presentation of a Serre weight (or Deligne–Lusztig representation) if the former is (λ-)compatible with $\zeta \in X^*(\underline{Z})$ and the latter is compatible with ζ.

Remark 2.5.5. A choice of algebraic central character lifting the central character of Serre weights or Deligne–Lusztig representations corresponds to a choice of lowest alcove presentations (see [Le+a, Lemma 2.2.4 and 2.3.2]). Later, we will study objects whose constructions depend on choices of lowest alcove presentations and their connections. The (λ-)compatibility is a notion to make such choices consistent.

Let $\lambda \in X^*(\underline{T})$ be a dominant character. We have the set of *admissible pairs* defined in [Le+a, §2.1]

$$\mathrm{AP}(\lambda+\eta) := \left\{ (\widetilde{w}_1, \widetilde{w}_2) \in (\underline{\widetilde{W}}_1^+ \times \underline{\widetilde{W}}^+)/X^0(\underline{T}) \mid \widetilde{w}_1 \uparrow t_\lambda \widetilde{w}_h^{-1} \widetilde{w}_2 \right\}.$$

Proposition 2.5.6 (Proposition 2.3.7 in [Le+a])**.** *Let $\lambda \in X^*(\underline{T})$ be a dominant character. Let m be an integer such that $m \geq \max\{h_\lambda + 3, 6\}$ if $G = \mathrm{GSp}_4$ and $m \geq \max\{h_\lambda + n - 1, 2(n-1)\}$ if $G = \mathrm{GL}_n$. For m-generic Deligne–Lusztig representation R, we have the following bijection*

$$\mathrm{AP}(\lambda + \eta) \simeq \mathrm{JH}(\overline{R} \otimes_{\mathbf{F}} W(\lambda))$$
$$(\widetilde{w}_1, \widetilde{w}_2) \mapsto F_{(\widetilde{w}_1, \widetilde{w}(R)\widetilde{w}_2^{-1}(0))}.$$

Moreover, every Jordan–Hölder factor are $(m - h_\lambda - 3)$-deep and the lowest alcove presentations $(\widetilde{w}_1, \widetilde{w}(R)\widetilde{w}_2^{-1}(0))$ of these Serre weights are λ-compatible with the lowest alcove presentation of R.

Remark 2.5.7. Suppose that we have $\lambda = 0$ in Proposition 2.5.6. By [Le+a, Proposition 2.1.6], the condition $\widetilde{w}_1 \uparrow \widetilde{w}_h^{-1}\widetilde{w}_2$ is equivalent to $\widetilde{w}_2 \uparrow \widetilde{w}_h\widetilde{w}_1$. Let us write

$$\nu = \pi^{-1}(\widetilde{w}_1) \cdot (\mu - \eta + s\widetilde{w}_2^{-1}(0))$$

for $(\widetilde{w}_1, \widetilde{w}_2) \in \mathrm{AP}(\eta)$ so that $F_{(\widetilde{w}_1, \widetilde{w}(R)\widetilde{w}_2^{-1}(0))} = F(\nu)$. Then

$$\pi^{-1}(\widetilde{w}_2) \cdot (\mu - \eta + s\widetilde{w}_2^{-1}(0)) \uparrow \widetilde{w}_h \cdot \nu$$

as in Proposition 2.2.6.

Proposition 2.5.8 (Proposition 2.6.2 in [Le+a])**.** *Let m be an integer such that $m \geq 6$ if $G = \mathrm{GSp}_4$ and $m \geq 2(n-1)$ if $G = \mathrm{GL}_n$. Let $\overline{\rho}$ be a tame inertial L-parameter over $\mathbf{F}$ with a m-generic lowest alcove presentation. Then there is a bijection*

$$\{(\widetilde{w}, \widetilde{w}_2) \in (\widetilde{\underline{W}}_1^+ \times \widetilde{\underline{W}}^+)/X^0(\underline{T}) \mid \widetilde{w}_2 \uparrow \widetilde{w}\} \xrightarrow{\sim} W^?(\overline{\rho})$$
$$(\widetilde{w}, \widetilde{w}_2) \mapsto F_{(\widetilde{w}, \widetilde{w}(\overline{\rho})\widetilde{w}_2^{-1}(0))}.$$

Moreover, every Jordan–Hölder factor are $(m-3)$-deep and the lowest alcove presentations $(\widetilde{w}, \widetilde{w}(\overline{\rho})\widetilde{w}_2^{-1}(0))$ of these Serre weights are compatible with the lowest alcove presentation of $\widetilde{w}(\overline{\rho})$.

Definition 2.5.9. Let σ and κ be Serre weights. We write $\sigma \uparrow \kappa$ if there exist $\lambda, \lambda' \in X_1^*(\underline{T})$ such that $\sigma \simeq F(\lambda)$, $\kappa \simeq F(\lambda')$, and $\lambda \uparrow \lambda'$.

Lemma 2.5.10. *Let $\overline{\rho}$ be a tame inertial L-parameter over $\mathbf{F}$ that is 6-generic if $G = \mathrm{GSp}_4$ and $2(n-1)$-generic if $G = \mathrm{GL}_n$. Let $\sigma, \kappa \in W^?(\overline{\rho})$ be Serre weights. Then $\sigma \uparrow \kappa$ if and only if $\sigma \simeq F_{(\widetilde{w}, \widetilde{w}(\overline{\rho})\widetilde{w}_2^{-1}(0))}$ and $\kappa \simeq F_{(\widetilde{w}', \widetilde{w}(\overline{\rho})\widetilde{w}_2^{-1}(0))}$ for $\widetilde{w}, \widetilde{w}' \in \widetilde{\underline{W}}_1^+$ and $\widetilde{w}_2 \in \widetilde{\underline{W}}^+$ such that $\widetilde{w}_2 \uparrow \widetilde{w} \uparrow \widetilde{w}'$.*

Proof. Suppose that $\sigma \simeq F_{(\widetilde{w}, \widetilde{w}(\overline{\rho})\widetilde{w}_2^{-1}(0))}$ and $\kappa \simeq F_{(\widetilde{w}', \widetilde{w}(\overline{\rho})\widetilde{w}_2'^{-1}(0))}$ for $\widetilde{w}, \widetilde{w}' \in \widetilde{\underline{W}}_1^+$ and $\widetilde{w}_2, \widetilde{w}_2' \in \widetilde{\underline{W}}^+$ such that $\widetilde{w}_2 \uparrow \widetilde{w}$ and $\widetilde{w}_2' \uparrow \widetilde{w}'$. By changing $\widetilde{w}_2$ up to $X^0(\underline{T})$ (and $\widetilde{w}$ accordingly), we can guarantee that $\pi^{-1}(\widetilde{w}_2) \cdot (\widetilde{w}(\overline{\rho})\widetilde{w}_2^{-1}(0) - \eta)$ and $\pi^{-1}(\widetilde{w}_2') \cdot (\widetilde{w}(\overline{\rho})\widetilde{w}_2'^{-1}(0) - \eta)$ are in the same $\underline{W}_a$-orbit (under the p-dot action). Let $\widetilde{w}_2 = wt_\nu$ and $\widetilde{w}_2' = w't_{\nu'}$. Then $(p\pi^{-1} - 1)\nu \equiv (p\pi^{-1} - 1)\nu' \bmod \underline{\Lambda}_R$. By applying $(1 + p\pi^{-1} + \dots (p\pi^{-1})^{r-1})$ to this equation where r is an integer such that $\pi^r = \mathrm{id}$, we deduce that $\nu \equiv \nu' \bmod \underline{\Lambda}_R$. Then $p\pi^{-1}(\nu) - \widetilde{w}(\overline{\rho})\nu - \eta$ and $p\pi^{-1}(\nu) - \widetilde{w}(\overline{\rho})\nu' - \eta$ are in the same $\underline{W}_a$-orbit. Since they are in the same p-alcove $\underline{C}_0 + \pi^{-1}(p\nu)$, they have to be equal. Thus, we conclude that $\nu = \nu'$, which implies that $\widetilde{w}_2 = \widetilde{w}_2'$. Then $\widetilde{w} \uparrow \widetilde{w}'$ follows immediately from $\sigma \uparrow \kappa$. $\square$

For the remaining of this section, we let $G = \mathrm{GSp}_4$. The following lemmas will be useful in proving main results in §6.

Lemma 2.5.11 (Corollary 2.6.5 in [Le+a]). *Let $\lambda \in X^*(\underline{T})$ be a dominant character. Let $\overline{\rho}$ and τ be tame inertial L-parameters over* $\mathbf{F}$ *and E, with λ-compatible 6-generic and $\max\{6, h_\lambda + 3\}$-generic lowest alcove presentation respectively, such that $\widetilde{w}(\overline{\rho}, \tau) = t_{w^{-1}(\lambda+\eta)}$ for some $w \in \underline{W}$. Then we have*

$$\mathrm{JH}(\overline{\sigma}(\lambda, \tau)) \cap W^?(\overline{\rho}) = \{(F_{\overline{\rho}}(w))\}\,.$$

Lemma 2.5.12. *Let $\lambda \in X_1^*(\underline{T})$ be a 12-deep dominant character. Let $\overline{\rho}$ be a tame inertial L-parameter over* $\mathbf{F}$ *with a 12-generic lowest alcove presentation $(s_{\overline{\rho}}, \mu_{\overline{\rho}})$. For each $s \in \underline{W}$, we consider a tame inertial type $\tau(s, \widetilde{w}_h \cdot \lambda + \eta)$. Suppose that for each $s \in \underline{W}$, $\tau(s, \widetilde{w}_h \cdot \lambda + \eta)$ admits a lowest alcove presentation such that $\widetilde{w}(\overline{\rho}, \tau(s, \widetilde{w}_h \cdot \lambda + \eta)) \in \mathrm{Adm}(\eta)$. Then $F(\lambda) \in W^?(\overline{\rho})$.*

Proof. This can be proven as [LLL19, Lemma 4.1.10]. □

Remark 2.5.13. Let $(s, \mu + \eta)$ be the lowest alcove presentation of $\tau(s, \widetilde{w}_h \cdot \lambda + \eta)$ as in the previous Lemma. We remark that the condition $\widetilde{w}(\overline{\rho}, \tau(s, \widetilde{w}_h \cdot \lambda + \eta)) \in \mathrm{Adm}(\eta)$ implies that the images of $st_{\mu+\eta}$ and $s_{\overline{\rho}} t_{\mu_{\overline{\rho}}}$ under (2.5.3) are equal, or equivalently, $\mu_{\overline{\rho}} - \mu \in \underline{\Lambda}_R$ (cf. the condition **(P3)** in [LLL19, §4.1]). This condition uniquely determines the lowest alcove presentation $(s, \mu + \eta)$ of $\tau(s, \widetilde{w}_h \cdot \lambda + \eta)$ (if it exists).

Lemma 2.5.14. *Let $\overline{\rho} : I_{\mathbf{Q}_p} \to {}^L\underline{G}(\mathbf{F})$ be a 6-generic tame inertial L-parameter. Let $\sigma \in W^?(\overline{\rho})$ be a Serre weight with a lowest alcove presentation $(\widetilde{w}_h^{-1}\widetilde{w}, \widetilde{w}_2)$ for some $(\widetilde{w}, \widetilde{w}_2) \in \widetilde{\underline{W}}_1^+ \times \widetilde{\underline{W}}^+$ such that $\widetilde{w}_2 \uparrow \widetilde{w}_h^{-1}\widetilde{w}$. Suppose that τ is a tame type with a 6-generic lowest alcove representation such that $\widetilde{w}(\overline{\rho}, \tau) = \widetilde{w}^{-1} w_0 \widetilde{w}_2$. Then,*

1. *$\sigma \in \mathrm{JH}(\overline{\sigma}(\tau))$; and*

2. *if $\kappa \in W^?(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\tau))$ and $\sigma \uparrow \kappa$, then $\sigma = \kappa$.*

Proof. Since $\widetilde{w}(\overline{\rho})\widetilde{w}_2^{-1}(0) = \widetilde{w}(\tau)\widetilde{w}^{-1}(0)$, we have $\sigma = F_{(\widetilde{w}_h^{-1}\widetilde{w}, \widetilde{w}(\tau)\widetilde{w}^{-1}(0))}$ and $F_{(\widetilde{w}_h^{-1}\widetilde{w}, \widetilde{w}(\tau)\widetilde{w}^{-1}(0))} \in \mathrm{JH}(\overline{\sigma}(\tau))$ by Proposition 2.5.6. Suppose that $\kappa \in W^?(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\tau))$ and $\sigma \uparrow \kappa$. Let $(\widetilde{s}, \omega)$ be a lowest alcove presentation of κ compatible with $\overline{\rho}$. By Proposition 2.5.6, there exists $\widetilde{s}_1 \in \widetilde{W}^+$ such that $\widetilde{s} \uparrow \widetilde{w}_h^{-1}\widetilde{s}_1$ and $\omega = \widetilde{w}(\tau)\widetilde{s}_1^{-1}(0)$. By Lemma 2.5.10, $\sigma \uparrow \kappa$ implies that $\widetilde{s}_1 = \widetilde{w}$ and $\widetilde{w}_h^{-1}\widetilde{w} \uparrow \widetilde{s}$. By [Le+a, Proposition 2.1.6], $\widetilde{w}_h^{-1}\widetilde{w} \uparrow \widetilde{s}$ is equivalent to $\widetilde{w}_h\widetilde{s} \uparrow \widetilde{w}$ and $\widetilde{s} \uparrow \widetilde{w}_h^{-1}\widetilde{s}_1$ is equivalent to $\widetilde{s}_1 = \widetilde{w} \uparrow \widetilde{w}_h\widetilde{s}$. Thus $\widetilde{s} = \widetilde{w}_h^{-1}\widetilde{w}$ and $\sigma = \kappa$. □

Chapter 3

The theory of local models

In this chapter, we generalize the theory of local models in [Le+a] to the group GSp_4. Note that when we write GSp_4 in this chapter, we mean the dual group $\mathrm{GSp}_4^\vee$. In particular, the set of coroots $\Phi^\vee$ of GSp_4 is identified with a set of roots of $\mathrm{GSp}_4^\vee$ by the duality isomorphism ϕ. If $\alpha^\vee \in \Phi^\vee$, we let $U_{\alpha^\vee} \subset \mathrm{GSp}_4^\vee$ denote the root subgroup associated to the root $\phi(\alpha^\vee)$ of $\mathrm{GSp}_4^\vee$. In other words, $U_\alpha^\vee \subset \mathrm{GSp}_4^\vee$ is a subgroup such that $U_{\alpha^\vee} \simeq \mathbf{G}_a$ and $tut^{-1} = \phi(\alpha^\vee)(t)u$ for any $t \in T^\vee$ and $u \in U_{\alpha^\vee}$.

We let $G = \mathrm{GSp}_4$ for the remainder of this thesis.

Let $X = \mathbb{A}^1_{\mathbf{Z}}$ be an affine line with coordinate function v. We denote by $X_0 = \operatorname{Spec} \mathbf{Z}$ the zero section of X and $X^0 = \operatorname{Spec} \mathbf{Z}[v, v^{-1}]$. We often write $t : \operatorname{Spec} R \to X$ to denote $\mathbf{Z}[v]$-algebra R such that v is mapped to $t \in R$.

3.1 Global affine Grassmannians

Let $\mathcal{G}$ be the Neron blowup of $\mathrm{GSp}_{4/X}$ in $B_{/X}$ along X_0 defined in [MRR, Definition 3.1]. By Theorem 3.2 of *loc. cit.*, it is a smooth affine group scheme over X with connected fibers. For $t : \operatorname{Spec} R \to X$ with t regular in R, the set of R-points is given by

$$\mathcal{G}(R) = \{g \in \mathrm{GSp}_4(R) \mid \ g \bmod t \in B(R/t)\}.$$

There is a morphism of X-group schemes $\mathcal{G} \to \mathrm{GSp}_{4/X}$. If $g \in \mathcal{G}(R)$, we denote by $\overline{g}$ its image in $\mathrm{GSp}_4(R)$. We also have a similitude character $\mathrm{sim} : \mathcal{G} \to \mathbf{G}_m$ sending g to $\mathrm{sim}(\overline{g})$.

The base change $\mathcal{G} \times_X X^0$ is isomorphic to GSp_{4/X^0} and the base change along $\varpi : \operatorname{Spec} \mathcal{O} \to X$ is isomorphic to standard Iwahori group scheme $\mathcal{I}$ whose R-points for any $\mathcal{O}$-algebra R are given by

$$\mathcal{I}(R) = \{g \in \mathrm{GSp}_4(R[\![v]\!]) \mid g \bmod v \in B(R)\}\,.$$

In particular, $\mathcal{G} \times_{X,\varpi} \operatorname{Spec} \mathcal{O}$ coincides with the group scheme constructed in [PZ13, Corollary 4.2] for GSp_4 and $\mathcal{I}$ (see also [MRR, Example 3.3]).

We also define a functor $L^+\mathcal{M}$ whose R-points, for $t : \operatorname{Spec} R \to X$, are given by

$$L^+\mathcal{M}(R) := \{g \in \operatorname{Lie} \mathrm{GSp}_4(R[\![v-t]\!]) \mid g \text{ is upper triangular modulo } v\}.$$

Let $\mathcal{G}_4$ be the Bruhat–Tits group for GL_4 defined as in [Le+a, §3.1]. Note that the map $\mathrm{std} : \mathrm{GSp}_4 \to \mathrm{GL}_4$ induces a morphism X-group schemes $\mathcal{G} \to \mathcal{G}_4$ (see [MRR, §2.4]) which we denote by std as well. It is easy to see that $\mathrm{std} : \mathcal{G} \to \mathcal{G}_4$ is a closed immersion.

We write $L\mathcal{G}$ for the loop group and $L^+\mathcal{G}$ for the positive loop group of $\mathcal{G}$. For $t : \operatorname{Spec} R \to X$, their R-points are given by

$$
\begin{aligned}
L\mathcal{G}(R) &= \mathcal{G}(R(\!(v-t)\!)) \\
L^+\mathcal{G}(R) &= \mathcal{G}(R[\![v-t]\!])
\end{aligned}
$$

where we consider $R[\![v-t]\!]$ and $R(\!(v-t)\!)$ as $\mathbf{Z}[v]$-algebra by sending v to v. It is known that $L^+\mathcal{G}$ is representable by a (not finite type) group scheme and $L\mathcal{G}$ is representable by an ind-group scheme.

Remark 3.1.1. When R is Noetherian, v is regular in both $R[\![v-t]\!]$ and $R(\!(v-t)\!)$. Thus we have the following description:

$$
\begin{aligned}
L\mathcal{G}(R) &= \{g \in \mathrm{GSp}_4(R(\!(v-t)\!)) \mid g \bmod v \in B(R(\!(v-t)\!)/v)\} \\
L^+\mathcal{G}(R) &= \{g \in \mathrm{GSp}_4(R[\![v-t]\!]) \mid g \bmod v \in B(R[\![v-t]\!]/v)\}\,.
\end{aligned}
$$

We define $\mathrm{Gr}_{\mathcal{G},X}$ to be the fpqc quotient sheaf $L^+\mathcal{G}\backslash L\mathcal{G}$. By [PZ13, Proposition 6.5], $\mathrm{Gr}_{\mathcal{G},X}$ is representable by an ind-projective ind-scheme. By the properties of $\mathcal{G}$, the generic fiber $\mathrm{Gr}_{\mathcal{G},X} \times_X X^0$ is isomorphic to $\mathrm{Gr}_{\mathrm{GSp}_4} \times_{\mathbf{Z}} X^0$, a constant familiy of affine Grassmannian for the group GSp_4 over X^0, and its special fiber $\mathrm{Gr}_{\mathcal{G},X} \times_X X_0$ is the affine flag variety $\mathrm{Fl} := \mathcal{I}\backslash \mathrm{GSp}_4$.

Let $d \in \mathbf{Z}$ and $h \in \mathbf{Z}_{\geq 0}$. We define subfunctors $L\mathcal{G}^{\mathrm{sim}=d}, L\mathcal{G}^{\mathrm{sim}=d,\leq h} \subset L\mathcal{G}$ by

$$
\begin{aligned}
L\mathcal{G}^{\mathrm{sim}=d}(R) &= \left\{ g \in L\mathcal{G}(R) \mid \mathrm{sim}(\overline{g}) \in (v-t)^d (R[\![v-t]\!])^\times \right\} \\
L\mathcal{G}^{\mathrm{sim}=d,\leq h}(R) &= \left\{ g \in L\mathcal{G}^{\mathrm{sim}=d}(R) \mid \overline{g} \in \frac{1}{(v-t)^h} M_4(R[\![v-t]\!]) \right\}.
\end{aligned}
$$

Both of them are stable under left multiplication by $L^+\mathcal{G}$ and induce fpqc quotient subsheafs

$$
\mathrm{Gr}_{\mathcal{G},X}^{\mathrm{sim}=d,\leq h} := L^+\mathcal{G}\backslash L\mathcal{G}^{\mathrm{sim}=d,\leq h} \subset \mathrm{Gr}_{\mathcal{G},X}^{\mathrm{sim}=d} := L^+\mathcal{G}\backslash L\mathcal{G}^{\mathrm{sim}=d} \subset \mathrm{Gr}_{\mathcal{G},X}.
$$

The sheaf $\mathrm{Gr}_{\mathcal{G},X}^{\mathrm{sim}=d,\leq h}$ is representable by a projective scheme over X and $\mathrm{Gr}_{\mathcal{G},X}^{\mathrm{sim}=d} = \varinjlim_h \mathrm{Gr}_{\mathcal{G},X}^{\mathrm{sim}=d,\leq h}$.

Our next goal is describing affine open charts of the projective scheme $\mathrm{Gr}_{\mathcal{G},X}^{\mathrm{sim}=d,\leq h}$. We define the negative loop group to be a subfunctor $L^{--}\mathcal{G} \subset L\mathcal{G}$ such that for $t : \operatorname{Spec} R \to X$,

$$
L^{--}\mathcal{G}(R) = \left\{ g \in \mathcal{G}(R(\!(v-t)\!)) \;\middle|\; \begin{array}{c} \overline{g} \in \mathrm{GSp}_4\left(R\left[\frac{1}{v-t}\right]\right) \\ \overline{g} \bmod \frac{1}{v-t} \in \overline{U}(R) \\ \overline{g} \bmod \frac{v}{v-t} \in B\left(R\left[\frac{1}{v-t}\right] / \left(\frac{v}{v-t}\right)\right) \end{array} \right\}.
$$

Lemma 3.1.2. *The multiplication map*

$$
L^+\mathcal{G} \times_X L^{--}\mathcal{G} \to L\mathcal{G}
$$

is formally étale in the sense of [Le+a, Definition 3.2.4], and so is the map $L^{--}\mathcal{G} \to \mathrm{Gr}_{\mathcal{G},X}$.

Proof. Since this map is a restriction of a monomorphism $L^+\mathcal{G}_4 \times_X L^{--}\mathcal{G}_4 \to L\mathcal{G}_4$ ([Le+a, Lemma 3.2.2]), it is a monomorphism. Then we can show that it is formally étale following the argument in [Le+a, Lemma 3.2.6] using a version of [Le+a, Lemma 3.2.3] for our setup and the formal smoothness of $L^+\mathcal{G}$ and $L^{--}\mathcal{G}$. Note that formal smoothness easily follows from the smoothness of $\mathcal{G}$. ☐

Let $\widetilde{z} = wt_\nu \in \widetilde{W}^\vee$. We define $\mathcal{U}(\widetilde{z})$ to be the subfunctor of $L\mathcal{G}$ given by

$$\mathcal{U}(\widetilde{z})(R) = \left\{ g \in L\mathcal{G}(R) \,\middle|\, \begin{array}{c} \overline{g}(v-t)^{-\nu} \in \mathrm{GSp}_4(R[\frac{1}{v-t}]), \\ \overline{g}(v-t)^{-\nu}w^{-1} \bmod \frac{1}{v-t} \in \overline{U}(R), \\ \overline{g}(v-t)^{-\nu} \bmod \frac{v}{v-t} \in B(R[\frac{1}{v-t}]/(\frac{v}{v-t})) \end{array} \right\}.$$

Lemma 3.1.3. *The subfunctor $\mathcal{U}(\widetilde{z}) \subset L\mathcal{G}$ is stable under left multiplication by $L^{--}\mathcal{G}$ and is left $L^{--}\mathcal{G}$-torsor. The natural map $\mathcal{U}(\widetilde{z}) \to \mathrm{Gr}_{\mathcal{G},X}$ is monomorphism.*

Proof. The first claim follows from the definition of $\mathcal{U}(\widetilde{z})$ and the second claim follows from Lemma 3.1.2. ☐

For $d = \mathrm{sim}(\nu)$ and $h \in \mathbf{Z}_{\geq 0}$, we define $\mathcal{U}(\widetilde{z})^{\mathrm{sim},\leq h}$ to be the base change $\mathcal{U}(\widetilde{z}) \times_{L\mathcal{G}} L\mathcal{G}^{\mathrm{sim}=d,\leq h}$. The following Proposition shows that $\mathcal{U}(\widetilde{z})^{\mathrm{sim},\leq h}$ is represented by a finite type affine scheme over X.

Proposition 3.1.4. *For a Noetherian $\mathbf{Z}[v]$-algebra R, $\mathcal{U}(\widetilde{z})^{\mathrm{sim},\leq h}(R)$ is the set of matrices $A \in \mathrm{GSp}_4(R[(v-t)^{\pm 1}])$ satisfying:*

- *For $1 \leq i, j \leq 4$,*

$$A_{ij} = v^{\delta_{i>j}} \left(\sum_{k=-h}^{\nu'_j - \delta_{i>j} - \delta_{i<w'(j)}} c_{ij,k}(v-t)^k \right)$$

and $c_{w'(j)j,\nu'_j - \delta_{w'(j)>j}} = 1$ where $(\nu'_1, \nu'_2, \nu'_3, \nu'_4) = \mathrm{std}(\nu)$ and $w' = \mathrm{std}(w)$,

- $\mathrm{sim}(A) = \mathrm{sim}(w)(v-t)^d$.

Proof. This follows from the fact that $\mathcal{U}(\widetilde{z})^{\mathrm{sim},\leq h} = \mathcal{U}_4(\mathrm{std}(\widetilde{z}))^{\det,\leq h} \times_{L\mathcal{G}_4} L\mathcal{G}$ and Proposition 3.2.8 in [Le+a] (where $\mathcal{U}_4(\mathrm{std}(\widetilde{z}))^{\det,\leq h}$ denotes the affine chart defined in §3.2 of *loc. cit.*). ☐

Proposition 3.1.5. *The map $\mathcal{U}(\widetilde{z})^{\mathrm{sim},\leq h} \to \mathrm{Gr}_{\mathcal{G},X}^{\mathrm{sim}=d,\leq h}$ is an open immersion.*

Proof. We claim that $\mathcal{U}(\widetilde{z})^{\mathrm{sim},\leq h} \to \mathrm{Gr}_{\mathcal{G},X}^{\mathrm{sim}=d,\leq h}$ is formally étale. Since a formally étale monomorphism between finite type schemes is an open immersion ([Le+a, Remark 3.2.5] and [Stacks, Tag 025G]), this completes the proof.

Let A be an Artinian local ring with residue field k. Suppose we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\, k \arrow[r] \arrow[d] & \mathcal{U}(\widetilde{z})^{\mathrm{sim},\leq h} \arrow[d] \\ \mathrm{Spec}\, A \arrow[r] \arrow[ur, dashed] & \mathrm{Gr}_{\mathcal{G},X}^{\mathrm{sim}=d,\leq h}. \end{tikzcd}$$

We write $\overline{t} \in k$ and $t \in A$ for the image of the coordinate v of X. Since k and A are Noetherian, we can interpret the image of $\mathrm{Spec}\, k$ (and $\mathrm{Spec}\, A$) in $\mathcal{U}(\widetilde{z})$ as a matrix $g_k \in \mathrm{GSp}(k((v-\overline{t})))$ satisfying certain conditions. We claim that there exists $g_A \in \mathcal{U}(\widetilde{z})(A)$ lifting g_k. Since $g_k(v-\bar{t})^{-\nu} \in \mathrm{GSp}_4(k[\frac{1}{v-\bar{t}}])$, we can find $g_A \in L\mathcal{G}(A)$ lifting g_k and $g_A(v-t)^{-\nu} \in \mathrm{GSp}_4(A[\frac{1}{v-t}])$ by the smoothness of GSp_4. The set of such g_A is $\mathrm{Lie}\,\mathrm{GSp}_4(\mathfrak{m}_A[\frac{1}{v-t}])(v-t)^{\nu}$-coset. To show that there exists $g'_A \in \mathcal{U}(\widetilde{z})(A)$ lifting g_k, we need to find g'_A satisfying two conditions:

$$g'_A(v-t)^{-\nu}w^{-1} \bmod \frac{1}{v-t} \in \overline{U}(A),\ \ g'_A(v-t)^{-\nu} \bmod \frac{v}{v-t} \in B(A[\frac{1}{v-t}]/(\frac{v}{v-t})).$$

Thus, we need to find $N \in \mathrm{Lie}\,\mathrm{GSp}_4(\mathfrak{m}_A[\frac{1}{v-t}])$ such that

$$g_A(v-t)^{-\nu}w^{-1} + Nw^{-1} \bmod \frac{1}{v-t} \in \overline{U}(A),\ \ g_A(v-t)^{-\nu} + N \bmod \frac{v}{v-t} \in B(A[\frac{1}{v-t}]/(\frac{v}{v-t})).$$

The existence of such N follows from the surjectivity of the quotient map

$$\mathrm{Lie}\,\mathrm{GSp}_4(\mathfrak{m}_A[\frac{1}{v-t}]) \to \mathrm{Lie}\,\mathrm{GSp}_4(\mathfrak{m}_A) \times \mathrm{Lie}\,\mathrm{GSp}_4(\mathfrak{m}_A[\frac{1}{v-t}]/(\frac{v}{v-t})).$$

Once we know that $\mathcal{U}(\widetilde{z})(A) \neq \emptyset$, we can use Lemma 3.1.2 to find $\mathrm{Spec}\,A \to \mathcal{U}(\widetilde{z})$ which lifts g_k and when composed with $\mathcal{U}(\widetilde{z}) \to \mathrm{Gr}_{\mathcal{G},X}$ provides the A-point of $\mathrm{Gr}_{\mathcal{G},X}^{\mathrm{sim}=d,\leq h}$ given in the above diagram. Then it has to factor through $\mathcal{U}(\widetilde{z})^{\mathrm{sim},\leq h}$. Thus $\mathcal{U}(\widetilde{z})^{\mathrm{sim},\leq h} \to \mathrm{Gr}_{\mathcal{G},X}^{\mathrm{sim}=d,\leq h}$ is formally étale. □

3.2 Geometry of universal local models

We introduce universal local models and discuss their basic properties.

3.2.1 Schubert varieties

Given a dominant cocharacter $\lambda \in X_*(T^\vee)$, we denote by s_λ a section $X \to \mathrm{Gr}_{\mathcal{G},X}$ induced by the element $(v-t)^\lambda \in L\mathcal{G}(R)$ for any $\mathbf{Z}[v]$-algebra R. A *global Schubert variety* $\mathcal{S}_X(\lambda)$ is defined as the minimal irreducible closed subscheme of $\mathrm{Gr}_{\mathcal{G},X}$ containing the section s_λ and stable under the right multiplication of $L^+\mathcal{G}$ (cf. [Zhu14, Definition 3.1]). The map $S_X(\lambda) \to X$ is proper. We also write $\mathcal{S}_{X^0}(\lambda) = \mathcal{S}_X(\lambda) \times_X X^0$. As $\mathrm{Gr}_{\mathcal{G},X} \times_X X^0 \simeq \mathrm{Gr}_{\mathrm{GSp}_4} \times_{\mathbf{Z}} X^0$, $\mathcal{S}_{X^0}(\lambda)$ is the constant family of the Schubert variety in $\mathrm{Gr}_{\mathrm{GSp}_4}$ for λ over X^0.

Let $\mathrm{Conv}(\lambda)$ be the convex hull of the subset $W\lambda \subset X_*(T^\vee)$. A *open Schubert cell* $\mathcal{S}^\circ(X)$ is defined as an open subscheme

$$\mathcal{S}^\circ_X(\lambda) = \mathcal{S}_X(\lambda) \backslash \cup_{\lambda' \in \mathrm{Conv}(\lambda), \lambda' \notin W\lambda} \mathcal{S}_X(\lambda') \subset \mathcal{S}_X(\lambda).$$

Again, the base change $\mathcal{S}^\circ_{X^0}(\lambda) = \mathcal{S}^\circ_X(\lambda) \times_X X^0$ is the constant family of open Schubert cells of $\mathrm{Gr}_{\mathrm{GSp}_4}$ for λ over X^0.

We have a map $L^+\mathcal{G} \to \mathrm{Gr}_{\mathcal{G},X}$ given by the orbit map $g \mapsto s_\lambda g$. Note that it factors through a subscheme $\mathrm{Gr}_{\mathcal{G},X}^{\mathrm{sim}=d,\leq h}$ with $d = \mathrm{sim}(\lambda)$ and h sufficiently large. The stabilizer subgroup scheme $L^+\mathcal{G}_\lambda \subset L^+\mathcal{G}$ of s_λ is given by

$$L^+\mathcal{G}_\lambda(R) = L^+\mathcal{G}(R) \cap \mathrm{Ad}((v-t)^{-\lambda})(L^+\mathcal{G}(R))$$

for $t : \operatorname{Spec} R \to X$. Thus we have a monomorphism

$$L^+\mathcal{G}_\lambda \backslash L^+\mathcal{G} \hookrightarrow \mathrm{Gr}_{\mathcal{G},X}^{\mathrm{sim}=d,\leq h}$$

whose scheme-theoretic image is $\mathcal{S}_\lambda(X)$. Over X^0, we have an isomorphism $(L^+\mathcal{G}_\lambda\backslash L^+\mathcal{G}) \times_X X^0 \simeq \mathcal{S}^\circ_{X^0}(\lambda)$. Note that there is a map from $L^+\mathcal{G}_\lambda \times_X X^0$ to $P_\lambda \times X^0$ sending $g \mapsto g \bmod (v-t)$ where $P_\lambda \subset \mathrm{GSp}_4$ is a parabolic subgroup associated with λ and containing B. Thus, we have a natural map $(L^+\mathcal{G}_\lambda\backslash L^+\mathcal{G}) \times_X X^0 \to P_\lambda\backslash \mathrm{GSp}_4 \times_{\mathbf{Z}} X^0$ given by $g \mapsto g \bmod (v-t)$. When it is composed with the previous isomorphism, we get a map

$$\pi_\lambda : \mathcal{S}^\circ_{X^0}(\lambda) \to (P_\lambda\backslash \mathrm{GSp}_4) \times_{\mathbf{Z}} X^0.$$

3.2.2 Universal local models

For convenience, we let $\mathrm{std} : \mathbb{A}^3 \to \mathbb{A}^4$ be the morphism sending (a_1, a_2, a_3) to $(a_1, a_2, a_3 - a_2, a_3 - a_1)$. Note that this matches with the description of $\mathrm{std} : X_*(T^\vee) \to X_*(T_4^\vee)$.

We define a subfunctor $L\mathcal{G}^\nabla$ of $L\mathcal{G} \times_{\mathbf{Z}} \mathbb{A}^3$ given by

$$L\mathcal{G}^\nabla(R) = \left\{ (g, \mathbf{a}) \mid g \in L\mathcal{G}(R), \mathbf{a} \in \mathbb{A}^3, v\frac{d\overline{g}}{dv}\overline{g}^{-1} + \overline{g}\mathrm{Diag}(\mathrm{std}(\mathbf{a}))\overline{g}^{-1} \in \frac{1}{v-t} L^+\mathcal{M}(R) \right\}.$$

for $t : \operatorname{Spec} R \to X$. Since $L\mathcal{G}^\nabla$ is stable under left multiplication by $L^+\mathcal{G}$, it defines a closed sub-ind-scheme $\mathrm{Gr}^\nabla_{\mathcal{G},X} := L^+\mathcal{G}\backslash L\mathcal{G}^\nabla \subset \mathrm{Gr}_{\mathcal{G},X} \times_{\mathbf{Z}} \mathbb{A}^3$ which is ind-proper over $X \times_{\mathbf{Z}} \mathbb{A}^3$.

Definition 3.2.3. Let $\lambda \in X_*(T^\vee)$ be a dominant cocharacter. We define the *naive universal local model* $\mathcal{M}^{\mathrm{nv}}_X(\leq \lambda, \nabla)$ as $\mathrm{Gr}^\nabla_{\mathcal{G},X} \cap (\mathcal{S}_X(\lambda) \times_{\mathbf{Z}} \mathbb{A}^3)$.

We write $\mathcal{M}^{\mathrm{nv}}_{X^0}(\leq \lambda, \nabla) = \mathcal{M}^{\mathrm{nv}}_X(\leq \lambda, \nabla) \times_X X^0$. It is a proper scheme over $X^0 \times \mathbb{A}^3$.

Proposition 3.2.4. *Let $\lambda \in X_*(T^\vee)$ be a dominant cocharacter. The map π_λ induces an isomorphism*

$$\left(\mathcal{M}^{\mathrm{nv}}_{X^0}(\leq \lambda, \nabla) \cap (\mathcal{S}^\circ_{X^0}(\lambda) \times_{\mathbf{Z}} \mathbb{A}^3)\right)\left[\frac{1}{h_\lambda!}\right] \simeq (P_\lambda\backslash \mathrm{GSp}_4) \times_{\mathbf{Z}} X^0 \times_{\mathbf{Z}} \mathbb{A}^3 \left[\frac{1}{h_\lambda!}\right].$$

Proof. Let $\overline{N}_\lambda$ be the unipotent radical of opposite parabolic to P_λ. Recall the decomposition $\overline{N}_\lambda \simeq \prod U_{\alpha^\vee}$ where the product runs over $\alpha^\vee \in \Phi^\vee$ satisfying $\langle \phi(\alpha^\vee), \lambda \rangle < 0$. For any commutative ring R and $N \in \overline{N}_\lambda(R)$, we can write N as a product of $N_{\alpha^\vee} \in U_{\alpha^\vee}(R)$ over the same set of $\alpha^\vee$.

Note that $\{\overline{N}_\lambda w\}_{w \in W}$ forms an affine open cover of $P_\lambda\backslash \mathrm{GL}_n$. By pulling back along π_λ, we get an affine open cover of $\mathcal{S}^\circ_{X^0}(\lambda)$. More precisely, we have $\pi_\lambda^{-1}(\overline{N}_\lambda w) = \widetilde{N}_\lambda w$ where $\widetilde{N}_\lambda$ is the affine scheme over X^0 whose R-points for for $t : \operatorname{Spec} R \to X$ are given by

$$\widetilde{N}_\lambda(R) = \left\{ (v-t)^\lambda N \mid N \in \overline{N}_\lambda(R[v-t]), \deg N_{\alpha^\vee} \leq \langle \phi(\alpha^\vee), -\lambda \rangle - 1 \right\}.$$

Let us write $N_{\alpha^\vee} = \sum_{j=0}^{\langle \phi(\alpha^\vee), -\lambda \rangle - 1} X_{\alpha^\vee, j}(v-t)^j$ with $X_{\alpha^\vee, j} \in R$. From the proof of [Le+a, Proposition 3.3.4], we can deduce that taking the intersection $\mathrm{Gr}^\nabla_{\mathcal{G},X} \cap (\widetilde{N}_\lambda w \times_{\mathbf{Z}} \mathbb{A}^3)\left[\frac{1}{h_\lambda!}\right]$ imposes conditions that $X_{\alpha^\vee, j}$ for $\alpha^\vee$ and $j > 0$ is determined by $X_{\alpha'^\vee, 0}$ for $\alpha' \leq \alpha < 0$. Since the map

$$\pi_\lambda : \mathrm{Gr}^\nabla_{\mathcal{G},X} \cap (\widetilde{N}_\lambda w \times_{\mathbf{Z}} \mathbb{A}^3)\left[\frac{1}{h_\lambda!}\right] \to \overline{N}_\lambda w \times_{\mathbf{Z}} X^0 \times_{\mathbf{Z}} \mathbb{A}^3 \left[\frac{1}{h_\lambda!}\right]$$

takes $(v-t)^{\lambda}Nw$ to $(N \bmod (v-t))w$, this proves that π_λ is an isomorphism. $\square$

Remark 3.2.5. The above Proposition implies that $\mathcal{M}^{\mathrm{nv}}_{X^0}(\leq \lambda, \nabla) \cap (\mathcal{S}^\circ_{X^0}(\lambda) \times_{\mathbf{Z}} \mathbb{A}^3)\left[\frac{1}{h_\lambda!}\right]$ is indeed proper over $X^0 \times_{\mathbf{Z}} \mathbb{A}^3\left[\frac{1}{h_\lambda!}\right]$ (by the properness of partial flag varieties). Thus the map

$$\mathcal{M}^{\mathrm{nv}}_{X^0}(\leq \lambda, \nabla) \cap (\mathcal{S}^\circ_{X^0}(\lambda) \times_{\mathbf{Z}} \mathbb{A}^3)\left[\frac{1}{h_\lambda!}\right] \hookrightarrow \mathcal{M}^{\mathrm{nv}}_{X^0}(\leq \lambda, \nabla)\left[\frac{1}{h_\lambda!}\right]$$

is a proper open immersion. Since proper morphism is universally closed, this map is an inclusion of a connected component. Using this inductively, we obtain an isomorphism (cf. [Le+a, Corollary 3.3.5])

$$\left(\mathcal{M}^{\mathrm{nv}}_{X^0}(\leq \lambda, \nabla)\left[\frac{1}{h_\lambda!}\right]\right)_{\mathrm{red}} \xrightarrow{\sim} \coprod_{\lambda' \leq \lambda, \lambda' \in X_*^+(T^\vee)} (P_{\lambda'}\backslash \mathrm{GSp}_4) \times_{\mathbf{Z}} X^0 \times \mathbf{Z}\mathbb{A}^3\left[\frac{1}{h_\lambda!}\right].$$

Definition 3.2.6. Let $\lambda \in X_*(T^\vee)$ be a dominant cocharacter. The *universal local model* $\mathcal{M}_X(\lambda, \nabla)$ is the closure of the connected component $\mathcal{M}^{\mathrm{nv}}_{X^0}(\leq \lambda, \nabla) \cap (\mathcal{S}^\circ_{X^0}(\lambda) \times_{\mathbf{Z}} \mathbb{A}^3)$ of $\mathcal{M}^{\mathrm{nv}}_{X^0}(\leq \lambda, \nabla)$ inside $\mathcal{M}^{\mathrm{nv}}_{X}(\leq \lambda, \nabla)$. In particular, it is v-flat.

Proposition 3.2.7. *Let $\lambda \in X_*(T^\vee)$ be a dominant cocharacter. Then $\mathcal{M}^{\mathrm{nv}}_{X^0}(\leq \lambda, \nabla)\left[\frac{1}{(2h_\lambda)!}\right]$ is smooth over $X^0 \times_{\mathbf{Z}} \mathbb{A}^3\left[\frac{1}{(2h_\lambda)!}\right]$.*

Proof. Since the proof of [Le+a, Proposition 3.2.6] generalizes to our situation quite straight forwardly, we only sketch the idea of the proof. We follow the notation in *loc. cit.* and write $Y = \mathcal{M}^{\mathrm{nv}}_{X^0}(\leq \lambda, \nabla)\left[\frac{1}{(2h_\lambda)!}\right]$, $S = X^0 \times_{\mathbf{Z}} \mathbb{A}^3\left[\frac{1}{(2h_\lambda)!}\right]$, and pr for the natural projection map $Y \to S$.

There is a $T^\vee$-action on Y induced by the right multiplication by $T^\vee$ on $\mathcal{G}$. Note that this preserves the non-smooth locus in Y. Since pr is proper and the non-smooth locus in Y is closed, the image of the non-smooth locus in S is closed as well. If the non-smooth locus is non-empty, it has a non-zero geometric fiber which is a proper variety over a field with $T^\vee$-action. Thus, such fiber contains a $T^\vee$-fixed point. Since it is contained in a geometric fiber of $\mathcal{S}_{X^0}(\lambda)$, it is known to be contained in the support of a section $s_\mu : X^0 \to \mathcal{S}_{X^0}(\lambda)$ for some $\mu \in \mathrm{Conv}(\lambda)$ (see Lemma 3.3.7 in *loc. cit.*).

Thus, it suffices to prove the smoothness at a closed point $x \in \operatorname{Spec} k \to Y$ lying in the support of s_μ. Let $s = \mathrm{pr}(x)$. We claim that $\dim T_x Y/S \leq \dim_x Y_s$. If we have this bound, the completion $\widehat{\mathcal{O}}_{Y,x}$ is generated over $\widehat{\mathcal{O}}_{S,s}$ by $\dim_x Y_s$ many variables. Since S is regular, $\dim \widehat{\mathcal{O}}_{Y,x} \leq \dim \widehat{\mathcal{O}}_{S,s} + \dim_x Y_s$ where the equality holds if and only if $\widehat{\mathcal{O}}_{Y,x}$ is a power series ring over $\widehat{\mathcal{O}}_{S,s}$ with $\dim_x Y_s$ many variables. By Remark 3.2.5 and the choice of x, we have

$$\dim_x Y_s = \dim P_\mu \backslash \mathrm{GSp}_4, \ \ \dim_x Y = \dim P_\mu \backslash \mathrm{GSp}_4 + 4$$

Thus, the equality holds and pr is smooth at x by [Stacks, Tag 07VH].

Finally, we prove the claimed inequality. Following the argument in [Le+a, Proposition 3.3.6] and using Lemma 3.1.4, we can show that $T_x Y_s$ is a subspace of the space of matrices of the form $(1+\epsilon X)(v-t)^\mu$ where $X \in \operatorname{Lie} \mathrm{GSp}_4(k(\!(v-t)\!))$ whose entries are polynomials with degree bounds, and X satisfies certain equation imposed by $\mathrm{Gr}^{\nabla}_{\mathcal{G},X^0}$. More precisely, the computation in *loc. cit.* shows that all diagonal entries of X are zero, and for each root α, the α-th entry of X is zero if $\langle \mu, \alpha^\vee \rangle \geq 0$ and is determined by the coefficient of the lowest degree term if $\langle \mu, \alpha^\vee \rangle < 0$. Thus the space of such X has dimension at most $\dim P_\mu \backslash \mathrm{GSp}_4$. This completes the proof. $\square$

We finish this section with two lemmas regarding the normalization of the universal local model. For an integral scheme Y, we let $Y^{\mathrm{nm}} \to Y$ denote the normalization of Y.

Let $l > 0$ be an integer. It will be useful to consider the base change

$$\mathcal{M}_X(\lambda, \nabla)_l := \mathcal{M}_X(\lambda, \nabla) \times_{X, v \mapsto v^l} X$$

and its normalization $(\mathcal{M}_X(\lambda, \nabla)_l)^{\mathrm{nm}}$. When $l = e$ and if we can take $\varpi = (-p)^{1/e}$ as the uniformizer in $\mathcal{O}$, this has the advantage that the map $\operatorname{Spec} \mathcal{O} \to X \times \mathbb{A}^3$ sending v to $-p$ is the composite of the maps $\operatorname{Spec} \mathcal{O} \to X \times \mathbb{A}^3$ sending $v \mapsto \varpi$ and $X \times \mathbb{A}^3 \to X \times \mathbb{A}^3$ sending $v \mapsto v^e$. In particular, we have

$$\mathcal{M}_X(\lambda, \nabla) \times_{X \times \mathbb{A}^3, v \mapsto -p} \operatorname{Spec} \mathcal{O} = \mathcal{M}_X(\lambda, \nabla)_e \times_{X \times \mathbb{A}^3, v \mapsto \varpi} \operatorname{Spec} \mathcal{O}.$$

Lemma 3.2.8. *There is an open subscheme $U \subset \mathbb{A}^3$ only depending on λ and l, such that $\mathcal{M}_X(\lambda, \nabla)_l \times_{\mathbb{A}^3} U \to X \times_{\mathbf{Z}} U$ and $(\mathcal{M}_X(\lambda, \nabla)_l)^{\mathrm{nm}} \times_{\mathbb{A}^3} U \to X \times_{\mathbf{Z}} U$ are flat.*

Proof. This follows from [Le+a, Lemma 3.5.5, Remark 3.5.6]. □

Lemma 3.2.9. *There is an open subscheme $U \subset \mathbb{A}^3$ only depending on λ and l, such that if R is complete DVR and $f : \operatorname{Spec} R \to X \times \mathbb{A}^3$ is a morphism sending v to a uniformizer of R and factors through $X \times U$, the base change*

$$(\mathcal{M}_X(\lambda, \nabla)_l)^{\mathrm{nm}}_R := (\mathcal{M}_X(\lambda, \nabla)_l)^{\mathrm{nm}} \times_{X \times \mathbb{A}^3} \operatorname{Spec} R$$

is flat over $\operatorname{Spec} R$, and $(\mathcal{M}_X(\lambda, \nabla)_l)^{\mathrm{nm}}_R$ is normal.

Proof. This follows from [Le+a, Proposition 3.5.2]. To satisfy Setup 3.5.1 of *loc. cit.*, take $S = \mathbb{A}^3[\frac{1}{(2h_\lambda)!}]$, $M = (\mathcal{M}_X(\lambda, \nabla)_l)^{\mathrm{nm}} \times_{\mathbb{A}^3} S$, and use Proposition 3.2.7. □

3.3 Local models in mixed characteristic

We specialize to objects over $\mathcal{O}$ by taking base change $\operatorname{Spec} \mathcal{O} \to X$ sending v to $-p$. For a fpqc sheaf $Y \to X$, we write $Y_{\mathcal{O}} = Y \times_X \operatorname{Spec} \mathcal{O}$. For example, we have $L\mathcal{G}_{\mathcal{O}} = L\mathcal{G} \times_X \operatorname{Spec} \mathcal{O}$ and $L^+\mathcal{G}_{\mathcal{O}}, L^+\mathcal{M}_{\mathcal{O}}$ similarly.

We have the global affine Grassmannian $\mathrm{Gr}_{\mathcal{G},\mathcal{O}} = L^+\mathcal{G}_{\mathcal{O}} \backslash L\mathcal{G}_{\mathcal{O}}$. Its generic fiber $\mathrm{Gr}_{\mathcal{G},E}$ is equal to the usual affine Grassmannian associated to GSp_4 over E. For $\lambda \in X_*(T^\vee)$ a dominant cocharacter, we write $S^\circ_E(\lambda) \subset \mathrm{Gr}_{\mathrm{GSp}_4,E}$ for the open affine Schubert cell and $S_E(\lambda)$ for its reduced closure. The Zariski clousre of $S_E(\lambda)$ in $\mathrm{Gr}_{\mathcal{G},\mathcal{O}}$ is the Pappas–Zhu local model $M(\leq \lambda)$ associated to the group GSp_4, the conjugacy class of λ, and the Iwahori subgroup $\mathcal{I}$ ([PZ13]). It is known that $M(\leq \lambda)$ is projective over $\operatorname{Spec} \mathcal{O}$ (§7.1 in *loc. cit.*).

Let $\mathbf{a} \in \mathcal{O}^3$. We define $L\mathcal{G}^{\nabla_{\mathbf{a}}}_{\mathcal{O}} \subset L\mathcal{G}_{\mathcal{O}}$ to be the subfunctor given by

$$L\mathcal{G}^{\nabla_{\mathbf{a}}}_{\mathcal{O}}(R) := \left\{ g \in L\mathcal{G}_{\mathcal{O}}(R) \mid v\frac{d\overline{g}}{dv}\overline{g}^{-1} + \overline{g}\mathrm{Diag}(\mathrm{std}(\mathbf{a}))\overline{g}^{-1} \in \frac{1}{v+p} L^+\mathcal{M}_4(R) \right\}$$

for an $\mathcal{O}$-algebra R. It is stable under left $L^+\mathcal{G}_{\mathcal{O}}$ multiplication and induces a closed sub-ind-scheme $\mathrm{Gr}^{\nabla_{\mathbf{a}}}_{\mathcal{G},\mathcal{O}} \subset \mathrm{Gr}_{\mathcal{G},\mathcal{O}}$.

Definition 3.3.1. We define the *naive local model* as $M^{\mathrm{nv}}(\leq \lambda, \nabla_{\mathbf{a}}) = M(\leq \lambda) \cap \mathrm{Gr}_{\mathcal{G},\mathcal{O}}^{\nabla_{\mathbf{a}}}$. The *(mixed characteristic) local model* $M(\lambda, \nabla_{\mathbf{a}})$ is defined to be the Zariski closure of $S_E^{\circ}(\lambda) \cap \mathrm{Gr}_{\mathcal{G},\mathcal{O}}^{\nabla_{\mathbf{a}}}$ in $M(\leq \lambda)$. By its construction, $M(\lambda, \nabla_{\mathbf{a}})$ is projective and flat over $\operatorname{Spec} \mathcal{O}$.

We define $U(\widetilde{z}) := \mathcal{U}(\widetilde{z}) \times_X \operatorname{Spec} \mathcal{O}$ and $U(\widetilde{z}, \lambda, \nabla_{\mathbf{a}}) := U(\widetilde{z}) \cap M(\lambda, \nabla_{\mathbf{a}})$ where the intersection is taken inside $\mathrm{Gr}_{\mathcal{G},\mathcal{O}}$. Note that the latter is equal to $(\mathcal{U}(\widetilde{z})^{\mathrm{sim}, \leq h} \times_X \operatorname{Spec} \mathcal{O}) \cap M(\lambda, \nabla_{\mathbf{a}})$ for large enough $h \geq 0$. Thus, it is an open affine subscheme of $M(\lambda, \nabla_{\mathbf{a}})$ by Proposition 3.1.5.

Our main interest in the geometry of $M(\lambda, \nabla_{\mathbf{a}})$ is whether a completed local ring $\mathcal{O}^{\wedge}_{M(\lambda, \nabla_{\mathbf{a}}), x}$ is a domain for some $x \in M(\lambda, \nabla_{\mathbf{a}})$. To understand this property more geometrically, we introduce the following definition.

