MV polytopes and reduced double Bruhat cells

by

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Abstract

When G is a complex reductive algebraic group, MV polytopes are in bijection with the non-negative tropical points of the unipotent group of G. By fixing w from the Weyl group, we can define MV polytopes whose highest vertex is labelled by w. We show that these polytopes are in bijection with the non-negative tropical points of the reduced double Bruhat cell labelled by w^{-1}. To do this, we define a collection of generalized minor functions $\Delta_\gamma^{\mathrm{new}}$ which tropicalize on the reduced Bruhat cell to the BZ data of an MV polytope of highest vertex w.

We also describe the combinatorial structure of MV polytopes of highest vertex w. We explicitly describe the map from the Weyl group to the subset of elements bounded by w in the Bruhat order which sends $u \mapsto v$ if the vertex labelled by u coincides with the vertex labelled by v for every MV polytope of highest vertex w. As a consequence of this map, we prove that these polytopes have vertices labelled by Weyl group elements less than w in the Bruhat order.

A motivation for studying MV polytopes of highest vertex w is that they are the finite-type equivalent of lower affine MV polytopes for $\widehat{SL_2}$. We show that for $\ell(w) \leq 3$, lower affine MV polytopes with highest vertex w are in bijection with the non-negative tropical points of the reduced double Bruhat cell labelled by w^{-1} for $\widehat{SL_2}$.

Finally, MV polytopes in the finite case are defined by the tropical Plücker relations while rank 2 affine MV polytopes are defined by "diagonal relations". We prove that for B_2 polytopes, these diagonal relations hold and are equivalent to the tropical Plücker relations.

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Chapter 1

Introduction

For G a complex reductive algebraic group, the irreducible representations are highest weight representations. To understand the tensor products of these irreducible representations, Lusztig defined a canonical basis for each $V(\omega_i)$, which behaves nicely with the decomposition of these tensor products into their irreducible subrepresentations [Lus90]. In [MV07], Mirkovíc and Vilonen provide another basis using the geometric Satake correspondence, which relates the representation theory of the Langlands dual group $G^\vee$ with the intersection homology of the affine Grassmanian, $\mathcal{G}r$.

Under this correspondence, the bases of the representations correspond to certain subvarieties of $\mathcal{G}r$, called Mirkovíc-Vilonen (MV) cycles. These MV cycles are the irreducible components of the intersection of infinite cells and as such, are difficult to understand as geometric objects. Anderson first conjectured that MV cycles could be analysed by studying their moment polytopes [And03] and in [Kam10], Kamnitzer gives a combinatorial description of MV cycles using these moment polytopes, called MV polytopes. Goncharov and Shen [GS15] take this one step further by explicitly showing that the set of MV polytopes are the tropical points of the unipotent subgroup of G.

1.1 MV polytopes as tropical points

In [Kam10], Kamnitzer gives a combinatorial characterization of MV polytopes via their hyperplane data. In this characterization, MV polytopes are convex polytopes with edges in the root directions and the positions of the hyperplanes satisfy certain relations called *the tropical Plücker relations*; if the hyperplane data satisfies these conditions, we call it a *BZ data*. The vertices of such a polytope come with an automatic labelling by the Weyl group, W. For example, the vertex of lowest weight is labelled by e and the vertex of highest weight is labelled by the longest Weyl element, w_0. A pair of vertices share an edge if the vertices are labelled by w and ws_i for some $w \in W$ and some simple reflection s_i.

Tropical geometry provides a perspective that associates MV polytopes with the tropical points of the unipotent subgroup N of G. The benefit to this point of view is that the tropical Plücker relations come from the Plücker relations on N, which arise naturally by studying the transition maps of Lusztig's positive atlas [BZ97].

To define the tropical points of N, first consider the torus T of G. Define the tropical points of T as $T(\mathbb{Z}^{\mathrm{trop}}) = X_*(T)$. A *positive structure* on a variety is an atlas of toric charts where the transition maps are subtraction-free. For a variety X with a positive structure, there is a unique way to define the tropical points $X(\mathbb{Z}^{\text{trop}})$, as outlined in [FG09].

The variety N has a positive structure, called *Lusztig's positive atlas*. Consider a reduced word $\underline{i} = (i_1, \dots, i_m)$ of w_0. With respect to $\underline{i}$, the coordinates $x_{\underline{i}} : \mathbb{C}^m \to N$ are in Lusztig's positive atlas, where $x_{\underline{i}}(a_1, \dots, a_m) = x_{i_1}(a_1) \dots x_{i_m}(a_m)$ and the image of $x_i : \mathbb{C} \to G$ is the Chevalley subgroup associated to the simple root α_i. Consider the *potential function* $\chi : N \to \mathbb{C}$ by

$$\chi(x_{\underline{i}}(a_1, \dots, a_m)) = \sum_{k=1}^{m} a_k.$$

Goncharov and Shen [GS15] define the *non-negative* tropical points by *tropicalizing* the function χ; in fact, any function on N which can be written as a subtraction-free expression in some coordinates of the positive atlas can be tropicalized to a function on the tropical points. The non-negative tropical points $N(\mathbb{Z}^{\text{trop}})_{\geq}$ are the points $\ell \in N(\mathbb{Z}^{\text{trop}})$ such that $\chi^{\text{trop}}(\ell) \geq 0$.

By Theorem 5.4 of [GS15] and Theorem 4.5 of [Kam10], there is a bijection between the non-negative tropical points $N(\mathbb{Z}^{\text{trop}})_{\geq}$ and the set of MV polytopes $\mathcal{P}$. The main result is to prove a similar theorem for a subset $\mathcal{P}_w \subseteq \mathcal{P}$.

1.2 MV polytopes of highest vertex w

In this thesis, for $w \in W$, we investigate a subset of MV polytopes called *MV polytopes of highest vertex* w, denoted by $\mathcal{P}_w$. These polytopes are MV polytopes whose vertex labelled by w is equal to the vertex labelled by w_0.

The original motivation to study these polytopes was to develop a better understanding of affine MV polytopes. In Chapter 5, we see that $\widehat{SL_2}$ affine MV polytopes naturally split into a lower polytope, an upper polytope and a middle polytope. The set of lower affine MV polytopes are exactly polytopes of highest vertex w for some w in the affine Weyl group. By studying the finite-type class $\mathcal{P}_w$, we are better able to understand these lower affine MV polytopes.

In addition to their connection to the affine case, these MV polytopes are also of interest due to their connection to preprojective algebra modules and MV cycles. In [BKT14], the authors define a class of preprojective algebra modules of interest, $\mathscr{T}^w$ and in [Mén22], Ménard proves that $\mathcal{P}_w$ is exactly set of MV polytopes associated to these modules. Additionally, the set of MV cycles which correspond to $\mathcal{P}_w$ are also of interest in recent work of Gaitsgory.

First, we study the structure of $\mathcal{P}_w$ by studying the vertices of $P \in \mathcal{P}_w$.

1.3 Vertex data of $\mathcal{P}_w$

We define the vertex data of $P \in \mathcal{P}_w$ to be the set $(\mu_v)_{v \in W}$ such that $P = \text{conv}\{\mu_v : v \in W\}$. A useful tool in studying the vertex data is the Lusztig data: for a reduced word $\underline{i} = (i_1, \dots, i_m)$, the *Lusztig data* $n_\bullet^{\underline{i}}$ is the m-tuple of the "lengths" between the vertices $\mu_{s_{i_1} \cdots s_{i_k}}$ and $\mu_{s_{i_1} \cdots s_{i_{k-1}}}$ for $k = 1, \dots, m$.

An immediate consequence of the definition of $\mathcal{P}_w$ is that $\mu_v = \mu_w$ for any vertex μ_v labelled by $v \in W$ such that $w \leq_R v$. This is equivalent to the condition that for any reduced word $\underline{i} = (i_1, \dots, i_m)$ of w_0 such that $s_{i_1} \dots s_{i_{\ell(w)}} = w$, the Lusztig data $n_\bullet^{\underline{i}}$ is zero in the coordinates

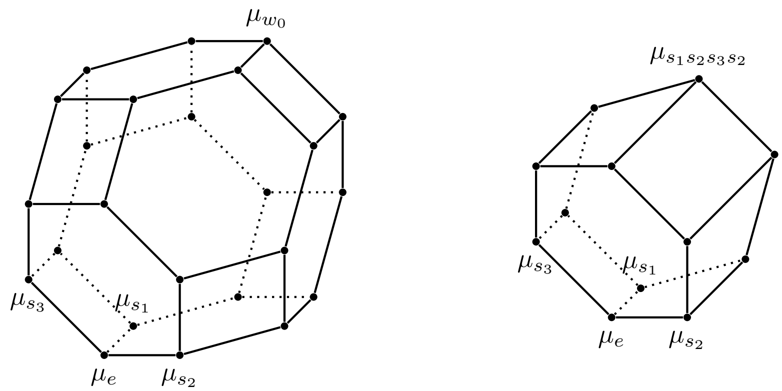


Figure 1.1: An A_3 MV polytope (left) and an A_3 MV polytope of highest vertex $s_1s_2s_3s_2$ (right)

$j \geq \ell(w) + 1$. For a general reduced word $\underline{i}$ of w_0, we determine which coordinates in the Lusztig data $n_\bullet^{\underline{i}}$ are zero.

Proposition A (Proposition 3.1.24)**.** *For a reduced word $\underline{i}$ of w_0, the zeros in the Lusztig data of an MV polytopes of highest vertex w is the rightmost subword of $\underline{i}$ which is a reduced word of $w^{-1}w_0$.*

From this proposition, we obtain the first main result. Define v_w to be the maximal length element such that $v_w \leq_R v$ and $v_w \leq w$; this element exists and is unique.

Theorem B (Theorem 3.1.26, Corollary 3.1.27)**.** *For every $P \in \mathcal{P}_w$ with vertex data $(\mu_v)_{v \in W}$, $\mu_v = \mu_{v_w}$ for every $v \in W$ and hence $P = conv\{\mu_v : v \in W, v \leq w\}$.*

1.4 MV polytopes of highest vertex w as tropical points

We would like to realize $\mathcal{P}_w$ as the non-negative tropical points on some subvariety of N such that the tropicalized generalized minors functions send a non-negative tropical point to the BZ data of an MV polytope of highest vertex w. The candidate for this subvariety is the reduced double Bruhat cell, $L^{w^{-1}} = N \cap B_- w^{-1} B_-$.

The reduced double Bruhat cell has a positive structure given by the Lusztig parameterization, where we take the reduced word $\underline{i}$ to be a reduced word of w^{-1} and the torus to be $(\mathbb{C}^\times)^m$. In these coordinates, the potential function χ is still a subtraction-free and thus the non-negative tropical points $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})_\geq$ are well-defined with respect to Lusztig's positive atlas and the potential χ.

For ω_i a fundamental weight and $u_1, u_2 \in W$, define the *generalized minor functions* $\Delta_{u_1\omega_i, u_2\omega_i} : G \to \mathbb{C}$ by

$$\Delta_{u_1\omega_i, u_2\omega_i}(g) = \langle g \cdot v_{u_2\omega_i}, v_{u_1\omega_i} \rangle \tag{1.1}$$

where $v_{u_j\omega_i}$ is the vector of weight $u_j\omega_i$ in $V(\omega_i)$, the fundamental representation of weight ω_i. Restricted to N, the collection of the tropicalized generalized minors functions send non-negative tropical points to the BZ data of an MV polytope.

Restricted to $L^{w^{-1}}$, if $v \not\leq w$, then the generalized minor functions $\Delta_{\omega_i, v\omega_i}$ are zero. Denote $\Gamma^w = \{v\omega_i : v \in W, v \leq_R w, i \in I\}$ to be the set of weights such that the corrseponding minor is nonzero on $L^{w^{-1}}$. For any $\gamma \in \Gamma \setminus \Gamma^w$, the tropicalized generalized minor associated to this weight will give an infinite function and hence the collection of the tropicalized generalized minors will not send a non-negative tropical point to a BZ datum. Instead, we redefine these minors by $\Delta^{\text{new}}_{v\omega_i} := \Delta_{(v_w^{-1}v)\omega_i, v\omega_i}$. The choice of $(v_w^{-1}v)\omega_i$ will be the smallest weight γ such that $\Delta_{\gamma, v\omega_i} \neq 0$ on $L^{w^{-1}}$. Consider the collection of tropical functions $M_\gamma = (\Delta^{\text{new}}_\gamma \circ \eta^{-1}_{w^{-1}})^{\text{trop}}$ for $\gamma \in \Gamma$, where $\eta_{w^{-1}}$ is a necessary change of coordinates.

Theorem C (Proposition 3.3.12)**.** *On $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})_\geq$, the collection $(M_\gamma)_{\gamma\in\Gamma}$ satisfies the following conditions:*

(i) the edge inequalities,

(ii) the tropical Plücker relations on the subcollection $(M_\gamma)_{\gamma\in\Gamma^w}$,

(iii) the edge equalities on the subcollection $(M_\gamma)_{\gamma\in\Gamma\setminus\Gamma^w}$.

Using this new collection of tropical functions $(M_\gamma)_{\gamma\in\Gamma}$, we obtain an identical result to the case of N.

Theorem D (Theorem 3.3.13)**.** *There is a bijection $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})_\geq \longrightarrow \mathcal{P}_w$ by $\ell \to (M_\gamma(\ell))_{\gamma\in\Gamma}$.*

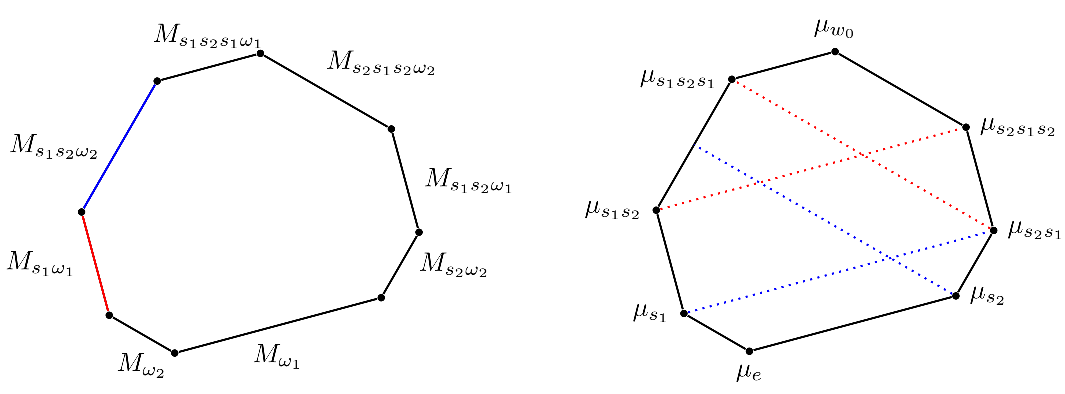
Remark E. The result (iii) of Theorem C depends on Conjecture 3.3.10, which states certain vanishing conditions of the generalized minor functions on $L^{w^{-1}}$.

1.5 Detropical diagonal relations

When the root system is of rank 2, MV polytopes are 2-dimensional polygons. In the finite case, there are only three distinct rank 2 root systems: A_2, B_2 and G_2. As in Section 1.1, the BZ datum of an MV polytope satisfies the tropical Plücker relations. On the other hand, in the affine rank 2 case, the BZ datum satisfies the *diagonal relations*, which are relations of the form

$$\min\{\langle \mu_w - \mu_{s_i w}, \omega_j\rangle, \langle \mu_{ws_j} - \mu_{s_i w s_j}, \omega_i\rangle\} = 0. \tag{1.2}$$

The tropical Plücker relations are closely related to the relations which describe how the Lusztig data $n^{\underline{i}}_\bullet$ changes as the reduced word $\underline{i}$ changes and therefore, given the hyperplanes along one side



(a) The hyperplanes of a B_2 polytope

(b) The diagonal relations of a B_2 polytope

Figure 1.2: The hyperplanes and diagonal relations of a B_2 MV polytope

of the polytope, the tropical Plücker relations can be used to calculate the hyperplanes on the other side of the polytope. For example, in the case of B_2, if we have the Lusztig data along the right side of the polytope in Figure 1.2a, this is equivalent to knowing the values for the hyperplanes M_{ω_2}, M_{ω_1}, $M_{s_2\omega_2}$, $M_{s_1s_2\omega_1}$, $M_{s_2s_1s_2\omega_2}$, and $M_{s_1s_2s_1\omega_1}$. The first tropical Plücker relation does not include $M_{s_1\omega_1}$, and thus we can determine the value of $M_{s_1s_2\omega_2}$ from the known hyperplanes. The second relation does not include $M_{s_1s_2\omega_1}$ and hence can be used to determine $M_{s_1\omega_1}$.

The difficulty of the tropical Plücker relations is that they are not uniform across the three rank 2 root systems. Instead, we would like to consider the diagonal relations on rank 2 polytopes, which do have a uniform expression across the three cases. They also have a simple visual interpretation. For example, in the case of B_2, these diagonals relations require that the vertex $\mu_{s_1s_2}$ is on or below the line $\mu_{s_2} + x\alpha_1^\vee$ and $\mu_{s_2s_1}$ is on or below the line $\mu_{s_1} + x\alpha_2^\vee$ with at least one of these vertices falling on the corresponding line. See Figure 1.2b for the two diagonal relations on B_2 polytopes.

We conjecture that the rank 2 polytopes given by the diagonal relations in (1.2) are equivalent to the rank 2 polytopes given by the tropical Plücker relations. The A_2 case is simple: the tropical Plücker relation is exactly the diagonal relation. For B_2 polygons, we have two diagonal relations:

$$M_{s_1\omega_1} + 2M_{s_2\omega_2} = \min\{M_{\omega_1} + M_{s_1s_2\omega_2} + M_{s_2\omega_2}, M_{s_2s_1\omega_1} + 2M_{\omega_2}\},$$
$$M_{s_2s_1\omega_1} + 2M_{s_1s_2\omega_2} = \min\{M_{s_1\omega_1} + 2M_{s_2s_1s_2\omega_2}, M_{s_1s_2s_1\omega_1} + M_{s_2\omega_2} + M_{s_1s_2\omega_2}\}.$$

Using the representation theory of B_2, we show the following relations on generalized minors:

$$\Delta_{s_1\omega_1}\Delta^2_{s_2\omega_2} = \Delta_{\omega_1}\Delta_{s_2\omega_2}\Delta_{s_1s_2\omega_2} + \Delta_{s_2s_1\omega_1}\Delta^2_{\omega_2} + \Delta_{\omega_1}\Delta_{\omega_2}\Delta_{s_2s_1s_2\omega_2},$$
$$\Delta_{s_2s_1\omega_1}\Delta^2_{s_1s_2\omega_2} = \Delta_{s_1\omega_1}\Delta^2_{s_2s_1s_2\omega_2} + \Delta_{s_1s_2s_1\omega_1}\Delta_{s_2\omega_2}\Delta_{s_1s_2\omega21} + \Delta_{s_1s_2s_1\omega_1}\Delta_{\omega_2}\Delta_{s_2s_1s_2\omega_2}.$$

By tropicalizing this system, we prove that the last term does not contribute to the tropicalization and thus, the diagonal relations hold for B_2 MV polytopes.

Theorem F (Corollary 4.3.2)**.** *The diagonal relations hold on B_2 MV polytopes.*

We expect that the diagonal relations will hold on G_2 as well.

Conjecture G. *The diagonal relations hold on G_2 MV polytopes.*

1.6 Rank 2 affine MV polytopes

When $G = \widehat{SL_2}$, we consider the affine MV polytopes $P \subset \text{span}_{\mathbb{R}}\{\alpha_0, \alpha_1\}$ as defined in [BDKT13]. In this case, we have two separate sets of vertices μ_w, μ^w which are labelled by the affine Weyl group (W); these vertices are called "top" and "bottom" vertices respectively. Affine polytopes decompose into three sets of data: a lower polytope defined as $\text{conv}\{\mu_w : w \in W\}$, an upper polytope defined as $\text{conv}\{\mu^w : w \in W\}$ and a decorated four sided middle polytope. More specifically, μ_w vertices are of the form μ_k or $\overline{\mu}_k$ and μ^w are of the form μ^k and $\overline{\mu}^k$ for $k \in \mathbb{N}$. We require that the limit as $k \to \infty$ of each set of vertices exists. These four limits will be the four vertices of the middle polytope.

Lower polytopes can be defined as the convex hull of a finite number of vertices, so for any given affine MV polytope, there exists some $w \in W$ such that highest weight vertex of the lower polytope is labelled by w. Denote $\mathcal{P}_w$ as lower affine polytopes of highest vertex at most w. We are interested in showing that these polytopes are in bijection with the tropical points of some subvariety of $\widehat{SL_2}$.

Note that the upper polytopes are obtained by applying a reflection to the lower polytopes, and thus by characterizing lower affine MV polytopes, we also characterize upper affine MV polytopes.

1.7 Lower affine MV polytopes

We define the BZ data of a lower affine MV polytope $(M_\gamma)_{\gamma\in\Gamma}$ in an identical way to the finite case in [Kam10]. These BZ data satisfy similar edge inequalities and edge equalities to the finite case, but the tropical Plücker relations are replaced by diagonal relations similar to (1.2).

The first result is that lower affine MV polytopes are completely characterized by their lower BZ data. Let $\mathcal{M}_\Gamma^w$ be the set of all such BZ data $(M_\gamma)_{\gamma\in\Gamma}$ that satisfy the edge inequalities, the edge equalities and the diagonal relations. Then we have a bijection between these BZ datum and polytopes of highest vertex w.

Theorem H (Theorem 5.2.12)**.** *There is a bijection $\mathcal{P}_w \to \mathcal{M}_\Gamma^w$ by $P \mapsto (M_\gamma)_{\gamma\in\Gamma}$. Hence a lower MV polytope of highest vertex w is completely determined by its lower BZ data.*

Consider the reduced double Bruhat cell $L^{w^{-1}} := N \cap B_- w^{-1} B_-$. By [Wil13], Lusztig's positive atlas is indeed an atlas on $L^{w^{-1}}$, so the reduced double Bruhat cell is a positive space and the tropical points $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})$ are defined.

As in the finite case, for $u_1, u_2 \in W$ and ω_i a fundamental weight, we define the *generalized minors* as functions $\Delta_{u_1\omega_i, u_2\omega_i} : G \to \mathbb{C}$ which acts on G as in (1.1). For γ too large, $\Delta_{\omega_i,\gamma} = 0$ on $L^{w^{-1}}$ and the resulting tropical function will be infinite. Set $\Gamma^w = \{v\omega_i : v \leq_R w, i \in I\}$ to be the set of weights such that $\Delta_{\omega_i,\gamma} \neq 0$ on $L^{w^{-1}}$. For $v\omega_i \in \Gamma \setminus \Gamma^w$, we want to define a new minor which is nonzero on the reduced Bruhat cell. Set $\Delta^{\text{new}}_{v\omega_i} := \Delta_{v_w^{-1}v\omega_i, v\omega_i}$ where v_w is as defined in the finite case. Consider the collection of tropical functions $M_\gamma = \left(\Delta^{\text{new}}_\gamma\right)^{\text{trop}}$ for $\gamma \in \Gamma$. We expect that these functions satisfy the diagonal relations.

Conjecture I. *For any $w \in W$, the collection $(M_\gamma)_{\gamma\in\Gamma^w}$ satisfy the diagonal relations on the tropical points $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})$.*

To show that these M_γ satisfy the other defining properties of a lower BZ data of highest vertex w, we need to restrict to the non-negative tropical points. For $\underline{i}$ a reduced word of w, set $w^{\underline{i}}_k = s_{i_1} s_{i_2} \cdots s_{i_k}$. Define a new potential on $L^{w^{-1}}$ by

$$\tau = \Delta^{-1}_{\omega_{i_1}} \Delta^{-1}_{s_{i_1}\omega_{i_1}} \Delta^2_{\omega_{i_2}} + \Delta^{-1}_{\omega_{i_2}} \Delta^{-1}_{w^{\underline{i}}_2\omega_{i_2}} \Delta^2_{s_{i_1}\omega_{i_1}} + \sum_{k=2}^{m-1} \Delta^{-1}_{w^{\underline{i}}_{k-1}\omega_{i_{k-1}}} \Delta^{-1}_{w^{\underline{i}}_{k+1}\omega_{i_{k+1}}} \Delta^2_{w^{\underline{i}}_k\omega_{i_k}}.$$

The non-negative tropical points of $L^{w^{-1}}$ with respect to τ are $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})^\tau_\geq := \{\ell \in L^{w^{-1}}(\mathbb{Z}^{\text{trop}}) : \tau^{\text{trop}}(\ell) \geq 0\}$.

Theorem J (Lemma 5.3.12, Corollary 5.3.17)**.** *If the diagonal relations hold for $(M_\gamma)_{\gamma\in\Gamma^w}$ on the tropical points $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})$, then*

(i) $(M_\gamma)_{\gamma\in\Gamma^w}$ satisfies the edge inequalities on $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})^\tau_\geq$,

(ii) $(M_\gamma)_{\gamma\in\Gamma\setminus\Gamma^w}$ satisfies the edge equalities on $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})$.

By combining Conjecture I and Theorem J, for ℓ a non-negative tropical point, we expect the collection $(M_\gamma(\ell))_{\gamma\in\Gamma}$ to satisfy the defining relations of a lower BZ datum of highest vertex w.

Conjecture K. *For any $w \in W$, the map $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})^{\tau}_{\geq} \longrightarrow \mathcal{P}^w$ defined by $\ell \mapsto (M_\gamma(\ell))_{\gamma\in\Gamma}$ is a bijection.*

We prove these conjectures are true for small enough w.

Theorem L (Lemma 5.3.8, Theorem 5.3.18)**.** *For $w \in W$ with $\ell(w) \leq 3$, the collection $(M_\gamma)_{\gamma\in\Gamma^w}$ satisfies the diagonal relations on $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})$ and the map $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})^{\tau}_{\geq} \longrightarrow \mathcal{P}^w$ defined by $\ell \mapsto (M_\gamma(\ell))_{\gamma\in\Gamma}$ is a bijection.*

Chapter 2

MV polytopes

MV polytopes were originally defined by Anderson [And03] as the moment polytopes of certain subvarieties of the affine Grassmanian called MV cycles. In [Kam10], Kamnitzer gave a completely combinatorial description of MV polytopes using their hyperplane data. In particular, a GGMS polytope is an MV polytope exactly when the hyperplane data is a BZ datum. In this chapter, we define MV polytopes as combinatorial objects and outline their relation to preprojective algebra modules. We describe the crystal structure on the set of MV polytopes and define the Saito crystal reflection. In the final section, we use tropical geometry to express the set of MV polytopes as the tropical points of some variety.

2.1 Notation

Let G be a complex reductive algebraic group. Let T be a maximal torus of G. We define the weight and coweight lattice as $X^* = \mathrm{Hom}(T, \mathbb{C}^\times)$ and $X_* = \mathrm{Hom}(T, \mathbb{C}^\times)$ respectively. Let $W = N_G(T)/T$ be the Weyl group.

Fix B be a Borel subgroup of G such that $T \subset B$. Let N be the unipotent subgroup of B. Let I be the index set of the simple roots and denote α_i as the simple root associated to the index i while $\alpha_i^\vee$ is the simple coroot. Let Δ be the set of roots and Δ_+ the set of positive roots while $\Delta^\vee$ is set of coroots and $\Delta_+^\vee$ the set of positive coroots. Let $\langle \cdot, \cdot \rangle : X_* \times X^* \to \mathbb{C}$ be the pairing of the weight and the coweight lattice and set $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$. Denote by $Q = \mathbb{N}\Delta$ the root lattice so $Q_+ = \mathbb{N}\Delta_+$ is the positive root cone. Similarly, let $Q^\vee = \mathbb{N}\Delta^\vee$ be the coroot lattice and $Q_+ = \mathbb{N}\Delta_+^\vee$ the positive coroot cone. Let ω_i be the fundamental weights, which form a basis of the weight space X^* such that $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{i,j}$.

Consider the space $\mathfrak{t}_\mathbb{R} = X_* \otimes \mathbb{R}$ and $\mathfrak{t}_\mathbb{R}^* = X^* \otimes \mathbb{R}$. Define a partial order on X_* by $\mu \leq \lambda \iff \lambda - \mu \in Q_+^\vee$ and a partial order on X^* by $\mu \leq \lambda \iff \lambda - \mu \in Q_+$. Define the twisted partial order $\geq_w$ on $\mathfrak{t}_\mathbb{R}$ by $\beta \leq_w \alpha \iff \langle \beta - \alpha, w\omega_i \rangle \geq 0$ for all $i \in I$.

2.2 The Weyl group

The Weyl group is the group of reflections on the root space generated by the simple reflections s_i defined by the action $s_i(\alpha) = \alpha - \langle \alpha_i^\vee, \alpha \rangle \alpha_i$ for $\alpha \in \Delta$. The action of the Weyl group can be extended to the weight lattice by $s_i(\beta) = \beta - \langle \alpha_i^\vee, \beta \rangle \alpha_i$ for $\beta \in X_*$. Similarly, W acts on the coroots and the coweight lattice by $s_i(\beta) = \beta - \langle \beta, \alpha_i \rangle \alpha_i^\vee$ for $\beta \in X^*$.

The Weyl group is a Coxeter group with presentation

$$W = \langle s_i : (s_i s_j)^{m_{ij}} = 1 \rangle$$

where $m_{ii} = 1$ and $m_{ij} \in \mathbb{N}$. The relation $(s_i s_j)^{m_{ij}}$ is equivalent to the *braid relation* $s_i s_j s_i \cdots = s_j s_i s_j$ of length m_{ij}.

Example 2.2.1. When $G = SL_3$, the Lie algebra of SL_3 is $\mathfrak{g} = sl_3$. The simple roots can be realized as $\alpha_1 = (1, -1, 0), \alpha_2 = (0, 1, -1)$. Then

$$s_1(\alpha_1) = -\alpha_1, \qquad s_1(\alpha_2) = \alpha_1 + \alpha_2, \qquad s_2(\alpha_1) = \alpha_1 + \alpha_2, \qquad s_2(\alpha_2) = -\alpha_2.$$

On the vector $a = (a_1, a_2, a_3)$, $s_1(a) = (a_2, a_1, a_3)$ while $s_2(a) = (a_1, a_3, a_2)$ so that s_1 and s_2 are the permutations (12) and (23) respectively. Thus the Weyl group of G is S_3, the permutation group on 3 elements. The relation $(s_1 s_2)^3 = 1$ is equivalent to the order 3-braid relation $s_1 s_2 s_1 = s_2 s_1 s_2$.

Denote the set of simple reflections as $S = \{s_i : i \in I\}$. By definition, any $w \in W$ can be written as a product of generators. It is well-known that S is a minimal generating set for W, so there exists a unique minimal k such that $w = s_{i_1} \cdots s_{i_k}$. We call this k the *length of* w and denote it by $\ell(w) = k$. The product $s_{i_1} \cdots s_{i_k}$ is *reduced* if $k = \ell(w)$. The tuple of indices $\underline{i} = (i_1, \ldots, i_k)$ a *word* of w if $w = s_{i_1} \cdots s_{i_k}$. A *reduced word* of w is $\underline{i}$ such that $w = s_{i_1} \cdots s_{i_k}$ is reduced.

For convenience, we will say for $v, w \in W$, $v \cdot w$ is a *reduced product* if $\ell(vw) = \ell(v) + \ell(w)$.

Corollary 2.2.2 ([BB05, Corollary 1.4.8])**.** *Any word $\underline{i} = (i_1, \ldots, i_k)$ for w contains a reduced word for w as a subword and is obtained by deleting an even number of indices.*

Using the length, we can define the *left descent set* and *right descent set* respectively as

$$D_L(w) = \{s_i \in S : \ell(s_i w) < \ell(w)\}, \qquad D_R(w) = \{s_i \in S : \ell(w s_i) < \ell(w).$$

There are two standard partial orders on the Weyl group, the strong Bruhat order and the weak Bruhat orders. We will outline these orders here.

Definition 2.2.3. We say $u \leq w$ in the *strong Bruhat order* if for some reduced word $s_{i_1} \cdots s_{i_m}$ of w, there is a subword $s_{i_{j_1}} \cdots s_{i_{j_k}}$ with $1 \leq j_1 < j_2 < \cdots < j_k \leq m$ which is a reduced word for u. Define the *strong Bruhat interval* as $[v, w] = \{x : v \leq x \leq w\}$.

It is clear from the definition that $u \leq w \iff u^{-1} \leq w^{-1}$. From now on, we will refer to this as the Bruhat order. In fact, the Bruhat order is independent of the choice of reduced word for w.

Corollary 2.2.4 ([BB05, Corollary 2.2.3])**.** *If $u \leq w$, then every reduced word of w has a subword that is a reduced word for u.*

The Bruhat order is a directed poset, i.e. for any pair of elements, there exists an element larger than both of them.

Proposition 2.2.5 ([BB05, Proposition 2.2.9])**.** *For every $u, v \in W$, there exists a $w \in W$ such that $u \leq w$ and $v \leq w$.*

The Weyl group of G is a finite group and thus there is a unique element of longest length, denoted by w_0. The longest element has a few useful properties.

Proposition 2.2.6 ([BB05, Proposition 2.3.1, 2.3.2, 2.3.4])**.** *For W a finite Coxeter group,*

1) for every $u \in W$, $u \leq w_0$

2) $w_0^{-1} = w_0$

3) $\ell(ww_0) = \ell(w_0) - \ell(w)$

4) $u \leq w \iff uw_0 \geq ww_0 \iff w_0u \geq ww_0 \iff w_0uw_0 \leq w_0ww_0$

There are two weak Bruhat orders, the left Bruhat order and the right Bruhat order.

Definition 2.2.7. We say $u \leq_R w$ in the *right weak Bruhat order* if $w = us_{i_1} \cdots s_{i_k}$ for some $s_{i_j} \in S$ such that $\ell(w) = \ell(u) + k$. We say $u \leq_L w$ in the *left weak Bruhat order* if $w = s_{i_1} \cdots s_{i_k} u$ for some $s_{i_j} \in S$ such that $\ell(w) = \ell(u) + k$. Define the weak Bruhat intervals as $[v, w]_R = \{x : v \leq_R x \leq_R w\}$ and $[v, w]_L = \{x : v \leq_L x \leq_L w\}$

We state a simplified version of a result of Björner and Wachs. First, for $x, y \in W$, we will say z is a *minimal upper bound* for x and y if $x, y \leq z$ and for any $z' \in W$ such that $x, y \leq z' \leq z$, then $z = z'$.

Theorem 2.2.8 ([BW88, Theorem 3.7, Theorem 4.4])**.** *Fix $v \in W$. Let $x, y \in [e, v]_R$ and suppose z is a minimal upper bound of x and y. Then $z \in [e, v]_R$.*

From now on, we drop the prefix "weak" and refer to these orders as the right and left Bruhat orders. Note that the left and right descent sets can be defined via these orders: $s_i \in D_L(w) \iff s_i \leq_R w$ and $s_i \in D_R(w) \iff s_i \leq_L w$.

The right Bruhat order has the **prefix property**: $u \leq_R w \iff$ there exists a reduced word $\underline{i} = (i_1, \ldots, i_k)$ of w such that $(i_1, \ldots, i_{\ell(u)})$ is a reduced word for u. Unlike the Bruhat order, it is not true that for any reduced word $\underline{i}$ of w, there is an initial word $(i_1, \ldots, i_m)$ that is a reduced word for u. If $u \leq_R w$, we will say that u is an *initial word* of w. Similarly, the left Bruhat order has the **suffix property** and we will say that u is a *terminal word* of w if $u \leq_L w$.

These two orders are isomorphic under the map $w \mapsto w^{-1}$, so that $u \leq_R w \iff u^{-1} \leq_L w^{-1}$. We state the next results in terms of the right Bruhat order but similar results hold for the left Bruhat order by this isomorphism.

Proposition 2.2.9 ([BB05, Proposition 3.1.2])**.** *For $u, w \in W$,*

1) $u \leq_R w \iff \ell(u) + \ell(u^{-1}w) = \ell(w)$

2) $w \leq_R w_0$ for all $w \in W$

3) $s_i \in D_L(u) \cap D_L(w)$ then $u \leq_R w \iff s_iu \leq_R s_iw$

In the weak order, the longest element still has many useful properties. First, condition 4) of Proposition 2.2.5 holds for both of the weak orders. Also, as the longest element $w_0 = w \cdot (w^{-1}w_0)$ is a reduced product for any w, then $w^{-1}w_0$ multiplied on the left with any terminal word of w will also be reduced. Similarly, w multiplied on the right with any initial word of $w^{-1}w_0$ will also be reduced. One application of this fact is the following lemma.

Lemma 2.2.10. *For $u, v \in W$, the following conditions are equivalent:*

1) $u \cdot v$ is a reduced product

2) $v \leq_R u^{-1}w_0$

3) $u \leq_L w_0v^{-1}$

Proof. By definition, $u \cdot v$ is reduced if and only if $\ell(uv) = \ell(u) + \ell(v)$. Then

$$\begin{aligned}\ell(v) + \ell(v^{-1}u^{-1}w_0) &= \ell(v) + \ell(w_0) - \ell(v^{-1}u^{-1}) = \ell(v) + \ell(w_0) - \ell(uv) \\ &= \ell(w_0) - \ell(u) = \ell(u^{-1}w_0)\end{aligned}$$

so by 1) of Proposition 2.2.9, $v \leq_R u^{-1}w_0$. A similar proof works for $u \leq_L w_0v^{-1}$. □

We end this section with three important properties that we will make use of multiple times throughout Chapter 3.

Theorem 2.2.11 ([BB05, Proposition 2.2.7, Theorem 1.5.1, Theorem 3.3.1])**.** *For W a Coxeter group,*

Lifting Property: *Suppose $u < w$ and $s_i \in D_L(w)\backslash D_L(u)$. Then $u \leq s_iw$ and $s_iu \leq w$.*

Exchange Property: *Let $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ be a reduced expression. If $\ell(s_iw) \leq \ell(w)$ for $s_i \in S$, then $s_iw = s_{i_1}s_{i_2}\cdots\widehat{s_{i_j}}\cdots s_{i_k}$, where $\widehat{s_{i_j}}$ means that this term is deleted.*

Word Property: *Every two reduced words for w can be connected via a sequence of braid relations.*

The Weyl group has another distinct product. For $w \in W$ and $s_i \in S$, let $s_i * w := \max\{w, s_iw\}$, where the maximum is the element in the set of maximal length. The *Demazure product* can be defined recursively by $s_{i_1} * \cdots * s_{i_k} := s_{i_1} * (s_{i_2} * \cdots * s_{i_k})$. This product is associative and well-defined by [BM15, Proposition 3.1].

Proposition 2.2.12 ([BJK22, Proposition 6.4])**.** *For $v, w \in W$, $v * w = \max\{xw : x \leq v\}$.*

Using the same proof technique as Proposition 6.4, we can relate the Demazure product to the weak orders.

Proposition 2.2.13. *For $v, w \in W$,*

$$v * w = \max\{xw : x \leq v \text{ and } \ell(xw) = \ell(x) + \ell(w)\} \tag{2.1}$$
$$= \max\{vy : y \leq w \text{ and } \ell(vy) = \ell(v) + \ell(y)\}. \tag{2.2}$$

Moreover, $v * w = xw = vy$ where x is the maximal length element such that $x \leq v$ and xw is reduced and y is the maximal length element such that $y \leq w$ and vy is reduced.

Proof. If $v = e$, then $e * w = w$ and clearly (2.1) holds. We proceed by induction.

For $v \neq e$, there exists $s_i \in D_L(v)$. As $\ell(s_iv) = \ell(v) - 1$, then by induction $(s_iv) * w = xw$ for some $x \leq s_iv$ and $\ell(xw) = \ell(x) + \ell(w)$. Since $s_i * (s_iv) = v$, then

$$v * w = s_i * (s_iv) * w = s_i * (xw)$$

because $*$ is associative. If $\ell(xw) > \ell(s_ixw)$, then $v*w = xw$ for $x \leq s_iv \leq v$ and $\ell(xw) = \ell(x)+\ell(w)$. Otherwise, $\ell(s_ixw) = \ell(xw) + 1$ so $v * w = s_ixw$. Note that $\ell(s_ixw) = 1 + \ell(xw) = \ell(x) + \ell(w) + 1$ so $s_i \notin D_L(x)$ and $\ell(s_ix) = \ell(x) + 1$. This implies $\ell(s_ixw) = \ell(s_ix) + \ell(w)$. Finally, by the Lifting Property, $s_ix \leq v$ and (2.1) holds. A similar proof works for (2.2).

