HAMILTONIAN GEOMETRY OF GENERALIZED KÄHLER METRICS

BY

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A thesis submitted in conformity with
the requirements for the degree of
Doctor of Philosophy

Graduate Department of Mathematics
University of Toronto

ABSTRACT

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2022

abstract

To my grandmother Margarete,
who could not live up to her mathematical potential because of a war she had to flee from.

ACKNOWLEDGEMENTS

CONTENTS

1

THE GENERALIZED KÄHLER CLASS

1.1 INTRODUCTION AND MOTIVATION

In this thesis we reveal new structures on the space of generalized Kähler metrics in a fixed generalized Kähler class. We introduce the new concept of a holomorphic family of branes to study an infinite dimensional nonabelian principal bundle of groupoid bisections. With this we construct the correct framework to analyze a Calabi-like conjecture for generalized Kähler metrics and provide a gradient flow towards the solution.

Kähler metrics have been studied intensively ever since their discovery at the beginning of the last century. They are a special kind of Riemannian metric with particularly nice properties. Usually, a Riemannian metric g on an $2n$-dimensional manifold M is a tensor with $(2n \cdot 2n)$ degrees of freedom g_{ij}. However, if it is Kähler, this means the manifold has a complex structure and the Kähler form

$$\omega = g(I-,-)$$

is a closed differential 2-form, the metric can be descried, at least locally, by a single real-valued function $f \in C^\infty_M(\mathbb{R})$ degree of freedom.

$$g_{i\bar{j}} = \frac{\partial f}{\partial z^i \partial \bar{z}^j}$$

An important application is the study of Einstein's theory of gravity, a theory of manifolds with a (pseudo-) Riemannian metric. In the simplified case of a vacuum, the Einstein field equations relate the metric g to its *Ricci curvature* Ric_g by its *scalar curvature* R_g which is the trace of Ric_g.

$$Ric_g = \frac{1}{2} R_g \cdot g$$

An immediate solution would be a metric with vanishing Ricci curvature. On a compact Kähler manifold (M, I, g) with trivial canonical bundle the Calabi conjecture which has now been proven by Yau [25] asserts the existence of a solution of a Ricci-flat Kähler metric in every Kähler class. A Kähler class is the deRham cohomology class of the Kähler form ω in $H^2_M(\mathbb{R})$. Given a holomorphic volume form $\Omega \in \Omega^{(n,0)}_X$, Ricci flatness is achieved by a Kähler metric g if the n-th power of its Kähler form ω can be prescribed to be $\theta = \Omega \wedge \overline{\Omega}$. The $\partial\bar{\partial}$-Lemma on compact Kähler manifolds implies that any two Kähler forms ω_0, ω_1 in the same Kähler class differ by

$$\omega_1 - \omega_0 = \partial\bar{\partial}\phi$$

for a real smooth function $\phi \in C^{\infty}(\mathbb{R})$. But this means the entire Kähler class of ω_0 is of the form

$$\mathcal{H}_{\omega_0} = \left\{ \varphi \in C^{\infty}_{M}(\mathbb{R}) \;\middle|\; \omega_0 + i\partial\bar{\partial}\varphi > 0 \right\}$$

Yau's theorem is a very powerful result that provides a large number of examples of Ricci flat Kähler metrics. Because Ricci flat Kähler metrics satisfy the Einstein field equations they are examples of Kähler-Einstein metrics on manifolds with vanishing first Chern class. Since the original proof by Yau, other works have shed light on the Calabi-Yau theorem from different points of view. Particularly interesting for our work is Joel Fine's geometric interpretation [8] of the Calabi conjecture as a Hamiltonian group action on an infinite dimensional Kähler manifold. It is a quite popular approach to interpret conditions like this one as the moment map for a Hamiltonian action for if there is a complexification of the action, complex reduction opens up a door to powerful tools to analyze the zero levelset of the moment map. This zero levelset of the moment map contains solutions to the imposed condition.

Ricci-flatness is far from being the only distinguishing property a Kähler metric can have. The scalar curvature of a metric g is the trace of the Ricci curvature. It too can be described as a moment map. Fujiki and Donaldson observed that the scalar curvature is a moment map on $\mathcal{J}$, the space of almost complex structures on a symplectic manifold (M, ω) for the action of symplectomorphpisms. The search for constant scalar curvature Kähler metrics can then be understood as a Kähler quotient.

Kähler manifolds also made an important appearance as tagret spaces in the study of nonlinear sigma models. This is a theory of maps from a pseudo Riemannian surface to a Riemannian manifold.

$$\Sigma \xrightarrow{\varphi} (M, g)$$

The Riemannian metric tensor is used in the definition of an action functional on the space of maps, that, roughly speaking, measures the size of the embedded surface. If the metric g is Kähler, Zumino [26] has shown that one can extend the symmetries of this model to include supersymmetry. It was in this context that Gates, Hull and Roček considered a more general type of metric on the target manifold that we call nowadays a *generalized Kähler metric*. While it is not a Kähler metric, it is not too far removed and still gives rise to supersymmetry in the nonlinear sigma model. The geometry of this type of metric forms the focus of this thesis. Natural questions to consider are: What are special generalized Kähler metrics in a generalized Kähler class? What should the notion of generalized Kähler class be? Do all classes admit such a metric and is it unique?

This thesis discusses the generalized Kähler metrics in the perspective of particular kinds of Lagrangian submanifolds of a symplectic manifolds that forms a Morita equivalence between two Poisson manfolds. This perspective is due to [2] and extends an observation made by Donaldson [6]. Our first main result is the construction of a formal almost Kähler structure on the space of prequantized generalized Kähler metrics. The inspiration was derived from another paper by Donaldson [6] and lectures by Nigel Hitchin [15], especially the transgression of a symlectic form to a mapping space.

Theorem 1.1 (c.f. theorem (5.7))**.** *Let Z, Ω be a holomorphic Morita equivalence between holomorphic Poisson manifolds (X_+, σ_+) and (X_-, σ_-). The space of smooth LS bisections $B\Gamma(Z)$ is formally an infinite dimensional almost Kähler manifold for $H^1_{X_-} = 0$.*

$$(B\Gamma(Z), \mathbb{I}, \varpi)$$

We show that in Generalized Kähler geometry, too, there is a Hamiltonian group action on prequantied generalized Kähler metrics in a fixed generalized Kähler class. We describe a Calabi-like condition on the generalized Kähler metrics (I_+, I_-, F, Q) by prescribing the n-th power of F by a real volume form θ. In this setup we work out a gradient flow towards the this levelset with teqchniques from GIT, even though many aspects of GIT do not hold in this general framework.

Theorem 1.2 (c.f. theorem (5.17))**.** *Let (Z, Ω) be a holomorphic Morita equivalence between holomorphic Poisson manifolds (X_+, σ_+) and (X_-, σ_-) with a fixed LS bisection ℓ_0. For any $\theta \in \Omega^{2n}_{X_+}$ there exists a functional on the space of generalized Kähler metrics $L\Gamma(Z) \to \mathbb{R}$ that is extremized by an LS bisection ℓ such that $(\ell^* B)^{[n]} = \theta$.*

As a byproduct we define holomorphic families of branes to abtract the transgression of a symplectic form from an ambient symplectic target manifold to the mapping space of Lagrangians into the target. This is the tool that allows us to define the the almost Kähler form on $B\Gamma(Z)$. We work out the explicit conditions for a holomorphic family of Lagrangian branes in a symplectic background

Theorem 1.3 (c.f. theorem (3.8))**.** *A smooth family of Lagrangian branes $(\widehat{L}, U, \nabla) \xrightarrow{\pi} X$ over (X, I) in a sympletic manifold (M, ω) is holomorphic if and only if ∇ can be extended to a full connetion $\widehat{\nabla}$ on U and in a local trivialization $W \times L$ of $\widehat{L}$ over $W \subseteq X$ the following conditions hold for $F = F^{\widehat{\nabla}}$.*

(a) $F^{VV} = 0$
(b) $(I^* \otimes 1)F^{HV} = (\hat{\iota}^* \omega)^{HV}$
(c) $(I^* \otimes 1)(F^{HH})^{(20+02)} = ((\hat{\iota}^* \omega)^{HH})^{(20+02)}$

In more generality, it turns out that under certain conditions every holomorphic family gives rise to a Kähler structure on its parameter space. We explore this phenomenon and describe Kähler structures on $\mathbb{C}^2$ in this manner.

1.2 OUTLINE OF THE THESIS

This thesis is devided into the following parts: The remainder of this chapter 1 outlines findings leading up to the start of our work. Based on this discussion we propose a new notion of a generalized Kähler class that extends the previous concept of a Kähler class. We explain the advantages of this new way of thinking and argue why it is indispensible in moving forward in this thesis. In chapter 2 we review some necessary background material for the content of this thesis. We start this discussion with holomorphic Poisson manifolds as found in classical complex geometry. We then present a neat reformulation of this geometry in the language of generalized complex geometry. In the same vein we cross paths with generalized complex branes and an enlarged automorphism group that will play a major role in this story. Chapter 3 investigates the occurance of Lagrangian branes in a family over a complex parameter space. This is a new concept that we define to get a grasp on the local geometry on the space of generalized Kähler metrics. To familiarize the reader with this new concept we explore examples. In chapter 4 we apply the concept of holomorphic families of complex branes to the problem at hand. That is, we approach the question of understanding the space of generalized Kähler metrics in a fixed Kähler class. Analogous to prequantizing a Kähler form, we consider the larger space of brane bisections, which one can call prequantized generalized Kähler metrics, that surjects onto the space in question. Our first step is to set up a universal holomorphic familiy of complex branes over the parameter space of brane bisections. The next step follows in chapter 5. We construct with the material developed in chapter 3 an almost Kähler metric on the parameter space of brane bisections from the universal holomorphic family and show that it receives a Hamiltonian group action. Finally, we describe the GIT interpretation of a Calabi-like condition and explicitly construct a functional whose downward gradient flow leads to the zero level set of the moment map. Lastly, we propose a construction that equipps any complex parameter space of a holomorphic family of Lagrangian branes in a symplectic background with a Kähler form. As an example we construct new Kähler metrics on the complex plane.

1.3 PREVIOUS WORK

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To give an outline of the contents and goal of this thesis we dive right into the heart of the material previously developed by others. A more careful introduction and setup of notation conventions can be found in chapter 2. In particular the notions of holomorphpic Poisson structures, symplectic groupoids and Morita equivalences between these are explained from the ground up.

A type of metric, broarder than the kind of Kähler metrics, was first discovered by Gates, Hull and Rocek [**?**] in their quest to loosen the conditions on metrics on the target space of nonlinear sigma models that still give rise to supersymmetry in the model. They discribe a Riemannian manifold (M, g) with two complex structures I_+, I_- whose Hermitian forms ω_+, ω_- satisfy a joint closure condition.

$$d_+^c \omega_+ + d_-^c \omega_- = 0$$

$$dd_\pm^c \omega_\pm = 0$$

There are many reformulations of this triplet of structures for example as a pair of commuting generalized complex structures [13] $\mathbb{J}_A, \mathbb{J}_B$ with a positivty condition on their product $\mathbb{G} = -\mathbb{J}_A \mathbb{J}_B$. That is, $\langle \mathbb{G}-, - \rangle$ has to define a positive definite metric on $T \oplus T^*$. Another presentation is possible through the $+i$-eigenbundles of $\mathbb{J}_A, \mathbb{J}_B$ which form complex Dirac structures L_A, L_B. The compatibility condition for this pair to define a GK structure has been worked out in [10].

Proposition 1.4 ([10], Proposition 4.3). *A pair $L_A, L_B \subseteq \mathbb{T}M \otimes \mathbb{C}$ of complex Dirac structures defines a GK structure precisely when the following three properties are satisfied:* RN8EDF

1. *Both Dirac structures define GC structures, which is to say that $L_A \cap \overline{L}_A = 0$ and $L_B \cap \overline{L}_B = 0$*
2. *The complex Dirac structures*

$$L_{\sigma_+} \frac{i}{2}\left(\overline{L}_B - \overline{L}_A\right), \qquad L_{\sigma_-} = \frac{i}{2}\left(\overline{L}_B - L_A\right)$$

   define holomorphic Poisson structures $(I_+, \sigma_+), (I_-, \sigma_-)$ respectively.

3. *For all nonzero $u \in L_A \cap L_B$, we have $\langle u, \bar{u} \rangle > 0$*

■

Hitchin discovered in [14] that underlying every Generalized Kähler structure is $Q = \frac{1}{2}[I_+, I_-]g^{-1}$, a smooth Poisson structure. This is the imaginary part shared by $\sigma_\pm = -\frac{1}{4}(I_\pm Q + iQ)$. We are interested in a particular presentation worked out in [2] targeting Generalized Kähler metrics with the following property

Definition 1.5 ([2], Definition 2.2). A generalized Kähler structure $(\mathbb{J}_A, \mathbb{J}_B)$ is of *symplectic type* if $\mathbb{J}_A$ is gauge equivalent to a symplectic structure. That is, there is a closed 2-form β such that 8QJSTV

$$e^{\beta}\mathbb{J}_A e^{-\beta} = \begin{pmatrix} 0 & -F^{-1} \\ F & 0 \end{pmatrix}$$

where F is a symplectic form. ■

This condition is equivalent to the requirement that the holomorphic Poisson structures found in proposition (1.4) are B-field equivalent $L_{\sigma_+} = e^F L_{\sigma_-}$. Explicitly, this may be formulated as the conditions from [12] where I_+, I_- are the same complex structures present in the bihermitian formulation.

$$I_+ - I_- = QF \tag{1.1}$$

$$I_+^* F + F I_- = 0 \tag{1.2}$$

The meaning of this pair of equations can be described as follows. A GC structure $\mathbb{J}$ has the form of a holomorphic Poisson structure if and only if there is a $\mathbb{J}$-invariant isotropic integrable splitting of the Courant algebroid, compare with lemma (2.10) and equation (2.5). The above equations describes the graph Γ_F in the first splitting as a second such splitting with the holomorphic Poisson structure of $I_+ = I_- + QF$.

To relax the nondegenracy condition of F, a *degenerate* GK structure is defined in [2] as a GK structure with possibly degenerate symplectic form F. The metric g and B-field underlying the GK structure may be recovered as

$$F I_\pm = -g \mp b$$

Before we can present a degenerate GK structure of symplectic type as done in [2], we recall a notion of equivalence between Poisson manifolds defined by Ping Xu [23] as follows. In section (**??**) we will revisit this notion in more detail.

Definition 1.6 (Morita equivalence)**.** Let $(X_\pm, \sigma_\pm)$ be holomorphic Poisson manifolds. A *Morita* 2UUWJY *equivalence* from (X_-, σ_-) to (X_+, σ_+) is a holomorphic symplectic manifold (Z, Ω) with surjective submersions $\pi_\pm$

$$\begin{tikzcd} & (Z,\Omega) \arrow[dl, "\pi_+"'] \arrow[dr, "\pi_-"] & \\ (X_+,\sigma_+) & & (X_-,\sigma_-) \end{tikzcd} \tag{1.3}$$

with connected and simply-connected fibers such that

1. π_+ is Poisson and π_- is anti Poisson
2. the vertical distributions $\ker(d\pi_+)$ and $\ker(d\pi_-)$ are symplectic complements and
3. π_- and π_+ are complete

■

An LS bisection of (Z, Ω), short for Lagrangian-Symplectic bisection, is a submanifold of in (Z, Ω) that intersects every π_+ and π_- fiber uniquely and to which $\mathrm{Im}(\Omega)$ restricts to zero and $\mathrm{Re}(\Omega)$ restricts to be symplectic. LS bisections are called "brane bisections" in [2] but in this thesis, the notion of a brane bisection is reserved for an LS bisection with a flat unitary connection on the trivial bundle over the bisection. With this we are ready to state the final incarnation of a degenerate generalized Kähler structure of symplectic type.

Theorem 1.7 ([2], Theorem 5.3)**.** *A degrate GK structure of symplectic type defined by the data* 3AGOVR (I_+, I_-, Q, F) *is equivalent to a holomorphic symplectic Morita equivalence with brane bisection* $(Z, \Omega, \mathcal{L})$ *between the holomorphic Poisson structures* $\sigma_\pm = -\frac{1}{4}(I_\pm Q + iQ)$.

This insight is the fundamental ingredient for the results developed in this thesis and akin to an observation of Donaldson [7]. He found that classical Kähler structures can be encoded as Lagrangian submanifolds of a holomorphic symplectic affine bundle for the cotangent bundle. Details are carefully laied out in section 3 of [2].

To stress a particular point of view, we treat the real part of the holomorphic symplectic structure Ω on Z as a spacefilling brane for its imaginary part as explained later in (3.12). This encodes the degenerate GK structure of symplectic type as the difference between a Lagrangian and a spacefilling brane in a real symplectic background.

Definition 1.8 (Difference of branes)**.** Let $(M, \mathbb{J})$ be a generalized complex manifold with two generalized complex branes 507DHP

$$\begin{array}{c} (L_i, \nabla_i) \\ \downarrow \\ b_i \longrightarrow M \end{array}$$

for $i \in \{1, 2\}$. The *difference* $F = b_1 \triangleleft b_2$ of the branes is the curvature of the tensor product of the bundles pulled back to the intersection of the branes.

$$\begin{tikzcd} & i_1^*(L_1, \nabla_1) \otimes i_2^*(L_2, \nabla_2)^\vee \arrow[d] & \\ b_2 & b_1 \cap b_2 \arrow[l, hook', "i_1"'] \arrow[r, hook, "i_2"] & b_2 \end{tikzcd}$$

■

We can make sense of this difference also without the presence of prequantization data for the generalized complex branes by simply subracting the curvatures from one another when pulled back to the intersection.

The generalized Kähler class $[Z, \Omega]$ provides a spacefilling brane $B = \mathrm{Re}(\Omega)$ for the real symplectic manifold $(Z, \omega = \mathrm{Im}(\Omega))$. A generalized Kähler metric ℓ in this class as a brane bisection gives another brane in (Z, ω). The following lemma explains our interest in the intersection of these two branes.

Lemma 1.9. *The generalized Kähler metric $F = \ell^*\Omega$ determined by the brane bisection ℓ in the the GK class $[(Z, \Omega)]$ is the difference between the spacefilling brane $(Z, Re(\Omega))$ and the Lagrangian brane l.* ZOEGVW

1.4 UNDERSTANDING KÄHLER GEOMETRY

Now that the context has been established, we can properly pose and outline the leading question we would like to address in this work. We first revist results that have previously been discovered.

Donaldson has constructed in [5] a formal symmetric space structure on the space of Kähler metrics in a fixed Kähler class using a metric previously defined by Mabuchi [18]. Similar work has previously been done by Semmes and Bourguignon. The significance of understanding the space of Kähler metrics goes back to the question of understanding deformations to nearby metrics and finding distinguished metrics with preferable proporties in a given class of metrics. For example, one can ask for an extremal Kähler metric that extremizes a certain functional without leaving the Kähler class.

One such functional of a Riemannian metric g is its *Scalar curvature* S_g. Fujiki and Donaldson have both described this as a function on the space compatible complex structures I on a fixed symplectic manifold (M, F) and in turn the space of Riemannian metrics with fixed Kähler form F. The scalar curvature plays the role of a moment map for a Hamiltonian action by Hamiltonian diffeomorphisms $\mathrm{Ham}(M, F)$. This not only interprets the locus of metrics with vanishing scalar curvature as the symplectic reduction, but also opens up a whole range of tools from algebraic geometry to answer the question: Does a such a metric in the given class even exist? Is there a direct path from the current metric to this desired metric? Is the desired metric unique?

An even stronger assumption is for the *Ricci curvature* Ric_g for a metric g to vanish. This has famously been conjectured to be true for a unique metric in a given Kähler class on a Calabi Yau n-fold (X, Ω) by Calabi [4] and proven later by Yau [25]. This conjecture is implied by the more general statement: "For a fixed top-degree form θ in the class $[F_0]^n$, there is a Kähler form $F \in [F]$ such that $F^n = \theta$."

This, too, has been phrased as a smplectic reduction by Joel Fine [8]. Although, it is very important to stress that the setup is quite different to the considerations made by Mabuchi, Donaldson and Fujiki. On a fixed complex manifold $X = (M, I)$ with a Hermitian line bundle (L, h) the space of unitary connections whose curvature is of type $(1, 1)$ is Kähler. It recevies an action by the Hermitian gauge transformations $\mathcal{G}(L, h)$ which is Hamiltonian with an equivariant moment map. Its value on a connection is the n-th power of the curvature of the connection. As such, prescribing the volume form can be seen as finding an element in the correct level set of the moment map. Again, powerful tools from GIT are therefore applicable in this context and Fine is able to construct a Kempf-Ness functional $\mathcal{F}$ on the space of Kähler potentials for a fixed top-degree form θ which is minimized at precisely the θ-levelset of the moment map. This is the famous F_0-functional previously described by Tian [] and others. Even more, this functional $\mathcal{F}$ can be used to gradient flow down to the levelset from any inital Kähler metric in the class. Furthermore, the space of Kähler potentials in this setup carries a symmetric space structure with geodesics between any two points. Because $\mathcal{F}$ is convex along geodesics the minimum providing the solution to the Calabi conjecture is necessarily unique. However, the symmetric space structure differs from the one considered above by Donaldson. Section (5.5) will dive deeper into this work and compare it to our results generalizing it.

The most important attempts that have been made to adapt and extend these arguments to the realm of generalized Kähler metrics are summarized below. In the special case of compact symplectic toric manifolds Boulanger [3] has computed a moment map on the space of almost generalized complex structures compatible with a fixed symplectic form (M, ω) for the group action of toric symplectomorphpisms. This moment map approach leads to his definition of a scalar cuvature for generalized Kähler metrics and further to the construction of a gradient flow to extremal metrics for the scalar curvature. Goto **[?]** has extended the setup of a moment map interpretation for generalized Kähler metrics to metrics of symplectic type.

These two works focus on the a generalization of the scalar curvature along the lines of Donaldson and Mabuchi. In contrast, we study a generalization of the work of Joel Fine to address a Calabi-like statement for generalized Kähler metrics.

Gualtieri has proposed in [11] the generalized Calabi-Yau equation that requires the norms of the pure spinors ρ_A, ρ_B representing the generalized complex structures $\mathbb{J}_A, \mathbb{J}_B$ to be related by a constant

$$(\rho_A, \bar{\rho}_A) = c \cdot (\rho_B, \bar{\rho}_B) \tag{1.4}$$

This recoveres the classical Calabi Yau condition of a compact Kähler manifold (X, I, F) with nonvanishing holomorphic n-form Ω in terms of the spinors $\rho_A = e^{iF}$ and $\rho_B = \Omega$.

Apostolov and Streets study in [1] generalized Kähler metrics where the holomoprhic Poisson structures $\sigma_\pm$ are invertible with inverses $\omega_{I_\pm}$. They look for representatives satisfying Gualtieri's Calabi-Yau condition in the variational class of generalized Kähler metrics fixing $\Omega = \mathrm{Re}(\omega_\pm)$ as well the deRham cohomology classes of $\mathrm{Im}(\omega_\pm)$. They find that such a solution is unique and necessarily of hyper-Kähler-type and provide a formal GIT interpretation with a Kempf-Ness functional to flow towards the zero of the moment map. The role of the compact group in their work is filled by the group of ω-Hamiltonian symplectomorphpisms. In contrast to their work, we don't require the Hitchin Poisson structure Q to be invertible. Moreover, our proposal for the notion of a generalized Kähler class fixes the isomorphism class of the Morita equivalence (Z, Ω) instead of fixing the deRham classes of the closed symplectic forms inverse to $\sigma_\pm$. Another difference to our work is the group that is acting. Instead of Q-Hamiltonian diffeomorphisms, which form formally a nonabelian group with Lie bracket given by the Poisson bracket $\{\ \}$ of Q, we use gauge transformations of the trivial unitary line bundle on X_- which is an abelian group.

