SPECIAL CORRESPONDENCES OF ABELIAN VARIETIES AND EISENSTEIN SERIES

BY

ALI CHERAGHI

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Department of Mathematics
University of Toronto

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Ali Cheraghi

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Department of Mathematics

University of Toronto

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Gross and Zagier were able to prove a relation between special values of derivatives of L-functions and algebro-geometrical data such as Heegner points and intersection numbers. A program initiated by Kudla generalized this to higher-dimensional cases and this has been a fruitful program of research for years. Here we prove two results in this direction. First, we consider the Shimura variety parametrizing pairs of CM abelian varieties of the same dimension with the same CM-field $K$ acting on them and define special cycles on this variety (over the integral model considered as a Deligne-Mumford stack) and relate the Arakelov degree of these special cycles to Fourier coefficients of totally positive index of Hilbert Eisenstein series. Then, we define Green functions on this Shimura variety and relate the degrees of Green functions to the Fourier coefficients of the same Eisenstein series with indices that are not totally positive (or the constant coefficient). In our second result, we do the same for the moduli space of pairs of abelian varieties first of which have CM by ring of integers $O_{K_0}$ of a CM-field $K_0$ and the second one is higher-dimensional and has an action by $O_{K_0}$. Then we use the same methods as in the first result to prove that the degrees of special cycles on this Shimura variety is related to Fourier coefficients of Eisenstein series.
To My Parents
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INTRODUCTION

Siegel-Weil formula was formulated in [21], [22], [24], [25] and is a relation between Eisenstein series and theta series. Specifically, this formula expresses an Eisenstein series as an average of theta series. This formula has been extensively generalized to different situations, specifically geometric Siegel-Weil formulas as in [7], [6], etc., arithmetic Siegel-Weil formula (i.e. Arakelov-theoretic Siegel-Weil formula) as in [17], [9], [10], [11], [2], [18] etc. and has been fruitful in relating cuspidal automorphic representations of different reductive groups (e.g. dual reductive groups).

In these generalizations, there have been beautiful theories such as local theta correspondence [12], Gan-Gross-Prasad conjecture and results towards it [3], [4], [5].

These generalizations, especially the so called Kudla programme has been useful in proving some important instances of Colmez conjecture on periods of CM abelian varieties, for example the average case of Colmez conjecture which was proven independently by Andreatta-Goren-Howard-Madapusi Pera in [1] and Yuan-Zhang in [31].

1.1 SUM OF FOUR SQUARES

By a result of Lagrange, we know that we can write every positive integer as a sum of four squares. Now the question arises that in how many ways can we write a positive integer as a sum of four squares (including 0 and the squares of negative numbers counted). The following table has the result for $1 \leq n \leq 10$. Let’s call this function $f(n)$. 

\[ \begin{array}{|c|c|}
\hline
n & f(n) \\
\hline
1 & 1 \\
2 & 1 \\
3 & 2 \\
4 & 4 \\
5 & 2 \\
6 & 5 \\
7 & 2 \\
8 & 4 \\
9 & 5 \\
10 & 6 \\
\hline
\end{array} \]
If we write the generating series for the function \( f(n) \), we would get the following
\[
\sum_{n \geq 0} f(n)q^n = (1 + 2q + 2q^4 + 2q^9 + \cdots)^4
\]
because if we write a number \( n \) as \( a^2 + b^2 + c^2 + d^2 \), then we multiply the powers \( q^{a^2}, q^{b^2}, q^{c^2}, q^{d^2} \) and we would add one to the coefficient of \( q^n \) (the fact that we have the coefficient 2 appearing in the series above is because for each nonzero \( a^2 \) there are two representations as a square, namely \((\pm a)^2\)). Now the important thing about the series \( \theta(\tau) = \sum_{n \in \mathbb{Z}} e^{2\pi i n \tau} = 1 + 2q + 2q^4 + 2q^9 + \cdots \) \((q = e^{2\pi i \tau} \text{ and } \tau \text{ changes in the upper half-plane})\) is the modular property below which can be proven by Poisson summation formula:
\[
\theta\left(-\frac{1}{4\tau}\right) = \sqrt{-2i\tau}\theta(\tau)
\]
and so the fourth power has the property
\[
\theta^4\left(-\frac{1}{4\tau}\right) = -4\tau^2\theta^4(\tau)
\]
and with some algebra we would get
\[
\theta^4\left(\frac{\tau}{4\tau + 1}\right) = (4\tau + 1)^2\theta^4(\tau)
\]
so that we have
\[
\theta^4\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2\theta^4(\tau) \quad (1.1)
\]
for \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \) and also for \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) (due to the fact that \( \theta(\tau) \) is given by its Fourier expansion), and a simple algebraic manipu-
lation shows that the identity $1.1$ is true for the subgroup of matrices in $\text{SL}_2(\mathbb{Z})$ generated by $\pm \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$ and $\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ which we call $\Gamma_0(4)$.