The maximal length element in the set $\{xw : x \leq v \text{ and } \ell(xw) = \ell(x) + \ell(w)\}$ must occur when $\ell(x)$ is of maximal length. Thus $v * w = xw$ where x is the maximal length element such that $x \leq v$ and xw is reduced. By an identical argument, $v * w = vy$ for y the maximal length element such that $y \leq w$ and vy is reduced. □

It immediately follows that $w \leq_L v * w$ and $v \leq_R v * w$ by Proposition 2.2.9 1).

2.3 Finite-type MV polytopes

To define MV polytopes, we first consider GGMS polytopes.

Definition 2.3.1. Consider a collection $\mu_\bullet = (\mu_w)_{w\in W}$ in the coroot lattice $Q^\vee$ such that $\mu_v \leq_w \mu_w$ for all $v, w \in W$. A Gelfand-Goresky-MacPherson-Serganova (GGMS) polytope is a convex polytope $P(\mu_\bullet)$ of the form $P(\mu_\bullet) = \bigcap_{w\in W} C_w^{\mu_w}$ where

$$C_w^{\mu_w} = \{x \in \mathfrak{t}_\mathbb{R} : \langle x, w \cdot \omega_i\rangle \geq \langle \mu_w, w \cdot \omega_i\rangle, \forall i\}.$$

By [Kam10, Proposition 2.2], $P(\mu_\bullet) = \text{conv}\{\mu_w : w \in W\}$. We call $(\mu_\bullet)$ the *vertex data* of the polytope.

We can also define a GGMS polytope using the hyperplane data. The hyperplanes are indexed by weights of the form $w\omega_i$. Define the set of *chamber weights* $\Gamma = \{w\omega_i : w \in W, i \in I\}$. Let $M_\bullet = (M_\gamma)_{\gamma\in\Gamma}$ be a collection of integers that satisfy the *edge inequalities* for each $w \in W$ and $i \in I$:

$$M_{ws_i\omega_i} + M_{w\omega_i} + \sum_{j\neq i} a_{ji} M_{w\omega_j} \leq 0 \tag{2.3}$$

where $a_{ji} = \langle\alpha_j^\vee, \alpha_i\rangle$. Then the polytope $P(M_\bullet)$ defined by the hyperplane data is

$$P(M_\bullet) = \{x \in \mathfrak{t}_\mathbb{R} : \langle x, \gamma\rangle \geq M_\gamma, \forall\gamma \in \Gamma\}.$$

By [Kam10, Proposition 2.2], these two definitions are equivalent in the following way. If $P = P(\mu_\bullet)$, then $P = P(M_\bullet)$ where we set $M_{w\omega_i} = \langle\mu_w, w \cdot \omega_i\rangle$. If $P = P(M_\bullet)$ then $P = P(\mu_\bullet)$ where we set $\mu_w = \sum_{i\in I} M_{w\omega_i} w \cdot \alpha_i^\vee$. From now on, for a GGMS polytope P, we will denote $(\mu_\bullet)$ as the vertex data and $(M_\bullet)$ as the hyperplane data.

Consider $w \in W$ and $s_i \notin D_R(w)$. There is an edge in $P(\mu_\bullet)$ connecting μ_w and μ_{ws_i} where

$$\mu_{ws_i} - \mu_w = cw \cdot \alpha_i^\vee \tag{2.4}$$

and $c = -M_{w\omega_i} - M_{ws_i\omega_i} - \sum_{j\neq i} a_{ji}M_{w\omega_j}$. Note that from the edge inequalities (2.3) $c \geq 0$. We call c the *length* of the edge from μ_w to μ_{ws_i}. The next lemma follows directly from (2.4).

Lemma 2.3.2. *For P a GGMS polytope with vertex data $(\mu_\bullet)$ and hyperplane data $(M_\bullet)$,* $\mu_{ws_i} - \mu_w = 0 \iff M_{w\omega_i} + M_{ws_i\omega_i} = \sum_{j\neq i} a_{ji} M_{w\omega_j}$.

Example 2.3.3. For $G = SL_3$, the simple coroots are given by $\alpha_1^\vee = (1,-1,0)$, $\alpha_2^\vee = (0,1,-1)$ so these GGMS polytopes are actually polygons. See Figure 2.1 for an example.

The fundamental weights are $\omega_1 = (1,0,0)$, $\omega_2 = (1,1,0)$ and the chamber weights are $\Gamma = \{\omega_1, \omega_2, s_1\omega_1, s_2\omega_2, s_2s_1\omega_1, s_1s_2\omega_2\}$. These chamber weights index the hyperplanes $(M_\bullet)$ as in Figure 2.1.

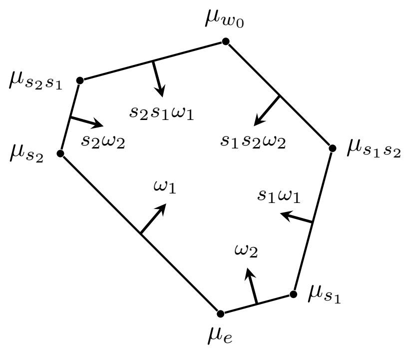


Figure 2.1: An A_2 MV polytope

2.3.1 BZ data

When a GGMS polytope is an MV polytope, the hyperplane data satisfies certain relations. First, we recall the tropical Plücker relations, which come from the tropicalization of the Plücker relations of [BZ97].

Definition 2.3.4. The collection $(M_\gamma)_{\gamma\in\Gamma}$ satisfies the tropical Plücker relations if for each $w \in W$ and every $i, j \in I$ such that $i \neq j$ and $s_i, s_j \notin D_R(w)$, then either $a_{ij} = 0$ or the following holds:

1) If $a_{ij} = a_{ji} = -1$, then

$$M_{ws_i\omega_i} + M_{ws_j\omega_j} = \min\{M_{w\omega_i} + M_{ws_is_j\omega_j}, M_{ws_js_i\omega_i} + M_{w\omega_j}\}$$

2) If $a_{ij} = -1, a_{ji} = -2$, then

$$\begin{aligned} M_{ws_j\omega_j} + M_{ws_is_j\omega_j} + M_{ws_i\omega_i} &= \min\{2M_{ws_is_j\omega_j} + M_{w\omega_i}, 2M_{w\omega_j} + M_{ws_is_js_i\omega_i}, \\ &\qquad M_{\omega_j} + M_{ws_js_is_j\omega_j} + M_{ws_i\omega_i}\} \\ M_{ws_js_i\omega_i} + 2M_{ws_is_j\omega_j} + M_{ws_i\omega_i} &= \min\{2M_{w\omega_j} + 2M_{ws_is_js_i\omega_i}, 2M_{ws_js_is_j\omega_j} + M_{ws_i\omega_i}, \\ &\qquad M_{ws_is_js_i\omega_i} + 2M_{ws_is_j\omega_j} + M_{w\omega_i}\} \end{aligned}$$

3) If $a_{ij} = -2, a_{ji} = -1$, then

$$\begin{aligned} M_{ws_js_i\omega_i} + M_{ws_i\omega_i} + M_{ws_is_j\omega_j} &= \min\{2M_{ws_i\omega_i} + M_{ws_js_is_j\omega_j}, 2M_{ws_is_js_i\omega_i} + M_{w\omega_j}, \\ &\qquad M_{ws_is_js_i\omega_i} + M_{w\omega_i} + M_{ws_is_j\omega_j}\} \end{aligned}$$

$$M_{ws_j\omega_j} + 2M_{ws_i\omega_i} + M_{ws_is_j\omega_j} = \min\{2M_{ws_is_js_i\omega_i} + 2M_{w\omega_j}, 2M_{w\omega_i} + 2M_{ws_is_j\omega_j}, M_{w\omega_j} + 2M_{ws_i\omega_i} + M_{ws_js_is_j\omega_j}\}$$

If $a_{ij} = -3$ or $a_{ji} = -3$, the tropical Plücker relations are given in [BZ97, Proposition 4.2]. We omit them here due to length.

Note that the tropical Plücker relations impose conditions on each 2-face of P.

Definition 2.3.5. The collection $(M_\gamma)_{\gamma\in\Gamma}$ is a Bernstein-Zelevinsky (BZ) datum of coweight λ if:

(i) $(M_\bullet)$ satisfies the tropical Plücker relations,

(ii) $(M_\bullet)$ satisfies the edge inequalities (2.3),

(iii) $M_{\omega_i} = 0$ and $M_{w_0\cdot\omega_i} = \langle\lambda, w_0\cdot\omega_i\rangle$.

We define an MV polytope as GGMS polytope P whose hyperplane data $(M_\bullet)$ is a BZ datum. This definition is equivalent to the original definition of MV polytopes as the moment polytopes of MV cycles.

Theorem 2.3.6 ([Kam10, Theorem 3.1])**.** *A GGMS polytope $P(M_\bullet)$ is an MV polytope if and only if it is the moment polytope of a stable MV cycle.*

Denote by $\mathcal{P}$ the set of MV polytopes. For any $P \in \mathcal{P}$, the polytope is determined by its vertex data $(\mu_\bullet)$, which is a collection of points in $Q^\vee$, or its BZ data $(M_\bullet)$, which is a collection of integers. There is one more set of combinatorial data which determines P, closely related to the vertex data.

2.3.2 Lusztig data

In this section, assume P is an MV polytope with vertex data $(\mu_\bullet)$ and BZ data $(M_\bullet)$. For each reduced word $\underline{i} = (i_1, \ldots, i_m)$ of w_0, we will define the Lusztig data $n_\bullet^{\underline{i}}(P)$ of the polytope.

First, define the Weyl group elements $w_k^{\underline{i}} = s_{i_1}\cdots s_{i_k}$ for $1 \le k \le m$ and set $w_0^{\underline{i}} = e$. The reduced word $\underline{i}$ gives a path μ_e, $\mu_{w_1^{\underline{i}}}$, $\mu_{w_2^{\underline{i}}}$, $\ldots$, $\mu_{w_{m-1}^{\underline{i}}}$, μ_{w_0} in the 1-skeleton of P. From (2.4),

$$\mu_{w_k^{\underline{i}}} - \mu_{w_{k-1}^{\underline{i}}} = \left(-M_{w_{k-1}^{\underline{i}}\omega_{i_k}} - M_{w_k^{\underline{i}}\omega_{i_k}} - \sum_{j\neq i_k} a_{ji_k} M_{w_{k-1}^{\underline{i}}\omega_j}\right) w_{k-1}^{\underline{i}}\alpha_{i_k}^\vee$$

Set $n_k^{\underline{i}} = -M_{w_{k-1}^{\underline{i}}\omega_{i_k}} - M_{w_k^{\underline{i}}\omega_{i_k}} - \sum_{j\neq i_k} a_{ji_k} M_{w_{k-1}^{\underline{i}}\omega_j}$. By the edge inequalities (2.3), $n_k^{\underline{i}} \ge 0$ for $1 \le k \le m$. We call the tuple $n_\bullet^{\underline{i}}(P) = (n_1, \ldots, n_m)$ the *Lusztig data of P with respect to $\underline{i}$*. The Lusztig data corresponds to the lengths of the edges along the path determined by $\underline{i}$ above. Note that for any $P \in \mathcal{P}$ and any $\underline{i}$, $n_\bullet^{\underline{i}}(P) \in \mathbb{N}^m$.

For convenience, we will call the path μ_e, $\mu_{s_{i_1}}$, $\ldots$, $\mu_{s_{i_1}\cdots s_{i_{m-1}}}$, μ_{w_0} determined by a reduced word $\underline{i}$ of w_0 a *minimal path from μ_e to μ_{w_0}* in P. We will also use the shorthand $n_\bullet^{\underline{i}} := n_\bullet^{\underline{i}}(P)$ when it is clear what P is.

Example 2.3.7. For the A_2 polytope in Figure 2.1, the reduced word $\underline{i} = (1, 2, 1)$ gives the Lusztig data $n_\bullet^{121} = (1, 2, 2)$, which are the lengths of the edges on the right side of the polytope. For $\underline{i} = (2, 1, 2)$, $n_\bullet^{212} = (3, 1, 2)$ which are the lengths of the edges on the left side of the polytope.

Any MV polytope is completely determined by its Lusztig data along one minimal path.

Theorem 2.3.8 ([Kam10, Theorem 7.1])**.** *Let $\underline{i}$ be any reduced word of w_0. The Lusztig data with respect to $\underline{i}$ gives a bijection $\mathcal{P} \to \mathbb{N}^m$.*

2.3.3 Dual fan of a GGMS polytope

A GGMS polytope can be characterized by its dual fan in relation to a standard fan, called the Weyl fan. To describe this relationship, first we define fans and dual fans of polytopes.

Let V be a real vector space and let V^* be the dual space. A *polyhedral cone* in V is an finite intersection of closed linear half spaces. A *fan* $\mathcal{F}$ of V^* is a collection of polyhedral cones with the following properties:

(i) Every nonempty face of a cone in $\mathcal{F}$ is also a cone in $\mathcal{F}$,

(ii) The intersection of any two cones in $\mathcal{F}$ is a face of both,

(iii) The union $\bigcup \mathcal{F} = V^*$.

A fan $\mathcal{F}_1$ is a *coarsening* of $\mathcal{F}_2$ if every cone of $\mathcal{F}_1$ is a union of cones in $\mathcal{F}_2$.

Define the Weyl fan $\mathcal{W}$ in $\mathfrak{t}^*_{\mathbb{R}}$ as the fan generated by the maximal cones

$$C^*_w = \{\alpha \in \mathfrak{t}^*_{\mathbb{R}} : \langle w \cdot \alpha_i^\vee, \alpha \rangle \geq 0, \forall i \in I\}.$$

For any convex polytope $P \subset V$, we can define the support function of P as $\psi_P : V^* \to \mathbb{R}$ by $\psi_P(\alpha) = \min_{x \in P} \langle x, \alpha \rangle$. Define the dual fan $\mathcal{N}(P) = \{C^*_F : F \text{ is a face of } P\}$ in V^*, where

$$C^*_F = \{\alpha \in V^* : \langle v, \alpha \rangle = \psi_P(\alpha), \forall v \in F\}$$

Corollary 2.3.9 ([Kam10, Corollary A.4])**.** *A GGMS polytope P is a polytope in $Q^\vee$ whose dual fan $\mathcal{N}(P)$ is a coarsening of the Weyl fan $\mathcal{W}$.*

The dual fan is a useful tool to study the vertices and hyperplanes of P. Maximal cones of the dual fan correspond to vertices of the polytope. If $\mathcal{N}(P)$ is a coarsening of $\mathcal{W}$, then there is an surjection from W to the set of vertices of P; in fact, this surjection determines the choice of labelling on the vertices μ_w.

Additionally, the defining rays of the maximal cones of the dual fan correspond to the codimension 1 faces of P. This correspondence defines a surjection from the chamber weights Γ to the defining rays of the maximal cones of $\mathcal{N}(P)$.

2.4 Crystal structure and the Saito reflection

The set of MV polytopes has a bicrystal structure and hence a reflection of the crystal will result in an action on the set of MV polytopes. First, we define a crystal structure as in [Kas95, Section 7.2].

Definition 2.4.1. A *crystal* is a set B along with the maps

$$\mathrm{wt} : B \to X_*, \quad \tilde{e}_i : B \to B \sqcup \{0\} \quad \tilde{f}_i : B \to B \sqcup \{0\}, \quad \varepsilon_i : B \to \mathbb{Z} \cup \{-\infty\}, \quad \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$$

for each $i \in I$ with the following axioms:

1) For all $b \in B, i \in I$, $\varphi_i(b) = \varepsilon_i(b) + \langle \mathrm{wt}(b), \alpha_i \rangle$

2) If $b \in B, i \in I$ and $\tilde{e}_i(b) \neq 0$, then

$$\mathrm{wt}(\tilde{e}_i(b)) = \mathrm{wt}(b) + \alpha_i^\vee, \qquad \varepsilon_i(\tilde{e}_i(b)) = \varepsilon_i(b) - 1, \qquad \varphi_i(e_i(b)) = \varphi_i(b) + 1$$

3) If $b \in B, i \in I$ and $\tilde{f}_i(b) \neq 0$, then

$$\mathrm{wt}(\tilde{f}_i(b)) = \mathrm{wt}(b) - \alpha_i^\vee, \qquad \varepsilon_i(\tilde{f}_i(b)) = \varepsilon_i(b) + 1, \qquad \varphi_i(f_i(b)) = \varphi(b) - 1$$

4) $b' = \tilde{e}_i(b) \iff f_i(b') = b$

A *highest weight crystal* has a unique element b_0 such that b_0 can be obtained by any element $b \in B$ by applying a sequence of $\tilde{e}_i$ for different $i \in I$.

In particular, we are interested in the crystal $B(\infty)$. This is the highest weight crystal determined by the relations $\mathrm{wt}(b_0) = 0$ and $\varepsilon_i(b) = \max\{n : \tilde{e}_i^n b \neq 0\}$.

Let $*$ denote Kashiwara's involution on $B(\infty)$ [Kas93]. Define $\tilde{e}_i^* = * \circ \tilde{e}_i \circ *$, $\tilde{f}_i^* = * \circ \tilde{f}_i \circ *$, $\varepsilon_i^*(b) = \varepsilon(*b)$ and $\varphi_i^* = \varphi_i(*b)$ for every $i \in I, b \in B(\infty)$. Then $(B(\infty), \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \tilde{e}_i^*, \tilde{f}_i^*)$ is also a crystal. We call $B(\infty)$ a *bicrystal* with these two crystal structures where the weight functions agree and $\mathrm{wt}(b) \in -Q_+$ for every $b \in B(\infty)$. In [Kam07], Kamnitzer defines the bicrystal structure on the set of MV polytopes and proves that this structure is isomorphic to the $B(\infty)$ bicrystal.

Theorem 2.4.2 ([Kam07, Theorem 6.2, Corollary 6.3])**.** *Let P be an MV polytope with vertex data $(\mu_\bullet)$.*

1) $\tilde{f}_j(P)$ is the unique MV polytope with vertex data $(\mu'_\bullet)$ where

$$\mu'_e = \mu_e \text{ and } \mu'_w = \mu_w + \alpha_j^\vee \text{ if } s_j w < w.$$

2) $\tilde{e}_j(P) = 0 \iff \mu_e = \mu_{s_j}$. Otherwise, $\tilde{e}_j(P)$ is the unique MV polytope with vertex data $(\mu'_\bullet)$ where

$$\mu'_e = \mu_e \text{ and } \mu'_w = \mu_w - \alpha_j^\vee \text{ if } s_j w < w.$$

3) $\tilde{f}_j^*(P)$ is the unique MV polytope with vertex data $(\mu'_\bullet)$ where

$$\mu'_{w_0} = \mu_{w_0} + \alpha_j^\vee \text{ and } \mu'_w = \mu_w \text{ for } s_j w > w.$$

4) $\tilde{e}_j^*(P) = 0 \iff \mu_{w_0 s_j} = \mu_{w_0}$. Otherwise, $\tilde{e}_j^*(P)$ is the unique MV polytope with vertex data $(\mu'_\bullet)$ where

$$\mu'_{w_0} = \mu_{w_0} - \alpha_j^\vee \text{ and } \mu'_w = \mu_w \text{ for } s_j w > w.$$

Example 2.4.3. When $G = SL_3$, consider the polytope P given by the Lusztig data $n_\bullet^{(1,2,1)}(P) = (1, 0, 2)$ and $n_\bullet^{(2,1,2)}(P) = (1, 1, 0)$. Then the crystal operators act as follows:

$$n_\bullet^{(1,2,1)}(\tilde{f}_1(P)) = (2, 0, 2), \qquad n_\bullet^{(1,2,1)}(\tilde{e}_1(P)) = (0, 0, 2),$$

$$n_\bullet^{(1,2,1)}(\tilde{f}_2^*(P)) = (1,0,3), \qquad n_\bullet^{(1,2,1)}(\tilde{e}_2^*(P)) = (1,0,1)$$
$$n_\bullet^{(2,1,2)}(\tilde{f}_2(P)) = (2,1,0), \qquad n_\bullet^{(2,1,2)}(\tilde{e}_2(P)) = (0,1,0),$$
$$n_\bullet^{(2,1,2)}(\tilde{f}_1^*(P)) = (1,1,1), \qquad n_\bullet^{(2,1,2)}(\tilde{e}_1^*(P)) = 0.$$

Note that $\tilde{f}_2(P) = \tilde{f}_2^*(P)$ and $\tilde{e}_2(P) = \tilde{e}_2^*(P)$. To see how these operators act on P, see Figure 2.2.

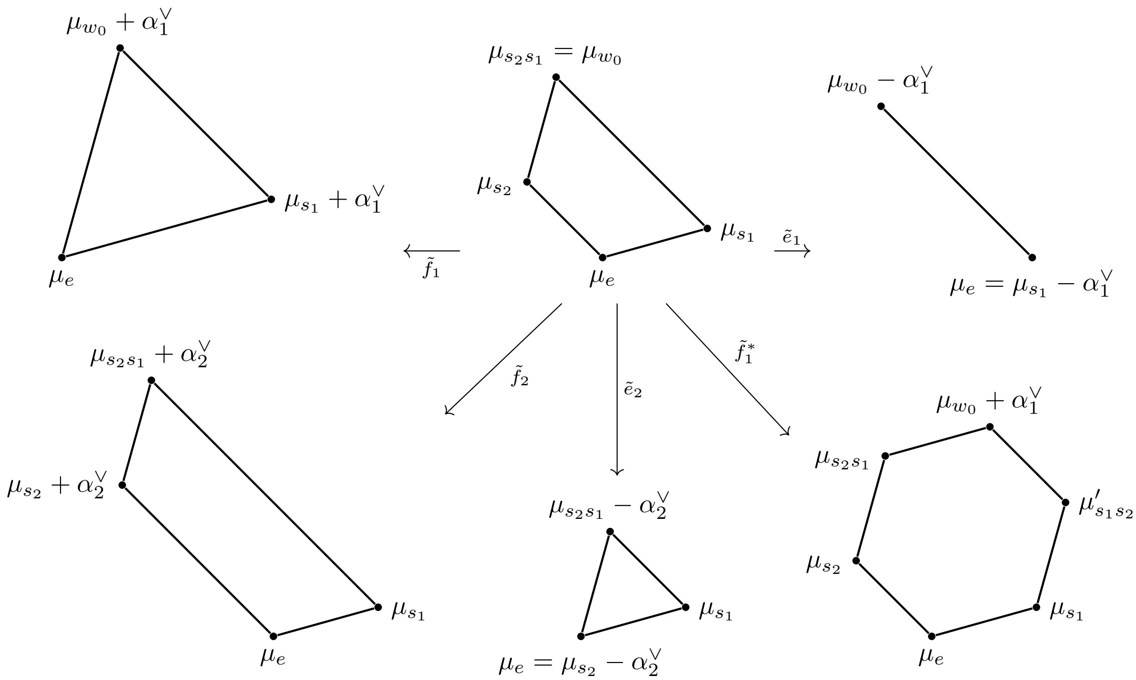


Figure 2.2: The crystal action on the polytope P in Example 2.4.3 with vertex data $(\mu_\bullet)$

For each j, define j^* to be the index such that $s_{j^*}w_0 = w_0 s_j$. Note that for the reduced word $\underline{i} = (i_1, \ldots, i_m)$ of w_0, $s_{i_1} \cdots s_{i_{m-1}} \alpha_{i_m} = \alpha_{i_m^*}$ so that $f_{i_m^*}^*$ and $e_{i_m^*}^*$ will change the last component of the Lusztig data with respect to $\underline{i}$.

We can explicitly see how these operators act on the Lusztig data of a polytope. Suppose $n_\bullet^{\underline{i}}(P)$ is the Lusztig data with respect to $\underline{i}$ for a polytope $P \in \mathcal{P}$. Then

$$n_\bullet^{\underline{i}}(\tilde{f}_{i_1}(P)) = (n_1+1, n_2, \ldots, n_m), \qquad n_\bullet^{\underline{i}}(\tilde{e}_{i_1}(P)) = (n_1-1, n_2, \ldots, n_m),$$
$$n_\bullet^{\underline{i}}(\tilde{f}_{i_m^*}^*(P)) = (n_1, \ldots, n_{m-1}, n_m+1), \qquad n_\bullet^{\underline{i}}(\tilde{e}_{i_m^*}^*(P)) = (n_1, \ldots, n_{m-1}, n_m-1).$$

The value of the crystal operators ε_i can be easily determined by the Lusztig data.

Corollary 2.4.4. *For a reduced word $\underline{i} = (i_1, \ldots, i_m)$ of w_0 and $P \in \mathcal{P}$, if P has Lusztig data $n_\bullet^{\underline{i}}$, then $\varepsilon_{i_1}(P) = n_1^{\underline{i}}$ and $\varepsilon_{i_m^*}^*(P) = n_m^{\underline{i}}$.*

Theorem 2.4.2 associates a unique MV polytope $\mathrm{Pol}(b)$ to each $b \in B(\infty)$, where $\mathrm{Pol}(b_0)$ is the polytope consisting of the point μ_e. We can also use the Saito reflection on the bicrystal $B(\infty)$ to describe the polytope $\mathrm{Pol}(b)$. For the rest of the subsection, we follow [BKK21, Section 3.3].

Definition 2.4.5. Define the map $\tilde{\sigma}_i : \{b \in B(\infty) : \varepsilon_i(b) = 0\} \to \{b \in B(\infty) : \varepsilon_i^*(b) = 0\}$ by $\tilde{\sigma}_i(b) = \left(\tilde{f}_i\right)^{\varphi_i^*(b)} (\tilde{e}_i^*)^{\varepsilon_i^*(b)} (b)$. The *Saito reflection* is the map

$$\sigma_i : B(\infty) \to \{b \in B(\infty) : \varepsilon_i^*(b) = 0\}$$

defined by $\sigma_i(b) = \tilde{\sigma}_i((\tilde{e}_i)^{\varepsilon_i(b)} b)$.

Similarly, define $\tilde{\sigma}_i^* : \{b \in B(\infty) : \varepsilon_i^*(b) = 0\} \to \{b \in B(\infty) : \varepsilon_i(b) = 0\}$ by $\tilde{\sigma_i}^*(b) = \left(\tilde{f}_i^*\right)^{\varphi_i(b)} (\tilde{e}_i)^{\varepsilon_i(b)} (b)$. Define the $*$-*Saito reflection* as the map

$$\sigma_i^* : B(\infty) \to \{b \in B(\infty) : \varepsilon_i(b) = 0\}$$

defined by $\sigma_i^*(b) = \tilde{\sigma}_i^* \left((\tilde{e}_i^*)^{\varepsilon_i^*(b)} b\right)$.

Note that by definition, $\varepsilon_i^*(\sigma_i(b)) = 0$ and $\varepsilon_i(\sigma_i^*(b)) = 0$. Also, $\tilde{\sigma}_i^* = \tilde{\sigma}_i^{-1}$ by [Sai94, Corollary 3.4.8]. The operators σ_i, σ_i^* satisfy the same braid relations as the simple reflections s_i thus for any $w \in W$, it is well defined to set $\sigma_w := \sigma_{i_1} \cdots \sigma_{i_m}$ where $\underline{i}$ is a reduced word of w.

Lemma 2.4.6 ([Sai94, Proposition 3.4.7], [BK12, Property (L3)])**.** *Let $b \in B(\infty)$ and let $n_\bullet^{\underline{i}}$ be the Lusztig data of* Pol(b) *with respect to $\underline{i} = (i_1, \ldots, i_m)$. Consider $\underline{j} = (i_2, \ldots, i_m, i_1^*)$. Then*

$$n_\bullet^{\underline{j}}(\mathrm{Pol}(\sigma_{i_1}(b))) = (n_2, n_3, \ldots, n_m, 0)$$

Consider $\underline{k} = (i_m^*, i_1, i_2, \ldots, i_{m-1})$. Then

$$n_\bullet^{\underline{k}}(\mathrm{Pol}(\sigma_{i_m^*}^*(b)) = (0, n_1, \ldots, n_{m-1})$$

Using this lemma, the Lusztig data can be computed by composing crystal operators with certain Saito reflections.

Corollary 2.4.7. *For $b \in B(\infty)$ suppose* Pol(b) *has vertex data $(\mu_\bullet)$. Then for every $w \in W$ and $j \in I$ such that $ws_j > w$,*

$$\mu_{ws_j} - \mu_w = \varepsilon_j(\sigma_{w^{-1}}(b)) w\alpha_j^\vee$$

Proof. Consider Pol(b) with vertex data $(\mu_\bullet)$. For $w \in W$ and $j \in I$ such that $ws_j > w$, there is a reduced word $\underline{i}$ such that $s_{i_1} \cdots s_{i_{\ell(w)}} = w$ and $s_{i_{\ell w+1}} = s_j$. By definition, $\mu_{ws_j} - \mu_w = n_{\ell(w)+1}(\mathrm{Pol}(b)) \cdot w\alpha_j^\vee$, where $n_\bullet^{\underline{i}}(P) = (n_1, \ldots, n_m)$.

Recall that $\sigma_{w^{-1}} = \sigma_{s_{i_{\ell(w)}}} \sigma_{s_{i_{\ell(w)-1}}} \cdots \sigma_{s_{i_2}} \sigma_{s_{i_1}}$. By Lemma 2.4.6, then Pol$(\sigma_{w^{-1}}(b))$ has Lusztig data $(n_{\ell(w)+1}, \ldots, n_m, 0, \ldots, 0)$ with respect to the reduced word $(j, i_{\ell(w)+2}, \ldots, i_m, i_1^*, \ldots, i_{\ell(w)}^*)$. By Corollary 2.4.4, $\varepsilon_j(\sigma_{w^{-1}}(b)) = \varepsilon_j(\mathrm{Pol}(\sigma_{w^{-1}}(b)) = n_{\ell(w)+1}(\mathrm{Pol}(b))$. □

This corollary allows us to write $\mu_w(b)$ in a closed form. Note that the map $\tilde{\sigma}_i$ has the property that $\mathrm{wt}(\tilde{\sigma}_i(b)) = s_i \mathrm{wt}(b)$ so it follows that $\mathrm{wt}(\sigma_i(b)) = s_i(\mathrm{wt}(e_i^{*\varepsilon_i^*(b)}(b))) = s_i(\mathrm{wt}(b) + \varepsilon_i(b)\alpha_i^\vee)$. For non-trivial $w = s_{i_1} \cdots s_{i_m}$, by inductively applying this equality we have

$$\mathrm{wt}(\sigma_{w^{-1}}(b)) = \sum_{k=1}^{m} \varepsilon_{i_k}(\sigma_{s_{i_m} \cdots s_{i_k}}(b)) s_{i_k} \cdots s_{i_m} \alpha_{i_k}^\vee.$$

As $\mu_e = 0$, it follows from Corollary 2.4.7 that

$$\mu_{s_{i_1}\cdots s_{i_m}}(b) = \sum_{k=1}^{m} \mu_{s_{i_1}\cdots s_{i_k}} - \mu_{s_{i_1}\cdots s_{i_{k-1}}} = \sum_{k=1}^{m} \varepsilon_{i_k}(\sigma_{s_{i_m}\cdots s_{i_{k-1}}}(b)) s_{i_1}\cdots s_{i_{k-1}}\alpha_{i_k}^{\vee} = w \cdot \mathrm{wt}(\sigma_{w^{-1}}(b)).$$

Thus the vertex data $(\mu_\bullet(b))$ of $\mathrm{Pol}(b)$ can be explicitly determined by the Saito reflection where

$$\mu_w(b) = w \cdot \mathrm{wt}(\sigma_{w^{-1}}(b)) - \mathrm{wt}(b). \tag{2.5}$$

Note that we shift by $\mathrm{wt}(b)$ so that $\mu_e(b) = \mathrm{wt}(b) - \mathrm{wt}(b) = 0$.

2.5 Preprojective algebra modules and Dynkin diagram folding

We give a very brief background on preprojective algebra modules and the associated MV polytope. This section is needed to prove the *generalized diagonal relations* of Section 3.1.2.

2.5.1 Preprojective modules

We will define the preprojective algebra associated to a simply-laced algebraic group. First, we start with some general definitions.

Definition 2.5.1. A *quiver* $Q = (I, E, s, t)$ consists of a vertex set I, an arrow set E, a source map $s : E \to I$ and a target map $t : E \to I$. We write the arrow $\alpha \in E$ as $\alpha : i \to j$, where $i = s(\alpha)$ and $j = t(\alpha)$.

Define $E^* = \{\alpha^* : \alpha \in E\}$ where $s(\alpha^*) = t(\alpha)$ and $t(\alpha^*) = s(\alpha)$. Let $\overline{Q} = (I, E \sqcup E^*, s, t)$ be the double quiver.

The *path algebra* of $\overline{Q}$ over $\mathbb{C}$ is the algebra $\mathbb{C}\overline{Q}$. Consider the ideal J generated by $\sum_{\alpha\in E}(\alpha\alpha^* - \alpha^*\alpha)$.

Definition 2.5.2. The *preprojective* of $\overline{Q}$ over $\mathbb{C}$, denoted by $\Lambda(Q)$, is the quotient of $\mathbb{C}\overline{Q}$ by the ideal J.

A $\Lambda(Q)$-module M is an I-graded vector space $\bigoplus_{i\in I} M_i$ with maps $M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ for each $\alpha \in E \sqcup E^*$ which satisfy

$$\sum_{\alpha\in E,\, t(\alpha)=i} M_\alpha M_{\alpha^*} - M_{\alpha^*} M_\alpha = 0$$

for each $i \in I$.

We consider a few special $\Lambda(Q)$-modules. For $i \in I$, let S_i be the 1-dimensional module concentrated at the vertex i, where all arrows act as zero. Let I_i be the annihilator of S_i. For any $w \in W$ we define $I_w := I_{i_1}\cdots I_{i_m}$, where $\underline{i}$ is a reduced word of w. Note that this is independent of the choice of $\underline{i}$ and thus is well-defined.

For a module M, the i-socle is the largest submodule of M which is isomorphic to $S_i^{\oplus k}$ for some $k \in \mathbb{N}$, while the i-head is the largest quotient of M which is isomorphic to $S_i^{\oplus k}$ for some $k \in \mathbb{N}$. In fact, $\mathrm{soc}_i M \cong \mathrm{Hom}_{\Lambda(Q)}(\Lambda(Q)/I_i, M)$ and $\mathrm{hd}_i M \cong (\Lambda(Q)/I_i) \otimes_{\Lambda(Q)} M$.

Let G be a simply-laced complex algebraic group. Fix Q to be an orientation of the Dynkin diagram associated to the simple coroots of G and set $\Lambda := \Lambda(Q)$. For M a Λ-module, we can define *dimension vector* as $\underline{\dim} M = \sum_{i \in I} \dim M_i \alpha_i^\vee$, which is contained in the coroot lattice $Q^\vee$. By [BK12], we can associated a GGMS polytope to a Λ-module M by

$$\mathrm{Pol}(M) := \mathrm{conv}\{\underline{\dim} M - \underline{\dim} N : N \subset M \text{ is a submodule}\}.$$

By [BKT14, Theorem 5.4], for any $w \in W$ we define the submodules $M^w \subseteq M$ as the image of the map $I_w \otimes_\Lambda \mathrm{Hom}_\Lambda(I_w, M) \to M$. By Remark 5.19 (i), $\mathrm{Pol}(M)$ will have vertex data $(\mu_w)_{w \in W}$ where $\mu_w = \underline{\dim} M - \underline{\dim} M^w$. For certain modules M, M^w and $M^{s_i w}$ are closely related.

Lemma 2.5.3 ([BGK12, Lemma 2])**.** *For $w \in W$, consider $i \in I$ such that $s_i w > w$. For M a finite-dimensional Λ-module, if $\mathrm{Ext}^1_\Lambda(S_i, M) = 0$ then $M^{s_i w} \cong I_i \otimes_\Lambda M^w$.*

Finally, a result of Crawley-Boevey tells us that we can switch the rolls of S_i and M in the previous lemma.

Lemma 2.5.4 ([CB00, Lemma 1])**.** *For any Λ-modules X, Y, $\dim \mathrm{Ext}^1_\Lambda(X, Y) = \dim \mathrm{Ext}^1_\Lambda(Y, X)$.*

Finally, we define the subset of Λ-modules $\mathscr{T}^w$.

Definition 2.5.5. Let $\mathscr{T}^w$ to be the set of Λ-modules M such that $M^w = M$.

By [BKT14, Remark 5.5 (ii)], this is the same category $\mathscr{T}^w$ defined in [BKT14] and is also the category of modules $\mathcal{C}_{w^{-1} w_0}$ defined in [Mén22, Definition 2.5]. Note that for $M \in \mathscr{T}^w$, the vertex data of $\mathrm{Pol}(M)$ will satisfy $\mu_w = \mu_e$.

2.5.2 Dynkin diagram folding

To extend a result from the simply-laced case to the non-simply-laced case, we use the technique of folding. We will follow the notation used in [JS17].

Let G be a simply-laced algebraic group. Consider a bijection $\sigma : I \to I$ with $a_{ij} = a_{\sigma(i)\sigma(j)}$. This induces a *Dynkin diagram automorphism* on G by $\sigma : G \to G$ such that $\sigma(x_{\pm i}(a)) = x_{\pm \sigma(i)}(a)$. Let G^σ be the fixed point group on G and call the pair (G, G^σ) a *symmetric pair.*

Example 2.5.6. With the appropriate choice of σ, we have the symmetric pairs (A_{2k-1}, C_k), (D_{k+1}, B_k), (D_4, G_2), and (E_6, F_4)

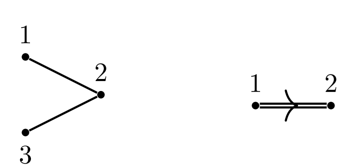


Figure 2.3: The A_3 and C_2 Dynkin diagrams of the simple coroots

Example 2.5.7. Using the labelling in Figure 2.3, the bijection $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$ is defined by $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 1$. This will induce the symmetric pair (A_3, C_2).

A Dynkin diagram automorphism σ induces an action on the weight and coweight lattices by $\sigma(\alpha_i) = \alpha_{\sigma(i)}$ and $\sigma(\alpha_i^\vee) = \alpha_{\sigma(i)}^\vee$. It also induces a group automorphism on W by $\sigma(s_i) = s_{\sigma(i)}$. We set $W^\sigma := \{w \in W : \sigma(w) = w\}$.

Denote by $\bar{\mathfrak{g}}$ to be the Lie algebra of G^σ. Let $\overline{W}$ be the Weyl Group of $\bar{\mathfrak{g}}$, generated by simple reflections $\bar{s}_i$. There is a group isomorphism $\Theta : \overline{W} \to W^\sigma$ defined by

$$\Theta(\bar{s}_i) = s_i^\sigma := \prod_{t=0}^{k_i-1} s_{\sigma^t(i)}$$

where k_i is the number of elements in the σ-orbit of i.

Example 2.5.8. Continuing Example 2.5.7, the Weyl group of A_3 is given by the presentation $\langle s_1, s_2, s_3 : (s_1 s_2)^3 = 1, (s_1 s_3)^2 = 1, (s_2 s_3)^3 = 1, s_1^2 = s_2^2 = s_3^2 = 1\rangle$, while $\overline{W} = \langle \bar{s}_1, \bar{s}_2 : (\bar{s}_1 \bar{s}_2)^4 = 1, \bar{s}_1^2 = \bar{s}_2^2 = 1\rangle$. The map Θ is given by:

$$\Theta(\bar{s}_1) = s_1 s_3, \qquad\qquad \Theta(\bar{s}_2) = s_2.$$

Now, we will consider the σ-invariant MV polytopes of G. Denote $\mathcal{P}$ to be the set of MV polytopes for $\mathfrak{g}$. The diagram automorphism σ induces an action on $\mathcal{P}$ by

$$\sigma(P) := \text{conv}\{\sigma^{-1}(\mu_{\sigma(w)}) : w \in W\}.$$

If $\sigma(P) = P$, we call P *σ-invariant.* Denote the set of σ-invariant MV polytopes by $\mathcal{P}^\sigma$ and let $\overline{\mathcal{P}}$ be the set of MV polytopes for $\bar{\mathfrak{g}}$. There is an identification between these two sets of polytopes.

Theorem 2.5.9 ([Hon09, Theorem 3.10], [JS17, Theorem 6.2])**.** *For $P \in \mathcal{P}^\sigma$ with vertex data $(\mu_w)_{w\in W}$, define $\Phi(P) = conv\{\overline{\mu}_{\overline{w}} : \overline{w} \in \overline{W}\}$, where $\overline{\mu}_{\overline{w}} := \mu_{\Theta(\overline{w})}$. The map $\Phi : \mathcal{P}^\sigma \to \overline{\mathcal{P}}$ is a bijection.*

Example 2.5.10. Continuing Example 2.5.8, the A_3 polytope on the left side of Figure 2.4 is mapped to the C_2 polytope on the right side of the figure.

