RANDOM CANONICAL PRODUCTS AND THE SECULAR FUNCTION OF THE STOCHASTIC AIRY OPERATOR

BY

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Abstract

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Secular functions of random matrices and their limits are of recent interest in random matrix theory. Such functions are entire with zeros the spectra of the corresponding operators. For example, the general beta ensembles, extending the joint eigenvalue law of classical random matrix ensembles, have a universal soft edge limit upon rescaling called the Airy beta point process. This process also arises as eigenvalues of a random operator called the stochastic Airy operator. It is proven here that secular functions of the general beta ensembles converge in distribution to that of the stochastic Airy operator. Furthermore, this convergence is realized in the context of regularized determinants of operators. This is done by proving new asymptotics of the Airy process and rigidity estimates of the general beta ensembles and establishing this convergence for more general random sequences. These results extend the currently known case for the Gaussian ensembles in Lambert and Paquette (2020). Growth asymptotics are proven for the secular function of the stochastic Airy operator, and as an application some open questions in Lambert and Paquette (2020) are answered. By applying and extending the work in Valkó and Virág (2020) in the bulk case, the secular function is proven to be a unique limiting solution of an ordinary differential equation. Additionally, new convergence laws for discrete matrix models limiting to the stochastic Airy operator are proven, including convergence of the derivatives of eigenfunctions.

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1 Introduction and statement of main results

One area of study in random matrix theory is the convergence of random matrices as their matrix dimension n goes to infinity. For example, one of the earliest examples of a random matrix was given by Eugene Wigner in the 1950s. Consider an $n \times n$ matrix A_n with entries as iid standard normals. The matrix $B_n = 2^{-1/2}\left(A_n + A_n^T\right)$, whose distribution is known as the **Gaussian orthogonal ensemble** (GOE), was one item of Wigner's study. For example, it is well known that the joint density of the eigenvalues $\mu_k^{(n)}$, $1 \leq k \leq n$ of B_n is

$$\frac{1}{Z_\beta^{(n)}} \exp\left(-\beta \sum_{k=1}^{n} \frac{\left(\mu_k^{(n)}\right)^2}{4}\right) \prod_{1 \leq i < j \leq n} \left|\mu_i^{(n)} - \mu_j^{(n)}\right|^\beta \tag{1}$$

where $\beta = 1$ and $Z_\beta^{(n)}$ is the normalization. Upon the rescaling $\tilde{\mu}_k^{(n)} = \mu_k^{(n)}/\sqrt{2n}$ of the eigenvalues $\mu_k^{(n)}$ one obtains the classical Wigner semicircle law:

$$\mathbb{E}\frac{1}{n}\sum_{k=1}^{n} \delta_{\tilde{\mu}_k^{(n)}} \Longrightarrow \rho_{\text{sc}}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\mathbb{1}_{|x| \leq 2}. \tag{2}$$

Here $\delta_{\tilde{\mu}_k^{(n)}}$ denotes a Dirac measure and $\mathbb{1}$ an indicator function.

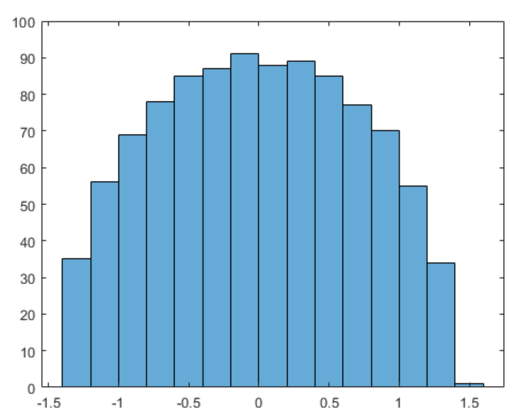

Figure 1: Histogram of the rescaled eigenvalues $\tilde{\mu}_k^{(1000)}$ of a GOE B_n

In fact, the weak convergence (2) is not unique to $\beta = 1$: β may be any positive number, and the joint distribution has the name of a **Gaussian beta ensemble**. Even more, there is a further generalization in replacing the $x^2/4$ term in (1) with a function $V(x)$, called the **potential**, satisfying certain assumptions, which then yields the **general beta ensembles**. Convergence (2) then holds with ρ_{sc} replaced by a more general continuous function ρ called the **equilibrium density**.

Convergence of random matrices can take may different forms in addition to (2): convergence to a limiting random operator, of eigenvalues or eigenvectors and operations on them, or of objects associated to random matrices. For example, in Ramirez, Rider, and Virág (2011) it is shown that upon a certain rescaling $\tilde{\lambda}_k^{(n)}$ of a Gaussian beta ensemble there is the convergence $\tilde{\lambda}_k^{(n)} \Longrightarrow \Lambda_k$ as $n \to \infty$ in the sense of finite-dimensional distributions. Here $\Lambda_k = \Lambda_{\beta,k}$, $k \geq 1$, known as the **Airy beta point process** (short: **Airy process**), are the eigenvalues of the limiting random operator, known as the **stochastic Airy operator**,

$$\mathcal{H} = \mathcal{H}_\beta = -\partial_x^2 + x + \frac{2}{\sqrt{\beta}} db_x. \tag{3}$$

This operator acts on functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with Dirichlet boundary condition $f(0) = 0$ and b is a standard Brownian motion. In fact, in Ramirez, Rider, and Virág (2011) random matrices $\mathcal{H}_n$ are constructed as discrete models of $\mathcal{H}$ to yield the convergence $\tilde{\lambda}_k^{(n)} \Longrightarrow \Lambda_k$. Furthermore, this convergence continues to hold for the general beta ensembles provided $\beta \geq 1$.

One main item of this thesis is proving the open conjecture of the convergence of random complex entire functions with zeros the rescaled general beta ensembles $\lambda_k^{(n)}$ to a limiting entire function with zeros Λ_k, again provided $\beta \geq 1$:

$$\prod_{k=1}^{n} \left(1 - \frac{z}{\lambda_k^{(n)}}\right) e^{z/\lambda_k^{(n)}} \Longrightarrow \prod_{k=1}^{\infty} \left(1 - \frac{z}{\Lambda_k}\right) e^{z/\Lambda_k} \tag{4}$$

in the topology of compact convergence in $\mathbb{C}$ as $n \to \infty$. This may be viewed as convergence of "secular functions" or "infinite characteristic polynomials" of random operators with spectrum the corresponding zeros. For example, in the Gaussian case the operators are $\mathcal{H}_n$ and $\mathcal{H}$, and (4) may be rewritten as

$$\det{}_2 \left(I - z\mathcal{H}_n^{-1}\right) \Longrightarrow \det{}_2 \left(I - z\mathcal{H}^{-1}\right)$$

where $\det_2$ is a regularized determinant on the Hilbert-Schmidt operators. Here, the normalization to be 1 at $z = 0$ is chosen, and the exponential terms are necessary corrections for convergence. For example, almost surely $1/\Lambda_k$, $k \geq 1$ is in ℓ^2 but not ℓ^1.

This kind of convergence–looking at the objects of random functions with zeros the spectra of convergent random matrices–has been of recent interest in random matrix theory. In Valkó and Virág (2020), instead of the Airy process they consider a different scaling limit of the Gaussian beta ensembles called the sine beta process. Again there is a limiting random operator, called the sine beta operator, and a convergence as in (4). There they call the limiting entire function the **secular function** $\zeta(z)$ of the sine beta operator. There are interesting connections between $\zeta(z)$ and the Riemann zeta function, discussed for example in Chhaibi, Najnudel, and Nikeghbali (2014). Additionally, in Chhaibi, Hovhannisyan, Najnudel, Nikeghbali, and Rodgers (2019) the same convergence for the sine beta operator is considered. This work is the bulk case, essentially working in the interior of the spectrum support of random matrices, where the Airy process is the edge case, in particular the soft edge. In Lambert and Paquette (2020) they show essentially (4) in the special case of the Gaussian beta ensembles.

Also, in an informal sense, in taking $\beta \to \infty$ in (3) the white noise term db_x vanishes and the deterministic, classical Airy operator

$$\mathcal{A} = -\partial_x^2 + x$$

is left. In Vallée and Soares (2010) they remark that the classical Airy function has the form

$$\mathrm{Ai}(z) = \mathrm{Ai}(0)\, e^{-\left|\frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)}\right| z} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\Lambda_k^0}\right) e^{z/\Lambda_k^0}$$

where Λ_k^0 are the eigenvalues of $\mathcal{A}$.

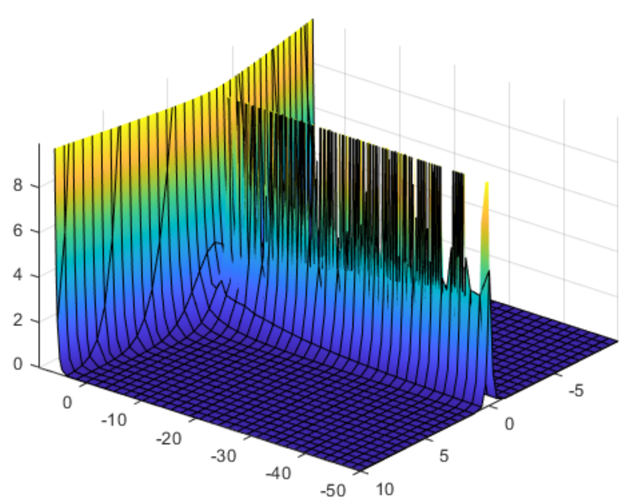

Figure 2: Plot of $1/\left|\mathrm{Ai}(z)\right|$

The organization of this thesis and the main results are as follows. In Section 3, the general beta ensembles $\mu_k^{(n)}$ and its related notions are carefully defined following Bourgade, Erdös, and Yau (2014). Throughout this thesis "general beta ensembles" refers to the notion in Bourgade, Erdös, and Yau (2014) with some additional assumptions which are given in Section 3.1. The rescaling $\lambda_k^{(n)}$ of the ensembles to obtain $\lambda_k^{(n)} \Longrightarrow \Lambda_k$ is given. The statement of edge universality, which allows comparing different ensembles, is provided. In particular, a presently unpublished variant is stated which was confirmed with the authors in Bourgade, Erdös, and Yau (2014).

In Section 4, the definition of the stochastic Airy operator $\mathcal{H}$ is reviewed following Ramirez, Rider, and Virág (2011). In particular, the bilinear form $\langle \mathcal{H}f, f\rangle$ defined there is carefully reviewed here, as well as the resulting Courant-Fischer characterization of the eigenvalues Λ_k. Various known properties of the Airy process Λ_k are collected and some proven. Extending the work in Ramirez, Rider, and Virág (2011) it is proven that with a slight change in the definition of the bilinear form more precise asymptotics can be obtained on Λ_k as $k \to \infty$. The main results are the following:

Theorem 1. *For any* $0 < \varepsilon < 1/6$ *there exists* $k_0 = k_0(\varepsilon)$ *large so that*

$$\mathbb{P}\left(\left|\Lambda_k - \Lambda_k^0\right| \leq k^{1/6+\varepsilon}\right) \geq 1 - 2^{-k^{\chi}}, \quad \textit{for all} \quad k \geq k_0$$

where $\chi = \chi(\varepsilon) > 0$*. In particular, for any* $\delta > 0$*,* $k_0 = k_0(\varepsilon, \delta)$ *may be chosen so that*

$$\mathbb{P}\left(\left|\Lambda_k - \Lambda_k^0\right| \leq k^{1/6+\varepsilon},\, k \geq k_0\right) \geq 1 - \delta.$$

Corollary 2. *For any* $0 < \varepsilon < 1/6$ *there exists* $k_0 = k_0(\varepsilon)$ *large and* $\chi = \chi(\varepsilon) > 0$ *so that for* $k \geq k_0$*, with probability at least* $1 - 2^{-k^{\chi}}$*,*

$$\Lambda_k = \Lambda_k^0 + O\left(k^{1/6+\varepsilon}\right) = \left(\frac{3\pi}{2}\right)^{2/3} k^{2/3} + O\left(k^{1/6+\varepsilon}\right) \tag{5}$$

where $O\left(k^{1/6+\varepsilon}\right)$ *is deterministic.*

In Section 5, the construction of the tridiagonal matrix models $\mathcal{H}_n$ having eigenvalues the rescaled Gaussian beta ensembles done in Bloemendal and Virág (2013) and Ramirez, Rider, and Virág (2011) is reviewed. In fact, the more general construction seen there is done where $\mathcal{H}_n$ are discrete models converging to a more general limiting random operator

$$-\partial_x^2 + y'(x)$$

where $y(x)$ is a certain continuous random process. By a slight abuse of notation Λ_k will also denote the eigenvalues of this more general limiting operator. Various facts are recalled including convergence of the eigenvalues $\lambda_k^{(n)}$ and eigenfunctions $v_k^{(n)}$ to those of the limiting operator Λ_k, f_k. Adding to these convergence results the following is proven:

Theorem 3. *If $y_{n,i} \Longrightarrow y_i$, $i = 1, 2$ in the topology of compact convergence for some random continuous processes y_i then $D_n v_k^{(n)} \Longrightarrow f_k'$ in the same topology as $n \to \infty$.*

Here $y_{n,i}$, y_i are certain random processes defining $\mathcal{H}_n$, and D_n is a discrete derivative. To establish Theorem 3, additional, new convergence results are proven in L_n^*, the subspace of L^2 that these matrix models act on.

In Section 6, the notion of a random entire function is given. In particular, the notion of a random complex canonical product is given. The essential example is the random entire function with zeros Λ_k:

$$\mathfrak{p}(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{\Lambda_k}\right) e^{z/\Lambda_k}.$$

Following Valkó and Virág (2020) and their work in the bulk case, this function is called the **secular function** of the stochastic Airy operator. Various concepts from the theory of entire functions and their growth are reviewed and applied to $\mathfrak{p}(z)$ to prove almost sure asymptotics of its growth. The notions of p-Schatten-von Neumann operators $A \in \mathcal{S}_p$, generalizing trace ($p = 1$) and Hilbert-Schmidt ($p = 2$) operators, and their regularized determinants $\det_p (I - zA)$ are given. Furthermore, their natural connection to entire functions is explained. From Dumaz, Li, and Valkó (2020) it is reviewed that the stochastic Airy operator $\mathcal{H}$ almost surely has inverse $\mathcal{H}^{-1} \in \mathcal{S}_2$ a Hilbert-Schmidt integral operator. Furthermore, it is shown that

$$\det{}_2 \left(I - z\mathcal{H}^{-1}\right) = \mathfrak{p}(z) \quad \text{and} \quad \det{}_2 \left(I - z\mathcal{H}_n^{-1}\right) = \prod_{k=1}^{n} \left(1 - \frac{z}{\lambda_k^{(n)}}\right) e^{z/\lambda_k^{(n)}} \tag{6}$$

where the latter is in the Gaussian case. Viewing the latter as the **secular function** of $\mathcal{H}_n$, now (4) can be expressed as a convergence of operator secular functions in the topology of compact convergence: as $n \to \infty$,

$$\det{}_2 \left(I - z\mathcal{H}_n^{-1}\right) \Longrightarrow \det{}_2 \left(I - z\mathcal{H}^{-1}\right) \tag{7}$$

Finally, some convergence in law operations are shown for random entire functions.

In Section 7, it is shown that there is a natural convergence law for random entire functions (and regularized determinants). The new notion of **product convergent** random sequences $a_k^{(n)}$, $1 \le k \le n$, $n \ge 1$, A_k, $k \ge 1$ of **genus** p is introduced. Briefly, the notion requires that $a_k^{(n)} \Longrightarrow A_k$ as $n \to \infty$ in the sense of finite-dimensional distributions, almost surely $A_k \neq 0$ for all k, and $a_k^{(n)}$, $n \geq 1$ are tight in $\ell^{p+1}(\mathbb{Z}_{\geq 1}, \mathbb{C})$ where $p \in \mathbb{Z}_{\geq 0}$. The following is proven:

Theorem 4. *If $a_k^{(n)}$, A_k are product convergent random sequences of genus p then $P_n(z) \Longrightarrow P(z)$ as $n \to \infty$ in the topology of compact convergence where*

$$P_n(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{a_k^{(n)}}\right) \exp\left(\sum_{j=1}^{p} \frac{z^j}{j\left(a_k^{(n)}\right)^j}\right)$$

and

$$P(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{A_k}\right) \exp\left(\sum_{j=1}^{p} \frac{z^j}{jA_k^j}\right).$$

Various precise asymptotics are proven for the general beta ensembles when $\beta \geq 1$ which culminate in the following:

Theorem 5. *For $\beta \geq 1$ the rescaled general beta ensembles $\lambda_k^{(n)}$ and Airy beta point process Λ_k are product convergent of genus 1.*

One such precise asymptotic that leads to Theorem 5 is the following **rigidity estimate** at the particle level. Such estimates are probabilistic bounds on the distance between $\lambda_k^{(n)}$ and the rescaled n-**quantiles** of the equilibrium density ρ. The (unscaled) n-quantiles $\gamma_k^{(n)}$ are defined by

$$\frac{k}{n} = \int_{-\infty}^{\gamma_k^{(n)}} \rho$$

and "particle level" refers to the bound being in terms of k.

Theorem 6. *For $\beta \geq 1$ there exists $0 < \kappa_0 \leq 2/5$ so that for any $0 < \kappa < \kappa_0$, $0 < \delta < 1/2$ and $\varepsilon > 0$ there exists $n_0 = n_0(\kappa, \varepsilon)$ and $k_0 = k_0(\delta, \varepsilon)$ large so that for $n \geq n_0$,*

$$\mathbb{P}\left(\left|\lambda_k^{(n)} - (\pi s_A)^{2/3} n^{2/3}\left(\gamma_k^{(n)} - A\right)\right| \leq \pi^{2/3} k^{1/6+\delta}, \textit{ for all } k_0 \leq k \leq n^{\kappa}\right) \geq 1 - \varepsilon.$$

Similar bound are also proved for $k \geq n^{\kappa}$ which results in the following:

Corollary 7. *For $\beta \geq 1$ there exists a deterministic constant $c > 0$ so that for any $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ and $k_0 = k_0(\varepsilon)$ large so that for $n \geq n_0$,*

$$\mathbb{P}\left(\lambda_k^{(n)} \geq ck^{2/3}, \quad \textit{for all} \quad k \geq k_0\right) \geq 1 - \varepsilon.$$

Theorems 5 and 4 solve the open problem of (4). Recall in Lambert and Paquette (2020) this was essentially shown in the Gaussian case. As remarked earlier, the secular function convergence in the Gaussian case (7) now follows as an application by using (6). More generally, the following secular function convergence is true:

Corollary 8. *If A_n, A are random $p+1$-Schatten-von Neumann operators, $p \geq 1$ with spectra $a_k^{(n)}$, A_k respectively which are product convergent random sequences of genus p then*

$$\det{}_{p+1}\left(I - zA_n^{-1}\right) \Longrightarrow \det{}_{p+1}\left(I - zA^{-1}\right)$$

as $n \to \infty$ in the topology of compact convergence.

Another application of the work in this section is to the work in Lambert and Paquette (2020). Their method for establishing secular function convergence in the Gaussian case produced a limiting random function $\mathrm{SAi}_z(t)$, $(t,z) \in \mathbb{R} \times \mathbb{C}$ that was not defined as a canonical product. Consequently, they had some open questions about the growth of $\mathrm{SAi}_z(t)$ as an entire function. The following is proven which provides some partial answers:

Proposition 9. *As random real analytic functions with the topology of compact convergence,*

$$\frac{SAi_x(0)}{SAi_0(0)} \exp\left(-x \times \frac{SAi_0{}'(0)}{SAi_0(0)}\right) \overset{d}{=} \mathfrak{p}(-x).$$

It the following a C^0*-set* C_j in $\mathbb{C}$ is essentially a sequence of disks to avoid because they may contain function zeros, and is defined in detail in Section 6.3.

Corollary 10. *Almost surely, outside $C_j \cap \mathbb{R}$ where C_j is some C^0-set in $\mathbb{C}$,*

$$\log SAi_x(0) = \log SAi_0(0) - \frac{3}{2}x^{3/2} + x \times \frac{SAi_0{}'(0)}{SAi_0(0)} + o\left(x^{3/2}\right), \quad x \to \infty$$

and

$$\log SAi_x(0) = \log SAi_0(0) + x \times \frac{SAi_0{}'(0)}{SAi_0(0)} + o\left(|x|^{3/2}\right), \quad x \to -\infty.$$

Finally, in Section 8 a vector-valued ordinary differential equation is developed for the stochastic Airy operator $\mathcal{H}$. This is accomplished by extending the work in Valkó and Virág (2020) in the bulk case to the edge case. The earlier remarked realization of $\mathcal{H}^{-1}$ as a Hilbert-Schmidt integral integral operator in Dumaz, Li, and Valkó (2020) takes the particular form

$$\mathcal{H}^{-1}f(x) = \int_0^\infty k(x,y)\, f(y)\, dy,\ f \in L^2([0,\infty), \mathbb{R})$$

with integral kernel

$$k(x,y) = \psi_\infty(x)\,\psi_d(y)\,1_{x \geq y} + \psi_d(x)\,\psi_\infty(y)\,1_{x<y},\ (x,y) \in [0,\infty)^2$$

where ψ_∞, ψ_d are certain absolutely continuous solutions of $\mathcal{H}\psi = 0$. With

$$H(t,z) = \sum_{n=0}^{\infty} (-1)^n z^n d_n(t) \in \mathbb{C}^2, \ (t,z) \in [0,\infty) \times \mathbb{C}$$

where

$$d_n(t)^T = \int_{0 \le t_1 < \cdots < t_n \le t} [1,0] R(t_1) J \cdots R(t_n) J dt_1 \cdots dt_n$$

and

$$R(t) = \begin{bmatrix} \psi_d(t)^2 & \psi_\infty(t)\psi_d(t) \\ \psi_\infty(t)\psi_d(t) & \psi_\infty(t)^2 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

the following is proven:

Theorem 11. *Almost surely $H(t,z)$ is the unique solution of the ordinary differential equation*

$$J\frac{\partial}{\partial t} H(t,z) = -zR(t) H(t,z), \ t \in [0,\infty), \ H(0,z) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\det{}_2 \left(I - z\mathcal{H}^{-1}\right) = \mathfrak{p}(z) = \lim_{t\to\infty} e^{z\int_0^t k(x,x)dx} H(t,z)[1,0] \tag{8}$$

where the limit holds uniformly in z on compacts as $t \to \infty$.

More specifically,

$$H(t,z) = \begin{bmatrix} 1 + \sum_1^\infty \frac{(-1)^n z^n}{n!} \int_{[0,t]^n} \det [k(t_i,t_j)]_{i,j=1}^n dt_1 \cdots dt_n \\ z\int_0^t \psi_d(s)^2 ds - \sum_2^\infty \frac{(-1)^n z^n}{(n-1)!} \int_0^t \int_{[0,t]^{n-1}} \tilde{\mathcal{K}}(t_1,\ldots,t_{n-1},t_n) dt_1 \cdots dt_{n-1} dt_n \end{bmatrix}$$

where

$$\tilde{\mathcal{K}}(t_1,\ldots,t_{n-1},t_n) = \det \begin{bmatrix} [k(t_1,t_j)]_{j=1}^n \\ \vdots \\ [k(t_{n-1},t_j)]_{j=1}^n \\ [\psi_d(t_j)\psi_d(t_n)]_{j=1}^n \end{bmatrix}.$$

Also, both coordinates of $H(t,z)$ are almost surely entire in z for fixed t. Theorem 11 is in fact proven for a more general family of integral kernels: real-valued k of the form

$$k(x,y) = k_1(x) k_2(y) 1_{x\ge y} + k_1(y) k_2(x) 1_{x<y}, \ (x,y) \in [0,\infty)^2$$

for some functions $k_i : [0,\infty) \to \mathbb{R}$ satisfying that they are measurable, bounded on compacts, $k_2(0) = 0$ and k_2 is continuous at 0. Furthermore, (8) holds with $\mathfrak{p}(z)$ replaced by $\det{}_2(I - zK)$ where K is the corresponding integral operator, provided it is Hilbert-

Schmidt, and the limit may be dropped if the support of k is contained in $[0,t]^2$ for some $t>0$. Finally, the connections and differences between the objects and arguments here and in Valkó and Virág (2020) in the bulk case are remarked.

2 Notation

Functions and measures:

$$1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

denotes an indicator function. $\delta_x(A) = 1_A(x)$ denotes a Dirac measure. A^T denotes matrix transpose. $\wedge$ denotes minimum and $\vee$ maximum. $O(g(x))$ is the standard big-oh notation, e.g. $|O(g(x))| \leq C|g(x)|$ for large x and some $C > 0$. Given a sequence x_n and a function $f(x)$, $x_n \in o(f(n))$ means $x_n/f(n) \to 0$ as $n \to \infty$. In particular, $x_n \in o(1)$ means $x_n \to 0$. Similarly $f(x) \in o(g(x))$ as $x \to \infty$ is defined. The complex logarithm log is taken as the principal branch cut along the negative real axis where the principle argument arg satisfies $-\pi < \theta \leq \pi$. Im and Re respectively denote the imaginary and real parts of complex numbers. $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ respectively denote the floor and ceiling functions. $\sqcup$ denotes disjoint union. ∂ denotes set boundary.

Spaces and sets: $C(X,Y)$, $C_b(X,Y)$, $C_c(X,Y)$, $C^n(X,Y)$ respectively denote the continuous, bounded, compactly supported, continuous nth derivative functions $X \to Y$, where if Y is omitted then $X = Y$. $L^p(X)$ denotes an L^p space with domain X and codomain either $\mathbb{R}$ or $\mathbb{C}$. L^p_{loc} are the functions which are L^p on compact subsets. For $n \geq 1$, $[n] = \{1, \ldots, n\}$. $AC(X,Y)$ is the set of absolutely continuous functions $X \to Y$. $B(X,Y)$ is the bounded linear operators $X \to Y$. Y^X all functions $X \to Y$. $H(X,Y)$ is the holomorphic functions $X \to Y$ where X, $Y \subseteq \mathbb{C}$. In the case X, $Y \subseteq \mathbb{R}$, $H(X,Y)$ are the real analytic functions (i.e. anywhere locally a convergent power series). For $x \in (M,d)$ some metric space, $B(x,\varepsilon) = \{y \in M : d(x,y) < \varepsilon\}$ and $\overline{B}(x,\varepsilon)$ is with $<$ replaced with $\leq$. $\mathcal{S}_p$ denotes the p-Schatten-von Neumann operators.

Norms: $\|\cdot\|_\infty$ is the supremum norm.

Other symbols: $:=$ denotes "define."

3 General beta ensembles

3.1 Basic definitions

The foundational object in this thesis are the general beta ensembles, a distribution on n points generalizing the joint eigenvalue densities of classical random matrix ensembles. Following Bourgade, Erdös, and Yau (2014) their basic definitions are given here. Let $\beta > 0$, which in general is fixed and is omitted from notation. For each $n \geq 1$ let

$$\Xi^{(n)} = \{x = (x_1, \ldots, x_n) : x_1 < \cdots < x_n\} \subseteq \mathbb{R}^n$$

and let $\mu_{\beta,V}^{(n)} = \left(\mu_1^{(n)}, \ldots, \mu_n^{(n)}\right)$ be a real-valued random variable with density on $\mathbb{R}^n$,

$$\frac{1}{Z_{\beta,V}^{(n)}} \exp\left(-\beta n \sum_1^n \frac{V(x_k)}{2}\right) \prod_{1 \leq i < j \leq n} |x_i - x_j|^\beta 1_{\Xi^{(n)}}(x)\, dx \tag{9}$$

where $Z_{\beta,V}^{(n)}$ is the normalization. Continuing, $V : \mathbb{R} \to \mathbb{R}$ has continuous fourth derivative satisfying that for some constant $W \geq 0$,

$$\inf_{x \in \mathbb{R}} V''(x) \geq -2W \tag{10}$$

and for some $\alpha > 0$, for $|x|$ large,

$$V(x) > (2 + \alpha) \ln(1 + |x|). \tag{11}$$

In de Monvel, Pastur, and Shcherbina 1995 it is shown that under these conditions the density (9) is well-defined. $\mu_{\beta,V}^{(n)}$ is referred to as a **general beta ensemble** with **particles** $\mu_k^{(n)}$ and **potential** V. Throughout the notation for β, V is often omitted, and sometimes also for n. Intersecting with statistical mechanics μ may be viewed as a Gibbs measure on N particles in $\mathbb{R}$ with logarithmic interaction and inverse temperature β.

It is moreover shown in de Monvel, Pastur, and Shcherbina 1995 that another consequence of the conditions on V is that the **averaged density of the empirical spectral measure**

$$\mathbb{E}\frac{1}{n} \sum_{k=1}^n \delta_{\mu_k^{(n)}} \tag{12}$$

converges weakly to a continuous function $\rho = \rho_V$ of compact support called the **equilibrium density**. Here $\frac{1}{n}\sum_{k=1}^n \delta_{\mu_k}$ is defined as the unique Radon measure on $\mathbb{R}$ given by an application of the Riesz representation theorem to the positive linear functional

$$\mathbb{E}\frac{1}{n}\sum_{k=1}^{n}\varphi\left(\mu_k\right), \quad \varphi\in C_c\left(\mathbb{R}\right).$$

In particular,

$$\int_{\mathbb{R}}\varphi d\left(\mathbb{E}\frac{1}{n}\sum_{k=1}^{n}\delta_{\mu_k}\right)=\mathbb{E}\frac{1}{n}\sum_{k=1}^{n}\varphi\left(\mu_k\right), \quad \varphi\in C_c\left(\mathbb{R}\right),$$

and the stated weak convergence can be characterized as follows:

Proposition 12. *(12) converges weakly to ρ iff for all $\varphi\in C_c\left(\mathbb{R}\right)$,*

$$\mathbb{E}\frac{1}{n}\sum_{k=1}^{n}\varphi\left(\mu_k^{(n)}\right)\to\int_{\mathbb{R}}\varphi\left(x\right)\rho\left(x\right)dx \tag{13}$$

as $n\to\infty$.

Proof. Applying the Riesz representation theorem to each ensemble yields Radon measures ν_n where

$$\int_{\mathbb{R}}\varphi d\nu_n=\mathbb{E}\frac{1}{n}\sum_{k=1}^{n}\varphi\left(\mu_k^{(n)}\right), \quad \varphi\in C_c\left(\mathbb{R}\right).$$

Each ν_n is in fact a probability measure: furthermore by Riesz,

$$\nu_n\left(\mathbb{R}\right)\geq\sup\left\{\mathbb{E}\frac{1}{n}\sum_{k=1}^{n}\varphi_m\left(\mu_k^{(n)}\right):m\geq1\right\}$$

ranging over bump functions $\varphi_m\in C_c\left(\mathbb{R}\right)$ with values in $[0,1]$ which are identically 1 on $[-m,m]$. Then, with $A_m^{(n)}=\left\{\mu_k^{(n)}\in[-m,m],\forall k\right\}$,

$$\mathbb{E}\frac{1}{n}\sum_{k=1}^{n}\varphi_m\left(\mu_k^{(n)}\right)\geq\mathbb{P}\left(A_m^{(n)}\right).$$

Since $A_m^{(n)}\uparrow\Omega$ as $m\to\infty$ therefore by continuity from below, $1=\lim_m\mathbb{P}\left(A_m^{(n)}\right)$. Hence, $\nu_n\left(\mathbb{R}\right)=1$. As probability measures the claim now follows from Lemma 82. □

In regards to ρ there are some further assumptions on V. One is that V is taken so that the support of ρ is of the form $[A,B]\subseteq\mathbb{R}$. The second is that V is **regular** in the sense of

Kuijlaars and McLaughlin 2000: ρ is positive on (A, B) and

$$\rho(x) = s_A\sqrt{x-A}\,(1 + O(x-A)), \quad x \to A^+$$
$$\rho(x) = s_B\sqrt{B-x}\,(1 + O(B-x)), \quad x \to B^-$$

for some constants $s_A,\ s_B > 0$. In Kuijlaars and McLaughlin 2000 it is shown that the assumption of regularity is dense with a natural topology. Finally, the **limiting classical location** $\gamma_k^{(n)}$ of the kth particle $\mu_k^{(n)}$ is the n-quantile of the equilibrium measure:

$$\int_A^{\gamma_k^{(n)}} \rho = \frac{k}{n}.$$

Throughout this thesis a "general beta ensemble" refers to one with all the stated assumptions in this section. Following Bourgade, Erdös, and Yau (2014) the below fact is known as a **rigidity estimate**:

Fact 13. *(Theorem 2.4 Bourgade, Erdös, and Yau (2014)) For any $\xi > 0$ there exists a constant $c = c(\xi) > 0$ and $n_0 = n_0(\xi)$ large so that for $n \geq n_0$, for all $1 \leq k \leq n$,*

$$\mathbb{P}\left(\left|\mu_k^{(n)} - \gamma_k^{(n)}\right| > n^{-2/3+\xi}\hat{k}^{-1/3}\right) \leq e^{-n^c}$$

where $\hat{k} = \min\{k, n+1-k\}$.

Furthermore,

$$\gamma_k^{(n)} \sim \left(\frac{k}{n}\right)^{2/3}\left(1 + O\left(\left(\frac{k}{n}\right)^{2/3}\right)\right) \tag{14}$$

where $A \sim B$ means $c_1 \leq A/B \leq c_2$ for some constants $c_1,\ c_2$ (Equation 2.10 Bourgade, Erdös, and Yau (2014)).

3.2 Gaussian beta ensembles

In the case of the potential $\tilde{V}(x) = x^2/2$ the ensembles $\tilde{\mu}$ are called the **Gaussian beta ensembles**. For example, when $\beta = 1$ the ensemble is precisely the classical Gaussian orthogonal ensemble considered in Section 1. A central theme of this thesis is exploiting what is known about the Gaussian beta ensembles in theorems about the general ones. This is because a lot is known about the Gaussian ones and a fundamental result known as edge universality, discussed in Section 3.4, permits transference of certain properties. The following is classical:

Lemma 14. *$\tilde{V}(x) = x^2/2$ is a well-defined potential and has corresponding equilibrium density the classical Wigner semicircle law*

$$\rho_{sc}(x) = \frac{1}{2\pi}\sqrt{4-x^2}1_{|x|\leq 2}$$

where $A=-2$, $B=2$ and $s_A = s_B = 1/\pi$.

3.3 Rescaling general beta ensembles

It is often useful to rescale general beta ensembles, i.e. shift or stretch the equilibrium measure and thereby the parameters A, s_A, etc. For example, when comparing two different ensembles it can be useful to require their A, s_A parameters to match or be normalized to $A=0$, $s_A=1$. The general idea is the following, and can be essentially obtained by working with the formulation of weak convergence given by Proposition 12 and performing a change of variables:

Proposition 15. *If μ is a general beta ensemble with potential V and support $[A,B]$ then $s_A^{2/3}(\mu - A)$ is a general beta ensemble with potential*

$$W(x) = V\left(s_A^{-2/3}x + A\right),$$

equilibrium measure

$$\rho_W(x) = s_A^{-2/3}\rho_V\left(s_A^{-2/3}x + A\right)$$

with support $\left[0, s_A^{2/3}(B-A)\right]$ and satisfying $s_0 = 1$, $s_{s_A^{2/3}(B-A)} = s_B/s_A$, and n-quantiles $s_A^{2/3}\left(\gamma_k^{(n)} - A\right)$ where $\gamma_k^{(n)}$ are the n-quantiles of μ.