After realizing that $\theta^4(\tau)$ has modular properties, we can consider the vector space of all functions $f : \mathbb{H} \rightarrow \mathbb{C}$ with the same modular property ($\mathbb{H}$ is the upper half-plane):

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 f(\tau)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$. The important property of this vector space (not obvious a priori) is that it is 2-dimensional and can be generated by two linearly independent “Eisenstein series”:

$$G_{2,2} = -\frac{\pi^2}{3} (1 + 24 \sum_{n \geq 1} \left( \sum_{0 < d | n \atop d \text{ odd}} d \right) q^n)$$

$$G_{2,4} = -\pi^2 (1 + 8 \sum_{n \geq 1} \left( \sum_{0 < d | n \atop 4 | d} d \right) q^n)$$

and so we get that the “theta series” $\theta^4$ is a linear combination of these two and looking at the first few Fourier coefficients, we would get

$$\theta^4(\tau) = -\frac{1}{\pi^2} G_{2,4}(\tau)$$

which would in turn give us all the Fourier coefficients of $\theta^4(\tau)$ in terms of that of $G_{2,4}$:

$$f(n) = 8 \sum_{0 < d | n \atop 4 | d} d$$

(1.2)

and we get the number of ways we can write a number in terms of a sum of four squares in terms of a sum of divisors function. This can also be generalized to other cases of representation of numbers as a sum of two squares, six squares, etc. and for each of these cases we have a new theta series. This is one of the instances (and historically one of the first instances) that the relation of theta series ($\theta^4$ above) and Eisenstein series ($G_{2,4}$ above) helped to solve number theoretic problems (in particular you can see that $1.2$ shows that we can write every positive number as a sum of four squares).
1.2 SIEGEL-WEIL FORMULA

In this section, we are going to see the Siegel-Weil formula and in later sections, we would see different instances and generalizations of this formula to other situations.

Siegel-Weil formula was obtained in \cite{21,22,24,25} and essentially relates the average of some theta series in a fixed genus to Eisenstein series. Consider the abelian group $\Lambda = \mathbb{Z}^n$ and a quadratic form $Q : \Lambda \rightarrow \mathbb{Z}$ on $\Lambda$ (so this quadratic form on $(x_1, x_2, \cdots, x_n) \in \mathbb{Z}^n = \Lambda$ can be written as $Q(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} a_{ij} x_i x_j$ for some integers $a_i$) and we are going to attach a theta function to this quadratic form by

$$\theta_{\Lambda}(\tau) = \sum_T r_{\Lambda}(T)e^{2\pi i \text{tr}(T\tau)}$$

where $T$ goes over the positive semi-definite symmetric matrices with integer coefficients and where $r_{\Lambda}(T)$ is the number of $(x_1, x_2, \cdots, x_n) \in \mathbb{Z}^n = \Lambda$ with the property $\frac{1}{2}(\langle x_i, x_j \rangle)_{i,j} = T$ where $\langle x, y \rangle = Q(x + y) - Q(x) - Q(y)$ and $\tau$ is a variable in the $n$-dimensional Siegel half-space

$$\mathbb{H}_n = \{ x + iy | x \in \text{Sym}_n(\mathbb{R}), y \in \text{Sym}_n(\mathbb{R}) > 0 \}$$

This theta series has “modular properties” similar to the one we showed for $\theta^4$ in the previous section and Siegel was able to show that an average of these theta series would be equal to an Eisenstein series. For stating these results we need on more definition:

**Definition 1.1.** We say that two quadratic lattices $\Lambda, \Lambda'$ are in the same genus and show by $\Lambda \sim \Lambda'$ if $\Lambda \otimes \mathbb{R} \cong \Lambda' \otimes \mathbb{R}$ and $\Lambda \otimes \mathbb{Q}_p \cong \Lambda' \otimes \mathbb{Q}_p$ for all primes $p$ where the isomorphism is an isomorphism of quadratic spaces.