Definition 3.3.2. Let Y be a scheme. A point $y \in Y$ is called *unibranch* if the normalization of the local ring $(\mathcal{O}_{Y,y})_{\mathrm{red}}$ is local.

Remark 3.3.3. Suppose that Y is Noetherian and excellent. The followings are equivalent:

1. $y \in Y$ is unibranch;
2. the fiber above y of the normalization map $Y^{\mathrm{nm}} \to Y$ is a single point ([Stacks, Tag 0C3B]);
3. the completed local ring $\mathcal{O}^{\wedge}_{Y,y}$ is a domain ([Stacks, Tag 0C2E]).

Let $\widetilde{z} = wt_{\nu} \in \widetilde{W}^{\vee}$. There is a constant section $\widetilde{z} : \operatorname{Spec} \mathbf{Z} \hookrightarrow \mathcal{U}(\widetilde{z})^{\mathrm{sim}, \leq h} \times_X X_0$ for h large enough, given by $wv^{\nu} \in \mathrm{GSp}_4(\mathbf{Z}(\!(v)\!))$. We denote its composition with $\mathcal{U}(\widetilde{z})^{\mathrm{sim}, \leq h} \times_X X_0 \hookrightarrow \mathrm{Gr}_{\mathcal{G},X} \times_X X_0$ again by $\widetilde{z}$.

Let $\mathbf{a} \in \mathbb{A}^3(\mathcal{O})$. We write the induced $\mathbf{F}$ point by $\mathbf{a}$ as well. We write $\widetilde{z}_{\mathbf{F},\mathbf{a}}$ for the $\mathbf{F}$-point $\operatorname{Spec} \mathbf{F} \to (\mathrm{Gr}_{\mathcal{G},X} \times_X X_0) \times_{\mathbf{Z}} \mathbb{A}^3$ given by $(\widetilde{z}, \mathbf{a})$.

Recall that e denotes the ramification index of the extension $\mathcal{O}/\mathbf{Z}_p$.

Theorem 3.3.4. *There exists a non-empty open subscheme $U \subset \mathbb{A}^3$ depending only on λ and e, such that if $\mathbf{a} \in U(\mathcal{O})$, then $M(\lambda, \nabla_{\mathbf{a}})$ is unibranch at any point $\widetilde{z}_{\mathbf{F},\mathbf{a}}$ contained in the special fiber. In addition, $\mathcal{O}(U(\widetilde{z}, \lambda, \nabla_{\mathbf{a}}))^{\wedge_p}$ is a domain.*

To prove this Theorem, we introduce the local model in equal characteristic and prove its unibranch property.

Definition 3.3.5. Let $\mathbf{a} \in \mathbb{A}^3(\mathbf{F})$. We define $\mathcal{M}^{\mathrm{nv}}(\leq \lambda, \nabla_{\mathbf{a}}) := \mathcal{M}_X^{\mathrm{nv}}(\leq \lambda, \nabla) \times_{\mathbb{A}^3} \mathbf{a}$. We define the *equal characteristic local model* $\mathcal{M}(\lambda, \nabla_{\mathbf{a}})$ by the Zariski closure of

$$\left(\mathcal{M}_{X^0}^{\mathrm{nv}}(\leq \lambda, \nabla) \cap (\mathcal{S}_{X^0}^{\circ}(\lambda) \times_{\mathbf{Z}} \mathbb{A}^3)\right) \times_{\mathbb{A}^3} \mathbf{a}$$

in $\mathcal{M}^{\mathrm{nv}}(\leq \lambda, \nabla_{\mathbf{a}})$.

Remark 3.3.6. Although mixed or equal characteristic local models are generally *not* equal to base changes of universal local models, their generic fibers can be obtained by taking base changes of the generic fibers of universal local models. Suppose that $h_{\lambda} < p$. By taking fiber product at E-point $(-p, \mathbf{a}) \in X \times \mathbb{A}^3$ (resp. at $\mathbf{F}$-point $\mathbf{a}$), we also have

$$\mathcal{M}_{X^0}(\lambda, \nabla) \times_{X \times \mathbb{A}^3} \operatorname{Spec} E = M(\lambda, \nabla_{\mathbf{a}}) \times_{\mathcal{O}} \operatorname{Spec} E$$
$$\mathcal{M}_{X^0}(\lambda, \nabla) \times_{\mathbb{A}^3} \operatorname{Spec} \mathbf{F} = \mathcal{M}(\lambda, \nabla_{\mathbf{a}}) \times_X X^0.$$

Moreover, $M(\lambda, \nabla_{\mathbf{a}})$ (resp. $\mathcal{M}(\lambda, \nabla_{\mathbf{a}})$) is characterized as p-flat (resp. v-flat) closure of $\mathcal{M}_{X^0}(\lambda, \nabla) \times_{X \times \mathbb{A}^3} \operatorname{Spec} E$ inside $\mathcal{M}_X(\lambda, \nabla) \times_{X \times \mathbb{A}^3} \operatorname{Spec} \mathcal{O}$ (resp. $\mathcal{M}_{X^0}(\lambda, \nabla) \times_{X \times \mathbb{A}^3} \operatorname{Spec} E$ inside $\mathcal{M}_X(\lambda, \nabla) \times_{\mathbb{A}^3} \operatorname{Spec} \mathbf{F}$).

For $l \in \mathbf{Z}_{>0}$ and an X-scheme Y, we write Y_l for the base change $Y \times_{X, v \mapsto v^l} X$.

Proposition 3.3.7 (cf. Proposition 3.4.4 in [Le+a])**.** *Let $l > 0$ be an integer. Then $\mathcal{M}(\lambda, \nabla_{\mathbf{a}})_l$ is unibranch at any $\widetilde{z}_{\mathbf{F},\mathbf{a}}$ contained in its special fiber. Moreover, the preimage of $\mathcal{U}(\widetilde{z})$ in $(\mathcal{M}(\lambda, \nabla_{\mathbf{a}})_l)^{\mathrm{nm}} \times_X X_0$ is connected.*

Proof. This mainly follows from [Le+a, Lemma 3.4.7, 3.4.8]. Indeed, we can take a one-parameter subgroup in Lemma 3.4.7 in *loc. cit.* valued in $T^\vee \times \mathbf{G}_m$ (instead of $T_4^\vee \times \mathbf{G}_m$) whose induced action on $\mathcal{U}(\widetilde{z})^{\mathrm{sim}, \leq h}$ (instead of $\mathcal{U}_4(\widetilde{z})^{\det, \leq h}$) satisfies the conditions stated in Lemma 3.4.7 in *loc. cit.*. Using this, the proof of [Le+a, Proposition 3.4.4] can be applied in our case too. $\square$

Proof of Theorem 3.3.4. The novel idea in [Le+a] is comparing the mixed characteristic and equal characteristic local models inside the universal local model. Let U be an open subscheme $U \subset \mathbb{A}^3$ satisfying Lemma 3.2.8 and 3.2.9 for $l = e$ and $(2h_\lambda)!e!$ is invertible. Note that this implies $U(\mathcal{O}) = \emptyset$ unless $p > (2h_\lambda)!e!$. Thus we can assume that $\mathcal{O}/\mathbf{Z}_p$ is tame and take $\varpi = (-p)^{1/e}$ (after enlarging the residue field if necessary). Let $\mathbf{a} \in U(\mathcal{O})$. The following diagram explains how to make such comparison.

$$\begin{tikzcd}
 & M(\lambda, \nabla_{\mathbf{a}})^{\mathrm{nm}} \arrow[dl, dashed, "\sim"'] \arrow[d] & \\
(\mathcal{M}_X(\lambda, \nabla)_e)^{\mathrm{nm}} \times_{X \times U} \operatorname{Spec} \mathcal{O} \arrow[r] \arrow[d] & M(\lambda, \nabla_{\mathbf{a}}) \arrow[r] \arrow[d] & \operatorname{Spec} \mathcal{O} \arrow[d, "(v \mapsto \varpi, \mathbf{a})"] \\
(\mathcal{M}_X(\lambda, \nabla)_e)^{\mathrm{nm}} \arrow[r] & \mathcal{M}_X(\lambda, \nabla)_e \arrow[r] & X \times U \\
(\mathcal{M}_X(\lambda, \nabla)_e)^{\mathrm{nm}} \times_{X \times U} \mathbb{A}^1_{\mathbf{F}} \arrow[r] \arrow[u] & \mathcal{M}(\lambda, \nabla_{\mathbf{a}})_e \arrow[r] \arrow[u] & \mathbb{A}^1_{\mathbf{F}} \arrow[u, "(v \mapsto v, \mathbf{a})"'] \\
 & (\mathcal{M}(\lambda, \nabla_{\mathbf{a}})_e)^{\mathrm{nm}} \arrow[ul, dashed, two heads] \arrow[u] &
\end{tikzcd}$$

By Lemma 3.2.8, $\mathcal{M}_X(\lambda, \nabla)_e$ is flat over $X \times U$. Thus, all rectangles are cartesian by Remark 3.3.6. Moreover, two base changes of normalization map $(\mathcal{M}_X(\lambda, \nabla)_e)^{\mathrm{nm}} \to \mathcal{M}_X(\lambda, \nabla)_e$ are finite and birational. Finiteness is obvious, and birationality is preserved by base change because the dense open subscheme $\mathcal{M}_{X^0}(\lambda, \nabla)_e \subset \mathcal{M}_X(\lambda, \nabla)_e$, which is already normal by Proposition 3.2.7, is still dense after each base change. This induces two surjective dashed arrows by [Stacks, Tag 035Q], and the top dashed arrow is an isomorphism by Lemma 3.2.9.

Note that $M(\lambda, \nabla_{\mathbf{a}})$ and $\mathcal{M}(\lambda, \nabla_{\mathbf{a}})$ share the same special fiber. Suppose that $M(\lambda, \nabla_{\mathbf{a}})$ is *not* unibranch at $\widetilde{z}_{\mathbf{F},\mathbf{a}}$. Then there are at least two points in the preimage of $\widetilde{z}_{\mathbf{F},\mathbf{a}}$ in $M(\lambda, \nabla_{\mathbf{a}})^{\mathrm{nm}}$. Therefore, the preimage of $\widetilde{z}_{\mathbf{F},\mathbf{a}}$ in $(\mathcal{M}(\lambda, \nabla)_e)^{\mathrm{nm}}$ contains at least two points. In turn, this implies that the preimage of $\widetilde{z}_{\mathbf{F},\mathbf{a}}$ in $(\mathcal{M}_X(\lambda, \nabla)_e)^{\mathrm{nm}} \times_{X \times U} \mathbb{A}^1_{\mathbf{F}}$ contains at least two points. By the surjectivity of the bottom dashed map, this contradicts Proposition 3.3.7. This proves that $M(\lambda, \nabla_{\mathbf{a}})$ is unibranch at $\widetilde{z}_{\mathbf{F},\mathbf{a}}$.

Note that $\mathcal{O}(U(\widetilde{z}, \lambda, \nabla_{\mathbf{a}}))[1/\varpi]$ is regular domain by Proposition 3.2.7. Also, the preimage of $U(\widetilde{z}, \lambda, \nabla_{\mathbf{a}})$ in the special fiber of $M(\lambda, \nabla_{\mathbf{a}})^{\mathrm{nm}}$ is connected by Proposition 3.3.7 and the above diagram. Then $\mathcal{O}(U(\widetilde{z}, \lambda, \nabla_{\mathbf{a}}))^{\wedge_p}$ is a domain by [Le+a, Lemma 3.7.2]. $\square$

3.4 Products of local models

We now generalize Theorem 3.3.4 to products of local models. Let $\mathcal{J}$ be a finite set. Let $\lambda = (\lambda_j)_{j\in\mathcal{J}} \in X_*(T^\vee)^{\mathcal{J}}$ be a dominant cocharacter and $\widetilde{z} = (\widetilde{z}_j)_{j\in\mathcal{J}} \in \widetilde{W}^{\vee,\mathcal{J}}$. We define

$$\mathcal{M}_{X,\mathcal{J}}(\lambda,\nabla) = \prod_{j\in\mathcal{J}} \mathcal{M}_X(\lambda_j,\nabla) \subset (\mathrm{Gr}_{\mathcal{G},X} \times_{\mathbf{Z}} \mathbb{A}^3)^{\mathcal{J}}.$$

Let $\mathbf{a} = (\mathbf{a}_j)_{j\in\mathcal{J}} \in (\mathbb{A}^3)^{\mathcal{J}}(\mathcal{O})$. We also define $M_{\mathcal{J}}(\lambda,\nabla_{\mathbf{a}}) := \prod_{j\in\mathcal{J}} M(\lambda_j,\nabla_{\mathbf{a}_j})$ and $U_{\mathcal{J}}(\widetilde{z},\lambda,\nabla_{\mathbf{a}})) := \prod_{j\in\mathcal{J}} U(\widetilde{z}_j,\lambda_j,\nabla_{\mathbf{a}_j})$. The following is the main result of this chapter.

Theorem 3.4.1. *There exists a non-empty open subscheme $U \subset (\mathbb{A}^3)^{\mathcal{J}}$ depending only on λ and e, such that if $\mathbf{a} \in U(\mathcal{O})$, then $M_{\mathcal{J}}(\lambda,\nabla_{\mathbf{a}})$ is unibranch at any point $\widetilde{z}_{\mathbf{F},\mathbf{a}}$ contained in its special fiber. In addition, $\mathcal{O}(U_{\mathcal{J}}(\widetilde{z},\lambda,\nabla_{\mathbf{a}}))^{\wedge_p}$ is a domain.*

Proof. Let $M_j := \mathcal{M}_X(\lambda_j,\nabla) \to X \times \mathbb{A}^3$ and $\widetilde{z}_j \in \widetilde{W}^\vee$. The special fiber $M_j \times_X X_0$ intersect with a section $\widetilde{z}_j$ only if $\widetilde{z}_j = wt_\nu$ for some $w \in W$ and $\nu \in \mathrm{Conv}(\lambda_j)$. Note that there are finitely many $\widetilde{z}_j$ satisfying these conditions.

Let η be the generic point of $\mathbb{A}^3$. We define Fix_j to be the set of $\widetilde{z}_j \in \widetilde{W}^\vee$ which intersects with $M_j \times_{X\times\mathbb{A}^3} (X_0 \times \eta)$. For any $\widetilde{z}_j$ such that $\widetilde{z}_j = wt_\nu$ for some $w \in W$ and $\nu \in \mathrm{Conv}(\lambda_j)$, consider a map

$$f_{\widetilde{z}_j} : (M_j \times_X X_0) \cap \widetilde{z} \to \mathbb{A}^3.$$

(Here, the intersection is taken inside $\mathrm{Gr}_{\mathcal{G},X} \times_X X_0 \times \mathbb{A}^3$.) The image of $f_{\widetilde{z}_j}$ is constructible, and it contains η if and only if $\widetilde{z}_j \in Fix_j$. Note that any constructible set containing the generic point of irreducible scheme contains an open neighborhood of the generic point. Thus, there exists an open neighborhood V'_j of η which is contained in the image of $f_{\widetilde{z}_j}$ for all $\widetilde{z}_j \in Fix_j$ and the complement of the image of $f_{\widetilde{z}_j}$ for all $\widetilde{z}_j \notin Fix_j$.

Let V_j be the intersection of V'_j with the open subscheme of $\mathbb{A}^3$ satisfying the conclusion of Lemma 3.2.8 for λ_j and e. Then $M_j|_{V_j} \to X \times V_j$ and $\widetilde{z}_j \in Fix_j$ satisfy the assumptions of [Le+a, Corollary 3.6.2]. As a result, there exists an integer e' and an open subscheme $U'_j \subset V_j$ satisfying the following: for any $\widetilde{z}_j \in Fix_j$ and $\mathbf{a}_j \in U_j(\mathcal{O})$, there exists a finite DVR extension $\mathcal{O}'$ of $\mathcal{O}$ of degree $\leq e'$ and an $\mathcal{O}'$-point of M_j lifting $(\widetilde{z}_j, \mathbf{a}_j) \in M(\lambda_j,\nabla_{\mathbf{a}_j})(\mathbf{F})$.

Let $U_j \subset U'_j$ be the open subscheme in which Theorem 3.3.4 holds for λ_j and $ee'!$. Then we define $U = \prod_{j\in\mathcal{J}} U_j$. Let $\mathcal{O}'/\mathcal{O}$ be the extension obtained by adjoining $e'!$-th root of uniformizer and $e'!$-th root of unity. Note that $\mathcal{O}'$ contains any extension of $\mathcal{O}$ of degree $\leq e'$. For $\widetilde{z} \in \prod_{j\in\mathcal{J}} Fix_j$ and $\mathbf{a} \in U(\mathcal{O})$, $M_{\mathcal{J}}(\lambda,\nabla_{\mathbf{a}})$ is unibranch at $\widetilde{z}_{\mathbf{F},\mathbf{a}}$ if and only if the completed local ring of $M_{\mathcal{J}}(\lambda,\nabla_{\mathbf{a}})$ at $\widetilde{z}_{\mathbf{F},\mathbf{a}}$ is a domain, which is a subring of the completed local ring of $M_{\mathcal{J}}(\lambda,\nabla_{\mathbf{a}}) \times_{\mathcal{O}} \mathrm{Spec}\,\mathcal{O}'$ at $\widetilde{z}_{\mathbf{F},\mathbf{a}}$. Similarly, $\mathcal{O}(U_{\mathcal{J}}(\widetilde{z},\lambda,\nabla_{\mathbf{a}}))^{\wedge_p}$ is a subring of $\mathcal{O}(U_{\mathcal{J}}(\widetilde{z},\lambda,\nabla_{\mathbf{a}} \times_{\mathcal{O}} \mathrm{Spec}\,\mathcal{O}'))^{\wedge_p}$. Thus, it suffices to prove the claim after replacing $\mathcal{O}$ by $\mathcal{O}'$. Note that both the completed local ring of $M_{\mathcal{J}}(\lambda,\nabla_{\mathbf{a}}) \times_{\mathcal{O}} \mathrm{Spec}\,\mathcal{O}'$ at $\widetilde{z}_{\mathbf{F},\mathbf{a}}$ and $\mathcal{O}(U_{\mathcal{J}}(\widetilde{z},\lambda,\nabla_{\mathbf{a}} \times_{\mathcal{O}} \mathrm{Spec}\,\mathcal{O}'))^{\wedge_p}$ are completed tensor products of the completed local ring of $M(\lambda_j,\nabla_{\mathbf{a}_j}) \times_{\mathcal{O}} \mathrm{Spec}\,\mathcal{O}'$ at $\widetilde{z}_{j,\mathbf{F},\mathbf{a}_j}$ and $\mathcal{O}(U_{\mathcal{J}}(\widetilde{z}_j,\lambda_j,\nabla_{\mathbf{a}_j} \times_{\mathcal{O}} \mathrm{Spec}\,\mathcal{O}'))^{\wedge_p}$, which are known to be domain by Theorem 3.3.4. By the choice of $\mathcal{O}'$, each $M_j \times_{\mathcal{O}} \mathrm{Spec}\,\mathcal{O}'$ has $\mathcal{O}'$-point lifting $(\widetilde{z}_j, \mathbf{a}_j)$ in its special fiber. Then the two claims follow from [KW09b, Proposition 2.2], and [Bar+14, Lemma A.1.1] respectively (as explained in the last two paragraphs of the proof of [Le+a, Theorem 3.7.1]). $\square$

3.5 Special fiber of naive local models in mixed characteristic

In the remainder of this chapter, we study the special fiber of the naive local models in mixed characteristic. Recall that the naive local models in mixed characteristic are defined by imposing the monodromy condition on the Pappas–Zhu local models inside $\mathrm{Gr}_{\mathcal{G},\mathcal{O}} = L^+\mathcal{G}_{\mathcal{O}}\backslash L\mathcal{G}_{\mathcal{O}}$. The special fiber $\mathrm{Gr}_{\mathcal{G},\mathcal{O}} \times_{\mathcal{O}} \operatorname{Spec} \mathbf{F}$ is the affine flag variety $\mathrm{Fl} := \mathcal{I}_{\mathbf{F}}\backslash L(\mathrm{GSp}_4)_{\mathbf{F}}$. Given $\mathbf{a} \in \mathcal{O}^3$, we define $\mathrm{Fl}^{\nabla_{\mathbf{a}}} := \mathrm{Fl} \times_{\mathrm{Gr}_{\mathcal{G},\mathcal{O}}} \mathrm{Gr}_{\mathcal{G},\mathcal{O}}^{\nabla_{\mathbf{a}}}$.

It is known by [PZ13, Theorem 9.3] that the special fiber $M(\leq \lambda)_{\mathbf{F}}$ is the reduced union of the affine Schubert cells $S^\circ_{\mathbf{F}}(\widetilde{z})$ for $\widetilde{z} \in \mathrm{Adm}^\vee(\lambda)$. Thus, the underlying reduced closed subscheme of $M^{\mathrm{nv}}(\leq \lambda, \nabla_{\mathbf{a}})_{\mathbf{F}}$ is equal to the reduced union of $S^\circ_{\mathbf{F}}(\widetilde{z}) \cap \mathrm{Fl}^{\nabla_{\mathbf{a}}}$ for $\widetilde{z} \in \mathrm{Adm}^\vee(\lambda)$.

For $\alpha \in \Phi$, we define $H_\alpha^{(0,1)} = \{x \in X^*(T) \times_{\mathbf{Z}} \mathbf{R} \mid 0 < \langle x, \alpha^\vee\rangle < 1\}$. The following is the main result of this section.

Theorem 3.5.1. *Let $h > 0$ be an integer, $\widetilde{w} \in \widetilde{W}$, and $\mathbf{a} \in \mathcal{O}^3$. Suppose that $\widetilde{w}$ is h-small and $\mathbf{a} \bmod \varpi \in \mathbf{F}^3$ is h-generic (Definition 2.1.8). Then $S^\circ_{\mathbf{F}}(\widetilde{w}^*) \cap \mathrm{Fl}^{\nabla_{\mathbf{a}}}$ is an affine space of dimension $4 - \#\left\{\alpha \in \Phi^+ \mid \widetilde{w}(A_0) \subset H_\alpha^{(0,1)}\right\}$.*

As a Corollary, we get a classification of top($=4$)-dimensional irreducible components of $M^{\mathrm{nv}}(\leq \lambda, \nabla_{\mathbf{a}})_{\mathbf{F}}$. If $d \in \mathbf{Z}_{\geq 0}$ and X is a reduced scheme, we let $\mathrm{Irr}_d X$ denote the set of d-dimensional irreducible closed subschemes of X.

Definition 3.5.2. We say that $\widetilde{w} \in \widetilde{W}$ is *regular* if $\widetilde{w}^*(A_0) \not\subseteq H_\alpha^{(0,1)}$ for any $\alpha \in \Phi^+$. Let $\lambda \in X_*(T^\vee)$. We define $\mathrm{Adm}^\vee_{\mathrm{reg}}(\lambda) \subset \mathrm{Adm}^\vee(\lambda)$ to be the subset of $\widetilde{z} \in \mathrm{Adm}^\vee(\lambda)$ such that $\widetilde{z}^*$ is regular. We similarly define $\mathrm{Adm}_{\mathrm{reg}}(\nu)$ for $\nu \in X^*(T)$. Then $(-)^*$ induces a bijection between $\mathrm{Adm}_{\mathrm{reg}}(\nu)$ and $\mathrm{Adm}^\vee_{\mathrm{reg}}(\phi(\nu))$.

Corollary 3.5.3. *Let $\lambda \in X^*(T^\vee)$ be a dominant cocharacter. Suppose that $\mathbf{a} \bmod \varpi \in \mathbf{F}^n$ is h_λ-generic. There is a bijection*

$$\begin{aligned} \mathrm{Adm}^\vee_{\mathrm{reg}}(\lambda) &\xrightarrow{\sim} \mathrm{Irr}_4 M^{\mathrm{nv}}(\leq \lambda, \nabla_{\mathbf{a}})_{\mathbf{F}} \\ \widetilde{z} &\mapsto \overline{(S^\circ_{\mathbf{F}}(\widetilde{z}) \cap \mathrm{Fl}^{\nabla_{\mathbf{a}}})}. \end{aligned}$$

Proof. This follows from [PZ13, Theorem 9.3] and Theorem 3.5.1. To apply Theorem 3.5.1, note that $\widetilde{z} \in \mathrm{Adm}^\vee_{\mathrm{reg}}(\lambda)$ is h_λ-small. ☐

To prove the Theorem 3.5.1, we describe $S^\circ_{\mathbf{F}}(\widetilde{z})$ using explicit coordinates. A version of Bruhat decomposition says that we have a double coset decomposition

$$(L\mathrm{GSp}_4)_{\mathbf{F}} = \cup_{\widetilde{z} \in \widetilde{W}^\vee} \mathcal{I}_{\mathbf{F}} \widetilde{z} \mathcal{I}_{\mathbf{F}}$$

and the open Schubert cell $S^\circ_{\mathbf{F}}(\widetilde{z})$ can be identified with $\mathcal{I}_{\mathbf{F}}\backslash \mathcal{I}_{\mathbf{F}}\widetilde{z}\mathcal{I}_{\mathbf{F}}$.

Let $L^{--}\mathcal{G}_{\mathbf{F}} \subset L\mathcal{G}_{\mathbf{F}}$ be the subfunctor given by

$$L^{--}\mathcal{G}_{\mathbf{F}}(R) = \left\{ g \in \mathrm{GSp}_4\left(R\left[\frac{1}{v}\right]\right) \mid g \bmod \frac{1}{v} \in \overline{B}(R)\right\}$$

for any $\mathbf{F}$-algebra R. We define $N_{\widetilde{z}} := \widetilde{z}^{-1} L^{--}\mathcal{G}_{\mathbf{F}} \widetilde{z} \cap \mathcal{I}_{\mathbf{F}}$.

Recall that the duality isomorphism ϕ identifies a coroot $\alpha^\vee$ of G and a root $\phi(\alpha^\vee)$ of $G^\vee$, and we write $U_{\alpha^\vee}$ for the root subgroup of $G^\vee$ associated to $\phi(\alpha^\vee)$. Given $(\alpha^\vee, m) \in \Phi^\vee \times \mathbf{Z}$, we let $U_{\alpha^\vee,m} \subset LU_{\alpha^\vee}$ denote the subfunctor such that for any $\mathbf{F}$-algebra R, $U_{\alpha^\vee,m}(R) \subset U_{\alpha^\vee}(R(\!(v)\!))$ is identified with $v^m R \subset R(\!(v)\!)$ under the isomorphism $U_{\alpha^\vee} \simeq \mathbf{G}_a$.

In what follows, we fix $x \in A_0$. Note that $\lceil \langle x, \alpha^\vee \rangle \rceil = \delta_{\alpha^\vee > 0}$ and $\lfloor \langle x, \alpha^\vee \rangle \rfloor = -\delta_{\alpha^\vee < 0}$.

Proposition 3.5.4. *Let $\widetilde{z} \in \widetilde{W}^\vee$.*

1. *We have the following decomposition*

$$N_{\widetilde{z}} \simeq \prod_{(\alpha^\vee, m) \in \Phi^\vee_{\widetilde{z}}} U_{-\alpha^\vee, m}$$

where the product runs over the set $\Phi^\vee_{\widetilde{z}} = \{(\alpha^\vee, m) \in \Phi^\vee \times \mathbf{Z} \mid \langle x, \alpha^\vee \rangle < m < \langle \widetilde{z}^*(x), \alpha^\vee \rangle\}$.

2. *There is an isomorphism*

$$\mathcal{I}_{\mathbf{F}} \times_{\mathbf{F}} \{\widetilde{z}\} \times_{\mathbf{F}} N_{\widetilde{z}} \simeq \mathcal{I}_{\mathbf{F}} \widetilde{z} \mathcal{I}_{\mathbf{F}}$$

given by multiplication. This induces an isomorphism $\widetilde{z} N_{\widetilde{z}} \simeq S^\circ_{\mathbf{F}}(\widetilde{z})$.

Proof. 1. We claim that $U_{-\alpha^\vee, m} \subset N_{\widetilde{z}}$ if and only if $(\alpha^\vee, m) \in \Phi^\vee_{\widetilde{z}}$. It is easy to check that $U_{-\alpha^\vee, m} \subset \mathcal{I}_{\mathbf{F}}$ if and only if $\delta_{\alpha^\vee > 0} \leq m$. Let $\widetilde{z}^* = s t_\nu \in \widetilde{W}$. Direct computation shows that $\widetilde{z} U_{-\alpha^\vee, m} \widetilde{z}^{-1} = U_{-s^{-1}(\alpha^\vee), m - \langle \nu, \alpha^\vee \rangle}$. Thus $U_{-\alpha, m} \subset \widetilde{z}^{-1} L^{--} \mathcal{G}_{\mathbf{F}} \widetilde{z}$ if and only if $m - \langle \nu, \alpha^\vee \rangle < \langle x, s^{-1}(\alpha^\vee) \rangle$, or equivalently $m < \langle s(x) + \nu, \alpha^\vee \rangle$. This proves the claim.

Observe that $U_{-\alpha^\vee, m} \subset N_{\widetilde{z}}$ only if $\langle \widetilde{z}^*(x), \alpha^\vee \rangle > 0$. Let $w \in W$ be the unique element such that $w \widetilde{z}^* \in \widetilde{W}^+$. If $\langle \widetilde{z}^*(x), \alpha^\vee \rangle = \langle w \widetilde{z}^*(x), w(\alpha^\vee) \rangle > 0$, then $w(\alpha^\vee) > 0$. In other words, $\phi(w) N_{\widetilde{z}} \phi(w)^{-1} \subset L\overline{U}$. Then the claimed decomposition follows from the decomposition of $\overline{U}$.

2. The injectivity of the multiplication map is obvious. The surjectivity follows from $\mathcal{I}_{\mathbf{F}} = ((\widetilde{z}^{-1} \mathcal{I}_{\mathbf{F}} \widetilde{z}) \cap \mathcal{I}_{\mathbf{F}}) N_{\widetilde{z}}$.

□

Let $d_{\alpha^\vee, \widetilde{z}} = \lfloor \langle \widetilde{z}^*(x), \alpha^\vee \rangle \rfloor - \lceil \langle x, \alpha^\vee \rangle \rceil$.

Corollary 3.5.5. *Let R be a Noetherian $\mathbf{F}$-algebra. If $M \in N_{\widetilde{z}}(R)$, we can write $M = \prod_{\alpha^\vee \in \Phi^\vee} v^{\delta_{\alpha^\vee > 0}} M_{\alpha^\vee}$ where $M_{\alpha^\vee} \in U_{-\alpha^\vee}(R[v]) \simeq R[v]$ is a polynomial of degree at most $d_{\alpha^\vee, \widetilde{z}}$.*

Proof of Theorem 3.5.1. Let $\widetilde{z} = \widetilde{w}^*$ and $w \in W$ be the unique element such that $w\widetilde{w} \in \widetilde{W}^+$. We follow the proof of [Le+a, Theorem 4.2.4] using the Corollary 3.5.5 instead of Corollary 4.2.12 of *loc. cit.*. Indeed, *loc. cit.* shows that the monodromy condition on $\widetilde{z}M$ for some $M \in N_{\widetilde{z}}(R)$ implies that coefficients in the polynomial $M_{\alpha^\vee}$ are determined by its top degree coefficient and coefficients of $M_{\alpha'^\vee}$ such that $w(\alpha'^\vee) < w(\alpha^\vee)$. Inductively, this shows that $S^\circ_{\mathbf{F}}(\widetilde{z}) \cap \mathrm{Fl}^{\nabla_{\mathbf{a}}}$ is an affine space with coordinates given by the top degree coefficients of $M_{\alpha^\vee}$. Thus, the dimension of $S^\circ_{\mathbf{F}}(\widetilde{z}) \cap \mathrm{Fl}^{\nabla_{\mathbf{a}}}$ is equal to

$$\#\left\{\alpha^\vee \in \Phi^\vee \mid U_{-\alpha^\vee, m} \subset N_{\widetilde{z}} \text{ for some } m \geq 0\right\}.$$

This is equal to the number of $\alpha^\vee \in \Phi^\vee$ such that $\langle \widetilde{z}^*(x), \alpha^\vee \rangle > 0$ and $\widetilde{z}^*(x)$ and x does not lie in the same α-strip. The first condition says that $w(\alpha^\vee) \in \Phi^{\vee,+}$, which has size 4. For such α, the second condition holds unless $\alpha^\vee \in \Phi^{\vee,+}$ and $\widetilde{z}^*(A_0) \subset H^{(0,1)}_\alpha$. This completes the proof. □

We now make comparison between irreducible components in $M^{\mathrm{nv}}(\leq \lambda, \nabla_{\mathbf{a}})_{\mathbf{F}}$ for different choices of λ and $\mathbf{a}$. We define $L\mathrm{GSp}_4^{\nabla_0} \subset L\mathrm{GSp}_4$ as the subsheaf given by

$$R \mapsto \left\{ A \in L\mathrm{GSp}_4(R) \mid v\frac{dA}{dv}A^{-1} \in \frac{1}{v}L^+\mathcal{M}_{\mathbf{F}}(R) \right\}$$

for $\mathbf{F}$-algebra R. Let Fl^{∇_0} denote the fpqc-quotient sheaf $\mathcal{I}_{\mathbf{F}} \backslash L\mathrm{GSp}_4^{\nabla_0}$.

Remark 3.5.6. Let $\mathbf{a} \in \mathcal{O}^3$ and $\widetilde{z} = s^{-1}t_\mu \in \widetilde{W}^\vee$ such that $\mathbf{a} \equiv s^{-1}(\mu) \bmod \varpi$. Choose a dominant cocharacter $\lambda \in X_*(T^\vee)$ and $\widetilde{w} \in \widetilde{W}$. A direct computation shows that (cf. [Le+a, Proposition 4.3.1])

$$\begin{aligned} M^{\mathrm{nv}}(\leq \lambda, \nabla_{\mathbf{a}})_{\mathbf{F}}\widetilde{z} &= M(\leq \lambda)_{\mathbf{F}}\widetilde{z} \cap \mathrm{Fl}^{\nabla_0} \\ (S^\circ_{\mathbf{F}}(\widetilde{w}^*) \cap \mathrm{Fl}^{\nabla_{\mathbf{a}}})\widetilde{z} &= (S^\circ_{\mathbf{F}}(\widetilde{w}^*)\widetilde{z}) \cap \mathrm{Fl}^{\nabla_0}. \end{aligned}$$

This allows us to compare $M^{\mathrm{nv}}(\leq \lambda, \nabla_{\mathbf{a}})_{\mathbf{F}}$ for different λ and $\mathbf{a}$ and its irreducible components inside Fl^{∇_0}.

Definition 3.5.7. Let $\widetilde{s} \in \widetilde{W}$ and $\widetilde{w}_1, \widetilde{w}_2 \in \widetilde{W}^+$. We define

$$\begin{aligned} S^\circ_{\mathbf{F}}(\widetilde{w}_1, \widetilde{w}_2, \widetilde{s}) &:= S^\circ_{\mathbf{F}}((\widetilde{w}_2^{-1}w_0\widetilde{w}_1)^*)\widetilde{s}^* \subset \mathrm{Fl} \\ S^\circ_{\mathbf{F}}(\widetilde{w}_1, \widetilde{w}_2, \widetilde{s})^{\nabla_0} &:= S^\circ_{\mathbf{F}}(\widetilde{w}_1, \widetilde{w}_2, \widetilde{s}) \cap \mathrm{Fl}^{\nabla_0} \subset \mathrm{Fl}^{\nabla_0} \\ S^{\nabla_0}_{\mathbf{F}}(\widetilde{w}_1, \widetilde{w}_2, \widetilde{s}) &:= \overline{S^\circ_{\mathbf{F}}(\widetilde{w}_1, \widetilde{w}_2, \widetilde{s})^{\nabla_0}} \end{aligned}$$

where the closure is taken in Fl^{∇_0}.

Lemma 3.5.8. *Let $\widetilde{s} = t_\mu s$, $\widetilde{w}_1$, and $\widetilde{w}_2$ be as above. If further $\widetilde{w}_2^{-1}w_0\widetilde{w}_1$ is m-small and $\widetilde{s}$ is m-generic for some integer m, then $S^{\nabla_0}_{\mathbf{F}}(\widetilde{w}_1, \widetilde{w}_2, \widetilde{s})$ is irreducible closed subvariety of Fl^{∇_0} of dimension 4.*

Proof. By [Le+a, Proposition 2.1.5], $\widetilde{w}_2^{-1}w_0\widetilde{w}_1$ is regular, and $S^{\nabla_0}_{\mathbf{F}}(\widetilde{w}_1, \widetilde{w}_2, \widetilde{s})$ is isomorphic to $S^\circ_{\mathbf{F}}(\widetilde{w}^*) \cap \mathrm{Fl}^{\nabla_{\mathbf{a}}}$ by Remark 3.5.6, where $\widetilde{w} = \widetilde{w}_2^{-1}w_0\widetilde{w}_1$ and $\mathbf{a} \in \mathbf{Z}^3$ such that $\mathbf{a} \equiv \phi(s^{-1}(\mu)) \bmod p$. Then the claim follows from Theorem 3.5.1. $\square$

In fact, many of $S^{\nabla_0}_{\mathbf{F}}(\widetilde{w}_1, \widetilde{w}_2, \widetilde{s})$ for different $\widetilde{s}, \widetilde{w}_1$, and $\widetilde{w}_2$ are equal.

Proposition 3.5.9. *Let $\widetilde{s} \in \widetilde{W}$ and $\widetilde{w}_1, \widetilde{w}_2 \in \widetilde{W}^+$.*

1. *Suppose that, for $i = 1, 2$, $\widetilde{w}_i$ is m_i-small for some integer m_i and $\widetilde{s}$ is $(m_1 + m_2)$-generic. Then we have*

$$S^{\nabla_0}_{\mathbf{F}}(\widetilde{w}_1, \widetilde{w}_2, \widetilde{s}) = S^{\nabla_0}_{\mathbf{F}}(\widetilde{w}_1, e, \widetilde{s}\widetilde{w}_2^{-1}).$$

2. *Suppose that $\widetilde{w}_1$ is in $\widetilde{W}_1^+$ and $\widetilde{s}$ is 3-generic. For all $w \in W$, we have*

$$S^{\nabla_0}_{\mathbf{F}}(\widetilde{w}_1, e, \widetilde{s}) = S^{\nabla_0}_{\mathbf{F}}(\widetilde{w}_1, e, \widetilde{s}w).$$

Proof. The first item is [Le+a, Proposition 4.3.5], and the second item is Proposition 4.3.6 in *loc. cit.*. Note that their proofs generalize to our setup straightforwardly. $\square$

Definition 3.5.10. Let $(\widetilde{w}_1, \omega) \in \widetilde{W}_1^+ \times X^*(T)$ with t_ω being 3-generic. We define $C_{(\widetilde{w}_1, \omega)} := S^{\nabla_0}_{\mathbf{F}}(\widetilde{w}_1, e, \widetilde{s})$ for any $\widetilde{s} \in \widetilde{W}$ such that $\widetilde{s}(0) = \omega$.

This is well-defined by Proposition 3.5.9. Since $\widetilde{w}_1$ is 3-small, $C_{(\widetilde{w}_1,\omega)}$ is irreducible of dimension 4 by Lemma 3.5.8.

Let $\lambda \in X_*(T^\vee)$ be a regular dominant cocharacter. Recall that the set of admissible pair

$$\mathrm{AP}(\phi^{-1}(\lambda)) = \left\{(\widetilde{w}_1, \widetilde{w}_2) \in (\widetilde{W}_1^+ \times \widetilde{W}^+)/X^0(T) \mid \widetilde{w}_1 \uparrow t_{\phi^{-1}(\lambda)-\eta}\widetilde{w}_h^{-1}\widetilde{w}_2\right\}.$$

By [Le+a, Corollary 2.1.7], there is a bijection

$$\begin{aligned} \mathrm{AP}(\phi^{-1}(\lambda)) &\xrightarrow{\sim} \mathrm{Adm}^\vee_{\mathrm{reg}}(\lambda) \\ (\widetilde{w}_1, \widetilde{w}_2) &\mapsto (\widetilde{w}_2^{-1} w_0 \widetilde{w}_1)^*. \end{aligned} \tag{3.5.11}$$

Theorem 3.5.12. *Let $\lambda \in X_*(T^\vee)$ be a regular dominant cocharacter and $\mathbf{a} \in \mathcal{O}^3$. Let $\widetilde{z} = s^{-1}t_\mu \in \widetilde{W}^\vee$ be $(h_\lambda + 3)$-generic such that $\mathbf{a} \equiv \phi(s^{-1}(\mu)) \bmod \varpi$. We have a bijection*

$$\begin{aligned} \mathrm{AP}(\phi^{-1}(\lambda)) &\xrightarrow{\sim} \mathrm{Irr}_4(M^{\mathrm{nv}}(\leq \lambda, \nabla_{\mathbf{a}})_{\mathbf{F}}\widetilde{z}) \\ (\widetilde{w}_1, \widetilde{w}_2) &\mapsto C_{(\widetilde{w}_1, \widetilde{z}^*\widetilde{w}_2^{-1}(0))}. \end{aligned}$$

Proof. By Corollary 3.5.3, Remark 3.5.6, and (3.5.11), there is a bijection

$$\begin{aligned} \mathrm{AP}(\phi^{-1}(\lambda)) &\xrightarrow{\sim} \mathrm{Irr}_4(M^{\mathrm{nv}}(\leq \lambda, \nabla_{\mathbf{a}})_{\mathbf{F}}\widetilde{z}) \\ (\widetilde{w}_1, \widetilde{w}_2) &\mapsto S_{\mathbf{F}}^{\nabla_0}(\widetilde{w}_1, \widetilde{w}_2, \widetilde{z}^*). \end{aligned}$$

Then the claim follows from the equlities

$$S_{\mathbf{F}}^{\nabla_0}(\widetilde{w}_1, \widetilde{w}_2, \widetilde{z}^*) = S_{\mathbf{F}}^{\nabla_0}(\widetilde{w}_1, e, \widetilde{z}^*\widetilde{w}_2^{-1}) = C_{(\widetilde{w}_1, \widetilde{z}^*\widetilde{w}_2^{-1}(0))}$$

by Proposition 3.5.9, the genericity assumption on $\widetilde{z}$, and noting that $\widetilde{w}_1$ is 3-small and $\widetilde{w}_2$ is h_λ-small. $\square$

3.6 Matching irreducible components and Serre weights

Let $\mathcal{I}_{1,F} \subset \mathcal{I}_{\mathbf{F}}$ be the subfunctor given by

$$R \mapsto \{A \in \mathrm{GSp}_4(R[\![v]\!]) \mid A \bmod v \in U(R)\}$$

for $\mathbf{F}$-algebra R. We define $\widetilde{\mathrm{Fl}}$ as the fpqc-quotient sheaf $\mathcal{I}_{1,\mathbf{F}} \backslash L\mathrm{GSp}_4$. Then the natural quotient map $\Psi : \widetilde{\mathrm{Fl}} \to \mathrm{Fl}$ is a $T_{\mathbf{F}}^\vee$-torsor.

If $X \subset \mathrm{Fl}$ is a closed subvariety, we let $\widetilde{X}$ denote the pullback $X \times_{\mathrm{Fl}} \widetilde{\mathrm{Fl}}$. Then $\widetilde{X} \to X$ is again a $T_{\mathbf{F}}^\vee$-torsor. Let $\lambda \in X_*(T^\vee)$ be a dominant cocharacter and $\mathbf{a} \in \mathcal{O}^3$. Then we have the following $T_{\mathbf{F}}^\vee$-torsors

$$\begin{aligned} \widetilde{M}(\leq \lambda)_{\mathbf{F}} &\to M(\leq \lambda)_{\mathbf{F}} \\ \widetilde{M}^{\mathrm{nv}}(\leq \lambda, \nabla_{\mathbf{a}})_{\mathbf{F}} &\to M^{\mathrm{nv}}(\leq \lambda, \nabla_{\mathbf{a}})_{\mathbf{F}} \\ \widetilde{M}(\lambda, \nabla_{\mathbf{a}})_{\mathbf{F}} &\to M(\lambda, \nabla_{\mathbf{a}})_{\mathbf{F}}. \end{aligned}$$

For $(\widetilde{w}, \omega) \in \widetilde{W}_1^+ \times X^*(T)$ such that t_ω is 3-generic, we also have a $T_{\mathbf{F}}^\vee$-torsor $\widetilde{C}_{(\widetilde{w},\omega)} \to C_{(\widetilde{w},\omega)}$.

Let $\mathcal{J} = \mathrm{Hom}(k, \mathbf{F})$. For $j \in \mathcal{J}$, if $X_j \subset \mathrm{Fl}$ (resp. $\widetilde{X}_j \subset \widetilde{\mathrm{Fl}}$) is a closed subvariety, we let $X_{\mathcal{J}} := \prod_{j\in\mathcal{J}} X_j$ (resp. $\widetilde{X}_{\mathcal{J}} := \prod_{j\in\mathcal{J}} \widetilde{X}_j$) to be the closed subvariety in $\mathrm{Fl}^{\mathcal{J}}$ (resp. $\widetilde{\mathrm{Fl}}^{\mathcal{J}}$). In particular, given a dominent cocharacter $\lambda = (\lambda_j)_{j\in\mathcal{J}} \in X_*(\underline{T}^\vee)$ and $\mathbf{a} = (\mathbf{a}_j)_{j\in\mathcal{J}} \in (\mathcal{O}^3)^{\mathcal{J}}$, we have

$$M^{\mathrm{nv}}_{\mathcal{J}}(\le \lambda, \nabla_{\mathbf{a}})_{\mathbf{F}} = \prod_{j\in\mathcal{J}} M(\le \lambda_j, \nabla_{\mathbf{a}_j})_{\mathbf{F}}$$
$$\widetilde{M}^{\mathrm{nv}}_{\mathcal{J}}(\le \lambda, \nabla_{\mathbf{a}})_{\mathbf{F}} = \prod_{j\in\mathcal{J}} \widetilde{M}(\le \lambda_j, \nabla_{\mathbf{a}_j})_{\mathbf{F}}.$$

Let $\widetilde{z} = (\widetilde{z}_j)_{j\in\mathcal{J}} \in \underline{\widetilde{W}}^\vee$ such that $\widetilde{z}_j = s_j^{-1} t_{\mu_j}$ and $\mathbf{a}_j \equiv s_j^{-1}(\mu_j) \bmod \varpi$. We have the following cartesian diagram

$$\begin{tikzcd}
\widetilde{M}^{\mathrm{nv}}_{\mathcal{J}}(\le \lambda, \nabla_{\mathbf{a}})_{\mathbf{F}} \arrow[r, hook, "r_{\widetilde{z}}"] \arrow[d] & \widetilde{\mathrm{Fl}}^{\nabla_0}_{\mathcal{J}} \arrow[d] \\
M^{\mathrm{nv}}_{\mathcal{J}}(\le \lambda, \nabla_{\mathbf{a}})_{\mathbf{F}} \arrow[r, hook, "r_{\widetilde{z}}"] & \mathrm{Fl}^{\nabla_0}_{\mathcal{J}}
\end{tikzcd}$$

where $r_{\widetilde{z}}$ is right translation by $\widetilde{z}$. Let $(\widetilde{w}_1, \widetilde{w}_2) \in \mathrm{AP}(\phi^{-1}(\lambda)) = \prod_{j\in\mathcal{J}} \mathrm{AP}(\phi^{-1}(\lambda_j))$. If $\widetilde{z}$ is $(h_\lambda + 3)$-generic, by Theorem 3.5.12, we have a $7f$-dimensional irreducible component of $\widetilde{M}^{\mathrm{nv}}(\le \lambda, \nabla_{\mathbf{a}})_{\mathbf{F}}\widetilde{z}$ given by

$$\widetilde{C}_{(\widetilde{w}_1, \widetilde{z}^* \widetilde{w}_2^{-1}(0))} := \prod_{j\in\mathcal{J}} \widetilde{C}_{(\widetilde{w}_{1,j}, \widetilde{z}_j^* \widetilde{w}_{2,j}^{-1}(0))}.$$

Definition 3.6.1. Let σ be a 3-deep Serre weight and ζ be an algebraic central character (Definition 2.5.4) lifting central character of σ. Let $(\widetilde{w}_1, \omega) \in \underline{\widetilde{W}}_1^+ \times X^*(\underline{T})$ be the lowest alcove presentation of σ corresponding to ζ. We define $C^\zeta_\sigma := C_{(\widetilde{w}_1, \omega)}$ and $\widetilde{C}^\zeta_\sigma := \widetilde{C}_{(\widetilde{w}_1, \omega)}$.

Theorem 3.6.2. *Let $\lambda \in X_*(\underline{T}^\vee)$ be a regular dominant cocharacter. Let R be a Deligne–Lusztig representation with $\max\{6, h_\lambda + 3\}$-generic lowest alcove presentation (s, μ) that is $(\lambda - \eta)$-compatible with $\zeta \in X^*(\underline{Z})$. Let $\mathbf{a} \in (\mathcal{O}^3)^{\mathcal{J}}$ such that $\mathbf{a} \equiv \phi(s^{-1}(\mu + \eta)) \bmod \varpi$. Then we have*

$$\mathrm{Irr}_{4f}\left((M^{\mathrm{nv}}_{\mathcal{J}}(\le \lambda, \nabla_{\mathbf{a}})_{\mathbf{F}})\phi(s^{-1}t_{\mu+\eta})\right) = \left\{C^\zeta_\sigma \mid \sigma \in \mathrm{JH}(\overline{R} \otimes W(\phi^{-1}(\lambda) - \eta))\right\}.$$

Proof. This follows from Proposition 2.5.6 and Theorem 3.5.12. □

3.7 Torus fixed points

The torus $T^{\vee,\mathcal{J}}$ acts on $\mathrm{Fl}^{\mathcal{J}}$ by right translation and stablizes $\mathrm{Fl}^{\nabla_0}_{\mathcal{J}}$. The natural inclusion $\widetilde{W}^{\vee,\mathcal{J}} \subset \mathrm{Fl}^{\mathcal{J}}$ (sending $wt_\nu \in \widetilde{W}^{\vee,\mathcal{J}}$ to $wv^\nu \in \mathrm{Fl}^{\mathcal{J}}(\mathbf{F})$) identifies $\widetilde{W}^{\vee,\mathcal{J}}$ and the set of $T^{\vee,\mathcal{J}}$-fixed points in $\mathrm{Fl}^{\mathcal{J}}$. Note that $\widetilde{W}^{\vee,\mathcal{J}}$ is contained in $\mathrm{Fl}^{\nabla_0}_{\mathcal{J}}$. In this section, we prove the following result on the $T^{\vee,\mathcal{J}}$-fixed points of irreducible components in $\mathrm{Fl}^{\nabla_0}_{\mathcal{J}}$.

Theorem 3.7.1. *Let σ be a 3-deep Serre weight with a lowest alcove presentation $(\widetilde{w}_1, \omega)$. Let $C^{T^{\vee,\mathcal{J}}}_{(\widetilde{w}_1,\omega)}$ be the set of $T^{\vee,\mathcal{J}}$-fixed points in $C_{(\widetilde{w}_1,\omega)}$.*

1. *We have $\left\{(t_\omega w \widetilde{w}_1)^* \mid w \in W^{\mathcal{J}}\right\} \subset C^{T^{\vee,\mathcal{J}}}_{(\widetilde{w}_1,\omega)} \subset \{(t_\omega \widetilde{w})^* \mid \widetilde{w} \in \widetilde{W}^{\mathcal{J}}, \widetilde{w} \le w_0 \widetilde{w}_1\}$.*

2. *For each $j \in \mathcal{J}$, there is a polynomial $P_{\widetilde{w}_{1,j}} \in \mathbf{Z}[X_1, X_2, X_3]$ depending only on $\widetilde{w}_{1,j} \in \widetilde{W}_1^+$ such that if $P_{\widetilde{w}_{1,j}}(\omega_j) \not\equiv 0 \mod p$, then $C^{T^\vee,\mathcal{J}}_{(\widetilde{w}_1,\omega)} = \{(t_\omega \widetilde{w})^* \mid \widetilde{w} \in \widetilde{W}^{\mathcal{J}}, \widetilde{w} \leq w_0\widetilde{w}_1\}$*

Proof. It suffices to prove the claim for $\mathcal{J} = \{*\}$. By definition, $C_{(\widetilde{w}_1,\omega)} = S_{\mathbf{F}}^{\nabla_0}(\widetilde{w}_1, e, t_\omega)$. By Proposition 3.5.9, for any $w \in W$, the latter is equal to $S_{\mathbf{F}}^{\nabla_0}(\widetilde{w}_1, e, t_\omega w w_0)$ which contains $(t_\omega w \widetilde{w}_1)^*$. This proves the first inclusion in item (1).

Since $C_{(\widetilde{w}_1,\omega)}$ is contained in $S_{\mathbf{F}}(\widetilde{w}_1^* w_0 t_\omega)$, the second inclusion in the item (2) follows from the standard result on torus fixed points in Schubert varieties.

Finally, we prove item (2) by following the proof of [Le+a, Proposition 4.7.3]. Since the proof of *loc. cit.* generalizes to our setup, we only sketch the argument. Let $\widetilde{w} \in \widetilde{W}$ and $\widetilde{w} \leq w_0\widetilde{w}_1$. By Lemma 4.7.1 in *loc. cit.*, $(t_\omega\widetilde{w})^* \in C^{T^\vee}_{(\widetilde{w}_1,\omega)}$ if and only if $S^\circ_{\mathbf{F}}(\widetilde{w}_1^* w_0) \cap \mathrm{Fl}^{\nabla_{\phi(\omega)}}$ has nonempty intersection with the open neighborhood $L^{--}\mathcal{G}\widetilde{w}^*$. Note that here we interpret $\phi(\omega)$ as an element in $\mathbf{Z}^3$.