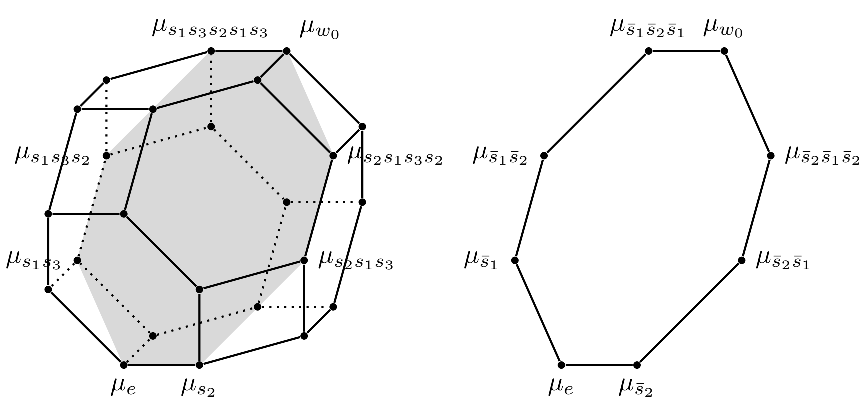


Figure 2.4: A σ-invariant A_3 polytope and the corresponding C_2 polytope

2.6 Tropical geometry

In this section, following [FG09] and [GS15], we outline the basic theory of tropical geometry and describe the correspondence between MV polytopes and the non-negative tropical points of the unipotent subgroup $N \subseteq G$.

2.6.1 Tropical points

On an algebraic torus T, define the tropical points as the cocharacters of T, i.e $T(\mathbb{Z}^{\text{trop}}) = X_*(T)$.

Example 2.6.1. 1) In SL_3, the set of diagonal matrices $D \cong (\mathbb{C}^\times)^2$ is a torus. Then the tropical points of D are given by

$$D(\mathbb{Z}^{\text{trop}}) = \left\{ z \to \begin{bmatrix} z^{n_1} & 0 & 0 \\ 0 & z^{n_2} & 0 \\ 0 & 0 & z^{-n_1-n_2} \end{bmatrix} : n_1, n_2 \in \mathbb{Z} \right\} \cong \mathbb{Z}^2.$$

2) For $T = (\mathbb{C}^\times)^n$, then $T(\mathbb{Z}^{\text{trop}}) = X_*(T) = \text{Hom}(\mathbb{C}^\times, {\mathbb{C}^\times}^n) = \mathbb{Z}^n$.

We recall the concepts of positive spaces and tropical points as in [FG09, Section 1].

Definition 2.6.2. Let X be an irreducible variety. A *positive atlas* on X is a collection of birational isomorphisms $\{\alpha\}_{\alpha \in \mathcal{C}_X}$ over $\mathbb{Q}$ where $\alpha : T \to X$ and T is a split algebraic torus. These coordinate systems satisfy:

(i) Each α is regular on the complement of a positive divisor in T and is given by a positive rational function,

(ii) For any pair α, β of coordinate systems, $\beta^{-1} \circ \alpha$ is a positive birational isomorphism of T.

If X has a positive atlas, we call X a *positive space.*

Using a positive atlas, there is a unique way to define the $\mathbb{Z}$-tropical points of the variety X.

Definition 2.6.3. The tropical points of a positive space X is defined as

$$X(\mathbb{Z}^{\text{trop}}) = \bigsqcup_{\alpha} T(\mathbb{Z}^{\text{trop}})/(\text{identifications } (\beta^{-1} \circ \alpha)^{\text{trop}}).$$

For a subtraction-free function F on T, we can tropicalize it to a function F^{trop} on the tropical points. To see how a tropical function is related to the original function, consider the following example:

$$F(x, y, z) = \frac{xy}{z} + 2z \longmapsto F^{\text{trop}}(x, y, z) = \min\{x + y - z, z\}$$

We will call a function F on X *positive* if it can be written as a subtraction-free expression in the coordinates of a positive atlas of X. We will denote the tropical function by F^{trop}.

2.6.2 MV polytopes as tropical points

For G a reductive complex algebraic group, let T be a torus of G, B a Borel subgroup containing T and N its unipotent subgroup. Consider the map $x_i : \mathbb{C} \to N$ with image in the Chevalley subgroup of α_i by $x_i(a) = \exp(aE_i)$. For the tuple $\underline{i} = (i_1, \dots, i_k)$, we define $x_{\underline{i}}(a_1, \dots, a_k) = x_{i_1}(a_1) \cdots x_{i_k}(a_k)$.

We will show that the variety N is a positive space. For a reduced word $\underline{i}$ of w_0, define the Lusztig parameterization associated to $\underline{i}$ as the map $x_{\underline{i}} : (\mathbb{C}^\times)^m \to N$ by $(a_1, \dots, a_m) \mapsto x_{\underline{i}}(a_1, \dots a_m)$, where $m = \ell(w_0)$. This map is a birational isomorphism by [Lus94] and hence gives a coordinate system on N. In fact, the collection of the charts $(x_{\underline{i}})$ form a positive atlas of N, called *Lusztig's positive atlas* [Lus94]. Thus the tropical points of N are defined and $N(\mathbb{Z}^{\text{trop}}) \cong \mathbb{Z}^m$.

Example 2.6.4. Let $G = SL_3$. Since $w_0 = s_1s_2s_1 = s_2s_1s_2$, we have the two coordinates on N:

$$x_1(a_1)x_2(a_2)x_1(a_3) = \begin{bmatrix} 1 & a_1 + a_3 & a_1a_2 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{bmatrix} \qquad x_2(b_1)x_1(b_2)x_2(b_3) = \begin{bmatrix} 1 & b_2 & b_2b_3 \\ 0 & 1 & b_1 + b_3 \\ 0 & 0 & 1 \end{bmatrix}.$$

The transition maps are subtraction-free:

$$x_{212}^{-1} \circ x_{121}(a_1, a_2, a_3) = \left(\frac{a_2a_3}{a_1 + a_3}, a_1 + a_3, \frac{a_1a_2}{a_1 + a_3} \right).$$

Thus $N(\mathbb{Z}^{\text{trop}}) \cong \mathbb{C}^3(\mathbb{Z}^{\text{trop}}) = \mathbb{Z}^3$.

In Section 2.3.2, we saw that the set of MV polytopes are in bijection with $\mathbb{N}^m$ by fixing a reduced word $\underline{i}$ and considering the Lusztig data of P with respect to $\underline{i}$. We would like to show that the set of MV polytopes of G are in bijection with the non-negative tropical points of N. First, we define a positive function to pick out the "non-negative" points.

Define the *potential function* $\chi : N \to \mathbb{C}$ by

$$\chi(x_{\underline{i}}(a_1, \dots, a_m)) = \sum_{k=1}^{m} a_k.$$

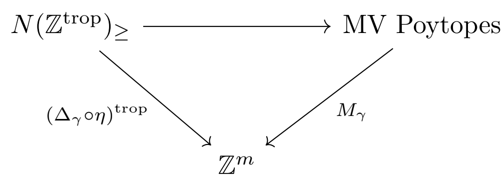
This function is subtraction-free in the Luzstig coordinates $x_{\underline{i}}$. In fact, χ is independent of $\underline{i}$ and thus positive on Lusztig's positive atlas. Hence we have a tropical function χ^{trop} acting on $N(\mathbb{Z}^{\text{trop}})$. Define the non-negative points as

$$N(\mathbb{Z}^{\text{trop}})_{\geq} = \{\ell \in N(\mathbb{Z}^{\text{trop}}) : \chi^{\text{trop}}(\ell) \geq 0\}.$$

Under the correspondence $N(\mathbb{Z}^{\text{trop}}) \cong \mathbb{Z}^m$, we can write $\ell \in N(\mathbb{Z}^{\text{trop}})$ as $\ell = (A_1, \dots, A_m)$ for some $A_i \in \mathbb{Z}$. Then $\chi^{\text{trop}}(\ell) \geq 0 \iff \min\{A_1, \dots, A_m\} \geq 0 \iff A_i \geq 0$ for all i. Thus, $N(\mathbb{Z}^{\text{trop}})_{\geq} \cong \mathbb{N}^m$. By the correspondence between $\mathcal{P} \cong \mathbb{N}^m$, we can find a bijection $N(\mathbb{Z}^{\text{trop}})_{\geq} \cong \mathcal{P}$.

Theorem 2.6.5 ([GS15, Theorem 5.4], [Kam10, Theorem 4.5])**.** *For G a reductive algebraic group, there is a bijection between the non-negative tropical points $N(\mathbb{Z}^{\text{trop}})_{\geq}$ and the set of MV polytopes $\mathcal{P}$.*

As Lusztig's positive atlas consists of $x_{\underline{i}}$ for all reduced words, this bijection is independent of the reduced word used for the Lusztig data in the bijection $\mathcal{P} \to \mathbb{N}^m$. This bijection is also compatible with the hyperplane data in the sense that the following diagram commutes:



As these $\Delta_\gamma \circ \eta$ functions satisfy the Plücker relations, a consequence of this diagram commuting is that the tropical Plücker relations are automatically satisfied by the tropical functions M_γ.

For a generic MV polytope, the highest vertex is labelled by the longest element of the Weyl group, w_0. In this thesis, we will consider MV polytopes where the highest vertex is labelled by w for some $w \in W$. We will prove that Theorem 2.6.5 is also true for this class of polytopes, where N is replaced by a subvariety of N.

Chapter 3

Finite-type MV polytopes of highest vertex w

We define a subset of MV polytopes, called MV polytopes of highest vertex w, and show that there is a bijection between the non-negative tropical points of $L^{w^{-1}}$ and these polytopes.

3.1 Combinatorial data of MV polytopes of highest vertex w

First, we introduce the definition of an MV polytope of highest vertex w.

Definition 3.1.1. Fix $w \in W$. Let P be an MV polytope with vertex data $(\mu_\bullet)$. We say P is an MV polytope of highest vertex at most w if $\mu_w = \mu_{w_0}$. Denote by $\mathcal{P}_w$ the set of MV polytopes of highest vertex w.

Remark 3.1.2. Recall in Section 2.5.1 we define the set of MV polytopes associated to $\mathscr{T}^w$ as the set of Λ-modules M such that $M^w = M$. By [Mén22, Proposition 5.33], $\mathcal{P}_{ww_0}$ is the set of MV polytopes associated to the modules in $\mathscr{T}^w$ (under a reflection by w_0 and a shift to make $\mu_e = 0$).

Example 3.1.3. Consider MV polytopes associated to the group of type B_2. For $w = s_2s_1s_2$, $\mathcal{P}_{s_2s_1s_2}$ is the set of polytopes such that $\mu_{s_2s_1s_2} = \mu_{w_0}$, see Figure 3.1 for an example. This condition will also imply that $\mu_{s_1s_2} = \mu_{s_1s_2s_1}$. In Section 3.1.2 we will explore how the condition $\mu_w = \mu_{w_0}$ affects the vertex data of a rank 2 polytope.

A reduced word $\underline{i}$ for w_0 gives a minimal path in the polytope of P beginning at μ_e and ending at μ_{w_0}. If this path passes through μ_w, then the condition $\mu_w = \mu_{w_0}$ forces the Lusztig data $n_\bullet^{\underline{i}}(P)$ to be zero in the coordinates after $\ell(w)$. More precisely, we can show that every vertex which appears after μ_w in such a minimal path will necessarily be equal to μ_w.

Lemma 3.1.4. *Let P be an MV polytope with vertex data $(\mu_w)_{w\in W}$. For $v, w \in W$, if $w \leq_R v$ then $\mu_v - \mu_w \in Q_+^\vee$.*

Proof. If $w \leq_R v$, then there exists a reduced word $(i_1, \dots, i_{\ell(v)-\ell(w)})$ such that $ws_{i_1}\dots s_{i_{\ell(v)-\ell(w)}} = v$. By (2.4) in Section 2.3,

$$\mu_v - \mu_w = \sum_{k=1}^{\ell(v)-\ell(w)} c_k w s_{i_1}\dots s_{i_{k-1}}\alpha_{i_k}^\vee$$

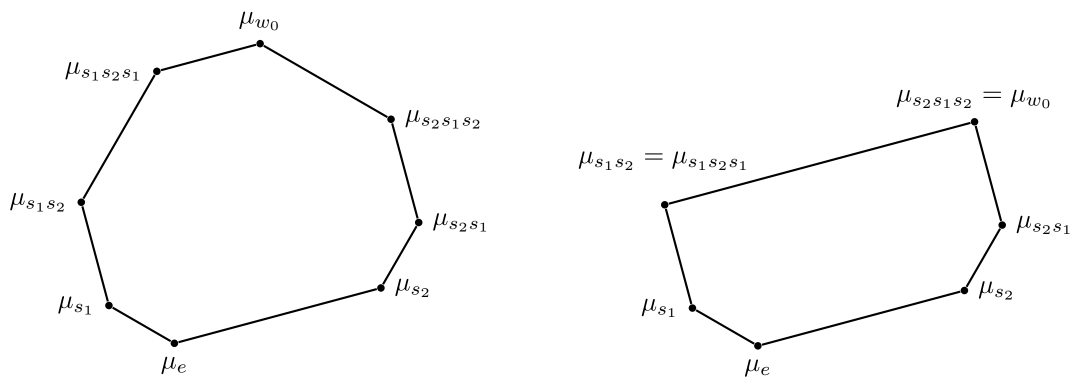


Figure 3.1: A standard B_2 polytope (left) and a B_2 polytope of highest vertex $s_2s_1s_2$ (right)

for coefficients $c_k \geq 0$. Since each $ws_{i_1}\dots s_{i_{k-1}}$ is reduced, $ws_{i_1}\dots s_{i_{k-1}}\alpha_{i_k}^\vee$ is a positive coroot. Thus $\mu_v - \mu_w$ is a non-negative sum of positive coroots so $\mu_v - \mu_w \in Q_+^\vee$. □

Lemma 3.1.5. *Fix $w \in W$ and suppose $P \in \mathcal{P}_w$. For $v \in W$, if $w \leq_R v$, then $\mu_w = \mu_v$.*

Proof. Suppose v is such that $w \leq_R v$. By Lemma 3.1.4, $\mu_v - \mu_w \in Q_+^\vee$ and $\mu_{w_0} - \mu_v \in Q_+^\vee$. Thus

$$0 = \mu_{w_0} - \mu_w = (\mu_{w_0} - \mu_v) + (\mu_v - \mu_w)$$

But the sum of non-zero points in $Q_+^\vee$ is still a non-zero point in $Q_+^\vee$ and hence the only possible values of $\mu_{w_0} - \mu_v$ and $\mu_v - \mu_w$ are zero. Thus $\mu_w = \mu_v = \mu_{w_0}$. □

By the definition of the Lusztig data and its relation to the vertices (see (2.4)), this lemma allows us to characterize $\mathcal{P}_w$ in terms of its Lusztig data with respect to certain reduced words.

Corollary 3.1.6. *Fix $w \in W$. The following conditions are equivalent:*

(i) $P \in \mathcal{P}_w$,

(ii) There exists a reduced word $\underline{i}$ of w_0 with $(i_1, \dots, i_{\ell(w)})$ a reduced word for w such that the Lusztig data $n_\bullet^{\underline{i}}(P)$ has $n_k = 0$ for $k \geq \ell(w)$,

(iii) For every reduced word $\underline{i}$ of w_0 with $(i_1, \dots, i_{\ell(w)})$ a reduced word for w, the Lusztig data $n_\bullet^{\underline{i}}(P)$ has $n_k = 0$ for $k \geq \ell(w)$.

Recall that an MV polytope P is determined by its BZ data $(M_\gamma)_{\gamma\in\Gamma}$. We characterize the BZ data for $P \in \mathcal{P}_w$.

Lemma 3.1.7. *The collection $(M_\gamma)_{\gamma\in\Gamma}$ is the BZ datum of an MV polytope with highest vertex w exactly when*

(i) $(M_\gamma)_{\gamma\in\Gamma}$ is the BZ datum of an MV polytope,

(ii) There exists a reduced word $\underline{j} = (j_1, \dots, j_k)$ of $w^{-1}w_0$ such that for $\ell = 0, \dots, k-1$,

$$M_{w\cdot w_\ell^{\underline{j}} s_{i_{\ell+1}}\omega_{i_{\ell+1}}} + M_{w\cdot w_\ell^{\underline{j}}\omega_{i_{\ell+1}}} = -\sum_{j\neq i_{\ell+1}} a_{j,i_{\ell+1}} M_{w\cdot w_{\ell+1}^{\underline{j}}\omega_j}. \tag{3.1}$$

Proof. Consider $P \in \mathcal{P}$ with vertex data $(\mu_\bullet)$ and BZ data $(M_\bullet)$. The only thing we need to show is that (ii) is equivalent to $\mu_w = \mu_{w_0}$.

Suppose that $\mu_w = \mu_{w_0}$. For any reduced word $\underline{j}$ of $w^{-1}w_0$, $w \leq_R w \cdot w_\ell^{\underline{j}}$ for $0 \leq \ell \leq k$. By Lemma 3.1.5 it follows that $\mu_w = \mu_{ww_1^{\underline{j}}} = \cdots = \mu_{ww_{k-1}^{\underline{j}}} = \mu_{w_0}$. Thus $\mu_{ww_\ell^{\underline{j}}} = \mu_{ww_{\ell+1}^{\underline{j}}}$ for $0 \leq \ell \leq k-1$, which is equivalent to (3.1) by Lemma 2.3.2.

Suppose $\underline{j} = (j_1, \ldots, j_k)$ is a reduced word for $w^{-1}w_0$ such that (ii) holds. As (3.1) is equivalent to $\mu_{ww_\ell^{\underline{j}}} = \mu_{ww_{\ell+1}^{\underline{j}}}$ and this holds for $0 \leq \ell \leq k-1$, then $\mu_w = \mu_{ww_1^{\underline{j}}} = \cdots = \mu_{ww_k^{\underline{j}}} = \mu_{w_0}$ and so $P \in \mathcal{P}_w$. □

Lemma 3.1.7 and Corollary 3.1.6 both only give information about the structure of the polytope $P \in \mathcal{P}_w$ along the minimal paths from μ_e to μ_{w_0} that pass through the vertex μ_w. To understand the whole structure of P, we need to understand the Lusztig data along any minimal path from μ_e to μ_{w_0}.

We will prove that for every $P \in \mathcal{P}_w$ with vertex data $(\mu_\bullet)$, $\mu_v = \mu_{v_w}$ for some well defined element v_w. The proof is organized as follows. In Section 3.1.1, we define this element v_w for $v, w \in W$. In Section 3.1.2, we outline the generalized diagonal relations and see how these relations completely determine the vertex data for rank 2 polytopes. In Section 3.1.3, we show that the Saito reflection acts on $\mathcal{P}_w$ in a useful way and finally, in Section 3.1.4 we will show exactly where the Lusztig data is zero for an arbitrary reduced word of w_0.

3.1.1 Intersections of Bruhat intervals

We use some general properties of Coxeter groups as detailed in Section 2.2. For two tuples, we denote the concatenation by $(x_1, \ldots, x_m) \mathbin{+\!\!+} (y_1, \ldots, y_n) = (x_1, \ldots, x_m, y_1, \ldots, y_n)$. We say that $v \cdot w$ is a reduced product for $v, w \in W$ if $\ell(vw) = \ell(v) + \ell(w)$. Recall that for W, we denote S to be the set of simple reflections while

$$D_R(v) = \{s_i \in S : s_i \leq_L v\}, \qquad D_L(v) = \{s_i \in S : s_i \leq_R v\}.$$

As W is finite, there is a unique longest element w_0. This element has a special property for any reduced decomposition into two elements.

Lemma 3.1.8. *For any $x, y \in W$ such that $w_0 = x \cdot y$ is a reduced product, then $D_R(x) \cap D_L(y) = \emptyset$ and $D_R(x) \cup D_L(w^{-1}w_0) = S$.*

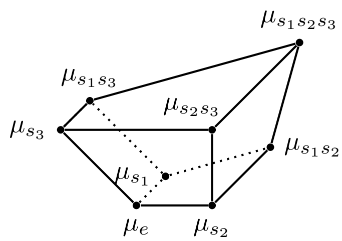
Proof. By the conditions of $\ell(x) + \ell(y) = \ell(w_0)$ and $w_0 = x \cdot y$, then $D_R(x) \cap D_L(y) = \emptyset$.

Suppose there exists $s \notin D_R(x) \cup D_L(y)$. Then $x \cdot s \cdot y$ is an element of length $\ell(w_0) + 1$, which contradicts the maximality of w_0. □

In Section 3.1.4, we will prove that for $P \in \mathcal{P}_w$ with vertex data $(\mu_v)_{v \in W}$, $P = \operatorname{conv}\{\mu_v : v \in W, v \leq w\}$. The main result will be to explicitly describe the map $W \to [e, w]$ which arises by sending $v \mapsto u$ if for every $P \in \mathcal{P}_w$, $\mu_v = \mu_u$. By Lemma 3.1.5, we know that for $v \geq_R w$, $v \mapsto w$. When $v \not\geq_R w$, this map is slightly more complicated.

Example 3.1.9. Consider G of type A_3. The simple coroots are $\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee$ and the Weyl group is given by the presentation $W = \langle s_1, s_2, s_3 : (s_1s_3)^2 = 1, (s_1s_2)^3 = 1, (s_2s_3)^3 = 1, s_1^2 = s_2^2 = s_3^2 = 1\rangle$.

Let $w = s_1s_2s_3$ and consider $P \in \mathcal{P}_w$ with Lusztig data $(1,1,1,0,0,0)$ associated to the reduced word $\underline{i} = (1,2,3,1,2,1)$. This polytope has the following form:



Note that the vertices are indeed labelled by the set $\{v \in W : v \leq w\}$. For $v \in W$ larger than w, the relations on the vertices μ_v are:

$$\mu_{s_2s_3s_1s_2s_1} = \mu_{s_2s_3}, \quad \mu_{s_1s_3s_2s_1} = \mu_{s_1s_3}, \quad \mu_{s_3s_2s_1} = \mu_{s_3}, \quad \mu_{s_1s_2s_1} = \mu_{s_1s_2}, \quad \mu_{s_2s_1} = \mu_{s_2}.$$

Notice that if $\mu_v = \mu_u$ then $u \leq_R v$.

Suppose for $v \in W$, u is the Weyl group element such that $u \leq w$ and $\mu_v = \mu_u$ for every $P \in \mathcal{P}_w$. By examining the previous example, we expect two conditions on u: first, we expect that $u \leq_R v$; equivalently, this says there must be a minimal path from μ_e to μ_{w_0} in the polytope that passes through both the vertices μ_u and μ_v. Second, we expect that u is the longest element such that $u \leq w$ and $u \leq_R v$. First we prove that this element is well-defined.

Lemma 3.1.10. *For every $v, w \in W$, the set $[e,v]_R \cap [e,w]$ has a unique element of longest length.*

Proof. As $[e,v]_R \cap [e,w]$ is a finite set, there exists an element of longest length. Suppose there exists two distinct elements x, y of longest length.

Consider the set $[x,w] \cap [y,w]$. As this set is finite, there exists an element z (not necessarily unique) of minimal length. This element z has the property that for any $z' \in W$ such that $z' \leq z$, $x \leq z'$ and $y \leq z'$, then $\ell(z) = \ell(z')$ by the minimality of z and hence $z = z'$. We apply [BW88, Theorem 3.7] (see Theorem 2.2.8), so $z \leq_R v$ as well. Thus $z \in [e,v]_R \cap [e,w]$ but $\ell(z) > \ell(x) = \ell(y)$, which contradicts that x, y are of longest length. $\square$

Since $x \leq_R v \iff x^{-1} \leq_L v^{-1}$ and $x \leq w \iff x^{-1} \leq w^{-1}$ then there is a bijection $[e,v^{-1}]_R \cap [e,w^{-1}] \to [e,v]_L \cap [e,w]$ by $x \mapsto x^{-1}$. As $\ell(x^{-1}) = \ell(x)$, this lemma also holds for the left Bruhat order.

Corollary 3.1.11. *For every $v, w \in W$, the set $[e,v]_L \cap [e,w]$ has a unique element of longest length.*

Lemma 3.1.10 ensures the following definition is well-defined.

Definition 3.1.12. For any $v, w \in W$, denote v_w to be the unique element of maximal length in $[e,v]_R \cap [e,w]$.

This element is closely related to the Demazure product (see Section 2.2).

Proposition 3.1.13. *Fix $w \in W$. For any $v \in W$, $v_w = (vw_0)((w_0v^{-1}) * w)$, where $*$ is the Demazure product.*

Proof. By Proposition 2.2.13, $(w_0v^{-1})*w = (w_0v^{-1})\cdot x$ where x is the maximal length element such that $x \leq w$ and $(w_0v^{-1})\cdot x$ is reduced. By Lemma 2.2.10, $w_0v^{-1}\cdot x$ is reduced if and only if $x \leq_R v$. Thus x is the maximal length element such that $x \leq w$ and $x \leq_R v$ so by definition, $x = v_w$. □

We could alternatively take $(vw_0)((w_0v^{-1})*w)$ as the definition of v_w and the uniqueness of v_w will be automatic as this product is well-defined. For our purposes, it is useful to use the Bruhat orders to define v_w but this connection to the Demazure product simplifies some of the proofs.

As v_w is an initial word of v, then $v = v_w \cdot (v_w^{-1}v)$ is a reduced product. The next lemma shows how the terminal word $v_w^{-1}v$ of v relates to $w^{-1}w_0$.

Lemma 3.1.14. *Fix $w \in W$. For every $v \in W$, $v_w^{-1}v \leq_R w^{-1}w_0$.*

Proof. Note that $v_w^{-1}v \leq_R w^{-1}w_0 \iff v_w^{-1}vw_0 \geq_R w^{-1} \iff (vw_0)^{-1}v_w \geq_L w$. But $v_w = (vw_0)((vw_0)^{-1}*w)$ so that $(vw_0)^{-1}v_w = (vw_0)^{-1}*w$. A consequence of Proposition 2.2.13 is that $(vw_0)^{-1}v_w = (vw_0)^{-1}*w \geq_L w$ as desired. □

3.1.2 Generalized diagonals

In this section we prove two technical lemmas, which state the *generalized diagonal relations* on MV polytopes. These relations are inspired by the diagonal relations in the rank 2 case, see Chapter 4 for more details. These inequalities are interesting because they relate vertices of the form μ_w and μ_{s_jw} which are vertices that do not necessarily share a face of the polytope (see Figure 3.2). On the other had, the tropical Plücker relations only give relations amongst vertices with a shared face.

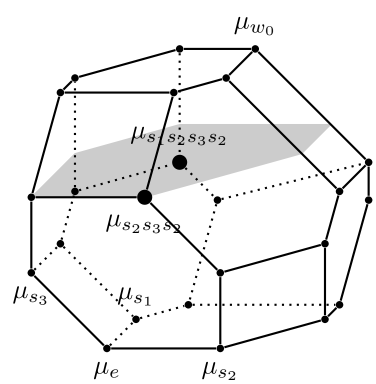


Figure 3.2: A generalized diagonal with strict inequality

The first lemma requires the use of preprojective algebra modules, see Section 2.5.1 for more details. We recall a few definitions. For G a simply-laced complex algebraic group, let Λ be the preprojective algebra associated to the double quiver of an orientation of the Dynkin diagram of the coroots of $\mathfrak{g}$. For a Λ-module M, define the submodules M^w as the image of the map $I_w \otimes_\Lambda \mathrm{Hom}_\Lambda(I_w, M) \to M$. The associated MV polytope is given by

$$\mathrm{Pol}(M) = \mathrm{conv}\{\mu_w : w \in W\}, \text{ where } \mu_w = \underline{\dim} M - \underline{\dim} M^w.$$

The following proof of the simply-laced case is due to Pierre Baumann. We thank him for allowing its inclusion in this text.

Lemma 3.1.15. *Assume G is simply-laced. Let P be an MV polytope with vertex data $(\mu_w)_{w\in W}$. For every $w \in W$, $s_j \in D_L(w)$, the inequality $\langle \mu_w - \mu_{s_jw}, \omega_k\rangle \leq 0$ holds for every $k \neq j$.*

Proof. Let M be the Λ-module associated to the polytope P, i.e. $\underline{\dim}M - \underline{\dim}M^w = \mu_w$ for $w \in W$. We want to prove that $\mu_w - \mu_{s_jw} = \underline{\dim}M^{s_jw} - \underline{\dim}M^w = n\alpha_j^\vee - \beta$ for $n \in \mathbb{N}$ and $\beta \in Q_+^\vee$.

First, suppose that $\mathrm{Ext}^1(M, S_j) = 0$. By [BGK12, Lemma 2], $M^w = I_j \otimes_\Lambda M^{s_jw}$. Consider the short exact sequence

$$0 \to I_j \to \Lambda \to \Lambda/I_j \to 0.$$

As the tensor product is right exact, by applying the functor $\otimes_\Lambda M^{s_jw}$ we get the long exact sequence $\cdots \to I_j \otimes_\Lambda M^{s_jw} \to \Lambda \otimes_\Lambda M^{s_jw} \to (\Lambda/I_j) \otimes_\Lambda M^{s_jw}$. Note that $(\Lambda/I_j) \otimes_\Lambda M^{s_jw} = M^{s_iw}/I_j$; by definition, $M^{s_jw}/I_j = \mathrm{hd}_j(M^{s_jw}) = S_j^{\oplus n}$ for some $n \in \mathbb{N}$. Thus we have the resulting exact sequence

$$0 \to \ker(\phi) \to M^w \xrightarrow{\phi} M^{s_jw} \to S_j^{\oplus n} \to 0,$$

with the dimension vectors $\underline{\dim}M^{s_jw} + \underline{\dim}\ker(\phi) = \underline{\dim}M^w + n\alpha_i$. As $\underline{\dim}\ker(\phi) \in Q_+^\vee$, the claim holds for M of this form.

Suppose M is a general Λ-module. Let N be the maximal extension of M by S_j, i.e. take $m \in \mathbb{N}$ such that

$$0 \to S_j^{\oplus m} \to N \to M \to 0.$$

By the proof of [BGK12, Lemma 2], $0 \to S_j^{\oplus m} \to N^{s_jw} \to M^{s_jw} \to 0$ is exact and thus $\underline{\dim}N^{s_jw} = \underline{\dim}M^{s_jw} + m\alpha_j^\vee$. Also, as the composition $N^w \to N \to M \to M/M^w$ is zero, then $\underline{\dim}N/N^w \geq \underline{\dim}M/M^w$. Thus there exists $\gamma \in Q_+^\vee$ such that $\underline{\dim}N/N^w - \underline{\dim}M/M^w = \gamma$. Then

$$\underline{\dim}M^w - \underline{\dim}N^w = \gamma + \underline{\dim}M - \underline{\dim}N = \gamma - m\alpha_j^\vee$$

Finally, the difference between the dimension vector of M^w and M^{s_jw} is as follows:

$$\begin{aligned}\underline{\dim}M^{s_jw} - \underline{\dim}M^w &= \underline{\dim}N^{s_jw} - \underline{\dim}N^w - (\underline{\dim}M^w - \underline{\dim}N^w) - (\underline{\dim}N^{s_jw} - \underline{\dim}M^{s_jw}) \\ &= n\alpha_j^\vee - \beta - (\gamma - m\alpha_j^\vee) - (m\alpha_j^\vee) = n\alpha_j^\vee - \beta - \gamma\end{aligned}$$

Since $\mu_w = \underline{\dim}M - \underline{\dim}M^w$, then $\mu_w - \mu_{s_jw} = n\alpha_j^\vee - \beta - \gamma$ so $\langle \mu_w - \mu_{s_jw}, \omega_k\rangle = -\langle \beta + \gamma, \omega_k\rangle \leq 0$ for every $k \neq j$. □

Now, we implement the folding technique of Section 2.5.2 to prove the general case.

Lemma 3.1.16. *Assume G is non-simply-laced. Let P be an MV polytope with vertex data $(\mu_w)_{w\in W}$. For every $w \in W$, $s_j \in D_L(w)$, the inequality $\langle \mu_w - \mu_{s_jw}, \omega_k\rangle \leq 0$ holds for every $k \neq j$.*

Proof. Let σ be a Dynkin diagram automorphism of G and let $\mathcal{P}^\sigma$ be the set of σ-invariant MV polytopes of G. Let $\overline{\mathcal{P}}$ be the set of MV polytopes associated to G^σ, the fixed point group of σ. Recall by Theorem 2.5.9, $\Phi : \mathcal{P}^\sigma \to \overline{\mathcal{P}}$ is a bijection, where P with vertex data $(\mu_w)_{w\in W}$ is sent to $\overline{P}$ with vertex data $(\mu_{\Theta(\overline{w})}) = (\overline{\mu}_{\overline{w}})$.

Let $\overline{P} \in \overline{\mathcal{P}}$. Consider $\overline{w} \in \overline{W}$ arbitrary. Let $\bar{s}_j \in D_L(\overline{w})$ and $k \in \overline{I}$ such that $k \neq j$. We want to show that $\langle \overline{\mu}_{\overline{w}} - \overline{\mu}_{\bar{s}_j\overline{w}}, \overline{\omega}_k\rangle \leq 0$.

Since Φ is a bijection, there exists a $P \in \mathcal{P}^\sigma$ such that $\overline{\mu}_{\overline{w}} = \mu_{\Theta(\overline{w})}$. Then for the vertices of $\overline{P}$,

$$\overline{\mu}_{\overline{w}} - \overline{\mu}_{\bar{s}_j \overline{w}} = \mu_{\Theta(\overline{w})} - \mu_{\Theta(\bar{s}_j \overline{w})} = \mu_{\Theta(\overline{w})} - \mu_{s_j^\sigma \Theta(\overline{w})}.$$

Note that s_i^σ depends on the number of elements in the σ-orbit of i, which can only equal $1, 2$ or 3. We consider the case where there are 3 elements in the orbit. Then

$$\begin{aligned}
\mu_{\Theta(\overline{w})} - \mu_{s_j^\sigma \Theta(\overline{w})} &= \mu_{\Theta(\overline{w})} - \mu_{s_{\sigma^2(j)}\Theta(\overline{w})} + \mu_{s_{\sigma^2(j)}\Theta(\overline{w})} - \mu_{s_{\sigma(j)}s_{\sigma^2(j)}\Theta(\overline{w})} \\
&\quad + \mu_{s_{\sigma(j)}s_{\sigma^2(j)}\Theta(\overline{w})} - \mu_{s_j s_{\sigma(j)} s_{\sigma^2(j)}\Theta(\overline{w})} \\
\implies \langle \mu_{\Theta(\overline{w})} - \mu_{s_j^\sigma \Theta(\overline{w})}, \omega_k \rangle &= \langle \mu_{\Theta(\overline{w})} - \mu_{s_{\sigma^2(j)}\Theta(\overline{w})}, \omega_k \rangle + \langle \mu_{s_{\sigma^2(j)}\Theta(\overline{w})} - \mu_{s_{\sigma(j)}s_{\sigma^2(j)}\Theta(\overline{w})}, \omega_k \rangle \\
&\quad + \langle \mu_{s_{\sigma(j)}s_{\sigma^2(j)}\Theta(\overline{w})} - \mu_{s_j s_{\sigma(j)} s_{\sigma^2(j)}\Theta(\overline{w})}, \omega_k \rangle
\end{aligned}$$

Since $k \in \overline{I}$ but $\{j, \sigma(j), \sigma^2(j)\} \cap \overline{I} = \{j\}$, then $k \neq \sigma(j)$ or $\sigma^2(j)$. Thus we can apply the simply-laced case to each term on the right side of the above equation, and hence $\langle \mu_{\Theta(\overline{w})} - \mu_{s_j^\sigma \Theta(\overline{w})}, \omega_k \rangle \leq 0$. As $\overline{\omega}_k$ is the restriction of ω_k to the subspace $\mathfrak{h}^\sigma$, then $\langle \overline{\mu}_{\overline{w}} - \overline{\mu}_{\bar{s}_j \overline{w}}, \overline{\omega}_k \rangle = \langle \mu_{\Theta(\overline{w})} - \mu_{s_i^\sigma \Theta(\overline{w})}, \omega_k \rangle \leq 0$. For the cases with 1 or 2 elements in the σ-orbit, we will simply have fewer terms on the right side of the above equation. ☐

Recall we define $* : I \to I$ where i^* is the index such that $s_i w_0 = w_0 s_{i^*}$. For $w = s_{i_1} \cdots s_{i_m}$, we define $w^* = s_{i_1^*} \cdots s_{i_m^*}$.

Lemma 3.1.17. *Fix $w \in W$. For every $P \in \mathcal{P}_w$, and for every $s_j \in D_L(w)$, $\mu_{s_j w} = \mu_{w_0 s_{j^*}}$.*

Proof. First, as $w \leq_R w_0$ then $s_j w \leq_R s_j w_0 = w_0 s_{j^*}$ so by Lemma 3.1.4, $\mu_{w_0 s_{j^*}} - \mu_{s_j w} \in Q_+^\vee$ and for a reduced word $\underline{i}$ of $w^{-1} w_0$,

$$\mu_{w_0 s_{j^*}} - \mu_{s_j w} = \sum_{r=1}^{\ell(w_0)-\ell(w)-1} c_r (s_j w) s_{i_1} \dots s_{i_{r-1}} \alpha_{i_r}^\vee \tag{3.2}$$

for $c_r \geq 0$ and positive coroots $(s_j w) s_{i_1} \cdots s_{i_{r-1}} \alpha_{i_r}^\vee$. It follows that $\langle \mu_{w_0 s_{j^*}} - \mu_{s_j w}, \omega_k \rangle \geq 0$ for all $k \in I$.

By the generalized diagonal relations, $\langle \mu_w - \mu_{s_j w}, \omega_k \rangle \leq 0$ for $k \neq j$ and hence $\langle \mu_{w_0} - \mu_{s_j w}, \omega_k \rangle \leq 0$ as well. As $(w_0 s_{j^*}) \cdot \alpha_{j^*}^\vee = \alpha_j^\vee$, then $\mu_{w_0} - \mu_{w_0 s_{j^*}} \in \mathbb{Z}\alpha_j^\vee$, so for all $k \neq j$,

$$0 = \langle \mu_{w_0} - \mu_{w_0 s_{j^*}}, \omega_k \rangle = \langle \mu_{w_0} - \mu_{s_j w}, \omega_k \rangle - \langle \mu_{w_0 s_{j^*}} - \mu_{s_j w}, \omega_k \rangle.$$

Both of these terms must be zero for all $k \neq j$, so $\mu_{w_0 s_{j^*}} - \mu_{s_j w} \in \mathbb{Z}\alpha_j^\vee$. As each $(s_j w) s_{i_1} \cdots s_{i_{r-1}} \alpha_{i_r}^\vee$ is a distinct positive coroot for every r and $(s_j w) w^{-1} w_0 = (w_0 s_{j^*}) \alpha_{j^*}^\vee = \alpha_j^\vee$, then it follows that $(s_j w) s_{i_1} \cdots s_{i_{r-1}} \alpha_{i_r}^\vee \neq \alpha_{j^*}^\vee$ for every $r < \ell(w_0) - \ell(w)$. If any $c_r \neq 0$ in (3.2), then $\alpha_j^\vee$ is a positive sum of positive coroots, which contradicts that $\alpha_j^\vee$ is a simple coroot. Thus $c_r = 0$ for all $1 \leq r \leq \ell(w_0) - \ell(w) - 1$ and $\mu_{s_0 s_{j^*}} = \mu_{s_j w}$. ☐

When G is of rank 2, then Lemma 3.1.17 completely determines the vertex data of a polytope in $\mathcal{P}_w$. To see this, consider $w = s_{i_1} s_{i_2} \dots s_{i_m} \in W$. As G has two simple roots, there are only two simple reflections and so w is an alternating product of s_1 and s_2.

The existence of only two simple roots means that MV polytopes are 2-dimensional polygons. The two simple reflections generate two distinct reduced words for w_0: the alternating product $s_1s_2s_1\dots$ of length $\ell(w_0)$ and the alternating product $s_2s_1s_2\dots$ of length $\ell(w_0)$. These two reduced words give two minimals paths from μ_e to μ_{w_0} and correspond to the two sides of the polygon.

For $P \in \mathcal{P}_w$, μ_w is on one side of the polygon and the vertex data for any vertex along this minimal path is described by Lemma 3.1.5, i.e. if $v \geq_R w$, then $\mu_v = \mu_w$, otherwise $v \leq_R w$ and μ_v can be distinct. For $v \not\leq_R w$ and $v \not\geq_R w$, then μ_v is necessarily on the minimal path from μ_e to μ_{w_0} which does not contain μ_w, and hence either $v \leq_R s_{i_1}w$ or $v \geq_R s_{i_1}w$. By Lemma 3.1.17, $D_L(w) = s_{i_1}$ and so $\mu_{s_{i_1}w} = \mu_{w_0s_{i_1^*}}$.