Now we state the theorem of Siegel:

**Theorem 1.2.** (Siegel) Fix a quadratic lattice $\Lambda$, then

$$\frac{\sum_{\Lambda' \sim \Lambda} \theta_{\Lambda'}(\tau)}{\#\text{Aut}(\Lambda')} = E_{\Lambda}(\tau)$$

where $E_{\Lambda}(\tau)$ is an Eisenstein series on $\mathbb{H}_n$.

If one compares the constant terms of the two sides in the above theorem one would get a mass formula which is a formula for $\frac{1}{\#\text{Aut}(\Lambda')}$ in terms of special values of $L$-functions (this is also known as the Minkowski-Siegel-Smith mass formula). For example, for a rank $n$ even unimodular lattice $\Lambda$
(i.e. a lattice in which \( \det((x_i, x_j))_{i,j} = \pm 1 \) for \( x_1, x_2, \cdots, x_n \) a basis of \( \Lambda \) and \( 2|Q(x) \) for all \( x \in \Lambda \) the mass formula becomes \( \frac{1}{|\text{Aut}(\Lambda)|} = \frac{B_a}{n} \prod_{1 \leq i \leq n} \frac{B_1}{4} \).

To explain the Siegel-Weil formula in the general case we need to define the Weil representation and use it to define theta functions and Eisenstein series in the general case and then change the discrete average of theta functions by theta integrals, which we are going to explain now.

To define the Weil representation for the archimedean places, we take a quadratic space \((V, q)\) of dimension \(d\) over \(\mathbb{R}\). Then we let \(S(V)\) be the space of Schwartz functions (i.e. an infinitely differentiable complex-valued function on \(V\) such that all the partial derivatives of any order are of rapid decay). Let \(\psi : \mathbb{R} \to \mathbb{C}^\times\) be the character \(\psi(x) = e^{2\pi itx}\) and let \(\chi : \mathbb{R}^\times \to \mathbb{C}^\times\) take values in \(\{\pm 1\}\) and let \(\gamma\) be an \(8t\) root of unity. Also let \(\hat{\Phi}\) be the Fourier transform of \(\Phi\) given by

\[
\hat{\Phi}(x) = \int_V \Phi(y)\psi(\langle x, y \rangle)dy
\]

where \(\langle x, y \rangle = q(x + y) - q(x) - q(y)\). Then the Weil representation is a representation of the group \(Mp_2(\mathbb{R}) \times O(V)\) on \(S(V)\) \((r : Mp_2(\mathbb{R}) \times O(V) \to \text{Aut}(S(V))\) defined by the following formulas (\(\Phi\) is an element of \(S(V)\)):

1. \((r(h)\Phi)(x) = \Phi(h^{-1}x) \quad \forall h \in O(V)\)
2. \((r(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix})\Phi)(x) = \chi(a)|a|^{d/2}\Phi(ax) \quad \forall a \in \mathbb{R}^\times\)
3. \((r(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix})\Phi)(x) = \psi(bq(x))\Phi(x) \quad \forall b \in \mathbb{R}\)
4. \((r(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})\Phi)(x) = (\pm 1)^d \gamma \hat{\Phi} \).

Weil representation for quadratic spaces over \(\mathbb{C}\) is defined similarly (\(\psi : \mathbb{C} \to \mathbb{C}^\times\) defined to be \(e^{4\pi i \text{Re}(x)}\)).