We claim that the intersection is nonempty if $\phi(\omega) \mod p \in \mathbf{F}^3$ is contained in some nonempty open subscheme $U \subset \mathbb{A}^3$. Then we take $P_{\widetilde{w}_1}(X_1, X_2, X_3)$ to be a polynomial such that $P_{\widetilde{w}_1} \circ \phi^{-1}$ vanishes on the complement of U (here, we view ϕ as a map $\mathbb{A}^3 \to \mathbb{A}^3$). The idea is that one can consider affine flag variety $\mathcal{F}l_{\mathbf{Z}}$ and its open Schubert variety $\mathcal{S}^\circ_{\mathbf{Z}}(\widetilde{w}_1^* w_0)$ defined over $\mathbf{Z}$. Then we impose the monodromy condition with parameters $(b, \mathbf{a}) \in \mathbb{A}^1 \times_{\mathbf{Z}} \mathbb{A}^3$ given by

$$bv\frac{dg}{dv}g^{-1} + g\mathrm{Diag}(\mathrm{std}(\mathbf{a}))g^{-1} \in \frac{1}{v}L^+\mathcal{M}$$

which induces a closed subscheme $\mathcal{Y}^\circ$ in $\mathcal{S}^\circ_{\mathbf{Z}}(\widetilde{w}_1^* w_0) \times \mathbb{A}^1 \times_{\mathbf{Z}} \mathbb{A}^3$. Its fiber at $(1, \phi(\omega)) \in \mathbb{A}^1 \times_{\mathbf{Z}} \mathbb{A}^3(\mathbf{F})$ is equal to $S^\circ_{\mathbf{F}}(\widetilde{w}_1^* w_0) \cap \mathrm{Fl}^{\nabla_{\phi(\omega)}}$. Meanwhile the reduced fiber at $(0, \mathbf{a}) \in \mathbb{A}^1 \times_{\mathbf{Z}} \mathbb{A}^3(\mathbf{C})$ is an open dense subscheme of an irreducible component $\mathrm{Fl}_{v\mathbf{a},\widetilde{w}_1}$ of a certain affine Springer fiber considered in Appendix A. By Theorem A.3, this reduced fiber has nonempty intersection with $L^{--}\mathcal{G}_k\widetilde{w}^*$. Thus $\mathcal{Y}^\circ \cap L^{--}\mathcal{G}_{\mathbf{Z}}\widetilde{w}^* \subset \mathcal{Y}^\circ$ is a nonempty open subscheme. An easy generalization of Theorem 3.5.1 shows that there is an open subscheme $V \subset \mathbb{A}^1 \times_{\mathbf{Z}} \mathbb{A}^3$ such that $\mathcal{Y}^\circ|_V \to V$ is a trivial vector bundle over V. Thus the subspace of parameters $(b, \mathbf{a}) \in V$ at which the fiber of $\mathcal{Y}^\circ$ has nonempty intersection with $L^{--}\mathcal{G}_{\mathbf{Z}}\widetilde{w}^*$ is open. By observing that such subspace is stable under scalar multiplication (acting on $\mathbb{A}^1 \times_{\mathbf{Z}} \mathbb{A}^3$ diagonally), the subspace of parameters $(1, \mathbf{a})$ at which the fiber of $\mathcal{Y}^\circ$ (which is $S^\circ_{\mathbf{F}}(\widetilde{w}_1^* w_0) \cap \mathrm{Fl}^{\nabla_{\mathbf{a}}}$) has nonempty intersection with $L^{--}\mathcal{G}_{\mathbf{Z}}\widetilde{w}^*$ in $\mathbb{A}^3$ is also open in $\mathbb{A}^3$. This proves our claim. $\square$

Let $\overline{\rho}$ be a tame inertial L-parameter over $\mathbf{F}$ with 6-generic lowest alcove presentation (s, μ) compatible with $\zeta \in X^*(\underline{Z})$. The following Theorem gives a geometric interpretation of the set $W^?(\overline{\rho})$ under polynomial genericity assumptions.

Theorem 3.7.2. *Let $\overline{\rho}$ be as above. Let $W_g^\zeta(\overline{\rho})$ be a set of 3-deep Serre weights σ with lowest alcove presentation compatible with ζ such that $\widetilde{w}^*(\overline{\rho}) \in C^\zeta_\sigma$.*

1. *We have $W_{\mathrm{obv}}(\overline{\rho}) \subset W_g^\zeta(\overline{\rho}) \subset W^?(\overline{\rho})$.*

2. *Let $(\widetilde{w}_1, \omega)$ be a lowest alcove presentation of $\sigma \in W^?(\overline{\rho})$ compatible with ζ. For each $j \in \mathcal{J}$, let $P_{\widetilde{w}_{1,j}}$ be the polynomial in Theorem 3.7.1. If $P_{\widetilde{w}_{1,j}}(\omega_j) \neq 0 \mod p$ for all $j \in \mathcal{J}$, then $\sigma \in W_g^\zeta(\overline{\rho})$*

Proof. Let $\sigma \in W^?(\overline{\rho})$. By Proposition 2.5.8, σ is 3-deep with a lowest alcove presentation $(\widetilde{w}_1, \widetilde{w}(\overline{\rho})\widetilde{w}_2^{-1}(0))$ for some $(\widetilde{w}_1, \widetilde{w}_2) \in (\widetilde{\underline{W}}_1^+ \times \widetilde{\underline{W}}^+)/X^0(\underline{T})$ satisfying $\widetilde{w}_2 \uparrow \widetilde{w}_1$. We write $\widetilde{w}(\overline{\rho})\widetilde{w}_2 = t_\omega w$ so that $\widetilde{w}(\overline{\rho})\widetilde{w}_2^{-1}(0) = \omega$. Also $\sigma \in W_{\mathrm{obv}}(\overline{\rho})$ implies $\widetilde{w}_1 = \widetilde{w}_2$. Then both item (1) and (2) follow from the corresponding items in

Theorem 3.7.1 and noting that

$$\{(t_\omega \widetilde{w})^* \mid \widetilde{w} \in \widetilde{W}^{\mathcal{J}}, \widetilde{w} \leq w_0 \widetilde{w}_1\} = \{(t_\omega w \widetilde{w})^* \mid w \in W^{\mathcal{J}}, \widetilde{w} \in \widetilde{W}^+, \widetilde{w} \uparrow \widetilde{w}_1\}.$$

□

We end this section by recording the following Proposition, which will be used in §6.2.

Proposition 3.7.3. *$C_{(\widetilde{w}_1,\omega)}$ is unibranch at its $\underline{T}^\vee$-fixed points.*

Proof. This can be proven as the corresponding result for GL_4 [Le+a, Proposition 4.7.5] similarly to Proposition 3.3.7. □

Chapter 4

Moduli stacks in p-adic Hodge theory

In this chapter, we construct a symplectic variant of the moduli stacks of étale (φ, Γ)-modules constructed in [EG23] and prove its properties including the existence of closed substacks parameterizing potentially crystalline representations. We also construct symplectic variants for the p-adic formal algebraic stacks considered in [Le+a, §5], namely, the moduli stacks of Breuil–Kisin modules, and étale ϕ-modules. Then we study the geometric properties of potentially crystalline substacks using local models in §3.

4.1 Sympletic étale (φ, Γ)-modules

Only in this section, we allow $K/\mathbf{Q}_p$ to be ramified.

For a p-adically complete $\mathcal{O}$-algebra R, we write $\mathcal{X}_n(R)$ for the groupoid of projective étale (φ, Γ)-modules of rank n with R-coefficients, in the sense of [EG23, Definition 2.7.2]. Then $\mathcal{X}_n$ is a Noetherian formal algebraic stack over $\operatorname{Spf} \mathcal{O}$ (Corollary 5.5.17 in *loc. cit.*). When R is a finite $\mathcal{O}$-algebra, there is an equivalence of categories between the category of projective étale (φ, Γ)-modules with R-coefficient and the category of continuous representations of G_K on finite projective R-modules. The goal of this section is to construct a moduli stack of *symplectic* (φ, Γ)-modules over $\mathcal{O}$ using moduli stacks of Emerton–Gee.

We start by defining a symplectic variant of étale (φ, Γ)-modules.

Definition 4.1.1. Let R be a p-adically complete $\mathcal{O}$-algebra. A *symplectic* projective étale (φ, Γ)-module (of rank 4) with R-coefficients is a triple (M, N, α) where $M \in \mathcal{X}_4(R)$, $N \in \mathcal{X}_1(R)$, and $\alpha : M \simeq M^\vee \otimes N$ is an isomorphism between rank 4 étale (φ, Γ)-modules satisfying the alternating condition $(\alpha^\vee \otimes N)^{-1} \circ \alpha = -1_M$. We write $\mathcal{X}_{\mathrm{Sym}}(R)$ for the groupoid of projective symplectic étale (φ, Γ)-modules with R-coefficients.

Lemma 4.1.2. *Let R be a commutative ring with identity and G be an abstract group. There is an equivalence of groupoids*

$$\{\rho : G \to \mathrm{GSp}_4(R)\} \simeq \left\{ (\rho', \chi, \alpha) \mid \begin{array}{c} \rho' : G \to \mathrm{GL}_4(R), \chi : G \to \mathrm{GL}_1(R), \\ \alpha : \rho' \simeq (\rho')^\vee \otimes \chi, (\alpha^\vee \otimes \chi)^{-1} \circ \alpha = -1 \end{array} \right\}$$
$$\rho \mapsto (\mathrm{std}(\rho), \mathrm{sim}(\rho), \alpha_\rho)$$

where morphisms in the latter groupoid is given by isomorphisms of ρ' and χ commuting with α, and the isomorphism α_ρ is given by the matrix J with respect to the standard basis of $\mathrm{std}(\rho)$ *and its the dual basis.*

Proof. Note that α defines a non-degenerate alternating R-bilinear pairing $\langle -, - \rangle_\alpha : \rho' \times \rho' \to R$ given by $\langle x, y \rangle_\alpha = \alpha(y)(x)$. Then the essential surjectivity follows from the presentation of standard symplectic space. More precisely, one can find two vectors $v_1, v_4 \in \rho'$ such that $\langle v_1, v_4 \rangle_\alpha = 1$, and two vectors v_2, v_3 such that $\langle v_2, v_3 \rangle_\alpha = 1$ in the complement of the span of v_1, v_4. Then the basis $\{v_1, v_2, v_3, v_4\}$ gives the standard symplectic space. The proof of fully faithfulness is elementary. □

Corollary 4.1.3. *Let R be a finite local $\mathcal{O}$-algebra. There is an equivalence of groupoids*

$$\{\rho : G_K \to \mathrm{GSp}_4(R)\} \simeq \mathcal{X}_{\mathrm{Sym}}(A).$$

Proof. This follows from Lemma 4.1.2 and Fontaine's equivalence between Galois representations and étale (φ, Γ)-modules. □

We recall some definitions from [Eme]. A formal algebraic stack is *quasi-compact* if it admits a morphism from a quasi-compact formal algebraic space which is representable by algebraic spaces, smooth, and surjective. A morphism $\mathcal{X} \to \mathcal{Y}$ of formal algebraic stacks is *quasi-compact* if for every morphism $Z \to \mathcal{Y}$ whose source is an affine scheme, the fiber product $\mathcal{X} \times_{\mathcal{Y}} Z$ is a quasi-compact formal algebraic stack. A formal algebraic stack $\mathcal{X}$ is *quasi-separated* if its diagonal morphism (which is representable by algebraic spaces) is quasi-compact and quasi-separated. We have the following obvious generalization.

Definition 4.1.4. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of formal algebraic stacks is *quasi-separated* if the relative diagonal $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact and quasi-separated.

It is easy to see that if $f : \mathcal{X} \to \mathcal{Y}$ is a quasi-separated morphism of formal algebraic stacks whose target is quasi-separated, then $\mathcal{X}$ is quasi-separated.

A formal algebraic stack $\mathcal{X}$ is *locally Noetherian* if it admits a morphism from a disjoint union of affine formal algebraic spaces which is representable by algebraic spaces, smooth, and surjective. Moreover, $\mathcal{X}$ is *Noetherian* if it is locally Noetherian, quasi-compact, and quasi-separated.

We denote by $\mathrm{std} : \mathcal{X}_{\mathrm{Sym}} \to \mathcal{X}_4$ and $\mathrm{sim} : \mathcal{X}_{\mathrm{Sym}} \to \mathcal{X}_1$ morphisms sending (M, N, α) to M and N respectively.

Theorem 4.1.5. *The morphism $\mathrm{std} \times \mathrm{sim} : \mathcal{X}_{\mathrm{Sym}} \to \mathcal{X}_4 \times_{\mathrm{Spf}\,\mathcal{O}} \mathcal{X}_1$ is representable by algebraic spaces, quasi-compact, and quasi-separated. In particular, the stack $\mathcal{X}_{\mathrm{Sym}}$ is a Noetherian formal algebraic stack over $\mathrm{Spf}\,\mathcal{O}$.*

Proof. We first prove that $\mathrm{std} \times \mathrm{sim}$ is representable by algebraic spaces and quasi-compact. Let $S \to \mathrm{Spf}\,\mathcal{O}$ be a test scheme and $S \to \mathcal{X}_4 \times_{\mathrm{Spf}\,\mathcal{O}} \mathcal{X}_1$ be a morphism corresponding to étale (φ, Γ)-modules M_S and N_S (of rank 4 and 1, respectively). If S' is a S-scheme, we write $(M_{S'}, N_{S'})$ for the pullback of (M_S, N_S) to S'. The fiber product

$$\underline{\alpha}_S := \mathcal{X}_{\mathrm{Sym}} \times_{\mathcal{X}_4 \times \mathcal{X}_1} S$$

is given by the following subsheaf of $\underline{Isom}(M_S, M_S^\vee \otimes N_S)$

$$\underline{\alpha}_S : S' \mapsto \{\alpha' \in Isom(M_{S'}, M_{S'}^\vee \otimes N_{S'}) \mid ((\alpha')^\vee \otimes N_{S'}) \circ \alpha' = -1_{M_S}\}\,.$$

We have the following cartesian diagram

$$\begin{tikzcd} \underline{\alpha}_S \arrow[rr] \arrow[d] & & S \arrow[d, "-1_{M_S}"] \\ \underline{Isom}(M_S, M_S^\vee \otimes N_S) \arrow[r] & \underline{Isom}(M_S, M_S^\vee \otimes N_S) \times_S \underline{Isom}(M_S^\vee \otimes N_S, M_S) \arrow[r, "c"] & \underline{Aut}(M_S) \end{tikzcd},$$

where the right vertical map is a constant section given by $-1_{M_S} \in \underline{Aut}(M_S)$, the left bottom horizontal map sends α to $(\alpha, (\alpha^\vee \otimes N_S)^{-1})$, and c is the composition. Since $\mathcal{X}_4$ is quasi-separated, $\underline{Aut}(M_S) \to S$ is quasi-compact and quasi-separated. This implies that the section -1_{M_S} is quasi-compact, and so is $\underline{\alpha}_S \to \underline{Isom}(M_S, M_S^\vee \otimes N_S)$. In particular, $\underline{\alpha}_S \to S$ is quasi-compact. This proves that $\mathrm{std} \times \mathrm{sim}$ is representable by algebraic spaces and quasi-compact.

We now prove that $\mathrm{std} \times \mathrm{sim}$ is quasi-separated. By the following cartesian diagram

$$\begin{tikzcd} \underline{Isom}((M_1, N_1, \alpha_1), (M_2, N_2, \alpha_2)) \arrow[r] \arrow[d] & \underline{Isom}(M_1, M_2) \times \underline{Isom}(N_1, N_2) \arrow[r] \arrow[d] & S \arrow[d] \\ \mathcal{X}_{\mathrm{Sym}} \arrow[r] & \mathcal{X}_{\mathrm{Sym}} \times_{\mathcal{X}_4 \times \mathcal{X}_1} \mathcal{X}_{\mathrm{Sym}} \arrow[r] & \mathcal{X}_{\mathrm{Sym}} \times \mathcal{X}_{\mathrm{Sym}} \end{tikzcd},$$

it is suffice to show that

$$\underline{Isom}((M_1, N_1, \alpha_1), (M_2, N_2, \alpha_2)) \to \underline{Isom}(M_1, M_2) \times_S \underline{Isom}(N_1, N_2)$$

is quasi-compact and quasi-separated. It is clearly a monomorphism, and thus is quasi-separated. Note that there is a morphism

$$\begin{aligned} d : \underline{Isom}(M_1, M_2) \times_S \underline{Isom}(N_1, N_2) &\to \underline{Isom}(M_1, M_2) \times_S \underline{Isom}(N_1, N_2) \\ (f, g) &\mapsto (\alpha_2^{-1} \circ ((f^\vee)^{-1} \otimes g) \circ \alpha_1, g) \end{aligned}$$

which makes the following diagram cartesian

$$\begin{tikzcd} \underline{Isom}((M_1, N_1, \alpha_1), (M_2, N_2, \alpha_2)) \arrow[r] \arrow[d] & \underline{Isom}(M_1, M_2) \times_S \underline{Isom}(N_1, N_2) \arrow[d, "(\mathrm{id}, d)"] \\ \underline{Isom}(M_1, M_2) \times_S \underline{Isom}(N_1, N_2) \arrow[r, "\Delta"] & (\underline{Isom}(M_1, M_2) \times_S \underline{Isom}(N_1, N_2))^2 \end{tikzcd}.$$

Then the top horizontal arrow is quasi-compact because $\underline{Isom}(M_1, M_2) \times_S \underline{Isom}(N_1, N_2)$ is quasi-separated.

Finally, $\mathcal{X}_{\mathrm{Sym}}$ is a formal algebraic stack over $\mathrm{Spf}\, \mathcal{O}$ by [Eme, Lemma 5.19]. It is quasi-compact and quasi-separated because $\mathrm{std} \times \mathrm{sim}$ and $\mathcal{X}_4 \times_{\mathrm{Spf}\, \mathcal{O}} \mathcal{X}_1$ are. It is also locally Noetherian because $\mathcal{X}_4 \times_{\mathrm{Spf}\, \mathcal{O}} \mathcal{X}_1$ is, and $\mathrm{std} \times \mathrm{sim}$ is quasi-compact. $\square$

Let $\tau : I_K \to T^\vee(E)$ be an inertial K-type and $\lambda \in X_*(\underline{T}^\vee)$ be a dominant cocharacter. Recall the potentially crystalline (resp. semistable) substack $\mathcal{X}_4^{\lambda', \tau'}$ (resp. $\mathcal{X}_4^{\mathrm{ss}, \lambda', \tau'}$) of $\mathcal{X}_4$ with Hodge type $\lambda' := \mathrm{std}(\lambda)$ and descent type $\tau' := \mathrm{std}(\tau)$ ([EG23, Theorem 4.8.12]). It is a $\mathcal{O}$-flat p-adic formal algebraic substack. For finite flat $\mathcal{O}$-algebra A, $\mathcal{X}_4^{\lambda', \tau'}(A)$ (resp. $\mathcal{X}_4^{\mathrm{ss}, \lambda', \tau'}(A)$) is the full subgroupoid of $\mathcal{X}_4(A)$ consisting of lattices in potentially crystalline (resp. semistable) G_K-representations of Hodge type λ' and inertia type τ'. For convenience, we write $\mathcal{X}_4^{\lambda, \tau}$ and $\mathcal{X}_4^{\mathrm{ss}, \lambda, \tau}$ instead of $\mathcal{X}_4^{\lambda', \tau'}$ and $\mathcal{X}_4^{\mathrm{ss}, \lambda', \tau'}$. We define $\mathcal{X}_{\mathrm{Sym}}^{\lambda, \tau}$ (resp. $\mathcal{X}_{\mathrm{Sym}}^{\mathrm{ss}, \lambda, \tau}$) to be the $\mathcal{O}$-flat part (see [Eme, Example 9.11]) of $\mathcal{X}_{\mathrm{Sym}} \times_{\mathcal{X}_4} \mathcal{X}_4^{\lambda',\tau'}$ (resp. $\mathcal{X}_{\mathrm{Sym}} \times_{\mathcal{X}_4} \mathcal{X}_4^{\mathrm{ss},\lambda',\tau'}$).

Proposition 4.1.6. *Let λ, τ be as above.*

1. *The stack $\mathcal{X}_{\mathrm{Sym}}^{\lambda,\tau}$ (resp. $\mathcal{X}_{\mathrm{Sym}}^{\mathrm{ss},\lambda,\tau}$) is a p-adic formal algebraic stack and uniquely determined as $\mathcal{O}$-flat closed substack of $\mathcal{X}_{\mathrm{Sym}}$ such that for finite flat $\mathcal{O}$-algebra A, $\mathcal{X}_{\mathrm{Sym}}^{\lambda,\tau}(A) \subset \mathcal{X}_{\mathrm{Sym}}(A)$ (resp. $\mathcal{X}_{\mathrm{Sym}}^{\mathrm{ss},\lambda,\tau}(A) \subset \mathcal{X}_{\mathrm{Sym}}(A)$) is precisely the subgroupoid consisting of lattices in potentially crystalline (resp. semistable) G_K-representations of Hodge type λ and inertia type τ.*

2. *The algebraic stacks $\mathcal{X}_{\mathrm{Sym}}^{\lambda,\tau} \times_{\mathrm{Spf}\,\mathcal{O}} \operatorname{Spec} \mathbf{F}$ (resp. $\mathcal{X}_{\mathrm{Sym}}^{\mathrm{ss},\lambda,\tau} \times_{\mathrm{Spf}\,\mathcal{O}} \operatorname{Spec} \mathbf{F}$) are equidimensional of dimension $d_\lambda := \sum_{j\in\mathcal{J}} \dim P_{\lambda_j} \backslash \mathrm{GSp}_4$.*

Proof. The first claim follows from the construction. For the second claim, the proof is identical to that of [EG23, Theorem 4.8.14], using GSp_4 instead of GL_4 and the dimension formula for the potentially crystalline (or semistable) symplectic deformation ring ([BG19, Theorem A]). □

4.1.7 Irreducible components in $\mathcal{X}_{\mathrm{Sym,red}}$

Let $\mathcal{X}_{4,\mathrm{red}}$ be the underlying reduced substack of $\mathcal{X}_4$. It is an algebraic stack of finite presentation over $\mathbf{F}$ and equidimensional of dimension $6[K : \mathbf{Q}_p]$ ([EG23, Theorem 5.5.11 and 6.5.1]). Moreover, its irreducible components are labelled by Serre weights (of $\mathrm{GL}_4(k)$). Our goal is to prove analogous results for the underlying reduced substack $\mathcal{X}_{\mathrm{Sym,red}}$ of $\mathcal{X}_{\mathrm{Sym}}$.

Let σ be a Serre weight (of $\mathrm{GSp}_4(k)$). Then there exists $\lambda \in X_1^*(\underline{T})$ such that $\sigma \simeq F(\lambda)$. For each $j \in \mathcal{J}$, we identify λ_j with a triple of integers $(\lambda_{j,1}, \lambda_{j,2}; \lambda_{j,3})$ such that $0 \le \lambda_{j,1} - \lambda_{j,2}, \lambda_{j,2} \le p-1$ as explained in §2.1.

Definition 4.1.8. We say that $\overline{\rho} : G_K \to \mathrm{GSp}_4(\mathbf{F})$ is *maximally non-split of niveau 1 and of weight σ* if

$$\overline{\rho} = \begin{pmatrix} \chi_1 & * & * & * \\ 0 & \chi_2 & * & * \\ 0 & 0 & \chi_3 & * \\ 0 & 0 & 0 & \chi_4 \end{pmatrix} \tag{4.1.9}$$

where

- $\overline{\rho}$ is maximally non-split of niveau 1, i.e. it has a unique G_K-stable complete flag;
- $\bigoplus_{i=1}^4 \chi_i|_{I_K} = \prod_{j\in\mathcal{J}} \overline{\omega}_{K,\sigma_j}^{\phi(\lambda_j+\eta_j)}$;
- If $\chi_1\chi_2^{-1}|_{I_K} = \overline{\varepsilon}^{-1}$ (resp. $\chi_2\chi_3^{-1}|_{I_K} = \overline{\varepsilon}^{-1}$), then $\lambda_{j,2} = p-1$ (resp. $\lambda_{j,1} - \lambda_{j,2} = p-1$) for all $j \in \mathcal{J}$ if and only $\chi_1\chi_2^{-1} = \overline{\varepsilon}^{-1}$ (resp. $\chi_2\chi_3^{-1} = \overline{\varepsilon}^{-1}$) and the element of $\mathrm{Ext}^1_{G_K}(\chi_1,\chi_2)$ (resp. $\mathrm{Ext}^1_{G_K}(\chi_2,\chi_3)$) determined by $\overline{\rho}$ is très ramifiée. Otherwise, $\lambda_{j,1} - \lambda_{j,2} = 0$ (resp. $\lambda_{j,2} - \lambda_{j,3} = 0$) for all $j \in \mathcal{J}$. (Note that $\chi_1\chi_2^{-1} = \chi_3\chi_4^{-1}$.)

The following is the main Theorem of this section.

Theorem 4.1.10. *The stack $\mathcal{X}_{\mathrm{Sym,red}}$ is an algebraic stack over $\operatorname{Spec} \mathbf{F}$ of finite presentation and equidimensional of dimension $4[K : \mathbf{Q}_p]$. For each (isomorphism class of) Serre weight σ, there exists an irreducible component $\mathcal{C}_\sigma \subset \mathcal{X}_{\mathrm{Sym,red}}$ containing a dense locus of $\overline{\rho}$ maximally non-split of niveau 1 and of weight σ. This induces a bijection between the set of isomorphism classes of Serre weights and the set of irreducible components of $\mathcal{X}_{\mathrm{Sym,red}}$.*

Before we proceed, we define *potentially diagonalizable representations*. Let $\mathrm{std}' : \mathrm{GSp}_4 \to \mathrm{GL}_5$ denote the composite of the projection $\mathrm{GSp}_4 \twoheadrightarrow \mathrm{SO}_5$ and the standard representation $\mathrm{SO}_5 \to \mathrm{GL}_5$.

Definition 4.1.11. A continuous representation $\rho : G_K \to \mathrm{GSp}_4(\mathcal{O})$ is *potentially diagonalizable* if $\rho \otimes_{\mathcal{O}} E$ is potentially crystalline and $\mathrm{std}(\rho)$ is potentially diagonalizable in the sense of [Bar+14, §1.4].

Example 4.1.12. Suppose that $\rho : G_K \to \mathrm{GSp}_4(\mathcal{O})$ is potentially crystalline and ordinary (in the sense of [Bar+14, §1.4]). Then ρ is potentially diagonalizable by Lemma 1.4.3 of *loc. cit.*.

The following two results will be proven in §4.1.15 and §4.1.19 respectively. We write $\mathcal{X}_{\mathrm{Sym,red},\overline{\mathbf{F}}_p}$ for $\mathcal{X}_{\mathrm{Sym,red}} \times_{\mathbf{F}} \operatorname{Spec} \overline{\mathbf{F}}_p$.

Proposition 4.1.13. *For each Serre weight σ, there exists an algebraic stack $\mathcal{C}_{\sigma,\overline{\mathbf{F}}_p} \subset \mathcal{X}_{\mathrm{Sym,red},\overline{\mathbf{F}}_p}$ irreducible of dimension $4[K : \mathbf{Q}_p]$ containing a dense locus of $\overline{\rho}$ maximally non-split of niveau 1 and of weight σ. Furthermore, there exists a closed substack $\mathcal{C}_{small} \subset \mathcal{X}_{\mathrm{Sym,red},\overline{\mathbf{F}}_p}$ of dimension strictly less than $4[K : \mathbf{Q}_p]$ such that*

$$\mathcal{X}_{\mathrm{Sym,red},\overline{\mathbf{F}}_p} = \bigcup_{\sigma} \mathcal{C}_{\sigma,\overline{\mathbf{F}}_p} \cup \mathcal{C}_{small}.$$

Theorem 4.1.14. *Any continuous representation $\overline{\rho} : G_K \to \mathrm{GSp}_4(\overline{\mathbf{F}}_p)$ admits a lift $\rho : G_K \to \mathrm{GSp}_4(\overline{\mathbf{Z}}_p)$ such that $\rho \otimes_{\overline{\mathbf{Z}}_p} \overline{\mathbf{Q}}_p$ is crystalline with regular Hodge–Tate weights. Furthermore, ρ can be taken to be potentially diagonalizable.*

Granting the Proposition 4.1.13 and Theorem 4.1.14, we prove:

Proof of Theorem 4.1.10. If a closed immersion of reduced algebraic stacks that are locally of finite type over $\operatorname{Spec} \overline{\mathbf{F}}_p$ is surjective on finite type points (see [EG23, §6.6]), then it is an isomorphism. Thus, if we prove that any $\overline{\rho} : G_K \to \mathrm{GSp}_4(\overline{\mathbf{F}}_p)$ is contained in some $\mathcal{C}_{\sigma,\overline{\mathbf{F}}_p}$, then the closed immersion $\cup_\sigma \mathcal{C}_{\sigma,\overline{\mathbf{F}}_p} \hookrightarrow \mathcal{X}_{\mathrm{Sym,red},\overline{\mathbf{F}}_p}$ from Proposition 4.1.13 is an isomorphism. By Theorem 4.1.14, $\overline{\rho}$ is contained in the reduction of a crystalline stack $\mathcal{X}^{\lambda}_{\mathrm{Sym}} \times_{\operatorname{Spf} \mathcal{O}} \operatorname{Spec} \overline{\mathbf{F}}_p$ for some regular cocharacter λ. By Proposition 4.1.6, the algebraic stack $\mathcal{X}^{\lambda}_{\mathrm{Sym}} \times_{\operatorname{Spf} \mathcal{O}} \operatorname{Spec} \overline{\mathbf{F}}_p$ is equidimensional of dimension $4[K : \mathbf{Q}_p]$. So its underlying space is a union of $\mathcal{C}_{\sigma,\overline{\mathbf{F}}_p}$, and one of them contains $\overline{\rho}$.

It remains to show that $\mathcal{C}_{\sigma,\overline{\mathbf{F}}_p}$ descends to $\mathcal{C}_\sigma$ over $\operatorname{Spec} \mathbf{F}$. We need to show that $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$ stabilizes each component $\mathcal{C}_{\sigma,\overline{\mathbf{F}}_p} \subset \mathcal{X}_{\mathrm{Sym,red},\overline{\mathbf{F}}_p}$. This follows because $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$-action preserves the property of being maximally non-split of niveau 1 and of weight σ. $\square$

4.1.15 Families of extensions

We prove the Proposition 4.1.13. The essential ingredient is generalizations of the Proposition 5.4.4 in [EG23] which computes the dimension of families of extensions inside $\mathcal{X}_{n,\mathrm{red},\overline{\mathbf{F}}_p}$. We start by introducing several notations.

Recall that S (resp. Q) denotes the Siegel (resp. Klingen) parabolic subgroup of GSp_4. We write U_S (resp. U_Q) for its unipotent radical and L_S (resp. L_Q) for its Levi component. Then $U_S \simeq \mathbf{G}_a^{\oplus 3}$ and U_Q is an extension of $\mathbf{G}_a^{\oplus 2}$ by $\mathbf{G}_a$.

We also need the following auxiliary groups. Let Q_4 be the minimal parabolic of GL_4 containing $\mathrm{std}(Q)$ and U_{Q_4} be its unipotent radical. We define the following subgroups of U_{Q_4};

$$U_{Q_3} = \left\{ \begin{pmatrix} 1 & * & * & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \; U_{Q_3'} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \; U_{P_4} = \left\{ \begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

Let $\iota : \mathrm{GL}_4 \to \mathrm{GL}_4$ be an involution given by $\iota(A) = J^{-1} A^{-\top} J$. Then $(U_{Q_4})^\iota = U_Q$ and $\iota : U_{Q_3} \xrightarrow{\sim} U_{Q_3'}$.

Let R be a finite $\overline{\mathbf{Z}}_p$-algebra. Let $\theta : G_K \to \mathrm{GL}_2(R)$ be a continuous representation and $\chi : G_K \to R^\times$ be a continuous character. Then $\theta \oplus (\theta^\vee \otimes \chi)$ is L_S-valued representation of G_K and $\chi \oplus \theta \oplus \det(\theta)\chi^{-1}$ is L_Q-valued representation of G_K. We write

$$\begin{aligned} \mathrm{Ad}_S(\theta, \chi) : G_K &\xrightarrow{\theta \oplus (\theta^\vee \otimes \chi)} L_S(R) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(U_S(R)) \\ \mathrm{Ad}_Q(\chi, \theta) : G_K &\xrightarrow{\chi \oplus \theta \oplus \det(\theta)\chi^{-1}} L_Q(R) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(U_Q(R)) \\ \mathrm{Ad}_{Q_4}(\chi, \theta) : G_K &\xrightarrow{\chi \oplus \theta \oplus \det(\theta)\chi^{-1}} L_{Q_4}(R) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(U_{Q_4}(R)). \end{aligned}$$

Note that G_K-action on U_{Q_3} and $U_{Q_3'}$ induced by $\mathrm{Ad}_{Q_4}(\chi, \theta)$ define subrepresentations of $\mathrm{Ad}_{Q_4}(\chi, \theta)$, which we denote by $\mathrm{Ad}_{Q_3}(\chi, \theta)$ and $\mathrm{Ad}_{Q_3'}(\chi, \theta)$ respectively. The isomorphism $\iota : U_{Q_3} \xrightarrow{\sim} U_{Q_3'}$ induces an isomorphism of G_K-modules

$$\mathrm{Ad}_{Q_3}(\chi, \theta) \xrightarrow{\sim} \mathrm{Ad}_{Q_3'}(\chi, \theta).$$

There is an obvious quotient map $U_{P_4} \twoheadrightarrow U_{Q_3'}$. We can view U_{P_4} as the unipotent of standard parabolic subgroup of GL_4 whose Levi factor is $\mathrm{GL}_3 \times \mathrm{GL}_1$. If $\psi : G_K \to \mathrm{GL}_3(R)$ is a continuous representation, we write

$$\mathrm{Ad}_{P_4}(\psi, \chi) : G_K \xrightarrow{\psi \oplus \chi} \mathrm{GL}_3(R) \times \mathrm{GL}_1(R) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(U_{P_4}(R)).$$

Finally, we remark that there is an auxiliary embedding

$$\begin{aligned} \iota_{\mathrm{aux}} : \{(g,h) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \mid \det(g) = \det(h)\} &\to \mathrm{GSp}_4 \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x & y \\ z & x \end{pmatrix} \right) &\mapsto \begin{pmatrix} a & & & b \\ & x & y & \\ & z & w & \\ c & & & d \end{pmatrix}. \end{aligned}$$

We now consider families of extensions valued in the Siegel parabolic subgroup. Let T be a reduced finite type $\overline{\mathbf{F}}_p$-scheme with a morphism $T \to \mathcal{X}_{2,\mathrm{red},\overline{\mathbf{F}}_p}$ whose scheme-theoretic image is of pure dimension d. We let $\chi : G_K \to \overline{\mathbf{F}}_p^\times$ be a continuous character. We write $\overline{\theta}_T$ for the family of G_K-representations corresponding to $T \to \mathcal{X}_{2,\mathrm{red}}$. Following the convention of [EG23], we write $H^2(G_K, \mathrm{Ad}_S(\overline{\theta}_T, \chi))$ for the coherent sheaf on T defined as the cohomology group of the *Herr complex* $H^2(\mathcal{C}^\bullet(M))$ on T where M is the rank 3 étale (φ, Γ)-module corresponding to the family $\mathrm{Ad}_S(\overline{\theta}_T, \chi)$ (see Remark 5.1.30 of *loc. cit.*).

We suppose that $H^2(G_K, \mathrm{Ad}_S(\overline{\theta}_T, \chi))$ is locally free of constant rank r. Following the discussion before

Proposition 5.4.4 in *loc. cit.*, we find a complex of finite rank locally free $\mathcal{O}_T$-modules

$$C_T^0 \to Z_T^1$$

such that $\operatorname{coker}(C_T^0 \to Z_T^1) \simeq H^1(G_K, \operatorname{Ad}_S(\overline{\theta}_T, \chi))$. Let $V = \underline{Spec}(\operatorname{Sym}((Z_T^1)^\vee))$. Then V parameterizes a universal family of extension

$$0 \to \overline{\theta} \to \overline{\rho}_V \to \overline{\theta}^\vee \otimes \chi \to 0$$

such that $\overline{\rho}_V$ is valued in GSp_4. This induces a morphism $V \to \mathcal{X}_{\mathrm{Sym},\mathrm{red},\overline{\mathbf{F}}_p}$.

Lemma 4.1.16. *Let $T \to \mathcal{X}_{2,\mathrm{red},\overline{\mathbf{F}}_p}, \chi$, and V be as above. In particular, the scheme-theoretic image of $T \to \mathcal{X}_{2,\mathrm{red},\overline{\mathbf{F}}_p}$ is of pure dimension d, and $H^2(G_K, \operatorname{Ad}_S(\overline{\theta}_T, \chi))$ is locally free of constant rank r. Then the scheme-theoretic image of $V \to \mathcal{X}_{\mathrm{Sym},\mathrm{red},\overline{\mathbf{F}}_p}$ is of dimension $\leq d + 3[K : \mathbf{Q}_p] + r - 1$. Furthermore, if T is generically maximally non-split of niveau 1, then the equality holds.*

Proof. This follows from the proof of [EG23, Proposition 5.4.4]. □

We now consider the Klingen parabolic case. Let $\overline{\theta} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ be a continuous irreducible representation. Let T be a reduced finite type $\overline{\mathbf{F}}_p$-scheme with a morphism $T \to \mathcal{X}_{1,\mathrm{red},\overline{\mathbf{F}}_p}$ whose scheme-theoretic image is of pure dimension d. For each $t \in T(\overline{\mathbf{F}}_p)$, $H^1(G_K, \operatorname{Ad}_Q(\chi_t, \overline{\theta}))$ parameterizes $\overline{\rho} : G_K \to Q(\overline{\mathbf{F}}_p)$ whose projection onto $L_Q(\overline{\mathbf{F}}_p)$ is $\chi_t \oplus \overline{\theta} \oplus \det(\overline{\theta})\chi^{-1}$. However, the unipotent radical U_Q is non-abelian. To avoid using non-abelian cohomology, we construct a family of extensions in two steps.

Since θ is irreducible, we have $H^2(G_K, \operatorname{Ad}_{Q_3}(\chi_T, \overline{\theta})) = 0$. Similar to Siegel parabolic case, there exists a locally free $\mathcal{O}_T$-module Z_T^1 of constant rank with a surjection onto $H^1(G_K, \operatorname{Ad}_{Q_3}(\chi_T, \overline{\theta}))$. The vector bundle $V := \underline{Spec}(\operatorname{Sym}((Z_T^1)^\vee))$ parameterizes a universal family of extension

$$0 \to \chi_T \to \overline{\psi}_V \to \overline{\theta} \to 0.$$

By [EG23, Proposition 5.4.4], the scheme-theoretic image of $V \to \mathcal{X}_{3,\mathrm{red},\overline{\mathbf{F}}_p}$ is of dimension $\leq d + 2[K : \mathbf{Q}_p] - 1$.

We can and do replace V by its open dense locus of non-split extensions. Let $\delta = \det(\overline{\theta})$ and χ_V be the pullback of χ_T to V. By local Tate duality, we have $H^2(G_K, \operatorname{Ad}_{P_4}(\overline{\psi}_V, \chi_V^{-1}\delta)) = 0$. Thus, there is a locally free $\mathcal{O}_V$-module Z_V^1 of constant rank with a surjection onto $H^1(G_K, \operatorname{Ad}_{P_4}(\overline{\psi}_V, \chi_V^{-1}\delta))$, and the vector bundle $W := \underline{Spec}(\operatorname{Sym}((Z_V^1)^\vee))$ parameterizes a universal family of extensions

$$0 \to \overline{\psi}_V \to \overline{\rho}_W \to \det(\overline{\rho})\chi_T \to 0.$$

This induces a morphism $f : W \to \mathcal{X}_{4,\mathrm{red},\overline{\mathbf{F}}_p}$ whose scheme-theoretic image is of dimension $\leq d + 5[K : \mathbf{Q}_p] - 2$ by *loc. cit.*. We replace W by its open dense locus of non-split extensions. Then $\widetilde{W} = W \times_{\mathcal{X}_{4,\mathrm{red},\overline{\mathbf{F}}_p}} \mathcal{X}_{\mathrm{Sym},\mathrm{red},\overline{\mathbf{F}}_p}$ is again a scheme. We write $\widetilde{f} : \widetilde{W} \to \mathcal{X}_{\mathrm{Sym},\mathrm{red},\overline{\mathbf{F}}_p}$.

Lemma 4.1.17. *Let $\overline{\theta}, T \to \mathcal{X}_{1,\mathrm{red},\overline{\mathbf{F}}_p}$, and $\widetilde{f} : \widetilde{W} \to \mathcal{X}_{\mathrm{Sym},\mathrm{red},\overline{\mathbf{F}}_p}$ be as above. In particular, the scheme-theoretic image of $T \to \mathcal{X}_{1,\mathrm{red},\overline{\mathbf{F}}_p}$ is of pure dimension d. We further assume that $H^2(G_K, \chi_T^2\delta^{-1})$ is locally free of constant rank r. Then the scheme-theoretic image of $\widetilde{f}$ is of dimension $\leq d + 3[K : \mathbf{Q}_p] + r - 1$.*

Proof. For $w : \operatorname{Spec} \mathbf{F} \to \mathcal{X}_{\mathrm{Sym,red},\overline{\mathbf{F}}_p}$, we write $\widetilde{W}_{\tilde{f}(w)}$ (resp. $W_{f(w)}$) for the fiber of $\tilde{f}$ at w (resp. of f at $\operatorname{std}(w)$). We have the following commutative diagram

$$\begin{tikzcd}
\widetilde{W}_{\tilde{f}(w)} \arrow[r] \arrow[d, equal] \arrow[drr] & \operatorname{Spec} \overline{\mathbf{F}}_p \arrow[drr, "w"] & & \\
W_{f(w)} \arrow[drr] & & \widetilde{W} \arrow[r, "\tilde{f}"] \arrow[d] & \mathcal{X}_{\mathrm{Sym,red},\overline{\mathbf{F}}_p} \arrow[d, "\operatorname{std}"] \\
& & W \arrow[r, "f"] & \mathcal{X}_{4,\mathrm{red},\overline{\mathbf{F}}_p}.
\end{tikzcd}$$

By [Stacks, Tag 0DS4], we have

$$\dim W - \dim W_{f(w)} \le d + 5[K : \mathbf{Q}_p] - 2.$$

By the same Lemma, it is suffice to show $\dim \widetilde{W} \le \dim W - 2[K : \mathbf{Q}_p] + r + 1$ in order to get the bound on the dimension of scheme-theoretic image of $\tilde{f}$. Recall that W is a dense open subscheme of a vector bundle over V. For $v : \operatorname{Spec} \overline{\mathbf{F}}_p \to V$, we define $W_v := W \times_{V,v} \operatorname{Spec} \overline{\mathbf{F}}_p$ and $\widetilde{W}_v := \widetilde{W} \times_{V,v} \operatorname{Spec} \overline{\mathbf{F}}_p$. To prove the claimed inequality, we prove

$$\dim \widetilde{W}_v \le \dim W_v - 2[K : \mathbf{Q}_p] + r + 1. \tag{4.1.18}$$

We first claim that the image of $\widetilde{W}_v$ in W_v is contained in a codimension $2[K : \mathbf{Q}_p] - r - 1$ closed subscheme of W_v. By our construction, v correspond to a non-split extension $\overline{\psi}_v$ of $\overline{\theta}$ by χ_t, where t is the image of v under $V \to T$. Then $\overline{\psi}_v$ determines a class $[c] \in H^1(G_K, \mathrm{Ad}_{Q_3}(\chi_t, \overline{\theta}))$. By the isomorphism $\iota : \mathrm{Ad}_{Q_3}(\chi_t, \overline{\theta}) \xrightarrow{\sim} \mathrm{Ad}_{Q_3'}(\chi_t, \overline{\theta})$, we get $[c'] := \iota([c])$. Any $w \in W_v(\overline{\mathbf{F}}_p)$ determines a class $[b] \in H^1(G_K, \mathrm{Ad}_{P_4}(\overline{\psi}_v, \chi_t^{-1}\delta))$. Then w is in the image of $\widetilde{W}_v$ if and only if $[b]$ is mapped to a line $\overline{\mathbf{F}}_p[c']$ under the map

$$H^1(G_K, \mathrm{Ad}_{P_4}(\overline{\psi}_v, \chi_t^{-1}\delta)) \to H^1(G_K, \mathrm{Ad}_{Q_3'}(\chi_t, \overline{\theta})).$$

We take the long exact sequence of Galois cohomology to the following short exact sequence

$$0 \to \chi_t^2\delta^{-1} \to \mathrm{Ad}_{P_4}(\overline{\psi}_v, \chi_t^{-1}\delta) \to \mathrm{Ad}_{Q_3'}(\chi_t, \overline{\theta}) \to 0.$$

Since $\dim_{\overline{\mathbf{F}}_p} H^2(G_K, \chi_t^2\delta^{-1}) = r$, it shows that the set of $[b]$ mapped to $\overline{\mathbf{F}}_p[c']$ is a subspace $H' \subset H^1(G_K, \mathrm{Ad}_{P_4}(\overline{\psi}_v, \chi_t^{-1}\delta))$ of codimension $2[K : \mathbf{Q}_p] - r - 1$. Recall that $W = \underline{Spec}((Z_V^1)^\vee)$ where Z_V^1 is locally free $\mathcal{O}_V$-module with a surjection onto $H^1(G_K, \mathrm{Ad}_{P_4}(\overline{\psi}_V, \chi_V^{-1}\delta))$. Then $W_v = \underline{Spec}((Z_v^1)^\vee)$ where $Z_v^1 := v^*(Z_V^1)$, and the image of $\widetilde{W}_v$ in W_v is given by the inverse image of H' in Z_v^1. This proves the claim.

Take $w \in W_v(\overline{\mathbf{F}}_p)$ in the image of $\widetilde{W}_v$. We compute the dimension of the fiber

$$\widetilde{W}_w := \widetilde{W}_v \times_{W_v,w} \operatorname{Spec} \overline{\mathbf{F}}_p.$$

If the dimension of $\widetilde{W}_w$ is zero, we get the inequality (4.1.18). Let $\overline{\rho}_w : G_K \to \mathrm{GL}_4(\overline{\mathbf{F}}_p)$ be the continuous representation corresponding to w. Then $\widetilde{W}_w$ parameterizes triples $(\overline{\rho}_w, \zeta, \alpha)$ satisfying the conditions in

Lemma 4.1.2. One can easily see that $\zeta \simeq \det(\overline{\theta})$. Suppose that $(\overline{\rho}_w, \det(\overline{\theta}), \alpha_1)$ and $(\overline{\rho}_w, \det(\overline{\theta}), \alpha_2)$ are such triples. Since $\mathrm{Aut}(\overline{\rho}_w) = \overline{\mathbf{F}}_p^\times$, $\alpha_2^{-1}\alpha_1 = c$ for some $c \in \overline{\mathbf{F}}_p^\times$. Then map $(\overline{\rho}_w, \det(\overline{\theta}), \alpha_1) \to (\overline{\rho}_w, \det(\overline{\theta}), \alpha_2)$ identity on $\overline{\rho}_w$ and multiplication by c^{-1} on $\det(\overline{\theta})$ is then an isomorphism. Thus, there is the unique up to isomorphism triple $(\overline{\rho}, \det(\overline{\theta}), \alpha)$ parameterized by $\widetilde{W}_w$. This shows that $\widetilde{W}_w$ is of dimension 0. □

Proof of Proposition 4.1.13. Let σ be a Serre weight. Then $\sigma \simeq F(\lambda)$ for $\lambda \in X_1^*(\underline{T})$ well-defined up to $(p-\pi)X^0(\underline{T})$. We identify λ_j to a triple $(\lambda_{j,1}, \lambda_{j,2}; \lambda_{j,3})$ as in §2.1. Then $\lambda' := (\lambda_{j,1}+\lambda_{j,2}+\lambda_{j,3}, \lambda_{j,1}+\lambda_{j,3})$ defines an element in $X_1^*(\underline{T}_2)/(p-\pi)X^0(\underline{T}_2)$, and thus $\sigma' := F(\lambda')$ is a well-defined Serre weight of $\mathrm{GL}_2(k)$.

There exists an irreducible component $\mathcal{C}_{\sigma',\overline{\mathbf{F}}_p} \subset \mathcal{X}_{2,\mathrm{red},\overline{\mathbf{F}}_p}$ of dimension $[K : \mathbf{Q}_p]$ characterized as a closure of the locus of G_K-representations of the form

$$\overline{\theta} = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

such that $\chi_1 \mathrm{ur}_{t_1^{-1}} \oplus \chi_2 \mathrm{ur}_{t_2^{-1}} = \prod_{j\in\mathcal{J}} \varpi_{K,\sigma_j}^{\lambda_j'+(1,0)}$ for some $t_1, t_2 \in \overline{\mathbf{F}}_p^\times$, and moreover, if $\chi_1\chi_2^{-1}|_{I_K} = \overline{\varepsilon}^{-1}$, then $\lambda_{j,2} = p-1$ for all $j \in \mathcal{J}$ if and only if $\chi_1\chi_2^{-1} = \overline{\varepsilon}^{-1}$ and the extension class is très ramifiée (see [Le+a, §7.4]). We call such nonsplit $\overline{\theta}$ as of weight σ'. We recall some of its properties from the proof of [EG23, Theorem 5.5.12] (corrected in [EG]). There exists an open substack $\mathcal{U}_{\sigma'} \subset \mathcal{C}_{\sigma',\overline{\mathbf{F}}_p}$ consisting of nonsplit $\overline{\theta}$ of weight σ'. Also, there exists a codimension 1 closed substack $\mathcal{C}_{\sigma'}^{\mathrm{fixed}} \subset \mathcal{C}_{\sigma',\overline{\mathbf{F}}_p}$ containing dense locus of nonsplit $\overline{\theta}$ of weight σ' as above with $t_2 = 1$. We let $\mathcal{U}_{\sigma'}^{\mathrm{fixed}} = \mathcal{C}_{\sigma'}^{\mathrm{fixed}} \cap \mathcal{U}_{\sigma'}$. We can and do assume that $\mathcal{C}_{\sigma',\overline{\mathbf{F}}_p}$ (resp. $\mathcal{U}_{\sigma'}$) is obtained from $\mathcal{C}_{\sigma'}^{\mathrm{fixed}}$ (resp. $\mathcal{U}_{\sigma'}^{\mathrm{fixed}}$) by taking unramified twists.

We write χ_σ for the continuous character $\prod_{j\in\mathcal{J}} \varpi_{K,\sigma_j}^{\mathrm{sim}(\phi(\lambda+\eta))} : G_K \to \overline{\mathbf{F}}_p^\times$. For nonsplit $\overline{\theta}$ of weight σ' as above, a direct computation shows that the dimension of $H^2(G_K, \mathrm{Ad}_S(\overline{\theta}, \chi_\sigma))$ is equal to 1 if $\chi_\sigma\chi_2^{-2} = \overline{\varepsilon}^{-1}$ and 0 otherwise. Note that $\chi_\sigma\chi_2^{-2} = \overline{\varepsilon}^{-1}$ if and only if $\lambda_{j,1} - \lambda_{j,2}$ is equal to 0 for all $j \in \mathcal{J}$ or equal to $p-1$ for all $j \in \mathcal{J}$, and $t_2^2 = 1$.

Suppose that $\lambda_{j,1} - \lambda_{j,2} = p-1$ for all $j \in \mathcal{J}$, which we call très ramifiée case. We let T be a $\overline{\mathbf{F}}_p$-scheme smoothly covering $\mathcal{U}_{\sigma'}^{\mathrm{fixed}}$. Then $H^2(G_K, \mathrm{Ad}_S(\overline{\theta}_T, \chi_\sigma))$ is locally free of constant rank 1. Otherwise, we replace $\mathcal{U}_{\sigma'}$ by its open substack so that $t_2^2 \neq 1$, and let T be a $\overline{\mathbf{F}}_p$-scheme smoothly covering $\mathcal{U}_{\sigma'}$. Then $H^2(G_K, \mathrm{Ad}_S(\overline{\theta}_T, \chi_\sigma))$ is locally free of constant rank 0. In both cases, T is necessarily irreducible, reduced, and of finite type over $\overline{\mathbf{F}}_p$. By applying Lemma 4.1.16 and the preceding discussion, we obtain a vector bundle V over T parameterizing extensions $\overline{\rho}_V$ of $\overline{\theta}_T^\vee \otimes \chi_\sigma$ by $\overline{\theta}_T$, and the induced morphism $V \to \mathcal{X}_{\mathrm{Sym},\mathrm{red},\overline{\mathbf{F}}_p}$ has the scheme-theoretic image of dimension $4[K : \mathbf{Q}_p] - 1$. We let $\mathcal{C}_{\sigma,\overline{\mathbf{F}}_p}$ be the scheme-theoretic image of the unramified twists of $\overline{\rho}_V$. By Lemma 5.3.2 in [EG23], $\mathcal{C}_{\sigma,\overline{\mathbf{F}}_p}$ has dimension $4[K : \mathbf{Q}_p]$.

It remains to construct $\mathcal{C}_{small}$ and prove that $\mathcal{X}_{\mathrm{Sym},\mathrm{red},\overline{\mathbf{F}}_p} = \bigcup_\sigma \mathcal{C}_{\sigma,\overline{\mathbf{F}}_p} \cup \mathcal{C}_{small}$. If the union $\bigcup_\sigma \mathcal{C}_{\sigma,\overline{\mathbf{F}}_p} \cup \mathcal{C}_{small}$ exhausts all $\overline{\mathbf{F}}_p$-points of $\mathcal{X}_{\mathrm{Sym},\mathrm{red},\overline{\mathbf{F}}_p}$, then the equality follows. Thus, we construct finitely many closed substacks of $\mathcal{X}_{\mathrm{Sym},\mathrm{red},\overline{\mathbf{F}}_p}$ containing $\overline{\mathbf{F}}_p$-points of $\mathcal{X}_{\mathrm{Sym},\mathrm{red},\overline{\mathbf{F}}_p}$ which are not contained in all $\mathcal{C}_{\sigma,\overline{\mathbf{F}}_p}$ and take $\mathcal{C}_{small}$ to be the union of such closed substacks.