Example 16. By Lemma 14, if $\tilde{\mu}$ is a Gaussian beta ensemble then Proposition 15 shows that $\pi^{-2/3}(\tilde{\mu}+2)$ is a general beta ensemble with potential

$$W(x) = \left(\pi^{2/3}x - 2\right)^2/2,$$

equilibrium measure

$$\rho_W(x) = \frac{1}{2}\sqrt{x}\sqrt{4-\pi^{2/3}x}1_{[0,4\pi^{-2/3}]}$$

satisfying $s_0 = 1$, $s_{4\pi^{-2/3}} = 1$ and n-quantiles $\pi^{-2/3}\left(\gamma_k^{(n)}+2\right)$ where $\gamma_k^{(n)}$ are the n-quantiles of $\tilde{\mu}$.

3.4 Edge universality

A fundamental result shown in Bourgade, Erdös, and Yau (2014) is that there is a certain universality tying together many of the general beta ensembles. As noted in Section 3.2, a central theme in this thesis is transferring the wealth of knowledge for Gaussian beta ensembles to general ones. A natural way of introducing this universality is first establishing the universal rescaling limit of the general beta ensembles. First in the Gaussian case, the shifted Gaussian beta ensemble $\tilde{\lambda} = n^{2/3}\left(\tilde{\mu} - 2\right)$ is known to converge weakly to a random real sequence $\Lambda_1 \leq \Lambda_2 \leq \cdots$ in the sense of finite-dimensional distributions (Theorem 1.1 Ramirez, Rider, and Virág 2011): for each $m \geq 1$,

$$\left(\tilde{\lambda}_1^{(n)}, \ldots, \tilde{\lambda}_m^{(n)}\right) \Longrightarrow \left(\Lambda_1, \ldots, \Lambda_m\right) \tag{15}$$

as $n \to \infty$. The sequence $\Lambda_k = \Lambda_{\beta,k}$ is called the **Airy beta point process** (short: **Airy process**). The sequence is fundamental in this thesis and may be realized as the spectrum of a random operator which is defined in detail in Section 4, and additional properties of the sequence are discussed there. Beyond the Gaussian case, Λ_k is actually universal in the following sense:

Proposition 17. *(Corollary 2.2 Bourgade, Erdös, and Yau (2014)) For any $\beta \geq 1$ and any general beta ensemble μ the rescaled ensemble*

$$\lambda_k^{(n)} = \left(\pi s_A\right)^{2/3} n^{2/3} \left(\mu_k^{(n)} - A\right)$$

with parameters coming from μ has finite-dimensional convergence to Λ_k.

Note that the statement of Proposition 17 is slightly different than in Bourgade, Erdös, and Yau (2014). To obtain this result, the discussed notion of *edge universality* is used:

Fact 18. *(Theorem 2.1 Bourgade, Erdös, and Yau (2014)) Let $\beta \geq 1$ and μ, ν be two general beta ensembles whose equilibrium densities both have support with left endpoint 0 and satisfying $s_0 = 1$. For any constant $\kappa < 2/5$ there exists $\chi > 0$ so that for any fixed $m \geq 1$ and any* **observable** *$O \in C_c^1\left(\mathbb{R}^m, \mathbb{R}\right)$ there exists a constant $c > 0$ so that for any n and any $A \subseteq [1, n^\kappa] \cap \mathbb{Z}$ with $|A| = m$,*

$$\left|\mathbb{E} O\left(\left(n^{2/3} k^{1/3}\left(\mu_k^{(n)} - \gamma_{\mu,k}^{(n)}\right)\right)_{k\in A}\right) - \mathbb{E} O\left(\left(n^{2/3} k^{1/3}\left(\nu_k^{(n)} - \gamma_{\nu,k}^{(n)}\right)\right)_{k\in A}\right)\right| \leq c n^{-\chi}$$

where $\gamma_{\mu,k}^{(n)}$, $\gamma_{\nu,k}^{(n)}$ are resp. the n-quantiles of μ, ν. Moreover, the n-quantiles may be taken to be the same, e.g. replace $\gamma_{\nu,k}^{(n)}$ with $\gamma_{\mu,k}^{(n)}$.

The following result is a useful asymptotic formula for the rescaled n-quantiles of the rescaled Gaussian ensemble $\tilde{\lambda}_k$ in Example (20):

Fact 19. *(Equation 3.10 Landon (2020))There exists $\alpha > 0$ small so that for $1 \leq k \leq \alpha n$,*

$$n^{2/3}\left(\tilde{\gamma}_k^{(n)} + 2\right) = \left(\frac{3\pi}{2}\right)^{2/3} k^{2/3} + O\left(\frac{k^{5/3}}{n}\right)$$

where $\tilde{\gamma}_k^{(n)}$ are the n-quantiles of the Gaussian beta ensemble $\tilde{\mu}$.

Proof of Proposition 17. Let $m \geq 1$. By Lemma 82 it suffices to show that for $\varphi \in C_c^\infty(\mathbb{R}^m, \mathbb{R})$,

$$\mathbb{E}\varphi(\lambda_k)_{k=1}^m \to \mathbb{E}\varphi(\Lambda_k)_{k=1}^m.$$

In particular,

$$\psi(x) = \varphi\left(\pi^{2/3}k^{-1/3}x_k + \Lambda_k^0 - o(1)\right)_{k=1}^m \in C_c^1(\mathbb{R}^m, \mathbb{R})$$

where by Fact 26, $\Lambda_k^0 = (3\pi/2)^{2/3} k^{2/3} + o(1)$ are the negatives of the classical Airy function eigenvalues. Let $\tilde{\mu}$ be the Gaussian beta ensemble as stated in Example 16. By Proposition 15 and Example 16 the ensembles $s_A^{2/3}(\mu - A)$ and $\pi^{-2/3}(\tilde{\mu} + 2)$ satisfy the requirements of Fact 18. Hence, there exists χ, $c > 0$ such that for n sufficiently large, in choosing the n-quantiles to be the same,

$$\Big|\mathbb{E}\psi\left(n^{2/3}k^{1/3}\left(s_A^{2/3}(\mu_k - A) - \pi^{-2/3}(\tilde{\gamma}_k + 2)\right)\right)_{k=1}^m -$$
$$- \mathbb{E}\psi\left(n^{2/3}k^{1/3}\left(\pi^{-2/3}(\tilde{\mu}_k + 2) - \pi^{-2/3}(\tilde{\gamma}_k + 2)\right)\right)_{k=1}^m\Big| \leq cn^{-\chi}.$$

From this it follows that for large n,

$$\left|\mathbb{E}\varphi(\lambda_k + o_k(n))_{k=1}^m - \mathbb{E}\varphi\left(\tilde{\lambda}_k + o_k(n)\right)_{k=1}^m\right| \leq cn^{-\chi} \tag{16}$$

where $\lambda_k = n^{2/3}(\pi s_A)^{2/3}(\mu_k - A)$, $\tilde{\lambda}_k = n^{2/3}(\tilde{\mu}_k + 2)$ and $o_k(n) = \Lambda_k^0 - o(1) - n^{2/3}(\tilde{\gamma}_k + 2)$. Here $o_k(n) \to 0$ as $n \to \infty$ because from the classical asymptotics

$$n^{2/3}(\tilde{\gamma}_k + 2) = \left(\frac{3\pi}{2}\right)^{2/3} k^{2/3} + O\left(\frac{k^{5/3}}{n}\right)$$

from Fact 19 it follows that $o_k(n) = O\left(k^{5/3}/n\right)$. Since φ is in particular Lipschitz continuous therefore

$$\left|\mathbb{E}\varphi(\lambda_k + o_k(n))_{k=1}^m - \mathbb{E}\varphi(\lambda_k)_{k=1}^m\right| = o(n) \tag{17}$$

and likewise for $\tilde{\lambda}$. Consequently, by a triangle inequality with (16) and (17), for large n,

$$\left|\mathbb{E}\varphi\left(\lambda_k\right)_{k=1}^m - \mathbb{E}\varphi\left(\tilde{\lambda}_k\right)_{k=1}^m\right| \leq cn^{-\chi} + o\left(n\right).$$

Hence, for large n,

$$\left|\mathbb{E}\varphi\left(\lambda_k\right)_{k=1}^m - \mathbb{E}\varphi\left(\Lambda_k\right)_{k=1}^m\right| \leq cn^{-\chi} + o\left(n\right) + \left|\mathbb{E}\varphi\left(\tilde{\lambda}_k\right)_{k=1}^m - \mathbb{E}\varphi\left(\Lambda_k\right)_{k=1}^m\right|$$

which vanishes with n by (15) and Lemma 82. □

Example 20. By Lemma 14, universality for the Gaussian beta ensemble takes the form

$$\tilde{\lambda}_k^{(n)} = n^{2/3}\left(\tilde{\mu}_k^{(n)} + 2\right) \Longrightarrow \Lambda_k$$

in the sense of finite-dimensional distributions.

Additionally, there is the following variant of Fact 18 in the specialized case $m = 1$. This alternative statement was verified by the authors in Bourgade, Erdös, and Yau (2014). It is extremely useful for comparing many individual particles at once, e.g. for a whole family of observables. For example, this result is essential in establishing the asymptotics in probability for the general beta ensembles seen in Section 7.2 that are required for establishing the secular function convergence (4) in Section 7.3.

Fact 21. *(Theorem 2.1 Bourgade, Erdös, and Yau 2014) Let $\beta \geq 1$ and μ, ν be two general beta ensembles whose equilibrium densities both have support with left endpoint 0 satisfying $s_0 = 1$. For any constant $\kappa < 2/5$ there exists positive constants ξ, χ, c_1 such that for a given collection of observables*

$$\left\{O_{n,k} \mid n \geq 1, 1 \leq k \leq n^\kappa\right\}$$

where $O_{n,k} \in C_c^2\left(\mathbb{R}\right)$, have support size up to n^ξ and satisfy that $\left\|O_{n,k}\right\|_\infty$, $\left\|O'_{n,k}\right\|_\infty$, $\left\|O''_{n,k}\right\|_\infty \leq c_2$ for some single constant $c_2 > 0$, then for any n and any $1 \leq k \leq n^\kappa$,

$$\left|\mathbb{E}O_{n,k}\left(n^{2/3}k^{1/3}\left(\mu_k^{(n)} - \gamma_{\mu,k}^{(n)}\right)\right) - \mathbb{E}O_{n,k}\left(n^{2/3}k^{1/3}\left(\nu_k^{(n)} - \gamma_{\nu,k}^{(n)}\right)\right)\right| \leq c_1 n^{-\chi}$$

where $\gamma_{\mu,k}^{(n)}$, $\gamma_{\nu,k}^{(n)}$ are the n-quantiles of μ, ν. Moreover the n-quantiles may be taken to be the same, e.g. replace $\gamma_{\nu,k}^{(n)}$ with $\gamma_{\mu,k}^{(n)}$.

4 The stochastic Airy operator and the Airy beta point process

4.1 Distributional definition of the stochastic Airy operator

This section follows the work in Holcomb and Virág (2019), Ramirez, Rider, and Virág (2011), and Minami (2014) to formally define the stochastic Airy operator $\mathcal{H}$, informally defined in (3), in terms of distributions. This construction will also yield a bilinear form which will be used to develop asymptotics on the Airy process Λ_k in Section 4.3. Fix $\beta > 0$. Let $\mathcal{D} = \mathcal{D}'(\mathbb{R}_{>0})$ be the space of all distributions on $\mathbb{R}_{>0}$. Let H^1_{loc} be the space of $f \in AC(\mathbb{R}_{\geq 0}, \mathbb{R})$ satisfying that $f'1_I \in L^2$ for any compact $I \subseteq \mathbb{R}_{\geq 0}$. Note that functions in H^1_{loc} are a.e. differentiable and are continuous. Furthermore, the space L^1_{loc} is a subspace of H^1_{loc}. The **stochastic Airy operator with parameter** β is defined as the random linear map $\mathcal{H} = \mathcal{H}_\beta : H^1_{\text{loc}} \to \mathcal{D}$ where $\mathcal{H}(f)$ is the distribution

$$-\partial_x^2 f + xf + \frac{2}{\sqrt{\beta}} f b'_x \tag{18}$$

and b is a standard Brownian motion. Here $-\partial_x^2 f$ is defined as a distributional derivative of the distribution

$$\langle f, \phi \rangle = \int_0^\infty f(x)\phi(x), \quad \phi \in C_c^\infty.$$

Continuing, the latter distribution is well-defined because $L^1_{\text{loc}} \subseteq H^1_{\text{loc}}$: $\langle -\partial_x^2 f, \phi \rangle = \langle f, \partial_x^2 \phi \rangle$. Next, xf is the distribution $\langle xf, \phi \rangle = \langle f, x\phi \rangle$. The last term fb' is defined as the distributional derivative of the continuous function

$$\int_0^x f_t b'_t dt := -\int_0^x b_t f'(t)\, dt + f(x)\, b_x. \tag{19}$$

Define L^* to be the space of $f \in AC(\mathbb{R}_{\geq 0}, \mathbb{R})$ with Dirichlet boundary condition $f(0) = 0$ and satisfying

$$\|f\|_*^2 = \int_0^\infty f'(x)^2 + xf(x)^2\, dx < \infty. \tag{20}$$

An eigenvalue, eigenfunction pair of $\mathcal{H}$ is $(\Lambda, f) \in \mathbb{R} \times L^*$ so that $\mathcal{H}f = \Lambda f$ in the sense of distributions. This choice of definition for $\|\cdot\|_*$ is from Holcomb and Virág (2019) but another common choice, for example seen in Ramirez, Rider, and Virág (2011), is replacing only $xf(x)^2$ with $(1+x)f(x)^2$. Later in Section 4.2 it is remarked that the difference is non-consequential. Also, note that L^* contains $\phi \in C_c^\infty(\mathbb{R}_{\geq 0})$, $\phi(0) = 0$. The following are some properties about L^* and its norm:

Lemma 22. *If $f \in L^*$ then f, $\sqrt{x}f$ and $f' \in L^2(\mathbb{R}_{\geq 0}, \mathbb{R})$ and hence*

$$\|f\|_*^2 = \|f'\|_2^2 + \left\|\sqrt{x}f\right\|_2^2.$$

Proof. $\|\sqrt{x}f\|_2^2$ and $\|f'\|_2^2 \leq \|f\|_*^2$, and since f is continuous therefore $\int_0^1 f(x)^2\,dx < \infty$. Hence,

$$\int_0^\infty f(x)^2\,dx \leq \int_0^1 f(x)^2\,dx + \int_1^\infty f(x)^2\,xdx < \infty.$$

□

Lemma 23. *L^* is a real normed vector space with norm $\|\cdot\|_*$ and inner product*

$$\langle f, g\rangle_* = \langle f', g'\rangle_2 + \left\langle \sqrt{x}f, \sqrt{x}g\right\rangle_2.$$

Proof. It is immediate that $\langle\cdot,\cdot\rangle_*$ defines an inner product and hence induces the norm $\|\cdot\|_*$. □

4.2 A bilinear form and the Airy beta point process

Following Holcomb and Virág (2019), Ramirez, Rider, and Virág (2011), and Minami (2014) a symmetric bilinear form $\langle \mathcal{H}f, f\rangle$ will now be defined on L^*, which as noted earlier will be used to develop asymptotics on the Airy process Λ_k. First, define for $f \in L^*$,

$$\langle \mathcal{A}f, f\rangle = \|f\|_*^2 \tag{21}$$

where

$$\mathcal{A} = -\partial_x^2 + x \tag{22}$$

is the classical Airy operator. Note that this agrees with the distributional definition when $f = \phi \in L^* \cap C_c^\infty$ by performing integration by parts:

$$\left\langle -\partial_x^2\phi, \phi\right\rangle = \left\langle \phi, -\partial_x^2\phi\right\rangle = \langle \phi', \phi'\rangle_2$$

and $\langle x\phi, \phi\rangle = \langle \sqrt{x}\phi, \sqrt{x}\phi\rangle_2$, so that by Lemma 23, $\langle \phi', \phi'\rangle_2 + \langle \sqrt{x}\phi, \sqrt{x}\phi\rangle_2 = \|\phi\|_*^2$.

Second, define

$$\langle fb', f\rangle = \left\langle f, \bar{b}' f\right\rangle_2 - 2\left\langle f', \tilde{b} f\right\rangle_2 \tag{23}$$

where

$$b = \bar{b} + \tilde{b} \quad \text{where} \quad \bar{b}_x = \int_x^{x+1} b_t\,dt \quad \text{and} \quad \tilde{b} = b - \bar{b}. \tag{24}$$

Note the following:

Fact 24. *(Lemma 2.3 Ramirez, Rider, and Virág 2011) There is a random constant $C > 0$ so that almost surely*

$$\left|\overline{b}_x'\right|, \left|\tilde{b}_x\right| \leq C\sqrt{\log(2+x)}, \quad x > 0.$$

(23) is now well-defined: in using Lemma 22, for some constant $x_0 > 0$, almost surely

$$\int_{x_0}^{\infty} f(x)^2 \left|\overline{b}_x'\right| dx \leq C \int_0^{\infty} f(x)^2 x dx < \infty,$$

and by Hölder's inequality,

$$\int_{x_0}^{\infty} \left|f'(x) f(x) \tilde{b}_x\right| dx \leq C \int_0^{\infty} \left|f'(x) f(x) \sqrt{x}\right| dx \leq C \left\|f'\right\|_2^2 \left\|\sqrt{x} f\right\|_2^2 < \infty.$$

Furthermore, note that definition (23) also agrees with the distributional one when $f = \phi \in L^* \cap C_c^\infty$: by definition (19), performing integration by parts,

$$\langle \phi b', \phi \rangle = \left\langle \partial_x \left(-\int_0^x b_t \phi'(t) dt + \phi(x) b_x \right), \phi \right\rangle = -2 \langle \phi', \phi b \rangle_2,$$

and similarly this can be shown to equal $\left\langle \phi, \overline{b}' \phi \right\rangle_2 - 2 \left\langle \phi', \tilde{b} \phi \right\rangle_2$. Finally, define

$$\langle \mathcal{H} f, f \rangle = \langle \mathcal{A} f, f \rangle + \frac{2}{\sqrt{\beta}} \langle f b', f \rangle \tag{25}$$

which is now almost surely well-defined. In particular,

$$\langle \mathcal{H} f, f \rangle = \left\|f'\right\|_2^2 + \left\|\sqrt{x} f\right\|_2^2 + \frac{2}{\sqrt{\beta}} \left\langle f, \overline{b}' f \right\rangle_2 - \frac{4}{\sqrt{\beta}} \left\langle f', \tilde{b} f \right\rangle_2. \tag{26}$$

Note that this is the same formula in Ramirez, Rider, and Virág (2011), thereby verifying the earlier comment in Section 4.1 about the choice of norm $\|\cdot\|_*$ in (20). A benefit of this choice is the explicit action of the classical Airy operator $\mathcal{A}$ and the white noise term b' in the bilinear form. For example, this will be helpful in developing asymptotics on the Airy process due to the Courant-Fischer characterization now being available to identify eigenvalues of the operators:

Fact 25. *(Section 4.3 Holcomb and Virág (2019), Remark 2 Minami (2014)) The eigenvalues*

Λ_k^0 *of the deterministic Airy operator* $\mathcal{A}$ *satisfy*

$$\Lambda_k^0 = \inf_{\dim B = k} \sup_{\substack{f \in B \\ \|f\|_2 = 1}} \langle \mathcal{A} f, f \rangle \tag{27}$$

where the infimum is over all linear subspaces B *of* L^**, and similarly for the stochastic Airy operator* $\mathcal{H}$*,*

$$\Lambda_k = \inf_{\dim B = k} \sup_{\substack{f \in B \\ \|f\|_2 = 1}} \langle \mathcal{H} f, f \rangle . \tag{28}$$

Using that Λ_k^0 is the negative of the kth zero of the classical Airy function Ai, there are the following classical asymptotics which will be useful throughout:

Fact 26. *(Equation 2.52 Vallée and Soares (2010))*

$$\Lambda_k^0 = \left(\frac{3\pi}{2}\right)^{2/3} k^{2/3} + o\left(1\right).$$

The following are some additional properties of the Airy process that will also be useful throughout:

Fact 27. *(Proposition 3.5 Ramirez, Rider, and Virág (2011)) Almost surely* $\Lambda_1 < \Lambda_2 < \cdots$*,* $\Lambda_k \to \infty$ *and* Λ_k *has no finite accumulation point.*

Proposition 28. *For any deterministic constant* $c \in \mathbb{R}$*, almost surely* $c \notin \{\Lambda_k\}$*.*

Proof. If $\mathcal{H} f = cf$ for $f \in L^*$ then almost surely $f'(x) / (\sqrt{x} f(x)) \to 1$ as $x \to \infty$ (p. 929 Ramirez, Rider, and Virág (2011)). Consequently, for each $\varepsilon > 0$ there exists $x_0 > 0$ large so that for all $x \geq x_0$,

$$(1 - \varepsilon) x^{1/2} < \frac{f'(x)}{f(x)} < (1 + \varepsilon) x^{1/2}.$$

Integrating then yields, for some random constants $0 < C_1, C_2 < \infty$,

$$C_1 e^{\frac{2}{3}(1-\varepsilon) x^{3/2}} \leq f(x) \leq C_2 e^{\frac{2}{3}(1+\varepsilon) x^{3/2}}, \quad x \geq x_0.$$

In particular, in choosing $\varepsilon < 1/4$, $f(x)^2 \geq \exp\left(x^{3/2}\right)$ for large x and hence $f \notin L^2$. By Lemma 22 a contradiction is reached. $\square$

4.3 Asymptotics of the Airy beta point process

The bilinear form for the stochastic Airy operator in Section 4.2 will now be used to develop asymptotics of the Airy process Λ_k. Namely these are the new results Theorem 1 and

Corollary 2, asymptotics in probability, and the known result Corollary 33, almost surely asymptotics. These will be essential in later sections, such as for proving the secular function convergence in Theorem 4 in Section 7 for the general beta ensembles and stochastic Airy operator.

The arguments follow and expand upon the work in Ramirez, Rider, and Virág (2011) and Holcomb and Virág (2019). In particular, for this section $\overline{b}$, originally defined in (24) as an average over an interval of length 1, is instead defined on one of length $k^{-\alpha}$: for a fixed integer $k \geq 1$ and fixed $\alpha > 0$ define

$$b = \overline{b} + \tilde{b} \quad \text{where} \quad \overline{b}_x = k^{\alpha} \int_x^{x+k^{-\alpha}} b_t dt \quad \text{and} \quad \tilde{b} = b - \overline{b}. \tag{29}$$

Lemma 29 below shows that the bilinear form $\langle \mathcal{H} f, f \rangle$ defined in Section 4.2, now with the decomposition (29) of b, is still almost surely well-defined for the same reasons.

The following is a probabilistic version of Fact 24 with more precise estimates, including an almost sure statement. The argument there is extended to the new decomposition (29) here to yield these new results.

Lemma 29. *For any $\alpha > 0$ and $r > 0$ with $2r < 1$ there exists $k_0 = k_0(\alpha, r) \geq 1$ large so that for $k \geq k_0$, with α, k determining (29),*

$$\mathbb{P}\left(\left| \overline{b}'_x \right| \leq k^{\alpha(1-r)} \sqrt{\log(2+x)} \quad \text{and} \quad \left| \tilde{b}_x \right| \leq k^{-\alpha r} \sqrt{\log(2+x)} \quad \text{for all} \quad x > 0 \right)$$
$$\geq 1 - 2^{-k^{\alpha(1-2r)}/19}. \tag{30}$$

In particular, there exists a random constant $C = C(\alpha, k) > 0$ so that almost surely,

$$\left| \overline{b}'_x \right|, \ \left| \tilde{b}_x \right| \leq C \sqrt{\log(2+x)} \quad \text{for all} \quad x > 0.$$

Remark 30. Consequently, a sharp bound on $\tilde{b}$ can be obtained at the cost of the bound on $\overline{b}'$. As seen later, the choice of the sharpness on $\tilde{b}$ is useful.

Proof. Consider the first claim. Choose $k_0 = k_0(\alpha, r) \geq 1$ large so that for $k \geq k_0$, $k^{\alpha(1-2r)}/18 \geq 2$ and

$$2k^{2\alpha} 2^{-k^{\alpha(1-2r)}/18} + \frac{k^{\alpha}}{k^{\alpha(1-2r)}/18 - 1} 2^{1-k^{\alpha(1-2r)}/18} \leq 2^{-k^{\alpha(1-2r)}/19}. \tag{31}$$

Fix $k \geq k_0$ so that α, k determine (29). For the first claim it suffices to show

$$\mathbb{P}\left(\sup_{x>0}\sup_{0\leq t\leq k^{-\alpha}}\frac{|b_{x+t}-b_x|}{\sqrt{\log(2+x)}}\leq k^{-\alpha r}\right)\geq 1-2^{-k^{\alpha(1-2r)}/19}. \tag{32}$$

This is because within this event, for all $x>0$,

$$\left|\overline{b}'_x\right| = k^\alpha\left|b_{x+k^{-\alpha}}-b_x\right| \leq k^{\alpha(1-r)}\sqrt{\log(2+x)}$$

and

$$\left|\widetilde{b}_x\right| = \left|\overline{b}_x - b_x\right| \leq k^\alpha\int_0^{k^{-\alpha}}\left|b_{x+t}-b_x\right|dt \leq k^{-\alpha r}\sqrt{\log(2+x)}.$$

Define

$$X_n = \sup_{0\leq t\leq k^{-\alpha}}\left|b_{nk^{-\alpha}+t}-b_{nk^{-\alpha}}\right|, \quad n\geq 0$$

which are iid by independence of Brownian increments. For $x \geq 0$ define

$$[x] = \max\left\{nk^{-\alpha} : nk^{-\alpha}\leq x, \quad n\in\mathbb{Z}_{\geq 0}\right\}.$$

Note that $[x]\,k^\alpha \in \mathbb{Z}_{\geq 0}$ and for $0\leq t\leq k^{-\alpha}$, $[x+t]$ is either $[x]$ or $[x]+k^{-\alpha}$. Choose $n_0 = \lceil 2k^\alpha\rceil$ so that $n_0k^{-\alpha}\geq 2$, and let

$$Y_n = \frac{X_n}{\sqrt{\log(nk^{-\alpha})}}, \quad n\geq n_0$$

and $Y_n = X_n/\sqrt{\log 2}$ for $0\leq n<n_0$. Note that for $x>0$,

$$Y_{[x]k^\alpha} = \frac{X_{[x]k^\alpha}}{\sqrt{\log[x]}},\ [x]\,k^\alpha\geq n_0 \quad\text{and}\quad Y_{[x]k^\alpha} = \frac{X_{[x]k^\alpha}}{\sqrt{\log 2}},\ [x]\,k^\alpha < n_0.$$

Since for x, $t\geq 0$,

$$|b_{x+t}-b_x| \leq \left|b_{x+t}-b_{[x+t]}\right| + \left|b_{[x+t]}-b_{[x]}\right| + \left|b_{[x]}-b_x\right|,$$

to establish (32) it further suffices to show that

$$\mathbb{P}\left(\sup_{n\geq 0}Y_n \leq \frac{1}{3}k^{-\alpha r}\right)\geq 1-2^{-k^{\alpha(1-2r)}/19} \tag{33}$$

and that almost surely,

$$\sup_{x>0}\sup_{0\le t\le k^{-\alpha}}\frac{|b_{x+t}-b_x|}{\sqrt{\log(2+x)}}\le 3\sup_{n\ge 0}Y_n. \tag{34}$$

To illustrate establishing (34), for $x>0$ with $[x]\,k^{\alpha}\ge n_0$,

$$\sup_{0\le t\le k^{-\alpha}}\frac{\left|b_{x+t}-b_{[x+t]}\right|}{\sqrt{\log(2+x)}}\le\max\left\{\sup_{\substack{0\le t\le k^{-\alpha}\\ [x+t]=[x]}}\frac{\left|b_{x+t}-b_{[x+t]}\right|}{\sqrt{\log[x]}},\ \sup_{\substack{0\le t\le k^{-\alpha}\\ [x+t]=[x]+k^{-\alpha}}}\frac{\left|b_{x+t}-b_{[x+t]}\right|}{\sqrt{\log([x]+k^{-\alpha})}}\right\}.$$

Write

$$\sup_{\substack{0\le t\le k^{-\alpha}\\ [x+t]=[x]}}\frac{\left|b_{x+t}-b_{[x+t]}\right|}{\sqrt{\log[x]}}=\sup_{\substack{0\le t\le k^{-\alpha}\\ [x+t]=[x]}}\frac{\left|b_{[x]+s+t}-b_{[x]}\right|}{\sqrt{\log[x]}}$$

where $x=[x]+s$ for $0\le s<k^{-\alpha}$. For such t in the supremum, $[x]=[x+t]=[[x]+s+t]$ and hence $s+t<k^{-\alpha}$. Therefore, the above is less than or equal to

$$\sup_{0\le t\le k^{-\alpha}}\frac{\left|b_{[x]+t}-b_{[x]}\right|}{\sqrt{\log[x]}}=Y_{[x]k^{\alpha}}.$$

The rest of the argument for (34) is similar. Next, to establish (33) and thereby complete the proof of the first claim, first note that for any $c>0$,

$$\mathbb{P}(X_n>c)\le\mathbb{P}\left(\sup_{0\le t\le k^{-\alpha}}|b_t|\ge c\right)\le 2\mathbb{P}\left(\sup_{0\le t\le k^{-\alpha}}b_t\ge c\right).$$

Furthermore, by the reflection principle the latter equals

$$4\mathbb{P}\left(b_{k^{-\alpha}}\ge c\right)=\frac{4}{\sqrt{2\pi}}k^{\alpha/2}\int_c^{\infty}e^{-x^2/\left(2k^{-\alpha}\right)}dx.$$

With $\operatorname{erfc}(x)=2\pi^{-1/2}\int_x^{\infty}e^{-t^2}dt$ the complementary error function, a substitution yields

$$\frac{\sqrt{\pi}}{2\sqrt{2k^{-\alpha}}}\operatorname{erfc}\left(\frac{c}{\sqrt{2k^{-\alpha}}}\right)=\int_c^{\infty}e^{-x^2/\left(2k^{-\alpha}\right)}dx.$$

Hence, using the inequality $\operatorname{erfc}(x)\le\exp(-x^2)$ (Equation 5 in Chiani, Dardari, and Simon (2003)),

$$\int_c^{\infty}e^{-x^2/\left(2k^{-\alpha}\right)}dx\le\frac{\sqrt{\pi}}{2\sqrt{2}}k^{\alpha/2}e^{-k^{\alpha}c^2/2}.$$

In summary,

$$\mathbb{P}\left(X_n > c\right) \leq k^\alpha e^{-k^\alpha c^2/2}.$$

Now,

$$\mathbb{P}\left(\sup_{n\geq 0} Y_n > c\right) \leq \left(n_0 - 1\right)\mathbb{P}\left(X_0 > c\sqrt{\log 2}\right) + \sum_{n_0}^{\infty}\mathbb{P}\left(X_n > c\sqrt{\log\left(nk^{-\alpha}\right)}\right)$$

$$\leq \left(n_0 - 1\right) k^\alpha 2^{-k^\alpha c^2/2} + k^\alpha \sum_{n_0}^{\infty}\left(nk^{-\alpha}\right)^{-k^\alpha c^2/2}.$$

Choose $c = k^{-\alpha r}/3$. Then, using that $n_0 k^{-\alpha} \geq 2$ and that $k^{\alpha(1-2r)}/18 \geq 2$,

$$\mathbb{P}\left(\sup_{n\geq 0} Y_n > \frac{1}{3}k^{-\alpha r}\right) \leq \left(n_0 - 1\right) k^\alpha 2^{-k^{\alpha(1-2r)}/18} + k^\alpha \int_{n_0 k^{-\alpha}}^{\infty} x^{-k^{\alpha(1-2r)}/18} dx$$

$$\leq \left(n_0 - 1\right) k^\alpha 2^{-k^{\alpha(1-2r)}/18} + \frac{k^\alpha}{k^{\alpha(1-2r)}/18 - 1} 2^{1-k^{\alpha(1-2r)}/18}.$$

Hence, by (31) and that $n_0 = \lceil 2k^\alpha \rceil$,

$$\mathbb{P}\left(\sup_{n\geq 0} Y_n > \frac{1}{3}k^{-\alpha r}\right) \leq 2^{-k^{\alpha(1-2r)}/19}.$$

For the second claim, the argument is mostly similar. Choose k_0 large so that for $k \geq k_0$, $c \geq 1$, $k^\alpha c^2/2 \geq 2$ and

$$2k^{2\alpha} 2^{-k^\alpha c^2/2} + \frac{2k^\alpha}{k^\alpha c^2/2 - 1} 2^{-k^\alpha c^2/2} \leq 2^{-k^\alpha c^2/3}.$$

Fix α and $k \geq k_0$ determining (29). Again, $\sup_{n\geq 0} Y_n \leq c$ implies

$$\left|\overline{b}_x'\right| \leq 3ck^\alpha \sqrt{\log\left(2+x\right)} \quad \text{and} \quad \left|\tilde{b}_x\right| \leq 3c\sqrt{\log\left(2+x\right)}, \quad x > 0$$

and

$$\mathbb{P}\left(\sup_{n\geq 0} Y_n > c\right) \leq \left(n_0 - 1\right) k^\alpha 2^{-k^\alpha c^2/2} + k^\alpha \int_2^\infty x^{-k^\alpha c^2/2} dx \leq 2^{-k^\alpha c^2/3}.$$

Hence, in letting $Z = \sup_{n\geq 0} Y_n$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(Z > n\right) < \infty,$$

so that by Borel-Cantelli,

$$\mathbb{P}\left(\bigcup_{m=1}^{\infty}\bigcap_{n\geq m}\{Z\leq n\}\right)=1.$$

□

Before continuing it is important to note that with the new decomposition (29), the Airy process as given in Fact 25, originally with the $k^{-\alpha}=1$ decomposition, does not change. It is clear that the argument in Corollary 2.6 in Ramirez, Rider, and Virág (2011) in establishing (28) in the $k^{-\alpha}=1$ case, now combined with Lemma 29, continues to work the same way. For example, for verifying (28) for Λ_1, one item of verification is that still

$$\left.\frac{d}{d\varepsilon}\frac{\langle\mathcal{H}(f+\varepsilon\varphi),f+\varepsilon\varphi\rangle}{\|f+\varepsilon\varphi\|_2^2}\right|_{\varepsilon=0}=0$$

implies $\mathcal{H}f=\Lambda_1 f$ where $\varphi\in C_c^{\infty}$ and $f\in L^*$, $\|f\|_2=1$ is a minimizer of

$$\langle\mathcal{H}f,f\rangle=\tilde{\Lambda}_1=\inf\left\{\langle\mathcal{H}g,g\rangle : g\in L^*,\|g\|_2=1\right\}.$$

By direct calculation in using (26), in only using the abstract relations $b=\bar{b}+\tilde{b}$,

$$\left.\frac{d}{d\varepsilon}\langle\mathcal{H}(f+\varepsilon\varphi),f+\varepsilon\varphi\rangle\right|_{\varepsilon=0}$$

$$=2\langle f',\varphi'\rangle_2+2\left\langle\sqrt{x}f,\sqrt{x}\varphi\right\rangle_2+\frac{4}{\sqrt{\beta}}\left(\left\langle f\bar{b}',\varphi\right\rangle_2-\left\langle f\tilde{b},\varphi'\right\rangle_2-\left\langle f'\tilde{b},\varphi\right\rangle_2\right).$$

By using integration by parts on the $\left\langle f\bar{b}',\varphi\right\rangle_2$ term the above simplifies to

$$2\langle f',\varphi'\rangle_2+2\left\langle\sqrt{x}f,\sqrt{x}\varphi\right\rangle_2-\frac{4}{\sqrt{\beta}}\left(\langle f'b,\varphi\rangle_2+\langle fb,\varphi'\rangle_2\right).$$

Continuing, the distributional definitions in Section 4.1 shows that this equals

$$2\left\langle-\partial_x^2 f,\varphi\right\rangle+2\langle xf,\varphi\rangle+\frac{4}{\sqrt{\beta}}\langle fb',\varphi\rangle=2\langle Hf,\varphi\rangle.$$

Similarly,

$$\left.\frac{d}{d\varepsilon}\|f+\varepsilon\varphi\|_2^2\right|_{\varepsilon=0}=2\langle f,\varphi\rangle.$$

and consequently,

$$\frac{d}{d\varepsilon}\frac{\langle\mathcal{H}(f+\varepsilon\varphi),f+\varepsilon\varphi\rangle}{\|f+\varepsilon\varphi\|_2^2}\Bigg|_{\varepsilon=0}=\frac{2\langle\mathcal{H}f,\varphi\rangle\times 1-\langle\mathcal{H}f,f\rangle\times 2\langle f,\varphi\rangle}{1}.$$

This set to zero then yields $\langle\mathcal{H}f,\varphi\rangle = \tilde{\Lambda}_1\langle f,\varphi\rangle$. Hence, $\mathcal{H}f = \tilde{\Lambda}_1 f$ in the sense of distributions so that $\tilde{\Lambda}_1=\Lambda_1$. Another technical lemma:

Lemma 31. *For any $\delta>0$ there exists $c_0=c_0(\delta)$ large so that for $c\geq c_0$,*

$$c\log(2+x)\leq x+c^{1+\delta},\quad x>0.$$

Proof. Assume $c\geq 2$. Rewrite the inequality as

$$\log(2+x)\leq\frac{1}{c}x+c^{\delta},\quad x>0. \tag{35}$$

Note that $\log(2+x)$, $c^{-1}x+c^{\delta}$, $x>0$ are increasing functions and their derivatives satisfy

$$\frac{1}{2+x}\leq\frac{1}{c}\Longleftrightarrow x\geq c-2.$$

Consequently, if the inequality holds for $0<x<c-2$ then it also holds for $x\geq c-2$. To show (35) for $0<x<c-2$ it suffices to show that

$$\log(2+x)\leq c^{\delta},\quad 0<x<c-2.$$

Since $\log(2+x)$ is increasing it further suffices to show that

$$\log(2+(c-2))=\log c\leq c^{\delta}$$

which trivially has a solution for large c. □

Lemmas 29 and 31 may now be combined to produce a certain linear bound in probability for $\left|\bar{b}'\right|$ and $\tilde{b}^2$ in the following lemma. Coming back to Remark 30 it is now seen why the sharpness of bound is chosen for $\tilde{b}$. In fact, it appears that the optimal bound in Theorem 1 requires precisely the inequality in the following lemma.

