Semidirect Products of ∞–Operads

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Abstract

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We provide a construction of an ∞–operad from a functor $BG \longrightarrow \mathrm{Op}_\infty$ encoding the action of a group G on a given unital ∞–operad whose underlying ∞–category is a Kan complex. This construction, restricted to classical operads in Set viewed as ∞–operads, coincides with the semidirect product construction. Taking this as the definition of semidirect product of ∞–operads, we show that the action of G on the given ∞–operad is equivalent to the trivial action if and only if the corresponding semidirect products are equivalent. We then outline how one might generalize this result to operads in Top and use this to show that the semidirect product of the real version of the little n–disks operad with $SO(n-1)$ or $SO(n-2)$ for n even or odd respectively corresponding to the usual action is equivalent to the semidirect product corresponding to the trivial action.

To my parents

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Contents

Notation and Conventions

We establish some initial notation and conventions here, closely following [Lur08; Lur17]. We will introduce additional ones as needed in the main text.

Classical Categories

We will write ordinary categories as 1–categories. These will be denoted by sans serif letters as $\mathsf{A}, \mathsf{B}, \mathsf{C}$, etc. We will denote the set of objects of a 1–category A as $\mathsf{ob}(\mathsf{A})$. Given objects $x, y \in \mathsf{ob}(\mathsf{A})$, we will write the set of morphisms from x to y in A as $\mathsf{A}(x, y)$ or $\mathrm{Hom}_{\mathsf{A}}(x, y)$ or $\mathrm{Map}_{\mathsf{A}}(x, y)$ as convenient. We will also denote $\mathsf{A}(x, x)$ as $\mathrm{End}_{\mathsf{A}}(x)$. When there is no ambiguity, we will ignore subscripts and simply write $\mathrm{Hom}(x, y)$, $\mathrm{Map}(x, y)$, $\mathrm{End}(x)$, etc. Given 1–categories A, B, we will denote both the set and the 1–category of functors from A to B as $\mathcal{F}\mathrm{un}(\mathsf{A}, \mathsf{B})$. Given a functor $F : \mathsf{A} \longrightarrow \mathsf{B}$, we will write the slice categories over and under F as $\mathsf{B}_{/F}$ and $\mathsf{B}_{F/}$ respectively; in particular, slices over and under objects $x \in \mathsf{B}$ will be written $\mathsf{B}_{/x}$ and $\mathsf{B}_{x/}$ respectively, viewing x as a functor $* \longrightarrow \mathsf{B}$, where $*$ is the terminal 1–category. For a functor $F : \mathsf{A} \longrightarrow \mathsf{B}$, objects $x, y \in \mathsf{ob}(\mathsf{A})$ and a morphism $f \in \mathsf{B}(Fx, Fy)$, $\mathsf{A}^f(x, y)$ will denote the pullback of the span $\mathsf{A}(x, y) \xrightarrow{F} \mathsf{B}(Fx, Fy) \xleftarrow{f} *$. We list notation for some standard 1–categories or bicategories:

Set	1–category of (small) sets and functions
Top	1–category of topological spaces and continuous functions
Δ	simplicial indexing category: objects are the finite linearly ordered sets $[n] = \{0 < 1 < 2 < \cdots < n\}$ for $n = 0, 1, 2, \ldots$ and morphisms are order-preserving functions
sSet	1–category of simplicial sets and natural transformations which is simply $\mathcal{F}\mathrm{un}\left(\Delta^{\mathrm{op}}, \mathsf{Set}\right)$
$\mathcal{F}\mathrm{in}_*$	1–category with objects pointed finite sets $\langle n \rangle = \{*, 1, \ldots, n\}$, $*$ being the base-point, and morphisms, base-point preserving maps
Γ^*	1–category whose objects are pairs $(\langle n \rangle, i)$ with $i \in \langle n \rangle^\circ$, where $\langle n \rangle^\circ = \langle n \rangle \setminus \{*\}$, and morphisms $(\langle m \rangle, i) \longrightarrow (\langle n \rangle, j)$ are base-point preserving maps $\alpha : \langle m \rangle \longrightarrow \langle n \rangle$ with $\alpha(i) = j$.
$\mathcal{C}\mathrm{at}$	1–category of small 1–categories and functors or strict bicategory of small 1–categories, functors, and natural transformations, as needed
$\mathcal{C}\mathrm{at}_{\mathsf{V}}$	1–categories enriched in a monoidal category V
$\mathrm{Cart}(\mathsf{C})$	1–category or bicategory of Cartesian fibrations over a given 1–category C and morphisms thereof
$\mathrm{CoCart}(\mathsf{C})$	1–category or bicategory of coCartesian fibrations over a given 1–category C and morphisms thereof

Higher Categories

We will write the nerve of a 1–category C as $N(\mathsf{C})$. Arbitrary simplicial sets will be denoted by ordinary letters A, B, C, etc. By a simplicial category, we will mean an sSet–enriched category and we will denote the category of simplicial categories as $\mathcal{C}\mathrm{at}_{\mathsf{sSet}}$ or $\mathsf{s}\mathcal{C}\mathrm{at}$. We will always consider $\mathcal{C}\mathrm{at}_{\mathsf{sSet}}$ to be equipped with the Bergner model structure where a simplicial category is fibrant if all its morphism simplicial sets are Kan complexes. We will not fix notation for arbitrary simplicial categories as we will rarely need them. Neverthless, given a simplicial category S, its simplicial nerve will be denoted $\mathcal{N}(S)$. By an ∞–category, we will mean a quasicategory – that is, a simplicial set that has a filler for every inner horn inclusion $\Lambda^n_i \longrightarrow \Delta^n, 0 < i < n$. We will denote ∞–categories by script letters such as $\mathcal{A}, \mathcal{B}, \mathcal{C}$, etc. Morphism simplicial sets and functor simplicial sets will be denoted similar to the 1–categorical case – that is, $\mathcal{A}(x, y)$, $\mathrm{Map}_{\mathcal{A}}(x, y)$, $\mathrm{Hom}_{\mathcal{A}}(x, y)$, $\mathrm{End}_{\mathcal{A}}(x)$, $\mathcal{F}\mathrm{un}(\mathcal{A}, \mathcal{B})$, etc. Slices will also be denoted similarly – $\mathcal{B}_{/F}$ and $\mathcal{B}_{F/}$ for a map of simplicial sets $F : \mathcal{A} \longrightarrow \mathcal{B}$. For a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$, objects $x, y \in \mathcal{A}_0$ and a morphism $f \in \mathcal{B}(x, y)_0$, $\mathcal{A}^f(x, y)$ is defined similarly as in the case of 1–categories. We then list some standard ∞–categories and simplicial categories:

sSet	simplicial category whose objects are simplicial sets and morphism simplicial sets are the functor simplicial sets
sSet^+	simplicial category whose objects are marked simplicial sets and morphism simplicial sets are the full subcategories of functor simplicial sets generated by functors that preserve the markings
$\mathcal{C}\mathrm{at}^{\Delta}_{\infty}$	simplicial category whose objects are quasicategories and morphism simplicial sets are the maximal Kan complexes contained in the functor simplicial sets
$\mathcal{C}\mathrm{at}_{\infty}$	the simplicial nerve $\mathcal{N}\left(\mathcal{C}\mathrm{at}^{\Delta}_{\infty}\right)$, called the ∞–category (quasicategory) of ∞–categories (quasicategories)
$\mathrm{Op}^{\Delta}_{\infty}$	simplicial category whose objects are ∞–operads and morphism simplicial sets $\mathrm{Op}^{\Delta}_{\infty}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes})$ are the maximal Kan complexes of the morphism simplicial sets $\mathcal{F}\mathrm{un}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes})$ generated by the maps of ∞–operads
Op_{∞}	the simplicial nerve $\mathcal{N}\left(\mathrm{Op}^{\Delta}_{\infty}\right)$, called the ∞–category of ∞–operads; this is also the subcategory of $\mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$ generated by the ∞–operads and morphisms of ∞–operads
$\mathrm{Cart}(\mathcal{C})$	∞–category of Cartesian fibrations over a given ∞–category $\mathcal{C}$ and morphisms thereof
$\mathrm{CoCart}(\mathcal{C})$	∞–category of coCartesian fibrations over a given ∞–category $\mathcal{C}$ and morphisms thereof

1 Introduction

The notion of a semidirect product of a classical operad first appeared in [Wah01] where it was used to study the framed little disks operad introduced in [Get94] and higher dimensional versions thereof. Wahl [Wah01] exhibited the framed little n–disks operad $f\mathcal{D}_n$ as the semidirect product of the ordinary little n–disks operad $\mathcal{D}_n$ with $SO(n)$. One application of viewing the framed little n–disks operad this way is that it leads to an $SO(n)$ equivariant version of the famous recognition principle of May [May72]: just as a connected space is weak homotopy equivalent to a loop space if and only if it is a $\mathcal{D}_n$–algebra, a connected $SO(n)$–space is weak homotopy equivalent a loop space on an $SO(n)$–space if and only if it is an $f\mathcal{D}_n$–algebra [SW03, p. 2]. Framed disks operads feature in other important developments in topology including manifold calculus [BW13] and the study of configuration spaces [MSS02, §4.2]. Thus, it is interesting to import the theory of semidirect products of operads to the setting of ∞–categories and ∞–operads. In the present work, we take some first steps in this direction and comment on a potential application of the theory.

In [Lur17, 5.4.2.9, p. 931], Lurie remarks that a specific ∞–operad $\mathrm{BTop}(k)^{\otimes}$ which he constructs can be "regarded as a kind of semidirect product" of the little k–cubes operad E_k. He further remarks that it has one of the main properties of semidirect products: roughly, that $\mathrm{BTop}(k)^{\otimes}$–algebras in a symmetric monoidal ∞–category $\mathcal{C}$ are E_k–algebras in $\mathcal{C}$ with a compatible action of the group $\mathcal{T}\mathrm{op}(k)$ – see [Wah01, 1.3.5, p. 18] for a more precise description of this property. However, it is not clear from the discussion what the definition of a semidirect product should be in the context of ∞–operads. Nevertheless, an essential point in Lurie's discussion is that $\mathrm{BTop}(k)^{\otimes}$ is an assembly of a family of ∞–operads. We may thus ask whether the assembly construction generalizes semidirect products more broadly. Recall that an action of a group or a monoid G on some ∞–operad can be identified with a functor $F : BG \longrightarrow \mathrm{Op}_\infty$, where Op_∞ is the ∞–category of ∞–operads. Then, observe that Op_∞ is a subcategory of $\mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$, the ∞–category of ∞–categories over the category of pointed finite sets, so that one may take the unstraightening of this functor in a suitable sense. If we can then show that this process gives a family of ∞–operads as defined in [Lur17, 2.3.2.10, p. 253], then we may take the assembly of this family. The natural question then is: does this construction applied to F give the semidirect product corresponding to F?

En route to answering this question, we first explore in §2 the relationship between the Grothendieck construction of a functor valued in the category of classical Set–operads and families of classical operads in order to understand better the analogous relationship between unstraightening and families of ∞–operads. The main results that we show here are that the Grothendieck construction of a functor does result in a family of operads and conversely, a family of operads satisfying a technical condition gives a functor into the category of operads. These facts were discussed in outline in [httb] and we have made precise the arguments given there. The main statement we have proven in this section is:

Theorem 2.3.13. *Given a pseudofunctor $F : \mathsf{C} \longrightarrow \mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}$, the Grothendieck construction*

$\int \mu F$, *where* $\mu : \mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*} \longrightarrow \mathcal{C}\mathrm{at}$ *is the the forgetful functor* $(\mathsf{A} \xrightarrow{f} \mathcal{F}\mathrm{in}_*) \longmapsto \mathsf{A}$, *equipped with the map to* $\mathsf{C} \times \mathcal{F}\mathrm{in}_*$ *is a* C*–family of* 1*–operads if and only if* F *factors through* Op.

Here, Op is the category of operads viewed as their respective categories of operators. Based on the 1– and 2–categorical ideas used in showing this, in §3, we develop some ∞–categorical foundations to translate these results to the setting of ∞–operads. With these foundations, in §4.1, we relate the unstraightening construction of [Lur08, §3.2] with functors valued in Op_∞ and families of ∞–operads. In particular, we show that if we start with a functor valued in Op_∞ and unstraighten it – making sense of what unstraightening means for Op_∞ – we obtain a family of ∞–operads over the indexing category. On the other hand, if we have a family of ∞–operads over some ∞–category $\mathcal{C}$ satisfying technical conditions to be made precise later, then the straightening of that family – again, making sense of straightening in this context – is a functor that factors through Op_∞. The formal statement is as follows:

Theorem 4.1.5. *Let* $F : \mathcal{K} \longrightarrow \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$ *be a functor of* ∞*–categories and* $p : \mathcal{E} \longrightarrow \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*)$, *the sliced unstraightening of* F *over* $\pi_1 : \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*) \longrightarrow \mathcal{K}$. p *is a* $\mathcal{K}$*–family of* ∞*–operads if and only if* F *factors through the inclusion* $\mu : \mathrm{Op}_\infty \hookrightarrow \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$.

In proving this, we have noticed the following sufficient condition for a map $\mathcal{O}^\otimes \longrightarrow \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*)$ to be a $\mathcal{K}$–family of ∞–operads that we did not find in the existing literature:

Theorem 4.1.3. *If a map* $p : \mathcal{E} \longrightarrow \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*)$ *satisfies:*

(i) the map $p_1 = \pi_1 p : \mathcal{E} \longrightarrow \mathcal{K}$ *is a coCartesian fibration (where* $\pi_1 : \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*) \longrightarrow \mathcal{K}$ *is the projection onto the first factor)*

(ii) p is a morphism of coCartesian fibrations $p_1 \longrightarrow \pi_1$

(iii) the fibre $\mathcal{E}_C = p_1^{-1}(C)$ *above each object* $C \in \mathcal{K}_0$ *is an* ∞*–operad*

(iv) fibre transport in $\mathcal{E}$ *along any map* $f \in \mathcal{K}_1$ *is a map of* ∞*–operads*

then $\mathcal{E}$ *is a* $\mathcal{K}$*–family of* ∞*–operads.*

Through §4.2 to §4.4 we establish that classical semidirect products of unital Set operads – unital, in the sense of [Lur17, 2.3.1.1, p. 244] – whose underlying categories are groupoids arise as assemblies of unstraightened Op_∞–valued functors:

Theorem 4.4.3. *Let a group* G *act on a* Set*–valued unital* 1*–operad* $\mathcal{O}^\otimes$ *whose underlying category is a groupoid. Then, the nerve of the semidirect product* $(G \ltimes \mathcal{O})^\otimes$ *is an assembly of the nerve of the Grothendieck construction of the functor* $BG \longrightarrow \mathrm{Op}$ *corresponding to the action.*

We should note that we have not checked the analogous statements for topological or simplicial operads but we have speculated on how the argument might proceed. In §4.5, we define semidirect products of ∞–operads and prove some results relating them to the functors representing the respective actions. Our main result in this direction is the following:

Theorem 4.5.7. *Let $F, F' : BG \longrightarrow \mathrm{Op}_\infty^{\mathrm{rd}}$ be functors for some group G such that both send the unique object of BG to a fixed ∞–operad $\mathcal{O}^\otimes$ – that is, they represent two different actions of G on $\mathcal{O}^\otimes$. Then, $(G \ltimes_F \mathcal{O})^\otimes \simeq (G \ltimes_{F'} \mathcal{O})^\otimes$ if and only $F \simeq F'$ as objects of $\mathcal{F}\mathrm{un}\,(BG, \mathrm{Op}_\infty)$.*

Finally, in §5, we comment on a potential application of semidirect product ∞–operads to a phenomenon observed by Khoroshkin and Willwacher [KW17] concerning framed little disks operads.

2 Classical Operads and Unstraightening

Given a coloured 1–operad $\mathcal{O}$ in Set equipped with the Cartesian monoidal structure [Lur17, 2.1.1.1, p. 171], which we will simply call an operad throughout this section, we can construct its category of operators $\mathcal{O}^{\otimes}$ equipped with a canonical map to $\mathcal{F}\mathrm{in}_*$ [Lur17, 2.1.1.7, p. 173]. As a matter of convenience, in this section, we will call the map $\mathcal{O}^{\otimes} \longrightarrow \mathcal{F}\mathrm{in}_*$ the category of operators of $\mathcal{O}$ as opposed to the domain $\mathcal{O}^{\otimes}$. In particular, each such category of operators is a category over $\mathcal{F}\mathrm{in}_*$. Consider then a functor F from some 1–category C into the category $\mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}$ of categories over $\mathcal{F}\mathrm{in}_*$. If the image of F consists of categories of operators and some sufficiently nice maps between them that correspond in some way to maps of their underlying operads, a natural question to ask is what kind of object the Grothendieck construction of the functor $\mu F : \mathsf{C} \longrightarrow \mathcal{C}\mathrm{at}$ is, where $\mu : \mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*} \longrightarrow \mathcal{C}\mathrm{at}$ is the forgetful functor sending maps into $\mathcal{F}\mathrm{in}_*$ to their respective domains. The goal of this section is to provide a description of this object and relate it to functors valued in the category of operads. This will provide us with some concrete intuition as to how Lurie's straightening and unstraightening constructions interact with functors valued in ∞–operads.

In §2.1, we will establish some definitions and elementary results needed to make this idea precise. We will define the category of operads to be a category Op consisting of categories of operators and sketch a proof that this is equivalent to the classical category of operads. We will also define a 1–categorical analogue of a family of ∞–operads – that is, a collection of operads varying covariantly with a base category. To make sense of this last statement, we then establish in §2.2 that starting with such a family of operads with a nice property, if we apply the inverse Grothendieck construction to it – of course, after making sense of this – we obtain a functor into $\mathcal{C}\mathrm{at}$ that must factor through the functor $\mathrm{Op} \longrightarrow \mathcal{C}\mathrm{at}$ that sends a category of operators $\mathcal{O}^{\otimes} \longrightarrow \mathcal{F}\mathrm{in}_*$ to the domain $\mathcal{O}^{\otimes}$. That is, families of operads give rise to Op–valued functors. In §2.3, we then show that applying the Grothendieck construction to a functor $\mathsf{C} \longrightarrow \mathrm{Op} \longrightarrow \mathcal{C}\mathrm{at}$ results in a family of operads, showing that Op–valued functors encode essentially the same data as operad families.

2.1 Some Preliminaries on Operads

We first show that morphisms between two operads are in one–to–one correspondence with morphisms between their categories of operators over $\mathcal{F}\mathrm{in}_*$ that preserve coCartesian lifts of inert morphisms. To do so, we first recall some relevant definitions and basic results.

Definition 2.1.1 (Morphism of Operads (cf. [Yau16, 11.6.8, p. 196]))**.** Let $\mathcal{O}$ and $\mathcal{P}$ be operads. A morphism $\mathfrak{F} : \mathcal{O} \longrightarrow \mathcal{P}$ of coloured operads consists of a function of colours $\mathsf{ob}(\mathcal{O}) \longrightarrow \mathsf{ob}(\mathcal{P})$ which we also denote by $\mathfrak{F}$ and for each finite set I, for every sequence of colours $\{a_i\}_{i\in I}$ in $\mathsf{ob}(\mathcal{O})$ and every colour b in $\mathsf{ob}(\mathcal{P})$, a function:

$$\mathfrak{F}_{\{a_i\}_{i\in I};b} : \mathcal{O}\left(\{a_i\}_{i\in I}\,; b\right) \longrightarrow \mathcal{P}\left(\{\mathfrak{F}a_i\}_{i\in I}\,; b\right)$$

When the inputs and outputs are clear from context, we will simply write $\mathfrak{F}$ for such maps. Then, this data is required to satisfy that all diagrams of the following forms commute:

$$\begin{tikzcd} & \{*\} \arrow[dl, "\mathrm{id}_a"'] \arrow[dr, "\mathrm{id}_{\mathfrak{F}a}"] & \\ \mathcal{O}(a;a) \arrow[rr, "\mathfrak{F}"'] & & \mathcal{P}(\mathfrak{F}a;\mathfrak{F}a) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}\left(\{a_i\}_{i\in I};b\right) \arrow[r, "\sigma"] \arrow[d, "\mathfrak{F}"'] & \mathcal{O}\left(\left\{a_{\sigma(i)}\right\}_{i\in I};b\right) \arrow[d, "\mathfrak{F}"] \\ \mathcal{P}\left(\{\mathfrak{F}a_i\}_{i\in I};b\right) \arrow[r, "\sigma"'] & \mathcal{P}\left(\left\{\mathfrak{F}a_{\sigma(i)}\right\}_{i\in I};b\right) \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{O}\left(\{a_i\}_{i\in I};a\right)\times\prod_{i\in I}\mathcal{O}\left(\left\{b^i_j\right\}_{j\in J_i};a_i\right) \arrow[r, "\circ"] \arrow[d, "\mathfrak{F}\times\prod_{i\in I}\mathfrak{F}"'] & \mathcal{O}\left(\left\{b^i_j\right\}_{j\in J_i,i\in I};a\right) \arrow[d, "\mathfrak{F}"] \\ \mathcal{P}\left(\{\mathfrak{F}a_i\}_{i\in I};\mathfrak{F}a\right)\times\prod_{i\in I}\mathcal{P}\left(\left\{\mathfrak{F}b^i_j\right\}_{j\in J_i};\mathfrak{F}a_i\right) \arrow[r, "\circ"'] & \mathcal{P}\left(\left\{\mathfrak{F}b^i_j\right\}_{j\in J_i,i\in I};\mathfrak{F}a\right) \end{tikzcd}$$

where the $\circ$ maps are compositions associated to any function $J \longrightarrow I$ with fibres $J_i, i \in I$ and the σ maps are actions of the automorphism group of I on operation objects. We will write $\mathrm{Op}'(\mathcal{O},\mathcal{P})$ to denote the set of morphisms of operads $\mathcal{O} \longrightarrow \mathcal{P}$. $\diamond$

Remark 2.1.2. In the above definition, the top left diagram expresses the perservation of units, the top right diagram expresses equivariance, and the bottom diagram expresses preservation of composition. $\diamond$

Remark 2.1.3. It is easy to see that there is a category whose objects are operads and morphisms are morphisms of operads as defined above. We denote this category Op'. $\diamond$

Definition 2.1.4. Let $\mathcal{O}$ and $\mathcal{P}$ be two operads with categories of operators $p : \mathcal{O}^{\otimes} \longrightarrow \mathcal{F}\mathrm{in}_*$ and $q : \mathcal{P}^{\otimes} \longrightarrow \mathcal{F}\mathrm{in}_*$. Let $f : \mathcal{O}^{\otimes} \longrightarrow \mathcal{P}^{\otimes}$ satisfying $qf = p$ and that for every inert morphism α in $\mathcal{F}\mathrm{in}_*$ and every p–coCartesian lift $\overline{\alpha}$ of α in $\mathcal{O}^{\otimes}$, $f(\overline{\alpha})$ is a q–coCartesian lift of α in $\mathcal{P}^{\otimes}$. Then, f is called a morphism of categories of operators and is written $f : p \longrightarrow q$ or $f : \mathcal{O}^{\otimes} \longrightarrow \mathcal{P}^{\otimes}$, when p and q are clear from context. We will write $\mathrm{Alg}_p(q)$ – or, simply $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$, when p and q are clear from context – to denote the set of morphisms of categories of operators from p to q. $\diamond$

We recall the definition of an ∞–operad [Lur17, 2.1.1.10, p. 174] at this point for the purpose of showing that all 1–categories over $\mathcal{F}\mathrm{in}_*$ satisfying the 1–categorical analogue of the definition of an ∞–operad is equivalent to a category of operators.

Definition 2.1.5 (∞–Operad)**.** Let $p : \mathcal{E} \longrightarrow N(\mathcal{F}\mathrm{in}_*)$ be a functor of ∞–categories satisfying:

(i) **Inert Lifting:** Every inert map $f : \langle m\rangle \longrightarrow \langle n\rangle$ in $N(\mathcal{F}\mathrm{in}_*)$ has a p–coCartesian lift $f : X \longrightarrow Y$ starting at X for every $X \in \mathcal{E}_{\langle m\rangle}$ which we call an inert morphism in $\mathcal{E}$.

(ii) **Decomposition of Morphisms:** Let $X \in \mathcal{E}_{\langle m\rangle}, Y \in \mathcal{E}_{\langle n\rangle}$, $\alpha : \langle m\rangle \longrightarrow \langle n\rangle$. Choose p–coCartesian lifts $g^i : Y \longrightarrow Y_i$ of the Segal maps $\rho^i : \langle n\rangle \longrightarrow \langle 1\rangle$ starting at Y inducing maps $g^i_* : \mathcal{E}^{\alpha}(X,Y) \longrightarrow \mathcal{E}^{\rho^i\alpha}(X,Y_i)$ by post-composition. Then, the map

$$\mathcal{E}^{\alpha}(X,Y) \xrightarrow{\{g^i_*\}_{i=1}^n} \prod_{i=1}^n \mathcal{E}^{\rho^i\alpha}(X,Y)$$

is a homotopy equivalence.

(iii) **Formation of Tuple Objects:** For each collection of objects $X_1,\dots,X_n \in \mathcal{E}_{\langle 1\rangle}$, there exists an object $X \in \mathcal{E}_{\langle n\rangle}$ with p–coCartesian lifts $g^i : X \longrightarrow X_i$ of the Segal maps $\rho^i : \langle n\rangle \longrightarrow \langle 1\rangle$ starting at X.

Then, p is called an ∞–operad. $\diamond$

Lemma 2.1.6. *Let $p : \mathsf{E} \longrightarrow \mathcal{F}\mathrm{in}_*$ be a functor of 1–categories satisfying the properties of an ∞–operad. Then there exists a category of operators $q : \mathcal{O}^{\otimes} \longrightarrow \mathcal{F}\mathrm{in}_*$ with an equivalence of categories $G : \mathcal{O}^{\otimes} \longleftrightarrow \mathsf{E} : F$ over $\mathcal{F}\mathrm{in}_*$ such that both F and G preserve coCartesian lifts of inert morphisms.*

Proof Sketch. We will define the operad $\mathcal{O}$ as follows. The set of colours of $\mathcal{O}$ is defined to be $\mathsf{ob}\left(\mathsf{E}_{\langle 1\rangle}\right)$. Let $X_1,\dots,X_n,Y$ be colours of $\mathcal{O}$. We define $\mathcal{O}\left(\{X_i\}_{i\in\langle n\rangle^\circ};Y\right)$ as follows. By the formation of tuple objects in an ∞–operad, pick an object $X \in \mathsf{E}_{\langle n\rangle}$ with p–coCartesian lifts $f_i : X \longrightarrow X_i$ of the Segal maps $\rho^i : \langle n\rangle \longrightarrow \langle 1\rangle$ and define $\mathcal{O}\left(\{X_i\}_{i\in\langle n\rangle^\circ};Y\right)$ to be $\mathsf{E}(X,Y)$. Denote this choice X as $\bigoplus_{i=1}^n X_i = X_1 \oplus \cdots \oplus X_n$. One verifies that $\mathcal{O}$ is an operad. This uses the fact that we are dealing with 1-categories, so that uniqueness of composites up to equivalence gives uniqueness, making everything well-defined and ensuring the axioms hold.

We define a map $G : \mathcal{O}^{\otimes} \longrightarrow \mathsf{E}$ over $\mathcal{F}\mathrm{in}_*$ as follows. G sends each $(\langle n\rangle, X_1,\dots,X_n) \in \mathcal{O}^{\otimes}$ to $\bigoplus_i X_i \in \mathsf{E}_{\langle n\rangle}$. A morphism in $\mathcal{O}^{\otimes}$ $f : (\langle m\rangle, Y_1,\dots,Y_m) \longrightarrow (\langle n\rangle, Z_1,\dots,Z_n)$ consists of a map $\alpha : \langle m\rangle \longrightarrow \langle n\rangle$ and for each $j \in \langle n\rangle^\circ$ an operation $f_j \in \mathcal{O}\left(\{Y_i\}_{i\in\alpha^{-1}(j)};Z_j\right)$ which is a map $f_j : \bigoplus_{i\in\alpha^{-1}(j)} Y_i \longrightarrow Z_j$ in E. By the decomposition of morphisms in an ∞–operad, this gives a unique map in $\mathsf{E}^{\alpha}\left(\bigoplus Y_i, \bigoplus Z_j\right)$ which we take to be Gf. After verifying that G is functorial, we observe that G is over $\mathcal{F}\mathrm{in}_*$ and that it is fully faithful. That it preserves inert morphisms is immediate. For essential surjectivity, we take an arbitrary object $X \in \mathcal{E}$ over some $\langle n\rangle$, pick p–coCartesian lifts $X \longrightarrow X_i$ of the Segal maps ρ^i and construct, using decomposition of morphisms, an isomorphism $X \longrightarrow \bigoplus_i X_i$. **q.e.d.**

Remark 2.1.7. We notice that there exists a category whose objects are categories of operators and whose morphisms are morphisms thereof. We denote this category Op. $\diamond$

We next show that morphisms of operads $\mathcal{O} \longrightarrow \mathcal{P}$ are in one–to–one correspondence with morphisms of categories of operators $\mathcal{O}^{\otimes} \longrightarrow \mathcal{P}^{\otimes}$.

Theorem 2.1.8. *Let $p : \mathcal{O}^{\otimes} \longrightarrow \mathcal{F}\mathrm{in}_*$ and $q : \mathcal{P}^{\otimes} \longrightarrow \mathcal{F}\mathrm{in}_*$ be categories of operators of operads $\mathcal{O}$ and $\mathcal{P}$. There is a bijection*

$$\mathrm{Op}'(\mathcal{O}, \mathcal{P}) \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$$

Proof Sketch. Let $\mathfrak{F} : \mathcal{O} \longrightarrow \mathcal{P}$ be a morphism of operads. Let $(\langle m \rangle, (x_1, \ldots, x_m))$ be an object of $\mathcal{O}^{\otimes}$ and define

$$\phi(\mathfrak{F})(\langle m \rangle, (x_1, \ldots, x_m)) = (\langle m \rangle, (\mathfrak{F}(x_1), \ldots, \mathfrak{F}(x_m)))$$

Let

$$\left(\alpha, \left\{f_j \in \mathcal{O}\left(\{a_i\}_{i \in \alpha^{-1}(j)}; y_j\right)\right\}_{j \in \langle n \rangle^{\circ}}\right) : (\langle m \rangle, (x_1, \ldots, x_m)) \longrightarrow (\langle n \rangle, (y_1, \ldots, y_n))$$

be a morphism in $\mathcal{O}^{\otimes}$ and define:

$$\phi(\mathfrak{F})(\alpha; \{f_j\}) = \left(\alpha, \left\{\mathfrak{F}(f_j) \in \mathcal{P}\left(\{\mathfrak{F}(a_i)\}_{i \in \alpha^{-1}(j)}; \mathfrak{F}(y_j)\right)\right\}_{j \in \langle n \rangle^{\circ}}\right)$$

It is immediate that $\phi(\mathfrak{F}) : \mathcal{O}^{\otimes} \longrightarrow \mathcal{P}^{\otimes}$ is a functor over $\mathcal{F}\mathrm{in}_*$. Now, let $\alpha : \langle m \rangle \longrightarrow \langle n \rangle$ be an inert morphism in $\mathcal{F}\mathrm{in}_*$ with a lift $\overline{\alpha} = \left(\alpha, \{f_j\}_{j \in \langle n \rangle^{\circ}}\right)$. Denote by $\mathcal{O}^{\otimes}_{\langle 1 \rangle}$ the fibre of $\mathcal{O}^{\otimes}$ over $\{\langle 1 \rangle\} \subset \mathcal{F}\mathrm{in}_*$. This is a category with objects of the form $(\langle 1 \rangle, x)$, where x is a colour of $\mathcal{O}$ and morphisms are of the form $\left(\mathrm{id}_{\langle 1 \rangle}, f\right) : (\langle 1 \rangle, x) \longrightarrow (\langle 1 \rangle, y)$, where f is a 1-ary operations in $\mathcal{O}(\{x\}; y)$. We will write such objects and morphisms of $\mathcal{O}^{\otimes}_{\langle 1 \rangle}$ as simply x and f, respectively. It is then easy to check that $\overline{\alpha}$ is p–coCartesian if and only if the f_j are isomorphisms in $\mathcal{O}^{\otimes}_{\langle n \rangle}$. Now, $\phi(\mathfrak{F})(\overline{\alpha}) = (\alpha, \{\mathfrak{F}(f_j)\})$ where each $\mathfrak{F}(f_j)$ is an isomorphism since $\mathfrak{F}$ restricts to a functor $\mathcal{O}^{\otimes}_{\langle 1 \rangle} \longrightarrow \mathcal{P}^{\otimes}_{\langle 1 \rangle}$. Thus, $\phi(\mathfrak{F})(\overline{\alpha})$ is q–coCartesian.