Now for defining the Weil representation for nonarchimedean places, we take a nonarchimedean characteristic \(0\) field \(k\) and a quadratic space \((V, q)\) over \(k\). Let \(O(V)\) be the orthogonal group of \(V\) and \(Mp_2(k)\) be the double cover of \(SL_2(k)\) which is called the metaplectic group. Then the Weil representation in this situation is the representation of \(Mp_2(k) \times O(V)\) on \(S(V)\) which is defined by the same formulas as above changing \(\mathbb{R}\) by \(k\) (If \(k\) is an extension of \(\mathbb{Q}_p\), \(\psi\) is defined to be \(\psi = \psi_{\mathbb{Q}_p} \circ tr_{k/\mathbb{Q}_p}\) where \(\psi_{\mathbb{Q}_p}(x) = e^{-2\pi il(x)}\) and \(l(x) : \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z}\) is the natural embedding).

So we have defined the Weil representation for both archimedean and nonarchimedean situation and we are going to use the Weil representations
Let $V$ be a quadratic space over a number field $k$. Take an adelic Schwartz function $\Phi \in S(V \otimes_k \mathbb{A}_k)$. We form a theta series by

$$
\theta(g,h,\Phi) = \sum_{x \in V} \langle r(g,h)\Phi(x) \rangle \quad (g,h) \in Mp_2(\mathbb{A}_k) \times O(V \otimes_k \mathbb{A}_k)
$$

This function is invariant under the action of $Mp_2(k) \times O(V)$. Now we define the Eisenstein series for some adelic Schwartz function (which we again call $\Phi$). Define $\delta : GL_2(\mathbb{A}_k) \to \mathbb{R}^\times$ to be defined on each $GL_2(k_v)$ for a place $v$ to be as follows: Take a matrix $A \in GL_2(k_v)$ and take its parabolic part as in Iwasawa decomposition and let it be $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, then define the Siegel Eisenstein series for $s \in \mathbb{C}$ to be

$$
E(s,g,\Phi) = \sum_{\gamma \in P(k) \setminus SL_2(k)} \delta(\gamma g)^s \langle r(\gamma g)\Phi(0) \rangle
$$

where $P(k)$ is the Borel subgroup of $SL_2(k)$. This series is convergent for $\text{Re}(s)$ large enough. Also the Eisenstein series satisfies a functional equation and extends to $\mathbb{C}$ as a meromorphic function.

Finally we can state the Siegel-Weil formula which writes a special value of Eisenstein series as an integral of theta functions:

**Theorem 1.3.** (Siegel-Weil formula) Let $(V,q)$ be a “nice” quadratic space (for example $(V,q)$ be anisotropic) then

$$
E(0,g,\Phi) = \frac{\lambda}{\text{vol}(SO(V) \setminus SO(V \otimes_k \mathbb{A}_k))} \int_{SO(V) \setminus SO(V \otimes_k \mathbb{A}_k)} \theta(g,h,\Phi)dh
$$

where $\lambda = 1$ if $\dim V > 2$ and $\lambda = 2$ if $\dim V = 2$ or $1$.

It is this formula that we are going to see other instances in geometric situations (namely Gross-Keating formula) and arithmetic situation (namely Kudla-Rapoport-Yang formula).

### 1.3 Gross-Keating Formula

In this section we are going to see the Gross-Keating formula [6] which is one case of geometric version of Siegel-Weil formula.

First we are going to define the Hurwitz class number of a positive integer. Hurwitz class number $H(d)$ is defined similar to the class number of imaginary quadratic fields with discriminant $-d$, the difference being that we take into account the inverse number of automorphisms of the binary quadratic forms. For example $H(3) = \frac{1}{3}$ because the only binary quadratic form of discriminant $-3$ up to $SL_2(\mathbb{Z})$-equivalence is $x^2 + xy +$
Theorem 1.4. (Hurwitz) If $m$ is not a perfect square, then we have

$$\sum_{dd' = m} \max\{d, d'\} = \sum_{t \in \mathbb{Z}, 4m - t^2 > 0} H(4m - t^2)$$