Lemma 32. *For any $0<\varepsilon<1/6$ there exists $k_0=k_0(\varepsilon)$ large so that for $k\geq k_0$, for some determination of (29) dependent on ε and k,*

$$\mathbb{P}\left(\left|\bar{b}'_x\right|+\frac{2}{\sqrt{\beta}}k^{1/3+\varepsilon}\tilde{b}_x^2\leq\frac{\sqrt{\beta}}{2}k^{-1/3-\varepsilon}x+\frac{\sqrt{\beta}}{2}k^{1/3-\varepsilon}\right)\geq 1-2^{-k^{\chi}}$$

where $\chi = \chi(\varepsilon) > 0$.

Proof. Choose α, $r > 0$ to satisfy $2r < 1$ and $\alpha(1+r) - 1/3 < \varepsilon$ so that

$$1/3 + \varepsilon + \alpha(1-r) < 2/3 + 2\varepsilon - 2\alpha r. \tag{36}$$

Assume $\varepsilon < \alpha r$ so that the choices of ε, α, r are now characterized by $2r < 1$ and

$$\alpha(1+r) - \frac{1}{3} < \varepsilon < \alpha r \Longleftrightarrow 0 < \alpha r - \varepsilon < \frac{1}{3} - \alpha. \tag{37}$$

Later it will be shown that the solution is well-defined. Next, choose $\delta > 0$ so that

$$\left(\frac{2}{3} + 2\varepsilon - 2\alpha r\right) + \delta\left(\frac{2}{3} + 2\varepsilon - 2\alpha r\right) < \frac{2}{3}. \tag{38}$$

By Lemma 31 there exists $c_0 = c_0(\delta)$ large so that for $c \geq c_0$,

$$c\log(2+x) \leq x + c^{1+\delta}, \quad x > 0. \tag{39}$$

Choose $k_0 = k_0(\alpha, r, \varepsilon, \delta) \geq 1$ large so that for $k \geq k_0$,

$$\frac{2}{\sqrt{\beta}}k^{1/3+\varepsilon+\alpha(1-r)}\sqrt{\log(2+x)} + \frac{4}{\beta}k^{2/3+2\varepsilon-2\alpha r}\log(2+x)$$

$$\leq \frac{8}{\beta}k^{2/3+2\varepsilon-2\alpha r}\log(2+x), \quad x > 0, \tag{40}$$

$$\frac{8}{\beta}k^{2/3+2\varepsilon-2\alpha r}\log(2+x) \leq x + \left(\frac{8}{\beta}\right)^{1+\delta}k^{(2/3+2\varepsilon-2\alpha r)+\delta(2/3+2\varepsilon-2\alpha r)}, \quad x > 0, \tag{41}$$

and

$$\left(\frac{8}{\beta}\right)^{1+\delta}k^{(2/3+2\varepsilon-2\alpha r)+\delta(2/3+2\varepsilon-2\alpha r)} \leq k^{2/3}. \tag{42}$$

For example, (40) follows from applying

$$\sqrt{\log(2+x)} \leq \frac{1}{\sqrt{\log 2}}\log(2+x), \quad x > 0,$$

and (36), (41) is due to (37) implying

$$2\alpha r - 2\varepsilon < \frac{2}{3} \Longrightarrow \frac{2}{3} + 2\varepsilon - 2\alpha r > 0$$

and then using (39), and (42) is immediate from (38). Furthermore, choose k_0 so that

Lemma 29 holds with α, r chosen here. Fix $k \geq k_0$ and choose a realization of (30), with α, k determining (29), with probability at least $1 - 2^{-k^{\alpha(1-2r)}/19}$. Then,

$$\left|\overline{b}'_x\right| + \frac{2}{\sqrt{\beta}} k^{1/3+\varepsilon} \widetilde{b}_x^2 \leq k^{\alpha(1-r)} \sqrt{\log(2+x)} + \frac{2}{\sqrt{\beta}} k^{1/3+\varepsilon-2\alpha r} \log(2+x), \quad x > 0.$$

It now suffices to show

$$k^{\alpha(1-r)} \sqrt{\log(2+x)} + \frac{2}{\sqrt{\beta}} k^{1/3+\varepsilon-2\alpha r} \log(2+x) \leq \frac{\sqrt{\beta}}{2} k^{-1/3-\varepsilon} x + \frac{\sqrt{\beta}}{2} k^{1/3-\varepsilon}$$

or equivalently

$$\frac{2}{\sqrt{\beta}} k^{1/3+\varepsilon+\alpha(1-r)} \sqrt{\log(2+x)} + \frac{4}{\beta} k^{2/3+2\varepsilon-2\alpha r} \log(2+x) \leq x + k^{2/3}, \quad x > 0.$$

This is immediate from (40), (41) and (42). To check existence of such ε, α, r first note that by (37), necessarily $\alpha < 1/3$, and since $r < 1/2$, therefore $\varepsilon < 1/6$. Let $\alpha = 1/3 - x$, $r = 1/2 - y$ and $\varepsilon = 1/6 - z$ for small x, y, $z > 0$. Inequality (37) then becomes

$$0 < -\frac{1}{2}x - \frac{1}{3}y + xy + z < x.$$

Given any x, $y > 0$ there is a solution of the above inequality since

$$\frac{1}{2}x + \frac{1}{3}y - xy < z < \frac{3}{2}x + \frac{1}{3}y - xy$$

is equivalent to it. Furthermore, it shows that in taking x, $y \to 0^+$ then $z \to 0^+$ (i.e. $\varepsilon = (1/6)^-$), and $x \to (1/3)^-$, $y \to (1/2)^-$ has $\frac{1}{2}x + \frac{1}{3}y - xy \to (1/6)^-$ (i.e. $\varepsilon = 0^+$). Finally, let $\chi = \alpha(1-2r) > 0$ where it is clear that the division by 19 can be removed in Lemma 29. □

Recall the statement of Theorem 1: *For any* $0 < \varepsilon < 1/6$ *there exists* $k_0 = k_0(\varepsilon)$ *large so that*

$$\mathbb{P}\left(\left|\Lambda_k - \Lambda_k^0\right| \leq k^{1/6+\varepsilon}\right) \geq 1 - 2^{-k^{\chi}}, \quad \textit{for all} \quad k \geq k_0$$

where $\chi = \chi(\varepsilon) > 0$. *In particular, for any* $\delta > 0$, $k_0 = k_0(\varepsilon, \delta)$ *may be chosen so that*

$$\mathbb{P}\left(\left|\Lambda_k - \Lambda_k^0\right| \leq k^{1/6+\varepsilon},\ k \geq k_0\right) \geq 1 - \delta.$$

Proof of Theorem 1. Consider the first claim. Applying Lemma 32 with $1/6 - \varepsilon$ yields $k_0 = k_0(\varepsilon)$ large and $\chi = \chi(\varepsilon) > 0$ so that for $k \geq k_0$, with ε, k determining (29), and with probability at least $1-2^{-k^{\chi}}$,

$$-\overline{b}'_x + \frac{2}{\sqrt{\beta}}k^{1/2-\varepsilon}\tilde{b}_x^2 \leq \frac{\sqrt{\beta}}{2}k^{-1/2+\varepsilon}x + \frac{\sqrt{\beta}}{2}k^{1/6+\varepsilon}. \tag{43}$$

Fix k and choose a realization of (43). It suffices to show that for all $f \in L^*$,

$$-k^{1/6+\varepsilon}\left\|f\right\|_2^2 + \left(1-k^{-1/2+\varepsilon}\right)\langle \mathcal{A}f, f\rangle \leq \langle \mathcal{H}f, f\rangle \leq \left(1+k^{-1/2+\varepsilon}\right)\langle \mathcal{A}f, f\rangle + k^{1/6+\varepsilon}\left\|f\right\|_2^2. \tag{44}$$

This is because for such k it would then follow from the Courant-Fischer characterizations of Λ_k, Λ_k^0 in Fact 25 that

$$-k^{1/6+\varepsilon} + \left(1-k^{-1/2+\varepsilon}\right)\Lambda_k^0 \leq \Lambda_k \leq \left(1+k^{-1/2+\varepsilon}\right)\Lambda_k^0 + k^{1/6+\varepsilon}.$$

Using the classical asymptotics $\Lambda_k^0 = \left(3\pi/2\right)^{2/3}k^{2/3} + o\left(1\right)$ where $o\left(1\right)$ vanishes with k from Fact 26 then shows that, in possibly enlarging k_0,

$$\left|\Lambda_k - \Lambda_k^0\right| \leq k^{1/6+\varepsilon}.$$

To that end, consider the inequality

$$\langle \mathcal{H}f, f\rangle \geq \left(1-k^{-1/2+\varepsilon}\right)\langle \mathcal{A}f, f\rangle - k^{1/6+\varepsilon}\left\|f\right\|_2^2$$

where the remaining inequality of (44) is similar. By (25) the above inequality is equivalent to

$$-\langle fb', f\rangle \leq \frac{\sqrt{\beta}}{2}k^{-1/2+\varepsilon}\langle \mathcal{A}f, f\rangle + \frac{\sqrt{\beta}}{2}k^{1/6+\varepsilon}\left\|f\right\|_2^2. \tag{45}$$

Recall that by (23),

$$-\langle fb', f\rangle = \left\langle f, -\overline{b}'f\right\rangle_2 - 2\left\langle f', -\tilde{b}f\right\rangle_2. \tag{46}$$

Using the inequality

$$-2yz \leq \left(\frac{\sqrt{\beta}}{2}k^{-1/2+\varepsilon}\right)^{-1}y^2 + \left(\frac{\sqrt{\beta}}{2}k^{-1/2+\varepsilon}\right)z^2$$

with $y = -\tilde{b}_x f\left(x\right)$ and $z = f'\left(x\right)$ yields

$$-2\left\langle f', -\tilde{b}f\right\rangle_2 \leq \frac{2}{\sqrt{\beta}}k^{1/2-\varepsilon}\left\langle f, \tilde{b}^2 f\right\rangle_2 + \frac{\sqrt{\beta}}{2}k^{-1/2+\varepsilon}\left\|f'\right\|_2^2.$$

Substituting this into (46) then yields

$$-\langle fb', f\rangle \le \left\langle f, \left(-\bar{b}' + \frac{2}{\sqrt{\beta}} k^{1/2-\varepsilon}\tilde{b}^2\right) f \right\rangle_2 + \frac{\sqrt{\beta}}{2} k^{-1/2+\varepsilon} \|f'\|_2^2 .$$

It then follows from (43) that

$$-\langle fb', f\rangle \le \left\langle f, \left(\frac{\sqrt{\beta}}{2} k^{-1/2+\varepsilon} x + \frac{\sqrt{\beta}}{2} k^{1/6+\varepsilon}\right) f \right\rangle_2 + \frac{\sqrt{\beta}}{2} k^{-1/2+\varepsilon} \|f'\|_2^2 .$$

By (21) this rewrites as

$$\begin{aligned} -\langle fb', f\rangle &\le \frac{\sqrt{\beta}}{2} k^{-1/2+\varepsilon} \left(\left\|\sqrt{x} f\right\|_2^2 + \|f'\|_2^2 \right) + \frac{\sqrt{\beta}}{2} k^{1/6+\varepsilon} \|f\|_2^2 \\ &= \frac{\sqrt{\beta}}{2} k^{-1/2+\varepsilon} \langle \mathcal{A} f, f\rangle + \frac{\sqrt{\beta}}{2} k^{1/6+\varepsilon} \|f\|_2^2 . \end{aligned}$$

This is precisely the required inequality (45). For the second claim, now

$$\mathbb{P}\left(\left\{ \left|\Lambda_k - \Lambda_k^0\right| \le k^{1/6+\varepsilon},\, k \ge k_0 \right\}^C \right) \le \sum_{k=k_0}^{\infty} \frac{1}{2^{k^\chi}} < \infty$$

and hence taking $k_0 \to \infty$ minimizes the probability. □

Recall Fact 26 says that the eigenvalues Λ_k^0 of the deterministic Airy operator $\mathcal{A} = -\partial_x^2 + x$, or the "$\beta = \infty$" stochastic Airy operator, have the asymptotics $\Lambda_k^0 = (3\pi/2)^{2/3} k^{2/3} + o(1)$ where $o(1)$ vanishes as $k \to \infty$. As in Holcomb and Virág (2019) it may now be concluded that almost surely $\Lambda_k / k^{2/3}$ has the same asymptotics. Furthermore, it may be newly shown that in probability Λ_k is at most $k^{1/6}$ away from Λ_k^0. Recall Corollary 2: *For any* $0 < \varepsilon < 1/6$ *there exists* $k_0 = k_0(\varepsilon)$ *large and* $\chi = \chi(\varepsilon) > 0$ *so that for* $k \ge k_0$*, with probability at least* $1 - 2^{-k^\chi}$*,*

$$\Lambda_k = \Lambda_k^0 + O\left(k^{1/6+\varepsilon}\right) = \left(\frac{3\pi}{2}\right)^{2/3} k^{2/3} + O\left(k^{1/6+\varepsilon}\right)$$

where $O\left(k^{1/6+\varepsilon}\right)$ *is deterministic.*

Proof of Corollary 2. Simply apply Theorem 1 to

$$\Lambda_k = \Lambda_k^0 + \left(\Lambda_k - \Lambda_k^0\right).$$

□

The following known law of large numbers for Λ_k (e.g. Corollary 4.3.8 Holcomb and Virág (2019)) may now be immediately recovered:

Corollary 33. *Almost surely,*

$$\frac{\Lambda_k}{k^{2/3}} \to \left(\frac{3\pi}{2}\right)^{2/3}.$$

5 Discrete models

Here the discrete random matrix models $\mathcal{H}_n$ which converge to a limiting random operator

$$\mathcal{H} = -\partial_x^2 + y'(x)$$

are defined following Ramirez, Rider, and Virág (2011) and Bloemendal and Virág (2013). For example, when

$$y'(x) = x + \frac{2}{\sqrt{\beta}} b'_x,$$

$\mathcal{H}$ is the stochastic Airy operator and $\mathcal{H}_n$ has eigenvalues the Gaussian beta ensembles. In the general setting some known convergence properties around $\mathcal{H}_n$ and $\mathcal{H}$ are reviewed and new ones are proven. This culminates in a proof of the convergence of the derivatives of the eigenfunctions of $\mathcal{H}_n$ to those of $\mathcal{H}$.

5.1 Embedding and operators

To define the discrete matrix models $\mathcal{H}_n$ limiting to the random operator

$$\mathcal{H} = -\partial_x^2 + y'(x)$$

where $y(x)$ is a continuous random process, which is more carefully defined in Section 5.2, first the setup from Ramirez, Rider, and Virág (2011) and Bloemendal and Virág (2013) is given. This includes defining the relevant spaces and operators and will be done briefly. Although some demonstration and context is given later, note that these operators are discrete versions of the limiting ones seen in $\mathcal{H}$.

Fix $m_n \to \infty$ with $m_n = o(n)$. Consider the sequence space and accompanying norm

$$\ell_n^2 = \{(v_0, v_1, \ldots) : v_j \in \mathbb{R}\}, \quad \|v\|_n^2 = \frac{1}{m_n} \sum_0^\infty v_j^2.$$

Then

$$\ell_n^2 \to L^2(\mathbb{R}_{\geq 0}, \mathbb{R}), \quad v \mapsto v(x) = v_{\lfloor m_n x \rfloor}$$

is an isometric embedding of step functions

$$v(x) = \sum_{j=0}^\infty v_j 1_{\left[\frac{j}{m_n}, \frac{j+1}{m_n}\right)}(x)$$

where $L^2(\mathbb{R}_{\geq 0}, \mathbb{R})$ has the usual $\|\cdot\|_2$ norm. Identify

$$\mathbb{R}^n = \left\{ v \in \ell_n^2 : v_j = 0, \quad j \geq n \right\}$$

and define L_n^* as the isometric image in $L^2(\mathbb{R}_{\geq 0}, \mathbb{R})$ which is then closed:

$$v(x) = \sum_{j=0}^{n-1} v_j 1_{\left[\frac{j}{m_n}, \frac{j+1}{m_n}\right)}(x).$$

To now define the relevant operators, let

$$T_n : L^2(\mathbb{R}_{\geq 0}, \mathbb{R}) \to L^2(\mathbb{R}_{\geq 0}, \mathbb{R}), \quad T_n f(x) = f\left(x + \frac{1}{m_n}\right),$$

extending the left shift on ℓ_n^2,

$$T_n v(x) = \sum_0^\infty v_{j+1} 1_{\left[\frac{j}{m_n}, \frac{j+1}{m_n}\right)}(x),$$

and let

$$T_n^\dagger f(x) = f\left(x - \frac{1}{m_n}\right) 1_{x \geq \frac{1}{m_n}},$$

extending the right shift on ℓ_n^2,

$$T_n^\dagger v(x) = \sum_1^\infty v_{j-1} 1_{\left[\frac{j}{m_n}, \frac{j+1}{m_n}\right)}(x).$$

Let, where $If(x) = f(x)$,

$$D_n = m_n (T_n - I),$$

extending the forward difference on ℓ_n^2,

$$D_n v(x) = m_n \sum_0^\infty (v_{j+1} - v_j) 1_{\left[\frac{j}{m_n}, \frac{j+1}{m_n}\right)}(x),$$

and let

$$D_n^\dagger = m_n \left(T_n^\dagger - I\right),$$

extending the backward difference on ℓ_n^2,

$$D_n^\dagger v(x) = m_n \left(-v_0 1_{\left[0, \frac{1}{m_n}\right)}(x) + \sum_1^\infty (v_{j-1} - v_j) 1_{\left[\frac{j}{m_n}, \frac{j+1}{m_n}\right)}(x) \right).$$

Define $E_n = m_n 1_{[0,m_n^{-1})}$ extending a "discrete delta function at the origin" so that $E_n v(x) = m_n v_0 1_{[0,m_n^{-1})}(x)$ and let $R_n = 1_{[0,nm_n^{-1})}$ extending orthogonal projection of ℓ_n^2 onto $\mathbb{R}^n$ so that

$$R_n v(x) = \sum_0^{n-1} v_j 1_{\left[\frac{j}{m_n}, \frac{j+1}{m_n}\right)}(x).$$

Fix two discrete-time real-valued random processes $(y_{n,i;j})_{j=0,\dots,n}$, $i = 1, 2$ with $y_{n,i;0} = 0$, and let

$$y_{n,i}(x) = \sum_0^{n} y_{n,i;j} 1_{\left[\frac{j}{m_n}, \frac{j+1}{m_n}\right)}(x).$$

Define

$$\operatorname{diag}(D_n y_{n,i}) f(x) = (D_n y_{n,i}(x)) \times f(x)$$

extending

$$\operatorname{diag}(D_n y_{n,i}) v(x) = m_n \sum_0^{\infty} v_j (y_{n,i;j+1} - y_{n,i;j}) 1_{\left[\frac{j}{m_n}, \frac{j+1}{m_n}\right)}(x).$$

Let

$$V_n = \operatorname{diag}(D_n y_{n,1}) + \frac{1}{2}\left(\operatorname{diag}(D_n y_{n,2}) T_n + T_n^\dagger \operatorname{diag}(D_n y_{n,2})\right)$$

where products are compositions. Finally, define the discrete models

$$\mathcal{H}_n = R_n \left(D_n^\dagger D_n + V_n + m_n E_n\right).$$

5.2 Tridiagonal matrix models and limiting operator

Taking the standard matrix representation with respect to $\mathbb{R}^n$ of $\mathcal{H}_n$ yields an $n \times n$ tridiagonal matrix with jth diagonal entry

$$a_j = 2m_n^2 + m_n (y_{n,1;j} - y_{n,1;j-1}), \ 1 \leq j \leq n$$

and symmetric jth off-diagonal entry

$$b_j = -m_n^2 + \frac{1}{2} m_n (y_{n,2;j} - y_{n,2;j-1}), \ 1 \leq j \leq n-1.$$

That is,

$$\mathcal{H}_n = \begin{bmatrix} a_1 & b_1 & & & 0 \\ b_1 & a_2 & b_2 & & \\ & b_2 & a_3 & & \\ & & & \ddots & b_{n-1} \\ 0 & & & b_{n-1} & a_n \end{bmatrix}.$$

As a real symmetric matrix it has real eigenvalues $\lambda_1^{(n)} \leq \cdots \leq \lambda_n^{(n)}$. Let $v_k^{(n)} \in L_n^*$ be the corresponding embedded eigenfunctions so that $\mathcal{H}_n v_k^{(n)} = \lambda_k^{(n)} v_k^{(n)}$ and they are normalized by $\|v\|_n = \|v\|_2 = 1$. With certain tightness, convergence, growth and oscillation bounds on the random processes $y_{n,i}$, known as Assumptions 1 and 2 in Ramirez, Rider, and Virág (2011) and Bloemendal and Virág (2013), a limiting operator arises as follows. Included in these assumptions is the existence of a continuous random process $y(x)$, $x \geq 0$, $y(0) = 0$ so that

$$y_{n,1} + y_{n,2} \Longrightarrow y$$

in the topology of compact convergence, and a deterministic, unbounded, nondecreasing continuous function $\overline{\eta}(x) > 0$. With a slight abuse of notation, let

$$\mathcal{H} = -\partial_x^2 + y'(x) \tag{47}$$

and consider the Dirichlet case of the eigenvalue problem $\mathcal{H}f = \Lambda f$, $f(0) = 0$ on $[0, \infty)$. The kth bottom eigenfunction, eigenvalue pair of $\mathcal{H}$ is $(f_k, \Lambda_k) \in L^* \times \mathbb{R}$, defined in a certain distributional sense, where $\|f\|_2 = 1$ and L^* is a certain subspace of $L^2([0, \infty), \mathbb{R})$ with norm

$$\|f\|_*^2 = \|f'\|_2^2 + \left\| f\sqrt{1+\overline{\eta}} \right\|_2^2.$$

These definitions are precisely formulated essentially as they were in the case of the stochastic Airy operator in Section 4.1. See Bloemendal and Virág (2013) and Ramirez, Rider, and Virág (2011) for more. Note the slight abuse of notation with Λ_k which is remarked in the following.

Example 34. The special case where

$$\overline{\eta}(x) = x \quad \text{and} \quad y'(x) = x + \frac{2}{\sqrt{\beta}} b'_x$$

yields the stochastic Airy operator which was considered in detail in Section 4.1. Here Λ_k is the stochastic Airy point process and $\lambda_k^{(n)}$, $1 \leq k \leq n$ is a rescaled Gaussian beta ensemble as in Example 20.

The following are some collected known results for later reference:

Fact 35. *Referencing from Bloemendal and Virág (2013):*

1. *(Fact 2.1) L^* functions are Hölder-$\frac{1}{2}$ continuous and satisfy $f(0) = 0$.*
2. *(Equation 2.14)*
$$f_k'(x) - f_k'(0) = -\Lambda_k \int_0^x f_k(t)\,dt + y(x) f_k(x) - \int_0^x y(t) f_k'(t)\,dt.$$
3. *(Theorem 2.10) For each k, $\lambda_k^{(n)} \Longrightarrow \Lambda_k$ and $v_k^{(n)} \Longrightarrow f_k$ in L^2 as $n \to \infty$.*

5.3 Uniform convergence of the derivatives of eigenfunctions

With the constructions done for the discrete models $\mathcal{H}_n$ and the limiting operator $\mathcal{H}$ generalizing the stochastic Airy operator, the proof of Theorem 3 can now be begun. As in Ramirez, Rider, and Virág (2011) and Bloemendal and Virág (2013) a reduction to a deterministic setting is done. There it is explained that the assumptions placed on $y_{n,i}$ yield a tightness so that for every subsequence there is a further subsequential limit, and hence may be realized almost surely on some single probability space by Skorokhod's representation theorem. In particular, $y_{n,i} \to y_i$ uniformly on compacts for some continuous functions y_i so that $y_1 + y_2 = y$ and y is continuous, and the conclusions of Fact 35 hold. Note that assuming y_i is continuous is not done in Ramirez, Rider, and Virág (2011) and Bloemendal and Virág (2013) and is an extra assumption. In the special case of the stochastic Airy operator in Example 34 the y_i may be chosen to be continuous as in Section 6 of Ramirez, Rider, and Virág (2011).

Following Ramirez, Rider, and Virág (2011) and Bloemendal and Virág (2013), define the L_n^*-norm
$$\|v\|_{*n}^2 = \|D_n v\|_2^2 + \left\| v(x)\sqrt{1+\overline{\eta}_n(x)} \right\|_2^2 + m_n v(0)^2 \tag{48}$$
where
$$\overline{\eta}_n(x) = \sum_0^{n-1} \overline{\eta}\left(\frac{j}{m_n}\right) 1_{\left[\frac{j}{m_n}, \frac{j+1}{m_n}\right)}.$$
Some additional known facts are recorded for later use:

Fact 36. *Referencing from Bloemendal and Virág (2013):*

1. *(Lemma 2.13) There exists constants c_1, $c_2 > 0$ so that for all n and $v \in L_n^*$,*
$$c_1 \|v\|_{*n}^2 - c_2 \|v\|_2^2 \le \langle v, \mathcal{H}_n v\rangle_2 \le c_2 \|v\|_{*n}^2.$$

2. *(Lemma 2.15) If $f_n \in L_n^*$ and $\|f_n\|_{*n}$ are uniformly bounded in n then there exists $f \in L^*$ so that along some subsequence $f_n \to f$ uniformly on compacts and in L^2 and $D_n f_n \to f'$ weakly in L^2.*

Just as $\mathcal{H}_n$ is defined with operators discretizing the limiting operator $\mathcal{H}$, the proof of Theorem 3 proceeds by a discrete version of

$$f_k'(x) - f_k'(0) = -\Lambda_k \int_0^x f_k(t)\,dt + y(x) f_k(x) - \int_0^x y(t) f_k'(t)\,dt \tag{49}$$

from Fact 35 and handling the individual limits. Some of these are proven as more general limiting properties in L_n^* as follows. First the $\int_0^x f_k(t)\,dt$ term in (49):

Lemma 37. *If $f_n \in L_n^*$, $\|f_n\|_{*n}$ are uniformly bounded in n, $f_n \to f$ uniformly on compacts for some $f \in L^*$ then*

$$\frac{1}{m_n} \sum_{1}^{\lfloor x m_n \rfloor + 1} f_n\left(\frac{j}{m_n}\right) \to \int_0^x f(t)\,dt$$

uniformly in x on compacts as $n \to \infty$.

Proof. Fix a compact subset I and let $x \in I$. Since $f_n \in L_n^*$ therefore $f_n(t) = f_n(j m_n^{-1})$ on $[j m_n^{-1}, (j+1) m_n^{-1})$ and so

$$\frac{1}{m_n} \sum_{1}^{\lfloor x m_n \rfloor + 1} f_n\left(\frac{j}{m_n}\right) = \sum_{1}^{\lfloor x m_n \rfloor + 1} \int_{j m_n^{-1}}^{(j+1) m_n^{-1}} f_n(t)\,dt.$$

Note that when $j = \lfloor x m_n \rfloor$, $x \in [j m_n^{-1}, (j+1) m_n^{-1})$. Hence,

$$\left| \frac{1}{m_n} \sum_{1}^{\lfloor x m_n \rfloor + 1} f_n\left(\frac{j}{m_n}\right) - \int_0^x f(t)\,dt \right|$$

$$\leq \sum_{1}^{\lfloor x m_n \rfloor - 1} \int_{j m_n^{-1}}^{(j+1) m_n^{-1}} |f_n(t) - f(t)|\,dt + r_1(n) + r_2(x,n) + r_3(x,n)$$

where

$$r_1(n) = \int_0^{m_n^{-1}} |f(t)|\,dt,\ r_2(x,n) = \int_{\lfloor x m_n \rfloor m_n^{-1}}^{x} |f_n(t) - f(t)|\,dt$$

and

$$r_3(x,n) = \int_x^{(\lfloor x m_n \rfloor + 2) m_n^{-1}} |f_n(t)|\,dt.$$

Since $f \in L^*$ is continuous by Fact 35 therefore $r_1(n) \to 0$ as $n \to \infty$. Because $x \in$

$[jm_n^{-1},(j+1)\,m_n^{-1})$ therefore

$$\left|x-\frac{\lfloor xm_n\rfloor}{m_n}\right|\le\frac{1}{m_n}.$$

Continuing, since $f_n\to f$ uniformly on compacts therefore $r_2(x,n)\to 0$ uniformly in x on I. Next, by definition (48) of $\|\cdot\|_{*n}$,

$$\frac{1}{m_n}\sum_0^{n-1}f_n\left(\frac{j}{m_n}\right)^2\left(1+\overline{\eta}\left(\frac{j}{m_n}\right)\right)=\left\|f_n(t)\sqrt{1+\overline{\eta}_n(t)}\right\|_2^2\le\|f_n\|_{*n}^2 \tag{50}$$

is uniformly bounded in n since this is the case for $\|f_n\|_{*n}$. Consequently, since $\overline{\eta}>0$ therefore

$$r_3(x,n)\le\left|x-\frac{\lfloor xm_n\rfloor+1}{m_n}\right|\left|f_n\left(\frac{\lfloor xm_n\rfloor}{m_n}\right)\right|+\frac{1}{m_n}\left|f_n\left(\frac{\lfloor xm_n\rfloor+1}{m_n}\right)\right|$$

vanishes with n uniformly in x on I. For example,

$$\left|x-\frac{\lfloor xm_n\rfloor+1}{m_n}\right|\left|f_n\left(\frac{\lfloor xm_n\rfloor}{m_n}\right)\right|\le\frac{1}{m_n}\left|f_n\left(\frac{\lfloor xm_n\rfloor}{m_n}\right)\right|$$

and

$$\frac{1}{m_n}\sum_0^{n-1}f_n\left(\frac{j}{m_n}\right)^2$$

is uniformly bounded in n by (50). So by the monotone convergence theorem,

$$\lim_{n\to\infty}\frac{1}{m_n}\sum_0^{n-1}f_n\left(\frac{j}{m_n}\right)^2<\infty$$

and hence has vanishing tails. Similarly,

$$\sum_1^{\lfloor xm_n\rfloor-1}\int_{jm_n^{-1}}^{(j+1)m_n^{-1}}|f_n(t)-f(t)|\,dt\le\frac{\lfloor xm_n\rfloor-1}{m_n}\sup_J|f_n(t)-f(t)|$$

vanishes with n uniformly in x on I where $J\supseteq I$ is some compact set. □

Continuing the discretization of (49), now consider the y_1 half of the $y(x)\,f_k(x)-\int_0^x y(t)\,f_k'(t)\,dt$ term:

Lemma 38. *If the assumptions of Lemma 37 hold and $D_nf_n\to f'$ weakly in L^2 then*

$$\frac{1}{m_n}\sum_1^{\lfloor xm_n\rfloor}diag\left(D_ny_{n,1}\right)f_n\left(\frac{j}{m_n}\right)\to y_1(x)\,f(x)-\int_0^x y_1(t)\,f'(t)\,dt$$

uniformly in x on compacts as $n\to\infty$.