1.5 GENERALIZED KÄHLER CLASS

In light of the previous section, [2] have remarked on the separation of holomorphic and smooth data required to encode a GK structure. While the holomorphic symplectic Morita equivalence can be defined purely in the category $\mathcal{PG}$ of holomorphic Poisson manifolds, see section (2.1) for details. The LS bisection, in contrast, is a smooth object and does not fit into this category. Motivated by this separation, we define the holomorphic background underlying a GK structure as the generalized Kähler class.

Definition 1.10. A *generalized Kähler class* is an arrow in the category $\mathcal{PG}$ of holomorphic Poisson manifolds. That is, an isomorphism class of a holomorphic symplectic Morita equivalence as in (1.3). ■ CGKIAZ

We interpret the LS bisections of Z as the GK metrics in the class that (Z, Ω) determines. This suggests another definition to adapt the notion of the *Kähler cone*. This should describe the set of all generalized Kähler classes between two given holomophic Poisson manifolds.

Definition 1.11. The *complex Kähler moduli* for two holomorphic Poisson manifolds (X_+, σ_+) and (X_-, σ_-) is the set of isomorphisms classes that admit smooth LS-bisections.

$$\mathcal{K}(X_+, X_-) \subseteq \mathrm{Hom}_{\mathcal{PG}}((X_+, \sigma_+), (X_-, \sigma_-))$$

■

The existence of a smooth bisection immediately forces the underlying smooth spaces of X_+ and X_- to be diffeomorphic. We can, however, not expect for this locus to form a cone or affine linear space in general. This can readily be observed by noticing that two GK classes cannot be added, only composed as arrows in $\mathcal{PG}$. We will for now have a closer look at deformations of GK classes described by the deformation complex $\Omega^{\geq 1} \to \mathfrak{X}^{\geq 1}$. It is, however, to be expected that the Maurer Cartan equation cuts out a highly nonlinear space of the hom set $\mathcal{PG}^{\ell}(X_+, X_-)$ in the presence of the Koszul bracket. We leave this question open for future work and only address infinitesimal deformations here.

The set of morphisms $\mathcal{PG}^{\ell}(X_+, X_-)$ is a bitorsor for $\mathcal{PG}^{\ell}(X_+)$ and $\mathcal{PG}^{\ell}(X_-)$ by composition on the left and right, respectively. It was shown [2] that $\mathcal{PG}^{\ell}(X, \sigma)$ is isomorphic as the group to Courant automorphisms

$$\mathrm{Aut}(I, \sigma) = \left\{ (\varphi, F) \in \mathrm{Diff}_Q(M) \times \Omega^{2,cl} \middle| FI + IF + FQF = 0 \text{ and } \varphi_*(I^F) = I \right\}$$

where $Q = Im(\sigma)$. From theorem (1.7) it is known that an element $[Z, \Omega, \ell] \in \mathcal{PG}^{\ell}(X_+, X_-)$ determines precisely a degenerate GK metric of symplectic type with the complex structures I_+, I_- on X_+ and X_- as well as $F = \ell^*\Omega \in \Omega^{2,cl}_M$ on the smooth Poisson manifold (M, Q) underlying both (X_+, σ_+) and (X_-, σ_-). The imaginary parts $(X_+, \mathrm{Im}(\sigma_+))$ and $(X_-, \mathrm{Im}(\sigma_-))$ are identified by the smooth Poisson diffeomorphism $\varphi_\ell = \pi_- \circ \ell : X_+ \to X_-$ as ℓ is an Im-Lagrangian bisection viewed as a section of π_+.

The conditions of a degenerate GK metric of symplectic type $I_+ - I_- = QF$ and $I_+^* F + FI_- = 0$ may be used to define in analogy to Courant automorphisms a corresponding torsor of *Courant Isomorphisms*

$$\mathfrak{Iso}_{\mathcal{C}}(X_+, X_-) = \left\{ (\varphi, F) \in \mathrm{Diff}_Q(X_-, X_+) \times \Omega^{2,cl} \middle| \; FI_+ + (I_-^{\varphi^{-1}})^* F = 0 \text{ and } \varphi_*(I_+ + QF) = I_- \right\}$$

There is a forgetful map

$$\mathcal{PG}^{\ell}(X_+, X_-) \to \mathcal{PG}(X_+, X_-)$$

where each fiber is a torsor for Lagrangian bisections of the Weinstein groupoid (G_-, Ω_-) of (X_-, σ_-).

Theorem 1.12. *Infinitesimal deformations of the generalized Kähler class in the complex Kähler moduli are* 8W48AF
given by

$$T_{[Z]}\mathcal{K}(X_+, X_-) = \ker \left(H^{(1,1)}_{\mathbb{R}} \xrightarrow{\Lambda^\bullet \sigma} \mathbb{H}^1(\mathfrak{X}^{\geq 1}, d_\sigma) \right)$$

under the assumption that (X_-, σ_-) does not permit global holomorphic Poisson vector fields.

Proof. A deformation of the generalized Kähler class $[Z]$ is a deformation of the Morita equivalence (Z, Ω) to a genuinely new isomorphism class. Because (Z, Ω) is a principal bibundle, such a deformation can be described by composition with a deformation of the Weinstein groupoid

$(\mathcal{G}_-, \Omega_-)$ integrating (X_-, σ_-) as a holomophic symplectic right principal affine bundle of itself that fixes the holomorphic Poisson structure on the base.

$$T_{[Z]}\mathcal{PG}(X_+, X_-) \cong T_{[G_-]}\mathcal{PG}(X_-, X_-)$$

These deformations are controlled by the deformation complex

$$\left[\Omega^{\geq 1}_{X_-} \xrightarrow{\Lambda^\bullet \sigma} \mathfrak{X}^{\geq 1}_{X_-}\right]$$

To compute its first hypercohomology, we resolve this complex by

$$\left[0 \to \mathfrak{X}^{\geq 1}_{X_-}\right] \to \left[\Omega^{\geq 1}_{X_-} \to \mathfrak{X}^{\geq 1}_{X_-}\right] \to \Omega^{\geq 1}_{X_-}$$

which turns into the long exact sequence

$$\cdots \to H^0(\mathfrak{X}^{\geq 1}) \to T_{[G_-]}\mathrm{Pic}(X_-, \sigma_-) \to \mathbb{H}^1(\Omega^{\geq 1}) \to \mathbb{H}^1(\mathfrak{X}^{\geq 1}) \to \cdots$$

We may assume that there are no global holomorphic Poisson vector fields. We resolve further the sheaf of closed holomophic 1-forms by the double complex.

$$\begin{tikzcd}
\Omega^{1,2} & & \\
\Omega^{1,1} \arrow[u] \arrow[r] & \Omega^{2,1} & \\
\Omega^{1,0} \arrow[u, "\bar\partial"] \arrow[r, "\partial"'] & \Omega^{2,0} \arrow[u] \arrow[r] & \Omega^{3,0}
\end{tikzcd}$$

A class in $\mathbb{H}^1$ is a pair (ω, β) of $\omega \in \Omega^{1,1}$ and $\beta \in \Omega^{2,0}$ such that $\bar\partial\omega = 0$, $\partial\omega + \bar\partial\beta = 0$ and $\partial\beta = 0$. The effect of this pair on the groupoid (G_-, Ω_-) is an infinitesimal magnetic deformation

$$\Omega_- \mapsto \Omega_- + s^*(\omega + \beta)$$

Requiring that the deformed bibundle admits any Im-Lagrangian bisections forces ω and β to be real. For the $(2,0)$-form β this means it is identically zero. This leaves $\omega \in \Omega^{(1,1)}(\mathbb{R})$ which is d-closed modulo $\bar\partial$-exact $(1,1)$-forms $\bar\partial\alpha$.

Since we impose the deformation class $[(\omega, 0)]$ to be trivial for (X_-, σ_-) its image under the map $\Lambda^\bullet\sigma$ has to be exact in $\mathbb{H}^1$. In other words, $\sigma\omega = \bar\partial V$ for some $(1,0)$ Poisson vector field V. ■

Remark 1.13. The classical Kähler class of a Kähler metric F on (M, I) is typically thought of as its TRZ1N8
class $[F] \in H^2_{dR}(X, \mathbb{R})$ in second deRham cohomology. It can be imported into our discussion by viewing it as defining the holomorphic symplectic affine bundle $(Z, \Omega) = (T^*X, \Omega_0 + \pi^*F)$ where Ω_0 is the canonical holomorphic symplectic form on T^*X. The zero section of T^*X represents in

this presentation of the isomorphism class $[Z, \Omega]$ the particular Kähler metric F. It is important to remark that it is no longer a holomorphic submanifold and (Z, Ω) is no longer a holomorphic symplectic groupoid. All other Kähler metrics $F + dd^c f$ in this class are graphs of exact 1-forms $\Gamma_{df} \subseteq T^*M$.

The discussion above simplifies in the presence of the $\partial\bar{\partial}$-Lemma when $H^2_{dR}(X, \mathbb{C})$ decomposes into $H^{2,0} + H^{1,1} + H^{0,2}$. Furthermore, the vanishing of the Poisson tensor $\sigma = 0$ simplifies the Maurer Cartan equation on $\Omega^{\geq 1}_X$ as the Koszul bracket $[\ ]_Q$ is zero in this case. Deformations of the Kähler class within the complex generalized Kähler moduli, following the above theorem (**??**), are given by classes $[\omega] \in H^{(1,1)}(M, \mathbb{R})$. This precisely agrees with deformations of $[F]$ in the *Kähler cone* $\mathcal{K}_{(M,I)}$ which forms an open convex cone in $H^{(1,1)}(M, \mathbb{R})$. Interestingly, if we did allow the existence of holomorphic vectorfields on (M, I), these would appear in Maurer Cartan elements of the complex $\Omega^{\geq 1} \xrightarrow{0} \mathfrak{X}^{\geq 1}$. ∎

2 BACKGROUND

In this chapter we lay out the foundations for the content of this thesis. This moreover serves the purpose of introducing notation and conventions. We follow [12, 13] for concepts of generalized complex geometry and refer to [21, 19] as a standard reference on symplectic groupoids. We use a series of papers by Xu and Weinstein to introduce a Morita Equivalence [], for everhthing else, we refer the reader to [9, 16]. None of the material presented in this chapter is original, everything is well known in the literature.

2.1 HOLOMORPHIC POISSON MANIFOLDS

The fundamental building block of the theory developed in this work is a smooth $2n$-dimensional manifold M that carries an integrable complex structure I and a smooth Poisson structure Q with the compatibility G0N27K

$$IQ - QI^* = 0$$

Definition 2.1 (Poisson manifold)**.** A *Poisson structure* on a manifold M is a bivector $Q \in \Gamma(\Lambda^2 TM)$ that satisfies the integrability condition with respect to the Schouten bracket. TLMSIG

$$[Q, Q] = 0$$

■

The notion of a Poisson structure on a manifold makes sense in both the smooth category of smooth manifolds as well as in the holomorphic category. We denote by $X = (M, I)$ the smooth manifold together with the complex structure I. The real and imaginary parts of a holomorphic Poisson structure $\sigma = I^*Q + iQ$ are smooth Poisson structures in their own right.

A Poisson structure that is invertible is a symplectic structure $F = Q^{-1}$. This is a 2-form $F \in \Gamma(\Lambda^2 T^*M)$ that satisfies the integrability condition $dF = 0$.

We may think of the Poisson bivector as a morphism $T^*M \xrightarrow{Q} TM$. The image $\mathrm{Im}(Q)$ is a possibly singular distribution in the tangent bundle. Nonetheless, it gives rise to a foliation of the manifold with leaves on which the Poisson tensor becomes nondegenerate by construction. For that reason one refers to the *symplectic leaves* of the *characteristic foliation* of a Poisson manifold. Weinstein has shown in [**?**] that every point of a Poisson manifold belongs to a unique leaf. As the rank of $\mathrm{Im}(Q)$ may vary, so does possibly also the dimension of the leaves.

Brackets from Poisson structures

The Poisson bivector Q induces a bracket on the algebra of functions $f, g \in C^\infty_M(\mathbb{R})$

$$\{f, g\} = Q(df, dg)$$

which is a Lie bracket, namely skew, bilinear and a derivation of itself (Jacobi identity), plus a derivation of the pre-existing product by multiplication of functions (Leibniz identity). One calls this for obvious reasons the *Poisson bracket*. The original application of Poisson geometry in classical mechanics is to produce Hamiltonian symmetries from functions by precomposition as follows

$$C^\infty_M(\mathbb{R}) \xrightarrow{d} \Omega^1 \xrightarrow{Q} \mathfrak{X}_M \tag{2.1}$$

This associates to any function $f \in C^\infty_M(\mathbb{R})$ its *Hamiltonian vector field* $X_f = Q(df)$. The *Koszul bracket* $[\ ]_Q$ introduced in [] is the unique Lie bracket on 1-forms Ω^1_M for which (2.1) are morphisms of Lie algebras. We dive deeper into this structure surrounding the discussion of Lie algebroids in section 2.3. Explicitly, for $a, b \in \Omega^1_M$

$$[a, b]_Q = L_{Q(a)}b - L_{Q(b)}a - dQ(a, b)$$

A closer inspection reveals that the Koszul bracket applied to two closed 1-forms is exact.

Complex Identities

In view of the complex structure I as a bundle endomorphism $TM \xrightarrow{I} TM$, just like a symplectic form $TM \xrightarrow{F} T^*M$, we write FI for their composition. This amounts to $FI(-,-) = F(I-,-)$ when written as a bicovector. Analogously for the dual I^* we write $I^*F(-,-) = F(-, I-)$. The complex structure I on M decomposes complex differential forms into a double complex $\Omega^{p,q}_M$. The projection of a 2-form F into this decomposition satisfies

$$2F^{(2,0)+(0,2)}I = (FI + I^*F) \qquad 2F^{(1,1)}I = (FI - I^*F) \tag{2.2}$$

The same discussion can be repeated for bivector fields $Q \in \mathfrak{X}^2_M$.

90 years ago Erich Kähler came across a particularly remarkable kind of metric built from a complex structure and a symplcectic form [17].

Definition 2.2 (Kähler manifold)**.** A complex manifold $X = (M, I)$ with a symplectic structure F is 5MFQXP
a *Kähler manifold* if F is of type (1,1) for I and $g = -FI$ defines a Riemannian metric. ■

On a Kähler manifold (X, I, F) the *Lefschetz operator* L_F on differential forms $\Omega^\bullet_M$ is $Fa \mapsto F \wedge a$. This forms an isomorphism on forms of degree $k \leq n$ onto its image. Since we have a Riemannian metric at our disposal on a Kähler manifold, $\Omega^\bullet_M(\mathbb{R})$ obtains an inner pairing that we can use to define a formal adjoint operator Λ_F. The commutator of these operators has on k-forms the eigenvalue

$$[L_F, \Lambda_F]\alpha = (k-n)\alpha$$

as proven for example in the book [16]. More generally, on k-forms we can compute the commutator of the i-th power of L with Λ.

$$[L^i, \Lambda] = i(k-n+i-1)L^{i-1}$$

We will use time and time again the special case of $k=2$ and $i=n$ applied to $a \wedge b$ for $a, b \in \Omega^1$

$$n \cdot a \wedge b \wedge F^{n-1} = F^{-1}(a,b)F^n \tag{2.3}$$

where we have also used $\Lambda_F(a \wedge b) = -F^{-1}(a,b)$.

A map between two Poisson manifolds $(M,Q) \xrightarrow{f} (N,P)$is said to be a *Poisson map* if the Poisson structures are f-related.

$$f_*Q = P$$

At first glance, this appears to be a reasonable choice of morphisms to construct a category of Poisson manifolds. A slightly more relaxed and generalized notion of morphism is that of a *Morita equivalence*. We say that two Poisson manifolds (M,Q) and (N,P) are Morita equivalent if there is a symplectic manifold (S,ω) mapping to (M,Q) with a Poisson map and to (N,P) with an Anti-Poisson map

$$\begin{tikzcd} & (S,\omega) \arrow[dl, "\pi_M"'] \arrow[dr, "\pi_N"] & \\ (M,Q) & & (N,P) \end{tikzcd} \tag{2.4}$$

satisfying the conditions listed in (1.6).

Definition 2.3 ([2], Definition 6.1)**.** The *holomorphic Picard groupoid* $\mathcal{PG}$ is the category whose objects URGYKK
are integrable holomorphic Poisson manifolds and whose morphisms are isomorphism classes of holomorphic symplectic Morita equivalences. The *Picard group* of a holomorphic Poisson manifold (X,σ) is the automorphism gorup of (X,σ) in $\mathcal{PG}$:

$$\mathrm{Pic}(X,\sigma) = \hom_{\mathcal{PG}}((X,\sigma),(X,\sigma)).$$

2.2 GENERALIZED COMPLEX GEOMETRY

This section is devoted to the introduction of generalized geometry with the goal of reaching a working definition of generalized complex branes which represent the machinary to encode generalized Kähler metrics. The field of generalized geometry has only come to life in the past two decades during which it has experienced a sweeping success as the native language to formulate higher geometric structures. Initiated by Nigel Hitchin and studied by his students Marco Gualtieri and Gil Cavalcanti, it has flourished through the work of many more contributors.

The central object in this realm is the *generalized tangent bundle* $\mathbb{T}M = TM \oplus T^*M$ of a smooth manifold M. This is a Courant algebroid when equipped with the pairing

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}\left(\xi(Y) + \eta(X)\right)$$

the *Courant bracket*

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + L_X \eta - \iota_Y d\xi$$

and the projection $\mathbb{T}M \xrightarrow{\rho} TM$ as the *anchor*. Note that this bracket fails to be a Lie bracket because its Jacobiator $\mathrm{Jac}_{[\![\ ]\!]}$ is nonzero, more precisely, the failure is measured by

$$\mathrm{Jac}_{[\![\ ]\!]}(e_1, e_2, e_3) = \langle [\![e_1, e_2]\!], e_3 \rangle$$

where $e_1, e_2, e_3 \in \Gamma(\mathbb{T}M)$. A *generalized complex structure* $\mathbb{J}$ on M is an orthogonal endomorphism of $\mathbb{T}M$ that squares to $-\mathbb{1}$ and is integrable with respect to the Courant bracket $[\![\ ]\!]$. The integrability condition is, as in classical complex geometry, the requirement that the $+i$ eigenbundle

$$L_{\mathbb{J}} = \ker(\mathbb{1} - i\mathbb{J})$$

in $\mathbb{T}_{\mathbb{C}}M$ is closed under the Courant bracket.

Generalized complex structures recover the classical notions of complex and symplectic geometry. A complex manifold (M, I) may be viewed as the generalized complex structure

$$\mathbb{J}_I = \begin{pmatrix} I & 0 \\ 0 & I^* \end{pmatrix}$$

Indeed, integrability of I is equivalent to the integrability of $\mathbb{J}_I$. For a symplectic manifold (M, F) can can construct the generalized complex structure

$$\mathbb{J}_F = \begin{pmatrix} 0 & -F^{-1} \\ F & 0 \end{pmatrix}$$

Again, the integrability of $\mathbb{J}_F$ captures precisely the integrability condition of F, namely $dF = 0$. Unlike classical geometry, the symmetries of generalized complex structures include new kinds of elements, namely closed 2-forms $B \in \Omega^{2,cl}$, so called *B-field transformations*. Automorphisms of $\mathbb{T}M$ as a Courant algebroid are structure preservng bundle automorphisms encoded by a pair $(\varphi, B) \in \mathrm{Diff}(M) \times \Omega^{2,cl}_M$ that acts as

$$(\varphi, e^B) \cdot (X + \xi) = \varphi_* X + (\varphi^{-1})^* \xi + \iota_{\varphi_* X} B$$

Derivations of $\mathbb{T}$, infinitesimal symmetries, are elements in the product $\mathfrak{X}_M \times \Omega^{2,cl}$. Sections $X + \xi \in \Gamma(\mathbb{T})$ act by the adjoint action

$$\mathrm{ad}_{X+\xi} = (X, d\xi) \in \mathrm{Der}(\mathbb{T})$$

in which covectors act trivially. In the presence of a generalized complex structure $(\mathbb{T}, \mathbb{J})$ we call the pair (φ, B) a symmetry of $\mathbb{J}$ if it commutes with $\mathbb{J}$. In the case of a complex structure $\mathbb{J}_I$ these are precisely $(\varphi, B) \in \mathrm{Aut}(I) \times \Omega^{(1,1),cl}$. Leaving aside the question of integrability for a moment, symmetries of $\mathbb{J}_I$ of the form $(0, F)$ are precisely closed $(1,1)$-forms which define Kähler metrics on (M, I). It will become clear in the discussion around branes in (**??**) why a symmetry like this gives rise to a Kähler structure.

The generalized symmetries of $\mathbb{J}_F$ are symplectomorphisms with vanishing B-field $(\varphi, 0) \in \mathrm{Sym}(M, F) \times \Omega^{2,cl}_M$. In this context we see that symplectic and complex structures have a more comparable group of symmetries. In particular, we can generate a *Hamiltonian* symmetry from a smooth function $f \in C^\infty_M(\mathbb{R})$.

$$X + \xi = -\mathbb{J} d_L f$$

For symplectic structures this is nothing new and we obtain the classical Hamiltonian vector field $X_f = F^{-1} df$, but for a complex structure this produces

$$-\mathbb{J}_I df = d^c f$$

Generalized Kähler structure

At this point we can properly introduce the generalization of a Kähler structure (definition 2.2) in the context of generalized geometry which was already heavily discussed in the introduction (**??**).

Definition 2.4 (Generalized Kähler structure [11])**.** A *Generalized Kähler Structure* on M is a pair of 28E4MY
commuting generalized complex structures $\mathbb{J}_A, \mathbb{J}_B$ such that $\mathbb{G} = -\mathbb{J}_A \mathbb{J}_B$ defines a positive metric

$$\langle \mathbb{G} e_1, e_2 \rangle$$

on sections $e_1, e_2 \in \Gamma(\mathbb{E})$. ∎

The classical notion of a Kähler structure (I, F, g) on M is captured in this way when we encode both the symplectic and compex structure as generalized complex structures $\mathbb{J}_I$ and $\mathbb{J}_F$ as above. Commutativity places the condition $FI + I^* F = 0$ on the pair (I, F) and positivity of

$$\mathbb{G} = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$$

is the usual positivity condition on the Riemannian metric $g = -FI$.

Generalized Complex Branes

Let M be a manifold with a generalized complex structure $\mathbb{J}$. The correct sub-object to consider in generalized geometry is that of a generalized complex brane introduced in [11]. A *generalized* *submanifold* (S, F) is a pair of a submanifold $S \hookrightarrow M$ with a closed 2-form F. The *generalized tangent bundle* $\tau_{(S,F)}$ to the generalized submanifold (S, F) is the subbundle

$$\tau_{(S,F)} = \{X + \xi \in TS \oplus T^*M \mid i^*\xi = \iota_X F\}$$

of $\mathbb{T}M$. A *generalized complex submanifold* (S, F) is a generalized submanifold of $(M, \mathbb{J})$ for which the generalized tangent bundle is $\mathbb{J}$ invariant. A generalized complex submanifold for which F is integral and prequantized by a unitary hermitian line bundle $(L, \nabla) \to S$ is called a *generalized complex brane*.