Now let $f : \mathcal{O}^{\otimes} \longrightarrow \mathcal{P}^{\otimes}$ be a morphism of categories of operators. It is easy to see that the object function of $f|_{\mathcal{O}^{\otimes}_{\langle 1 \rangle}}$ yields function of colours $\mathfrak{F}_1 : \mathsf{ob}(\mathcal{O}) \longrightarrow \mathsf{ob}(\mathcal{P})$. Then, let $g \in \mathcal{O}\left(\{a_i\}_{i \in I}; b\right)$ be an operation in $\mathcal{O}$, with $|I| = n$. Picking an ordering of I and writing $I = \{1, \ldots, n\}$, this determines a unique morphism $(\gamma_n, \{g\}) : (\langle n \rangle, (a_1, \ldots, a_n)) \longrightarrow (\langle 1 \rangle, b)$ in $\mathcal{O}^{\otimes}$ where $\gamma_n : \langle n \rangle \longrightarrow \langle 1 \rangle$ is the morphism in $\mathcal{F}\mathrm{in}_*$ given by $\gamma_n(i) = 1$ if $i \in \langle n \rangle^{\circ}$ and $\gamma_n(*) = *$. Then, $f(\gamma_n, g) = (\gamma_n, g')$ for some operation $g' \in \mathcal{P}\left(\{f(a_i)\}_{i \in I}; f(b)\right)$. We define $\psi(f)(g) = g'$. It is lengthy but straightforward to verify that the functoriality of f over $\mathcal{F}\mathrm{in}_*$ and its preservation of coCartesian lifts of inert morphisms, $\psi(f)$ is a morphism of operads. **q.e.d.**

Remark 2.1.9. Note that the above proof involved a choice of ordering for each finite set I. If we made different set of such choices, we would obtain a different map ψ' but one can show that this map is equivalent in the following sense. For each choice of $f \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$, there exist isomorphisms

$\mathcal{O} \xrightarrow{r} \mathcal{O}$ and $\mathcal{P} \xrightarrow{s} \mathcal{P}$ making the following diagram commute in Op':

$$\begin{tikzcd}
\mathcal{O} \arrow[r, "\psi(f)"] \arrow[d, "r"'] & \mathcal{P} \arrow[d, "s"] \\
\mathcal{O} \arrow[r, "\psi'(f)"'] & \mathcal{P}
\end{tikzcd}$$

$\diamond$

Without going into more details, we claim that the above construction yields:

Theorem 2.1.10. *There is an equivalence of categories* $\mathrm{Op} \cong \mathrm{Op}'$.

justifying the following (re)definition:

Definition 2.1.11. We will call the category Op the category of operads; its objects, operads and its morphisms, morphisms of operads. $\diamond$

Remark 2.1.12. Consider the lax slice bicategory $\mathrm{id}_{\mathcal{C}\mathrm{at}} \downarrow \mathcal{F}\mathrm{in}_*$ of 1–categories over $\mathcal{F}\mathrm{in}_*$ [JY21, 4.1.2, p. 148]. The objects of this bicategory are 1–functors $\mathsf{A} \longrightarrow \mathcal{F}\mathrm{in}_*$ while 1–morphisms are 1–functors $f : \mathsf{A} \longrightarrow \mathsf{B}$ along with a natural transformation $\theta_f : p \Longrightarrow qf$, where $p : \mathsf{A} \longrightarrow \mathcal{F}\mathrm{in}_*$ and $q : \mathsf{B} \longrightarrow \mathcal{F}\mathrm{in}_*$ are objects. 2–morphisms are between 1–morphisms $(f, \theta_f), (g, \theta_g) : p \longrightarrow q$ is a natural transformation $\alpha : f \Longrightarrow g$ such that the following equality of diagrams holds in the bicategory $\mathcal{C}\mathrm{at}$:

$$\begin{tikzcd}
\mathsf{A} \arrow[rr, bend left, "g"] \arrow[rr, bend right, "f"'] \arrow[rr, phantom, "\alpha\Uparrow"] \arrow[dr, "p"'] & & \mathsf{B} \arrow[dl, "q"] \\
& \mathcal{F}\mathrm{in}_* \arrow[u, phantom, "\theta_f \Nearrow"] &
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{A} \arrow[rr, "g"] \arrow[dr, "p"'] & & \mathsf{B} \arrow[dl, "q"] \\
& \mathcal{F}\mathrm{in}_* \arrow[u, phantom, "\theta_g \Nearrow"] &
\end{tikzcd}$$

Now, consider the "sub–bicategory" of $\mathrm{id}_{\mathcal{C}\mathrm{at}} \downarrow \mathcal{F}\mathrm{in}_*$ consisting of all objects, all morphisms and only the 2–isomorphisms. We will denote this also as $\mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}$ when convenient.

Suppose, now, we have (f, θ_f) as above and suppose $\overline{\alpha} : X \longrightarrow Y$ is p–coCartesian lift of an inert morphism $\alpha : \langle m \rangle \longrightarrow \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$. Consider the morphisms $\overline{\alpha}' := f(\overline{\alpha})$ $\alpha' := qf(\overline{\alpha})$. If α' is inert and $\overline{\alpha}'$ is coCartesian, we say that the 1–morphism (f, θ_f) perserves inerts. Then, we may consider the bicategory whose objects are objects of $\mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}$ equivalent in $\mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}$ to some operad (which is an object of $\mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}$) via an equivalence that preserves inerts, 1–morphisms are 1–morphisms in $\mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}$ which preserve coCartesian lifts of inert morphisms and 2–morphisms are all the ones between the 1–morphisms. Again, we will denote this "sub-bicategory" simply as Op, when convenient. $\diamond$

Next, we will describe the notion of a family of operads which we will show that the Grothendieck construction of Op–valued functors result in. Hence, we recall the definition of a $\mathcal{C}$–family of

∞–operads [Lur17, p. 2.3.2.10] and then interpret the definition for 1–categories and coloured Set–valued 1–operads.

Definition 2.1.13 ($\mathcal{C}$–Family of ∞–Operads)**.** A map $p : \mathcal{O}^{\otimes} \longrightarrow \mathcal{C} \times N(\mathcal{F}\mathrm{in}_*)$ of ∞–categories is called a $\mathcal{C}$–family of ∞–operads if it satisfies the following properties:

(i) **Categorical Fibration:** The map p is a categorical fibration.

(ii) **Fibres are ∞–Operads:** For all $C \in \mathcal{C}_0$, the fibre $\mathcal{O}^{\otimes}_C = \mathcal{O}^{\otimes} \times_{\{C\}\times N(\mathcal{F}\mathrm{in}_*)} (\mathcal{C} \times N(\mathcal{F}\mathrm{in}_*))$ with the projection $p_C : \mathcal{O}^{\otimes}_C \longrightarrow \mathcal{C} \times N(\mathcal{F}\mathrm{in}_*) \longrightarrow N(\mathcal{F}\mathrm{in}_*)$ is an ∞–operad.

(iii) **Inert Lifting:** For each inert map $\alpha : \langle m \rangle \longrightarrow \langle n \rangle$, each object $C \in \mathcal{C}_0$ and each object X in the fibre $\mathcal{O}^{\otimes}_{C,\langle m \rangle} = \mathcal{O}^{\otimes} \times_{\{(C,\langle m \rangle)\}} (\mathcal{C} \times N(\mathcal{F}\mathrm{in}_*))$, there exists a p–coCartesian (not just p_C–coCartesian) lift $\overline{\alpha} : X \longrightarrow Y$ of (id_C, α).

(iv) **Formation of Tuple Objects:** The diagram $q : \Delta^0 * \langle n \rangle^\circ = \Delta^0 * \left(\coprod_{i=1}^n \Delta^0\right) \longrightarrow \mathcal{O}^{\otimes}$ formed by the p–coCartesian lifts $f_i : X \longrightarrow X_i$ of the Segal maps $\rho^i : \langle m \rangle \longrightarrow \langle 1 \rangle$ as shown below:

$$\begin{tikzcd}
 & & X \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\
X_1 & X_2 & \cdots & X_{n-1} & X_n
\end{tikzcd}$$

is a p–limit diagram.

$\diamond$

Remark 2.1.14. Point (ii) above is not a corollary of point (i) for we require the coCartesian factorization property to hold for all maps $X \longrightarrow Z$ in $\mathcal{O}^{\otimes}$ and not just $\mathcal{O}^{\otimes}_C$. $\diamond$

Definition 2.1.15 (C–Family of 1–Operads)**.** To interpret the above definition for coloured Set–valued 1–operads, we need only understand point (iii) above in the strict setting for the other points have direct specializations. Recall that the diagram q above being a p–limit [Lur22, 7.1.5.1, 7.1.4.1] means that the map of slices $p_{/q|_{\langle n \rangle^\circ}} : \mathcal{O}^{\otimes}_{/q|_{\langle n \rangle^\circ}} \longrightarrow (\mathcal{C} \times N(\mathcal{F}\mathrm{in}_*))_{p \circ q|_{\langle n \rangle^\circ}}$ given by p induces, for all other objects f over $q|_{\langle n \rangle^\circ}$, a homotopy equivalence:

$$\mathcal{O}^{\otimes}_{/q|_{\langle n \rangle^\circ}}(f, q) \simeq (\mathcal{C} \times N(\mathcal{F}\mathrm{in}_*))_{/pq|_{\langle n \rangle^\circ}}(pf, pq)$$

In the case of operads (coloured 1–operads in Set), we will simply demand the above to be a bijection. $\diamond$

Corollary 2.1.16. *Given a C–family $p : \mathcal{O}^{\otimes} \longrightarrow \mathsf{C} \times \mathcal{F}\mathrm{in}_*$ of 1–operads, the nerve $N(p) : N(\mathcal{O}^{\otimes}) \longrightarrow N(\mathsf{C} \times \mathcal{F}\mathrm{in}_*) \cong N(\mathsf{C}) \times N(\mathcal{F}\mathrm{in}_*)$ an $N(\mathsf{C})$–family of ∞–operads.*

Remark 2.1.17. One can try to develop the entire theory we will describe in this section in terms of coloured 1–operads in an arbitrary symmetric monoidal category and an enriched version of coCartesian fibrations and the Grothendieck construction as given in [BW19], but we avoid this here since examining Set–operads is sufficient to gain intuition about the ∞–operadic case. $\diamond$

With this foundation, we can now examine how the Grothendieck construction interacts with functors valued in Op, which we do in the next two sections.

2.2 From Operad Families to Op–Valued Functors

Let C be a 1–category and suppose we have C–family of 1–operads $p : \mathcal{O}^{\otimes} \longrightarrow \mathsf{C} \times \mathcal{F}\mathrm{in}_*$ such that $p_1 = \pi_1 p : \mathcal{O}^{\otimes} \longrightarrow \mathsf{C}$ is a coCartesian fibration or Grothendieck opfibration, where $\pi_1 : \mathsf{C} \times \mathcal{F}\mathrm{in}_* \longrightarrow \mathsf{C}$ is the projection onto the first factor. Let $F : \mathsf{C} \longrightarrow \mathcal{C}\mathrm{at}$ be the pseudofunctor [JY21, Defn. 4.1.2, p. 148] obtained by the inverse Grothendieck construction of p_1 in the coCartesian case as defined by dualizing the construction in [Joh02, Thm. 1.3.5, p. 267] – here $\mathcal{C}\mathrm{at}$ is taken as the bicategory of small categories, functors and natrual transformations. We will show that under a suitable condition which we will impose later, this requires F to factor (up to natural isomorphism of pseudofunctors) through the map $\iota : \mathrm{Op} \xhookrightarrow{\iota'} \mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*} \xrightarrow{\mu} \mathcal{C}\mathrm{at}$, where ι' is the inclusion that sends a category of operators to its underlying map to $\mathcal{F}\mathrm{in}_*$ and μ is the forgetful functor that sends a map into $\mathcal{F}\mathrm{in}_*$ to its domain – see Remark 2.1.12 for the bicategorical formulation of $\mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}$ and Op. We will note, however, that while the bicategorical details of the constructions we will describe can be made precise, we will avoid full detail as our goal is to gain intuition for the case of ∞–operads.

As a first step, recall from the definition of the inverse Grothendieck construction in the coCartesian case that Fc is defined to be the fibre $p_1^{-1}(c) = \mathcal{O}_c^{\otimes}$ above c in $\mathcal{O}^{\otimes}$. Let $p_2 = \pi_2 p$, where π_2 is the second projection $\mathsf{C} \times \mathcal{F}\mathrm{in}_* \longrightarrow \mathcal{F}\mathrm{in}_*$. By definition of a C–family, the map $p_2|_{\mathcal{O}_c^{\otimes}} : Fc = \mathcal{O}_c^{\otimes} \longrightarrow \mathcal{F}\mathrm{in}_*$ is an object of Op. Hence, the object function of F factors through ι. For convenience, from now on we will write $p_2|_{\mathcal{O}_c^{\otimes}}$ as $p_{2,c}$.

For showing that the morphism function of F factors through Op, consider a morphism $f : c \longrightarrow d$ in C. We wish to show that $Ff : Fc = \mathcal{O}_c^{\otimes} \longrightarrow Fd = \mathcal{O}_d^{\otimes}$ is a map of 1–operads. To clarify, we have 1–operads $p_{2,x} : \mathcal{O}_x^{\otimes} = Fx \longrightarrow \mathcal{F}\mathrm{in}_*$ for $x \in \{c, d\}$ for which we wish to show that there exists a natural isomorphism $\eta_f : p_{2,c} \Longrightarrow p_{2,d} \circ Ff$, and given a $p_{2,c}$–coCartesian lift $\overline{\alpha} : X \longrightarrow Y$ in $\mathcal{O}_c^{\otimes}$ of an inert morphism $\alpha : \langle m \rangle \longrightarrow \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$, we have that $Ff(\overline{\alpha})$ is a $p_{2,d}$–coCartesian lift in $\mathcal{O}_d^{\otimes}$ of α or a conjugate thereof by isomorphisms in $\mathcal{F}\mathrm{in}_*$. For this, we recall that the inverse Grothendieck construction of p_1 sends the morphism f in C to a functor, unique up to natural equivalence, given by fibre transport along f in $\mathcal{O}^{\otimes}$. For objects of $Fc = \mathcal{O}_c^{\otimes}$, this means the following. Given an object X in $\mathcal{O}^{\otimes}$, we pick a p_1–coCartesian lift $\overline{f}_X : X \longrightarrow X'$ in $\mathcal{O}^{\otimes}$ and take $Ff(X) = X'$ which is well-defined up to canonical isomorphism for coCartesian lifts are unique up to canonical isomorphism. We then wish to produce an isomorphism $\eta(f)_X : p_{2,c}(X) = \langle m \rangle \xrightarrow{\cong} p_{2,d}(X') = \langle m' \rangle$ in $\mathcal{F}\mathrm{in}_*$ – of course, this isomorphism, if it exists, necessitates $m = m'$.

Consider the image of the map $\overline{f}_X$ under p:

$$\begin{tikzcd}
X \arrow[r, "\overline{f}_X"] \arrow[d, mapsto, "p"] & X' \arrow[d, mapsto, "p"] \\
(c, \langle m \rangle) \arrow[r, "(f, p_2(\overline{f}_X))"] & (d, \langle m' \rangle)
\end{tikzcd}$$

It is not immediate that $p_2\left(\overline{f}_X\right)$ is an isomorphism but we will now impose the condition that it is – that is, we assume that each p_1–coCartesian morphism is sent to an isomorphism in $\mathcal{F}\mathrm{in}_*$ by p_2. With this, we may take $\eta(f)_X = p_2\left(\overline{f}_X\right)$. We wish to show that $\eta(f)_X$ is natural in X but before this we note an interesting implication of our condition.

The assumption we made on $p_2\left(\overline{f}_X\right)$ guarantees that $p\left(\overline{f}_X\right) = \left(f, p_2\left(\overline{f}_X\right)\right)$ is π_1–coCartesian as the π_1–coCartesian morphisms in $\mathsf{C} \times \mathcal{F}\mathrm{in}_*$ are precisely the morphisms of the form (f', γ') where f' is any morphism of C and γ' is an isomorphism in $\mathcal{F}\mathrm{in}_*$. That is, our condition requires p to be a map of coCartesian fibrations $p_1 \longrightarrow \pi_1$. If, on the other hand, we required p to be a map of coCartesian fibrations to begin with, the same observation about the π_1–coCartesian morphisms would imply that p_2 sends p_1–coCartesian morphisms to isomorphisms. Furthermore, in proving these last two claims we used nothing special about $\mathcal{O}^{\otimes}$ or $\mathcal{F}\mathrm{in}_*$ so that we have, in passing, proved the following general result:

Theorem 2.2.1. *Let $p : \mathsf{E} \longrightarrow \mathsf{C} \times \mathsf{D}$ be a functor of 1–categories and denote $p_W = \pi_W p$, where π_W is the projection onto the $W \in \{\mathsf{C}, \mathsf{D}\}$ factor of $\mathsf{C} \times \mathsf{D}$. Assume, in addition, that p_C is a coCartesian fibration. Then, p is a map of coCartesian fibrations $p_\mathsf{C} \longrightarrow \pi_\mathsf{C}$ if and only if p_D sends p_C–coCartesian morphisms to isomorphisms in D.*

For showing the naturality of $\eta(f)_X$ in X, we recall that F acts on morphisms of C also by fibre transport along f in $\mathcal{O}^{\otimes}$. What this means is that for a morphism $\overline{\alpha} : X \longrightarrow Y$ in $\mathcal{O}^{\otimes}_c$, we choose p_1–coCartesian lifts $\overline{f}_X : X \longrightarrow X'$ and $\overline{f}_Y : Y \longrightarrow Y'$ of $f : c \longrightarrow d$ starting at X and Y, respectively, and consider the following diagram:

$$\begin{tikzcd}
& Y \arrow[rr, "\overline{f}_Y"] \arrow[dd, mapsto, "p_1"] & & Y' \arrow[dd, mapsto, "p_1"] \\
X \arrow[ur, "\overline{\alpha}"] \arrow[rr, "\overline{f}_X"] \arrow[dd, mapsto, "p_1"] & & X' \arrow[ur, dashed, "\overline{\alpha}'"] \arrow[dd, mapsto, "p_1"] & \\
& c \arrow[rr, "f"] & & d \\
c \arrow[ur, equal] \arrow[rr, "f"] & & d \arrow[ur, equal] &
\end{tikzcd}$$

where a unique $\overline{\alpha}'$ exists making the top square commute since $\overline{f}_X$ is p_1–coCartesian and the bottom square commutes. Then, $Ff(\overline{\alpha})$ is defined to be $\overline{\alpha}'$ which is well defined up to canonical isomorphism since the choice involving the two coCartesian lifts of f is. Let

$$\alpha = p_{2,c}\left(\overline{\alpha}\right) : p_{2,c}(X) = \langle m\rangle \longrightarrow p_{2,c}(Y) = \langle n\rangle$$

and

$$\alpha' = p_{2,d}\left(\overline{\alpha}'\right) : p_{2,d}(X') = \left\langle m'\right\rangle \longrightarrow p_{2,d}(Y') = \left\langle n'\right\rangle$$

where, as before, $p_2\left(\overline{f}_X\right)$ and $p_2\left(\overline{f}_Y\right)$ are isomorphisms, necessitating $m' = m$ and $n' = n$. Then, if we take the image of the top square under p_2, we obtain the following commuting diagram in $\mathcal{F}\mathrm{in}_*$:

$$\begin{tikzcd}
p_{2,c}(Y) = \langle n\rangle \arrow[r, "\eta(f)_Y = p_{2,c}\left(\overline{f}_Y\right)"] & p_{2,d}(Ff(Y)) = \langle n\rangle \\
p_{2,c}(X) = \langle m\rangle \arrow[u, "p_{2,c}(\overline{\alpha}) = \alpha"] \arrow[r, "\eta(f)_X = p_2\left(\overline{f}_X\right)"'] & p_{2,d}(Ff(X)) = \langle m\rangle \arrow[u, "p_{2,d}(Ff(\overline{\alpha})) = \alpha'"']
\end{tikzcd}$$

showing that the $\eta(f)_X$ are the components of a natural isomorphism $p_{2,c} \Longrightarrow Ff \circ p_{2,d}$.

It remains to show that if α as above is, in addition, an inert morphism in $\mathcal{F}\mathrm{in}_*$ and $\overline{\alpha}$ is $p_{2,c}$–coCartesian, then α' must be inert as well and $\overline{\alpha}'$ must be $p_{2,d}$–coCartesian – that is Ff takes inert morphisms to inert morphisms. Since $\eta(f)_X$ and $\eta(f)_Y$ are isomorphisms, $\alpha' = \eta(f)_Y \circ \alpha \circ \eta(f)_X^{-1}$ must be inert, since α is. We need only show that $\overline{\alpha}'$ is $\pi_{2,d}$–coCartesian. To proceed with this, let $\overline{\beta} : X' \longrightarrow Z'$ be any morphism in $\mathcal{O}_d^{\otimes} = Fd$ with a factorization $\psi \circ \alpha' = \beta$ for $p_{2,d}\left(\overline{\beta}\right) = \beta$ and some $\psi : \langle n\rangle \longrightarrow \langle k\rangle \in \mathcal{F}\mathrm{in}_*$. The previous statement means we have the following diagram, where the vertical arrows are p and the dashed vertical lines indicate pairs of the form $(a, \langle l\rangle) \in \mathsf{C} \times \mathcal{F}\mathrm{in}_*$

– we will explain how g and h are obtained here shortly:

$$
\begin{tikzcd}
& Y \arrow[rr, "\overline{f}_Y"] \arrow[dd, "p"] \arrow[rrr, dashed, bend left, "\exists! g"] & & Y' \arrow[r, dashed, "\exists! h"] \arrow[dd, "p"] & Z' \arrow[dd, "p"] \\
X \arrow[ur, "\overline{\alpha}"] \arrow[rr, "\overline{f}_X"] \arrow[dd, "p"] & & X' \arrow[ur, "\overline{\alpha}'"] \arrow[urr, "\overline{\beta}"] \arrow[dd, "p"] & & \\
& c \arrow[rr, "f"] \arrow[dd, dashed, no head] & & d \arrow[r, equal] \arrow[dd, dashed, no head] & d \arrow[dd, dashed, no head] \\
c \arrow[ur, equal] \arrow[rr, "f"] \arrow[dd, dashed, no head] & & d \arrow[ur, equal] \arrow[urr, equal] \arrow[dd, dashed, no head] & & \\
& \langle n \rangle \arrow[rr, "\eta(f)_Y"] & & \langle n \rangle \arrow[r, "\psi"] & \langle k \rangle \\
\langle m \rangle \arrow[ur, "\alpha"] \arrow[rr, "\eta(f)_X"] & & \langle m \rangle \arrow[ur, "\alpha'"] \arrow[urr, "\beta"] & &
\end{tikzcd}
$$

Using the fact that p is a C–family of 1–operads and α is inert, there exists a p–coCartesian lift φ of (id_c, α) but we notice that every p–coCartesian lift of (id_c, α) is, in particular, a $p_{2,c}$–coCartesian lift of α. Hence, there is a unique automorphism $\omega : Y \longrightarrow Y$ over id_c such that $\omega\varphi = \overline{\alpha}$ and $p_2(\omega) = p_{2,c}(\omega) = \mathrm{id}_{\langle n \rangle}$. It is then easy to show, using the p–coCartesian property of φ and the automorphism ω, that $\overline{\alpha}$ is p–coCartesian. This and all "levels" in the above diagram (excluding g and h) commuting, yields a unique morphism $g : Y \longrightarrow Z'$ satisfying $g\overline{\alpha} = \overline{\beta f}_X$ and $p(g) = (f, \psi\eta(f)_Y)$. By virtue of $\overline{f}_Y$ being p_1–coCartesian and commutativity at all levels of the diagram (now excluding only h), we obtain a unique map $h : Y' \longrightarrow Z'$ satisfying $h\overline{f}_Y = g$ and $p_1(h) = \mathrm{id}_d$. We then observe that

$$p_2\left(h\overline{f}_Y\right) = p_2(g) \iff p_2(h) \circ \eta(f)_Y = \psi \circ \eta(f)_Y \iff p_2(h) = \psi$$

since $\eta(f)_Y$ is an isomorphism. This shows that $\overline{\alpha}'$ is, in fact, $p_{2,d}$–coCartesian. Thus, we have proved the following result:

Theorem 2.2.2. *Let $p : \mathcal{O}^{\otimes} \longrightarrow \mathsf{C} \times \mathcal{F}\mathrm{in}_*$ be C–family of 1–operads and $\pi_W : \mathsf{C} \times \mathcal{F}\mathrm{in}_* \longrightarrow W$ be the projection onto the $W \in \{\mathsf{C}, \mathcal{F}\mathrm{in}_*\}$ factor. If $p_{\mathsf{C}} = \pi_{\mathsf{C}} p$ is a coCartesian fibration and p is a map of coCartesian fibrations, or equivalently if $p_{\mathcal{F}\mathrm{in}_*}$ carries p_{C}–coCartesian morphisms to isomorphisms, then, the inverse Grothendieck construction of p_1 factors through $\iota : \mathrm{Op} \hookrightarrow \mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*} \longrightarrow \mathcal{C}\mathrm{at}$, up to equivalence.*

Suppose, now, that we have a pseudofunctor $F : \mathsf{C} \longrightarrow \mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}$. Recalling that $\mu : \mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*} \longrightarrow \mathcal{C}\mathrm{at}$ is the map that sends an object $\mathsf{D} \longrightarrow \mathcal{F}\mathrm{in}_*$ in $\mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}$ to D, the Grothendieck construction of μF consists of the following basic data [Joh02, Defn. 1.3.1, p. 264]:

(i) objects are pairs (c, X) for objects $c \in \mathsf{C}$ and $X \in \mu F c$

(ii) morphisms $(c, X) \longrightarrow (d, Y)$ are pairs $(f, \overline{f})$, where $f : c \longrightarrow d$ is morphism in C and $\overline{f} : \mu F f(X) \longrightarrow Y$ is a morphism in $\mu F d$

Given morphisms $(c, X) \xrightarrow{(f,\overline{f})} (d, Y) \xrightarrow{(g,\overline{g})} (e, Z)$, the composite is $(g \circ f, \overline{g} \cdot \overline{f})$ where $\overline{g} \cdot \overline{f}$ is defined as the following composite in Fe:

$$\mu F(g \circ f)(X) \xrightarrow{\quad\cong\quad} (\mu F g \circ \mu F f)(X) \xrightarrow{\mu F g(\overline{f})} \mu F(g)(Y) \xrightarrow{\quad\overline{g}\quad} Z$$

with the first morphism in this sequence being the natural isomorphism coming from the pseudofunctoriality of F. Of course, $\int \mu F$ is equipped with a map $p_1 : \int \mu F \longrightarrow \mathsf{C}$, given by projection onto the first factor for both objects and morphisms, which is a coCartesian fibration. It is easy to see that each fibre $\left(\int \mu F\right)_c = p^{-1}(c)$ – that is, the pullback of p_1 along the inclusion $\{c\} \hookrightarrow \mathsf{C}$ – of $\int \mu F$ is isomorphic to $\mu F c$ and is hence provided with a map $p_2 : \left(\int \mu F\right)_c \cong \mu F c \longrightarrow \mathcal{F}\mathrm{in}_*$ which is precisely the map $Fc : \mu F c \longrightarrow \mathcal{F}\mathrm{in}_*$. We then observe that there is a map $\int \mu F \longrightarrow \mathcal{F}\mathrm{in}_*$ which restricts to Fc on the fibre above c for each object $c \in \mathsf{C}$. The object map of this functor is, of course, defined fibrewise:

$$p_2(c, X) := Fc(X) \in \mathcal{F}\mathrm{in}_*$$

Let $(f, \overline{f}) : (c, X) \longrightarrow (d, Y)$. We need to define a map $p_2(f, \overline{f}) : Fc(X) \longrightarrow Fd(Y) \in \mathcal{F}\mathrm{in}_*$. We have morphism $\overline{f} : \mu F f(X) \longrightarrow Y$ which allows us to define:

$$p_2(f, \overline{f}) := Fd(\overline{f}) : Fd(\mu F f(X)) \longrightarrow Fd(Y)$$

As written, it might not be clear that $p_2(f, \overline{f})$ has the right domain and hence, we notice that we have a natural isomorphism $Fd \circ \mu F f = Fd \circ Ff \cong Fc$ due to the fact that $Ff : Fc \longrightarrow Fd$ is a map of categories $\mu F c \longrightarrow \mu F d$ over $\mathcal{F}\mathrm{in}_*$ up to equivalence and $\mu F f$ is this underlying map. This means that we have the equality $Fc(X) = Fd(\mu F f(X))$ in $\mathcal{F}\mathrm{in}_*$ since $\mathcal{F}\mathrm{in}_*$ is skeletal and hence dom $p_2(f, \overline{f})$ is, in fact, $Fc(X)$, as required. That p_2 is functorial is straightforward to check and we omit the details of this verification.

Of course, the above construction yields a map $\int \mu F \xrightarrow{p} \mathsf{C} \times \mathcal{F}\mathrm{in}_*$ such that $\pi_i p = p_i, i \in \{1, 2\}$. It is easy to check that for any morphism $f : c \longrightarrow d$ in C, the morphism $\left(f, \mathrm{id}_{Ff(X)}\right) : (c, X) \longrightarrow (d, Ff(X))$ in $\int \mu F$ is a coCartesian lift of f. Then, we have that $p_2(f, \mathrm{id}_{Ff(X)}) = Fd(\mathrm{id}_{Ff(X)}) = \mathrm{id}_{Fd(Ff(X))} = \mathrm{id}_{Fc(X)}$ is an isomorphism. Since this is true for all morphisms in C, by Theorem 2.2.1, p is a map of coCartesian fibrations. Then, by Theorem 2.2.2, we have that F must factor through the forgetful functor $\iota : \mathrm{Op} \xhookrightarrow{\iota'} \mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*} \xrightarrow{\mu} \mathcal{C}\mathrm{at}$, up to equivalence. We record this observation as the following result:

Theorem 2.2.3. *Any pseudofunctor* $\mathsf{C} \longrightarrow \mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*} \longrightarrow \mathcal{C}\mathrm{at}$ *that yields a* C*–family of* 1*–operads under the Grothendieck construction must factor through* $\iota : \mathrm{Op} \longrightarrow \mathcal{C}\mathrm{at}$*, up to equivalence.*

In the next subsection, we will explore the converse to this statement.

2.3 Grothendieck Construction for Operads

Consider a pseudofunctor $F : \mathsf{C} \longrightarrow \mathrm{Op}$ for some 1–category C. Let $\iota : \mathrm{Op} \hookrightarrow \mathcal{C}\mathrm{at}_{\mathcal{F}\mathrm{in}_*} \longrightarrow \mathcal{C}\mathrm{at}$ be the forgetful functor that sends an object $p : \mathcal{O}^{\otimes} \longrightarrow \mathcal{F}\mathrm{in}_*$ of Op to $\mathcal{O}^{\otimes}$, as before. Consider, then, the Grothendieck construction $\mathsf{E} = \int(\iota F) \longrightarrow \mathsf{C}$ of $\iota \circ F$. Recall from the previous subsection that E is naturally equipped with a map to $\mathsf{C} \times \mathcal{F}\mathrm{in}_*$. We will now show that this map is a C–family of operads.

Corollary 2.3.1. *Let* $p : \mathsf{E} \longrightarrow \mathsf{C} \times \mathcal{F}\mathrm{in}_*$ *be as above. Then,* p *is an isofibration.*

Proof. The first projection projection $\pi_1 : \mathsf{C} \times \mathcal{F}\mathrm{in}_* \longrightarrow \mathsf{C}$ is a categorical fibration and $p_1 = \pi_1 p : \mathsf{E} \longrightarrow \mathsf{C}$ is a categorical fibration since it, arising from a Grothendieck construction, is a coCartesian fibration. It is easy to check – or recall from the Joyal model structure – that categorical fibrations satisfy 2-out-of-3 and hence p is a categorical fibration. **q.e.d.**

Corollary 2.3.2. *Let* $p : \mathsf{E} \longrightarrow \mathsf{C} \times \mathcal{F}\mathrm{in}_*$ *be as above. Then, each fibre* $\mathsf{E}_c = \mathsf{E} \times_{\{c\} \times \mathcal{F}\mathrm{in}_*} (\mathsf{C} \times \mathcal{F}\mathrm{in}_*)$ *equipped with the projection* $p_c : \mathsf{E}_c \longrightarrow \mathsf{C} \times \mathcal{F}\mathrm{in}_* \longrightarrow \mathcal{F}\mathrm{in}_*$ *is an operad isomorphic to* Fc*.*

Proof. This is immediate from the facts that $\mathsf{E} = \int \iota F$ and F has codomain Op. **q.e.d.**

Lemma 2.3.3. *Let* $\alpha : \langle m \rangle \longrightarrow \langle n \rangle$ *be an inert map in* $\mathcal{F}\mathrm{in}_*$*. Then, for each* $c \in \mathsf{ob}(\mathsf{C})$*,* $x = (x_1, \ldots, x_m) \in \mathsf{ob}(\mathsf{E}_c)$*,* (id_c, α) *admits a* p*–coCartesian lift (not merely* p_c*–coCartesian, where* p_c *is as in the above corollary) to a map in* $\mathsf{E}_c \subset \mathsf{E}$*.*

Proof. Since E_c is an operad, we have a p_c–coCartesian lift $\overline{\alpha} : X \longrightarrow Y$ in $\mathsf{E}_c \subset \mathsf{E}$ of α – of course, here, X is the tuple $(c, \langle m \rangle, x)$. We will show that this $\overline{\alpha}$ is, in fact, p–coCartesian.