If $v \not\leq w$, then $v \not\leq_R w$ and $v \not\leq_R s_{i_1}w$, so it must follow that either $v \geq_R w$ or $v \geq_R s_{i_1}w$. For the first case, we have already shown $\mu_v = \mu_w$. For the second case, as μ_v is between the vertices $\mu_{s_{i_1}w}$ and $\mu_{w_0s_{i_1^*}}$, the equality $\mu_{s_{i_1}w} = \mu_{w_0s_{i_1^*}}$ forces $\mu_v = \mu_{s_{i_1}w}$. It follows that on the side of the polygon which does not contain μ_w, the highest vertex is labelled by $\mu_{s_{i_1}w}$ and the only possible distinct vertices are labelled by $v \leq w$. Hence $P = \mathrm{conv}\{\mu_v : v \in W, v \leq w\}$.

3.1.3 Crystal action on $\mathcal{P}_w$

In this section, we will show that the Saito reflection behaves well with $\mathcal{P}_w$. First, we briefly recall the crystal structure of MV polytopes and the Saito reflection, see Section 2.4 for more details.

The set of MV polytopes has crystal structure $B(\infty)$. The MV polytope associated to $b \in B(\infty)$, denoted $\mathrm{Pol}(b)$, is given by the vertex data $(\mu_w(b))_{w\in W}$, where

$$\mu_w(b) = w \cdot \mathrm{wt}(\sigma_{w^{-1}}(b)) - \mathrm{wt}(b). \tag{3.3}$$

The action of the Saito reflection on the crystal $B(\infty)$ has a known effect on the Lusztig data of $\mathrm{Pol}(b)$ for $b \in B(\infty)$. If $\mathrm{Pol}(b)$ has Lusztig data $(n_1, \dots n_m)$ associated to the reduced word $(i_1, \dots, i_m)$, then $\mathrm{Pol}(\sigma_{i_1}(b))$ has Lusztig data $(n_2, \dots, n_m, 0)$ associated to the reduced word $(i_2, \dots, i_m, i_1^*)$ while $\mathrm{Pol}(\sigma^*_{i_m}(b))$ has Lusztig data $(0, n_1, \dots, n_{m-1})$ associated to the reduced word $(i_m^*, i_1, \cdots, i_{m-1})$. The crystal operators are also related to the Lusztig data $n_\bullet^{\underline{i}}(P)$ by $\varepsilon_{i_m}(b) = n_m$ and $\varepsilon^*_{i_1^*}(b) = n_1$.

Lemma 3.1.18. *Fix $w \in W$. Let $\underline{i} = (i_1, \dots, i_m)$ be a reduced word for $w^{-1}w_0$. For $b \in B(\infty)$, the following are equivalent:*

(i) $\mathrm{Pol}(b) \in \mathcal{P}_w$,

(ii) $\varepsilon^*_{i_m^*}(b) = 0$ *and* $\varepsilon^*_{i_k^*}(\sigma^*_{s_{i^*_{k+1}}\dots s_{i^*_m}}(b)) = 0$ *for* $1 \leq k < m$,

(iii) $\sigma_{w^{-1}}(b) = b_0$.

Proof. Extend $\underline{i}$ to a reduced word $\underline{i}' = (j_1, \dots, j_{\ell(w_0)-m}, i_1, \dots, i_m)$ of w_0. Denote the Lusztig data of $\mathrm{Pol}(b)$ associated to $\underline{i}'$ by $(n_1, \dots, n_{\ell(w_0)-m}, N_1, \dots, N_m)$. By Corollary 2.4.4, we know that $\varepsilon^*_{i_m^*}(b) = N_m$. For $1 \leq k < m$, consider the polytope $\mathrm{Pol}(\sigma^*_{s_{i^*_{k+1}}\dots s_{i^*_m}}(b))$. By Lemma 2.4.6, this polytope has Lusztig data $(0, \dots, 0, n_1, \dots, n_{\ell(w_0)-m}, N_1, \dots, N_k)$ associated to the reduced word $(i^*_{k+1}, \dots, i^*_m, j_1, \dots, j_{\ell(w_0)-m}, i_1, \dots, i_k)$. Then $\varepsilon^*_{i_k^*}(\sigma^*_{s_{i_{k+1}}\dots s_{i_m}}(b)) = N_k$.

Thus $\varepsilon^*_{i_k^*}(\sigma_{s_{i^*_{k+1}}\dots s_{i^*_m}}(b)) = 0$ for $1 \leq k \leq m$ if and only if $N_k = 0$ for $1 \leq k \leq m$. By Corollary 3.1.6, this is equivalent to $\mathrm{Pol}(b) \in \mathcal{P}_w$ so (i) is equivalent to (ii).

As Pol(b) has vertex data $\mu_v = v \cdot \mathrm{wt}(\sigma_{v^{-1}}(b)) - \mathrm{wt}(b)$ then

$$\mu_w = \mu_{w_0} \iff w \cdot \mathrm{wt}(\sigma_{w^{-1}}(b)) = w_0 \cdot \mathrm{wt}(\sigma_{w_0^{-1}}(b)) \iff w \cdot \mathrm{wt}(\sigma_{w^{-1}}(b)) = 0$$

since $\sigma_{w_0}(b') = b_0$ for every $b' \in B(\infty)$. As b_0 is the unique element of weight zero, then the weight $\mathrm{wt}(\sigma_{w^{-1}}(b)) = 0 \iff \sigma_{w^{-1}}(b) = b_0$. Thus (i) is equivalent to (iii). □

In the next two lemmas, we will show how the Saito reflection $\sigma^*_{j^*}$ acts on $P \in \mathcal{P}_w$.

Corollary 3.1.19. *Fix $w \in W$. For $b \in B(\infty)$, if* Pol(b) $\in \mathcal{P}_w$ *and $s_j \in D_R(w^{-1}w_0)$, then* Pol($\sigma^*_{j^*}(b)$) $\in \mathcal{P}_{s_{j^*}w}$.

Proof. First, notice that the condition $s_j \in D_R(w^{-1}w_0)$ ensures there exists a reduced word $\underline{i} = (i_1, \dots, i_{\ell(w)}, k_1, \dots, k_{m-\ell(w)-1}, j)$ of w_0 such that $(i_1, \dots, i_{\ell(w)})$ is a reduced word of w and $(k_1, \dots, k_{m-\ell(w)-1}, j)$ is a reduced word of $w^{-1}w_0$.

Suppose that Pol(b) $\in \mathcal{P}_w$. By Corollary 3.1.6, the Lusztig data of Pol(b) with respect to $\underline{i}$ is $(n_1, \dots, n_{\ell(w)}, 0, \dots, 0)$. By Lemma 2.4.6, Pol($\sigma^*_{j^*}(b)$) has Lusztig data $(0, n_1, \dots, n_{\ell(w)}, 0, \dots, 0)$ with respect to the reduced word $(j^*, i_1, \dots, i_{\ell(w)}, k_1, \dots, k_{m-\ell(w)-1})$. Hence Pol($\sigma^*_{j^*}(b)$) $\in \mathcal{P}_{s^*_j w}$ by Corollary 3.1.6. □

Lemma 3.1.20. *Fix $w \in W$. For $b \in B(\infty)$, if* Pol(b) $\in \mathcal{P}_w$ *and $s_j \notin D_R(w^{-1}w_0)$, then* Pol($\sigma^*_{j^*}(b)$) $\in \mathcal{P}_w$.

Proof. As $w_0 = w \cdot w^{-1}w_0$ is reduced, we also have the reduced product $w_0 = (w^{-1}w_0) \cdot w^*$. By setting $x = w^{-1}w_0$ and $y = w^*$, we can apply Lemma 3.1.8 so that $D_R(w^{-1}w_0) \cap D_L(w^*) = \emptyset$ and $D_R(w^{-1}w_0) \cup D_L(w^*) = S$. Thus $s_j \notin D_R(w^{-1}w_0) \iff s_{j^*} \in D_L(w)$.

Consider $b \in B(\infty)$ such that Pol(b) $\in \mathcal{P}_w$. Let $s_{j^*} \in D_L(w)$. By Lemma 3.1.18, to show Pol($\sigma^*_{j^*}(b)$) $\in \mathcal{P}_w$, it is enough to show that $\sigma_{w^{-1}}(\sigma^*_{j^*}(b)) = b_0$.

Let $\underline{i} = (i_1, \dots, i_{m-1}, j)$ be a reduced word of w_0 such that $w = s_{j^*} s_{i_1} \cdots s_{i_k}$ for $k = \ell(w) - 1$. Let $(n_1, \dots, n_{m-1}, n_m)$ be the Lusztig data of Pol(b) with respect to $\underline{i}$. The polytope Pol$\left(\sigma^*_{j^*}(b)\right)$ has Lusztig data $(0, n_1, \cdots, n_{m-1})$ for reduced word $(j^*, i_1, \dots, i_{m-1})$ so that Pol$\left(\sigma_{j^*}\sigma^*_{j^*}(b)\right)$ has Lusztig data $(n_1, \dots, n_{m-1}, 0)$ with respect to $\underline{i}$. Notice that

$$\sigma_{w^{-1}}(\sigma^*_{j^*}(b)) = \sigma_{(s_{j^*}w)^{-1}}(\sigma_{j^*}\sigma^*_j(b)) = \sigma_{s_{i_k}\cdots s_{i_2}s_{i_1}}(\sigma_{j^*}\sigma^*_j(b))$$

so that Pol$\left(\sigma_{w^{-1}}(\sigma^*_{j^*}(b))\right)$ has Lusztig data $(n_{k+1}, \dots, n_{m-1}, 0, \dots, 0)$ with respect to the reduced word $(i_{k+1}, \dots, i_{m-1}, j, i^*_1, \dots, i^*_k)$.

Since Pol(b) $\in \mathcal{P}_w$, then $\mu_{s_{i_1}\cdots s_{i_k}} = \mu_{s_{j^*}w} = \mu_{w_0 s_j}$ by by the generalized diagonal relations of Lemma 3.1.17. The relation between Lusztig data and vertices (see (2.4)) implies that $n_{k+1} = \cdots n_{m-1} = 0$ in this Lusztig data. Thus Pol$\left(\sigma_{w^{-1}}(\sigma^*_{j^*}(b))\right) =$ Pol(b_0). By the uniqueness of Pol(b), $\sigma_{w^{-1}}(\sigma^*_{j^*}(b)) = b_0$ as desired. □

3.1.4 Lusztig and vertex data of $\mathcal{P}_w$

The goal of this section is to show that for any $P \in \mathcal{P}_w$, $\mu_v = \mu_{v_w}$ for every $v \in W$. First, we need to investigate where the zeros in the Lusztig data are located.

Let $\underline{i} = (i_1, \ldots, i_m)$ be a tuple. Consider two subwords of $\underline{i}$, $\underline{a} = (i_{a_1}, \ldots, i_{a_k})$ and $\underline{b} = (i_{b_1} \ldots, i_{b_k})$. We say the subword $\underline{a}$ comes after $\underline{b}$ in the reverse-lexicographical order if for some n, $a_n < b_n$ and $a_j = b_j$ for every $j \geq n$.

Definition 3.1.21. Let $\underline{i}$ be a reduced word of w_0. For $w \in W$, define the **rightmost** subword $\underline{i}^w$ as the first subword in the reverse-lexicographical ordering that is a reduced word of w.

The next two lemmas will show that this rightmost word for $w^{-1}w_0$ will always start with a reduced word for $v_w^{-1}v$.

Lemma 3.1.22. *Fix $w \in W$. Let $\underline{i} = (i_1, \ldots, i_m)$ be a reduced word for w_0 and let $i^w = (i_{j_1}, \ldots, i_{j_{\ell(w)}})$.*

For any terminal subword $\underline{i}' = (i_k, i_{k+1}, \ldots, i_m)$ of $\underline{i}$, the subword of $\underline{i}$ indexed by the intersection $\{k, k+1, \ldots, m\} \cap \{j_1, \ldots, j_{\ell(w)}\}$ is a reduced word for the maximal length element in $[e, w]_L \cap [e, s_{i_k} \cdots s_{i_m}]$.

Proof. We proceed by induction on $m + 1 - k$, the length of the terminal subword $\underline{i}'$.

Suppose $k = m$. If $[e, w]_L \cap [e, s_{i_m}] \neq \{e\}$, then the maximal element in this set is s_{i_m}. Then $s_{i_m} \in D_R(w)$ and hence $j_{\ell(w)} = m$ by definition. Thus the intersection $\{m\} \cap \{j_1, \ldots, j_{\ell(w)}\} = m$ and the reduced word (i_m) is a reduced word for s_{i_m}. If $[e, w]_L \cap [e, s_{i_m}] = \{e\}$, then $s_{i_m} \notin D_R(w)$ and so $\{m\} \cap \{j_1, \ldots, j_{\ell(w)}\} = \emptyset$ which is a reduced word for the maximal element.

Assume the hypothesis holds for the subword $(i_{k+1}, \ldots, i_m)$ and let y' be Weyl element given by the subword of $\underline{i}$ indexed by $\{k+1, \ldots, m\} \cap \{j_1, \cdots, j_{\ell(w)}\}$. By assumption, this is also the maximal element in $[e, w]_L \cap [e, s_{i_{k+1}} \cdots s_{i_m}]$.

For $\underline{i}' = (i_k, \ldots, i_m)$, let y be the Weyl element given by the subword of $\underline{i}$ indexed by $\{k, k+1, \ldots, m\} \cap \{j_1, \ldots, j_{\ell(w)}\}$. Then either $y = y'$ or $y = s_{i_k} \cdot y'$ is a reduced product and so $\ell(y') \leq \ell(y)$. By definition of $\underline{i}^w$, $y \leq_L w$ and hence $y \in [e, w]_L \cap [e, s_{i_k} \ldots s_{i_m}]$.

For any $x \in [e, w]_L \cap [e, s_{i_k} \ldots s_{i_m}]$, either $x \in [e, w]_L \cap [e, s_{i_{k+1}} \ldots s_{i_m}]$ or $x = s_{i_k} \cdot x'$ is a reduced product for some $x' \in [e, w]_L \cap [e, s_{i_{k+1}} \ldots s_{i_m}]$. In the first case, $\ell(x) \leq \ell(y') \leq \ell(y)$. In the second case, if $x' \neq y'$, then $\ell(x) = \ell(x') + 1 < \ell(y') + 1$ so that $\ell(x) \leq \ell(y') \leq \ell(y)$. If $x' = y'$, then necessarily $x = s_{i_k} y' = y$ by above and $\ell(x) = \ell(y)$. Thus the length of every element in this intersection is bounded above by $\ell(y)$ and hence y must be the unique of the maximal length element. □

Note that by the definition of $\underline{i}^w$, the phrase "the subword of $\underline{i}$ indexed by" in the previous lemma can be replaced with "the terminal subword of $\underline{i}^w$ indexed by".

Lemma 3.1.23. *Let $v, w \in W$. For every reduced word $\underline{i} = (i_1, \ldots, i_m)$ of w_0 such that $v_w = s_{i_1} \ldots s_{i_{\ell(v_w)}}$ and $v = s_{i_1} \ldots s_{i_{\ell(v)}}$, $\underline{i}^{w^{-1}w_0} = (i_{\ell(v_w)+1}, \ldots, i_{\ell(v)}, i_{j_{\ell(v)+1}} \ldots, i_{j_{m+\ell(v_w)-\ell(w)}})$ for some indices $\ell(v) + 1 \leq j_{\ell(v)+1} \leq \cdots \leq j_{m+\ell(v_w)-\ell(w)} \leq m$.*

Proof. Let $\underline{i}$ be a reduced word of w_0 as in the statement of the lemma. To show that the word $\underline{i}^{w^{-1}w_0}$ begins with a reduced word for $v_w^{-1}v$, we will show that $(v_w^{-1}v)^{-1}w^{-1}w_0$ is the longest length element of $[e, w^{-1}w_0]_L \cap [e, v^{-1}w_0]$. By Lemma 3.1.22, this says that the length $\ell(w^{-1}w_0) - \ell(v_w^{-1}v)$ terminal word of $\underline{i}^{w^{-1}w_0}$ is a reduced word of $(v_w^{-1}v)^{-1}w^{-1}w_0$. But as $w^{-1}w_0 = \left(v_w^{-1}v\right) \cdot \left((v_w^{-1}v)^{-1}w^{-1}w_0\right)$ is a reduced product, this will imply the initial word of length $\ell(v_w^{-1}v)$ of $\underline{i}^{w^{-1}w_0}$ must be a reduced word for $v_w^{-1}v$.

Claim 1. $(v_w^{-1}v)^{-1}w^{-1}w_0$ is the longest length element of $[e, w^{-1}w_0]_L \cap [e, v^{-1}w_0]$.

Proof: Note that $x \in [e, w^{-1}w_0]_L \cap [e, v^{-1}w_0] \iff x^{-1} \in [e, w_0w]_R \cap [e, w_0v]$. The longest element in this intersection is the Demazure product $(w_0ww_0)((w_0w^{-1}w_0) * (w_0v))$ by Proposition 3.1.13. We want to show that this product is equal to $w_0w(v_w^{-1}v)$.

By Proposition 2.2.13, $((w_0w^{-1}w_0)*(w_0v)) = w_0w^{-1}w_0x$ where x is the maximal length element such that $x \leq w_0v$ and $(w_0w^{-1}w_0) \cdot x$ is reduced. Recall that $a \mapsto w_0aw_0$ is an automorphism of the weak and strong Bruhat orders. Then $x \leq w_0v \iff w_0xw_0 \leq vw_0$. Also, by Lemma 2.2.10,

$$(w_0w^{-1}w_0) \cdot x \text{ is reduced } \iff x \leq_R w_0w \iff w_0xw_0 \leq_R ww_0 \iff w^{-1} \cdot (w_0xw_0) \text{ is reduced}$$

Since $\ell(x) = \ell(w_0xw_0)$, then $w^{-1} * (vw_0) = w^{-1}(w_0xw_0)$ by the maximality of x. Thus

$$x = w_0w(w^{-1} * (vw_0))w_0 = w_0w((vw_0)^{-1} * w)^{-1}w_0 = w_0w(v_w^{-1}(vw_0))w_0 = (w_0w)(v_w^{-1}v)$$

Hence $w_0w(v_w^{-1}v)$ is the longest length element in $[e, w_0w]_R \cap [e, w_0v]$ and so $(v_w^{-1}v)^{-1}w^{-1}w_0$ is the longest length element in $[e, w^{-1}w_0]_L \cap [e, v^{-1}w_0]$. ■

Now, by applying Lemma 3.1.22, the terminal subword of $\underline{i}^{w^{-1}w_0}$ indexed by the intersection of the indices of $\underline{i}^{w^{-1}w_0}$ with $\{\ell(v) + 1, \ldots, m\}$ is a reduced word of $(v_w^{-1}v)^{-1}w^{-1}w_0$. Thus this intersection is of length $\ell(w_0)+\ell(v_w)-\ell(w)-\ell(v)$ and is equal to $\{j_{\ell(v)+1}, j_{\ell(v)+2}, \ldots, j_{m+\ell(v_w)-\ell(w)}\}$ for some indices $\ell(v) + 1 \leq j_{\ell(v)+1} \leq \cdots \leq j_{m+\ell(v_w)-\ell(w)} \leq m$. As $(i_{\ell(v_w)+1}, \cdots, i_{\ell(v)})$ is a reduced word of $v_w^{-1}v$, then the word $(i_{\ell(v_w)+1}, \cdots, i_{\ell(v)}) +\!\!+ (i_{j_{\ell(v)+1}}, \ldots, i_{j_{m-\ell(w)}})$ is a reduced word for $w^{-1}w_0$ and must be the rightmost such word. □

We will show that for any $P \in \mathcal{P}_w$, the Lusztig data of P with respect to the reduced word $\underline{i}$ will have zeros in the position of the subword $\underline{i}^{w^{-1}w_0}$.

Proposition 3.1.24. *Let $\underline{i} = (i_1, \ldots, i_m)$ by any reduced word of w_0. For any $w \in W$ and any $P \in \mathcal{P}_w$, the Lusztig data of P with respect to $\underline{i}$ will have zeros in the position of the subword $\underline{i}^{w^{-1}w_0}$.*

Proof. Fix a reduced word $\underline{i} = (i_1, \ldots, i_m)$ of w_0. We proceed by induction on $\ell(w^{-1}w_0)$.

When $\ell(w^{-1}w_0) = 1$, then $\underline{i}^{w^{-1}w_0} = i_j$ for some j. If $j = m$, then $(i_1, \ldots, i_{m-1})$ is a reduced word for w and by Lemma 3.1.5, $n_m^{\underline{i}} = 0$. If $j \neq m$, then $\sigma^*_{s_{i^*_{j+1}} \cdots s_{i^*_m}}(b) \in \mathcal{P}_w$ by Lemma 3.1.20 so the reduced word $\underline{i}' = (i^*_{j+1}, \ldots, i^*_m, i_1, \ldots, i_j)$ has $\underline{i}'^{w^{-1}w_0}$ in the last position, hence $n_{i_j} = 0$ by above.

Assume for $\ell(w^{-1}w_0) = k$, the zeros of the Lusztig data $n^{\underline{i}}_\bullet$ are in the position $\underline{i}^{w^{-1}w_0}$. Suppose w is such that $\ell(w^{-1}w_0) = k+1$ and $n^{\underline{i}}_\bullet$ is the Lusztig data with respect to $\underline{i}$. If i_j is the final coordinate of $\underline{i}^{w^{-1}w_0}$, then $s_{i_{j+1}}, \ldots, s_{i_m} \notin D_R(w^{-1}w_0)$ so we can apply Lemma 3.1.20 $j - 1$ times so that the Lusztig data with respect to $(i^*_{j+1}, \ldots, i^*_m, i_1, \ldots, i_j)$ of the resulting polytope is $(0, \ldots, 0, n_1, \ldots, n_j)$. By the base case, $n_j = 0$ and hence the Lusztig data in the position of the final term of $\underline{i}^{w^{-1}w_0}$ is zero. Now, apply $\sigma^*_{i^*_j}$ so that the Lusztig data with respect to $(i^*_j, i^*_{j+1}, \ldots, i^*_m, i_1, \ldots, i_{j-1})$ of the resulting polytope is $(0, 0, \ldots, 0, n_1, \ldots, n_{j-1})$. By Corollary 3.1.19, this polytope is in $\mathcal{P}_{s_{i^*_j}w}$, where $\ell(s_{i^*_j}w) = \ell(w) - 1$. Thus, by the induction assumption, the Lusztig data corresponding to the rest of the coordinates of $\underline{i}^{w^{-1}w_0}$ will be zero. □

Example 3.1.25. Continuing Example 3.1.9, we have $w = s_1s_2s_3$ and $w^{-1}w_0 = s_1s_2s_1 = s_2s_1s_2$.

For a word $\underline{i}$, the zeros in the Lusztig data are given by the rightmost appearance of a word of $w^{-1}w_0$. The location of these zeros in various reduced words of w_0 prove the vertex equalities in Example 3.1.9.

Reduced word	Lusztig data	Equality of vertices
$(1,2,3,1,2,1)$	$(n_1,n_2,n_3,0,0,0)$	$\mu_w = \mu_{ws_1} = \mu_{ws_1s_2} = \mu_{w_0}$
$(2,3,1,2,1,3)$	$(n_1,n_2,0,0,0,n_6)$	$\mu_{s_2s_3} = \mu_{s_2s_3s_1} = \mu_{s_2s_3s_1s_2} = \mu_{s_2s_3s_1s_2s_1}$
$(1,3,2,1,3,2)$	$(n_1,n_2,0,0,n_5,0)$	$\mu_{s_1s_3} = \mu_{s_1s_3s_2} = \mu_{s_1s_3s_2s_1}$, $\mu_{ws_2s_1} = \mu_{w_0}$
$(3,2,1,3,2,3)$	$(n_1,0,0,n_4,0,n_6)$	$\mu_{s_3} = \mu_{s_3s_1} = \mu_{s_3s_2s_1}$, $\mu_{s_3s_2s_1s_3} = \mu_{s_3s_2s_1s_3s_2}$
$(1,2,1,3,2,1)$	$(n_1,n_2,0,n_4,0,0)$	$\mu_{s_1s_2} = \mu_{s_1s_2s_1}$, $\mu_{ws_1} = \mu_{ws_1s_2} = \mu_{w_0}$
$(2,1,3,2,1,3)$	$(n_1,0,n_3,0,0,n_6)$	$\mu_{s_2} = \mu_{s_2s_1}$, $\mu_{s_2s_1s_3} = \mu_{s_2s_1s_3s_2} = \mu_{s_2s_1s_3s_2s_1}$

Table 3.1: The zeros in the Lusztig data for A_3 MV polytopes

Finally, we can prove that the Lusztig data will have zeros in the positions between μ_{v_w} and μ_v for every Weyl group element v.

Theorem 3.1.26. *Fix $w \in W$. For every $P \in \mathcal{P}_w$ with vertex data $(\mu_v)_{v\in W}$, $\mu_v = \mu_{v_w}$ for every $v \in W$.*

Proof. Consider $P \in \mathcal{P}_w$. For $v \in W$, take a reduced word $\underline{i}$ of w_0 such that both v_w and v are initial words, i.e. $v_w = s_{i_1} \dots s_{i_{\ell(v_w)}}$ and $v = s_{i_1} \dots s_{i_{\ell(v)}}$. Then by Lemma 3.1.23 and Lemma 3.1.24, we know that the Lusztig data associated to $\underline{i}$ will have zeros in the subword $\underline{i}^{w^{-1}w_0} = (i_{\ell(v_w)+1}, \dots, i_{\ell(v)}, i_{j_{\ell(v)+1}}, \dots, i_{j_{m-\ell(v)}})$. Hence $\mu_{v_w} = \mu_v$. ☐

Corollary 3.1.27. *Fix $w \in W$. For every $P \in \mathcal{P}_w$ with vertex data $(\mu_v)_{v\in W}$, $P = \operatorname{conv}\{\mu_v : v \le w\}$.*

Remark 3.1.28. The description of $\mathcal{P}_w$ given by Corollary 3.1.27 suggests a relationship between $\mathcal{P}_w$ and extremal MV polytopes defined by [NS09]. Naito and Sagaki prove that extremal MV polytopes can be explicitly described as $P_{w\cdot\lambda} = \operatorname{conv}\{v \cdot \lambda : v \le w\}$, where λ is a dominant coweight. Using the Lusztig data description of these extremal MV polytopes in that paper, we can see (up to a reflection by w_0 and a shift to make $\mu_e = 0$), these polytopes are in $\mathcal{P}_w$.

In [BJK22], Besson, Jeralds and Keirs study the weight polytopes of Demazure modules and prove they are extremal MV polytopes. These polytopes can be described in the following ways:

$$P^w_\lambda = \operatorname{conv}\{g(v)\lambda : v \in W\} = \operatorname{conv}\{v\lambda : v \le W\} = \bigcap_{v\in W} C_v^{g(v)\lambda}$$

where $g(v) = v(v^{-1} * w)$. By Proposition 3.1.13 proved above, $g(vw_0) = v_w$. Thus, under the identification $\mu_v = g(vw_0)\lambda - g(w_0)\lambda$, $P^w_\lambda \in \mathcal{P}_w$.

Another related concept are the polytopes defined in [TW15]. For $w \in S_n$, the *Bruhat interval polytope* $Q_{e,w}$ is the polytope with vertex data $(\mu_v)_{v\le w}$ given by $\mu_v = (v(1), \dots, v(n))$. By extending the vertex data to $(\mu_v)_{v\in W}$ by $\mu_v = \mu_{v_w}$, we see this is a polytope of highest vertex w.

3.2 The dual fan of $\mathcal{P}_w$

Using Theorem 3.1.26, we would like to study the dual fan of polytopes in $\mathcal{P}_w$. Recall from Section 2.3.3 that the Weyl fan $\mathcal{W}$ is the fan of $\mathfrak{t}_{\mathbb{R}}^*$ with maximal cones C_v^* for $v \in W$ defined by

$$C_v^* = \{\beta \in \mathfrak{t}_{\mathbb{R}}^* : \langle v \cdot \alpha_i^\vee, \beta\rangle \geq 0, \forall i\}.$$

Any GGMS polytope P with vertex data $(\mu_\bullet)$ is given by $P = \bigcap_{v\in W} C_v^{\mu_v}$ where

$$C_v^{\mu_v} = \{x \in \mathfrak{t}_{\mathbb{R}} : \langle x, v \cdot \omega_i\rangle \geq \langle \mu_v, v \cdot \omega_i\rangle, \forall i\}.$$

The dual fan of a GGMS polytope P is $\mathcal{N}(P) = \{C_{F,P}^* : F \text{ is a face of } P\}$ such that

$$C_{F,P}^* = \{\beta \in \mathfrak{t}_{\mathbb{R}}^* : \langle x, \beta\rangle = \psi_P(\beta), \forall x \in F\}.$$

where $\psi_P(\beta) = \min_{y\in P}\langle y, \beta\rangle$. By [Kam10, Corollary A.4], (see Corollary 2.3.9) P is a coarsening of the Weyl fan $\mathcal{W}$ and the following corollary is immediate.

Corollary 3.2.1. *For any GGMS polytope P with vertex data $(\mu_\bullet)$, $C_v^* \subseteq C_{\mu_v,P}^*$ for every $v \in W$.*

Definition 3.2.2. Fix $w \in W$. Let $\mathcal{F}^w$ be the fan of $\mathfrak{t}_{\mathbb{R}}^*$ defined by the maximal cones for $v \in W$

$$D_v^* := \bigcup_{\substack{u\in W\\ u_w = v}} C_u^*.$$

D_v^* is indexed by $v \in W$ such that $v \leq w$. Clearly, $\mathcal{F}^w$ is a coarsening of the Weyl fan.

Proposition 3.2.3. *Let $w \in W$ and suppose P is an MV polytope. $P \in \mathcal{P}_w$ if and only if $\mathcal{N}(P)$ is a coarsening of $\mathcal{F}^w$.*

Proof. Consider a polytope $P \in \mathcal{P}_w$ with vertex data $(\mu_\bullet)$ and $v \in W$ arbitrary. For every $\beta \in \mathfrak{t}_{\mathbb{R}}^*$, $\langle \mu_v, \beta\rangle = \langle \mu_{v_w}, \beta\rangle$ so by definition $C_{\mu_v,P}^* = C_{\mu_{v_w},P}^*$. By Corollary 3.2.1, it follows that $C_v^* \subseteq C_{\mu_{v_w},P}^*$ for every $v \in W$, hence $D_v^* \subseteq C_{\mu_{v_w},P}^*$ and $\mathcal{N}(P)$ is a coarsening of $\mathcal{F}^w$.

For the converse, consider an MV polytope P such that the dual fan is a coarsening of $\mathcal{F}^w$. By Corollary 3.2.1, $C_w^* \subseteq C_{\mu_w,P}^*$. As $\mathcal{N}(P)$ is a coarsening of $\mathcal{F}^w$, then $D_w^* \subseteq C_{\mu_w,P}^*$ as well so $C_{w_0}^* \subseteq D_w^*$ implies $C_{w_0}^* \subseteq C_{\mu_w,P}^*$. Thus for every $\beta \in C_{w_0}^*$, $\langle \mu_w, \beta\rangle = \langle \mu_{w_0}, \beta\rangle$. But this is only possible when $\mu_w = \mu_{w_0}$ so $P \in \mathcal{P}_w$. □

As a result of this correspondence, the cones of the dual fan of P correspond with the vertices of P while the defining rays of the maximal cones of $\mathcal{N}(P)$ will correspond with the codimension 1 faces of P. In the standard case, these codimension 1 faces are exactly the hyperplanes M_γ for every $\gamma \in \Gamma$. When $w \neq w_0$, some of the hyperplanes M_γ of $\mathcal{P}_w$ may have larger codimension and hence $\mathcal{P}_w$ can have fewer than $|\Gamma|$ codimension 1 faces. An interesting question would be to find exactly which chamber weights label these codimension 1 faces in $\mathcal{P}_w$.

Question 3.2.4. What are the defining rays of the maximal cones of $\mathcal{F}^w$?

These rays will correspond to some subset of the chamber weights $\Gamma_{\mathcal{P}_w}$. This subset will give us the defining hyperplanes of P, i.e. $P = \{x : \langle x, \gamma\rangle \leq M_\gamma, \forall\gamma \in \Gamma_{\mathcal{P}_w}\}$.

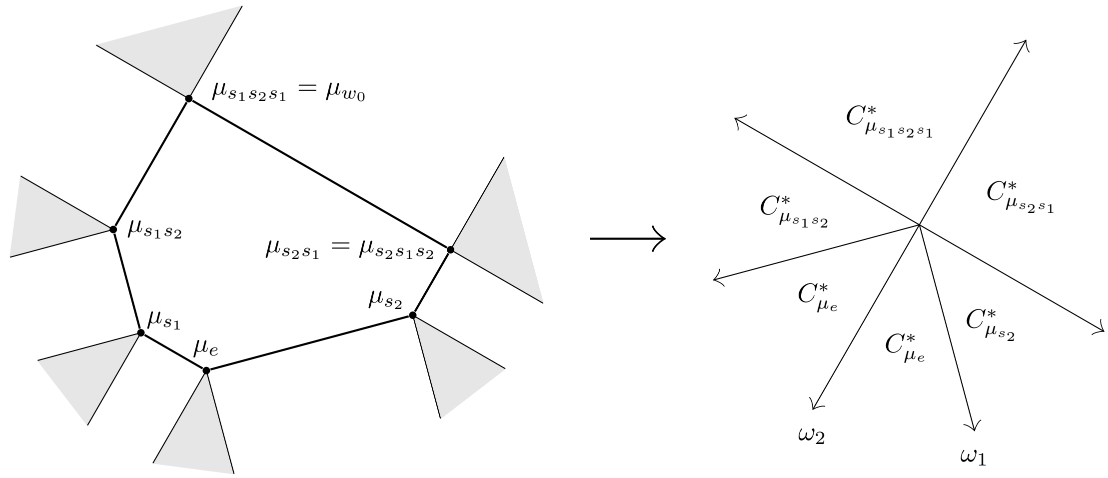


Figure 3.3: The dual fan of a B_2 polytope of highest vertex $s_1s_2s_1$

Example 3.2.5. For B_2 polytopes in $\mathcal{P}_{s_1s_2s_1}$, the hyperplanes labelled by $s_2s_1\omega_1$ and $s_1s_2s_1\omega_1$ are not defining hyperplanes of these polytopes. See Figure 3.3 for the dual fan of such a polytope and see Figure 3.4 for the defining hyperplanes.

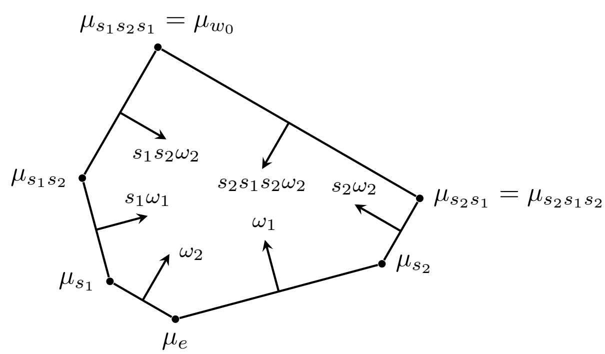


Figure 3.4: The hyperplanes of a B_2 polytope of highest vertex $s_1s_2s_1$

Consider the set $\Gamma^{w^{-1}} = \{v\omega_j : v \leq_R w^{-1}, j \in I\}$. It seems that defining hyperplane data is related to the set $\Gamma^{w^{-1}}$ though it is not easy to see exactly what this relationship is.

Example 3.2.6. For A_3, the codimension 1 hyperplane data is related to $\Gamma^{w^{-1}}$ by Table 3.2. Here, the defining hyperplanes are index by $\Gamma_{\mathcal{P}_w} \subset \Gamma$.

3.3 Combinatorial data of $L^{w^{-1}}$

We will define functions M_γ on the tropical points of the reduced double Bruhat cell $L^{w^{-1}}$ that will send non-negative tropical points to the BZ data associated to MV polytopes of highest vertex w. These functions will come from the tropicalization of the generalized minors functions.

Weyl element	$\Gamma_{\mathcal{P}_w}$ vs. $\Gamma^{w^{-1}}$
$w = s_1s_3s_2s_1$	$\Gamma_{\mathcal{P}_w} = \{\omega_1, \omega_2, \omega_3, s_1\omega_1, s_3\omega_3, s_2s_3\omega_3, s_1s_2s_3\omega_3, s_2s_1s_3s_2\omega_2\}$ $\Gamma^{w^{-1}} = \{\omega_1, \omega_2, \omega_3, s_1\omega_1, s_3\omega_3, s_2s_1\omega_1, s_2s_3\omega_3, s_1s_2s_3\omega_3\}$
$w = s_1s_2s_3s_2$	$\Gamma_{\mathcal{P}_w} = \{\omega_1, \omega_2, \omega_3, s_1\omega_1, s_3\omega_3, s_2s_1\omega_1, s_3s_2s_1\omega_1, s_2s_1s_3s_2\omega_2\}$ $\Gamma^{w^{-1}} = \{\omega_1, \omega_2, \omega_3, s_1\omega_1, s_3\omega_3, s_2s_1\omega_1, s_3s_2s_1\omega_1, s_2s_3\omega_3\}$
$w = s_2s_1s_2s_3$	$\Gamma_{\mathcal{P}_w} = \Gamma^{w^{-1}}$ $\Gamma^{w^{-1}} = \{\omega_1, \omega_2, \omega_3, s_1\omega_1, s_2\omega_2, s_2s_1\omega_1, s_2s_1\omega_2, s_3s_2s_1\omega_1\}$
$w = s_2s_1s_3s_2$	$\Gamma_{\mathcal{P}_w} = \{\omega_1, \omega_2, \omega_3, s_1\omega_1, s_2\omega_2, s_3\omega_3, s_2s_1\omega_1, s_2s_3\omega_3, s_3s_2s_1\omega_1, s_1s_2s_3\omega_3\}$ $\Gamma^{w^{-1}} = \{\omega_1, \omega_2, \omega_3, s_2\omega_2, s_1s_2\omega_2, s_3s_2\omega_2, s_3s_1s_2\omega_2, s_2s_1s_3s_2\omega_2\}$
$w = s_2s_3s_2s_1$	$\Gamma_{\mathcal{P}_w} = \{\omega_1, \omega_2, \omega_3, s_2\omega_2, s_3\omega_3, s_3s_2\omega_2, s_2s_3\omega_3, s_3s_1s_2\omega_2\}$ $\Gamma^{w^{-1}} = \{\omega_1, \omega_2, \omega_3, s_2\omega_2, s_3\omega_3, s_3s_2\omega_2, s_2s_3\omega_3, s_1s_2s_3\omega_3, s_1s_3s_2\omega_2\}$

Table 3.2: The defining hyperplanes of $\mathcal{P}_w$

3.3.1 Tropicalized generalized minors of $L^{w^{-1}}$

Recall we define the maps $x_i : \mathbb{C} \to N$ by $x_i(a) = \exp(aE_i)$. We similarly define $y_i : \mathbb{C} \to N$ by $y_i(a) = \exp(aF_i)$. For $i \in I$, we fix a representative of $s_i \in G$ by $s_i = y_i(1)x_i(-1)y_i(1)$ and thus for any $w \in W$ we can fix a representation for $w \in G$ by $w = s_{i_1} \cdots s_{i_m}$ where $\underline{i} = (i_1, \ldots, i_m)$ is a reduced word for w.

Definition 3.3.1. For $u, v \in W$, the reduced double Bruhat cell is $L^{u,v} := NuN \cap B_-vB_-$.

In particular, we are interested in the reduced double Bruhat cell $L^{w^{-1}} := L^{e,w^{-1}} = N \cap B_-w^{-1}B_-$. Following [GS15, Section 5], we have a positive structure on $L^{w^{-1}}$ described as follows. Let $x_i : \mathbb{C}^\times \to L^{w^{-1}}$ be defined as in Section 2.6. For the reduced word $\underline{i} = (i_1, \ldots, i_m)$ of w^{-1}, define $x_{\underline{i}} : (\mathbb{C}^\times)^m \to L^{w^{-1}}$ by $x_{\underline{i}}(a_1, \ldots, a_m) = x_{i_1}(a_1) \ldots x_{i_m}(a_m)$. From the application of [FZ99, Theorem 1.2], this is a coordinate system on $L^{w^{-1}}$. Consider the atlas given by the charts $(x_{\underline{i}})$ where $\underline{i}$ runs over all reduced words of w^{-1}. This atlas gives a positive structure on $L^{w^{-1}}$ which we will still call *Lusztig's positive atlas.* As in the case of N, define the potential function

$$\chi(x_{\underline{i}}(a_1, \ldots, a_m)) = \sum_{i=1}^{m} a_i.$$

The potential χ is still independent of $\underline{i}$ and is positive on this atlas, so we can define the non-negative tropical points

$$L^{w^{-1}}(\mathbb{Z}^{\text{trop}})_{\geq} = \{a \in L^{w^{-1}}(\mathbb{Z}^{\text{trop}}) : \chi^t(a) \geq 0\}.$$

To define the functions M_γ on $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})$, we need to introduce the generalized minors.