A symplectic manifold (M, ω) viewed as a generalized complex manifold with $\mathbb{J}_\omega$ has two kinds of GC branes.

Example 2.5. Consider $\mathcal{L} \hookrightarrow M$ as a GC brane with with a flat unitary bundle (L, ∇). The generalized tangent bundle in this case is

$$\tau_{\mathcal{L},0} = T\mathcal{L} \oplus \mathrm{Ann}(T\mathcal{L})$$

Requiring that this bundle is preserved by $\mathbb{J}_\omega$ is equivalent to the folowing two conditions. First, $i^*\omega(X) = 0$ which means $\mathcal{L}$ is isotropic, Second, $-\omega^{-1}\xi \in TL$ which means that L is coisotropic. In summary, the GC brane condition for symplectic manifolds recovers Lagrangian submanifolds that carry a flat unitary line bundle. ■

While on the topic of Lagrangian branes in a symplectic manifold, we digress and take a look at infinitesimally close Lagrangian branes. Due two Weinstein it is known that deformations of the Lagrangian submanifold are controlled by closed 1-forms on the Lagrangian $a \in \Omega^{1,cl}_{\mathcal{L}}$. This deforms $\mathcal{L}$ into the normal direction $[\omega^{-1}(a)] \in N\mathcal{L}$. The same degree of freedom controls deformations of unitary connections. We combine these observations to the following lemma.

Lemma 2.6. *The tangent space to Lagrangian branes $\mathcal{L}Brn$ at $(\mathcal{L}, \nabla)$ is identified with closed complex* I4BL2F
1-forms on $\mathcal{L}$

$$T_{(\mathcal{L},\nabla)} = \Omega^{1,cl}_{\mathcal{L}}(\mathbb{C})$$

Example 2.7. The other kind of GC brane present in a symplectic manifold is that of a *coisotropic* C398KY
A-Brane found by Kapustin and Orlov []. Consider (M, ∇) as a spacefilling brane in (M, ω) with curvature F^∇. The generalized tangent bundle of (M, ∇) is the graph of F^∇ in $\mathbb{T}M$.

$$\tau_{M,F^\nabla} = \Gamma_{F^\nabla}$$

This bundle is $\mathbb{J}_\omega$ stable if and only if $\omega^{-1}F$ defines an almost complex structure. In particular, $F + i\omega$ is a holomorphic symplectic form for $I = \omega^{-1}F$. ■

We record this observation that is going to be important later in the following lemma.

Lemma 2.8. *A holomorphic symplectic manifold $(Z, I, B + i\omega)$ is equivalent to a spacefilling brane (Z, B) in the generalized complex manifold (Z, ω).* KN1VRG

Example 2.9. A complex manifold (M, I) with brane $(S, \nabla) \hookrightarrow M$ turns out to be a complex submanifold that supports a holomorphic hermitian line bundle. Requiring that $\tau_{S,\nabla}$ is $\mathbb{J}_I$ invariant first places the constraint on S that $I(TS) \subseteq TS$ which means it is a complex submanifold. The application of $\mathbb{J}_I$ to $X + \xi$ is $IX - I^*\xi$ which satisfies

$$\iota_{IX}F = i^*(-I^*\xi) = -I^*F(X)$$

for all $X \in TS$ precisely if F is of type $(1, 1)$ for I. But the vanishing of $F^{2,0}$ implies that $\nabla^{(0,1)}$ defines a holomorhpic structure on L for which F^∇ is the Chern curvature.

If we assume that this brane is supported on the entire manifold and that the curvature F^∇ is nondegenerate, we have found a Kähler structure (M, I, F^∇). This is a crucial point of view that we will generalize below in (**??**). ∎

We now turn to the question of existence of a spacefilling brane (M, F) in $(M, \mathbb{J})$ in general as investigated in in [12]. We may assume $F = 0$ without loss of generality for if it is nonzero, $(M, 0)$ is a brane in $(M, \mathbb{J}' = e^{-F}\mathbb{J}e^F)$ if and only if (M, F) is a brane in $(M, \mathbb{J})$. The generalized tangent bundle of $(M, 0)$ is $TM \hookrightarrow \mathbb{T}M$ and its preservation by

$$\mathbb{J} = \begin{pmatrix} I & Q \\ k & -I^* \end{pmatrix}$$

requires first of all $k = 0$. Because $\mathbb{J}^2 = -\mathbb{1}$ and $\mathbb{J}$ is orthogonal for the pairing $\langle\ \rangle$ I has to be an almost complex structure for which Q is a bivector of type $(2, 0) + (0, 2)$. The integrability of both I and Q is a consequence of the integrability of $\mathbb{J}$. This discussion can be captures as the following.

Lemma 2.10 ([12], Proposition 5)**.** *$(M, \mathbb{J})$ carries a spacefilling brane if and only if it is a holomorhpic Poisson manifold (M, I, Q).* R3MW8M

This is actually a slight alteration of the more general Proposition 5 in [12] as we already assume that T plays the role of an integrable, coisotropic splitting of the short exact sequence

$$T^*M \to TM \oplus T^*M \to TM$$

and the only remaining proporty to check is the $\mathbb{J}$ invariance of this splitting. More importantly, this is yet another classic geometric structure that can be encoded by generalized complex geometry. We denote a generalized complex structure of such form by $\mathbb{J}_{I,Q}$.

The existence of a spacefilling brane is great, but can we do better than that? Can $(M, \mathbb{J})$ accommodate a second spacefilling brane? This is the same as asking for $\mathbb{J}_{I,Q}$ to preserve the graph Γ_F of a closed 2-form F. We find that this is equivalent to

$$FI + I^*F + FQF = 0 \tag{2.5}$$

For the same reason as above, $I + QF$ is a holomorphic symplectic Poisson structure on (M, Q) and rewriting (2.5) with $I_- = I$ and $I_+ = I + QF$ recovers the equations 1.1

$$
\begin{aligned}
I_+ - I_- &= QF \\
I_+^* F + FI_- &= 0
\end{aligned}
$$

considered in section (1.3) when we introduced degenerate generalized Kähler structures of symplectic type. Hence, in this way we can think of a GK structure of symplectic type as a pair of integrable, coisotropic $\mathbb{J}$-invariant splittings of a single GC structure $(M, \mathbb{J})$.

But why stop at two spacefilling branes if there might be more? Holomorphic poisson structures (I_i, σ_i) with a common imaginary part $Q = \mathrm{Im}(\sigma_i)$ on M form the objects of a groupoid for which arrows are the B-field transforms relating the spacefilling branes. To conclude our discussion of generalized complex geometry, we explain a method discovered in [] to generate new spacefilling branes called the *flow constuction*.

Flow Construction

We start with a holomorphic Poisson manifold (M, I, Q) as discussed above. A smooth function 2HOIT4
$f \in C^\infty(M, \mathbb{R})$ gives rise to a family of new holomorphic Poisson structures (I_t, Q) related by F_t to (M, I, Q).

Let φ_t be the flow of the Hamiltonian vector field $X_f = Q(df)$. We use this flow to propagate

$$I_t = I^{\varphi_t} = (\varphi_t)_* \circ I \circ (\varphi_t^{-1})_*$$

To construct F_t such that $I_t - I = QF_t$ for all t, we notice that $\dot{I}_t = Q(dd_t^c f)$ and integrate this to

$$F_t = \int_0^t dd_s^c f ds = \int \varphi_s^* dd^c f ds$$

This results in (I_t, I, Q, F_t) which is a pair of holomorphic poisson manifolds connected by the arrow F_t. In particular, this construction can be used to deform a given generalized Kähler structure (I_+, I_-, Q, F) by adding F_t to F and flowing I_+ by the Hamiltonian flow φ_t.

Another significant application of this construction can be demonstrated with a quick computation. We apply the pair (φ_t, F_t) a symmetry an automorphism of the Courant algebroid to $\mathbb{J}_{I,Q}$

$$e^{-F_t} \begin{pmatrix} I_t & Q \\ 0 & -I_t^* \end{pmatrix} e^{F_t} = \begin{pmatrix} I & Q \\ 0 & -I^* \end{pmatrix}$$

This means (φ_t, F_t) furnishes an example of an automorphism of (M, I, Q).

2.3 LIE GROUPOIDS

The primary motivation for the introduction of Lie groupoids in this thesis is their role as the global MB9VA4
objects integrating Poisson manifolds, more precisely the Poisson Lie algebroid $T^*_Q M$ introduced in (**??**). They are instrumental in the presentation of a generalized Kähler structure as a bisection in a Morita equivalence as done in [2].

A groupoid cosists of a space M of points and another space G of arrows which connect points in M. An arrow $g \in G$ starts at a *source* point $s(g)$ in M and ends at a *target* point $t(g)$ in M. This association gives two maps s and t from G to M.

$$\begin{tikzcd} G \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ M \end{tikzcd} \tag{2.6}$$

Every element in $x \in M$ needs to have an *identity* arrow whose source and target is x. Associating this unique arrow to every element in M defines the map $M \xrightarrow{\varepsilon} G$. Thinking of arrows as paths from their source to their target, we can define a product of arrows by concatenation, however, only if the target of the second arrow agrees with the source of the first arrow. In this case we call these arrows composable and denote the subset of all ordered pairs of composable arrows

$$G^{(2)} = \left\{ (g_1, g_2) \in G \times G \middle| s(g_1) = t(g_2) \right\}$$

The product $g_1 g_2 = m(g_1, g_2)$ is then an arrow that shares its source with g_2 and its target with g_1. To complete the introduction of groupoids, we require every arrow $g \in G$ to have an inverse arrow $i(g)$ that is composes with g to the identity arrow of $s(g)$, respectively $t(g)$, on the left, respectively on the right.

The name *groupoid* suggests a group like behaviour of G with the important distinction that not any two elements may be composed. A group is a groupoid for which the space of points contains only one element $G \rightrightarrows *$

Definition 2.11. A *Lie groupoid* is a groupoid for which both G and M are smooth manifolds the source and target morphisms s, t are surjective submersions and inversion i and multiplication m are smooth maps

$$G^{(2)} \xrightarrow{m} G \xleftarrow{i} G$$

■

Example 2.12. The *pair groupoid* $\mathcal{P}_M \rightrightarrows M$ of any smooth manifold M is the groupoid where the space of arrows $\mathcal{P}_M = M \times M$ are ordered pairs (p, q) of points in M representing an arrow from q to p. As such the source map is projection to the second factor and the target is given by projection to the first factor. The inverse of (p, q) is the pair (q, p), the identity bisection is the embedding of the diagonal $\Delta M \hookrightarrow M \times M$. Two arrows (p, q) and (r, s) are composable if and only if $q = r$ and the product in this case is given by (p, s). ■

Example 2.13. The *action groupoid* of a group action $M \times G \xrightarrow{\rho} M$ is the groupoid

$$\begin{tikzcd} M \times G \arrow[d, shift right, "\rho"'] \arrow[d, shift left, "\pi_M"] \\ M \end{tikzcd} \tag{2.7}$$

Two arrows (p, g) and (q, h) are composable if and only if $q.h = p$ in which case the product is (q, hg). The inverse of (p, g) is $(p.g, g^{-1})$ and the identity bisection is $M \times \{e\}$. ■

A *morphism of groupoids* from $G \rightrightarrows M$ to $\mathcal{H} \rightrightarrows N$ consists of a pair of morphisms (f_1, f_0), one for arrows and one for points, such that they intertwine with the groupoid structure maps s, t, ε, i, m. This means, in particular, that the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "f_1"] \arrow[d, shift right] \arrow[d, shift left] & \mathcal{H} \arrow[d, shift right] \arrow[d, shift left] \\ M \arrow[r, "f_0"'] & N \end{tikzcd}$$

Given a groupoid $G \rightrightarrows M$, we define the orbit of a point $p \in M$ as all points $q \in M$ for which there is $g \in G$ sending p to $g(p) = q$. The orbit space M/G of a groupoid $G \rightrightarrows M$ is not guaranteed to be a smooth manifold, let alone exist as a topological space. An extreme example is the orbit space of a group $*/G$.

Bisection of Groupoid

A *bisection* ℓ of a groupoid $G \rightrightarrows M$ is a submanifold ℓ in G that uniquely intersects each fiber of s and t. Another perspective on a bisection ℓ is a section of the target map $G \xrightarrow{t} M$ in such a way that $s \circ \ell$ defines a diffeomorphism on M. It can be advantageous to think of a bisection to be especially a section of t and define ℓ as a map that associates to $x \in M$ the unique arrow in the intersection of $\ell \hookrightarrow G$ and $t^{-1}(x)$.

$$\begin{tikzcd} G \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ M \arrow[u, bend right, "\ell"'] \end{tikzcd} \tag{2.8}$$

The multiplication m on the groupoid induces a multiplication operation of two bisection ℓ_1, ℓ_2.

$$(\ell_1 * \ell_2)(x) = m(l_1(x), l_2((s \circ \ell_1)(x))$$

This forms the product on the group $\Gamma(G)$ of bisections. We write $\Gamma_t(G)$ to specify that we view ℓ in particular as a section of t, respectively $\Gamma_s(G)$ for the source.

Definition 2.14. The tangent space to $\Gamma(G)$ at ℓ consists of sections $[X]$ of the normal bundle $N\ell$. ■ DB4XQ7

When we do choose a preferred point of view of ℓ as a section of t, there is a unique vector field in every normal class $[X]$ that is tangent to the target fibers. From every bisection ℓ of G we obtain a diffeomorphism φ_ℓ on M which applies the unique arrow $\ell(x)$ which ℓ singles out to the point x

$$\varphi_\ell = s \circ \ell$$

Due to notational convention we trace the arrow backwards from its target $t(\ell)$ to its source $s(\ell)$. This assignment is a group homomorphism $\Gamma(G) \xrightarrow{\varphi} \mathrm{Diff}(M)$ because it satisfies $\varphi_{\ell_2} \circ \varphi_{\ell_1} = \varphi_{\ell_1 * \ell_2}$. A bisection ℓ also gives rise to a diffeomorphism $\widetilde{\varphi}_\ell$ on G itself by multiplying $g \in G$ on the right with the unique arrow in ℓ that is composable with g on the right. In formulas this is

$$\widetilde{\varphi}_\ell(g) = m(g, \ell(s(g)))$$

which is also consistent with the product of bisections

$$(\ell_1 * \ell_2)(x) = \widetilde{\varphi}_{\ell_2}(\ell_1(x))$$

It preserves the target of an arrow and covers φ_ℓ on M.

$$\begin{tikzcd} G \arrow[r, "\widetilde{\varphi}_\ell"] \arrow[d, "s"'] & G \arrow[d, "s"'] \\ M \arrow[r, "\varphi_\ell"] & M \end{tikzcd} \tag{2.9}$$

The *transpose* of a bisection $\ell \in \Gamma_t(G)$ is the same bisection $\ell^T \in \Gamma_s(G)$, but viewed as a section of s and vice versa. It satisfies the property

$$\ell^T \circ \varphi_\ell = \ell. \tag{2.10}$$

Lie Algebroids

In the same way that infinitesimally the multiplication on a Lie group is captured by a Lie algebra, there is an infinitesimal counterpart to a Lie groupoid.

Definition 2.15. A *Lie algebroid* over M is a vector bundle $A \to M$ with a Lie bracket $[\,]$ on its sections $\Gamma(A)$ and an *anchor* map $A \xrightarrow{\rho} TM$ such that the Leibniz rule holds for $a, b \in \Gamma(A)$, $f \in C^\infty_M(\mathbb{R})$

$$[a, fb] = \rho(a)f \cdot b + f[a, b]$$

and the anchor is a morphism of Lie algebras between sections of A and vector fields with their standard Lie bracket. ■

The Lie algebroid $\mathcal{A}$ of a Lie groupoid G is the normal bundle of the identity bisection $M \xrightarrow{\varepsilon} G$.

$$\mathcal{A} = Lie(G) = N\varepsilon(M) = \varepsilon^* TG/TM$$

The difference of tangent maps $ds - dt : TG \to TM$ is well defined on the normal bundle and descends to give the anchor map of $\mathcal{A}$. We may view a section $a \in \Gamma(\mathcal{A})$ as first order differential of a family of bisection $g_t \in \Gamma(G)$ with $g_0 = \varepsilon$

$$a = \frac{d}{dt}\bigg|_0 g_t$$

This identifies section of the Lie algebroid with the Lie algebra of bisections of the groupoid. In formulas, $\mathrm{Lie}(\Gamma(G)) = \Gamma(\mathrm{Lie}(G))$. This lets us define the adjoint action of a bisection $h \in \Gamma(G)$ on $a \in \Gamma(\mathcal{A})$ by

$$\mathrm{Ad}_h a = \frac{d}{dt}\bigg|_0 h g_t h^{-1}$$

And consequently we can define the Lie bracket as

$$[a, b] = \mathrm{ad}_a b = \frac{d}{dt}\bigg|_0 \mathrm{Ad}_{g_t} b$$

where g_t is a family of bisections integrating a.

Example 2.16. An example of a Lie algebroid that we have come across in disguise in section (**??**) is that of the cotangent bundle of a Poisson manifold (M, Q). The Koszul bracket $[\ ]_Q$ is a Lie bracket on 1-forms, the space of sections of T^*M, and Q acts as the anchor $T^*M \xrightarrow{Q} TM$. This is indeed a bracket preserving morphism as discussed around equation 2.1 and $[\ ]_Q$ is built to satisfy the Leibniz indentity. We denote this *Poisson Lie algebroid* by $T^*_Q M$. ■ V7R6TP

A section $a \in \Gamma(\mathcal{A})$ gives rise to a unique vector field X_a along the identity bisection that is tangent to the target distribution dt. In this way we identify

$$N\varepsilon(M) \cong \ker(dt)\big|_M$$

By left translation this gives rise to a unique left invariant vector field on the groupoid tangent to the target fibers.

$$(X_a^L)_g = dL_g(X_a)$$

The integration map $\Gamma(\mathcal{A}) \xrightarrow{\exp} \Gamma(G)$ is constructed by flowing the identity bisection by the left invariant vector field determined by a Lie algebroid bisection $a \in \Gamma(\mathcal{A})$.

$$g_t = e^{tX_a^L} \circ \varepsilon$$

The question of integrability, namely if there exists a Lie groupoid for a given Lie algebroid of which it is the Lie algebroid, does not always have a positive or necessarily unique answer. If it does, we call $\mathcal{A}$ *integrable* and Moerdijk-Marcun have shown in **[?]** that uniqueness can be guaranteed if we assume for the source fibers to be connected and simply connected. The case of integrating the Poisson Lie algebroid T_Q^*M from example (2.16) will be addressed below in the section on *symplectic Lie groupoids*.

Groupoid module

In similar fashion to group actions, we want to introduce the action of a groupoid $G \rightrightarrows M$ on a space P. The distinguishing feature is, as for composition of groupoid arrows, that not every element $g \in G$ can act on every element $p \in P$. To sort this out, we specify a moment map J that picks out the arrows that are allowed to act on a given point

$$\begin{tikzcd} & G \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ P \arrow[r, "J"'] & M \end{tikzcd} \tag{2.11}$$

A *groupoid action* of $G \rightrightarrows M$ on P consists of a pair (J, ρ) of a moment map J as above and an action morphism

$$P \times_{(J,t)} G \xrightarrow{\rho} P$$

such that

(a) J is G-equivariant for the action of G on M, that is,

$$J(p \cdot g) = s(g)$$

(b) The action is a group homomorphism

$$(p \cdot g) \cdot h = p \cdot (gh)$$

(c) The identity arrow acts trivally

$$p \cdot \varepsilon(J(p)) = p$$

Definition 2.17 ([20])**.** A principal groupoid bundle for $G \rightrightarrows M$ is a G space (P, J, ρ) with a surjective submersion $P \xrightarrow{\pi} X$ such that

(a) The action of G is preserving the fibers of π

(b) The action on each fiber $\pi^{-1}(x)$ is effective, that is,

$$P \times_{J,t} G \xrightarrow{\mathrm{id} \times \rho} P \times_X P$$

is a diffeomorphism.

$$
\begin{tikzcd}[column sep=small]
 & P \arrow[dl, "\pi"'] \arrow[dr, "J"] & & G \arrow[dl, shift left] \arrow[dl, shift right] \\
X & & M &
\end{tikzcd}
\tag{2.12}
$$

The notion of *bisections* goes over to groupoid principal bundles. A bisection of P in (2.12) is a section ℓ of π such that $\mathrm{Im}(\ell) \hookrightarrow P$ intersects every J-fiber uniquely.

The set $\Gamma(P)$ of bisections of P does not form a group, but a right $\Gamma(G)$-set. Because the action of G on P is principal, $\Gamma(P)$ forms a torsor for $\Gamma(G)$. As before, every bisection ℓ gives rise to a diffeomorphism $X \xrightarrow{\varphi_\ell} M$ by composition $\varphi_\ell = J \circ \ell$.

Example 2.18. An example is the action of a groupoid on itself by right, respectively left, multiplication. In this case the moment map is the source, respectively the target, map. We call the action morphism for the left, respectively right self action, L, respectively R.

$$
\begin{tikzcd}[column sep=small]
 & G \arrow[dl, "t"'] \arrow[dr, "s"] & & G \arrow[dl, shift right, "t"'] \arrow[dl, shift left, "s"] \\
M & & M &
\end{tikzcd}
$$

■

A *Morita equialence* is a weaker sense of morphism between groupoids. We say that two groupoids $G \rightrightarrows M$ and $H \rightrightarrows N$ are Morita equivalent if there exists a groupoid bimodule with a left H and a right G action such that

(a) Both actions are principal
(b) The left H-action and right G-action commute.

Example 2.19. To illustrate this with an example, we will continue to consider a standard Lie group action $G \circlearrowright X$ on a smooth manifold. The action groupoid $G \times M \rightrightarrows M$ is Morita equivalent to the quotient M/G as the trivial groupoid.

$$
\begin{tikzcd}[column sep=small]
M/G \arrow[dr, shift left] \arrow[dr, shift right] & & M \arrow[dl] \arrow[dr] & & G \times M \arrow[dl, shift left] \arrow[dl, shift right] \\
 & M/G & & M &
\end{tikzcd}
$$

■

Symplectic Lie Groupoids

In order to integrate the Poisson Lie algebroid of a Poisson structure (M, Q) in a meaningful way that incorporates the Koszul bracket $[\ ]_Q$, we enhance groupoids in this section with a symplectic structure on the space of arrows. As we will discover, many structures on a groupoid have a particular flavour if the groupoid is symplectic.

Definition 2.20. A *symplectic groupoid* $(G, \omega) \rightrightarrows M$ is a Lie groupoid which carries a *multiplicative* symplectic form ω on the space of arrows, that is, ω is compatible with the multiplication in the sense that on $G^{(2)}$

$$p_1^*\omega + p_2^*\omega - m^*\omega = 0$$

Another way to phrase this compaibility is to say that the graph of the groupoid multiplication $G^{(2)} \xrightarrow{m} G$ is a Lagrangian in $G \times \overline{G} \times \overline{G}$. ■

The space of points M of a symplectic groupoid carries a Poisson structure for which the target map is Poisson and the source map is Anti-Poisson. If the Poisson Lie groupoid T_Q^*M is integrable, the unique source simply connected groupoid integrating it is a symplectic groupoid and the induced Poisson structure on M agrees with Q.