Suppose we a have morphism $\rho : X \longrightarrow W$ in $\mathcal{O}^{\otimes}$ such that $p(\rho) = (f, \beta) : p(X) = (c, \langle m \rangle) \longrightarrow p(W) = (d, \langle k \rangle)$ and there exists $(g, \psi) : (c, \langle n \rangle) \longrightarrow (d, \langle m \rangle)$ satisfying $(g, \psi) \circ p(\overline{\alpha}) = p(\rho)$. We wish to produce a lift $\overline{\psi} : Y \longrightarrow W$ with $p\left(\overline{\psi}\right) = (g, \psi)$ and $\overline{\psi} \circ \overline{\alpha} = \rho$. In other words, we wish to show the existence of a $\overline{\psi}$ making the following diagram commute:

$$\begin{tikzcd}
& Y \arrow[rr, dashed, "\overline{\psi}"] \arrow[dd, mapsto, "p"] & & W \arrow[dd, mapsto, "p"] \\
X \arrow[ur, "\overline{\alpha}"] \arrow[urrr, "\rho"] \arrow[dd, mapsto, "p"] & & & \\
& (c, \langle n \rangle) \arrow[rr, "(g,\psi)"] & & (d, \langle k \rangle) \\
(c, \langle m \rangle) \arrow[ur, "(\mathrm{id}_c,\alpha)"] \arrow[urrr, "(f,\beta)"'] & & &
\end{tikzcd} \tag{1}$$

Note that we immediately have $g = f$.

Now, since $\mathsf{E} = \int \iota F$ arises from a Grothendieck construction, it has a canonical cleavage – see, for instance, [Joh02, p. 267]. That is, there is a canonical choice of a p_1–coCartesian lift for each morphism h in C and each object A in E above dom h. Let $\overline{f}_X : X \longrightarrow X'$ and $\overline{f}_Y : Y \longrightarrow Y'$ be these canonical p_1–coCartesian lifts of f starting at X and Y respectively. These give a fibre transport $\overline{\alpha}' : X' \longrightarrow Y'$ of $\overline{\alpha}$ over (id_d, α') for some morphism $\alpha' : \langle m' \rangle \longrightarrow \langle n' \rangle$. Then, since $\overline{f}_X$ is p_1–coCartesian, we have a morphism $\overline{\zeta} : X' \longrightarrow W$ with $p_1\left(\overline{\zeta}\right) = \mathrm{id}_d$ and $\overline{\zeta} \circ \overline{f}_X = \rho$. Let ζ be the second coordinate of $p\left(\overline{\zeta}\right)$. Let $\gamma_R : \langle r \rangle \longrightarrow \langle r' \rangle$ be the second coordinate of $p\left(\overline{f}_R\right)$ for $(R, r) \in \{(X, m), (Y, n)\}$ for convenience. Combining these, we have the following diagram commuting at all levels:

$$\begin{tikzcd}
& Y \arrow[rrr, "\overline{f}_Y"] \arrow[ddd, mapsto, "p"] & & & Y' \arrow[ddd, mapsto, "p"] \\
& & W \arrow[ddd, mapsto, "p"] & & \\
X \arrow[ur, "\overline{\alpha}"] \arrow[urr, "\rho"] \arrow[rrr, "\overline{f}_X"] \arrow[ddd, mapsto, "p"] & & & X' \arrow[ul, "\overline{\zeta}"] \arrow[uur, "\overline{\alpha}'"] \arrow[ddd, mapsto, "p"] & \\
& (c, \langle n \rangle) \arrow[rrr, "(f,\gamma_Y)"] \arrow[dr, "(f,\psi)"] & & & (d, \langle n' \rangle) \\
& & (d, \langle k \rangle) & & \\
(c, \langle m \rangle) \arrow[uur, "(\mathrm{id}_c,\alpha)"] \arrow[urr, "(f,\beta)"] \arrow[rrr, "(f,\gamma_X)"] & & & (d, \langle m' \rangle) \arrow[ul, "(\mathrm{id}_d,\zeta)"] \arrow[uur, "(\mathrm{id}_d,\alpha')"] &
\end{tikzcd} \tag{2}$$

Recall from definition of the canonical cleavage associated to a Grothendieck construction that fibre transport along f corresponds precisely to the functor $\iota Ff : \iota Fc \longrightarrow \iota Fd$ – that is, X', Y' and $\overline{\alpha}'$ are necessarily $Ff(X) = \iota Ff(X)$, $Ff = \iota Ff(Y)$ and $Ff\left(\overline{\alpha}\right) = \iota Ff\left(\overline{\alpha}\right)$ respectively, noting that ιFf is the functor underlying Ff. In particular, since Ff is a map of operads, $\overline{\alpha}'$ must be a p_d–coCartesian of an inert morphism – that is, α' must be inert and $\overline{\alpha}'$ must be p_d–coCartesian.

Since F is a pseudofunctor into $\mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}$, the map Ff is a map $\iota Fc \longrightarrow \iota Fd$ in $\mathcal{C}\mathrm{at}$ with a natural isomorphism $\eta : p_c \Longrightarrow p_d \circ \iota Ff$. By the definition of the canonical cleavage again, the component $\eta_X : p_c(X) = \langle m \rangle \longrightarrow p_d(\iota Ff(X)) = p_d(X') = \langle m' \rangle$ is precisely the map γ_X. This implies that γ_X is an isomorphism and $m' = m$. By the same argument, γ_Y is also an isomorphisms and $n' = n$. This gives us a map

$$\left(\mathrm{id}_d, \psi \circ \gamma_Y^{-1}\right) : (d, \langle n \rangle) = \left(d, \langle n' \rangle\right) \longrightarrow (d, \langle k \rangle)$$

That is, the diagram (2) commutes at all levels with $\left(\mathrm{id}_d, \psi \circ \gamma_Y^{-1}\right)$ included. By the p_d–coCartesian property of $\overline{\alpha}'$, we have a unique lift $\overline{\xi} : Y' \longrightarrow W$ of $\left(\mathrm{id}_d, \psi \circ \gamma_Y^{-1}\right)$ making diagram (2) still commute when added in. Then, taking $\overline{\psi} = \overline{\xi} \circ \overline{f}_Y : Y \longrightarrow W$ makes (2) commute. In particular, this $\overline{\psi}$ makes (1) commute. **q.e.d.**

Remark 2.3.4. In the above argument, we could directly construct $\overline{\alpha}$ as follows. We first recall that the Grothendieck construction of the pseudofunctor $\iota F : \mathsf{C} \longrightarrow \mathrm{Op} \longrightarrow \mathcal{C}\mathrm{at}$ consists of the following data:

(i) objects are $(c, \langle n \rangle, x)$ with $c \in \mathsf{ob}(\mathsf{C})$, $n \in \mathbb{N}$, $x = (x_1, \ldots, x_n) \in \mathsf{ob}\left(\mathcal{O}^{\otimes}_{\langle 1 \rangle}\right)^n$

(ii) a morphism $f : (a, \langle m \rangle, x) \longrightarrow (b, \langle n \rangle, y)$ consists of:

- a morphism $f_{\mathsf{C}} : a \longrightarrow b$ in C
- a morphism $f_{\mathcal{F}\mathrm{in}_*} : \langle m \rangle \longrightarrow \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$
- a morphism $f_b : \iota F f_{\mathsf{C}}(x) \longrightarrow y$ in $\iota F b$ over $f_{\mathcal{F}\mathrm{in}_*}$, which in turn consists of an operation $f_j : \{F f_{\mathsf{C}}(x_k)\}_{k \in f^{-1}_{\mathcal{F}\mathrm{in}_*}(j)} \longrightarrow y_j$, for each $j \in \langle n \rangle^{\circ}$, in the operad Fb.

When we write $\iota F f_{\mathsf{C}}(x)$, what we really mean is $\iota F f_{\mathsf{C}}(\langle n \rangle, x)$ and similarly, by $F f_{\mathsf{C}}(x_k)$, we mean $F f_{\mathsf{C}}(\langle 1 \rangle, x_k) = \iota F_{\mathsf{C}}(\langle 1 \rangle, x_k)$. Visibly, E comes equipped with a map to $\mathsf{C} \times \mathcal{F}\mathrm{in}_*$. This means that the object X in the proof of the previous theorem is a tuple $(c, \langle m \rangle, (x_1, \ldots, x_m))$ for colours $x_1, \ldots, x_m$ in Fc. We take Y to be the tuple $\left(c, \langle n \rangle, \left(x_{\alpha^{-1}(1)}, \ldots, x_{\alpha^{-1}(n)}\right)\right)$, using that α being inert requires α^{-1} to be an injective function $\langle n \rangle \longrightarrow \langle m \rangle$. We then take $\overline{\alpha}$ to be the map:

$$\left(\mathrm{id}_c, \alpha, \left\{\mathrm{id}_{x_{\alpha^{-1}(j)}}\right\}_{j \in \langle n \rangle^{\circ}}\right) : (c, \langle m \rangle, (x_1, \ldots, x_m)) \longrightarrow \left(c, \langle n \rangle, \left(x_{\alpha^{-1}(1)}, \ldots, x_{\alpha^{-1}(n)}\right)\right)$$

in E and check by hand that this is p–coCartesian. $\diamond$

We have shown that $p : \mathsf{E} \longrightarrow \mathsf{C} \times \mathcal{F}\mathrm{in}_*$ satisfies two out of the three properties of a C–family of operads. For the third property, we first observe that it is satisfied within each fibre $\mathsf{E}_c, c \in \mathsf{ob}(\mathsf{C})$.

Lemma 2.3.5. *Let $r : \mathcal{P}^{\otimes} \longrightarrow \mathcal{F}\mathrm{in}_*$ be an operad. Let $Y \in \mathsf{ob}(\mathcal{P}^{\otimes})$ with $r(Y) = \langle n \rangle$ with r–coCartesian lifts $f_i : Y \longrightarrow Y_i$ of the Segal maps $\rho^i : \langle n \rangle \longrightarrow \langle 1 \rangle$. For every $X \in \mathsf{ob}(\mathcal{P}^{\otimes})$ equipped with maps $g_i : X \longrightarrow Y_i$ in $\mathcal{P}^{\otimes}$ and with $r(X) = \langle m \rangle$, the map*

$$\mathcal{P}^{\otimes}_{/\{Y_i\}_{i=1}^n}(X, Y) \longrightarrow \mathcal{F}\mathrm{in}_{*/\{r(Y_i)\}}(r(X), r(Y)) = \mathcal{F}\mathrm{in}_{*/\{\langle 1 \rangle\}^n}(\langle m \rangle, \langle n \rangle)$$

induced by r is an equivalence (that is, a bijection).

Proof. Let $\alpha \in \mathcal{F}\mathrm{in}_{*/\{\langle 1 \rangle\}}(\langle m \rangle, \langle n \rangle)$ so that $\rho^i \alpha = p(g_i)$. Denote by $(\mathcal{P}^{\otimes})^{\alpha}(X, Y)$ the subset of $\mathcal{P}^{\otimes}(X, Y)$ mapping to α under p. $(\mathcal{P}^{\otimes})^{\rho^i \alpha}(X, Y)$ is defined similarly. Recall from the second point in the definition of an ∞–operad [Lur17, 2.1.1.10(2), p. 174], that the map

$$(\mathcal{P}^{\otimes})^{\alpha}(X, Y) \xrightarrow{\phi} \prod_{i=1}^{n} (\mathcal{P}^{\otimes})^{\rho^i \alpha}(X, Y_i)$$

induced by post-composition with the f_i in $\mathcal{P}^{\otimes}$ is a homotopy equivalence – that is, a bijection in our case, since we are dealing with discrete morphism spaces (or simplicial sets). This means precisely that the maps $g : X \longrightarrow Y$ with $p(g) = \alpha$ are in bijective correspondence with the collections of their composites $\{f_i g\}_{i \in \langle n \rangle^{\circ}}$. However, since the object X comes with a choice of maps $\{g_i\}_{i \in \langle n \rangle^{\circ}}$ satisfying $p(g_i) = \rho^i \alpha$, we have a unique map $g = \phi^{-1}\left(\{g_i\}_{i \in \langle n \rangle^{\circ}}\right) : X \longrightarrow Y$ with $p(g) = \alpha$ and $f_i g = g_i$ for each $i \in \langle n \rangle^{\circ}$. That is, we have a unique $g \in \mathcal{P}^{\otimes}_{/\{Y_i\}}(X, Y)$ with $p(g) = \alpha \in \mathcal{F}\mathrm{in}_{*/\{\langle 1 \rangle\}^n}(\langle m \rangle, \langle n \rangle)$ for each such α. **q.e.d.**

Remark 2.3.6. We could show the above directly by observing that in a category of operators $\mathcal{P}^{\otimes}$, objects X, Y are of the forms $(\langle m \rangle, (x_1, \ldots, x_m)), (\langle n \rangle, (y_1, \ldots, y_n))$ respectively, the objects Y_i can be taken to be $(\langle 1 \rangle, y_i)$ and the morphisms $f_i : Y \longrightarrow Y_i$ can be taken to be $(\rho^i, \{\mathrm{id}_{y_i}\})$. Instead, we proved the result in a way more similar to the ∞–categorical setting. $\diamond$

Remark 2.3.7. Observe that if an α as in the above proof exists, it is necessarily unique and hence the g with $p(g) = \alpha$ is also the unique element of $\mathcal{P}^{\otimes}_{\{Y_i\}}(X, Y)$. In the case of an ∞–operad, the relative limit condition of Definition 2.1.13(iv) is actually a contractibility condition. $\diamond$

Lemma 2.3.8. *For any d in C and an object Y in $\mathsf{E}_d \subset \mathsf{E}$ over $(d, \langle n \rangle)$ in $\mathsf{C} \times \mathcal{F}\mathrm{in}_*$, the diagram $q : \Delta^0 * \langle n \rangle^{\circ} \longrightarrow \mathsf{E}$ formed by the p–coCartesian lifts $f_i : Y \longrightarrow Y_i$ within E_d of the Segal maps $\rho^i : \langle n \rangle \longrightarrow \langle 1 \rangle$ as shown below:*

$$\begin{tikzcd}
 & & & Y \arrow[llld, "f_1"'] \arrow[lld, "f_2"] \arrow[rd, "f_{n-1}"] \arrow[rrd, "f_n"] & & \\
Y_1 & Y_2 & & \cdots & Y_{n-1} & Y_n
\end{tikzcd}$$

is a p–limit diagram.

Proof. For convenience, we denote $q^{\circ} = q|_{\langle n \rangle^{\circ}}$ and, as before, we take $p_j = \pi_j p$ where π_i is the j–th projection of $\mathsf{C} \times \mathcal{F}\mathrm{in}_*$. Let X be any object in $\mathsf{E}_{/q^{\circ}}$ – that is, X is an object in E equipped with maps $g_i : X \longrightarrow Y_i$ for each $i \in \langle n \rangle^{\circ}$. Let $p(X) = (c, \langle m \rangle)$ and $p(g_i) = (p_1(g_i), p_2(g_i)) : (c, \langle m \rangle) \longrightarrow (d, \langle 1 \rangle)$. We would like to show that p induces a bijection:

$$\mathsf{E}_{/q^{\circ}}(X, Y) \xrightarrow{\phi} (\mathsf{C} \times \mathcal{F}\mathrm{in}_*)_{/pq^{\circ}}((c, \langle m \rangle), (d, \langle n \rangle))$$

Let $(r, \alpha) : (c, \langle m \rangle) \longrightarrow (d, \langle n \rangle)$ be an element of the codomain of ϕ so that $\rho^i \alpha = p_2(f_i)\alpha = p_2(g_i)$ and $r = \mathrm{id}_d r = p_1(f_i) f = p_1(g_i)$. Then, we have the canonical p_1–coCartesian lift $\overline{r} : X \longrightarrow X'$ of r from the canonical cleavage associated to $p_1 : \mathsf{E} \longrightarrow \mathsf{C}$ arising from a Grothendieck construction. This gives unique maps $s_i : X' \longrightarrow Y_i$ with $p_1(s_i) = \mathrm{id}_d$, $s_i \overline{r} = g_i$. Since $\overline{r}$ is p_1–coCartesian and p is a morphism of coCartesian fibrations, $p(\overline{r})$ is π_1–coCartesian and hence $p_2(\overline{r})$ is an isomorphism. Hence, we have:

$$p_2(s_i) p_2(\overline{r}) = p_2(g_i) \implies p_2(s_i) = p_2(g_i) p_2(\overline{r})^{-1} = \rho^i \left(\alpha p_2(\overline{r})^{-1}\right) = p_2(f_i) \left(\alpha p_2(\overline{r})^{-1}\right)$$

That is, we have an element $\beta = \alpha p_2\left(\overline{r}\right)^{-1} \in (\{d\} \times \mathcal{F}\mathrm{in}_*)_{/\{p(Y_i)\}_{i=1}^n}((d, \langle m \rangle), (d, \langle n \rangle))$. Now, using the fact that E_d is an operad, by the previous lemma, there exists a unique map $s : X' \longrightarrow Y$ with $f_i s = s_i$ for each $i \in \langle n \rangle^\circ$ and $p(s) = (\mathrm{id}_d, \beta)$. Then, we have that

$$p_2\left(s\overline{r}\right) = p_2(s) p_2\left(\overline{r}\right) = \alpha p_2\left(\overline{r}\right) p_2\left(\overline{r}\right)^{-1} = \alpha$$

and hence $p\left(s\overline{r}\right) = (r, \alpha)$. That is $\phi\left(s\overline{r}\right) = (r, \alpha)$. Since (r, α) were arbitrary, every element of codom ϕ has a preimage.

We would like that this preimage is unique. Hence, let $u : X \longrightarrow Y$ be any other map in $\mathsf{E}_{/q^\circ}(X, Y)$ with $\phi\left(u\right) = (r, \alpha)$. By the p_1–coCartesian property of $\overline{r}$, there exists a unique map $v : X' \longrightarrow Y$ with $p_1(v) = \mathrm{id}_d$ and $v\overline{r} = u$. This means $p_2(v) = p_2(u) p_2\left(\overline{r}\right)^{-1} = \alpha p_2\left(\overline{r}\right)^{-1} = \beta$. This necessitates $v = s$ by the uniqueness of s. Thus, $u = v\overline{r} = s\overline{r}$. **q.e.d.**

Remark 2.3.9. We could also show the above theorem by direct verification using the concrete description of a Grothendieck construction but the argument we gave is more easily generalized to the ∞–categorical setting. $\diamond$

Remark 2.3.10. Notice that there is a more general categorical statement not involving operads that we have implicitly proved in the previous lemma: let $p : \mathsf{E} \longrightarrow \mathsf{C} \times \mathsf{D}$ be a map of 1–categories such that $p_1 = \pi_1 p : \mathsf{E} \longrightarrow \mathsf{C}$ is a coCartesian fibration and p is a map of coCartesian fibrations. For some object $d \in \mathsf{ob}(\mathsf{C})$, let Y be an object equipped with p–coCartesian morphisms $f_i : Y \longrightarrow Y_i$ for $i = 1, \ldots, n$, with $p(f_i) = (\mathrm{id}_d, p_2(f_i))$, forming a diagram $q : \Delta^0 * \langle n \rangle^\circ \longrightarrow \mathsf{E}_d$ that is a p_d–limit, where $p_d = p|_{\mathsf{E}_d}$. Then, q is also a p–limit. This will the be the viewpoint for the proof in the ∞–categorical setting. $\diamond$

Remark 2.3.11. Note that the map (r, α) in the above proof must be the unique element of codom ϕ if it exists so that ∞–categorical version of the above lemma is again a contractibility condition. $\diamond$

We have thus shown the following result:

Theorem 2.3.12. *For any pseudofunctor $F : \mathsf{C} \longrightarrow \mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}$ that factors through the inclusion $\mathrm{Op} \xrightarrow{\iota'} \mathcal{C}\mathrm{at}_{\infty/\mathcal{F}\mathrm{in}_*}$, the map $\int \iota F \longrightarrow \mathsf{C} \times \mathcal{F}\mathrm{in}_*$ is a classical C–family of 1–operads.*

We can now collect the results of this and the previous subsection into the following theorem:

Theorem 2.3.13. *Given a pseudofunctor $F : \mathsf{C} \longrightarrow \mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}$, the Grothendieck construction $\int \mu F$, where $\mu : \mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*} \longrightarrow \mathcal{C}\mathrm{at}$ is the the forgetful functor $(\mathsf{A} \xrightarrow{f} \mathcal{F}\mathrm{in}_*) \longmapsto \mathsf{A}$, equipped with the map to $\mathsf{C} \times \mathcal{F}\mathrm{in}_*$ is a C–family of 1–operads if and only if F factors through* Op.

We conclude this section by noting that instead of post-composing functors $\mathsf{C} \longrightarrow \mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}$ by μ and then taking the Grothendieck construction, we could consider how the Grothendieck construction functor interacts with slices. We will see that this is one way to make the ideas of this section precise in the context of ∞–operads. Nevertheless, post-composing with μ first in this section made the computations direct and gave a concrete sense of how the underlying machinery should work in the ∞–categorical setting where such computations are difficult.

3 ∞–Categorical Preliminaries

Here we shall develop the ∞–categorical foundations needed to make sense of unstraightening a functor valued in Op_∞ and showing that the result is a family of ∞–operads. Functors from an ∞–category $\mathcal{K}$ into the slice ∞–category $\mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$ of ∞–categories over $N(\mathcal{F}\mathrm{in}_*)$ admit an unstraightening construction obtained by taking a sliced verstion of the original unstraightening functor $\mathrm{USt} : \mathcal{F}\mathrm{un}(\mathcal{K}, \mathcal{C}\mathrm{at}_\infty) \longrightarrow \mathrm{CoCart}(\mathcal{K})$, which we do in §3.1. We will show that this construction gives an equivalence $\mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}\right) \simeq \mathrm{CoCart}\left(\mathcal{K}\right)_{/\pi}$, where $\pi : \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*) \longrightarrow \mathcal{K}$ is the projection onto the first factor – a coCartesian fibration. For a Kan complex $\mathcal{K}$ this equivalence extends to $\left(\mathcal{C}\mathrm{at}_\infty\right)_{/\pi}$ since any map into a Kan complex from an ∞–category can be shown to be equivalent to a coCartesian fibration. In §3.2, we will examine how this sliced version of the straightening construction acts on objects and morphisms in $\mathcal{K}$. Some special properties of Kan complexes will be essential for dealing with fibre transports in a family of ∞–operads which is essential to proving that Op_∞–valued functors correspond to families of ∞–operads via the above unstraightening construction – we describe these properties of Kan complexes in §3.3. We will need some lemmas involving pullback squares and cubes with certain faces forming pullback squares and we develop these in §3.4. In §3.5, we develop the main results that allow us to deduce that Op_∞–valued functors "unstraighten" to ∞–operad families and ∞–operad families that can be "straightened" give Op_∞–valued functors under the process.

3.1 Slice ∞–Categories

Before describing the passage from functors $\mathcal{K} \longrightarrow \mathrm{Op}_\infty$ to $\mathcal{K}$–families $\mathcal{P}^\otimes \longrightarrow \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*)$, we will recall some basic results on slice ∞–categories since Op_∞ and $\mathcal{K}$–families are both contained in slices of $\mathcal{C}\mathrm{at}_\infty$. That is, we will show the relation between $\mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}\right)$ and $\mathcal{C}\mathrm{at}_{\infty/(\mathcal{K}\times N(\mathcal{F}\mathrm{in}_*))}$. We would like to apply the unstraightening functor to $\mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}\right)$ to pass to a subcategory of $\mathcal{C}\mathrm{at}_{\infty/(\mathcal{K}\times N(\mathcal{F}\mathrm{in}_*))}$ but this does not make sense as is so a first step is to pass to a slice over $\mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_\infty\right)$ and then apply unstraightening to this slice which is well-defined. To this end, we first recall the definition of a slice [Lan21, 1.4.13, p. 86] and then prove a statement giving an alternative definition of slices over a single object since these are the slices we are interested in.

Definition 3.1.1 (Slice ∞–Category)**.** For any map of simplicial sets $p : I \longrightarrow \mathcal{A}$, $\mathcal{A}_{/p}$ is defined as follows:

$$\begin{aligned}\left(\mathcal{A}_{/p}\right)_n &= \mathrm{Hom}_{\mathsf{sSet}_{I/}}\left(\begin{array}{cc} I & I \\ \big\downarrow{\scriptstyle i} & \big\downarrow{\scriptstyle p} \\ \Delta^n * I & \mathcal{A}\end{array}\right) \\ &= \{\text{maps } f : \Delta^n * I \longrightarrow \mathcal{A} \text{ such that } f|_I = f \circ i = p\}\end{aligned}$$

with face and degeneracy maps induced by the maps $\Delta^n * I \longrightarrow \Delta^m * I$ that are in turn induced by the face and deneracy maps $\Delta^n \longrightarrow \Delta^m$. $\mathcal{A}_{/p}$ is called the slice of $\mathcal{A}$ over p. ◇

Remark 3.1.2. By the fact that joins preserve colimits, Hom functors takes colimits in the domain argument to limits and every simplicial set is a colimit of representables, we have, for all $J \in \mathsf{sSet}$:

$$\begin{aligned}\mathrm{Hom}_{\mathsf{sSet}}\left(J, \mathcal{A}_{/p}\right) &= \mathrm{Hom}_{\mathsf{sSet}_{I/}}\left(\begin{array}{ccc} I & & I \\ \big\downarrow{\scriptstyle i} & , & \big\downarrow{\scriptstyle p} \\ J * I & & \mathcal{A}\end{array}\right) \\ &= \{\text{maps } f : J * I \longrightarrow \mathcal{A} \text{ such that } f|_I = f \circ i = p\}\end{aligned}$$

This is how slices are defined in [Lur08, §1.2.9]. ◇

Lemma 3.1.3. *For an object $a \in \mathcal{A}_0$ of an ∞–category $\mathcal{A}$, the slice $\mathcal{A}_{/a}$ is the following pullback in* sSet*:*

$$\begin{tikzcd} \mathcal{A}_{/a} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A}^{\Delta^1} \arrow[d, "\mathrm{ev}_1"] \\ \Delta^0 \arrow[r, "a"'] & \mathcal{A} \end{tikzcd}$$

where ev_1 *is the composite*

$$\mathcal{A}^{\Delta^1} \xrightarrow{\simeq} \mathcal{A}^{\Delta^1} \times \Delta^{\{1\}} \xrightarrow{\mathrm{id} \times 1} \mathcal{A}^{\Delta^1} \times \Delta^1 \xrightarrow{\mathrm{ev}} \mathcal{A}$$

Proof. Take $I = \Delta^0$ and $p = a$ in the definition of a slice $\mathcal{A}_{/p}$. Unwrapping the definition of ev_1, we observe that it sends a simplex $\Delta^1 \times \Delta^n \xrightarrow{f} \mathcal{A}$ in $\mathcal{A}_n^{\Delta^1}$ to the simplex $f|_{\Delta^{\{1\}} \times \Delta^n} : \Delta^n \cong \Delta^{\{1\}} \times \Delta^n \longrightarrow \mathcal{A}$ in $\mathcal{A}_n \cong \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, \mathcal{A})$. Then, $\mathrm{ev}_1 f = a$ if and only if f factors as $\Delta^1 \times \Delta^n \xrightarrow{g} \Delta^n * \Delta^0 \xrightarrow{h} \mathcal{A}$ where g is given by collapsing $\Delta^{\{1\}} \times \Delta^n$ to a point and h is a map in $\left(\mathcal{A}_{/a}\right)_n$. **q.e.d.**

The next result relates the functor ∞–category $\mathcal{F}\mathrm{un}(\mathcal{A}, \mathcal{B}_{/b})$ to a slice of $\mathcal{F}\mathrm{un}(\mathcal{A}, \mathcal{B})$ giving the desired passage from $\mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}\right)$ to a slice of $\mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_\infty\right)$. We will use the following notation when showing the relation.

Notation 3.1.4. Let A and B be simplicial sets. Let $b : \Delta^0 \longrightarrow B$ be an object of B. Then, we denote by $\mathrm{const}_b : A \longrightarrow B$ the composite $A \xrightarrow{!} \Delta^0 \xrightarrow{b} B$. ◇

Remark 3.1.5. For all $s \in A_n$, $\mathrm{const}_b(s) = b^n$ where b^n is the degenerate n–simplex of B with all vertices taken as b. ◇

Lemma 3.1.6. *Consider the functor ∞–category $\mathcal{F}\mathrm{un}(\mathcal{A}, \mathcal{B}_{/b})$ for arbitrary ∞–categories $\mathcal{A}$ and $\mathcal{B}$ with an object $b : \Delta^0 \longrightarrow \mathcal{B}$ of $\mathcal{B}$. Then,*

$$\mathcal{F}\mathrm{un}\left(\mathcal{A}, \mathcal{B}_{/b}\right) \simeq \mathcal{F}\mathrm{un}\left(\mathcal{A}, \mathcal{B}\right)_{/\mathrm{const}_b}$$

Proof. We have the following pullback diagram defining the slice $\mathcal{B}_{/b}$:

$$\begin{tikzcd} \mathcal{B}_{/b} \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{B}^{\Delta^1} \arrow[d, "\mathrm{ev}_1"] \\ \Delta^0 \arrow[r, "b"'] & \mathcal{B} \end{tikzcd}$$

We have that $\mathcal{F}\mathrm{un}(\mathcal{A}, -) = (-)^{\mathcal{A}} : \mathsf{sSet} \longrightarrow \mathsf{sSet}$ is right adjoint to $- \times \mathcal{A} : \mathsf{sSet} \longrightarrow \mathsf{sSet}$ so that it preserves limits and hence we have that the following is a pullback diagram in sSet:

$$\begin{tikzcd} \mathcal{F}\mathrm{un}\left(\mathcal{A}, \mathcal{B}_{/b}\right) \arrow[r, "f_*"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{F}\mathrm{un}\left(\mathcal{A}, \mathcal{B}^{\Delta^1}\right) \arrow[d, "(\mathrm{ev}_1)_*"] \\ \mathcal{F}\mathrm{un}\left(\mathcal{A}, \Delta^0\right) \arrow[r, "b_*"'] & \mathcal{F}\mathrm{un}\left(\mathcal{A}, \mathcal{B}\right) \end{tikzcd}$$

We then observe that, by the $- \times X \dashv (-)^X$ adjunction:

$$\mathcal{F}\mathrm{un}\left(\mathcal{A}, \mathcal{B}^{\Delta^1}\right) \simeq \mathcal{F}\mathrm{un}\left(\mathcal{A} \times \Delta^1, \mathcal{B}\right) \simeq \mathcal{F}\mathrm{un}\left(\Delta^1 \times \mathcal{A}, \mathcal{B}\right) \simeq \mathcal{F}\mathrm{un}\left(\Delta^1, \mathcal{B}^{\mathcal{A}}\right) \simeq \mathcal{F}\mathrm{un}\left(\mathcal{A}, \mathcal{B}\right)^{\Delta^1}$$

and by the same chain of equivalences, $(\mathrm{ev}_1)_* = \mathrm{ev}_1 : \mathcal{F}\mathrm{un}(\mathcal{A}, \mathcal{B})^{\Delta^1} \longrightarrow \mathcal{F}\mathrm{un}(\mathcal{A}, \mathcal{B})$. We then observe that for the unique map $!_n : \mathcal{A} \times \Delta^n \longrightarrow \Delta^0 \in \mathcal{F}\mathrm{un}\left(\mathcal{A}, \Delta^0\right)_n$, we have that $b_*(!_n)$ is the composite $\mathcal{A} \times \Delta^n \xrightarrow{!_n} \Delta^0 \xrightarrow{b} \mathcal{B}$ which is the degenerate simplex with all vertices equal to const_b. Then, under the equivalence $\Delta^0 \simeq \mathcal{F}\mathrm{un}\left(\mathcal{A}, \Delta^0\right)$, we have that the bottom row of the above diagram becomes $\Delta^0 \xrightarrow{\mathrm{const}_b} \mathcal{F}\mathrm{un}\left(\mathcal{A}, \mathcal{B}\right)$. Thus, by the previous lemma, we have the desired result. **q.e.d.**

Corollary 3.1.7. *For all ∞–categories $\mathcal{K}$ and $\mathcal{C}$, we have an equivalence*

$$\mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_{\infty/\mathcal{C}}\right) \simeq \mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_\infty\right)_{/\mathrm{const}_{\mathcal{C}}}$$

We now make sense of applying unstraightening to the right side of the above equivalence to obtain an equivalence to a slice of $\mathrm{CoCart}\left(\mathcal{K}\right)$, the ∞–category of coCartesian fibrations over $\mathcal{K}$.