Proof. Fix a compact set I and let $x\in I$. By direct computation,

$$\frac{1}{m_n}\sum_{1}^{\lfloor xm_n\rfloor}\operatorname{diag}\left(D_n y_{n,1}\right) f_n\left(\frac{j}{m_n}\right)=\sum_{1}^{\lfloor xm_n\rfloor} f_n\left(\frac{j}{m_n}\right)\left(y_{n,1;j+1}-y_{n,1;j}\right)$$

$$=y_{n,1}\left(\frac{\lfloor xm_n\rfloor+1}{m_n}\right)f_n\left(x\right)-\frac{1}{m_n}\sum_{0}^{\lfloor xm_n\rfloor-1}\left(T_n y_{n,1}\times D_n f_n\right)\left(\frac{j}{m_n}\right)-y_{n,1}\left(\frac{1}{m_n}\right)f_n\left(0\right).$$

First it is shown that $y_{n,1}\left(m_n^{-1}\right)f_n\left(0\right)\to 0$ and that

$$y_{n,1}\left(\frac{\lfloor xm_n\rfloor+1}{m_n}\right)f_n\left(x\right)\to y_1\left(x\right)f\left(x\right)$$

uniformly in $x\in I$. The first limit is due to $f_n\to f$ and $y_{n,1}\to y_1$ uniformly on compacts, and $f\left(0\right)=0$ since $f\in L^*$. For the second limit, again since $f_n\to f$ and $y_{n,1}\to y_1$ uniformly on compacts therefore

$$\left|y_{n,1}\left(\frac{\lfloor xm_n\rfloor+1}{m_n}\right)-y_1\left(x\right)\right|,\quad\left|f_n\left(x\right)-f\left(x\right)\right|\to 0$$

uniformly in $x\in I$. For example, for $x\in I$, x and $m_n^{-1}\left(\lfloor xm_n\rfloor+1\right)$ belong to a compact set as $n\to\infty$ and $\left|x-m_n^{-1}\left(\lfloor xm_n\rfloor+1\right)\right|\to 0$. The second limit now follows since y_1 and f, due to Fact 35, are continuous.

It now remains to show that

$$\frac{1}{m_n}\sum_{0}^{\lfloor xm_n\rfloor-1}\left(T_n y_{n,1}\times D_n f_n\right)\left(\frac{j}{m_n}\right)\to\int_0^x y_1\left(t\right)f'\left(t\right)dt$$

uniformly in $x\in I$. Let $\varepsilon>0$. Since on $[jm_n^{-1},\left(j+1\right)m_n^{-1})$, $T_n y_{n,1}\left(t\right)$ and $D_n f_n\left(t\right)$ are constant in t therefore

$$\frac{1}{m_n}\sum_{0}^{\lfloor xm_n\rfloor-1}\left(T_n y_{n,1}\times D_n f_n\right)\left(\frac{j}{m_n}\right)=\sum_{0}^{\lfloor xm_n\rfloor-1}\int_{jm_n^{-1}}^{\left(j+1\right)m_n^{-1}}T_n y_{n,1}\left(t\right)D_n f_n\left(t\right)dt$$

$$=\int_0^\infty T_n y_{n,1}\left(t\right)1_{\left[0,\frac{\lfloor xm_n\rfloor}{m_n}\right]}\left(t\right)D_n f_n\left(t\right)dt.$$

Then,

$$\left|\frac{1}{m_n}\sum_{0}^{\lfloor xm_n\rfloor-1}\left(T_n y_{n,1}\times D_n f_n\right)\left(\frac{j}{m_n}\right)-\int_0^x y_1\left(t\right)f'\left(t\right)dt\right|$$

$$\leq \left| \int_0^\infty T_n y_{n,1}(t)\, 1_{\left[0, \frac{\lfloor x m_n \rfloor}{m_n}\right]}(t)\, D_n f_n(t)\, dt - \int_0^\infty y_1(t)\, 1_{[0,x]}(t)\, D_n f_n(t)\, dt \right| +$$

$$+ \left| \int_0^\infty y_1(t)\, 1_{[0,x]}(t)\, D_n f_n(t)\, dt - \int_0^\infty y_1(t)\, 1_{[0,x]}(t)\, f'(t)\, dt \right|. \tag{51}$$

For the first term of (51), Hölder's inequality yields the upper bound

$$\left\| T_n y_{n,1} 1_{\left[0, \frac{\lfloor x m_n \rfloor}{m_n}\right]} - y_1 1_{[0,x]} \right\|_2 \| D_n f_n \|_2 . \tag{52}$$

Since $\|f_n\|_{*n}$ is uniformly bounded in n therefore by definition (48) of $\|\cdot\|_{*n}$,

$$\| D_n f_n \|_2^2 \leq \sup_n \| f_n \|_{*n}^2 < \infty. \tag{53}$$

Furthermore, since $x \in [\lfloor x m_n \rfloor m_n^{-1}, (\lfloor x m_n \rfloor + 1)\, m_n^{-1})$,

$$\left\| T_n y_{n,1} 1_{\left[0, \frac{\lfloor x m_n \rfloor}{m_n}\right]} - y_1 1_{[0,x]} \right\|_2^2 \leq \sum_0^{\lfloor x m_n \rfloor - 1} \int_{j m_n^{-1}}^{(j+1) m_n^{-1}} (T_n y_{n,1}(t) - y_1(t))^2\, dt + r(x,n)$$

where

$$r(x,n) = \int_{\lfloor x m_n \rfloor m_n^{-1}}^{x} y_1(t)^2\, dt.$$

Since $y_{n,1} \to y_1$ uniformly on compacts and y_1 is continuous therefore for large n,

$$\sum_0^{\lfloor x m_n \rfloor - 1} \int_{j m_n^{-1}}^{(j+1) m_n^{-1}} (T_n y_{n,1}(t) - y_1(t))^2\, dt \leq \varepsilon^2 \frac{\lfloor x m_n \rfloor}{m_n}$$

where $\lfloor x m_n \rfloor / m_n$ is bounded uniformly in n and $x \in I$. Similarly,

$$r(x,n) \leq c \left(x - \frac{\lfloor x m_n \rfloor}{m_n} \right) \leq \frac{c}{m_n}$$

where $c > 0$ is some constant. Hence, by (52) the first term of (51) vanishes with n uniformly in $x \in I$. For the second term of (51), since $D_n f_n \to f'$ weakly in L^2,

$$\left| \int_0^\infty y_1(t)\, 1_{[0,x]}(t)\, D_n f_n(t)\, dt - \int_0^\infty y_1(t)\, 1_{[0,x]}(t)\, f'(t)\, dt \right| \to 0$$

pointwise in $x \in I$. To extend this is uniform convergence, by Arzelà-Ascoli it suffices to show that

$$g_n(x) = \int_0^\infty y_1(t)\, 1_{[0,x]}(t)\, D_n f_n(t)\, dt$$

where x is restricted to a compact interval containing all $[0,x]$, $x \in I$, is equicontinuous and pointwise bounded. This is because the topology of compact convergence on $\mathbb{R}^{\mathbb{R}}$ is metrizable (e.g. see Section 6.1), and by Arzelà-Ascoli any subsequence of g_n has a convergent subsequence in this topology to

$$g(x) = \int_0^\infty y_1(t)\, 1_{[0,x]}(t)\, f'(t)\, dt.$$

To that end, by Hölder's inequality, again since $\|f_n\|_{*n}$ is uniformly bounded and using (53),

$$|g_n(x)| \le \left\|y_1 1_{[0,x]}\right\|_2 \sup_n \|D_n f_n\|_2 < \infty,$$

and for $x < y$,

$$|g_n(x) - g_n(y)| \le \left\|y_1 1_{[x,y]}\right\|_2 \sup_n \|D_n f_n\|_2,$$

thereby showing moreover uniform equicontinuity. Consequently, the second term of (51) vanishes uniformly in $x \in I$. □

Next, the y_2 half of the $y(x) f_k(x) - \int_0^x y(t) f_k'(t)\, dt$ term of (49):

Corollary 39. *If the assumptions of Lemma 37 hold and $D_n f_n \to f'$ weakly in L^2 then*

$$\frac{1}{m_n}\sum_1^{\lfloor x m_n\rfloor} diag(D_n y_{n,2})\, T_n f_n\left(\frac{j}{m_n}\right) \quad \textit{and} \quad \frac{1}{m_n}\sum_1^{\lfloor x m_n\rfloor} T_n^\dagger diag(D_n y_{n,2})\, f_n\left(\frac{j}{m_n}\right)$$

each converge to

$$y_2(x) f(x) - \int_0^x y_2(t) f'(t)\, dt$$

uniformly in x on compacts as $n \to \infty$.

Proof. By direct computation,

$$\frac{1}{m_n}\sum_1^{\lfloor x m_n\rfloor} \mathrm{diag}(D_n y_{n,2})\, T_n f_n\left(\frac{j}{m_n}\right) = \sum_1^{\lfloor x m_n\rfloor} f_n\left(\frac{j+1}{m_n}\right)(y_{n,2;j+1} - y_{n,2;j})$$

$$= -y_{n,2;1} f_n\left(\frac{1}{m_n}\right) - \frac{1}{m_n}\sum_0^{\lfloor x m_n\rfloor} y_{n,2} \times D_n f_n\left(\frac{j}{m_n}\right) + y_{n,2;\lfloor x m_n\rfloor+1} f_n\left(\frac{\lfloor x m_n\rfloor + 1}{m_n}\right)$$

and

$$\frac{1}{m_n}\sum_1^{\lfloor x m_n\rfloor} T_n^\dagger \mathrm{diag}(D_n y_{n,2})\, f_n\left(\frac{j}{m_n}\right) = \sum_1^{\lfloor x m_n\rfloor} f_n\left(\frac{j-1}{m_n}\right)(y_{n,2;j} - y_{n,2;j-1})$$

$$= -\frac{1}{m_n}\sum_{0}^{\lfloor xm_n\rfloor -2} T_n y_{n,2} \times D_n v^{(n)}\left(\frac{j}{m_n}\right) + y_{n,2;\lfloor xm_n\rfloor} f_n\left(\frac{\lfloor xm_n\rfloor - 1}{m_n}\right).$$

It is clear that the work in Lemma 38 also shows that uniformly on compacts as $n \to \infty$, $-y_{n,2;1}v_1^{(n)} \to 0$,

$$y_{n,2;\lfloor xm_n\rfloor+1} f_n\left(\frac{\lfloor xm_n\rfloor + 1}{m_n}\right), \; y_{n,2;\lfloor xm_n\rfloor} f_n\left(\frac{\lfloor xm_n\rfloor - 1}{m_n}\right) \to y_2(x) f(x),$$

and

$$\frac{1}{m_n}\sum_{0}^{\lfloor xm_n\rfloor} y_{n,2} \times D_n f_n\left(\frac{j}{m_n}\right), \; \frac{1}{m_n}\sum_{0}^{\lfloor xm_n\rfloor -2} T_n y_{n,2} \times D_n v^{(n)}\left(\frac{j}{m_n}\right) \to \int_0^x y_2(t) f'(t)\, dt.$$

□

The final step of the discretization of (49) is the $f_k'(x) - f_k'(0)$ term:

Lemma 40. *If $f_n \in L_n^*$, $f_n \to f$ pointwise for some $f \in L^*$, and*

$$D_n f_n(x) - D_n f_n(0) \to f'(x) - f'(0)$$

uniformly in x on compacts then as $n \to \infty$, $D_n f_n(0) \to f'(0)$ and hence $D_n f_n(x) \to D_n f_n(0)$ uniformly in x on compacts.

Proof. It suffices to show $D_n f_n(0) \to f'(0)$. Fix $x > 0$ and apply summation by parts

$$\frac{1}{m_n}\sum_{0}^{\lfloor xm_n\rfloor -1} \cdot \left(\frac{j}{m_n}\right)$$

to $D_n f_n(x) - D_n f_n(0)$ to yield, using that $f_n(\lfloor xm_n\rfloor m_n^{-1}) = f_n(x)$,

$$\frac{1}{m_n}\sum_{0}^{\lfloor xm_n\rfloor -1}\left(D_n f_n\left(\frac{j}{m_n}\right) - D_n f_n(0)\right) = f_n(x) - \frac{\lfloor xm_n\rfloor}{m_n} D_n f_n(0) - f_n(0).$$

By Fact 35, $f(0) = 0$ and so $f_n(0) = 0$. Continuing, f' is continuous, so

$$f(x) - xf'(0) = \int_0^x f'(t) - f'(0)\, dt = \int_0^{\lfloor xm_n\rfloor m_n^{-1}} f'(t) - f'(0)\, dt + r(x, n)$$

where

$$r(x, n) = \int_{\lfloor xm_n\rfloor m_n^{-1}}^{x} f'(t) - f'(0)\, dt$$

vanishes with n. Additionally, since $D_n f_n(t) - D_n f_n(0)$ is constant on the intervals $[jm_n^{-1}, (j+1)m_n^{-1})$,

$$\frac{1}{m_n}\sum_{0}^{\lfloor xm_n\rfloor -1}\left(D_n f_n\left(\frac{j}{m_n}\right) - D_n f_n(0)\right) = \int_0^{\lfloor xm_n\rfloor m_n^{-1}} D_n f_n(t) - D_n f_n(0)\, dt.$$

Consequently,

$$\left|\left(f_n(x) - \frac{\lfloor xm_n\rfloor}{m_n} D_n f_n(0)\right) - (f(x) - xf'(0))\right|$$

$$\leq \int_0^{\lfloor xm_n\rfloor m_n^{-1}} |(D_n f_n(t) - D_n f_n(0)) - (f'(t) - f'(0))|\, dt + |f_n(0)| + |r(x,n)|.$$

By assumption on the integrand this vanishes as $n \to \infty$ and hence

$$f_n(x) - \frac{\lfloor xm_n\rfloor}{m_n} D_n f_n(0) \to f(x) - xf'(0).$$

Since $f_n(x) \to f(x)$ and $|\lfloor xm_n\rfloor m_n^{-1} - x| \leq m_n^{-1}$ therefore $D_n f_n(0) \to f'(0)$. □

With these various L_n^* convergence results in hand effectively discretizing 49 the proof of Theorem 3 may now be done. Recall its statement: *If $y_{n,i} \Longrightarrow y_i$, $i = 1, 2$ in the topology of compact convergence for some random continuous processes y_i then $D_n v_k^{(n)} \Longrightarrow f_k'$ in the same topology as $n \to \infty$.*

Proof of Theorem 3. Recall $v_k^{(n)}$, $\lambda_k^{(n)}$ are the eigenfunction, eigenvalue pairs of $\mathcal{H}_n$, and likewise f_k, Λ_k for $\mathcal{H}$. Now in a deterministic setting it must be shown that

$$D_n v_k^{(n)} \to f_k'$$

uniformly on compacts for fixed k. Fix a compact and let x be an element. Since $\lfloor xm_n\rfloor m_n^{-1} < nm_n^{-1}$ uniformly in x for large n, and $\mathcal{H}_n$ is acting on eigenfunctions in L_n^* with support $[0, nm_n^{-1})$, therefore the $R_n = 1_{[0,nm_n^{-1})}$ term may now be removed from $\mathcal{H}_n$ to yield

$$\mathcal{H}_n = D_n^\dagger D_n + V_n + m_n E_n.$$

Hence,

$$-\left(D_n^\dagger D_n + m_n E_n\right) v_k^{(n)} = -\lambda_k^{(n)} v_k^{(n)} + V_n v_k^{(n)}.$$

Summation by parts

$$\frac{1}{m_n}\sum_{1}^{\lfloor xm_n\rfloor} \cdot \left(\frac{j}{m_n}\right)$$

may now be applied to yield

$$D_n v_k^{(n)}(x) - D_n v_k^{(n)}(0) = -\lambda_k^{(n)} \frac{1}{m_n} \sum_1^{\lfloor x m_n \rfloor} v_k^{(n)}\left(\frac{j}{m_n}\right) + \frac{1}{m_n} \sum_1^{\lfloor x m_n \rfloor} V_n v_k^{(n)}\left(\frac{j}{m_n}\right). \tag{54}$$

For example, direct computation shows

$$D_n^\dagger D_n v_k^{(n)}(t) = m_n^2 \left(- \left(v_k^{(n)}\left(\frac{j}{m_n}\right) - v_k^{(n)}(0)\right) 1_{\left[0, \frac{1}{m_n}\right)}(t) + \right.$$

$$\left. + \sum_1^{n-1} \left(-v_k^{(n)}\left(\frac{j-1}{m_n}\right) + 2 v_k^{(n)}\left(\frac{j}{m_n}\right) - v_k^{(n)}\left(\frac{j+1}{m_n}\right)\right) 1_{\left[\frac{j}{m_n}, \frac{j+1}{m_n}\right)}(t) \right)$$

and E_n is identically zero in $\sum_1^{\lfloor x m_n \rfloor} \cdot (j m_n^{-1})$. Continuing with (54),

$$\frac{1}{m_n} \sum_1^{\lfloor x m_n \rfloor} V_n v_k^{(n)}\left(\frac{j}{m_n}\right) = \frac{1}{m_n} \sum_1^{\lfloor x m_n \rfloor} \operatorname{diag}(D_n y_{n,1}) v_k^{(n)}\left(\frac{j}{m_n}\right) +$$

$$+ \frac{1}{2}\left(\frac{1}{m_n} \sum_1^{\lfloor x m_n \rfloor} \operatorname{diag}(D_n y_{n,2}) T_n v_k^{(n)}\left(\frac{j}{m_n}\right) + \frac{1}{m_n} \sum_1^{\lfloor x m_n \rfloor} T_n^\dagger \operatorname{diag}(D_n y_{n,2}) v_k^{(n)}\left(\frac{j}{m_n}\right)\right). \tag{55}$$

Similarly in the limit, by Fact 35,

$$f_k'(x) - f_k'(0) = -\Lambda_k \int_0^x f_k(t)\, dt + \left(y(x) f_k(x) - \int_0^x y(t) f_k'(t)\, dt \right). \tag{56}$$

By Fact 36 there exists constants c_1, $c_2 > 0$ so that for all n,

$$c_1 \left\| v_k^{(n)} \right\|_{*n}^2 - c_2 \left\| v_k^{(n)} \right\|_2^2 \leq \left\langle v_k^{(n)}, \mathcal{H}_n v_k^{(n)} \right\rangle_2.$$

Using that $v_k^{(n)}$ are $\|\cdot\|_2$-normalized eigenfunctions it follows that

$$c_1 \left\| v_k^{(n)} \right\|_{*n}^2 - c_2 \leq \lambda_k^{(n)} \Longrightarrow \left\| v_k^{(n)} \right\|_{*n}^2 \leq \frac{1}{c_1} \lambda_k^{(n)} + \frac{c_2}{c_1}.$$

Fact 35 in the deterministic setting says $\lambda_k^{(n)} \to \Lambda_k$ and so

$$\sup_n \left(\frac{1}{c_1} \lambda_k^{(n)} + \frac{c_2}{c_1} \right) \in \mathbb{R}.$$

Consequently, $\left\| v_k^{(n)} \right\|_{*n}$ is uniformly bounded in n. Hence, by Fact 36 there exists $f \in$

L^* such that along some subsequence $v_k^{(n)} \to f$ uniformly on compacts and in L^2, and $D_n v_k^{(n)} \to f'$ weakly in L^2. Moreover, f is the desired eigenfunction $f = f_k$ by Fact 35. By putting together equations (54), (55), (56), Lemmas 37, 38 and Corollary 39, and using that $y_1 + y_2 = y$ it now follows that along this subsequence,

$$D_n v_k^{(n)}(x) - D_n v_k^{(n)}(0) \to f_k'(x) - f_k'(0)$$

uniformly in x on compacts. Consequently, by Lemma 40, along this subsequence,

$$D_n v_k^{(n)}(x) \to f_k'(x)$$

uniformly in x on compacts. In summary, using that $\mathbb{R}^{\mathbb{R}}$ with the topology of compact convergence is metrizable (e.g. see Section 6.1), given any subsequence of $D_n v_k^{(n)}$ there is a further subsequence along which the above limit holds, and hence it holds for the whole sequence. □

6 Random entire functions

Recall that a fundamental object in this thesis is the random entire function

$$\mathfrak{p}(z) = \prod_{1}^{\infty} \left(1 - \frac{z}{\Lambda_k}\right) e^{z/\Lambda_k},\ z \in \mathbb{C},$$

a complex infinite product with zeros the Airy process Λ_k. This function may also be viewed as the secular function of the stochastic Airy operator whose spectrum is Λ_k. Also, recall the convergence in distribution (4) of secular functions to $\mathfrak{p}(z)$ that will be proved in Section 7, and almost sure growth asymptotics of $\mathfrak{p}(z)$ in Section 6.3. To accomplish these results, this section reviews the relevant theory of complex entire functions and their growth and its application in probability. The notion of regularized determinants is introduced, serving as the connection between secular functions and operators. Furthermore, some probabilistic tools are reviewed and new ones developed and proven.

6.1 Basic definitions and relevant spaces

Let S be a metric space with metric $d(\cdot,\cdot)$ and consider $S^{\mathbb{C}}$ with the topology of uniform convergence on compact subsets of $\mathbb{C}$. This is also known as the "topology of compact convergence," and for the subspace $C(\mathbb{C}, S)$, the "compact-open topology." The topology of compact convergence for $S^{\mathbb{C}}$ has basis consisting of all sets

$$V(K, f, \varepsilon) = \left\{ g \in S^{\mathbb{C}} : \sup_{z \in K} d(g(z), f(z)) < \varepsilon \right\}$$

where $K \subseteq \mathbb{C}$ is compact, $f \in S^{\mathbb{C}}$ and $\varepsilon > 0$. Moreover,

$$\rho(f, g) = \sum_{j=1}^{\infty} 2^{-j} \sup_{z \in K_j} \frac{d(f(z), g(z))}{1 + d(f(z), g(z))}$$

is a metric on $S^{\mathbb{C}}$ where $K_1 \subseteq K_2 \subseteq \cdots$ is a **compact exhaustion** of $\mathbb{C}$, i.e. compact subsets of $\mathbb{C}$ so that for any compact subset $K \subseteq \mathbb{C}$, $K \subseteq K_j$ for some j. In fact, this metric induces the topology of compact convergence for $S^{\mathbb{C}}$, and $S^{\mathbb{C}}$ is complete with this metric when S is complete. See Section 46 in Munkres (2000) and Section 5.5.2 in Ahlfors (1979) for more on these topics.

With the subspace topology on $C(\mathbb{C}, S)$ this space is also complete by being closed: uniform convergence of continuous functions on a compact yields a continuous limit, and then extend to $\mathbb{C}$ using the compact exhaustion. In the case $S = \mathbb{C}$, the holomorphic function space $H(\mathbb{C})$ is given the subspace topology from $C(\mathbb{C})$ and is also closed and hence complete. In the case $S = \mathbb{R}$, the real analytic function space $H(\mathbb{R})$ is likewise given the subspace topology from $C(\mathbb{C})$. A **random (complex) entire function** $f(z)$, $z \in \mathbb{C}$ is a random function

$$f(z) : \Omega \to H(\mathbb{C})$$

where Ω is a probability space. A **random real analytic function** $f(x)$, $x \in \mathbb{R}$ is similarly a random function $f(x) : \Omega \to H(\mathbb{R})$. Unless otherwise stated the topology of compact convergence is given to each of $H(\mathbb{C})$ and $H(\mathbb{R})$.

Another choice of topology for $H(\mathbb{R})$ is the topology of uniform convergence on compact subsets of functions and all their derivatives. This is done by giving this topology to $C^\infty(\mathbb{R})$, which is again metrizable, and asserting the subspace topology onto $H(\mathbb{R})$. Note that this topology is coarser than the topology of compact convergence.

6.2 Random canonical products and the secular function of the stochastic Airy operator

A particular form of entire functions is often used throughout this thesis, defined as follows. Let $a_k \in \mathbb{C}$ be a deterministic sequence so that:

1. $a_k \neq 0$ for all k.
2. For some $p \in \mathbb{Z}_{\geq 0}$,
$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^{p+1}} < \infty.$$

Then the function

$$P(z) = \prod_{1}^{\infty} p_k(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) \exp\left(\sum_{j=1}^{p} \frac{z^j}{j a_k^j}\right)$$

where the exp term is removed if $p = 0$, is called a (**Weierstrass**) **canonical product** of **genus** p. Often, p is taken to be minimal. Note $|a_k| \to \infty$. $P(z)$ has roots precisely a_k (repeated with multiplicity) and it converges uniformly and absolutely on compact subsets K of $\mathbb{C}$ to an entire function. This means $p_k(z) \to 1$ uniformly on K and, in possibly excluding finitely many $p_k(z)$,

$$\sum_{k=1}^{\infty} \log p_k(z)$$

converges uniformly and absolutely on K where log is the principal branch. If $z \notin \{a_k\}$ then

$$P(z) = \exp\left(\sum_{k=1}^{\infty} \log p_k(z)\right).$$

Moreover,

$$\prod_{k=1}^{n} p_k(z) \to P(z)$$

uniformly on compacts as $n \to \infty$. Note that the ordering of the zeros a_k is not important: on any compact subset, only finitely many a_k lie in it, and for any $n \geq 1$,

$$P(z) = p_1(z) \cdots p_n(z) \prod_{k=n+1}^{\infty} p_k(z). \tag{57}$$

See Levin (1996) for more on these topics.

Now let $A_k : \Omega \to \mathbb{C}$ be sequence of random variables on some probability space, where almost surely A_k satisfies the conditions (1) and (2). Then, the random function

$$P(z) = \prod_{k=1}^{\infty} p_k(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{A_k}\right) \exp\left(\sum_{j=1}^{p} \frac{z^j}{jA_k^j}\right)$$

where $p \in \mathbb{Z}_{\geq 0}$ is deterministic is called a **random canonical product** of **genus** p. To be technically precise, for any (probability zero) realization where any of the requirements (1) and (2) are not all satisfied, define $P(z)$ to be the identically 1 function. This particular choice is mostly arbitrary but it will be convenient in Section 7 when showing product convergence. Also, this convention is used throughout for random functions for any realization where they may not be well-defined. The following is elementary:

Lemma 41. *$P(z)$ is a well-defined random entire function.*

In the case that almost surely $P(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$, $P(x)$ may be viewed as a random real analytic function by again defining any (probability zero) realization where $P(x) \notin \mathbb{R}$ to be the identically 1 function:

Corollary 42. *$P(x)$ is a well-defined random real analytic function.*

The necessary tools are now in place to construct $\mathfrak{p}(z)$, the **secular function** of the stochastic Airy operator $\mathcal{H}$. This is a canonical product associated to $\mathcal{H}$, namely its spectrum Λ_k or the Airy process, and the name "secular function" comes from Valkó and Virág (2020) where the sine beta operator was instead considered. Note that the terminology "secular function" was historically used for "characteristic polynomial." The function $\mathfrak{p}(z)$ may also be viewed as an "infinite characteristic polynomial" of $\mathcal{H}$.

Example 43. Recall from Section 4.1 the stochastic Airy operator with parameter $\beta > 0$,

$$\mathcal{H} = \mathcal{H}_\beta = -\partial_x^2 + x + \frac{2}{\sqrt{\beta}} b'_x$$

which has eigenvalues Λ_k, $k \geq 1$, the Airy process. Let

$$\mathfrak{p}(z) = \mathfrak{p}_\beta(z) = \prod_1^n \left(1 - \frac{z}{\Lambda_k}\right) e^{z/\Lambda_k}.$$

By Corollary 33, almost surely

$$\frac{\Lambda_k}{k^{2/3}} \to \left(\frac{3\pi}{2}\right)^{2/3}.$$

Consequently, $\mathfrak{p}(z)$ has genus 1 (and this is minimal): for any $p > 0$,

$$\frac{k^{-2p/3}}{|\Lambda_k|^{-p}} \to \left(\frac{3\pi}{2}\right)^{2p/3} > 0$$

and so by the limit comparison test $\sum |\Lambda_k|^{-p} < \infty$ iff $p > 3/2$. Furthermore, by Proposition 28 the remaining requirement (1) holds. Hence, $\mathfrak{p}(z)$ is a random canonical product of genus 1.

6.3 Growth properties and asymptotics

The theory of entire functions can be used to infer almost sure asymptotics on the growth of random canonical products and provide further characterizations. This sections reviews these concepts by following Levin (1996) and it is then applied to the secular function of the stochastic Airy operator. Consider a deterministic entire function $f(z) \in H(\mathbb{C})$. It has **order**

$$\rho = \inf \left\{ k > 0 : \max_{|z|=r} |f(z)| < \exp\left(r^k\right) \quad \text{for } r \text{ sufficiently large} \right\} \tag{58}$$

and **type**

$$\sigma = \inf \left\{ A > 0 : \max_{|z|=r} |f(z)| < \exp\left(A r^\rho\right) \quad \text{for } r \text{ sufficiently large} \right\}$$

where $f(z)$ is of **normal type** if $0 < \sigma < \infty$. Now assume $f(z)$ is a canonical product with zeros a_k. The **convergence exponent** of a_k is

$$\inf\left\{r > 0 : \sum_{k=1}^{\infty} \frac{1}{|a_k|^r} < \infty\right\} = \rho$$

which agrees with the order of $f(z)$. Let

$$n(r) = |\{a_k : |a_k| \leq r\}|,$$

the counting function of the zeros of $f(z)$. It has **order**

$$\limsup_{r\to\infty} \frac{\log n(r)}{\log r} = \rho,$$

again agreeing with the order of $f(z)$. The **upper density** of a_k is

$$\overline{\Delta} = \limsup_{r\to\infty} \frac{n(r)}{r^\rho} = \limsup_{k\to\infty} \frac{k}{|a_k|^\rho},$$

the **lower density** $\underline{\Delta}$ is with a lim inf, and when $\overline{\Delta} = \underline{\Delta}$ the **density** is

$$\Delta = \lim_{r\to\infty} \frac{n(r)}{r^\rho} = \lim_{k\to\infty} \frac{k}{|a_k|^\rho}. \tag{59}$$

When $\rho \notin \mathbb{Z}$ (called **non-integral order**) and $0 < \Delta < \infty$ then $f(z)$ is of normal type.

Example 44. Consider $\mathfrak{p}(z)$ as defined in Example 43. It is clear that the argument there in establishing that almost surely $\mathfrak{p}(z)$ has genus 1 also shows that almost surely $\mathfrak{p}(z)$ has order, and hence the Airy process Λ_k has convergence exponent, $\rho = 3/2$. By Corollary 33, almost surely $\mathfrak{p}(z)$ has density $\Delta = 2/(3\pi)$. Consequently, the zero counting function $n(r)$ of the Airy process satisfies

$$\lim_{r\to\infty} \frac{n(r)}{r^{3/2}} = \frac{2}{3\pi}.$$

Since $\rho \notin \mathbb{Z}$ and $0 < \Delta < \infty$ therefore almost surely $\mathfrak{p}(z)$ is of normal type.

It is important to note that in general a deterministic entire function $f(z)$ with zeros a_k and finite order ρ can be written (non-uniquely) as

$$f(z) = z^m e^{g(z)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) \exp\left(\sum_{j=1}^{p} \frac{z^j}{j a_k^j}\right)$$

where $p \leq \rho$, $m \geq 0$ is the order of the zero at $z = 0$ and $g(z)$ is a polynomial of degree at most ρ. This theorem is often called Hadamard's theorem.

Example 45. Any random canonical product $f(z)$ with zeros the Airy process Λ_k necessarily almost surely has order $\rho = 3/2$. Since almost surely $0 \notin \{\Lambda_k\}$ by Proposition 28 therefore the most general expression for $f(z)$ is

$$f(z) = e^{A+Bz} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\Lambda_k}\right) e^{z/\Lambda_k} = e^{A+Bz} \mathfrak{p}(z)$$

where A, B are random variables. Consequently, $\mathfrak{p}(z)$ is the choice where there is the normalization $\mathfrak{p}(0) = 1$, and such a canonical product may be called **normalized**. For example, in Lambert and Paquette (2020) it is asked whether a random function that is constructed there from the stochastic Airy operator is a canonical product. Later in Section 7.4.2 it is proven that in some sense this is the case where the function there takes the form $e^{A+Bz}\mathfrak{p}(z)$ for nonzero A, B.

Consider now when the zeros a_k of a deterministic entire function $f(z)$ has density Δ, non-integral order $\rho \notin \mathbb{Z}$ and $a_k > 0$ always. Then,

$$\sup_{0<\theta<2\pi} \left| \log f\left(re^{i\theta}\right) - \frac{\pi\Delta}{\sin(\pi\rho)} e^{i\rho(\theta-\pi)} r^{\rho} \right| \sin\frac{\theta}{2} = o\left(r^{\rho}\right), \quad r \to \infty.$$

Alternatively, in removing the supremum the resulting asymptotic in r is said to hold uniformly in θ, $0 < \theta < 2\pi$. This provides an asymptotic of $f(z)$ along any such ray $z = re^{i\theta}$. In particular, by taking real parts,

$$\log\left|f\left(re^{i\theta}\right)\right| = \frac{\pi\Delta}{\sin(\pi\rho)} \cos\left(\rho(\theta - \pi)\right) r^{\rho} + \frac{o\left(r^{\rho}\right)}{\sin(\theta/2)}, \quad r \to \infty.$$

To extend this to $\theta = 0, 2\pi$, certain regions of the plane containing the zeros a_k must be excluded. A set of disks C_j, $j \in \mathbb{Z}_{\geq 1}$ in the complex plane with centres z_j and radii r_j is a **C^0-set** if

$$\lim_{R\to\infty} \frac{1}{R} \sum_{|z_j|<R} r_j = 0.$$

For such $f(z)$ there exists a C^0-set outside of which

$$\log\left|f\left(re^{i\theta}\right)\right| = \frac{\pi\Delta}{\sin(\pi\rho)} \cos\left(\rho(\theta - \pi)\right) r^{\rho} + o\left(r^{\rho}\right), \quad r \to \infty$$

holds uniformly in θ, $0 \leq \theta \leq 2\pi$. To be precise, "outside" can be taken to mean $\mathbb{C} - \bigcup_j C_j$ where C_j are taken as open disks.