Definition 3.3.2. Consider the highest weight representation $V(\lambda)$ of G. Let γ and δ be an extremal weights of $V(\lambda)$ and let v_γ and v_δ be vectors in $V(\lambda)$ of weight λ and δ respectively. Let $\langle \cdot, \cdot \rangle$ denote the Shapovalov form [Sha72], i.e. $\langle F_iv, w\rangle = \langle v, E_iw\rangle$. The *generalized minors* are functions $\Delta_{\delta,\gamma} : G \to \mathbb{C}$ such that

$$\Delta_{\delta,\gamma}(g) = \langle g \cdot v_\gamma, v_\delta\rangle.$$

We use the shorthand Δ_γ when $\delta = \lambda$.

Denote the subset of chamber weights $\Gamma^w = \{v\omega_j : j \in I, v \in W \text{ such that } v \leq_R w\} \subseteq \Gamma$. By [BFZ05, Proposition 2.8], $L^{w^{-1}}$ can be defined by the vanishing conditions of generalized minors:

$$L^{w^{-1}} = \{g \in N : \Delta_{\omega_i,\omega_i}(g) = 1, \Delta_{\omega_i,w\omega_i}(g) \neq 0, \Delta_{\omega_i,v\omega_i}(g) = 0 \text{ for } v\omega_i \notin \Gamma^w\}. \tag{3.4}$$

Example 3.3.3. Let $G = SL_3$. The fundamental weights can be realized as $\omega_1 = (1,0,0)$ and $\omega_2 = (1,1,0)$. We use the shorthand $\omega_1 = 1$ and $\omega_2 = 12$, where each number indicates which coordinate is equal to 1. The simple reflections act as the transposition $s_1 = (12)$ and $s_2 = (23)$ on the fundamental weights.

When $w = s_1 s_2$, the reduced Bruhat cell is given by

$$L^{s_2 s_1} = \left\{ x_2(\beta)x_1(\alpha) = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} : \alpha, \beta \in \mathbb{C}^\times \right\}.$$

Note that $\Gamma^{s_1 s_2} = \{1, 2, 12, 13, 23\}$. Indeed, $\Delta_1 = 1$, $\Delta_{12} = 1$, $\Delta_2 = \alpha$, $\Delta_{23} = \alpha\beta$, $\Delta_{13} = \beta$ are all nonzero but $\Delta_3 = 0$ as $3 \notin \Gamma^{s_1 s_2}$.

Define the map $\eta_{w^{-1}}$ on $L^{w^{-1}}$ by setting $\eta_{w^{-1}}(x)$ to be the unique element in $N \cap B_- \overline{w^{-1}} x^T$. By [BZ97, Theorem 1.2, Proposition 1.3], $\eta_{w^{-1}}$ is a regular automorphism of $L^{w^{-1}}$ and $\eta^{-1}_{w^{-1}}(z) = (\eta_w(z^\iota))^\iota$, where ι is the anti-automorphism that sends $x_{i_1}(t_1) \dots x_{i_m}(t_m) \mapsto x_{i_m}(t_m) \dots x_{i_1}(t_1)$.

Define the y-coordinates $y_{\underline{i}}(b_\bullet) := \eta^{-1}_{w^{-1}}(x_{\underline{i}}(b_\bullet)) = \iota \circ \eta_w(x_{\underline{i}}(b_\bullet)^\iota)$; these are the coordinates used in the proof of [Kam10, Theorem 7.1] (see Theorem 2.3.8). For $\gamma \in \Gamma^w$, define $M_\gamma = (\Delta_\gamma \circ \eta^{-1}_{w^{-1}})^{\text{trop}}$. Then $(M_\gamma)_{\gamma \in \Gamma^w}$ is a collection of functions on the tropical points of $L^{w^{-1}}$.

Example 3.3.4. Continuing Example 3.3.3, the y-coordinates on $L^{s_2 s_1}$ are given by

$$y_{21}(\alpha^{-1}, \beta^{-1}) = \eta^{-1}_{w^{-1}}(x_2(\beta)x_1(\alpha)) = \begin{bmatrix} 1 & \alpha^{-1} & 0 \\ 0 & 1 & \beta^{-1} \\ 0 & 0 & 1 \end{bmatrix}.$$

For $(a, b) \in L^{s_2 s_1}(\mathbb{Z}^{\text{trop}})$, the functions M_γ take on the values

$$M_1(a,b) = 0, \quad M_{12}(a,b) = 0, \quad M_2(a,b) = -a, \quad M_{23}(a,b) = -a-b, \quad M_{13}(a,b) = -b.$$

Consider $\gamma \notin \Gamma^w$. By (3.4), $\Delta_\gamma = 0$. We would like to redefine these generalized minors so that we have functions which are non-zero on $L^{w^{-1}}$.

Definition 3.3.5. For $v \in W$ and $s_i \in D_L(v)$, define $\Delta^{\text{new}}_{v\omega_i} := \Delta_{v_w^{-1} v\omega_i, v\omega_i}$.

Example 3.3.6. Continuing 3.3.4, we redefine the minor $\Delta^{\text{new}}_3 = \Delta_{s_1\omega_1, w_0\omega_1} = \Delta_{2,3} = \beta$. Note that this is the smallest row set which results is a nonzero minor with the column set.

For $\gamma \in \Gamma$, define $M_\gamma = (\Delta^{\text{new}}_\gamma \circ \eta^{-1}_{w^{-1}})^{\text{trop}}$. Note that $M_\gamma = (\Delta_\gamma \circ \eta^{-1}_{w^{-1}})^{\text{trop}}$ for $\gamma \in \Gamma^w$. We will show that for each $\ell \in L^{w^{-1}}(\mathbb{Z}^{\text{trop}})_\geq$, $(M_\gamma(\ell))_{\gamma \in \Gamma}$ is the BZ data of some $P \in \mathcal{P}_w$. To do this, we need to show that $(M_\gamma(\ell))_{\gamma \in \Gamma}$ is a BZ datum and that the edge equalities in (ii) of Lemma 3.1.7 hold.

First, we will show that this $\Delta_\gamma^{\text{new}}$ is the "smallest" non-zero minor; this will imply the edge equalities. We start with a technical lemma. Recall the partial ordering on X^* where $a \leq b \iff b - a \in Q_+$.

Lemma 3.3.7. *For $b \in B$, $\alpha \in X^*$, $u \in W$, and λ a dominant weight, if $\langle v_\alpha, buv_\lambda\rangle \neq 0$, then $\alpha = u\lambda + \beta$ for $\beta \in \Delta_+ \cap u\Delta_-$.*

Proof. Consider the Lie algebra of G, $\mathfrak{g}$. Recall the root space decomposition of $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$, where $\mathfrak{h}$ is the Cartan subalgebra and $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}\}$. Set $\mathfrak{b} = \mathfrak{h} \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$.

For λ a dominant weight, consider the representation $V(\lambda)$. The Demazure module is defined as $V_u(\lambda) = U(\mathfrak{b}) \cdot v_{u\lambda}$, where $v_{u\lambda}$ is a vector of weight $u\lambda$ in the 1-dimensional $u\lambda$-weight space of $V(\lambda)$. We will show that the weights of $V_u(\lambda)$ is the set $u\lambda + \Delta_+ \cap u\Delta_-$.

First, consider $\mathfrak{n}_u = \bigoplus_{\alpha\in\Delta_+\cap u\Delta_-} \mathfrak{g}_\alpha$ and $\mathfrak{n}_{-u} = \bigoplus_{\alpha\in\Delta_+\cap u\Delta_+} \mathfrak{g}_\alpha$. Then $\mathfrak{b} = \mathfrak{n}_u \oplus \mathfrak{n}_{-u} \oplus \mathfrak{h}$ so we have a PBW basis ABC, where A is a product of vectors from $\mathfrak{n}_u$, B is a product of vectors in $\mathfrak{n}_{-u}$ and C is a product of vectors from $\mathfrak{h}$.

Suppose that $\beta \in \Delta_+ \cap u\Delta_+$. Then $u^{-1}\beta \in \Delta_+$ as well. Consider $X_\beta \in B$. Then

$$X_\beta \cdot v_{u\lambda} = X_\beta u \cdot v_\lambda = u(u^{-1}X_\beta u)v_\lambda.$$

Note that $u^{-1}X_\beta u = (\text{ad}_u(X_\beta)) \cdot v = X_{u^{-1}\beta}$. But as $u^{-1}(\beta) \in \Delta_+$ and v_λ is a highest weight vector of $V(\lambda)$, then $X_{u^{-1}\beta} \cdot v_\lambda = 0$.

Thus for every $b \in B$ with $b \neq 1$, $b \cdot v_{u\lambda} = 0$. As C does not affect the weight of $v_{u\lambda}$, then $BCv_{u\lambda}$ only has weights $u\lambda$. Thus $U(\mathfrak{b}) \cdot v_{u\lambda} = ABC \cdot v_{u\lambda}$ has weights $u\lambda - \Delta_+ \cap u\Delta_-$ as desired.

□

Lemma 3.3.8. *Let λ be a dominant weight and let $g \in L^{w^{-1}}$. Then $\Delta_{u\lambda,\mu}(g) = 0$ for any $\mu \ngeq wu\lambda$.*

Proof. Take $g \in L^{w^{-1}}$, then $g = b_1 w^{-1} b_2$ for $b_1, b_2 \in B_-$. Let $\mu \in X^*$ be a weight of $V(\lambda)$ so $\Delta_{u\lambda,\mu}(g) = \langle gv_\mu, uv_\lambda\rangle$ for vectors $v_\lambda, v_\mu \in V(\lambda)$ of weights λ and μ respectively. By the definition of the Shapovalov form, $\langle gv_\mu, uv_\lambda\rangle = \langle w^{-1}b_2v_\mu, b_2^t uv_\lambda\rangle = \langle v_\mu, g^t uv_\lambda\rangle$, where $g^t = b_2^t w b_1^t$.

Note that $g^t uv_\lambda = \sum_{\alpha\in Q}\langle v_\alpha, g^t uv_\lambda\rangle v_\alpha$ for weight vectors $v_\alpha \in V(\lambda)$ of weight α. In fact,

$$g^t uv_\lambda = \sum_{\alpha\in X^*} \langle w^{-1}b_2v_\alpha, b_1^t uv_\lambda\rangle v_\alpha.$$

Let $w^{-1}b_2v_\alpha = \sum_{\eta\in X^*}\langle w^{-1}b_2v_\alpha, v_\eta\rangle v_\eta$. Then

$$g^t uv_\lambda = \sum_{\alpha,\eta\in X^*} \langle w^{-1}b_2v_\alpha, v_\eta\rangle\langle v_\eta, b_1^t uv_\lambda\rangle v_\alpha.$$

By Lemma 3.3.7, $\langle v_\eta, b_1^t uv_\lambda\rangle \neq 0 \iff \eta = u\lambda + \beta$ for $\beta \in \Delta_+ \cap u\Delta_-$. Since B_- always lowers the weights, $\langle w^{-1}b_2v_\alpha, v_\eta\rangle \neq 0 \iff w\eta = \alpha - \gamma$ for some $\gamma \in Q_+$. Thus

$$\alpha = wu\lambda + w\beta + \gamma$$

By [BZ97, Corollary 2.3], as wu is reduced then $\Delta_+ \cap u^{-1}(\Delta_-) \subseteq \Delta_+ \cap (wu)^{-1}(\Delta_-)$. Note that

$\alpha \in \Delta_+ \cap xy(\Delta_-) \iff -x^{-1}\beta \in y\Delta_+ \cap x^{-1}(\Delta_-)$. Thus we also have the inclusion

$$\Delta_+ \cap u(\Delta_-) \subseteq w^{-1}\Delta_+ \cap u(\Delta_-).$$

Then $\beta \in w^{-1}\Delta_+ \cap u(\Delta_-)$ as well so there exists $\delta \in \Delta_+$ such that $\beta = w^{-1}\delta \iff w\beta = \delta$. Thus $w\beta \in \Delta_+$ and $\alpha = wu\lambda + \delta + \gamma \in wu\lambda + Q_+$. Hence

$$g^t uv_\lambda = \sum_{\gamma \in Q_+} \langle v_{wu\lambda+\gamma}, g^t uv_\lambda \rangle v_{wu\lambda+\gamma}$$

and so $\Delta_{u\lambda,\mu}(g) = 0$ for $\mu \not\geq wu\lambda$. □

Corollary 3.3.9. *Fix $w \in W$ and let $u \leq_R w^{-1}w_0$. For every $s_i \in S$ such that wus_i is a reduced product, then $\Delta_{u\omega_i, wus_i\omega_i} = 0$ on $L^{w^{-1}}$.*

Proof. Since wus_i is reduced, then $wu\alpha_i \in \Delta_+$. As $s_i\omega_i = \omega_i - \alpha_i$, then $wus_i\omega_i = wu\omega_i - wu\alpha_i$ and hence $wus_i\omega_i \leq wu\omega_i$. By setting $\lambda = \omega_i$ and $\mu = wus_i\omega_i$, we can apply Lemma 3.3.8 to see that $\Delta_{u\omega_i, wus_i\omega_i}(g) = 0$ for $g \in L^{w^{-1}}$. □

Conjecture 3.3.10. *Fix $w \in W$, let $v \in W$. Set $u = v_w^{-1}v$ and let $s_i \notin D_R(v)$ such that $(vs_i)_w = v_w$. Then $\Delta_{u\omega_i, vs_i\omega_i} = 0$ on $L^{w^{-1}}$.*

Remark 3.3.11. This conjecture is known for a few special cases. When $v_w = w$, then $u = w^{-1}v$ is an initial word of $w^{-1}w_0$ and hence the conjecture is equivalent to Corollary 3.3.9. On the other hand, when $v_w = v$ then $u = e$ so the generalized minor of interest is of the form $\Delta_{\omega_i, vs_i\omega_i}$. By assumption, $v \leq w$ but s_i is such that $(vs_i)_w = v$ and hence $vs_i \not\leq w$ by maximality of $(vs_i)_w$. Thus the conjecture follows from [BFZ05, Proposition 2.8] (see (3.4)).

These two results imply the edge equalities are satisfied for large enough γ.

Proposition 3.3.12. *$(M_\gamma)_{\gamma \in \Gamma}$ satisfy the edge equalities (ii) of Lemma 3.1.7. In other words, for every $v \in W$ and $s_i \notin D_R(v)$ such that $\mu_{vs_i} = \mu_v$,*

$$M_{v\omega_i} + M_{vs_i\omega_i} = -\sum_{j \neq i} a_{j,i} M_{v\omega_j}.$$

Proof. By [BZ01, Proposition 4.1], for every u, w such that $\ell(us_i) = \ell(u)+1$, $\ell(wus_i) = \ell(w)+\ell(u)+1$,

$$\Delta_{u\omega_i, wu\omega_i}\Delta_{us_i\omega_i, wus_i\omega_i} = \Delta_{us_i\omega_i, wu\omega_i}\Delta_{u\omega_i, wus_i\omega_i} + \prod_{j \neq i} \Delta_{u\omega_j, wu\omega_j}^{-a_{j,i}}$$

By Corollary 3.3.9, $\Delta_{u\omega_i, wus_i\omega_i} = 0$ on $L^{w^{-1}}$ and by tropicalizing, we obtain

$$M_{wu\omega_i} + M_{wus_i\omega_i} = \sum_{j \neq i} (-a_{j,i}) M_{wu\omega_j}$$

which are exactly the edge equalities.

For $v \in W$ arbitrary, set $u = v_w^{-1}v$. For s_i such that $\ell(vs_i) = \ell(v) + 1$ and $(vs_i)_w = v_w$, then by

[BZ01, Proposition 4.1]

$$\Delta^{\text{new}}_{v\omega_i}\Delta^{\text{new}}_{vs_i\omega_i} = \Delta_{us_i\omega_i,v\omega_i}\Delta_{u\omega_i,vs_i\omega_i} + \prod_{j\neq i}\left(\Delta^{\text{new}}_{v\omega_j}\right)^{-a_{j,i}}$$

By Conjecture 3.3.10, $\Delta_{u\omega_i,vs_i\omega_i} = 0$ on $L^{w^{-1}}$ and hence by tropicalizing,

$$M_{v\omega_i} + M_{vs_i\omega_i} = \sum_{j\neq i}(-a_{ji})M_{v\omega_j}.$$

So we have proved that for every $v \in V$ such that $\mu_{vs_i} = \mu_v = \mu_{v_w}$, the edge equalities $M_{v\omega_i} + M_{vs_i\omega_i} = \sum_{j\neq i}(-a_{ji})M_{v\omega_j}$ are satisfied. □

Theorem 3.3.13. *There is a bijection* $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})_{\geq} \longrightarrow \mathcal{P}_w$ *by*

$$\ell \to (M_\gamma(\ell))_{\gamma\in\Gamma}.$$

Proof. First, we show is that the collection $(M_\gamma)_{\gamma\in\Gamma}$ is the BZ data of an MV polytope in $\mathcal{P}_w$. The collection $(M_\gamma)_{\gamma\in\Gamma^w}$ satisfies the tropical Plücker relations as Δ_γ satisfies the Plücker relations. The collection $(M_\gamma)_{\gamma\in\Gamma}$ satisfies the edge equalities (ii) of Lemma 3.1.7 by Proposition 3.3.12 and thus we can recursively define these tropical functions by the collection $(M_\gamma)_{\gamma\in\Gamma^w}$ using the relation (3.1). It is easy to see that these $(M_\gamma)_{\gamma\in\Gamma\setminus\Gamma^w}$ will also satisfy the tropical Plücker relations, thus $(M_\gamma)_{\gamma\in\Gamma}$ is the BZ data of some MV polytope, $P \in \mathcal{P}$. Finally, by Lemma 3.1.7, $P \in \mathcal{P}_w$ and so this map is well defined.

To show this map is a bijection, fix a reduced word $\underline{i} = (i_1, \dots, i_m)$ of w_0 such that a $(i_1, \dots, i_{\ell(w)})$ is a reduced word for w. The map

$$(M_\gamma(\ell))_{\gamma\in\Gamma} \mapsto \left(-M_{w^{\underline{i}}_k s_{k+1}\omega_{k+1}} - M_{w^{\underline{i}}_k\omega_{k+1}} + \sum_{j\neq i_{k+1}} a_{j,i_{k+1}} M_{w^{\underline{i}}_{k+1}\omega_j}\right)_{k=0}^{m-1} \tag{3.5}$$

sends the BZ data of P to the Lusztig data of P with respect to the reduced word of $\underline{i}$. By Proposition 3.1.24, $n_k = 0$ for $k > \ell(w)$ so by Theorem 2.3.8, (3.5) is a bijection from $\mathcal{P}_w \to \mathbb{N}^{\ell(w)}$. But $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})_{\geq} \cong \mathbb{N}^{\ell(w)}$ so by composing these maps, the inverse $(M_\gamma(\ell))_{\gamma\in\Gamma} \mapsto \ell$ is a bijection. □

Chapter 4

Diagonals Relations

As seen in Chapter 2, an MV polytope can be described by the combinatorial data $(M_\gamma)_{\gamma\in\Gamma}$ which satisfy the edge inequalities and the tropical Plücker relations. In the case of A_2, the tropical Plücker relation induces the condition that the length of A is equal to the maximum of the lengths of B and C (see Figure 4.1). This is equivalent to the condition that the $\mu_{s_2s_1}$ vertex is on or below the line in the α_1 root direction from the μ_{s_1} vertex and the $\mu_{s_1s_2}$ vertex is on or below the line in the α_2 root direction from the μ_{s_2} vertex, with at least one of the vertices $\mu_{s_2s_1}$ or $\mu_{s_1s_2}$ on these lines. This can be written as the *diagonal relation*:

$$\min\{\langle \mu_{s_2} - \mu_{s_1s_2}, \omega_1\rangle, \langle \mu_{s_1} - \mu_{s_2s_1}, \omega_2\rangle\} = 0.$$

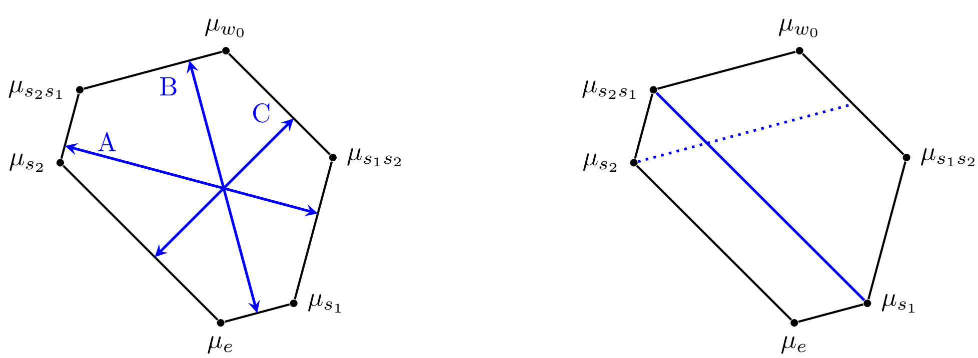


Figure 4.1: Two different visualizations of the tropical Plücker/diagonal relations of A_2 polytopes

In this chapter, we will prove that similar diagonal relations hold for the B_2 case. We do this by proving some "detropical diagonal relations" hold on the generalized minors using the representation theory of B_2. We then show these relations tropicalize to the diagonal relations on the BZ data.

4.1 Notation

Consider the B_2 root system and let α_1 be the long root, α_2 be the short root so that $\alpha_1^\vee$ is the short coroot and $\alpha_2^\vee$ is the long coroot. Then $a_{12} = \langle \alpha_1^\vee, \alpha_2\rangle = -1$ and $a_{21} = \langle \alpha_2^\vee, \alpha_1\rangle = -2$. The action of the Weyl group on the coroots are explicitly given by:

$$s_1(\alpha_1^\vee) = -\alpha_1^\vee, \qquad s_1(\alpha_2^\vee) = \alpha_2^\vee + 2\alpha_1^\vee, \qquad s_2(\alpha_1^\vee) = \alpha_1^\vee + \alpha_2^\vee, \qquad s_2(\alpha_2^\vee) = -\alpha_2^\vee.$$

Let ω_1, ω_2 be the fundamental weights. The Weyl group is D_4 so the longest word is length 4 and the MV polytopes are octagons. See Figure 4.2 for an example.

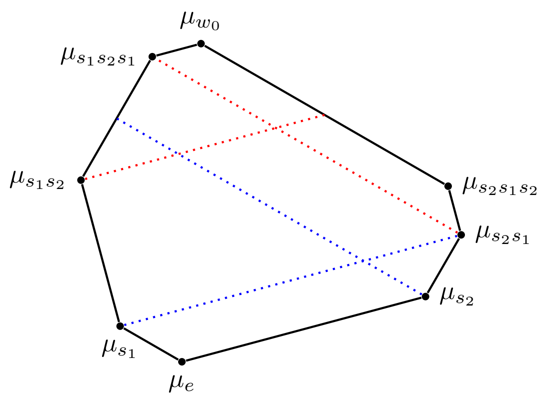


Figure 4.2: The diagonals of a B_2 polytope

For a B_2 MV polytope P with vertex data $(\mu_\bullet)$, the diagonal relations are given by

$$0 = \min\{\langle \mu_{s_1} - \mu_{s_2 s_1}, \omega_1\rangle, \langle \mu_{s_2} - \mu_{s_1 s_2}, \omega_2\rangle\},$$
$$0 = \min\{\langle \mu_{s_1 s_2} - \mu_{s_2 s_1 s_2}, \omega_1\rangle, \langle \mu_{s_2 s_1} - \mu_{s_1 s_2 s_1}, \omega_2\rangle\}.$$

As in the A_2 case, these diagonals can be visually seen in the polytope as a condition on the relation between the line in the simple root direction from a vertex and the next vertex on the opposite side of the polytope. In Figure 4.2, the blue lines show the first diagonal relation while the red lines show the second diagonal relation. On the BZ data $(M_\bullet)$ of P, these diagonal relations are given by:

$$M_{s_1\omega_1} + 2M_{s_2\omega_2} = \min\{M_{\omega_1} + M_{s_1 s_2\omega_2} + M_{s_2\omega_2}, M_{s_2 s_1\omega_1} + 2M_{\omega_2}\}, \tag{4.1}$$
$$M_{s_2 s_1\omega_1} + 2M_{s_1 s_2\omega_2} = \min\{M_{s_1\omega_1} + 2M_{s_2 s_1 s_2\omega_2}, M_{s_1 s_2 s_1\omega_1} + M_{s_2\omega_2} + M_{s_1 s_2\omega_2}\}. \tag{4.2}$$

Recall that the tropical Plücker relations are

$$M_{s_1\omega_1} + M_{s_2 s_1\omega_1} + M_{s_2\omega_2} = \min\{2M_{s_2 s_1\omega_1} + M_{\omega_2}, 2M_{\omega_1} + M_{s_2 s_1 s_2\omega_2}, M_{\omega_1} + M_{s_1 s_2 s_1\omega_1} + M_{s_2\omega_2}\}, \tag{P1}$$
$$M_{s_1 s_2\omega_2} + 2M_{s_2 s_1\omega_1} + M_{s_2\omega_2} = \min\{M_{s_2 s_1 s_2\omega_2} + 2M_{s_2 s_1\omega_1} + M_{\omega_2}, 2M_{\omega_1} + 2M_{s_2 s_1 s_2\omega_2}, 2M_{s_1 s_2 s_1\omega_1} + 2M_{s_2\omega_2}\}. \tag{P2}$$

Notice that the tropical Plücker relations are order 3 and order 4 respectively, while the diagonals are both order 3. To prove the diagonal relations hold, we will find detropical diagonal relations on the generalized minors which tropicalize to the diagonal relations.

Remark 4.1.1. A tropical function has many possible detropical functions which tropicalize to the original function. For example, the tropical expression $\min\{2x, 2y\}$ can have detropicalization

$x^2 + y^2$ or $(x+y)^2$. The first detropicalization we call the *naive detropicalization*: this equation assumes every term in the detropicalization contributes to the tropical equation.

In particular, the naive detropicalization of the diagonal relations are

$$\Delta_{s_1\omega_1}\Delta^2_{s_2\omega_2} = \Delta_{\omega_1}\Delta_{s_1s_2\omega_2}\Delta_{s_2\omega_2} + \Delta_{s_2s_1\omega_1}\Delta^2_{\omega_2},$$
$$\Delta_{s_2s_1\omega_1}\Delta^2_{s_1s_2\omega_2} = \Delta_{s_1\omega_1}\Delta^2_{s_2s_1s_2\omega_2} + \Delta_{s_1s_2s_1\omega_1}\Delta_{s_2\omega_2}\Delta_{s_1s_2\omega_2}.$$

We will prove that these relations do not hold. To detropicalize the diagonal relations, we need extra terms which will not contribute to the tropicalization. First we will consider the representation theory of the Lie algebra of type B_2.

4.2 Detropical diagonal relations from representation theory

As per [BZ97, Remark 6.8], the Plücker relations come from the kernels of the map from the symmetric algebra of the fundamental representations to the ring of regular functions $\mathbb{C}[N]$. To find other relations, we will calculate these kernels (for the tensor product), which we denote by

$$0 \to K_{i,j} \to V(\omega_i) \otimes V(\omega_j) \to V(\omega_i + \omega_j) \to 0.$$

We want to consider how tensor products of the fundamental representations relate to relations on the generalized minors.

In this section, we extend the definition of generalized minors from Definition 3.3.2 to include arbitrary vectors in $V(\lambda)$. For $v \in V(\lambda)$, define the generalized minor $\Delta_{v,\lambda} : N \to \mathbb{C}$ by $\Delta_{v,\lambda}(g) = \langle g \cdot v, v_\lambda \rangle$. We will denote this by Δ_v when it is obvious which representation we are working in.

Lemma 4.2.1. *For $n, m \in \mathbb{N}$, let $\Phi_{n,m} : V(\omega_i)^n \otimes V(\omega_j)^m \to V(n\omega_i + m\omega_j)$ be the natural projection map. For $v \in \ker(\Phi_{n,m})$, $\Delta_v = 0$.*

Proof. The idea of the proof is that $V(n\omega_i + m\omega_j)$ is the subrepresentation of $V(\omega_i)^n \otimes V(\omega_j)^m$ generated by the highest weight vector $v^n_{\omega_i} \otimes v^m_{\omega_j}$. Any vector of $V(\omega_i)^n \otimes V(\omega_j)^m$ not contained in the highest weight subrepresentation must only contain vectors of weight strictly less that $n\omega_i + m\omega_j$ and hence will pair trivially with $v^n_{\omega_i} \otimes v^m_{\omega_j}$.

First, note that $\langle \Phi_{n,m}(g \cdot v), \Phi_{n,m}(v^n_{\omega_i} \otimes v^m_{\omega_j})\rangle = \langle g \cdot v, v^n_{\omega_i} \otimes v^m_{\omega_j}\rangle$. As $\Phi_{n,m}$ is a homomorphism of representations, $\Phi_{n,m}(g \cdot v) = g \cdot (\Phi_{n,m}(v))$ and hence $\Delta_{\Phi_{n,m}(v)} = \Delta_v$ Thus if $v \in \ker(\Phi_{n,m})$, then $\Delta_v = 0$. □

For $v = x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m \in V(\omega_i)^n \otimes V(\omega_j)^m$, then

$$\begin{aligned}\Delta_v = \langle g \cdot v, v^n_{\omega_i} \otimes v^n_{\omega_j}\rangle &= \langle g \cdot x_1, v_{\omega_i}\rangle \ldots \langle g \cdot x_n, v_{\omega_i}\rangle\langle g \cdot y_1, v_{\omega_j}\rangle \ldots \langle g \cdot y_m, v_{\omega_j}\rangle \\ &= \Delta_{x_1}(g) \ldots \Delta_{x_n}(g)\Delta_{y_1}(g) \ldots \Delta_{y_m}(g).\end{aligned}$$

Thus the equality $\Delta_v = 0$ results in a relation on the generalized minors labelled by vectors in the fundamental representations.

Let $\mathfrak{g} = sp_4(\mathbb{C})$ be the complex simple Lie algebra of type B_2. Consider $V(\omega_1)$ and $V(\omega_2)$, the fundamental weight representations. By [FH91, Section 16.2], $V(\omega_1)$ is the simple 5-dimensional

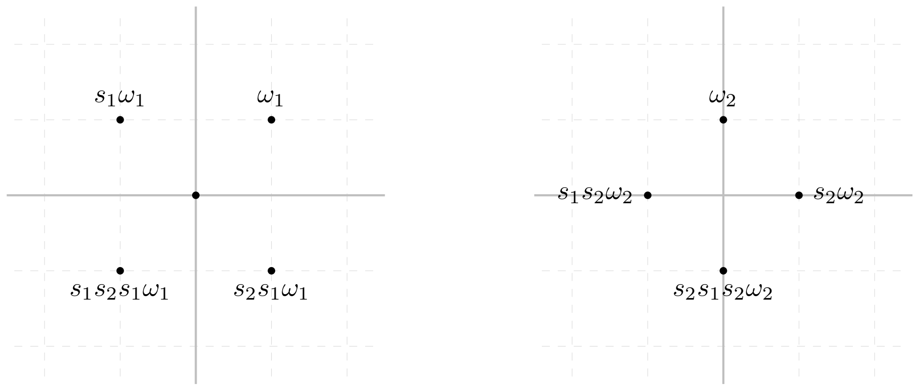


Figure 4.3: The weight diagrams of $V(\omega_1)$ (left) and $V(\omega_2)$ (right)

representation with 1-dimensional weight spaces at

$$\omega_1, \qquad s_1\omega_1 = 2\omega_2 - \omega_1, \qquad s_2s_1\omega_1 = \omega_1 - 2\omega_2, \qquad s_1s_2s_1\omega_1 = -\omega_1, \qquad 0.$$

$V(\omega_2)$ is the simple 4-dimensional representation with 1-dimensional weight spaces at

$$\omega_2, \qquad s_2\omega_2 = \omega_1 - \omega_2, \qquad s_1s_2\omega_2 = \omega_2 - \omega_1, \qquad s_1s_2s_1\omega_2 = -\omega_2.$$

Fix $v_{w\omega_i} \in V(\omega_i)_{w\omega_i}$ to be the unit vectors in the extremal weight spaces. Set $u \in V(\omega_1)_0$ such that u is a unit vector. Note that this is a non-extremal weight vector.

As $V(\omega_2) \otimes V(\omega_1) \cong V(\omega_1 + \omega_2) \oplus V(\omega_2)$, then $K_{21} \cong V(\omega_2)$. The weight space decomposition is

$$\begin{aligned}
(K_{21})_{\omega_2} &= \mathbb{C}(v_{\omega_2} \otimes u - 2v_{s_2\omega_2} \otimes v_{s_1\omega_1} + 2v_{s_1s_2\omega_2} \otimes v_{\omega_1}),\\
(K_{21})_{\omega_1-\omega_2} &= \mathbb{C}(v_{s_2\omega_2} \otimes u - 2v_{\omega_2} \otimes v_{s_2s_1\omega_1} - 2v_{s_2s_1s_2\omega_2} \otimes v_{\omega_1}),\\
(K_{21})_{-\omega_2} &= \mathbb{C}(-v_{s_2s_1s_2\omega_2} \otimes u - 2v_{s_2\omega_2} \otimes v_{s_1s_2s_1\omega_1} + 2v_{s_1s_2\omega_2} \otimes v_{s_2s_1\omega_1}),\\
(K_{21})_{-\omega_1+\omega_2} &= \mathbb{C}(v_{s_1s_2\omega_2} \otimes u - 2v_{\omega_2} \otimes v_{s_1s_2s_1\omega_1} - 2v_{s_2s_1s_2\omega_2} \otimes v_{s_1\omega_1}).
\end{aligned}$$

Consider $V(\omega_2) \otimes K_{21} \cong V(\omega_2) \otimes V(\omega_2) \cong V(2\omega_2) \oplus V(\omega_1) \oplus \mathbb{C}$. Thus the weight space $(V(\omega_2) \otimes K_{21})_{\omega_1}$ is 2-dimensional and spanned by the vectors

$$\begin{aligned}
x_1 &= v_{s_2\omega_2} \otimes (v_{\omega_2} \otimes u - 2v_{s_2\omega_2} \otimes v_{s_1\omega_1} + 2v_{s_1s_2\omega_2} \otimes v_{\omega_1}),\\
x_2 &= v_{\omega_2} \otimes (v_{s_2\omega_2} \otimes u - 2v_{\omega_2} \otimes v_{s_2s_1\omega_1} - 2v_{s_2\omega_2} \otimes v_{\omega_1}).
\end{aligned}$$

In fact, the weight space $V(2\omega_2)_{\omega_1} = \operatorname{span}\{x_1 - x_2\}$. Similarly, the weight space of $(V(\omega_2) \otimes K_{21})_{-\omega_1}$ is spanned by vectors

$$\begin{aligned}
y_1 &= v_{s_1s_2\omega_2} \otimes (v_{s_2s_1s_2\omega_2} \otimes u + 2v_{s_2\omega_2} \otimes v_{s_1s_2s_1\omega_1} - 2v_{s_1s_2\omega_2} \otimes v_{s_2s_1\omega_1}),\\
y_2 &= v_{s_2s_1s_2\omega_2} \otimes (v_{s_1s_2\omega_2} \otimes u - 2v_{\omega_2} \otimes v_{s_1s_2s_1\omega_1} - 2v_{s_2s_1s_2\omega_2} \otimes v_{s_1\omega_1}),
\end{aligned}$$

and $V(2\omega_2)_{-\omega_1} = \operatorname{span}\{y_1 - y_2\}$.

Consider the map $\Phi_{2,1} : V(\omega_2)^2 \otimes V(\omega_1) \to V(2\omega_2 + \omega_1)$. The kernel of this map contains $V(\omega_2) \otimes K_{21}$ and thus the vectors $x_1 - x_2, y_1 - y_2$ are both in $\ker(\Phi)$. Hence $\frac{1}{2}\Delta_{x_1-x_2} = \frac{1}{2}\Delta_{y_1-y_2} = 0$, which results in the relations:

$$\Delta_{s_1\omega_1}\Delta^2_{s_2\omega_2} = \Delta_{\omega_1}\Delta_{s_2\omega_2}\Delta_{s_1s_2\omega_2} + \Delta_{s_2s_1\omega_1}\Delta^2_{\omega_2} + \Delta_{\omega_1}\Delta_{\omega_2}\Delta_{s_2s_1s_2\omega_2}, \tag{4.3}$$

$$\Delta_{s_2s_1\omega_1}\Delta^2_{s_1s_2\omega_2} = \Delta_{s_1\omega_1}\Delta^2_{s_2s_1s_2\omega_2} + \Delta_{s_1s_2s_1\omega_1}\Delta_{s_2\omega_2}\Delta_{s_1s_2\omega_2} + \Delta_{s_1s_2s_1\omega_1}\Delta_{\omega_2}\Delta_{s_2s_1s_2\omega_2}. \tag{4.4}$$

In the next section, we will show that the tropicalization of these equations are the diagonal relations. Note these are indeed not the naive detropicalization (as per Remark 4.1.1) but include an extra non-zero term that will not contribute to the tropicalization.

4.3 Tropical relations

Consider the tropicalization of the system (4.3) and (4.4):

$$M_{s_1\omega_1} + 2M_{s_2\omega_2} = \min\{M_{\omega_1} + M_{s_2\omega_2} + M_{s_1s_2\omega_2}, M_{s_2s_1\omega_1} + 2M_{\omega_2} \\ M_{\omega_1} + M_{\omega_2} + M_{s_2s_1s_2\omega_2}\}, \tag{4.5}$$

$$M_{s_2s_1\omega_1} + 2M_{s_1s_2\omega_2} = \min\{M_{s_1\omega_1} + 2M_{s_2s_1s_2\omega_2}, M_{s_1s_2s_1\omega_1} + M_{s_2\omega_2} + M_{s_1s_2\omega_2}, \\ M_{s_1s_2s_1\omega_1} + M_{\omega_2} + M_{s_2s_1s_2\omega_2}\}. \tag{4.6}$$

Lemma 4.3.1. *The tropical system (4.5) and (4.6) is equivalent to the diagonal relations.*

Proof. From (4.5) and (4.6), we know that

$$M_{s_1\omega_1} + 2M_{s_2\omega_2} \leq M_{s_2s_1\omega_1} + 2M_{\omega_2}, \qquad M_{s_2s_1\omega_1} + 2M_{s_1s_2\omega_2} \leq M_{s_1\omega_1} + 2M_{s_2s_1s_2\omega_2}$$

respectively. By adding these inequalities, it follows that

$$M_{s_2\omega_2} + M_{s_1s_2\omega_2} \leq M_{\omega_2} + M_{s_2s_1s_2\omega_2}.$$

Hence $M_{\omega_1} + M_{s_2\omega_2} + M_{s_1s_2\omega_2} \leq M_{\omega_1} + M_{\omega_2} + M_{s_2s_1s_2\omega_2}$ so that the last term of (4.5) does not contribute to the minimum. Similarly, $M_{s_1s_2s_1\omega_1} + M_{s_2\omega_2} + M_{s_1s_2\omega_2} \leq M_{s_1s_2s_1\omega_1} + M_{\omega_2} + M_{s_2s_1s_2\omega_2}$ so that the last term of (4.6) does not contribute to the minimum. □

Corollary 4.3.2. *The diagonal relations hold on B_2 MV polytopes.*

4.4 The case of G_2

Consider the root system of type G_2. Let the long root be α_1 while α_2 is the short root so that $\alpha_1^\vee$ is the short coroot and $\alpha_2^\vee$ is the long coroot. Then MV polytopes are 12-sided polytopes, see Figure 4.4 for an example. The diagonal relations for G_2 polytopes are given by the tropical equations:

$$M_{s_1\omega_1} + 3M_{s_2\omega_2} = \min\{M_{s_2s_1\omega_1} + 3M_{\omega_2}, M_{\omega_1} + M_{s_1s_2\omega_2} + 2M_{s_2\omega_2}\}, \tag{4.7}$$

$$2M_{s_2s_1\omega_1} + 3M_{s_1s_2\omega_2} = \min\{M_{s_1s_2s_1\omega_1} + M_{s_2s_1\omega_1} + 2M_{s_2\omega_2} + M_{s_1s_2\omega_2}, \\ 2M_{s_1\omega_1} + 3M_{s_2s_1s_2\omega_2}\}, \tag{4.8}$$

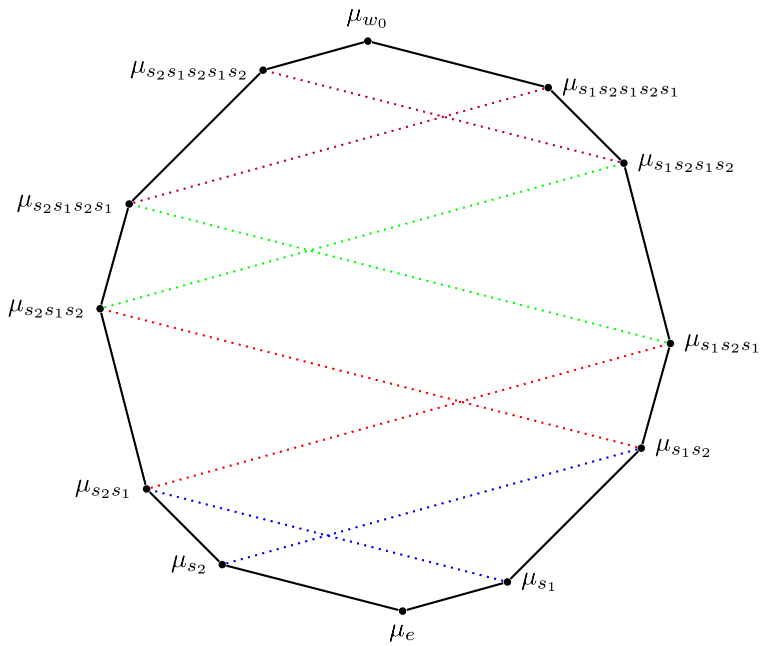


Figure 4.4: The diagonals of a G_2 polytope

$$2M_{s_1s_2s_1\omega_1} + 3M_{s_2s_1s_2\omega_2} = \min\{2M_{s_2s_1s_2s_1\omega_1} + 3M_{s_1s_2\omega_2}, \\ M_{s_2s_1\omega_1} + M_{s_1s_2s_1\omega_1} + 2M_{s_1s_2s_1s_2\omega_2} + M_{s_2s_1s_2\omega_2}\}, \tag{4.9}$$

$$M_{s_2s_1s_2s_1\omega_1} + 3M_{s_1s_2s_1s_2\omega_2} = \min\{M_{-\omega_1} + M_{s_2s_1s_2\omega_2} + 2M_{s_1s_2s_1s_2\omega_2}, M_{s_1s_2s_1\omega_1} + 3M_{-\omega_2}\}. \tag{4.10}$$

Compared to the tropical Plücker relations (see [BZ97]) which are of degrees 6, 9, 10 and 15, the diagonal relations are only degree 4 ((4.7) and (4.10)) or degree 5 ((4.8) and (4.9)).