Lemma 3.1.8. *For any ∞–categories $\mathcal{A}$ and $\mathcal{B}$, some simplicial set I, and a diagram $p : I \longrightarrow \mathcal{A}$, each functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ gives a functor $F_{/p} : \mathcal{A}_{/p} \longrightarrow \mathcal{B}_{/F\circ p}$.*

Proof. Let $f : \Delta^n * I \longrightarrow \mathcal{A} \in \left(\mathcal{A}_{/p}\right)_n$ such that $fi = p$ where $i : I \hookrightarrow \Delta^n * I$ is the inclusion. Let $F_{/p}f = F \circ f : \Delta^n * I \longrightarrow \mathcal{B}$ and observe that $F_{/p}f \circ i = F \circ f \circ i = F \circ p$. Hence, $F_{/p}$ is a map $\left(\mathcal{A}_{/p}\right)_n \longrightarrow \left(\mathcal{B}_{/F\circ p}\right)_n$ for all n. That it commutes with face and degeneracy maps follows from the fact that all maps involved in its definition do so. **q.e.d.**

Given the above setting, we also have the following result from [Lur08, §2.4.5] whose proof we omit:

Theorem 3.1.9. *Let $F, \mathcal{A}, \mathcal{B}, p$ be as in the previous lemma. Then, if F is an equivalence, so is $F_{/p}$.*

Corollary 3.1.10. *For any ∞–categories $\mathcal{K}$ and $\mathcal{C}$, if $\pi : \mathcal{K} \times \mathcal{C} \longrightarrow \mathcal{K}$ is the projection onto the first factor, we have an equivalence*

$$\mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_\infty\right)_{/\mathrm{const}_\mathcal{C}} \simeq \mathrm{CoCart}\left(\mathcal{K}\right)_{/\pi}$$

Proof. Let $\mathrm{USt} : \mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_\infty\right) \longrightarrow \mathrm{CoCart}\left(\mathcal{K}\right)$ be the unstraigtening functor, which is an equivalence. This gives an equivalence of slices by the above theorem. Then, it suffices to show that $\mathrm{USt}\left(\mathrm{const}_\mathcal{C}\right) = \pi$. Observe that $\mathrm{const}_\mathcal{C}$ factors as $\mathcal{K} \longrightarrow \Delta^0 \xrightarrow{\mathcal{C}} \mathcal{C}\mathrm{at}_\infty$. Recall, then, that the unstraightening of $\Delta^0 \xrightarrow{\mathcal{C}} \mathcal{C}\mathrm{at}_\infty$ arises as a pullback of a universal coCartesian fibration $U\mathcal{C}\mathrm{at}_\infty \longrightarrow \mathcal{C}\mathrm{at}_\infty$ along $\Delta^0 \xrightarrow{\mathcal{C}} \mathcal{C}\mathrm{at}_\infty$ [Lan21, 3.3.14, p. 200] which is the unique map $\mathcal{C} \longrightarrow \Delta^0$:

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & U\mathcal{C}\mathrm{at}_\infty \arrow[d] \\
\Delta^0 \arrow[r, "\mathcal{C}"'] & \mathcal{C}\mathrm{at}_\infty
\end{tikzcd}$$

The pullback of this map along the unique map $\mathcal{K} \longrightarrow \Delta^0$ is the first projection $\pi : \mathcal{K} \times \mathcal{C} \longrightarrow \mathcal{K}$:

$$\begin{tikzcd}
\mathcal{K} \times \mathcal{C} \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[r] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & U\mathcal{C}\mathrm{at}_\infty \arrow[d] \\
\mathcal{K} \arrow[r, "!"'] \arrow[rr, bend right, "\mathrm{const}_\mathcal{C}"'] & \Delta^0 \arrow[r, "\mathcal{C}"'] & \mathcal{C}\mathrm{at}_\infty
\end{tikzcd}$$

Since pullbacks of pullbacks are again pullbacks, we have that π is the unstraigtening of $\mathrm{const}_\mathcal{C}$. **q.e.d.**

$\mathrm{CoCart}(\mathcal{K})$ is the subcategory of $\mathcal{C}\mathrm{at}_{\infty/\mathcal{K}}$ generated by the coCartesian fibrations over $\mathcal{K}$ so that the inclusion $\mathrm{CoCart}(\mathcal{K}) \hookrightarrow \mathcal{C}\mathrm{at}_{\infty/\mathcal{K}}$ induces an inclusion $\mathrm{CoCart}(\mathcal{K})_{/\pi} \hookrightarrow \left(\mathcal{C}\mathrm{at}_{\infty/\mathcal{K}}\right)_{/\pi}$. The following two results [Lan21, Exercise 142, p. 292] will then allow us to relate $\left(\mathcal{C}\mathrm{at}_{\infty/\mathcal{K}}\right)_{/\pi}$ with $\mathcal{C}\mathrm{at}_{\infty/(\mathcal{K}\times\mathcal{C})}$.

Lemma 3.1.11. *Let $p : I \longrightarrow \mathcal{A}$ be a diagram in an ∞–category $\mathcal{A}$ and $q : J \longrightarrow \mathcal{A}_{/p}$ another diagram in the slice over p for simplicial sets I, J. Let $\overline{q} : J * I \longrightarrow \mathcal{A}$ be the map corresponding to q. Then, there exists an isomorphism of simplicial sets:*

$$\left(\mathcal{A}_{/p}\right)_{/q} \cong \mathcal{A}_{/\overline{q}}$$

Proof. By unwrapping the definition of slices, we have that

$$\left(\mathcal{A}_{/\overline{q}}\right)_n = \left\{ \mathsf{sSet}\text{ diagrams } \begin{tikzcd} J * I \arrow[d, hook] \arrow[dr, "\overline{q}"] & \\ \Delta^n * J * I \arrow[r] & A \end{tikzcd} \right\}$$

Building on an element of the above set, we obtain the following diagram where the face opposite I commutes by assumption, that opposite A commutes because it is a chain of inclusions, and that opposite $\Delta^n * J * I$ commutes by definition of $\overline{q}$ so that the whole diagram commutes:

$$\begin{tikzcd} J * I \arrow[d, hook] \arrow[dr, dashed, "\overline{q}"] & I \arrow[l, hook'] \arrow[dl, hook'] \arrow[d, "p"] \\ \Delta^n * J * I \arrow[r] & A \end{tikzcd}$$

However, this means that the bottom map is precisely a map $\Delta^n * J \longrightarrow A$ which restricts to q on J. Thus, the n–simplices of $\mathcal{A}_{/\overline{q}}$ are precisely the n–simplices of $\left(\mathcal{A}_{/p}\right)_{/q}$. The face and degeneracy maps for both slices are given by the face and deneracy maps $\Delta^n \longrightarrow \Delta^m$ under the functor $- * J * I$. Hence, the identification of simplices given before commutes with the face and degeneracy maps. **q.e.d.**

Theorem 3.1.12. *For any ∞–category $\mathcal{A}$ with objects $a, b \in \mathcal{A}_0$ and a morphism $b \xrightarrow{f} a$ in $\mathcal{A}$ whose corresponding object in $\mathcal{A}_{/a}$ is f', we have an equivalence:*

$$\left(\mathcal{A}_{/a}\right)_{/f'} \simeq \mathcal{A}_{/b}$$

Proof. Take $p = a : \Delta^0 \longrightarrow \mathcal{A}$, $q = f' : \Delta^0 \longrightarrow \mathcal{A}_{/a}$ and $\overline{q} = f : \Delta^1 \cong \Delta^0 * \Delta^0 \longrightarrow \mathcal{A}$ in the above theorem to obtain

$$\left(\mathcal{A}_{/a}\right)_{/f'} \cong \mathcal{A}_{/f}$$

Given a simplex $\Delta^n * \Delta^1 \xrightarrow{s} \mathcal{A}$ in $\mathcal{A}_{/f}$, we have a simplex $\Delta^n * \Delta^0 \longrightarrow \Delta^n * \Delta^1 \xrightarrow{s} \mathcal{A}$ induced by the inclusion $j : \Delta^0 \hookrightarrow \Delta^1$ sending the unique vertex to the 0 vertex of Δ^1 which gives a map $\mathcal{A}_{/f} \longrightarrow \mathcal{A}_{/b}$. Since the map j is, in fact, a horn inclusion $\Lambda^1_0 \hookrightarrow \Delta^1$, it has the left lifting property with respect to all left fibrations and is hence left anodyne [Lur22, 4.2.4.10]. Then, the map j^{op} is a right fibration and is still an inclusion $\left(\Delta^0\right)^{\text{op}} \hookrightarrow \left(\Delta^1\right)^{\text{op}}$. Thus, by [Lur08, 4.1.1.3, p. 186], j^{op} is cofinal and by [Lur08, 4.1.1.8, p. 188], the induced map $\mathcal{A}^{\text{op}}_{/f} \longrightarrow \mathcal{A}^{\text{op}}_{/b}$ is an equivalence, giving an equivalence $\mathcal{A}_{/f} \simeq \mathcal{A}_{/b}$. Thus, we have

$$\left(\mathcal{A}_{/a}\right)_{/f'} \cong \mathcal{A}_{/f} \simeq \mathcal{A}_{/b}$$

q.e.d.

Corollary 3.1.13. *For ∞–categories $\mathcal{K}$ and $\mathcal{C}$ with $\pi : \mathcal{K} \times \mathcal{C} \longrightarrow \mathcal{K}$ the projection to the first* *factor, we have*

$$\left(\mathcal{C}\mathrm{at}_{\infty/\mathcal{K}}\right)_{/\pi} \simeq \mathcal{C}\mathrm{at}_{\infty/(\mathcal{K}\times\mathcal{C})}$$

Proof. Take $\mathcal{A} = \mathcal{C}\mathrm{at}_\infty, a = \mathcal{K}, b = \mathcal{K}\times\mathcal{C}$ and $f' = \pi$ in the previous theorem. **q.e.d.**

We may now apply the above results in sequence to the case of $\mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}\right)$ to obtain the main result of this subsection which was originally stated in [httb]:

Theorem 3.1.14. *For each ∞–category $\mathcal{K}$ and $\mathcal{C}$, there exists a sequence of maps:*

$$\begin{aligned}
&\mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_{\infty/\mathcal{C}}\right)\\
\xrightarrow{\simeq}\,&\mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_\infty\right)_{/\mathrm{const}_\mathcal{C}}\\
\xrightarrow{\simeq}\,&\mathrm{CoCart}\left(\mathcal{K}\right)_{/\pi}\\
\hookrightarrow\,&\left(\mathcal{C}\mathrm{at}_{\infty/\mathcal{K}}\right)_{/\pi}\\
\xrightarrow{\simeq}\,&\mathcal{C}\mathrm{at}_{\infty/(\mathcal{K}\times\mathcal{C})}
\end{aligned}$$

showing that $\mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_{\infty/\mathcal{C}}\right)$ is equivalent to a subcategory of $\mathcal{C}\mathrm{at}_{\infty/(\mathcal{K}\times\mathcal{C})}$.

Theorem 3.1.15. *When $\mathcal{K}$ is a Kan complex, we have an equivalence:*

$$\mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_{\infty/\mathcal{C}}\right) \simeq \mathcal{C}\mathrm{at}_{\infty/(\mathcal{K}\times\mathcal{C})}$$

Proof. It suffices to show that the inclusion $\mathrm{CoCart}(\mathcal{K}) \hookrightarrow \mathcal{C}\mathrm{at}_{\infty/\mathcal{K}}$ is an equivalence when $\mathcal{K}$ is a Kan complex, since this equivalence will give an equivalence of slices $\mathrm{CoCart}(\mathcal{K})_{/\pi} \hookrightarrow \left(\mathcal{C}\mathrm{at}_{\infty/\mathcal{K}}\right)_\pi$ by Theorem 3.1.9. For any inner fibration $\mathcal{E} \xrightarrow{p} \mathcal{K}$ and any morphism $f' : x' \longrightarrow y'$ in $\mathcal{E}$, f' is an equivalence if and only if it is a p–coCartesian lift of an equivalence [Lan21, 3.1.6, p. 166]. If p is a coCartesian fibration, then it is a categorical fibration since p–coCartesian lifts always exist and for equivalences in $\mathcal{K}$, these lifts are also equivalences by the previous statement.

Now, if p is a categorical fibration and $\mathcal{K}$ is a Kan complex, then all morphisms in $\mathcal{K}$ being equivalences admit lifts to equivalences and these must be p–coCartesian lifts so that p must be a coCartesian fibration. Thus, for a Kan complex $\mathcal{K}$, $\mathrm{CoCart}(\mathcal{K})$ is strictly equal to the ∞–category $\mathcal{C}\mathrm{at}^{\mathrm{fib}}_{\infty/\mathcal{K}}$ of categorical fibrations over $\mathcal{K}$. The latter is equivalent to $\mathcal{C}\mathrm{at}_{\infty/\mathcal{K}}$ by fibrant replacement in the Joyal model structure since the categorical fibrations are precisely the fibrant objects in this model structure. **q.e.d.**

Definition 3.1.16 (Sliced Straightening/Unstraightening)**.** We will call the map

$$\mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_{\infty/\mathcal{C}}\right) \simeq \mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_\infty\right)_{/\mathrm{const}_\mathcal{C}} \longrightarrow \mathrm{CoCart}(\mathcal{K})_{/\pi}$$

of Theorem 3.1.14 the sliced unstraightening functor over $\mathcal{C}$ and denote it $\mathrm{USt}_{/\mathcal{C}}$. We will denote the inverse of this by $\mathrm{St}_{/\mathcal{C}}$ and call it the sliced straightening functor over $\mathcal{C}$. ⋄

The results of this subsection establish that unstraightening the functor $\mathcal{K} \longrightarrow \mathcal{C}\mathrm{at}_\infty$ underlying a functor $\mathcal{K} \longrightarrow \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$ results in a functor $\mathcal{E} \longrightarrow \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*)$ which is a map of coCartesian fibrations over $\mathcal{K}$. We are now equipped to consider the unstraightening of functors $\mathcal{K} \longrightarrow \mathrm{Op}_\infty \hookrightarrow \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$ the result of which will be shown to be precisely the $\mathcal{K}$–families of ∞–operads. However, we will require some basic facts about how the straightening functor acts on 1–simplices which we establish in the next subsection.

3.2 Straightening

By the straightening–unstraightening equivalence for slices as stated in Theorem 3.1.14, to understand the unstraightening of a functor $\mathcal{K} \longrightarrow \mathrm{Op}_\infty \hookrightarrow \mathcal{C}\mathrm{at}_{\infty/\mathcal{C}}$ it suffices to study the sliced straightening of a map of coCartesian fibrations $\mathcal{E} \longrightarrow \mathcal{K} \times \mathcal{C}$ over $\mathcal{K}$, where the coCartesian fibration $\pi : \mathcal{K} \times \mathcal{C} \longrightarrow \mathcal{K}$ is the first projection. Specializing to a $\mathcal{K}$–family of ∞–operads $p : \mathcal{O}^\otimes \longrightarrow \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*)$, we may take the sliced straightening of p over $\mathcal{K} \times N(\mathcal{F}\mathrm{in}_*) \xrightarrow{\pi} \mathcal{K}$ when the map $\mathcal{O}^\otimes \xrightarrow{p} \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*) \xrightarrow{\pi} \mathcal{K}$ is a coCartesian fibration. To do so we will need some preliminary results about how the sliced straightening functor acts on morphisms.

We recall how the straightening functor $\mathrm{CoCart}(S) \longrightarrow \mathcal{F}\mathrm{un}(S, \mathcal{C}\mathrm{at}_\infty)$ arises from the staright-ening functor for marked simplicial sets over a given simplicial set S: $\mathrm{St}^+_S : \mathsf{sSet}^+_{/S} \longrightarrow \left(\mathsf{sSet}^+\right)^{\mathfrak{C}[S]}$ where we view S as the marked simplicial set $S^\sharp$ which is S with all its edges marked. For a simplicially enriched category $\mathcal{C}$ and a morphism $\phi : \mathfrak{C}[S] \longrightarrow \mathcal{C}^{\mathrm{op}}$ of simplicially enriched categories, we have a corresponding straightening functor $\mathrm{St}^+_\phi : \mathsf{sSet}^+_{/S} \longrightarrow \left(\mathsf{sSet}^+\right)^{\mathcal{C}}$ [Lur08, §3.2.1]. Taking $\mathcal{C} = \mathfrak{C}[S]^{\mathrm{op}}$ and $\phi = \mathrm{id}_{\mathfrak{C}[S]}$, we obtain the straightening functor $\mathrm{St}^+_S : \mathsf{sSet}^+_{/S} \longrightarrow \left(\mathsf{sSet}^+\right)^{\mathfrak{C}[S]^{\mathrm{op}}}$. Recall, then, that Cartesian fibrations with Cartesian morphisms marked are precisely the fibrant objects in the Cartesian model structure on $\mathsf{sSet}^+_{/S}$ [Lur08, 3.1.4.1, p. 136]. If we have two Cartesian fibrations $E_1 \longrightarrow S$ and $E_2 \longrightarrow S$, and a map of marked simplicial sets $E_1 \longrightarrow E_2$ over S, then this is necessarily a map of Cartesian fibrations. Thus, by fibrant replacement we have an equivalence $\widehat{\mathrm{Cart}}(S) \simeq \mathsf{sSet}^+_{/S}$, where $\widehat{\mathrm{Cart}}(S)$ is the full simplicial subcategory of $\mathsf{sSet}^+_{/S}$ spanned by the Cartesian fibrations. The simplicial nerve of $\widehat{\mathrm{Cart}}(S)$ is precisely $\mathrm{Cart}(S)$ – the ∞–category of Cartesian fibrations over S. Note that $\widehat{\mathrm{Cart}}(S)$ is a fibrant simplicial category – that is, all its morphism simplicial sets are Kan complexes – for the following reason. $\widehat{\mathrm{Cart}}(S)$ consists of the fibrant objects of $\mathsf{sSet}^+_{/S}$ in the Cartesian model structure. However, in this model structure, every object is also cofibrant since the cofibrations are monomorphisms. Hence, we may apply [Lur08, A.3.1.7, p. 671] to obtain a Kan fibration as follows for each objects $A, B \in \widehat{\mathrm{Cart}}(S)$:

$$\widehat{\mathrm{Cart}}(S)(A,B) \longrightarrow \widehat{\mathrm{Cart}}(S)(\varnothing, B) \times_{\widehat{\mathrm{Cart}}(S)(\varnothing,*)} \widehat{\mathrm{Cart}}(S)(A,*) \cong *$$

making $\widehat{\mathrm{Cart}}(S)(A,B)$ a fibrant simplicial set – a Kan complex.

Now, denote by $\left(\left(\mathsf{sSet}^+\right)^{\mathfrak{C}[S]^{\mathrm{op}}}\right)^\circ$ the subcategory of $\left(\mathsf{sSet}^+\right)^{\mathfrak{C}[S]^{\mathrm{op}}}$ obtained by fibrant-cofibrant replacement. $\left(\mathsf{sSet}^+\right)^\circ$ is defined similarly. Then, by [Lur08, 4.2.4.4 , p. 212], we have an equivalence:

$$\mathcal{N}\left(\left(\left(\mathsf{sSet}^+\right)^{\mathfrak{C}[S]^{\mathrm{op}}}\right)^{\circ}\right) \simeq \mathcal{F}\mathrm{un}\left(S^{\mathrm{op}}, \mathcal{N}\left(\left(\mathsf{sSet}^+\right)^{\circ}\right)\right)$$

Then, observe that $\mathsf{sSet}^+ \simeq \mathsf{sSet}^+_{/*}$ which implies $\left(\mathsf{sSet}^+\right)^{\circ} \simeq \left(\mathsf{sSet}^+_{/*}\right)^{\circ}$. This, in turn, implies $\mathcal{N}\left(\left(\mathsf{sSet}^+\right)^{\circ}\right) \simeq \mathcal{N}\left(\left(\mathsf{sSet}^+_{/*}\right)^{\circ}\right) \simeq \mathrm{CoCart}(*)$ and we have $\mathrm{CoCart}(*) \simeq \mathcal{F}\mathrm{un}(*, \mathcal{C}\mathrm{at}_\infty) \simeq \mathcal{C}\mathrm{at}_\infty$, by the straightening-unstraightening equivalence. Putting these equivalences together, we have and equivalence $\mathcal{N}\left(\left(\mathsf{sSet}^+\right)^{\circ}\right) \simeq \mathcal{C}\mathrm{at}_\infty$. Applying $\mathcal{F}\mathrm{un}\left(S^{\mathrm{op}}, -\right)$ to this equivalence, we obtain:

$$\mathcal{N}\left(\left(\left(\mathsf{sSet}^+\right)^{\mathfrak{C}[S]^{\mathrm{op}}}\right)^{\circ}\right) \simeq \mathcal{F}\mathrm{un}\left(S^{\mathrm{op}}, \mathcal{N}\left(\left(\mathsf{sSet}^+\right)^{\circ}\right)\right) \simeq \mathcal{F}\mathrm{un}\left(S^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty\right)$$

Hence, we may view St^+_S as a map $\widehat{\mathrm{Cart}}(S) \longrightarrow \left(\left(\mathsf{sSet}^+\right)^{\mathfrak{C}[S]^{\mathrm{op}}}\right)^{\circ}$ and take its simplicial nerve to obtain the functor $\mathrm{St}_S : \mathrm{Cart}(S) \longrightarrow \mathcal{F}\mathrm{un}\left(S^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty\right)$. The coCartesian straightening functor is defined by the dual construction. This gives us the following two lemmas.

Lemma 3.2.1. *The map*

$$\mathcal{C}\mathrm{at}_\infty \simeq \mathrm{CoCart}\left(\Delta^0\right) \xrightarrow{\mathrm{St}} \mathcal{F}\mathrm{un}\left(\Delta^0, \mathcal{C}\mathrm{at}_\infty\right) \simeq \mathcal{C}\mathrm{at}_\infty$$

is equivalent to the identity on $\mathcal{C}\mathrm{at}_\infty$.

Proof. We will show this in the case of Cartesian fibrations over Δ^0 and functors $(\Delta^0)^{\mathrm{op}} \simeq \Delta^0 \longrightarrow \mathcal{C}\mathrm{at}_\infty$, and the coCartesian case will follow by duality. By [Lur08, 3.2.1.14, p. 149], the map $\mathrm{St}^+_{\Delta^0} : \mathsf{sSet}^+ \simeq \mathsf{sSet}^+_{/\Delta^0} \longrightarrow \left(\mathsf{sSet}^+\right)^{\mathfrak{C}[\Delta^0]^{\mathrm{op}}} \simeq \mathsf{sSet}^+$ is equivalent to the identity. Replacing sSet^+ by $\left(\mathsf{sSet}^+\right)^{\circ}$ and $\mathsf{sSet}^+_{/\Delta^0}$ by $\widehat{\mathrm{Cart}}(\Delta^0)$, and then taking the simplicial nerve we obtain that the map

$$\mathrm{St} = \mathcal{N}\left(\mathrm{St}^+_{\Delta^0}\right) : \mathcal{C}\mathrm{at}_\infty \simeq \mathrm{Cart}(\Delta^0) \longrightarrow \mathcal{F}\mathrm{un}\left(\left(\Delta^0\right)^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty\right) \simeq \mathcal{C}\mathrm{at}_\infty$$

is equivalent to the identity. **q.e.d.**

Lemma 3.2.2. *Straightening is natural in the sense that the following diagram commutes for all functors* $f : \mathcal{A} \longrightarrow \mathcal{B}$*:*

$$\begin{tikzcd}
\mathrm{CoCart}(\mathcal{B}) \arrow[r, "\mathrm{St}"] \arrow[d, "f^*"'] & \mathcal{F}\mathrm{un}(\mathcal{B}, \mathcal{C}\mathrm{at}_\infty) \arrow[d, "f^*"] \\
\mathrm{CoCart}(\mathcal{A}) \arrow[r, "\mathrm{St}"'] & \mathcal{F}\mathrm{un}(\mathcal{A}, \mathcal{C}\mathrm{at}_\infty)
\end{tikzcd}$$

where f^* on the left is taking pullbacks along f and f^* on the right is precomposition.

Proof. We first show the analogous result for the Cartesian case. Taking $S' = \mathcal{A}$, $S = \mathcal{B}$, $\mathcal{C} = \mathfrak{C}[\mathcal{B}]^{\mathrm{op}}$, $\phi = \mathrm{id}_{\mathfrak{C}[\mathcal{B}]} : \mathfrak{C}[S] = \mathfrak{C}[B] \longrightarrow \mathcal{C}^{\mathrm{op}}$, $p = f : \mathcal{A} \longrightarrow \mathcal{B}$, in [Lur08, 3.2.1.4(2), p. 145] to obtain the following up-to-natural-equivalence commuting diagram of simplicial categories:

$$\begin{tikzcd}
\mathsf{sSet}^+_{/\mathcal{A}} \arrow[d, "p_! = f_!"'] \arrow[dr, "\mathrm{St}^+_{\phi\circ\mathfrak{C}[p]} = \mathrm{St}^+_{\mathfrak{C}[f]}"] & \\
\mathsf{sSet}^+_{/\mathcal{B}} \arrow[r, "\mathrm{St}^+_{\phi} = \mathrm{St}^+_{\mathcal{B}}"'] & \left(\mathsf{sSet}^+\right)^{\mathfrak{C}[\mathcal{B}]^{\mathrm{op}}}
\end{tikzcd}$$

where $f_!$ is the post-composition map induced by f. Then, taking $S = \mathcal{A}$, $\phi = \mathrm{id}_{\mathfrak{C}[\mathcal{A}]}$, $\mathcal{C} = \mathfrak{C}[\mathcal{A}]^{\mathrm{op}}$, $\mathcal{C}' = \mathfrak{C}[\mathcal{B}]^{\mathrm{op}}$, $\pi = \mathfrak{C}[f]^{\mathrm{op}} : \mathcal{C} = \mathfrak{C}[\mathcal{A}]^{\mathrm{op}} \longrightarrow \mathfrak{C}[\mathcal{B}]^{\mathrm{op}} = \mathcal{C}'$ in [Lur08, 3.2.1.4(3), p. 145], we get the up-to-natural-equivalence commuting diagram:

$$\begin{tikzcd}
\mathsf{sSet}^+_{/\mathcal{A}} \arrow[r, "\mathrm{St}^+_{\phi} = \mathrm{St}^+_{\mathcal{A}}"] \arrow[dr, "\mathrm{St}^+_{\pi\circ\phi} = \mathrm{St}^+_{\mathfrak{C}[f]}"'] & \left(\mathsf{sSet}^+\right)^{\mathfrak{C}[\mathcal{A}]^{\mathrm{op}}} \arrow[d, "\pi_!"] \\
& \left(\mathsf{sSet}^+\right)^{\mathfrak{C}[\mathcal{B}]^{\mathrm{op}}}
\end{tikzcd}$$

and $\pi_!$ is the left adjoint of the pre-composition map π^* induced by π. Gluing these diagrams along the diagonal, we obtain the commuting square:

$$\begin{tikzcd}
\mathsf{sSet}^+_{/\mathcal{A}} \arrow[r, "\mathrm{St}^+_{\mathcal{A}}"] \arrow[d, "f_!"'] & \left(\mathsf{sSet}^+\right)^{\mathfrak{C}[\mathcal{A}]^{\mathrm{op}}} \arrow[d, "\pi_!"] \\
\mathsf{sSet}^+_{/\mathcal{B}} \arrow[r, "\mathrm{St}^+_{\mathcal{B}}"'] & \left(\mathsf{sSet}^+\right)^{\mathfrak{C}[\mathcal{B}]^{\mathrm{op}}}
\end{tikzcd} \tag{3}$$

Taking the simplicial nerve of this diagram after passing to appropriate subcategories of fibrant–cofibrant objects, we obtain the following commuting diagram of ∞–categories:

$$\begin{tikzcd}
\mathrm{Cart}(\mathcal{A}) \arrow[r, "\mathrm{St}"] \arrow[d, "\mathcal{N}(f_!)"'] & \mathcal{F}\mathrm{un}(\mathcal{A}^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty) \arrow[d, "\mathcal{N}(\pi_!)"] \\
\mathrm{Cart}(\mathcal{B}) \arrow[r, "\mathrm{St}"'] & \mathcal{F}\mathrm{un}(\mathcal{B}^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty)
\end{tikzcd} \tag{4}$$

Using the universal property of pullbacks to produce unit and counit morphisms, it is easy to show that post-composition by f is left adjoint to taking pullbacks along f. We will denote this pullback functor by f^*. We also know that $\mathrm{St}^+_{\mathcal{A}}$ is left adjoint to $\mathrm{USt}^+_{\mathcal{A}}$ and same for the corresponding functors for $\mathcal{B}$. Thus, by replacing all arrows in the square (3) by their right adjoints, we obtain the following diagram (which may or may not commute):

$$\begin{tikzcd} \mathsf{sSet}^+_{/\mathcal{A}} & \left(\mathsf{sSet}^+\right)^{\mathfrak{C}[\mathcal{A}]^{\mathrm{op}}} \arrow[l, "\mathrm{USt}^+_{\mathcal{A}}"'] \\ \mathsf{sSet}^+_{/\mathcal{B}} \arrow[u, "f^*"] & \left(\mathsf{sSet}^+\right)^{\mathfrak{C}[\mathcal{B}]^{\mathrm{op}}} \arrow[l, "\mathrm{USt}^+_{\mathcal{B}}"] \arrow[u, "\pi^*"'] \end{tikzcd} \tag{5}$$

Applying the simplicial nerve after passing to appropriate fibrant–cofibrant replacements, we obtain the following diagram of ∞–categories (again, which we do not know to commute yet):

$$\begin{tikzcd} \mathrm{Cart}(\mathcal{A}) & \mathcal{F}\mathrm{un}(\mathcal{A}^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty) \arrow[l, "\mathrm{USt}"'] \\ \mathrm{Cart}(\mathcal{B}) \arrow[u, "\mathcal{N}(f^*)"] & \mathcal{F}\mathrm{un}(\mathcal{B}^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty) \arrow[l, "\mathrm{USt}"] \arrow[u, "\mathcal{N}(\pi^*)"'] \end{tikzcd} \tag{6}$$

However, the arrows in the above diagram are right adjoint to the arrows in the square (4), by [Lur08, 5.2.4.5, p. 285], and hence, by [Lur08, 5.2.2.6, p. 277], the square (6) must commute. Then, by the straightening–unstraightening equivalence, the following diagram commutes:

$$\begin{tikzcd} \mathrm{Cart}(\mathcal{A}) \arrow[r, "\mathrm{St}"] & \mathcal{F}\mathrm{un}(\mathcal{A}^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty) \\ \mathrm{Cart}(\mathcal{B}) \arrow[u, "\mathcal{N}(f^*)"] \arrow[r, "\mathrm{St}"'] & \mathcal{F}\mathrm{un}(\mathcal{B}^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty) \arrow[u, "\mathcal{N}(\pi^*)"'] \end{tikzcd}$$

The vertical maps can be taken to be f^* and $(f^*)^{\mathrm{op}}$ in the sense of the statement of the lemma. By duality, we obtain the coCartesian case. **q.e.d.**

We then prove a result that will allow us to show that straightening a family of ∞–operads gives a functor into the ∞–category of ∞–operads.