Proposition 46. *Almost surely, uniformly in* θ*,* $0 < \theta < 2\pi$*,*

$$\left|\log \mathfrak{p}\left(re^{i\theta}\right) + \frac{2}{3} i e^{3i\theta/2} r^{3/2}\right| \sin\frac{\theta}{2} = o\left(r^{3/2}\right), \quad r \to \infty$$

and

$$\log\left|\mathfrak{p}\left(re^{i\theta}\right)\right| = \frac{2}{3}\sin\frac{3\theta}{2} r^{3/2} + \frac{o\left(r^{3/2}\right)}{\sin\left(\theta/2\right)}, \quad r \to \infty.$$

Moreover outside some C^0*-set, uniformly in* θ*,* $0 \leq \theta \leq 2\pi$*,*

$$\log\left|\mathfrak{p}\left(re^{i\theta}\right)\right| = \frac{2}{3}\sin\frac{3\theta}{2} r^{3/2} + o\left(r^{3/2}\right), \quad r \to \infty.$$

Proof. Choose a realization of $\mathfrak{p}(z)$. Since Λ_k increase to ∞ therefore either the asymptotics immediately hold or there exists $k_0 \geq 1$ so that $\Lambda_{k_0} < 0$ and $\Lambda_k > 0$ for $k > k_0$. In this case, using that

$$\log\left(z_1 z_2\right) = \log z_1 + \log z_2 + i\left(\arg\left(z_1 z_2\right) - \arg z_1 - \arg z_2\right),$$

for $z = re^{i\theta}$,

$$\log \mathfrak{p}(z) = \log \prod_{1}^{k_0}\left(1 - \frac{z}{\Lambda_k}\right) e^{z/\Lambda_k} + \log \prod_{k=k_0+1}^{\infty}\left(1 - \frac{z}{\Lambda_k}\right) e^{z/\Lambda_k} + c$$

for some $c \in \mathbb{C}$ with $|c| \leq 6\pi$. Since the definitions of order and density are limits therefore the truncated infinite product is also a canonical product of the same genus, order and density of 1, $3/2$ and $2/(3\pi)$. Consequently, all the stated asymptotics hold for it. Using that $|e^z| = \left|e^{\mathrm{Re} z}\right| \leq e^{|z|}$, the logarithm of the finite truncated product is absolutely bounded by $O(r)$, $r \to \infty$. Hence in the limit as $r \to \infty$ both c and the truncated log vanish uniformly in θ thereby leaving only the desired asymptotics. □

6.4 Schatten-von Neumann operators and regularized determinants

There is a natural representation of canonical products as a certain determinant of linear operators. Through this connection the definition of "secular function" for the normalized canonical product $\mathfrak{p}(z)$ of the stochastic Airy operator $\mathcal{H}$ gains more appreciation. In this section, following Gohberg, Goldberg, and Krupnik (2012) these determinants are defined and the relevant operator theory is reviewed. Furthermore, additional properties are proven and applications are done to $\mathfrak{p}(z)$, including reviewing the realization of $\mathcal{H}^{-1}$ as a Hilbert-Schmidt integral operator done in Dumaz, Li, and Valkó (2020).

Let H be a separable Hilbert space and $A : H \to H$ a compact operator. The kth

singular number or s-**number** $s_k = s_k(A)$ of A is defined as $(\lambda_k(A^*A))^{1/2}$ where

$$\lambda_1(A^*A) \geq \lambda_2(A^*A) \geq \cdots$$

are the non-zero eigenvalues of A^*A repeated by their multiplicity. For $p \geq 1$ the **Schatten-von Neumann algebra** $\mathcal{S}_p$ is the complete algebra of all compact operators A on H so that the **Schatten-von Neumann norm** satisfies

$$\|A\|_p = \|s_j(A)\|_{\ell^p} < \infty.$$

In particular, $p = 1$ yields the **trace class** or **nuclear** operators.

For finite rank operators F on H, there is a decomposition $H = M + N$ where M is finite-dimensional, $F(M) \subseteq M$ and $N \subseteq \ker F$. Letting F_1 be the restriction of F to M, define

$$\mathrm{tr}F = \mathrm{tr}F_1 = \sum_k \lambda_k(F_1) \quad \text{and} \quad \det(I+F) = \det(I_1+F_1) = \prod_k (1+\lambda_k(F_1))$$

where I, I_1 are respectively the identity operators on H and M. The **regularized determinant** $\det_m(I+A)$, $m = \lfloor p \rfloor$ is defined on $\mathcal{S}_p$ by first defining for finite rank operators F,

$$\det{}_m(I+F) = \det(I+F)\exp\left(\sum_{k=1}^{m-1} \frac{(-1)^k}{k}\mathrm{tr}\left(F^k\right)\right).$$

An alternative expression is

$$\det{}_m(I+F) = \prod_k (1+\lambda_k(F))\exp\left(\sum_{j=1}^{m-1} \frac{(-1)^j \lambda_k(F)^j}{j}\right) \tag{60}$$

where the eigenvalues are repeated by their multiplicities. There is then a continuous extension of $\det_m$ in $\|\cdot\|_p$ from the finite rank operators to all of $\mathcal{S}_p$ (Theorem 1.1, p. 187 and p. 194 Gohberg, Goldberg, and Krupnik (2012)). In particular:

Lemma 47. *Identity (60) holds on all of $\mathcal{S}_p$.*

Proof. For $A \in \mathcal{S}_p$ the operator

$$R_A = (I+A)\exp\left(\sum_{k=1}^{m-1} \frac{(-1)^k}{k}A^k\right) - I$$

where for a bounded operator B, $\exp B = \sum_1^\infty (n!)^{-1} B^n$, is trace class and $\det_1(I+R_A) =$

$\det_m(I+A)$ (Equation 2.11, p. 195 Gohberg, Goldberg, and Krupnik (2012)). Then, the Lidskii trace theorem (Theorem 6.1, p. 63 Gohberg, Goldberg, and Krupnik (2012)) states that

$$\det{}_1(I+R_A) = \prod_k (1+\lambda_k(R_A))$$

where $\lambda_k(R_A)$ are the nonzero eigenvalues of R_A repeated with multiplicity. By definition of R_A this product equals

$$\prod_k (1+\lambda_k(A)) \exp\left(\sum_{j=1}^{m-1} \frac{(-1)^j \lambda_k(A)^j}{j}\right).$$

□

Formula (60) now provides a connection between Schatten-von Neumann operators and canonical products:

Corollary 48. *If $A \in \mathcal{S}_p$, $p \in \mathbb{Z}_{\geq 1}$ has eigenvalues $1/\lambda_k$ repeated with multiplicity, then*

$$\det{}_p(I-zA) = \prod_k \left(1-\frac{z}{\lambda_k}\right) \exp\left(\sum_{j=1}^{p-1} \frac{z^j}{j\lambda_k^j}\right)$$

is a normalized canonical product of genus $p-1$ with zeros λ_k.

Proof. Since $A \in \mathcal{S}_p$ therefore for each n,

$$\sum_1^n \frac{1}{|\lambda_k|^p} \leq \sum_1^n s_k(A)^p$$

(Corollary 3.4, p. 54 Gohberg, Goldberg, and Krupnik (2012)). Hence, in taking $n \to \infty$,

$$\sum_1^\infty \frac{1}{|\lambda_k|^p} \leq \|A\|_p^p < \infty.$$

Hence, the product defines an entire function. □

A **random Schatten-von Neumann operator** is a random variable $A : \Omega \to \mathcal{S}_p$ where $\mathcal{S}_p$ is the metric space induced by its norm and Ω is some probability space. In the case of Corollary 48, $f(z) = \det_p(I-zA)$ is then a random entire function which is well-defined by the following:

Lemma 49. *For $p \in \mathbb{Z}$, the mapping*

$$\det{}_p\left(1 - z\cdot\right) : \mathcal{S}_p \to H\left(\mathbb{C}\right)$$

is continuous.

Proof. For A, $B \in \mathcal{S}_p$,

$$\left|\det{}_p\left(I - zA\right) - \det{}_p\left(I - zB\right)\right| \leq \left\|zA - zB\right\|_p \exp\left(\Gamma_p\left(\left\|zA\right\|_p + \left\|zB\right\|_p + 1\right)^p\right)$$

where $\Gamma_p > 0$ is some constant (Theorem 2.2, p. 194 Gohberg, Goldberg, and Krupnik (2012)). Consequently, for z in a compact subset $K \subseteq \mathbb{C}$ there exists a constant $c > 0$ so that

$$\left|\det{}_p\left(I - zA\right) - \det{}_p\left(I - zB\right)\right| \leq c\left\|A - B\right\|_p \exp\left(\Gamma_p\left(c\left\|A\right\|_p + c\left\|B\right\|_p + 1\right)^p\right).$$

Hence, if $A_n \to A$ in $\mathcal{S}_p$ then $\det{}_p\left(I - zA_n\right) \to \det{}_p\left(I - zA\right)$ uniformly on compacts, the topology of $H\left(\mathbb{C}\right)$. □

In fact, the following more general fact is true, which will be useful in Section 7 for showing product convergence:

Lemma 50. *For $p \in \mathbb{Z}_{\geq 1}$ the mapping*

$$\det{}_p : \ell^p\left(\mathbb{Z}_{\geq 1}, \mathbb{C}\right) \to H\left(\mathbb{C}\right), \quad \det{}_p\left(x\right)\left(z\right) = \prod_{k=1}^{\infty}\left(1 - zx_k\right)\exp\left(\sum_{j=1}^{p-1}\frac{z^j x_k^j}{j}\right)$$

is continuous.

Proof. Let $x = \left(x_k\right) \in \ell^p$ and $f\left(z\right) = \det{}_p\left(x\right)\left(z\right)$. Note that

$$f_k\left(z\right) = \left(1 - zx_k\right)\exp\left(\sum_{j=1}^{p-1}\frac{z^j x_k^j}{j}\right) \to 1 \quad \text{as} \quad k \to \infty$$

uniformly on compacts because $x_k \to 0$. Additionally, on a fixed compact set $K \subseteq \mathbb{C}$, there is a constant $c_0 > 0$ and large k_0 so that for $k \geq k_0$,

$$\left|\log f_k\left(z\right)\right| \leq c_0\left|x_k\right|^p. \tag{61}$$

Hence, $\sum_{k_0}^{\infty}\log f_k\left(z\right)$ converges uniformly and absolutely on K. This shows $f\left(z\right) \in H\left(\mathbb{C}\right)$.

Now, let $x^{(n)} \in \ell^p$ so that $x^{(n)} \to x$ in ℓ^p, and let $f^{(n)}(z) = \det_p\left(x^{(n)}\right)(z)$. On a fixed compact set $K \subseteq \mathbb{C}$, since $x_k \to 0$ therefore there exists k_0 large so that for $k \geq k_0$, $1/x_k$, when defined, is not in K. Continuing, since $\left\|x^{(n)} - x\right\|_{\ell^p} \to 0$ as $n \to \infty$, therefore there is n_0 large so that for $n \geq n_0$ and $k \geq k_0$, $1/x_k^{(n)}$, when defined, is not in K. Hence, for $n \geq n_0$ the following is well-defined on K:

$$f^{(n)}(z) = \prod_{k=1}^{k_0-1} f_k^{(n)}(z) \prod_{k=k_0}^{\infty} f_k^{(n)}(z)$$

with

$$\prod_{k=k_0}^{\infty} f_k^{(n)}(z) = \exp\left(\sum_{k=k_0}^{\infty} \log f_k^{(n)}(z)\right) \tag{62}$$

and similarly for $f(z)$. Since $x_k^{(n)} \to x_k$ for each $1 \leq k < k_0$,

$$\prod_{k=1}^{k_0-1} f_k^{(n)}(z) \to \prod_{k=1}^{k_0-1} f_k(z) \quad \text{as} \quad n \to \infty$$

uniformly on K. To establish that $f^{(n)}(z) \to f(z)$ uniformly on K and thereby conclude the proof, it now suffices to show this convergence for the truncated products $\prod_{k_0}^{\infty}$. Moreover, by (62) it suffices to show

$$\sum_{k=k_0}^{\infty} \log f_k^{(n)}(z) \to \sum_{k=k_0}^{\infty} \log f_k(z)$$

uniformly on K as $n \to \infty$. There exists a constant $c_1 > 0$ so that on K, for $n \geq n_0$,

$$\left|\sum_{k=k_0}^{\infty} \log f_k^{(n)}(z) - \sum_{k=k_0}^{\infty} \log f_k(z)\right| \leq c_1 \sum_{k_0}^{\infty} \left|\left(x_k^{(n)}\right)^p - x_k^p\right|$$

$$\leq c_1 \left\|\left(\left(x_k^{(n)}\right)^p - x_k^p\right)\right\|_{\ell^1}.$$

If $p = 1$ the argument is done, so consider when $p \geq 2$. Since

$$\left(x_k^{(n)}\right)^p - x_k^p = \left(x_k^{(n)} - x_k\right)\left(\left(x_k^{(n)}\right)^{p-1} + \left(x_k^{(n)}\right)^{p-2} x_k + \cdots + x_k^{p-1}\right)$$

therefore

$$\left|\left(x_k^{(n)}\right)^p - x_k^p\right| \leq \left|x_k^{(n)} - x_k\right| \left(\left|x_k^{(n)}\right|^{p-1} + \left|x_k^{(n)}\right|^{p-2} |x_k| + \cdots + |x_k|^{p-1}\right)$$

$$\leq p\left|x_k^{(n)}-x_k\right|\left(\left|x_k^{(n)}\right|^{p-1}+|x_k|^{p-1}\right).$$

Hence, by Hölder's inequality, for $n\geq n_0$,

$$\left\|\left(\left(x_k^{(n)}\right)^p-x_k^p\right)\right\|_{\ell^1}\leq p\left\|x^{(n)}-x\right\|_{\ell^p}\left(\left\|\left(x^{(n)}\right)^{p-1}\right\|_{\ell^{p/(p-1)}}+\left\|x^{p-1}\right\|_{\ell^{p/(p-1)}}\right)$$

$$=p\left\|x^{(n)}-x\right\|_{\ell^p}\left(\left\|x^{(n)}\right\|_{\ell^p}^{p-1}+\|x\|_{\ell^p}^{p-1}\right).$$

Continuing, since $x^{(n)}\to x$ in ℓ^p, there is a constant $c_2>0$ so that the above is further bounded by

$$c_2p\left\|x^{(n)}-x\right\|_{\ell^p}\|x\|_{\ell^p}^{p-1}.$$

In summary, on K, for $n\geq n_0$,

$$\left|\prod_{k=k_0}^{\infty}f_k^{(n)}(z)-\prod_{k=k_0}^{\infty}f_k(z)\right|\leq c_1c_2p\left\|x^{(n)}-x\right\|_{\ell^p}\|x\|_{\ell^p}^{p-1}.$$

□

To apply this to the stochastic Airy operator $\mathcal{H}$, its known construction as a Schatten-von Neumann operator must be reviewed. **Hilbert-Schmidt integral operators** A are elements of $\mathcal{S}_2$, called **Hilbert-Schmidt operators**, that have the form

$$Af(x)=\int_X k(x,y)\,f(y)\,d\mu(y),\quad f\in L_2(X,\mu)$$

where k is a measurable function on $X\times X$ satisfying

$$\int_{X\times X}|k|^2<\infty.$$

In fact, if A is only defined by the above formula and k satisfies the above then A is Hilbert-Schmidt (Theorem 7.7, p. 70 Gohberg, Goldberg, and Krupnik (2012)). Special to the Hilbert-Schmidt integral operators is a Taylor series expression for its regularized determinant:

$$\det\nolimits_2(I-zA)=1+\sum_2^{\infty}\frac{(-1)^n z^n}{n!}\int_{X^n}\det\left[k(t_i,t_j)\,1_{i\neq j}\right]_{i,j=1}^n. \tag{63}$$

This is essentially Theorem 3.1, p. 176 in Gohberg, Goldberg, and Krupnik (2012) where the formula is given for another kind of determinant called a Hilbert-Carleman determinant, and within the proof it is remarked that the two determinants agree for Hilbert-Schmidt integral operators.

Fact 51. *(Proposition 9 Dumaz, Li, and Valkó (2020)) Almost surely the stochastic Airy operator $\mathcal{H} = \mathcal{H}_\beta$, $\beta > 0$ has an inverse operator $\mathcal{H}^{-1} = \mathcal{H}_\beta^{-1}$ which takes the form of a Hilbert-Schmidt integral operator on $L^2\left([0,\infty),\mathbb{R}\right)$,*

$$\mathcal{H}^{-1} f(x) = \int_0^\infty k(x,y)\, f(y)\, dy$$

where

$$k(x,y) = \psi_\infty(x)\,\psi_d(y)\,1_{x\geq y} + \psi_d(x)\,\psi_\infty(y)\,1_{x<y},\ x,\ y \in [0,\infty).$$

Here $\psi_d \notin L^2\left([0,\infty),\mathbb{R}\right)$ is the solution of $\mathcal{H}\psi = 0$ with initial condition $\psi(0) = 0$, $\psi'(0) = 1$, and $\psi_\infty \in L^2\left([0,\infty),\mathbb{R}\right)$ is the unique function satisfying $\mathcal{H}\psi_\infty = 0$, $\psi_\infty(0) = 1$. In particular, ψ_d, $\psi_\infty \in AC\left([0,\infty),\mathbb{R}\right)$.

Example 52. By Fact 51, almost surely $\mathcal{H}^{-1} \in \mathcal{S}_2$ with eigenvalues $1/\Lambda_k$. Consequently, by Example 43 and Corollary 48,

$$\det{}_2\left(I - z\mathcal{H}^{-1}\right) = \mathfrak{p}(z).$$

This connection serves as a bridge between the viewpoints of the Airy process Λ_k as the zeros of an entire function and as the eigenvalues of an operator. To realize this connection further, recall from Section 5 and Example 34 the random tridiagonal matrices $\mathcal{H}_n$ on the subspaces L_n^* of $L^2(\mathbb{R}_{\geq 0},\mathbb{R})$ with eigenvalues $\lambda_1^{(n)} \leq \cdots \leq \lambda_n^{(n)}$, a rescaled Gaussian beta ensemble, converging to Λ_k in the sense of finite-dimensional distributions. Note that L_n^* has the orthonormal basis $e_j = 1_{\left[jm_n^{-1},(j+1)m_n^{-1}\right)}$, $0 \leq j < n$: by definition L_n^* is the linear span of this set and being orthonormal (in L^2) is immediate by the disjointness of the intervals. By expanding $\mathcal{H}_n v$ for $v \in L_n^*$ and collecting like terms based on $\langle v, e_j\rangle$, $0 \leq j < n$ the expression

$$\mathcal{H}_n v = \sum_{j=0}^{n-1} \langle v, e_j\rangle_2\, h_j^{(n)}, \quad v \in L_n^*$$

is obtained where $h_j^{(n)} \in L_n^*$ come from $\mathcal{H}_n$. Furthermore, this expression extends to $L^2(\mathbb{R}_{\geq 0},\mathbb{R})$. With this extension $\mathcal{H}_n$ has range L_n^* and is bounded by Hölder's inequality:

$$\|\mathcal{H}_n f\|_2 \leq \left(\sum_{j=0}^{n-1} \|e_j\|_2 \left\|h_j^{(n)}\right\|_2\right) \|f\|_2 .$$

Hence, $\mathcal{H}_n$ is finite rank. Additionally, when no $\lambda_k^{(n)}$, $1 \le k \le n$ is zero this argument may be repeated to show that $\mathcal{H}_n^{-1}$ is also finite rank on $L^2(\mathbb{R}_{\geq 0}, \mathbb{R})$. Since L_n^* is closed therefore $L^2(\mathbb{R}_{\geq 0}, \mathbb{R}) = L_n^* \oplus (L_n^*)^{\perp}$ and consequently, by the definition of regularized determinants for finite rank operators,

$$\det{}_2\left(I - z\mathcal{H}_n^{-1}\right) = \prod_{k=1}^{n}\left(1 - \frac{z}{\lambda_k^{(n)}}\right)e^{z/\lambda_k^{(n)}}.$$

This identity may be extended by defining the above product to be the identically 1 function for a realization where some $\lambda_k^{(n)} = 0$. Later in Section 7 it will be shown that

$$\prod_{k=1}^{n}\left(1 - \frac{z}{\lambda_k^{(n)}}\right)e^{z/\lambda_k^{(n)}} \Longrightarrow \mathfrak{p}(z)$$

in the topology of compact convergence, thereby having the dual perspective

$$\det{}_2\left(I - z\mathcal{H}_n^{-1}\right) \Longrightarrow \det{}_2\left(I - z\mathcal{H}^{-1}\right)$$

of convergence of secular functions of operators.

In general, if $\lambda_k^{(n)}$ is a rescaled general beta ensemble converging to Λ_k then call the random entire function

$$\prod_{k=1}^{n}\left(1 - \frac{z}{\lambda_k^{(n)}}\right)e^{z/\lambda_k^{(n)}}$$

(which is the identically 1 function for any realization where $0 \in \left\{\lambda_k^{(n)}\right\}$) the **secular function** of the rescaled ensemble.

6.5 Convergence operations in distribution

In this section some convergence laws for random entire functions are stated, including logarithmic derivatives. These will be useful in Section 7.4.2. The proofs are essentially elementary applications of the continuous mapping theorem. For example, recall that the secular functions for rescaled general beta ensembles

$$\prod_{1}^{n}\left(1 - \frac{z}{\lambda_k^{(n)}}\right)e^{z/\lambda_k^{(n)}}$$

are defined to be the identically 1 function for realizations where some $\lambda_k^{(n)} = 0$. Since

$$\{g \in H(\mathbb{C}) : g(z_0) \neq 0\} = \bigcup_{m=1}^{\infty} V\left(\{z_0\}, 0, \frac{1}{m}\right)^C$$

is a measurable subset of $H(\mathbb{C})$, Lemma 85 may be used to deal with technicalities like the above.

Lemma 53. *Let f_n, f be random entire functions on probability spaces Ω_n, Ω and $z_0 \in \mathbb{C}$ a deterministic constant with $\mathbb{P}(f(z_0) \neq 0) = 1$. If $f_n \Longrightarrow f$ in the topology of compact convergence then*

$$\frac{f_n}{f_n(z_0)} \Longrightarrow \frac{f}{f(z_0)}$$

in the same topology, where $f/f_n(z_0)$ is the identically 1 function for any realization where $f_n(z_0) = 0$, and likewise for $f/f(z_0)$.

Lemma 54. *Let f_n, f be random real analytic functions on probability spaces Ω_n, Ω and $x_0 \in \mathbb{R}$ a deterministic constant with $\mathbb{P}(f(x_0) \neq 0) = 1$. If $f_n \Longrightarrow f$ in the topology of uniform convergence on compact subsets of $\mathbb{R}$ of functions and all their derivatives then*

$$\frac{f_n}{f_n(x_0)} \Longrightarrow \frac{f}{f(x_0)}$$

in the topology of compact convergence, for any $m \geq 0$,

$$\frac{f_n^{(m)}(x_0)}{f_n(x_0)} \Longrightarrow \frac{f^{(m)}(x_0)}{f(x_0)},$$

and

$$\frac{f_n}{f_n(x_0)} \exp\left(-x\frac{f_n'(x_0)}{f_n(x_0)}\right) \Longrightarrow \frac{f}{f(x_0)} \exp\left(-x\frac{f'(x_0)}{f(x_0)}\right)$$

in the topology compact convergence, where all expressions are respectively the identically 1 function whenever $f_n(x_0) = 0$ or $f(x_0) = 0$.

7 Limiting random canonical products

Recall in Section 5 the discrete models $\mathcal{H}_n$ with eigenvalues a rescaled Gaussian beta ensemble $\lambda_k^{(n)}$ that limit to the stochastic Airy operator $\mathcal{H}$. And, in Example 52 it shown that there are the secular functions

$$\det{}_2\left(I - z\mathcal{H}_n^{-1}\right) = \prod_1^n \left(1 - \frac{z}{\lambda_k^{(n)}}\right) e^{z/\lambda_k^{(n)}}$$

and

$$\det{}_2\left(I - z\mathcal{H}^{-1}\right) = \prod_1^\infty \left(1 - \frac{z}{\Lambda_k}\right) e^{z/\Lambda_k}.$$

Continuing, it is claimed that

$$\det{}_2\left(I - z\mathcal{H}_n^{-1}\right) \Longrightarrow \det{}_2\left(I - z\mathcal{H}^{-1}\right)$$

in the topology of compact convergence. This section develops a framework for proving this claim, which more generally proves secular function convergence of Schatten-von Neumann operators satisfying certain assumptions. This will be a consequence of proving convergence for certain random canonical products. One main ingredient is the definition of a special kind of random sequence, and it is shown that the rescaled general beta ensembles and Airy process are an example when $\beta \geq 1$. This involves establishing precise asymptotics on the ensembles. Furthermore, some applications are considered to questions posed in Lambert and Paquette (2020) which considered the Gaussian case.

7.1 Product convergent random sequences

In this section the random sequences are defined for which the convergence of random canonical products stated in Theorem 4 holds:

$$\prod_{k=1}^n \left(1 - \frac{z}{a_k^{(n)}}\right) \exp\left(\sum_{j=1}^p \frac{z^j}{j\left(a_k^{(n)}\right)^j}\right) \Longrightarrow \prod_{k=1}^\infty \left(1 - \frac{z}{A_k}\right) \exp\left(\sum_{j=1}^p \frac{z^j}{jA_k^j}\right).$$

Such random sequences will include the general beta ensembles and Airy process when $\beta \geq 1$, this being shown in Section 7.2. Consider random variables $a_k^{(n)} : \Omega_n \to \mathbb{C}$, $1 \leq k \leq n$, $n \geq 1$ and $A_k : \Omega \to \mathbb{C}$, $k \geq 1$ for probability spaces Ω_n, $n \geq 1$, Ω. Alternatively, let $a^{(n)} = \left(a_1^{(n)}, \ldots, a_n^{(n)}, 0, \ldots\right) \in \mathbb{C}^{\mathbb{Z}_{\geq 1}}$ and likewise for A, which are well-defined $\mathbb{C}^{\mathbb{Z}_{\geq 1}}$-valued random variables with the product topology. Assume the following:

1. $a^{(n)} \Longrightarrow A$ in the sense of finite-dimensional distributions.

2. Almost surely $A_k \neq 0$ for all k.

3. $1/a^{(n)}$, $n \geq 1$ are tight as $\ell^{p+1}(\mathbb{Z}_{\geq 1}, \mathbb{C})$-valued random variables.

Here the convention is that $1/A = (0, \ldots)$ for any realization where (2) fails. That is, $1/A = f(A)$ where

$$f(x) = \begin{cases} (1/x_k) & x_k \neq 0 \ \forall k \\ 0 & \text{otherwise} \end{cases}, \quad x \in \mathbb{C}^{\mathbb{Z}_{\geq 1}}. \tag{64}$$

The same convention goes for $1/a^{(n)}$ for any realization where $a_k^{(n)} = 0$ for some k. There are some important remarks:

Remark 55.

1. The mapping (64) is measurable: with π_k the kth projection,

$$S = \left\{x \in \mathbb{C}^{\mathbb{Z}_{\geq 1}} : x_k \neq 0 \ \forall k\right\} = \bigcap_k \pi_k^{-1}\left(\{0\}\right)^C$$

   is measurable, the mapping

$$g : S \to \mathbb{C}^{\mathbb{Z}_{\geq 1}}, \quad g(x) = \frac{1}{x}$$

   is continuous and hence $f = g 1_S$ is measurable by Lemma 85. Hence, $1/a^{(n)}$ and $1/A$ are well-defined $\mathbb{C}^{\mathbb{Z}_{\geq 1}}$-valued random variables.

2. $1/a^{(n)} \Longrightarrow 1/A$ in $\mathbb{C}^{\mathbb{Z}_{\geq 1}}$: the mapping (64) has continuity points all x satisfying $x_k \neq 0$ for all k, so by (1) and (2) the continuous mapping theorem may be applied.

3. Moreover, $1/a^{(n)} \Longrightarrow 1/A$ in $\ell^{p+1}(\mathbb{Z}_{\geq 1}, \mathbb{C})$ with the norm topology, as follows. Since this topology is coarser than that of $\mathbb{C}^{\mathbb{Z}_{\geq 1}}$ therefore $1/a^{(n)}$ are well-defined $\ell^{p+1}(\mathbb{Z}_{\geq 1}, \mathbb{C})$-valued random variables. By (2), given any subsequence of n there is a further subsequence so that along it, $1/a^{(n)} \Longrightarrow 1/A$ in $\mathbb{C}^{\mathbb{Z}_{\geq 1}}$. By tightness (3), there is a further subsequence and a $\ell^{p+1}(\mathbb{Z}_{\geq 1}, \mathbb{C})$-valued random variable X so that along this subsequence, $1/a^{(n)} \Longrightarrow X$ in $\ell^{p+1}(\mathbb{Z}_{\geq 1}, \mathbb{C})$. Since the projections

$$\pi_m : \ell^{p+1}(\mathbb{Z}_{\geq 1}, \mathbb{C}) \to \mathbb{C}^m, \quad x \mapsto (x_1, \ldots, x_m)$$

   are continuous therefore by the continuous mapping theorem, $1/a^{(n)} \Longrightarrow X$ in $\mathbb{C}^{\mathbb{Z}_{\geq 1}}$ also. Hence, $1/A \stackrel{d}{=} X$ in $\mathbb{C}^{\mathbb{Z}_{\geq 1}}$, and then also in $\ell^{p+1}(\mathbb{Z}_{\geq 1}, \mathbb{C})$ since its topology is coarser. Consequently, the whole sequence satisfies $1/a^{(n)} \Longrightarrow 1/A$ in $\ell^{p+1}(\mathbb{Z}_{\geq 1}, \mathbb{C})$.

4. By (3) and (2) the random canonical product

$$P(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{A_k}\right) \exp\left(\sum_{j=1}^{p} \frac{z^j}{j A_k^j}\right)$$

is now well-defined.

Random sequences $a_k^{(n)}$, A_k that satisfy the above assumptions are collectively called **product convergent** with **genus** p. Note that for any realization of A where $A_k = 0$ for some k, by the above convention $P(z)$ is the identically 1 function, thereby agreeing with the previous conventions placed in Section 6. Likewise for the random canonical products

$$P_n(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{a_k^{(n)}}\right) \exp\left(\sum_{j=1}^{p} \frac{z^j}{j \left(a_k^{(n)}\right)^j}\right).$$

The definition comes from a natural extension of what holds for the secular function $\mathfrak{p}(z)$ of the stochastic Airy operator, seen in detail in Example 44. Note that when $P(z)$ has non-integral order then it can be guaranteed that $P(z)$ is non-trivial by the almost sure growth statements in Section 6.3. Furthermore, in that section it is shown that the secular function $\mathfrak{p}(z)$ of the stochastic Airy operator is such an example.

7.2 Product convergence of the Airy beta point process

This section is dedicated to verifying Theorem 5, namely that the rescaled general beta ensembles $\lambda_k^{(n)}$ and Airy process Λ_k with $\beta \geq 1$ are product convergent with genus 1, with corresponding random canonical product $P(z) = \mathfrak{p}(z)$ the secular function of the stochastic Airy operator. Recall that $\lambda_k^{(n)}$ are defined Proposition 17 and Λ_k in Fact 25. Properties (1) and (2) have already been shown in Propositions 17 and 28. This section is dedicated to verifying (3). The method of proof is instead developing precise asymptotics on the particles $\lambda_k^{(n)}$. This is done by separating the particles indexed by k before and after n^κ for small $\kappa > 0$. For $k \leq n^\kappa$ precise rigidity estimates for $\lambda_k^{(n)}$ are developed, and for $k \geq n^\kappa$, rigidity estimates as discussed in Section 3.1 are mostly sufficient.

7.2.1 Rigidity estimates at the particle level for rescaled general beta ensembles

Recall from Section 3 that

$$\lambda_k^{(n)} = (\pi s_A)^{2/3} n^{2/3} \left(\mu_k^{(n)} - A\right)$$

where μ is a general beta ensemble with equilibrium density $\rho(x)$ having support with left endpoint A and satisfying

$$\rho(x) = s_A \sqrt{x - A}\,(1 + O(x - A)), \quad x \to A^+$$

for $s_A > 0$, and $\gamma_k^{(n)}$ is the n-quantile of $\rho(x)$. Furthermore, $s_A^{2/3}\left(\mu_k^{(n)} - A\right)$ is a general beta ensemble with $s_0 = 1$ and n-quantiles $s_A^{2/3}\left(\gamma_k^{(n)} - A\right)$. The following contains a consequence of the rigidity estimate in Fact 13 and a rigidity estimate particular to the rescaled general beta ensembles:

Lemma 56. *For any $\xi > 0$ there exists $c = c(\xi) > 0$ and $n_0 = n_0(\xi, c)$ large so that for $n \geq n_0$,*

$$\mathbb{P}\left(\left|\mu_k^{(n)} - \gamma_k^{(n)}\right| \leq n^{-2/3+\xi}, \quad \textit{for all } k\right) \geq 1 - e^{-n^c}.$$

In particular, n_0 may be chosen so that for $n \geq n_0$,

$$\mathbb{P}\left(\left|\lambda_k^{(n)} - (\pi s_A)^{2/3} n^{2/3}\left(\gamma_k^{(n)} - A\right)\right| \leq \pi^{2/3} n^{\xi}, \quad \textit{for all } k\right) \geq 1 - e^{-n^c}.$$

Proof. By Fact 13 there exists $c = c(\xi) > 0$ and $n_0 = n_0(\xi)$ large so that for $n \geq n_0$ and any k,

$$\mathbb{P}\left(|\mu_k - \gamma_k| > n^{-2/3+\xi}\hat{k}^{-1/3}\right) \leq e^{-n^c}$$

where $\hat{k} = \min\{k, n+1-k\}$. Note that $\hat{k} \geq 1$ and so $\hat{k}^{-1/3} \leq 1$. Hence, $n^{-2/3+\xi}\hat{k}^{-1/3} \leq n^{-2/3+\xi}$ and so for $n \geq n_0$,

$$\mathbb{P}\left(|\mu_k - \gamma_k| > n^{-2/3+\xi}\right) \leq e^{-n^c}.$$

Then, for $n \geq n_0$,

$$\mathbb{P}\left(|\mu_k - \gamma_k| \leq n^{-2/3+\xi}, \quad \text{for all } k\right) \geq 1 - ne^{-n^c}.$$

In possibly enlarging $n_0 = n_0(\xi, c)$ it then follows that for $n \geq n_0$,

$$\mathbb{P}\left(|\mu_k - \gamma_k| \leq n^{-2/3+\xi}, \quad \text{for all } k\right) \geq 1 - e^{-n^{c/2}}.$$

This is because $x^{c/2}/\left(x^c - \log x\right) \to 0$ as $x \to \infty$ and hence eventually

$$n^c - \log n \geq n^{c/2} \iff 1 - ne^{-n^c} \geq 1 - e^{-n^{c/2}}.$$

Since this holds for any general beta ensemble, it in particular holds for the rescaled one: for $n \geq n_0$,

$$\mathbb{P}\left(\left|s_A^{2/3}\left(\mu_k - A\right) - s_A^{2/3}\left(\gamma_k - A\right)\right| \leq n^{-2/3+\xi}, \quad \text{for all } k\right) \geq 1 - e^{-n^{c/2}}.$$

Consequently,

$$\mathbb{P}\left(\left|\lambda_k - \left(\pi s_A\right)^{2/3} n^{2/3}\left(\gamma_k - A\right)\right| \leq \pi^{2/3} n^{\xi}, \quad \text{for all } k\right) \geq 1 - e^{-n^{c/2}}.$$

□

Although Lemma 56 has its uses, the n^ξ bound is problematic for looking at individual particles as $n \to \infty$. The goal is now to obtain rigidity estimates at the particle level, which essentially means replacing n with k in the bound. First this is done for the Gaussian beta ensembles $\tilde{\mu}$ as defined in Section 3.2, and then is extended in general via edge universality in Section 3.4. Recall from Example 20 that the rescaled Gaussian beta ensemble $\tilde{\lambda}$ satisfies

$$\tilde{\lambda}_k^{(n)} = n^{2/3}\left(\tilde{\mu}_k^{(n)} + 2\right) \Longrightarrow \Lambda_k$$

in the sense of finite-dimensional distributions. Let $\tilde{\gamma}_k^{(n)}$ be the n-quantiles for $\tilde{\mu}$. The following fact is required:

Fact 57. *(Theorem 1.2 Landon (2020)) For $\beta \geq 1$ there is a coupling of $\tilde{\lambda}^{(n)}$, $n \geq 1$ and Λ_k, $k \geq 1$ so that exists $\alpha > 0$ and $c_0 > 0$ so that for any $n \geq 1$,*

$$\mathbb{P}\left(\sup_{1 \leq k \leq n^\alpha}\left|\tilde{\lambda}_k^{(n)} - \Lambda_k\right| \geq n^{-\alpha}\right) \leq c_0 n^{-\alpha}.$$

Proposition 58. *For $\beta \geq 1$ there exists $\kappa_0 \leq 3/5$ and $c_0 > 0$ such that for any $0 < \kappa < \kappa_0$ and any $0 < \varepsilon < 1/3$ there exists $n_0 = n_0(\kappa)$ and $k_0 = k_0(\varepsilon)$ large so that for $n \geq n_0$, $k_0 \leq k \leq n^\kappa$,*

$$\mathbb{P}\left(\left|\tilde{\lambda}_k^{(n)} - n^{2/3}\left(\tilde{\gamma}_k^{(n)} + 2\right)\right| > k^{1/6+\varepsilon}\right) \leq c_0 n^{-\alpha} + 2^{-k^\chi}$$

where $\chi = \chi(\varepsilon) > 0$. In particular, for any $r > 0$, for any $0 < \kappa < \min\{\kappa_0, \kappa_0/r\}$ there exists $n_0 = n_0(\varepsilon, \kappa, r)$ and $k_0 = k_0(\varepsilon, r)$ large so that the probability bound may be replaced with k^{-r}.