Gross and Keating [6] were able to give a geometric interpretation of this formula using intersection number of modular correspondences. They consider the space $Y(1)(\mathbb{C}) \times Y(1)(\mathbb{C})$ where $Y(1) = \mathbb{H}/\text{SL}_2(\mathbb{Z})$ where $\mathbb{H}$ is the upper half-plane. Now consider the coordinate ring of above to be $\mathbb{C}[j, j']$ for independent variables $j, j'$, now consider the polynomial in $j(\tau), j'(\tau')$ (not a priori obvious that this is a polynomial in $j(\tau), j'(\tau')$):

$$\Phi_m(j(\tau), j'(\tau')) = \prod_{\det A = m} (j(\tau) - j(A\tau'))$$

where the product goes over the $2 \times 2$ matrices with integers coefficients $A$ with determinant $m$ ($\text{SL}_2(\mathbb{Z})$)-equivalence means that $A$ and $B$ are equivalent if $AB^{-1} \in \text{SL}_2(\mathbb{Z})$). Considering $j = j'$ in $\Phi_m$ (i.e. considering $f_m(j) = \Phi_m(j, j)$) for $m$ nonsquare, then we get that

$$\deg f_m = \sum_{dd' = m} \max\{d, d'\}$$

which is the left hand side of the Hurwitz theorem. So if we assume that $T_m$ is the divisor defined by $\Phi_m = 0$. Then we have the intersection formula

$$\langle T_m, T_1 \rangle = \sum_{dd' = m} \max\{d, d'\} \quad (1.3)$$

if $m$ is not a square, then Gross and Keating are able to compute the intersection numbers of $T_m$ and $T_1$ using arithmetic geometry, namely the fact that one can write the points of $Y(1)(\mathbb{C}) \times Y(1)(\mathbb{C})$ as pairs $(E, E')$ for elliptic curves $E, E'$ over $\mathbb{C}$ and then the $T_m \cap T_1$ will be supported on the pairs of elliptic curves with complex multiplication with orders of discriminant $\geq -4m$, and they use this representation to prove the following (this is a special case of prop. 2.4. of [6]):

Theorem 1.5. $T_m$ and $T_1$ intersect properly if $m$ is not a square and

$$\langle T_m, T_1 \rangle = \sum_{t \in \mathbb{Z}, 4m - t^2 > 0} H(4m - t^2)$$
Using the identity 1.3 given the Hurwitz theorem. Another result they have among others is that they are able to relate the intersection number above to coefficient of the special value of an Eisenstein series. What they consider is a Siegel Eisenstein series defined classically by

$$E(Z, s) = \sum \det(CZ + D)^{-2} \frac{(\det Y)^s}{|\det(CZ + D)|^{2s}}$$

where $s$ is a complex variable, $Z = X + iY$ is an element of $\mathbb{H}_g$ and the sum is over the representatives with $A, B, C, D$ being $g \times g$ matrices. Then this Eisenstein series has a Fourier expansion as follows:

$$E(Z, s) = \sum_{M \in \text{Sym}_x(Z)} c_M(Y, s) e^{\pi \text{itr}(MZ)}$$

This Siegel Eisenstein can be shown to have a functional equation and can be meromorphically continued to the entire $s$-plane. Then “after some calculation” one finds that

**Proposition 1.6.** We have

$$\langle T_m, T_1 \rangle = c_2 \sum_{\text{det} M = (1, m) \, M > 0} c_M(Y, 0)$$

for $c_M(Y, 0)$ being the $M^{th}$ Fourier coefficient of a genus 2 ($g = 2$) Eisenstein evaluated at $s = 0$ where $c_2$ is a rational constant which does not depend on $m$. Also the coefficient $c_M(Y, 0)$ does not depend on $Y$.

An interesting comment in the paper is that if one considers the triple intersection numbers $\langle T_{m_1}, T_{m_2}, T_{m_3} \rangle$, then one has

$$\langle T_{m_1}, T_{m_2}, T_{m_3} \rangle = c_3 \sum_{\text{diag} M = (m_1, m_2, m_3) \, M > 0} c'_M(Y, 0)$$

where $c_3$ is a rational constant independent of $m_1, m_2, m_3$ and $c'_M(Y, 0)$ is the $M^{th}$ Fourier coefficient of $\frac{\partial E}{\partial s}(Z, 0)$ where $E$ is a genus 6 non-holomorphic Eisenstein series. As this Eisenstein series is non-holomorphic, it has some “negative” indexed Fourier coefficients as well. It would be an interesting quest to try to find a geometric meaning for those coefficients. This is essentially what happens in the arithmetic formulation of the Siegel-Weil formula for which the Kudla-Rapoport-Yang formula (see next section) is a special case.
1.4 KUDLA-RAPOPORT-YANG FORMULA

In this section, we are going to talk about generalizations of geometric Siegel-Weil formula to the arithmetic (i.e. Arakelov-theoretic) case.