Lemma 3.2.3. *For any coCartesian fibrations of ∞–categories $p : \mathcal{E} \longrightarrow \mathcal{K}$ and $q : \mathcal{E}' \longrightarrow \mathcal{K}$ with a map $f : \mathcal{E} \longrightarrow \mathcal{E}'$ of coCartesian fibrations,* $\mathrm{St}(f)$ *is a map* $\mathcal{K} \times \Delta^1 \longrightarrow \mathcal{C}\mathrm{at}_\infty$ *with* $\mathrm{St}(f)|_{\mathcal{K}\times\Delta^{\{0\}}} = \mathrm{St}(p)$ *and* $\mathrm{St}(f)|_{\mathcal{K}\times\Delta^{\{1\}}} = \mathrm{St}(q)$*, and for each object $a \in \mathcal{K}_0$, we have that* $\mathrm{St}(f)|_{\{a\}\times\Delta^1} \simeq f|_{\mathcal{E}_a} : \mathcal{E}_a \longrightarrow \mathcal{E}'_a$.

Proof. In the previous lemma, take $\mathcal{A} = \Delta^0$, $\mathcal{B} = \mathcal{K}$ and $f = a : \Delta^0 \longrightarrow \mathcal{K}$, and use Lemma 3.2.1 to obtain the following commuting diagram:

$$\begin{tikzcd} \mathrm{CoCart}(\mathcal{K}) \arrow[r, "\mathrm{St}"] \arrow[d, "a^*"'] & \mathcal{F}\mathrm{un}(\mathcal{K}, \mathcal{C}\mathrm{at}_\infty) \arrow[d, "a^*"] \\ \mathrm{CoCart}(\Delta^0) \simeq \mathcal{C}\mathrm{at}_\infty \arrow[r, "\mathrm{St}\simeq\mathrm{id}"'] & \mathcal{C}\mathrm{at}_\infty \simeq \mathcal{F}\mathrm{un}(\Delta^0, \mathcal{C}\mathrm{at}_\infty) \end{tikzcd}$$

Consider f as a map $f : \Delta^1 \longrightarrow \mathrm{CoCart}(\mathcal{K})$. Then, $a^* f : \Delta^1 \longrightarrow \mathrm{CoCart}(\Delta^0)$ is precisely the map

$f|_{\mathcal{E}_a} : \mathcal{E}_a \longrightarrow \mathcal{E}_{a'}$ since $\mathcal{E}_a$ and $\mathcal{E}'_a$ are pullbacks along a, and a^*f is the induced map of pullbacks. On the other hand, $a^*\mathrm{St}(f)$ is the composite $\Delta^0 \times \Delta^1 \xrightarrow{a\times\mathrm{id}} \mathcal{K} \times \Delta^1 \xrightarrow{\mathrm{St}(f)} \mathcal{C}\mathrm{at}_\infty$ which is precisely the map $\mathrm{St}(f)|_{\{a\}\times\Delta^1}$. The bottom map being equivalent to $\mathrm{id}_{\mathcal{C}\mathrm{at}_\infty}$ means that these two maps are the same, up to equivalence. **q.e.d.**

Corollary 3.2.4. *Let $\pi : \mathcal{K} \times \mathcal{C} \longrightarrow \mathcal{K}$ be the projection onto the first factor for ∞–categories $\mathcal{K}$ and $\mathcal{C}$. Let $f : \mathcal{E} \longrightarrow \mathcal{K}$ be a coCartesian fibration and $p : \mathcal{E} \longrightarrow \mathcal{K}\times\mathcal{C}$, a morphism of coCartesian fibrations over $\mathcal{K}$. Consider the sliced straightening functor*

$$\mathrm{St}_{/\pi} : \mathrm{CoCart}(\mathcal{K})_{/\pi} \longrightarrow \mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_\infty\right)_{/\mathrm{const}_\mathcal{C}} \simeq \mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_{\infty/\mathcal{C}}\right)$$

Then, for each object $a \in \mathcal{K}_0$, $\mathrm{St}_{/\pi}(p)(a)$ is the map

$$f|_{\mathcal{E}_a} : \mathcal{E}_a \longrightarrow (\{a\} \times \mathcal{C}) \simeq \mathcal{C}$$

Proof. This is a matter of tracing through the definition of slices and then applying the previous lemma. **q.e.d.**

3.3 Special Properties of Kan Complexes

Kan complexes enjoy especially nice properties that are essential to proving our main results needed to define semidirect products of ∞–operads. For instance, homotopy invariant constructions on Kan complexes coincide in the Joyal and Quillen model structures. In addition, some properties of diagrams Kan complexes have convenient methods of verifying. We shall now discuss some of these properties. These are standard facts but we inculde them here for ease of referencing.

Corollary 3.3.1. *Cofibrations of Kan complexes in the Joyal model structure are precisely those in the Quillen model structure.*

Proof. In both model structures, cofibrations are precisely the monomorphisms. **q.e.d.**

Lemma 3.3.2. *Weak homotopy equivalences of Kan complexes are precisely the categorical equivalences.*

Proof. Let $f : X \longrightarrow Y$ be a weak homotopy equivalence of Kan complexes. Then, f being a weak homotopy equivalence of simplicial sets means it induces, by precomposition, a bijection $\pi_0(\mathcal{F}\mathrm{un}(Y, Z)) \longrightarrow \pi_0(\mathcal{F}\mathrm{un}(X, Z))$ by [Lur22, 3.1.6.12] for every Kan complex Z. Now, let $\mathcal{C}$ be any ∞–category. For any functor $g : X \longrightarrow \mathcal{C}$, im g is a Kan complex contained in $\mathcal{C}$ and hence we must have $\mathcal{F}\mathrm{un}(X, \mathcal{C}) = \mathcal{F}\mathrm{un}(X, \mathcal{C}^\simeq)$. Similarly, $\mathcal{F}\mathrm{un}(Y, \mathcal{C}) = \mathcal{F}\mathrm{un}(Y, \mathcal{C}^\simeq)$. $\mathcal{F}\mathrm{un}(X, \mathcal{C}^\simeq)$ and $\mathcal{F}\mathrm{un}(Y, \mathcal{C}^\simeq)$ are both Kan complexes by [Lur22, 3.1.3.4] and hence they are exactly their cores. Thus, f induces a bijection:

$$\pi_0(\mathcal{F}\mathrm{un}(Y, \mathcal{C})^\simeq) = \pi_0(\mathcal{F}\mathrm{un}(Y, \mathcal{C}^\simeq)) \longrightarrow \pi_0(\mathcal{F}\mathrm{un}(X, \mathcal{C}^\simeq)) = \pi_0(\mathcal{F}\mathrm{un}(X, \mathcal{C})^\simeq)$$

Thus, by [Lur22, 4.5.3.1], f is a categorical equivalence.

Let $f : X \longrightarrow Y$ be a categorical equivalence of Kan complexes. Then for each ∞–category $\mathcal{C}$, f induces a bijection $\pi_0(\mathcal{F}\mathrm{un}(Y, \mathcal{C})) \longrightarrow \pi_0(\mathcal{F}\mathrm{un}(X, \mathcal{C}))$. In particular, this holds for Kan complexes $\mathcal{C}$ and hence f is a weak homotopy equivalence of simplicial sets. **q.e.d.**

Lemma 3.3.3. *Categorical fibrations of Kan complexes are precisely the Kan fibrations.*

Proof. Let f be a categorical fibration of Kan complexes so that its restriction to the cores of its domain and codomain is a Kan fibration [Lur22, 4.4.3.7]. However, since the domain and codomain are both Kan complexes, this implies f itself is a Kan fibration.

Let f be a Kan fibration so that it has the right lifting property with respect to all horn inclusions. That is, f is an inner fibration, in particular. Then, the inclusion $\{0\} \hookrightarrow \Delta^1$ is precisely the horn inclusion $\Lambda^1_1 \hookrightarrow \Delta^1$ and having all lifts with respect to this implies that f is a categorical fibration by [Lur08, 2.4.6.5, p. 119]. **q.e.d.**

Corollary 3.3.4. *Kan complexes are fibrant and cofibrant in the Quillen model structure and the Joyal model structure.*

Proof. Every object is cofibrant in both model structures since the map $\varnothing \longrightarrow X$ is a monomorphism for all simplicial sets. Kan complexes are fibrant in the Quillen model structure because the horn lifting property defining them realizes the unique map to the terminal simplicial set as a Kan fibration. Kan complexes have lifts for inner horn inclusions, in particular, and hence are ∞–categories and thus fibrant in the Joyal model structure. **q.e.d.**

These statements together imply the full subcategory of simplicial sets spanned by Kan complexes has a model structure that is the restriction of both the Quillen and Joyal model structures and in which all objects are fibrant and cofibrant. We may use these basic observations to deduce useful characterizations of weak equivalences and homotopy pullbacks of Kan complexes in the Joyal model structure.

Theorem 3.3.5. *A homotopy pullback of Kan complexes in the Quillen model structure is a Kan complex itself and is a homotopy pullback in the Joyal model structure.*

Proof. Let $X \xrightarrow{f} Z \xleftarrow{g} Y$ be a span of Kan complexes. Factorize g as $Y \xrightarrow{g_e} Y' \xrightarrow{g_f} Z$ where g_e is a weak homotopy (equivalently, categorical) equivalence and g_f is a Kan (equivalently, categorical) fibration. Then, the strict pullback P of $X \xrightarrow{f} Z \xleftarrow{g_f} Y'$ is a homotopy pullback of the original span by [Lur22, 3.4.1.2]. It is a straightforward diagram chase to see that a strict pullback of Kan complexes is a Kan complex so to show that P is a Kan complex, it suffices to show that Y' is a Kan complex but this is immediate as Y' is weak homotopy equivalent and hence categorically equivalent to Y, a Kan complex. Since the maps g_e and g_f are a weak equivalence and a fibration in both model structures, the homotopy pullback P is a homotopy pullback in both model structures. **q.e.d.**

Corollary 3.3.6. *Homotopy fibres of maps of Kan complexes coincide for the Quillen and the Joyal model structures.*

Proof. Homotopy fibres are homotopy pullbacks of spans of the form $* \longrightarrow X \longleftarrow Y$. **q.e.d.**

Theorem 3.3.7. *A map $f : X \longrightarrow Y$ of Kan complexes is a categorical (equivalently, weak homotopy) equivalence if and only if for all objects $y \in Y_0$, its homotopy fibre of f above y is weakly contractible.*

Proof. Our proof will follow the sketch given in [htta]. Factorize f as $X \xrightarrow{w} X' \xrightarrow{k} Y$ for a choice of weak homotopy equivalence w and Kan fibration k. The homotopy fibre of f above y is the strict fibre of f' above y by [Lur22, 3.4.1.2], since X and Y are fibrant objects. Let this fibre be $F \overset{i}{\hookrightarrow} X'$. Then, for some $x \in F_0 \subset X'_0$, we have a long exact sequence of homotopy groups [GJ09, 7.3, p. 28]:

$$\cdots \longrightarrow \pi_n(F,x) \xrightarrow{i_*} \pi_n(X',x) \xrightarrow{k_*} \pi_n(Y,y) \longrightarrow \pi_{n-1}(F,x) \longrightarrow \cdots$$

Since F is weakly contractible $\pi_n(F,x) = 0$ for all n which implies that k_* is an isomorphism for all n. Thus, k is a weak homotopy equivalence. Since weak homotopy equivalences compose, $f = wk$ is as well. Conversely, k being a weak homotopy equivalence requires each k_* to be an isomorphism which, in turn, requires $\pi_n(F,x)$ to be 0 for all n by exactness – that is, F must be weakly contractible. **q.e.d.**

Theorem 3.3.8. *Suppose all vertices are Kan complexes in a square of the following form:*

$$\begin{tikzcd}
W \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\
X \arrow[r, "h"'] & Z
\end{tikzcd}$$

Such a square is homotopy Cartesian in the Quillen or, equivalently, Joyal, model structure if and only if for each object $x : * \longrightarrow X$, q induces a weak homotopy equivalence or, equivalently, categorical equivalence of homotopy fibres: $q_x : W_x \xrightarrow{\simeq} Y_{hx}$.

Proof. We shall follow the proof given for topological spaces in [MV15, 3.3.18, p. 121]. Factorize $Y \xrightarrow{g} Z$ as $Y \xrightarrow{g_e} Y' \xrightarrow{g_f} Z$ and $W \xrightarrow{p} X$ as $W \xrightarrow{p_e} W' \xrightarrow{p_f} X$, where g_e, p_e are weak homotopy equivalences and g_f, p_f are fibrations. By functoriality of factorization, we have a map $W' \xrightarrow{r} Y'$, making the following diagram commute (by definition [Rie14, 12.1.1, p. 191]):

$$\begin{tikzcd}
W' \arrow[rr, "r"] \arrow[dd, "p_f"'] & & Y' \arrow[dd, "g_f"] & \\
& W \arrow[ul, "p_e"'] \arrow[rr, "q"] \arrow[dl, "p"] & & Y \arrow[ul, "g_e"'] \arrow[dl, "g"] \\
X \arrow[rr, "h"] & & Z &
\end{tikzcd}$$

The homotopy fibre W_x can be taken to be the strict pullback of the span $* \xrightarrow{x} X \xleftarrow{p_f} W'$ and, similarly, Y_{hx} can be taken to be the strict pullback of the span $* \xrightarrow{hx} Z \xleftarrow{g_f} Y'$. The homotopy pullback of the span $X \xrightarrow{h} Z \xleftarrow{g} Y$ can be taken to be the strict pullback P of the span $X \xrightarrow{h} Z \xleftarrow{g_f} Y'$. Now, it is easy to see that the homotopy fibre Y_{hx} is also actually the strict fibre P_x of the span $* \xrightarrow{x} X \longleftarrow P$ from the following pasting of strict pullback squares:

$$\begin{tikzcd}
Y_{hx} \cong P_x \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & Y' \arrow[d] \\
* \arrow[r, "x"'] & X \arrow[r, "h"'] & Z
\end{tikzcd}$$

Also, the maps $W \xrightarrow{g_e q} Y'$ and $W \xrightarrow{p} X$ give a canonical map $W \xrightarrow{s} P$ while $W' \xrightarrow{r} Y'$ and $W' \xrightarrow{p_f} X$ give a canonical map $W' \xrightarrow{s'} P$ which satisfy that $s'p_e = s$.

Using [CK22, A.4, p. 35], it suffices to show that the induced map $W \xrightarrow{s} P$ is a weak equivalence if and only if for each x as above, the induced map $W_x \xrightarrow{q_x} P_x$ is a weak homotopy equivalence. However, since $W \xrightarrow{p_e} W'$ is a weak equivalence, $W \xrightarrow{s} P$ is a weak equivalence if and only if s' is, by 2–out–of–3.

Suppose now that q_x is a weak homotopy equivalence. Then, we have the following diagram of simplicial homotopy groups for each element $* \xrightarrow{w} W'$:

$$\begin{tikzcd}
\pi_{n+1}(X,x) \arrow[r] \arrow[d] & \pi_n(W_x,w) \arrow[r] \arrow[d] & \pi_n(W',w) \arrow[r] \arrow[d] & \pi_n(X,x) \arrow[r] \arrow[d] & \pi_{n-1}(W_x,w) \arrow[d] \\
\pi_{n+1}(X,x) \arrow[r] & \pi_n(P_x,s'w) \arrow[r] & \pi_n(P,s'w) \arrow[r] & \pi_n(X,x) \arrow[r] & \pi_{n-1}(P_x,s'w)
\end{tikzcd}$$

where the top and bottom rows are exact by [GJ09, 7.3, p. 28], noting that $P \longrightarrow X$, being a pullback of a fibration, is a fibration itself by [Bal21, 2.1.9, p. 14]. Here, by assumption, the second and fourth vertical maps from the left are isomorphisms. The first and fifth vertical maps from the left are identity maps. Thus, by the five lemma, the middle map is an isomorphism.

Suppose, then, that s' is a weak homotopy equivalence. Then, the same argument applied to a frame of the same diagram of exact sequences shifted by one to the left gives that the map $\pi_n(W_x,w) \longrightarrow \pi_n(P_x,s'w)$ is an isomorphism.

Since this argument is valid for all choices of base point w, changing x as needed, we have shown that the given square is a homotopy pullback in the Quillen model structure if and only if the induced maps of fibres are weak homotopy equivalences. The result for the Joyal model structure, which we want, follows from the observation that the two model structures coincide for Kan complexes and that homotopy fibres of Kan complexes are again Kan complexes. **q.e.d.**

3.4 Homotopy Pullbacks and Cubical Diagrams

A convenient way to reason about coCartesian morphisms in the ∞–categorical setting is through their characterization in terms of homotopy pullback diagrams of mapping spaces. Recall from the dual situations of [Lan21, 3.1.16, 3.1.18, p. 170 – 172] that:

Theorem 3.4.1. *Given an inner fibration $p : \mathcal{E} \longrightarrow \mathcal{C}$ of ∞–categories and a morphism $f : X \longrightarrow X'$ in $\mathcal{E}$, f is p–coCartesian if and only if for every object Z of $\mathcal{E}$, the following square is homotopy Cartesian in the Joyal model structure:*

$$\begin{tikzcd} \mathcal{E}(X', Z) \arrow[r, "f^*"] \arrow[d, "p"'] & \mathcal{E}(X, Z) \arrow[d, "p"] \\ \mathcal{C}(p(X'), p(Z)) \arrow[r, "p(f)^*"'] & \mathcal{C}(p(X), p(Z)) \end{tikzcd}$$

where the horizontal maps are induced by precomposition.

We will need to glue homotopy Cartesian squares as the one above to deduce that certain other squares are homotopy Cartesian. Hence, we start by establishing some basic gluing lemmas for homotopy Cartesian squares. For instance, the basic pasting lemma for two pullback squares holds for homotopy pullbacks as well, as shown in [CK22, A.6, p. 35]. We sketch the proof here for the convenience of the reader:

Lemma 3.4.2. *Consider a diagram of fibrant objects of the following form in some model category:*

$$\begin{tikzcd} U \arrow[r, "p"] \arrow[d, "u"] & V \arrow[r, "q"] \arrow[d, "v"] & W \arrow[d, "w"] \\ U' \arrow[r, "p'"'] & V' \arrow[r, "q'"'] & W' \end{tikzcd}$$

If the right square is homotopy Cartesian, then the outer rectangle is homotopy Cartesian if and only if the left square is.

Proof Sketch. Funtorially factorize the two right-most vertical morphisms into weak equivalences followed by fibrations, take the relevant strict pullbacks to obtain homotopy pullbacks and paste the diagrams in a suitable manner. Observe that we have canonical maps from U and V to the relevant homotopy pullbacks. Show that these maps are weak equivalences by an easy diagram chase in each case and then appealing to [CK22, A.5, p. 35]. **q.e.d.**

We then note two simple facts about squares where the vertical morphisms are either weak equivalences or isomorphisms.

Lemma 3.4.3. *Every commuting square of the following form in any 1–category is strictly Cartesian:*

$$\begin{tikzcd} A \arrow[r, "\cong"] \arrow[d] & B \arrow[d] \\ C \arrow[r, tail] & D \end{tikzcd}$$

where the top horizontal map is an isomorphism and the bottom horizontal map is a monomorphism.

Proof. This follows by a straightforward diagram chase. **q.e.d.**

Lemma 3.4.4. *Consider a commuting square of the following form in a right proper model category:*

$$\begin{tikzcd}
A \arrow[r, "\simeq"] \arrow[d] & B \arrow[d] \\
C \arrow[r, "\simeq"] & D
\end{tikzcd}$$

where the horizontal morphisms are weak equivalences. Then the square is homotopy Cartesian.

Proof. Factorize $B \longrightarrow D$ as a weak equivalence followed by a fibration to obtain the following diagram:

$$\begin{tikzcd}
A \arrow[rr, "\simeq"] \arrow[dr] \arrow[dd] & & B \arrow[dr, "\simeq"] \arrow[dd] & \\
& P \arrow[rr, "\simeq"] \arrow[dd] & & B' \arrow[dd] \\
C \arrow[rr, "\simeq"] \arrow[dr, equal] & & D \arrow[dr, equal] & \\
& C \arrow[rr, "\simeq"] & & D
\end{tikzcd}$$

where the front face is a strict pullback and hence $P \longrightarrow B'$ is a weak equivalence by the closure of weak equivalences under pullbacks along fibrations in a right proper model category [Bal21, 3.5.1, p. 34]. Hence, the map $A \longrightarrow P$ is a weak equivalence by 2–out-of–3. The result now follows by [CK22, A.4, p. 35]. **q.e.d.**

Next, we record some results about cubes with certain faces homotopy Cartesian.

Lemma 3.4.5. *Consider a commuting cube of fibrant objects of the following form in a model category:*

$$\begin{tikzcd}
U \arrow[rr, "p"] \arrow[dr, "q"] \arrow[dd, "u"] & & V \arrow[dr, "r"] \arrow[dd, "v"] & \\
& W \arrow[rr, "s"] \arrow[dd, "w"] & & X \arrow[dd, "x"] \\
U' \arrow[rr, "p'"] \arrow[dr, "q'"] & & V' \arrow[dr, "r'"] & \\
& W' \arrow[rr, "s'"] & & X'
\end{tikzcd}$$

If the right, left and front faces are homotopy pullback diagrams, then the back face is as well.

Proof. Apply Lemma 3.4.2 to the pasting of the left and front faces to obtain that the diagonal square formed by U, U', X, X' is homotopy Cartesian. This diagonal square is also the pasting of the back and the right faces of the cube. Now, apply Lemma 3.4.2 to the diagonal square considered as this pasting – that is, take the back square to be the left square in the pasting and the right square to be the right square in the pasting – to obtain that the back face is homotopy Cartesian. **q.e.d.**

Remark 3.4.6. Results analogous to the previous two hold for strictly Cartesian squares in any category by the same arguments. ◇

Lemma 3.4.7. *Consider a cube as in the previous lemma in* sSet *with all vertices Kan complexes but such that the right, left and back (as opposed to the front) faces are homotopy Cartesian with respect to the Joyal (equivalently, Quillen) model structure. Let* C *be an object of* W' *such that there exists an object* C_0 *in* U' *with* $q'(C_0) = C$*. Then,* s *induces a categorical (equivalently, weak homotopy) equivalence of homotopy fibres* $W_C \xrightarrow{\simeq} X_{s(C)}$*.*

Proof. Since the left face is homotopy Cartesian, by Theorem 3.3.8, q induces a categorical equivalence of homotopy fibres $U_{C_0} \xrightarrow{q} W_C$. Similarly, we have categorical equivalences $U_{C_0} \xrightarrow{p} V_{p'(C_0)}$ and $V_{p'(C_0)} \xrightarrow{r} X_{r'p'(C_0)} = X_{s'(C)}$. By the commutativity of the diagram and 2–out–of–3, the map $W_c \xrightarrow{s} X_{s'(C)}$ must also be a categorical (equivalently, weak homotopy) equivalence. **q.e.d.**

Lemma 3.4.8. *Let* $f : E \longrightarrow D, p : E \longrightarrow C$*,* $q : D \longrightarrow C$ *be morphisms in a category such that* $qf = p$*. Let* $*$ *be a terminal object of the ambient category and let* E_c *denote the strict fibre of* p *above* $c : * \longrightarrow E$ *defined by the pullback of the span* $* \xrightarrow{*} C \xleftarrow{p} E$*.* D_c *is defined similarly. Then, the following square is strictly Cartesian:*

$$\begin{tikzcd} E_c \arrow[r, "p"] \arrow[d, hook] & D_c \arrow[d, hook] \\ E \arrow[r, "p"] & D \end{tikzcd}$$

If all objects are fibrant and any two of the maps in the span $D_c \hookrightarrow D \xleftarrow{p} E$ *are fibrations in some model structure on the ambient category, then the square is homotopy Cartesian in that model structure.*

Proof. The given square forms the back face of the following cube:

$$\begin{tikzcd} E_c \arrow[rr] \arrow[dr] \arrow[dd] & & D_c \arrow[dr] \arrow[dd] & \\ & * \arrow[rr, equal, crossing over] & & * \arrow[dd] \\ E \arrow[rr] \arrow[dr] & & D \arrow[dr] & \\ & C \arrow[rr, equal] \arrow[from=uu, crossing over] & & C \end{tikzcd}$$

where the left and right faces are the strict pullback squares defining E_c and D_c respectively and the front face is Cartesian by Lemma 3.4.3. Finally, Lemma 3.4.5 in the strict case gives the desired result in the strict case. If any two of the maps are fibrations, then so is the third by 2–out–of–3. Now, if all objects are fibrant, then strict pullbacks along fibrations correspond to homotopy pullbacks and we are done. **q.e.d.**

We shall now apply these results to deduce some useful facts about homotopy Cartesian squares of mapping spaces that will be used in our main results concerning coCartesian morphisms and fibre transport in coCartesian fibrations.

Lemma 3.4.9. *Let*

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "p"] \arrow[dr, "f"'] & & \mathcal{D} \arrow[dl, "q"] \\ & \mathcal{C} & \end{tikzcd}$$

be a commuting diagram of ∞–categories. Then, for any object C of $\mathcal{C}$ and any two objects X and Y in the strict fibre $\mathcal{E}_C$ above C, the following square of right mapping spaces is strictly Cartesian:

$$\begin{tikzcd} \mathcal{E}^R_C(X,Y) \arrow[r, "p"] \arrow[d, hook] & \mathcal{D}^R_{p(C)}(p(X),p(Y)) \arrow[d, hook] \\ \mathcal{E}^R(X,Y) \arrow[r, "p"'] & \mathcal{D}^R(p(X),p(Y)) \end{tikzcd}$$

Proof. Recall the definition of right mapping spaces as strict pullbacks of the form [Lan21, 2.5.28, p. 157]:

$$\begin{tikzcd} \mathcal{A}^R(A,B) \arrow[r] \arrow[d] & \mathcal{A}_{/B} \arrow[d] \\ * \arrow[r] & \mathcal{A} \end{tikzcd}$$

where the right vertical map is the one given by restricting the n–simplices $\Delta^n * \Delta^0 \longrightarrow \mathcal{A}$ of $\mathcal{A}_{/B}$ to Δ^n. Using this characterization, we deduce that our square of interest forms the back face of the following cube:

$$\begin{tikzcd} \mathcal{E}^R_C(X,Y) \arrow[rr, "p"] \arrow[dr] \arrow[dd] & & \mathcal{D}^R_{p(C)}(p(X),p(Y)) \arrow[dr] \arrow[dd] & \\ & \mathcal{E}_{C/Y} \arrow[rr, "p_{/Y}"] \arrow[dd] & & \mathcal{D}_{C/p(Y)} \arrow[dd] \\ \mathcal{E}^R(X,Y) \arrow[rr, "p"] \arrow[dr] & & \mathcal{D}^R(p(X),p(Y)) \arrow[dr] & \\ & \mathcal{E}_{/Y} \arrow[rr, "p_{/Y}"] & & \mathcal{D}_{/p(Y)} \end{tikzcd} \tag{7}$$

Hence, it suffices to show that the front, left and right faces are strictly Cartesian by Lemma 3.4.5 in the strict case. The front face of our cube is the image under the functor $F : \mathsf{sSet}_{/\Delta^0} \longrightarrow \mathsf{sSet}$:

$(Z : \Delta^0 \longrightarrow S) \longmapsto S_{/Z}$ of the square:

$$\begin{tikzcd}
\mathcal{E}_C \arrow[r, "p"] \arrow[d, hook] & \mathcal{D}_C \arrow[d, hook] \\
\mathcal{E} \arrow[r, "p"] & \mathcal{D}
\end{tikzcd}$$

which is strictly Cartesian by Lemma 3.4.8 and since F is a right adjoint (to the join functor $- * \Delta^0$) [Lan21, 1.4.17, p. 89], the front face of the cube (7) is strictly Cartesian.

The left face, in turn, forms the back face of the following cube:

$$\begin{tikzcd}
\mathcal{E}^R_C(X,Y) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{E}^R(X,Y) \arrow[dr] \arrow[dd] & \\
& * \arrow[rr, equal] \arrow[dd] & & * \arrow[dd] \\
\mathcal{E}_{C/Y} \arrow[rr] \arrow[dr] & & \mathcal{E}_{/Y} \arrow[dr] & \\
& \mathcal{E}_C \arrow[rr, hook] & & \mathcal{E}
\end{tikzcd}$$

whose left and right faces are precisely the strictly Cartesian squares defining the respective right mapping spaces and whose front face is strictly Cartesian by Lemma 3.4.3. Thus, the back face of the above cube which is the left face of (7), our cube of interest, is strictly Cartesian. The right face of (7) is strictly Cartesian by an identical argument. **q.e.d.**

Remark 3.4.10. A similar argument shows the analogous results as above for left mapping spaces and ordinary mapping spaces by using the following definitions of these mapping spaces as pullbacks:

$$\begin{tikzcd}
\mathcal{A}^L(A,B) \arrow[r] \arrow[d] & \mathcal{A}_{A/} \arrow[d] \\
* \arrow[r] & \mathcal{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}(A,B) \arrow[r] \arrow[d] & \mathcal{A}^{\Delta^1} \arrow[d] \\
* \arrow[r] & \mathcal{A}^{\Delta^{\{0\}} \amalg \Delta^{\{1\}}}
\end{tikzcd}$$

$\diamond$

We then observe a basic fact about how inner fibrations interact with mapping spaces in order to give a condition for the strictly Cartesian squares of the two previous statements to be homotopy Cartesian. We include an elementary argument of this fact here although a more elegant argument can be found in [Lur08, 2.4.4.1, p. 114].

Lemma 3.4.11. *For any inner fibration $f : \mathcal{A} \longrightarrow \mathcal{B}$ and objects $X, Y \in \mathcal{A}_0$, the induced map $f : \mathcal{A}^R(X,Y) \longrightarrow \mathcal{B}^R(f(X), f(Y))$ is a categorical fibration.*

Proof. Recall that a morphism $\alpha : g \Longrightarrow h$ of $\mathcal{B}^R(f(X), f(Y))$ is a 2–simplex in $\mathcal{B}$ of the following form [Rie14, §15.4, 274]:

$$\begin{tikzcd} f(X) \arrow[rr, "g"] \arrow[dr, equal] & & f(Y) \\ & f(X) \arrow[ur, "h"'] & \end{tikzcd}$$

Observe, then, that the following square is strictly Cartesian and hence also homotopy Cartesian in the Joyal model structure since the bottom horizontal map is a categorical fibration:

$$\begin{tikzcd} \mathcal{A}(X,W) \arrow[r, "\mathrm{id}_X^*"] \arrow[d, "f"'] & \mathcal{A}(X,W) \arrow[d, "f"] \\ \mathcal{B}(f(X),f(W)) \arrow[r, "\mathrm{id}_{f(X)}^*"'] & \mathcal{B}(f(X),f(W)) \end{tikzcd}$$

noting that the mapping spaces here are the ordinary mapping spaces. This shows that id_X is an f–coCartesian lift of $\mathrm{id}_{f(X)}$. Thus, given a lift of g in $\mathcal{A}$, we immediately have a lift of α by the f–coCartesian property of id_X. This shows that every morphism in $\mathcal{B}^R(f(X),f(Y))$ with a lift of its domain along f itself admits a lift. It then suffices to show that $f : \mathcal{A}^R(X,Y) \longrightarrow \mathcal{B}^R(f(X),f(Y))$ is an inner fibration. However, this is immediate from the fact that f is an inner fibration, for the n–simplices and n–horns of $\mathcal{A}^R(X,Y)$ and $\mathcal{B}^R(f(X),f(Y))$ are $(n+1)$–simplices and $(n+1)$–horns of $\mathcal{A}$ and $\mathcal{B}$ respectively [Rie14, 15.4, p. 274]. **q.e.d.**

Lemma 3.4.12. *Let p,q,f be as in Lemma 3.4.9 but with the additional requirement that p is an inner fibration. Then, the square in that lemma is homotopy Cartesian.*

Proof. By Lemma 3.4.11, the bottom horizontal map in that square is a categorical fibration and hence the square being a strict pullback while all objects being ∞–categories ensures that it is homotopy Cartesian. **q.e.d.**

Lemma 3.4.13. *If we replace right mapping spaces with ordinary (symmetric) mapping spaces in Lemma 3.4.12, the square is still a homotopy pullback.*

Proof. Each right mapping spaces is categorically equivalent to the corresponding ordinary mapping space and the result follows by [CK22, A.4, p. 35]. **q.e.d.**

3.5 Fibrewise Properties of CoCartesian Fibrations

For much of this subsection, we will consider a morphism of coCartesian fibrations of ∞–categories of the form:

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "p"] \arrow[dr, "p_1"'] & & \mathcal{K} \times \mathcal{C} \arrow[dl, "\pi_1"] \\ & \mathcal{K} & \end{tikzcd}$$

where π_i is the i–th projection of $\mathcal{K} \times \mathcal{C}$ and $p_i = \pi_i p$. We note that both p_1 and π_1 are categorical fibrations since coCartesian fibrations are categorical fibrations [Lur08, 2.0.0.5, p. 53]. By 2–out–of–3, p must also be a categorical fibration. Furthermore, these imply that homotopy fibres of p_1 and π_1 in the Joyal model structure are precisely the strict fibres. Hence, we will simply refer to these homotopy/strict fibres as fibres. Given this context, we shall now develop some lemmas that allow us to use fibre transport along morphisms in $\mathcal{K}$ to reduce various global properties of morphisms or diagrams in $\mathcal{E}$ to fibrewise properties. These results when applied to the case of $\mathcal{C} = N(\mathcal{F}\mathrm{in}_*)$ and $\mathcal{K}$–families of ∞–operads that are coCartesian fibrations over $\mathcal{K}$ allow us to use the additional structure provided by the coCartesian lifts to reduce the checking several properties of a $\mathcal{K}$–family of ∞–operads to its fibres, the ∞–operads themselves.