Proof. Choose $\alpha > 0$ and $c_0 > 0$ so that the conclusion of Fact 57 holds. Let κ satisfy $0 < \kappa < \kappa_0 = \min\{\alpha, 3/5\}$, and $0 < \varepsilon < 1/3$. By Fact 19 there exists $n_0 = n_0(\kappa)$ large so that for $n \geq n_0$, $1 \leq k \leq n^\kappa$,

$$n^{2/3}(\tilde{\gamma}_k + 2) = \left(\frac{3\pi}{2}\right)^{2/3} k^{2/3} + O\left(\frac{k^{5/3}}{n}\right).$$

By Fact 26 it now follows that for $n \geq n_0$, $1 \leq k \leq n^\kappa$,

$$\Lambda_k^0 - n^{2/3}(\tilde{\gamma}_k + 2) = o(1) - O\left(\frac{k^{5/3}}{n}\right)$$

where Λ_k^0 are the eigenvalues of the classical Airy operator $\mathcal{A} = -\partial_x^2 + x$. Note that $O\left(n^{-1}k^{5/3}\right)$ vanishes with n uniformly in k, $1 \leq k \leq n^\kappa$ because $5\kappa/3 < 1$ by definition of κ_0. Consequently, there exists k_0 large so that, in possibly enlarging n_0, for $n \geq n_0$, $k_0 \leq k \leq n^\kappa$,

$$\left|\Lambda_k^0 - n^{2/3}(\tilde{\gamma}_k + 2)\right| \leq \frac{1}{4}. \tag{65}$$

Possibly enlarge n_0 and $k_0 = k_0(\varepsilon)$ further so that for $n \geq n_0$, $n^{-\alpha} \leq 1/2$, for $k_0 \leq k \leq n^\kappa$, $4^{-1}k^{1/6+\varepsilon} \geq k^{1/6+\varepsilon/2}$, and by Theorem 1, for $k \geq k_0$,

$$\mathbb{P}\left(\left|\Lambda_k - \Lambda_k^0\right| \leq k^{1/6+\varepsilon/2}\right) \geq 1 - 2^{-k^\chi} \tag{66}$$

where $\chi = \chi(\varepsilon) > 0$. For the remainder of this proof assume $n \geq n_0$ and $k_0 \leq k \leq n^\kappa$. Write

$$\begin{aligned}&\mathbb{P}\left(\left|\tilde{\lambda}_k - n^{2/3}(\tilde{\gamma}_k + 2)\right| > k^{1/6+\varepsilon}\right)\\ &\leq \mathbb{P}\left(\left|\tilde{\lambda}_k - \Lambda_k\right| > \frac{1}{2}k^{1/6+\varepsilon}\right) + \mathbb{P}\left(\left|\Lambda_k - n^{2/3}(\tilde{\gamma}_k + 2)\right| > \frac{1}{2}k^{1/6+\varepsilon}\right).\end{aligned}$$

Since $n^{-\alpha} \leq 1/2$ and $\kappa < \alpha$ it follows from Fact 57 that

$$\mathbb{P}\left(\left|\tilde{\lambda}_k - \Lambda_k\right| > \frac{1}{2}k^{1/6+\varepsilon}\right) \leq c_0 n^{-\alpha}.$$

Next, write

$$\begin{aligned}&\mathbb{P}\left(\left|\Lambda_k - n^{2/3}(\tilde{\gamma}_k + 2)\right| > \frac{1}{2}k^{1/6+\varepsilon}\right)\\ &\leq \mathbb{P}\left(\left|\Lambda_k - \Lambda_k^0\right| > \frac{1}{4}k^{1/6+\varepsilon}\right) + \mathbb{P}\left(\left|\Lambda_k^0 - n^{2/3}(\tilde{\gamma}_k + 2)\right| > \frac{1}{4}k^{1/6+\varepsilon}\right).\end{aligned}$$

Since $\left|\Lambda_k^0 - n^{2/3}(\tilde{\gamma}_k + 2)\right| \leq 1/4$ by (65) therefore

$$\mathbb{P}\left(\left|\Lambda_k^0 - n^{2/3}(\tilde{\gamma}_k + 2)\right| > \frac{1}{4}k^{1/6+\varepsilon}\right) = 0.$$

Since $4^{-1}k^{1/6+\varepsilon} \geq k^{1/6+\varepsilon/2}$ it follows from (66) that

$$\mathbb{P}\left(\left|\Lambda_k - \Lambda_k^0\right| > \frac{1}{4}k^{1/6+\varepsilon}\right) \leq 2^{-k^\chi}.$$

In summary, by collecting all the inequalities together, for $n \geq n_0$, $k_0 \leq k \leq n^\kappa$,

$$\mathbb{P}\left(\left|\tilde{\lambda}_k - n^{2/3}(\tilde{\gamma}_k + 2)\right| > k^{1/6+\varepsilon}\right) \leq c_0 n^{-\alpha} + 2^{-k^\chi}.$$

For the second claim, let $r > 0$ and assume $\kappa < \min\{\kappa_0/r, \kappa_0\}$. Since

$$2^{-k^\chi} \leq \frac{1}{2}k^{-r} \iff 1 + r\log_2 k \leq k^\chi$$

and $(1 + r\log_2 t)/t^\chi \to 0$ as $t \to \infty$ therefore $k_0 = k_0(\varepsilon, r)$ may be possibly enlarged so that the above holds for $k \geq k_0$. Furthermore, since $k \leq n^\kappa$,

$$\frac{c_0 n^{-\alpha}}{\frac{1}{2}k^{-r}} \leq 2c_0 n^{r\kappa-\alpha}$$

and $r\kappa < \kappa_0 \leq \alpha$, so $n_0 = n_0(\kappa, r)$ may be possibly enlarged so that $c_0 n^{-\alpha} \leq k^{-r}/2$ for $n \geq n_0$, $k \leq n^\kappa$. Hence, for $n \geq n_0$, $k_0 \leq k \leq n^\kappa$,

$$\mathbb{P}\left(\left|\tilde{\lambda}_k - n^{2/3}(\tilde{\gamma}_k + 2)\right| > k^{1/6+\varepsilon}\right) \leq k^{-r}.$$

□

Similar to the rescaled general beta ensemble $s_A^{2/3}(\mu - A)$, recall from Example 16 that the rescaled Gaussian beta ensemble $\pi^{-2/3}(\tilde{\mu} + 2)$ satisfy $s_0 = 1$ and have n-quantiles $\pi^{-2/3}(\tilde{\gamma}_k + 2)$. Consequently, edge universality from Section 3.4 may now be applied in the following way to generalize Proposition 58 to general beta ensembles and thereby prove Theorem 6: *For $\beta \geq 1$ there exists $0 < \kappa_0 \leq 2/5$ so that for any $0 < \kappa < \kappa_0$, $0 < \delta < 1/2$ and $\varepsilon > 0$ there exists $n_0 = n_0(\kappa, \varepsilon)$ and $k_0 = k_0(\delta, \varepsilon)$ large so that for $n \geq n_0$,*

$$\mathbb{P}\left(\left|\lambda_k^{(n)} - (\pi s_A)^{2/3} n^{2/3}\left(\gamma_k^{(n)} - A\right)\right| \leq \pi^{2/3}k^{1/6+\delta}, \quad \textit{for all } k_0 \leq k \leq n^\kappa\right) \geq 1 - \varepsilon.$$

Proof of Theorem 6. Let $\kappa_1 < \min\{\kappa_0/2, 2/5\}$ where κ_0 is from Proposition 58. By applying

Fact 21 to the ensembles $s_A^{2/3}(\mu - A)$ and $\pi^{-2/3}(\tilde{\mu}+2)$, there exists ξ, χ, $c_1 > 0$ so that for a given collection of observables

$$\{O_{n,k} \mid n \geq 1, 1 \leq k \leq n^{\kappa_1}\}$$

where $O_{n,k} \in C_c^2(\mathbb{R})$, have support size up to n^ξ and satisfy that $\|O_{n,k}\|_\infty$, $\|O'_{n,k}\|_\infty$, $\|O''_{n,k}\|_\infty \leq c_2$ for some single constant $c_2 > 0$, then for any n and any $1 \leq k \leq n^{\kappa_1}$,

$$\left|\mathbb{E}O_{n,k}\left(s_A^{2/3} n^{2/3} k^{1/3}(\mu_k - \gamma_k)\right) - \mathbb{E}O_{n,k}\left(\pi^{-2/3} n^{2/3} k^{1/3}(\tilde{\mu}_k - \tilde{\gamma}_k)\right)\right| \leq c_1 n^{-\chi}. \tag{67}$$

Let $\kappa < \min\{\kappa_1, \chi, \xi\}$ so that (67) in particular holds for $1 \leq k \leq n^\kappa$. Note "κ_0" in the claim of this theorem is chosen as $\min\{\kappa_1, \chi, \xi\}$. First it will be shown that for all n and all $2 \leq k \leq n^\kappa$ that

$$\begin{aligned}&\mathbb{P}\left(s_A^{2/3} n^{2/3} k^{1/3}(\mu_k - \gamma_k) \in \left[-k^{1/2+\delta}, k^{1/2+\delta}\right]\right)\\ &\geq \mathbb{P}\left(\pi^{-2/3} n^{2/3} k^{1/3}(\tilde{\mu}_k - \tilde{\gamma}_k) \in \left[-k^{1/2+\delta/2}, k^{1/2+\delta/2}\right]\right) - c_1 n^{-\chi}.\end{aligned} \tag{68}$$

A standard argument shows that the observables $O_{n,k}$, $n \geq 1$, $1 \leq k \leq n^\kappa$ can be chosen as bump functions satisfying the following: they are nonnegative, have derivatives of all orders, are identically 1 on $\left[-k^{1/2+\delta/2}, k^{1/2+\delta/2}\right]$ and vanish outside the superset $\left(-k^{1/2+\delta}, k^{1/2+\delta}\right)$. Since $\delta < 1/2$ the support size is at most n^ξ. Consequently, by (67), for all n and all $2 \leq k \leq n^\kappa$,

$$\begin{aligned}\mathbb{P}\left(s_A^{2/3} n^{2/3} k^{1/3}(\mu_k - \gamma_k) \in \left[-k^{1/2+\delta}, k^{1/2+\delta}\right]\right) &\geq \mathbb{E}O_{n,k}\left(s_A^{2/3} n^{2/3} k^{1/3}(\mu_k - \gamma_k)\right)\\ &\geq \mathbb{E}O_{n,k}\left(\pi^{-2/3} n^{2/3} k^{1/3}(\tilde{\mu}_k - \tilde{\gamma}_k)\right) - c_1 n^{-\chi}\\ &\geq \mathbb{P}\left(\pi^{-2/3} n^{2/3} k^{1/3}(\tilde{\mu}_k - \tilde{\gamma}_k) \in \left[-k^{1/2+\delta/2}, k^{1/2+\delta/2}\right]\right) - c_1 n^{-\chi}.\end{aligned}$$

This shows (68). Next, recalling that $\lambda_k = (\pi s_A)^{2/3} n^{2/3}(\mu_k - A)$ and $\tilde{\lambda}_k = n^{2/3}(\tilde{\mu}_k + 2)$ it now follows that for all n and all $2 \leq k \leq n^\kappa$,

$$\begin{aligned}&\mathbb{P}\left(\left|\lambda_k - (\pi s_A)^{2/3} n^{2/3}(\gamma_k - A)\right| \leq \pi^{2/3} k^{1/6+\delta}\right)\\ &\geq \mathbb{P}\left(\left|\tilde{\lambda}_k - n^{2/3}(\tilde{\gamma}_k + 2)\right| \leq \pi^{2/3} k^{1/6+\delta/2}\right) - c_1 n^{-\chi}.\end{aligned}$$

Then, for $k_0 \geq 2$ to be determined, for all n,

$$\mathbb{P}\left(\left\{\left|\lambda_k - (\pi s_A)^{2/3} n^{2/3} (\gamma_k - A)\right| \leq \pi^{2/3} k^{1/6+\delta}, \quad \text{for all } k_0 \leq k \leq n^\kappa\right\}^C\right)$$

$$\leq \sum_{k=k_0}^{n^\kappa} \mathbb{P}\left(\left|\tilde{\lambda}_k - n^{2/3} (\tilde{\gamma}_k + 2)\right| > k^{1/6+\delta/2}\right) + c_1 n^{-(\chi-\kappa)} \tag{69}$$

with $\chi - \kappa > 0$. Since $\kappa < \kappa_1 < \kappa_0/2$ where κ_0 is from Proposition 58 it follows that there exists $n_0 = n_0(\kappa)$ and $k_0 = k_0(\delta)$ large such that for $n \geq n_0$, (69) is less than or equal to

$$\sum_{k=k_0}^{\infty} k^{-2} + c_1 n^{-(\chi-\kappa)}.$$

Possibly enlarging $n_0 = n_0(\kappa, \varepsilon)$ and $k_0 = k_0(\delta, \varepsilon)$ the above bound can be replaced with ε, thereby showing the claim. □

7.2.2 Asymptotics of the rescaled general beta ensembles

In this section the rigidity estimates in Section 7.2.1 are now applied to yield lower bound asymptotics for the rescaled general beta ensembles λ_k to establish the tightness requirement (3).

Corollary 59. *Let $\beta \geq 1$ and κ_0 be from Theorem 6. There exists a deterministic constant $c > 0$ so that for any $0 < \kappa < \kappa_0$ and $\varepsilon > 0$ there exists $n_0 = n_0(\kappa, \varepsilon)$ and $k_0 = k_0(\varepsilon)$ large so that for $n \geq n_0$,*

$$\mathbb{P}\left(\lambda_k^{(n)} \geq c k^{2/3}, \quad \textit{for all } k_0 \leq k \leq n^\kappa\right) \geq 1 - \varepsilon.$$

Proof. By Theorem 6 with fixed $0 < \delta < 1/2$ there exists $n_0 = n_0(\kappa, \varepsilon)$ and $k_0 = k_0(\varepsilon)$ large so that for $n \geq n_0$,

$$\mathbb{P}\left(\lambda_k \geq (\pi s_A)^{2/3} n^{2/3} (\gamma_k - A) - \pi^{2/3} k^{1/6+\delta}, \quad \text{for all } k_0 \leq k \leq n^\kappa\right) \geq 1 - \varepsilon.$$

By (**??**) there exists a constant $c_3 > 0$ depending only on μ, the general beta ensemble which λ is the rescaling of, and possibly larger n_0 so that for $n \geq n_0$,

$$s_A^{2/3} (\gamma_k - A) \geq c_3 \left(\frac{k}{n}\right)^{2/3}$$

and hence

$$(\pi s_A)^{2/3} n^{2/3} (\gamma_k - A) \geq \pi^{2/3} c_3 k^{2/3}.$$

The claim now immediately follows. □

The less precise rigidity estimates, namely Lemma 56, are sufficient to establish the lower bound asymptotics for particles $k \geq n^\kappa$:

Proposition 60. *There is a deterministic constant $c > 0$ so that for any $0 < \kappa < 1$ and any $\varepsilon > 0$ there exists $n_0 = n_0(\kappa, \varepsilon)$ large so that for $n \geq n_0$,*

$$\mathbb{P}\left(\lambda_k^{(n)} \geq c k^{2/3}, \quad \textit{for all } k \geq n^\kappa\right) \geq 1 - \varepsilon.$$

Proof. Recall by Proposition 15 that the equilibrium density $\rho_s(t)$ of the rescaled ensemble $s_A^{2/3}(\mu - A)$ satisfies

$$\rho_s(t) = \sqrt{t}\,(1 + O(t)), \quad t \to 0^+$$

and is continuous with compact support. Consequently, there exists constants c_1, c_2 and $t_0 > 0$ so that $\rho_s(t) \leq c_1$ for all t and $\rho_s(t) \leq (1 + c_2)\sqrt{t}$ for $0 \leq t \leq t_0$. Then for $c_3 = \max\left\{c_1/\sqrt{t_0}, c_1, 1 + c_2\right\}$, $\rho_s(t) \leq c_3\sqrt{t}$ for all t. Hence, by Proposition 15,

$$\frac{k}{n} = \int_0^{s_A^{2/3}(\gamma_k - A)} \rho_s(t)\,dt \leq \int_0^{s_A^{2/3}(\gamma_k - A)} c_3\sqrt{t}dt = \frac{2c_3}{3}\left(s_A^{2/3}(\gamma_k - A)\right)^{3/2},$$

and so, with $c_4 = (3/(2c_3))^{2/3}$,

$$(\pi s_A)^{2/3} n^{2/3} (\gamma_k - A) \geq \pi^{2/3} c_4 k^{2/3}. \tag{70}$$

Note that c_4 is deterministic. Then, for $k \geq n^\kappa$,

$$\frac{1}{(\pi s_A)^{2/3} n^{2/3} (\gamma_k - A)} \leq \pi^{-2/3} c_4^{-1} n^{-2\kappa/3}. \tag{71}$$

By Lemma **??** there exists $k_0 = k_0(\varepsilon)$ and $n_0 = n_0(k_0, \varepsilon)$ large so that for $n \geq n_0$,

$$1 - \varepsilon \leq \mathbb{P}(\lambda_k \geq 0, \quad \text{for all } k \geq k_0). \tag{72}$$

By Lemma 56, possibly enlarge $n_0 = n_0(k_0, \varepsilon, \kappa)$ so that for $n \geq n_0$ the following holds: $n_0^\kappa \geq k_0$, $c_4^{-1} n^{-\kappa/3} < 1/2$ and

$$\mathbb{P}\left(\left|\lambda_k - (\pi s_A)^{2/3} n^{2/3} (\gamma_k - A)\right| \leq \pi^{2/3} n^{\kappa/3}, \quad \text{for all } k\right) \geq 1 - \varepsilon. \tag{73}$$

For $n \geq n_0$ choose a realization of (72) and (73) with probability at least $1-2\varepsilon$. Then, by (71), for $k \geq n^\kappa$,

$$\left|\frac{\lambda_k - (\pi s_A)^{2/3} n^{2/3} (\gamma_k - A)}{(\pi s_A)^{2/3} n^{2/3} (\gamma_k - A)}\right| \leq c_4^{-1} n^{-\kappa/3} < \frac{1}{2}.$$

Consequently, by (70) and (73), for $k \geq n^\kappa$, noting that $k \geq n_0^\kappa \geq k_0$ since $n \geq n_0$,

$$\lambda_k = |\lambda_k| = \left|(\pi s_A)^{2/3} n^{2/3} (\gamma_k - A) \left(1 + \frac{\lambda_k - (\pi s_A)^{2/3} n^{2/3} (\gamma_k - A)}{(\pi s_A)^{2/3} n^{2/3} (\gamma_k - A)}\right)\right| \geq \frac{\pi^{2/3} c_4}{2} k^{2/3}.$$

□

Corollary 7 is now immediate.

7.2.3 Application to product convergence

The previous sections may now be applied to prove Theorem 5:

Proof of Theorem 5. As noted at the beginning of this section, properties (1) and (2) have already been shown in Propositions 17 and 28., leaving only (3) for $p=1$. By Corollary 7 there exists a deterministic constant $c_1 > 0$ so that for any $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ and $k_0 = k_0(\varepsilon)$ large so that for $n \geq n_0$,

$$\mathbb{P}\left(\lambda_k^{(n)} \geq c_1 k^{2/3}, \quad \text{for all } k \geq k_0\right) \geq 1 - \varepsilon. \tag{74}$$

For $n \geq n_0$ and a realization of (74),

$$\frac{1}{\left|\lambda_k^{(n)}\right|} \leq \frac{1}{c_1} \frac{1}{k^{2/3}} \quad \text{for} \quad k \geq k_0.$$

By Remark (2), $1/\lambda^{(n)} \Longrightarrow 1/\Lambda$ in the sense of finite-dimensional distributions, recalling the convention that $1/\lambda^{(n)} = (0, \ldots)$ whenever $\lambda_k^{(n)} = 0$ for some k, and likewise for $1/\Lambda$. Therefore, n_0 may be enlarged so that for some constant $c_2 \geq 1/c_1$, for $n \geq n_0$, with probability at least $1-\varepsilon$,

$$\frac{1}{\left|\lambda_k^{(n)}\right|} \leq c_2 \frac{1}{k^{2/3}} \quad \text{for} \quad k < k_0.$$

It is elementary that the subset

$$K = \left\{x \in \ell^2(\mathbb{Z}_{\geq 1}, \mathbb{C}) : |x_k| \leq c_2 \frac{1}{k^{2/3}}, \quad \forall k\right\}$$

is compact since $\left(c_2 k^{-2/3}\right) \in \ell^2$. And, for $n \geq n_0$, with probability at least $1-2\varepsilon$, $1/\lambda^{(n)} \in K$. For $n < n_0$, since $1/\lambda^{(n)}$ are finitely many random variables, c_2 may be enlarged so that they belong to K also with probability at least $1-\varepsilon$. In summary, for all n,

$$\mathbb{P}\left(\frac{1}{\lambda^{(n)}} \in K\right) \geq 1 - 3\varepsilon.$$

□

7.3 Convergence to the limiting random canonical product

Recall the statement of Theorem 4: *If $a_k^{(n)}$, A_k are product convergent random sequences of genus p then $P_n(z) \Longrightarrow P(z)$ as $n \to \infty$ in the topology of compact convergence where*

$$P_n(z) = \prod_{k=1}^{n}\left(1 - \frac{z}{a_k^{(n)}}\right)\exp\left(\sum_{j=1}^{p}\frac{z^j}{j\left(a_k^{(n)}\right)^j}\right)$$

and

$$P(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z}{A_k}\right)\exp\left(\sum_{j=1}^{p}\frac{z^j}{jA_k^j}\right).$$

Proof of Theorem 4. By Remark (4), $P(z)$ is a well-defined random canonical product. By Remark (3), $1/a^{(n)} \Longrightarrow 1/A$ in $\ell^{p+1}(\mathbb{Z}_{\geq 1}, \mathbb{C})$, following the conventions from Section 7.1. Consequently, by Lemma 50 and the continuous mapping theorem,

$$\det{}_{p+1}\left(\frac{1}{a^{(n)}}\right) = P_n(z) \Longrightarrow \det{}_{p+1}\left(\frac{1}{A}\right) = P(z)$$

as $n \to \infty$ in the topology of compact convergence. □

7.4 Applications

7.4.1 Convergence as secular functions

Recall from Example 52 the discrete models $\mathcal{H}_n$ with eigenvalues $\lambda_k^{(n)}$, a rescaled Gaussian beta ensemble, converging to the stochastic Airy operator $\mathcal{H}$. Continuing, it was shown that $\mathcal{H}_n^{-1}$ may be viewed as a finite rank operator on $L^2(\mathbb{R}_{\geq 0}, \mathbb{R})$ with secular function

$$\det{}_2\left(I - z\mathcal{H}_n^{-1}\right) = \prod_{k=1}^{n}\left(1 - \frac{z}{\lambda_k^{(n)}}\right)e^{z/\lambda_k^{(n)}}.$$

Consequently, Theorems 5 and 4 yield Corollary 8 in the Gaussian case when $\beta \geq 1$: as $n \to \infty$,

$$\det{}_2\left(I - z\mathcal{H}_n^{-1}\right) \Longrightarrow \det{}_2\left(I - z\mathcal{H}^{-1}\right)$$

in the topology of compact convergence. More generally, Corollary 8 follows from Lemma 47 and Theorem 4.

7.4.2 The work of Lambert and Paquette

In Lambert and Paquette (2020) a different proof of secular function convergence is provided in the case of the Gaussian beta ensembles. By combining their results and the results here some open questions they posed are answered, as well as yielding some corollaries. There a random $C^1(\mathbb{R} \times \mathbb{C}, \mathbb{C})$ function $\mathrm{SAi}_z(t)$, $(t, z) \in \mathbb{R} \times \mathbb{C}$ which is entire in z and is defined as a certain solution of the system of stochastic differential equations,

$$d\phi_z'(t) = (t + z)\,\phi_z(t)\,dt + \phi_z(t)\,dB(t), \quad z \in \mathbb{C}$$

where B is a two-sided Brownian motion with $B(0) = 0$ and $\mathbb{E}\left(B(t)^2\right) = 4\beta^{-1}|t|$, $t \in \mathbb{R}$ and $\beta > 0$. A statement of their secular function convergence is as follows. Note that the minuses in the following is due to their definition of the stochastic Airy operator being the minus of the one stated here.

Fact 61. *(Theorem 1.1 Lambert and Paquette 2020) Let $-\mu = (-\mu_1, \ldots, -\mu_n)$ be a general beta ensemble with $V(x) = x^2$, i.e. -2μ is a Gaussian beta ensemble. With*

$$\varphi_n(x) = \prod_1^n (x - \mu_k), \quad w_n(x) = \left((2\pi)^{1/4} e^{nx^2} 2^{-n} \left(nx^2\right)^{-1/12} \sqrt{\frac{n!}{n^n}}\right)^{-1}$$

there exists centered Gaussian variables G_n so that as random real analytic functions with the topology of uniform convergence on compact subsets of $\mathbb{R}$ of the function and all its derivatives,

$$(w_n \times \varphi_n)\left(1 + \frac{x}{2n^{2/3}}\right) \frac{\mathbb{E}\exp G_n}{\exp G_n} \Longrightarrow SAi_x(0).$$

Note that by Theorems 5 and 4, and Corollary 42 that the secular function of the stochastic Airy operator

$$\mathfrak{p}(z) = \prod_1^\infty \left(1 - \frac{z}{\Lambda_k}\right) e^{z/\Lambda_k}, \quad z \in \mathbb{C}$$

yields a random real analytic function $\mathfrak{p}(x)$ with secular function convergence uniformly on compact subsets of $\mathbb{R}$. Furthermore, Fact 61 may be reduced to this finer topology. The relationship between $\mathfrak{p}(x)$ and $\mathrm{SAi}_x(0)$ stated in Proposition 9 is then established: *As random real analytic functions with the topology of compact convergence,*

$$\frac{SAi_x(0)}{SAi_0(0)}\exp\left(-x\times\frac{SAi_0{}'(0)}{SAi_0(0)}\right)\stackrel{d}{=}\mathfrak{p}(-x).$$

Proof of Proposition 9. Let

$$\tilde{\Psi}_n(x)=\Psi_n(x)\frac{\mathbb{E}\exp G_n}{\exp G_n}$$

where $\Psi_n(x)=(w_n\times\varphi_n)\left(1+x/\left(2n^{2/3}\right)\right)$. Assume $\tilde{\Psi}_n(0)\neq 0$. Then,

$$\frac{\tilde{\Psi}_n(x)}{\tilde{\Psi}_n(0)}=\frac{w_n\left(1+\frac{x}{2n^{2/3}}\right)\varphi_n\left(1+\frac{x}{n^{2/3}}\right)}{w_n(1)\,\varphi_n(1)}.$$

Write

$$w_n(x)=\left((2\pi)^{1/4}\,2^{-n}n^{-1/12}\sqrt{\frac{n!}{n^n}}\right)^{-1}e^{-nx^2}\,|x|^{1/6}.$$

Then,

$$\frac{w_n\left(1+\frac{x}{2n^{2/3}}\right)}{w_n(1)}=\frac{e^{-n\left(1+\frac{x}{2n^{2/3}}\right)^2}\left|1+\frac{x}{2n^{2/3}}\right|^{1/6}}{e^{-n}}=e^{-n^{1/3}x}\times e^{-\frac{x^2}{4n^{1/3}}}\left|1+\frac{x}{2n^{2/3}}\right|^{1/6}.$$

Also write

$$\frac{\varphi_n\left(1+\frac{x}{n^{2/3}}\right)}{\varphi_n(1)}=\prod_1^n\left(\frac{\frac{x}{2n^{2/3}}}{1-\mu_k}+1\right)=\prod_1^n\left(1-\frac{-x}{\tilde{\lambda}_k}\right)$$

where $\tilde{\lambda}_k=n^{2/3}(-2\mu_k+2)$. Note that $\tilde{\Psi}_n(0)=0$ iff $\mu_k=1$ for some k iff $\tilde{\lambda}_k=0$ for some k. Since -2μ is a Gaussian beta ensemble therefore by Example 20, $\tilde{\lambda}_k\Longrightarrow\Lambda_k$ in the sense of finite-dimensional distributions. In particular, as noted earlier,

$$\mathfrak{p}_n(-x)=\prod_1^n\left(1-\frac{-x}{\tilde{\lambda}_k}\right)\prod_1^n e^{-x/\tilde{\lambda}_k}$$

satisfies $\mathfrak{p}_n(x)\Longrightarrow\mathfrak{p}(x)$ in the topology compact convergence. By the continuous mapping theorem with $f(x)\mapsto f(-x)$ it follows that $\mathfrak{p}_n(-x)\Longrightarrow\mathfrak{p}(-x)$ also satisfies this convergence. Noting that $\prod_1^n e^{x/\tilde{\lambda}_k}=\exp\left(x\sum_1^n\tilde{\lambda}_k^{-1}\right)$,

$$\frac{\tilde{\Psi}_n(x)}{\tilde{\Psi}_n(0)}=\mathfrak{p}_n(-x)\exp\left(x\left(\sum_1^n\frac{1}{\tilde{\lambda}_k}-n^{1/3}\right)\right)\times e^{-\frac{x^2}{4n^{1/3}}}\left|1+\frac{x}{2n^{2/3}}\right|^{1/6}.$$

Furthermore, note that

$$\mathfrak{p}_n'(0) = \sum_{k=1}^{n} \frac{x}{\tilde{\lambda}_k^2} e^{-x/\tilde{\lambda}_k} \prod_{\substack{j=1 \\ j\neq k}}^{n} \left(1 - \frac{-x}{\tilde{\lambda}_k}\right) e^{-x/\tilde{\lambda}_k} \Bigg|_{x=0} = 0,$$

$$\frac{d}{dx} \exp\left(x\left(\sum_1^n \frac{1}{\tilde{\lambda}_k} - n^{1/3}\right)\right)\Bigg|_{x=0} = \sum_1^n \frac{1}{\tilde{\lambda}_k} - n^{1/3},$$

and

$$\frac{d}{dx}\left|1 + \frac{x}{2n^{2/3}}\right|^{1/6}\Bigg|_{x=0} = \frac{1}{12n^{2/3}}.$$

Consequently,

$$\frac{\tilde{\Psi}_n'(0)}{\tilde{\Psi}_n(0)} = \frac{d}{dx}\frac{\tilde{\Psi}_n(x)}{\tilde{\Psi}_n(0)}\Bigg|_{x=0} = \sum_1^n \frac{1}{\tilde{\lambda}_k} - n^{1/3} + \frac{1}{12n^{2/3}}. \tag{75}$$

Note that also

$$\mathbb{P}(\mathfrak{p}(0) = 0) \le \mathbb{P}(\Lambda_k = 0 \text{ for some } k) = 0.$$

Hence, by Lemma 54,

$$\mathfrak{p}_n(-x)\exp\left(-\frac{x^2}{4n^{1/3}} - \frac{x}{12n^{2/3}}\right)\left|1 + \frac{x}{2n^{2/3}}\right|^{1/6} \Longrightarrow \frac{\mathrm{SAi}_x(0)}{\mathrm{SAi}_0(0)}\exp\left(-x \times \frac{\mathrm{SAi}_0{}'(0)}{\mathrm{SAi}_0(0)}\right)$$

in the topology of compact convergence. Since

$$\exp\left(\frac{x^2}{4n^{1/3}} + \frac{x}{12n^{2/3}}\right)\left|1 + \frac{x}{2n^{2/3}}\right|^{-1/6} \to 1$$

uniformly on compacts therefore

$$p_n(-x) \Longrightarrow \frac{\mathrm{SAi}_x(0)}{\mathrm{SAi}_0(0)}\exp\left(-x \times \frac{\mathrm{SAi}_0{}'(0)}{\mathrm{SAi}_0(0)}\right).$$

□

In Question 1.13 in Lambert and Paquette 2020 they posed the open question of whether $\mathrm{SAi}_z(t)$ is an entire function in z of order $3/2$. Since $\mathfrak{p}(z)$ is of order $3/2$, Proposition 9 is then a partial answer. Furthermore, Corollary 10 provides partial answers to Questions 1.18 and 1.19 about asymptotics of $\mathrm{SAi}_z(t)$: *Almost surely, outside $C_j \cap \mathbb{R}$ where C_j is some*

C^0-set in $\mathbb{C}$ (see Section 6.3),

$$\log SAi_x(0) = \log SAi_0(0) - \frac{3}{2}x^{3/2} + x \times \frac{SAi_0'(0)}{SAi_0(0)} + o\left(x^{3/2}\right), \quad x \to \infty$$

and

$$\log SAi_x(0) = \log SAi_0(0) + x \times \frac{SAi_0'(0)}{SAi_0(0)} + o\left(|x|^{3/2}\right), \quad x \to -\infty.$$

Proof of Corollary 10. Since $\mathfrak{p}$ maps $\mathbb{R}$ to $\mathbb{R}$, by Proposition 46 there is some C^0-set C_j so that outside it,

$$\log \mathfrak{p}(r) = o\left(r^{3/2}\right), \quad r \to \infty$$

and

$$\log \mathfrak{p}(-r) = -\frac{3}{2}r^{3/2} + o\left(r^{3/2}\right), \quad r \to \infty.$$

Consider the second asymptotic. Since the C_j may be taken to be open disks therefore $C = \bigcup_j C_j$ has that C^C is closed. So the asymptotic is equivalent to

$$\mathfrak{p}(-r) \in \bigcap_n \bigcup_m \bigcap_{k \geq m} V\left([k, k+1] \cap C^C, \exp\left(-\frac{3}{2}r^{3/2} + o\left(r^{3/2}\right)\right), \frac{1}{n}\right),$$

noting that the right hand-side is measurable in both $H(\mathbb{R})$ and $H(\mathbb{C})$. Hence, it also occurs with probability 1 for

$$\frac{\mathrm{SAi}_r(0)}{\mathrm{SAi}_0(0)} \exp\left(-r \times \frac{\mathrm{SAi}_0'(0)}{\mathrm{SAi}_0(0)}\right),$$

i.e. outside C,

$$\log \mathrm{SAi}_r(0) = \log \mathrm{SAi}_0(0) + r \times \frac{\mathrm{SAi}_0'(0)}{\mathrm{SAi}_0(0)} - \frac{3}{2}r^{3/2} + o\left(r^{3/2}\right), \quad r \to \infty.$$

It is clear that the same argument works for the first asymptotic. □

Lastly, the following corollary falls out of the embedding of the secular function convergence as stated in this thesis in that of Lambert and Paquette (2020):

Corollary 62.

$$\sum_{k=1}^{n} \frac{1}{\tilde{\lambda}_k} - n^{1/3} \Longrightarrow \frac{SAi_0'(0)}{SAi_0(0)}.$$

In particular, if $\mathcal{H}_n$ is the discrete model with eigenvalues $\tilde{\lambda}_k$,

$$tr\left(\mathcal{H}_n^{-1} - n^{-2/3} I_n\right) \Longrightarrow \frac{SAi_0'(0)}{SAi_0(0)}.$$

Proof. In Proposition 9 a logarithmic derivative at 0 was taken to obtain (75). By Lemma 54 it then follows that

$$\sum_{1}^{n} \frac{1}{\tilde{\lambda}_k} - n^{1/3} + \frac{1}{12n^{2/3}} \Longrightarrow \frac{\mathrm{SAi}_0{}'(0)}{\mathrm{SAi}_0(0)}.$$

Since $1/\left(12n^{2/3}\right) \Longrightarrow 0$ the claim then follows. □

8 Ordinary differential equation

Recall in Fact 51 it is explained that almost surely the stochastic Airy operator $\mathcal{H}$ has inverse operator $\mathcal{H}^{-1}$, a Hilbert-Schmidt integral operator. Furthermore, in Example 52 it is shown that

$$\det{}_2\left(I - z\mathcal{H}^{-1}\right) = \mathfrak{p}(z)$$

where $\mathfrak{p}(z)$ is the secular function of $\mathcal{H}$. In this section it is shown that, from the above relationship, $\mathfrak{p}(z)$ arises as a unique limit of an ordinary differential equation coming from the kernel for $\mathcal{H}^{-1}$. More generally, a deterministic ordinary differential equation is developed and solved that arises from more general integral kernels extending the one for $\mathcal{H}^{-1}$. This is inspired from and builds on the work done for the bulk case in Valkó and Virág (2020). Section 8.4 goes over precisely how the work here uses the work in Valkó and Virág (2020) and why the arguments here are necessary to have similar results for the stochastic Airy operator.