In a symplectic groupoid, in addition to a diffeomorphism, every bisection also defines a 2-form F_ℓ by pulling back the symplectic form ω on G. Here it is important to note that the convention of viewing ℓ as a section of t or s will make a difference for the identification of $F = \ell^*\omega$ as a 2-form on M. The choices differ by pullback along φ_ℓ according to (2.10), more precisely, $\ell^*\omega = \varphi_\ell^*(\ell^T)^*\omega$ The kernel of this assignment consists of *Lagrangian bisections*

$$L\Gamma(G) \to \Gamma(G) \to \Omega_M^{2,cl}$$

One can show the following.

Lemma 2.21. *For $\ell \in \Gamma(G)$ the 2-form $F_\ell = \ell^*\omega$ is the failure of φ_ℓ to be a Poisson diffeomorphism of (M, Q) in the sense that*

$$(\varphi_\ell)_* Q^{F_\ell} = Q$$

It is therefore natural to keep track of the diffeomorphism and its failure to preserve the Poisson structure together.

Definition 2.22. The map

$$\Gamma(G, \omega) \xrightarrow{(\varphi, F)} \mathrm{Diff}(M) \ltimes \Omega_M^{2,cl}$$

associates to $\ell \in \Gamma(G, \omega)$ the pair (φ_ℓ, F_ℓ) of the diffeomorphism induced by ℓ and pullback of ω to ℓ. ■

The left invariant vector field on (G, ω) to a Lie algebroid section, that is 1-form, $a \in \Omega_M^1$ is

$$X_a^L = \omega^{-1} s^* a \tag{2.13}$$

This justifies to identify the tangent space of a bisection $\ell \in \Gamma(G)$ with 1-forms on M. More is true.

Lemma 2.23 ([24])**.** *If ℓ is a Lagrangian bisection, the tangent space to $L\Gamma(G)$ is, under the identification as in (2.13), isomorphic to* I4BL2F

$$T_\ell L\Gamma(G) \cong \Omega_M^{1,cl}$$

The adjoint action has a particularly nice description for Lagrangian bisections in symplectic groupoids.

Lemma 2.24. *Let $\ell \in L\Gamma(G)$ be a Lagrangian bisections and $a \in \Omega^{1,cl}_M(\mathbb{R})$ a Lie algebroid section, then*

$$Ad_\ell\ a = \varphi_\ell^* a$$

Proof. We begin by observing that for any two Lagrangian bisections ℓ_1, ℓ_2 and the flow of the left invariant vector field $\omega^{-1}s^*a$ on (G) satisfies

$$\ell_1 * (e^{t\omega^{-1}s^*a} \circ \ell_2) = e^{t\omega^{-1}s^*a} \circ (\ell_1 * \ell_2)$$

If we flow ℓ_1 by this vector field and multiply by ℓ_2 on the right, we can either find the corresponding right invariant vector field to flow ℓ_1 by flow ℓ_2.

$$(e^{t\omega^{-1}s^*a} \circ \ell_1) * \ell_2 = \ell_1 * (e^{t\omega^{-1}t^*a} \circ \ell_2) = \ell_1 * (e^{t\omega^{-1}s^*(\varphi_{\ell_2}^{-1})^*a} \circ \ell_2) = e^{t\omega^{-1}s^*(\varphi_{\ell_2}^{-1})^*a} \circ (\ell_1 * \ell_2)$$

This computation uses the fact that the normal classes of $\omega^{-1}s^*a$ and $\omega^{-1}t^*\varphi_\ell^*a$ agree in $\Gamma^{cl}(N\ell)$.

We compute with this $\ell \circ (e^{t\omega^{-1}s^*a} \circ \varepsilon) \circ \ell^{-1}$ where ℓ^{-1} is the inversion i of G composed with ℓ. One can show that $\varphi_{i(\ell)} = \varphi_\ell^{-1}$. Finally,

$$\ell * e^{t\omega^{-1}s^*a} * \ell^{-1} = (e^{t\omega^{-1}s^*a} \circ \ell) * i(\ell) = \varepsilon^{t\omega^{-1}s^*\varphi_\ell^*a} \circ (\ell * i(\ell)) = \varepsilon^{t\omega^{-1}s^*\varphi_\ell^*a} \circ \varepsilon$$

∎

In the symplectic category we can also enhance the notion of a groupoid module to include a symplectic structure (P, ω_P). We say that the groupoid action of $(G, \omega) \rightrightarrows M$ is *symplectic* if the graph of the action morphism is Lagrangian in $P \times \overline{P} \times \overline{G}$. In the case of the self action of a symplectic groupoid by multiplication, this is, of course, the same as multiplicativity of the symplectic form.

A Morita equivalence as in (**??**) is called *symplectic* if the bimodule P carries a symplectic form ω_P and both, the action of G on the right and H on the left are symplectic groupoid actions with the additional requirement that

(a) π_1 is Poisson and π_2 is Anti-Poisson for the natural Poisson structures Q_M, Q_N.
(b) the vertical bundles $\ker(\pi_{1*})$ and $\ker(\pi_{2*})$ are ω_P-complementary.
(c) π_1 and π_2 re complete.

Ping Xu [**?**] has shown that this integrates well with the notion of Morita equialence of Poisson manifolds

Lemma 2.25. *Two symplectic groupoids are symplectically Morita equivalent if and only if their identity bisections are Morita equivalent as Poisson manifolds as defined in (2.4).*

With this setup we can now properly understand a Generalized Kähler structure as described in [2] and quickly reviewed in chapter 1. All discussions from this section go over to the holomorphic category.

Let $(G,\Omega) \rightrightarrows (X,\sigma)$ be the holomorphic symplectic Weinstein groupoid of (X,σ), a holomorphic symplectic Poisson manifold. Furthermore, let (Z,Ω_Z) be a holomorhpic symplectic principal bundle for (G,Ω). In addition, we introduce a new class of bisections called *Lagrangian-symplectic* (short LS) bisections.

Definition 2.26. An *LS-bisection* of a holomorphic symplectic groupoid (G,Ω), respectively a holomorphic symplectic groupoid principal bundle (Z,Ω_Z), is a smooth bisection which is Lagrangian for the imaginary part of Ω, respectively Ω_Z, and symplectic for the real part. ■ T082XR

For this discussion it is often useful to assume the viewpoint of a holomophic symplectic groupoid (G,Ω) as the smooth symplectic groupoid of the underlying Imaginary part (G,ω) with an integrable complex structure compatible with ω in the sense that ω is of type $(20)+(02)$ for I. Then the discussion of LS-bisections simplifies to the familiar group of $L\Gamma(G,\omega)$ and we impose the open condition that $\ell^*(I^*\omega)$ is nondegenerate after the fact as we can recover $(G,\Omega) = (G, I^*\omega + i\omega)$ as well as $(X,\sigma) = (M, I, IQ + iQ)$.

Theorem 2.27 ([2], Theorem 5.3)**.** *A degenerate GK structure of symplectic type defined by the data (I_+, I_-, Q, F) is equivalent to a holomorphic symplectic Morita equivalence with brane bisection $(Z,\Omega,\mathcal{L})$ between the holomorphic Poisson structures $\sigma_\pm = -\frac{1}{4}(I_\pm Q + iQ)$.*

$$
\begin{tikzcd}
& (Z,\Omega_Z) \arrow[dl] \arrow[dr] & \\
(X_+,\sigma_+) \arrow[ur, bend left, "\ell"] & & (X_-,\sigma_-)
\end{tikzcd}
\tag{2.14}
$$

Note that the existence of a smooth bisection of this diagram means there is one unique smooth manifold M underlying both X_+ and X_-. Since this bisection is moreover Lagrangian for ω, this diffeomorphism identifies the real Poisson structures $\mathrm{Im}(\sigma_+)$ and $\mathrm{Im}(\sigma_-)$ as a unique smooth Poisson structure Q.

Applying instead the real functor to the holomorphic symplectic Morita equivalence in (2.14), we instead obtain smooth Poisson structures $P_\pm$ on M related by a B-field transformation.

Lemma 2.28. *For a degenerate GK structure (I_+, I_-, Q, F) the real Poisson structures $P_+ = I_+Q$ and $P_- = I_-Q$ are related by a B-field transform of $-F$.* K7KDJF

Proof. Two Poisson structures R and P are related by a 2-form B if $(1+BP)$ is invertible and $R = P^B = P(1+BP)^{-1}$. This is to say that $R - P = -RBP$. In this case we compute with (1.1)

$$P_+ - P_- = (I_+ - I_-)Q = QFQ = P_+ I_+^* F I_- P_- = P_+(-FI_-)I_-P_- = P_+FP_-$$

which concludes the proof. ∎

2.4 FLOW CONSTRUCTION REVISITED

As is beautifully laied out in chapter 7 of [2], the flow construction on a holomorhpic Poisson manifold (M, I, Q) as described in section (2.2) has a particularly nice description in terms of its integrating holomorphic symplectic groupoid (G, Ω).

Let $f \in C^\infty_M(\mathbb{R})$ and view df as a section of the Lie algebroid $T^*_Q M$. Let $\widetilde{X}_{df}$ be the unique left invariant vector field on (G, ω) that df defines. It is s-related to $X_{df} = -Q(df)$ on M. Denote the flow of $\widetilde{X}_{df}$ by $\widetilde{\varphi}_t$ which covers the flow φ_t of X_{df} on M. The exponential map takes df to the 1-parameter family of Lagrangian bisections

$$g_t = \widetilde{\varphi}_t \circ \varepsilon$$

Now we bring in the real part $B = I^*\omega$ on (G, Ω) and find that

Lemma 2.29 ([]). *The family of a the pair of a closed 2-form F_t and complex structure I_t*

$$F_t = g_t^* B = \int_0^t (\varphi_s^* dd^c f) ds$$

$$I_t = I^{\varphi_t}$$

form a family of degenerate GK structures of symplectic type (I, I_t, Q, F_t) on M.

This recovers precisely the flow construction of (I, Q) on M by f described above in section 2.2. The purpose of the flow construction when it was introduced in [?] was to deform existing GK structures (I_+, I_-, Q, F). We have previously understood this as a composition of arrows

$$\begin{tikzcd}
I_+ & & I_- \arrow[ll, bend right, "F"'] \\
& I_t \arrow[ul, bend left, "F+F_t"] \arrow[ur, bend right, "F_t"'] &
\end{tikzcd}$$

However, a degenerate GK structure may also be encoded as a holomoprhic Morita equivalence (Z, Ω_Z) between holomorphic Poisson manioflds $(X_\pm, \sigma_\pm)$ with a brane bisection ℓ.

$$\begin{tikzcd}
& (Z, \Omega_Z) \arrow[dl] \arrow[dr] & \\
(X_+, \sigma_+) \arrow[ur, bend left, "\ell"] & & (X_-, \sigma_-)
\end{tikzcd}$$

We apply the flow construction to the imaginary part of the Weinstein groupoid of (X_-, σ_-) which we denote by $(G, \omega) \rightrightarrows (M, Q)$ for $f \in C^\infty_{X_-}(\mathbb{R})$ as above and obtain a family of Lagrangian bisection g_t of (G, ω) which is a (possibly degenerate) LS bisection in (G, Ω). The corresponding family of 2-forms on X_- is $F_t = g_t^* \Omega$. Composing this self Morita equivalence with (Z, Ω_Z, ℓ) moves the bisection ℓ in the fixed background.

$$\begin{tikzcd}
& (Z, \Omega_Z) \arrow[dl] \arrow[dr] & \\
(X_+, \sigma_+) \arrow[ur, bend left, "\ell.g_t"] & & (X_-, \sigma_-)
\end{tikzcd}$$

This has precisely the effect of $(\ell.g_t)^* \Omega = \ell^* \Omega + \varphi_\ell^* g_t^* \Omega_-$.

Alternatively, we can magnetically deform the background and keep the bisection ℓ fixed as in ([2], Proposition 7.3).

$$\begin{tikzcd}
& (Z, \Omega_Z + s^* F_t) \arrow[dl, "\pi_+"'] \arrow[dr, "\varphi_t \circ \pi_-"] & & (G, \Omega) \arrow[dl, shift left] \arrow[dl, shift right] \\
(X_+, \sigma_+) \arrow[ur, bend left, "\ell"] & & (X_-, \sigma_-) \arrow[ur, bend left, "g_t"] &
\end{tikzcd}$$

This diagram encodes a family of GK metrics of symplectic type in the generalized Kähler class of $[Z, \Omega_Z]$. The conclusion of the present discussion is that the deformation of a GK structure within its GK class has a particularly nice description as the 1-parameter subgroup exponentiating a Lie algebroid section. This perspective moreover conveniently handles the highly nonlinear nature of the flow construction compared to adding $dd^c f$ to a Kähler form to move about in its Kähler class. It is a consequence of the nonabelian multiplication of groupoid bisections of the symplectic groupoid integrating the nontrival Lie bracket $[\ ]_Q$ on $T^*_Q M$. A very helpful corollary of the above lemma is the infinitesimal change of the holomorphic symplectic form.

Corollary 2.30. *Let $df \in \Omega^{1,cl}_{X_-}$ represent the left invariant vector field $\omega^{-1} \pi_-^* df$ on Z. Then* 9SL4E4

$$L_{\omega^{-1}\pi_-^* df} B = -\pi_-^* dd^c f$$

Moreover, the pullback of B along ℓ_t^T to X_- for a time dependent family of LS bisections such that $\dot{\ell}_t^T$ satisfies

$$\dot{F}_{\ell_t^T} = -dd^c f$$

Proof. (Z, Ω_Z) is holomorphic symplectic for a complex structure I_Z, hence $\Omega = I_Z^* \omega + i\omega$ and we know that π_- is a holomorphic map $(Z, I_Z) \to (X_-, I_-)$.

$$L_{\omega^{-1}\pi_-^* df} B = d(I_Z^* \omega)(\omega^{-1} \pi_-^* df) = dI_Z^* \pi_-^* df = \pi_-^* dI_-^* df = -\pi_-^* dd^c f$$

The second assertion follows because

$$\frac{d}{dt}\bigg|_0 F_t = \frac{d}{dt}\bigg|_0 (\ell_t^T)^* B = (\ell_0^T)^* \frac{d}{dt}\bigg|_0 (e^{\omega^{-1}\pi_-^* df})^* B = (\ell_0^T)^* L_{\omega^{-1}\pi_-^* df} B = -dd^c f$$

■

3 HOLOMORPHIC FAMILIES OF BRANES

From the discussion in the preceeding chapters it is clear that we are quite invested in understanding very well how Lagrangian bisections in a symplectic Morita equivalence can be deformed. This means we are interested in a 1-parameter family of Lagrangian bisections and even more generally in a family of Lagrangian bisections parametrized by any complex parameter space. In this chapter we abstract this notion to single out particular families of Lagrangian branes which we will denote *holomorphic families of branes*. These families are distinguished in that they include a 2-form F that is finely tuned to the embedding of the family into the ambient symplectic space.

It is common practice to use transgression arguments to induce geometric structures on mapping spaces. We view biscections $\ell \in L\Gamma_{\pi_+}(Z)$ of the holomorphic Morita equivalence between $X_\pm$ as maps $X_+ \xrightarrow{\ell} Z$. In this context the story would unfold similar to Hitchin's description of this construction concerning special Lagrangian submanifolds [15]. There is an evaluation map and a projection p to the first factor, the mapping space.

$$\begin{array}{ccc} L\Gamma_{\pi_+}(Z) \times X_+ & \xrightarrow{\text{ev}} & (Z, B + i\omega) \\ {\scriptstyle p}\big\downarrow & & \\ L\Gamma_{\pi_+}(Z) & & \end{array}$$

Under the assumption that X_+ is compact without boundary we can integrate the evaluation-pullback of B^{n+1} along the fibers of p to obtain a 2-form on the mapping space of LS bisections $L\Gamma_{\pi_+}(Z)$.

$$\widetilde{\Omega} = p_* \left(\text{ev}^* B^{[n+1]}\right)$$

On the space of LS bisections this produces an interesting 2-form. It is not even necessary to require the bisection to be $\text{Im}(\Omega)$-Lagrangian. On the bigger space of $\text{Re}(\Omega)$-symplectic bisections this formalism induces the 2-form

$$\widetilde{\Omega}(a, b) = \int_{X_-} P(a, b) F_{\ell^T}^{[n]} - I_-^* a \wedge I_-^* b \wedge F_{\ell^T}^{[n-1]}$$

for tangent vectors $a, b \in \Omega^1(\mathbb{R})$ to an $\text{Re}(\Omega)$-symplectic bisection ℓ. This form reproduces the symplectic form on the space of unitary connections on a hermitian Line bundle with nondegenerate cuvature of type $(1, 1)$ as considered in Joel Fine's work. We will go into more detail later in section (5.5) where we adapt the original notion of Ω_A for the symplectic form used in [8].

This construction, as is, has a fundamental shortcoming for the discussion of Lagrangian brane bisections as it does not capture the degree of freedom of the connection on a Lagrangian bisection. How could it, the evaluation map on brane bisections (ℓ, ∇) disregards the connection entirely when evaluating this pair on a point $x \in X_+$.

$$\begin{tikzcd} B\Gamma_{\pi_+}(Z) \times X_+ \arrow[r, "\mathrm{ev}"] \arrow[d, "p"'] & (Z, B + i\omega) \\ B\Gamma_{\pi_+}(Z) & \end{tikzcd}$$

To solve this problem we equip the domain of the evaluation map with a tautological line bundle constructed from the line bundle intrinsic to every point $(\ell, \nabla) \in B\Gamma(Z)$. But the tautology only goes as far as providing a partial connection along the vertical fibers of p. There are many ways in which we could complete this to a full connection $\widehat{\nabla}$, but the correct connection to consider for the transgression has to be finely attuned to the embedding of each fiber into the target Z. From there the transgression of B is modified by subtracting the curvature of $\widehat{\nabla}$ and integrating the $(n+1)$ power over the fibers.

$$\widetilde{\Omega} = p_* \left(\mathrm{ev}^* B - F^{\widehat{\nabla}} \right)^{[n+1]}$$

This chapter is devoted to precisely this finely tuned compatibility condition to find the right connection $\widehat{\nabla}$. We start by defining a slight abstraction of the mapping space and evaluation map construction in the case of branes which we call *smooth family of branes*. The families of branes that satisfy the correct distinguishing condition will be called *holomorphic families of branes*.

3.1 DEFINITIONS

Definition 3.1 (Smooth family of branes)**.** Let $(M, \mathbb{J})$ be a generalized complex manifold and X a smooth manifold. A smooth family of branes of type L over X in $(M, \mathbb{J})$ consists of a commuting diagram JU9R1L

$$\begin{tikzcd} \widehat{L} \arrow[rr, "\hat{\iota}"] \arrow[dr, "\pi"'] & & X \times M \arrow[dl, "p_1"] \\ & X & \end{tikzcd}$$

such that

(a) π is a surjective submersion and locally around any $x \in X$ there exists a neighborhood $W \subseteq X$ with a trivialization $\pi^{-1}(W) \cong W \times L$.

(b) The fiber $\pi^{-1}(x_0) \xrightarrow{\hat{\iota}} \{x_0\} \times M$ over every $x_0 \in X$ is a brane. In particular, every fiber of π carries a unitary line bundle (U_{x_0}, ∇_{x_0}) which assemble into hermitian line bundle with a partial fiberwise connection only $(U, \nabla) \to \widehat{L}$.

■

Remark 3.2. Notice that the definition of $\hat{\imath}$ forces us inevitably to choose a lift of the class of normal vector to L, which defines a tangent vector to it in the space of Lagrangians, to an actual vector field along L. ■

Definition 3.3. (Holomorphicity condition) A smooth family of branes $\widehat{L} \to X$ of type L over X in M is a *holomorphic family of branes* if the parameter space (X, I) is a complex manifold and $\widehat{L}$ carries a *full* unitary line bundle $(\widehat{U}, \widehat{\nabla})$ such that

$$\widehat{L} \xrightarrow{\hat{\imath}} X \times M$$

is a generalized complex brane for the product generalized complex structure $\mathbb{J}_I \times \mathbb{J}_\omega$. ■

Remark 3.4. We may extend the deinition to include *almost* holomorphic families of branes which are smooth families of branes over almost complex manifolds (X, I) that satisfy preciesly the same conditions in (3.3). ■

Definition 3.5 (Equivalence of holomorphic branes)**.** An equivalence from a holomorphic family of branes $(\widehat{L}_X, U_X, \nabla_X) \to (X, I)$ to a holomorphic family of branes $(\widehat{L}_Y, U_Y, \nabla_Y) \to (Y, J)$ consists of the following.

(a) A generalized holomorphic isomorphism

$$\begin{tikzcd}
(U_f, \nabla_f) \arrow[d] & \\
(X, I) \arrow[r, "f"] & (Y, J)
\end{tikzcd}$$

(b) An isomorphism of fiber bundles

$$\widehat{L}_X \xrightarrow{\hat{f}} \widehat{L}_Y$$

such that the following diagram commutes

$$\begin{tikzcd}
\widehat{L}_X \arrow[rr, "\hat{\imath}_X"] \arrow[dr, "\hat{f}"] \arrow[dd] & & X \times M \arrow[dr, "f \times \mathrm{id}_M"] \arrow[ddll] & \\
& \widehat{L}_Y \arrow[rr, "\hat{\imath}_Y"] \arrow[dd] & & Y \times M \arrow[ddll] \\
(X, I) \arrow[dr, "f"] & & & \\
& (Y, J) & &
\end{tikzcd}$$

and $(\widehat{L}_X, U_X, \nabla_X)$ agrees with the pullback of $(\widehat{L}_Y, U_Y, \nabla_Y)$ along the generalized holomorphic map $(f \times \mathrm{id}_M, p_1^*(U_f, \nabla_f))$. ■

Remark 3.6. There are other notions of equivalence which would be weaker. For instance one that allows a Hamiltonian generalized complex isomorphism of $(M, \mathbb{J})$. But for the purpose of our application to generalized Kähler metrics, it is important that two holomorphic famimlies of Lagrangian branes are equivalent precisely when they give rise to the same family of generalized Kähler metrics. ■

3.2 HOLOMORPHIC FAMILIES OF LAGRANGIAN BRANES

We now investigate the condition of holomirphicity of a family of branes in the specific case of a complex family of Lagrangian branes in a symplectic manifold. 9Q0JRO

Remark 3.7. Upon choosing a local trivialization $\pi^{-1}(W) \cong W \times L$ of $\widehat{L} \to X$ we can decompose a 2-form ρ on the total space of $\widehat{L}$ as C7XW32

$$\rho = \rho^{HH} + \rho^{HV} + \rho^{VV} \in \Lambda^2 T^*W \oplus (T^*W \otimes T^*L) \oplus \Lambda^2 T^*L$$

■

Theorem 3.8. *A smooth family of Lagrangian branes $(\widehat{L}, U, \nabla) \xrightarrow{\pi} X$ over (X, I) in a sympletic manifold (M, ω) is holomorphic if and only if ∇ can be extended to a full connetion $\widehat{\nabla}$ on U and in a local trivialization $W \times L$ of $\widehat{L}$ over $W \subseteq X$ the following conditions hold for $F = F^{\widehat{\nabla}}$.* NF54T1

(a) $F^{VV} = 0$
(b) $(I^* \otimes 1)F^{HV} = (\hat{\iota}^*\omega)^{HV}$
(c) $(I^* \otimes 1)(F^{HH})^{(20+02)} = ((\hat{\iota}^*\omega)^{HH})^{(20+02)}$

Proof. Denote the product GC structure on $\widehat{M}$ by $\mathbb{J} = \mathbb{J}_I \times \mathbb{J}_\omega$. We can decompose the tangent bundle of both $\widehat{L}$ and $\widehat{M}$ into horizontal and vertical directions. The pullback to $\widehat{L}$ has the components

$$\hat{\iota}^* = \begin{pmatrix} \mathrm{id}_{T^*X} & \psi \\ 0 & \iota^* \end{pmatrix}$$

where $\psi \in H^* \otimes V$ is the slope of embedding a horizontal tangent vector to $\widehat{L}$. Likewise we can decompose

$$\hat{\iota}_* = \begin{pmatrix} \mathrm{id}_{TX} & 0 \\ \psi & \iota_* \end{pmatrix}$$

The generalized tangent bundle $\tau_{\widehat{L},F}$ of $\widehat{L}$ is an extension of $T\widehat{L}$ by $N^*\widehat{L}$. Recall that $N^*\widehat{L}$ is the subbundle of $T^*\widehat{M}$ that is given by the kernel of $\hat{\iota}^*$. Here this consists of a pair of covectors

$$\begin{pmatrix} -\psi\xi_M \\ \xi_M \end{pmatrix} \in T^*\widehat{M} \tag{3.1}$$

for $\xi_M \in N^*_{rel}\widehat{L}$.