Corollary 3.5.1. *Any p_1–coCartesian lift of any morphism $f : C \longrightarrow D$ in $\mathcal{K}$ is p–coCartesian.*

Proof. Let $\overline{f} : X \longrightarrow Y$ be a p_1–coCartesian lift of f. Then, $p\left(\overline{f}\right)$ is π_1–coCartesian. By [Lan21, 3.1.4, p. 165], $\overline{f}$ is also p–coCartesian. **q.e.d.**

Lemma 3.5.2. *Let $p_C : \mathcal{E}_C \longrightarrow \{C\} \times \mathcal{C}$ be the map of fibres above some object $C \in \mathcal{K}_0$ induced by p. Then, if $\overline{\alpha} : X \longrightarrow Y$ is a p–coCartesian morphism in $\mathcal{E}$ with $p(\overline{\alpha}) = (\mathrm{id}_C, \alpha) : (C, M) \longrightarrow (C, N)$, we must have that $\overline{\alpha}$ is also p_C–coCartesian.*

Proof. In order for $\overline{\alpha}$ to be p_C–coCartesian, we require a dashed lift in every possible diagram of the following form:

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r, hook] \arrow[rr, bend left, "\overline{\alpha}"] & \Lambda^n_0 \arrow[r] \arrow[d, hook] & \mathcal{E}_C \arrow[d, "p_C"] \\
& \Delta^n \arrow[r] \arrow[ur, dashed, "?"] & \{C\} \times \mathcal{C}
\end{tikzcd}$$

Composing with the inclusions $\mathcal{E}_C \hookrightarrow \mathcal{E}$ and $\{C\} \times \mathcal{C} \hookrightarrow \mathcal{K} \times \mathcal{C}$, we have the following diagram:

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r, hook] \arrow[rr, bend left, "\overline{\alpha}"] & \Lambda^n_0 \arrow[r] \arrow[d, hook] & \mathcal{E}_C \arrow[r, hook] \arrow[d, "p_C"] & \mathcal{E} \arrow[d, "p"] \\
& \Delta^n \arrow[r] \arrow[ur, dashed, "?"] & \{C\} \times \mathcal{C} \arrow[r, hook] & \mathcal{C}
\end{tikzcd}$$

where a lift to $\mathcal{E}$ exists by the p–coCartesian property of $\overline{\alpha}$. Since the right-most square is a strict pullback, we have a unique lift $\Delta^n \longrightarrow \mathcal{E}_C$. That this makes the two triangles in the left-most square commute is a straightforward diagram argument using the fact that the right-most horizontal maps are monomorphisms. **q.e.d.**

Lemma 3.5.3. *Let $p_C : \mathcal{E}_C \longrightarrow \{C\} \times \mathcal{C}$ be the map of fibres over some object $C \in \mathcal{K}_0$ induced by p and $\overline{\alpha} : X \longrightarrow Y$ be a p_C–coCartesian morphism such that $p(\overline{\alpha}) = (\mathrm{id}_C, \alpha) : (C, M) \longrightarrow (D, N)$. If (id_C, α) has some p–coCartesian lift, then $\overline{\alpha}$ itself is p–coCartesian.*

Proof. Let $\overline{\beta} : X \longrightarrow Y'$ be a p–coCartesian lift of (id_C, α). In particular, it is p_C–coCartesian by Lemma 3.5.2. It is then easy to show that there exists an equivalence $\overline{\gamma} : Y' \longrightarrow Y$ with $p\left(\overline{\gamma}\right) = (\mathrm{id}_C, \gamma)$ and a 2–simplex:

$$\begin{tikzcd} X \arrow[rr, "\overline{\beta}"] \arrow[dr, "\overline{\alpha}"'] & & Y' \arrow[dl, "\overline{\gamma}"] \\ & Y & \end{tikzcd}$$

in $\mathcal{E}$. We then have the following pasting of squares:

$$\begin{tikzcd} \mathcal{E}(Y,W) \arrow[r, "\overline{\gamma}^*"] \arrow[d, "p"] & \mathcal{E}(Y',W) \arrow[r, "\overline{\beta}^*"] \arrow[d, "p"] & \mathcal{E}(X,W) \arrow[d, "p"] \\ \mathcal{K}(C,D) \times \mathcal{C}(N,K) \arrow[r, "(\mathrm{id}_C,\gamma)^*"] & \mathcal{K}(C,D) \times \mathcal{C}(p_2(Y'),K) \arrow[r, "(\mathrm{id}_C,\beta)^*"] & \mathcal{K}(C,D) \times \mathcal{C}(M,K) \end{tikzcd}$$

where the right square is homotopy Cartesian by the p–coCartesian property of $\overline{\beta}$. The horizontal maps on the left square are equivalences since $\overline{\gamma}$ is an equivalence and hence by the fact that sSet with Quillen model structure is right proper [Lur08, p. 653], the fact that for Kan complexes the Joyal model structure coincides with the Quillen model structure (note that the Joyal model structure is not right proper for all of sSet) and Lemma 3.4.4, the left square is also homotopy Cartesian. Thus, by Lemma 3.4.2, the outer rectangle is homotopy Cartesian but this is precisely saying that $\overline{\alpha}^*$ is p–coCartesian. **q.e.d.**

Lemma 3.5.4. *Let $\overline{\alpha} : X \longrightarrow Y$ be a p–coCartesian morphism in $\mathcal{E}$ such that $p\left(\overline{\alpha}\right) = (\mathrm{id}_C, \alpha) : (C, M) \longrightarrow (C, N)$. Then, the fibre transport of $\overline{\alpha}$ along any morphism $f : C \longrightarrow D$ in $\mathcal{K}$ is also p–coCartesian.*

Proof. Choose p_1–coCartesian lifts $\overline{f}_X, \overline{f}_Y$ of f starting at X and Y respectively. By Corollary 3.5.1, these are also p–coCartesian. Let $p\left(\overline{f}_Z\right) = (f, \beta_Z)$ for $Z \in \{X, Y\}$. Let $\overline{\alpha}' : X' \longrightarrow Y'$ be the fibre transport of $\overline{\alpha}$ along f using the lifts $\overline{f}_X$ and $\overline{f}_Y$ and let $p\left(\overline{\alpha}'\right) = (\mathrm{id}_D, \alpha')$. We then have the following commuting cube, for an arbitrary object W of $\mathcal{E}$:

$$\begin{tikzcd} \mathcal{E}(Y',W) \arrow[rr, "(\overline{\alpha}')^*"] \arrow[dr, "\overline{f}_Y^*"] \arrow[dd, "p"] & & \mathcal{E}(X',W) \arrow[dr, "\overline{f}_X^*"] \arrow[dd, "p"] & \\ & \mathcal{E}(Y,W) \arrow[rr, "\overline{\alpha}^*"] \arrow[dd, "p"] & & \mathcal{E}(X,W) \arrow[dd, "p"] \\ \mathcal{K}(D,D) \times \mathcal{C}(N',K) \arrow[rr, "(\mathrm{id}_D,\alpha')^*"] \arrow[dr, "(f,\beta_Y)^*"] & & \mathcal{K}(D,D) \times \mathcal{C}(M',K) \arrow[dr, "(f,\beta_X)^*"] & \\ & \mathcal{K}(C,D) \times \mathcal{C}(N,K) \arrow[rr, "(\mathrm{id}_C,\alpha)^*"] & & \mathcal{K}(C,D) \times \mathcal{C}(M,K) \end{tikzcd}$$

Since $\overline{f}_X, \overline{f}_Y$ and $\overline{\alpha}$ are all p–coCartesian, the right, left and front faces are homotopy Cartesian by Theorem 3.4.1, which implies that the back face is homotopy Cartesian by Lemma 3.4.5. Therefore, by Theorem 3.4.1 again, $\overline{\alpha}'$ is p–coCartesian. **q.e.d.**

Lemma 3.5.5. *Let $p_C : \mathcal{E}_C \longrightarrow \{C\} \times \mathcal{C}$ be the map of fibres over some object $C \in \mathcal{K}_0$ induced by p and $\overline{\alpha} : X \longrightarrow Y$ be a p_C–coCartesian morphism such that $p(\overline{\alpha}) = (\mathrm{id}_C, \alpha) : (C, M) \longrightarrow (D, N)$. p_D is defined similarly for any other object $D \in \mathcal{K}_0$. If for any morphism $f : C \longrightarrow D$ in $\mathcal{K}$, the fibre transport of $\overline{\alpha}$ along f is p_D–coCartesian, then $\overline{\alpha}$ is p–coCartesian.*

Proof. For an arbitrary object W in $\mathcal{E}$ with $p(W) = (D, K)$, we would like to show that the following square is homotopy Cartesian in the Joyal model structure:

$$\begin{tikzcd}
\mathcal{E}(Y,W) \arrow[r, "\overline{\alpha}^*"] \arrow[d, "p"'] & \mathcal{E}(X,W) \arrow[d, "p"] \\
\mathcal{K}(C,D) \times \mathcal{C}(N,K) \arrow[r, "(\mathrm{id}_C,\alpha)^*"'] & \mathcal{K}(C,D) \times \mathcal{C}(M,K)
\end{tikzcd}$$

where $p(\overline{\alpha}) = (\mathrm{id}_C, \alpha) : (C, M) \longrightarrow (D, N)$. Since all vertices in the above square are Kan complexes, by Theorem 3.3.8, it suffices to show that for each $(f, \beta) \in \mathcal{K}(C, D) \times \mathcal{C}(N, K)$, $\overline{\alpha}^*$ induces a homotopy equivalence of fibres.

Choose p_1–coCartesian lifts $\overline{f}_Z : Z \longrightarrow Z', Z \in \{X, Y\}$ of f starting at X and Y respectively. These are p–coCartesian by Corollary 3.5.1 and thus we have corresponding homotopy Cartesian squares:

$$\begin{tikzcd}
\mathcal{E}(Z',W) \arrow[r, "\overline{f}_Z^*"] \arrow[d, "p"'] & \mathcal{E}(Z,W) \arrow[d, "p"] \\
\mathcal{K}(D,D) \times \mathcal{C}(p_2(Z'),K) \arrow[r, "(f,p_2(\overline{f}))^*"'] & \mathcal{K}(C,D) \times \mathcal{C}(p_2(Z),K)
\end{tikzcd} \tag{8}$$

In addition, we have the following homotopy Cartesian squares by Lemma 3.4.13:

$$\begin{tikzcd}
\mathcal{E}_D(Z',W) \arrow[r, hook] \arrow[d] & \mathcal{E}(Z',W) \arrow[d] \\
\{D\} \times \mathcal{C}(p_2(Z'),K) \arrow[r, hook] & \mathcal{K}(D,D) \times \mathcal{C}(p_2(Z'),K)
\end{tikzcd} \tag{9}$$

which composes with (8) to give a homotopy Cartesian square:

$$\begin{tikzcd}
\mathcal{E}_D(Z',W) \arrow[r] \arrow[d] & \mathcal{E}(Z,W) \arrow[d] \\
\{D\} \times \mathcal{C}(p_2(Z'),K) \arrow[r] & \mathcal{K}(C,D) \times \mathcal{C}(p_2(Z'),K)
\end{tikzcd} \tag{10}$$

Then, by the assumption that the fibre transport $\overline{\alpha}'$ of $\overline{\alpha}$ along f is p_D–coCartesian, we have the following homotopy Cartesian square:

$$\begin{tikzcd}
\mathcal{E}_D(Y',W) \arrow[r] \arrow[d] & \mathcal{E}_D(X',W) \arrow[d] \\
\{D\} \times \mathcal{C}(p_2(Y'),K) \arrow[r] & \{D\} \times \mathcal{C}(p_2(X'),K)
\end{tikzcd} \tag{11}$$

Pasting the squares (10) and (11) together gives the following cube:

$$\begin{tikzcd}
\mathcal{E}_D(Y',W) \arrow[rr, "(\overline{\alpha}')^*"] \arrow[dr] \arrow[dd, "p"] & & \mathcal{E}_D(X',W) \arrow[dr] \arrow[dd, "p"] & \\
& \mathcal{E}(Y,W) \arrow[rr, "\overline{\alpha}^*"] \arrow[dd, "p"] & & \mathcal{E}(X,W) \arrow[dd, "p"] \\
\{D\} \times \mathcal{C}(p_2(Y'),K) \arrow[rr, "(\mathrm{id}_D,\alpha')^*"] \arrow[dr] & & \{D\} \times \mathcal{C}(p_2(X'),K) \arrow[dr] & \\
& \mathcal{K}(C,D) \times \mathcal{C}(N,K) \arrow[rr, "(\mathrm{id}_C,\alpha)^*"] & & \mathcal{K}(C,D) \times \mathcal{C}(M,K)
\end{tikzcd}$$

whose right, left and back faces are homotopy Cartesian by construction. Now, observe that the maps $p_2\left(\overline{f}_Z\right) : p_2(Z) \longrightarrow p_2(Z')$ must be equivalences since $p\left(\overline{f}_Z\right)$ is π_1–coCartesian. This means there exists some map $\gamma : p_2(Y') \longrightarrow K$ such that we have a 2–simplex giving the composite $\gamma p_2\left(\overline{f}_Y\right) \sim \beta$. That is, $\beta \in \mathcal{C}(N,K)$ has a preimage $\gamma \in \mathcal{C}(p_2(Y'),K)$. Also, we have that id_D maps to $f \in \mathcal{K}(C,D)$. Thus, we may now apply Lemma 3.4.7 to deduce that the map of fibres above (f,β) in the front square is a weak homotopy equivalence, as required. **q.e.d.**

Lemma 3.5.6. *Let $q : \Delta^0 * S \longrightarrow \mathcal{E}$ be some diagram in $\mathcal{E}_D$ for some simplicial set S and some object $D \in \mathcal{K}_0$. Then, q is a p_D–limit if and only if it is a p–limit.*

Proof. Let q be a p_D–limit. Let $q|_{\Delta^0}(0) = Y$ be the apex of the cone given by q with $p(Y) = (D,N)$. Consider any object X over $q^\circ = q|_S$ with $p(X) = (C,M)$. We would like to show that the induced map $p : \mathcal{E}_{/q^\circ}(X,Y) \longrightarrow \mathcal{K}(C,D) \times \mathcal{C}(M,N)$ is a homotopy equivalence [Lur22, 7.1.5.1 and 7.1.4.1]. For this, it suffices to show that for each $(f,\alpha) \in \mathcal{K}(C,D) \times \mathcal{C}(M,N)$, the homotopy fibre of p above (f,α) is weakly contractible by Theorem 3.3.7, since the mapping spaces are Kan complexes. Choose a p_1–coCartesian lift $\overline{f} : X \longrightarrow X'$ of f starting at X and recall that this must also be p–coCartesian by Corollary 3.5.1. Furthermore, f is p–coCartesian in $\mathcal{E}_{/q^\circ}$ by [Lur08, 2.4.3.3, p. 111] giving a homotopy Cartesian square:

$$\begin{tikzcd}
\mathcal{E}_{/q^\circ}(X',Y) \arrow[r, "\overline{f}^*"] \arrow[d, "p"] & \mathcal{E}_{/q^\circ}(X,Y) \arrow[d, "p"] \\
\mathcal{K}(D,D)_{/p_1 q^\circ} \times \mathcal{C}_{/p_2 q^\circ}(M',N) \arrow[r, "(f,p_2(\overline{f}))^*"'] & \mathcal{K}(C,D)_{/p_1 q^\circ} \times \mathcal{C}_{/p_2 q^\circ}(M,N)
\end{tikzcd} \tag{12}$$

Notice, then, that by applying the functor $(g : \Delta^0 \longrightarrow S) \longmapsto S_{/g}$ to the pullback of the span $\Delta^0 \xrightarrow{D} \mathcal{K} \xleftarrow{p_1} \mathcal{E}$, we have an isomorphism $\mathcal{E}_{D/q^\circ} \cong (\mathcal{E}_{/q^\circ})_D$, where the subscript D on the right means the fibre of $p_1 : \mathcal{E}_{/q^\circ} \longrightarrow \mathcal{K}_{/p_1 q^\circ}$ above the following object of $\mathcal{K}_{/p_1 q^\circ}$:

$$\begin{tikzcd}
 & & & D \arrow[llld, "\mathrm{id}_D"'] \arrow[lld, "\mathrm{id}_D"] \arrow[rd, "\mathrm{id}_D"'] \arrow[rrd, "\mathrm{id}_D"] & & \\
D & D & & \cdots & D & D
\end{tikzcd}$$

We then have the following homotopy Cartesian square by Lemma 3.4.13:

$$\begin{tikzcd}
\mathcal{E}_{D/q^\circ}(X', Y) \cong \left(\mathcal{E}_{/q^\circ}\right)_D (X', Y) \arrow[r, hook] \arrow[d, "p_D"'] & \mathcal{E}_{/q^\circ}(X', Y) \arrow[d, "p"] \\
\{D\} \times \mathcal{C}_{/p_2 q^\circ}(M', N) \arrow[r, hook] & \mathcal{K}_{/p_1 q^\circ}(D, D) \times \mathcal{C}_{/p_2 q^\circ}(M', N)
\end{tikzcd} \tag{13}$$

which composes with (12) to give a homotopy Cartesian square:

$$\begin{tikzcd}
\mathcal{E}_{D/q^\circ}(X', Y) \arrow[r] \arrow[d, "p_D"'] & \mathcal{E}_{/q^\circ}(X, Y) \arrow[d, "p"] \\
\{D\} \times \mathcal{C}_{/p_2 q^\circ}(M', N) \arrow[r] & \mathcal{K}_{/p_1 q^\circ}(C, D) \times \mathcal{C}_{/p_2 q^\circ}(M, N)
\end{tikzcd}$$

Now, note that (f, α) is the image of (id_D, α') under the bottom horizontal map, where α' is equivalent to some "composite" $\alpha p_2 \left(\overline{f}\right)^{-1}$ – note that $p_2 \left(\overline{f}\right)$ is an equivalence in $\mathcal{C}$ since $\overline{f}$ is p–coCartesian – so that the top map induces a weak homotopy equivalence of the homotopy fibres above these by Theorem 3.3.8. However, since q is p_D–limit, the left-most vertical map in the above square is a homotopy equivalence and hence its homotopy fibre above (id_D, α) in $\mathcal{E}_{D/q^\circ}(X', Y)$ is weakly contractible. Thus, the homotopy fibre above (f, α) is also weakly contractible.

For the other direction, observe that we still have the homotopy Cartesian square (13), but this time for some arbitrary object $X' \in \mathcal{E}_D$ and apply an argument similar to the previous paragraph to conclude that the homotopy fibres of the map $\mathcal{E}_{D/q^\circ}(X', Y) \longrightarrow \{D\} \times \mathcal{C}_{p_2 q^\circ}(M', N)$ are weakly contractible ensuring that the map is a weak homotopy equivalence and thus showing that q is a p_D–limit. **q.e.d.**

4 Semidirect Products of ∞–Operads

The main result of the previous section was that 1–functors $F : \mathsf{C} \longrightarrow \mathrm{Op}$ valued in coloured 1–operads are equivalent to C–families of coloured 1–operads that are coCartesian fibrations over C. We proved that post-composing F with the forgetful functor $\iota : \mathrm{Op} \longrightarrow \mathcal{C}\mathrm{at}$ and taking the Grothendieck construction resulted in a C–family. Conversely, we proved that if we start with a C–family whose underlying map to $\mathsf{C} \times \mathcal{F}\mathrm{in}_*$ is a map of coCartesian fibrations, then its inverse Grothendieck construction factors through Op. We shall now prove analogous results where we replace 1–operads by ∞–operads, 1–categories by ∞–categories, $\mathcal{F}\mathrm{in}_*$ by its nerve, $\mathcal{C}\mathrm{at}$ by $\mathcal{C}\mathrm{at}_\infty$, the Grothendieck construction by Luries unstraightening construction and its inverse by straightening [Lur08, §3.2]. We shall deal with this in §4.1 which will extensively use intermediary results from §3. This will then allow us to show that semidirect products of classical Set operads satisfying some hypotheses are subsumed by Lurie's assembly construction of [Lur17, §2.3.3] which we do in §4.2, §4.3 and §4.4. Then, in §4.5, we define semidirect products of ∞–operads as assemblies of families of ∞–operads arising from Op_∞–valued functors and show that under some hypotheses, two semidirect products of ∞–operads being equivalent implies the functors from which they arise are equivalent.

4.1 ∞–Operad Families and Op_∞ Diagrams

Let us begin with a functor of ∞–categories $F : \mathcal{K} \longrightarrow \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$. The sliced unstraightening (Definition 3.1.16) of this functor over $\mathcal{K}$ gives a morphism of coCartesian fibrations

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "p"] \arrow[dr, "p_1"'] & & \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*) \arrow[dl, "\pi_1"] \\ & \mathcal{K} & \end{tikzcd}$$

where p_1 is a coCartesian fibration given by the regular unstraightening of $\tau F : \mathcal{K} \longrightarrow \mathcal{C}\mathrm{at}_\infty$, for the forgetful functor $\tau : \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)} \longrightarrow \mathcal{C}\mathrm{at}_\infty$, and π_1 is the projection onto the first factor. We would like to show that if this map p happens to exhibit $\mathcal{E}$ as a $\mathcal{K}$–family of ∞–operads, then F must factor through the inclusion $\mu : \mathrm{Op}_\infty \hookrightarrow \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$, where we view Op_∞ as a subcategory of $N(\mathcal{F}\mathrm{in}_*)$. By definition of a subcategory, it suffices to show that the objects and morphisms of $\mathcal{K}$ are taken to objects and morphisms of Op_∞ respectively by F.

For objects, first observe that $\mathrm{St}_{/\pi_1}(p) \simeq F$ takes objects of $\mathcal{K}$ to ∞–operads. By Corollary 3.2.4, we have that for each object $C \in \mathcal{K}_0$, $\mathrm{St}_{/\pi_1}(p)(C)$ is the map

$$p|_{\mathcal{E}_C} : \mathcal{E}_C \longrightarrow (\{C\} \times N(\mathcal{F}\mathrm{in}_*)) \simeq N(\mathcal{F}\mathrm{in}_*)$$

By the definition of a $\mathcal{K}$–family of ∞–operads, $p|_{\mathcal{E}_C} : \mathcal{E}_C \longrightarrow N(\mathcal{F}\mathrm{in}_*)$ is an an ∞–operad. We would like that this makes $F(C)$ an ∞–operad as well. For this, notice that we have an equivalence of ∞–categories $\zeta_C : F(C) \simeq \mathcal{E}_C : \psi_C$ which commutes with the maps to $N(\mathcal{F}\mathrm{in}_*)$ by the fact that $\mathrm{St}_{/\pi_1} : \mathrm{CoCart}(\mathcal{K})_{/\pi_1} \longrightarrow \mathcal{F}\mathrm{un}\left(\mathcal{K}, \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}\right)$ is a categorical equivalence by the usual straightening-unstraightening equivalence and [Lur08, 1.2.9.3, p. 44]. This means that given a p_C–coCartesian lift of an inert morphism in $\mathcal{E}_C$, we obtain a p_C–coCartesian lift of that same morphism in $F(C)$ by the following pasting of commuting diagrams:

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left] & \Lambda^n_0 \arrow[r] \arrow[d] & F(C) \arrow[r, shift left, "\simeq"] \arrow[d] & \mathcal{E}_C \arrow[l, shift left] \arrow[d] \\
 & \Delta^n \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed] & N(\mathcal{F}\mathrm{in}_*) \arrow[r, equal] & N(\mathcal{F}\mathrm{in}_*)
\end{tikzcd}$$

giving the "inert lifting" condition of Definition 2.1.5. For the "decomposition of morphisms" condition on mapping spaces in that definition, observe that we have the following commuting diagram:

$$\begin{tikzcd}
\mathcal{E}_C^{\alpha}(X, Y) \arrow[r] \arrow[d] & \prod_{i=1}^n \mathcal{E}^{\rho^i\alpha}(X, Y) \arrow[d] \\
F(C)^{\alpha}(\psi_C(X), \psi_C(Y)) \arrow[r] & \prod_{i=1}^n F(C)^{\rho^i\alpha}(\psi_C(X), \psi_C(Y))
\end{tikzcd}$$

where the vertical arrows are homotopy equivalences of Kan complexes by the fact that categorical equivalences are fully faithful and essentially surjective. Thus, the bottom arrow is also a categorical equivalence by 2–out–of–3. For the "formation of tuple objects" condition in Definition 2.1.5, let $X_1, \ldots, X_n$ be objects of $F(C)$, and consider the objects $\zeta_C(X_i)$ in $\mathcal{E}_C$. There exists an object X' in $\mathcal{E}_C$ with coCartesian lifts $g^i : X' \longrightarrow \zeta_C(X_i)$ of the Segal maps ρ^i. Consider then the maps $\psi_C g^i : \psi_C(X') \longrightarrow \psi_C(\zeta_C(X_i)) \simeq X_i$. These maps are then easily checked to be coCartesian lifts of the Segal maps by an argument similar to the argument for inert lifting.

Remark 4.1.1. Note that in the above argument we have shown that any object of $\mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$ having a categorical equivalence to an ∞–operad over $N(\mathcal{F}\mathrm{in}_*)$ is itself an ∞–operad. In particular, any ∞–category categorically equivalent to an ∞–operad is also an ∞–operad. $\diamond$

We then shift focus to morphisms in $\mathcal{K}$ and their images in $\mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$. We want to show that F sends morphisms in $\mathcal{K}$ to morphisms of ∞–operads. Let $f : C \longrightarrow D$ be a morphism in $\mathcal{K}$ and observe that $Ff : F(C) \longrightarrow F(D)$ is a morphism over $N(\mathcal{F}\mathrm{in}_*)$. Then, by definition of unstraightening Ff is given by fibre transport along f in $\mathcal{E}$ [Lan21, §3.3, p. 192]. Since $\mathcal{E}$ is a $\mathcal{K}$–family of ∞–operads, for each inert morphism $\alpha : \langle m \rangle \longrightarrow \langle n \rangle$ in $N(\mathcal{F}\mathrm{in}_*)$ and each object $X \in \mathcal{E}_0$ with $p(X) = (C, \langle m \rangle)$, we have a p–coCartesian lift of (id_C, α) in $\mathcal{E}$. At the same time, $\mathcal{E}$ being a $\mathcal{K}$–family of ∞–operads implies that $\mathcal{E}_C$ is an ∞–operad and hence admits a p_C–coCartesian $\overline{\alpha} : X \longrightarrow X'$ of α within $\mathcal{E}_C$. By Lemma 3.5.3, this $\overline{\alpha}$ is, in fact, p–coCartesian. Applying Lemma 3.5.4, we have that the fibre transport $Ff\,(\overline{\alpha}) \in \mathcal{E}$ of $\overline{\alpha}$ along f is p–coCartesian which, by Lemma 3.5.2, implies that $Ff\,(\overline{\alpha})$ is p_D–coCartesian. This shows that Ff preserves coCartesian lifts of inert morphisms and is hence a morphism of ∞–operads. Thus, we have shown:

Theorem 4.1.2. *Let $F : \mathcal{K} \longrightarrow \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$ be a functor of ∞–categories and $p : \mathcal{E} \longrightarrow \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*)$, the sliced unstraightening of F over $\pi_1 : \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*) \longrightarrow \mathcal{K}$. If p is a $\mathcal{K}$–family of ∞–operads, then F factors through the inclusion $\mu : \mathrm{Op}_\infty \hookrightarrow \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$.*

Next, we will show the converse to the above result. That is, given that $F : \mathcal{K} \longrightarrow \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$ factors through $\mu : \mathrm{Op}_\infty \longrightarrow \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$. We would first like to show that each fibre $p_1^{-1}(\{C\}) = \mathcal{E}_C$ for $C \in \mathcal{K}_0$ is an ∞–operad. This again follows from Corollary 3.2.4 and the hypothesis that F sends objects of $\mathcal{K}$ to ∞–operads. We would then like to show that for each object $C \in \mathcal{K}_0$, each inert morphism $\alpha : \langle m \rangle \longrightarrow \langle n \rangle$ in $N(\mathcal{F}\mathrm{in}_*)$ and each $X \in \mathcal{E}_C$ with $p(X) = (C, \langle m \rangle)$, there exists a p–coCartesian lift $\overline{\alpha} : X \longrightarrow X'$ of (id_C, α) in $\mathcal{E}$. We take $\overline{\alpha}$ to be the p_C–coCartesian lift of α in $\mathcal{E}_C$ and appeal to Lemma 3.5.5 to deduce that $\overline{\alpha}$ is, in fact, p–coCartesian. We now claim that this is enough to guarantee that $\mathcal{E}$ is a $\mathcal{K}$–family of ∞–operads. The general result is as follows:

Theorem 4.1.3. *If a map $p : \mathcal{E} \longrightarrow \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*)$ satisfies:*

(i) the map $p_1 = \pi_1 p : \mathcal{E} \longrightarrow \mathcal{K}$ is a coCartesian fibration

(ii) p is a morphism of coCartesian fibrations $p_1 \longrightarrow \pi_1$

(iii) the fibre $\mathcal{E}_C = p_1^{-1}(C)$ above each object $C \in \mathcal{K}_0$ is an ∞–operad

(iv) fibre transport in $\mathcal{E}$ along any map $f \in \mathcal{K}_1$ is a map of ∞–operads

then $\mathcal{E}$ is a $\mathcal{K}$–family of ∞–operads.

Proof. From the above argument, it remains to show "formation of tuple objects" condition in Definition 2.1.13 holds but this is an immediate consequence of Lemma 3.5.6, the hypotheses and the fact that the relevant relative limit condition is satisfied within each fibre separately by [Lur17, 2.3.2.12, p. 254]. **q.e.d.**

As a consequence, we have shown the central result needed for defining semidirect products of ∞–operads:

Theorem 4.1.4. *Let $F : \mathcal{K} \longrightarrow \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$ be a functor of ∞–categories and $p : \mathcal{E} \longrightarrow \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*)$, the sliced unstraightening of F over $\pi_1 : \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*) \longrightarrow \mathcal{K}$. If F factors through the inclusion $\mu : \mathrm{Op}_\infty \hookrightarrow \mathcal{C}\mathrm{at}_{/N(\mathcal{F}\mathrm{in}_*)}$, then p exhibits $\mathcal{E}$ as a $\mathcal{K}$–family of ∞–operads.*

Theorem 4.1.2 and Theorem 4.1.4 together give:

Theorem 4.1.5. *Let $F : \mathcal{K} \longrightarrow \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$ be a functor of ∞–categories and $p : \mathcal{E} \longrightarrow \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*)$, the sliced unstraightening of F over $\pi_1 : \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*) \longrightarrow \mathcal{K}$. p is a $\mathcal{K}$–family of ∞–operads if and only if F factors through the inclusion $\mu : \mathrm{Op}_\infty \hookrightarrow \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$.*

4.2 Classical Semidirect Products

For some 1–category K , Consider a pseudofunctor $F : \mathsf{K} \longrightarrow \mathrm{Op}$, where Op is taken to be the bicategory of Set operads. Take the Grothendieck construction $\int \iota F$ where $\iota : \mathrm{Op} \longrightarrow \mathcal{C}\mathrm{at}$ is the forgetful functor $(\mathcal{O}^{\otimes} \longrightarrow \mathcal{F}\mathrm{in}_*) \longmapsto \mathcal{O}^{\otimes}$. We have shown that $\int \iota F$ is a K–family of 1–operads – observe that $\int \iota F \longrightarrow \mathcal{K} \times N(\mathcal{F}\mathrm{in}_*)$ and the sliced unstraightening $\mathrm{St}_{/\pi}(F)$ are the same functor. In the case that $\mathsf{K} = BM$ for some monoid M, there exists a 1–operad $\mathcal{O}^{\otimes}$ which is the image of the unique object of BM with F giving an action of M on $\mathcal{O}$. If we also suppose that $\mathcal{O}^{\otimes}$ has a unique colour so that it is an operad in the classical sense, we have a definition of a semidirect product operad $M \ltimes \mathcal{O}$, due to Wahl [Wah01], in the case that $\mathcal{O}$ has one colour. In this subsection, we wish to explore how the category of operators of this semidirect product operad relates to $\int \iota F$.