For each Serre weight σ, we define $\mathcal{Z}_{\sigma'} := \mathcal{C}_{\sigma',\overline{\mathbf{F}}_p} \backslash \mathcal{U}_{\sigma'}$. It is an algebraic stack over $\overline{\mathbf{F}}_p$ of dimension $\leq [K : \mathbf{Q}_p] - 1$. For each integer $r \geq 0$, there exists a locally closed substack $\mathcal{Z}_{\sigma',r} \subset \mathcal{Z}_{\sigma'}$ such that $H^2(G_K, \mathrm{Ad}_S(\overline{\theta}_{\mathcal{Z}_{\sigma',r}}, \chi_\sigma))$ is locally free of constant rank r (by upper-semicontinuity of fiber dimension). It is easy to check that the dimension of $\mathcal{Z}_{\sigma',r}$ is at most $[K : \mathbf{Q}_p] - r - 1$. When $r = 0$, there is nothing to prove. When $r = 1$, the locus of nonsplit $\overline{\theta}$ in $\mathcal{Z}_{\sigma',r}$ is contained in $\mathcal{C}_{\sigma'}^{\mathrm{fixed}} \backslash \mathcal{U}_{\sigma'}^{\mathrm{fixed}}$ which has dimension

$\leq [K:\mathbf{Q}_p]-2$, and the locus of semisimple $\overline{\theta}$ in $\mathcal{Z}_{\sigma',r}$ can be checked to have dimension ≤ -1. When $r=2$, $\mathcal{Z}_{\sigma',r}$ only contains finitely many reducible split and non-scalar $\overline{\theta}$ and thus has dimension ≤ -2. Finally, when $r=4$, $\mathcal{Z}_{\sigma',r}$ only contains finitely many scalar $\overline{\theta}$ and thus has dimension ≤ -4. Let T be a $\overline{\mathbf{F}}_p$-scheme smoothly covering $\mathcal{Z}_{\sigma,r}$. By applying Lemma 4.1.16 and the preceding discussion, we get a morphism $V\to\mathcal{X}_{\mathrm{Sym,red}}$, and the scheme-theoretic image of its unramified twists is a closed substack $\mathcal{C}_{\sigma,r}\subset\mathcal{X}_{\mathrm{Sym,red},\overline{\mathbf{F}}_p}$ of dimension $\leq 4[K:\mathbf{Q}_p]-1$. It contains the locus of $\overline{\rho}$ in $\mathcal{X}_{\mathrm{Sym,red},\overline{\mathbf{F}}_p}$ valued in the Siegel parabolic subgroup whose Levi factor is up to unramified twist of the form $\overline{\theta}\oplus\chi_\sigma\overline{\theta}^\vee$ for some $\overline{\theta}$ in $\mathcal{Z}_{\sigma',r}$. The union of all $\mathcal{C}_{\sigma,r}$ and $\mathcal{C}_{\sigma,\overline{\mathbf{F}}_p}$ exhausts $\overline{\mathbf{F}}_p$-points of $\mathcal{X}_{\mathrm{Sym,red},\overline{\mathbf{F}}_p}$ valued in the Siegel parabolic subgroup up to conjugation.

Let $\{\overline{\theta}_i:G_K\to\mathrm{GL}_2(\overline{\mathbf{F}}_p)\}$ be a finite set of irreducible representations such that any 2-dimensional irreducible G_K-representation with coefficients $\overline{\mathbf{F}}_p$ is an unramified twist of some $\overline{\theta}_i$. For each $0\leq a<p^f-1$, we have a morphism $\chi_a:\mathbf{G}_m\to\mathcal{X}_{1,\mathrm{red},\overline{\mathbf{F}}_p}$ corresponding to unramified twists of ϖ^a. Note that the scheme-theoretic image of χ_a is 0-dimensional. There exists a nonempty open subscheme $T_{a,i}\subset\mathbf{G}_m$ and the (possibly empty) closed subscheme $T^c_{a,i}:=\mathbf{G}_m\backslash T_{a,i}$ such that $H^2(G_K,\chi^2_{T_{a,i}}\det\left(\overline{\theta}\right)^{-1})$ (resp. $H^2(G_K,\chi^2_{T^c_{a,i}}\det\left(\overline{\theta}\right)^{-1})$) is locally free of rank 0, (resp. of rank 1). By Lemma 4.1.17 and its preceding discussion with $T=T_{a,i}$ (resp. $T=T^c_{a,i}$), we obtain a morphism $\widetilde{f}:\widetilde{W}\to\mathcal{X}_{\mathrm{Sym,red},\overline{\mathbf{F}}_p}$ (resp. $\widetilde{f}^c:\widetilde{W}^c\to\mathcal{X}_{\mathrm{Sym,red},\overline{\mathbf{F}}_p}$), depending on a and i, and the scheme-theoretic image of its unramified twists is a closed substack $\mathcal{C}_{a,\overline{\theta}_i}$ (resp. $\mathcal{C}_{a,\overline{\theta}^c}$) of dimension $\leq 3[K:\mathbf{Q}_p]$. By its construction, $\mathcal{C}_{a,\overline{\theta}_i}$ contains all $\overline{\mathbf{F}}_p$-points of $\mathcal{X}_{\mathrm{Sym,red},\overline{\mathbf{F}}_p}$ which are unramified twists of

$$\overline{\rho}=\begin{pmatrix}\varpi^a\otimes\mathrm{ur}_t & *_1 & *_2\\ & \overline{\theta}_i & *_3\\ & & \varpi^{-a}\otimes\mathrm{ur}_{t^{-1}}\otimes\det\left(\overline{\theta}_i\right)\end{pmatrix}$$

such that the extension class $*_1$ is nonsplit (equivalently, $*_3$ is nonsplit).

Let $\mathcal{Z}\subset\mathcal{X}_{2,\mathrm{red},\overline{\mathbf{F}}_p}\times\mathcal{X}_{2,\mathrm{red},\overline{\mathbf{F}}_p}$ be the codimension 1 closed substack consisting of $(\overline{\theta}_1,\overline{\theta}_2)$ such that $\det\left(\overline{\theta}_1\right)=\det\left(\overline{\theta}_2\right)$. Since $\iota_{\mathrm{aux}}(\overline{\theta}_1\oplus\overline{\theta}_2)$ is valued in GSp_4 after conjugation, we get an induced morphism $\mathcal{Z}\to\mathcal{X}_{\mathrm{Sym,red},\overline{\mathbf{F}}_p}$ whose scheme-theoretic image denoted by $\mathcal{C}_2$ has dimension $\leq 2[K:\mathbf{Q}_p]-1$. Note that $\mathcal{Z}$ contains all $\overline{\rho}$ as in the previous paragraph with the condition that $*_1=0$ (instead of $*_1$ nonsplit).

The only remaining G_K-representations valued in $\mathrm{GSp}_4(\overline{\mathbf{F}}_p)$ are unramified twists of 4-dimensional irreducible representations of the form $\mathrm{Ind}_{G_{K^4}}^{G_K}\varpi_{K^4,\iota}^{a(p+1)}$ where K^4/K is the unramified extension of degree 4, $\iota:K^4\hookrightarrow E$ is an embedding, and a is an integer. By taking the union of such finitely many families, we obtain 0-dimensional closed substack $\mathcal{C}_{irred}$ in $\mathcal{X}_{\mathrm{Sym,red},\overline{\mathbf{F}}_p}$.

Finally, we take $\mathcal{C}_{small}$ to be the finite union of closed substacks $\mathcal{C}_{irred}$, $\mathcal{C}_2$, $\mathcal{C}_{a,\overline{\theta}_i}$, $\mathcal{C}^c_{a,\overline{\theta}_i}$ for all a and i, and $\mathcal{C}_{\sigma,r}$ for all σ and r. It has dimension $<4[K:\mathbf{Q}_p]$, and $\bigcup_\sigma\mathcal{C}_{\sigma,\overline{\mathbf{F}}_p}\cup\mathcal{C}_{small}$ exhausts all $\overline{\mathbf{F}}_p$-points of $\mathcal{X}_{\mathrm{Sym,red},\overline{\mathbf{F}}_p}$. This completes the proof. □

4.1.19 Existence of crystalline lifts

In his thesis [Lina], Zhongyipan Lin developed an obstruction theory for lifting G_K-representations valued in reductive groups with mod p coefficients and applied it to prove the existence of crystalline lifts of G_K-representations valued in the exceptional group G_2. As already mentioned in *loc. cit.*, the existence of crystalline lifts for classical groups follows from certain codimension estimates on the moduli stack of (φ,Γ)-modules. We briefly recall Lin's results and prove the Theorem 4.1.14.

Recall that $G = \mathrm{GSp}_4$. We say that a G_K-representation $\overline{\rho} : G_K \to \mathrm{GSp}_4(\mathbf{F})$ is *G-completely reducible* if for any parabolic subgroup P of G containing the image of $\overline{\rho}$, a Levi subgroup L of P also contains the image of $\overline{\rho}$. We can easily list all G-completely reducible G_K-representations. For $i = 1, 2, 3$, let $\chi_i : G_K \to \mathbf{F}^\times$ be a continuous character and let $\overline{\theta}_i : G_K \to \mathrm{GL}_2(\mathbf{F})$ be a continuous irreducible representation. Then any G-completely reducible G_K-representations are up to conjugation one of the following:

1. $\chi_1 \oplus \chi_2 \oplus \chi_3\chi_2^{-1} \oplus \chi_3\chi_1^{-1}$ valued in $T(\mathbf{F})$;
2. $\overline{\theta}_1 \oplus \chi_3\overline{\theta}_1^\vee$ valued in $L_S(\mathbf{F})$;
3. $\chi_1 \oplus \overline{\theta}_1\chi_3\chi_1^{-1}$ where $\chi_3 = \det\big(\overline{\theta}_1\big)$ valued in $L_Q(\mathbf{F})$;
4. $\iota_{\mathrm{aux}}(\overline{\theta}_1 \oplus \overline{\theta}_2)$ where $\det\big(\overline{\theta}_1\big) = \det\big(\overline{\theta}_2\big)$ and;
5. $\overline{\rho} : G_K \to \mathrm{GSp}_4(\mathbf{F})$ irreducible as 4-dimensional representation.

In all cases, one can easily find explicit crystalline lifts using the crystalline lifts of irreducible G_K-representations (e.g. the proof of [EG23, Theorem 6.4.4]).

It remains to prove the existence of crystalline lifts of $\overline{\rho} : G_K \to \mathrm{GSp}_4(\mathbf{F})$ that factors through either $S(\mathbf{F})$ or $Q(\mathbf{F})$. Suppose that $\overline{\rho}$ factors through $S(\mathbf{F})$ and denote its Levi factor by $\overline{\theta} \oplus \overline{\chi}\overline{\theta}^\vee$ where $\overline{\theta} : G_K \to \mathrm{GL}_2(\mathbf{F})$ is a continuous representation and $\overline{\chi} : G_K \to \mathbf{F}^\times$ is a character. Then $\overline{\rho}$ corresponds to an extension class $[\overline{c}] \in H^1(G_K, \mathrm{Ad}_S(\overline{\theta}, \overline{\chi}))$. In this case, the G_K-module $\mathrm{Ad}_S(\overline{\theta}, \overline{\chi})$ is abelian. By [EG23, Theorem 6.4.4], we can find crystalline lifts $\theta : G_K \to \mathrm{GL}_2(\mathcal{O})$ of $\overline{\theta}$ and $\chi : G_K \to \mathcal{O}^\times$ of $\overline{\chi}$ with Hodge–Tate weights given by regular dominant cocharacters $\lambda = (\lambda_{j,1}, \lambda_{j,2})_{j\in\mathcal{J}} \in X_*(\underline{T}_2^\vee)$ and $\mu = (\mu_j)_{j\in\mathcal{J}} \in X_*(\underline{T}_1^\vee)$ respectively. Then $\chi\theta^\vee$ has Hodge–Tate weights $\lambda' = (\mu_j - \lambda_{j,2}, \mu_j - \lambda_{j,1})_{j\in\mathcal{J}}$. Twisting θ by a crystalline character with sufficiently large Hodge–Tate weights with trivial mod p reduction, we may assume that λ is *slightly larger than* λ' (i.e. $\lambda_{j,2} \geq -\lambda_{j,1} + \mu_j + 1$ for all $j \in \mathcal{J}$, and the inequality is strict for at least one j). Let $R = R^\lambda_{\overline{\theta}}$ be the crystalline framed deformation ring of $\overline{\theta}$ with Hodge–Tate weights λ. Then the proof of Theorem 6.3.2 in *loc. cit.* easily generalizes to our setup and shows that there exists a crystalline lift $\theta' : G_K \to \mathrm{GL}_2(\mathcal{O})$ of $\overline{\theta}$ with Hodge–Tate weights λ lying on the same irreducible component of $\operatorname{Spec} R$ that θ does, and a lift $\rho : G_K \to S(\mathcal{O})$ of $\overline{\rho}$ with Levi factor $\theta' \oplus \chi\theta'^\vee$. By Lemma 6.3.1 in *loc. cit.*, ρ is crystalline.

Finally, we suppose that $\overline{\rho} : G_K \to \mathrm{GSp}_4(\mathbf{F})$ factors through $Q(\mathbf{F})$ with Levi factor $\overline{\rho}^{\mathrm{ss}} = \overline{\chi} \oplus \overline{\theta}\det\big(\overline{\theta}\big)\overline{\chi}^{-1}$. We may assume $\overline{\theta}$ is irreducible; otherwise $\overline{\rho}$ factors through $S(\mathbf{F})$ and thus its crystalline lift exists by the previous paragraph. As in the previous paragraph, we can find crystalline lifts $\theta : G_K \to \mathrm{GL}_2(\mathcal{O})$ of $\overline{\theta}$ and $\chi : G_K \to \mathcal{O}^\times$ of $\overline{\chi}$ with Hodge–Tate weights given by regular dominant cocharacters $\lambda = (\lambda_{j,1}, \lambda_{j,2})_{j\in\mathcal{J}} \in X_*(\underline{T}_2^\vee)$ and $\mu = (\mu_j)_{j\in\mathcal{J}} \in X_*(\underline{T}_1^\vee)$ respectively. Then $\mathrm{Ad}_Q(\chi, \theta)/\chi^2\det(\theta)^{-1} \simeq \chi\theta^\vee$ has Hodge–Tate weights given by $(\mu_j - \lambda_{j,2}, \mu_j - \lambda_{j,1})_{j\in\mathcal{J}}$. By twisting χ, we may assume that $\mu_j \geq \lambda_{j,1}$ for all $j \in \mathcal{J}$ and the inequality is strict for at least one j.

Let $R^\lambda_{\overline{\theta}}$ (resp. $R^\mu_{\overline{\chi}}$) be the crystalline framed deformation ring of $\overline{\theta}$ (resp. $\overline{\chi}$) of Hodge–Tate weight λ (resp. μ). We let $X = \operatorname{Spec} R$ be an irreducible component of $\operatorname{Spec} R^\lambda_{\overline{\theta}}\widehat{\otimes}_{\mathcal{O}} R^\mu_{\overline{\chi}}$. We let $\theta^{\mathrm{univ}} : G_K \to \mathrm{GL}_2(R)$ and $\chi^{\mathrm{univ}} : G_K \to R^\times$ be the universal families.

Theorem 4.1.20 (Theorem A in [Lina])**.** *Let $[\overline{c}] \in H^1(G_K, \mathrm{Ad}_Q(\overline{\chi}, \overline{\theta}))$ be the class corresponding to $\overline{\rho}$. Recall that we assume $p > 2$. Suppose that*

1. *for each* $s \geq 1$*, the locus*

$$X_s := \{x \in \operatorname{Spec} R \mid \dim k(x) \otimes_R H^2(G_K, \chi^{\mathrm{univ}}(\theta^{\mathrm{univ}})^\vee) \geq s\}$$

   has codimension $\geq s+1$ *in* X*;*

2. *there exists a finite extension* K'/K *of degree prime-to-*p *such that* $\overline{\rho}^{\mathrm{ss}}(G_{K'})$ *has* p*-power order and;*

3. *there exists a* $\overline{\mathbf{Z}}_p$*-point of* $\operatorname{Spec} R$ *whose restriction to* $G_{K'}$ *is mildly regular in the sense of [Lina, Definition 3.0.1].*

Then there exists a $\overline{\mathbf{Z}}_p$*-point of* $\operatorname{Spec} R$ *corresponding to* $\theta' : G_K \to \mathrm{GL}_2(\overline{\mathbf{Z}}_p)$ *and* $\chi' : G_K \to \overline{\mathbf{Z}}_p^\times$ *and a class* $[c] \in H^1(G_K, \mathrm{Ad}_Q(\chi', \theta'))$ *lifting* $[\overline{c}]$.

We now explain how the assumptions in the previous Theorem hold in our setting. By the irreducibility of $\overline{\theta}$, $H^2(G_K, \chi^{\mathrm{univ}}(\theta^{\mathrm{univ}})^\vee) = 0$ and item (1) follows. Item (2) follows from the fact that both $\overline{\chi}$ and $\overline{\theta}$ have images of order prime-to-p. For item (3), we take the $\mathcal{O}$-point of $\operatorname{Spec} R$ given by θ and χ. By the condition on Hodge–Tate weights, we have $H^0(G_{K'}, \chi^2\theta^\vee) = 0$ (this is the condition (MR1) in [Lina, Definition 3.0.1]). To verify the condition (MR2) in *loc. cit.*, which asserts the non-degeneracy of the quadratic form defining the cup product on Lyndon–Demuškin complex, we follow §A in *loc. cit.*. Since $\overline{\rho}^{\mathrm{ss}}|_{G_{K'}}$ is a trivial representation, the computation is much simpler. If we identify the underlying 2-dimensional vector space of $\chi\theta^\vee$ as

$$\begin{aligned} \mathbf{G}_a \oplus \mathbf{G}_a &\simeq U_Q/Z(U_Q) \\ (x,y) &\mapsto \begin{pmatrix} 1 & x & y & \\ & 1 & & y \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \bmod Z(U_Q), \end{aligned}$$

then the quadratic form defining the cup product on Lyndon–Demuškin complex (see §2 and §A in *loc. cit.*) is given by a matrix $\begin{pmatrix} & M_{12} \\ M_{21} & \end{pmatrix}$ where

$$M_{12} = {}^tM_{21} = \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \cdots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}.$$

Thus, the quadratic form is non-degenerate. By Theorem 4.1.20, there exists a lift of $\overline{\rho}$ which is further crystalline by the condition on Hodge–Tate weights.

In all cases of $\overline{\rho}$, the crystalline lift ρ is ordinary when restricted to $G_{K'}$ for some finite unramified extension K'/K. Thus ρ is potentially diagonalizable as explained in Example 4.1.12. □

4.2 Symplectic Breuil–Kisin modules

We start by recalling the definition of Breuil–Kisin modules with tame descent data. We refer [Le+a, §5.1] for further detail.

Let $\tau : I_{\mathbf{Q}_p} \to \underline{T}_n^\vee(\mathcal{O})$ be a tame inertial L-parameter with 1-generic lowest alcove presentation (s, μ). Let r be the order of s_τ. We write $f' = fr$ and let K'/K be the unramified extension of degree r. We fix a choice $\pi' = (-p)^{1/(p^{f'}-1)}$. We let $L' = K'(\pi')$ and $\Delta = \mathrm{Gal}(L'/K)$. For an $\mathcal{O}$-algebra R, define $\mathfrak{S}_{L',R} := (\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} R)[\![u']\!]$. It is equipped with an endomorphism φ and Δ-action such that

$$\mathfrak{S}_{L',R}^\Delta = \mathfrak{S}_{K,R} := (\mathcal{O}_K \otimes_{\mathbf{Z}_p} R)[\![v]\!]$$

where $v = (u')^{p^{f'}-1}$. Let $\mathcal{J}' = \mathrm{Hom}_{\mathbf{Z}_p}(\mathcal{O}_{K'}, \mathcal{O})$. We have a decomposition $\mathfrak{S}_{L',R} = \bigoplus_{j' \in \mathcal{J}'} R[\![u']\!]$ where $\mathcal{O}_{K'}$ acts on j'-summand through the map j'. For any $\mathfrak{S}_{L',R}$-module M, we let $M^{(j')}$ be the $R[\![u']\!]$-submodule of M such that $\mathcal{O}_{K'}$ acts by j'.

Let $h \geq 0$ be an integer and R be an $\mathcal{O}$-algebra. A Breuil–Kisin module of rank n, height in $[0, h]$, and type τ with R-coefficients is a projective $\mathfrak{S}_{L',R}$-module $\mathfrak{M}$ of rank n equipped with

- an injective $\mathfrak{S}_{L',R}$-linear map $\phi_{\mathfrak{M}} : \varphi^*(\mathfrak{M}) \to \mathfrak{M}$ whose cokernel is annihilated by $E(v)^h$ where $E(v) = (v + p)^h$;
- a semilinear Δ-action commuting with $\phi_{\mathfrak{M}}$ such that

  $$\mathfrak{M}^{(j')} \bmod u' \simeq \tau^\vee \otimes_{\mathcal{O}} R$$

  as $\Delta' := \mathrm{Gal}(L'/K')$-representation for each $j' \in \mathcal{J}'$.

We write $Y_n^{[0,h],\tau}(R)$ for the groupoid of Breuil–Kisin modules of rank n, height in $[0, h]$, and type τ with R-coefficients.

Remark 4.2.1. Note that we can change the r above by any integer divisible by the order of s_τ. Using this, we can always interpret Breuil–Kisin modules with different descent data as modules over a same base.

Definition 4.2.2 (cf. Definition 2.1.9 in [EL])**.** For $\mathfrak{M} \in Y_n^{[0,h],\tau}(R)$, we define $\mathfrak{M}^\vee \in Y_n^{[0,h],\tau^\vee}(R)$ by setting

$$\mathfrak{M}^\vee = \mathrm{Hom}_{\mathfrak{S}_{L',R}}(\mathfrak{M}, \mathfrak{S}_{L',R})$$

with induced semilinear Δ-action and $\phi_{\mathfrak{M}^\vee} : \varphi^*(\mathfrak{M}^\vee) \to \mathfrak{M}^\vee$ given by the formula

$$\phi_{\mathfrak{M}^\vee}(f)(m) = \varphi(f(\phi_{\mathfrak{M}}^{-1}(E(v)^h m)))$$

for all $f \in \mathfrak{M}^\vee$ and $m \in \mathfrak{M}$. Here $\phi_{\mathfrak{M}}^{-1}(E(v)^h m)$ makes sense because of the height condition and that $\phi_{\mathfrak{M}}$ is injective.

Definition 4.2.3. For $i = 1, 2$, let $n_i \geq 1$ and $h_i \geq 0$ be integers and τ_i be a tame inertial L-parameter valued in $\underline{T}_{n_i}^\vee$. For $\mathfrak{M}_i \in Y_{n_i}^{[0,h_i],\tau_i}(R)$, we define a Breuil–Kisin module

$$\mathfrak{M}_1 \otimes_{\mathfrak{S}_{L',R}} \mathfrak{M}_2 \in Y_{n_1 n_2}^{[0,h_1+h_2],\tau_1\otimes\tau_2}(R)$$

by defining $\phi_{\mathfrak{M}_1\otimes\mathfrak{M}_2} := \phi_{\mathfrak{M}_1}\otimes\phi_{\mathfrak{M}_1}$ and taking Δ-action induced by Δ-actions on $\mathfrak{M}_1$ and $\mathfrak{M}_2$.

When $n=1$ and $h=0$, we write $Y_1^{0,\tau} := Y_1^{[0,0],\tau}$. For $\mathfrak{N}\in Y_1^{0,\tau}$, the evaluation map $\mathfrak{N}\otimes\mathfrak{N}^\vee\to\mathfrak{S}_{L',R}$ is an isomorphism of Breuil–Kisin modules.

Let $\tau : I_{\mathbf{Q}_p}\to\underline{T}^\vee(\mathcal{O})$ be a tame inertial L-parameter. For simplicity, we write $Y_4^{[0,h],\tau}$ to denote $Y_4^{[0,h],\mathrm{std}(\tau)}$.

Definition 4.2.4 (cf. Definition 2.1.17 in [EL]). A *symplectic* Breuil–Kisin module of height in $[0,h]$, and type τ with R-coefficients is a triple $(\mathfrak{M},\mathfrak{N},\alpha)$ where $\mathfrak{M}\in Y_4^{[0,h],\tau}(R)$, $\mathfrak{N}\in Y_1^{0,\mathrm{sim}(\tau)}$, and $\alpha:\mathfrak{M}\simeq\mathfrak{M}^\vee\otimes\mathfrak{N}$ satisfying the *alternating condition*

$$(\alpha^\vee\otimes\mathfrak{N})^{-1}\circ\alpha = -1_{\mathfrak{M}}.$$

We write $Y_{\mathrm{Sym}}^{[0,h],\tau}(R)$ for the groupoid of symplectic Breuil–Kisin modules of height in $[0,h]$ and type τ with R-coefficients.

Proposition 4.2.5. *The map $Y_{\mathrm{Sym}}^{[0,h],\tau}\to Y_4^{[0,h],\tau}\times_{\mathcal{O}}Y_1^{0,\mathrm{sim}(\tau)}$ sending $(\mathfrak{M},\mathfrak{N},\alpha)$ to $(\mathfrak{M},\mathfrak{N})$ is representable by schemes. In particular, $Y_{\mathrm{Sym}}^{[0,h],\tau}$ is p-adic formal algebraic stack.*

Proof. Let R be a p-adically complete $\mathcal{O}$-algebra and $S=\operatorname{Spec}R$. Consider a S-point $(\mathfrak{M}_S,\mathfrak{N}_S)\in Y_4^{[0,h],\tau}\times_{\mathcal{O}}Y_1^{0,\mathrm{sim}(\tau)}(S)$. For any S-scheme S', we write $(\mathfrak{M}_{S'},\mathfrak{N}_{S'})$ for the pullback of $(\mathfrak{M}_S,\mathfrak{N}_S)$ to S'. The fiber product $Y_{\mathrm{Sym}}^{[0,h],\tau}\times_{Y_4^{[0,h],\tau}\times_{\mathcal{O}}Y_1^{0,\mathrm{sim}(\tau)}}S$ is the subsheaf $\underline{\alpha}_S$ of $\underline{Isom}(\mathfrak{M}_S,\mathfrak{M}_S^\vee\otimes\mathfrak{N}_S)$ given by

$$\underline{\alpha}_S : S'\mapsto\left\{\alpha'\in Isom(\mathfrak{M}_{S'},\mathfrak{M}_{S'}^\vee\otimes\mathfrak{N}_{S'}) \mid ((\alpha')^\vee\otimes\mathfrak{N}_{S'})^{-1}\circ\alpha' = -1_{\mathfrak{M}_{S'}}\right\}$$

for any S-scheme S'. The sheaf $\underline{Isom}(\mathfrak{M}_S,\mathfrak{M}_S^\vee\otimes\mathfrak{N}_S)$ is representable by affine scheme by [Car+22, Proposition 3.1.3]. There is a cartesian square

$$\begin{tikzcd}
\underline{\alpha}_S \arrow[rr] \arrow[d] & & S \arrow[d, "-1_{\mathfrak{M}_S}"] \\
\underline{Isom}(\mathfrak{M}_S,\mathfrak{M}_S^\vee\otimes\mathfrak{N}_S) \arrow[r] & \underline{Isom}(\mathfrak{M}_S,\mathfrak{M}_S^\vee\otimes\mathfrak{N}_S)\times_S\underline{Isom}(\mathfrak{M}_S^\vee\otimes\mathfrak{N}_S,\mathfrak{M}_S) \arrow[r, "c"] & \underline{Aut}(\mathfrak{M}_S)
\end{tikzcd}$$

where the right verticle map is a constant section given by $-1_{\mathfrak{M}_S}\in\underline{Aut}(\mathfrak{M}_S)$, the left bottom horizontal map sends α to $(\alpha,(\alpha^\vee\otimes\mathfrak{N})^{-1})$, and c is the composition. This shows that $\underline{\alpha}_S$ is representable by scheme. The claim that $Y_{\mathrm{Sym}}^{[0,h],\tau}$ is a p-adic formal algebraic stack follows from [Eme, Lemma 5.19]. ☐

We recall several notions regarding Breuil–Kisin modules. We refer [Le+a, §5] and the references therein for more detailed discussions.

Let $\beta=(\beta^{(j')})_{j'\in\mathcal{J}'}$ be an eigenbasis of $\mathfrak{M}\in Y_n^{[0,h],\tau}$ (in the sense of [Le+a, Definition 5.1.6]). Then the dual basis $\beta^\vee=((\beta^{(j')})^\vee)_{j'\in\mathcal{J}'}$ is an eigenbasis of $\mathfrak{M}^\vee$.

Definition 4.2.6. Let $(\mathfrak{M},\mathfrak{N},\alpha)\in Y_{\mathrm{Sym}}^{[0,h],\tau}(R)$. An *eigenbasis* of $(\mathfrak{M},\mathfrak{N},\alpha)$ is a pair (β,γ) of eigenbasis $\beta=(\beta^{(j')})_{j'\in\mathcal{J}'}$ of $\mathfrak{M}$ and $\gamma=(\gamma^{(j')})_{j'\in\mathcal{J}'}$ of $\mathfrak{N}$ satisfying

$$\alpha(\beta^{(j')}) = ((\beta^{(j')})^\vee\otimes\gamma^{(j')})J.$$

Let $(\mathfrak{M}, \mathfrak{N}, \alpha) \in Y^{[0,h],\tau}_{\mathrm{Sym}}(R)$ be a symplectic Breuil–Kisin module with eigenbasis (β, γ). For $j' \in \mathcal{J}'$, we write $C^{(j')}_{\mathfrak{M},\beta} \in M_4(R[\![u']\!])$ for the matrix representation of $\phi^{(j')}_{\mathfrak{M}}$ with respect to $\varphi^*(\beta^{(j'-1)})$ and $\beta^{(j')}$. The height condition implies $E(u')^h(C^{(j')}_{\mathfrak{M},\beta})^{-1} \in M_4(R[\![u']\!])$.

Recall that (s, μ) is a 1-generic lowest alcove presentation of τ. For $j' \in \mathcal{J}'$, we write $j' = j_0 + fk$ for unique $0 \le j \le f-1$ and $0 \le k \le r-1$. Define

$$\begin{aligned} \boldsymbol{\alpha}'_{j'} &= \begin{cases} s_\tau^{-k}(s_{f-1}^{-1}s_{f-2}^{-1}\dots s_{f-j}^{-1})(\mu_{f-j}+\eta_{f-j}) & \text{if } j' \neq 0 \\ \mu_0 + \eta_0 & \text{if } j' = 0 \end{cases} \\ (\mathbf{a}')^{(j')} &= \sum_{i=0}^{f'-1} \boldsymbol{\alpha}'_{-j'+i} p^i \\ s'_{\mathrm{or},j'} &= s_\tau^{k+1}(s_{f-1}^{-1}s_{f-2}^{-1}\dots s_{j+1}^{-1}). \end{aligned} \tag{4.2.7}$$

Then we “remove the descent datum” and get the matrix

$$A^{(j')}_{\mathfrak{M},\beta} := \mathrm{Ad}(\phi(s'_{\mathrm{or},j'})^{-1}\phi(-(\mathbf{a}')^{(j')})(u'))(C^{(j')}_{\mathfrak{M},\beta}) \in M_4(R[\![v]\!])$$

that is upper triangular modulo v for each $j' \in \mathcal{J}'$. By height condition, we have $E(v)^h(A^{(j')}_{\mathfrak{M},\beta})^{-1} \in M_4(R[\![v]\!])$ which is upper triangular modulo v. Both $C^{(j')}_{\mathfrak{M},\beta}$ and $A^{(j')}_{\mathfrak{M},\beta}$ only depend on $j' \bmod f$. Since $\alpha(\beta^{(j')}) = ((\beta^{(j')})^\vee \otimes \gamma^{(j')})J$, we have the following symplecticity property of matrices $C^{(j')}_{\mathfrak{M},\beta}$ and $A^{(j')}_{\mathfrak{M},\beta}$.

Lemma 4.2.8. *Let $(\mathfrak{M}, \mathfrak{N}, \alpha) \in Y^{[0,h],\tau}_{\mathrm{Sym}}(R)$ with an eigenbasis (β, γ).*

1. *The partial Frobenius matrix $C^{(j')}_{\mathfrak{M},\beta}$ satisfies*

$$(C^{(j')}_{\mathfrak{M},\beta})^\top J C^{(j')}_{\mathfrak{M},\beta} = C^{(j')}_{\mathfrak{N},\gamma} E(v)^h J$$

   In particular, we have $C^{(j')}_{\mathfrak{M},\beta} \in \mathrm{GSp}_4(R[\![u']\!][\frac{1}{E(v)}])$.

2. *Similarly, we have*

$$(A^{(j')}_{\mathfrak{M},\beta})^\top J A^{(j')}_{\mathfrak{M},\beta} = A^{(j')}_{\mathfrak{N},\gamma} E(v)^h J.$$

   In particular, for p-adically complete R, we have $A^{(j')}_{\mathfrak{M},\beta} \in L\mathcal{G}_{\mathcal{O}}(R)$.

Proof. We can show that $(\beta^\vee \otimes \gamma)J$ is an eigenbasis of $\mathfrak{M}^\vee \otimes \mathfrak{N}$ and

$$C^{(j')}_{\mathfrak{M}^\vee \otimes \mathfrak{N}, (\beta^\vee \otimes \gamma)J} = C^{(j')}_{\mathfrak{N},\gamma} E(v)^h J^{-1} (C^{(j')}_{\mathfrak{M},\beta})^{-\top} J$$

and similarly for $A^{(j')}_{\mathfrak{M}^\vee \otimes \mathfrak{N}, (\beta^\vee \otimes \gamma)J}$. Then the claim follows from the definition of symplectic Breuil–Kisin module and its eigenbasis. ☐

We record how $A^{(j')}_{\mathfrak{M},\beta}$ changes under the change of basis.

Proposition 4.2.9. *Let $(\mathfrak{M}, \mathfrak{N}, \alpha) \in Y^{[0,h],\tau}_{\mathrm{Sym}}(R)$ with an eigenbasis (β_1, γ_1). Suppose that (β_2, γ_2) is another eigenbasis such that, for each $j' \in \mathcal{J}'$, $\beta_2^{(j')} = \beta_1^{(j')} D^{(j')}$ and $\gamma_2^{(j')} = \gamma_1^{(j')} \mathrm{sim}(D^{(j')})$ for some $D^{(j')} \in$*

$\mathrm{GSp}_4(R[\![u']\!])$. *Define*

$$I^{(j')} := \mathrm{Ad}(\phi(s'_{\mathrm{or},j'})^{-1}\phi(-(\mathbf{a}')^{(j')})(u'))(D^{(j')}).$$

Then $I^{(j')} \in \mathcal{I}(R)$ *and it satisfies*

$$A^{(j')}_{\mathfrak{M},\beta_1} = I^{(j')}A^{(j')}_{\mathfrak{M},\beta_2}(\mathrm{Ad}(\phi(s_j^{-1})\phi(\mu_j+\eta_j)(v))(\varphi(I^{(j'-1)})^{-1}))$$

where $j \in \mathcal{J}$, $j = j' \bmod f$.

Conversely, for any $I^{(j')} \in \mathcal{I}(R)$ *only depending on* $j' \bmod f$*, then the matrix*

$$D^{(j')} := \mathrm{Ad}(\phi((\mathbf{a}')^{(j')})(u')\phi(s'_{\mathrm{or},j'}))(I^{(j')}) \in \mathrm{GSp}_4(R[\![u']\!]),$$

and (β_2, γ_2) *given by* $\beta_{2,j'} = \beta_{1,j'}D^{(j')}$ *and* $\gamma_{2,j'} = \gamma_{1,j}\,\mathrm{sim}(D^{(j')})$ *is an eigenbasis.*

Proof. This essentially follows from [Le+a, Proposition 5.1.8]. Note that the conditions $\beta_2^{(j')} = \beta_1^{(j')}D^{(j')}$ and $\gamma_2^{(j')} = \gamma_1^{(j')}\,\mathrm{sim}(D^{(j')})$ imply that (β_2, γ_2) is an eigenbasis. □

Definition 4.2.10. Let $(\mathfrak{M}, \mathfrak{N}, \alpha) \in Y^{[0,h],\tau}_{\mathrm{Sym}}(\mathbf{F})$. We define the *shape* of $(\mathfrak{M}, \mathfrak{N}, \alpha)$ to be the unique element $\widetilde{z} \in \widetilde{W}^{\vee,\mathcal{J}}$ such that $A^{(j)}_{\mathfrak{M},\beta} \in \mathcal{I}(\mathbf{F})\widetilde{z}_j\mathcal{I}(\mathbf{F})$ for some eigenbasis (β, γ) and for each $j \in \mathcal{J}$. By Proposition 4.2.9, this is independent of the choice of eigenbasis.

For integers $a \leq b$, we define $L^{[a,b]}\mathcal{G}_{\mathcal{O}}$ to be the subfunctor of $L\mathcal{G}_{\mathcal{O}}$ given by

$$L^{[a,b]}\mathcal{G}_{\mathcal{O}}(R) = \left\{g \in L\mathcal{G}_{\mathcal{O}}(R) \mid E(v)^{-a}\overline{g}, E(v)^{b}\overline{g}^{-1} \in M_4(R[\![v+p]\!])\right\}.$$

for $\mathcal{O}$-algebra R. It is preserved by left and right multiplication by $L^+\mathcal{G}_{\mathcal{O}}$. We write

$$\mathrm{Gr}^{[a,b]}_{\mathcal{G},\mathcal{O}} = L^+\mathcal{G}_{\mathcal{O}}\backslash L^{[a,b]}\mathcal{G}_{\mathcal{O}}$$

for the induced sub-ind-scheme of $\mathrm{Gr}_{\mathcal{G},\mathcal{O}}$. For $(s,\mu) \in W^{\mathcal{J}} \times X^*(T)^{\mathcal{J}}$, we define (s,μ)-twisted φ-conjugation action of $(L^+\mathcal{G}_{\mathcal{O}})^{\mathcal{J}}$ on $(L^{[a,b]}\mathcal{G}_{\mathcal{O}})^{\mathcal{J}}$ by

$$(I^{(j)})_{j\in\mathcal{J}} \cdot (A^{(j)})_{j\in\mathcal{J}} := \Big(I^{(j)}A^{(j)}((\mathrm{Ad}(\phi(s_j^{-1})\phi(\mu_j+\eta_j)(v))(\varphi(I^{(j-1)})^{-1}))\Big)_{j\in\mathcal{J}}.$$

Similarly, we define (s,μ)-twisted conjugation action by the above formula, but with the φ dropped. We denote by $[(L^{[a,b]}\mathcal{G}_{\mathcal{O}})^{\mathcal{J}}/_{\varphi,(s,\mu)}(L^+\mathcal{G}_{\mathcal{O}})^{\mathcal{J}}]$ the fpqc quotient stack using (s,μ)-twisted φ-conjugation action.

Let $Y^{[0,h],\tau,\beta}_{\mathrm{Sym}}$ be the fpqc-stackification of a category fibered in groupoids over $\mathrm{Spf}\,\mathcal{O}$ sending R to a groupoid of tuples $(\mathfrak{M}, \mathfrak{N}, \alpha, \beta, \gamma)$ where $(\mathfrak{M}, \mathfrak{N}, \alpha) \in Y^{[0,h],\tau}_{\mathrm{Sym}}(R)$ and (β, γ) is an eigenbasis of $(\mathfrak{M}, \mathfrak{N}, \alpha)$. By Proposition 4.2.9, $Y^{[0,h],\tau,\beta}_{\mathrm{Sym}}$ is $L^+\mathcal{G}^{\mathcal{J}}_{\mathcal{O}}$-torsor over $Y^{[0,h],\tau}_{\mathrm{Sym}}$.

Proposition 4.2.11. *The morphism* $Y^{[0,h],\tau,\beta}_{\mathrm{Sym}} \to (L^{[0,h]}\mathcal{G}_{\mathcal{O}})^{\mathcal{J}}$ *sending* $(\mathfrak{M}, \mathfrak{N}, \alpha, \beta, \gamma)$ *to* $(A^{(j)}_{\mathfrak{M},\beta})_{\mathcal{J}}$ *induces an isomorphism of p-adic formal algebraic stacks* $Y^{[0,h],\tau}_{\mathrm{Sym}} \simeq [(L^{[0,h]}\mathcal{G}_{\mathcal{O}})^{\mathcal{J}}/_{\varphi,(s,\mu)}(L^+\mathcal{G}_{\mathcal{O}})^{\mathcal{J}}]^{\wedge_p}$. *Here,* $\wedge_p$ *denotes the p-adic completion.*

Proof. This can be proven as [Le+a, Proposition 5.2.1] using Proposition 4.2.9. □

Recall that $\mathrm{Gr}_{\mathcal{G},\mathbf{F}}$ is equal to the affine flag variety $\mathrm{Fl} = \mathcal{I}_{\mathbf{F}}\backslash L(\mathrm{GSp}_4)_{\mathbf{F}}$. For integers $a \leq b$, let $L^{[a,b]}(\mathrm{GSp}_4)_{\mathbf{F}}$ denote the functor defined on $\mathbf{F}$-algebras R by

$$L^{[a,b]}(\mathrm{GSp}_4)_{\mathbf{F}} : R \to \left\{A \in L\mathrm{GSp}_4(R) \mid Av^{-a}, A^{-1}v^b \in M_4(R[\![v]\!])\right\}.$$

The fpqc quotient $[\mathcal{I}_{\mathbf{F}}\backslash L^{[a,b]}(\mathrm{GSp}_4)_{\mathbf{F}}]$ is isomorphic to a Noetherian closed subscheme $\mathrm{Fl}^{[a,b]} \subset \mathrm{Fl}$. Note that $\mathrm{Gr}_{\mathcal{G},\mathbf{F}}^{[a,b]} \subset \mathrm{Fl}^{[a,b]}$ where the inclusion is strict.

We define

$$\mathcal{I}_{1,\mathbf{F}} : R \mapsto \{A \in \mathrm{GSp}_4(R[\![v]\!]) \mid A \bmod v \in U(R)\}$$

and $\widetilde{\mathrm{Gr}}_{\mathcal{G},\mathbf{F}}^{[a,b]} := [\mathcal{I}_{1,\mathbf{F}}\backslash L^{[a,b]}\mathcal{G}_{\mathbf{F}}]$. It is a $T_{\mathbf{F}}^{\vee}$-torsor over $\mathrm{Gr}_{\mathcal{G},\mathbf{F}}^{[a,b]}$. Similarly, we have $\widetilde{\mathrm{Fl}}^{[a,b]} := \mathcal{I}_{1,\mathbf{F}}\backslash L^{[a,b]}(\mathrm{GSp}_4)_{\mathbf{F}}$ which is a $T_{\mathbf{F}}^{\vee}$-torsor over $\mathrm{Fl}^{[a,b]}$.

Proposition 4.2.12. *Suppose that the lowest alcove presentation (s,μ) of τ is $(h+1)$-generic. There is an isomorphism between algebraic stacks $\pi_{(s,\mu)} : Y_{\mathrm{Sym},\mathbf{F}}^{[0,h],\tau} \xrightarrow{\sim} [(\widetilde{\mathrm{Gr}}_{\mathcal{G},\mathbf{F}}^{[0,h]})^{\mathcal{J}}/_{(s,\mu)}T_{\mathbf{F}}^{\vee,\mathcal{J}}]$.*

Proof. This follows from Proposition 4.2.11 and [Le+a, Lemma 5.2.2]. Note that $(I^{(j)})_{j\in\mathcal{J}}$ in *loc. cit.* is an element in $\mathrm{GSp}_4(R[\![v]\!])^{\mathcal{J}}$ if and only if $(X_j)_{j\in\mathcal{J}}$ in *loc. cit.* is an element in $\mathrm{GSp}_4(R[\![v]\!])^{\mathcal{J}}$. □

Recall that, for each $\widetilde{z} \in \widetilde{W}^{\vee,\mathcal{J}}$, there is a subfunctor $\mathcal{U}(\widetilde{z}) = \prod_{j\in\mathcal{J}}\mathcal{U}(\widetilde{z}_j) \subset L\mathcal{G}^{\mathcal{J}}$. For any integers $a \leq b$, we define

$$U^{[a,b]}(\widetilde{z}) := \mathcal{U}(\widetilde{z})_{\mathcal{O}} \cap L^{[a,b]}\mathcal{G}_{\mathcal{O}}^{\mathcal{J}}.$$

The natural projection map $U^{[a,b]}(\widetilde{z}) \to (\mathrm{Gr}_{\mathcal{G},\mathcal{O}}^{[a,b]})^{\mathcal{J}}$ is an open immersion by Proposition 3.1.5. Since $\mathrm{Gr}_{\mathcal{G},\mathbf{F}}^{\mathcal{J}} = \mathrm{Fl}^{\mathcal{J}}$ is ind-proper and its $T^{\vee,\mathcal{J}}$-fixed points are exactly $\widetilde{z} \in \widetilde{W}^{\vee,\mathcal{J}}$, $\mathcal{U}(\widetilde{z})_{\mathbf{F}}$ form an open cover of $\mathrm{Gr}_{\mathcal{G},\mathbf{F}}^{\mathcal{J}}$. Also we have $T^{\vee,\mathcal{J}}$-torsors

$$\widetilde{\mathcal{U}}(\widetilde{z}) := T^{\vee,\mathcal{J}}\mathcal{U}(\widetilde{z}) \to \mathcal{U}(\widetilde{z})$$
$$\widetilde{U}^{[a,b]}(\widetilde{z}) := T_{\mathcal{O}}^{\vee,\mathcal{J}}U(\widetilde{z})^{[a,b]} \to U(\widetilde{z})^{[a,b]}.$$

The images of $U(\widetilde{z})_{\mathbf{F}}^{[a,b]}$ (resp. $\widetilde{U}(\widetilde{z})_{\mathbf{F}}^{[a,b]}$) form an open cover of $(\mathrm{Gr}_{\mathcal{G},\mathbf{F}}^{[a,b]})^{\mathcal{J}}$ (resp. $(\widetilde{\mathrm{Gr}}_{\mathcal{G},\mathbf{F}}^{[a,b]})^{\mathcal{J}}$).

Recall that $Y_{\mathrm{Sym}}^{[0,h],\tau}$ is a p-adic formal algebraic stack, which implies $Y_{\mathrm{Sym}}^{[0,h],\tau} = \varinjlim_i Y_{\mathrm{Sym}}^{[0,h],\tau} \times_{\mathrm{Spf}\,\mathcal{O}} \mathrm{Spec}\,\mathcal{O}/\varpi^i$. The algebraic stack $Y_{\mathrm{Sym}}^{[0,h],\tau} \times_{\mathrm{Spf}\,\mathcal{O}} \mathrm{Spec}\,\mathcal{O}/\varpi^i$ has the same underlying topological spaces for all $i \in \mathbf{Z}_{>0}$. There is a bijection between open substack of given algebraic stack and open subsets of its underlying topological spaces (see [Stacks, Tag 06FJ]). Thus there is bijection between open substacks of $Y_{\mathrm{Sym},\mathcal{O}/\varpi^i}^{[0,h],\tau} := Y_{\mathrm{Sym}}^{[0,h],\tau} \times_{\mathrm{Spf}\,\mathcal{O}} \mathrm{Spec}\,\mathcal{O}/\varpi^i$ and open substacks of $Y_{\mathrm{Sym},\mathbf{F}}^{[0,h],\tau}$.

Definition 4.2.13 (cf. Definition 5.2.4 in [Le+a])**.** Let $\widetilde{z} \in \widetilde{W}^{\vee,\mathcal{J}}$.

1. We define $Y_{\mathrm{Sym},\mathbf{F}}^{[0,h],\tau}(\widetilde{z})$ to be the open substack of $Y_{\mathrm{Sym},\mathbf{F}}^{[0,h],\tau}$ corresponding to

$$[\widetilde{U}^{[0,h]}(\widetilde{z})_{\mathbf{F}}/_{(s,\mu)}T_{\mathbf{F}}^{\vee,\mathcal{J}}] \subset [(\widetilde{\mathrm{Gr}}_{\mathcal{G},\mathbf{F}}^{[0,h]})^{\mathcal{J}}/_{(s,\mu)}T_{\mathbf{F}}^{\vee,\mathcal{J}}]$$

under the isomorphism $\pi_{(s,\mu)}$. We write $Y_{\mathrm{Sym},\mathcal{O}/\varpi^i}^{[0,h],\tau}(\widetilde{z})$ to be the open substack of $Y_{\mathrm{Sym},\mathcal{O}/\varpi^i}^{[0,h],\tau}$ induced by $Y_{\mathrm{Sym},\mathbf{F}}^{[0,h],\tau}(\widetilde{z})$.

2. We define the p-adic formal open substack $Y^{[0,h],\tau}_{\mathrm{Sym}}(\widetilde{z}) := \varinjlim_i Y^{[0,h],\tau}_{\mathrm{Sym},\mathcal{O}/\varpi^i}(\widetilde{z})$ of $Y^{[0,h],\tau}_{\mathrm{Sym}}$.

3. Let R be a p-adically complete $\mathcal{O}$-algebra. We say that $(\mathfrak{M},\mathfrak{N},\alpha) \in Y^{[0,h],\tau}_{\mathrm{Sym}}(R)$ *admits $\widetilde{z}$-gauge* if $(\mathfrak{M},\mathfrak{N},\alpha) \in Y^{[0,h],\tau}_{\mathrm{Sym}}(\widetilde{z})(R)$.

4. For $(\mathfrak{M},\mathfrak{N},\alpha) \in Y^{[0,h],\tau}_{\mathrm{Sym}}(\widetilde{z})(R)$, we define a *$\widetilde{z}$-gauge basis* of $(\mathfrak{M},\mathfrak{N},\alpha)$ as an eigenbasis (β,γ) such that $A^{(j)}_{\mathfrak{M},\beta} \in \widetilde{\mathcal{U}}(\widetilde{z}_j)(R)$ for all $j \in \mathcal{J}$.

Proposition 4.2.14. *Let (s,μ) be a $(h+1)$-generic lowest alcove presentation of τ. Let R be a p-adically complete $\mathcal{O}$-algebra and $(\mathfrak{M},\mathfrak{N},\alpha) \in Y^{[0,h],\tau}_{\mathrm{Sym}}(\widetilde{z})(R)$. If $\mathfrak{M}$ admits an eigenbasis, then $\mathfrak{M}$ admits a $\widetilde{z}$-gauge basis, and the set of $\widetilde{z}$-gauge bases is a $T^{\vee,\mathcal{J}}(R)$-torsor.*

Proof. Note that if (β,γ) is a $\widetilde{z}$-gauge basis for $(\mathfrak{M},\mathfrak{N},\alpha)$, then β has to be a $\widetilde{z}$-gauge basis for $\mathfrak{M}$ in the sense of [Le+a, Definition 5.2.6]. By Proposition 5.2.7 of *loc. cit.*, $\mathfrak{M}$ admits a $\widetilde{z}$-gauge basis β, and the set of such basis is $T_4^{\vee,\mathcal{J}}$-torsor. Let γ be any eigenbasis of $\mathfrak{N}$. Since $(\beta^\vee \otimes \gamma)J$ and $\alpha(\beta)$ are $\widetilde{z}$-gauge basis of $\mathfrak{M}^\vee \otimes \mathfrak{N}$, there exists $t \in T_4^{\vee,\mathcal{J}}(R)$ such that $\alpha(\beta_j) = (\beta_j^\vee \otimes \gamma_j)Jt_j$ for $j \in \mathcal{J}$. Then the alternating condition implies that $t_j J t_j^{-1} = J$. Let $\beta' = \beta t'$ for some $t' \in T_4^{\vee,\mathcal{J}}(R)$ and $\gamma' = \gamma c$ for some $c \in (R[\![v]\!]^\times)^{\mathcal{J}}$. Then $(\beta')^\vee = \beta^\vee (t')^{-1}$ and

$$\alpha(\beta'_j) = ((\beta'_j)^\vee \otimes \gamma'_j)t'_j J t'_j t_j c_j^{-1} \text{ for each } j \in \mathcal{J}.$$

The condition on t implies that there exists t'_j and c_j such that $t'_j J t'_j = c_j J t_j^{-1}$ for each $j \in \mathcal{J}$, and the solution gives a $\widetilde{z}$-gauge basis (β',γ') of $(\mathfrak{M},\mathfrak{N},\alpha)$. Since the set of solutions is a $T^{\vee,\mathcal{J}}$-torsor, the set of $\widetilde{z}$-gauge basis is a $T^{\vee,\mathcal{J}}$-torsor. $\square$

Let $\lambda \in X_*(T^\vee)^{\mathcal{J}}$ be a dominant cocharacter such that $\mathrm{std}(\lambda_j) \in [0,h]^4$ for all $j \in \mathcal{J}$. There is a $\mathcal{O}$-flat closed p-adic formal substack $Y_4^{\le \mathrm{std}(\lambda),\tau} \subset Y_4^{[0,h],\tau}$ (e.g. [Le+a, §5.3]). For simplicity, we write $Y_4^{\le\lambda,\tau}$ instead of $Y_4^{\le \mathrm{std}(\lambda),\tau}$. We have its symplectic variant $Y^{\le\lambda,\tau}_{\mathrm{Sym}}$ which is defined as a $\mathcal{O}$-flat part of $Y^{[0,h],\tau}_{\mathrm{Sym}} \times_{Y_4^{[0,h],\tau}} Y_4^{\le\lambda,\tau}$. For any finite extension E'/E with the ring of integer $\mathcal{O}'$, $(\mathfrak{M},\mathfrak{N},\alpha) \in Y^{[0,h],\tau}_{\mathrm{Sym}}(\mathcal{O}')$ belongs to $(\mathfrak{M},\mathfrak{N},\alpha) \in Y^{\le\lambda,\tau}_{\mathrm{Sym}}(\mathcal{O}')$ if and only if the elementary divisors of $A^{(j)}_{\mathfrak{M},\beta} \in \mathrm{GSp}_4(E'(\!(v+p)\!))$ (for any eigenbasis (β,γ)) is bounded by $E(v)^{\lambda_j}$ for each $j \in \mathcal{J}$. In particular, $(A^{(j)}_{\mathfrak{M},\beta})_{j\in\mathcal{J}}$ gives a E'-point in $S_E(\lambda) \subset \mathrm{Gr}^{\mathcal{J}}_{\mathcal{G},E}$. We have its open substack

$$Y^{\le\lambda,\tau}_{\mathrm{Sym}}(\widetilde{z}) = Y^{\le\lambda,\tau}_{\mathrm{Sym}} \times_{Y^{[0,h],\tau}_{\mathrm{Sym}}} Y^{[0,h],\tau}_{\mathrm{Sym}}(\widetilde{z}).$$

Remark 4.2.15. For any finite extension $\mathbf{F}'/\mathbf{F}$, $Y_4^{\le\lambda,\tau}(\mathbf{F}')$ is the full subgroupoid of $Y_4^{[0,h],\tau}(\mathbf{F}')$ consisting of Breuil–Kisin modules whose shapes lie in the set $\mathrm{Adm}^\vee(\mathrm{std}(\lambda))$ by [CL18, Proposition 5.4]. By Lemma 2.1.6, $Y^{\le\lambda,\tau}_{\mathrm{Sym}}(\mathbf{F}')$ is the full subgroupoid of $Y^{[0,h],\tau}_{\mathrm{Sym}}(\mathbf{F}')$ consisting of Breuil–Kisin modules whose shapes lie in the set $\mathrm{Adm}^\vee(\lambda) \simeq \mathrm{Adm}^\vee(\mathrm{std}(\lambda))^\Theta$.