To find the conditions on F for which $\tau_{\widehat{L},F}$ is $\mathbb{J}$ invariant, we consider the following cases. Note that $(\hat{\iota}^*\omega)^{VV} = 0$ by the assumption that $\widehat{L}$ is relative Lagrangian. This implies in particular that $F^{VV} = 0$.

1. To begin, we take an element from $\tau_{\widehat{L},F}$ that is purely in $N^*\widehat{L}$ as in (3.1). To guarantee that

$$\mathbb{J}\begin{pmatrix} -\psi\xi_M \\ \xi_M \end{pmatrix} = \begin{pmatrix} I^*\psi\xi_M \\ -\omega^{-1}\xi_M \end{pmatrix} \in \tau_{\widehat{L},F}$$

the vector component $\omega^{-1}\xi_M$ has to lie in $T\widehat{L}$. Because the vector component is entirely vertical, and ξ_M was chosen arbitrarily, the first condition that arises is

$$\omega^{-1}\left(N^*_{rel}\widehat{L}\right) \subseteq T_{rel}\widehat{L} \tag{3.2}$$

This is, however, trivially satisfied by the Lagrangian family, because every fiber is a Lagrangian in its own right in M. Further, it remains to verify

$$-F\omega^{-1}\xi_M = \hat{\iota}^*\left(I^*\psi\xi_M\right)$$

Under our Lagrangian assumption (3.2) is an equality and we can find a unique vertical $V \in T\widehat{L}$ such that $\xi_M = \omega(\iota_* V)$. This simplifies the computation to

$$F\iota_* V = \hat{\iota}^*\left(I^*\psi\omega\iota_* V\right)$$

Notice that $\psi\omega\iota_*$ is equal to the HV component of $\hat{\iota}^*\omega$. If we take into account that $\widehat{L}$ is relative Lagrangian, which means that $(\hat{\iota}^*\omega)^{VV} = 0$, we arrive at condition (a).

2. To proceed, we consier elements of the form

$$W + \xi_X + \xi_M$$

for a horizontal vector $W \in TX$. They lie in $\tau_{\widehat{L},F}$ under the condition that

$$\begin{pmatrix} F^{HH}W \\ F^{HV}W \end{pmatrix} = \begin{pmatrix} \xi_X + \psi\xi_M \\ \iota^*\xi_M \end{pmatrix}$$

The vertical component of this equality prescribes

$$\iota^*\xi_M = F^{HV}W = -\iota^*\omega(\psi I W) \tag{3.3}$$

This determines ξ_M up a relative conormal component. The horizontal component in turn forces

$$\xi_X = F^{HH}W - \psi\xi_M \tag{3.4}$$

We now compute

$$\mathbb{J}\begin{pmatrix} W+\xi_X \\ \psi W+\xi_M \end{pmatrix} = \begin{pmatrix} IW - I^*\xi_X \\ \omega\psi W - \omega^{-1}\xi_M \end{pmatrix} = \begin{pmatrix} IW - I^*\xi_X \\ \omega\psi W + \psi IW \end{pmatrix} = \hat{\iota}^* \begin{pmatrix} IW \\ 0 \end{pmatrix} + \begin{pmatrix} -I^*\xi_X \\ \omega\psi W \end{pmatrix}$$

and impose the condition

$$\begin{pmatrix} F^{HH}IW \\ F^{HV}IW \end{pmatrix} = \hat{\iota}^* \begin{pmatrix} -I^*\xi_X \\ \omega\psi W \end{pmatrix} = \begin{pmatrix} -I^*\xi_X + \psi\omega\psi W \\ \iota^*\omega\psi W \end{pmatrix}$$

The vertical component of this equality is vacuous as it repeats condition (a). Substituting 3.3 and 3.4 into the horizontal component we obtain

$$F^{HH}IW = -I^*\left(F^{HH}W + \psi\omega(\psi IW)\right) + \psi\omega\psi W$$

Separating the terms involving ω and F^{HH} as well as noting that $\psi\omega\psi = (\hat{\iota}^*\omega)^{HH}$ and recalling the projection to the $(20+02)$ component from equation (2.2) finally results in condition (b).

■

Remark 3.9. It is worth remarking at this point that the fiberwise brane condition, in this case $F^{VV} = 0$, is not enough to guaruantee hat the total space of $\widehat{L}$ embeds into $X \times M$ as a generalized complex brane. This is a genuinely stronger condition on how the branes in the family vary over X. ■

3.3 EXAMPLES

To illustrate the above theorem we digress for a moment and demonstrate it in a few examples. C12L6W

Example 3.10. Consider the cylinder $\mathbb{R} \times S^1$ with coordinates (t, θ) and symplectic form $\omega =$ CTWJD9
$dt \wedge d\theta$. Let $\widetilde{X}$ be the family fo Lagrangian circles constant in the first factor.

$$S^1 \hookrightarrow \mathbb{R} \times S^1$$

We equip every such Lagrangian circle with a flat connection of the form $d + iAd\theta$ on the trivial bundle where $A \in \mathbb{R}$. We identify elements that differ by a gauge transformation of the form $g(z) = z^k$ on S^1 which leaves us with an $X = \mathbb{R} \times S^1$ worth of Lagrangian branes indexed by coordinates (t, A).

In these coordinates the complex structure sends $\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial A}$ and $\frac{\partial}{\partial A} \mapsto -\frac{\partial}{\partial t}$. (X, I) is biholomorphic to $\mathbb{C}^*$. We choose coordinates (τ, φ, θ) for the tautological brane $\widehat{S^1}$ over $X \cong \mathbb{C}^*$ and denote the coordinates on $X \times M$ by $(\tau, \varphi, t, \theta)$. The tautologoical bundle on $\widehat{S^1}$ has curvature form $\widehat{A} = \frac{\varphi}{2\pi} d\theta$

One now verifies that

$$\hat{\imath}^*(dt \wedge d\theta) = d\tau \wedge d\theta$$

and a direct computation of the curvature of the tautological connection shows that

$$(d\widehat{A})^{HV} \circ I = (\frac{1}{2\pi} d\varphi \wedge d\theta) \circ I = \frac{1}{2\pi} d\tau \wedge d\theta$$

which means consition (a) of theorem (3.8) is already met. The remaining condition (b) is vacuous as our parameter space X in this example is of complex dimension 1 and therefore has no $(20+02)$ forms. It is worth noting that we immediately chose a distinguished vector field along a circle for a given class of a normal vector field, namely the one tangent to the $\mathbb{R}$-direction of the cyclinder. ■

Example 3.11. We now turn to an example of a holomorphic family of branes in a holomorphic symplectic background which is a rich source of interesting examples. Consider $\mathbb{C}^2$ with its canonical holomorphic symplectic form $\Omega = du \wedge dv$ in coordinates (u, v). Let (M, ω) be the real symplectic manifold which is $\mathbb{C}^2$ equipped with the imaginary part $\omega = \mathrm{Im}(du \wedge dv)$ only. It is clear that every holomorphic Lagrangian in $(\mathbb{C}^2, \Omega)$ is in particular a real Lagrangian in (M, ω). Y4PQ7U

Specifically, we want to consider here the family of complex lines through the origin parametrized by $\mathbb{P}^1$ with trivial connection on the trivial bundle. This family is described by the total space of the tautological bundle.

$$\begin{tikzcd}
\mathrm{tot}(\mathcal{O}(-1)) \arrow[rr, "\hat{\imath}"] \arrow[dr] & & \mathbb{P}^1 \times \mathbb{C}^2 \arrow[dl] \\
& \mathbb{P}^1 &
\end{tikzcd}$$

In the standard chart $[z:1]$ of $\mathbb{P}^1$ and with fibre coordinate w of $\mathcal{O}(-1)$, the embedding is $([z:1], w) \mapsto ([z:1], (wz, w))$. This family does, however, not satisfy the holomorphicity condition (3.3) as condition (a) of theorem (3.8) is not met when $\hat{\imath}^*\omega = \mathrm{Im}(wdz \wedge dw)$ is nonzero while

$$\begin{tikzcd}
(\underline{\mathbb{C}}, \widehat{\nabla} = d) \arrow[d, "\pi"'] \\
\mathrm{tot}(\mathcal{O}(-1))
\end{tikzcd} \tag{3.5}$$

is entirely flat.

To produce a holomorphic family, we turn on the real part $B = \mathrm{Re}(\Omega)$ on (M, ω) which is a B-field symmetry of the generalized complex background. $\mathrm{Re}(du \wedge dv)$ can be prequantized by $d + iA$ on the trivial bundle over $\mathbb{C}^2$ where $A = \mathrm{Re}(udv)$. The B-field transformation tensors every brane in (M, ω) with this bundle which places on the Lagrangian $\ell \in \mathbb{P}^1$ the connection

$$(\underline{\mathbb{C}}, d + i\iota_\ell^* A) \to \ell$$

The complex family of branes (3.5) now carries the connection $\widehat{\nabla} = d + i\widehat{A}$ where $\widehat{A} = \hat{\iota}^* A$. It is needless to choose an explicit representative of a normal vector field to lift a tangent vector, as the restrictive nature of this complex family uniquely determines a lift of a vector field $W = W^z \frac{\partial}{\partial z}$

This connection changes in horizontal $\frac{\partial}{\partial z}$ direction by

$$L_{\frac{\partial}{\partial z}} \widehat{A} = \iota_{\frac{\partial}{\partial z}} d\mathrm{Re}(wzdw) = \mathrm{Re}(\iota_{\frac{\partial}{\partial z}} wdz \wedge dw) = \iota_{\frac{\partial}{\partial z}} \hat{\iota}^* B$$

As a result, (b) follows because B and ω are the real and imaginary parts of a holomorphic symplectic form $F^{HV} = d\hat{\iota}^* A = -(I^* \otimes 1)\hat{\iota}^*\omega$. As before the next condition (c) is vacuous in this dimension.

Because the space of Lagrangians $\mathbb{P}^1$ is already complex before adding the flat bundle degree of freedom, the complex structure on the parameter space of this family of branes is not induced from $\mathbb{J}_\omega$ as in the example (3.10) or as explained in more detail below in definition (4.5). Moreover, enforcing the flat bundle on every Lagrangian in this family by its location immediately determines the unique brane on any Lagrangian. Instead, one can observe that from a vector $\frac{\partial}{\partial z}$ on $\mathbb{P}^1$ which determines a vector field along the embedded $\ell \hookrightarrow \mathbb{C}^2$ when pushed forward, it is precisely the application of $i\mathbb{J}_\omega$ that computes the corresponding closed 1-form $a \in T_\nabla \mathcal{A}(\underline{\mathbb{C}} \to \ell)$ that describes the infinitesimal motion of the connection so that the pair (ℓ, ∇) stays within this family of branes.

Notice that in this particular example we could have also added the connection $\widehat{\nabla} = d - i\mathrm{Re}(\frac{1}{2}w^2 dz)$ which restricts to the trivial flat connection on every fibre and satisfies (a). ∎

Example 3.12 (constant family of branes)**.** Let $(M, B + i\omega)$ be a holomorphic symplectic manifold for the complex stucture $J = \omega^{-1}B$. We may regard $(M, \mathbb{J}_\omega)$ as a generalized complex manifold. The generalized tangent bundle of (M, B) is $\tau_{(M,B)} = \Gamma_B$, the graph of $T \xrightarrow{B} T^*$. This is preserved by $\mathbb{J}_\omega$ precisely because $\omega^{-1}B = J$ squares to $-\mathrm{id}_T$, see example (2.7). We can now view this spacefilling brane to be a constant family of spacefilling branes over any complex manifold (S, I). ZQYR6S

$$\begin{tikzcd} S \times M \arrow[rr, "\hat{\iota}"] \arrow[dr] & & S \times M \arrow[dl] \\ & S & \end{tikzcd}$$

as the generalized cotangent bundle $\hat{\tau}_{(S\times M, \pi^* B)} = \Gamma_B \times TS$ is clearly preserved by the product generalized complex structure $\mathbb{J}_I \times \mathbb{J}_\omega$ of $S \times M$. ∎

4 THE UNIVERSAL BRANE BISECTION

The objective of this chapter is to describe the geometry of brane bisections in the holomorphic symplectic Morita equivalence (Z, Ω). A brane bisection is a prequantized LS bisection, that is, a LS bisection that supports a flat unitary connection on a hermitian line bundle (which we assume here to be trivial). They are to generalized Kähler metrics what a prequantum line bundle with unitary connection is to a classical Kähler metric. The space of all brane bisections $B\Gamma(Z)$ carries a tautological family of branes $\widehat{\ell}$. On top of this tautological family exist two canonical unitary line bundles and we find that the average connection satisfies the holomorphicity condition for this tautological family of Lagrangian branes. C5NDLG

4.1 THE SETUP

As before, this discussion evolves around the holomorphic symplectic Morita equivalence (Z, Ω) between holomorphic Poisson manifolds representing a fixed generalized Kähler class (definition 1.10).

$$\begin{tikzcd} & (Z, \Omega) \arrow[dl, "\pi_+"'] \arrow[dr, "\pi_-"] & \\ (X_+, \sigma_+) & & (X_-, \sigma_-) \end{tikzcd}$$

Definition 4.1 (Lagrangian brane bisection)**.** A *Lagrangian brane bisection* of a holomorphic symplectic Morita equivalence is an LS bisection, as defined in (2.26), equipped with a trivial flat line bundle $(\underline{\mathbb{C}}, \nabla)$. Denote the space of brane bisections of Z by $B\Gamma(Z)$. ■ F0UQDP

Remark 4.2. There is a forgetful map from Lagrangian brane bisections to LS bisections that forgets about the connection data and the trivial bundle. The fiber of each LS bisection ℓ consists of the flat unitary connections on the trivial bundle supported on it. We denot this space by $\mathcal{A}(\underline{\mathbb{C}}_\ell)$ E1Z36R

$$0 \to \mathcal{A}(\underline{\mathbb{C}}_\ell) \to B\Gamma(Z) \to L\Gamma(Z) \to 0$$

Notice that this sequence splits by equipping every bisection with the trivial connection on the trivial bundle. Moreover, we can view $B\Gamma(Z)$ as a principal bundle over $L\Gamma(Z)$ for the action of $\mathcal{A}(\underline{\mathbb{C}}_{X_-})$ which is nothing more than closed 1-forms $\Omega^{1,cl}_{X_-}$ on X_-. ■

The tangent space of each (ℓ, ∇) to $B\Gamma(Z)$ is an extension of the tangent space of ℓ by closed 1-forms on ℓ

$$\Omega^{1,cl}_{\ell}(\mathbb{R}) \to T_{\ell,\nabla}B\Gamma(Z) \to T_{\ell}L\Gamma(Z) \cong \Gamma^{cl}(N\ell)$$

where $\Gamma^{cl}(N\ell) = \{[X] \in \Gamma(N\ell) \quad | \quad d\ell^*(\iota_X\omega) = 0\}$. This sequence does not naturally split as a normal direction to ℓ is sufficient to describe its infinitesimal motion as a submanifold, but not to compare the unitary connections on it with those on nearby ones. To choose a splitting of this sequence is to choose a splitting of $T\ell \to TZ \to N\ell$ for every ℓ. A consistent such choice is given for example by a fibration K on Z transverse to $T\ell \hookrightarrow TZ$ to lift any class of normal vector fields along ℓ to a unique vector field along ℓ tangent to K.

Lemma 4.3. *The tangent space to brane bisections $B\Gamma(Z)$ at (ℓ, ∇) can be identified with* INEJE3

$$T_{(l,\nabla)}B\Gamma(Z) = \Gamma^{cl}(N\ell \oplus T^*\ell)$$

combining the tangent space of bisections described in (2.14) and Lagrangian branes (2.23).

Proof. To make this identification we may choose the π_+-fibers in Z. This harmonizes particularly well with the point of view of ℓ as sections of π_+. ■

Remark 4.4. Notice that the tangent space at (ℓ, ∇) fits into the exact sequence 688L6R

$$T\ell \oplus \mathrm{Ann}(T\ell) \longrightarrow TZ \oplus T^*Z \longrightarrow N\ell \oplus T^*\ell$$

As usual, $T\ell \oplus \mathrm{Ann}(T\ell)$ is the generalized tangent bundle of the Lagrangian brane ℓ. It is therefore natural to define $\eta_\ell = N\ell \oplus T^*\ell$, the cokernel of the embedding of the generalized tangent bundle, as the generalized cotangent bundle of the Lagrangian brane. A tangent vector has the form $\xi = ([X], \beta)$ for $[X] \in \Gamma^{cl}(N\ell)$ and $\beta \in \Omega^{1,cl}_{\ell}$. ■

Definition 4.5. $B\Gamma(Z)$ carries the almost complex structure $\mathbb{I}$ which, at every point $\ell \hookrightarrow Z$, interchanges the Lagrangian and connection degree of freedom through the isomorphism $T^*\ell \to N\ell$. X2PFB8

$$\mathbb{I}([X], \beta) = \left(-\omega^{-1}\beta\big|_{\ell}\,,\ \omega(\widetilde{X})\right)$$

where $\widetilde{X} \in \Gamma^{cl}(\ell, TZ)$ is any vector field along ℓ lifting the class $[X] \in \Gamma^{cl}(N\ell)$. ■

Remark 4.6. Remark (4.4) informs us that $T_{(\ell,\nabla)}$ is the cokernel of the embedding of the generalized tangent bundle of the brane into the generalized tangent bundle of the ambient generalized complex manifold $(Z, \mathbb{J}_\omega)$. As a brane, this subbundle is $\mathbb{J}_\omega$ invariant and therefore induces an endomorphism on the quotient that squares to $-\mathbb{1}$ that is the almost complex structure $\mathbb{I}$. ■ X0IOVQ

Definition 4.7 (Tautological bundle)**.** The tautological bundle $\widehat{\ell}$ places over each brane bisection (ℓ, ∇) the bisection ℓ itself. As such, an element of $\widehat{\ell}$ comprises a brane bisection with a chosen point on it. ■

Lemma 4.8. $\widehat{\ell} \xrightarrow{\widehat{\pi}} B\Gamma(Z)$ *is a smooth family of Lagrangian branes in Z as defined in (3.1).* 4U24VW

Proof. The tautological nature of $\widehat{\ell}$ provides us with an evaluation map $\widehat{\ell} \xrightarrow{\text{ev}} Z$ that can be unfurled to evaluate each point $(l, \nabla, p) \in \widehat{\ell}$ in its designated copy of Z as a fiber in the product $B\Gamma(Z) \times Z$. It is clear from the construction that $\widehat{\ell}$ maps onto $B\Gamma(Z)$ as a submersion which leaves to verify the existence of local trivializations. Every bisection is inherently diffeomorphic to both $X_\pm$ and we can trivialize $\widehat{\ell} \cong X_\pm \times B\Gamma(Z)$. ∎

The remainder for this chapter is devoted to the task of equipping $\widehat{\ell}$ with a unitary connection to promote it to a holomorphic family of branes.

Definition 4.9. The total space $\widehat{\ell}$ carries a tautological partial connection $\widetilde{\nabla}$ on the trivial bundle $\underline{\mathbb{C}}$ which restricts to the connection ∇ over the fibre of $\widehat{\ell}$ over (ℓ, ∇). ∎ K2M3LI

In order to turn this fiberwise connection into a full connection on the total space of $\widehat{\ell}$ we need a subbundle of $T\widehat{\ell}$ transverse to the vertical subbundle $\ker d\widehat{\pi}$. With that we would be able to extent $\widetilde{\nabla}$ trivially in the direction of the transverse subbundle. The fibers of $\pi_\pm$ of the bibundle induce naturally two canonical subbundles on $\widehat{\ell}$.

Definition 4.10. Let $p_\pm$ be the composition of $\pi_\pm$ with the evaluatiuon map 4QXSE9

$$\begin{tikzcd} & & X_- \\ \widehat{\ell} \arrow[r, "\text{ev}"] & Z \arrow[ur, "\pi_-"] \arrow[dr, "\pi_+"'] & \\ & & X_+ \end{tikzcd} \tag{4.1}$$

and let $\widehat{K}_\pm$ be the fibers of $p_\pm$. ∎

Lemma 4.11. *The horizontal distributions $H_\pm = \ker(dp_\pm)$ define flat Ehresmann connections of the fiber bundle*

$$\begin{tikzcd} \widehat{\ell} \arrow[d, "\widehat{\pi}"] \\ B\Gamma \end{tikzcd}$$

Proof. The lift of a vector $\xi = ([X], \beta)$ to a horizontal vector $\widehat{\xi}^\pm$ in $H_\pm$ is uniquely identified by the property that it has to be killed by $(p_\pm)_*$. This means $\text{ev}_*\widehat{\xi}^\pm$ is a vector in TZ at $p \in \ell$ which is killed by $(\pi_\pm)_*$ and projects down to ξ which is a normal vector field $[X]$ along ℓ. For this reason $\widehat{\xi}^+$ has to be the unique vector field $X^\pm$ along $\ell \hookrightarrow Z$ in the class $[X]$ tangent to the fibers of $\pi_\pm$ in Z. ∎

4.2 TECHNICAL CHOICES AND NOTATION

Remark 4.12. Notice that the choices of a horizontal distributions in lemma (4.11) have three immediate consequences. SBXA2L

(i) It automatically identifies all fibres of $\widehat{\ell}$, which are embedded submanifolds $\ell \hookrightarrow Z$, with the quotient $Z/K_{\pm} \simeq X_{\pm}$ and therefore trivializes $\widehat{\ell}$ as $B\Gamma \times X_{\pm}$.

(ii) It moreover suggests to view the bisection preferentially as a section of π_+ (respectively π_-) which we discussed previously.