We recall the definition of a semidirect product for Set–valued operads with a single colour:

Definition 4.2.1 (Semidirect Product Operad with One Colour)**.** Let $F : BM \longrightarrow \mathrm{Op}$ and $\mathcal{O}^{\otimes}$ be as above and let $*$ denote the unique colour of $\mathcal{O}$. Define:

$$(M \ltimes \mathcal{O}) \left(\{*\}^k ; * \right) := M^k \times \mathcal{O} \left(\{*\}^k ; * \right)$$

A composition map

$$\circ : (M \ltimes \mathcal{O}) \left(\{*\}^k ; * \right) \times \prod_{i=1}^{k} (M \ltimes \mathcal{O}) \left(\{*\}^{q_i} ; * \right) \longrightarrow (M \ltimes \mathcal{O}) \left(\{*\}^{q_1 + \cdots + q_k} ; * \right)$$

is defined as follows, given operations $(g_1, \ldots, g_k, r) \in (M \ltimes \mathcal{O}) \left(\{*\}^k ; * \right)$ and $(h^i_1, \ldots, h^i_{q_i}, s_i) \in (M \ltimes \mathcal{O}) \left(\{*\}^{q_i} ; * \right)$ for $i \in \langle k \rangle^{\circ}$:

$$\begin{aligned} &(g_1, \ldots, g_k, r) \circ \left\{ (h^i_1, \ldots, h^i_{q_i}, s_i) \right\}_{i \in \langle k \rangle^{\circ}} \\ := &\left(g_1 h^1_1, \ldots, g_1 h^1_{q_1}, \ldots, g_k h^k_1, \ldots, g_k h^k_{q_k}, r \circ (g_1 s_1, \ldots, g_i s_i) \right) \end{aligned}$$

where $g_i s_i$ denotes the operation $F(g_i)(s_i)$. One checks that this makes $M \ltimes \mathcal{O}$ an operad. We will call this the semidirect product of $\mathcal{O}$ by M. $\diamond$

We wish to generalize this definition to coloured operads so that it fits with the theory we have built so far. The simplest idea in this regard might be to replace $(M \ltimes \mathcal{O}) \left(\{*\}^k ; * \right)$ and each $(M \ltimes \mathcal{O}) \left(\{*\}^{q_i} ; * \right)$ with $(M \ltimes \mathcal{O}) \left(\{b_i\}_{i \in \langle k \rangle^{\circ}} ; a \right)$ and $(M \ltimes \mathcal{O}) \left(\left\{ c^i_j \right\}_{j \in \langle q_i \rangle^{\circ}} ; b_i \right)$ respectively for colours c^i_j, b_i, a of $\mathcal{O}$. Notice, however, that this requires r to be an operation in $\mathcal{O} \left(\{g_i b_i\}_{i \in \langle k \rangle^{\circ}} ; a \right)$ for the composition as written above to be defined. This leads to the following definition of a semidirect product of a coloured operad with some monoid:

Definition 4.2.2 (Semidirect Product Operad)**.** Let $F : BM \longrightarrow \mathrm{Op}$ for some monoid M such that $\mathcal{O}^{\otimes}$ is the image of the unique object of BM under F. We define:

$$(M \ltimes \mathcal{O})\left(\{b_i\}_{i\in\langle k\rangle^\circ}\,;a\right) := \coprod_{(g_1,\dots,g_k)\in M^k} \{(g_1,\dots,g_k)\} \times \mathcal{O}\left(\{g_ib_i\}_{i\in\langle k\rangle^\circ}\,;a\right)$$

where g_ib_i denotes $F(g_i)(b_i)$. For an operation $(g_1,\dots,g_k,r) \in (M \ltimes \mathcal{O})\left(\{b_i\}_{i\in\langle k\rangle^\circ}\,;b\right)$, we have that $r \in \mathcal{O}\left(\{g_ib_i\}_{i\in\langle k\rangle^\circ}\right)$. Similarly, for operations $(h_1^i,\dots,h_{q_i}^i,s_i) \in (M \ltimes \mathcal{O})\left(\left\{c_j^i\right\}_{j\in\langle q_i\rangle^\circ}\,;b_i\right)$, $s_i \in \mathcal{O}\left(\left\{h_j^ic_j^i\right\}_{j\in\langle q_i\rangle^\circ}\,;b_i\right)$ such that $g_is_i \in \mathcal{O}\left(\left\{g_ih_j^ic_j^i\right\}_{j\in\langle q_i\rangle^\circ}\,;g_ib_i\right)$. Thus, we can define the composition:

$$(M \ltimes \mathcal{O})\left(\{b_i\}_{i\in\langle k\rangle^\circ}\,;a\right) \times \prod_{i=1}^{k}(M \ltimes \mathcal{O})\left(\{c_j^i\}_{j\in\langle q_i\rangle^\circ}\,;b_i\right) \longrightarrow (M \ltimes \mathcal{O})\left(\left\{\{c_j^i\}_{j\in\langle q_i\rangle^\circ}\right\}_{i\in\langle k\rangle^\circ}\,;a\right)$$

by the exact same formula as in the previous definition. Then, $M \ltimes \mathcal{O}$ is a coloured operad with the same colour set as $\mathcal{O}$ which we will call the semidirect product of $\mathcal{O}$ by M. $\diamond$

Remark 4.2.3. The colours of $M \ltimes \mathcal{O}$ are the same as those of $\mathcal{O}$ and hence the objects of $(M \ltimes \mathcal{O})^{\otimes}$ are precisely the object of $\mathcal{O}^{\otimes}$. $\diamond$

One cannot help notice the similarity of this definition of semidirect product operad with the Grothendieck construction $p : \int \iota F \longrightarrow BM \times \mathcal{F}\mathrm{in}_*$. Indeed, we can define a map $\gamma : \int \iota F \longrightarrow (M \ltimes \mathcal{O})^{\otimes}$ as follows:

(i) Let an object of $\int \iota F$ is of the form $(*, X)$ where $*$ is the unique object of BM and X is an object of the fibre $\left(\int \iota F\right)_* = \mathcal{O}^{\otimes}$. X is also an object of $(M \ltimes \mathcal{O})^{\otimes}$. Thus, we may set:

$$\gamma(*, X) = X$$

(ii) Let $f : X \longrightarrow Y$ be a morphism in $\int \iota F$ such that $p(f) = (g, \alpha)$ for some morphism $g : * \longrightarrow *$ in BM (which we may identify with an element $g \in M$) and some morphism $\alpha : \langle m\rangle \longrightarrow \langle n\rangle$ in $\mathcal{F}\mathrm{in}_*$. Now, f consists of a morphism $f' : gX = Fg(X) \longrightarrow Y$ in $\mathcal{O}^{\otimes}$ covering α. Since X and Y are objects of $(M \ltimes \mathcal{O})^{\otimes}$, we may choose lifts of Segal maps in $(M \ltimes \mathcal{O})^{\otimes}$ to realize X and Y as objects $\bigoplus_{i=1}^{m} X_i$ and $\bigoplus_{j=1}^{n} Y_j$ respectively – in other words, X and Y are tuples of colours of $M \ltimes \mathcal{O}$. We then have that $Fg(X) \simeq \bigoplus_{i=1}^{m} Fg(X_i)$ by the fact that Fg is a functor over $\mathcal{F}\mathrm{in}_*$ that preserves coCartesian lifts of inert morphisms (up to equivalence). If we then unwrap the definition of $(M \ltimes \mathcal{O})^{\otimes}$, we see that (α, f') is precisely a morphism $X \longrightarrow Y$ in $(M \ltimes \mathcal{O})^{\otimes}$. So, we may set:

$$\gamma(f) = (\alpha, f')$$

One easily verifies that γ is functor over $\mathcal{F}\mathrm{in}_*$ that preserves coCartesian lifts of inert morphisms.

We would eventually like to show that this map assembles $\int \iota F$ to $(M \ltimes \mathcal{O})^{\otimes}$ in the sense of [Lur17, 2.3.3.1, p. 259], recalling that $N\left(\int \iota F\right)$ is an $N(\mathsf{K})$–family of ∞–operads by Corollary 2.1.16 and hence a generalized ∞–operad by [Lur17, 2.3.2.13, p. 254]. However, we will have to defer the proof of this claim until we develop some machinery for the task. For now, we describe what $\int \iota F$, $(M \ltimes \mathcal{O})^{\otimes}$ and γ are a bit more concretely.

Remark 4.2.4. Strictly speaking, an object of $\int \iota F$ is a tuple $(*, \langle m\rangle, X_1, \ldots, X_m)$ where $*$ is the unique object of BM, $\langle m\rangle \in \mathsf{ob}(\mathcal{F}\mathrm{in}_*)$ and $X_i, i \in \langle m\rangle^{\circ}$ are colours of $\mathcal{O}^{\otimes}$ while objects of $(M \ltimes \mathcal{O})^{\otimes}$ are tuples $(\langle m\rangle, X_1, \ldots, X_m)$ for $\langle m\rangle \in \mathcal{F}\mathrm{in}_*$ and $X_i, i \in \langle m\rangle^{\circ}$ are again colours of $\mathcal{O}^{\otimes}$. Thus,

$$\gamma(*, \langle m\rangle, X_1, \ldots, X_m) = (\langle m\rangle, X_1, \ldots, X_m)$$

Hence, we may take the objects of both $\int \iota F$ and $(M \ltimes \mathcal{O}^{\otimes})$ as simply all possible tuples of colours $(X_1, \ldots, X_m)$ of $\mathcal{O}^{\otimes}$. Let $X = \{X_i\}_{i\in\langle m\rangle^{\circ}}$ and $Y = \{Y_j\}_{j\in\langle n\rangle^{\circ}}$. A morphism of $f : X \longrightarrow Y$ consists of a morphism $g : * \longrightarrow *$ in BM, a morphism $\alpha : \langle m\rangle \longrightarrow \langle n\rangle$ and operations $f_j \in \mathcal{O}\left(\{gX_i = Fg(X_i)\}_{i\in\alpha^{-1}(j)}; Y_j\right)$ for each $j \in \langle n\rangle^{\circ}$. A morphism $f : X \longrightarrow Y$ consists of, again, a morphism $\alpha : \langle m\rangle \longrightarrow \langle n\rangle$, sequences of elements $\left\{g_i^j\right\}_{i\in\alpha^{-1}(j)}$ of M and operations $f'_j \in \mathcal{O}\left(\left\{g_i^j X_i = F g_i^j(X_i)\right\}_{i\in\alpha^{-1}(j)}; Y_j\right)$ for each $j \in \langle n\rangle^{\circ}$. That is, the difference is that the g_i^j are potentially not all the same in the case of $(M \ltimes \mathcal{O})^{\otimes}$. Thus, taking each $g_i^j = g$, we may write:

$$\gamma\left(g, \alpha, \{f_j\}_{j\in\langle n\rangle^{\circ}}\right) = \left(\alpha, \left\{\left(\left\{g_i^j = g\right\}_{i\in\alpha^{-1}(j)}, f_j\right)\right\}_{j\in\langle n\rangle^{\circ}}\right)$$

It is easy to see that the map γ preserves coCartesian lifts of inert morphisms. $\diamond$

4.3 CoCartesian Operads

To proceed with showing that the map $\gamma : \int \iota F \longrightarrow (M \ltimes \mathcal{O})^{\otimes}$, defined in the previous subsection for a monoid M and a functor $F : BM \longrightarrow \mathsf{Op}$ taking the unique object of BM to $\mathcal{O}^{\otimes}$, assembles the domain to the codomain, we will require the notion of a coCartesian 1–operad which we define next.

Definition 4.3.1 (CoCartesian Operad)**.** Let A be a 1–category and for objects $X_1, \ldots, X_m, Y \in \mathsf{ob}(\mathsf{A})$, define:

$$\mathsf{A}_c\left(\{X_i\}_{i\in\langle m\rangle^{\circ}}; Y\right) = \prod_{i=1}^{m} \mathsf{A}(X_i, Y)$$

Given $f = \{f_i\} \in \mathsf{A}_c\left(\{X_i\}_{i\in\langle n\rangle^{\circ}}; Y\right)$ and $g = g^i = \left\{g_j^i\right\} \in \mathsf{A}_c\left(\left\{W_j^i\right\}_{j\in\langle m_i\rangle^{\circ}}; X_i\right)$, define a composite:

$$f \circ \left\{g^i\right\}_{i\in\langle n\rangle^{\circ}} = \left\{\left\{f_i g_j^i\right\}_{j\in\langle m_i\rangle^{\circ}}\right\}_{i\in\langle n\rangle}$$

It easy easy to check that this makes A_c into a 1–operad in Set whose set of colours is $\mathsf{ob}\,(\mathsf{A})$. We will denote the category of operators of A_c by $\mathsf{A}^{\amalg}$. We will call A_c the cocompletion of A and any 1–operad equivalent to A_c for some 1–category A, a coCartesian operad. $\diamond$

We wish to show that this definition of coCartesian 1–operad satisfies the definition of coCartesian ∞–operad as given in [Lur17, 2.4.3.1, p. 297] when we take the nerve. That is, we would like to show that there is a bijection:

$$\mathsf{sSet}_{/N(\mathcal{F}\mathrm{in}_*)}\left(K, N\left(\mathsf{A}^{\amalg}\right)\right) \cong \mathsf{sSet}\left(K \times_{N(\mathcal{F}\mathrm{in}^*)} N(\Gamma^*), N(\mathsf{A})\right) \tag{14}$$

natural in K, for all simplicial sets K, where Γ^* is the category whose:

(i) objects are pairs $(\langle n\rangle, i)$ with $\langle n\rangle \in \mathsf{ob}\,(\mathcal{F}\mathrm{in}_*)$ and $i \in \langle n\rangle^\circ$

(ii) morphisms $(\langle m\rangle, i) \longrightarrow (\langle n\rangle, j)$ are morphisms $\alpha : \langle m\rangle \longrightarrow \langle n\rangle$ with $\alpha(i) = j$

equipped with a forgetful map $\Gamma^* \longrightarrow \mathcal{F}\mathrm{in}_*$ sending $(\langle n\rangle, i)$ to $\langle n\rangle$ and morphisms to themselves. We first show the desired bijection for all $K = \Delta^n$. Notice that $\Delta^n = \mathrm{Hom}_\Delta(-, [n]) = N([n])$ where $[n]$ is viewed as the category generated by a path graph with n vertices. Then, since the nerve functor is a right adjoint [Lur08, 1.2.3.1, p. 34], it preserves limits and hence, we must have $\Delta^n \times_{N(\mathcal{F}\mathrm{in}_*)} N(\Gamma^*) \cong N([n] \times_{\mathcal{F}\mathrm{in}_*} \Gamma^*)$. Furthermore, since the nerve functor is fully faithful [Lan21, 1.2.20, p. 32], it suffices to show that we have a bijection:

$$\mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}\left([n], \mathsf{A}^{\amalg}\right) \cong \mathcal{C}\mathrm{at}\left([n] \times_{\mathcal{F}\mathrm{in}_*} \Gamma^*, \mathsf{A}\right) \tag{15}$$

Let $p : \mathsf{A}^{\amalg} \longrightarrow \mathcal{F}\mathrm{in}_*$ be the map realizing $\mathsf{A}^{\amalg}$ as an operad. Fix a map $r : [n] \longrightarrow \mathcal{F}\mathrm{in}_*$ which gives a sequence of composeable morphisms

$$\langle m_0\rangle \xrightarrow{\alpha_1} \langle m_1\rangle \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} \langle m_n\rangle$$

in $\mathcal{F}\mathrm{in}_*$. Consider a map $f : [n] \longrightarrow \mathsf{A}^{\amalg}$ such that $pf = r$. The data of f consists of the above sequence of maps in $\mathcal{F}\mathrm{in}_*$ given by r, along with a sequence of morphisms

$$X^0 \xrightarrow{f^1} X^1 \xrightarrow{f^2} \cdots \xrightarrow{f^n} X^n$$

in $\mathsf{A}^{\amalg}$ for $i = 1, \ldots, n$ such that $p(f^i) = \alpha_i$, where $X^q := \left\{X^q_j\right\}_{j \in \langle m_q\rangle^\circ}$ is an object of $\mathsf{A}^{\amalg}$ above $\langle m_q\rangle$ for $q \in \langle n\rangle^\circ$. The f^i, in turn, consist of operations $f^{i,k} \in \mathsf{A}_c\left(\left\{X^{i-1}_j\right\}_{j \in \alpha_i^{-1}(k)}; X^i_k\right)$ for each $i = 1, 2, \ldots, n$ and $k \in \langle m_i\rangle^\circ$. By definition of $\mathsf{A}^{\amalg}$, each $f^{i,k}$ as above is a collection of morphisms $\left\{f^{i,k}_j : X^{i-1}_j \longrightarrow X^i_k\right\}_{j \in \alpha_i^{-1}(k)}$ in A.

Now, $[n] \times_{\mathcal{F}\mathrm{in}_*} \Gamma^*$ is the pullback of the span $[n] \xrightarrow{pf} \mathcal{F}\mathrm{in}_* \longleftarrow \Gamma^*$ in $\mathcal{C}\mathrm{at}$ and hence has as objects tuples $(i, \langle m_i\rangle, j)$ for $i \in \langle n\rangle^\circ$ and $j \in \langle m_i\rangle^\circ$. For $i = 1, \ldots, n$, a morphism of $[n] \times_{\mathcal{F}\mathrm{in}_*} \Gamma^*$

of the form $(i-1, \langle m_{i-1}\rangle, j) \longrightarrow (i, \langle m_i\rangle, \alpha_i(j))$ consist of the unique morphism $u_i : (i-1) \longrightarrow i$ in $[n]$ and the morphism $\alpha_i : \langle m_{i-1}\rangle \longrightarrow \langle m_i\rangle$. All other morphisms are composites of such morphisms. This allows us to define the following mapping:

$$\begin{array}{rccc} \overline{f} & : & [n]\times_{\mathcal{F}\mathrm{in}_*}\Gamma^* & \longrightarrow & \mathsf{A} \\ & : & (i, \langle m_i\rangle, j) & \longmapsto & \left\{X_k^i\right\}_{k\in\langle m_i\rangle} \\ & : & (u_i,\alpha_i) : (i-1, \langle m_{i-1}\rangle, j) \longrightarrow (i, \langle m_i\rangle, \alpha_i(j)) & \longmapsto & f_j^{i,\alpha_i(j)} \end{array}$$

It is then straightforward to check that the mapping

$$\mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}\left([n], \mathsf{A}^{\amalg}\right) \longrightarrow \mathcal{C}\mathrm{at}\left([n]\times_{\mathcal{F}\mathrm{in}_*}\Gamma^*, \mathsf{A}\right) : f \longmapsto \overline{f}$$

is bijective.

Now, consider the coface map $\delta^i : [n-1] \longrightarrow [n]$. Then, $\mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}\left(\delta_i, \mathsf{A}^{\amalg}\right)$ sends $f : [n] \longrightarrow \mathsf{A}^{\amalg}$ to $f\delta^i : [n-1] \longrightarrow \mathsf{A}^{\amalg}$ which is the following sequence of composeable maps in $\mathsf{A}^{\amalg}$:

$$X^0 \xrightarrow{f^1} X^1 \xrightarrow{f^2} \cdots \xrightarrow{f^{i-1}} X^{i-1} \xrightarrow{f^{i+1}\circ f^i} X^{i+1} \xrightarrow{f^{i+2}} \cdots \xrightarrow{f^n} X^n$$

It is then straightforward to check that the following diagram commutes:

$$\begin{tikzcd} \mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}\left([n], \mathsf{A}^{\amalg}\right) \arrow[r, "\overline{(-)}"] \arrow[d, "(\delta^i)^*"'] & \mathcal{C}\mathrm{at}\left([n]\times_{\mathcal{F}\mathrm{in}_*}\Gamma^*, \mathsf{A}\right) \arrow[d, "(\delta^i\times_{\mathcal{F}\mathrm{in}_*}\Gamma^*)^*"] \\ \mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}\left([n-1], \mathsf{A}^{\amalg}\right) \arrow[r, "\overline{(-)}"'] & \mathcal{C}\mathrm{at}\left([n]\times_{\mathcal{F}\mathrm{in}_*}\Gamma^*, \mathsf{A}\right) \end{tikzcd}$$

We can similarly show that $\overline{(-)}$ commutes with codegeneracy maps. This, in turn, shows that the bijective correspondence (15) is natural in $[n]$. In addition, the fully faithful property of the nerve functor implies that we indeed have a bijection as (14) whenever $K = \Delta^n$ and this bijection is natural in the Δ^n argument.

Now, take an arbitrary simplicial set K and write it as a colimit of representables: $K \cong \mathrm{colim}_{[n]\in\Delta}\,\Delta^n$. It is well known that colimits in a topos are stable under pullback [Lur08, §6.1] so that $-\times_{\mathcal{F}\mathrm{in}_*}\Gamma^*$ preserves colimits. Thus, we have:

$$\begin{aligned} \mathsf{sSet}\left(K\times_{N(\mathcal{F}\mathrm{in}_*)} N\left(\Gamma^*\right), N(\mathsf{A})\right) &\cong \mathsf{sSet}\left(\left(\mathrm{colim}\,\Delta^n\right)\times_{N(\mathcal{F}\mathrm{in}_*)} N\left(\Gamma^*\right), N(\mathsf{A})\right) \\ &\cong \mathsf{sSet}\left(\mathrm{colim}\left(\Delta^n\times_{N(\mathcal{F}\mathrm{in}_*)} N\left(\Gamma^*\right)\right), N(\mathsf{A})\right) \\ &\cong \lim \mathsf{sSet}\left(\Delta^n\times_{N(\mathcal{F}\mathrm{in}_*)} N\left(\Gamma^*\right), N(\mathsf{A})\right) \\ &\cong \lim \mathsf{sSet}_{/N(\mathcal{F}\mathrm{in}_*)}\left(\Delta^n, N\left(\mathsf{A}^{\amalg}\right)\right) \\ &\cong \mathsf{sSet}_{/N(\mathcal{F}\mathrm{in}_*)}\left(\mathrm{colim}\,\Delta^n, N\left(\mathsf{A}^{\amalg}\right)\right) \\ &\cong \mathsf{sSet}_{/N(\mathcal{F}\mathrm{in}_*)}\left(K, N\left(\mathsf{A}^{\amalg}\right)\right) \end{aligned}$$

Hence we have shown:

Theorem 4.3.2. *For any* 1*–category* A*, we have an isomorphism* $N\left(\mathsf{A}^{\amalg}\right) \cong N(\mathsf{A})^{\amalg}$ *so that this is a coCartesian* ∞*–operad.*

Notation 4.3.3. Given a 1–operad $\mathcal{O}$, we can consider the underlying category $\mathcal{O}^{\otimes}_{\langle 1\rangle}$ of its category of operators. We can then form the cocompletion of the underlying category $\left(\mathcal{O}^{\otimes}_{\langle 1\rangle}\right)^{\amalg}$ to get a coCartesian 1–operad. We will denote this simply as $\mathcal{O}^{\amalg}$. $\diamond$

Corollary 4.3.4. *For any* 1*–operad* $\mathcal{O}^{\otimes}$*, the underlying categories of* $\mathcal{O}^{\amalg}$ *and* $\mathcal{O}^{\otimes}$ *are equal.*

The following lemma is an elementary fact that shows that maps of (categories of operators of) 1–operads are determined by underlying mappings of operation sets. This will be helpful to prove a result for checking when certain maps into coCartesian operads are categorical fibrations. This will in turn be a crucial point in proving the map $\gamma : \int \iota F \longrightarrow (M \ltimes \mathcal{O})^{\otimes}$ assembles $\int \iota F$ to $(M \ltimes \mathcal{O})^{\otimes}$.

Lemma 4.3.5. *Let* $\phi : \mathcal{A}^{\otimes} \longrightarrow \mathcal{B}^{\otimes}$ *be a map of* 1*–operads. Then the values of* ϕ *on morphisms covering active maps of the form* $\langle m\rangle \longrightarrow \langle 1\rangle$ *in* $\mathcal{F}\mathrm{in}_*$ *determine* f*.*

Proof. Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{A}^{\otimes}$ covering a map $\alpha : \langle m\rangle \longrightarrow \langle n\rangle$. Then X and Y are tuples $\{X_i\}_{i\in\langle m\rangle^{\circ}}$ and $\{Y_j\}_{j\in\langle n\rangle^{\circ}}$ respectively and f consists of operations $f_j \in \mathcal{A}\left(\{X_i\}_{i\in\alpha^{-1}(j)}; Y_j\right)$ for each $j \in \langle n\rangle^{\circ}$. Take the following morphisms in $\mathcal{A}^{\otimes}$, for each $j \in \langle n\rangle^{\circ}$:

(i) $g_j = \left(\rho^j, \{\mathrm{id}_{Y_j}\}\right)$ for each $j \in \langle m\rangle^{\circ}$ where $\rho^j : \langle m\rangle \longrightarrow \langle 1\rangle$ is the j–th Segal map

(ii) $h_j = \left(\beta_j, \{\mathrm{id}_{X_i}\}_{i\in\langle|\alpha^{-1}(j)|\rangle^{\circ}}\right)$, where $\beta_j : \langle|\alpha^{-1}(j)|\rangle \longrightarrow \langle m\rangle$ is the inert morphism defined as follows. Let $\alpha^{-1}(j) = \left\{k_1, \ldots, k_{|\alpha^{-1}(j)|}\right\}$ where $k_i < k_{i+1}$ and take $\beta_j(i) = k_i$ for $i \in \langle|\alpha^{-1}(j)|\rangle^{\circ}$, and $\beta_j(*) = *$.

For a fixed $j \in \langle n\rangle^{\circ}$, the composite $g_j f h_j : \left\{X_{\beta_j(i)}\right\}_{i\in\langle|\alpha^{-1}(j)|\rangle^{\circ}} \longrightarrow \{Y_j\}$ lies over the morphism $\rho^j\alpha\beta_j : \langle|\alpha^{-1}(j)|\rangle \longrightarrow \langle 1\rangle$ which is the unique active morphism of this form. Since ϕ is a map of 1–operads, we have that $\phi(g_j f h_j)$ also lies over $\rho^j\alpha\beta_j$. It is then easy to see that $g_j f h_j$ is equal to the pair $(\rho^j\alpha\beta_j, \{f_j\})$ so that $\phi(g_j f h_j) = \left(\rho^j\alpha\beta_j, \overline{\phi}(f_j)\right)$ for some operation $\overline{\phi}(f_j) \in \mathcal{B}\left(\left\{X_{\beta_j(i)}\right\}_{i\in\langle|\alpha^{-1}(j)|\rangle}; Y_j\right)$. Then, if $\phi(f) = \left(\alpha, \{g_j\}_{j\in\langle n\rangle^{\circ}}\right)$ for operations $g_j \in \mathcal{B}\left(\{\phi(X_i)\}_{i\in\alpha^{-1}(j)}; \phi(Y_j)\right)$, each g_j must be equal to $\overline{\phi}(f_j)$ by functoriality of ϕ. **q.e.d.**

Lemma 4.3.6. *Any map* $\phi : \mathcal{O}^{\otimes} \longrightarrow \mathcal{O}^{\amalg}$ *of* 1*–operads that restricts to an isofibration* $\mathcal{O}^{\otimes}_{\langle 1\rangle} \longrightarrow \mathcal{O}^{\amalg}_{\langle 1\rangle}$ *is itself an isofibration and* $N(\phi) : N\left(\mathcal{O}^{\otimes}\right) \longrightarrow N\left(\mathcal{O}^{\amalg}\right) \cong N\left(\mathcal{O}^{\otimes}_{\langle 1\rangle}\right)^{\amalg}$ *is a categorical fibration.*

Proof. An isomorphism $f : X = \{X_i\}_{i\in\langle m\rangle^{\circ}} \longrightarrow Y = \{Y_j\}_{j\in\langle m\rangle^{\circ}}$ in $\mathcal{O}^{\amalg}$ consists of a an isomorphism $\alpha : \langle m\rangle \longrightarrow \langle m\rangle$ along with morphisms $f_j : X_{\alpha^{-1}(j)} \longrightarrow Y_j$ for each $j \in \langle m\rangle^{\circ}$ in the underlying category $\mathcal{O}^{\amalg}_{\langle 1\rangle}$. In addition, each f_j is an isomorphism in the underlying category, call it C. Then, since $\phi|_{\mathsf{C}}$ is an isofibration, we have lifts $\overline{f}_j : \overline{X}_{\alpha^{-1}(j)} \longrightarrow \overline{Y}_j$ in $\mathcal{O}^{\otimes}$ of each f_j along ϕ, for each choice of lift $\overline{X}_{\alpha^{-1}(j)}$ of $X_{\alpha^{-1}(j)}$ along ϕ, for $j \in \langle m\rangle^{\circ}$. That is, we have $\phi\left(\mathrm{id}_{\langle 1\rangle}, \overline{f}_j\right) = \left(\mathrm{id}_{\langle 1\rangle}, f_j\right)$, $\phi\left(\overline{X}_i\right) = X_i$ and $\phi\left(\overline{Y}_j\right) = Y_j$ for each $i, j \in \langle m\rangle^{\circ}$. Consider then the morphism $\overline{f} = \left(\alpha, \left\{\overline{f}_j\right\}_{j\in\langle m\rangle^{\circ}}\right) : \overline{X} = \left\{\overline{X}_i\right\}_{i\in\langle m\rangle^{\circ}} \longrightarrow \overline{Y} = \left\{\overline{Y}_j\right\}_{j\in\langle m\rangle^{\circ}}$ in $\mathcal{O}^{\otimes}$. By the previous lemma, the image of this under ϕ is determined by the morphisms $\phi\left(\mathrm{id}_{\langle 1\rangle}, \left\{\overline{f}_j\right\}\right)$ and hence we have that $\phi\left(\overline{f}\right) = f$. Thus, since each $\overline{f}_j$ is an isomorphism and α is an isomorphism in the respective categories, $\overline{f}$ is an isomorphism lifting f for the choice of domain $\overline{X}$. Since f and $\overline{X}$ were arbitrary, ϕ is an isofibration. By [Lan21, 2.1.5, p. 98], $N(\phi)$ is a categorical fibration since all maps of 1–categories are inner fibrations. **q.e.d.**

4.4 Assembly of an Op–Valued Functor

The proof of [Lur17, 2.3.4.4, p. 275] gives a procedure to construct a a generalized ∞–operad or, equivalently a family of ∞–operads, that assembles to a given ∞–operad with some hypotheses on the ∞–operad. In this subsection, we follow this procedure to construct a family of 1–operads that assembles to a classical semidirect product operad. Suppose we have a functor $F : BM \longrightarrow \mathrm{Op}$ for a monoid M, as before, so that the image of the unique object of BM is some Set–valued 1–operad $\mathcal{O}^{\otimes}$. We then apply the procedure to the semidirect product $(M \ltimes \mathcal{O})^{\otimes}$. For convenience of notation, we will denote $(M \ltimes \mathcal{O})^{\otimes}$ as $\mathcal{A}^{\otimes}$.

The procudure requires that $\mathcal{A}^{\otimes}$ be a unital ∞–operad which means the space of nullary operations $\mathcal{A}(\varnothing; X)$ is contractible for each colour X. Since $\mathcal{A}^{\otimes}$ is a Set–valued operad, the operation spaces are discrete and the assumption amounts requiring $\mathcal{A}(\varnothing; X)$ to have a unique element. This can be ensured if $\mathcal{O}(\varnothing; X)$ has a unique element for $\mathcal{A}(\varnothing; X) = M^0 \times \mathcal{O}(\varnothing; X) \cong \mathcal{O}(\varnothing; X)$. Thus we place this assumption on $\mathcal{O}^{\otimes}$. Given that $\mathcal{A}^{\otimes}$ is unital, we may now invoke [Lur17, 2.4.3.9, p. 300] to obtain a map

$$u : \mathcal{A}^{\otimes} \longrightarrow \mathcal{A}^{\amalg}$$

that extends the identity functor on underlying categories: $\mathcal{A}^{\otimes}_{\langle 1\rangle} \longrightarrow \mathcal{A}^{\amalg}_{\langle 1\rangle}$. Strictly speaking, we must pass to the nerve $N(\mathcal{A}^{\otimes})$, check that this is a unital ∞–operad, apply the result to obtain a map $u_0 : N(\mathcal{A}^{\otimes}) \longrightarrow N\left(\mathcal{A}^{\amalg}\right) \cong N(\mathcal{A})^{\amalg}$ and recover the map u by using the fact that the nerve functor is fully faithful, but we skip this straightforward verification.