To prove the diagonal relations, we need to find the detropical diagonal relations on the generalized minors. Using the representation theory of G_2, we know that the naive detropicalization of these diagonals does not hold - the detropical relations must contain terms that should not contribute to the tropicalization. For the order 4 diagonals, the detropical diagonal relations are given by

$$\Delta_{s_1\omega_1}\Delta^3_{s_2\omega_2} = \Delta_{s_2s_1\omega_1}\Delta^3_{\omega_2} + \Delta_{\omega_1}\Delta_{s_1s_2\omega_2}\Delta^2_{s_2\omega_2} + \Delta_{\omega_1}\Delta^2_{\omega_2}\Delta_{s_2s_1s_2\omega_2} + \Delta_{\omega_1}\Delta_{\omega_2}\Delta_{s_2\omega_2}\Delta_u,$$

$$\Delta_{s_2s_1s_2s_1\omega_1}\Delta^3_{s_1s_2s_1s_2\omega_2} = \Delta_{s_1s_2s_1s_2s_1\omega_1}\Delta_{s_2s_1s_2\omega_2}\Delta^2_{s_1s_2s_1s_2\omega_2} + \Delta_{s_1s_2s_1s_2s_1\omega_1}\Delta^2_{s_2s_1s_2s_1s_2\omega_2}\Delta_{s_1s_2\omega_2} \\ + \Delta_{s_1s_2s_1\omega_1}\Delta^3_{s_2s_1s_2s_1s_2\omega_2} + \Delta_{s_1s_2s_1s_2s_1\omega_1}\Delta_{s_1s_2s_1s_2\omega_2}\Delta_{s_2s_1s_2s_1s_2\omega_2}\Delta_u.$$

where u is the non-extremal weight of $V(\omega_2)$ of weight 0. It is unclear why the terms containing Δ_u should not contribute to the tropicalization. For the order 5 diagonals, we similarly expect that the detropical relations will include terms with the non-extremal weight vector of $V(\omega_2)$.

Chapter 5

Affine MV polytopes

In this chapter, we will consider G an affine Kac-Moody group. The representation theory of these groups are very similar to the representation theory of algebraic groups, so there is hope that the geometric theory of MV cycles will extend.

Unfortunately, in this case the affine Grassmanian can no longer be viewed as a geometric object and the standard definition of an MV cycle is not well defined. In recent work, Braverman, Finkelberg and Nakajima conjecture that geometric spaces called *Coulomb branches* will replace the affine Grassmanian in the generalization of the geometric Satake correspondence to the affine case [BFN19]. There is an analogue of MV cycles, called double MV cycles, introduced by Muthiah [Mut13]. In type A, Muthiah gives a combinatorial description of double MV cycles, though how these combinatorics are related to affine MV polytopes are not known.

On the other hand, affine MV polytopes are well understood. They have been defined using preprojective algebras [BKT14], KLR algebras [TW16] and affine PBW bases [MT18]. As in the finite case, affine MV polytopes have a purely combinatorial definition, which is used in this chapter. The motivation for this research is to contribute to the work of realizing affine MV polytopes as the tropical points of some variety. More specifically, we would like to answer the following question.

Question 5.0.1. When we take G to be an affine Kac-Moody group, can we generalize Theorem 2.6.5 to find a bijection between the tropical points of a variety and affine MV polytopes?

In this chapter, we simplify this question to the case where G is of rank 2. As Baumann, Kamnitzer and Tingley proved that affine MV polytopes are characterized by the fact that their 2-faces are rank 2 MV polytopes [BKT14], an answer to this question for rank 2 MV polytopes will be the first step towards an understanding of the general case.

Even by considering only 2-dimensional MV polytopes, this question is complicated, particularly because of the required decoration on the MV polytope. Instead, we can split an MV polytope into three polytopes: a lower, a middle and an upper polytope. We answer this question for the lower and upper polytopes.

5.1 Background

We consider the affine Kac Moody group of type $A_1^{(1)}$, denoted by $G = \widehat{SL_2}$, and its corresponding affine Lie algebra $\widehat{sl_2}$. Let α_0, α_1 be the simple roots where α_1 is the root for SL_2 and α_0 is the affine root. Let $\alpha_0^\vee, \alpha_1^\vee$ be the simple coroots, set $\delta = \alpha_0^\vee + \alpha_1^\vee$ and denote $\mathfrak{t}_\mathbb{R} = \mathrm{span}_\mathbb{R}\{\alpha_0^\vee, \alpha_1^\vee\}$. Let $\mathfrak{t}_\mathbb{R}^*$ be the dual space and let $\langle \cdot, \cdot \rangle : \mathfrak{t}_\mathbb{R} \times \mathfrak{t}_\mathbb{R}^* \to \mathbb{R}$ be the natural pairing. Let ω_0, ω_1 be the fundamental weights, i.e. $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{i,j}$. Denote the set of roots by Δ and the coroots by $\Delta^\vee$. The set of positive coroots are $\Delta_+^\vee = \{\alpha_0^\vee + i\delta, \alpha_1^\vee + i\delta, (i+1)\delta : i \in \mathbb{Z}_{\geq 0}\}$ while the set of negative coroots are $\Delta_-^\vee = -\Delta_+^\vee$. It follows that $\Delta^\vee = \Delta_+^\vee \cup \Delta_-^\vee$.

Let W be the affine Weyl group associated to G. The Weyl group can be viewed as the reflection group on the coroots generated by the simple reflections, s_1, s_2. These reflections act on the coroots by:

$$s_0(\alpha_0^\vee) = -\alpha_0^\vee, \qquad s_0(\alpha_1^\vee) = \alpha_1^\vee + \delta, \qquad s_1(\alpha_0^\vee) = \alpha_0^\vee + \delta, \qquad s_1(\alpha_1^\vee) = -\alpha_1.$$

Definition 5.1.1. A convex polytope P in $\mathrm{span}_\mathbb{Z}\{\alpha_0^\vee, \alpha_1^\vee\}$ is an $\widehat{sl_2}$ *GGMS polytope* if all edges are parallel to coroots in $\Delta^\vee$.

We can label the vertices of a GGMS polytope P by $\mu_k, \mu^k, \overline{\mu}_k, \overline{\mu}^k$ for $k \in \mathbb{N}$. For $k \geq 1$, there exists $a_k, a^k, \bar{a}_k, \bar{a}^k \in \mathbb{N}$ such that

$$\mu_k - \mu_{k-1} = a_k(\alpha_1^\vee + (k-1)\delta), \qquad \mu^{k-1} - \mu^k = a^k(\alpha_0^\vee + (k-1)\delta)),$$
$$\overline{\mu}_k - \overline{\mu}_{k-1} = \bar{a}_k(\alpha_0^\vee + (k-1)\delta), \qquad \overline{\mu}^{k-1} - \overline{\mu}^k = \bar{a}^k(\alpha_1^\vee + (k-1)\delta).$$

The limits $\lim_{k\to\infty} \mu_k, \lim_{k\to\infty} \mu^k, \lim_{k\to\infty} \overline{\mu}_k, \lim_{k\to\infty} \overline{\mu}^k$ exist and we denote them by

$$\lim_{k\to\infty} \mu_k = \mu_\infty, \qquad \lim_{k\to\infty} \mu^k = \mu^\infty, \qquad \lim_{k\to\infty} \overline{\mu}_k = \overline{\mu}_\infty, \qquad \lim_{k\to\infty} \overline{\mu}^k = \overline{\mu}^\infty. \tag{5.1}$$

We define a *decorated GGMS polytope* as a GGMS polytope P along with two sequences of non-negative integers $\lambda = (\lambda_1 \geq \lambda_2, \dots)$ and $\overline{\lambda} = (\overline{\lambda}_1 \geq \overline{\lambda}_2, \dots)$ with $\mu^\infty - \mu_\infty = |\lambda|\delta$ and $\overline{\mu}^\infty - \overline{\mu}_\infty = |\overline{\lambda}|\delta$.

We consider the MV polytopes of P as defined in [BDKT13]:

Definition 5.1.2 ([BDKT13, Definition 3.4])**.** An $\widehat{sl_2}$ MV polytope P is a decorated GGMS polytope such that:

(i) If the lines between $\mu_\infty, \overline{\mu}_\infty$ and $\mu^\infty, \overline{\mu}^\infty$ are parallel then $\lambda = \overline{\lambda}$, otherwise one is obtained from the other by removing a part of size $\langle \mu_\infty - \overline{\mu}_\infty, \omega_0 + \omega_1 \rangle$,

(ii) $\lambda_1, \overline{\lambda}_1 \leq \langle \mu_\infty - \overline{\mu}_\infty, \omega_0 + \omega_1 \rangle$,

(iii) For $k \geq 2$, $\max\{\langle \overline{\mu}_k - \mu_{k-1}, \omega_1 \rangle, \langle \mu_k - \overline{\mu}_{k-1}, \omega_0 \rangle\} = 0$,

(iv) For $k \geq 2$, $\min\{\langle \overline{\mu}^k - \mu^{k-1}, \omega_0 \rangle, \langle \mu^k - \overline{\mu}^{k-1}, \omega_1 \rangle\} = 0$.

For an example of one of these polytopes, see Figure 5.1. Note that $\langle \mu_\infty - \overline{\mu}_\infty, \omega_0 + \omega_1 \rangle$ is the "width" of the polytope.

These polytopes naturally split into three sub-polytopes: a lower polytope, an upper polytope and a middle polytope. For an MV polytope P, we can define the lower and upper sub-polytopes of P as

$$L(P) = \mathrm{conv}\{\mu_k, \overline{\mu}_k : k \in \mathbb{N}\}, \qquad U(P) = \mathrm{conv}\{\mu^k, \overline{\mu}^k : k \in \mathbb{N}\}.$$

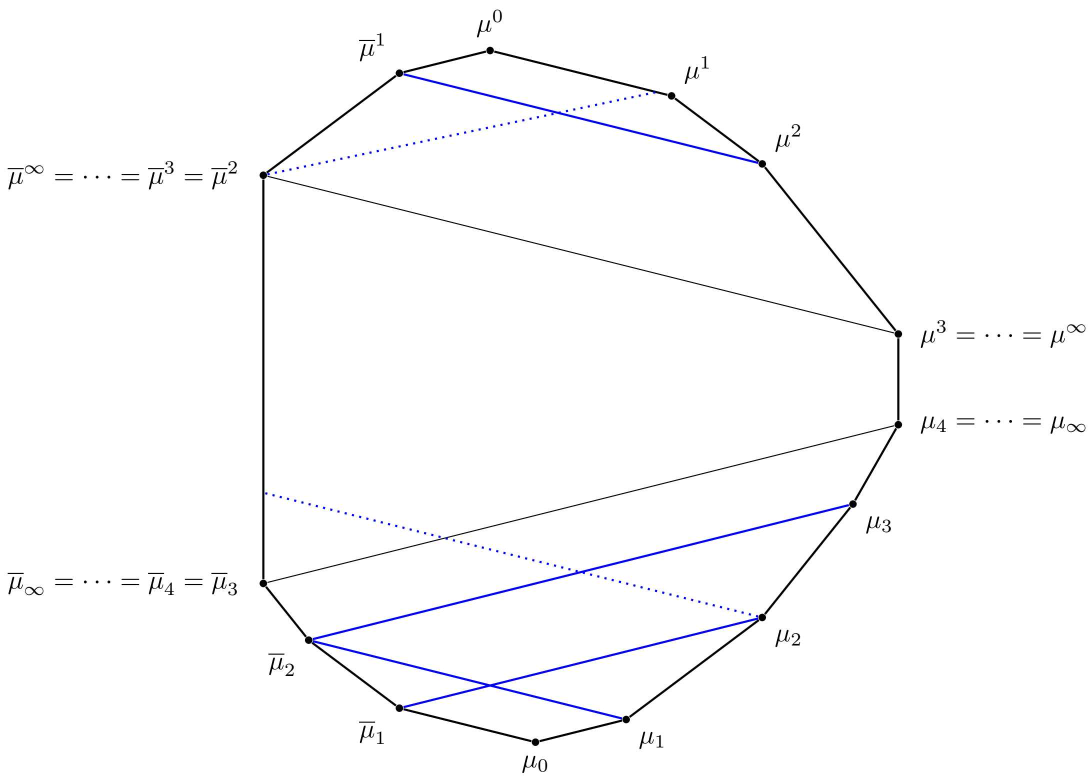


Figure 5.1: A rank 2 affine MV polytope

The middle polytope is the convex hull $M(P) = \text{conv}\{\mu_\infty, \mu^\infty, \overline{\mu}_\infty, \overline{\mu}^\infty\}$ along with the decorations $\lambda, \overline{\lambda}$.

Lemma 5.1.3. *For an MV polytope P, the subpolytopes $L(P), U(P)$ and $M(P)$ are GGMS polytopes.*

Proof. The only thing we need to show is that any new edges in these subpolytopes are still in the direction of a coroot and have integer length. In $L(P)$, the only new edge we introduce is $\mu_\infty - \overline{\mu}_\infty$. Let $k \in N$ be such that $\mu_k = \mu_\infty$ and $\overline{\mu}_\infty = \overline{\mu}_k$. First, notice that

$$\begin{aligned}\mu_\infty - \overline{\mu}_\infty = \mu_k - \overline{\mu}_k &= \sum_{i=1}^{k}(\mu_i - \mu_{i-1}) - \sum_{i=1}^{k}(\overline{\mu}_k - \overline{\mu}_{k-1}) \\ &= \sum_{i=1}^{k} a_i(\alpha_1^\vee + (i-1)\delta) - \sum_{i=1}^{k} \bar{a}_i(\alpha_0^\vee + (i-1)\delta)\end{aligned}$$

and thus $\langle \mu_\infty - \overline{\mu}_\infty, \omega_0 + \omega_1\rangle = \sum_{i=1}^k (2i-1)a_i - \sum_{i=1}^k (2i-1)\bar{a}_k \in \mathbb{Z}$.

The $k+1$ lower diagonal relation (iii) in Definition 5.2.2 gives $\min\{\langle \overline{\mu}_\infty - \mu_\infty, \omega_1\rangle, \langle \mu_\infty - \overline{\mu}_\infty, \omega_0\rangle\} = 0$. Thus one of these is zero and hence the edge $\mu_\infty - \overline{\mu}_\infty \in \mathbb{Z}\alpha_0^\vee$ or $\mathbb{Z}\alpha_1^\vee$.

A similar proof works to show that the edge $\mu^\infty - \overline{\mu}^\infty$ in $U(P)$ is also in $\mathbb{Z}\alpha_0^\vee$ or $\mathbb{Z}\alpha_1^\vee$. As $M(P)$ shares the edges $\mu_\infty - \overline{\mu}_\infty$ and $\mu^\infty - \overline{\mu}^\infty$ with $U(P)$ and $L(P)$, then all edges of $M(P)$ are in simple coroot directions with lengths in $\mathbb{Z}$ as well. Thus these three subpolytopes are GGMS polytopes. □

Note that the previous lemma does not necessarily hold for a GGMS polytope P as these polytopes may not satisfy the diagonal relations.

We can define subsets of affine MV polytopes inspired by these subpolytopes.

Definition 5.1.4. A *lower affine MV polytope* is an affine MV polytope such that $\mu_\infty = \mu^0$ or $\overline{\mu}_\infty = \mu^0$. An *upper affine MV polytope* is an affine MV polytope such that $\mu^\infty = \mu_0$ or $\overline{\mu}^\infty = \mu_0$.

A *middle affine MV polytope* is an affine MV polytope such that $\mu_\infty = \mu_0$ or $\mu^\infty = \mu^0$ and $\overline{\mu}_\infty = \mu_0$ or $\overline{\mu}^\infty = \mu^0$ along with the decoration $\lambda, \overline{\lambda}$.

We will call the equation (iii) in Definition 5.2.2 the *lower diagonals*, while (iv) will be called the *upper diagonals*. These conditions have a visual interpretation. Consider the line in the $\alpha_1^\vee$ direction passing through $\overline{\mu}_{k-1}$ and the line in the $\alpha_0^\vee$ direction passing through μ_{k-1}. The diagonal relation guarantees that μ_k is on or below the first line while $\overline{\mu}_k$ is on or below the second line, with at least one of these vertices on their respective lines. If a vertex is on the diagonal, we call this line an *active diagonal*. In Figure 5.1, the diagonal relations are shown as the blue lines, where the solid lines are active diagonals and the dotted lines are not active.

We call a_n the length of the edge between μ_n and μ_{n+1}, while $\bar{a}_n$ is the length of the edge between $\overline{\mu}_n$ and $\overline{\mu}_{n+1}$. Similarly, let a^n be the length of the edge between μ^n and μ^{n+1} and $\bar{a}^n$ the length of the edge between $\overline{\mu}^n$ and $\overline{\mu}^{n+1}$. As in the finite case, we can define the Lusztig data as the sequence of lengths along a minimal path from μ_0 to μ^0 in the polytope.

Definition 5.1.5. The *right Lusztig data* is $(a_n, \lambda_n, a^n)_{n\in\mathbb{N}}$. The *left Lusztig data* is $(\bar{a}_n, \overline{\lambda}_n, \bar{a}^n)_{n\in\mathbb{N}}$.

Although the vertex data $(\mu_k, \overline{\mu}_k, \mu^k, \overline{\mu}^k)_{k\in\mathbb{N}}$ of an affine MV polytope P is an infinite collection, the polytope P can be written as a convex hull of a finite set and hence only a finite subset of the vertex data will be distinct. As we will see in Section 5.2.1, all lower affine MV polytopes have highest vertex w for some Weyl group element w. Similarly, every upper affine MV polytope is an upper MV polytope of lowest vertex v for some $v \in W$.

In general, the techniques to answer Question 5.0.1 in the finite case do not extend to the affine case. Instead, we would like to answer a simpler question by looking at the lower and upper affine polytopes separately. In the next section, we will describe the BZ data of lower affine MV polytopes and then we will show that these polytopes correspond to the tropical points of a reduced double Bruhat cell.

5.2 BZ data of affine MV polytopes

We define the BZ data of affine polytopes in a similar way to the finite case in Chapter 3. First, suppose that $s_{i_1}s_{i_2}\ldots s_{i_k}$ is a reduced word in the Weyl group. For an integer $k \geq 1$, there are exactly two Weyl group elements of length k: one starting with s_0 and the other starting with s_1. Denote the unique Weyl group element of length k starting with s_0 as $\overline{w}_k$ and denote the length k element starting with s_1 as w_k.

We will call weights of the form $s_{i_1}s_{i_2}\ldots s_{i_k}\omega_{i_k}$ *chamber weights of level* i_k and label the set of chamber weights by Γ. We will denote the weight $s_{i_1}s_{i_2}\ldots s_{i_k}\omega_{i_k}$ by $\overline{\gamma}_{k+1}$ if $i_1 = 0$, γ_{k+1} when $i_1 = 1$ and set $\gamma_1 = \omega_0$, $\overline{\gamma}_1 = \omega_1$.

For a fixed $w \in W$, denote $\Gamma^w = \{s_{i_1}\cdots s_{i_k}\omega_{i_k} : s_{i_1}\cdots s_{i_k} \leq w\} \cup \{\omega_0, \omega_1\} \subset \Gamma$, where $\leq$ is the strong Bruhat order.

Remark 5.2.1. As the root system is rank 2, these subsets of chamber weights are easy to describe. For $w = w_m$, these sets are given by:

$$\Gamma^w = \{\gamma_{k_1}, \overline{\gamma}_{k_2} : 1 \leq k_1 \leq m+1, 1 \leq k_2 \leq m\}, \quad \Gamma \setminus \Gamma^w = \{\gamma_{k_1}, \overline{\gamma}_{k_2} : k_1 \geq m+2, k_2 \geq m+1\}.$$

Similarly for $w = \overline{w}_m$, these sets are given by:

$$\Gamma^w = \{\gamma_{k_1}, \overline{\gamma}_{k_2} : 1 \leq k_1 \leq m, 1 \leq k_2 \leq m+1\}, \quad \Gamma \setminus \Gamma^w = \{\gamma_{k_1}, \overline{\gamma}_{k_2} : k_1 \geq m+1, k_2 \geq m+2\}.$$

Definition 5.2.2. Let P be a GGMS polytope. Define the hyperplane data of P as the collection $(M_\gamma, M^\gamma)_{\gamma\in\Gamma}$ where we define the lower hyperplane data $(M_\gamma)_{\gamma\in\Gamma}$ by $M_{\overline{\gamma}_k} = \langle \overline{\mu}_k, \overline{\gamma}_k\rangle$ and $M_{\gamma_k} = \langle \mu_k, \gamma_k\rangle$. Define the upper hyperplane data $(M^\gamma)_{\gamma\in\Gamma}$ by $M^{\overline{\gamma}_k} = \langle \overline{\mu}^k, -\gamma_k\rangle$ and $M^{\gamma_k} = \langle \mu^k, -\overline{\gamma}_k\rangle$.

For ease of notation, we will denote $M_{\overline{\gamma}_k} = M_{\overline{k}}, M_{\gamma_k} = M_k, M^{\overline{\gamma}_k} = M^{\overline{k}}, M^{\gamma_k} = M^k$. Also, set $\gamma_k^* := \overline{\gamma}_k$ and $\overline{\gamma}_k^* := \gamma_k$. The hyperplane data of a polytope will define the polytope P.

Lemma 5.2.3. *For P a GGMS polytope, $P = \{x \in \mathfrak{t}_\mathbb{R} : \langle x, \gamma\rangle \leq M_\gamma, \langle x, -\gamma^*\rangle \leq M^\gamma, \forall\gamma\in\Gamma\}$.*

Proof. Let P be a GGMS polytope. Let $\psi_P : \mathfrak{t}_\mathbb{R}^* \to \mathbb{R}$ by $\psi_P(\theta) = \max_{x\in P}\langle x, \theta\rangle$. By [Kam10, Appendix],

$$P = \{x \in \mathfrak{t}_\mathbb{R} : \langle x, \theta\rangle \leq \psi_P(\theta), \forall\theta \in \mathfrak{t}_\mathbb{R}^*\}.$$

We can apply [BKT14, Proposition 2.5, 2.16] to see that

$$\psi_P(\gamma_k) = \langle \mu_k, \gamma_k\rangle, \qquad \psi_P(\overline{\gamma}_k) = \langle \overline{\mu}_k, \overline{\gamma}_k\rangle, \qquad \psi_P(-\gamma_k) = \langle \overline{\mu}^k, -\gamma_k\rangle, \qquad \psi_P(-\overline{\gamma}_k) = \langle \mu^k, -\overline{\gamma}_k\rangle$$

so that $P \subset \{x \in \mathfrak{t}_\mathbb{R} : \langle x, \gamma\rangle \leq M_\gamma, \langle x, -\gamma^*\rangle \leq M^\gamma, \forall\gamma\in\Gamma\}$.

To prove the other inclusion, consider the affine Weyl fan. This fan has cones given by

$$\begin{aligned} C_{w_k} &= \mathrm{span}_{\mathbb{R}_{\geq 0}}\{\gamma_k, \gamma_{k+1}\}, & C_{\overline{w}_k} &= \mathrm{span}_{\mathbb{R}_{\geq 0}}\{\overline{\gamma}_k, \overline{\gamma}_{k+1}\}, & C^{\overline{w}_k} &= \mathrm{span}_{\mathbb{R}_{\geq 0}}\{-\gamma_k, -\gamma_{k+1}\}, \\ C^{w_k} &= \mathrm{span}_{\mathbb{R}_{\geq 0}}\{-\overline{\gamma}_k, -\overline{\gamma}_{k+1}, \} & C_\infty &= \mathrm{span}_{\mathbb{R}_{\geq 0}}\{\omega_0 - \omega_1\}, & C_{\overline{\infty}} &= \mathrm{span}_{\mathbb{R}_{\geq 0}}\{-\omega_0 + \omega_1\}. \end{aligned} \tag{5.2}$$

The union of these cones is $\mathfrak{t}_\mathbb{R}^*$ and ψ_P is linear on each of these cones [BKT14]. Consider x such that $\langle x, \gamma\rangle \leq M_\gamma, \langle x, -\gamma^*\rangle \leq M^\gamma, \forall\gamma\in\Gamma$. If $\theta \in C_{w_k}$, then $\theta = n_1\gamma_k + n_2\gamma_{k+1}$. By linearity of ψ_P and the definition of M_γ, $\psi_P(\theta) = n_1M_k + n_2M_{k+1}$ so that

$$\langle x, \theta\rangle = n_1\langle x, \gamma_k\rangle + n_2\langle x, \gamma_{k+1}\rangle \leq n_1M_k + n_2M_{k+1} = \psi_P(\theta).$$

A similar argument works for $\theta \in C_{\overline{w}_k} \cup C^{w_k} \cup C^{\overline{w}_k}$.

For $\theta \in C_\infty$, then $\theta = n_1(\omega_0 - \omega_1)$ for some $n \in \mathbb{R}_{\geq 0}$. By linearity of ϕ_P and the definition of M_γ, $\psi_P(\theta) = nM_1 + nM^1$ so that

$$\langle x, \theta\rangle = n\langle x, \omega_0\rangle + n\langle x, -\omega_1\rangle \leq nM_1 + nM^1 = \psi_P(\theta).$$

A similar argument works for $\theta \in C_{\overline{\infty}}$. Thus $P = \{x \in \mathfrak{t}_\mathbb{R} : \langle x, \gamma\rangle \leq M_\gamma, \langle x, -\gamma^*\rangle \leq M^\gamma, \forall\gamma\}$. ☐

By the definition of the hyperplane data, the vertices are given by $\mu_k = \sum_{i=0}^1 M_{w_k\cdot\omega_i} w_k \cdot \alpha_i^\vee$ and $\overline{\mu}_k = \sum_{i=0}^1 M_{\overline{w}_k\cdot\omega_i}\overline{w}_k \cdot \alpha_i^\vee$, where $w_0 = \overline{w}_0 = e$ the identity. Thus, for $k \geq 0$, the vertices are given

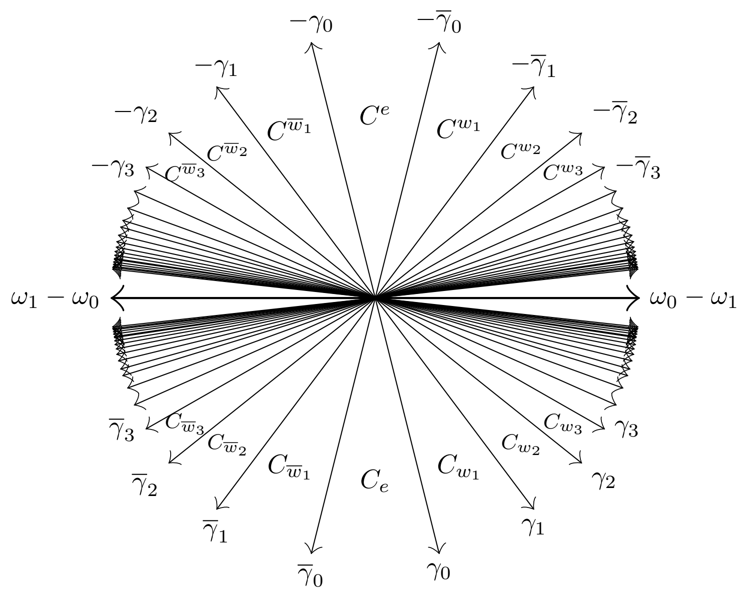


Figure 5.2: The affine Weyl fan

by

$$\mu_k = M_k(\alpha_1^\vee + k\delta) - M_{k+1}(\alpha_1^\vee + (k-1)\delta), \qquad \overline{\mu}_k = M_{\overline{k}}(\alpha_0^\vee + k\delta) - M_{\overline{k+1}}(\alpha_0^\vee + (k-1)\delta),$$
$$\mu^k = M^k(\alpha_0^\vee + k\delta) - M^{k+1}(\alpha_0^\vee + (k-1)\delta), \qquad \overline{\mu}^k = M^{\overline{k}}(\alpha_1^\vee + k\delta) - M^{\overline{k+1}}(\alpha_1^\vee + (k-1)\delta).$$

Note we use the notation $w_0 = e$ for convenience when we are listing the elements $\{w_k : k \in \mathbb{N}\}$. This is not to be confused the notation for the longest Weyl element w_0 in the finite case.

The relation between the Lusztig data and the hyperplane data is given by

$$a_k = 2M_k - M_{k-1} - M_{k+1}, \qquad \bar{a}_k = 2M_{\overline{k}} - M_{\overline{k-1}} - M_{\overline{k+1}},$$
$$a^k = -2M^k + M^{k-1} + M^{k+1}, \qquad \bar{a}^k = -2M^{\overline{k}} + M^{\overline{k-1}} + M^{\overline{k+1}}.$$

The diagonal relations can be rewritten in terms of the hyperplane data of the polytope.

Proposition 5.2.4. *Let P be a GGMS polytope which has lower hyperplane data $(M_\gamma)_{\gamma\in\Gamma}$ and upper hyperplane data $(M^\gamma)_{\gamma\in\Gamma}$. The condition (iii) in Definition 5.1.2 on the vertices of P is equivalent to equations:*

$$kM_k + kM_{\overline{k}} = \min\{kM_{k-1} + M_k + (k-1)M_{\overline{k+1}}, kM_{\overline{k-1}} + M_{\overline{k}} + (k-1)M_{k+1}\}, \text{ for } k \geq 2.$$

The condition (iv) in Definition 5.1.2 on the vertices of P is equivalent to the equations:

$$kM^k + kM^{\overline{k}} = \max\{kM^{k-1} + M^k + (k-1)M^{\overline{k+1}}, kM^{\overline{k-1}} + M^{\overline{k}} + (k-1)M^{k+1}\}, \text{ for } k \geq 2.$$

Proof. Let $k \geq 2$. First, notice that

$$\langle \overline{\mu}_k - \mu_{k-1}, \omega_1 \rangle = kM_{\overline{k}} - (k-1)M_{\overline{k+1}} - kM_{k-1} + (k-1)M_k,$$

$$\langle \mu_k - \overline{\mu}_{k-1}, \omega_0 \rangle = kM_k - (k-1)M_{k+1} - kM_{\overline{k-1}} + (k-1)M_{\overline{k}}.$$

The maximum of $\{\langle \overline{\mu}_k - \mu_{k-1}, \omega_1 \rangle, \langle \mu_k - \overline{\mu}_{k-1}, \omega_0 \rangle\}$ is equal to the minimum

$$-\min\{(k-1)M_{\overline{k+1}} + kM_{k-1} + M_k, (k-1)M_{k+1} + kM_{\overline{k-1}} + M_{\overline{k}}\} + kM_k + kM_{\overline{k}}$$

so that the condition $\max\{\langle \overline{\mu}_k - \mu_{k-1}, \omega_1 \rangle, \langle \mu_k - \overline{\mu}_{k-1}, \omega_0 \rangle\} = 0$ is equivalent to

$$\min\{(k-1)M_{\overline{k+1}} + kM_{k-1} + M_k, (k-1)M_{k+1} + kM_{\overline{k-1}} + M_{\overline{k}}\} = kM_k + kM_{\overline{k}}.$$

A similar proof works for the upper diagonal equations. □

When P is an MV polytope, the hyperplane data will satisfy the diagonal relations. We will call the lower hyperplane data $(M_\gamma)_{\gamma\in\Gamma}$ a *lower BZ datum* when the collection satisfies the lower diagonal relations (iii) in Definition 5.1.2. We call the upper hyperplane data $(M^\gamma)_{\gamma\in\Gamma}$ an *upper BZ datum* when the collection satisfies the upper diagonal relations (iv) in Definition 5.1.2. We call $(M_\gamma, M^\gamma)_{\gamma\in\Gamma}$ a *BZ datum* when $(M_\gamma)_{\gamma\in\Gamma}$ is a lower BZ datum and $(M^\gamma)_{\gamma\in\Gamma}$ is an upper BZ datum.

Similar to the finite case, the Lusztig data along one minimal path completely determines an MV polytope and every possible Lusztig data results in an MV polytope.

Theorem 5.2.5 ([BDKT13, Theorem 3.11])**.** *Consider a collection $(a_n, \lambda_n, a^n)_{n\in\mathbb{N}}$ such that*

(i) $(a_n, \lambda_n, a^n) \in \mathbb{Z}^3$ for all n

(ii) for large enough k, $a_k = \lambda_k = a^k = 0$

(iii) $\lambda_1 \geq \lambda_2 \geq \cdots$

Then there is a unique affine MV polytope P whose right Lusztig data is $(a_n, \lambda_n, a^n)_{n\in\mathbb{N}}$.

In the proof of this theorem, the left Lusztig data is explicitly constructed from the right Lusztig data. Moreover, by [BDKT13, Remark 3.22], the resulting polytope P with right Lusztig data (a_i, λ, a^i) has left Lusztig data $(\bar{a}_i, \overline{\lambda}, \bar{a}^i)$ where

$$\begin{aligned}\bar{a}_1 = \max\{&(k-1)a_k + (k-2)a_{k-1} - 2a_{k-2} - \cdots - 2a_2 - 2a_1, \text{ for } k \geq 2,\\ &\lambda_1 - \cdots - 2a_3 - 2a_2 - 2a_1,\\ &ka^k + (k+1)a^{k+1} + 2a^{k+2} + 2a^{k+3} + \cdots - 2a_{k-2} - \cdots - 2a_2 - 2a_1, \text{ for } k \geq 1\}.\end{aligned}$$

We will give an alternate proof of this equality for lower affine MV polytopes using the diagonal relations. To write the diagonals in terms of the Lusztig data, note that for $k \geq 2$,

$$\begin{aligned}\langle \overline{\mu}_k - \mu_{k-1}, \omega_1 \rangle &= \sum_{s=1}^{k} \langle \overline{\mu}_s - \overline{\mu}_{s-1}, \omega_1 \rangle - \sum_{s=1}^{k-1} \langle \mu_s - \mu_{s-1}, \omega_1 \rangle = \sum_{s=2}^{k} (s-1)\bar{a}_s - \sum_{s=1}^{k-1} s a_s,\\ \langle \mu_k - \overline{\mu}_{k-1}, \omega_0 \rangle &= \sum_{s=1}^{k} \langle \mu_s - \mu_{s-1}, \omega_0 \rangle - \sum_{s=1}^{k-1} \langle \overline{\mu}_s - \overline{\mu}_{s-1}, \omega_0 \rangle = \sum_{s=2}^{k} (s-1)a_s - \sum_{s=1}^{k-1} s\bar{a}_s.\end{aligned}$$

Thus the lower diagonal equations can be written as

$$\max\{(k-1)\bar{a}_k + (k-2)\bar{a}_{k-1} + \cdots + 2\bar{a}_3 + \bar{a}_2 - a_1 - 2a_2 - \cdots - (k-1)a_{k-1},$$

$$(k-1)a_k + (k-2)a_{k-1} + \cdots + 2a_3 + a_2 - \bar{a}_1 - 2\bar{a}_2 - \cdots - (k-1)\bar{a}_{k-1}\} = 0.$$

Lemma 5.2.6. *For an affine MV polytope P with right Lusztig data (a_i, λ, a^i),*

$$\bar{a}_1 \geq \max_{\substack{n\in\mathbb{N}\\ n\geq 2}}\{(n-1)a_n + (n-2)a_{n-1} - 2a_{n-2} - \cdots - 2a_2 - 2a_1\}. \tag{5.3}$$

If the right Lusztig data of the polytope has $\lambda = 0$ and $a^k = 0$ for all $k \geq 0$, then this is an equality.

Proof. Let $k \geq 2$. Consider the second term in the maximum of the $k+1$ diagonal. As this term is at most 0, we have

$$\bar{a}_1 \geq ka_{k+1} + (k-1)a_k + \cdots + 2a_3 + a_2 - 2\bar{a}_2 - \cdots - k\bar{a}_k.$$

Now consider the k^{th} diagonal. As the first term is at most 0, then isolating for $\bar{a}_k$, we have

$$\bar{a}_k \leq \frac{1}{k-1}\left(a_1 + 2a_2 + \cdots + (k-1)a_{k-1} - (k-2)\bar{a}_{k-1} - \cdots - 2\bar{a}_3 - \bar{a}_2\right) \tag{5.4}$$

Then

$$\bar{a}_1 \geq ka_{k+1} + (k-1)a_k + \sum_{i=1}^{k-1}\left((i-1) - \frac{ik}{k-1}\right)a_i - \sum_{i=2}^{k-1}\left(i - \frac{(i-1)k}{k-1}\right)\bar{a}_i.$$

By iterating this technique of removing the largest $\bar{a}_{k-n}$ by using the $k-n$ diagonal to find $\bar{a}_{k-n} \leq \frac{1}{k-n-1}\left(\sum_{i=1}^{k-n-1} ia_i - \sum_{i=2}^{k-n-1}(i-1)\bar{a}_i\right)$, this simplifies to

$$\bar{a}_1 \geq ka_{k+1} + (k-1)a_k - 2\sum_{i=1}^{k-1} a_i.$$

So we have shown $\bar{a}_1$ is larger than the maximum in (5.3).

Suppose the right Lusztig data is $(a_n, 0, 0)_{n\in\mathbb{N}}$. From the diagonal relations, either there exists $k \in \mathbb{N}$ such that $\langle \mu_k - \overline{\mu}_{k-1}, \omega_0\rangle = 0$ or $\langle \overline{\mu}_k - \mu_{k-1}, \omega_1\rangle = 0$ for every k.

Suppose we are in the first case and n is the smallest number such that $\langle \mu_n - \overline{\mu}_{n-1}, \omega_0\rangle = 0$. Using induction, it is easy to see that $\bar{a}_k = a_{k+1}$ for all $k \leq n$. By assumption, the second term in the n^{th} diagonal is zero and hence $\bar{a}_1 = (n-1)a_n + (n-2)a_{n-1} - 2a_{n-2} - \cdots - 2a_2 - 2a_1$.