On the local model side, recall the Pappas–Zhu local model $M_{\mathcal{J}}(\le\lambda) \subset \mathrm{Gr}^{[0,h],\mathcal{J}}_{\mathcal{J},\mathcal{O}}$. We have its open neighborhood at $\widetilde{z}$ defined by $U(\widetilde{z},\le\lambda) := M_{\mathcal{J}}(\le\lambda) \cap \mathcal{U}(\widetilde{z})_{\mathcal{O}}$. We also write $\widetilde{U}(\widetilde{z},\le\lambda) := T^{\vee,\mathcal{J}}_{\mathcal{O}} \times U(\widetilde{z},\le\lambda)$.

Theorem 4.2.16. *Let (s,μ) be a $(h+1)$-generic lowest alcove presentation of τ and let $\widetilde{z} \in \widetilde{W}^{\vee,\mathcal{J}}$. We have* *a local model diagram of p-adic formal algebraic stacks over $\mathcal{O}$*

$$\begin{tikzcd} & \widetilde{U}(\widetilde{z}, \leq \lambda)^{\wedge_p} \arrow[dl, "T_{\mathcal{O}}^{\vee,\mathcal{J}}"'] \arrow[dr, "T_{\mathcal{O}}^{\vee,\mathcal{J}}"] & \\ Y_{\mathrm{Sym}}^{\leq\lambda,\tau}(\widetilde{z}) = \left[\widetilde{U}(\widetilde{z}, \leq \lambda)/_{(s,\mu)}T_{\mathcal{O}}^{\vee,\mathcal{J}}\right]^{\wedge_p} & & U(\widetilde{z}, \leq \lambda)^{\wedge_p} \end{tikzcd}$$

where diagonal arrows are $T_{\mathcal{O}}^{\vee,\mathcal{J}}$-torsors. The superscript $\wedge_p$ means taking p-adic completion.

Proof. We need to show that $Y_{\mathrm{Sym}}^{\leq\lambda,\tau}(\widetilde{z}) = \left[\widetilde{U}(\widetilde{z}, \leq \lambda)/_{(s,\mu)}T_{\mathcal{O}}^{\vee,\mathcal{J}}\right]^{\wedge_p}$. Let $Y_{\mathrm{Sym}}^{\leq\lambda,\tau,\beta}(\widetilde{z})$ be the fpqc-stackification of a category fibered in groupoids over $\mathrm{Spf}\,\mathcal{O}$ sending R to a groupoid of tuples $(\mathfrak{M}, \mathfrak{N}, \alpha, \beta, \gamma)$ where $(\mathfrak{M}, \mathfrak{N}, \alpha) \in Y_{\mathrm{Sym}}^{\leq\lambda,\tau}(\widetilde{z})(R)$ and (β, γ) is a $\widetilde{z}$-gauge basis of $(\mathfrak{M}, \mathfrak{N}, \alpha)$. For a ring of integer $\mathcal{O}'$ in a finite extension E'/E and $(\mathfrak{M}, \mathfrak{N}, \alpha, \beta, \gamma) \in Y_{\mathrm{Sym}}^{\leq\lambda,\tau,\beta}(\widetilde{z})(\mathcal{O}')$, $(A_{\mathfrak{M},\beta}^{(j)})_{\mathcal{J}} \in \widetilde{U}(\widetilde{z}, \leq \lambda)(\mathcal{O}')$ by the characterization of $Y_{\mathrm{Sym}}^{\leq\lambda,\tau}$ and the definition of gauge bases. Since $Y_{\mathrm{Sym}}^{\leq\lambda,\tau,\beta}(\widetilde{z})$ and $\widetilde{U}(\widetilde{z}, \leq \lambda)^{\wedge_p}$ are $\mathcal{O}$-flat, this induces a map $Y_{\mathrm{Sym}}^{\leq\lambda,\tau,\beta}(\widetilde{z}) \to \widetilde{U}(\widetilde{z}, \leq \lambda)^{\wedge_p}$ sending $(\mathfrak{M}, \mathfrak{N}, \alpha, \beta, \gamma)$ to $(A_{\mathfrak{M},\beta}^{(j)})_{\mathcal{J}}$, which induces an isomorphism $Y_{\mathrm{Sym}}^{\leq\lambda,\tau}(\widetilde{z}) \xrightarrow{\sim} \left[\widetilde{U}(\widetilde{z}, \leq \lambda)/_{(s,\mu)}T_{\mathcal{O}}^{\vee,\mathcal{J}}\right]^{\wedge_p}$ by Proposition 4.2.9 and 4.2.14. □

Corollary 4.2.17. *Under the hypothesis in Theorem 4.2.16, $Y_{\mathrm{Sym}}^{\leq\lambda,\tau}(\widetilde{z}) \neq \emptyset$ if and only if $\widetilde{z} \in \mathrm{Adm}^{\vee}(\lambda)$.*

Proof. The stack $Y_{\mathrm{Sym}}^{\leq\lambda,\tau}(\widetilde{z})$ is nonempty if and only if $U(\widetilde{z}, \leq \lambda) \neq \emptyset$. By [PZ13, Theorem 9.3], $M_{\mathcal{J}}(\leq \lambda)_{\mathbf{F}} = \cup_{\widetilde{s} \in \mathrm{Adm}^{\vee}(\lambda)} S_{\mathbf{F}}^{\circ}(\widetilde{s})$. Then $U(\widetilde{z}, \leq \lambda) \neq \emptyset$ if and only if $S_{\mathbf{F}}^{\circ}(\widetilde{s}) \cap \mathcal{U}(\widetilde{z})_{\mathbf{F}}$ for some $\widetilde{s} \in \mathrm{Adm}^{\vee}(\lambda)$. The last condition is equivalent to $\widetilde{z} \in S_{\mathbf{F}}(\widetilde{s})$ for some $\widetilde{s} \in \mathrm{Adm}^{\vee}(\lambda)$ by [Le+a, Lemma 4.7.1] (which easily generalizes to our setup). Then the claim follows from the standard description of torus fixed points of affine Schubert varieties. □

4.3 Symplectic étale φ-modules

In this section, we fix a dominant cocharacter $\lambda \in X_*(T)^{\mathcal{J}}$ such that $\mathrm{std}(\lambda_j) \in [0, h]^4$ and a 1-generic inertial tame type τ.

Let $n > 0$ be an integer. The ring $\mathcal{O}_{\mathcal{E},K} := W(k)(\!(v)\!)^{\wedge_p}$ is equipped with Frobenius endomorphism φ extending usual Frobenius on $W(k)$ and sending v to v^p. For p-adically complete Noetherian $\mathcal{O}$-algebra R, $\Phi\text{-}\mathrm{Mod}_K^{\text{ét},n}(R)$ is defined as the groupoid of rank n étale φ-modules with R-coefficients. It is known that $\Phi\text{-}\mathrm{Mod}_K^{\text{ét},n}$ is an ind-algebraic fppf stack over $\mathrm{Spf}\,\mathcal{O}$ [EG23, Corollary 3.1.5].

Objects of $\Phi\text{-}\mathrm{Mod}_K^{\text{ét},n}(R)$ are given by rank n projective modules $\mathcal{M}$ over $\mathcal{O}_{\mathcal{E},K}\widehat{\otimes}_{\mathbf{Z}_p}R$ equipped with an isomorphism $\phi_{\mathcal{M}} : \varphi^*(\mathcal{M}) \xrightarrow{\sim} \mathcal{M}$. For each $j \in \mathcal{J}$, we have an induced morphism $\phi_{\mathcal{M}}^{(j)} : \mathcal{M}^{(j-1)} \to \mathcal{M}^{(j)}$. We also define $\mathcal{M}^{\vee} \in \Phi\text{-}\mathrm{Mod}_K^{\text{ét},n}(R)$ the dual étale φ-module of $\mathfrak{M}$ whose underlying module is $\mathrm{Hom}_{\mathcal{O}_{\mathcal{E},K}\widehat{\otimes}_{\mathbf{Z}_p}R}(\mathcal{M}, \mathcal{O}_{\mathcal{E},K}\widehat{\otimes}_{\mathbf{Z}_p}R)$ and $\phi_{\mathcal{M}^{\vee}} : \varphi^*(\mathcal{M}^{\vee}) \to \mathcal{M}^{\vee}$ given by a formula

$$\phi_{\mathcal{M}^{\vee}}(f)(m) = \varphi(f(\phi_{\mathcal{M}}^{-1}(m))$$

for all $f \in \varphi^*(\mathcal{M}^{\vee}) = \mathrm{Hom}_{\mathcal{O}_{\mathcal{E},K}\widehat{\otimes}_{\mathbf{Z}_p}R}(\varphi^*(\mathcal{M}), \mathcal{O}_{\mathcal{E},K}\widehat{\otimes}_{\mathbf{Z}_p}R)$ and $m \in \mathcal{M}$.

We define $\Phi\text{-}\mathrm{Mod}_K^{\text{ét},\mathrm{Sym}}$ as the moduli stack of *symplectic* étale φ-modules whose objects are given by triples $(\mathcal{M}, \mathcal{N}, \alpha) \in \Phi\text{-}\mathrm{Mod}_K^{\text{ét},\mathrm{Sym}}(R)$ where $\mathcal{M} \in \Phi\text{-}\mathrm{Mod}_K^{\text{ét},4}(R)$, $\mathcal{N} \in \Phi\text{-}\mathrm{Mod}_K^{\text{ét},1}(R)$, and $\alpha : \mathcal{M} \xrightarrow{\sim} \mathcal{M}^{\vee} \otimes_{\mathcal{O}_{\mathcal{E},K}\widehat{\otimes}_{\mathbf{Z}_p}R} \mathcal{N}$ satisfying alternating condition $(\alpha^{\vee} \otimes \mathcal{N})^{-1} \circ \alpha = -1$.

Proposition 4.3.1. *The map $\Phi\text{-Mod}_K^{\text{ét,Sym}} \to \Phi\text{-Mod}_K^{\text{ét},4} \times_{\mathcal{O}} \Phi\text{-Mod}_K^{\text{ét},1}$ sending $(\mathcal{M}, \mathcal{N}, \alpha)$ to $(\mathcal{M}, \mathcal{N})$ is representable by algebraic spaces. In particular, $\Phi\text{-Mod}_K^{\text{ét,Sym}}$ is an ind-algebraic fppf stack over $\operatorname{Spf}\mathcal{O}$.*

Proof. This can be proven as Proposition 4.2.5 using [EG23, Corollary 3.1.5] instead of [Car+22, Proposition 3.1.3]. $\square$

Definition 4.3.2. Let R be a p-adically complete $\mathcal{O}$-algebra and $(\mathcal{M}, \mathcal{N}, \alpha) \in \Phi\text{-Mod}_K^{\text{ét,Sym}}(R)$. A *basis* of $(\mathcal{M}, \mathcal{N}, \alpha)$ is a pair $(\beta, \gamma) = ((\beta^{(j)})_{j\in\mathcal{J}}, (\gamma^{(j)})_{j\in\mathcal{J}})$ where for each $j \in \mathcal{J}$, $\beta^{(j)}$ (resp. $\gamma^{(j)}$) is a basis for a rank 4 (resp. 1) free $R[\![v]\!][1/v]^{\wedge_p}$-module $\mathcal{M}^{(j)}$ (resp. $\mathcal{N}^{(j)}$) such that

$$\alpha(\beta^{(j)}) = ((\beta^{(j)})^\vee \otimes \gamma^{(j)})J.$$

Lemma 4.3.3. *Let R be a p-adically complete $\mathcal{O}$-algebra and $(\mathcal{M}, \mathcal{N}, \alpha) \in \Phi\text{-Mod}_K^{\text{ét,Sym}}(R)$.*

1. *If (β, γ) is a basis of $(\mathcal{M}, \mathcal{N}, \alpha)$, then the matrix representation of $\phi_{\mathcal{M}}^{(j)}$ (resp. $\phi_{\mathcal{N}}^{(j)}$) with respect to the basis $\beta^{(j-1)}$ and $\beta^{(j)}$ (resp. $\gamma^{(j-1)}$ and $\gamma^{(j)}$) is given by a matrix $A^{(j)} \in \mathrm{GSp}_4(R(\!(v)\!)^{\wedge_p})$ (resp. $\mathrm{sim}(A) \in (R(\!(v)\!)^{\wedge_p})^\times$).*
2. *If (β_1, γ_1) and (β_2, γ_2) are basis of $(\mathcal{M}, \mathcal{N}, \alpha)$ such that $\beta_2^{(j)} = \beta_1^{(j)} A_j$ and $\gamma_2^{(j)} = \gamma_1^{(j)} c_j$ for some $A_j \in \mathrm{GL}_4(R(\!(v)\!)^{\wedge_p})$ and $c_j \in (R(\!(v)\!)^{\wedge_p})^\times$, then $A_j \in \mathrm{GSp}_4(R(\!(v)\!)^{\wedge_p})$ and $c_j = \mathrm{sim}(A_j)$.*

Proof. Item (1) can be shown as Lemma 4.2.8. Item (2) follows from a direct computation using the condition $\alpha(\beta^{(j)}) = ((\beta^{(j)})^\vee \otimes \gamma^{(j)})J$. $\square$

For a tame inertial type τ' valued in $T_n^\vee(E)$, there is a morphism $\varepsilon_{\tau'} : Y_n^{[0,h],\tau'} \to \Phi\text{-Mod}_K^{\text{ét},n}$ representable by algebraic spaces, proper, and of finite presentation ([Le+a, Proposition 5.4.1]).

Given $\mathfrak{M} \in Y_n^{[0,h],\tau'}(R)$ and an integer $m \geq 0$, we define $\mathfrak{M}(m) \in Y_n^{[m,h+m],\tau'}(R)$ to be the Breuil–Kisin module whose underlying module is $\mathfrak{M}$ and the Frobenius endomorphism is given by $\phi_{\mathfrak{M}(m)} = E(v)^m \phi_{\mathfrak{M}}$.

We define a morphism of ind-algebraic fppf stacks over $\operatorname{Spf}\mathcal{O}$

$$\begin{aligned}\varepsilon_\tau : Y_{\mathrm{Sym}}^{[0,h],\tau} &\to \Phi\text{-Mod}_K^{\text{ét,Sym}}\\ (\mathfrak{M}, \mathfrak{N}, \alpha) &\mapsto (\varepsilon_{\mathrm{std}(\tau)}(\mathfrak{M}), \varepsilon_{\mathrm{sim}(\tau)}(\mathfrak{N}(h)), \varepsilon_{\mathrm{std}(\tau)}(\alpha)).\end{aligned}$$

Remark 4.3.4. Note that the dual of Breuil–Kisin module and étale φ-module are not compatible. This is because of $E(v)^h$ in the formula defining the Frobenius of dual Breuil–Kisin modules. This is why we use $\varepsilon_{\mathrm{sim}(\tau)}(\mathfrak{N}(h))$ instead of $\varepsilon_{\mathrm{sim}(\tau)}(\mathfrak{N})$ in the definition of ε_τ, so that we still have $\varepsilon_\tau(\mathfrak{M}^\vee \otimes \mathfrak{N}) = \varepsilon_{\mathrm{std}(\tau)}(\mathfrak{M})^\vee \otimes \varepsilon_{\mathrm{sim}(\tau)}(\mathfrak{N}(h))$. In particular, this explains that $\varepsilon_{\mathrm{std}(\tau)}(\alpha)$ is well-defined.

Proposition 4.3.5. *Suppose that τ is $(h+1)$-generic. The map $\varepsilon_\tau : Y_{\mathrm{Sym}}^{[0,h],\tau} \to \Phi\text{-Mod}_K^{\text{ét,Sym}}$ is a closed immersion.*

Proof. By [Le+a, Proposition 5.4.3], $\varepsilon_\tau : Y_4^{[0,h],\tau} \to \Phi\text{-Mod}_K^{\text{ét},4}$ is a monomorphism. Then ε_τ is fully faithful by [Stacks, Tag 04ZZ]. As a result, the diagram

$$\begin{tikzcd} Y_{\mathrm{Sym}}^{[0,h],\tau} \arrow[r, "\varepsilon_\tau"] \arrow[d] & \Phi\text{-Mod}_K^{\text{ét,Sym}} \arrow[d] \\ Y_4^{[0,h],\tau} \times_{\operatorname{Spf}\mathcal{O}} Y_1^{0,\mathrm{sim}(\tau)} \arrow[r, "\varepsilon_{\mathrm{std}(\tau)} \times \varepsilon_{\mathrm{sim}(\tau)}"] & \Phi\text{-Mod}_K^{\text{ét},4} \times_{\operatorname{Spf}\mathcal{O}} \Phi\text{-Mod}_K^{\text{ét},1} \end{tikzcd}$$

is cartesian, and the claim follows from [Le+a, Proposition 5.4.3]. $\square$

Lemma 4.3.6. *Let* $(\mathfrak{M}, \mathfrak{N}, \alpha) \in Y^{[0,h],\tau}_{\mathrm{Sym}}(R)$ *and* $(\mathcal{M}, \mathcal{N}, \varepsilon_\tau(\alpha)) = \varepsilon_\tau((\mathfrak{M}, \mathfrak{N}, \alpha))$. *If* (β, γ) *is an eigenbasis of* $(\mathfrak{M}, \mathfrak{N}, \alpha)$*, there exists a basis* $(\mathfrak{b}, \mathfrak{c})$ *of* $(\mathcal{M}, \mathcal{N}, \varepsilon_\tau(\alpha))$ *determined by* (β, γ) *such that* $\phi^{(j)}_{\mathcal{M}}$ *with respect to* $\mathfrak{b}$ *is given by* $A^{(j)}_{\mathfrak{M},\beta} s_j^{-1} v^{\mu_j + \eta_j}$.

Proof. The existence of $\mathfrak{b}$ follows from [Le+a, Proposition 5.4.2]. Then there is a unique $\mathfrak{c}$ such that $(\mathfrak{b}, \mathfrak{c})$ is a basis of $(\mathcal{M}, \mathcal{N}, \varepsilon_\tau(\alpha))$. $\square$

Let $a \leq b$ be integers. There is a natural map

$$\begin{aligned} \iota'_{\widetilde{z}} : \prod_{j \in \mathcal{J}} (L^{[a,b]}\mathrm{GSp}_4)_{\mathbf{F}} \widetilde{z}_j &\to \Phi\text{-}\mathrm{Mod}^{\text{ét},\mathrm{Sym}}_K \\ (A^{(j)}\widetilde{z}_j)_{j\in\mathcal{J}} &\mapsto (\mathcal{M}, \mathcal{N}, \alpha) \end{aligned}$$

where $\mathcal{M}$ (resp. $\mathcal{N}$) is a free rank 4 (resp. 1) étale φ-module such that $\phi^{(j)}_{\mathcal{M}}$ (resp. $\phi^{(j)}_{\mathcal{N}}$) with respect to the standard basis is given by $A^{(j)}\widetilde{z}_j$ (resp. $\mathrm{sim}(A^{(j)}\widetilde{z}_j)$) and α is given by the matrix J with respect to the standard basis of $\mathcal{M}$ and its dual basis. We also define a closed subscheme

$$\widetilde{\mathrm{Fl}}^{[a,b]}_{\mathcal{J},\widetilde{z}} := \prod_{j\in\mathcal{J}} (\mathcal{I}_{1,\mathbf{F}} \backslash (L^{[a,b]}\mathrm{GSp}_4)_{\mathbf{F}} \widetilde{z}_j) \subset \widetilde{\mathrm{Fl}}^{\mathcal{J}}.$$

We denote by $T^{\vee,\mathcal{J}}_{\mathbf{F}}$-conj a $T^{\vee,\mathcal{J}}_{\mathbf{F}}$-action on $\widetilde{\mathrm{Fl}}^{[a,b]}_{\mathcal{J},\widetilde{z}}$ given by

$$(D_j)_{j\in\mathcal{J}} \cdot (\mathcal{I}_{1,\mathbf{F}} A^{(j)}\widetilde{z}_j)_{j\in\mathcal{J}} = (D_j \mathcal{I}_{1,\mathbf{F}} A^{(j)} \widetilde{z}_j D_{j-1}^{-1})_{j\in\mathcal{J}}$$

for $(D_j)_{j\in\mathcal{J}} \in T^{\vee,\mathcal{J}}_{\mathbf{F}}$ and $(\mathcal{I}_{1,\mathbf{F}} A^{(j)}\widetilde{z}_j)_{j\in\mathcal{J}} \in \widetilde{\mathrm{Fl}}^{[a,b]}_{\mathcal{J},\widetilde{z}}$.

Proposition 4.3.7. *Suppose that* $\widetilde{z} = \sigma^{-1} t_{\nu+\eta}$ *where* ν *is* $(b-a+1)$*-deep in* $\underline{C}_0$. *Then the morphism* $\iota'_{\widetilde{z}}$ *induces a monomorphism*

$$\iota_{\widetilde{z}} : [\widetilde{\mathrm{Fl}}^{[a,b]}_{\mathcal{J},\widetilde{z}} / T^{\vee,\mathcal{J}}_{\mathbf{F}}\text{-conj}] \hookrightarrow \Phi\text{-}\mathrm{Mod}^{\text{ét},\mathrm{Sym}}_K.$$

Proof. By Lemma 4.3.3 and [Le+a, Lemma 5.4.4], the morphism $\iota'_{\widetilde{z}}$ factors through a monomorphism

$$[\prod_{j\in\mathcal{J}} (L^{[a,b]}\mathrm{GSp}_4)_{\mathbf{F}} \widetilde{z}_j /_{\varphi} \mathcal{I}_{\mathbf{F}}] \hookrightarrow \Phi\text{-}\mathrm{Mod}^{\text{ét},\mathrm{Sym}}_K,$$

and the source is isomorphic to $[\widetilde{\mathrm{Fl}}^{[a,b]}_{\mathcal{J},\widetilde{z}} / T^{\vee,\mathcal{J}}_{\mathbf{F}}\text{-conj}]$ by Lemma 5.2.2 in *loc. cit.*. $\square$

By combining Proposition 4.2.12, Lemma 4.3.6, and Proposition 4.3.7, we obtain the following result.

Proposition 4.3.8. *Let* $a \leq b$, $h \geq 0$ *be integers and* $\widetilde{z} = \sigma^{-1} t_{\nu+\eta} \in \widetilde{W}^{\vee,\mathcal{J}}$ *such that* ν *is* $(b-a+1)$*-deep* *in $\underline{C}_0$. Suppose that $(\mathrm{Gr}^{[0,h],\mathcal{J}}_{\mathcal{G},\mathbf{F}})\widetilde{w}^*(\tau) \subset \mathrm{Fl}^{[a,b]}_{\mathcal{J},\widetilde{z}}$. Then we have a commutative diagram*

$$\begin{tikzcd}
\widetilde{M}_{\mathcal{J}}(\leq \lambda)_{\mathbf{F}} \arrow[r, hook] \arrow[d] & \widetilde{\mathrm{Gr}}^{[0,h],\mathcal{J}}_{\mathcal{G},\mathbf{F}} \arrow[r, "r_{\widetilde{w}^*(\tau)}"] \arrow[d, "\pi_{(s,\mu)}"] & \widetilde{\mathrm{Fl}}^{[a,b]}_{\mathcal{J},\widetilde{z}} \arrow[r] & \left[\widetilde{\mathrm{Fl}}^{[a,b]}_{\mathcal{J},\widetilde{z}}/T^{\vee,\mathcal{J}}_{\mathbf{F}}\text{-conj}\right] \arrow[d, "\iota_{\widetilde{z}}"] \\
Y^{\leq\lambda,\tau}_{\mathrm{Sym},\mathbf{F}} \arrow[r, hook] & Y^{[0,h],\tau}_{\mathrm{Sym},\mathbf{F}} \arrow[rr, hook, "\varepsilon_\tau"] & & \Phi\text{-}\mathrm{Mod}^{\text{ét},\mathrm{Sym}}_K
\end{tikzcd}$$

where all hooked arrows are closed immersions.

4.4 Local models for potentially crystalline stacks

We have a canonical morphism $\varepsilon_\infty : \mathcal{X}_n \to \Phi\text{-}\mathrm{Mod}^{\text{ét},n}_K$ constructed in [EG23, Proposition 3.7.2] which corresponds to restricting G_K-representations to G_{K_∞}-representations when evaluated at complete Noetherian finite local $\mathcal{O}$-algebras. We have an induced map $\varepsilon_\infty : \mathcal{X}_{\mathrm{Sym}} \to \Phi\text{-}\mathrm{Mod}^{\text{ét},\mathrm{Sym}}_K$.

Proposition 4.4.1. *Suppose τ is $(h+2)$-generic. Then the map $\varepsilon_\infty : \mathcal{X}^{[0,h],\tau}_{\mathrm{Sym}} \to \Phi\text{-}\mathrm{Mod}^{\text{ét},\mathrm{Sym}}_K$ is a monomorphism.*

Proof. This follows from [Le+a, Proposition 7.2.11] using the argument given in the proof of Proposition 4.3.5 □

We define $\mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym}}$ (resp. $\mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym,reg}}$) to be the union of $\mathcal{X}^{\lambda',\tau}_{\mathrm{Sym}}$ for all dominant (resp. regular dominant) cocharacters $\lambda' \leq \lambda$. Similarly, we have $\mathcal{X}^{\leq\lambda',\tau'}_4$.

Recall the $\mathcal{O}$-flat closed substack $Y^{[0,h],\tau,\nabla_\infty}_4$ (resp. $Y^{\leq\lambda,\tau,\nabla_\infty}_4$) of $Y^{[0,h],\tau}_4$ (resp. $Y^{\leq\lambda,\tau}_4$) defined in [Le+a, §7.2]. It's key property is that if τ is $(h+2)$-generic, then $\mathcal{X}^{[0,h],\tau}_4 \to \Phi\text{-}\mathrm{Mod}^{\text{ét},4}_K$ factors through $Y^{[0,h],\tau,\nabla_\infty}_4$ and induces isomorphisms $\mathcal{X}^{[0,h],\tau}_4 \simeq Y^{[0,h],\tau,\nabla_\infty}_4$ and $\mathcal{X}^{\leq\lambda,\tau}_4 \simeq Y^{\leq\lambda,\tau,\nabla_\infty}_4$. We define $Y^{[0,h],\tau,\nabla_\infty}_{\mathrm{Sym}}$ (resp. $Y^{\leq\lambda,\tau,\nabla_\infty}_{\mathrm{Sym}}$) to be the $\mathcal{O}$-flat part of $Y^{[0,h],\tau}_{\mathrm{Sym}} \times_{Y^{[0,h],\tau}_4} Y^{[0,h],\tau,\nabla_\infty}_4$ (resp. $Y^{\leq\lambda,\tau}_{\mathrm{Sym}} \times_{Y^{\leq\lambda,\tau}_4} Y^{\leq\lambda,\tau,\nabla_\infty}_4$). We have the following result.

Proposition 4.4.2. *Suppose that τ is $(h+2)$-generic. The map ε_∞ factors through $Y^{[0,h],\tau,\nabla_\infty}_{\mathrm{Sym}}$ and induces isomorphisms $\mathcal{X}^{[0,h],\tau}_{\mathrm{Sym}} \simeq Y^{[0,h],\tau,\nabla_\infty}_{\mathrm{Sym}}$ and $\mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym}} \simeq Y^{\leq\lambda,\tau,\nabla_\infty}_{\mathrm{Sym}}$.*

Proof. Since ε_∞ and ε_τ are fully faithful, we have

$$\mathcal{X}^{[0,h],\tau}_{\mathrm{Sym}} \simeq \left((\mathcal{X}^{[0,h],\tau}_4 \times \mathcal{X}^{[0,2h],\mathrm{sim}(\tau)}_1) \times_{\Phi\text{-}\mathrm{Mod}^{\text{ét},4}_K \times \Phi\text{-}\mathrm{Mod}^{\text{ét},1}_K} \Phi\text{-}\mathrm{Mod}^{\text{ét},\mathrm{Sym}}_K\right)_{\mathcal{O}\text{-flat}}$$
$$Y^{[0,h],\tau,\nabla_\infty}_{\mathrm{Sym}} \simeq \left((Y^{[0,h],\tau,\nabla_\infty}_4 \times Y^{[0,2h],\mathrm{sim}(\tau)}_1) \times_{\Phi\text{-}\mathrm{Mod}^{\text{ét},4}_K \times \Phi\text{-}\mathrm{Mod}^{\text{ét},1}_K} \Phi\text{-}\mathrm{Mod}^{\text{ét},\mathrm{Sym}}_K\right)_{\mathcal{O}\text{-flat}}$$

where the subscript $\mathcal{O}$-flat means taking $\mathcal{O}$-flat part. Then the claim follows from the corresponding result for $\mathrm{GL}_4 \times \mathrm{GL}_1$ in [Le+a, Prop 7.2.3]. □

We assume that τ is $(h+2)$-generic until the end of this section. By using the previous Proposition, we consider $\mathcal{X}^{[0,h],\tau}_{\mathrm{Sym}}$ as a closed substack of $Y^{[0,h],\tau}_{\mathrm{Sym}}$.

Before stating the main result of this section, we introduce some notations. Recall that $Y^{\leq\lambda,\tau}_{\mathrm{Sym}}$ is the union of $Y^{\leq\lambda,\tau}_{\mathrm{Sym}}(\widetilde{z})$ for $\widetilde{z} \in \mathrm{Adm}^\vee(\lambda)$. We define

$$Y^{\leq\lambda,\tau,\nabla_\infty}_{\mathrm{Sym}}(\widetilde{z}) \subset Y^{\leq\lambda,\tau,\nabla_\infty}_{\mathrm{Sym}}, \quad \mathcal{X}^{\lambda,\tau}_{\mathrm{Sym}}(\widetilde{z}) \subset \mathcal{X}^{\lambda,\tau}_{\mathrm{Sym}}, \quad \mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym,reg}}(\widetilde{z}) \subset \mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym,reg}}$$

to be the open substacks obtained by taking intersection with $Y^{\leq\lambda,\tau}_{\mathrm{Sym}}(\widetilde{z})$ inside $Y^{\leq\lambda,\tau}_{\mathrm{Sym}}$. We also define $T^{\vee,\mathcal{J}}_{\mathcal{O}}$-torsors

$$\widetilde{U}(\widetilde{z},\leq\lambda,\nabla_{\tau,\infty})\to Y^{\leq\lambda,\tau,\nabla_\infty}_{\mathrm{Sym}}(\widetilde{z}),\ \ \widetilde{\mathcal{X}}^{\lambda,\tau}_{\mathrm{Sym}}(\widetilde{z})\to\mathcal{X}^{\lambda,\tau}_{\mathrm{Sym}}(\widetilde{z}),\ \ \widetilde{\mathcal{X}}^{\lambda,\tau}_{\mathrm{Sym,reg}}(\widetilde{z})\to\mathcal{X}^{\lambda,\tau}_{\mathrm{Sym,reg}}(\widetilde{z})$$

by taking pullback along the $T^{\vee,\mathcal{J}}_{\mathcal{O}}$-torsor $\widetilde{U}(\widetilde{z},\leq\lambda)^{\wedge_p}\to Y^{\leq\lambda,\tau}_{\mathrm{Sym}}(\widetilde{z})$.

We define $M_{\mathcal{J},\mathrm{reg}}(\leq\lambda,\nabla_{\mathbf{a}})$ to be the union of $M_{\mathcal{J}}(\lambda',\nabla_{\mathbf{a}})$ for all regular dominant cocharacters $\lambda'\leq\lambda$. Note that $M_{\mathcal{J},\mathrm{reg}}(\leq\lambda,\nabla_{\mathbf{a}})$ is a closed subvariety of $M^{\mathrm{nv}}_{\mathcal{J}}(\leq\lambda,\nabla_{\mathbf{a}})$ which is in turn contained in $M_{\mathcal{J}}(\leq\lambda)$. We define open subschemes

$$U^{\mathrm{nv}}(\widetilde{z},\leq\lambda,\nabla_{\mathbf{a}})\subset M^{\mathrm{nv}}_{\mathcal{J}}(\leq\lambda,\nabla_{\mathbf{a}})\ \text{ and }\ U_{\mathrm{reg}}(\widetilde{z},\leq\lambda,\nabla_{\mathbf{a}})\subset M_{\mathcal{J},\mathrm{reg}}(\leq\lambda,\nabla_{\mathbf{a}})$$

by taking intersection with the open subscheme $U(\widetilde{z},\leq\lambda)\subset M_{\mathcal{J}}(\leq\lambda)$. We also define $T^{\vee,\mathcal{J}}_{\mathcal{O}}$-torsors

$$\widetilde{U}^{\mathrm{nv}}(\widetilde{z},\leq\lambda,\nabla_{\mathbf{a}})\to U^{\mathrm{nv}}(\widetilde{z},\leq\lambda,\nabla_{\mathbf{a}})\ \text{ and }\ U_{\mathrm{reg}}(\widetilde{z},\leq\lambda,\nabla_{\mathbf{a}})\to\widetilde{U}_{\mathrm{reg}}(\widetilde{z},\leq\lambda,\nabla_{\mathbf{a}})$$

by taking pullback along the trivial $T^{\vee,\mathcal{J}}_{\mathcal{O}}$-torsor $\widetilde{U}(\widetilde{z},\leq\lambda)\to U(\widetilde{z},\leq\lambda)$.

To the lowest alcove presentation (s,μ) of τ, we associate tuples of integers $\mathbf{a}_\tau=(\mathbf{a}_{\tau,j'})_{j'\in\mathcal{J}'}\in(\mathcal{O}^3)^{\mathcal{J}'}$ given by $\mathbf{a}_{\tau,j'}:=(s'_{\mathrm{or},j'})^{-1}(\mathbf{a}')^{(j')}/(1-p^{f'})$. We remark that if $j\in\mathcal{J}$ and $j\equiv j'\bmod f$, then $\mathbf{a}_{\tau,j'}\equiv s_j^{-1}(\mu_j+\eta_j)\bmod\varpi$ (see [Le+a, Lemma 7.3.1]).

Theorem 4.4.3. *1. We have the following commutative diagram of p-adic formal algebraic stacks*

$$\begin{tikzcd}
\widetilde{\mathcal{X}}^{\leq\lambda,\tau}_{\mathrm{Sym,reg}}(\widetilde{z}) \arrow[r, hook] \arrow[rrrr, dashed, bend left] & \widetilde{U}(\widetilde{z},\leq\lambda,\nabla_{\tau,\infty}) \arrow[r, hook] \arrow[dl, "T^{\vee,\mathcal{J}}_{\mathcal{O}}"'] \arrow[rr, dashed, bend left] & \widetilde{U}(\widetilde{z},\leq\lambda)^{\wedge_p} \arrow[dl, "T^{\vee,\mathcal{J}}_{\mathcal{O}}"'] \arrow[dr] & \widetilde{U}^{\mathrm{nv}}(\widetilde{z},\leq\lambda,\nabla_{\mathbf{a}_\tau})^{\wedge_p} \arrow[l, hook'] \arrow[dr, "T^{\vee,\mathcal{J}}_{\mathcal{O}}"'] & \widetilde{U}_{\mathrm{reg}}(\widetilde{z},\leq\lambda,\nabla_{\mathbf{a}_\tau})^{\wedge_p} \arrow[l, hook'] \\
Y^{\leq\lambda,\tau,\nabla_\infty}_{\mathrm{Sym}}(\widetilde{z}) \arrow[r, hook] & Y^{\leq\lambda,\tau}_{\mathrm{Sym}}(\widetilde{z}) & & U(\widetilde{z},\leq\lambda)^{\wedge_p} & U^{\mathrm{nv}}(\widetilde{z},\leq\lambda,\nabla_{\mathbf{a}_\tau})^{\wedge_p}. \arrow[l, hook']
\end{tikzcd}$$

where the two parallelograms are cartesian, hooked arrows are closed immersions, and diagonal arrows are $T^{\vee,\mathcal{J}}_{\mathcal{O}}$-torsors. Moreover, there exists an integer $N_{\mathrm{sing}}=N(\{\lambda_j\}_{j\in\mathcal{J}})$ only depending on $\{\lambda_j\}_{j\in\mathcal{J}}\subset\mathbf{Z}^3$ such that if μ is N_{sing}-deep in $\underline{C}_0$, then the dotted arrows exist and are closed immersions for all $\widetilde{z}\in\mathrm{Adm}^\vee(\lambda)$.

2. *There exists a nonzero polynomial $P=P_{\{\lambda_j\}_{j\in\mathcal{J}}}(X_1,X_2,X_3)\in\mathbf{Z}[X_1,X_2,X_3]$ depending only on $\{\lambda_j\}_{j\in\mathcal{J}}\subset\mathbf{Z}^3$ and the ramification index e of $\mathcal{O}$ such that if μ is P-generic, then the longer dotted arrow is an isomorphism. In particular, we obtain the following local model diagram*

$$\begin{tikzcd}
& \widetilde{\mathcal{X}}^{\leq\lambda,\tau}_{\mathrm{Sym,reg}}(\widetilde{z})\simeq\widetilde{U}_{\mathrm{reg}}(\widetilde{z},\leq\lambda,\nabla_{\mathbf{a}_\tau})^{\wedge_p} \arrow[dl, "T^{\vee,\mathcal{J}}_{\mathcal{O}}"'] \arrow[dr, "T^{\vee,\mathcal{J}}_{\mathcal{O}}"] & \\
\mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym,reg}}(\widetilde{z}) & & U_{\mathrm{reg}}(\widetilde{z},\leq\lambda,\nabla_{\mathbf{a}_\tau})^{\wedge_p}
\end{tikzcd}.$$

Furthermore, for any tame $\overline{\rho}\in\mathcal{X}_{\mathrm{Sym}}(\mathbf{F})$ and regular dominant $\lambda'\leq\lambda$, the versal rings to $\mathcal{X}^{\lambda',\tau}_{\mathrm{Sym}}$ at $\overline{\rho}$ are domains (if is nonzero).

For the proof of Theorem 4.4.3, we need the following Lemma. Its proof is given in §5.4.

Lemma 4.4.4. *Suppose that τ is $(h+16)$-generic. Then $\mathcal{X}^{\lambda,\tau}_{\mathrm{Sym}}(\widetilde{z}) \neq \emptyset$ if and only if $\widetilde{z} \in \mathrm{Adm}^\vee(\lambda)$.*

Proof of Theorem 4.4.3. Since the proof is completely analogous to that of [Le+a, Theorem 7.3.2], we briefly sketch the proof.

1. Let $\widetilde{U}(\widetilde{z}, \leq \lambda)^{\wedge_p} = \mathrm{Spf}\, R$. By Proposition 7.1.6 in *loc. cit.*, the composite $\mathrm{Spf}\, R \to Y^{\leq\lambda,\tau}_{\mathrm{Sym}} \to Y^{\leq\lambda',\tau'}_4$ provides an ideal $I_{\nabla_\infty} \leq R$ such that $\widetilde{U}(\widetilde{z}, \leq \lambda, \nabla_\infty) = \mathrm{Spf}\, R/I_{\nabla_\infty}$. By Proposition 7.1.10 of *loc. cit.*, there is an inclusion $\widetilde{U}(\widetilde{z}, \leq \lambda, \nabla_{\tau,\infty}) \times_{\mathcal{O}} \mathrm{Spec}\, \mathcal{O}/p \subset \widetilde{U}^{\mathrm{nv}}(\widetilde{z}, \leq \lambda, \nabla_{\mathbf{a}_\tau})$ of subschemes in $\widetilde{U}(\widetilde{z}, \leq \lambda)$. As an application of Proposition 3.3.9 in *loc. cit.*, we can lift the map
$$\widetilde{U}(\widetilde{z}, \leq \lambda, \nabla_{\tau,\infty}) \times_{\mathcal{O}} \mathrm{Spec}\, \mathcal{O}/p \to \widetilde{U}^{\mathrm{nv}}(\widetilde{z}, \leq \lambda, \nabla_{\mathbf{a}_\tau}) \to U^{\mathrm{nv}}(\widetilde{z}, \leq \lambda, \nabla_{\mathbf{a}_\tau})$$
to $\mathrm{Spec}\, R/I_{\nabla_\infty} \to U^{\mathrm{nv}}(\widetilde{z}, \leq \lambda, \nabla_{\mathbf{a}_\tau})$ at the cost of a genericity assumption on μ. (The Proposition 3.3.9 in *loc. cit.*, which uses Elkik's approximation theorem, easily generalizes to our setup; its proof only uses the fact that open charts of universal local models are affine and smooth after inverting v which follows from Proposition 3.2.7 in our case.) We can also lift the map
$$\widetilde{U}(\widetilde{z}, \leq \lambda, \nabla_{\tau,\infty}) \times_{\mathcal{O}} \mathrm{Spec}\, \mathcal{O}/p \to \widetilde{U}^{\mathrm{nv}}(\widetilde{z}, \leq \lambda, \nabla_{\mathbf{a}_\tau}) \to T^{\vee,\mathcal{J}}_{\mathcal{O}}$$
to $\mathrm{Spec}\, R/I_{\nabla_\infty} \to T^{\vee,\mathcal{J}}_{\mathcal{O}}$ because of smoothness of $T^{\vee,\mathcal{J}}_{\mathcal{O}}$. By combining with the previous map, we obtain a map $\mathrm{Spec}\, R/I_{\nabla_\infty} \to \widetilde{U}^{\mathrm{nv}}(\widetilde{z}, \leq \lambda, \nabla_{\mathbf{a}_\tau})$ which induces the shorter dotted arrow because R/I_{∇_∞} is p-adically complete. It is moreover a closed immersion because it is closed immersion modulo p. Since $\widetilde{\mathcal{X}}^{\leq\lambda,\tau}_{\mathrm{Sym,reg}}(\widetilde{z})$ (resp. $\widetilde{U}_{\mathrm{reg}}(\widetilde{z}, \leq \lambda, \nabla_{\mathbf{a}_\tau})^{\wedge_p}$) is $\mathcal{O}$-flat and equals to the union of top($=1+7\#\mathcal{J}$)-dimensional irreducible components of $\widetilde{U}(\widetilde{z}, \leq \lambda, \nabla_{\tau,\infty})$ (resp. the $\mathcal{O}$-flat locus of $\widetilde{U}^{\mathrm{nv}}(\widetilde{z}, \leq \lambda, \nabla_{\mathbf{a}_\tau})$), the longer dotted arrow is induced by the shorter dotted arrow.

2. Since both $\widetilde{\mathcal{X}}^{\leq\lambda,\tau}_{\mathrm{Sym,reg}}(\widetilde{z})$ and $\widetilde{U}_{\mathrm{reg}}(\widetilde{z}, \leq \lambda, \nabla_{\mathbf{a}_\tau})^{\wedge_p}$ are reduced $\mathcal{O}$-flat p-adic formal schemes equidimensional of dimension $1+7\#\mathcal{J}$, the longer dotted arrow is an isomorphism if both $\widetilde{\mathcal{X}}^{\leq\lambda,\tau}_{\mathrm{Sym,reg}}(\widetilde{z})$ and $\widetilde{U}_{\mathrm{reg}}(\widetilde{z}, \leq \lambda, \nabla_{\mathbf{a}_\tau})^{\wedge_p}$ have the same number of irreducible component. We apply Theorem 3.4.1 for all regular dominant $\lambda' \leq \lambda$. Then there exists a polynomial $P \in \mathbf{Z}[X_1, X_2, X_3]$ only depending on $\{\lambda_j\}_{j\in\mathcal{J}}$ and the ramification index e of $\mathcal{O}/\mathbf{Z}_p$ such that if $P(\mathbf{a}_{\tau,j}) \neq 0 \bmod p$ for all $j \in \mathcal{J}$, $U(\widetilde{z}, \lambda', \nabla_{\mathbf{a}_\tau})$ is unibranch at $\widetilde{z}$ and $U(\widetilde{z}, \lambda', \nabla_{\mathbf{a}_\tau})^{\wedge_p}$ is irreducible for all regular dominant $\lambda' \leq \lambda$. By Corollary 4.2.17. this implies that the number of irreducible component of $\widetilde{U}_{\mathrm{reg}}(\widetilde{z}, \leq \lambda, \nabla_{\mathbf{a}_\tau})^{\wedge_p}$ is at most
$$\#\{\lambda' \leq \lambda \mid \lambda' \text{ regular dominant, } \widetilde{z} \in \mathrm{Adm}^\vee(\lambda')\}.$$
On the other hand, Lemma 4.4.4 implies that the number of irreducible component of $\widetilde{\mathcal{X}}^{\leq\lambda,\tau}_{\mathrm{Sym,reg}}(\widetilde{z})$ is at least
$$\#\{\lambda' \leq \lambda \mid \lambda' \text{ regular dominant, } \widetilde{z} \in \mathrm{Adm}^\vee(\lambda')\}.$$
Thus the longer dotted arrow is an isomorphism.

Suppose that $\overline{\rho} \in \mathcal{X}_{\mathrm{Sym}}(\mathbf{F})$. For regular dominant $\lambda' \leq \lambda$, we let $n_{\lambda'}$ be the number of minimal primes in any versal ring to $\mathcal{X}^{\lambda',\tau}_{\mathrm{Sym}}$ at $\overline{\rho}$. If $\overline{\rho} \in \mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym}}(\mathbf{F})$ is tame, it's image in $Y^{\leq\lambda,\tau}_{\mathrm{Sym}}(\mathbf{F})$ is a triple

$(\mathfrak{M}_{\overline{\rho}}, \mathfrak{N}_{\overline{\rho}}, \alpha_{\overline{\rho}})$ where $\mathfrak{M}_{\overline{\rho}}$ is a semisimple Breuil–Kisin module of some shape $\widetilde{z}$ by [Le+a, Proposition 5.5.7]. Then $\mathfrak{M}_{\overline{\rho}} \in Y^{\leq\lambda,\tau}_{\mathrm{Sym}}(\widetilde{z})(\mathbf{F})$ lifts to an element in $T^{\vee,\mathcal{J}}(\mathbf{F})\widetilde{z} \subset \widetilde{U}(\widetilde{z}, \leq \lambda)(\mathbf{F})$. Thus, we can take a versal ring to $\mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym}}$ at $\overline{\rho}$ which is also a versal ring to $\widetilde{U}_{\mathrm{reg}}(\widetilde{z}, \leq \lambda', \nabla_{\mathbf{a}_\tau})$ at $\widetilde{z}$. By the previous paragraph, the number of minimum primes of this versal ring is exactly

$$\sum_{\lambda' \leq \lambda} n_{\lambda'} = \#\{\lambda' \leq \lambda \mid \lambda' \text{ regular dominant, } \widetilde{z} \in \mathrm{Adm}^\vee(\lambda')\}.$$

By induction on λ, we can show that $n_\lambda = 1$ if and only if $\widetilde{z} \in \mathrm{Adm}^\vee(\lambda)$ and $n_\lambda = 0$ otherwise.

□

4.5 Potentially crystalline stacks modulo p

Let $\lambda \in X_*(T^\vee)^{\mathcal{J}}$ be a regular dominant cocharacter such that $\mathrm{std}(\lambda) \subset ([0,h]^4)^{\mathcal{J}}$. Let τ be a tame inertial type with $(h+2)$-generic lowest alcove presentation (s,μ) which is λ-compatible with $\zeta \in X^*(Z)^{\mathcal{J}}$. Recall that both $\mathcal{X}_{\mathrm{Sym,red}}$ and $\mathcal{X}^{\lambda,\tau}_{\mathrm{Sym},\mathbf{F}}$ are equidimensional algebraic stacks over $\mathbf{F}$ of dimension $4f$. Thus $\mathcal{X}^{\lambda,\tau}_{\mathrm{Sym},\mathbf{F}}$ is topologically a union of irreducible components of $\mathcal{X}_{\mathrm{Sym,red}}$, which are labelled by Serre weights.

Suppose that $\mathcal{C}_\sigma \subset \mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym},\mathbf{F}}$. We define algebraic stacks $\widetilde{\mathcal{C}}_\sigma$ and $\widetilde{\mathcal{X}}^{\leq\lambda,\tau}_{\mathrm{Sym},\mathbf{F}}$ by the following cartesian diagram

$$\begin{tikzcd} \widetilde{\mathcal{C}}_\sigma \arrow[r, hook] \arrow[d] & \widetilde{\mathcal{X}}^{\leq\lambda,\tau}_{\mathrm{Sym},\mathbf{F}} \arrow[r, hook] \arrow[d] & \widetilde{M}_{\mathcal{J}}(\leq\lambda)_{\mathbf{F}} \arrow[d, "\pi_{(s,\mu)}"] \\ \mathcal{C}_\sigma \arrow[r, hook] & \mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym},\mathbf{F}} \arrow[r, hook] & Y^{\leq\lambda,\tau}_{\mathrm{Sym},\mathbf{F}}. \end{tikzcd} \tag{4.5.1}$$

If μ is $(2h-2)$-deep in $\underline{C}_0$, then the closed immersion $\widetilde{\mathcal{X}}^{\leq\lambda,\tau}_{\mathrm{Sym},\mathbf{F}} \hookrightarrow \widetilde{M}_{\mathcal{J}}(\leq\lambda)_{\mathbf{F}}$ factors through $\widetilde{M}^{\mathrm{nv}}_{\mathcal{J}}(\leq \lambda, \nabla_{\mathbf{a}_\tau})_{\mathbf{F}}$ by [Le+a, Proposition 7.1.10]. Recall that by Theorem 3.6.2, $7f$-dimensional irreducible components of $\widetilde{M}^{\mathrm{nv}}_{\mathcal{J}}(\leq\lambda, \nabla_{\mathbf{a}_\tau})_{\mathbf{F}}$ are exactly $\widetilde{C}^{\zeta}_{\sigma}\widetilde{w}(\tau)^*$ for all $\sigma \in \mathrm{JH}(W(\phi^{-1}(\lambda)-\eta)\otimes\overline{\sigma}(\tau))$.

The following Theorem describes the underlying reduced substack of $\mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym},\mathbf{F}}$.

Theorem 4.5.2. *Let λ, τ be as above. We assume that μ is $(h+16)$-deep in $\underline{C}_0$. Then we have*

$$\left(\mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym,reg}}\right)_{\mathrm{red}} = \left(\mathcal{X}^{\lambda,\tau}_{\mathrm{Sym}}\right)_{\mathrm{red}} = \bigcup_{\sigma\in\mathrm{JH}(W(\phi^{-1}(\lambda)-\eta)\otimes\overline{\sigma}(\tau))} \mathcal{C}_\sigma.$$

We need the following Lemma, whose proof is given in §5.4.