The next ingredient in the construction is a specific map $d : \mathcal{A}^{\otimes}_{\langle 1\rangle} \times \mathcal{F}\mathrm{in}_* \longrightarrow \mathcal{A}^{\amalg}$ corresponding to the projection

$$\pi : \left(\mathcal{A}^{\otimes}_{\langle 1\rangle} \times \mathcal{F}\mathrm{in}_*\right) \times_{\mathcal{F}\mathrm{in}_*} \Gamma^* \longrightarrow \mathcal{A}^{\otimes}_{\langle 1\rangle}$$

under the bijection:

$$\mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}\left(\mathcal{A}^{\otimes}_{\langle 1\rangle} \times \mathcal{F}\mathrm{in}_*, \mathcal{A}^{\amalg}\right) \cong \mathcal{C}\mathrm{at}\left(\left(\mathcal{A}^{\otimes}_{\langle 1\rangle} \times \mathcal{F}\mathrm{in}_*\right) \times_{\mathcal{F}\mathrm{in}_*} \Gamma^*, \mathcal{A}^{\otimes}_{\langle 1\rangle}\right)$$

For convenience, we denote $\mathcal{A}^{\otimes}_{\langle 1\rangle} \times \mathcal{F}\mathrm{in}_*$ as C. Then, for a vertex $X : [0] \longrightarrow \mathsf{C}$, we have the precomposition map $X^* : \mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}*}\left(\mathsf{C}, \mathcal{A}^{\amalg}\right) \longrightarrow \mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}\left([0], \mathcal{A}^{\amalg}\right)$ as well as the map induced by precompsition with X: $\widetilde{X} : \mathcal{C}\mathrm{at}\left(\mathsf{C} \times_{\mathcal{F}\mathrm{in}_*} \Gamma^*, \mathcal{A}^{\otimes}_{\langle 1\rangle}\right) \longrightarrow \mathcal{C}\mathrm{at}\left([0] \times_{\mathcal{F}\mathrm{in}_*} \Gamma^*, \mathcal{A}^{\otimes}_{\langle 1\rangle}\right)$. These make the following diagram commute by naturality:

$$\begin{tikzcd}
\mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}*}\left(\mathsf{C}, \mathcal{A}^{\amalg}\right) \arrow[r, "\cong"] \arrow[d, "X^*"'] & \mathcal{C}\mathrm{at}\left(\mathsf{C} \times_{\mathcal{F}\mathrm{in}_*} \Gamma^*, \mathcal{A}^{\otimes}_{\langle 1\rangle}\right) \arrow[d, "\widetilde{X}"] \\
\mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*}\left([0], \mathcal{A}^{\amalg}\right) \arrow[r, "\cong"'] & \mathcal{C}\mathrm{at}\left([0] \times_{\mathcal{F}\mathrm{in}_*} \Gamma^*, \mathcal{A}^{\otimes}_{\langle 1\rangle}\right)
\end{tikzcd}$$

The map $X' = X \times_{\mathcal{F}\mathrm{in}_*} \Gamma^* : [0] \times_{\mathcal{F}\mathrm{in}_*} \Gamma^* \longrightarrow \mathsf{C} \times_{\mathcal{F}\mathrm{in}_*} \Gamma^*$ has image $\{(X_0, \langle m\rangle, i) : i \in \langle m\rangle^\circ\}$ where $X(0) = (X_0, \langle m\rangle)$. Then, $\pi \circ X'$ sends all objects of $[0] \times_{\mathcal{F}\mathrm{in}_*} \Gamma^*$ to the single object $X_0 \in \mathcal{A}^{\otimes}_{\langle 1\rangle}$. If we unwrap the definition of the correspondence, this means that

$$d(X(0)) = (\overbrace{X_0, \ldots, X_0}^{m \text{ times}})$$

By a similar argument, we can show that for all $f : X \longrightarrow Y \in \mathcal{A}^{\otimes}_{\langle 1\rangle}, \alpha : \langle m\rangle \longrightarrow \langle n\rangle \in \mathcal{F}\mathrm{in}_*$:

$$d(f, \alpha) = \left(\alpha, \left\{\coprod_{i \in \alpha^{-1}(j)} \{f\}\right\}_{j \in \langle n\rangle^\circ}\right)$$

That is, d is an analogue of a diagonal map.

Finally, we take the following pullback, where we will show that the map v_0 assembles $\mathcal{P}^{\otimes}_0$ to $N\left(\mathcal{A}^{\otimes}\right)$:

$$\begin{tikzcd}
\mathcal{P}^{\otimes}_0 \arrow[rr, "v_0"] \arrow[d, "w_0"'] & & N\left(\mathcal{A}^{\otimes}\right) \arrow[d, "N(u)"] \\
N\left(\mathcal{A}^{\otimes}_{\langle 1\rangle}\right) \times N(\mathcal{F}\mathrm{in}_*) \cong N\left(\mathcal{A}^{\otimes}_{\langle 1\rangle} \times \mathcal{F}\mathrm{in}_*\right) \arrow[rr, "N(d)"'] & & N\left(\mathcal{A}^{\amalg}\right) \cong N\left(\mathcal{A}^{\otimes}_{\langle 1\rangle}\right)^{\amalg}
\end{tikzcd}$$

First, we observe that $\mathcal{P}^{\otimes}_0$ is a generalized ∞–operad. To see this, recall that $N(u)$ is a categorical fibration by Lemma 4.3.6 and hence, a fibration in the generalized ∞–operadic model structure [Lur17, 2.3.2.4(3), p. 251]. Then, since fibrations are closed under pullback in any model category [Bal21, 2.1.9, p. 14], w_0 is a fibration, and since the codomain of w is fibrant by [Lur17, 2.3.2.9(1), p. 252], we must have that $\mathcal{P}^{\otimes}_0$ is fibrant, and hence, a generalized ∞–operad.

Remark 4.4.1. We have that all objects in the above span whose pullback we took are generalized ∞–operads. Then, since $N(u)$ is a fibration in the generalized ∞–operadic model structure as noted before, by [Lur08, A.2.4.5, p. 644], the above pullback diagram is a homotopy pullback diagram in the generalized ∞–operadic model structure. The same argument in the Joyal model structure shows that it is a homotopy pullback diagram in this model structure as well. $\diamond$

Furthermore, since the nerve functor preserves limits, being a right adjoint, we can take the above pullback diagram to be the nerve of the following 1–categorical pullback diagram:

$$\begin{tikzcd} \mathcal{P}^{\otimes} \arrow[r, "v"] \arrow[d, "w"'] & \mathcal{A}^{\otimes} \arrow[d, "u"] \\ \mathcal{A}^{\otimes}_{\langle 1 \rangle} \times \mathcal{F}\mathrm{in}_{*} \arrow[r, "d"'] & \mathcal{A}^{\amalg} \end{tikzcd}$$

We observe that $\mathcal{P}^{\otimes}$ consists of the following data:

(i) objects of the form $((X, \langle m \rangle), (\langle n \rangle, X_1, \ldots, X_n))$, where each X and X_j, $j \in \langle n \rangle^{\circ}$ are colours of $\mathcal{A}^{\otimes}$ and $d(X, \langle m \rangle) = u(\langle n \rangle, X_1, \ldots, X_n) = (\langle n \rangle, X_1, \ldots, X_n)$ which necessitates $m = n$ and for each $j \in \langle n \rangle^{\circ}$, $X_j = X$. That is, we can take the objects of $\mathcal{P}^{\otimes}$ to be tuples $(X, \langle m \rangle)$ for colours X of $\mathcal{A}^{\otimes}$ and objects $\langle m \rangle$ in $\mathcal{F}\mathrm{in}_{*}$.

(ii) morphisms $f : (X, \langle m \rangle) \longrightarrow (Y, \langle n \rangle)$ consist of a morphism $(f_0, f_1) : (X, \langle m \rangle) \longrightarrow (Y, \langle n \rangle)$ in $\mathcal{A}^{\otimes}_{\langle 1 \rangle} \times \mathcal{F}\mathrm{in}_{*}$ and a morphism in $\mathcal{A}^{\otimes}$ consisting of a morphism $\alpha : \langle m \rangle \longrightarrow \langle n \rangle$ in $\mathcal{F}\mathrm{in}_{*}$ and a collection of operations $\left\{ r_j \in \mathcal{A}\left(\{X_i\}_{i \in \alpha^{-1}(j)} ; Y_j \right) \right\}_{j \in \langle n \rangle^{\circ}}$, satisfying

$$d(f_0, f_1) = u\left(\alpha, \{r_j\}_{j \in \langle n \rangle^{\circ}}\right) = \left(\alpha, \{r_{j,i}\}_{i \in \alpha^{-1}(j), j \in \langle n \rangle^{\circ}}\right)$$

for morphisms $r_{j,i} : X_i \longrightarrow Y_j$ in $\mathcal{A}^{\otimes}_{\langle 1 \rangle}$ where u sends each operation r_j in $\mathcal{A}^{\otimes}$ to the operation $\{r_{j,i}\}_{i \in \langle \alpha^{-1}(j) \rangle}$ in $\mathcal{A}^{\amalg}$. This however, requires $\alpha = f_1$ and each $r_{j,i} = f_0 : X \longrightarrow Y$. That is, we can take the morphism f to be the tuple:

$$\left(f_0, f_1, \{r_j\}_{j \in \langle n \rangle^{\circ}}\right)$$

Notice, in particular, that the maps $d|_{s\mathcal{A}^{\otimes}_{\langle 1 \rangle} \times \{\langle 1 \rangle\}}$ and $u|_{\mathcal{A}^{\otimes}_{\langle 1 \rangle}}$ are isomorphisms onto $\mathcal{A}^{\amalg}_{\langle 1 \rangle}$ so that $v|_{\mathcal{P}^{\otimes}_{\langle 1 \rangle}} : \mathcal{P}^{\otimes}_{\langle 1 \rangle} \longrightarrow \mathcal{A}^{\otimes}_{\langle 1 \rangle}$ is an isomorphism of 1–categories and hence, under the nerve, gets taken to an isomorphism of simplicial sets. We then wish to show that $v_0 = N(v)$ is an approximation as defined in [Lur17, 2.3.3.6, p. 260]. We have that $N(d)$ is an approximation by [Lur17, 2.4.3.6, p. 299] and since v_0 is a pullback of this along a categorical fibration, v_0 is also an approximation by [Lur17, 2.3.3.9, p. 260].

We wish to relate the map $v : \mathcal{P}^{\otimes} \longrightarrow \mathcal{A}^{\otimes}$ with the map $\gamma : \int \iota F \longrightarrow \mathcal{A}^{\otimes} = (M \ltimes \mathcal{O})^{\otimes}$ in the hope of showing that γ realizes $\mathcal{A}^{\otimes}$ as an assembly of $\int \iota F$. To proceed with this, Observe further that each $r_{j,i}$ is a morphism in $(M \ltimes \mathcal{O})^{\otimes}_{\langle 1 \rangle}$ and hence consists of an element $g_{j,i} \in M$ along with a morphism $r'_{j,i} : F g_{j,i}(X_i) \longrightarrow Y_j$. Since all the $r_{i,j}$ must be the same we must have each $g_{j,i} = g$ for some fixed $g \in M$; of course, we also have that each $X_i = X$ and each $Y_j = Y$ for the fixed colours X, Y of $\mathcal{O}^{\otimes}$. However, notice that this implies each r_j consists of the tuple $(g, \ldots, g)$

with $|\alpha^{-1}(j)|$ copies of g and an operation $r'_j \in \mathcal{O}\left(\coprod_{i\in\alpha^{-1}(j)} \{Fg(X)\}\,;Y\right)$. Thus, r_j is precisely an operation in $\int \iota F$. Furthermore, since $\mathcal{A}^{\otimes}$ and $\int \iota F$ have the same object sets, we must have that v factors through γ. That is, there exists a functor $y : \mathcal{P}^{\otimes} \longrightarrow \int \iota F$ such that the following triangle commutes:

$$\begin{tikzcd} & \int \iota F \arrow[dr, "\gamma"] & \\ \mathcal{P}^{\otimes} \arrow[ur, "y"] \arrow[rr, "v"'] & & (M \ltimes \mathcal{O})^{\otimes} = \mathcal{A}^{\otimes} \end{tikzcd}$$

It is then easy to see that all maps in the above triangle are faithful and, at the same time injective, on objects. Thus, when we pass to the nerves, we have that all morphisms are monomorphisms of simplicial sets and hence are cofibrations in the generalized ∞–operadic model structure. We will see that this is enough to show that $N(\gamma)$ is also an approximation.

We first observe that $v : \mathcal{P}^{\otimes} \longrightarrow \mathcal{A}^{\otimes}$ is not an isofibration, in general, and hence $v_0 = N(v)$ is not a categorical fibration. Thus, v_0 being an approximation really means that we can factorize v_0 as

$$N\left(\mathcal{P}^{\otimes}\right) \xrightarrow{v'_0} \overline{\mathcal{P}}^{\otimes} \xrightarrow{v''_0} N\left(\mathcal{A}^{\otimes}\right)$$

where v'_0 is an equivalence of ∞–categories and v''_0 is a categorical fibration which is also an approximation. We similarly observe that $N(\gamma)$ is not, in general, a categorical fibration. So, we are led to consider a factorization of $N(\gamma)$ as

$$N\left(\int \iota F\right) \xrightarrow{\gamma'} \mathcal{B}^{\otimes} \xrightarrow{\gamma''} N\left(\mathcal{A}^{\otimes}\right)$$

where γ' is an equivalence of ∞–categories and γ' is a categorical fibration. By applying functoriality of factorization [Rie14, 12.1.1, p. 191] in the Joyal model structure we obtain a map $\overline{v} : \overline{\mathcal{P}}^{\otimes} \longrightarrow \mathcal{B}^{\otimes}$ making the following diagram commute:

$$\begin{tikzcd} \overline{\mathcal{P}}^{\otimes} \arrow[rr, "\overline{v}"] \arrow[dd, "v''_0"'] & & \mathcal{B}^{\otimes} \arrow[dd, "\gamma''"] & \\ & N\left(\mathcal{P}^{\otimes}\right) \arrow[ul, "v'_0"] \arrow[dl, "v_0"] \arrow[rr, "N(y)"] & & N\left(\int \iota F\right) \arrow[ul, "\gamma'"'] \arrow[dl, "N(\gamma)"] \\ N\left(\mathcal{A}^{\otimes}\right) \arrow[rr, equal] & & N\left(\mathcal{A}^{\otimes}\right) & \end{tikzcd}$$

By definition of a model category [Lur08, A.2.1.1, p. 639], we can take v'_0 to be an acyclic cofibration in the Joyal model structure which means it is a monomorphism of simplicial sets [Lur08, 2.2.5.1, p. 80]. By 2–out–of–3, we must have that v''_0 is also a monomorphism of simplicial sets. Similarly γ'' is a monomorphism as well. This finally guarantees that $\overline{v}$ is a monomorphism. Thus, all lifts required to make γ'' an approximation is contained in the image of $\overline{v}$ which is isomorphic to $\overline{\mathcal{P}}^{\otimes}$, which, in turn, has all these required lifts by the fact that v''_0 is an approximation.

Thus, $N(\gamma)$ is an approximation as well. In particular, it is also a weak approximation [Lur17, 2.3.3.10, p. 260]. That is, the argument roughly is that since $N(\mathcal{P}^{\otimes})$ has "enough structure to approximate" $N(\mathcal{A}^{\otimes})$ any ∞–category between the two – in particular, $N\left(\int \iota F\right)$ – should have that structure too. We have shown one of our central results:

Lemma 4.4.2. *Let $F : BM \longrightarrow \mathrm{Op}$ be a functor for some monoid M and ι, the forgetful map $\mathrm{Op} \longrightarrow \mathcal{C}\mathrm{at}_{/\mathcal{F}\mathrm{in}_*} \longrightarrow \mathcal{C}\mathrm{at}$. Let $\mathcal{O}^{\otimes}$ be the 1–operad that is the image of the unique object of BM. If $\mathcal{O}^{\otimes}$ is unital in the sense that it has a unique nullary operation for each colour, then the canonical map $\gamma : \int \iota F \longrightarrow (M \ltimes \mathcal{O})^{\otimes}$, under the nerve functor, is an approximation (and, also, a weak approximation) to the ∞–operad $N\left((M \ltimes \mathcal{O})^{\otimes}\right)$.*

If we now add the assumption that the underlying ∞–category $N\left(\mathcal{O}^{\otimes}_{\langle 1\rangle}\right)$ is a Kan complex and M is a group, then we have that $N\left((M \ltimes \mathcal{O})^{\otimes}\right)_{\langle 1\rangle}$ is a Kan complex as well. $\int \iota F$ has the same underlying category as $(M \ltimes \mathcal{O})^{\otimes}$ and hence $N\left(\int \iota F\right)_{\langle 1\rangle}$ is also a Kan complex. Finally, we observe that $\left(\int \iota F\right)_{\langle 0\rangle}$ consists of a single object – $(\langle 0\rangle, \varnothing)$ – and a unique morphism $(\langle 0\rangle, \varnothing) \longrightarrow (\langle 0\rangle, \varnothing)$ so that $N\left(\int \iota F\right)_{\langle 0\rangle}$ is a Kan complex. Thus, in [Lur17, 2.3.4.5(1), p. 275], if we substitute $\mathcal{O}^{\otimes} = N\left(\int \iota F\right)$ and $\mathcal{O}'^{\otimes} = N\left((M \ltimes \mathcal{O})^{\otimes}\right)$, and $f = N(\gamma)$, we obtain that $N(\gamma)$ realizes $N\left((M \ltimes \mathcal{O})^{\otimes}\right)$ as an assembly of $N\left(\int \iota F\right)$, yielding the second main result of this section:

Theorem 4.4.3. *Let a group G act on a Set–valued unital 1–operad $\mathcal{O}^{\otimes}$ whose underlying category is a groupoid. Then, the nerve of the semidirect product $(G \ltimes \mathcal{O})^{\otimes}$ is an assembly of the nerve of the Grothendieck construction of the functor $BG \longrightarrow \mathrm{Op}$ corresponding to the action.*

Even though we have shown what we had set out to, it is desirable to show the same result for the categories of operators of classical topological operads. While we have not proved the desired statement, we will still record it and discuss ideas for proving it along with the main obstacles:

Claim 4.4.4. *Let G be a topological group acting on a topological operad $\mathcal{O}^{\otimes}$ with $F : BG \longrightarrow \mathrm{Op}_\infty$, the functor corresponding to the action, where the image of the unique object of BG under F is $\mathcal{N}(\mathrm{Sing}(\mathcal{O}^{\otimes}))$. Then, $\mathrm{USt}_{/\pi}(\mu F)$ assembles to $\mathcal{N}\left(\mathrm{Sing}\left((G \ltimes \mathcal{O})^{\otimes}\right)\right)$, where $\mu : \mathrm{Op}_\infty \hookrightarrow \mathcal{C}\mathrm{at}_{/N(\mathcal{F}\mathrm{in}_*)}$ is the subcategory inclusion.*

Proof Idea. We try to follow the same line of reasoning as for the case of Set operads. Observe that taking $\mathcal{A}^{\otimes} = (G \ltimes \mathcal{O})^{\otimes}$, we can form the pullback of topological categories:

$$\begin{tikzcd} \mathcal{P}^{\otimes} \arrow[r, "v"] \arrow[d, "w"'] & \mathcal{A}^{\otimes} \arrow[d, "u"] \\ \mathcal{A}^{\otimes}_{\langle 1\rangle} \times \mathcal{F}\mathrm{in}_* \arrow[r, "d"'] & \mathcal{A}^{\amalg} \end{tikzcd}$$

Taking the topological nerve $\mathcal{N} \circ \mathrm{Sing}$ of this diagram, we still get a pullback diagram of simplicial sets since $\mathcal{N}$ and Sing are both right adjoints. We would like to conclude that $\mathcal{N}(\mathrm{Sing}(v)) : \mathcal{N}(\mathrm{Sing}(\mathcal{P}^{\otimes})) \longrightarrow \mathcal{N}(\mathrm{Sing}(\mathcal{A}^{\otimes}))$ is a weak approximation. For this we would require $\mathcal{N}(\mathrm{Sing}(u))$ is a categorical fibration. This step is not immediate as, even though we are guaranteed that u is an isofibration, it is not clear that $\mathcal{N}(\mathrm{Sing}(u))$ is an inner fibration of simplicial sets – this is the main difficulty that needs to be overcome. One can try to apply [Lur08, 1.1.5.11, p. 30] to conclude that $\mathcal{N}(\mathrm{Sing}(u))$ is an inner fibration after showing that the map u induces Serre fibrations of mapping spaces, but it is not clear why this should be true.

Once we have this however, notice that the proof in the case of Set operads used no other properties of the ordinary nerve not possessed by the topological nerve so that the rest of the argument goes through as is and shows that $\mathcal{N}\left(\mathrm{Sing}\left(\int \iota F\right)\right)$ assembles to $\mathcal{N}\left(\mathrm{Sing}\left((G \ltimes \mathcal{O})^{\otimes}\right)\right)$. One then needs to show that $\mathcal{N}\left(\mathrm{Sing}\left(\int \iota F\right)\right) \simeq \mathrm{USt}_{/\pi}(F)$ – the second main technical point.

Another approach might be to factorize u as a weak equivalence followed by a fibration of topological categories and then try to carry out the argument but in this case the second part of the argument does not seem as straightforward an adaptation. **q.e.d.**

4.5 Semidirect Product ∞–Operads

In the previous subsection, we have shown that semidirect products with groups of unital Set–valued operads whose underlying categories are groupoids all arise as assemblies of families obtained by the Grothendieck construction of the corresponding functors. That is, we have that given a group G acting on a Set–valued 1–operad $\mathcal{O}^{\otimes}$, we have an equivalence:

$$N\left((G \ltimes \mathcal{O})^{\otimes}\right) \simeq \mathrm{Asm}\left(N\left(\int \iota F\right)\right)$$

where $F : BG \longrightarrow \mathrm{Op}$ is the functor corresponding to the action of G on $\mathcal{O}^{\otimes}$ and $\iota : \mathrm{Op} \longrightarrow \mathcal{C}\mathrm{at}$ is the forgetful functor. This justifies the following definitions.

Definition 4.5.1 (Simple ∞–Operads)**.** We will call an ∞–operad simple if it is unital according to definition [Lur17, 2.3.1.1, p. 244] and its underlying ∞–category is a Kan complex. We will denote the full subcategory of Op_{∞} spanned by the simple ∞–operads as $\mathrm{Op}_{\infty}^{\mathrm{s}}$. $\diamond$

Definition 4.5.2 (Semidirect Product ∞–Operad)**.** Given a functor $F : BG \longrightarrow \mathrm{Op}_{\infty}^{\mathrm{s}}$ for some group G such that the unique object of BG is sent to some simple ∞–operad $\mathcal{O}^{\otimes}$, we will denote

$$(G \ltimes_F \mathcal{O})^{\otimes} := \mathrm{Asm}\left(\mathrm{USt}_{/N(\mathcal{F}\mathrm{in}_*)}(\mu_*(F))\right)$$

where $\mathrm{USt}_{/N(\mathcal{F}\mathrm{in}_*)} : \mathcal{F}\mathrm{un}\left(BG, \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}\right) \longrightarrow \mathrm{CoCart}(BG)_{/\pi}$ is the sliced unstraightening functor for $\pi : BG \times N(\mathcal{F}\mathrm{in}_*) \longrightarrow BG$, the projection onto the first factor, and

$$\mu_* : \mathcal{F}\mathrm{un}\left(BG, \mathrm{Op}_{\infty}\right) \longrightarrow \mathcal{F}\mathrm{un}\left(BG, \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}\right) \simeq \mathcal{F}\mathrm{un}\left(BG, \mathcal{C}\mathrm{at}_{\infty}\right)_{/\mathrm{const}_{N(\mathcal{F}\mathrm{in}_*)}}$$

is the functor induced by post composition with the inclusion $\mu : \mathrm{Op}_{\infty} \longrightarrow \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}$. We call $(G \ltimes_F \mathcal{O})^{\otimes}$ the semidirect product of G with $\mathcal{O}^{\otimes}$ with respect to F. $\diamond$

Remark 4.5.3. Recall that $\mathrm{USt}_{/N(\mathcal{F}\mathrm{in}_*)}(\mu_*(F))$ is a BG–family by Theorem 4.1.4 and hence it is valid to apply the assembly functor $\mathrm{Asm} : \mathrm{Op}_\infty^{\mathrm{gn}} \longrightarrow \mathrm{Op}_\infty$ to this, since every family of ∞–operads is also a generalized ∞–operad by [Lur17, 2.3.2.13, p. 254]. $\diamond$

We will now bring together all the previous work in proving some results about semidirect product ∞–operads that we wish to apply to various actions of $SO(n)$ on the little n–disks operads in the next section to prove some claims of topological interest.

Definition 4.5.4. We denote by $\mathrm{Op}_\infty^{\mathrm{rd}}$ the full subcategory of Op_∞ spanned by the reduced ∞–operads – that is, those that are unital and have contractible Kan complexes for underlying ∞–categories. $\diamond$

Corollary 4.5.5. *$\mathrm{Op}_\infty^{\mathrm{rd}}$ is a full subcategory of $\mathrm{Op}_\infty^{\mathrm{s}}$.*

Lemma 4.5.6. *Let $F : BG \longrightarrow \mathrm{Op}_\infty^{\mathrm{rd}} \subset \mathrm{Op}_\infty^{\mathrm{s}}$ be a functor with $\mathcal{O}^\otimes$ the image of the unique object of BG, then $(G \ltimes_F \mathcal{O})^\otimes$ is a simple ∞–operad.*

Proof. It suffices to show that $\mathcal{O}'^\otimes = \mathrm{USt}_{/N(\mathcal{F}\mathrm{in}_*)}(\mu_*(F))$ is a reduced BG–family for, then, we may apply [Lur17, 2.3.4.4, p. 275] to obtain that the assembly of this lies in $\mathrm{Op}_\infty^{\mathrm{s}}$. However, this follows from the fact that BG is a Kan complex and for the unique object $*$ in BG, the fibre $\mathcal{O}_*'^\otimes \simeq \mathcal{O}^\otimes$ is a reduced ∞–operad by [Lur17, 2.3.4.3, p. 275]. **q.e.d.**

Theorem 4.5.7. *Let $F, F' : BG \longrightarrow \mathrm{Op}_\infty^{\mathrm{rd}}$ be functors for some group G such that both send the unique object of BG to a fixed ∞–operad $\mathcal{O}^\otimes$ – that is, they represent two different actions of G on $\mathcal{O}^\otimes$. Then, $(G \ltimes_F \mathcal{O})^\otimes \simeq (G \ltimes_{F'} \mathcal{O})^\otimes$ if and only $F \simeq F'$ as objects of $\mathcal{F}\mathrm{un}\,(BG, \mathrm{Op}_\infty)$.*

Proof. We recall from [Lur17, 2.3.4.4, p. 260] that $\mathrm{Asm} : \mathrm{Op}_\infty^{\mathrm{gn}} \longrightarrow \mathrm{Op}_\infty$ induces an equivalence from $\mathrm{Op}_\infty^{\mathrm{gn,rd}}$, the full subcategory of $\mathrm{Op}_\infty^{\mathrm{gn}}$ spanned by the generalized ∞–operads that are reduced according to definition [Lur17, 2.3.4.2, p. 274] when regarded as families of ∞–operads, to $\mathrm{Op}_\infty^{\mathrm{s}}$. Thus, there is a disintegration functor $\mathrm{Dsnt} : \mathrm{Op}_\infty^{\mathrm{s}} \longrightarrow \mathrm{Op}_\infty^{\mathrm{gn,rd}}$ inverse to $\mathrm{Asm}|_{\mathrm{Op}_\infty^{\mathrm{rn,rd}}} : \mathrm{Op}_\infty^{\mathrm{gn,rd}} \longrightarrow \mathrm{Op}_\infty^{\mathrm{s}}$.

Suppose that the two semidirect products are equivalent. Since both semidirect products are simple ∞–operads by the previous lemma, we may apply Dsnt to obtain and equivalence $\mathrm{USt}_{/N(\mathcal{F}\mathrm{in}_*)}(\mu_*(F)) \simeq \mathrm{USt}_{/N(\mathcal{F}\mathrm{in}_*)}(\mu_*(F'))$ in $\mathrm{CoCart}(BG)_{/\pi}$. Note that this is also an equivalence in $\mathrm{Op}_\infty^{\mathrm{gn}}$ which is the codomain of Dsnt and hence is an equivalence of BG–families. This, in particular, means that fibre transport in each coCartesian fibration is a map of ∞–operads. This, in turn, implies that, sliced straightening-unstraightening being an equivalence, we can straighten these coCartesian fibrations to obtain an equivalence $\mu_*(F) \simeq \mu_*(F')$ in $\mathcal{F}\mathrm{un}\,(BG, \mathrm{Op}_\infty) \subset \mathcal{F}\mathrm{un}\left(BG, \mathcal{C}\mathrm{at}_{\infty/N(\mathcal{F}\mathrm{in}_*)}\right)$. This means that F and F' are equivalent.

For the other direction, suppose both functors are equivalent. Then, it is immediate that $\mathrm{Asm} \circ \mathrm{USt}_{/N(\mathcal{F}\mathrm{in}_*)} \circ \mu_*$ carries the equivalence between the functors to an equivalence of semidirect products. **q.e.d.**

5 Applications

We are now in a position to comment on a potential application of the above results to a phenomenon of topological and geometric importance. Notice that the $SO(n)$ action on $\mathcal{D}_n$ gives an action of $SO(k)$ on $\mathcal{D}_n$ for all $k < n$ by viewing $SO(k)$ as a subgroup of $SO(n)$. Hence, we may form the semidirect products $SO(k) \ltimes \mathcal{D}_n$. At the same time, we may form semidirect products $SO(k) \ltimes_{\mathrm{triv}} \mathcal{D}_n$ with the trivial action, which we will call the trivial semidirect products. In [KW17, 7.2, p. 28] have shown that the action of $SO(n-1)$ or $SO(n-2)$ for n even or odd respectively on the real version of $\mathcal{D}_n$ are, in some sense to be made precise, equivalent to the trivial actions. If this can be shown to imply that maps $B(SO(n-1)) \longrightarrow \mathrm{Op}_\infty$ or $B(SO(n-2)) \longrightarrow \mathrm{Op}_\infty$ corresponding to the actions are, in fact, equivalent to the maps corresponding to the trivial action then, we may apply Theorem 4.5.7 to deduce that the respective semidirect products are homotopy equivalent. On the other hand, if their finding can be shown to imply the semidirect products are equivalent to the trivial semidirect products, we can deduce that the actions were equivalent to the trivial actions to begin with. There are, however, gaps to fill in order for this argument to work in full, as we discuss below:

(i) The proof of our Claim 4.4.4 must be completed if we are to make any claims about topological operads.

(ii) Their definition of "homotopy operad" is quite a bit different from ∞–operads. Most notably, it seems that they do not require the existence of identity morphisms in the categories of operators of operads – see [KW17, Defn. 3.1, p. 6 and §3.4, p. 7]. We require some result stating either that their results hold for operads with identity morphisms or that at least in the case of little disks operads, this distinction does not matter.

(iii) It is not known to us that even with their unital variant as given in [KW17, §3.4, p. 7], their homotopy operads are equivalent to any of the models for ∞–operads we are used to dealing with, such as the ∞–operads of Lurie or the Dendroidal set model of [HHM13]. Hence, we are not yet able to conclude that their results carry over to ∞–operads as is.

(iv) One must prove the results of Khoroshkin and Willwacher [KW17] for the ordinary version of $\mathcal{D}_n$ and not just their real models. The sense in which they have shown their result is as follows. The action of a group G on $\mathcal{D}_n$ is a map $G \times \mathcal{D}_n(k) \longrightarrow \mathcal{D}_n(k)$ for each k satisfying some compatibility conditions with the composition of $\mathcal{D}_n$. Passing to homologies $H_*(BG)$ and $H_*(\mathcal{D}_n)$, this gives a "homotopy action" of the associative algebra object $H_*(BG)$ in commutative differential graded algebras on some CDGA model of $\mathcal{D}_n$ in the sense of [KW17, 3.6, p. 7]. From this observation, they claim that the homotopy type of this model for the action corresponds to the gauge equivalence class of a Maurer-Cartan element in a certain Lie algebra. They have shown the gauge equivalence classes of the Maurer-Cartan elements corresponding to the actions we are interested in to be trivial. However, it is not clear that this implies the actions to be homotopy equivalent to the trivial actions.

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