8.1 Kernel type

The upcoming ordinary differential equation will be in terms of integral kernels k with the following assumptions: k is real-valued and of the form

$$k(x,y) = k_1(x)\,k_2(y)\,1_{x\geq y} + k_1(y)\,k_2(x)\,1_{x<y}, \quad (x,y) \in [0,\infty)^2 \tag{76}$$

for some functions $k_i : [0,\infty) \to \mathbb{R}$ satisfying that they are measurable, bounded on compacts, $k_2(0) = 0$ and k_2 is continuous at 0. Alone the differential equation will only depend on the integral kernel and not the corresponding integral operator, but it will additionally be shown that in the Hilbert-Schmidt case there is a connection via regularized determinants from Section 6.4.

Example 63. By Fact 51, almost surely the stochastic Airy operator $\mathcal{H}$ has inverse $\mathcal{H}^{-1}$ a Hilbert-Schmidt integral operator with kernel k given by (76) with $k_1 = \psi_\infty \in L^2$ and $k_2 = \psi_d \notin L^2$.

Note that a similar family of integral kernels are the semi-separable ones (e.g. see Gohberg, Goldberg, and Krupnik (2012)), however they require the k_i to all be L^2 which fails for the stochastic Airy operator as seen in Example 63. Note that (76) defines a symmetric kernel k with $k(x,y) = k(y,x)$: if $x < y$ then $k(y,x) = k_1(y)\,k_2(x) = k(x,y)$ and if $x > y$ then $k(y,x) = k_1(x)\,k_2(y) = k(x,y)$.

8.2 Vector-valued ordinary differential equation

This section develops and solves the ordinary differential equation in Theorem 11 in the deterministic setting. Recall that the following arguments and differential equation extend the work in Valkó and Virág (2020) which considered the sine beta operator, and the precise similarities and differences are discussed in Section 8.4. Let k be an integral kernel as in Section 8.1. For $t_i \in [0, \infty)$, $n \geq 1$ let

$$\mathcal{K}(t_1, \ldots, t_n) = \det \left[k(t_i, t_j) \right]_{i,j=1}^{n}.$$

Lemma 64. *For $t < \infty$ and $n \geq 1$,*

$$\frac{1}{n!} \int_{[0,t]^n} \mathcal{K}(t_1, \ldots, t_n)\, dt_1 \cdots dt_n = \int_{0 \leq t_1 < \cdots < t_n \leq t} \mathcal{K}(t_1, \ldots, t_n)\, dt_1 \cdots dt_n.$$

Proof. Note that these integrals are well-defined because $k_i \in L^1_{\text{loc}}$. Let $\sigma \in S_n$. Since

$$\mathcal{K}\left(t_{\sigma(1)}, \ldots, t_{\sigma(n)}\right) = \det \begin{bmatrix} k\left(t_{\sigma(1)}, t_{\sigma(1)}\right) & k\left(t_{\sigma(1)}, t_{\sigma(2)}\right) & \cdots & k\left(t_{\sigma(1)}, t_{\sigma(n)}\right) \\ k\left(t_{\sigma(2)}, t_{\sigma(1)}\right) & \ddots & & \\ \vdots & & & \\ k\left(t_{\sigma(n)}, t_{\sigma(1)}\right) & \cdots & & k\left(t_{\sigma(n)}, t_{\sigma(n)}\right) \end{bmatrix}$$

the same row and column permutations can be made to obtain $\mathcal{K}(t_1, \ldots, t_n)$. By the change of variable $(t_1, \ldots, t_n) \mapsto \left(t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(n)}\right)$,

$$\int_{0 \leq t_{\sigma(1)} < \cdots < t_{\sigma(n)} \leq t} \mathcal{K}(t_1, \ldots, t_n)\, dt_1 \cdots dt_n$$

$$= \int_{0 \leq t_1 < \cdots < t_n \leq t} \mathcal{K}\left(t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(n)}\right) dt_1 \cdots dt_n,$$

which as noted then equals

$$\int_{0 \leq t_1 < \cdots < t_n \leq t} \mathcal{K}(t_1, \ldots, t_n)\, dt_1 \cdots dt_n.$$

Hence, by disjointness of the sets $\left\{0 \leq t_{\sigma(1)} < \cdots < t_{\sigma(n)} \leq t\right\}$ over $\sigma \in S_n$,

$$\int_{[0,t]^n} \mathcal{K}(t_1, \ldots, t_n)\, dt_1 \cdots dt_n = \sum_{\sigma \in S_n} \int_{0 \leq t_{\sigma(1)} < \cdots < t_{\sigma(n)} \leq t} \mathcal{K}(t_1, \ldots, t_n)\, dt_1 \cdots dt_n$$

$$= n! \int_{0 \le t_1 < \cdots < t_n \le t} \mathcal{K}(t_1, \ldots, t_n)\, dt_1 \cdots dt_n.$$

□

Lemma 65. *For $t < \infty$, $0 \le t_1 < \cdots < t_n \le t$ and $n \ge 1$,*

$$\mathcal{K}(t_1, \ldots, t_n) = \det\left[p_{i \wedge j} q_{i \vee j}\right]_{i,j=1}^{n} = p_1 q_n \prod_{k=1}^{n-1} (p_{k+1} q_k - p_k q_{k+1})$$

where $p_k = k_2(t_k)$ and $q_k = k_1(t_k)$.

Proof. Since $t_1 < \cdots < t_n$ therefore $k(t_i, t_j) = p_{i \wedge j} q_{i \vee j}$: if $i < j$ then

$$k(t_i, t_j) = k_1(t_j)\, k_2(t_i) = q_{i \vee j} p_{i \wedge j}$$

and likewise if $i \ge j$. Hence,

$$\mathcal{K}(t_1, \ldots, t_n) = \left[p_{i \wedge j} q_{i \vee j}\right]_{i,j=1}^{n}.$$

For $n = 1$ the identity reduces trivially to

$$\mathcal{K}(t_1) = k(t_1, t_1) = p_1 q_1,$$

and $n = 2$, 3 may be checked by direct computation. For example,

$$\mathcal{K}(t_1, t_2) = \det \begin{bmatrix} p_1 q_1 & p_1 q_2 \\ p_1 q_2 & p_2 q_2 \end{bmatrix} = p_1 q_2 (p_2 q_1 - p_1 q_2).$$

Consider $n \ge 3$. The last two rows of $\left[p_{i \wedge j} q_{i \vee j}\right]_{i,j=1}^{n}$ are

$$\begin{bmatrix} p_1 q_{n-1} & p_2 q_{n-1} & \cdots & p_{n-1} q_{n-1} & p_{n-1} q_n \\ p_1 q_n & p_2 q_n & \cdots & p_{n-1} q_n & p_n q_n \end{bmatrix}.$$

Consider when $q_{n-1} \ne 0$. Multiplying row $n-1$ by $-q_n / q_{n-1}$ and adding it to row n yields the new row n,

$$\left[0, \ldots, 0, p_n q_n - p_{n-1} q_n^2 / q_{n-1}\right].$$

Note that removing row and column n of $\left[p_{i \wedge j} q_{i \vee j}\right]_{i,j=1}^{n}$ yields $\left[p_{i \wedge j} q_{i \vee j}\right]_{i,j=1}^{n-1}$. Hence,

$$\mathcal{K}(t_1, \ldots, t_n) = \left(p_n q_n - \frac{p_{n-1} q_n^2}{q_{n-1}}\right) \det\left[p_{i \wedge j} q_{i \vee j}\right]_{i,j=1}^{n-1}$$

$$= \left(p_n q_n - \frac{p_{n-1} q_n^2}{q_{n-1}} \right) p_1 q_{n-1} \prod_{k=1}^{n-2} \left(p_{k+1} q_k - p_k q_{k+1} \right)$$

$$= p_1 q_n \prod_{k=1}^{n-1} \left(p_{k+1} q_k - p_k q_{k+1} \right).$$

Now, consider when $q_{n-1} = 0$. Then,

$$[p_{i \wedge j} q_{i \vee j}]_{i,j=1}^n = \begin{bmatrix} p_1 q_1 & p_1 q_2 & \cdots & p_1 q_{n-2} & 0 & p_1 q_n \\ & & \vdots & & & \\ p_1 q_{n-2} & p_2 q_{n-2} & \cdots & p_{n-2} q_{n-2} & 0 & p_{n-2} q_n \\ 0 & 0 & \cdots & 0 & 0 & p_{n-1} q_n \\ p_1 q_n & p_2 q_n & \cdots & p_{n-2} q_n & p_{n-1} q_n & p_n q_n \end{bmatrix}.$$

Consequently,

$$\mathcal{K}(t_1, \ldots, t_n) = -p_{n-1} q_n \det \begin{bmatrix} p_1 q_1 & p_1 q_2 & \cdots & p_1 q_{n-2} & 0 \\ & & \vdots & & \\ p_1 q_{n-2} & p_2 q_{n-2} & \cdots & p_{n-2} q_{n-2} & 0 \\ p_1 q_n & p_2 q_n & \cdots & p_{n-2} q_n & p_{n-1} q_n \end{bmatrix}$$

$$= -p_{n-1}^2 q_n^2 \det \begin{bmatrix} p_1 q_1 & p_1 q_2 & \cdots & p_1 q_{n-2} \\ & & \vdots & \\ p_1 q_{n-2} & p_2 q_{n-2} & \cdots & p_{n-2} q_{n-2} \end{bmatrix} = -p_{n-1}^2 q_n^2 \det [p_{i \wedge j} q_{i \vee j}]_{i,j=1}^{n-2}$$

$$= -p_{n-1}^2 q_n^2 p_1 q_{n-2} \prod_{k=1}^{n-3} \left(p_{k+1} q_k - p_k q_{k+1} \right).$$

Since $q_{n-1} = 0$ therefore the $k = n-2$, $n-1$ product terms satisfy

$$p_{n-1} q_{n-2} - p_{n-2} q_{n-1} = p_{n-1} q_{n-2} \quad \text{and} \quad p_n q_{n-1} - p_{n-1} q_n = -p_{n-1} q_n.$$

Therefore,

$$\mathcal{K}(t_1, \ldots, t_n) = p_1 q_n \prod_{k=1}^{n-1} \left(p_{k+1} q_k - p_k q_{k+1} \right).$$

□

Introduce the notation

$$\mathfrak{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathfrak{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Corollary 66. *For $t < \infty$, $0 \le t_1 < \cdots < t_n \le t$ and $n \ge 1$,*

$$\mathcal{K}(t_1, \ldots, t_n) = \mathfrak{i}^T R(t_1) J R(t_2) J \cdots R(t_n) J \mathfrak{i}$$

where

$$R(t_k) = \begin{bmatrix} k_2(t_k)^2 & k_1(t_k) k_2(t_k) \\ k_1(t_k) k_2(t_k) & k_1(t_k)^2 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Proof. Let p_k, q_k be as in Lemma 65 so that $k(t_i, t_j) = p_{i \wedge j} q_{i \vee j}$. Let

$$v(t_k) = \begin{bmatrix} p_k \\ q_k \end{bmatrix}.$$

Note that

$$v(t_k)^T J v(t_{k+1}) = [p_{k+1} q_k - p_k q_{k+1}]$$

and

$$v(t_1) v(t_n)^T J = \begin{bmatrix} p_1 q_n & \cdot \\ \cdot & \cdot \end{bmatrix}.$$

Then,

$$v(t_1) v(t_1)^T J v(t_2) v(t_2)^T J \cdots v(t_n) v(t_n)^T J$$

$$= v(t_1) \left[\prod_{k=1}^{n-1} (p_{k+1} q_k - p_k q_{k+1}) \right] v(t_n)^T J = \prod_{k=1}^{n-1} (p_{k+1} q_k - p_k q_{k+1}) \begin{bmatrix} p_1 q_n & \cdot \\ \cdot & \cdot \end{bmatrix}.$$

By Lemma 65 the above has top-left entry $\mathcal{K}(t_1, \ldots, t_n)$. Also,

$$v(t_k) v(t_k)^T = R(t_k).$$

□

An essential step in developing the upcoming ordinary differential equation is realizing the second coordinate of

$$\mathfrak{i}^T R(t_1) J R(t_2) J \cdots R(t_n) J$$

in Corollary 66 also as a determinant of a matrix. By trial-and-error the following is known:

Lemma 67. *For $t < \infty$, $0 \le t_1 < \cdots < t_n \le t$ and $n \ge 1$,*

$$\mathfrak{i}^T R(t_1) J R(t_2) J \cdots R(t_n) J = \left[\mathcal{K}(t_1, \ldots, t_n), -\tilde{\mathcal{K}}(t_1, \ldots, t_n) \right]$$

where, again with $p_k = k_2(t_k)$ *and* $q_k = k_1(t_k)$,

$$\tilde{\mathcal{K}}(t_1,\ldots,t_n) := \det\begin{bmatrix} [k(t_1,t_j)]_{j=1}^n \\ \vdots \\ [k(t_{n-1},t_j)]_{j=1}^n \\ [k_2(t_j)\,k_2(t_n)]_{j=1}^n \end{bmatrix} = \det\begin{bmatrix} [p_{1\wedge j}q_{1\vee j}]_{j=1}^n \\ \vdots \\ \left[p_{(n-1)\wedge j}q_{(n-1)\vee j}\right]_{j=1}^n \\ [p_j p_n]_{j=1}^n \end{bmatrix}$$

$$= p_1 p_n \prod_{k=1}^{n-1}\left(p_{k+1}q_k - p_k q_{k+1}\right).$$

Proof. The first coordinate is known be Corollary 66 so consider the second. For $n = 1$ the identity reduces trivially to $\tilde{\mathcal{K}}(t_1) = p_1^2$ and $n = 2$, 3 may be checked by direct computation. For example,

$$\tilde{\mathcal{K}}(t_1,t_2) = \det\begin{bmatrix} p_1q_1 & p_1q_2 \\ p_1p_2 & p_2^2 \end{bmatrix} = p_1p_2\left(p_2q_1 - p_1q_2\right).$$

Consider $n \geq 3$. Suppose $q_{n-1} \neq 0$. Note that

$$\tilde{\mathcal{K}}(t_1,\ldots,t_n) = \det\begin{bmatrix} p_1q_1 & p_1q_2 & \cdots & p_1q_{n-1} & p_1q_n \\ & & \vdots & & \\ p_1q_{n-1} & p_2q_{n-1} & \cdots & p_{n-1}q_{n-1} & p_{n-1}q_n \\ p_1p_n & p_2p_n & \cdots & p_{n-1}p_n & p_n^2 \end{bmatrix}.$$

Multiplying row $n-1$ by $-p_n/q_{n-1}$ and adding it to row n yields the new row n,

$$\left[0,0,\ldots,0,p_n^2 - p_{n-1}p_nq_n/q_{n-1}\right].$$

Then,

$$\tilde{\mathcal{K}}(t_1,\ldots,t_n) = \left(p_n^2 - \frac{p_{n-1}p_nq_n}{q_{n-1}}\right)\det\begin{bmatrix} p_1q_1 & p_1q_2 & \cdots & p_1q_{n-1} \\ & & \vdots & \\ p_1q_{n-1} & p_2q_{n-1} & \cdots & p_{n-1}q_{n-1} \end{bmatrix}.$$

Recognizing the latter determinant as that of the matrix $[p_{i\wedge j}q_{i\vee j}]_{i,j=1}^{n-1}$, by Lemma 65,

$$\tilde{\mathcal{K}}(t_1,\ldots,t_n) = \left(p_n^2 - \frac{p_{n-1}p_nq_n}{q_{n-1}}\right)p_1q_{n-1}\prod_{k=1}^{n-2}\left(p_{k+1}q_k - p_kq_{k+1}\right)$$

$$= p_1 p_n \prod_{k=1}^{n-1} \left(p_{k+1} q_k - p_k q_{k+1}\right).$$

Now, suppose that $q_{n-1} = 0$. Then,

$$\tilde{\mathcal{K}}\left(t_1, \ldots, t_n\right) = \det \begin{bmatrix} p_1 q_1 & p_1 q_2 & \cdots & p_1 q_{n-2} & 0 & p_1 q_n \\ & & \vdots & & & \\ p_1 q_{n-2} & p_2 q_{n-2} & \cdots & p_{n-2} q_{n-2} & 0 & p_{n-2} q_n \\ 0 & 0 & \cdots & 0 & 0 & p_{n-1} q_n \\ p_1 p_n & p_2 p_n & \cdots & p_{n-2} p_n & p_{n-1} p_n & p_n^2 \end{bmatrix}.$$

Hence,

$$\tilde{\mathcal{K}}\left(t_1, \ldots, t_n\right) = -p_{n-1} q_n \det \begin{bmatrix} p_1 q_1 & p_1 q_2 & \cdots & p_1 q_{n-2} & 0 \\ & & \vdots & & \\ p_1 q_{n-2} & p_2 q_{n-2} & \cdots & p_{n-2} q_{n-2} & 0 \\ p_1 p_n & p_2 p_n & \cdots & p_{n-2} p_n & p_{n-1} p_n \end{bmatrix}$$

$$= -p_{n-1}^2 p_n q_n \det \begin{bmatrix} p_1 q_1 & p_1 q_2 & \cdots & p_1 q_{n-2} \\ & & \vdots & \\ p_1 q_{n-2} & p_2 q_{n-2} & \cdots & p_{n-2} q_{n-2} \end{bmatrix}.$$

Using Lemma 65 again, the above equals

$$\left(-p_{n-1}^2 p_n q_n\right) p_1 q_{n-2} \prod_{k=1}^{n-3} \left(p_{k+1} q_k - p_k q_{k+1}\right).$$

Since $q_{n-1} = 0$ the $n-2$ and $n-1$ product terms are

$$p_{n-1} q_{n-2} - p_{n-2} q_{n-1} = p_{n-1} q_{n-2} \quad \text{and} \quad p_n q_{n-1} - p_{n-1} q_n = -p_{n-1} q_n.$$

Hence,

$$\tilde{\mathcal{K}}\left(t_1, \ldots, t_n\right) = p_1 p_n \prod_{k=1}^{n-1} \left(p_{k+1} q_k - p_k q_{k+1}\right).$$

Finally, in using the notation in the proof of Corollary 66,

$$v\left(t_1\right) v\left(t_n\right)^T J = \begin{bmatrix} p_1 q_n & -p_1 p_n \\ q_1 q_n & -q_1 p_n \end{bmatrix}$$

and hence,

$$\mathfrak{i}^T R(t_1) JR(t_2) J \cdots R(t_n) J = \prod_{k=1}^{n-1} (p_{k+1}q_k - p_k q_{k+1}) [1, 0] \begin{bmatrix} p_1 q_n & -p_1 p_n \\ q_1 q_n & -q_1 p_n \end{bmatrix}$$

$$= \left[p_1 q_n \prod_{k=1}^{n-1} (p_{k+1}q_k - p_k q_{k+1}), -p_1 p_n \prod_{k=1}^{n-1} (p_{k+1}q_k - p_k q_{k+1}) \right].$$

□

For $n \geq 0$ define column vectors $d_n : [0, \infty) \to \mathbb{R}^2$ by $d_0(t) = \mathfrak{i}$ and for $n \geq 1$,

$$d_n(t)^T = \int_{0 \leq t_1 < \cdots < t_n \leq t} \mathfrak{i}^T R(t_1) J \cdots R(t_n) J dt_1 \cdots dt_n.$$

Note that for $n \geq 1$, $d_n(0) = 0$ and by Lemma 67,

$$d_n(t) = \int_{0 \leq t_1 < \cdots < t_n \leq t} \begin{bmatrix} \mathcal{K}(t_1, \ldots, t_n) \\ -\tilde{\mathcal{K}}(t_1, \ldots, t_n) \end{bmatrix} dt_1 \cdots dt_n. \tag{77}$$

Define the column vector $H(t, z) : [0, \infty) \times \mathbb{C} \to \mathbb{C}^2$ by

$$H(t, z) = \sum_0^\infty (-1)^n z^n d_n(t).$$

This will be the solution of the upcoming differential equation. Additionally, for $n \geq 0$ define Φ_n, $\tilde{\Phi}_n$, $\Psi_n : [0, \infty) \to \mathbb{R}$ by $\Phi_0(t) = 1$, $\tilde{\Phi}_0(t) = 0$, $\Psi_0(t) = 1$,

$$\tilde{\Phi}_1(t) = \det \tilde{\mathcal{K}}(t) = k_2(t)^2,$$

$\Psi_1(t) = 0$, for $n \geq 1$,

$$\Phi_n(t) = \int_{[0,t]^n} \mathcal{K}(t_1, \ldots, t_n) dt_1 \cdots dt_n,$$

and for $n \geq 2$,

$$\Psi_n(t) = \int_{[0,t]^n} \det [k(t_i, t_j) 1_{i \neq j}]_{i,j=1}^n dt_1 \cdots dt_n \tag{78}$$

and

$$\tilde{\Phi}_n(t) = \int_{[0,t]^{n-1}} \tilde{\mathcal{K}}(t_1, \ldots, t_{n-1}, t) dt_1 \cdots dt_{n-1}.$$

All these integrals are well-defined because $k_i \in L^1_{\text{loc}}$ and are all 0 at 0.

Lemma 68. *As formal series,*

$$H(t,z)=\begin{bmatrix}1+\sum_1^\infty \frac{(-1)^n z^n}{n!}\Phi_n(t)\\ z\int_0^t k_2(s)^2\,ds-\sum_2^\infty \frac{(-1)^n z^n}{(n-1)!}\int_0^t \tilde{\Phi}_n(s)\,ds\end{bmatrix}.$$

Proof. By Corollary 66, for $n\geq 1$,

$$d_n(t)^T\mathfrak{i}=\int_{0\leq t_1<\cdots<t_n\leq t}\mathfrak{i}^T R(t_1)J\cdots R(t_n)J\mathfrak{i}dt_1\cdots dt_n$$

$$=\int_{0\leq t_1<\cdots<t_n\leq t}\mathcal{K}(t_1,\ldots,t_n)\,dt_1\cdots dt_n.$$

By Lemma 64 this equals

$$\frac{1}{n!}\int_{[0,t]^n}\mathcal{K}(t_1,\ldots,t_n)\,dt_1\cdots dt_n=\frac{1}{n!}\Phi_n(t).$$

Since $d_0(t)=\mathfrak{e}$ and $\Phi_0(t)=1$ therefore the first coordinate identity is shown. For the second, for $n\geq 2$, by Lemma 67,

$$d_n(t)^T\mathfrak{j}=\int_{0\leq t_1<\cdots<t_n\leq t}\mathfrak{i}^T R(t_1)J\cdots R(t_n)J\begin{bmatrix}0\\1\end{bmatrix}dt_1\cdots dt_n$$

$$=-\int_{0\leq t_1<\cdots<t_n\leq t}\tilde{\mathcal{K}}(t_1,\ldots,t_n)\,dt_1\cdots dt_n.$$

Since $k_i\in L^1_{\text{loc}}$ therefore by Lemma 84 this equals

$$-\int_0^t\int_{0\leq t_1<\cdots<t_{n-1}\leq t_n}\tilde{\mathcal{K}}(t_1,\ldots,t_{n-1},t_n)\,dt_1\cdots dt_{n-1}dt_n. \tag{79}$$

Applying the argument of Lemma 64 here, for $\sigma\in S_{n-1}$,

$$\tilde{\mathcal{K}}\left(t_{\sigma(1)},\ldots,t_{\sigma(n-1)},t_n\right)$$

$$=\det\begin{bmatrix}k\left(t_{\sigma(1)},t_{\sigma(1)}\right) & \cdots & k\left(t_{\sigma(1)},t_{\sigma(n-1)}\right) & k\left(t_{\sigma(1)},t_n\right)\\ & \vdots & & \\ k\left(t_{\sigma(n-1)},t_{\sigma(1)}\right) & \cdots & k\left(t_{\sigma(n-1)},t_{\sigma(n-1)}\right) & k\left(t_{\sigma(n-1)},t_n\right)\\ k_2\left(t_{\sigma(1)}\right)k_2(t_n) & \cdots & k_2\left(t_{\sigma(n-1)}\right)k_2(t_n) & k_2(t_n)k_2(t_n)\end{bmatrix}$$

$$=\tilde{\mathcal{K}}(t_1,\ldots,t_{n-1},t_n)$$

by applying the same operations on the first $n-1$ columns and rows. Continuing,

$$\int_{0\leq t_{\sigma(1)}<\cdots<t_{\sigma(n-1)}\leq t_n} \tilde{\mathcal{K}}\left(t_1,\ldots,t_{n-1},t_n\right)dt_1\cdots dt_{n-1}$$

$$=\int_{0\leq t_1<\cdots<t_{n-1}\leq t_n} \tilde{\mathcal{K}}\left(t_1,\ldots,t_{n-1},t_n\right)dt_1\cdots dt_{n-1}$$

and hence,

$$\int_{[0,t_n]^{n-1}} \tilde{\mathcal{K}}\left(t_1,\ldots,t_{n-1},t_n\right)dt_1\cdots dt_{n-1}$$

$$=(n-1)!\int_{0\leq t_1<\cdots<t_{n-1}\leq t_n} \tilde{\mathcal{K}}\left(t_1,\ldots,t_{n-1},t_n\right)dt_1\cdots dt_{n-1}.$$

Consequently, by (79), for $n\geq 2$,

$$d_n\left(t\right)^T\mathfrak{j}=-\frac{1}{(n-1)!}\int_0^t\int_{[0,t_n]^{n-1}} \tilde{\mathcal{K}}\left(t_1,\ldots,t_{n-1},t_n\right)dt_1\cdots dt_{n-1}dt_n$$

$$=-\frac{1}{(n-1)!}\int_0^t\tilde{\Phi}_n\left(t_n\right)dt_n.$$

Also, $d_0\left(t\right)=\mathfrak{e}$ matches with $\tilde{\Phi}_0\left(t\right)=0$, and by (77),

$$-zd_1\left(t\right)^T\begin{bmatrix}0\\1\end{bmatrix}=z\int_{0\leq t_1\leq t}\tilde{\mathcal{K}}\left(t_1\right)dt_1=z\int_0^t k_2\left(t_1\right)^2dt_1=z\int_0^t\tilde{\Phi}_1\left(t_1\right)dt_1.$$

□

Lemma 69. *The function* $\gamma:(0,\infty)\to(0,\infty)$,

$$\gamma\left(t\right)=\max\left\{\int_0^t k_1^2,\ \int_0^t k_2^2,\ \int_0^t\left|k_1k_2\right|,\ \sup_{0\leq s\leq t}\left|k_1\left(s\right)k_2\left(s\right)\right|,\ \sup_{0\leq s\leq t}k_2\left(s\right)^2,\ t\right\}$$

is increasing,

$$\left|\Phi_n\left(t\right)\right|\leq n!\left(2\gamma\left(t\right)\right)^n,\quad \textit{for all } n\geq 1,$$

and

$$\left|\Psi_n\left(t\right)\right|,\ \left|\tilde{\Phi}_n\left(t\right)\right|\leq n!\left(2\gamma\left(t\right)\right)^n,\quad \textit{for all } n\geq 2.$$

Proof. Since k_i are bounded on compacts therefore $\gamma\left(t\right)<\infty$. By expanding determinants,

$$\Phi_n\left(t\right)=\sum_{\sigma\in S_n}\operatorname{sgn}\left(\sigma\right)\int_{[0,t]^n}\prod_1^n k\left(t_i,t_{\sigma(i)}\right)dt_1\cdots dt_n.$$

Doing this also for $\Psi_n(t)$ except summing over only $\sigma \in S_n$ with no fixed points shows that

$$\left|\Phi_n(t)\right|,\ \left|\Psi_n(t)\right| \leq \sum_{\sigma \in S_n} \int_{[0,t]^n} \left| \prod_1^n k\left(t_i, t_{\sigma(i)}\right) \right| dt_1 \cdots dt_n.$$

For notational purposes let $f_1 = |k_1|$, $f_2 = |k_2|$ and $g_1 = |k_2|$, $g_2 = |k_1|$. Note that for $\sigma \in S_n$,

$$\left|k\left(t_i, t_{\sigma(i)}\right)\right| \leq \left|k_1(t_i)\, k_2\left(t_{\sigma(i)}\right)\right| + \left|k_1\left(t_{\sigma(i)}\right) k_2(t_i)\right|$$

and hence,

$$\int_{[0,t]^n} \left| \prod_1^n k\left(t_i, t_{\sigma(i)}\right) \right| dt_1 \cdots dt_n$$

$$\leq \int_{[0,t]^n} \prod_1^n \left(f_1(t_i)\, g_1\left(t_{\sigma(i)}\right) + f_2(t_i)\, g_2\left(t_{\sigma(i)}\right)\right) dt_1 \cdots dt_n$$

$$= \sum_{i_1,\ldots,i_n=1}^{2} \int_{[0,t]^n} \left(f_{i_1}(t_1)\, g_{i_1}\left(t_{\sigma(1)}\right)\right)\left(f_{i_2}(t_2)\, g_{i_2}\left(t_{\sigma(2)}\right)\right) \cdots \left(f_{i_n}(t_n)\, g_{i_n}\left(t_{\sigma(n)}\right)\right) dt_1 \cdots dt_n.$$

By pairing the f_{i_j}, g_{i_k} terms according to a common argument t_ℓ, using Fubini, and that

$$\int_0^t f_{i_j}(s)\, g_{i_k}(s)\, ds = \int_0^t k_1(s)^2\, ds,\ \int_0^t k_2(s)^2\, ds \text{ or } \int_0^t \left|k_1(s)\, k_2(s)\right| ds$$

which are each bounded by $\gamma(t)$, it follows that

$$\int_{[0,t]^n} \left| \prod_1^n k\left(t_i, t_{\sigma(i)}\right) \right| dt_1 \cdots dt_n \leq \sum_{i_1,\ldots,i_n=1}^{2} \gamma(t)^n = (2\gamma(t))^n.$$

Hence,

$$\left|\Phi_n(t)\right|,\ \left|\Psi_n(t)\right| \leq \sum_{\sigma \in S_n} (2\gamma(t))^n = n!\,(2\gamma(t))^n.$$

For $\tilde{\Phi}_n(t)$ the argument is similar. For notational purposes let $t_n = t$. Then,

$$\int_{[0,t]^{n-1}} \left|k_2\left(t_{\sigma(n)}\right) k_2(t_n)\right| \prod_1^{n-1} \left|k\left(t_i, t_{\sigma(i)}\right)\right| dt_1 \cdots dt_{n-1}$$

$$= \sum_{i_1,\ldots,i_{n-1}=1}^{2} \int_{[0,t]^{n-1}} \left(f_{i_1}(t_1)\, g_{i_1}\left(t_{\sigma(1)}\right)\right) \cdots \left(f_{i_{n-1}}(t_{n-1})\, g_{i_{n-1}}\left(t_{\sigma(n-1)}\right)\right)\left(f_2(t_n)\, g_1\left(t_{\sigma(n)}\right)\right) dt_1 \cdots dt_{n-1}.$$

Again pairing terms by a common argument t_ℓ, using that the $t_n = t$ term is either