Suppose we are in the second case so that $\langle \overline{\mu}_k - \mu_{k-1}, \omega_1\rangle = 0$ for every k and hence $\bar{a}_k = a_{k+1}$ for every k. As these polytopes are finite, there exists an n such that $a_n \neq 0$ but $a_{n+i} = 0$ for every $i \geq 1$. i.e. μ_n is the highest vertex on the right side of $L(P)$. It follows that $\bar{a}_{n-1} = a_n \neq 0$ but $\bar{a}_{n+i} = 0$ for all $i \geq 0$ so $\overline{\mu}_{n+i} = \overline{\mu}_{n-1}$ for all $i \geq 0$. Thus $\overline{\mu}_{n-1}$ is the highest vertex on the left side of $L(P)$.

Now, consider the upper left vertices. Let i be the smallest number such that $\overline{\mu}^i \neq \mu_n$ (as the right Lusztig data is $(a_n, 0, 0)_{n\in\mathbb{N}}$, $\overline{\mu}^0 = \mu_n$ so $i > 0$). Then $\mu_n - \overline{\mu}^i = \bar{a}^i(\alpha_1^\vee + (i-1)\delta)$ and thus $\langle \mu_n - \overline{\mu}^i, \omega_0\rangle \geq 0$.

Let j be the largest number such that $\overline{\mu}^j \neq \overline{\mu}_{n-1}$. Then $\overline{\mu}^j - \overline{\mu}_{n+1} = \bar{a}^{j+1}(\alpha_1^\vee + j\delta)$ and $\langle \overline{\mu}^j - \overline{\mu}_{n-1}, \omega_0\rangle \geq 0$. Similarly, for any k, $\overline{\mu}^{k-1} - \overline{\mu}^k = \bar{a}^k(\alpha_1^\vee + (k-1)\delta)$ so $\langle \overline{\mu}^{k-1} - \overline{\mu}^k, \omega_0\rangle \geq 0$. It follows that

$$\langle \mu_n - \overline{\mu}_{n-1}, \omega_0 \rangle = \left\langle \mu_n - \overline{\mu}^i + \sum_{k=i}^{j-1} \left(\overline{\mu}^k - \overline{\mu}^{k+1}\right) + \overline{\mu}^j - \overline{\mu}_{n-1}, \omega_0 \right\rangle \geq 0.$$

As the n^{th} diagonal relation guarantees $\langle \mu_n - \overline{\mu}_{n-1}, \omega_0 \rangle \leq 0$, then necessarily $\langle \mu_n - \overline{\mu}_{n-1}, \omega_0 \rangle = 0$. This means that only the first case is possible and so $\bar{a}_1 = (m-1)a_m + (m-2)a_{m-1} - 2a_{m-2} - \cdots 2a_2 - 2a_1$ for the smallest m such that $\langle \mu_m - \overline{\mu}_{m-1}, \omega_0 \rangle = 0$. □

5.2.1 Upper and lower polytopes

In this section, for $w \in W$, we will use the shorthand

$$\mu_w := \begin{cases} \mu_k, & \text{if } w = w_k \\ \overline{\mu}_k, & \text{if } w = \overline{w}_k \end{cases} \qquad \mu^w := \begin{cases} \overline{\mu}^k, & \text{if } w = w_k \\ \mu^k, & \text{if } w = \overline{w}_k \end{cases}$$

Definition 5.2.7. A *lower affine MV polytope of highest vertex at most* w is an affine MV polytope such that $\mu_w = \mu^0$. Denote the set of lower affine MV polytopes of highest vertex w by $\mathcal{P}_w$.

Definition 5.2.8. An *upper affine MV polytope of lowest vertex* w is an affine MV polytope such that $\mu^w = \mu_0$.

As upper and lower polytopes are closely related, we will only consider lower polytopes for the rest of the chapter. First, the condition $\mu_w = \mu^0$ will induce some conditions on the vertices on the other side of the polytope.

Lemma 5.2.9. *Consider $P \in \mathcal{P}_w$. If $w = w_m$ for some $m \in \mathbb{N}$, then $\overline{\mu}_{m-1} = \overline{\mu}^1$. If $w = \overline{w}_m$ for $m \in \mathbb{N}$, then $\mu_{m-1} = \mu^1$.*

Proof. Without loss of generality, suppose that $w = w_m$. First, notice that

$$\begin{aligned} \mu^0 - \overline{\mu}_{m-1} &= \sum_{s=1}^{\infty} (\overline{\mu}^{s-1} - \overline{\mu}^s) + (\overline{\mu}^\infty - \overline{\mu}_\infty) + \sum_{s=m-1}^{\infty} (\overline{\mu}_{s+1} - \overline{\mu}_s) \\ &= \sum_{s=1}^{\infty} (\bar{a}^s(\alpha_1^\vee + (s-1)\delta)) + |\overline{\lambda}|\delta + \sum_{s=m-1}^{\infty} (\bar{a}_{s+1}(\alpha_0^\vee + s\delta)). \end{aligned}$$

Thus, $\langle \mu^0 - \overline{\mu}_{m-1}, \omega_0 \rangle = \sum_{s=1}^{\infty} (s-1)\bar{a}^s + |\delta| + \sum_{s=k}^{\infty} (s+1)\bar{a}_{s+1} \geq 0$ as each term is non-negative. But by the lower diagonals (iii) in Definition 5.1.2,

$$\langle \mu^0 - \overline{\mu}_{m-1}, \omega_0 \rangle = \langle \mu_m - \overline{\mu}_{m-1}, \omega_0 \rangle \leq 0.$$

Thus $\langle \mu^0 - \overline{\mu}_{m-1}, \omega_0 \rangle = \sum_{s=1}^{\infty} (s-1)\bar{a}^s + |\delta| + \sum_{s=m-1}^{\infty} (s+1)\bar{a}_{s+1} = 0$ and hence $\bar{a}^s = 0$ for $s \geq 2$, $|\delta| = 0$ and $\bar{a}_t = 0$ for $t \geq m-1$. It follows that $\overline{\mu}_{m-1} = \overline{\mu}^1$. □

As in the finite case, we can consider the dual fan of $P \in \mathcal{P}_w$. Recall the affine Weyl fan defined by the maximal cones (5.2). Let $\ell(w) = m$. Define the cones

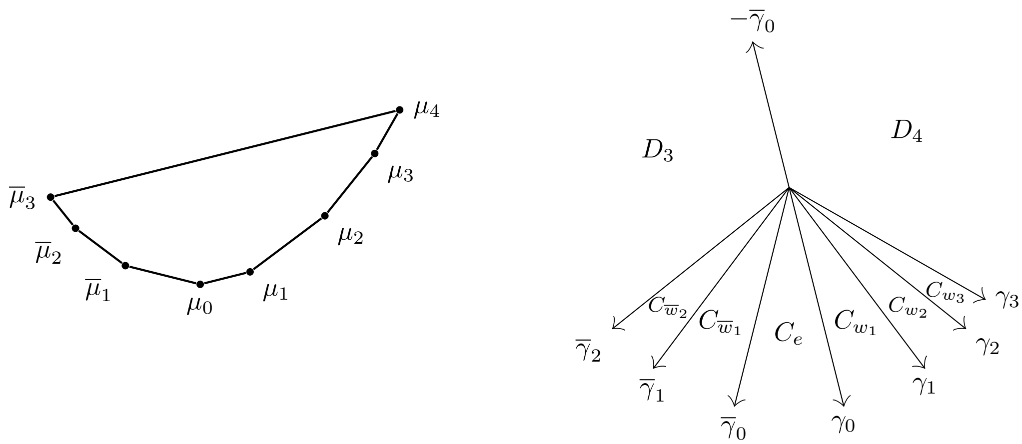
$$D_m = \begin{cases} \left(\bigcup_{k\geq m} C_{w_k}\right) \cup C_\infty \cup \left(\bigcup_{k\in\mathbb{N}} C^{w_k}\right), & \text{if } w = w_m \\ \left(\bigcup_{k\geq m} C_{\overline{w}_k}\right) \cup C_{\overline{\infty}} \cup \left(\bigcup_{k\in\mathbb{N}} C^{\overline{w}_k}\right), & \text{if } w = \overline{w}_m \end{cases}$$

$$D_{m-1} = \begin{cases} \left(\bigcup_{k\geq m-1} C_{\overline{w}_k}\right) \cup C_{\overline{\infty}} \cup \left(\bigcup_{k\geq 1} C^{\overline{w}_k}\right), & \text{if } w = w_m \\ \left(\bigcup_{k\geq m-1} C_{w_k}\right) \cup C_\infty \cup \left(\bigcup_{k\geq 1} C^{w_k}\right), & \text{if } w = \overline{w}_m \end{cases}$$

Note that D_m will always contain the cone C^e, as well as all the cones labelled by the Weyl elements associated to the vertices that lie between μ_w and μ^0 on a minimal path in P. Hence the dual fan $\mathcal{N}(P)$ is a coarsening of the fan generated by the maximal cones:

$$D_m, \qquad D_{m-1}, \qquad C_{w_k} \text{ for } 1 \leq k \leq m-1, \qquad C_{\overline{w}_k} \text{ for } 1 \leq k \leq m-2.$$

For an example, see Figure 5.3.



(a) A lower polytope in P_{w_4} (b) The dual fan of a polytope in P_{w_4}

Figure 5.3: A polytope in $\mathcal{P}_w$ and its dual fan

Remark 5.2.10. For P an affine MV polytope, every lower MV polytope $L(P) = \operatorname{conv}\{\mu_k, \overline{\mu}_k : k \in \mathbb{N}\}$ is a GGMS polytope by Lemma 5.1.3. It is easy to see that $L(P)$ is in fact an affine MV polytope as $L(P)$ will inherit all the conditions in Definition 5.1.2 from P.

As an affine MV polytope, the vertex μ^0 of $L(P)$ must be such that $\mu^0 = \mu_k$ or $\overline{\mu}_k$ for some $k \in \mathbb{N}$. This k must also be such that $\mu_\infty = \mu_k$ or $\overline{\mu}_\infty = \overline{\mu}_k$. Thus $L(P) \in \mathcal{P}_w$ for some $w \in W$.

Let $\underline{i} = (i_1, \ldots, i_m)$ be such that $w = s_{i_1} \ldots s_{i_m}$ is a reduced word.

Lemma 5.2.11. *The collection* $(M_\gamma)_{\gamma\in\Gamma}$ *is the lower BZ datum of a lower affine MV polytope of highest vertex* w *if each* $M_\gamma \in \mathbb{Z}$ *and:*

(i) For $1 \leq k \leq m$, $M_{\overline{k-1}} + M_{\overline{k+1}} \leq 2M_{\overline{k}}$ *and* $M_{k-1} + M_{k+1} \leq 2M_k$.

(ii) For $k \geq m+1$, $M_{\overline{k-1}} + M_{\overline{k+1}} = 2M_{\overline{k}}$ *and* $M_{k-1} + M_{k+1} = 2M_k$.

For $k = m$, *if* $i_1 = 1$, $M_{\overline{m-1}} + M_{\overline{m+1}} = 2M_{\overline{m}}$. *Otherwise,* $M_{m-1} + M_{m+1} = 2M_m$.

(iii) For $2 \leq k \leq m$

$$kM_{\overline{k}} + kM_k = \min\{kM_{\overline{k-1}} + M_{\overline{k}} + (k-1)M_{k+1}, kM_{k-1} + M_k + (k-1)M_{\overline{k+1}}\}.$$

Conversely, for an affine MV polytope P, the lower polytope $L(P)$ is a lower affine MV polytope of highest vertex w only if the lower BZ data $(M_\gamma)_{\gamma\in\Gamma}$ satisfies the conditions (i) - (iii).

Proof. Without loss of generality, suppose $w = w_m$ so that $i_1 = 1$.

Let P be an affine MV polytope and suppose that $L(P)$ is a lower affine MV polytope of highest vertex w. Then $(M_\gamma)_{\gamma\in\Gamma}$ is a lower BZ datum and the conditions (i) and (iii) follow directly. We need to show that the edge equalities (ii) hold exactly when $(M_\gamma)_{\gamma\in\Gamma}$ is the lower BZ datum of a lower polytope of highest vertex w.

By definition of $L(P)$, $\mu_w = \mu^0$, so that for every $k \geq m+1$, $\mu_k = \mu_w$. In particular, $\mu_k = \mu_{k-1}$ so that the Lusztig data $a_k = 0$. Thus $M_{k-1} + M_{k+1} = 2M_k$ for $k \geq m+1$.

If $k \geq m$, the edge equality $M_{\overline{k-1}} + M_{\overline{k+1}} = 2M_{\overline{k}}$ is equivalent to $\overline{\mu}_{k-1} = \overline{\mu}_k$. By Lemma 5.2.9, $\mu_w = \mu^0 \implies \overline{\mu}_{m-1} = \overline{\mu}^1$ and hence the rest of the edge equalities in (ii) hold.

Suppose that $(M_\gamma)_{\gamma\in\Gamma}$ satisfies (i) - (iii). All we need to show is that the edge equalities in (ii) imply the diagonal relations for (iii) for $k > m$ so that $(M_\gamma)_{\gamma\in\Gamma}$ is a lower BZ datum.

For $k \geq m$ suppose the diagonal relation holds:

$$-\min\{(k-1)M_{\overline{k+1}} + kM_{k-1} + M_k, (k-1)M_{k+1} + kM_{\overline{k-1}} + M_{\overline{k}}\} + kM_k + kM_{\overline{k}} = 0.$$

By adding $k(2M_{\overline{k+1}} - M_{\overline{k}} - M_{\overline{k+2}}) - k(2M_k - M_{k-1} - M_{k+1})$ to the first term in the diagonal, this term becomes

$$-kM_{\overline{k+2}} - (k+1)M_k - M_{k+1} + (k+1)M_{\overline{k+1}} + (k+1)M_{k+1}.$$

Similarly, by adding $k(2M_{k+1} - M_k - M_{k+2}) - k(2M_{\overline{k}} - M_{\overline{k-1}} - M_{\overline{k+1}})$ to the second term, we get

$$-kM_{k+2} - (k+1)M_{\overline{k}} - M_{\overline{k+1}} + (k+1)M_{k+1} + (k+1)M_{\overline{k+1}}.$$

Thus the $k+1^{\text{st}}$ diagonal equation also holds:

$$\min\{kM_{\overline{k+2}} + (k+1)M_k + M_{k+1}, kM_{k+2} + (k+1)M_{\overline{k}} + M_{\overline{k+1}}\} = (k+1)M_{k+1} + (k+1)M_{\overline{k+1}}$$

and by induction the diagonals hold for all k. □

Condition (i) guarantees that the sides of the polytope are non-negative and condition (ii) guarantees that the highest vertex is at most μ_w. Condition (iii) guarantees the diagonal relations are satisfied.

Let $\mathcal{M}_\Gamma^w$ be the set of all lower BZ data $(M_\gamma)_{\gamma\in\Gamma}$ that satisfy conditions (i), (ii) and (iii). Then we have a bijection between these BZ datum and polytopes of highest vertex w.

Theorem 5.2.12. *There is a bijection $\mathcal{P}_w \to \mathcal{M}_\Gamma^w$ by $P \mapsto (M_\gamma)_{\gamma\in\Gamma}$ as defined in Definition 5.2.2. Hence an MV polytope of highest vertex w is completely determined by its lower BZ data.*

Proof. By Lemma 5.2.11, the map that sends P to its lower BZ data is a map from $\mathcal{P}_w \to \mathcal{M}_\Gamma^w$. We need to show that a polytope $P \in \mathcal{P}_w$ is completely determined by its lower BZ data.

Suppose $(M_\gamma)_{\gamma\in\Gamma}$ is a collection as in Lemma 5.2.11. Without loss of generality, assume that $w = w_m$. Set

$$M^1 = -(m+1)M_m + (m-1)M_{m+1}, \qquad M^{\overline{1}} = -mM_{m+1} - mM_m,$$
$$M^{\overline{2}} = -(m+2)M_{m+1} - (m-2)M_m - M_{\overline{m}} + M_{\overline{m-1}}.$$

We recursively define $M^{k+1} = 2M^k - M^{k-1}$ for $k \geq 1$ and $M^{\overline{k+1}} = 2M^{\overline{k}} - M^{\overline{k-1}}$ for $k \geq 2$. Thus we have a collection $(M^\gamma)_{\gamma\in\Gamma}$ of integers which satisfy the edge equalities and diagonal relations trivially.

Define the map from $\mathcal{M}_\Gamma^w$ to convex polytopes where $(M_\gamma)_{\gamma\in\Gamma}$ is sent to the decorated polytope P with BZ data $(M_\gamma, M^\gamma)_{\gamma\in\Gamma}$ and decoration $\lambda = \lambda' = 0$. We need to prove that this polytope is a lower affine MV polytope of highest vertex w.

First, from the definition of $M^{\overline{2}}$, $(-2M^{\overline{1}}+M^{\overline{2}}+M^1)\alpha_1^\vee = \mu_m - \overline{\mu}_{m-1}$ so that $-2M^{\overline{1}}+M^{\overline{2}}+M^1 = \bar{a}^1 \in \mathbb{Z}$ and P is a GGMS polytope.

By Lemma 5.2.11, $(M_\gamma)_{\gamma\in\Gamma}$ is the lower BZ data of a lower affine MV polytope of highest vertex w and $(M^\gamma)_{\gamma\in\Gamma}$ is the upper BZ data of an affine MV polytope so P is in fact an MV polytope. By the definition of M^1 and $M^{\overline{1}}$, $\mu^0 = \mu_w$ thus P is a lower affine MV polytope of highest vertex w. □

Using the BZ data, we can now attempt to answer the following question:

Question 5.2.13. Can we find a natural variety X which is a positive space with a potential τ such that $X(\mathbb{Z}^{\text{trop}})^\tau_\geq$ will correspond to lower affine MV polytopes of highest vertex w?

Motivated by the finite case, we expect that $X = L^{w^{-1}}$. In the next section, we will find tropical functions M_γ that will take a non-negative tropical point to the BZ data of a lower affine MV polytope.

5.3 Tropical geometry of $L^{w^{-1}}$

Following [GLS11, Section 3.4], we can define the subgroups B_- and N of the Kac-Moody group $\widehat{SL_2}$. We define the reduced double Bruhat cell $L^{w^{-1}} := N \cap B_- w^{-1} B_-$.

Following Section 2.6, we want to define the tropical points of $L^{w^{-1}}$ using the Lusztig coordinates. Let x_i be the 1 parameter subgroup associated to α_i. Let $\underline{i} = (i_1, \ldots, i_m)$ be such that $w = s_{i_1} \ldots s_{i_m}$ is a reduced word, and define the function $x_{\underline{i}} : (C^\times)^m \to L^{w^{-1}}$ by $x_{\underline{i}}(a_1, \ldots, a_m) = x_{i_m}(a_m) \ldots x_{i_1}(a_1)$. Recall the following proposition of [Wil13]:

Proposition 5.3.1 ([Wil13, Proposition 4.3])**.** *The map $x_{\underline{i}} : (\mathbb{C}^\times)^m \to L^{w^{-1}}$ is a birational isomorphism.*

As we are working with w in the affine Weyl group of $\widehat{sl_2}$, there is only one choice of $\underline{i}$ and hence $x_{\underline{i}}$ forms a positive atlas on $L^{w^{-1}}$ as there are no transition functions. Thus the tropical points $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})$ are well defined.

Corollary 5.3.2. *$L^{w^{-1}}(\mathbb{Z}^{\text{trop}}) \cong \mathbb{Z}^m$.*

We can write $\ell \in L^{w^{-1}}(\mathbb{Z}^{\text{trop}})$ as $\ell = (P_1, \ldots, P_m)$, where $P_1, \ldots, P_m$ are indexed according to a reduced word $\underline{i} = (i_1, \ldots, i_m)$ for w. Note that the indices on these P_i are in the opposite order as the terms a_i in the Lusztig coordinates $x_{\underline{i}}(a_1, \ldots, a_m)$.

In the next sections, we define tropical functions M_γ and use a potential function to determine the non-negative tropical points of $L^{w^{-1}}$.

5.3.1 Tropical functions M_γ

Let λ be a dominant weight of $\widehat{SL_2}$ and let $V(\lambda)$ be the highest weight representation of weight λ, with highest weight vector v_λ. For δ, γ extremal weights of $V(\lambda)$, recall Definition 3.3.2 of the generalized minor function $\Delta_{\delta,\gamma}$. We denote $\Delta_{\omega_i,\gamma} = \Delta_\gamma$ when γ is a chamber weight of level i.

For $\gamma \in \Gamma$, the functions Δ_γ are defined with respect to extremal weight vectors in the fundamental representations of $\widehat{sl_2}$. These representations can be viewed as subrepresentations of the Fock space as in [Kac90]. We will use the combinatorial approach of [Tin11] to label vectors of the Fock space as partitions.

Definition 5.3.3. (Definition 2.2 of [Tin11]) A *charged partition* (λ, i) is a pair consisting of a partition λ and a integer i, where i is called the *charge*.

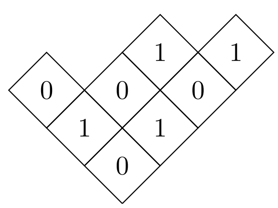
The *Fermionic Fock space* is the free span over all charged partitions. Define

$$F^{(m)} = \text{span}\{\text{charged partitions with charge } m\}.$$

We can write the charged partition (λ, k) as an upward-facing charged partition where each box of the partition is labelled by a 1 or a 0 by the following rules:

- the box in the first row is labelled by the charge
- the boxes in each row is constant
- the labelling on the rows alternates

For example, the charged partition ((3,2,2,1),0) is given by:



Let $\omega_i = (\emptyset, i) \in F^{(i)}$. By [Tin11, Proposition 3.34], the fundamental weight representations $V(\omega_i) \subset F^{(i)}$ are generated by the action of the elements F_1 and F_0 of $\widehat{sl_2}$ on the highest weight vector of weight ω_i. These actions are given by

$$F_i(\lambda, k) = \sum_{\substack{\mu \backslash \lambda \text{ is a} \\ i\text{-coloured box}}} (\mu, k), \qquad E_i(\lambda, k) = \sum_{\substack{\lambda \backslash \mu \text{ is a} \\ i\text{-coloured box}}} (\mu, k).$$

The chamber weights $\gamma = s_{i_1} \cdots s_{i_k} \omega_{i_k} \in \Gamma$ have extremal weight vectors v_γ given by the charged partition $((k, k-1, \ldots, 2, 1), i_k)$. Below, we denote extremal weight vectors v_γ by their weight γ.

To calculate the value of Δ_γ in the Lusztig coordinates of N, we need to understand how $x \in N$ acts on γ. Using that $x_i(p) = \exp(pE_i)$ and the series form of the exponential,

$$x_i(p) \cdot \gamma = \sum_{k=0}^{\infty} \frac{p^k}{k!} E_i^k \cdot \gamma = \sum_{\substack{\mu \text{ obtained by removing} \\ i\text{-coloured boxes from } \gamma}} \frac{p^{|\gamma \setminus \mu|}}{|\gamma \setminus \mu|!} \cdot \mu.$$

Then for an arbitrary $x \in N$,

$$x \cdot x_i(p) \cdot \gamma = \sum_{\substack{\mu \text{ obtained by removing} \\ i\text{-coloured boxes from } \gamma}} \frac{p^{|\gamma \setminus \mu|}}{|\gamma \setminus \mu|!} x \cdot \mu. \tag{5.5}$$

Inspired by the definition of $\underline{i}$-trails in [BZ01] on weights, we define an $\underline{i}$-trail for a partition γ to δ as follows.

Definition 5.3.4. Let $\underline{i} = (i_1, \dots, i_m)$. An $\underline{i}$-trail from γ to δ is a sequence of partitions μ_i such that $\gamma = \mu_0 \supseteq \mu_1 \supseteq \cdots \supseteq \mu_{m-1} \supseteq \mu_m = \delta$ and $\mu_{k-1} \setminus \mu_k$ are i_k-coloured boxes. Define the length of an $\underline{i}$-trail as the number of k such that $\mu_k \neq \mu_{k+1}$.

Lemma 5.3.5. *Let γ be a chamber weight of level i. In the Lusztig coordinates $x_{i_1}(p_1) \cdots x_{i_m}(p_m) \in N$ for a reduced word $\underline{i}$ of w^{-1}, the generalized minor takes on the value*

$$\Delta_\gamma(x_{i_1}(p_1) \cdots x_{i_m}(p_m)) = \sum_{\substack{(i_m, \dots, i_1)\text{-trails} \\ \text{from } \gamma \text{ to } \omega_i}} \prod_{s=1}^{m} \frac{p_{m+1-s}^{|\mu_{s-1} \setminus \mu_s|}}{|\mu_{s-1} \setminus \mu_s|!}.$$

Proof. Suppose that γ is a chamber weight of level i so that $\Delta_\gamma(g) = \langle g \cdot \gamma, \omega_i \rangle$. We proceed by induction on $\ell(w) = m$.

When $m = 1$, $\langle x_{i_1}(p_{i_1}) \cdot \gamma, \omega_i \rangle$ picks out the coefficient of ω_i. Thus $\Delta_\gamma(x_{i_1}(p_{i_1})) = p_{i_1}^{|\gamma \setminus \omega_i|}$ if there exists an i_1-trail from γ to ω_i, otherwise it is zero.

Suppose that for $x = x_{i_1}(p_1) \cdots x_{i_k}(p_k)$,

$$\Delta_\gamma(x) = \langle x \cdot \gamma, \omega_i \rangle = \sum_{\substack{(i_k, \dots, i_1)\text{-trails} \\ \text{from } \gamma \text{ to } \omega_i}} \prod_{s=1}^{k} \frac{p_{m+1-s}^{|\mu_{i_{s-1}} \setminus \mu_{i_s}|}}{|\mu_{i_{s-1}} \setminus \mu_{i_s}|!}.$$

Consider $x \cdot x_{i_{k+1}}(p_{k+1}) \cdot \gamma$. From (5.5),

$$\begin{aligned}
\langle x \cdot x_{i_{k+1}}(p_{k+1}) \cdot \gamma, \omega_i \rangle &= \sum_{\substack{i_{k+1}\text{-trails from} \\ \gamma \text{ to } \mu}} \frac{p_{m+1-(k+1)}^{|\gamma \setminus \mu|}}{|\gamma \setminus \mu|!} \langle x \cdot \mu, \omega_i \rangle \\
&= \sum_{\substack{i_{k+1}\text{-trails from} \\ \gamma \text{ to } \mu}} \left(\sum_{\substack{(i_k, \dots, i_1)\text{-trails} \\ \text{from } \mu \text{ to } \omega_i}} \prod_{s=1}^{k} \frac{p_{m+1-s}^{|\mu_{s-1} \setminus \mu_s|}}{|\mu_{s-1} \setminus \mu_s|!} \right) \frac{p_{m+1-(k+1)}^{|\gamma \setminus \mu|}}{|\gamma \setminus \mu|!} \\
&= \sum_{\substack{(i_{k+1}, i_k, \dots, i_1)\text{-trails} \\ \text{from } \gamma \text{ to } \omega_i}} \prod_{s=1}^{k+1} \frac{p_{m+1-s}^{|\mu_{s-1} \setminus \mu_s|}}{|\mu_{s-1} \setminus \mu_s|!}.
\end{aligned}$$

□

As the generalized minors can be written as subtraction-free expressions in the Lusztig coordinates, they are positive functions and thus can be tropicalized. Define the function $M_\gamma = \Delta_\gamma^{\mathrm{trop}}$ as the tropical function on $L^{w^{-1}}(\mathbb{Z}^{\mathrm{trop}})$. We will show that these tropical functions satisfy the same relations as in Lemma 5.2.11, and thus the set $(M_\gamma(\ell))_{\gamma\in\Gamma}$ will be the BZ datum of an affine MV polytope of highest vertex w for any non-negative tropical point $\ell \in L^{w^{-1}}$.

In the reduced word $\underline{i}$ for w, these generalized minor functions take on the value

$$\Delta_\gamma(x_{\underline{i}}(p_1,\ldots,p_m)) = \sum_{\substack{\underline{i}\text{-trails} \\ \text{from } \gamma \text{ to } \omega_i}} \prod_{s=1}^{m} \frac{p_s^{|\mu_{s-1}\setminus\mu_s|}}{|\mu_{s-1}\setminus\mu_s|!}.$$

By tropicalizing this relation, we give explicit formulae for these tropicalized functions.

Lemma 5.3.6. *Let $\underline{i} = (i_1,\ldots,i_m)$ be a reduced word for w and let $(P_1,\ldots,P_m) \in L^{w^{-1}}(\mathbb{Z}^{\mathrm{trop}})$. For any $k \in \mathbb{N}$,*

$$M_k(P_1,\ldots,P_m) = \min_{\substack{1\le j_1<j_2<\cdots<j_k\le m \\ i_{j_\ell}=0,\ \text{if } j_\ell=0 \mod 2 \\ i_{j_\ell}=1,\ \text{if } j_\ell=1 \mod 2}} \left\{\sum_{s=1}^{k}(k+1-s)P_{j_s}\right\},$$

$$M_{\bar{k}}(P_1,\ldots,P_m) = \min_{\substack{1\le j_1<j_2<\cdots<j_k\le m \\ i_{j_\ell}=0,\ \text{if } j_\ell=1 \mod 2 \\ i_{j_\ell}=1,\ \text{if } j_\ell=0 \mod 2}} \left\{\sum_{s=1}^{k}(k+1-s)P_{j_s}\right\}.$$

Note that when $\gamma_k, \overline{\gamma}_k \notin \Gamma^w$, then the set $\{(j_1,\ldots,j_k) : 1 \le j_1 < j_2 < \cdots < j_k \le m\}$ is empty. As the minimum is taken over an empty set, the corresponding function is equal to $-\infty$.

Proof. The tropicalization of Δ_γ is

$$M_\gamma(P_1,\ldots,P_m) = \min_{\substack{\underline{i}\text{-trails} \\ \text{from } \gamma \text{ to } \omega_i}} \left\{\sum_{s=1}^{m} |\mu_{s-1}\setminus\mu_s| P_s\right\}.$$

All we need to show is the minimum is only dependent on length k $\underline{i}$-trails from γ to ω_i.

Consider an $\underline{i}$-trail $\gamma = \mu_0 \supseteq \mu_1 \supseteq \cdots \supseteq \mu_{m-1} \supseteq \mu_m = \omega_i$ and the tropical value $\sum_{s=1}^{m} |\mu_{s-1} \setminus \mu_s|P_s$. We label the partition γ by labelling the boxes in $\mu_{s-1}\setminus\mu_s$ by P_s. Denote the labelling on box i in row j by $L(i,j)$. Define a path σ in γ as a collection of boxes from each row of the partition such that all the boxes are connected in γ, i.e. $\sigma = (\sigma(1),\cdots,\sigma(k))$ such that $1 \le \sigma(i) \le i$ is the choice of box in row i and $\sigma(i+1)$ is either $\sigma(i)$ or $\sigma(i)+1$. Then $L(\sigma(i),i)$ is the labelling in box $\sigma(i)$ in row i.

Let χ be a path in γ such that

$$L(\chi(1),1)+\cdots+L(\chi(k),k) = \min_{\sigma \text{ path in } \gamma}\{L(\sigma(1),1)+\cdots+L(\sigma(k),k)\},$$

the path with the minimal sum of its labelling. We will show that $\sum_{j=1}^{m} jL(\chi(j),j) \le \sum_{s=1}^{m} |\mu_{s-1} \setminus \mu_s|P_s$.

First, we define paths σ_i in γ. Let σ_i be the path such that $\sigma_i(j) = \chi(j)$ for $1 \le j \le i$. For $i+1 \le j \le k$, if $\chi(i+1) = \chi(i)$, then we set $\sigma_i(j) = \sigma_i(i) + (j-i)$. Otherwise, $\chi(i+1) = \chi(i)+1$

and we set $\sigma_i(j) = \sigma_i(i)$.

From this definition, $\sum_{j=1}^{i} L(\chi(j), j) = \sum_{j=1}^{i} L(\sigma_i(j), j)$. Since χ is a path with the minimal labelling, this implies that

$$\sum_{j=i+1}^{k} L(\chi(j), j) \leq \sum_{j=i+1}^{k} L(\sigma_i(j), j)$$

for every $1 \leq i < k$. It immediately follows that

$$\sum_{j=1}^{k}(j-1)L(\chi(j), j) = \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} L(\chi(j), j) \leq \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} L(\sigma_i(j), j).$$

Since $L(\chi(i), i) = L(\sigma_i(i), i)$ for $1 \leq i \leq k$, then $\sum_{j=1}^{k} jL(\chi(j), j) \leq \sum_{i=1}^{k}\sum_{j=i}^{k} L(\sigma_i(j), j)$. All that is left is to show that $\sum_{s=1}^{m} |\mu_{s-1} \setminus \mu_s| P_s = \sum_{s=1}^{k}\sum_{j=s}^{k} L(\sigma_s(j), j)$.

Claim 2. Let $S_i = \{(\sigma_i(j), j) : i \leq j \leq k\}$. The sets $\{S_i\}_{i=1}^{k}$ partition γ.

Proof: First, notice that the number of boxes in the set $\{(\sigma_i(j), j) : 1 \leq i \leq k, i \leq j \leq k\}$ is

$$\sum_{i=1}^{k}\sum_{j=i}^{k} 1 = \sum_{i=1}^{k}(k+1-i) = \sum_{i=1}^{k} i$$

which is exactly the number of boxes in γ.

Second, we show that no path σ_i contains a box $\sigma_i(j)$ in χ for $i+1 \leq j \leq k$. If $\sigma_i(j) = \sigma_i(i)$, then $\chi(i+1) = \chi(i)+1$ and so $\chi(j) \geq \chi(i)+1 > \sigma_i(j)$. If $\sigma_i(j) = \sigma_i(i)+(j-i)$, then $\chi(i+1) = \chi(i)$ so $\chi(j) < \chi(i) + (j-i) = \sigma_i(j)$. Thus σ_i doesn't intersect with χ above row i so $S_i \cap \{(\chi(j), j) : 1 \leq j \leq k\} = (\chi(i), i)$.

Third, we show for $i < j$, no two paths σ_i, σ_j intersect above row j. Let $j \leq \ell \leq k$ and consider $\sigma_i(\ell)$. If $\sigma_i(\ell) = \sigma_i(i)$, then $\chi(i+1) = \chi(i)+1$ so $\chi(s) > \chi(i)$ for every $s > i$. Then

$$\sigma_j(\ell) \geq \sigma_j(j) = \chi(j) > \chi(i) = \sigma_i(i) = \sigma_i(\ell)$$

If $\sigma_i(\ell) = \sigma_i(i) + (\ell - i)$, then $\chi(i+1) = \chi(i)$ so $\chi(s) < \chi(i) + (s-i)$ for every $s > i$. Then

$$\sigma_j(\ell) \leq \sigma_j(j) + (\ell - j) = \chi(j) + (\ell - j) < \chi(i) + (j-i) + (\ell - j) = \sigma_i(i) + (\ell - i) = \sigma_i(\ell)$$

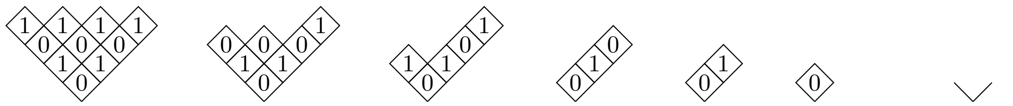
Thus $\sigma_i(\ell) \neq \sigma_j(\ell)$ as desired.

The second and third points tell us that $S_i \cap S_j = \emptyset$ for every $i \neq j$ and hence the sets S_i give a partition of γ. ■

As the sets S_i partition γ, then $\sum_{s=1}^{m} |\mu_{s-1} \setminus \mu_s| P_s = \sum_{i=1}^{k}\sum_{j=1}^{k} L(\sigma_s(j), j)$ as both are equal to the sum of all the labels on γ.

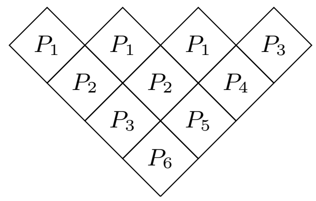
Consider a new labelling of γ by $L'(i, j) = L(\chi(j), j)$, i.e. we label each box in row j by $L(\chi(j), j)$. The corresponding $\underline{i}$-trail of length k for γ will give the tropical value $\sum_{j=1}^{m} jL(\chi(j), j)$. As we showed above, this is smaller than the value $\sum_{s=1}^{m} |\mu_{s-1} \setminus \mu_s| P_s$ and hence for any $\underline{i}$-trail, we can find a length k $\underline{i}$-trail which will give a smaller value. Thus the minimum in the tropical functions M_γ depend only on length k $\underline{i}$-trails. □

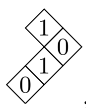
Example 5.3.7. Consider the reduced word $\underline{i} = (1, 0, 1, 0, 1, 0, 1)$ and the following $\underline{i}$-trail:



γ μ_1 μ_2 μ_3 μ_4 μ_5 $\mu_6 = \mu_7 = \omega_0$

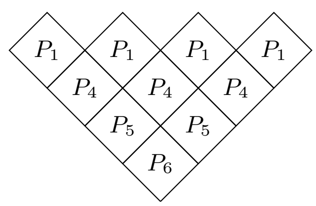
We label the partition γ as



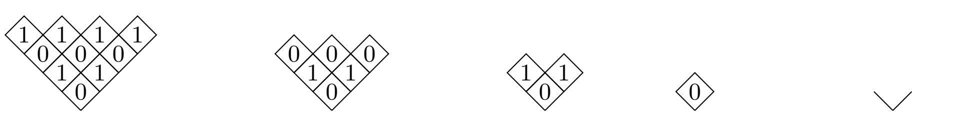
If $\chi =$  , then the paths σ_i in the proof of Lemma 5.3.6 are:

σ_1 σ_2 σ_3 $\sigma_4 = \chi$

Then the labelling of γ by



with the $\underline{i}$-trail



γ $\mu_1 = \mu_2 = \mu_3$ μ_4 μ_5 $\mu_6 = \mu_7 = \omega_0$

will give a smaller tropical value.

Using these equations for M_γ, we can show that the lower diagonals (iii) in Definition 5.2.2 are satisfied for small k.

Lemma 5.3.8. *On $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})$, the functions $M_k, M_{\overline{k}}$ satisfy*

$$kM_{\overline{k}} + kM_k = \min\{kM_{\overline{k-1}} + M_{\overline{k}} + (k-1)M_{k+1}, kM_{k-1} + M_k + (k-1)M_{\overline{k+1}}\} \tag{5.6}$$

for $k = 2, 3$.

Proof. Fix $(P_1, \ldots, P_m) \in L^{w^{-1}}(\mathbb{Z}^{\text{trop}})$. We will simplify the notation and drop the argument of M_γ by fixing $M_\gamma = M_\gamma(P_1, \ldots, P_m)$ in this proof.

Case $k = 2$: Using the formulae in Lemma 5.3.6, we know that

$$M_2 = \min_{\substack{j \\ i_j \text{ is odd}}} \{P_j\}, \qquad M_{\bar{2}} = \min_{\substack{j \\ i_j \text{ is even}}} \{P_j\},$$

$$M_3 = \min_{\substack{j_1 < j_2 \\ i_{j_1} \text{ is odd} \\ i_{j_2} \text{ is even}}} \{2P_{j_1} + P_{j_2}\}, \qquad M_{\bar{3}} = \min_{\substack{j_1 < j_2 \\ i_{j_1} \text{ is even} \\ i_{j_2} \text{ is odd}}} \{2P_{j_1} + P_{j_2}\}.$$

Let x, y be the indices such that $M_2 = P_x$ and $M_{\bar{2}} = P_y$.

Case A: Suppose $x < y$. Then $M_3 = 2P_x + P_y$ and $M_{\bar{3}} \geq P_x + 2P_y$ so that $\min\{P_x + M_{\bar{3}}, P_y + M_3\} = P_y + M_3 = 2P_x + 2P_y$.

Case B: Suppose that $y < x$. Then $M_{\bar{3}} = 2P_y + P_x$ and $M_3 \geq P_y + 2P_x$ so that $\min\{P_x + M_{\bar{3}}, P_y + M_3\} = P_x + M_{\bar{3}} = 2P_x + 2P_y$.

We have shown that $P_y + M_3 \geq 2P_x + P_y$ and $P_x + M_{\bar{3}} \geq 2P_x + P_y$ with at least one being equal and hence

$$\min\{P_x + M_{\bar{3}}, P_y + M_3\} = 2P_x + 2P_y.$$

Case $k = 3$: By the formulae in Lemma 5.3.6, there exists x_1, x_2, y_1, y_2 such that $M_3 = 2P_{x_1} + P_{x_2}$ and $M_{\overline{3}} = 2P_{y_1} + P_{y_2}$. Then $M_2 = \min\{P_{x_1}, P_{y_2}\}$, $M_{\overline{2}} = \min\{P_{x_2}, P_{y_1}\}$.