(iii) Once we attain the point of view of ℓ as a section of π_+ (respectively π_-), there is a distingushed vector field X^+ (respectively X^-) in the normal class of $[X]$ representing a tangent vector in $T_\ell B\Gamma(Z)$ that preserves ℓ as a section of π_+. Namely, $X^+ \in \Gamma^{cl}(\ell, K_+)$

Let us choose from here on forward to view ℓ as a section of π_+. A further choice can be made to parametrize the trivial flat bundle $(\underline{\mathbb{C}}, \nabla)$ over ℓ by a closed 1-form A on the opposite side $X_{\mp}$, in this case X_-. In this way we identify

$$B\Gamma(Z) \cong L\Gamma_{\pi_+}(Z) \times \Omega^{1,cl}_{X_-}(\mathbb{R})$$

The benefit of this is the trivialization of the tangent bundle $TB\Gamma(Z) \cong TL\Gamma_{\pi_+}(Z) \times T\Omega^{1,cl}_{X_-}(\mathbb{R}) = \Omega^{1,cl}_{X_-} \times \Omega^{1,cl}_{X_-}$ which in turn we can treat as complex valued closed 1-forms. The form $\alpha = a + ib$ representing the tangent vector $\xi_\alpha \in T_{(\ell,\nabla)}B\Gamma(Z)$ corresponds then to the element

$$\left([\omega^{-1}\pi_-^* a], \ell^*\pi_-^* b\right) \in \Gamma^{cl}(N\ell \oplus T^*\ell)$$

It is noteworthy that the almost complex structure on $B\Gamma(Z)$ defined in (4.5) can now be conveniently written as $\mathbb{I}\xi_\alpha = \xi_{i\alpha}$.

Definition 4.13. Denote $\widehat{\ell}$ together with the flat Ehresmann connection $H_{\pm}$ by $\widehat{X}_{\pm}$ ■ 80XY53

Following remark (4.12), we can decompose $\widehat{X}_{\pm} = X_{\pm} \times B\Gamma(Z)$ as well as $T\widehat{X}_{\pm} = TX_{\pm} \oplus \Omega^{1,cl}_{X_-}(\mathbb{C})$.

Definition 4.14. Let $\widehat{\xi}^{\pm}_\alpha$ be the lift of ξ_α to the horizontal $H_{\pm}$. ■ P1G2U8

It is clear that, by design, $\widehat{\xi}^+_{a+ib}$ is such that the evaluation map pushes this forward to

$$\mathrm{ev}_*\left(\widehat{\xi}^+_{a+ib}\right) = \omega^{-1}\pi_-^* a$$

Definition 4.15 (Universal diffeomorphism)**.** The amalgamation of all diffeomorphisms $X_+ \xrightarrow{\varphi_\ell} X_-$ iduced by a brane bisection ℓ arranged fiberwise over the appropriate element in $B\Gamma(Z)$ defines the *universal diffeomorphism*

$$\widehat{X}_+ \xrightarrow{\widehat{\varphi}} \widehat{X}_-$$

■

Lemma 4.16. *The tangent map of $\widehat{\varphi}$ has the form* KHC2CX

$$\widehat{\varphi}_*(\widehat{\xi}^+_{a+ib} + V) = \widehat{\xi}^-_{a+ib} - Q(a) + V$$

Proof. Regarding $\widehat{X}_+$ and $\widehat{X}_-$ as the products as in definition (4.13), the universal diffeomorphism takes the form

$$(\ell, \nabla, p) \mapsto (\ell^T, \nabla, \varphi_\ell(p))$$

Recall that the tangent bundles of $L\Gamma_{\pi_\pm}(Z)$ are closed sections of the subbundles $K_\pm = \ker(\pi_\pm) \subseteq TZ$ respectively. In particular, a vector $\widehat{\xi}^+_{(a,0)}$ corresponds to the vector field $\omega^{-1}\pi_-^* a$ along $\ell \hookrightarrow Z$. Leaving aside the connection component for a moment, which is unchanged, we are seeking a map

$$\Gamma(K_+) \times \Gamma(TX_+) \to \Gamma(K_-) \oplus \Gamma(TX_-)$$

which leaves the normal class in $N\ell$ unchanged and together with the component $TX_\pm$ evaluated at $\ell(p) = \ell^T(\varphi_\ell(p))$ is unchanged.
The former condition implies we ought to only change the section by a vector field tangent to ℓ, the latter condition implies we have to move the point on ℓ accordingly infinitesimally. To lie in K_- places on $\omega^{-1}\pi_-^* a + (\ell_-)_* W$ exactly the condition that

$$0 = (\pi_-)_*(\omega^{-1}\pi_-^* a + (\ell_-)_* W) = (\pi_-)_*(\omega^{-1})(a) + (\pi_-)_*(\ell_-)_* W = -Q(a) + W$$

With this we conclude that

$$\widehat{\xi}^+_a = \widehat{\xi}^+_a + Q(a) - Q(a) = \widehat{\xi}^-_a - Q(a)$$

Alternatively, we can interpret $(\ell_-)_* Q(a)$ as the difference between two splittings $s_\pm$ of the sequence

$$T\ell \to T\widehat{\ell} \to TB\Gamma(Z)$$

which takes values in $T\ell$. ∎

Other than in the examples (3.10), (3.11) of holomorphic families of Lagrangian branes there is usually no canonical choice of a horizontal lift. In this case, to complete the partial connection $\widetilde{\nabla}$ on $\widehat{\ell}$ from (4.9), we need to specify a distribution transverse to the vertical subbundle of $\widehat{\ell} \to B\Gamma(Z)$ to project out the vertical part of a vector field before applying $\widetilde{\nabla}$. As a result the extension of $\widetilde{\nabla}$ would be flat precisely along the leaves of the chosen horizontal distribution. Fortunately, we have not one, but two distinguished horizontal distributions arising from the two fibrations that a bibundle naturally has. Furthermore, as the set of all horizontal distributions is affine, we can extract a third horizontal which is the average of the beforementioned distributions.

Definition 4.17. 01Y432

(a) Let $\widehat{\nabla}^\pm$ be the extension of $\widetilde{\nabla}$ that is trivial along $H_\pm$ on $\widehat{X}_\pm$.

(b) Denote the average of these connections by

$$\overline{\nabla} = \frac{1}{2}\left(\widehat{\nabla}^{+} + \widehat{\varphi}^{*}\widehat{\nabla}^{-}\right) \tag{4.2}$$

(c) Denote the curvatures of $\widehat{\nabla}^{+}$, $\widehat{\nabla}^{-}$ and $\overline{\nabla}$ respectively by $\widehat{F}^{+}$, $\widehat{F}^{-}$ and $\overline{F}$.

■

In order to use these 2-forms on $\widehat{\ell}$ we need to understand what kind of components they have.

Lemma 4.18. *$\widehat{F}^{\pm}$ has only an $(H_{\pm}V)$ component and $\overline{F}$ has $(HH+HV)$ components, more precisely, for $f \in C^{\infty}_{X_-}(\mathbb{R})$ and $a, b, a_i, b_i \in \Omega^{1,cl}_{X_-}(\mathbb{R})$* YEYJBO

(a) $(\widehat{F}^{+})^{H_+H_+} = (\widehat{F}^{-})^{H_-H_-} = (\widehat{F}^{\pm})^{VV} = 0$
(b) $(\widehat{F}^{-})^{H_-V}(\widehat{\xi}^{-}_{a+ib}, V) = b(V)$
(c) $(\widehat{F}^{+})^{H_+V}(\widehat{\xi}^{+}_{a+ib}, V) = \widehat{\varphi}^{*}b(V)$
(d) $\iota_{\widehat{\xi}^{+}_{idf}}\overline{F} = \widehat{\varphi}^{*}df$
(e) $\overline{F}^{HH}(\widehat{\xi}^{+}_{a_1+ib_1}, \widehat{\xi}^{+}_{a_2+ib_2}) = \frac{1}{2}\widehat{\varphi}^{*}(Q(a_1,b_2) - Q(a_2,b_1))$

To reduce the notational load, by abuse of notation, we treat any function or 1-form on X_- to also stand for its pullback to $B\Gamma(Z) \times X_-$.

Proof. Part (a) follows from construction of $\widehat{\nabla}^{\pm}$ and the property that $\widehat{\nabla}\big|_{(\ell,\nabla)} = \nabla$ is flat. For part (b), let $\widehat{\nabla}^{-} = d + i\widehat{A}^{-}$ and consider formal coordinates $t_0, t_1, \ldots$ on $B\Gamma(Z)$ such that for some $b \in \Omega^{1,cl}_{X_-}$

$$\frac{\partial}{\partial t_0} = \xi_{ib}$$

We compute on $\widehat{X}^{-}$ the vertical and horizontal parts of the differential

$$\widehat{F}^{-} = d^{\widehat{X}_-}\widehat{A}^{-} = d^{X_-}\left(\widehat{A}^{-}\big|_{(\ell,\nabla)}\right) + \frac{\partial}{\partial t_i}\widehat{A}^{-} \wedge dt^{i}$$

The first term vanishes by assumption that $\widehat{\nabla}\big|_{(\ell,\nabla)} = \nabla$ is flat. Further, $\frac{\partial}{\partial t_0}\widehat{A}^{-} = b$ as per construction

$$e^{t\xi_{ib}}(\widehat{\nabla}^{-}) = \widehat{\nabla}^{-} + t \cdot ib$$

The only difference to the next part (c) is that b needs to be pulled back to $\widehat{X}^{+}$ in the equality $\frac{\partial}{\partial t_0}\widehat{A}^{-} = \widehat{\varphi}^{*}b$. For any generalized Kähler metric given by (ℓ, ∇), we identify the underlying smooth manifold of $X_\pm$ regardless as M and treat $I_\pm$ as complex structures on M. Part (d) is an immediate consequence of the previous parts and part (e) is proven by the computation

$$\begin{aligned}\overline{F}^{HH}(\widehat{\xi}^+_{a_1+ib_1}, \widehat{\xi}^+_{a_2+ib_2}) &= \frac{1}{2}\widehat{F}^+(\widehat{\xi}^+_{a_1+ib_1}, \widehat{\xi}^+_{a_2+ib_2}) + \frac{1}{2}\widehat{F}^-(\widehat{\xi}^-_{a_1+ib_1} - Q(a_1), \widehat{\xi}^-_{a_2+ib_2} - Q(a_2)) \\ &= \frac{1}{2}\widehat{\varphi}^*\left(\iota_{-Q(a_2)}b_1 - \iota_{Q(a_1)}b_2\right) \\ &= \frac{1}{2}\widehat{\varphi}^*\Big(Q(a_1,b_2) - Q(a_2,b_1)\Big)\end{aligned}$$

■

The setup alluded to this point can be summarized as the following. The universal bisection $\widehat{\ell}$ is a smooth family of branes over $B\Gamma(Z)$ in the symplectic bibundle (Z,ω).

4.3 BRANES OF THE OPERATION

We now come to the high point of this chapter, which is the observation that, in fact, this average connection $\overline{\nabla}$ is enhancing the universal family of brane bisetions to a holomorphic family. Even though the connection on every bisection ℓ is flat, the curvature of $\overline{\nabla}$ is nonzero.

Theorem 4.19. *Let $[(Z,\Omega)]$ be a generalized Kähler class from X_+ to X_-. The universal brane bisection* 9NQR75 *$\widehat{\ell}$ with the averaged tautological connection $(\mathbb{C},\overline{\nabla})$ and curvature $\overline{F} = \frac{1}{2}(\widehat{F}^+ + \widehat{\varphi}^*\widehat{F}^-)$ is an almost holomorphic family of branes over the space of all brane bisection in Z.*

$$\begin{tikzcd} \widehat{\ell} \arrow[r, "\hat{\iota}"] \arrow[dr] & B\Gamma(Z)\times Z \arrow[d] \\ & B\Gamma(Z) \end{tikzcd}$$

Proof. According to theorem (3.8) there are three conditions to check which we will walk through in the same order. As the choice of horizontal does not matter, we will apply this theorem with the global horizontal $H = H_+$.

(a) Since $\overline{F} = F^\nabla = 0$ along each fiber ℓ it is clear that $\overline{F}^{VV} = 0$ and condition ((a)) is met.

(b) To compute the mixed horizontal-vertical comnponent of $\overline{F}$, we choose an arbitrary $\widehat{\xi}^+_{a+ib} \in H_+$ and a vertical vector $V \in TX_+$.

$$\overline{F}(\widehat{\xi}^+_{a+ib}, V) = \widehat{\varphi}^* b(V)$$

The component $\widehat{F}^{\pm}(Q(a), V) = 0$ because the HH component of $\widehat{F}^{\pm}$ is the curvature of the brane (l, ∇) which is flat. The right hand side of the equation evaluates on a pair of horizontal and vertical vector as follows

$$\begin{aligned}
&-\mathbb{I}^*(\widehat{\ell}^*\omega)^{HV}(\widehat{\xi}^{+}_{a+ib}, V)\\
&= \omega\left(\widehat{\ell}_*(-\mathbb{I}\widehat{\xi}^{+}_{a+ib}), \widehat{\ell}_* V\right)\\
&= -\omega\left((\widehat{\ell})_*\widehat{\xi}^{+}_{-b+ia}, \widehat{\ell}_* V\right)\\
&= -\omega\left(\omega^{-1}\pi_-^*(-b), \ell_* V\right)\\
&= \pi_-^* b(\ell_* V)\\
&= \iota_V \varphi_\ell^* b
\end{aligned}$$

(c) In this last condition we check the equality $(\overline{F}^{HH})^{(20+02)} = -((\widehat{\ell}^*\widehat{\omega})^{HH})^{(2,0)+(0,2)}\mathbb{I}$. While $\overline{F}^{HH}$ does not have any (1,1) component, it is important to project out the (1,1) component of $\widehat{\ell}^*\widehat{\omega}$ with the use of equation (2.2).

Let $\widehat{\xi}^{+}_{a_1+ib_1}$ and $\widehat{\xi}^{+}_{a_2+ib_2}$ be horizontal vectors. Since $\widehat{F}^{+}$ vanishes entirely on horizontal vectors, the left-hand side simplifies to

$$\begin{aligned}
\overline{F}^{HH}(\widehat{\xi}^{+}_{a_1+ib_1}, \widehat{\xi}^{+}_{a_2+ib_2}) &= \frac{1}{2}\widehat{\varphi}^*\widehat{F}^{-}(\widehat{\xi}^{+}_{a_1+ib_1}, \widehat{\xi}^{+}_{a_2+ib_2})\\
&= \frac{1}{2}\widehat{F}^{-}(\widehat{\xi}^{-}_{a_1+ib_1} - Q(a_1), \widehat{\xi}^{-}_{a_2+ib_2} - Q(a_2))\\
&= -\frac{1}{2}\left(\iota_{Q(a_2)}b_1 - \iota_{Q(a_1)}b_2\right)\\
&= \frac{1}{2}\left(Q(a_1, b_2) - Q(a_2, b_1)\right)
\end{aligned}$$

Next, notice that the HH-component of $\widehat{\ell}^*\widehat{\omega}$ is

$$\widehat{\ell}^*\widehat{\omega}^{HH}(\widehat{\xi}^{+}_{a_1+ib_1}, \widehat{\xi}^{+}_{a_2+ib_2}) = p_-^* Q(a_1, a_2)$$

With this we conclude the computation

$$\begin{aligned}
-((\widehat{\ell}^*\widehat{\omega})^{HH})^{(2,0)+(0,2)}\mathbb{I}\left(\widehat{\xi}^{+}_{a_1+ib_1}, \widehat{\xi}^{+}_{a_2+ib_2}\right) &= -\frac{1}{2}\left((\widehat{\ell}^*\widehat{\omega})\mathbb{I} + \mathbb{I}^*\widehat{\ell}^*\widehat{\omega}\right)\left(\widehat{\xi}^{+}_{a_1+ib_1}, \widehat{\xi}^{+}_{a_2+ib_2}\right)\\
&= -\frac{1}{2}\left(\widehat{\ell}^*\widehat{\omega}(\widehat{\xi}^{+}_{-b_1+ia_1}, \widehat{\xi}^{+}_{a_2+ib_2}) + \widehat{\ell}^*\widehat{\omega}(\widehat{\xi}^{+}_{a_1+ib_1}, \widehat{\xi}^{-}_{b_2+ia_2})\right)\\
&= -\frac{1}{2}\left(Q(-b_1, a_2) + Q(a_1, -b_2)\right)\\
&= \frac{1}{2}\left(Q(a_1, b_2) - Q(a_2, b_1)\right)
\end{aligned}$$

∎

The parameter space $B\Gamma(Z)$ of this holomorphic family is infinite dimensional, but we can probe it with finite dimensional complex families of brane bisections over $\mathbb{C}^n$. In this sense, the following example provides a toy version of the construction from the previous theorem.

Example 4.20. Let (M, I, ω_M) be a compact Kähler manifold of real dimension $2n$. We will explore in this example finite dimensional subfamilies of the universal holomorphic family of branes. The Kähler class of ω_M is represented by the magnetically deformed cotangent bundle 7FNRLD

$$(Z, \Omega) = (T^*M, \Omega_0 + \pi^*\omega_M)$$

The imaginary part $\mathrm{Im}(\Omega) = \omega_0$ is the canonical smooth symplectic form on T^*M. Fix $\mathbb{C}^2$ with coordinate functions $z = (z_1, z_2)$ that have real and imaginary part $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. In addition, fix two real closed 1-forms $a_1, a_2 \in \Omega^{1,cl}_M(\mathbb{R})$ on M. The graph of every 1-form in T^*M is naturally identified with M itself. We consider the family of Lagrangian bisections of Z with generic fiber M over $\mathbb{C}^2$ that places over the point $z \in \mathbb{C}^2$ the graph of the linear combination $x_1 \cdot a_1 + x_2 \cdot a_2$ in Z.

$$\begin{tikzcd}
\mathbb{C}^2 \times M \arrow[rr, "\hat{\iota}"] \arrow[dr] & & \mathbb{C}^2 \times T^*M \arrow[dl] \\
& \mathbb{C}^2 &
\end{tikzcd}$$

$$\hat{\iota}((z_1, z_2), m) = (z, (m, x_1 \cdot a_1(m) + x_2 \cdot a_2(m)))$$

We equip this family of Lagrangians with the brane structure

$$\begin{tikzcd}
(\underline{\mathbb{C}}, \nabla = d + iA) \arrow[d] \\
\mathbb{C}^2 \times M
\end{tikzcd}$$

where $A = y_1 a_1 + y_2 a_2$. The conditions of holomorphicity found in theorem (3.8) are met as

$$F^\nabla = d(y_1 a_1 + y_2 a_2) = dy^1 \wedge a_1 + dy^2 \wedge a_2$$

and

$$\hat{\iota}^*\omega_0 = \hat{\iota}^* d\theta = d(x_1 a_1 + x_2 a_2) = dx^1 \wedge a_1 + dx^2 \wedge a_2$$

As a holomorphic 1-form dz satisfies $I^*dy = dx$. The second condition is vacuous as neiter $\hat{\iota}^*\omega$ nor F has any (HH) component.

The important take-away from this present discussion is the following. While the universal brane bisection constitutes a holomorphic family of Lagrangian branes over an infinite dimensional parameter space $B\Gamma$, we can probe its geometry locally by holomorphic families over finite dimensional parameter spaces. Universality of $B\Gamma$ suggests to view these finite dimensional families as embedding

$$X \hookrightarrow B\Gamma$$

which are holomorphic precisely when the family $\widehat{\ell} \to X$ is a holomorphic family of branes. ■

5 STRUCTURE ON $B\Gamma(Z)$

In this section we construct a Kähler metric on the parmeter space of the universal holomorphic family of brane bisections discussed in the previous chapter (4). Our first and foremost intend is to replicate the setup constructed by Joel Fine in his investigation into the Hamiltonian geometry of the space of unitary connections with symplectic curvature, more specifically in the case that the symplectic curvature is part of a Kähler metric [8]. Brane bisections of (Z, Ω) turn out to be the correct notion of a "prequantized" generalized Kähler metric with a forgetful map to the space of Kähler metrics. Further, we find that this mechanism works in a broader sense and can produce interesting Kähler metrics on any complex parameter space of a holomorphic family of Lagrangian branes.

5.1 DIFFERENCE OF BRANES

Next to the holomorphic family of branes $\widehat{\ell}$, explained in the previous chapter, (Z, ω) contains also a constant family of spacefilling branes (Z, B) as in exampple (3.12). This situation is, of course, reminiscent of the occurrence of a Lagrangian brane and a spacefilling brane in (Z, ω) whose difference is precisely what captures the generalized Kähler metric in the generalized Kähler class embodied by (Z, Ω). Hence we proceed in a similar fashion and compute the difference between these branes introduced in definition (1.8).

Definition 5.1. Let $\mathbb{F} = \widehat{\ell} \triangleleft \widehat{B}$ be the difference defined in (1.8) of the universal holomorphic family ZGKH4V
of Lagrangian branes in Z and the constant family of spacefilling branes $(Z, \widehat{B})$. ■

At this point it can be readily observed that $\mathbb{F} = \mathrm{ev}_+^* B - \overline{F}$ and more importantly that $\mathbb{F}$ is a closed 2-form on $\widehat{\ell}$. We refer to lemma (5.4)for an explicit computation of the components of $\mathbb{F}$.

5.2 FIBER INTEGRATION

For the ease of notation we will denote the n-th power of any differential form multiplied by $\frac{1}{n!}$ by $\alpha^{[n]}$. In other words, $\alpha^{[n]}$ is the $(n+1)$st term in the exponential e^{α}.

Definition 5.2. By rising $\mathbb{F}$ to its $n+1$st power and integrating it over the fibers of $\widehat{\ell} \to B\Gamma(Z)$, we TKOUAY
obtain a 2-form that we will denote by

$$\varpi = \pi_*(\mathbb{F}^{[n+1]})$$

■

It is clear from the construction that ϖ does not depend on any choice of horizontal lift from $B\Gamma(Z)$ to $\widehat{\ell}$. Therefore, unless stated otherwise, we will choose to work with H_+ on $\widehat{X}_+$ and denote this horizontal simply by H from now on. It is moreover immediate that ϖ is a closed 2-form on $B\Gamma(Z)$ for it was obtained by integrating a closed form over compact oriented fibers. More details on this matter can be found in [22].

5.3 EXPLICIT FORM

For reference sake we compute the different components $\mathbb{F} = \mathbb{F}^{HH} + \mathbb{F}^{HV} + \mathbb{F}^{VV}$ on $\widehat{X}_+$.

Definition 5.3. On a manifold M, for any integrable complex structure I, consider for $\alpha \in \Omega^{1,cl}_M(\mathbb{C})$ the map TEBPQ3

$$\alpha \mapsto \alpha_I = I^*\mathrm{Re}(\alpha) - \mathrm{Im}(\alpha) \in \Omega^1_M(\mathbb{R})$$

■

We continue to trivialize $T^*\widehat{X}_+$ as a bundle with fibers $\Omega^{1,cl}(\mathbb{C})$. Since X_- are the fibers of $\widehat{X}_-$, by abuse of notation we view these closed complex 1-forms are forms pulled back to $\widehat{X}_-$.