Lemma 4.5.3. *Suppose that μ is $(h+16)$-deep in $\underline{C}_0$. Then for each $\sigma \in \mathrm{JH}(W(\phi^{-1}(\lambda)-\eta)\otimes\overline{\sigma}(\tau))$, we have $\mathcal{C}_\sigma \subset \mathcal{X}^{\lambda,\tau}_{\mathrm{Sym},\mathbf{F}} \subset \mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym},\mathbf{F}}$.*

Proof of Theorem 4.5.2. Since μ is $(h+16)$-deep in $\underline{C}_0$, the Lemma 4.5.3 shows that $\mathcal{X}^{\lambda,\tau}_{\mathrm{Sym},\mathbf{F}}$ (resp. $\widetilde{\mathcal{X}}^{\lambda,\tau}_{\mathrm{Sym},\mathbf{F}}$) has at least $\#\mathrm{JH}(W(\phi^{-1}(\lambda)-\eta)\otimes\overline{\sigma}(\tau))$ many irreducible components of dimension $4f$ (resp. $7f$). On the other hand, the number of $7f$-dimensional irreducible components of $\widetilde{\mathcal{X}}^{\leq\lambda,\tau}_{\mathrm{Sym},\mathbf{F}}$ is at most that of $\widetilde{M}^{\mathrm{nv}}_{\mathcal{J}}(\leq \lambda, \nabla_{\mathbf{a}_\tau})_{\mathbf{F}}$, which is $\#\mathrm{JH}(W(\phi^{-1}(\lambda)-\eta)\otimes\overline{\sigma}(\tau))$ by Theorem 3.6.2. This proves our claim. □

Theorem 4.5.4. *Let (s,μ) be a $(h+3)$-generic lowest alcove presentation of τ which is $(\lambda-\eta)$-compatible with $\zeta \in X^*(\underline{Z})$. If σ is a Serre weight such that $\mathcal{C}_\sigma \subset \mathcal{X}^{\lambda,\tau}_{\mathrm{Sym}}$, then $\sigma \in \mathrm{JH}(W(\phi^{-1}(\lambda)-\eta)\otimes\overline{\sigma}(\tau))$ and we* *have a commutative diagram*

$$\begin{tikzcd}
 & & & & \widetilde{C}^{\zeta}_{\sigma} \arrow[d, hook] \\
\widetilde{\mathcal{C}}_{\sigma} \arrow[r, hook] \arrow[d, "T^{\vee,\mathcal{J}}_{\mathbf{F}}"] \arrow[urrrr, bend left, "\simeq"] & \widetilde{\mathcal{X}}^{\leq\lambda,\tau}_{\mathrm{Sym},\mathbf{F}} \arrow[d, "T^{\vee,\mathcal{J}}_{\mathbf{F}}"] & \widetilde{M}^{\mathrm{nv}}_{\mathcal{J}}(\leq\lambda,\nabla_{\mathbf{a}_{\tau}})_{\mathbf{F}} \arrow[l, hook'] \arrow[r, hook] & \widetilde{M}_{\mathcal{J}}(\leq\lambda)_{\mathbf{F}} \arrow[r, hook, "r_{\widetilde{w}^*(\tau)}"] \arrow[d, "T^{\vee,\mathcal{J}}_{\mathbf{F}}"] & \widetilde{\mathrm{Fl}}^{[0,h]}_{\mathcal{J},\widetilde{w}^*(\tau)} \arrow[d, "T^{\vee,\mathcal{J}}_{\mathbf{F}}"] \\
\mathcal{C}_{\sigma} \arrow[r, hook] \arrow[drrrr, bend right] & \mathcal{X}^{\leq\lambda,\tau}_{\mathrm{Sym},\mathbf{F}} \arrow[rr, hook] & & Y^{\leq\lambda,\tau}_{\mathrm{Sym},\mathbf{F}} \arrow[r, hook] & \left[\widetilde{\mathrm{Fl}}^{[0,h]}_{\mathcal{J},\widetilde{w}^*(\tau)}/T^{\vee,\mathcal{J}}_{\mathbf{F}}\text{-conj}\right] \arrow[d] \\
 & & & & \Phi\text{-}\mathrm{Mod}^{\text{ét},\mathrm{Sym}}_{K,\mathbf{F}}
\end{tikzcd}$$

where all rectangles are cartesian, all vertical arrows labelled by $T^{\vee,\mathcal{J}}_{\mathbf{F}}$ are $T^{\vee,\mathcal{J}}_{\mathbf{F}}$-torsors, and all hooked arrows are closed immersion. The bottom diagonal map is the composition of canonical morphisms $\mathcal{C}_\sigma \hookrightarrow \mathcal{X}_{\mathrm{Sym,red}} \xrightarrow{\varepsilon_\infty} \Phi\text{-}\mathrm{Mod}^{\text{ét},\mathrm{Sym}}_{K,\mathbf{F}}$. Furthermore, if (s,μ) is $(h+16)$-generic, then the above diagram holds for all $\sigma \in \mathrm{JH}(W(\phi^{-1}(\lambda)-\eta)\otimes\overline{\sigma}(\tau))$.

Proof. By Proposition 4.3.8 and (4.5.1), we get the diagram except the top diagonal arrow. The image of $\widetilde{\mathcal{C}}_\sigma$ in $\widetilde{M}^{\mathrm{nv}}_{\mathcal{J}}(\leq\lambda,\nabla_{\mathbf{a}_\tau})_{\mathbf{F}}$ is a top dimensional irreducible component. By Theorem 3.6.2, it is equal to $\widetilde{C}^{\zeta}_{\sigma'}\widetilde{w}^*(\tau)^{-1}$ for some $\sigma' \in \mathrm{JH}(W(\phi^{-1}(\lambda)-\eta)\otimes\overline{\sigma}(\tau))$. We claim that $\sigma'=\sigma$. Let $(\widetilde{w},\omega)$ be a lowest alcove presentation of σ' compatible with ζ. We write $\kappa = \pi^{-1}(\widetilde{w})\cdot(\omega-\eta)$ so that $\sigma' = F(\kappa)$. Note that κ is 3-deep in its alcove by Proposition 2.5.6. Also, we write $\widetilde{w} = t_{\nu_w}w$. We can and do choose a triple $(\widetilde{w}_1,\widetilde{w}_2,\widetilde{s})$ such that $\widetilde{s} = t_{\nu_s}s$ and

$$\widetilde{w}_1 = \widetilde{w},\ \widetilde{w}_2 = \widetilde{w}_h\widetilde{w}_1,\ \widetilde{s}\widetilde{w}_2^{-1}(0)=\omega,\ w_js_j^{-1}w_{j-1}^{-1}=1.$$

Then $C^{\zeta}_{\sigma'} = S^{\nabla_0}_{\mathbf{F}}(\widetilde{w}_1,\widetilde{w}_2,\widetilde{s})$ which contains a dense open subscheme $U^{\zeta}_{\sigma'} = S^{\circ}_{\mathbf{F}}(\widetilde{w}_1,\widetilde{w}_2,\widetilde{s})^{\nabla_0}$. Let $\widetilde{U}^{\zeta}_{\sigma'} \subset \widetilde{C}^{\zeta}_{\sigma'}$ be the preimage of $U^{\zeta}_{\sigma'}$. By [Le+a, Lemma 7.4.6], the image of $\widetilde{U}^{\zeta}_{\sigma'}$ in $\Phi\text{-}\mathrm{Mod}^{\text{ét},\mathrm{Sym}}_{K}$ consists of $(\mathcal{M},\mathcal{N},\alpha)$ such that $\mathbb{V}^*_K(\mathcal{M})$ has the form

$$\begin{pmatrix} \chi_1 & * & * & * \\ 0 & \chi_2 & * & * \\ 0 & 0 & \chi_3 & * \\ 0 & 0 & 0 & \chi_4 \end{pmatrix} \tag{4.5.5}$$

where $\chi_i = \prod_{j\in\mathcal{J}}\omega_{K,\sigma_j}^{(\kappa_j+\eta_j)_i}|_{I_{K_\infty}}$. On the other hand, $\widetilde{\mathcal{C}}_\sigma$ has an open dense subscheme $\widetilde{\mathcal{U}}_\sigma$ whose $\mathbf{F}$-points are maximally nonsplit of niveau 1 and of weight σ. Let $\overline{\rho}$ be an $\mathbf{F}$-point in $\widetilde{\mathcal{U}}_\sigma \cap \widetilde{U}'^{\zeta}_{\sigma}\widetilde{w}^*(\tau)^{-1}$. Since $\overline{\rho}$ is 3-generic, the G_{K_∞}-stable standard flag of $\overline{\rho}|_{G_{K_\infty}}$ given by (4.5.5) is also G_K-stable ([Le+a, Lemma 7.2.10(4)]). Then the condition on χ_i and $\overline{\rho}$ being maximally nonsplit of niveau 1 and of weight σ imply that $\sigma'=\sigma$. The last assertion follows from Theorem 4.5.2. $\square$

Proposition 4.5.6. *Let $\overline{\rho}$ be a 22-generic tame L-parameter over $\mathbf{F}$. Let $W_g(\overline{\rho})$ be a set of 3-deep Serre weights σ such that $\overline{\rho}\in\mathcal{C}_\sigma(\mathbf{F})$.*

1. *We have $W_{\mathrm{obv}}(\overline{\rho}) \subset W_g(\overline{\rho}) \subset W^?(\overline{\rho})$.*

2. *Let* $(\widetilde{w}_1, \omega)$ *be a lowest alcove presentation of* $\sigma \in W^?(\overline{\rho})$ *compatible with* ζ. *For each* $j \in \mathcal{J}$, *let* $P_{\widetilde{w}_{1,j}}$ *be the polynomial in Theorem 3.7.1. If* $P_{\widetilde{w}_{1,j}}(\omega_j) \neq 0 \mod p$ *for all* $j \in \mathcal{J}$, *then* $\sigma \in W_g(\overline{\rho})$

Proof. For any $\sigma \in W^?(\overline{\rho}|_{I_{\mathbf{Q}_p}})$, we can use Lemma 2.5.14 and [Le+a, Remark 2.1.8] to find a tame inertial L-parameter τ with a lowest alcove representation such that $\widetilde{w}(\overline{\rho}|_{I_{\mathbf{Q}_p}}, \tau) \in \mathrm{Adm}(\eta)$ and $\sigma \in \mathrm{JH}(\overline{\sigma}(\tau))$. Since $\overline{\rho}$ is 22-generic, such τ is 19-generic. By applying Theorem 4.5.4 to such τ, $\lambda = \eta$, and σ, we get $W_g(\overline{\rho}) = W_g^{\zeta}(\overline{\rho})$. Then the claim follows from Theorem 3.7.2. $\square$

Chapter 5

Global setup

5.1 Some local Galois deformation rings

Let $\widehat{\mathcal{C}}_{\mathcal{O}}$ be the category of completed Noetherian local $\mathcal{O}$-algebras with residue field $\mathbf{F}$.

Let F be either a number field or a local field. Let $\overline{\rho} : G_F \to \mathrm{GSp}_4(\mathbf{F})$ be a continuous representation. We let $R^{\square}_{\overline{\rho}}$ be the framed deformation ring representing the functor $D^{\square}_{\overline{\rho}}$ taking $A \in \widehat{\mathcal{C}}_{\mathcal{O}}$ to the set of $\mathrm{GSp}_4(A)$-valued lifts ρ of $\overline{\rho}$. If $\psi : G_F \to \mathcal{O}^{\times}$ is a character lifting $\mathrm{sim}(\overline{\rho})$, we write $R^{\square,\psi}_{\overline{\rho}}$ for the fixed similitude deformation ring representing $D^{\square,\psi}_{\overline{\rho}}$ taking $A \in \widehat{\mathcal{C}}_{\mathcal{O}}$ to the set of $\mathrm{GSp}_4(A)$-valued lifts ρ with $\mathrm{sim}(\rho) = \psi \otimes_{\mathcal{O}} A$.

5.1.1 Local deformations: $l = p$

Suppose that $F = K$. Given a type $(\lambda + \eta, \tau)$, we denote by $R^{\lambda+\eta,\tau}_{\overline{\rho}}$ the unique $\mathcal{O}$-flat quotient of $R^{\square}_{\overline{\rho}}$ whose A-points, for any $\mathcal{O}$-flat $A \in \widehat{\mathcal{C}}_{\mathcal{O}}$, are lattices in potentially crystalline representations with Hodge–Tate weight $\lambda + \eta$ and tame inertial type τ. We have its version with fixed similitude character $R^{\lambda+\eta,\tau,\psi}_{\overline{\rho}}$. Note that $R^{\lambda+\eta,\tau,\psi}_{\overline{\rho}}$ is non-zero only if ψ is potentially crystalline of type $(\mathrm{sim}(\lambda+\eta), \mathrm{sim}(\tau))$. We record the following Lemma relating a potentially crystalline deformation ring to its fixed similitude variants.

Lemma 5.1.2. *Recall that $p > 2$. Twisting by the universal unramified twist*

$$\mathrm{ur}_{1+x} : G_K \to \mathcal{O}[\![X]\!]$$

sending Frob_K to $1 + x$ induces an isomorphism $R^{\lambda+\eta,\tau}_{\overline{\rho}} \simeq R^{\lambda+\eta,\tau,\psi}_{\overline{\rho}}[\![X]\!]$.

Proof. This can be proven as [EG14, Lemma 4.3.1]. $\square$

We also write $R^{\lambda+\eta,\tau}_{\mathrm{std}(\overline{\rho})}$ to denote the potentially crystalline (GL_4-)deformation ring of $\mathrm{std}(\overline{\rho})$ and type $(\mathrm{std}(\lambda+\eta), \mathrm{std}(\tau))$.

We prove that certain potentially crystalline deformation ring of tame $\overline{\rho} : G_K \to \mathrm{GSp}_4(\mathbf{F})$ is nonzero and a domain. We have the following necessary condition for $R^{\lambda+\eta,\tau}_{\overline{\rho}} \neq 0$ under a mild assumption on $\overline{\rho}$ and τ.

Proposition 5.1.3. *Let $\overline{\rho} : G_K \to \mathrm{GSp}_4(\mathbf{F})$ be a continuous representation and $(\lambda + \eta, \tau)$ be a type.*

1. *If τ is not $(h_{\lambda+\eta} + 1)$-generic and $\overline{\rho}^{\mathrm{ss}}$ is $\max\{2h_{\lambda+\eta}, 22\}$-generic, then $R^{\lambda+\eta,\tau}_{\overline{\rho}} = 0$.*

2. *If τ has a lowest alcove presentation (s,μ) with μ $(h_{\lambda+\eta}+1)$-deep in $\underline{C}_0$ and $R_{\overline{\rho}}^{\lambda+\eta,\tau}\neq 0$, then $\overline{\rho}^{\mathrm{ss}}$ has a $\phi^{-1}(\lambda)$-compatible lowest alcove presentation such that $\widetilde{w}(\overline{\rho},\tau)\in \mathrm{Adm}(\phi^{-1}(\lambda)+\eta)$.*

Proof. By [Enn19, Lemma 5], if $R_{\overline{\rho}}^{\lambda+\eta,\tau}\neq 0$, then $R_{\overline{\rho}^{\mathrm{ss}}}^{\lambda+\eta,\tau}\neq 0$. Thus, we may assume that $\overline{\rho}=\overline{\rho}^{\mathrm{ss}}$.

1. Note that $R_{\overline{\rho}}^{\lambda+\eta,\tau}=0$ if $R_{\mathrm{std}(\overline{\rho})}^{\lambda+\eta,\tau}=0$. Then the latter follows from a mild strengthening of [Enn19, Proposition 7] (as explained in the proof of [Le+a, Corollary 8.5.2]).

2. If $R_{\overline{\rho}}^{\lambda+\eta,\tau}\neq 0$, then $R_{\mathrm{std}(\overline{\rho})}^{\lambda+\eta,\tau}\neq 0$. Then the claim follows from the corresponding result for GL_4 ([Le+a, Corollary 5.5.8]) and Lemma 2.1.6. □

There is a Coxeter length function l on $\underline{W}_a$ which can be extended to $\underline{\widetilde{W}}\simeq \underline{W}_a\rtimes\underline{\Omega}$ by setting $l(\widetilde{w}\delta)=l(\widetilde{w})$ for $\widetilde{w}\in\underline{W}_a$ and $\delta\in\underline{\Omega}$. It is expected that if $R_{\overline{\rho}}^{\lambda+\eta,\tau}\neq 0$, the complexity of $R_{\overline{\rho}}^{\lambda+\eta,\tau}$ increases as the length of $\widetilde{w}(\overline{\rho},\tau)$ decreases. In the special case that $\widetilde{w}(\overline{\rho},\tau)$ has the *maximal* length, i.e. $\widetilde{w}(\overline{\rho},\tau)\in \underline{W}(\phi^{-1}(\lambda)+\eta)$, we can compute $R_{\overline{\rho}}^{\lambda+\eta,\tau}$ explicitly under a genericity assumption.

Theorem 5.1.4. *Let $\overline{\rho}:G_K\to \mathrm{GSp}_4(\mathbf{F})$ be a tame representation. Let $(\lambda+\eta,\tau)$ be a type and (s,μ) be a $(2h_{\lambda+\eta}+1)$-generic lowest alcove presentation of τ. If there is a lowest alcove presentation $(s_{\overline{\rho}},\mu_{\overline{\rho}})$ of $\overline{\rho}|_{I_K}$ such that $\widetilde{w}(\overline{\rho},\tau)\in\underline{W}(\lambda+\eta)$, then $R_{\overline{\rho}}^{\lambda+\eta,\tau}$ is formally smooth over $\mathcal{O}$ with $4f+11$ variables. Moreover, any $\rho:G_K\to\mathrm{GSp}_4(\mathcal{O})$ of type $(\lambda+\eta,\tau)$ lifting $\overline{\rho}$ is potentially diagonalizable.*

An analogue of Theorem 5.1.4 for the group GL_n and $\lambda=0$ is proven in [LLL19, Theorem 3.4.1]. We first prove the following generalization of *loc. cit.* for any dominant λ. Then Theorem 5.1.4 follows almost immediately.

Theorem 5.1.5. *Let $\overline{\rho}:G_K\to\mathrm{GL}_n(\mathbf{F})$ be a semisimple representation. Let $(\lambda+\eta,\tau)$ be a type (for GL_n) and $(s,\mu)\in\underline{W}_n\times X^*(\underline{T}_n)$ be a $(2h_{\lambda+\eta}+1)$-generic lowest alcove presentation of τ. If there is a lowest alcove presentation $(s_{\overline{\rho}},\mu_{\overline{\rho}})$ of $\overline{\rho}|_{I_K}$ such that $\widetilde{w}(\overline{\rho},\tau)\in\underline{W_n}(\lambda+\eta')$, then $R_{\overline{\rho}}^{\lambda+\eta',\tau}$ is formally smooth over $\mathcal{O}$ with $\frac{n(n-1)f}{2}+n^2$ variables. Moreover, any $\rho:G_K\to\mathrm{GL}_n(\mathcal{O})$ of type $(\lambda+\eta,\tau)$ lifting $\overline{\rho}$ is potentially diagonalizable.*

Proof. We first claim that $\overline{\rho}\in\mathcal{X}_n^{\leq\lambda+\eta',\tau}(\mathbf{F})$. By [Le+a, Proposition 5.5.7], there exists $\overline{\mathfrak{M}}\in Y_n^{\leq\lambda+\eta',\tau}(\mathbf{F})$, semisimple of shape $\widetilde{w}(\overline{\rho},\tau)^*=w(\lambda+\eta')$ for some $w\in\underline{W}_n$, such that $\overline{\rho}|_{G_{K_\infty}}\simeq T_{dd}^*(\overline{\mathfrak{M}})$. Then the claim follows from Proposition 7.2.3 in *loc. cit.*.

Let $\overline{\beta}$ be a gauge basis for $\overline{\mathfrak{M}}$. Then we have

$$A^{(j)}_{\overline{\mathfrak{M}},\overline{\beta}}=D_j(v+p)^{w_j(\lambda+\eta')}$$

for some $D_j\in T_n^\vee(\mathbf{F})$. We let $R_{\overline{\mathfrak{M}}}^{\leq\lambda+\eta',\tau,\overline{\beta}}$ be the deformation ring that generalizes $R_{\overline{\mathfrak{M}}}^{\tau,\overline{\beta},\nabla}$ in [LLL19], which parametrizes lifts of $(\overline{\mathfrak{M}},\overline{\beta})$ of height bounded by η' and satisfying the monodromy condition over the generic fiber, by replacing η' with $\lambda+\eta'$. Then $R_{\overline{\mathfrak{M}}}^{\leq\lambda+\eta',\tau,\overline{\beta}}$ is a versal ring to $Y_n^{\leq\lambda+\eta',\tau,\nabla_\infty}$ at $\overline{\mathfrak{M}}$. As in Proposition 3.4.8 in *loc. cit.*, the universal partial Frobenius matrices has the form

$$A^{(j),\mathrm{univ},\nabla}=D_j^{\mathrm{univ}}(v+p)^{w_j(\lambda+\eta')}U^{(j),\mathrm{univ},\nabla}$$

for $j\in\mathcal{J}$ where $D_j^{\mathrm{univ}}\in T_n(R_{\overline{\mathfrak{M}}}^{\leq\lambda+\eta',\tau,\overline{\beta}})$ lifting D_j and $w_j^{-1}U^{(j),\mathrm{univ},\nabla}w_j\in\overline{U}_n(R_{\overline{\mathfrak{M}}}^{\leq\lambda+\eta',\tau,\overline{\beta}})$. In addition, for any root $\alpha<0$,

$$\left(w_j^{-1}U^{(j),\mathrm{univ},\nabla}w_j\right)_\alpha=v^{\delta_{w_j(\alpha)<0}}f_\alpha^{(j)}(v)$$

where $f_\alpha^{(j)} \in R_{\overline{\mathfrak{M}}}^{\leq\lambda+\eta',\tau,\overline{\beta}}[v]$ is a polynomial of degree $< -\langle\lambda+\eta',\alpha^\vee\rangle$ and is zero modulo the maximal ideal of $R_{\overline{\mathfrak{M}}}^{\leq\lambda+\eta',\tau,\overline{\beta}}$. Moreover, as in the proof of Proposition 3.4.12 in *loc. cit.* (this is where we need $(2h_{\lambda+\eta}+1)$-genericity of (s,μ)), $f_\alpha^{(j)} \mod \varpi$ is determined by its top degree coefficient. In particular, this implies that $R_{\overline{\mathfrak{M}}}^{\leq\lambda+\eta',\tau,\overline{\beta}}$ is a quotient of a power series ring over $\mathcal{O}$ in $\frac{n(n+1)f}{2}$ variables.

It follows from the diagram (3.16) in [LLL19] that there is an isomorphism

$$R_{\overline{\rho}}^{\leq\lambda+\eta'}[\![x_1,\dots,x_{nf}]\!] \simeq R_{\overline{\mathfrak{M}}}^{\leq\lambda+\eta',\tau,\overline{\beta}}[\![y_1,\dots,y_{n^2}]\!].$$

The former is of dimension $\frac{n(n+1)f}{2}+n^2+1$ where the latter is quotient of a power series ring over $\mathcal{O}$ in $\frac{n(n+1)f}{2}+n^2$.This shows that $R_{\overline{\rho}}^{\leq\lambda+\eta'}$ is formally smooth over $\mathcal{O}$ of relative dimension $\frac{n(n-1)f}{2}+n^2$. We have $R_{\overline{\rho}}^{\leq\lambda+\eta'} \simeq R_{\overline{\rho}}^{\lambda+\eta'}$ as the universal partial Frobenius matrices $A^{(j),\mathrm{univ},\nabla}$ has elementary divisor exactly $(v+p)^{\lambda+\eta'}$.

Finally, we prove that any $\rho : G_K \to \mathrm{GL}_n(\mathcal{O})$ of type $(\lambda+\eta',\tau)$ lifting $\overline{\rho}$ is potentially diagonalizable. Let $\rho' : G_K \to \mathrm{GL}_n(\mathcal{O})$ be a lift of $\overline{\rho}$ of type $(\lambda+\eta',\tau)$ whose associated Breuil–Kisin module is contained in the locus of $\mathrm{Spec}\, R_{\overline{\mathfrak{M}}}^{\leq\lambda+\eta',\tau,\overline{\beta}}$ given by the condition $U^{(j'),\mathrm{univ},\nabla}=1$. Let K'/K be an unramified extension of degree $n!$. Then $\rho'|_{G_{K'}}$ is a sum of characters. Since $R_{\overline{\rho}}^{\lambda+\eta'}$ is a domain, this shows that ρ is potentially diagonalizable. $\square$

Proof of Theorem 5.1.4. We can repeat the proof of Theorem 5.1.5 using Proposition 4.4.2 instead of the diagram (3.16) in [LLL19]. The symplecticity of $\overline{\rho}$ implies the symplecticity of $(\overline{\mathfrak{M}},\overline{\beta})$ (i.e. $D_j \in T^\vee(\mathbf{F})$ instead of $T_4^\vee(\mathbf{F})$). The appropriate symplectic deformation ring $R_{\overline{\mathfrak{M}}}^{\leq\lambda+\eta,\tau,\overline{\beta}}$ of $(\overline{\mathfrak{M}},\overline{\beta})$ can be obtained from the GL_4 case by imposing that $A^{(j),\mathrm{univ},\nabla}$ is valued in GSp_4, i.e. D_j^{univ} valued in $T^\vee$ and $w_j^{-1}U^{(j),\mathrm{univ},\nabla}w_j$ valued in $\overline{U}$. Then $R_{\overline{\mathfrak{M}}}^{\leq\lambda+\eta,\tau,\overline{\beta}}$ is a quotient of power series ring over $\mathcal{O}$ with $7f$ variables. On the other hand, $R_{\overline{\rho}}^{\lambda+\eta,\tau}$ has dimension $4f+11$. Then we have an isomorphism

$$R_{\overline{\rho}}^{\lambda+\eta,\tau}[\![x_1,\dots,x_{3f}]\!] \simeq R_{\overline{\mathfrak{M}}}^{\leq\lambda+\eta,\tau,\overline{\beta}}[\![y_1,\dots,y_{11}]\!]$$

which proves the claim. The potential diagonalizability can be proven similarly using Example 4.1.12. $\square$

We also record a lifting result for certain non-tame $\overline{\rho}$.

Lemma 5.1.6. *Let $\kappa \in X_1^*(\underline{T})$ be 0-deep. Let $\overline{\rho} : G_K \to \mathrm{GSp}_4(\mathbf{F})$ be of maximally nonsplit of niveau 1 and weight $\sigma = F(\kappa)$. Let a be an integer and let k be the unique integer such that $a-2k =: b \in \{3,4\}$. Let $(\lambda_a+\eta,\tau_a)$ be a type such that*

$$\lambda_a = \begin{cases} k(1,1;2) & \text{if } b=3 \\ (1,0;0)+k(1,1;2) & \text{if } b=4 \end{cases}$$

and $\tau_a = \tau(1,\kappa-\phi^{-1}(\lambda_a))$. Then there is a potentially crystalline lift $\rho : G_K \to \mathrm{GSp}_4(\mathcal{O})$ of $\overline{\rho}$ with Hodge type $\lambda_a+\eta$ and tame inertial type τ_a and of the form

$$\rho = \begin{pmatrix} \chi_1 & * & * & * \\ 0 & \chi_2 & * & * \\ 0 & 0 & \chi_3 & * \\ 0 & 0 & 0 & \chi_4 \end{pmatrix}$$

where $\oplus_{i=1}^4 \chi_i = \epsilon_p^{\lambda_a+\eta} \prod_{j\in\mathcal{J}} \omega_{K,\sigma_j}^{\phi(\kappa_j)-\lambda_{a,j}}$.

Proof. It is clear that χ_i lifts $\overline{\chi}_i$. Since κ is 0-deep, for $i < j$, $\overline{\chi}_i\overline{\chi}_j^{-1} \neq \overline{\epsilon}_p$. Thus, $\chi_i\chi_j^{-1} \neq \epsilon_p$. Then the existence of the lift ρ follows from the vanishing of cohomology $H^2(G_K, \chi_i\chi_j^{-1}) = 0$. It is potentially crystalline by [EG23, Lemma 6.3.1]. □

5.1.7 Local deformations: $l \neq p$

We record deformation problems for Ihara avoidance argument. Let l be a prime and $l \neq p$. Let $F/\mathbf{Q}_l$ be a finite extension with the ring of integers $\mathcal{O}_F$, a uniformizer ϖ_F, and the residue field k_F of size q_F. We assume that $q_F \equiv 1 \bmod p$. Let $\overline{\rho} : G_F \to \mathrm{GSp}_4(\mathbf{F})$ be a trivial representation and $\psi : G_F \to \mathcal{O}^\times$ be a continuous character trivial modulo ϖ.

Let $\zeta = (\zeta_1, \zeta_2)$ be a pair of continuous characters $\zeta_i : \mathcal{O}_F^\times \to \mathcal{O}^\times$ that are trivial modulo ϖ. We let $D_{\overline{\rho}}^\zeta$ be the functor taking $A \in \widehat{\mathcal{C}}_{\mathcal{O}}$ to the set of A-valued lifts $\rho : G_F \to \mathrm{GSp}_4(A)$ such that for any $\sigma \in I_F$, the characteristic polynomial of $\rho(\sigma)$ is

$$(X - \zeta_1(\mathrm{Art}_F^{-1}(\sigma)))(X - \zeta_2(\mathrm{Art}_F^{-1}(\sigma)))(X - \zeta_2(\mathrm{Art}_F^{-1}(\sigma)))^{-1}(X - \zeta_1(\mathrm{Art}_F^{-1}(\sigma)))^{-1}.$$

Then $D_{\overline{\rho}}^\zeta$ is a local deformation problem. We let $R_{\overline{\rho}}^\zeta \in \widehat{\mathcal{C}}_{\mathcal{O}}$ be an object representing $D_{\overline{\rho}}^\zeta$. We record the following results on $R_{\overline{\rho}}^\zeta$.

Proposition 5.1.8 (Proposition 7.4.7 and 7.4.8 in [Box+21])**.**

1. *Suppose that $\zeta = 1$ is the pair of trivial characters. Then $\mathrm{Spec}\, R_{\overline{\rho}}^1$ is equidimensional of dimension 11 and every generic point has characteristic zero. Moreover, every generic point of $\mathrm{Spec}\, R_{\overline{\rho}}^1/\varpi$ is the specialization of a unique generic point of $\mathrm{Spec}\, R_{\overline{\rho}}^1$.*

2. *Suppose that $\zeta_1, \zeta_2 \neq 1$ and $\zeta_1 \neq \zeta_2^{\pm 1}$. Then $\mathrm{Spec}\, R_{\overline{\rho}}^\zeta$ is irreducible of dimension 11, and its generic point has characteristic zero.*

5.2 Congruent patching functors

Recall the finite étale $\mathbf{Z}_p$-algebra $\mathcal{O}_p$. It can be written as a finite product of finite étale local $\mathbf{Z}_p$-algebras $\prod_{v\in S_p} \mathcal{O}_v$. We write $F_p = \mathcal{O}_p[1/p] \simeq \prod_{v\in S_p} F_v$. Let $\mathcal{J} = \mathrm{Hom}_{\mathbf{Z}_p}(\mathcal{O}_p, \mathcal{O})$.

Let $\overline{\rho} : G_{\mathbf{Q}_p} \to {}^L\underline{G}(\mathbf{F})$ be a tame L-parameter. We consider $\psi_p : G_{\mathbf{Q}_p} \to {}^L\underline{\mathbf{G}_m}(\mathcal{O})$ lifting $\mathrm{sim}(\overline{\rho})$. Note that ψ_p is equivalent to a collection $\{\psi_v : G_{F_v} \to \mathcal{O}^\times\}_{v\in S_p}$. For the applications in §6, we only consider ψ_p satisfying:

- ψ_p is potentially crystalline with tame descent data
- there exists an integer w such that $\mathrm{HT}_\sigma(\psi_v) = \{w\}$ for all $v \in S_p$ and $\sigma \in \mathrm{Hom}_{\mathbf{Q}_p}(F_v, E)$; and
- $\psi_v\epsilon_p^{-w}$ has finite image for all $v \in S_p$.

Note that the first and second conditions imply

$$\psi_p\epsilon_p^{-w}|_{I_{\mathbf{Q}_p}} = [\overline{\rho\epsilon_p}^{-w}|_{I_{\mathbf{Q}_p}}],$$

and the third condition implies $\psi_p\epsilon_p^{-w} = [\overline{\rho\epsilon_p}^{-w}]$.

Definition 5.2.1. 1. We define $\Psi(\overline{\rho})$ to be a set of $\psi_p : G_{\mathbf{Q}_p} \to {}^L\underline{\mathbf{G}_m}(\mathcal{O})$ lifting $\mathrm{sim}(\overline{\rho})$ and satisfying the above conditions. If $\psi_p \in \Psi(\overline{\rho})$, we say ψ_p and $\overline{\rho}$ are *compatible*.

2. We say a type $(\lambda+\eta,\tau)$ is *compatible with* ψ_p if $\mathrm{sim}(\lambda_j+\eta_j) = w$ for $j \in \mathcal{J}$ and

$$\mathrm{sim}(\tau)\epsilon_p^w|_{I_{\mathbf{Q}_p}} = \psi_p|_{I_{\mathbf{Q}_p}}.$$

Remark 5.2.2. It follows from the above discussion that given a tame L-parameter $\overline{\rho}$ and a type $(\lambda+\eta,\tau)$, there exists at most one $\psi_p \in \Phi(\overline{\rho})$ that is compatible with $(\lambda+\eta,\tau)$.

Let $\psi_p \in \Psi(\overline{\rho})$. We define

$$R_{\overline{\rho}}^{\psi_p} := \widehat{\otimes}_{v\in S_p,\mathcal{O}} R_{\overline{\rho}_v}^{\square,\psi_v}, \quad R_\infty^{\psi_p} := R_{\overline{\rho}}^{\psi_p}\widehat{\otimes}_{\mathcal{O}} R^p$$

where R^p is a complete Noetherian equidimensional flat $\mathcal{O}$-algebra. For a type $(\lambda+\eta,\tau)$ compatible with ψ_p, we define

$$R_{\overline{\rho}}^{\lambda+\eta,\tau,\psi_p} := \widehat{\otimes}_{v\in S_p,\mathcal{O}} R_{\overline{\rho}_v}^{\lambda_v+\eta_v,\tau_v,\psi_v}, \quad R_\infty^{\lambda+\eta,\tau,\psi_p} := R_\infty^{\psi_p} \otimes_{R_{\overline{\rho}}^{\psi_p}} R_{\overline{\rho}}^{\lambda+\eta,\tau,\psi_p}.$$

Let $\mathrm{Mod}(R_\infty^{\psi_p})$ be the category of finitely generated modules over $R_\infty^{\psi_p}$ and $\mathrm{Rep}_{\mathcal{O}}^{\psi_p}(\mathrm{GSp}_4(\mathcal{O}_p))$ be the category of topological $\mathcal{O}[\mathrm{GSp}_4(\mathcal{O}_p)]$-modules, which are finitely generated over $\mathcal{O}$, with fixed central character given by $\otimes_{v\in S_p}(\psi_v\epsilon_p^3|_{I_{F_v}})\circ(\mathrm{Art}_{F_v}|_{\mathcal{O}_{F_v}})$.

Definition 5.2.3. A *fixed similitude patching functor* for $\overline{\rho}$ is a triple $(\psi_p, R_\infty^{\psi_p}, M_\infty^{\psi_p})$ where $\psi_p \in \Psi(\overline{\rho})$, $R_\infty^{\psi_p}$ is as above, and

$$M_\infty^{\psi_p} : \mathrm{Rep}_{\mathcal{O}}^{\psi_p}(\mathrm{GSp}_4(\mathcal{O}_p)) \to \mathrm{Mod}(R_\infty^{\psi_p})$$

is an exact covariant functor satisfying the following conditions:

(1) for a type $(\lambda+\eta,\tau)$ compatible with ψ_p and a $\mathrm{GSp}_4(\mathcal{O}_p)$-stable $\mathcal{O}$-lattice $\sigma^\circ(\lambda,\tau)$ in $\sigma(\lambda,\tau)$, $M_\infty^{\psi_p}(\sigma^\circ(\lambda,\tau))$ is a maximal Cohen–Macaulay module over $R_\infty^{\lambda+\eta,\tau,\psi_p}$ if it is nonzero; and

(2) for all $\sigma \in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))$, $M_\infty^{\psi_p}(\sigma)$ is a maximal Cohen–Macaulay module over $R_\infty^{\lambda+\eta,\tau,\psi_p}/\varpi$ if it is nonzero.

Furthermore, we say that

(i) $M_\infty^{\psi_p}$ is *minimal* if R^p is formally smooth over $\mathcal{O}$ and $M_\infty^{\psi_p}(\sigma^\circ(\lambda,\tau))[p^{-1}]$ is locally free of rank at most one over $R_\infty^{\lambda+\eta,\tau,\psi_p}[p^{-1}]$; and

(ii) $M_\infty^{\psi_p}$ is *potentially diagonalizable* if $M_\infty^{\psi_p}(\sigma^\circ(\lambda,\tau))$ is nonzero for all $(\lambda+\eta,\tau)$ such that $\overline{\rho}_v$ has potentially diagonalizable lift of type $(\lambda_v+\eta_v,\tau_v)$ for each $v \in S_p$.

For the following definition, we let R_∞ be a complete local Noetherian $\mathcal{O}$-algebra with a surjection $R_\infty \to R_\infty^{\psi_p}$ for all $\psi_p \in \Phi(\overline{\rho})$. In the application, R_∞ will be a completed tensor product of local deformation rings and $R_\infty^{\psi_p}$ will be a quotient of R_∞ with fixed similitude character.

Definition 5.2.4. 1. We say a set

$$\mathcal{M}_\infty = \left\{(\psi_p, R_\infty^{\psi_p}, M_\infty^{\psi_p}) \mid (\psi_p, R_\infty^{\psi_p}, M_\infty^{\psi_p}) \text{ is a fixed similitude patching functor for } \overline{\rho}\right\}$$

is (or its elements are) *congruent* if the restrictions of $M_\infty^{\psi_p}$ to ϖ-torsion objects

$$M_\infty^{\psi_p}/\varpi : \mathrm{Rep}_{\mathbf{F}}^{\psi_p}(\mathrm{GSp}_4(\mathcal{O}_p)) \to \mathrm{Mod}(R_\infty^{\psi_p}/\varpi) \hookrightarrow \mathrm{Mod}(R_\infty/\varpi)$$

for all $(\psi_p, R_\infty^{\psi_p}, M_\infty^{\psi_p}) \in \mathcal{M}_\infty$ are equal.

2. We say that $\mathcal{M}_\infty$ is *minimal* (resp. *potentially diagonalizable*) if all $(\psi_p, R_\infty^{\psi_p}, M_\infty^{\psi_p}) \in \mathcal{M}_\infty$ are minimal (resp. potentially diagonalizable).

3. A *congruent family of fixed similitude patching functors* for $\overline{\rho}$ is a congruent set of fixed similitude patching functors $\mathcal{M}_\infty^\Phi$ such that the map

$$\begin{aligned} \mathcal{M}_\infty^\Phi &\to \Phi(\overline{\rho}) \\ (\psi_p, R_\infty^{\psi_p}, M_\infty^{\psi_p}) &\mapsto \psi_p \end{aligned}$$

is a bijection. In other words, $\mathcal{M}_\infty^\Phi$ consists of exactly one fixed similitude patching functor for each $\psi_p \in \Phi(\overline{\rho})$.

5.3 Algebraic automorphic forms

In this section, we recall the global setup in [EL, §4]. Let F be a totally real field. Suppose that $[F : \mathbf{Q}]$ is even. We also assume that p is unramified in F. Let $\mathcal{G}$ be the F-group $\mathrm{GU}_2(D)$ where D is a quaternion algebra over F ramified at all infinite places and split at all finite places. Such a D exists because $[F : \mathbf{Q}]$ is even. Then $\mathcal{G}$ is an inner form of GSp_4. The center $Z_\mathcal{G}$ is isomorphic to $\mathbf{G}_m$, and $\mathcal{G}$ is compact modulo center at infinity. Choose a maximal order $\mathcal{O}_D$ of D. It defines an $\mathcal{O}_F$-structure on $\mathcal{G}$. For each finite place v, we fix an isomorphism $\mathcal{O}_{D,v} \simeq M_2(\mathcal{O}_{F_v})$ which induces an isomorphism $\iota_v : \mathcal{G}_{/\mathcal{O}_{F_v}} \to \mathrm{GSp}_{4/\mathcal{O}_{F_v}}$.

Let $\chi : \mathbb{A}_F/F^\times \to \mathbf{C}^\times$ be a Hecke character. We write $\chi_{p,\iota} = \iota^{-1} \circ \chi$. Let $U = U_p U^{\infty,p} \leq \mathcal{G}(\mathcal{O}_p) \times \mathcal{G}(\mathbb{A}_F^{\infty,p})$ be a compact open subgroup. For a finite place v of F, we write $\mathrm{Iw}(v)$ (resp. $\mathrm{Iw}_1(v)$) for the (resp. pro-p) Iwahori subgroup of $\mathcal{G}(F_v) \simeq \mathrm{GSp}_4(F_v)$. Let W be an $\mathcal{O}$-module with a continuous action of U_p. We define $S_\chi(U, W)$ to be the $\mathcal{O}$-module of functions $f : \mathcal{G}(F^\times)\backslash\mathcal{G}(\mathbb{A}_F^\infty) \to W$ such that

$$f(zgu) = \chi_{p,\iota}(z) u_p^{-1} f(g) \quad \forall g \in \mathcal{G}(\mathbb{A}_F^\infty), u \in U, z \in Z_\mathcal{G}(\mathbb{A}_F^\infty).$$

Let $U \leq \mathcal{G}(\mathbb{A}_F^\infty)$ be a compact open subgroup. We assume that U is *sufficiently small* in the sense that it contains no element of order p. For our local applications, we will fix a finite place v_0 such that $q_{v_0} \not\equiv 1 \bmod p$ and has residue characteristic > 5 and assume that $U_{v_0} = \mathrm{Iw}_1(v_0)$. In this case, the choice of U_{v_0} ensures that U is sufficiently small. Let P_U be the finite set of finite places of F at which U is ramified. Let S_p be the set of places of F dividing p. For a finite set P, we define the *universal Hecke algebra* $\mathbb{T}^{P,\mathrm{univ}}$ to be the polynomial ring over $\mathcal{O}$ generated by $S_v, T_{v,1}, T_{v,2}$ for each $v \notin P$ and $v = v_0$ if $v_0 \in P$. When P contains $S_p \cup P_U$, there is a natural $\mathbb{T}^{P,\mathrm{univ}}$-action on $S_\chi(U, W)$ where $S_v, T_{v,1}, T_{v,2}$ act through the double coset operators

$$\left[U_v\begin{pmatrix}\varpi_v&&&\\&\varpi_v&&\\&&\varpi_v&\\&&&\varpi_v\end{pmatrix}U_v\right],\ \left[U_v\begin{pmatrix}\varpi_v&&&\\&\varpi_v&&\\&&1&\\&&&1\end{pmatrix}U_v\right],\ \left[U_v\begin{pmatrix}\varpi_v^2&&&\\&\varpi_v&&\\&&\varpi_v&\\&&&1\end{pmatrix}U_v\right]$$

respectively. We denote the image of $\mathbb{T}^{P,\mathrm{univ}}$ in $\mathrm{End}_{\mathcal{O}}(S_\chi(U,W))$ by $\mathbb{T}^P_\chi(U,W)$.

Let $\psi := \chi_{p,\iota}\epsilon_p^{-3}$. Let $\overline{r} : G_F \to \mathrm{GSp}_4(\mathbf{F})$ be an absolutely irreducible continuous representation and $\mathrm{sim}(\overline{r}) = \psi \bmod \varpi$. When we have a fixed place v_0, we assume that $\overline{r}|_{G_{F_{v_0}}}$ is a sum of unramified characters with no two eigenvalues of Frob_{v_0} have ratio q_{v_0}. In applications, we can always choose such v_0 ([Box+21, §7.7]). We write

$$\overline{r}|_{G_{F_{v_0}}} \simeq \mathrm{ur}_{c_1}\mathrm{ur}_{c_2}\mathrm{ur}_{c_0} \oplus \mathrm{ur}_{c_1}\mathrm{ur}_{c_0} \oplus \mathrm{ur}_{c_2}\mathrm{ur}_{c_0} \oplus \mathrm{ur}_{c_0}$$

for $c_1, c_2, c_0 \in \mathbf{F}^\times$. We denote by $P_{\overline{r}}$ the set of finite places either dividing p or at which $\overline{r}$ is ramified. For any finite set $P \supset P_{\overline{r}}$, we define a maximal ideal $\mathfrak{m}^P_{\overline{r},\chi} \leq \mathbb{T}^{P,\mathrm{univ}}$ with residue field $\mathbf{F}$ by demanding that

1. for each $v \notin P$,

$$S_v \bmod \mathfrak{m}^P_{\overline{r},\chi} = \overline{\psi}(\mathrm{Frob}_v)$$

   and the characteristic polynomial of $\overline{r}(\mathrm{Frob}_v)$ in $\mathbf{F}[X]$ is given by

$$X^4 - T_{v,1}X^3 + (q_vT_{v,2} + (q_v^3+q_v)S_v)X^2 - q_v^3T_{v,1}S_vX + q_v^6S_v^2 \bmod \mathfrak{m}^P_{\overline{r},\chi},$$

2. for $v = v_0$ (if $v_0 \in P$),

$$S_v \bmod \mathfrak{m}^P_{\overline{r},\chi} = c_0,\ T_{v,1} \bmod \mathfrak{m}^P_{\overline{r},\chi} = q_{v_0}c_1,\ \ T_{v,2} \bmod \mathfrak{m}^P_{\overline{r},\chi} = q_{v_0}^3c_2$$

(cf. [Box+21, §2.4.7]; note $\mathfrak{m}^P_{\overline{r},\chi}$ is well-defined by symplecticity of $\overline{r}$).

Definition 5.3.1. 1. We say that a pair $(\overline{r},\chi)$ as above is *automorphic of weight* $\mu \in (X^*(T)^+)^{S_p}$ *and level* U if there is a finite set of finite places P containing $P_U \cup P_{\overline{r}}$ such that $S_\chi(U, V(\mu)^\vee)_{\mathfrak{m}^P_{\overline{r},\chi}} \neq 0$.

2. Let σ be a Serre weight of $G_0(\mathbf{F}_p)$. We say that $\overline{r}$ is *modular of weight* σ (and level U), or equivalently, σ is a *modular weight* of $\overline{r}$ (at level U) if $S_\chi(U,\sigma^\vee)_{\mathfrak{m}^P_{\overline{r},\chi}} \neq 0$ for some χ. Note that $S_\chi(U,\sigma^\vee)_{\mathfrak{m}^P_{\overline{r},\chi}}$ does not depend on the choice of χ as long as $\chi_{p,\iota}\epsilon_p^{-3} \bmod \varpi = \mathrm{sim}(\overline{r})$.

3. We let $W(\overline{r})$ be the set of modular Serre weights of $\overline{r}$.

We remark that $(\overline{r},\chi)$ is automorphic for some χ if and only if $\overline{r}$ is a mod p reduction of the Galois representation attached to a regular algebraic cuspidal automorphic representation of $\mathcal{G}(\mathbb{A}_F)$ (equivalently, of $\mathrm{GSp}_4(\mathbb{A}_F)$); see [EL, Remark 4.2.4].

Let $(\overline{r},\chi)$ be automorphic of weight μ and level U and P be as above. Suppose that there is a subset $R \subset P$ disjoint from S_p such that for $v \in R$, $U_v = \mathrm{Iw}_1(v)$, $q_v \equiv 1 \bmod p$, and $\overline{r}|_{G_{F_v}}$ is trivial. For each $v \in R$, we choose a pair of characters $\zeta_v = (\zeta_{v,1}, \zeta_{v,2})$ such that

1. for $i = 1, 2$, $\zeta_{v,i} : \mathcal{O}_v^\times \to \mathcal{O}^\times$ is a continuous character trivial modulo ϖ;
2. either $\zeta_{v,1} = \zeta_{v,2} = 1$, or $\zeta_{v,1}, \zeta_{v,2} \neq 1$ and $\zeta_{v,1} \neq \zeta_{v,2}^{\pm 1}$.

Let $T_{\mathrm{der}} := \ker(T \xrightarrow{\mathrm{sim}} \mathbf{G}_m)$. The projection to the first two diagonal entries induces an isomorphism $T_{\mathrm{der}} \simeq \mathbf{G}_m^2$. We write ζ_R for the induced character

$$\zeta_R = \prod_{v \in R} [\overline{\zeta}_v] : \prod_{v \in R} T_{\mathrm{der}}(k_v) \to \mathcal{O}^\times.$$

We let $\mathbb{T}_\chi^P(U, V(\mu))_{\zeta_R}$ to be the image of $\mathbb{T}^{P,\mathrm{univ}}$ in $\mathrm{End}_{\mathcal{O}}(S_\chi(U, V(\mu))_{\zeta_R})$ where the subscript ζ_R denotes taking the ζ_R-coinvariant for the action of $\Pi_{v \in R} T_{\mathrm{der}}(k_v)$.

Proposition 5.3.2. *Keep the above notations and assumptions. We further assume that $\overline{r} : G_F \to \mathrm{GSp}_4(\mathbf{F})$ is absolutely irreducible. Then there exists a unique continuous representation*

$$r_{\chi,\mu}^P(U) : G_F \to \mathrm{GSp}_4(\mathbb{T}_\chi^P(U, V(\mu))_{\zeta_R, \mathfrak{m}_{\overline{r},\chi}^P})$$

such that

1. *$r_{\chi,\mu}^P(U)$ lifts $\overline{r}$;*
2. *$\mathrm{sim}(r_{\chi,\mu}^{P \cup Q}(U(Q))) = \psi$;*
3. *if $v \notin P \cup Q$, then $r_{\chi,\mu}^P(U)$ is unramified at v and the characteristic polynomial of $r_{\chi,\mu}^P(U)(\mathrm{Frob}_v)$ in $\mathbb{T}_\chi^P(U, V(\mu))_{\zeta_R, \mathfrak{m}_{\overline{r},\chi}^P}[X]$ is equal to*
$$X^4 - T_{v,1} X^3 + (q_v T_{v,2} + (q_v^3 + q_v)\chi_v(\varpi_v))X^2 - q_v^3 T_{v,1} \chi_v(\varpi_v) X + q_v^6 \chi_v(\varpi_v)^2,$$
4. *if $v \in R$, then the $\mathbb{T}_\chi^P(U, V(\mu))_{\zeta_R, \mathfrak{m}_{\overline{r},\chi}^P}$-point of $R_{\overline{r}|_{G_{F_v}}}^{\square}$ induced by $r_{\chi,\mu}^P(U)|_{G_{F_v}}$ factors through $R_{\overline{r}|_{G_{F_v}}}^{\zeta_v}$;*
5. *if $v_0 \in P$ and $v = v_0$, then*
$$r_{\chi,\mu}^P(U)|_{G_{F_v}} \simeq \mathrm{ur}_{T_{v,1}} \mathrm{ur}_{T_{v,2}} \mathrm{ur}_{S_v} \oplus \mathrm{ur}_{T_{v,1}} \mathrm{ur}_{S_v} \oplus \mathrm{ur}_{T_{v,2}} \mathrm{ur}_{S_v} \oplus \mathrm{ur}_{S_v};$$
6. *and for every $\mathcal{O}$-algebra homomorphism $f : \mathbb{T}_\chi^P(U, V(\mu))_{\zeta_R, \mathfrak{m}_{\overline{r},\chi}^P} \to E'$ where E' is a finite extension of E, the representation $f \circ r_{\chi,\mu}^P(U)|_{G_{F_v}}$ is de Rham of Hodge–Tate weights $\mu_v + \eta_v$ for all $v \in S_p$.*

Proof. This is proven in [EL, Proposition 4.2.6] except item (4) and (5). By the local-global compatibility in [Mok14, Theorem 3.5], item (4) follows from Proposition 2.4.28 [Box+21] and item (5) follows from Proposition 2.4.3, 2.4.4, and 7.4.2 in *loc. cit.*. ∎

5.4 Patching argument

In this section, we construct a congruent family of similitude patching functors in the global case (i.e. $\mathcal{O}_p = \mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p$) and in the local case (i.e. $\mathcal{O}_p = \mathcal{O}_K$).

Definition 5.4.1. 1. We say that a continuous representation $\overline{r} : G_F \to \mathrm{GSp}_4(\mathbf{F})$ is *odd* if for each infinite place v and corresponding choice of complex conjugation $c_v \in G_F$, $\mathrm{sim}(\overline{r})(c_v) = -1$.

2. We say that a continuous representation $\overline{r} : G_F \to \mathrm{GSp}_4(\mathbf{F})$ *satisfies Taylor–Wiles conditions* if $\overline{r}$ is absolutely irreducible, odd, and vast and tidy (in the sense of [Box+21, §7.5]).

Theorem 5.4.2. *1. Let $\overline{r} : G_F \to \mathrm{GSp}_4(\mathbf{F})$ be a continuous representation. Let χ be a Hecke character and $\psi := \chi\epsilon_p^{-3}$. Suppose that $\overline{r}$ satisfies Taylor–Wiles conditions and $(\overline{r}, \chi)$ is automorphic (of some weight μ' and level U'). Then there exists a congruent family of fixed similitude patching functors for $\overline{r}_p$.*

2. Let $\overline{\rho} : G_K \to \mathrm{GSp}_4(\mathbf{F})$ be a 16-generic continuous representation. If $\overline{\rho}$ is either tame or maximally nonsplit of niveau 1 and weight σ, there exists a congruent family of potentially diagonaliazable fixed similitude patching functors for $\overline{\rho}$. In the former case, it can be taken to be minimal.

Remark 5.4.3. 1. Let σ be a Serre weight of $\mathrm{GSp}_4(\mathcal{O}_F/p)$. Let $M_\infty^{\psi_p}$ be a fixed similitude patching functor in Theorem 5.4.2(1). It will be clear from the construction of $M_\infty^{\psi_p}$ that

$$(M_\infty^{\psi_p}(\sigma)/\mathfrak{m}_\infty)^\vee \simeq S_\psi(U, \sigma^\vee)_{\mathfrak{m}^P_{\overline{r},\chi}}.$$

In other words, $\sigma \in W(\overline{r})$ if and only if $M_\infty^{\psi_p}(\sigma) \neq 0$. Similarly, for a type (λ, τ) compatible with ψ_p, we have

$$(M_\infty^{\psi_p}(\sigma^\circ(\lambda,\tau))/\mathfrak{a}_\infty)^\vee = S_\psi(U, \sigma^\circ(\lambda,\tau)^\vee)_{\mathfrak{m}^P_{\overline{r},\chi}}.$$

In particular, if $M_\infty^{\psi_p}(\sigma^\circ(\lambda,\tau)) \neq 0$, then $\overline{r}_p$ has a potentially crystalline lift of type $(\lambda + \eta, \tau)$.

2. Let σ be a Serre weight of $\mathrm{GSp}_4(k)$ and $M_\infty^{\psi_p}$ be a fixed similitude patching functor in Theorem 5.4.2(2). It will be clear from the construction and Theorem 5.4.4 that if $M_\infty^{\psi_p}(\sigma^\circ(\lambda,\tau)) \neq 0$, then $\sigma \in W^?(\overline{\rho}|_{I_K})$. Also, for a type (λ,τ) compatible with ψ_p, if $M_\infty^{\psi_p}(\sigma^\circ(\lambda,\tau)) \neq 0$, then $\overline{\rho}$ has a potentially crystalline lift of type $(\lambda+\eta,\tau)$ by Proposition 5.3.2.

Granting item (1) in Theorem 5.4.2, we prove the following weight elimination result. It will be used to prove item (2) in Theorem 5.4.2.