Case A. Suppose $x_1 < x_2 = y_1 < y_2$. The three terms in the diagonal simplify to:

$$3M_3 + 3M_{\overline{3}} = 6P_{x_1} + 9P_{x_2} + 3P_{y_2}, \tag{5.7}$$

$$3M_2 + M_3 + 2M_{\overline{4}} = 3\min\{P_{x_1}, P_{y_2}\} + 2P_{x_1} + P_{x_2} + 2 \min_{s < x_1 < x_2 < y_2 < t} \begin{Bmatrix} 3P_s + 2P_{x_1} + P_{x_2}, \\ 3P_{x_2} + 2P_{y_2} + P_t \end{Bmatrix}, \tag{5.8}$$

$$3M_{\overline{2}} + M_{\overline{3}} + 2M_4 = 3P_{x_2} + 2P_{x_2} + P_{y_2} + 2 \min_{y_2 < u_1 < u_2} \begin{Bmatrix} 3P_{x_1} + 2P_{x_2} + P_{y_2}, \\ 3P_{y_2} + 2P_{u_1} + P_{u_2} \end{Bmatrix}. \tag{5.9}$$

Case A.1. $P_{x_1} \leq P_{y_2}$ so that $M_2 = P_{x_1}$. By the minimality of $M_{\overline{3}}$, $2P_{x_2} + P_{y_2} \leq 2P_{u_1} + P_{u_2}$, thus $M_4 = 3P_{x_1} + 2P_{x_2} + P_{y_2}$ and (5.9) is equal to (5.7). There are two cases for the value of (5.8) depending on the value of $M_{\overline{4}}$:

Case A.1.a $M_{\overline{4}} = 3P_s + 2P_{x_1} + P_{x_2}$ then (5.8) simplifies to:

$$3M_2 + M_3 + 2M_{\overline{4}} = 3(2P_s + P_{x_1}) + 6P_{x_1} + 3P_{x_2}.$$

By the minimality of $M_{\overline{3}}$, $2P_{x_2} + P_{y_2} \leq 2P_s + P_{x_1}$ so that (5.8) $\geq$ (5.7).

Case A.1.b $M_{\overline{4}} = 3P_{x_2} + 2P_{y_2} + P_t$ then (5.8) simplifies to:

$$3M_2 + M_3 + 2M_{\overline{4}} = 5P_{x_1} + 7P_{x_2} + 3P_{y_2} + (P_{y_2} + 2P_t).$$

By the minimality of M_3, $2P_{x_1} + P_{x_2} \leq 2P_{x_1} + P_t$ so that $P_{x_2} \leq P_t$. By assumption, $P_{x_1} \leq P_{y_2}$ so that $P_{x_1} + 2P_{x_2} \leq P_{y_2} + 2P_t$ and thus (5.8) $\geq$ (5.7).

Case A.2. $P_{y_2} \leq P_{x_1}$ so that $M_2 = P_{y_2}$. By the minimality of M_3 $2P_{x_1} + P_{x_2} \leq 2P_{y_2} + P_{u_1}$. Additionally, by minimality of M_2 and $M_{\overline{2}}$, $P_{y_2} \leq P_{u_2}$ and $P_{x_2} \leq P_{u_1}$ respectively. Thus

$$6P_{x_1} + 4P_{x_2} + 2P_{y_2} \leq 6P_{y_2} + 3P_{u_1} + P_{x_2} + 2P_{y_2} \leq 6P_{y_2} + 4P_{u_1} + 2P_{u_2}.$$

So $M_4 = 3P_{x_1} + 2P_{x_2} + P_{y_2}$ and (5.9) is equal to (5.7). There are two cases for the value of (5.8) depending on the value of $M_{\overline{4}}$.

Case A.2.a $M_{\overline{4}} = 3P_s + 2P_{x_1} + P_{x_2}$ then (5.8) simplifies to:

$$3M_2 + M_3 + 2M_{\overline{4}} = 6P_s + 6P_{x_1} + 3P_{x_2} + 3P_{y_2}.$$

By minimality of $M_{\overline{2}}$, $P_{x_2} \leq P_s$ so that (5.8) $\geq$ (5.7).

Case A.2.b $M_{\overline{4}} = 3P_{x_2} + 2P_{y_2} + P_t$ then (5.8) simplifies to:

$$3M_2 + M_3 + 2M_{\overline{4}} = 2P_{x_1} + 7P_{x_2} + 3P_{y_2} + 2(2P_{y_2} + P_t).$$

By the minimality of M_3, $2P_{x_1} + P_{x_2} \leq 2P_{y_2} + P_t$ so that (5.8) $\geq$ (5.7).

For this case, (5.7) = (5.9) and (5.7) $\leq$ (5.9) so that the diagonal relation holds.

Case B. Suppose $x_1 < y_1 < y_2 < x_2$. Then by minimality of M_3 and $M_{\overline{3}}$, $P_{x_1} \leq P_{y_2}$ and $P_{x_2} \leq P_{y_1}$. Thus $M_2 = P_{x_1}$ and $M_{\overline{2}} = P_{x_2}$. By the formulae in Lemma 5.3.6 and using the minimality of M_2, $M_{\overline{3}}$ and $M_{\overline{2}}$,

$$M_4 = 3P_{x_1} + 2P_{y_1} + P_{y_2}, \qquad M_{\overline{4}} = \min_{\substack{t_1 < t_2 < x_2 \\ x_2 < u_1 < u_2}} \left\{ \begin{matrix} 3P_{t_1} + 2P_{t_2} + P_{x_2}, \\ 3P_{x_2} + 2P_{u_1} + P_{u_2} \end{matrix} \right\}.$$

It follows that $3M_3 + 3M_{\overline{3}} = 6P_{x_1} + 3P_{x_2} + 6P_{y_1} + 3P_{y_2} = 3M_{\overline{2}} + M_{\overline{3}} + 2M_4$.

By the minimality of $M_{\overline{3}}$ and M_3, $2P_{y_1} + P_{y_2} \leq 2P_{t_1} + P_{t_2}$ and $2P_{x_1} + P_{x_2} \leq 2P_{u_1} + P_{u_2}$. Using these inequalities and the minimality of M_2 and $M_{\overline{2}}$,

$$6P_{t_1} + 4P_{t_2} + P_{x_2} \geq 6P_{y_1} + 3P_{y_2} + P_{t_2} + P_{x_2} \geq 6P_{y_1} + 3P_{y_2} + P_{x_1} + 2P_{x_2},$$
$$6P_{x_2} + 4P_{u_1} + 2P_{u_2} \geq 6P_{y_1} + 3P_{y_2} + P_{u_1} + 2P_{u_2} \geq 6P_{y_1} + 2P_{y_2} + P_{x_1} + 2P_{x_2}.$$

Hence $2M_{\overline{4}} \geq 6P_{y_1} + 2P_{y_2} + P_{x_1} + 2P_{x_2}$ and so

$$3M_2 + M_3 + 2M_{\overline{4}} \geq 6P_{x_1} + 3P_{x_2} + 6P_{y_1} + 3P_{y_2} = 3M_3 + 3M_{\overline{3}}.$$

Thus the diagonal relation holds.

Case C. The case of $y_1 < y_2 < x_1 < x_2$ is symmetric to Case A.

Case D. The case of $y_1 < x_1 < x_2 < y_2$ is symmetric to Case B. □

When $k > 3$, the number of cases becomes too large to manage. To prove the general case, we expect a inductive argument on the length of m will be necessary.

Remark 5.3.9. As in the B_2 case (see Remark 4.1.1), the naive detropicalization of these diagonal equations do not hold. To show this, we can find the detropicalized diagonal relation for $k = 2$ on the generalized minors by studying the representation theory of $\widehat{SL_2}$.

Define the map $K_{0,1} : V(\omega_0) \otimes V(\omega_1) \to V(\omega_0 + \omega_1)$. In the subrepresentation $V(\omega_0 + \omega_1 - \delta)$ of the domain, we have the weight vectors

$$\begin{gathered}\omega_0 \cdot F_0 F_1 \omega_1 + F_1 F_0 \omega_0 \cdot \omega_1 - 2 s_1 \omega_1 \cdot s_0 \omega_0, \\ \omega_0 \cdot F_1 F_0 F_1 \omega_1 + 2\omega_1 \cdot s_1 s_0 \omega_0 - F_1 F_0 \omega_0 \cdot s_1 \omega_1, \\ F_0 F_1 F_0 \omega_0 \cdot \omega_1 + 2\omega_0 \cdot s_0 s_1 \omega_1 - s_0 \omega_0 \cdot F_1 F_0 F_1 \omega_1.\end{gathered}$$

Thus, the representation $V(\omega_0) \otimes V(\omega_1) \otimes \ker(K_{01})$ contains the vector

$$\begin{aligned} v =& 2(\omega_0^2 \cdot s_1\omega_1 \cdot s_0 s_1 \omega_1 + \omega_1^2 \cdot s_0\omega_0 \cdot s_1 s_0 \omega_0 - s_0\omega_0^2 \cdot s_1 \omega_1^2) \\ &+ \omega_0 \cdot \omega_1 (s_0\omega_0 \cdot F_1 F_0 F_1 \omega_1 + F_0 F_1 F_0 \omega_0 \cdot s_1 \omega_1). \end{aligned} \tag{5.10}$$

As the kernel of the map $K : V(\omega_0)^{\otimes 2} \otimes V(\omega_1)^{\otimes 2} \to V(2\omega_0 + 2\omega_1)$ contains $V(\omega_0) \otimes V(\omega_1) \otimes \ker(K_{01})$, the vector (5.10) is in $\ker(K)$. Note that this vector is not contained in a irreducible subrepresentation of the kernel of K.

Let $x \in V(\omega_0)^{\otimes 2} \otimes V(\omega_1)^{\otimes 2}$. Consider the functions $\Delta_x : \widehat{SL_2} \to \mathbb{C}$ by

$$\Delta_x(g) = \langle g \cdot x, v_{\omega_0}^2 \otimes v_{\omega_1}^2 \rangle.$$

For $x \in \ker(K)$, $\Delta_x = 0$ as in the finite case (see Lemma 4.2.1). By using v from (5.10), this reduces to the following equation of generalized minors:

$$\Delta_1^2 \Delta_2 \Delta_{\bar{3}} + \Delta_{\bar{1}}^2 \Delta_{\bar{2}} \Delta_3 + \frac{1}{2} \Delta_1 \Delta_{\bar{1}} (\Delta_2 \Delta_{F_1 F_0 F_1 \omega_1} + \Delta_{F_0 F_1 F_0 \omega_0} \Delta_{\bar{2}}) = \Delta_2^2 \Delta_{\bar{2}}^2 \tag{5.11}$$

Note that the vectors $F_1 F_0 F_1 \omega_1$ and $F_0 F_1 F_0 \omega_0$ have weight which is not an extremal weight. As shown in Lemma 5.3.8, these terms do not contribute to the tropicalization, though the reason why is not clear from this detropical equation.

5.3.2 Non-negative tropical points

Let $(i_1, \ldots, i_m)$ be a reduced word for w. Recall the notation $w_k^{\underline{i}} = s_{i_1} s_{i_2} \cdots s_{i_k}$ from Chapter 2.3.2. Define the potential function on $L^{w^{-1}}$ as

$$\tau = \Delta_{\omega_{i_1}}^{-1} \Delta_{s_{i_1}\omega_{i_1}}^{-1} \Delta_{\omega_{i_2}}^2 + \Delta_{\omega_{i_2}}^{-1} \Delta_{w_2^{\underline{i}}\omega_{i_2}}^{-1} \Delta_{s_{i_1}\omega_{i_1}}^2 + \sum_{k=2}^{m-1} \Delta_{w_{k-1}^{\underline{i}}\omega_{i_{k-1}}}^{-1} \Delta_{w_{k+1}^{\underline{i}}\omega_{i_{k+1}}}^{-1} \Delta_{w_k^{\underline{i}}\omega_{i_k}}^2.$$

In the Lusztig coordinates, as the generalized minors are positive functions by Lemma 5.3.5 and τ is a subtraction-free expression of these minors, then τ is also a positive function. We will consider the non-negative tropical points $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})_{\geq}^{\tau} = \{\ell \in L^{w^{-1}}(\mathbb{Z}^{\text{trop}}) : \tau^{\text{trop}}(\ell) \geq 0\}$, or more explicitly, this is the set of $\ell \in L^{w^{-1}}(\mathbb{Z}^{\text{trop}})$ such that

$$\begin{aligned} 0 \leq \min_{k=2,\ldots,m-1} \{ & 2\Delta_{\omega_{i_2}}^{\text{trop}}(\ell) - \Delta_{\omega_{i_1}}^{\text{trop}}(\ell) - \Delta_{s_{i_1}\omega_{i_1}}^{\text{trop}}(\ell), 2\Delta_{s_{i_1}\omega_{i_1}}^{\text{trop}}(\ell) - \Delta_{\omega_{i_2}}^{\text{trop}}(\ell) - \Delta_{w_2^{\underline{i}}\omega_{i_2}}^{\text{trop}}(\ell), \\ & 2\Delta_{w_k^{\underline{i}}\omega_{i_k}}^{\text{trop}}(\ell) - \Delta_{w_{k-1}^{\underline{i}}\omega_{i_{k-1}}}^{\text{trop}}(\ell) - \Delta_{w_{k+1}^{\underline{i}}\omega_{i_{k+1}}}^{\text{trop}}(\ell)\}. \end{aligned}$$

We want to show that, under the correct coordinates, we can associate $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})_{\geq}^{\tau} \cong \mathbb{N}^m$. First, we show that using the terms of τ, we can define a different set of coordinates on $L^{w^{-1}}$.

Lemma 5.3.10. *The map* $\phi : L^{w^{-1}} \to (\mathbb{C}^\times)^m$ *defined by*

$$\phi(g) = \left(\frac{\Delta^2_{\omega_{i_2}}(g)}{\Delta_{\omega_{i_1}}(g)\Delta_{s_{i_1}\omega_{i_1}}(g)}, \frac{\Delta^2_{s_{i_1}\omega_{i_1}}(g)}{\Delta_{\omega_{i_2}}(g)\Delta_{w_2^{\underline{i}}\omega_{i_2}}(g)}, \frac{\Delta^2_{w_2^{\underline{i}}\omega_{i_2}}(g)}{\Delta_{s_{i_1}\omega_{i_1}}(g)\Delta_{w_3^{\underline{i}}\omega_{i_3}}(g)}, \ldots, \frac{\Delta^2_{w_{m-1}^{\underline{i}}\omega_{i_{m-1}}}(g)}{\Delta_{w_{m-2}^{\underline{i}}\omega_{i_{m-2}}}(g)\Delta_{w_m^{\underline{i}}\omega_{i_m}}(g)} \right)$$

is a birational isomorphism.

Proof. By [GLS11, Theorem 3.3, Proposition 9.1], for the reduced word $(i_1, \ldots, i_m)$ of w, the tuple

$$(\Delta_{s_{i_1}\omega_{i_1}}, \Delta_{s_{i_1}s_{i_2}\omega_{i_2}}, \ldots, \Delta_{w_{m-1}^{\underline{i}}\omega_{i_{m-1}}}, \Delta_{w_m^{\underline{i}}\omega_{i_m}})$$

is an initial seed for the cluster structure of $\mathbb{C}[L^{w^{-1}}]$. This seed induces an isomorphism on the function fields of $L^{w^{-1}}$ and $(\mathbb{C}^\times)^m$. Since the map between the function fields is an isomorphism, the map $\psi_1 : L^{w^{-1}} \to (\mathbb{C}^\times)^m$ given by

$$\psi_1(g) = \left(\Delta_{w_k^{\underline{i}}\omega_{i_k}}(g)\right)_{k=1}^m$$

is also a birational isomorphism.

Consider the map $\psi_2 : (\mathbb{C}^\times)^m \to (\mathbb{C}^\times)^m$ by

$$\psi_2(x_1, \ldots, x_m) \mapsto \left(\frac{1}{x_1}, \frac{x_1^2}{x_2}, \frac{x_2^2}{x_1 x_3}, \ldots, \frac{x_k^2}{x_{k-1}x_{k+1}}, \ldots, \frac{x_{m-1}^2}{x_{m-2}x_{m+1}} \right)$$

This map is an isomorphism so the composition $\psi_2 \circ \psi_1$ is still a birational isomorphism. As $\Delta_{\omega_i} = 1$ on $L^{w^{-1}}$, then $\psi_2 \circ \psi_1 = \phi$. □

By this lemma, we have that ϕ^{-1} gives a positive structure on $L^{w^{-1}}$. Since $\phi \circ x_{\underline{i}} = \tau$ is a positive function, then $x_{\underline{i}}$ and ϕ^{-1} are different coordinates in the same positive atlas on $L^{w^{-1}}$. Using the coordinates of ϕ^{-1}, then $\ell \in L^{w^{-1}}(\mathbb{Z}^{\text{trop}})^{\tau}_{\geq}$ if and only if $\ell \in \mathbb{N}^m$ by the definition of ϕ.

Corollary 5.3.11. $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})^{\tau}_{\geq} \cong \mathbb{N}^m$.

We can show that the collection $(M_\gamma)_{\gamma \in \Gamma^w}$ satisfies the edge inequalities (i) on $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})^{\tau}_{\geq}$.

Lemma 5.3.12. *If the diagonal relations hold on* $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})$ *for the collection* $(M_\gamma)_{\gamma \in \Gamma^w}$*, then the collection* $(M_\gamma)_{\gamma \in \Gamma^w}$ *also satisfies the edge inequalities on* $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})^{\tau}_{\geq}$*. More explicitly, for* $\ell \in L^{w^{-1}}(\mathbb{Z}^{\text{trop}})^{\tau}_{\geq}$,

(i) if $\gamma_{k+1} \in \Gamma^w$*, then* $M_{k-1}(\ell) + M_{k+1}(\ell) \leq 2M_k(\ell)$.

(ii) if $\overline{\gamma}_{k+1} \in \Gamma^w$*, then* $M_{\overline{k-1}}(\ell) + M_{\overline{k+1}}(\ell) \leq 2M_{\overline{k}}(\ell)$.

Proof. Without loss of generality, assume $w = w_m$. For convenience, we will denote $a_k = 2M_k - M_{k-1} - M_{k+1}$ and $\bar{a}_k = 2M_{\overline{k}} - M_{\overline{k-1}} - M_{\overline{k+1}}$.

In terms of chamber weights, $\tau = \Delta^{-1}_{\overline{\gamma}_1}\Delta^{-1}_{\gamma_2}\Delta^2_{\gamma_1} + \Delta^{-1}_{\gamma_1}\Delta^{-1}_{\gamma_3}\Delta^2_{\gamma_2} + \sum_{k=2}^{m-1}\Delta^{-1}_{\gamma_k}\Delta^{-1}_{\gamma_{k+2}}\Delta^2_{\gamma_{k+1}}$. Thus, $\tau^{\text{trop}} = \min_{1 \leq k \leq m}\{a_k\}$. For $\ell \in L^{w^{-1}}(\mathbb{Z}^{\text{trop}})^{\tau}_{\geq}$ then $\tau^{\text{trop}}(\ell) \geq 0$ or equivalently, $a_k(\ell) \geq 0$ for $1 \leq k \leq m$. Hence if $\gamma_{k+1} \in \Gamma^w$ the edge inequality (i) holds.

Suppose $\overline{\gamma}_{k+1} \in \Gamma^w$. By assumption, the $(k+1)^{\text{st}}$ diagonal holds:

$$\max\{k\bar{a}_{k+1} + (k-1)\bar{a}_k + \cdots + 2\bar{a}_3 + \bar{a}_2 - a_1 - 2a_2 - \cdots - ka_k,$$

$$ka_{k+1} + (k-1)a_k + \cdots + 2a_3 + a_2 - \bar{a}_1 - 2\bar{a}_2 - \cdots - k\bar{a}_k\} = 0.$$

If $k = 1$, then the second term gives us that $a_2 \leq \bar{a}_1$ and hence $\bar{a}_1$ is positive since $a_2 \geq 0$. Otherwise, by the k^{th} diagonal, one of the following equalities holds:

$$(k-1)\bar{a}_k + \cdots + 2\bar{a}_3 + \bar{a}_2 = a_1 + 2a_2 + \cdots + (k-1)a_{k-1}, \tag{5.12}$$

$$(k-1)a_k + \cdots + 2a_3 + a_2 = \bar{a}_1 + 2\bar{a}_2 + \cdots + (k-1)\bar{a}_{k-1}. \tag{5.13}$$

If (5.13) holds, then from the second term in the maximum of the $(k+1)^{\text{st}}$ diagonal, $ka_{k+1} \leq k\bar{a}_k$. Thus $\bar{a}_k$ is positive as we have already showed that $a_{k+1} \geq 0$.

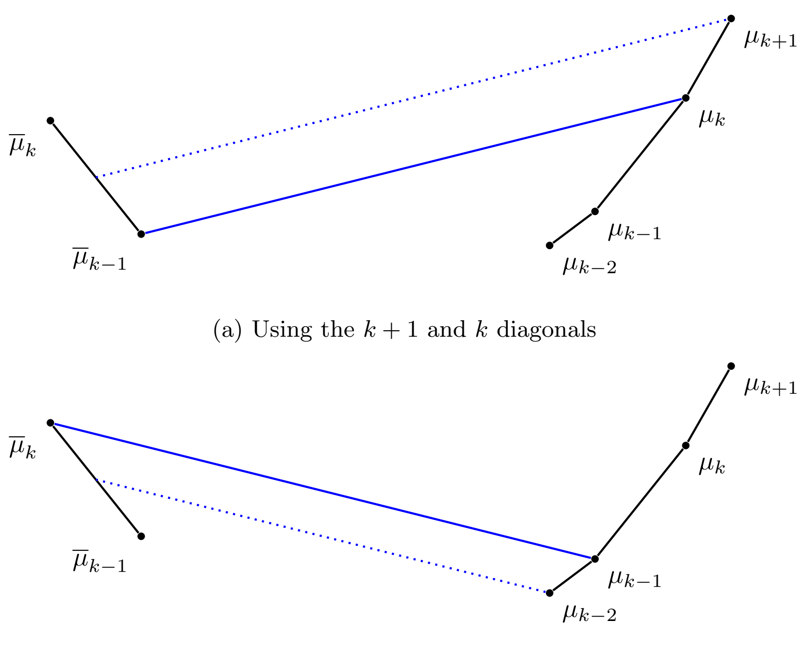
Suppose that (5.12) holds. This is equivalent to

$$(k-1)\bar{a}_k = -((k-2)\bar{a}_{k-1} + \cdots + 2\bar{a}_3 + \bar{a}_2 - a_1 - 2a_2 - \cdots - (k-2)a_{k-2}) + (k-1)a_{k-1}.$$

When $k = 2$, this equation gives $\bar{a}_2 = a_1$ and hence $\bar{a}_2$ is positive. Otherwise, by the $(k-1)^{\text{th}}$ diagonal, we have the inequality

$$(k-2)\bar{a}_{k-1} + \cdots + 2\bar{a}_3 + \bar{a}_2 - a_1 - 2a_2 - \cdots - (k-2)a_{k-2} \leq 0$$

and hence $(k-1)\bar{a}_k \geq (k-1)a_{k-1} \geq 0$ so $\bar{a}_k$ is positive. Thus if $\overline{\gamma}_{k+1} \in \Gamma^w$, then (ii) holds. □



(a) Using the $k+1$ and k diagonals

(b) Using the k and $k-1$ diagonals

Figure 5.4: Visualization of the proof of Lemma 5.3.12

Recall that the diagonals can be described as follows. Written as in the above proof, the first term in the k^{th} diagonal relation guarantees that k^{th} vertex on the left side of the polytope is on or below the line in α_0 direction from the $k-1^{\text{st}}$ vertex on the left side. The second term guarantees that the $k-1^{\text{st}}$ vertex on the left side is on or above the line in the $-\alpha_1$ direction from the k^{th}

vertex on the right side. The proof uses two diagonals to prove that $\bar{a}_k$ is larger than one of the edges on the left side of the polytope. The two cases can be visualized in Figure 5.4.

If $\gamma \notin \Gamma^w$, then M_γ is ∞ as it is the minimum of an empty set. We want to redefine the minors Δ_γ so that the tropicalized functions M_γ are a BZ datum.

Inspired by the finite case, for each $u \in W$, we define u_w as the maximal length element such that $u_w \leq_R v$ and $u_w \leq w$. In $\widehat{SL_2}$, it is easy to see this is well-defined and unique:

if $w = w_m$, then
$$u_w = \begin{cases} u, & \text{if } u \leq w \\ w_m, & \text{if } u = w_k \text{ and } u \not\leq w \\ \overline{w}_{m-1}, & \text{if } u = \overline{w}_k \text{ and } u \not\leq w \end{cases}$$
if $w = \overline{w}_m$, then
$$u_w = \begin{cases} u, & \text{if } u \leq w \\ w_{m-1}, & \text{if } u = w_k \text{ and } u \not\leq w \\ \overline{w}_m, & \text{if } u = \overline{w}_k \text{ and } u \not\leq w \end{cases}$$

This function takes the labelling on vertices above the highest vertex w and maps them to the largest Weyl group element smaller than w on the same side of the polytope. For example, when $w = w_m$, this function takes the labels of the vertices on the right side of the polytope above μ_w and maps them to w. On the left side, this function takes the vertices above $\mu_{\overline{w}_{m-1}}$ and maps the label to $\overline{w}_{m-1}$.

For $u\omega_i \in \Gamma$, we redefine the minors $\Delta^{\text{new}}_{u\omega_i} := \Delta_{u_w^{-1}u\omega_i, u\omega_i}$. Note that when $u\omega_i \in \Gamma^w$, then $\Delta^{\text{new}}_{u\omega_i} = \Delta_{u\omega_i}$. For $\gamma \notin \Gamma^w$, we can explicitly write out the new generalized minor functions.

Definition 5.3.13. Fix $w \in W$. If $\overline{\gamma}_k \notin \Gamma^w$, we define the minors

$$\Delta^{\text{new}}_{\overline{k}}(g) = \begin{cases} \langle g \cdot v_{\gamma_k}, v_{\overline{w}_{m-1}^{-1}\overline{\gamma}_k} \rangle, & \text{if } w = w_m \\ \langle g \cdot v_{\overline{\gamma}_k}, v_{\overline{w}_m^{-1}\overline{\gamma}_k} \rangle, & \text{if } w = \overline{w}_m \end{cases}$$

If $\gamma_k \notin \Gamma^w$, we define the minors

$$\Delta^{\text{new}}_{k}(g) = \begin{cases} \langle g \cdot v_{\gamma_k}, v_{w_m^{-1}\gamma_k} \rangle, & \text{if } w = w_m \\ \langle g \cdot v_{\gamma_k}, v_{w_{m-1}^{-1}\gamma_k} \rangle, & \text{if } w = \overline{w}_m \end{cases}$$

By definition of $u_w^{-1}u\omega_i$, there is only one $\underline{i}$-path from $u\omega_i$ to $u_w^{-1}u\omega_i$ and thus the new generalized minor $\Delta^{\text{new}}_{u\omega_i, u_w^{-1}u\omega_i}(x_{\underline{i}}(p_1, \ldots, p_m))$ is just a product of the p_i's.

Lemma 5.3.14. *Fix $w \in W$ and consider $x_{i_m}(p_m) \cdots x_{i_1}(p_1) \in L^{w^{-1}}$. If $\overline{\gamma}_k \notin \Gamma^w$, then*

$$\Delta^{\text{new}}_{\overline{k}}(x_{i_m}(p_m) \cdots x_{i_1}(p_1)) = \begin{cases} \displaystyle\prod_{s=2}^{m} \frac{p_s^{k-(s-2)}}{(k-(s-2))!}, & \text{if } w = w_m \\ \displaystyle\prod_{s=1}^{m} \frac{p_s^{k-(s-1)}}{(k-(s-1))!}, & \text{if } w = \overline{w}_m \end{cases}$$

If $\gamma_k \notin \Gamma^w$, then

$$\Delta^{\text{new}}_{k}(x_{i_m}(p_m) \cdots x_{i_1}(p_1)) = \begin{cases} \displaystyle\prod_{s=1}^{m} \frac{p_s^{k-(s-1)}}{(k-(s-1))!}, & \text{if } w = w_m \\ \displaystyle\prod_{s=2}^{m} \frac{p_s^{k-(s-2)}}{(k-(s-2))!} \rangle, & \text{if } w = \overline{w}_m \end{cases}$$

Proof. Suppose $w = w_m$ and let $x_{i_m}(p_m)\cdots x_{i_1}(p_1) \in L^{w^{-1}}$. For $\overline{\gamma}_k \notin \Gamma^w$, then $k \geq m$ and the only possible $\underline{i}$-trail is $\overline{w}_{m-1}^{-1}\overline{\gamma}_k \subset \overline{w}_{m-2}^{-1}\overline{\gamma}_k \subset \cdots \subset \overline{w}_1^{-1}\overline{\gamma}_k \subset \overline{\gamma}_k \subseteq \overline{\gamma}_k$, since $i_1 = 0$. Using this $\underline{i}$-trail,

$$\Delta_{\overline{k}}^{\text{new}}(x_{i_m}(p_m)\cdots x_{i_1}(p_1)) = \prod_{s=2}^{m} \frac{p_s^{k-(s-2)}}{(k-(s-2))!}.$$

For $\gamma_k \notin \Gamma^w$, then $k \geq m+1$ and the only possible $\underline{i}$-trail in this case is $w_m^{-1}\gamma_k \subset w_{m-1}^{-1}\gamma_k \subset \cdots \subset w_1^{-1}\gamma_k \subset \gamma_k$. Using this $\underline{i}$-trail,

$$\Delta_{k}^{\text{new}}(x_{i_m}(p_m)\cdots x_{i_1}(p_1)) = \prod_{s=1}^{m} \frac{p_s^{k-(s-1)}}{(k-(s-1))!}.$$

By switching the roles of 0 and 1, the case of $w = \overline{w}_m$ immediately follows. □

Denote $M_\gamma^{\text{new}} := (\Delta_\gamma^{\text{new}})^{\text{trop}}$. By tropicalizing these formulae, we have the following explicit equations for the new tropical minors.

Corollary 5.3.15. *Fix $w \in W$. For an arbitrary $(P_1, \ldots, P_m) \in L^{w^{-1}}(\mathbb{Z}^{\text{trop}})$, if $\overline{\gamma}_k \notin \Gamma^w$, then*

$$M_{\overline{k}}^{\text{new}}(P_1, \ldots, P_m) = \begin{cases} \displaystyle\sum_{s=2}^{m}(k-(s-2))P_s, & \textit{if } w = w_m \\ \displaystyle\sum_{s=1}^{m}(k-(s-1))P_s, & \textit{if } w = \overline{w}_m \end{cases}$$

If $\gamma_k \notin \Gamma^w$, then

$$M_{k}^{\text{new}}(P_1, \ldots, P_m) = \begin{cases} \displaystyle\sum_{s=1}^{m}(k-(s-1))P_s, & \textit{if } w = w_m \\ \displaystyle\sum_{s=2}^{m}(k-(s-2))P_s, & \textit{if } w = \overline{w}_m \end{cases}$$

These new generalized minors now satisfy the edge equalities.

Lemma 5.3.16. *On $L^{w^{-1}}$, the detropical edge equalities are satisfied:*

1) If $\overline{\gamma}_k \notin \Gamma^w$, then

a) if $w = w_m$, $(\Delta_{\overline{k}}^{\text{new}})^2 = \dfrac{k+1}{k-m+2}\Delta_{\overline{k-1}}^{\text{new}}\Delta_{\overline{k+1}}^{\text{new}}$

b) if $w = \overline{w}_m$, $(\Delta_{\overline{k}}^{\text{new}})^2 = \dfrac{k+1}{k-m+1}\Delta_{\overline{k-1}}^{\text{new}}\Delta_{\overline{k+1}}^{\text{new}}$

2) If $\gamma_k \notin \Gamma^w$, then

(a) if $w = w_m$, $(\Delta_{k}^{\text{new}})^2 = \dfrac{k+1}{k-m+1}\Delta_{k-1}^{\text{new}}\Delta_{k+1}^{\text{new}}$

(b) if $w = \overline{w}_m$, $(\Delta_{k}^{\text{new}})^2 = \dfrac{k+1}{k-m+1}\Delta_{k-1}^{\text{new}}\Delta_{k+1}^{\text{new}}$

Proof. Without loss of generality, assume $w = w_m$. If $\overline{\gamma}_k \notin \Gamma^w$, then $\overline{\gamma}_{k+1} \notin \Gamma^w$ and so Lemma 5.3.14 gives the values of $\Delta_{\overline{k}}$ and $\Delta_{\overline{k+1}}$. If $\overline{\gamma}_{k-1} \notin \Gamma^w$, Lemma 5.3.14 determines $\Delta_{\overline{k-1}}$. Otherwise,

$\overline{\gamma}_k = \overline{\gamma}_m$. As $\underline{i}$-trails are of length at most m, there is exactly one $\underline{i}$-trail from $\overline{\gamma}_m$ to ω_{i_m}, and so by Lemma 5.3.5, we can show that $\Delta_{\overline{k-1}}$ will be a monomial in the p_i's of the same form as in Lemma 5.3.14. Hence

$$\Delta_{\overline{k-1}}^{\text{new}}(x_{\underline{i}}(p_1,\ldots,p_m)) = \prod_{s=2}^{m} \frac{p_s^{k+1-s}}{(k+1-s)!}, \qquad \Delta_{\overline{k}}^{\text{new}}(x_{\underline{i}}(p_1,\ldots,p_m)) = \prod_{s=2}^{m} \frac{p_s^{k+2-s}}{(k+2-s)!},$$
$$\Delta_{\overline{k+1}}^{\text{new}}(x_{\underline{i}}(p_1,\ldots,p_m)) = \prod_{s=2}^{m} \frac{p_s^{k+3-s}}{(k+3-s)!}.$$

Thus

$$\begin{aligned}
\Delta_{\overline{k-1}}^{\text{new}}\Delta_{\overline{k+1}}^{\text{new}} &= \prod_{s=2}^{m} \frac{p_s^{k+1-s}}{(k+1-s)!} \cdot \prod_{t=2}^{m} \frac{p_t^{k+3-t}}{(k+3-t)!} \\
&= \prod_{s=3}^{m+1} \frac{1}{(k+2-s)!} \cdot \prod_{t=1}^{m-1} \frac{1}{(k+2-t)!} \left(\prod_{u=2}^{m} p_s^{2k+4-2u}\right) \\
&= \frac{k!}{(k+2-(m+1))!} \left(\prod_{s=2}^{m} \frac{1}{(k+2-s)!}\right) \frac{(k+2-m)!}{(k+1)!} \left(\prod_{s=2}^{m} \frac{1}{(k+2-s)!}\right) \prod_{u=2}^{m} p_s^{2k+4-2u} \\
&= \frac{k+2-m}{k+1} \left(\prod_{s=2}^{m} \frac{1}{(k+2-s)!} p_s^{k+2-s}\right)^2 = \frac{k+2-m}{k+1} \left(\Delta_{\overline{k}}^{\text{new}}\right)^2
\end{aligned}$$

If $\gamma_k \notin \Gamma^w$, then by a similar argument, the generalized minors are given by

$$\Delta_{k-1}^{\text{new}}(x_{\underline{i}}(p_1,\ldots,p_m)) = \prod_{s=1}^{m} \frac{p_s^{k-s}}{(k-s)!}, \qquad \Delta_{k}^{\text{new}}(x_{\underline{i}}(p_1,\ldots,p_m)) = \prod_{s=1}^{m} \frac{p_s^{k+1-s}}{(k+1-s)!},$$
$$\Delta_{k+1}^{\text{new}}(x_{\underline{i}}(p_1,\ldots,p_m)) = \prod_{s=1}^{m} \frac{p_s^{k+2-s}}{(k+2-s)!}.$$

Thus

$$\begin{aligned}
\Delta_{k-1}^{\text{new}}\Delta_{k+1}^{\text{new}} &= \prod_{s=1}^{m} \frac{p_s^{k-s}}{(k-s)!} \prod_{t=1}^{m} \frac{p_t^{k+2-t}}{(k+2-t)!} \\
&= \prod_{s=2}^{m+1} \frac{1}{(k+1-s)!} \prod_{t=0}^{m-1} \frac{1}{(k+1-t)!} \left(\prod_{u=1}^{m} p_u^{2k+2-2u}\right) \\
&= \frac{k!}{(k+1-(m+1))!} \left(\prod_{s=1}^{m} \frac{1}{(k+1-s)!}\right) \frac{(k+1-m)!}{(k+1)!} \left(\prod_{t=1}^{m} \frac{1}{(k+1-t)!}\right) \prod_{u=1}^{m} p_u^{2k+2-2u} \\
&= \frac{k+1-m}{k+1} \left(\prod_{s=1}^{m} \frac{1}{(k+1-s)!} p_u^{k+1-s}\right)^2 = \frac{k+1-m}{k+1} \left(\Delta_{k}^{\text{new}}\right)^2
\end{aligned}$$

□

Define the set $(M_\gamma)_{\gamma\in\Gamma}$ by $M_\gamma = \Delta_\gamma^{\text{trop}}$ if $\gamma \in \Gamma^w$ and $M_\gamma = \left(\Delta_\gamma^{\text{new}}\right)^{\text{trop}}$ if $\gamma \in \Gamma \setminus \Gamma^w$. By tropicalizing these equations, we immediately show the edge equalities hold.

Corollary 5.3.17. *The collection $(M_\gamma)_{\gamma\in\Gamma}$ satisfies the edge equalities (ii) in Lemma 5.2.11 on $L^{w^{-1}}$. More explicitly, for $\ell \in L^{w^{-1}}(\mathbb{Z}^{\text{trop}})$,* *(i) if* $\overline{\gamma}_k \notin \Gamma^w$*, then* $2M_{\overline{k}}(\ell) = M_{\overline{k-1}}(\ell) + M_{\overline{k+1}}(\ell)$

(ii) if $\gamma_k \notin \Gamma_w$*, then* $2M_k(\ell) = M_{k-1}(\ell) + M_{k+1}(\ell)$

Finally, for small enough w, we can construct a bijection between the set of lower affine MV polytopes and the non-negative tropical points with respect to τ of the reduced double Bruhat cell.

Theorem 5.3.18. *For* $w \in W$ *with* $\ell(w) \leq 3$*, the map* $L^{w^{-1}}(\mathbb{Z}^{\text{trop}})_{\geq} \longrightarrow \mathcal{P}_w$ *defined by*

$$\ell \mapsto (M_\gamma(\ell))_{\gamma \in \Gamma}$$

is a bijection.

Proof. Consider the functions $(M_\gamma)_{\gamma \in \Gamma}$. For $\gamma \in \Gamma^w$, these functions satisfy the diagonal relations by Lemma 5.3.8 and the edge inequalities by Corollary 5.3.17. For $\gamma \in \Gamma \setminus \Gamma^w$, these functions satisfy the edge equalities by Corollary 5.3.17. Hence, for any $\ell \in L^{w^{-1}}(\mathbb{Z}^{\text{trop}})$, $(M_\gamma(\ell))_{\gamma \in \Gamma}$ will satisfy the conditions (i) - (iii) in Lemma 5.2.11. Thus by Theorem 5.2.12, $(M_\gamma(\ell))_{\gamma \in \Gamma}$ is the BZ data of a lower affine MV polytope of highest vertex w and this map is well-defined.

To show this is in fact a bijection, consider the injective map which sends the BZ data $(M_\gamma(\ell))_{\gamma \in \Gamma}$ to $\left(2M_{s_{i_1}\cdots s_{i_k}\omega_{i_k}}(\ell) - M_{s_{i_1}\cdots s_{i_{k+1}}\omega_{i_{k+1}}}(\ell) - M_{s_{i_1}\cdots s_{i_{k-1}}\omega_{i_{k-1}}}(\ell)\right)_{k=1}^{m}$. By composing this map with the bijection in Lemma 5.3.10, there is a map which sends $(M_\gamma(\ell))_{\gamma \in \Gamma} \mapsto \ell$. Thus the map $\mathcal{P}_w \to L^{w^{-1}}(\mathbb{Z}^{\text{trop}})^{\tau}_{\geq}$ is a bijection. □

Remark 5.3.19. When G is a general affine Kac-Moody group, we expect a similar theorem to hold.

One crucial component to this theory would be the existence of a positive structure on $L^{w^{-1}}$. In the rank 2 case, the positive structure is trivial as there are no transition maps. For the general case, we would need to know that the transition functions $x_{\underline{i}} \circ x_{\underline{j}}^{-1}$ are subtraction free for two different reduced words $\underline{i}$ and $\underline{j}$ of w.

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