Lemma 5.4. *The components of $\mathbb{F}$ are as follows. Let $\widehat{\xi}_\alpha$, $\widehat{\xi}_\beta$ with $\alpha =, \beta =$ be horizontal tangent vectors to $\widehat{X}_+$ and $V \in \Gamma(TX_+)$.* VLV3BN

$$\mathbb{F}^{HH}(\widehat{\xi}^+_{a_0+ia_1}, \widehat{\xi}^+_{b_0+ib_1}) = \widehat{\varphi}^* Q(I^*a_0, b_0) - \frac{1}{2}\widehat{\varphi}^* \left(Q(a_0, b_1) - Q(b_0, a_1)\right)$$

$$\mathbb{F}^{HV}(\widehat{\xi}_\alpha, V) = \widehat{\varphi}^* \alpha_I(V)$$

Proof. For the (HH) component we combine the result of lemma (4.18) and the observation that

$$\mathrm{ev}^* B(\widehat{\xi}^+_{(a_0,a_1)}, \widehat{\xi}^+_{(b_0,b_1)}) = \widehat{\ell}^* B(\omega^{-1}\pi_-^* a_0, \omega^{-1}\pi_-^* b_0) = \widehat{\ell}^* \pi_-^* I^* a_0(\omega^{-1}\pi_-^* b_0) = \widehat{\varphi}^* I^* a_0(-Q(b_0)) = \widehat{\varphi}^* Q(I^* a_0, b_0)$$

For the HV component we combine again lemma (4.18) with a quick computation

$$B(\omega^{-1}\pi_-^*\mathrm{Re}(\alpha), V) = \pi_-^* I^* \mathrm{Re}(\alpha)((\ell_+)_* V) = (\varphi_\ell^* I^* \mathrm{Re}(\alpha))(V)$$

■

With the above preparation we are now able to express the 2-form ϖ in terms of the datum of the generalized Kähler structure on M from (**??**) which are (I_+, I_-, Q, F).

Proposition 5.5. *The value of ϖ on tangent vectors ξ_α and ξ_β at (ℓ, ∇) is* ZWL1K1

$$\varpi_{(l,\nabla)}(\xi_\alpha, \xi_\beta) = \frac{1}{2}\int_{X_-} \widehat{\varphi}^*(\alpha_I \wedge \beta_J + \alpha_J \wedge \beta_I) \wedge F_\ell^{[n-1]}$$

Proof. Denote the real and imaginary parts of of α and β be $\alpha = a_0 + ia_1$, $\beta = b_0 + ia_1$. We start with the definition of ϖ from (5.2) and choose the lifts $\widehat{\xi}_\alpha$ and $\widehat{\xi}_\beta$ to $\widehat{X}_+$ for convenience, though the expression is independent of the lift. The value of ϖ on ξ_α, ξ_β at (ℓ, ∇) is the integral over the fiber ℓ of the lifted tangent vectors contracted against $\mathbb{F}^{[n+1]}$.

$$\iota_{\widehat{\xi}_\beta}\iota_{\widehat{\xi}_\alpha}\mathbb{F}^{[n+1]} = \mathbb{F}(\widehat{\xi}_\alpha, \widehat{\xi}_\beta)\cdot \mathbb{F}^{[n]} - \mathbb{F}(\widehat{\xi}_\alpha) \wedge \mathbb{F}(\widehat{\xi}_\beta) \wedge \mathbb{F}^{[n-1]} \tag{5.1}$$

We will proceed to piece together the computation term by term with the help of previous lemmas.

1. To start, we notice that the only terms in the integral are the ones with $2n$ vertical legs. which is for any $k \in \mathbb{N}$
$$\mathbb{F}^k = F^k$$

2. The first summand has two terms coming from the two terms in $\mathbb{F} = \mathrm{ev}^* B - \overline{F}$ which we computed in (5.4)
$$\begin{aligned}\Big(Q(I_-^* a_0, b_0) - \frac{1}{2}\left(Q(a_0, b_1) - Q(b_0, a_1)\right)\Big) F^{[n]} \\ = \frac{1}{2}\Big(Q(\alpha_I, b_0) + Q(a_0, \beta_I)\Big) F^{[n]}\end{aligned}$$
of which we split the first summand in half and divided it among the latter two summands. Lastly, we force this term into the right shape to apply lemma (**??**)
$$\Big(F^{-1}(\alpha_I, FQ\beta_0) + F^{-1}(FQa_0, \beta_I)\Big) F^{[n]}$$
by using $Q(a,b) = F^{-1}(a, FQb) = F^{-1}(FQa, b)$. With said lemma (**??**) we finally obtain for the first summand of equation (5.1)
$$\Big((I^* a_0 - a_1) \wedge FQb_0 + FQa_0 \wedge (I^* b_0 - b_1)\Big) \wedge F^{[n-1]}$$

3. The second summand involves for each α, β
$$\mathbb{F}(\widehat{\xi}_\alpha) = \widehat{\varphi}^* \alpha_I$$
It is clear that of the terms $\mathbb{F} = \mathrm{ev}^* B - \overline{F}$ the first evaluates to $\mathrm{ev}^* B(\widehat{\xi}_\alpha) = B(\omega^{-1}\pi_-^*(\mathrm{Re}(\alpha))) = \pi_-^* I^*(\mathrm{Re}(\alpha))$ while we have already computed $\left(\overline{F}(\widehat{\xi}_\alpha)\right)^V = \left(\widehat{\varphi}^*\mathrm{Im}(\alpha)\right)^V$ in (4.18). The vertical component of the latter expression suffices as the integral over the fiber only contributes

with $2n$ vertical legs. As a word of warning though, $\overline{F}(\widehat{\xi}_\alpha)$ is not equal to $\widehat{\varphi}^*\mathrm{Im}(\alpha)$ as the former has a horizontal component involving $\mathrm{Re}(\alpha)$ and the latter doesn't keep track of it.

Putting this all together, we obtain from (5.1)

$$\begin{aligned}
&-\frac{1}{2}\Big((I_-^* a_0 - a_1)\wedge FQb_0 + FQa_0 \wedge (I_-^* b_0 - b_1) + 2(I_-^* a_0 - a_1)\wedge (I_-^* b_0 - b_1)\Big)\wedge F^{[n-1]} \\
&= -\frac{1}{2}\Big((I_-^* a_0 - a_1)\wedge (FQb_0 + I_-^* b_0 - b_1) + (FQa_0 + I_-^* a_0 - a_1)\wedge (I_-^* b_0 - b_1)\Big)\wedge F^{[n-1]} \\
&= -\frac{1}{2}\Big((I_-^* a_0 - a_1)\wedge (I_+^* b_0 - b_1) + (I_+^* a_0 - a_1)\wedge (I_-^* b_0 - b_1)\Big)\wedge F^{[n-1]}
\end{aligned}$$

where we have used $I_+^* = I_-^* + FQ$, the dual to equation (1.1). The result follows from here with the appropriate abbreviation in the proposition statement. ■

5.4 MAIN THEOREM

Lemma 5.6. *For a Generalized Kähler structure as introduced in (??) the generalized metric $\pm g + b$ is invertible if g is invertible.* 3E3WSM

Proof. Take a vector field $V \in \mathfrak{X}(M)$ and apply

$$(\pm g + b)(V,V) = \pm g(V,V) + b(V,V) = \pm g(V,V)$$

■

With the canonical complex structure on the space of brane bisections (4.5), we state the main result of this section.

Theorem 5.7. *Let Z,Ω be a holomorphic Morita equivalence between holomorphic Poisson manifolds (X_+,σ_+) and (X_-,σ_-). The space of smooth LS bisections $B\Gamma(Z)$ is formally an infinite dimensional almost Kähler manifold for $H^1_{X_-} = 0$.* 61H82T

$$(B\Gamma(Z), \mathbb{I}, \varpi)$$

Proof. From the construction as a push down of a closed form along closed compact fibers it immediately follows that ϖ is a closed 2-form.

To show nondegeneracy we use the complex structure $\mathbb{I}$ to compute with the explicit form (5.5) for any ξ_α with $\alpha = a + ib$.

$$\begin{aligned}
\varpi(\xi_\alpha, \mathbb{I}\xi_\alpha) &= \frac{1}{2}\int (\alpha_{I_-} \wedge I_+^* \alpha_{I_+} + \alpha_{I_+} \wedge I^* \alpha_I) \wedge F^{[n-1]} \\
&= -\frac{1}{2}\int F^{-1}(\alpha_{I_-}, I_+^*\alpha_{I_+}) + F^{-1}(\alpha_{I_+}, I_+^*\alpha_{I_-})F^{[n]} \\
&= -\frac{1}{2}\int F^{-1}(\alpha_{I_-}, I_+^*\alpha_{I_+}) - F^{-1}(I_+^*\alpha_{I_+}, \alpha_{I_-})F^{[n]} \\
&= -\int F^{-1}(\alpha_{I_-}, I_+^*\alpha_{I_+})F^{[n]} \\
&= -\int F^{-1}(I_-^* a - b, -I_+^* b - a)F^{[n]} \\
&= -\int \big(F^{-1}(I_-^* a, -I_+^* b) + F^{-1}(I_-^* a, -a) + F^{-1}(-b, -I_+^* b) + F^{-1}(-b,-a)\big)F^{[n]} \\
&= -\int \big(- F^{-1}(a,b) + I_+ F^{-1}(a,-a) + I_+ F^{-1}(b,b) - F^{-1}(a,b)\big)F^{[n]} \\
&= \int \big((g+b)^{-1}(a,a) + (g+b)^{-1}(b,b)\big)F^{[n]}
\end{aligned}$$

The first and last term are identical and vanish because $-F^{-1}(a,b)F^{[n]} = a \wedge b \wedge F^{[n-1]}$, a, b where are closed and we assume that $H_M^1 = 0$. Hence it equates to a total derivative and integrates to zero. The remaining terms involve $I_+ F^{-1} = -(FI_+)^{-1} = (g+b)^{-1}$ which is inertible because we assume g is.

Finally, compatibility with $\mathbb{I}$ can readily be verified with the property of $\alpha \mapsto \alpha_I$ as a complex map for any complex structure I on M. We morever use lemma (**??**) and the mixed $(1,1)$ condition (1.1) for F^{-1} which is $I_+ F^{-1} + F^{-1} I^* = 0$. It then follows that

$$\begin{aligned}
\varpi(\mathbb{I}\xi_\alpha, \xi_\beta) &= \int \big(I_-^* \alpha_{I_-} \wedge \beta_{I_+} + J^* \alpha_{I_+} \wedge \beta_{I_-}\big) \wedge F^{[n-1]} \\
&= -\int \big(\alpha_{I_-} \wedge I_+^* \beta_{I_+} + \alpha_{I_+} \wedge I_-^* \beta_{I_-}\big) \wedge F^{[n-1]} = -\varpi(\xi_\alpha, \mathbb{I}\xi_\beta)
\end{aligned}$$

■

Now that the formal almost Kähler manifold has been set up, we turn to the next ingredient for the Hamiltonian group action we aim to construct which is the group that is acting.

Definition 5.8 (Gauge group)**.** Let $\varepsilon \hookrightarrow G_-$ be the identity bisection of the holomorphic groupoid $(G_-, \Omega_-) \rightrightarrows X_-$. We define $\mathcal{G}(\underline{\mathbb{C}}_\varepsilon)$ as the unitary gauge transformations of the trivial hermitian bundle $(\underline{\mathbb{C}}, d) \to \varepsilon$. ■ JDZXIY

We can, of course, identify $\mathcal{G}(\underline{\mathbb{C}}_\varepsilon) \cong \mathrm{Maps}(X_-, S^1)$ and its Lie algebra $\mathfrak{k} = \mathrm{Lie}(\mathcal{G}(\underline{\mathbb{C}}_\varepsilon)) = C^\infty_{X_-}(\mathbb{R})$. The duallity pairing between $\mathfrak{k}$ and $\mathfrak{k}^* = \Omega^{2n}_{X_-}$ is the integration pairing

$$\theta(f) = \int_{X_-} f\theta$$

where $f \in \mathfrak{k}, \theta \in \mathfrak{k}^*$.

When we introduced $B\Gamma(Z)$ in the previous chapter, it appeared solely as a smooth space of infinite dimensions. We can also consider Lagrangian brane bisections of the groupoid $B\Gamma(G_-)$ which has in addition a binary product

Definition 5.9. Let $B\Gamma(G_-)$ be the space of smooth Lagrangian brane bisections of the holomorphic WG4EVB
groupoid $(G_-, \Omega_-) \rightrightarrows X_-$. The *product* of two brane bisections is

$$(\ell_1, \nabla_1) * (\ell_2, \nabla_2) = (\ell_1 * \ell_2, \nabla_{12})$$

where ∇_{12} is the sum of the connections ∇_1 and ∇_2 after pulling them back to $\ell_1 \times \ell_2 \subseteq G_- \times G_-$, intersecting with the space of composable arrows $G^{(2)}$ and reducing along the Lagrangian relation $G^{(2)} \xrightarrow{m} G$ of groupoid multiplication. ■

In the choice of trivialization of $B\Gamma(G_-)$ as $L\Gamma_t(G_-) \times \Omega^{1,cl}_{X_-}(\mathbb{R})$ where $(\ell, \nabla) \cong (\ell, d + is^*A)$ for $A \in \Omega^{1,cl}_{X_-}(\mathbb{R})$ this product can be written as

$$(\ell_1, A_1) * (\ell_2, A_2) = (\ell_1 * \ell_2, (\varphi_{\ell_2}^{-1})^* A_1 + A_2)$$

Remark 5.10. There is a natural embedding

$$\mathcal{G}(\underline{\mathbb{C}}_\varepsilon) \hookrightarrow B\Gamma(G_-) \tag{5.2}$$

as a normal subgroup $f \mapsto (\varepsilon, d + id\log(f))$. This subgroup is normal because conjugation by any brane bisection (g, ∇) of G_- preserves the identity bisection. If $H^1_{X_-}(\mathbb{R}) = 0$, the quotient of Lagrangian brane bisections by this subgroup removes the connection degree entirely and recovers Lagrangian bisections

$$B\Gamma(G_-)/\mathcal{G}(\underline{\mathbb{C}}_\varepsilon) \cong L\Gamma(G_-)$$

The decomposition of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(B\Gamma(G_-))$ into

$$\mathfrak{g} = \Omega^{1,cl}_{X_-}(\mathbb{C}) = \Omega^{1,cl}_{X_-}(\mathbb{R}) \oplus \Omega^{1,cl}_{X_-}(i\mathbb{R}) = \mathfrak{k} \oplus i\mathfrak{k}$$

is, of course, very reminiscent of a the embedding of a compact group K into its complexification $G = K^{\mathbb{C}}$.

$$K \hookrightarrow G \to G/K$$

In particular because the complex structure $\mathbb{I}$ is swapping elements in the two factors $\mathfrak{k} \oplus i\mathfrak{k}$. This naive point of view breaks down when we investigate the compatibility the complex structure and the Lie group structure on G. We will attempt to resolve this issue later in (**??**). For our current goal of implementing a moment map interpretation of a Calabi-like conjecture on the sapce of prequantized generalized Kähler structures this group $B\Gamma(G_-)$ satisfies the two importants requirements of containing the 'compact' group $\mathcal{G}(\underline{\mathbb{C}}_\varepsilon)$ and extending its action on $B\Gamma(Z)$. ■

Lemma 5.11. *$B\Gamma(G_-)$ is not a complex group, but an infinite dimensional almost complex manifold with a group multiplication.* LLGBMY

Proof. We check that the Nijenhuis tensor of the complex structure $\mathbb{I}$ on $B\Gamma(Z)$ is nonzero on two tangent vectors $\xi_a, \xi_{ib} \in T_{(\ell,\nabla)}B\Gamma(G)$.

$$N_{\mathbb{I}}(\xi_a, \xi_{ib}) = [\xi_a, \xi_{ib}] - [\xi_{ia}, \xi_{ib}] + \mathbb{I}\left([\xi_{ia}, \xi_{ib}] + [\xi_a, \xi_b]\right) = \mathbb{I}\xi_{-[a,b]_Q} = \xi_{-i[a,b]_Q}$$

This is a local property and does not depend on the global trivialization of the tangent bundle. ■

The action of Lagrangian bisections of G_- on Lagrangian bisection of a groupoid principal bundle Z can be extended to Lagrangian brane bisections in much the same way that multiplication of brane bisections of G_- was defiend.

$$B\Gamma(Z) \circlearrowleft B\Gamma(G_-) \tag{5.3}$$

Let $\ell \in L\Gamma(Z)$ and $g \in L\Gamma(G_-)$. We use the Lagrangian relation $Z \times_{X_-} G_- \xrightarrow{\rho} Z$ to tranport the connections on ℓ and g onto $\ell.g$ to convolve them.

Through the embedding (5.2) and the action of brane bisections (5.3) the gauge group $\mathcal{G}(\underline{\mathbb{C}}_\varepsilon)$ naturally acts on prequantized generalized Kähler metrics $B\Gamma(Z)$. The infinitesimal action in the trivialization $TB\Gamma(Z) \cong B\Gamma(Z) \times \Omega^{1,cl}_{X_-}(\mathbb{C})$ is simply the inclusion as imaginary 1-forms

$$\mathfrak{k} \hookrightarrow \Omega^{1,cl}_{X_-}(\mathbb{C}) \qquad f \mapsto idf$$

The next step on the path to a Hamiltonian group action is an equivariant moment map.

Definition 5.12. For $f \in C^\infty_{X_-}(\mathbb{R})$ we define a map $B\Gamma(Z) \to \mathbb{R}$ K97I8E

$$\mu_f = \pi_*(\widehat{\varphi}^* f \cdot \mathbb{F}^{[n]})$$

■

Remark 5.13. A computation reveals that μ can be equivalently viewed as a map $B\Gamma(Z) \xrightarrow{\mu} \Omega^{2n}_{X_-}$ that sends $(\ell, \nabla) \mapsto F^{[n]}_{\ell T}$. The duality pairing between $C^\infty_{X_-}(\mathbb{R})$ and $\Omega^{2n}_{X_-}(\mathbb{R})$ is the integration pairing. 80XY53

$$\mu_f(\ell, \nabla) = \int_\ell \widehat{\varphi}^* f \cdot B^{[n]} = \int_{X_+} \varphi_\ell^* f \cdot F^{[n]}_\ell = \int_{X_-} f \cdot F^{[n]}_{\ell T}$$

■

We may alter the moment map by a choice of smooth top-form θ on X_+.

$$\mu_\theta(\ell, \nabla) = (\varphi_\ell^{-1})^*(F_\ell^{[n]} - \theta) \tag{5.4}$$

The difference $F_\ell^{[n]} - \theta$ is a valid expression on X_+, as the moment map takes values in the dual of the Lie algebra $\mathfrak{k}^*$, we pull it back to X_-.

Theorem 5.14. *$(B\Gamma(Z), \varpi, \mathbb{I})$ is a $\mathcal{G}(\underline{\mathbb{C}}_\varepsilon)$-Hamiltonian almost Kähler manifold with equivariant moment map μ.*

Proof. The action vector field for $df \in \Omega^{1,cl}_{X_-} = \mathfrak{a}$ is ξ_{idf}. From lemma (4.18) we know

$$d\mu_f = d\pi_* \left(\widehat{\varphi}^* f \cdot \mathbb{F}^{[n]}\right) = \pi_* \left(\widehat{\varphi}^* df \cdot \mathbb{F}^{[n]}\right) = \pi_* \left(\iota_{\xi_{idf}} \mathbb{F} \cdot \mathbb{F}^{[n-1]}\right) = \iota_{\xi_{idf}} \varpi$$

Equivariance follows from the observation that μ_f is independent of ∇ and the coadjoint action of $\mathcal{G}(\underline{\mathbb{C}}_\varepsilon)$ on $\Omega^{2n}_{X_-}(\mathbb{R})$ is trivial as an abelian Lie group. ∎

The Kähler metric g on $B\Gamma(Z)$ which is $\varpi\mathbb{I}$, has the form

$$\mathrm{g}_{(\ell,\nabla)}(\xi_\alpha, \xi_\alpha) = \int_{X_-} \left(\| \operatorname{Re}(\alpha) \|^2_{(g+b)} + \| \operatorname{Im}(\alpha) \|^2_{(g+b)}\right) F_{\ell^T}^{[n]}$$

is gauge invariant as the gauge action only affects the connection ∇ on which F_{ℓ^T} does not depend. With the canonical horizontal from the splitting $T_{(\ell,\nabla)} = \Omega^{1,cl}_{X_-}(\mathbb{R}) \oplus \Omega^{1,cl}_{X_-}(\mathbb{R})$ this induces the generalized L^2 Riemannian metric

$$\widetilde{\mathrm{g}}_\ell(a, b) = \int_{X_-} (g+b)^{-1}(a, b)\; F_{\ell^T}^{[n]}$$

on the space of Lagrangian bisections $L\Gamma(Z)$ where $a, b \in \Omega^{1,cl}_{X_-}(\mathbb{R})$. This metric is not to be confused with the *Mabuchi metric* [18] on the space of Kähler metrics $\phi \in \mathcal{H}_{F_0}$.

$$\langle f, h\rangle_\phi = \int_M fh(F_0 + dd^c\phi)^{[n]}$$

for $f, h \in C^\infty$. This metric also plays a key role in Donaldson's work [5] as the symmetric space metric on the space $\mathcal{H}_{F_0}$ of Kähler potentials of (M, F_0). He describes this metric to be the formally negatively curved and establishes it as the dual symmetric space of Hamiltonian diffeomorphisms $\mathrm{Ham}(M, F_0)$ with the metric

$$\langle f, h\rangle_{F_0} = \int_M fhF_0^{[n]}$$

Another structure on $L\Gamma(Z)$ which is a direct analouge to the symmetric metric considered by Joel Fine [8] is given by a fixed $\theta_- \in \Omega^{2n}_{X_-}$.

$$(f, h)_{\theta_-} = \int fh\theta_- \tag{5.5}$$

In his case this is a metric on the Lie algebra with zero Lie bracket and therefore immediately furnishes an example of a biinvariant metric on the group. We, however, would have to assume that θ is invariant under all Hamiltonian flows of Q for this pairing to be Ad-invariant and hence define a biinvariant metric. This is a very strong assumption that would force (X_-, Q) to be unimodular, so we don't make this assumption as yet. Instead, we accept that $\langle\ \rangle_{\theta_-}$ is only left invariant.

The goal of the discussion in [8] is to interpret the condition of the Calabi conjecture as the level set of a moment map. Let us briefly recall the setup to understand which parts go over to our present situation.

5.5 HAMILTONIAN DYNAMICS OF KÄHLER METRICS

Let (M, I, F_0) be a Kähler manifold with a hermitian line bundle $(L, h) \to M$ such that $c_1(L) = [F_0]$. DB6JJY
The space $\mathcal{S}^{(1,1)}$ of unitary connections ∇ on L whose curvatures are of type $(1,1)$ surjects onto the space of Kähler metrics $\mathfrak{X}_{F_0}$ in the fixed class $[F_0]$ by taking the curvature of the connection.

To specialize the language we have created in this text to Kähler geometry, we set in our discussion $Q = 0$ which lets the holomorphic Poisson structures $\sigma_\pm$ vanish and identifies both (X_+, I_+) and (X_-, I_-) with a single complex manifold $X = (M, I)$. The holomorphic symplectic Morita equivalence simplifies to the affine cotangent principal bundle

$$(T^*M, \Omega_0 + \pi^* F_0) \to (M, I)$$

Assume, as before, that the real part $B = \mathrm{Re}(\Omega_0 + \pi^* F_0)$ is prequantized by $(U, \nabla_U) \to Z$. In the case of Kähler geometry we may pull back (L, h) and combine it with the real tautological 1-form prequantizing the canonical symplectic form $\omega_0 = \mathrm{Im}(\Omega_0)$ to

$$(U, \nabla_U) = (\pi^* L, \pi^* \nabla_0 - i I^* \theta)$$

where ∇_0 prequantizes F_0. We construct a map from imaginary Lagrangian brane bisections of (Z, Ω) to $\mathcal{S}^{(1,1)}$ by taking the difference of the Lagrangian brane and the spacefilling brane and pulling it back to M without yet computing the curvature.

$$\begin{array}{ccc}
B\Gamma(Z) & \xrightarrow{\quad\chi\quad} & \mathcal{S}^{(1,1)} \\
 & & \\
(\underline{\mathbb{C}}, \nabla) & & (U \otimes (\underline{\mathbb{C}})^*, \nabla_U - \nabla) \\
\downarrow & \longmapsto & \downarrow \\
\ell & & \ell
\end{array} \tag{5.6}$$

Notice further that the complex morphism of vector spaces $\Omega^{1,cl}(\mathbb{C}) \to V_I$ that assigns $\alpha \mapsto \alpha_I$ (definition 5.3) is precisely the tangent map of χ. We can show that it is, in fact, an isomorphism in a special case.