Theorem 5.4.4. *Let $\overline{r} : G_F \to \mathrm{GSp}_4(\mathbf{F})$, χ, and ψ be as in Theorem 5.4.2(1). Let $\sigma \in W(\overline{r})$. We suppose that either $(\overline{r}|_{G_{F_v}})^{\mathrm{ss}}$ is 22-generic for each $v|p$ or σ is 7-deep and $(\overline{r}|_{G_{F_v}})^{\mathrm{ss}}$ is 16-generic for each $v|p$. Then $\sigma \in W^?(\overline{r}_p^{\mathrm{ss}})$.*

Proof. By Theorem 5.4.2(1), there exists a congruent family of fixed similitude patching functors $\{M_\infty^{\psi_p}\}$ for $\overline{r}_p$. Let $\sigma = F(\lambda)$ be a modular Serre weight for $\overline{r}$. By the exactness of patching functors and Remark 5.4.3, $\overline{r}_p$ has a potentially crystalline lift of type (η,τ) for any 1-generic tame inertial type τ such that $\sigma \in \mathrm{JH}(\overline{\sigma}(\tau))$.

Suppose that λ is 12-deep. For each $s \in \underline{W}$, let $\tau_s = \tau(s, \widetilde{w}_h \cdot \lambda + \eta)$ be a 9-generic tame inertial L-parameter. Since $F(\lambda) \in \mathrm{JH}(\overline{\sigma}(\tau_s))$, for the unique $\psi_p \in \Phi(\overline{r}_p)$ compatible with (η, τ_s), we have

$$M_\infty^{\psi_p}(\sigma^\circ(\tau_s)) \neq 0.$$

Since $M_\infty^{\psi_p}(\sigma^\circ(\tau_s))$ is supported on $\operatorname{Spec} R_\infty^{\eta,\tau_s,\psi_p}$, $\overline{r}|_{G_{F_v}}$ admits a potentially crystalline lift with Hodge type η_v and tame inertial type $(\tau_s)_v$ for $v|p$. By Proposition 5.1.3, there exists a lowest alcove presentation of $\overline{r}_v^{\mathrm{ss}}$ (which does not depend on s by Remark 2.5.13) such that $\widetilde{w}(\overline{r}_v^{\mathrm{ss}},\tau_s)\in \mathrm{Adm}(\eta)$. By Lemma 2.5.12, $F(\lambda)\in W^?(\overline{r}_p^{\mathrm{ss}})$.

Suppose that λ is not 12-deep but 7-deep. Then the tame type $\tau_e=\tau(e,\widetilde{w}_h\cdot\lambda+\eta)$ is 4-generic but not 12-generic by [LLL19, Proposition 2.2.16] and its proof. Also, $F(\lambda)\in \mathrm{JH}(\overline{\sigma}(\tau_e))$. Then $\mathrm{std}(\overline{r}_p)$ does not admit a potentially crystalline lift of type $\mathcal{T}(\eta,\tau_e)$ by Proposition 3.3.2 in *loc. cit.*. This contradicts the first paragraph of this proof.

Finally, suppose that λ is not 7-deep and $(\overline{r}|_{G_{F_v}})^{\mathrm{ss}}$ is 22-generic for each $v|p$. For $v|p$, $\mathrm{std}(\overline{r}|_{G_{F_v}})$ does not have a potentially crystalline lift of type $(\eta',\mathrm{std}(\tau(e,\lambda)))$ by [Enn19, Theorem 8] as explained in the proof of [LLL19, Corollary 4.2.4]. Thus, $\overline{r}|_{G_{F_v}}$ does not have a potentially crystalline lift of type $(\eta,\tau(e,\lambda))$. This contradicts the assumption on $F(\lambda)$. □

Granting item (2) in Theorem 5.4.2, we prove Lemma 4.4.4 and 4.5.3 (thus finishing the proofs of Theorem 4.4.3 and 4.5.4). We first prove the following partial converse of Proposition 5.1.3.

Proposition 5.4.5. *Suppose that a tame $\overline{\rho}:G_K\to \mathrm{GSp}_4(\mathbf{F})$ is 16-generic. Let $(\lambda+\eta,\tau)$ be a type with a $(h_\lambda+1)$-generic lowest alcove presentation of τ. If $\overline{\rho}$ has a λ-compatible lowest alcove presentation such that $\widetilde{w}(\overline{\rho},\tau)^*\in \mathrm{Adm}^\vee(\lambda+\eta)$, then $R_{\overline{\rho}}^{\lambda+\eta,\tau}\neq 0$.*

Proof. By Theorem 5.4.2(2), there exists a congruent family of potentially diagonalizable minimal fixed similitude patching functors $\{M_\infty^{\psi_p}\}$ for $\overline{\rho}$. Choose a 16-generic lowest alcove presentation of $\overline{\rho}$ λ-compatible with τ. For each $s\in \underline{W}$, there exists a tame inertial type τ_s with 13-generic lowest alcove presentation compatible with $\overline{\rho}$ and $\widetilde{w}(\overline{\rho},\tau_s)=s^{-1}(\eta)$. By Lemma 2.5.11, we have

$$\mathrm{JH}(\overline{\sigma}(\tau_s))\cap W^?(\overline{\rho}|_{I_K})=\{F_{\overline{\rho}}(s)\}.$$

Note that any element in $\mathrm{JH}(\overline{\sigma}(\tau_s))$ is 10-deep by Proposition 2.5.6.

On the other hand, $\overline{\rho}$ admits a potentially diagonalizable lift of type (η,τ_s) by Theorem 5.1.4. Then for $\psi_p\in\Phi(\overline{\rho})$ compatible with (η,τ_s), $M_\infty^{\psi_p}(\sigma^\circ(\tau_s))\neq 0$. By Theorem 5.4.4, $\mathrm{JH}(\overline{\sigma}(\tau_s))\cap W(\overline{r})$ is nonempty and contained in $W^?(\overline{\rho}|_{I_K})$. Therefore, $M_\infty^{\psi_p}(F_{\overline{\rho}}(s))\neq 0$ and $\{F_{\overline{\rho}}(s)\}_{s\in\underline{W}}=W_{\mathrm{obv}}(\overline{\rho})$. For $(\lambda+\eta,\tau)$ as in the statement, $\mathrm{JH}(\overline{\sigma}(\lambda,\tau))\cap W_{\mathrm{obv}}(\overline{\rho})\neq\emptyset$ by [Le+a, Proposition 2.6.6]. Thus $M_\infty^{\psi_p}(\overline{\sigma}^\circ(\lambda,\tau))\neq 0$ and $R_{\overline{\rho}}^{\lambda+\eta,\tau}\neq 0$. □

Proof of Lemma 4.4.4. Suppose that $\widetilde{z}\in \mathrm{Adm}^\vee(\lambda)$. There exists a tame $\overline{\rho}\in\mathcal{X}_{\mathrm{Sym}}(\mathbf{F})$ with a 16-generic lowest alcove presentation such that $\widetilde{w}(\overline{\rho},\tau)^*=\widetilde{z}$. By Proposition 5.4.5, $\overline{\rho}\in\mathcal{X}_{\mathrm{Sym}}^{\lambda,\tau}(\mathbf{F})$. Furthermore, $\overline{\rho}$ is in $\mathcal{X}_{\mathrm{Sym}}^{\lambda,\tau}(\widetilde{z})$ because $\varepsilon_\infty(\overline{\rho})$ is equal to the image of $\widetilde{z}$ in $\Phi\text{-}\mathrm{Mod}_K^{\text{ét},\mathrm{Sym}}$ under $\iota'_{\widetilde{z}}$. Conversely, if $\mathcal{X}_{\mathrm{Sym}}^{\lambda,\tau}(\widetilde{z})\neq\emptyset$, then $Y_{\mathrm{Sym}}^{\leq\lambda,\tau}(\widetilde{z})\neq\emptyset$. This implies $\widetilde{z}\in\mathrm{Adm}^\vee(\lambda)$ by Corollary 4.2.17. □

Proof of Lemma 4.5.3. Let $(\widetilde{w},\omega)$ be a lowest alcove presentation of σ λ-compatible with τ. We also set $\kappa=\pi^{-1}(\widetilde{w})\cdot(\omega-\eta)$ so that $\sigma=F(\kappa)$. Then σ is 16-deep, which implies ω is 13-deep. Let $\overline{\rho}:G_K\to \mathrm{GSp}_4(\mathbf{F})$ be maximally nonsplit of niveau 1 and weight σ. Let $\widetilde{w}=t_{\eta_w}w$. Note that η_w is dominant and 3-small. Then a direct computation shows that $\overline{\rho}^{\mathrm{ss}}|_{I_K}$ admits a 10-generic lowest alcove presentation $(\pi^{-1}(w)^{-1}w,\omega+\pi^{-1}(w)^{-1}(\eta_w))$. Similarly, a tame inertial L-parameter $\tau(e,\kappa)$ admits a 10-generic lowest alcove presentation $(\pi^{-1}(w)^{-1}w,\omega+\pi^{-1}(w)^{-1}(\eta_w-\eta))$. By Lemma 5.1.6, $\overline{\rho}$ admits a potentially diagonalizable lift of type $(\eta,\tau(e,\kappa))$. Since $\widetilde{w}(\overline{\rho}^{\mathrm{ss}},\tau(e,\kappa))=t_\eta$, we have $W^?(\overline{\rho}^{\mathrm{ss}})\cap\mathrm{JH}(\overline{\sigma}(\tau(e,\kappa)))=\{\sigma\}$ by Proposition 2.5.11. Note that any element in $\mathrm{JH}(\overline{\sigma}(\tau(e,\kappa)))$ is 7-deep by Proposition 2.5.6.

By Theorem 5.4.2(2), $M_\infty^{\psi_p}(\overline{\sigma}(\tau(e,\kappa)) \neq 0$ for the unique $\psi_p \in \Phi(\overline{\rho})$ compatible with $(\eta, \tau(e,\kappa))$, and by Theorem 5.4.4, $M_\infty^{\psi_p}(\overline{\sigma}(\tau(e,\kappa)))/\varpi = M_\infty^{\psi_p}(\sigma)$. Then for any type $(\lambda+\eta,\tau)$ such that $\sigma \in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))$ and the unique $\psi'_p \in \Phi(\overline{\rho})$ compatible with $(\lambda+\eta,\tau)$, $M_\infty^{\psi'_p}(\overline{\sigma}(\lambda,\tau)) \neq 0$. Since it is supported on $R_\infty^{\lambda+\eta,\tau,\psi'_p}$, $\overline{\rho} \in \mathcal{X}^{\lambda,\tau}_{\mathrm{Sym},\mathbf{F}}(\mathbf{F})$. Since such $\overline{\rho}$ are dense in $\mathcal{C}_\sigma$, this implies that $\mathcal{C}_\sigma \subset \mathcal{X}^{\lambda,\tau}_{\mathrm{Sym},\mathbf{F}}$. $\square$

5.4.6 Construction of fixed similitude patching functors

We first provide a general construction of a fixed similitude patching functor by extending the construction in [EL, §4.4], which is based on [Car+16] and [Box+21]. Then we explain how the construction applies to particular cases.

Let F and $\overline{r}$ be as in Theorem 5.4.2(1). Given $\psi_p \in \Psi(\overline{r}_p)$, there exists a continuous character $\psi : G_F \to \mathcal{O}^\times$ lifting $\mathrm{sim}(\overline{r})$ and $\psi|_{G_{F_v}} = \psi_v$ for all $v \in S_p$ ([EL, Lemma 4.4.3]). Let χ be a Hecke character such that $\chi_{p,\iota} = \psi\epsilon_p^3$. We assume that $(\overline{r},\chi)$ is automorphic of weight μ and sufficiently small level U.

Let S_p be the set of places of F dividing p and S be a finite set of finite places containing S_p. We define

$$q = h^1(F_S/F, \mathrm{ad}(\overline{r})(1)), \quad g = 2q - 4[F:\mathbf{Q}_p] + |S| - 1.$$

For $T \subset S$, we define $\mathcal{T}_T := \mathcal{O}[\![y_1,\ldots,y_{11|T|-1}]\!]$. We also define $S_\infty := \mathcal{T}_S[\![\mathbf{Z}_p^{2q}]\!]$. Viewing S_∞ as an augmented $\mathcal{O}$-algebra, we let $\mathfrak{a}_\infty$ denote the augmentation ideal of S_∞.

Suppose that S contains $S_{\overline{r}} \cup S_U$. Let R be a subset of S disjoint from S_p. We assume that for each $v \in R$, $U_v = \mathrm{Iw}_1(v)$, $q_v \equiv 1 \bmod p$, and $\overline{r}|_{G_{F_v}}$ is trivial. The presence of R is necessary for the "Ihara avoidance" argument and is not necessary for applications to the Breuil–Mézard conjecture or the weight part of Serre's conjecture. For each $v \in R$, we choose a pair of characters $\zeta_v = (\zeta_{v,1},\zeta_{v,2})$ as in §5.3 which induce a character $\zeta_R : \prod_{v\in R} T_{\mathrm{der}}(k_v) \to \mathcal{O}^\times$.

For each $v \in S$, let $\mathcal{D}_v \subset \mathcal{D}_v^{\square,\psi|_{G_{F_v}}}$ be a local deformation problem represented by $R_v \in \widehat{\mathcal{C}}_{\mathcal{O}}$. If $v \in R$, we take $\mathcal{D}_v = \mathcal{D}_v^{\zeta_v}$. We consider a global deformation problem

$$\mathcal{S} = (S, \{\mathcal{D}_v\}_{v\in S}, \psi).$$

If T is a subset of S, we write $\mathcal{D}_{\mathcal{S}}^T$ for the functor of T-framed deformations of type $\mathcal{S}$. Since $\overline{r}$ is absolutely irreducible, $\mathcal{D}_{\mathcal{S}}$ (resp. $\mathcal{D}_{\mathcal{S}}^T$) is represented by $R_{\mathcal{S}}$ (resp. $R_{\mathcal{S}}^T$) in $\widehat{\mathcal{C}}_{\mathcal{O}}$. The choice of a universal lift $r_{\mathcal{S}} : G_F \to \mathrm{GSp}_4(R_{\mathcal{S}})$ gives an isomorphism $R_{\mathcal{S}} \otimes_{\mathcal{O}} \mathcal{T}_T \simeq R_{\mathcal{S}}^T$. Let $R_{\mathcal{S}}^{T,\mathrm{loc}} := \widehat{\otimes}_{v\in T} R_v$. Then there exists a natural map $R_{\mathcal{S}}^{T,\mathrm{loc}} \to R_{\mathcal{S}}^T$.

Given a Taylor–Wiles datum $(Q, (\overline{\alpha}_{v,1},\ldots,\overline{\alpha}_{v,4})_{v\in Q})$ ([EL, Definition 4.4.6]), we define the augmented deformation problem

$$\mathcal{S}_Q = (S \cup Q, \{\mathcal{D}_v\}_{v\in S} \cup \{\mathcal{D}_v^{\square,\psi|_{G_{F_v}}}\}_{v\in Q}, \psi).$$

For each $v \in Q$, let $\Delta_v = k_v^\times(p)^2$ where $k_v^\times(p)$ is the maximal p-power quotient of $k_v^\times$. Set $\Delta_Q = \prod_{v\in Q}\Delta_v$. Let $\mathfrak{a}_Q$ denote the augmentation ideal of $\mathcal{O}[\Delta_Q]$. Then there is a canonical local ring homomorphism $\mathcal{O}[\Delta_Q] \to R^S_{\mathcal{S}_Q}$ such that $R^S_{\mathcal{S}_Q}/\mathfrak{a}_Q \simeq R^S_{\mathcal{S}}$. Since $\overline{r}$ is odd and vast, [Box+21, Corollary 7.6.3] shows that for each $n \geq 1$, there exists a Taylor–Wiles datum Q_n disjoint from S such that

- $|Q_n| = q$;
- $q_v \equiv 1 \bmod p^n$ for each $v \in Q_n$;

- there exists a local $\mathcal{O}$-algebra surjection

$$\varphi_n^{\psi_p} : R_{\mathcal{S}}^{S,\mathrm{loc}}[\![x_1, \ldots, x_g]\!] \twoheadrightarrow R_{\mathcal{S}_{Q_n}}^S$$

with $g := 2q - 4[F : \mathbf{Q}] + |S| - 1$.

We define open compact subgroups $U_1^p(Q_n)$ of $\mathcal{G}(\mathbb{A}_F^{\infty,p})$ by setting $U_1^p(Q_n)_v = U_v^p$ if $v \notin Q \cup S$ and $U_1^p(Q_n)_v = \mathrm{Iw}_1(v)$ if $v \in Q$. We define a $\mathcal{G}(\mathcal{O}_p)$-patching datum in the sense of [EL, Definition 4.3.5]

$$\big(S_\infty, R_\infty^{\psi_p}, (R_n^{\psi_p}, \varphi_n^{\psi_p}, \{M_r^{\psi_p}(H)_n\}_{r\geq 1, H\leq_{\mathrm{c.o.}}\mathcal{G}(\mathcal{O}_p)}, \alpha_n^{\psi_p})_{n\geq 1}, \{M_r^{\psi_p}(H)_0\}_{r\geq 1, H\leq_{\mathrm{c.o.}}\mathcal{G}(\mathcal{O}_p)}\big)$$

by setting

- $R_\infty^{\psi_p} := R_{\mathcal{S}}^{S,\mathrm{loc}}[\![X_1, \ldots, X_g]\!]$;
- $R_n^{\psi_p} := R_{\mathcal{S}_{Q_n}}^S$ with S_∞-algebra structure induced by $S_\infty \twoheadrightarrow \mathcal{O}[\Delta_{Q_n}] \otimes_{\mathcal{O}} \mathcal{T}_S \to R_{\mathcal{S}_{Q_n}}^S$;
- $M_r^{\psi_p}(H)_0 := [S_\chi(H \cdot U^p, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{\overline{r},\chi}^S, \zeta_R}]^\vee$ where the subscript ζ_R denotes that we take the ζ_R-coinvariants for the action of $\prod_{v\in R} T_{\mathrm{der}}(k(v))$; and
- for $n \geq 1$,

$$\begin{aligned} M_r^{\psi_p}(H)_n &:= [S_\chi(H \cdot U_1^p(Q_n), \mathcal{O}/\varpi^r)_{\mathfrak{m}_{\overline{r},\chi}^{S\cup Q_n}, \mathfrak{m}_{Q_n}, \zeta_R}]^\vee \otimes_{R_{\mathcal{S}_{Q_n}}} R_n^{\psi_p} \\ \alpha_n^{\psi_p} &: M_r^{\psi_p}(H)_n/\mathfrak{a}_\infty \simeq M_r^{\psi_p}(H)_0 \end{aligned}$$

where $\mathfrak{m}_{Q_n}$ is the kernel of $\otimes_{v\in Q_n} \mathcal{O}[T(F_v)/T(\mathcal{O}_v)_1] \to \mathbf{F}$ sending

$$T(\mathcal{O}_v)/T(\mathcal{O}_v)_1 \mapsto 1, \beta_0(\varpi_v) \mapsto \chi_v(\varpi_v), \beta_1(\varpi_v) \mapsto \overline{\alpha}_{v,1}, \text{and } \beta_2(\varpi_v) \mapsto \overline{\alpha}_{v,1}\overline{\alpha}_{v,2},$$

and the isomorphism $\alpha_n^{\psi_p}$ follows from [Box+21, §2.4.29].

Fix a nonprincipal ultrafilter $\mathcal{F} \subset 2^{\mathbf{N}}$. By [EL, Lemma 4.3.9],

$$M_\infty^{\psi_p} := \varprojlim_{r,H} \mathcal{U}_{\mathcal{F}}(\{M_r^{\psi_p}(H)_n \otimes_{S_\infty} S_\infty/\mathfrak{m}_{S_\infty}^r\}_{n\geq 1})$$

is a finitely generated projective $S_\infty[\![\mathrm{GSp}_4(\mathcal{O}_p)]\!]$-module with compatible $S_\infty[\mathrm{GSp}_4(F_p)]$-action and the action of S_∞ factors through the map $S_\infty \to R_\infty^{\psi_p}$ induced by $S_\infty \to R_n^{\psi_p}$. We also let $M_\infty^{\psi_p}$ denote the covariant functor

$$\begin{aligned} \mathrm{Rep}_{\mathcal{O}}^{\psi_p}(\mathrm{GSp}_4(\mathcal{O}_p)) &\to \mathrm{Mod}(R_\infty^{\psi_p}) \\ V &\mapsto \mathrm{Hom}_{\mathcal{O}[\mathrm{GSp}_4(\mathcal{O}_p)]}(M_\infty^{\psi_p}, V^\vee)^\vee. \end{aligned}$$

Proposition 5.4.7. *The triple $(\psi_p, R_\infty^{\psi_p}, M_\infty^{\psi_p})$ is a fixed similitude patching functor for $\overline{r}_p$.*

Proof. Since $M_\infty^{\psi_p}$ is projective over $\mathcal{O}[\![\mathrm{GSp}_4(\mathcal{O}_p)]\!]$, $M_\infty^{\psi_p}$ is an exact functor. Let $(\lambda + \eta, \tau)$ be a type $(\lambda + \eta, \tau)$ compatible with ψ_p and $\sigma \in \mathrm{JH}(\overline{\sigma}(\lambda, \tau))$. It follows from [BG19, Theorem A] and [Box+21, Proposition 7.4.7, 7.4.8] that $\dim S_\infty = \dim R_\infty^{\lambda+\eta,\tau,\psi_p}$. Then the maximal Cohen–Macaulayness of $M_\infty^{\psi_p}(\sigma^\circ(\lambda, \tau))$ over $R_\infty^{\lambda+\eta,\tau,\psi_p}$ follows from the usual commutative algebra argument (e.g. [Car+16, Lemma 4.18]). The maximal Cohen–Macaulayness of $M_\infty^{\psi_p}(\sigma)$ over $R_\infty^{\lambda+\eta,\tau,\psi_p}/\varpi$ follows similarly. □

Proof of Theorem 5.4.2. We first prove item (1). Let $\psi_p \in \Psi(\overline{r}_p)$. We can assume that the level U' is sufficiently small by shrinking it if necessary. We apply the above construction for given $\overline{r}$, ψ_p, $S = S_{\overline{r}} \cup S_U$, $R = \emptyset$, and $\mathcal{D}_v = \mathcal{D}_v^{\square,\psi|_{G_{F_v}}}$ for each $v \in S$. We can and do choose Taylor–Wiles datum Q_n for each $n \geq 1$ and the nonprincipal ultrafilter $\mathcal{F}$ independent of ψ_p. Then all objects in the $\mathcal{G}(\mathcal{O}_p)$-patching datum reduced modulo ϖ in the construction in §5.4.6 are independent of ψ_p. Thus the set $\{M_\infty^{\psi_p}\}_{\psi_p \in \Phi(\overline{r}_p)}$ is a congruent family of fixed similitude patching functors for $\overline{r}_p$ by [EL, Lemma 4.3.4].

Suppose that we are in the setup of item (2). When $\overline{\rho}$ is tame, we call it the tame case, and when $\overline{\rho}$ is maximally nonsplit of niveau 1 and weight σ, we call it the maximally nonsplit case. In both cases, we apply [EL, Lemma 4.4.4] and the existence of potentially diagonalizable lifts (Theorem 4.1.14) to obtain a totally real field F, a continuous representation $\overline{r} : G_F \to \mathrm{GSp}_4(\mathbf{F})$, and a Hecke character χ such that

- $F_v \simeq K$ for all $v|p$;
- $\overline{r}|_{G_{F_v}} \simeq \overline{\rho}$;
- $\overline{r}$ is unramified at all finite places not dividing p;
- $\overline{r}$ satisfies Taylor–Wiles conditions; and
- $(\overline{r}, \chi)$ is potentially diagonalizably automorphic of level unramified outside p.

Since $\overline{r}$ is vast and tidy, there exists a place $v_0 \notin S_p$ such that $q_{v_0} \not\equiv 1 \bmod p$, no two eigenvalues of $\overline{r}(\mathrm{Frob}_{v_0})$ has ratio q_{v_0}, and the residue characteristic of v_0 is > 5 ([Box+21, §7.7]). We take $U^p \leq \mathcal{G}(\mathbb{A}_F^{\infty,p})$ by setting $U_v = \mathcal{G}(\mathcal{O}_v)$ for $v \neq v_0$ and $U_{v_0} = \mathrm{Iw}_1(v_0)$.

Fix a place $w|p$ of F. Let $\psi_w \in \Psi(\overline{\rho})$. Then there exists a unique $\psi_p \in \Psi(\overline{r}_p)$ such that $\psi_v \simeq \psi_w$ for each $v \in S_p$ as a character of $G_{F_v} \simeq G_K$. We apply the construction in §5.4.6 by taking $S = S_p \cup \{v_0\}$, $R = \emptyset$, and $\mathcal{D}_v = \mathcal{D}_v^{\square,\psi_v}$ for each $v \in S$. Again by choosing Taylor–Wiles datum and the nonprincipal ultrafilter independent of ψ_p, $\{M_\infty^{\psi_p}\}_{\Psi(\overline{\rho})}$ is a congruent set of fixed similitude patching functors for $\overline{r}_p$.

Let a be an integer such that $\mathrm{HT}_j(\psi_w) = a$ for all $j \in \mathcal{J}$. There exists a unique integer k such that $a = 2k + b$ where $b \in \{3, 4\}$. For each a, we can and do choose a type (λ_a, τ_a) such that

$$\lambda_a = \begin{cases} k(1,1;2) & \text{if } b = 3 \\ (1,0;0) + k(1,1;2) & \text{if } b = 4 \end{cases}$$

and $\widetilde{w}(\overline{\rho}^{\mathrm{ss}}, \tau_a) = t_{\lambda_a+\eta}$ (using Lemma 5.1.6 in the maximally nonsplit case). We take the ring R^p in §5.2 to be

$$(\widehat{\otimes}_{v \in S_p \setminus \{w\}} R_{\overline{r}|_{G_{F_v}}}^{\lambda_a+\eta,\tau_a,\psi_v}) \widehat{\otimes}_{\mathcal{O}} R_{v_0} [\![x_1, \dots, x_g]\!].$$

Note that in the tame case, R^p is formally smooth over $\mathcal{O}$ by Theorem 5.1.4 and [Box+21, Proposition 7.4.2]. Set $R_\infty^{\psi_w} = R_{\overline{\rho}}^{\square,\psi_w} \widehat{\otimes}_{\mathcal{O}} R^p$. We define a functor

$$\begin{aligned} M_\infty^{\psi_w} : \mathrm{Rep}_{\mathcal{O}}(\mathrm{GSp}_4(\mathcal{O}_K)) &\to \mathrm{Mod}(R_\infty^{\psi_w}) \\ V &\mapsto \mathrm{Hom}_{\mathcal{O}[\mathrm{GSp}_4(\mathcal{O}_p)]}(M_\infty^{\psi_p}, ((\otimes_{v \in S_p \setminus \{w\}, \mathcal{O}} \sigma^\circ(\lambda_a, \tau_a)) \otimes_{\mathcal{O}} V)^\vee)^\vee \end{aligned}$$

and claim that $\{M_\infty^{\psi_w}\}_{\psi_w \in \Phi(\overline{\rho})}$ is a congruent family of potentially diagonalizable minimal fixed similitude patching functors for $\overline{\rho}$.

For a type $(\lambda+\eta,\tau)$ (resp. a Serre weight σ), the maximal Cohen–Macaulay property of $M_\infty^{\psi_w}(\sigma^\circ(\lambda,\tau))$ (resp. $M_\infty^{\psi_w}(\sigma)$) follows from Proposition 5.4.7.

By Lemma 2.5.11, we have

$$\mathrm{JH}(\overline{\sigma}(\lambda_a,\tau_a))\cap W^?(\overline{\rho}^{\mathrm{ss}}|_{I_K})=\{F_{\overline{\rho}}(e)\}.$$

For $V\in\mathrm{Rep}_{\mathbf{F}}^{\psi_w}(\mathrm{GSp}_4(\mathcal{O}_K))$, Theorem 5.4.4 implies

$$\begin{aligned}M_\infty^{\psi_w}(V)&=\mathrm{Hom}_{\mathcal{O}[\mathrm{GSp}_4(\mathcal{O}_p)]}(M_\infty^{\psi_p},(\otimes_{v\in S_p\backslash\{w\}}\overline{\sigma}(\lambda_0,\tau_0))^\vee\otimes_{\mathcal{O}}V^\vee)\\&=\mathrm{Hom}_{\mathcal{O}[\mathrm{GSp}_4(\mathcal{O}_p)]}(M_\infty^{\psi_p}/\varpi,(\otimes_{v\in S_p\backslash\{w\}}F_{\overline{\rho}}(e))^\vee\otimes_{\mathbf{F}}V^\vee).\end{aligned}$$

Since $\{M_\infty^{\psi_p}\}_{\Psi(\overline{\rho})}$ is a congruent set of fixed similitude patching functors, this shows that $\left\{M_\infty^{\psi_w}\right\}_{\psi_w\in\Psi(\overline{\rho})}$ is a congruent family of fixed similitude patching functors for $\overline{\rho}$. In the tame case, $R_\infty^{\lambda_a+\eta,\tau_a,\psi_w}[1/p]$ is regular and thus $M_\infty^{\psi_w}(\sigma^\circ(\lambda,\tau))[1/p]$ is locally free over its support. Moreover, it has rank one; this can be checked at finite level, which follows from the multiplicity one assertion in Theorem 2.4.1, the choice of v_0 and Hecke operators at v_0, and [Box+21, Proposition 2.4.3, 2.4.4]. This proves that $\left\{M_\infty^{\psi_w}\right\}_{\psi_w\in\Psi(\overline{\rho})}$ is minimal in the tame case.

Finally, we show that $M_\infty^{\psi_w}$ is potentially diagonalizable. Note that, for each $v\in S_p\backslash\{w\}$, there exists a potentially diagonalizable lift of $\overline{r}|_{G_{F_v}}$ of type $(\lambda_a+\eta,\tau_a)$. If ρ is a potentially diagonalizable lift of $\overline{\rho}$ of type $(\lambda+\eta,\tau)$ compatible with ψ_w, we can apply [PT21, Theorem 3.4] to find a potentially diagonalizable lift r of $\overline{r}$ with $\mathrm{sim}(r)=\psi$. By [EL, Lemma 4.4.4], r is automorphic, which implies that $M_\infty^{\psi_w}(\sigma^\circ(\lambda,\tau))$ is nonzero. ☐

5.4.8 Ihara avoidance patching functors

We construct congruent patching functors for "Ihara avoidance" argument. We apply this to the modularity lifting result in §6.3.

Let $\overline{r}:G_F\to\mathrm{GSp}_4(\mathbf{F})$ be a continuous representation and χ be a Hecke character. We assume that

1. $\overline{r}$ satisfies Taylor–Wiles conditions;
2. $\overline{r}$ is unramified at all finite places not dividing p; and
3. $(\overline{r},\chi)$ is automorphic.

We apply the construction in §5.4.6 to two different setups: we let S to be a finite set containing $S_p\cup\{v_0\}$ and take $R=S\backslash(S_p\cup\{v_0\})$. For each $v\in R$, we take $\zeta_v=(\zeta_{v,1},\zeta_{v,2})$ to be either a pair of trivial characters for all $v\in R$, which we call the trivial case, or a pair of continuous characters such that $\zeta_{v,1},\zeta_{v,2}\neq 1$ $\zeta_{v,1}\neq\zeta_{v,2}^{\pm1}$ for all $v\in R$, which we call the non-trivial case. In the trivial case (resp. the non-trivial case), we obtain the following data

1. a global deformation problem $\mathcal{S}_1$ (resp. $\mathcal{S}_\zeta$) and a ring $R_{\mathcal{S}_1}$ (resp. $R_{\mathcal{S}_\zeta}$) representing it;
2. a fixed similitude patching functor $(\psi_p,R_\infty^{1,\psi_p},M_\infty^{1,\psi_p})$ (resp. $(\psi_p,R_\infty^{\zeta,\psi_p},M_\infty^{\zeta,\psi_p})$) for $\overline{r}_p$;
3. a $\mathcal{O}[\mathcal{G}(\mathcal{O}_p)]$-module

$$M^{1,\psi_p}:=\varprojlim_{H\le\mathcal{G}(\mathcal{O}_p)}(S_\chi(H\cdot U^p(R),\mathcal{O})_{\mathfrak{m}^S_{\overline{r},\psi},1})^\vee\quad(\text{resp. } M^{\zeta,\psi_p}:=\varprojlim_{H\le\mathcal{G}(\mathcal{O}_p)}(S_\chi(H\cdot U^p(R),\mathcal{O})_{\mathfrak{m}^S_{\overline{r},\psi},\zeta_R})^\vee).$$

Proposition 5.4.9. *We continue using the notation above.*

1. *The fixed similitude patching functors* $(\psi_p, R_\infty^{1,\psi_p}, M_\infty^{1,\psi_p})$ *and* $(\psi_p, R_\infty^{\zeta,\psi_p}, M_\infty^{\zeta,\psi_p})$ *are congruent.*

2. *The following diagrams of* S_∞*-algebras commute*

$$\begin{tikzcd} R_\infty^{1,\psi_p} \arrow[r] \arrow[d, "\mathrm{mod}\ \mathfrak{a}_\infty"] & \mathrm{End}_{S_\infty}(M_\infty^{1,\psi_p}) \arrow[d, "\mathrm{mod}\ \mathfrak{a}_\infty"] \\ R_{\mathcal{S}_1} \arrow[r] & \mathrm{End}_{\mathcal{O}}(M^{1,\psi_p}) \end{tikzcd} \qquad \begin{tikzcd} R_\infty^{\zeta,\psi_p} \arrow[r] \arrow[d, "\mathrm{mod}\ \mathfrak{a}_\infty"] & \mathrm{End}_{S_\infty}(M_\infty^{\zeta,\psi_p}) \arrow[d, "\mathrm{mod}\ \mathfrak{a}_\infty"] \\ R_{\mathcal{S}_\zeta} \arrow[r] & \mathrm{End}_{\mathcal{O}}(M^{\zeta,\psi_p}) \end{tikzcd}$$

   where the top rows modulo $\mathfrak{a}_\infty$ *are isomorphic to the corresponding bottom rows.*

3. *The isomorphism* $M_\infty^{1,\psi_p}/\varpi \simeq M_\infty^{\zeta,\psi_p}/\varpi$ *reduced modulo* $\mathfrak{a}_\infty$ *is the natural isomorphism* $M^{1,\psi_p}/\varpi \simeq M^{\zeta,\psi_p}/\varpi$.

Proof. Since all objects in the $\mathcal{G}(\mathcal{O}_p)$-patching data in the trivial and non-trivial cases are congruent modulo ϖ, item (1) follows immediately. Item (2) and (3) follow from the construction. $\square$

Chapter 6

Applications

6.1 The Breuil–Mézard conjecture

We first state the *geometric* and *versal* Breuil–Mezard conjectures for GSp_4 following [Le+a, §8.1].

Recall that the algebraic stack $\mathcal{X}_{\mathrm{Sym,red}}$ is equidimensional of dimension $4f$ and its irreducible components are labeled by Serre weights of $\mathrm{GSp}_4(k)$. We write $\mathbf{Z}[\mathcal{X}_{\mathrm{Sym,red}}]$ for the free abelian group generated by irreducible components of $\mathcal{X}_{\mathrm{Sym,red}}$. We call elements in $\mathbf{Z}[\mathcal{X}_{\mathrm{Sym,red}}]$ *cycles* and $\mathcal{C}_\sigma \in \mathbf{Z}[\mathcal{X}_{\mathrm{Sym,red}}]$ for a Serre weight σ an *irreducible cycle*. A cycle is *effective* if it is a sum of irreducible cycles with non-negative coefficients.

For a type $(\lambda+\eta,\tau)$, we define a cycle $\mathcal{Z}_{\lambda,\tau} := \sum_\sigma \mu_\sigma(\mathcal{X}^{\lambda+\eta,\tau}_{\mathrm{Sym},\mathbf{F}})\mathcal{C}_\sigma \in \mathbf{Z}[\mathcal{X}_{\mathrm{Sym,red}}]$ where $\mu_\sigma(\mathcal{X}^{\lambda+\eta,\tau}_{\mathrm{Sym},\mathbf{F}})$ is the multiplicity of $\mathcal{C}_\sigma$ as an irreducible component of $\mathcal{X}^{\lambda+\eta,\tau}_{\mathrm{Sym},\mathbf{F}}$ in the sense of [Stacks, Tag 0DR4].

Conjecture 6.1.1 (Geometric Breuil–Mezard conjecture for GSp_4)**.** *Let $\mathcal{S}$ be a set of types. For each $\sigma \in \mathrm{JH}(\overline{\sigma}(\mathcal{S})) := \cup_{(\lambda+\eta,\tau)\in\mathcal{S}}\mathrm{JH}(\overline{\sigma}(\lambda,\tau))$, there exists an effective cycle $\mathcal{Z}_\sigma \in \mathbf{Z}[\mathcal{X}_{\mathrm{Sym,red}}]$ such that for all $(\lambda+\eta,\tau)\in\mathcal{S}$,*

$$\mathcal{Z}_{\lambda,\tau} = \sum_{\sigma\in\mathrm{JH}(\overline{\sigma}(\lambda,\tau))} [\overline{\sigma}(\lambda,\tau):\sigma]\mathcal{Z}_\sigma.$$

Let $\overline{\rho} \in \mathcal{X}_{\mathrm{Sym}}(\mathbf{F})$. We fix a versal ring $R^{\mathrm{ver}}_{\overline{\rho}}$ for $\mathcal{X}_{\mathrm{Sym}}$ at $\overline{\rho}$. For example, we can take $R^{\mathrm{ver}}_{\overline{\rho}} = R^{\square}_{\overline{\rho}}$ the framed deformation ring. For a type $(\lambda+\eta,\tau)$, we define

$$\begin{aligned}\mathrm{Spf}\, R^{\mathrm{ver},\lambda+\eta,\tau}_{\overline{\rho}} &:= \mathrm{Spf}\, R^{\mathrm{ver}}_{\overline{\rho}} \times_{\mathcal{X}_{\mathrm{Sym}}} \mathcal{X}^{\lambda+\eta,\tau}_{\mathrm{Sym}} \\ \mathrm{Spf}\, R^{\mathrm{alg}}_{\overline{\rho}} &:= \mathrm{Spf}\, R^{\mathrm{ver}}_{\overline{\rho}} \times_{\mathcal{X}_{\mathrm{Sym}}} \mathcal{X}_{\mathrm{Sym,red}}.\end{aligned}$$

Note that the versal map $\mathrm{Spf}\, R^{\mathrm{alg}}_{\overline{\rho}} \to \mathcal{X}_{\mathrm{Sym,red}}$ is effective and arises from a map $i_{\overline{\rho}} : \mathrm{Spec}\, R^{\mathrm{alg}}_{\overline{\rho}} \to \mathcal{X}_{\mathrm{Sym,red}}$. The map $i_{\overline{\rho}}$ induces a surjective map from the set of irreducible components of $\mathrm{Spec}\, R^{\mathrm{alg}}_{\overline{\rho}}$ to the set of irreducible components of $\mathcal{X}_{\mathrm{Sym,red}}$ containing $\overline{\rho}$ ([Stacks, Tag 0DRB]). Define $\mathbf{Z}[\mathrm{Spec}\, R^{\mathrm{alg}}_{\overline{\rho}}]$ as the free abelian group generated by irreducible components of $\mathrm{Spec}\, R^{\mathrm{alg}}_{\overline{\rho}}$. We interpret $\mathbf{Z}[\mathcal{X}_{\mathrm{Sym,red}}]$ and $\mathbf{Z}[\mathrm{Spec}\, R^{\mathrm{alg}}_{\overline{\rho}}]$ as spaces of functions on sets of irreducible components. Then we have a pullback

$$i^*_{\overline{\rho}} : \mathbf{Z}[\mathcal{X}_{\mathrm{Sym,red}}] \to \mathbf{Z}[\mathrm{Spec}\, R^{\mathrm{alg}}_{\overline{\rho}}].$$

We define $\mathcal{Z}_{\lambda,\tau}(\overline{\rho}) := i^*_{\overline{\rho}}(\mathcal{Z}_{\lambda,\tau})$. The cycle $\mathcal{Z}_{\lambda,\tau}(\overline{\rho})$ is equal to the cycle corresponding to $\operatorname{Spec} R^{\mathrm{ver},\lambda+\eta,\tau}_{\overline{\rho}}/\varpi$ ([Stacks, Tag 0DRD]).

Conjecture 6.1.2 (Versal Breuil–Mezard conjecture for GSp_4)**.** *Let $\mathcal{S}$ be a set of types. For each $\sigma \in \mathrm{JH}(\overline{\sigma}(\mathcal{S}))$, there exists an effective cycle $\mathcal{Z}_\sigma(\overline{\rho}) \in \mathbf{Z}[\operatorname{Spec} R^{\mathrm{alg}}_{\overline{\rho}}]$ such that for all $(\lambda+\eta,\tau) \in \mathcal{S}$,*

$$\mathcal{Z}_{\lambda,\tau}(\overline{\rho}) = \sum_{\sigma \in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))} [\overline{\sigma}(\lambda,\tau):\sigma]\mathcal{Z}_\sigma(\overline{\rho}).$$

Let $\Lambda \subset X_*(\underline{T}^\vee)$ be a finite set of dominant cocharacters containing 0. We define $\mathcal{S}_\Lambda$ as the set of types $(\lambda'+\eta,\tau)$ where $\lambda' \leq \lambda$ and τ is $P_{\lambda+\eta,e}P_{22+h_{\lambda+\eta}}$-generic for some $\lambda \in \Lambda$. The following is the main result of this section.

Theorem 6.1.3. *Let $\overline{\rho}$ be a 22-generic semisimple L-homomorphism. For each Serre weight σ, there exists an effective cycle $\mathcal{Z}_\sigma(\overline{\rho}) \in \mathbf{Z}[\operatorname{Spec} R^{\mathrm{alg}}_{\overline{\rho}}]$ such that*

$$\mathcal{Z}_{\lambda,\tau}(\overline{\rho}) = \sum_{\sigma \in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))} [\overline{\sigma}(\lambda,\tau):\sigma]\mathcal{Z}_\sigma(\overline{\rho}).$$

for all $(\lambda+\eta,\tau) \in \mathcal{S}_\Lambda$.

Before we prove the Theorem, we discuss about cycles in the fixed similitude deformation ring. Suppose that C is an irreducible component in $\operatorname{Spec} R^{\lambda+\eta,\tau,\psi_p}_{\overline{\rho}}/\varpi$ for a type $(\lambda+\eta,\tau)$ and $\psi_p \in \Phi(\overline{\rho})$. By Lemma 5.1.2, unramified twists of C give an irreducible component $\widetilde{C} \subset \operatorname{Spec} R^{\lambda+\eta,\tau}_{\overline{\rho}}/\varpi$. Its image in $\operatorname{Spec} R^{\square}_{\overline{\rho}}/\varpi$ is independent of the choice of $(\lambda+\eta,\tau)$ and ψ_p.

Let $\mathbf{Z}[R^{\text{p-crys}}_{\overline{\rho}}]$ be a free abelian group generated by $4[K:\mathbf{Q}_p]+10$-dimensional cycles in $\operatorname{Spec} R^{\square}_{\overline{\rho}}/\varpi$ supported in the union of $\operatorname{Spec} R^{\lambda+\eta,\tau,\psi_p}_{\overline{\rho}}/\varpi$ for all choices of a dominant character $\lambda \in X^*(\underline{T})$, a (possibly trivial) tame inertial L-parameters τ, and the unique character $\psi_p \in \Psi(\overline{\rho})$ compatible with (λ,τ). By the previous paragraph, there is a natural injective group homomorphism

$$\widetilde{(\cdot)} : \mathbf{Z}[R^{\text{p-crys}}_{\overline{\rho}}] \hookrightarrow \mathbf{Z}[\operatorname{Spec} R^{\mathrm{alg}}_{\overline{\rho}}] \tag{6.1.4}$$

such that $\widetilde{Z}(R^{\lambda+\eta,\tau,\psi_p}_{\overline{\rho}}/\varpi) = \mathcal{Z}_{\lambda,\tau}(\overline{\rho})$ for all λ, τ, and ψ_p as above.

Let R be a Noetherian $\mathbf{F}$-algebra. Given a R-module M, we let $Z(M)$ denote the cycle

$$\sum_{\mathcal{C}} \mu_{\mathcal{C}}(M)\mathcal{C} \in \mathbf{Z}[\operatorname{Spec} R]$$

where $\mathcal{C}$ ranges over irreducible components of $\operatorname{Spec} R$, and $\mu_{\mathcal{C}}(M)$ denotes the length of the module $M_{\mathfrak{p}_{\mathcal{C}}}$ over $R_{\mathfrak{p}_{\mathcal{C}}}$ for the prime ideal $\mathfrak{p}_{\mathcal{C}}$ corresponding to $\mathcal{C}$.

Proof of Theorem 6.1.3. By Theorem 5.4.2, there is a congruent family of minimal fixed similitude patching functors $\{M^{\psi_p}_\infty\}_{\psi_p \in \Psi(\overline{\rho})}$ for $\overline{\rho}$. For a Serre weight σ, we define $\mathcal{Z}_\sigma(\overline{\rho}) := \widetilde{Z}(M^{\psi_p}_\infty(\sigma))$ for some choice of $\psi_p \in \Phi(\overline{\rho})$. Note that this is independent of the choice of ψ_p. For any $(\lambda+\eta,\tau) \in \mathcal{S}_\Lambda$, $R^{\lambda+\eta,\tau}_{\overline{\rho}}$ is domain by Theorem 4.4.3. Let $\psi_p \in \Phi(\overline{\rho})$ be the unique character compatible with $(\lambda+\eta,\tau)$. By [EG14, Lemma 2.2.7, 2.2.10] and the definition of (6.1.4), we have $\widetilde{Z}(M^{\psi_p}_\infty(\overline{\sigma}(\lambda,\tau))) = Z_{\lambda,\tau}(\overline{\rho})$ and

$$\widetilde{Z}(M^{\psi_p}_\infty(\overline{\sigma}(\lambda,\tau))) = \sum_{\sigma \in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))} [\overline{\sigma}(\lambda,\tau):\sigma]\widetilde{Z}(M^{\psi_p}_\infty(\sigma)). \qquad \square$$

We explain how one can interpolate Theorem 6.1.3 to prove a version of geometric Breuil–Mezard conjecture. We follow the axiomatic argument of [Le+a, §8.3]. Let $\mathcal{P}_{\mathrm{ss}}$ be the set of $\overline{\rho} \in \mathcal{X}_{\mathrm{Sym}}(\mathbf{F})$ such that $\overline{\rho}|_{I_K}$ is 22-generic tame inertial type for K over $\mathbf{F}$. Note that Theorem 6.1.3 holds for $\overline{\rho} \in \mathcal{P}_{\mathrm{ss}}$ and $(\lambda+\eta,\tau) \in \mathcal{S}_\Lambda$. We define a set $\widehat{\mathcal{S}}_\Lambda$ to be a union of $\mathcal{S}_\Lambda$ and a set of types of the form (η,τ) where τ is 6-generic. We also define a set $\widehat{\mathcal{S}}_{\Lambda,\mathrm{elim}}$ to be the union of $\widehat{\mathcal{S}}_\Lambda$ and the set of types (η,τ).

Definition 6.1.5. Let $\overline{\rho} \in \mathcal{P}_{\mathrm{ss}}$. We say that a Serre weight σ is $(\overline{\rho}, \widehat{\mathcal{S}}_{\Lambda,\mathrm{elim}})$*-irrelevant* if there exists a type $(\lambda+\eta,\tau) \in \widehat{\mathcal{S}}_{\Lambda,\mathrm{elim}}$ such that $\sigma \in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))$ and $R_{\overline{\rho}}^{\lambda+\eta,\tau} = 0$.

Lemma 6.1.6. *Any $\sigma \notin W^?(\overline{\rho}|_{I_K})$ is $(\overline{\rho}, \widehat{\mathcal{S}}_{\Lambda,\mathrm{elim}})$-irrelevant.*

Proof. Note that any Serre weight in $W^?(\overline{\rho}|_{I_K})$ is 19-deep by Proposition 2.5.8. If $\sigma \simeq F(\lambda)$ and λ is not 12-deep, the third and the last paragraphs of the proof of Theorem 5.4.4 shows that σ is $(\overline{\rho}, \widehat{\mathcal{S}}_{\Lambda,\mathrm{elim}})$-irrelevant. Suppose that $\sigma \simeq F(\lambda)$ is 12-deep and *not* $(\overline{\rho}, \widehat{\mathcal{S}}_{\Lambda,\mathrm{elim}})$-irrelevant, i.e. $R_{\overline{\rho}}^{\lambda+\eta,\tau} \neq 0$ for $(\lambda+\eta,\tau) \in \widehat{\mathcal{S}}_{\Lambda,\mathrm{elim}}$ such that $\sigma \in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))$. For each $s \in \underline{W}$, let $\tau_s = \tau(s, \widetilde{w}_h \cdot \lambda + \eta)$ be a 9-generic tame inertial L-parameter. Since $F(\lambda) \in \mathrm{JH}(\overline{\sigma}(\tau_s))$, we have $R_\infty^{\eta,\tau_s} \neq 0$. Then Proposition 5.1.3 and Lemma 2.5.12 imply that $F(\lambda) \in W^?(\overline{r}_p^{\mathrm{ss}})$ (as explained in the first paragraph of the proof of Theorem 5.4.4). ☐

Lemma 6.1.7.

1. *Suppose $\mathcal{C}_\sigma \subset \mathcal{X}_{\mathrm{Sym},\mathbf{F}}^{\lambda+\eta,\tau}$ for some $(\lambda+\eta,\tau) \in \mathcal{S}_\Lambda$. Then there exists $\overline{\rho} \in \mathcal{P}_{\mathrm{ss}}$ such that $\overline{\rho} \in \mathcal{C}_\sigma$.*

2. *Let $\mathbf{Z}[\mathcal{C}(\mathcal{S}_\Lambda)]$ be the $\mathbf{Z}$-span of all irreducible components in $\mathcal{X}_{\mathrm{Sym},\mathbf{F}}^{\lambda+\eta,\tau}$ for some $(\lambda+\eta,\tau) \in \mathcal{S}_\Lambda$. Then the map*

$$i^*_{\mathcal{P}_{\mathrm{ss}},\mathcal{S}_\Lambda} := \prod_{\overline{\rho}\in\mathcal{P}_{\mathrm{ss}}} i^*_{\overline{\rho}}|_{\mathbf{Z}[\mathcal{C}(\mathcal{S}_\Lambda)]} : \mathbf{Z}[\mathcal{C}(\mathcal{S}_\Lambda)] \to \prod_{\overline{\rho}\in\mathcal{P}_{\mathrm{ss}}} \mathbf{Z}[\mathrm{Spec}\, R_{\overline{\rho}}^{\mathrm{alg}}]$$

is injective. Moreover, $Z \in \mathbf{Z}[\mathcal{C}(\mathcal{S}_\Lambda)]$ is effective if and only if $i^*_{\overline{\rho}}(Z)$ is effective for all $\overline{\rho} \in \mathcal{P}_{\mathrm{ss}}$.

Proof.

1. By Theorem 4.5.2, $\mathcal{C}_\sigma \subset \mathcal{X}_{\mathrm{Sym},\mathbf{F}}^{\lambda+\eta,\tau}$ for some $(\lambda+\eta,\tau) \in \mathcal{S}_\Lambda$ implies that $\sigma \in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))$. By Proposition 4.5.6, it suffices to find $\overline{\rho} \in \mathcal{P}_{\mathrm{ss}}$ such that $\sigma \in W_{\mathrm{obv}}(\overline{\rho}|_{I_K})$. This can be proven as [Le+a, Lemma 8.4.9].

2. This easily follows from (1) (see Lemma 8.2.2 in *loc. cit.*).

☐

Lemma 6.1.8. *For any Serre weight σ, there exists integers $n^\sigma_{\lambda,\tau}$ such that*

$$[\sigma] - \sum_{(\lambda+\eta,\tau)\in\widehat{\mathcal{S}}_\Lambda} n^\sigma_{\lambda,\tau}[\overline{\sigma}(\lambda,\tau)]$$

is supported only at $(\mathcal{P}_{\mathrm{ss}}, \widehat{\mathcal{S}}_{\Lambda,\mathrm{elim}})$-irrelevant Serre weights. In other words, $\widehat{\mathcal{S}}_\Lambda$ is $(\mathcal{P}_{\mathrm{ss}}, \widehat{\mathcal{S}}_{\Lambda,\mathrm{elim}})$-Breuil–Mézard system in the sense of [Le+a, Definition 8.3.3].

Proof. This can be proven as [Le+a, Lemma 8.4.4] using Lemma 6.1.6. ☐

Lemma 6.1.7 shows that for each σ as above, if there exists $\mathcal{Z}_\sigma \in \mathbf{Z}[\mathcal{C}(\mathcal{S}_\Lambda)]$ such that $i^*_{\overline{\rho}}(\mathcal{Z}_\sigma) = \mathcal{Z}_\sigma(\overline{\rho})$ for all $\overline{\rho} \in \mathcal{P}_{\mathrm{ss}}$, then Conjecture 6.1.1 for $\mathcal{S} = \mathcal{S}_\Lambda$ follows from Theorem 6.1.3. Due to Lemma 6.1.8, one may hope to define $\mathcal{Z}_\sigma$ by

$$\sum_{(\lambda+\eta,\tau)\in\widehat{\mathcal{S}}_\Lambda} n^\sigma_{\lambda,\tau}\mathcal{Z}_{\lambda,\tau}.$$

However, the condition $i^*_{\overline{\rho}}(\mathcal{Z}_\sigma)=\mathcal{Z}_\sigma(\overline{\rho})$ may not hold in this case. This is because Theorem 6.1.3 does not apply to $(\lambda+\eta,\tau)\in\widehat{\mathcal{S}}_\Lambda\backslash\mathcal{S}_\Lambda$. This motivates the following definitions.

Definition 6.1.9 (cf. Definition 8.3.3 in [Le+a])**.** Let $\sigma\in\mathrm{JH}(\overline{\sigma}(\widehat{\mathcal{S}}_\Lambda))$ be a Serre weight.