$|k_1(t)\,k_2(t)|$ or $k_2(t)^2$, and that each of these is bounded by $\gamma(t)$ it now follows that

$$\left|\tilde{\Phi}_n(t)\right| \le \sum_{\sigma\in S_n} (2\gamma(t))^n = n!\,(2\gamma(t))^n .$$

□

Corollary 70.

$$\left|H(t,z)\,\mathfrak{i}^T\right| \le 1+\sum_1^\infty (2|z|\,\gamma(t))^n \quad \textit{and} \quad \left|H(t,z)\,\mathfrak{j}^T\right| \le t\sum_1^\infty n\,(2|z|\,\gamma(t))^n .$$

Proof. By Lemma 68,

$$H(t,z)\,\mathfrak{i}^T = 1+\sum_1^\infty \frac{(-1)^n z^n}{n!}\Phi_n(t)$$

and hence by Lemma 69,

$$\left|H(t,z)\,\mathfrak{i}^T\right| \le 1+\sum_1^\infty (2|z|\,\gamma(t))^n .$$

Similarly,

$$H(t,z)\,j^T = z\int_0^t k_2(s)^2\,ds - \sum_2^\infty \frac{(-1)^n z^n}{(n-1)!}\int_0^t \tilde{\Phi}_n(s)\,ds$$

and hence,

$$\left|H(t,z)\,j^T\right| \le |z|\int_0^t 2\gamma(t)\,ds + \sum_2^\infty \frac{|z|^n}{(n-1)!}\int_0^t n!\,(2\gamma(t))^n\,ds$$

$$\le 2t\,|z|\,\gamma(t) + \sum_2^\infty nt\,(2|z|\,\gamma(t))^n = t\sum_1^\infty n\,(2|z|\,\gamma(t))^n .$$

□

Corollary 71. *For $t<\infty$ and $0\le s\le t$, on the neighbourhood $|z|<1/(2\gamma(t))$, for fixed $u\in[s,t]$ both coordinates of $H(u,z)$ are absolutely convergent, are analytic in z and $H(u,z)\in\mathbb{R}^2$ when $Imz=0$.*

Proof. By Corollary 70 the coordinates of $H(u,z)$ are respectively absolutely bounded by

$$1+\sum_1^\infty (2|z|\,\gamma(u))^n \quad \text{and} \quad u\sum_1^\infty n\,(2|z|\,\gamma(u))^n .$$

Since γ is increasing by Lemma 69 therefore these series are respectively further bounded by

$$1+\sum_{1}^{\infty}\left(2\left|z\right|\gamma\left(t\right)\right)^{n} \quad \text{and} \quad t\sum_{1}^{\infty}n\left(2\left|z\right|\gamma\left(t\right)\right)^{n}.$$

Both of these series converge on $|z| < 1/\left(2\gamma\left(t\right)\right)$. As absolutely convergent series they are analytic, and when z is real, $H\left(u,z\right)$ is real by definition. □

Lemma 72. *For $t_i \geq 0$ and $n \geq 1$,*

$$\det\left[k\left(t_i,t_j\right)\right]_{i,j=1}^{n} = \sum_{\substack{A\subseteq[n]\\ 0\leq|A|\leq n-2}} \det\left[k\left(t_i,t_j\right)1_{i\neq j}\right]_{i,j\in[n]-A}\prod_{i\in A}k\left(t_i,t_i\right)+\prod_{1}^{n}k\left(t_i,t_i\right).$$

Proof. By expanding determinants, where F_σ denotes the fixed points of σ,

$$\det\left[k\left(t_i,t_j\right)\right]_{i,j=1}^{n} = \sum_{\sigma\in S_n}\mathrm{sgn}\sigma\prod_{1}^{n}k\left(t_i,t_{\sigma(i)}\right)$$

$$= \sum_{\sigma\in S_n-1}\mathrm{sgn}\sigma\prod_{i\notin F_\sigma}k\left(t_i,t_{\sigma(i)}\right)\prod_{i\in F_\sigma}k\left(t_i,t_i\right)+\prod_{1}^{n}k\left(t_i,t_i\right)$$

$$= \sum_{A\subseteq[n]}\sum_{\substack{\sigma\in S_n-1\\ F_\sigma=A}}\mathrm{sgn}\sigma\prod_{i\notin A}k\left(t_i,t_{\sigma(i)}\right)\prod_{i\in A}k\left(t_i,t_i\right)+\prod_{1}^{n}k\left(t_i,t_i\right).$$

Note that $|F_\sigma| \geq n-1$ iff $\sigma = 1$. Hence,

$$\det\left[k\left(t_i,t_j\right)\right]_{i,j=1}^{n} = \sum_{\substack{A\subseteq[n]\\ 0\leq|A|\leq n-2}}\sum_{\substack{\sigma\in S_n\\ F_\sigma=A}}\mathrm{sgn}\sigma\prod_{i\notin A}k\left(t_i,t_{\sigma(i)}\right)\prod_{i\in A}k\left(t_i,t_i\right)+\prod_{1}^{n}k\left(t_i,t_i\right). \tag{80}$$

Note that for $A \subseteq [n]$ with $0 \leq |A| \leq n-2$ the mapping

$$\left\{\sigma\in S_n, F_\sigma=A\right\}\to\left\{\psi\in S_{[n]-A},F_\psi=\emptyset\right\}, \quad \sigma\mapsto\psi=\sigma|_{[n]-A}$$

is well-defined, invertible are preserves sgn. For example, if $\sigma|_{[n]-A}\left(i\right)\in A$ for some $i\notin A$ then $\sigma^2\left(i\right)=\sigma\left(i\right)$ and hence $\sigma\left(i\right)=i$, a contradiction. Hence,

$$\sum_{\substack{\sigma\in S_n\\ F_\sigma=A}}\mathrm{sgn}\sigma\prod_{i\notin A}k\left(t_i,t_{\sigma(i)}\right) = \sum_{\substack{\sigma\in S_{[n]-A}\\ F_\sigma=\emptyset}}\mathrm{sgn}\sigma\prod_{i\in[n]-A}k\left(t_i,t_{\sigma(i)}\right).$$

Denote $\left[k\left(t_i,t_j\right)\right]_{i,j\in A}$ by writing the columns and rows according to A written in increasing order. Then,

$$\sum_{\substack{\sigma\in S_n\\ F_\sigma=A}} \operatorname{sgn}\sigma \prod_{i\notin A} k\left(t_i, t_{\sigma(i)}\right) = \det\left[k\left(t_i, t_j\right) 1_{i\neq j}\right]_{i,j\in[n]-A}.$$

Combined with (80) this completes the proof. □

For $(t,z)\in[0,\infty)\times\mathbb{C}$, define

$$h(t,z) = \sum_{0}^{\infty} \frac{(-1)^n z^n}{n!} \Psi_n(t) = 1 + \sum_{2}^{\infty} \frac{(-1)^n z^n}{n!} \Psi_n(t) \tag{81}$$

recalling the definition (78) of Ψ_n.

Corollary 73.

$$|h(t,z)| \le 1 + \sum_{2}^{\infty} \left(2|z|\gamma(t)\right)^n$$

and so for $0\le s\le t$ there exists a neighbourhood of $z=0$ on which, for fixed $u\in[s,t]$, $h(u,z)$ is absolutely convergent, analytic in z and $h(u,z)\in\mathbb{R}^2$ when $Imz=0$.

Proof. This follows from the same argument in Lemma 71. □

Lemma 74. *For $t\ge 0$ there exists a neighbourhood of $z=0$ on which*

$$h(t,z) = e^{z\int_0^t k(x,x)dx} H(t,z)\,\mathfrak{i}^T.$$

Proof. If $t=0$ then the claim reduces trivially to $1 = 1\times \mathfrak{i}^T\mathfrak{i}$. Consider $t>0$. By Lemma 72,

$$\det\left[k\left(t_i,t_j\right)\right]_{i,j=1}^n = \sum_{\substack{A\subseteq[n]\\ 0\le|A|\le n-2}} \det\left[k\left(t_i,t_j\right)1_{i\neq j}\right]_{i,j\in[n]-A} \prod_{i\in A} k\left(t_i,t_i\right) + \prod_{1}^{n} k\left(t_i,t_i\right).$$

Let $I = \int_0^t k(s,s)\,ds$. Then, for $n\ge 1$, by Fubini,

$$\Phi_n(t) = \int_{[0,t]^n} \det\left[k\left(t_i,t_j\right)\right]_{i,j=1}^n dt_1\cdots dt_n$$

$$= \sum_{\substack{A\subseteq[n]\\ 0\le|A|\le n-2}} \int_{[0,t]^n} \det\left[k\left(t_i,t_j\right)1_{i\neq j}\right]_{i,j\in[n]-A} \prod_{i\in A} k\left(t_i,t_i\right) dt_1\cdots dt_n + I^n.$$

Replacing A with $[n]-A$ yields

$$\Phi_n(t) = \sum_{\substack{A\subseteq[n]\\2\le|A|\le n}} \int_{[0,t]^n} \det\left[k(t_i,t_j)\,1_{i\neq j}\right]_{i,j\in A} \prod_{i\in[n]-A} k(t_i,t_i)\,dt_1\cdots dt_n + I^n$$

noting that correctly $\Phi_1(t) = \int_0^t k(s,s)\,ds = I$. Then, by Fubini,

$$\Phi_n(t) = \sum_{\substack{A\subseteq[n]\\2\le|A|\le n}} I^{|[n]-A|}\Psi_{|A|}(t) + I^n.$$

Consequently,

$$\Phi_n(t) = \sum_{i=2}^{n} \sum_{\substack{A\subseteq[n]\\|A|=i}} I^{n-i}\Psi_i(t) + I^n = \sum_{i=2}^{n} \binom{n}{i} I^{n-i}\Psi_i(t) + I^n. \tag{82}$$

By Corollary 73 there is a neighbourhood of $z=0$ on which $h(t,z)$ is analytic in z. Consider the series product

$$h(t,z)\,e^{-zI} = \left(1+\frac{z^2}{2}\Psi_2(t)-\cdots\right)\left(1-zI+\frac{z^2}{2}I^2-\cdots\right)$$

which is also analytic in z on this neighbourhood. The nth, $n\ge 2$ coefficients take the form

$$1\times\frac{(-1)^n}{n!}I^n + \sum_{i=2}^{n}\frac{(-1)^i}{i!}\Psi_i(t)\frac{(-1)^{n-i}}{(n-i)!}I^{n-i} = (-1)^n\left(\frac{I^n}{n!}+\sum_{i=2}^{n}\frac{\Psi_i(t)\,I^{n-i}}{i!\,(n-i)!}\right)$$

$$= \frac{(-1)^n}{n!}\left(I^n+\sum_{i=2}^{n}\binom{n}{i}\Psi_i(t)\,I^{n-i}\right) = \frac{(-1)^n}{n!}\Phi_n(t),$$

using (82) at the end. The $n=0,1$ coefficients are respectively $1=\Phi_0(t)$ and $-I=-\Phi_1(t)$. Hence on this neighbourhood,

$$h(t,z)\,e^{-zI} = \sum_{0}^{\infty}\frac{(-1)^n z^n}{n!}\Phi_n(t)$$

which is $H(t,z)\,\mathrm{i}^T$ by Lemma 68. □

Lemma 75. *There is a neighbourhood of $z=0$ on which*

$$\left.\frac{\partial}{\partial t}\right|_{t=0^+} H(t,z) = 0.$$

Proof. Since $H(0,z) = d_0(t)$ therefore

$$\frac{1}{t}\left(H(t,z) - H(0,z)\right) = \frac{1}{t}\sum_{n=1}^{\infty}(-1)^n z^n d_n(t).$$

By Corollary 71, on $|z| < 1/(2\gamma(1))$ the latter series is absolutely convergent uniformly in each $t \in [0,1]$. By (77) and Lemmas 65 and 67, where again $p_k = k_2(t_k)$, $q_k = k_1(t_k)$, for $n \geq 1$ the integrand of $d_n(t)$ is

$$\begin{bmatrix} \mathcal{K}(t_1,\ldots,t_n) \\ -\tilde{\mathcal{K}}(t_1,\ldots,t_n) \end{bmatrix} = \begin{bmatrix} p_1 q_n \prod_1^{n-1}(p_{k+1}q_k - p_k q_{k+1}) \\ -p_1 p_n \prod_1^{n-1}(p_{k+1}q_k - p_k q_{k+1}) \end{bmatrix}.$$

Since the k_i are bounded on compacts there exists $c > 0$ so that $|k_1(t)|$, $|k_2(t)| \leq c$ for $t \in [0,1]$. Let $\varepsilon > 0$. Since $k_2(0) = 0$ and k_2 is continuous at 0 there exists $\delta > 0$ small so that $|k_2(t)| \leq \varepsilon$ for $t \in [0,\delta]$, and that $2c\varepsilon\delta \leq 1$. Hence, for $t \in [0,\delta]$ each component of the integrand of $d_n(t)$ is absolutely bounded by $\varepsilon c(2c\varepsilon)^{n-1}$. Consequently, each component of

$$\frac{1}{t}\sum_{n=1}^{\infty}(-1)^n z^n d_n(t)$$

for $t \in [0,\delta]$ is absolutely bounded by

$$\frac{1}{t}\sum_{n=1}^{\infty}|z|^n \frac{\varepsilon c(2c\varepsilon)^{n-1}t^n}{n!} = \varepsilon\sum_{n=1}^{\infty}\frac{c|z|^n(2c\varepsilon\delta)^{n-1}}{n!}$$

$$\leq \varepsilon\sum_{n=1}^{\infty}\frac{c|z|^n}{n!} < \infty.$$

This vanishes with ε. □

The following elementary fact about ordinary differential equations will be required:

Fact 76. *(Theorem 2.1 Weidmann (2006), Remark 16 Valkó and Virág (2020)) Let $A(t)$ be a 2×2 matrix with components measurable in t and locally integrable on $[a,b)$. Then the $\mathbb{C}^2$-valued ordinary differential equation for $z \in \mathbb{C}$,*

$$\frac{\partial}{\partial t}G(t,z) = zA(t)G(t,z),\ t \in [a,b),\ G(a,z) = 0$$

has a unique solution which is entire in z for each $t \in [a,b)$.

Theorem 77. *For each $z \in \mathbb{C}$, $H(t,z)$ uniquely solves the ordinary differential equation*

$$J\frac{\partial}{\partial t}H(t,z) = -zR(t)H(t,z), \quad t \in [0,\infty), \quad H(0,z) = \mathfrak{i}$$

and both coordinates of $H(t,z)$ are entire in z for all fixed t.

Proof. First consider the claim where $t \in [0,\infty)$ is replaced with $t \in [0,b)$ for $0 < b < \infty$. By definition $H(0,z) = \mathfrak{i}$. First consider $t > 0$. By Lemma 84, for $n \geq 2$,

$$d_n(t)^T = \int_{0 \leq t_1 < \cdots < t_n \leq t} \mathfrak{i}^T R(t_1) J \cdots R(t_n) J dt_1 \cdots dt_n$$

$$= \int_0^t \int_{0 \leq t_1 < \cdots < t_{n-1} \leq t_n} \mathfrak{i}^T R(t_1) J \cdots R(t_{n-1}) J dt_1 \cdots dt_{n-1} R(t_n) J dt_n$$

$$= \int_0^t d_{n-1}(t_n)^T R(t_n) J dt_n.$$

This additionally holds for $n = 1$:

$$d_1(t)^T = \int_{0 \leq t_1 \leq t} \mathfrak{i}^T R(t_1) J dt_1 = \int_0^t d_0(t_1)^T R(t_1) J dt_1.$$

Multiplying both sides of

$$d_n(t)^T = \int_0^t d_{n-1}(t_n)^T R(t_n) J dt_n$$

by $(-1)^n z^n$ summing to $m \geq 1$ yields

$$\sum_{n=1}^m (-1)^n z^n d_n(t)^T = -\int_0^t z \sum_{n=1}^m (-1)^{n-1} z^{n-1} d_{n-1}(u)^T R(u) J du. \tag{83}$$

By Corollary 71, for any $u \in [0,t]$ both coordinates of $H(u,z)$ converge absolutely on $|z| < 1/(2\gamma(t))$ and in particular on $|z| < 1/(2\gamma(b))$. Fix such a z. Then by definition of $H(t,z)$,

$$H_m(u,z) = \sum_{n=1}^m (-1)^{n-1} z^{n-1} d_{n-1}(u)^T \tag{84}$$

converges pointwise in $u \in [0,t]$ to $H(u,z)^T$ as $m \to \infty$. Moreover, it is clear from Lemma 69 and Corollary 70 that both coordinates of (84) are respectively uniformly in $u \in [0,t]$

and m absolutely bounded by

$$1+\sum_{1}^{\infty}\left(2\left|z\right|\gamma\left(t\right)\right)^{n}\quad\text{and}\quad t\sum_{1}^{\infty}n\left(2\left|z\right|\gamma\left(t\right)\right)^{n}.$$

Hence, by the bounded convergence theorem applied to (83), taking $m\to\infty$ yields that for each $|z|<1/\left(2\gamma\left(b\right)\right)$,

$$H\left(t,z\right)^{T}-\mathfrak{i}^{T}=-\int_{0}^{t}zH\left(u,z\right)^{T}R\left(u\right)Jdu. \tag{85}$$

For example, the convergence theorem is applied to each coordinate of (83) and it is used that the k_i terms coming from $R\left(u\right)J$ are bounded on compacts. Since by definition $J^{T}=J^{-1}$ and $R\left(u\right)^{T}=R\left(u\right)$, transposing both sides of (85) yields

$$H\left(t,z\right)-\mathfrak{i}=-\int_{0}^{t}zJ^{-1}R\left(u\right)H\left(u,z\right)du.$$

Therefore, for $t>0$,

$$J\frac{\partial}{\partial t}H\left(t,z\right)=-zR\left(t\right)H\left(t,z\right)$$

To extend this to $t=0$, note that

$$R\left(0\right)H\left(0,z\right)=\begin{bmatrix}k_{2}\left(0\right)^{2} & k_{1}\left(0\right)k_{2}\left(0\right)\\ k_{1}\left(0\right)k_{2}\left(0\right) & k_{1}\left(0\right)^{2}\end{bmatrix}\mathfrak{i}=\begin{bmatrix}k_{2}\left(0\right)^{2}\\ k_{1}\left(0\right)k_{2}\left(0\right)\end{bmatrix}=0$$

by assumption, and that

$$J\left.\frac{\partial}{\partial t}\right|_{t=0^{+}}H\left(t,z\right)=\begin{bmatrix}-\frac{\partial}{\partial t}H\left(t,z\right)\mathfrak{j}^{T}\\ \frac{\partial}{\partial t}H\left(t,z\right)\mathfrak{i}^{T}\end{bmatrix}=0$$

by Lemma 75. In summary, it has been shown that for $b>0$ and $|z|<1/\left(2\gamma\left(b\right)\right)$,

$$J\frac{\partial}{\partial t}H\left(t,z\right)=-zR\left(t\right)H\left(t,z\right),\ t\in\left[0,b\right),\ H\left(0,z\right)=\mathfrak{i}. \tag{86}$$

Rewrite this equation as

$$\frac{\partial}{\partial t}H\left(t,z\right)=-zJ^{-1}R\left(t\right)H\left(t,z\right)=zA\left(t\right)H\left(t,z\right)$$

where

$$A(t) = -J^{-1}R(t) = \begin{bmatrix} -k_1(t)\,k_2(t) & -k_1(t)^2 \\ k_2(t)^2 & k_1(t)\,k_2(t) \end{bmatrix}.$$

Since the k_i are bounded on compacts, by Fact 76, (86) actually has a unique solution which is entire in z for $t \in [0,b)$. Hence, for $t \in [0,b)$, $H(t,z)$ agrees with this unique solution for $|z| < 1/(2\gamma(b))$ and hence everywhere. Taking $b \to \infty$ completes the proof. □

8.3 Application to Hilbert-Schmidt operators and the stochastic Airy operator

8.3.1 Hilbert-Schmidt kernel truncations

Let K be a Hilbert-Schmidt integral operator with kernel k as defined in Section 6.4 where k satisfies the additional assumptions in Section 8.1. Recall by Example 63 that the stochastic Airy operator provides an example of such an operator and kernel. For $0 < a < \infty$ define the truncated integral kernel

$$k_a(x,y) = k_1(x)\,k_2(y)\,1_{a \geq x \geq y} + k_1(y)\,k_2(x)\,1_{x<y\leq a},\ x,y \in [0,\infty)$$

and the corresponding integral operator

$$K_a f(x) = \int_0^\infty k_a(x,y)\,f(y)\,dy = \int_0^a k(x,y)\,f(y)\,dy,\ f \in L^2([0,\infty),\mathbb{R}).$$

Lemma 78. *K_a is a Hilbert-Schmidt integral operator with kernel k_a satisfying the assumptions in Section 8.1 and $K_a \to K$ in $\mathcal{S}_2$ as $a \to \infty$.*

Proof. Note that

$$\|k_a\|_2^2 = \int_0^a \int_0^a k_a(x,y)\,dxdy \leq \|k\|_2^2 < \infty.$$

Hence, by Section 6.4, $K_a \in \mathcal{S}_2$. Since

$$k_a(x,y) = k_1(x)\,1_{[0,a]}(x)\,k_2(y)\,1_{[0,a]}(y)\,1_{x\geq y} + k_1(y)\,1_{[0,a]}(y)\,k_2(x)\,1_{[0,a]}(x)\,1_{x<y}$$

therefore K_a moreover satisfies the conditions in Section 8.1. Next, with $K_a - K \in \mathcal{S}_2$,

$$\|K_a - K\|_{\mathcal{S}_2}^2 = \|k_a - k\|_2^2$$

$$= \int_a^\infty \int_0^\infty k(x,y)^2\,dxdy + \int_0^a \int_a^\infty k(x,y)^2\,dxdy.$$

The first integral vanishes as $a \to \infty$. For the second,

$$f_a(y) = \int_a^\infty k(x,y)^2\,dx < \infty, \quad y\text{-a.e.}$$

because $f_a(y) \le f_0(y)$ and $\int_0^\infty f_0(y)\,dy = \|k\|_2^2 < \infty$. So $f_a \to 0$ a.e. and therefore by the dominated convergence theorem,

$$\int_0^\infty f_a(y)\,dy \to 0.$$

□

Corollary 79. $\det_2(I - zK_a) \to \det_2(I - zK)$ *as entire functions in the topology of compact convergence.*

Proof. Apply Lemma 49 to Lemma 78. □

8.3.2 Regularized determinants and the stochastic Airy operator

The following shows that regularized determinants can be recovered from the ordinary differential equation in Theorem 77 in the special case of the Hilbert-Schmidt integral operators:

Proposition 80. *Let K be a Hilbert-Schmidt integral operator with kernel k satisfying the assumptions in Section 8.1. If the support of k is contained in $[0,t]^2$ for some t then*

$$\det_2(I - zK) = h(t,z)$$

where $h(t,z)$ is defined in (81), and

$$\det_2(I - zK) = e^{z\int_0^t k(x,x)dx} H(t,z)\,\mathbf{i}^T.$$

Otherwise,

$$\det_2(I - zK) = \lim_{t\to\infty} e^{z\int_0^t k(x,x)dx} H(t,z)\,\mathbf{i}^T$$

where the limit holds uniformly in z on compacts as $t \to \infty$.

Proof. For the first claim, by (63),

$$\det_2(I - zK) = 1 + \sum_2^\infty \frac{(-1)^n z^n}{n!} \int_{[0,t]^n} \det\left[k(t_i,t_j)\,1_{i\neq j}\right]_{i,j=1}^n dt_1\cdots dt_n$$

$$= 1 + \sum_{2}^{\infty} \frac{(-1)^n z^n}{n!} \Psi_n (t) = h (t, z) .$$

By Lemma 74 it then follows that

$$\det{}_2 (I - zK) = e^{z \int_0^t k(x,x) dx} H (t, z) \mathfrak{i}^T .$$

For the second claim, let $K_a \in \mathcal{S}_2$, k_a, $a > 0$ be truncations of K as defined in Section 8.3.1. Then, by Corollary 79, as $a \to \infty$,

$$\det{}_2 (I - zK_a) \to \det{}_2 (I - zK)$$

uniformly in z on compacts. Furthermore, by the first claim, for each $a > 0$,

$$\det{}_2 (I - zK_a) = e^{z \int_0^a k(x,x) dx} H (a, z) \mathfrak{i}^T .$$

□

By Example 63, in the case of the stochastic Airy operator,

$$R (t) = \begin{bmatrix} \psi_d (t)^2 & \psi_\infty (t) \psi_d (t) \\ \psi_\infty (t) \psi_d (t) & \psi_\infty (t)^2 \end{bmatrix}$$

and Theorem 11 follows immediately from Proposition 80. Note that

$$H (t, z) = \begin{bmatrix} 1 + \sum_1^\infty \frac{(-1)^n z^n}{n!} \int_{[0,t]^n} \det \left[k (t_i, t_j) \right]_{i,j=1}^n dt_1 \cdots dt_n \\ z \int_0^t \psi_d (s)^2 ds - \sum_2^\infty \frac{(-1)^n z^n}{(n-1)!} \int_0^t \int_{[0,t]^{n-1}} \tilde{\mathcal{K}} (t_1, \ldots, t_{n-1}, t_n) dt_1 \cdots dt_{n-1} dt_n \end{bmatrix}$$

where

$$\tilde{\mathcal{K}} (t_1, \ldots, t_{n-1}, t_n) = \det \begin{bmatrix} \left[k (t_1, t_j) \right]_{j=1}^n \\ \vdots \\ \left[k (t_{n-1}, t_j) \right]_{j=1}^n \\ \left[\psi_d (t_j) \psi_d (t_n) \right]_{j=1}^n \end{bmatrix} .$$

8.4 Relation to known bulk case of the sine beta operator

The work in Section 8.2 is an application and extension of the work in Valkó and Virág (2020) for the bulk case of the sine beta operator. There they consider a family of operators called Dirac operators,

$$\tau u = R^{-1} J u'$$

acting on $(0,\sigma]$ for $\sigma>0$ where $R(t)$, $t>0$ is similarly a 2×2 real matrix and J is as it is in Corollary 66. Note that by Theorem 11, almost surely $R(t)$ in the case of the stochastic Airy operator is not invertible:

$$\det R(t)=\psi_d(t)^2\,\psi_\infty(t)^2-\left(\psi_\infty(t)\,\psi_d(t)\right)^2=0.$$

Additionally, there the integral kernel has finite integral trace

$$\int_0^\sigma k_1(t)\,k_2(t)\,dt$$

however this integral diverges for the stochastic Airy operator when $\sigma=\infty$:

Lemma 81. *Almost surely,*

$$\int_0^\infty \psi_d(t)\,\psi_\infty(t)\,dt=\infty$$

Proof. In Proposition 9 and its proof in Dumaz, Li, and Valkó 2020 it is explained that for $a>0$ large and $t\geq a$,

$$\psi_\infty(t)=\psi_d(t)\int_t^\infty \psi_d(s)^{-2}\,ds\quad\text{and}\quad\frac{\psi_d'(t)}{\psi_d(t)}\leq\frac{3}{2}\sqrt{t}.$$

Then,

$$\psi_d(t)\,\psi_\infty(t)=\int_t^\infty\frac{\psi_d(t)^2}{\psi_d(s)^2}ds=\int_t^\infty\exp\left(-2\int_t^s\frac{\psi_d'(u)}{\psi_d(u)}du\right)ds.$$

Consequently,

$$\int_a^\infty\psi_d(t)\,\psi_\infty(t)\,dt\geq\int_a^\infty\int_t^\infty\exp\left(-2\left(s^{3/2}-t^{3/2}\right)\right)dsdt=\infty.$$

□

The statements and arguments in Section 8.2 relate to Valkó and Virág (2020) as follows. The statement and proof of Theorem 77 extends that of the differential equation in $H(t,z)$ in Proposition 13 in Valkó and Virág (2020), and the argument is very similar. The definition of $H(t,z)$ and $d_n(t)$ come from Proposition 19. Lemma 65 and Corollary 66 closely follow and extend the arguments in Proposition 9. Lemmas 72 and 74 also closely follow and extend the arguments in Lemma 10.

9 Elementary facts

The following are some required elementary results that are conveniently stated here and are proved due to lack of references.

Lemma 82. *If μ_n, μ be probability measures on $\mathbb{R}^n$ then, $\mu_n \Longrightarrow \mu$ iff*

$$\int f d\mu_n \to \int f d\mu \tag{87}$$

for all $f \in C_c(\mathbb{R}^n, \mathbb{R})$, or equivalently for all $f \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$.

Proof. Since $\mathbb{R}^n$ is separable and locally compact, Exercise 5.10 in Billingsley (1999) states that if (87) holds for $f \in C_c$ then $\mu_n \Longrightarrow \mu$. Hence, by the Portmanteau theorem $\mu_n \Longrightarrow \mu$ is equivalent to (87) for $f \in C_c$. So, if $\mu_n \Longrightarrow \mu$ then (87) holds for $f \in C_c \supseteq C_c^\infty$. Suppose (87) holds for all $f \in C_c^\infty$. Let $f \in C_c$ with compact support $K = \overline{B}(0, r) \subseteq \mathbb{R}^n$ so that it suffices to show (87) for f. Let $M = \sup_{x \in K} |f(x)|$, $\delta > 0$ and $\varepsilon > 0$. By the Stone-Weierstrass theorem there exists $\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ so that

$$\sup_{x \in K} |\phi(x) - f(x)| \leq \varepsilon,$$

the support of ϕ is contained in $\overline{B}(0, r+\delta)$, $m_{\mathbb{R}^n}\left(\overline{B}(0, r+\delta) - \overline{B}(0, \delta)\right) \leq \varepsilon$, and on $\overline{B}(0, r+\delta)$, ϕ is absolutely bounded by $2M$. Hence, for all n,

$$\begin{aligned}
\left|\int f d\mu_n - \int f d\mu\right| &\leq \int |f - \phi| 1_K d\mu_n + \int |\phi| 1_{\overline{B}(0,r+\varepsilon)-\overline{B}(0,r)} d\mu_n \\
&+ \int |\phi - f| 1_K d\mu + \int |\phi| 1_{\overline{B}(0,r+\varepsilon)-\overline{B}(0,r)} d\mu + \left|\int \phi d\mu_n - \int \phi d\mu\right| \\
&\leq 2\varepsilon + 4M\varepsilon + \left|\int \phi d\mu_n - \int \phi d\mu\right|.
\end{aligned}$$

Since $\phi \in C_c^\infty$, by assumption there is large n so that $\left|\int \phi d\mu_n - \int \phi d\mu\right| \leq \varepsilon$. □

Lemma 83. *If almost surely $X_n \to X$ then for any $\varepsilon > 0$,*

$$\lim_{m \to \infty} \mathbb{P}\left(\sup_{n \geq m} |X_m - X| > \varepsilon\right) = 0.$$

Proof. Since almost surely $X_n \to X$ therefore

$$0 = \mathbb{P}\left(\limsup_n \{|X_n - X| > \varepsilon\}\right) = \mathbb{P}\left(\bigcap_{m=1}^\infty \bigcup_{n \geq m} \{|X_n - X| > \varepsilon\}\right).$$

Then, by continuity from above,

$$\lim_{m\to\infty}\mathbb{P}\left(\bigcup_{n\geq m}\{|X_n-X|>\varepsilon\}\right)=0.$$

Consequently,

$$\lim_{m\to\infty}\mathbb{P}\left(\sup_{n\geq m}|X_n-X|>\varepsilon\right)\leq\lim_{m\to\infty}\mathbb{P}\left(\bigcup_{n\geq m}\{|X_n-X|>\varepsilon\}\right)=0.$$

$\square$

Lemma 84. *For $f\in L^1_{loc}(\mathbb{R}^n,\mathbb{C})$, for $t\geq 0$,*

$$\int_{0\leq t_1<\cdots<t_n\leq t} f(t_1,\ldots,t_n)\,dt_1\cdots dt_n$$

$$=\int_0^t\int_{0\leq t_1<\cdots<t_{n-1}\leq t_n} f(t_1,\ldots,t_n)\,dt_1\cdots dt_{n-1}dt_n.$$

Proof. Let

$$B_n(s)=\{(s_1,\ldots,s_n)\in\mathbb{R}^n:0\leq s_1<\cdots<s_n\leq s\}$$

and

$$\tilde{B}_n(s)=\{(s_1,\ldots,s_n)\in\mathbb{R}^n:0\leq s_1<\cdots<s_n<s\}$$

both bounded subsets. Note that

$$\int_0^t\int_{0\leq t_1<\cdots<t_{n-1}<t_n} f(t_1,\ldots,t_n)\,dt_1\cdots dt_{n-1}dt_n$$

$$=\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}1_{[0,t]}(t_n)\,1_{\tilde{B}_n(t_n)}(t_1,\ldots,t_{n-1})\,f(t_1,\ldots,t_n)\,dt_1\cdots dt_{n-1}dt_n$$

$$=\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}1_{B_n(t)}(t_1,\ldots,t_n)\,f(t_1,\ldots,t_n)\,dt_1\cdots dt_{n-1}dt_n.$$

By Fubini the latter equals

$$\int_{\mathbb{R}^n}1_{B_n(t)}(t_1,\ldots,t_n)\,f(t_1,\ldots,t_n)\,dt_1\cdots dt_n.$$

Moreover,

$$B_{n-1}(t_n)-\tilde{B}_{n-1}(t_n)=\left\{(t_1,\ldots,t_{n-1})\in\mathbb{R}^{n-1}:0\leq t_1<\cdots<t_{n-1}=t_n\right\}$$

$$= \tilde{B}_{n-2}(t_n) \times \{t_n\}$$

and hence

$$m_{\mathbb{R}^{n-1}}\left(B_{n-1}(t_n) - \tilde{B}_{n-1}(t_n)\right) = 0.$$

Consequently,

$$\int_0^t \int_{0 \le t_1 < \cdots < t_{n-1} < t_n} f(t_1, \ldots, t_n)\, dt_1 \cdots dt_{n-1} dt_n$$

$$= \int_0^t \int_{0 \le t_1 < \cdots < t_{n-1} \le t_n} f(t_1, \ldots, t_n)\, dt_1 \cdots dt_{n-1} dt_n.$$

□

Lemma 85. *If Y, Z are topological spaces with Borel σ-algebras, X is a measurable subset of Z and has the subspace topology and corresponding Borel σ-algebra, and $f : X \to Y$ is measurable then $g : Z \to Y$, $g = f1_X$ is measurable.*

Proof. For $B \in \mathcal{B}(Y)$,

$$g^{-1}(B) = \left(g^{-1}(B) \cap X\right) \sqcup \left(g^{-1}(B) \cap (Z - X)\right) = f^{-1}(B) \sqcup \begin{cases} \emptyset & 0 \notin B \\ Z - X & 0 \in B \end{cases}.$$

Since $X \in \mathcal{B}(Z)$ and so the σ-algebra on X is $\{C \cap X : C \in \mathcal{B}(Z)\}$ therefore $f^{-1}(B) \in \mathcal{B}(Z)$. □

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