Lemma 5.15. *On a complex Kähler manifold (M, I) with vanishing first cohomology group $H^1_M(\mathbb{R}) = 0$ the map $\alpha \mapsto \alpha_I$ is an isomorphism.* 0U63TV

$$\Omega^{1,cl}_M(\mathbb{C}) \to V_I := \left\{ a \in \Omega^1(\mathbb{R}) \;\middle|\; da \in \Omega^{1,1}_M \right\}$$

Proof. Let $\alpha = a + ib$, the map is well defined because $d(I^*a - b) = d^c a + I^*da - db = d^c a$. This is d-closed and d^c-exact and therefore $d^c a = dd^c f$ for some $f \in C^\infty(M, \mathbb{R})$ by the $\partial\bar{\partial}$-lemma, in particular of type (1,1). Suppose now that $\sigma \in \Omega^1$ and $d\sigma$ is of type (1,1). Because it is of type (1,1) and clearly closed, there is some $f \in C^\infty(M, \mathbb{R})$ such that $d\sigma = dd^c f$. The difference $\sigma - d^c f$ is closed an by cohomological assumptions exact, hence $\sigma = d^c f + dg$ for some $g \in C^\infty(M, \mathbb{R})$. But $d^c f + dg = -(I^*df - dg) = -(df + idg)_I$ and so the map is surjective. If $\alpha = a + ib$ maps to zero, $0 = I^*a - b$, implies that a is d^c-closed and d-closed and therefore $a - iI^*a$ a holomorphic 1-form. ∎

With this result we go ahead and prove even more.

Lemma 5.16. *The identification $B\Gamma(Z) \xrightarrow{\chi} \mathcal{S}^{(1,1)}$ for (Z, Ω) and (L, h) as defined above is bijective when $H^1_M(\mathbb{R}) = 0$.*

Proof. We explicitly construct the inverse. As before, ∇_0 is a fixed choice of connection on L prequantizing F_0 on M. Any $\nabla \in \mathcal{S}^{(1,1)}$ differs from ∇_0 by $\alpha = \nabla_1 - \nabla_0$ a real 1-form that is not closed, but da is of type $(1,1)$ for I. The previous lemma asserts that there are functions $f, g \in C^\infty_M(\mathbb{R})$ such that $a = d^c f + dg$. With this we build the Lagrangian brane bisection $(\ell = \Gamma_{df}, d + i\pi^* dg)$. One checks now that this is, indeed, is a map inverse to χ. The apparent choice of ∇_0 we made to magnetically deform (T^*M, Ω_0) into (Z, Ω) and to now construct the inverse of χ only determines the representative of the isomorphism class of (Z, Ω), not the class itself. ∎

This allows us to compare all pieces of the Hamiltonian group action on either space. We start with the groups that act. On $S^{(1,1)}$, since this is a space of certain connections, we naturally have a group action of τ, the group of bundle isometries of (L, h) covering the identity on M.

$$\tau = \mathrm{Map}(M, S^1)$$

The Lie algebra to this group is $\mathfrak{t} = \mathrm{Map}(M, \mathbb{R})$. This immediately identifies with with the gauge group $\mathcal{G}(\underline{\mathbb{C}}_\varepsilon)$ with Lie algebra $\mathfrak{k} = C^\infty_M(\mathbb{R})$ as M is diffeomorphic to the identity bisection $\varepsilon \hookrightarrow T^*M$.

The symlectic forms ϖ on $B\Gamma(Z)$ and Ω_A on $\mathcal{S}^{(1,1)}$ compare as follows. Let $A = \chi(\ell, \nabla)$ and ξ_α, ξ_β tangent vectors for $\alpha, \beta \in \Omega^{1,cl}_M(\mathbb{C})$.

$$(\chi^*\Omega_A)(\xi_\alpha, \xi_\beta) = \Omega_A(\alpha_I, \beta_I) = \int_M \alpha_I \wedge \beta_I \wedge \omega_A^{[n]} = \varpi_{(\ell,\nabla)}(\xi_\alpha, \xi_\beta)$$

The moment maps agree as well under the identification by χ.

$$\mu(A) = \omega_A^n / n! = \mu(\ell, \nabla)$$

To prescribe the value of the moment map to be θ, Fine uses GIT to flow a Kempf-Ness functional from any initial connection $A_0 \in \mathcal{S}^{(1,1)}$ to a desired connection A_1 whose curvature's n-th power is θ. The complexification of τ is $\tau^{\mathbb{C}} = \mathrm{Map}(M, \mathbb{C}^*)$. The Kempf-Ness functional $\mathcal{F}_A$ is described uniquely up to a constant on the symmetric space $\tau^{\mathbb{C}}/\tau = \mathrm{Map}(M, \mathbb{R})$ as the function whose derivative, when pulled back to $\tau^{\mathbb{C}}$, is given by the moment map μ. This means,

$$d_g \mathcal{F}_A(\phi) = \mu(g.A)(\phi) \tag{5.7}$$

Furthermore, Fine remarks that geodesics of the metric defines in (5.5) are affine lines

$$\omega(t) = e^{tf} \cdot \omega = \omega + t \cdot dd^c f$$

which can be constructed between any two points in the convex space of Kähler metrics $\mathcal{H}_{F_0}$. Because $\mathcal{F}$ is convex along these geodesic paths, its extrema are unique.

5.6 GRADIENT FLOW

It is natural to ask for this construction to carry over in the more general framework of generalized Kähler metrics, especially, given that a comparable Hamiltonian group action with an equivariant moment map on a space of prequantized GK metrics appears in our work.

However, these two stories significantly differ in the complexification of the compact group $K = \mathcal{G}(\underline{\mathbb{C}}_\varepsilon)$. Our prime candidate $B\Gamma(G_-)$ in the role of a 'complexification' of $\mathcal{G}(\underline{\mathbb{C}}_\varepsilon)$ has some obvious shortcomings. For starters, the symmetric space $B\Gamma(G_-)/\mathcal{G}(\underline{\mathbb{C}}_\varepsilon) = L\Gamma(G_-)$ has a nontrivial Lie bracket given by the Poisson bracket while the group of unitary gauge transformations $\mathcal{G}(\underline{\mathbb{C}}_\varepsilon)$ is abelian. Moreover, the Nijenhuis tensor of the complex structure $\mathbb{I}$ on $B\Gamma(G)$ is nonzero as we have shown in lemma (5.11).

Leaving these differences aside for a moment, we try to proceed in the same way as Fine. by constructing a canoincal 1-form Υ on $L\Gamma(G)$ from the moment map μ and a fixed initial bisection $\ell_0 \in L\Gamma(Z)$. By assuming $H^1_{X_-}(\mathbb{R}) = 0$ any tangent vector is of the form $df \in \Omega^{1,cl}_{X_-}(\mathbb{R})$. Let $g \in L\Gamma(G)$, then

$$\Upsilon_g(df) = \mu_\theta(\ell_0 \cdot g)(f) = \int_{X_+} \varphi_\ell^* f \cdot (F_\ell^{[n]} - \theta)$$

We have used here the canonical splitting of the sequence $\mathcal{A} \to B\Gamma(G) \to L\Gamma(G)$ to act by an element $g \in L\Gamma(G)$. While the 1-form from (5.7) is closed and therefore gives rise to a well defined integration $\mathcal{F}$, we find that our Υ is not closed. Let $f, h \in C^\infty_{X_-}(\mathbb{R})$ be two tangent vectors to $L\Gamma(G)$

$$\begin{aligned}
d\Upsilon(f,h) &= L_f\Upsilon(h) - L_h\Upsilon(f) - \Upsilon([f,h]) \\
&= (L_f\Upsilon)(h) + \Upsilon([f,g]) - (L_h\Upsilon)(f) - \Upsilon([h,f]) - \Upsilon([f,h]) \\
&= \int_{X_-} h(-dd^c f \wedge F_{\ell^T}^{[n-1]}) + \int_{X_-} \{f,h\}\,(F_{\ell^T}^{[n]} - \theta) - \int_{X_-} f(-dd^c h \wedge F_{\ell^T}^{[n-1]}) \\
&= \int_{X_-} dh \wedge d^c f \wedge F_{\ell^T}^{[n-1]} - \int_{X_-} df \wedge d^c h \wedge F_{\ell^T}^{[n-1]} + \int_{X_-} \{f,h\}\,(F_{\ell^T}^{[n]} - \theta) \\
&= \int_{X_-} -F_{\ell^T}^{-1}(dh, -I^* df) \wedge F_{\ell^T}^{[n]} + \int_{X_-} F_\ell^{-1}(df, -I^* dh) F_{\ell^T}^{[n]} + \int_{X_-} \{f,h\}\,(F_{\ell^T}^{[n]} - \theta) \\
&= \int_{X_-} IF^{-1}(dh, df) \wedge F^{[n]} - \int_{X_-} IF^{-1}(df, dh) F^{[n]} + \int_{X_-} \{f,h\}\,(F^{[n]} - \theta) \\
&= \int_{X_-} g^{-1}(dh, df) \wedge F_{\ell^T}^{[n]} - \int_{X_-} g^{-1}(df, dh) F_{\ell^T}^{[n]} + \int_{X_-} \{f,h\}\,(F_{\ell^T}^{[n]} - \theta) \\
&= \int_{X_-} \{f,h\}\,(F_{\ell^T}^{[n]} - \theta)
\end{aligned}$$

Nonetheless, in the presence of the θ_--metric on $L\Gamma(G)$, Υ still gives rise to a vector field that would be the gradient of $\mathcal{F}$ if it existed.

$$V = g_{\theta_-}^{-1}(\Upsilon) = \frac{F_{\ell^T}^{[n]} - (\varphi_\ell)_*\theta}{\theta_-} \tag{5.8}$$

Further, we remark that along the 1-parameter paths obtained by exponentiating a Lie algebroid section $df \in \Gamma(T^*_Q M)$ the 1-form Υ induced by the moment map is strictly decreasing which is reminiscent of convexity. Let $\ell(t)$ be the exponential path e^{tdf} in $L\Gamma(Z)$.

$$\begin{aligned}
\frac{d}{dt}\Upsilon_{\ell(t)}(f) &= \frac{d}{dt}\int_{X_-} f \cdot (F^{[n]}_{\ell^T_t} - \theta) \\
&= \int_{X_-} f \cdot (-dd^c f \wedge F^{[n-1]}_{\ell^T_t}) \\
&= \int_{X_-} df \wedge d^c f \wedge F^{[n-1]}_{\ell^T_t} \\
&= \int_{X_-} F^{-1}(df, I^*_- df) F^{[n]}_{\ell^T_t} \\
&= \int_{X_-} -(-g+b)^{-1}(df, df) F^{[n]}_{\ell^T_t} \\
&= \int_{X_-} -(-g+b)\Big((-g+b)^{-1}df, (-g+b)^{-1}df\Big) F^{[n]}_{\ell^T_t} \\
&= \int_{X_-} g\Big((-g+b)^{-1}df, (-g+b)^{-1}df\Big) F^{[n]}_{\ell^T_t}
\end{aligned}$$

Under the identification of $B\Gamma(G)$ with the its orbit of $(\ell_0, \nabla_0) \in B\Gamma(Z)$ this specifies as well a flow on the orbit sending (ℓ_0, ∇_0) closer to the 0-levelset of μ_θ. Long time existence and convergence of this flow is another question addressed by stability considerations. While Fine colcudes in [8] that satbility is trivially satisfied in Kähler geometry, it might be a non trivial condition on generalized Kähler metrics.

5.7 THE REMEDY

It is well known in the literature of GIT that another functional

$$\ell \mapsto \| \mu(\ell) \|^2$$

gives also rise to a flow towards the zero of the moment map. In fact, the gradient flows agree up to reparemetrization. In contrast to $\mathcal{F}$, this construction can not be obstructed by the non-closure of the canoically defined 1-form Υ. While $\mathcal{F}$ is a functional on the symmetric space, this is a functional on $B\Gamma(Z)$ itself and it makes use of the Kähler metric $\omega\mathbb{I}$ on $B\Gamma(Z)$. It also does not hinge on a chosen point ℓ_0, but we will use this chosen point to identify $B\Gamma(G)$ with the its orbit of ℓ_0 in $B\Gamma(Z)$ to compare the two flows on $B\Gamma(Z)$.

The gradient of $\| \mu \|^2$ at a bisection $\ell \in L\Gamma(Z)$ is

$$(\nabla \| \mu \|^2)_\ell = 2\mathbb{I}\xi_{i\mu(\ell)} = -2\xi_{\mu(\ell)} = -2\omega^{-1}\pi^*_- d\left(\frac{F^{[n]}_{\ell^T} - \varphi_{\ell *}\theta}{\theta_-}\right)$$

where we view μ as $\mathfrak{k}$-valued instead of $\mathfrak{k}^*$-valued in the presence of the pairing $\langle\ \rangle_{\theta_-}$. Indeed, this is a multiple of the vector field (5.8) on $L\Gamma(G)$.

We unpack the pde underlying this flow in terms of the generalized Kähler structure (I_+, I_-, Q, F) on X_+. For any bisection ℓ_t we obtain an identification by a smooth diffeomorphism $\varphi_t : X_+ \to X_-$. We use this to place the time dependent complex structure $I_t = \varphi_t^* I_-$ on the same space as $F_t = \ell_t^* B$ and I_+. We use the paring $\langle\ \rangle_{\theta_-}$ to convert μ_θ from a $\mathfrak{k}^*$-valued function into the $\mathfrak{k}$-valued function $\widetilde{\phi}$ on X_-. As a function pulled back to X_+ this is

$$\phi_t = \phi(\ell_t) = \frac{F_t^{[n]} - \theta}{\theta_t}$$

where $\theta_t = \varphi_\ell^* \theta_-$ is also pulled back to X_+. This is the function driving the flow towards the zero of the moment map μ_θ.

From corollary (2.30) we already know

$$\dot{F}_t = -dd^c_{I^t_-}\phi(t)$$

Let ℓ_t be the 1-parameter family flowing a Lie algebroid element ϕ_t starting at ℓ_0. Equivalently we can write this as $\ell_t = \ell_0.g_t$ for the exponential g_t integrating ϕ_t. In particular, the time dependent diffeomorphism of ℓ_t decomposes into

$$\varphi_{\ell_t} = \varphi_{g_t} \circ \varphi_{\ell_0}$$

where φ_{g_t} is the Hamiltonian flow of $-Q(d\phi_t)$, the vector field that $\omega^{-1}\pi_-^* d\widetilde{\phi}_t$ is π_--related to. The time derivative of the complex structure I_t is

$$\left.\frac{d}{ds}\right|_t I_s = \left.\frac{d}{ds}\right|_t \varphi_{\ell_t}^* \varphi_{g_s}^* I_- = L_{-Q(d\phi_t)} I_t = -Q(dd^c_{I_t}\phi(t))$$

θ_t evolves by a similar argument with

$$\dot{\theta}_t = L_{-Q(d\phi_t)}\theta_t = -\{\phi_t, \theta_t\}$$

which is the Poisson-module action on K_{X_+}. Lastly, we compute the time derivative of ϕ_t.

$$\begin{aligned}
\dot{\phi}_t &= \frac{\left(\dot{F_t^{[n]}}\right)}{\theta_t} - \frac{\left(F_t^{[n]} - \theta\right)\dot{\theta}_t}{\theta_t^2} \\
&= \frac{-dd^c_t\phi_t \wedge F_t^{[n-1]}}{\theta_t} + \phi_t\frac{\{\phi_t, \theta_t\}}{\theta_t} \\
&= F^{-1}(dd^c_t\phi_t)\frac{F_t^{[n]}}{\theta_t} + \phi_t X^{\theta_t}(\phi_t)
\end{aligned}$$

X^{θ_t} is the modular vector field of θ_t. We summarize the results of this section in the following theorem.

Theorem 5.17. *Let (Z,Ω) be a holomorphic Morita equivalence between holomorphic Poisson manifolds (X_+,σ_+) and (X_-,σ_-) with a fixed LS bisection ℓ_0. For any $\theta \in \Omega^{2n}_{X_+}$ there exists a functional on the space of generalized Kähler metrics $L\Gamma(Z) \to \mathbb{R}$ that is extremized by an LS bisection ℓ such that $(\ell^* B)^{[n]} = \theta$.* BF177J

Given the properties of $B\Gamma(G)$ it is clear that this is not the complexification of the gauge group $\mathcal{G}(\underline{\mathbb{C}}_\varepsilon)$ and this is likely to explain the difficulties we faced in this section to employ a GIT reduction. If it is not a complex group, but a Lie group with an almost complex structure that is involutive only up to a certain equivalence, the notion of GIT might have to be adapted to higher complex group actions.

Example 5.18 (Example 4.20 continued)**.** Recall the setup of the holomorphic family of Lagrangian branes over $\mathbb{C}^2$. We used its coordinate functions as coefficients in the linear cobination of closed real 1-forms a_1, a_2 on M to obtain a varying family of Lagrangians in T^M. CEU6BP

$$\begin{tikzcd} \widehat{\ell} = \mathbb{C}^2 \times M \arrow[r, "\hat{\iota}"] \arrow[dr, "\pi"'] & \mathbb{C}^2 \times T^*M \arrow[d] \\ & \mathbb{C}^2 \end{tikzcd} \tag{5.9}$$

We now make use of the Kähler structure ω_M on M which places a nontrivial spacefilling brane on it by magnetic deformation.

$$B = I^*\omega_0 + \pi_M^*\omega_M$$

To proceed as in the case of the universal family, we compute the intersection $\mathbb{F}$ of these two families of branes which is a 2-form on $\mathbb{C}^2 \times M$. For this, we first compute the pullback of B to $\widehat{\ell}$. It is helpful for this computation to verify that $I^*\omega_0 = dI^*(\theta)$ as well as $a^* I^*_{T^*M}(\theta) = (I^*a)^*\theta = I^*a$ for any 1-form on M that is not necessarily holomorphic. This property, however, only holds for the pullback of the tautological 1-form θ. We proceed to compute

$$\hat{\iota}^* B = (\sum x^j a_j)^*(dI^*_{T^*M}(\theta) + \pi_M^*\omega_M) = d(\sum x^j a_j)^*(I_{T^*M}(\theta)) + \omega_M = dI^*(x^j a_j) + \omega_M$$

and

$$\mathbb{F} = \hat{\iota}^* B - F^\nabla = dx^j \wedge I^* a_j + x^j d^c a_j - dy^j \wedge a_j + \omega_M \tag{5.10}$$

where Einstein summation is assumed. Let us label the terms by (X), (C), (Y) and (M) in the order of appearance in (5.10). When integrating $\mathbb{F}^{[n+1]}$ over the fibers M of π only terms with the right amount of vertical degree contribute. To be precise, since M is of real dimension $2n$, we need exactly $2n$ vertical legs and 2 horizontal legs in the summands of $\mathbb{F}^{[n+1]}$ to contribute. In the case that $n = 1$, those are exactly the terms (XX) + (XY) + (YY):

$$\begin{aligned}
\pi_*(\mathbb{F}^2/2) &= \frac{1}{2}\int_M (dx^i \wedge I^* a_i)^2 + dx^i \wedge I^* a_i \wedge (-dy^j \wedge a_j) + (-dy^j \wedge a_j) \wedge dx^i \wedge I^* a_i + (dy^j \wedge a_j)^2 \\
&= \frac{1}{2}\left(dx^i \wedge dx^j \int_M (-I^* a_i \wedge I^* a_j) + 2 \cdot dx^i \wedge dy^j \int_M I^* a_i \wedge a_j + dy^i \wedge dy^j \int_M (-a_i \wedge a_j) \right) \\
&= \frac{1}{2}\left(-dx^i \wedge dx^j \cdot \mathbb{W}(a_i, a_j) + 2 \cdot dx^i \wedge dy^j \cdot \mathbb{G}(a_i, a_j) - dy^i \wedge dy^j \cdot \mathbb{W}(a_i, a_j) \right) \\
&= -dx^1 \wedge dx^2 \cdot \mathbb{W}(a_1, a_2) + dx^i \wedge dy^j \cdot \mathbb{G}(a_i, a_j) - dy^1 \wedge dy^2 \cdot \mathbb{W}(a_1, a_2)
\end{aligned}$$

for appropriate definitions of $\mathbb{W}(a, b) = \int_M a \wedge b$ and $\mathbb{G}(a, b) = \int_M I^* a \wedge b = \int g_M^*(a, b)\omega_M$ which are a symplectic form and a metric on the space of 1-forms as we have seen them previously.

When we go to the case of $n = 2$, the number of terms that appear doubles, namely we have (XXM) + (XYM) + (YYM)+ (XXC) + (XYC) + (YYC). The first three terms don't change drastically, we only have to generalize $\mathbb{W}(a, b) = \int_M a \wedge b \wedge \omega_M^{[n-1]}$ and $\mathbb{G}(a, b) = \int_M I^* a \wedge b = \int g_M^*(a, b)\omega_M^{[n]}$. The latter three however will contain coefficients that are linear in x^1 and x^2. For example, the (XXC) term is

$$\begin{aligned}
&\int_M dx^1 \wedge I^* a_1 \wedge dx^2 \wedge I^* a_2 \wedge x^k \cdot d^c a_k \\
&= -dx^1 \wedge dx^2 \cdot x^k \int_M I^* a_1 \wedge I^* a_2 \wedge dd^c f_k \\
&= -dx^1 \wedge dx^2 \cdot x^k \int_M a_1 \wedge a_2 \wedge dd^c f_k \\
&= 0
\end{aligned}$$

As long as the $\partial\bar{\partial}$-Lemma holds $d^c a_k = dd^c f_k$ for some function f_k and is therefore of type (1,1) which is in the kernel of the Lie algebra action of I^*. Hence the entire term vanishes because a_1 and a_2 are closed. The same holds for the (YYC) and only the (XYC) terms contributes

$$2\int_M dx^i \wedge I^* a_i \wedge (-dy^j \wedge a_j) \wedge x^k d^c a_k = 2dx^i \wedge dy^j \cdot \int_M I^* a_i \wedge a_j \wedge d^c a_k$$

At this point it is not complicated to compute the polynomial coefficients $\mathcal{P}_{xy}(x^1, \ldots, x^m)$, $\mathcal{P}_{xy}(x^1, \ldots, x^m)$ and $\mathcal{P}_{yy}(x^1, \ldots, x^m)$ in a more general setup of holomorphic families in an $2n$-manifold M over $\mathbb{C}^m$. The polynomials will go up to degree $n - 1$ and the constant terms are precisely $-\mathbb{W}(a_i, a_j)$, $\mathbb{G}(a_i, a_j)$ and $-\mathbb{W}(a_i, a_j)$ respectively.

The purpose of this example is to help understand the mechanism by which the transgressed Kähler structure arises. It would be, of course, desirable to undestand the properties of the transgression directly through its construction from $\mathbb{F}$. **[?]** provides insight into the reduction of Dirac structures along Lagrangian branes which could shed more light on this matter. The setup in his paper and his description of the reduced spinor have many similarities with the way we construct ϖ. Future work will hopefully enlighten us.

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