1. We say that σ $\widehat{\mathcal{S}}_\Lambda$-*covers* σ' if for $(\lambda+\eta,\tau)\in\widehat{\mathcal{S}}_\Lambda$ such that $\sigma\in\mathrm{JH}(\overline{\sigma}(\lambda,\tau))$, $\mathcal{C}_{\sigma'}$ lies in $\mathcal{X}^{\lambda+\eta,\tau}_{\mathrm{Sym},\mathbf{F}}$.
2. We say that σ is $(\mathcal{S}_\Lambda,\widehat{\mathcal{S}}_\Lambda)$-*generic* if for all Serre weights σ' such that σ $\widehat{\mathcal{S}}_\Lambda$-covers σ', $\mathcal{C}_{\sigma'}$ does not lie in $\mathcal{X}^{\lambda+\eta,\tau}_{\mathrm{Sym},\mathbf{F}}$ for any $(\lambda+\eta,\tau)\in\widehat{\mathcal{S}}_\Lambda\backslash\mathcal{S}_\Lambda$.

Lemma 6.1.10. *Suppose σ and σ' are 9-deep Serre weights, and σ $\widehat{\mathcal{S}}_\Lambda$-covers σ'. Then $\sigma'\uparrow\sigma$.*

Proof. Let τ be a 6-generic tame inertial L-parameter such that $\sigma\in\mathrm{JH}(\overline{\sigma}(\tau)$. Since σ $\widehat{\mathcal{S}}_\Lambda$-covers σ', $\mathcal{C}_{\sigma'}\subset\mathcal{X}^{\eta,\tau}_{\mathrm{Sym},\mathbf{F}}$. By Theorem 4.5.2, we have $\sigma'\in\mathrm{JH}(\overline{\sigma}(\tau))$. In particular, σ covers σ' in the sense of [Le+a, Definition 2.3.10], i.e.

$$\sigma'\in\bigcap_{\substack{\tau\text{ 6-generic}\\ \sigma\in\mathrm{JH}(\overline{\sigma}(\tau))}}\mathrm{JH}(\overline{\sigma}(\tau)).$$

Then our claim follows from the equivalence between item (1) and (4) in Proposition 2.3.12 in *loc. cit.* using 9-deepness of σ and σ'. (Note that in our setup, if $\widetilde{w}\in\widetilde{W}^+$ and $\widetilde{w}_1\uparrow\widetilde{w}$ for some $\widetilde{w}\in\widetilde{W}^+_1$, then $\widetilde{w}_1\in\widetilde{W}^+_1$. Thus $L(\pi^{-1}(\widetilde{w}_1)\cdot(\omega-\eta))|_{\mathrm{G}}$ in item (4) in *loc. cit.* is equal to $F(\pi^{-1}(\widetilde{w}_1)\cdot(\omega-\eta))=F_{(\widetilde{w}_1,\omega)}$ and $F_{(\widetilde{w}_1,\omega)}\uparrow F_{(\widetilde{w},\omega)}$.) ☐

Let $\mathcal{S}_{\mathcal{P}_{\mathrm{ss}},\Lambda}\subset\mathcal{S}_\Lambda$ be the subset consisting of $(\lambda+\eta,\tau)$ such that all $\sigma\in\mathrm{JH}(\overline{\sigma}(\lambda,\tau))$ are $(\mathcal{S}_\Lambda,\widehat{\mathcal{S}}_\Lambda)$-generic.

For $\sigma\in\mathrm{JH}(\overline{\sigma}(\widehat{\mathcal{S}}_\Lambda))$, we define $\mathrm{tr}_{\sigma,\widehat{\mathcal{S}}_\Lambda}$ an idempotent endomorphism of $\mathbf{Z}[\mathcal{X}_{\mathrm{Sym,red}}]$ which maps $\mathcal{C}_{\sigma'}$ to itself if σ $\widehat{\mathcal{S}}_\Lambda$-covers σ' and to 0 otherwise. The following Lemma shows that $\mathrm{tr}_{\sigma,\widehat{\mathcal{S}}_\Lambda}$ eliminates the contribution of types that Theorem 6.1.3 does not apply to.

Lemma 6.1.11. *If σ is $(\mathcal{S}_\Lambda,\widehat{\mathcal{S}}_\Lambda)$-generic, we have $\mathrm{tr}_{\sigma,\widehat{\mathcal{S}}_\Lambda}(\mathcal{Z}_{\lambda,\tau})=0$ for any $(\lambda+\eta,\tau)\in\widehat{\mathcal{S}}_\Lambda\backslash\mathcal{S}_\Lambda$.*

Proof. This follows from the definition of $\mathrm{tr}_{\sigma,\widehat{\mathcal{S}}_\Lambda}$. ☐

For $\overline{\rho}\in\mathcal{P}_{\mathrm{ss}}$ and $(\mathcal{S}_\Lambda,\widehat{\mathcal{S}}_\Lambda)$-generic σ, there is an unique idempotent endomorphism $\mathrm{tr}_{\sigma,\widehat{\mathcal{S}}_\Lambda}(\overline{\rho})$ of $\mathbf{Z}[\mathrm{Spec}\,R^{\mathrm{alg}}_{\overline{\rho}}]$ satisfying the following condition (see [Stacks, Tag 0DRB, Tag 0DRD])

$$i^*_{\overline{\rho}}\circ\mathrm{tr}_{\sigma,\widehat{\mathcal{S}}_\Lambda}=\mathrm{tr}_{\sigma,\widehat{\mathcal{S}}_\Lambda}(\overline{\rho})\circ i^*_{\overline{\rho}}. \tag{6.1.12}$$

Then we have the following equality (see, [Le+a, Lemma 8.3.7])

$$\mathrm{tr}_{\sigma,\widehat{\mathcal{S}}_\Lambda}(\overline{\rho})(\mathcal{Z}_\sigma(\overline{\rho}))=\mathcal{Z}_\sigma(\overline{\rho}). \tag{6.1.13}$$

Theorem 6.1.14. *For each $(\mathcal{S}_\Lambda, \widehat{\mathcal{S}}_\Lambda)$-generic σ in $\mathrm{JH}(\overline{\sigma}(\mathcal{S}_{\mathcal{P}_{\mathrm{ss}},\Lambda}))$, there exists a unique effective cycle $\mathcal{Z}_\sigma$ in $\mathbf{Z}[\mathcal{X}_{\mathrm{Sym,red}}]$ with the support contained in $\{\mathcal{C}_\kappa | \kappa \uparrow \sigma\}$ and for each $\overline{\rho} \in \mathcal{P}_{\mathrm{ss}}$, $i^*_{\overline{\rho}}(\mathcal{Z}_\sigma) = \mathcal{Z}_\sigma(\overline{\rho})$. Moreover, for $(\lambda+\eta, \tau) \in \mathcal{S}_{\mathcal{P}_{\mathrm{ss}},\Lambda}$, we have*

$$\mathcal{Z}_{\lambda,\tau} = \sum_{\sigma \in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))} [\overline{\sigma}(\lambda,\tau) : \sigma] \mathcal{Z}_\sigma.$$

In particular, Conjecture 6.1.1 holds for $\mathcal{S} = \mathcal{S}_{\mathcal{P}_{\mathrm{ss}},\Lambda}$.

Proof. We define

$$\mathcal{Z}_\sigma := \mathrm{tr}_{\sigma,\widehat{\mathcal{S}}_\Lambda}\left(\sum_{(\lambda+\eta,\tau)\in\widehat{\mathcal{S}}_\Lambda} n^\sigma_{\lambda,\tau}\mathcal{Z}_{\lambda,\tau}\right).$$

For each $\overline{\rho} \in \mathcal{P}_{\mathrm{ss}}$, choose a fixed similitude patching functor $(\psi_{\overline{\rho}}, R_\infty^{\psi_{\overline{\rho}}}, M_\infty^{\psi_{\overline{\rho}}})$. We have

$$\begin{aligned}
\mathcal{Z}_\sigma(\overline{\rho}) &= \widetilde{Z}(M_\infty^{\psi_{\overline{\rho}}}(\sigma)) \\
&= \sum_{(\lambda+\eta,\tau)\in\widehat{\mathcal{S}}_\Lambda} n^\sigma_{\lambda,\tau}\widetilde{Z}(M_\infty^{\psi_{\overline{\rho}}}(\overline{\sigma}(\lambda,\tau))) \\
&= \sum_{(\lambda+\eta,\tau)\in\widehat{\mathcal{S}}_\Lambda} n^\sigma_{\lambda,\tau}\,\mathrm{tr}_{\sigma,\widehat{\mathcal{S}}_\Lambda}(\overline{\rho})(\widetilde{Z}(M_\infty^{\psi_{\overline{\rho}}}(\overline{\sigma}(\lambda,\tau)))) \\
&= \sum_{(\lambda+\eta,\tau)\in\mathcal{S}_\Lambda} n^\sigma_{\lambda,\tau}\,\mathrm{tr}_{\sigma,\widehat{\mathcal{S}}_\Lambda}(\overline{\rho})(\widetilde{Z}(M_\infty^{\psi_{\overline{\rho}}}(\overline{\sigma}(\lambda,\tau)))) \\
&= \sum_{(\lambda+\eta,\tau)\in\mathcal{S}_\Lambda} n^\sigma_{\lambda,\tau}\,\mathrm{tr}_{\sigma,\widehat{\mathcal{S}}_\Lambda}(\overline{\rho})(\mathcal{Z}_{\lambda,\tau}(\overline{\rho})) \\
&= i^*_{\overline{\rho}}(\mathcal{Z}_\sigma)
\end{aligned}$$

where the first equality is by definition, second equality follows from Lemma 6.1.8 and 6.1.6, the third equality follows from (6.1.13), the fourth equality follows from Lemma 6.1.11, the fifth equality follows from the proof of Theorem 6.1.3, and the final equality follows from the definition of $\mathcal{Z}_\sigma$ and (6.1.12). Now the uniquiness and effectivity of $\mathcal{Z}_\sigma$, as well as the claimed equality, follow from Lemma 6.1.7. $\square$

Remark 6.1.15. If there is a Breuil–Mezard system $\widehat{\mathcal{S}}'$ containing $\widehat{\mathcal{S}}_\Lambda$ such that $\widehat{\mathcal{S}}_\Lambda$ is a $(\mathcal{P}^{\mathrm{ss}}, \widehat{\mathcal{S}}')$-Breuil–Mezard system and Conjecture 6.1.1 holds for $\widehat{\mathcal{S}}'$, then the cycles constructed in Theorem 6.1.14 conincide with those in Conjecture 6.1.1 for $\widehat{\mathcal{S}}'$ (cf. [Le+a, Theorem 8.3.5(4)]).

Although the set $\mathcal{S}_{\mathcal{P}_{\mathrm{ss}},\Lambda}$ is not characterized by genericity conditions, there is a smaller subset that is characterized by genericity conditions. Given a polynomial $f \in \mathbf{Z}[X_1, X_2, X_3]$ and $\omega \in X^*(T) \simeq \mathbf{Z}^3$, define

$$f^\omega(X_1, X_2, X_3) := \prod_{\nu=(\nu_1,\nu_2;\nu_3)\in\mathrm{Conv}(\omega)} f(X_1-\nu_1, X_2-\nu_2, X_3-\nu_3)$$

We also define

$$P_{\mathcal{P}_{\mathrm{ss}},\Lambda,e} = \prod_{\lambda\in\Lambda}(P_{\lambda+\eta,e}\prod_{j\in\mathcal{J}}\widetilde{P}_{\eta,e}^{(\lambda+\eta-w_0(\eta))_j}).$$

The following Lemma is a straightforward generalization of [Le+a, Lemma 8.4.11, 8.5.1].

Lemma 6.1.16. *1. If τ is a tame inertial type for K with lowest alcove presentation $(s, \mu - \eta)$ with $P_{\mathcal{P}_{\mathrm{ss}},\Lambda,e}$-generic μ, then $(\lambda + \eta, \tau) \in \mathcal{S}_{\mathcal{P}_{\mathrm{ss}},\Lambda}$ for any $\lambda \in \Lambda$.*

2. If $\overline{\rho}$ is a tame inertial type for K over $\mathbf{F}$ with a lowest alcove presentation $(s, \mu - \eta)$ with $P_{\mathcal{P}_{\mathrm{ss}},\Lambda,e}$-generic μ, then for any tame inertial τ type for K with a lowest alcove presentation such that $\widetilde{w}(\overline{\rho}, \tau) \in \mathrm{Adm}^{\vee}(\lambda + \eta)$, then $(\lambda + \eta, \tau) \in \mathcal{S}_{\mathcal{P}_{\mathrm{ss}},\Lambda}$ for any $\lambda \in \Lambda$.

Corollary 6.1.17. *For each Serre weight σ, there exists an effective cycle $\mathcal{Z}_{\sigma}$ in $\mathbf{Z}[\mathcal{X}_{\mathrm{Sym,red}}]$ with the support contained in $\{\mathcal{C}_{\kappa} | \kappa \uparrow \sigma\}$ such that for any $\lambda \in \Lambda$ and a tame inertial type τ for K with a lowest alcove presentation (s, μ_{η}) with $P_{\mathcal{P}_{\mathrm{ss}},\Lambda,e}$-generic μ,*

$$\mathcal{Z}_{\lambda,\tau} = \sum_{\sigma \in \mathrm{JH}(\overline{\sigma}(\lambda,\tau))} [\overline{\sigma}(\lambda, \tau) : \sigma] \mathcal{Z}_{\sigma}.$$

Proof. By Lemma 6.1.16, for any λ and τ as above, all $\sigma \in \mathrm{JH}(\overline{\sigma}(\lambda, \tau))$ is $(\mathcal{S}_{\Lambda}, \widehat{\mathcal{S}}_{\Lambda})$-generic. Then the claim follows from Theorem 6.1.14. $\square$

Finally, we can use Corollary 6.1.17 to prove versal Breuil–Mézard conjectures for (not necessarily tame) $\overline{\rho}$ with polynomial genericity.

Corollary 6.1.18. *Let $\overline{\rho} : G_K \to \mathrm{GSp}_4(\mathbf{F})$ be a continuous representation. Suppose that $\overline{\rho}^{\mathrm{ss}}|_{I_K}$ has a lowest alcove presentation $(s, \mu - \eta)$ with $P_{\mathcal{P}_{\mathrm{ss}},\Lambda,e} P_{22}$-generic μ. Then for each Serre weight σ, there exists a cycle $\mathcal{Z}_{\sigma}(\overline{\rho}) \in \mathbf{Z}[\mathrm{Spec}\, R^{\mathrm{alg}}_{\overline{\rho}}]$ such that for any $\lambda \in \Lambda$ and any tame inertial type τ for K,*

$$Z(R^{\lambda+\eta,\tau}_{\overline{\rho}} / \varpi) = \sum_{\sigma} [\overline{\sigma}(\lambda, \tau) : \sigma] \mathcal{Z}_{\sigma}(\overline{\rho}).$$

Proof. If σ is $(\mathcal{S}_{\Lambda}, \widehat{\mathcal{S}}_{\Lambda})$-generic, we define $\mathcal{Z}_{\sigma}(\overline{\rho}) := i^*_{\overline{\rho}}(\mathcal{Z}_{\sigma})$ with $\mathcal{Z}_{\sigma}$ as in Theorem 6.1.14. Otherwise, we define $\mathcal{Z}_{\sigma}(\overline{\rho}) = 0$. If $(\lambda + \eta, \tau) \in \mathcal{S}_{\mathcal{P}_{\mathrm{ss}},\Lambda}$, then all $\sigma \in \mathrm{JH}(\overline{\sigma}(\lambda, \tau))$ are $(\mathcal{S}_{\Lambda}, \widehat{\mathcal{S}}_{\Lambda})$-generic, and the claim follows from Theorem 6.1.14.

If $(\lambda + \eta, \tau) \notin \mathcal{S}_{\mathcal{P}_{\mathrm{ss}},\Lambda}$, we have $\widetilde{w}(\overline{\rho}, \tau) \notin \mathrm{Adm}^{\vee}(\lambda + \eta)$ by Lemma 6.1.16. Then Proposition 5.1.3 shows that $R^{\lambda+\eta,\tau}_{\overline{\rho}} = 0$. It remains to show that $\mathcal{Z}_{\sigma}(\overline{\rho}) = 0$ for all $\sigma \in \mathrm{JH}(\overline{\sigma}(\lambda, \tau))$. If σ is not $(\mathcal{S}_{\Lambda}, \widehat{\mathcal{S}}_{\Lambda})$-generic, this is automatic. Suppose that σ is $(\mathcal{S}_{\Lambda}, \widehat{\mathcal{S}}_{\Lambda})$-generic and $\mathcal{Z}_{\sigma}(\overline{\rho}) \neq 0$. Then the support of the cycle $\mathcal{Z}_{\sigma}$ contains the point $\overline{\rho} \in \mathcal{X}_{\mathrm{Sym,red}}(\mathbf{F})$. This implies that $\overline{\rho}^{\mathrm{ss}}$ is in the support of $\mathcal{Z}_{\sigma}$. In particular, $\mathcal{Z}_{\sigma}(\overline{\rho}^{\mathrm{ss}}) = \widetilde{Z}(M^{\psi_{\overline{\rho}}}_{\infty}(\sigma)) \neq 0$. However, this implies that $R^{\lambda+\eta,\tau}_{\overline{\rho}^{\mathrm{ss}}} \neq 0$. Then $\widetilde{w}(\overline{\rho}, \tau) \in \mathrm{Adm}(\lambda + \eta)$ by Proposition 5.1.3(2). By Lemma 6.1.16(2), this contradicts our assumption $(\lambda + \eta, \tau) \notin \mathcal{S}_{\mathcal{P}_{\mathrm{ss}},\Lambda}$. This concludes the proof. $\square$

6.2 The weight part of a Serre's conjecture

Let $F, \chi, \overline{r}$ be as in §5.3. We further assume that $\overline{r}$ is automorphic of some weight μ and level U. Recall that $W(\overline{r})$ is the set of modular Serre weights of $\overline{r}$. The following conjecture is due to Gee–Herzig–Savitt ([GHS18]).

Conjecture 6.2.1. *If $\overline{r}|_{G_{F_v}}$ is tame and sufficiently generic at $v|p$, then $W(\overline{r}) = W^?(\overline{r}_p|_{I_{\mathbf{Q}_p}})$.*

Following [Le+a, §9.1], we formulate a version of this conjecture without the tameness hypothesis.

We say that a Serre weight σ is *generic* if σ is $(\mathcal{S}_{\{0\}}, \widehat{\mathcal{S}}_{\{0\}})$-generic. Note that if σ is generic, then it is $(\mathcal{S}_\Lambda, \widehat{\mathcal{S}}_\Lambda)$-generic for any finite set of dominant cocharacters $\Lambda \subset X_*(\underline{T}^\vee)$ (cf. [Le+a, Lemma 8.4.8]). We let $W_{\mathrm{gen}}(\overline{r})$ be the subset of generic Serre weights in $W(\overline{r})$.

Definition 6.2.2. Let $\overline{\rho} : G_{\mathbf{Q}_p} \to {}^L\underline{G}(\mathbf{F})$ be an L-parameter. We define $W^{\mathrm{BM}}_{\mathrm{gen}}(\overline{\rho})$ to be a set of $\sigma = \otimes_{v\in S_p}\sigma_v$ where σ_v is a generic Serre weight such that $\overline{\rho}_v$ is contained in the support of $\mathcal{Z}_{\sigma_v}$ defined in Theorem 6.1.14.

Using this recipe, we can formulate a generalization of Conjecture 6.2.1 for $\overline{r}$ not necessarily tame at places above p.

Conjecture 6.2.3. *The set $W_{\mathrm{gen}}(\overline{r})$ is equal to $W^{\mathrm{BM}}_{\mathrm{gen}}(\overline{r}_p)$.*

We also define $W_g(\overline{r}_p) = \{\sigma = \otimes_{v\in S_p}\sigma_v \mid \sigma_v \in W_g(\overline{r}_p|_{G_{F_v}})\}$, a set of Serre weights of $\mathrm{GSp}_4(\mathcal{O}_F/p)$, where $W_g(\overline{r}_p|_{G_{F_v}})$ is the set defined in Proposition 4.5.6. Note that any $\sigma \in W_g(\overline{r}_p)$ is 3-deep.

Let $\mathcal{X}_{\mathrm{Sym},F_p} := \prod_{v\in S_p,\mathcal{O}} \mathcal{X}_{\mathrm{Sym},F_v}$. By Theorem 4.1.10, there is a bijection between irreducible components in the underlying reduced substack $\mathcal{X}_{\mathrm{Sym},F_p,\mathrm{red}}$ and isomorphism classes of Serre weights of $\mathrm{GSp}_4(\mathcal{O}_F/p)$. If $\sigma = \otimes_{v\in S_p}\sigma_v$ is a Serre weight of $\mathrm{GSp}_4(\mathcal{O}_F/p)$, we denote its corresponding irreducible component by $\mathcal{C}_\sigma := \prod_{v\in S_p,\mathcal{O}} \mathcal{C}_{\sigma_v}$.

Lemma 6.2.4. *Let $\sigma \in W_g(\overline{r}_p)$. Let $R_{\overline{r}_p} := \widehat{\otimes}_{v\in S_p} R^{\square}_{\overline{r}|_{G_{F_v}}}$ with a versal morphism*

$$\iota_{\overline{r}_p} : R_{\overline{r}_p} \to \mathcal{X}_{\mathrm{Sym},F_p}.$$

If $\overline{r}_p|_{I_{\mathbf{Q}_p}}$ is 22-generic, then $\iota^*_{\overline{r}_p}(\mathcal{C}_\sigma)$ is an irreducible cycle corresponding to an irreducible subscheme $\mathcal{C}_\sigma(\overline{r}_p) \subset \operatorname{Spec} R_{\overline{r}_p}/\varpi$. Moreover, if $\mathcal{C}_\sigma \subset \prod_{v\in S_p,\mathcal{O}} \mathcal{X}^{\lambda_v+\eta_v,\tau_v}_{\mathrm{Sym},F_v,\mathbf{F}}$, then $\mathcal{C}_\sigma(\overline{r}_p)$ is contained in $\operatorname{Spec} R^{\lambda+\eta,\tau}_{\overline{r}_p}/\varpi$.

Proof. By Proposition 4.5.6, $W_g(\overline{r}_p) \subset W^?(\overline{r}_p|_{I_{\mathbf{Q}_p}})$. By Lemma 2.5.14 and [Le+a, Remark 2.1.8], there is a tame inertial L-parameter τ with a lowest alcove representation such that $\widetilde{w}(\overline{\rho}|_{I_{\mathbf{Q}_p}}, \tau) \in \mathrm{Adm}(\eta)$ and $\sigma \in \mathrm{JH}(\overline{\sigma}(\tau))$. Since $\overline{\rho}$ is 22-generic, such τ is 19-generic. Then the first claim follows from Theorem 4.5.4 and Proposition 3.7.3. The second claim follows from the definition of $\mathcal{C}_\sigma(\overline{r}_p)$. $\square$

The following is the main result of this section.

Theorem 6.2.5. *There exists a polynomial $P(X_1, X_2, X_3) \in \mathbf{Z}[X_1, X_2, X_3]$ independent of p such that if $\overline{r} : G_F \to \mathrm{GSp}_4(\mathbf{F})$ is automorphic, satisfies Taylor–Wiles conditions, and for each $v|p$, $\overline{r}|_{G_{F_v}}$ is tame with a lowest alcove presentation $(s_v, \mu_v - \eta)$ with P-generic μ_v, then the Conjecture 6.2.1 and 6.2.3 hold for $\overline{r}$.*

Proof. We claim that there exists a polynomial $Q(X_1, X_2, X_3)$ such that if μ_v is Q-generic, then $W^?(\overline{r}_p|_{I_{\mathbf{Q}_p}}) = W_g(\overline{r}_p)$. It follows from Proposition 4.5.6(1) that $W_g(\overline{r}_p) \subset W(\overline{r})$. Let

$$Q(X_1, X_2, X_3) := \prod_{\widetilde{w}\in \widetilde{W}_1^+/X^0(T)} \prod_{\widetilde{w}_2\uparrow\widetilde{w}, \widetilde{w}_2\in\widetilde{W}^+} \prod_{w\in W} P_{\widetilde{w}}(X + w\widetilde{w}_2^{-1}(0)).$$

with the polynomial $P_{\widetilde{w}}$ in Proposition 4.5.6. Let $\sigma \in W^?(\overline{r}_p|_{I_{\mathbf{Q}_p}})$ with a lowest alcove presentation $(\widetilde{w}, \omega)$ compatible with $(s, \mu - \eta)$. If μ_v is Q-generic, then ω_v is $P_{\widetilde{w}}$ (this can be checked directly using Proposition 2.5.8). Thus, Proposition 4.5.6(2) implies that $W^?(\overline{r}_p|_{I_{\mathbf{Q}_p}}) = W_g(\overline{r}_p)$.

We take P to be the product $P_{22}P_{2\eta,e}P^{\eta_0}_{\eta,e}Q$ and assume that μ_v is P-generic.

Step 1. We first show that $W(\overline{r}) \cap W_{\mathrm{obv}}(\overline{r}_p|_{I_{\mathbf{Q}_p}})$ is nonempty. Let τ be the tame inertial L-parameter with a lowest alcove presentation η-compatible with that of $\overline{r}_p$ and $\widetilde{w}(\tau) = \widetilde{w}(\overline{r}_p)t_{-\eta-w_0\eta}$ (note that this condition and $P_{2\eta,e}P_{22}$-genericity of μ_v imply that τ is $P_{2\eta,e}P_{22}$-generic). By [Le+a, Lemma 2.6.7], $W^?(\overline{r}_p|_{I_{\mathbf{Q}_p}}) \subset \mathrm{JH}(\overline{\sigma}(\eta,\tau))$. For the unique $\psi_p \in \Psi(\overline{r}_p)$ compatible with $(2\eta,\tau)$ and a fixed similitude patching functor $M_\infty^{\psi_p}$ for $\overline{r}_p$ (which exists by Theorem 5.4.2), the automorphy of $\overline{r}$ implies that $M_\infty^{\psi_p}(\sigma^\circ(\eta,\tau)) \neq 0$. By Theorem 4.4.3 and $P_{2\eta}$-genericity of τ, $M_\infty^{\psi_p}(\sigma^\circ(\eta,\tau))$ has full support on $\operatorname{Spec} R_{\overline{r}_p}^{2\eta,\tau,\psi_p}$. By Lemma 6.2.4, we have $\mathcal{C}_\sigma(\overline{r}_p) \subset \operatorname{Spec} R_{\overline{r}_p}^{2\eta,\tau}/\varpi$ for any $\sigma \in W^?(\overline{r}_p|_{I_{\mathbf{Q}_p}})$. Then there exists $\sigma' \in W^?(\overline{r}_p|_{I_{\mathbf{Q}_p}})$ such that $\widetilde{Z}(M_\infty^{\psi_p}(\sigma'))$ is supported on $\mathcal{C}_\sigma(\overline{r}_p)$. Since the support of $\widetilde{Z}(M_\infty^{\psi_p}(\sigma')) = \mathcal{Z}_{\sigma'}(\overline{\rho})$ is contained in $\{\mathcal{C}_\kappa | \kappa \uparrow \sigma'\}$, this in particular implies that $\sigma \uparrow \sigma'$. Take $\sigma \simeq F(\lambda) \in W_{\mathrm{obv}}(\overline{r}_p|_{I_{\mathbf{Q}_p}})$ such that λ is in the highest p-restricted alcove. Then $\sigma \uparrow \sigma'$ implies that $\sigma = \sigma'$, and σ is also contained in $W(\overline{r})$.

Step 2. We show that $W_{\mathrm{obv}}(\overline{r}_p|_{I_{\mathbf{Q}_p}}) \subset W(\overline{r})$. Let $\sigma \in W_{\mathrm{obv}}(\overline{r}_p|_{I_{\mathbf{Q}_p}})$. We can write $\sigma = F_{\overline{\rho}}(w^{-1})$ for some $w \in \underline{W}$. We take τ to be 19-generic tame inertial type such that $\widetilde{w}(\overline{r}_p,\tau) = t_{w(\eta)}$. By Lemma 2.5.11, $\sigma \in \mathrm{JH}(\overline{\sigma}(\tau))$.

There exists a unique $\psi_p \in \Psi(\overline{r}_p)$ compatible with (η,τ) and a fixed similitude patching functor $M_\infty^{\psi_p}$ for $\overline{r}_p$. By the previous step, we can choose $\sigma \in W_{\mathrm{obv}}(\overline{r}_p|_{I_{\mathbf{Q}_p}}) \cap W(\overline{r})$. Then we have $M_\infty^{\psi_p}(\sigma^\circ(\tau_0)) \neq 0$. Thus, there exists a lift r_0 of $\overline{r}$ such that is potentially crystalline of type $(\eta_v, \tau_{0,v})$ at $v|p$. By Theorem 5.1.4, this shows that $\overline{r}$ is potentially diagonalizably automorphic. For an arbitrary $\sigma \in W_{\mathrm{obv}}(\overline{r}_p|_{I_{\mathbf{Q}_p}})$, we can apply Theorem 5.1.4, [Boo19, Theorem 1.1], and [PT21, Theorem 3.4] to find a lift r of $\overline{r}$ that is potentially diagonalizable of type (η_v,τ_v) at $v|p$. By [EL, Lemma 4.4.4], r is automorphic and $M_\infty^{\psi_p}(\sigma^\circ(\tau)) \neq 0$. By reducing modulo p, $M_\infty^{\psi_p}(\sigma) \neq 0$ and thus $\sigma \in W(\overline{r})$.

Step 3. We show that $W(\overline{r}) = W^?(\overline{r}_p|_{I_{\mathbf{Q}_p}})$. Let $\sigma \in W^?(\overline{r}|_{I_{\mathbf{Q}_p}})$ and τ be the type satisfying conditions in Lemma 2.5.14. Since $\mathrm{JH}(\overline{\sigma}(\tau)) \cap W^?(\overline{r}|_{I_K})$ is nonempty if and only if $\mathrm{JH}(\overline{\sigma}(\tau)) \cap W_{\mathrm{obv}}(\overline{r}|_{I_K})$ is nonempty, the previous step implies that $M_\infty^{\psi_p}(\sigma^\circ(\tau)) \neq 0$ and it has full support on $\operatorname{Spec} R_{\overline{r}_p}^{\eta,\tau}$, by Theorem 4.4.3 and the $P_{\eta,e}^{\eta_0}$-genericity of μ. Since $\mathcal{C}_\sigma(\overline{r}_p) \subset \operatorname{Spec} R_{\overline{r}_p}^{\eta,\tau}$ by Lemma 6.2.4, there exists $\sigma' \in \mathrm{JH}(\overline{\sigma}(\tau))$ such that $M_\infty^{\psi_p}(\sigma')$ is supported on $\mathcal{C}_\sigma(\overline{r}_p)$. As we argued at the end of Step 1, we have $\sigma \uparrow \sigma'$, which implies $\sigma = \sigma'$ by Lemma 2.5.14 and thus $\sigma \in W(\overline{r})$. Thus the Conjecture 6.2.1 holds for $\overline{r}$.

Step 4. It remains to prove that $W_{\mathrm{gen}}(\overline{r}) = W_{\mathrm{gen}}^{\mathrm{BM}}(\overline{r}_p)$. Let σ be a generic Serre weight. Then σ is $(\mathcal{S}_\Lambda, \widehat{\mathcal{S}}_\Lambda)$-generic. Let $M_\infty^{\psi_p}$ be a fixed similitude patching functor for $\overline{r}_p$ constructed in Theorem 5.4.2(1). Then $M_\infty^{\psi_p}(\sigma) \neq 0$ if and only if $\sigma \in W(\overline{r})$. On the other hand, $\sigma \in W_{\mathrm{gen}}^{\mathrm{BM}}(\overline{r}_p)$ is equivalent to $\overline{r}_p \in \iota_{\overline{r}_p}^*(\mathcal{Z}_\sigma)$, which is equivalent to $M_\infty^{\psi_p}(\sigma) \neq 0$ by the construction of $\mathcal{Z}_\sigma$. Thus the Conjecture 6.2.3 holds for $\overline{r}$. $\square$

6.3 Modularity lifting in generic tamely potentially crystalline case

We prove the following modularity lifting result.

Theorem 6.3.1. *Let $r : G_F \to \mathrm{GSp}_4(E)$ be a continuous representation satisfying following conditions:*

1. *r is unramified at all but finitely many places;*
2. *$r|_{G_{F_v}}$ is potentially crystalline of type $(\lambda_v + \eta_v, \tau_v)$ with a lowest alcove presentation $(s_v, \mu_v - \eta_v)$ with $P_{\lambda+\eta,e}$-generic μ_v;*
3. *$\overline{r}$ satisfies Taylor–Wiles hypothesis;*
4. *$\overline{r}|_{G_{F_v}}$ is tame at $v|p$; and*

5. $\overline{r} \simeq \overline{r}_{\pi,p,\iota}$ *for some cuspidal automorphic representation* π *of* $\mathrm{GSp}_4(\mathbb{A}_F)$ *of weight* λ *and central character* χ *such that* $\sigma(\tau_v)$ *is a* K*-type for* π *at* $v|p$.

Then $r \simeq r_{\widetilde{\pi},p,\iota}$ *for some cuspidal automorphic representation* $\widetilde{\pi}$ *of* $\mathrm{GSp}_4(\mathbb{A}_F)$ *of weight* λ *central character* χ *such that* $\sigma(\tau_v)$ *is a* K*-type for* $\widetilde{\pi}$ *at* $v|p$.

The Theorem follows from the following Lemma and base change argument.

Lemma 6.3.2. *Let* $r : G_F \to \mathrm{GSp}_4(E)$ *be a continuous representation satisfying following conditions:*

1. r *is unramified at all but finitely many places;*
2. *if* r *is ramified at a place* $v \nmid p$*, then* $\overline{r}|_{G_{F_v}}$ *is trivial,* $r|_{G_{F_v}}$ *has only unipotent ramification, and* $q_w = 1 \bmod p$*;*
3. $r|_{G_{F_v}}$ *is potentially crystalline of type* $(\lambda_v + \eta_v, \tau_v)$ *with a lowest alcove presentation* $(s_v, \mu_v - \eta_v)$ *with* $P_{\lambda+\eta,e}$*-generic* μ_v*;*
4. $\overline{r}$ *satisfies Taylor–Wiles hypothesis;*
5. $\overline{r}|_{G_{F_v}}$ *is tame at* $v|p$*; and*
6. $\overline{r} \simeq \overline{r}_{\pi,p,\iota}$ *for some cuspidal automorphic representation* π *of* $\mathrm{GSp}_4(\mathbb{A}_F)$ *of weight* λ *central character* χ *such that* $\sigma(\tau_v)$ *is a* K*-type for* π *at* $v|p$ *and for all finite places* v *of* F*,* $(\pi_v)^{\mathrm{Iw}(v)} \neq 0$.

Then $r \simeq r_{\widetilde{\pi},p,\iota}$ *for some cuspidal automorphic representation* $\widetilde{\pi}$ *of* $\mathrm{GSp}_4(\mathbb{A}_F)$ *of weight* λ *central character* χ *such that* $\sigma(\tau_v)$ *is a* K*-type for* $\widetilde{\pi}$ *at* $v|p$.

Proof. We apply Proposition 5.4.9 to F, χ, $S = S_r \cup \{v_0\}$. By the assumptions on r, it gives an E-point of $\operatorname{Spec} R_{\mathcal{S}_1}$. On the other hand, $r_{\pi,p,\iota}$ gives an E-point of $\operatorname{Spec} R_{\mathcal{S}_\zeta}$. Thus the module M^{χ,ψ_p} is nonzero. By Theorem 4.4.3, $P_{\lambda+\eta,e}$-genericity of μ, [Box+21, Proposition 7.4.2], and Proposition 5.1.8(2), $M_\infty^{\zeta,\psi_p}(\sigma^\circ(\lambda,\tau))$ has full support over $R_\infty^{\zeta,\lambda+\eta,\tau,\psi_p}$. By the congruence between M_∞^{1,ψ_p} and M_∞^{ζ,ψ_p}, $M_\infty^{1,\psi_p}(\overline{\sigma}(\lambda,\tau))$ has full support over $R_\infty^{1,\lambda+\eta,\tau,\psi_p}/\varpi$. Then by Proposition 5.1.8(1), $M_\infty^{1,\psi_p}(\sigma^\circ(\lambda,\tau))$ has full support over $R_\infty^{1,\lambda+\eta,\tau,\psi_p}$. This proves the claim. ☐

Proof of Theorem 6.3.1. There exists a solvable extension of totally real fields F'/F satisfying the following conditions:

1. F'/F is linearly disjoint from $F^{\ker \overline{r}}$;
2. any place $v|p$ of F splits completely in F';
3. if $r|_{G_{F'}}$ is ramified at a place w of F' lying over a place $v \nmid p$ of F, then $\overline{r}|_{G_{F'_w}}$ is trivial, $r|_{G_{F'_w}}$ has only unipotent ramification, and $q_w = 1 \bmod p$;
4. there is a cuspidal automorphic representation π' of $\mathrm{GSp}_4(\mathbb{A}_{F'})$ of weight $\lambda' = (\lambda_w)$, where $\lambda_w = \lambda_v$ for a place v of F and $w|v$, which is a base change of π, and for all finite places w of F', $(\pi'_w)^{\mathrm{Iw}(w)} \neq 0$ (here we use [Mok14, Proposition 4.13]).

Then by [Box+21, Lemma 8.3.2] (which easily generalizes to our setup), it suffices to show that there exists a cuspidal automorphic representation $\widetilde{\pi}'$ of $\mathrm{GSp}_4(\mathbb{A}_{F'})$ of weight λ' such that for a place $w|p$ of F' above a place v of F, $\sigma(\tau_v)$ is a K-type for π at w, and $r|_{G_{F'}} \simeq r_{\widetilde{\pi}',p,\iota}$. This follows from Lemma 6.3.2. ☐

Appendix A

Torus fixed points of certain affine Springer fibers

In this chapter, we explain how the main result of [Boi] generalizes to the group GSp_4. Let $\mathrm{Fl}_{\mathbf{C}}$ be the affine flag variety over $\mathbf{C}$ whose $\mathbf{C}$-points are given by $\mathrm{GSp}_4(\mathbf{C}(\!(v)\!))/\mathcal{I}_{\mathbf{C}}$, where $\mathcal{I}_{\mathbf{C}} \subset \mathrm{GSp}_4(\mathbf{C}[\![v]\!])$ is the Iwahori subgroup. Let $\mathbf{a} \in \operatorname{Lie}\mathrm{GSp}_4(\mathbf{C})$ be a regular semisimple element. We consider the affine Springer fiber $\mathrm{Fl}_{v\mathbf{a}}$ associated to the element $v\mathbf{a} \in \operatorname{Lie}\mathrm{GSp}_4(\mathbf{C}(\!(v)\!))$. The $\mathbf{C}$-points of $\mathrm{Fl}_{\mathbf{C}}$ correspond to *Iwahori subalgebras* inside the loop algebra $\operatorname{Lie}\mathrm{GSp}_4(\mathbf{C}(\!(v)\!))$, i.e. $\mathrm{GSp}_4(\mathbf{C}(\!(v)\!))$-conjugates of the Lie algebra of the Iwahori subgroup. Under this correspondence, the $\mathbf{C}$-points of $\mathrm{Fl}_{v\mathbf{a}}$ are characterized as Iwahori subalgebras containing the element vs. If $g \in \mathrm{GSp}_4(\mathbf{C}(\!(v)\!))$, then its image in $\mathrm{Fl}_{\mathbf{C}}(\mathbf{C})$ is in $\mathrm{Fl}_{v\mathbf{a}}$ if and only if $v\mathbf{a} \in gL^+\mathcal{M}_{\mathbf{C}}g^{-1}$, or equivalently, $g^{-1}\mathbf{a}g \in \frac{1}{v}L^+\mathcal{M}_{\mathbf{C}}$.

Recall that $\mathrm{Fl}_{\mathbf{C}}$ admits T-action induced by the left multiplication on $L\mathrm{GSp}_4$. The T-fixed points of $\mathrm{Fl}_{\mathbf{C}}$ are given by the image of the map $\widetilde{W} \to \mathrm{Fl}_{\mathbf{C}}(\mathbf{C})$ sending wt_λ to $\phi(w)v^{-\phi(\lambda)}\mathcal{I}_{\mathbf{C}}$. Note that this is a composition of the map $(-)^* : \widetilde{W} \to \widetilde{W}^\vee$, taking inverse on $\widetilde{W}^\vee$, and the natural embedding of $\widetilde{W}^\vee$ into $\mathrm{Fl}_{\mathbf{C}}(\mathbf{C})$. For simplicity, we write the image of $\widetilde{w}$ under this map by $\widetilde{w}^{-*}$. We want to further understand which of these T-fixed points are contained in $\mathrm{Fl}_{v\mathbf{a}}$. Let $\widetilde{w} \in \widetilde{W}$. We define

$$\mathrm{Fl}_{v\mathbf{a}}(\widetilde{w}) := \mathrm{GSp}_4(\mathbf{C}[\![v]\!])\widetilde{w}^{-*}\mathcal{I}_{\mathbf{C}}/\mathcal{I}_{\mathbf{C}} \cap \mathrm{Fl}_{v\mathbf{a}}.$$

By [Boi, Proposition 2.1], if $\widetilde{w} \in \widetilde{W}_1^+$, then $\mathrm{Fl}_{v\mathbf{a}}(\widetilde{w})$ is irreducible of dimension 4. Let $\mathrm{Fl}_{v\mathbf{a},\widetilde{w}}$ be the closure of $\mathrm{Fl}_{v\mathbf{a}}(\widetilde{w})$ inside $\mathrm{Fl}_{v\mathbf{a}}$.

Remark A.1. Suppose that $\widetilde{w} \in \widetilde{W}$ is equal to $\widetilde{w}'\delta$ for some $\widetilde{w}' \in W_a$ and $\delta \in \Omega$. Since δ^{-*} normalizes $\mathcal{I}_{\mathbf{C}}$, we have $\mathrm{Fl}_{v\mathbf{a},\widetilde{w}} = \mathrm{Fl}_{v\mathbf{a},\widetilde{w}'}\delta$.

Remark A.2. Note that we use the right quotient by Iwahori to define the affine flag variety $\mathrm{Fl}_{\mathbf{C}}$, as opposed to the left quotient in the body of this thesis. These conventions can be compared by taking inverse at the level of $L\mathrm{GSp}_4$. This also explains the difference between the embedding of the affine Weyl group $\widetilde{W}$ into the loop group $L\mathrm{GSp}_4$ in this appendix and the body of this thesis (which are differed by taking inverse).

Theorem A.3. *The set of T-fixed points of $\mathrm{Fl}_{v\mathbf{a},\widetilde{w}}$ is given by*

$$\mathrm{Fl}^T_{v\mathbf{a},\widetilde{w}} = \{\widetilde{z}^{-*} \mid \widetilde{z} \in \widetilde{W}, \widetilde{z} \le w_0\widetilde{w}\}.$$

Proof. We follow the proof of [Boi, Theorem 3.1] with minor modifications.

By Remark A.1, it suffices to consider $\widetilde{w} \in W_a$. Let w be an element of $W^\vee$. By noting that $w\mathrm{Fl}_{v\mathbf{a},\widetilde{w}} = \mathrm{Fl}_{vw(\mathbf{a}),\widetilde{w}}$, $\widetilde{z}^{-*} \in \mathrm{Fl}_{v\mathbf{a},\widetilde{w}}$ if and only if $w\widetilde{z}^{-*} \in \mathrm{Fl}_{vw(\mathbf{a}),\widetilde{w}}$. Thus, it suffices to show that

$$\{\widetilde{z}^{-*} \mid \widetilde{z} \in \widetilde{W}^{+,\vee}, \widetilde{z} \le \widetilde{w}\} \subset \mathrm{Fl}^T_{v\mathbf{a},\widetilde{w}}.$$

Let $\widetilde{w}' := \mathcal{T}(\widetilde{w})$. Recall that there exists a 6-dimensional affine subvariety $M^{\widetilde{w}'}$ inside $L^+\overline{U}_4$ defined in [Boi, Appendix B.2]. We define $M^{\widetilde{w}} := M^{\widetilde{w}'} \cap L^+\mathrm{GSp}_4$. By Lemma B.2 in *loc. cit.*, we have

$$M^{\widetilde{w}}\widetilde{w}^{-*}\mathcal{I}_{\mathbf{C}}/\mathcal{I}_{\mathbf{C}} \subset \mathrm{Fl}_{v\mathbf{a},\widetilde{w}}.$$

Note that $M^{\widetilde{w}}\widetilde{w}^{-*}\mathcal{I}_{\mathbf{C}}/\mathcal{I}_{\mathbf{C}}$ is contained in the lower triangular Iwahori orbit, which is locally closed and affine by [GKM06, Theorem 0.2]. Since both are 4-dimensional affine spaces, they have to be the same. Moreover, since $\mathrm{Fl}_{v\mathbf{a},\widetilde{w}}$ is also 4-dimensional, $M^{\widetilde{w}}\widetilde{w}^{-*}\mathcal{I}_{\mathbf{C}}/\mathcal{I}_{\mathbf{C}}$ is open (and dense) in $\mathrm{Fl}_{v\mathbf{a},\widetilde{w}}$.

For $\widetilde{z} \in \widetilde{W}$, there exists a neighborhood $U_{\widetilde{z}} = \widetilde{z}^{-*}L^-U\mathcal{I}_{\mathbf{C}}/\mathcal{I}_{\mathbf{C}} \subset \mathrm{Fl}_{\mathbf{C}}$ such that $\widetilde{z}^{-*}$ is in $\mathrm{Fl}_{vs,\widetilde{w}}$ if and only if $U_{\widetilde{z}} \cap M^{\widetilde{w}}\widetilde{w}^*\mathcal{I}_{\mathbf{C}}/\mathcal{I}_{\mathbf{C}}$ is non-empty (see Lemma 3.1 in *loc. cit.*).

It remains to prove that the intersection $M^{\widetilde{w}}\widetilde{w}^{-*}\mathcal{I}_{\mathbf{C}}/\mathcal{I}_{\mathbf{C}} \cap U_{\widetilde{z}}$ is nonempty for $\widetilde{z} \in \widetilde{W}$ such that $\widetilde{z} \le \widetilde{w}$. Indeed, we prove a slightly stronger statement. Let $\widetilde{w}_0 \in W_a$ be the element corresponding to the highest restricted alcove. We show that

$$M^{\widetilde{w}_0}\widetilde{w}^{-*}\mathcal{I}_{\mathbf{C}}/\mathcal{I}_{\mathbf{C}} \cap U_{\widetilde{z}} \ne \emptyset. \tag{A.4}$$

As explained in Appendix B.3 of *loc. cit.*(also, see Lemma B.1), this follows from a non-vanishing of certain matrix determinant. To explain this, we introduce some notation.

We view $\mathrm{Fl}_{\mathbf{C}}$ as a subset of affine flags in a 4-dimensional $\mathbf{C}(\!(v)\!)$-vector space. Let $e_1, \dots, e_4$ be the standard basis of the $\mathbf{C}(\!(v)\!)$-vector space. For vectors $v_1, \dots, v_4$, we let $\langle v_1, \dots, v_4\rangle$ be the $\mathbf{C}[\![v]\!]$-span of $v_1, \dots, v_4$ and $\langle v_1, \dots, v_4\rangle_{v^{-1}}$ be the $\mathbf{C}[1/v]$-span of $v_1, \dots, v_4$. Then (A.4) holds if and only if

$$M^{\widetilde{w}_0}\widetilde{w}^{-*}\langle e_1, \dots, e_l, ve_{l+1}, \dots, ve_4\rangle \cap \widetilde{z}^{-*}\langle v^{-1}e_1, \dots, v^{-1}e_l, e_{l+1}, \dots, e_4\rangle_{v^{-1}} = 0 \tag{A.5}$$

for all $0 \le l \le 3$. In our case, we can check these conditions directly.

We have

$$M^{\widetilde{w}_0} = \begin{pmatrix} 1 & & & \\ vc_{12}A_{21} & 1 & & \\ c_{13}A_{31} + vc_{123}A_{21}A_{32} & c_{23}A_{32} & 1 & \\ v(c_{14}A_{41} + c_{124}A_{21}A_{42} + c_{134}A_{31}A_{43}) + v^2c_{1234}A_{21}A_{32}A_{43} & c_{24}A_{42} + vc_{234}A_{32}A_{43} & vc_{34}A_{43} & 1 \end{pmatrix}$$

where the condition on GSp_4 implies

$$\begin{aligned} c_{34}A_{43} &= -c_{12}A_{21} \\ c_{24}A_{42} &= c_{13}A_{31} \\ c_{123}A_{21}A_{32} &= c_{234}A_{32}A_{43}. \end{aligned}$$

We can write down the matrix representation of $M^{\widetilde{w}_0}$ with respect to v^ke_j for all $k \in \mathbf{Z}$, $1 \le j \le 4$.

However, to check (A.5), we only need to consider finitely many $v^k e_j$'s. Let $\omega_l = (1, \dots, 1, 0, \dots, 0)$ be the cocharacter of GL_4 whose first l entries are 1 and the remaining entries are 0. Note that $\widetilde{w}\langle e_1, \dots, e_l, ve_{l+1}, \dots, ve_4\rangle$ contains $v^{1-\widetilde{w}(\omega_l)_4}\langle e_1, \dots, e_4\rangle$ which is stabilized by $M^{\widetilde{w}_0}$. Similarly, $\widetilde{z}^{-*}\langle v^{-1}e_1, \dots, v^{-1}e_l, e_{l+1}, \dots, e_4\rangle_{v^{-1}}$ contains $v^{-\widetilde{z}^{-*}(\omega_l)_1}\langle e_1, \dots, e_4\rangle_{v^{-1}}$. Thus, we can only consider $v^k e_j$'s whose image under $M^{\widetilde{w}_0}$ is not contained in the union of $v^{1-\widetilde{w}(\omega_l)_4}\langle e_1, \dots, e_4\rangle$ and $v^{-\widetilde{z}^{-*}(\omega_l)_1}\langle e_1, \dots, e_4\rangle_{v^{-1}}$, and there are only finitely many of them.

Now we compute a block matrix whose ij-th block, for $1 \le i \le (1 - \widetilde{w}(\omega_l)_4) - (-\widetilde{z}^{-*}(\omega_l)_1) - 1$ and $1 \le j \le (1 - \widetilde{w}(\omega_l)_4) - (1 - \widetilde{w}(\omega_l)_1)$, is a matrix representation of $M^{\widetilde{w}_0}$ with respect to the linearly independent vectors

$$\begin{aligned} &v^{(1-\widetilde{w}(\omega_l)_1)+j-1}e_1, \dots, v^{(1-\widetilde{w}(\omega_l)_1)+j-1}e_{r_{lij}} \quad \text{in the domain} \\ &v^{(-\widetilde{z}^{-*}(\omega_l)_1)+i}e_1, \dots, v^{(-\widetilde{z}^{-*}(\omega_l)_1)+i}e_{s_{lij}} \quad \text{in the codomain,} \end{aligned}$$

where r_{lij} is the maximal integer such that $v^{(1-\widetilde{w}(\omega_l)_1)+j-1}e_{r_{lij}} \in \widetilde{w}\langle e_1, \dots, e_l, ve_{l+1}, \dots, ve_4\rangle$ and s_{lij} is the maximal integer such that $v^{(-\widetilde{z}^{-*}(\omega_l)_1)+i}e_{s_{lij}} \notin \widetilde{z}^{-*}\langle v^{-1}e_1, \dots, v^{-1}e_l, e_{l+1}, \dots, e_4\rangle_{v^{-1}}$. Then (A.5) holds if and only if this block matrix has a trivial kernel after specializing all A_{mn}'s to some complex numbers satisfying the required relations.

For simplicity, we check this for $\widetilde{w} = \widetilde{w}_0$ and $\widetilde{z} = e$, and the remaining cases will be a straightforward computation. When $l = 0$, the block matrix is given by

$$\left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ c_{12}A_{21} & 0 & 1 & 0 \\ c_{123}A_{21}A_{32} & c_{13}A_{31} & c_{23}A_{32} & 1 \\ c_{14}A_{41} + c_{124}A_{21}A_{42} + c_{134}A_{31}A_{43} & 0 & c_{24}A_{42} & 0 \end{array}\right).$$

(Note that in this case and the cases $l = 1, 2$ below, the first row of blocks is empty.) One can easily see that the determinant has a non-zero term $c_{14}A_{41}$ (which is not affected by the condition imposed by GSp_4).

When $l = 1$, the block matrix is given by

$$\left(\begin{array}{c|cc} 0 & 1 & \\ c_{12}A_{21} & 0 & 1 \\ c_{123}A_{21}A_{32} & c_{13}A_{31} & c_{23}A_{32} \end{array}\right)$$

whose determinant obvious does not vanish everywhere.

When $l = 2$, the block matrix is given by

$$\left(\begin{array}{c|cc|ccc} 0 & 1 & 0 & & & \\ c_{12}A_{21} & 0 & 1 & & & \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & c_{12}A_{21} & 0 & 0 & 1 & 0 \\ 0 & c_{123}A_{21}A_{32} & 0 & c_{13}A_{31} & c_{23}A_{32} & 1 \\ c_{1234}A_{21}A_{32}A_{43} & c_{14}A_{41} + c_{124}A_{21}A_{42} + c_{134}A_{31}A_{43} & c_{234}A_{32}A_{43} & 0 & c_{24}A_{42} & 0 \end{array}\right).$$

In this case, one can observe that the determinant has a non-zero monomial $c_{14}A_{41}$, which is not affected by the condition imposed by GSp_4. Thus the determinant does not vanish everywhere.

Finally, when $l = 3$, the block matrix is given by

$$\left(\begin{array}{cc|ccc}
1 & 0 & & & \\
\hline
0 & 0 & 1 & & \\
c_{12}A_{21} & 0 & 0 & 1 & \\
c_{123}A_{21}A_{32} & 0 & c_{13}A_{31} & c_{23}A_{32} & 1 \\
c_{14}A_{41} + c_{124}A_{21}A_{42} + c_{134}A_{31}A_{43} & c_{234}A_{32}A_{43} & 0 & c_{24}A_{42} & 0
\end{array}\right)$$

and its determinant is equal to $-c_{234}A_{32}A_{43}$ which does not vanish everywhere. $\square$

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