In a paper published in 1975 [32], Zagier considered the Hurwitz class numbers $H(N)$ and formed the series $\sum_{N \geq 0} H(N)q^N$ (where we let $H(0) = -\frac{1}{12}$) and was able to add a non-holomorphic part to this series and prove that it becomes a modular form of weight $\frac{3}{2}$.

**Theorem 1.7.** ([32], Theorem 2) Let $\beta(t) = \int_1^\infty u^{-3/2}e^{-ut}du$ be the Bessel function and $q = e^{2\pi i \tau}$. Then the series

$$F(z) = y^{-1/2} \sum_{n \in \mathbb{Z}} \beta(4\pi n^2 y)e^{-2\pi i n^2 z} + \sum_{N \geq 0} H(N)q^N \quad (z = x + iy, y > 0)$$

is a modular form of weight $\frac{3}{2}$ for the congruence subgroup $\Gamma_0(4)$:

$$F\left(\frac{az + b}{cz + d}\right) = (\frac{c}{d})\left(\frac{-1}{d}\right)^{1/2}(cz + d)^{3/2}F(z) \quad \forall \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \Gamma_0(4)$$

As we know by the previous section, the holomorphic part of this modular form is related to geometry by the modular correspondences on the modular curve. The question arises whether it is possible to relate the negative indexed Fourier coefficients of $F(z)$ (and more generally for various modular forms) to arithmetic geometry using Arakelov geometry and the use of Gillet-Soulé Chow groups and pairs of Green functions and Weil divisors.

One of the instances this was done is in the paper of Kudla, Rapoport and Yang [17]. The motivation came from a paper of Kudla [13] that a certain family of Siegel Eisenstein series of $GSp_{2g}$ was introduced and these Eisenstein series had an odd functional equations and so a zero at the center of symmetry. It was suggested in [13] that this Eisenstein series may have relations to arithmetic geometry. The paper showed this for the “simplest possible example of an incoherent Eisenstein series”.

What they did was to consider a quadratic imaginary field $K$ and the moduli scheme $M$ over $\text{Spec} \mathcal{O}_K$ of elliptic curves with complex multiplication by $O_K$. Then they defined the special cycles $Z(n)$ for every positive integer $n$ which is a 0-cycle on $M$, explicitly the locus on which the elliptic curves have an additional endomorphism $y$ with $y^2 = -n$ anticommuting with the action of $O_K$. Then they considered the Eisenstein series

$$E_-(\tau, s) = y^{s/2} \sum_{\gamma \in \Gamma_0 \setminus \Gamma} (c\tau + d)^{-1}\Phi_q^-(\gamma)$$
where \( v \) is the imaginary part of \( \tau \), \( K = \mathbb{Q}(\sqrt{-q}) \) for a prime \( q \) and \( \Gamma = \text{SL}_2(\mathbb{Z}) \) and \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \) and also

\[
\Phi_q^{-}(\gamma) = \begin{cases} \chi_q(a) & q|c \\ -iq^{-1/2}\chi_q(a) & q \nmid c \end{cases}
\]

This is an “incoherent” Eisenstein series and so the central value of it (i.e \( E_{-}(\tau, 0) \)) is equal to zero and so the central derivative of \( E_{-} \) at \( s = 0 \) would be important which we call

\[
\Phi(\tau) := -h_K \frac{\partial}{\partial s}(E_{-}(\tau, s))|_{s=0}
\]

where \( h_K \) is the class number of \( K \). The first theorem of [17] is to compute the Fourier coefficients of \( \Phi \):

**Theorem 1.8.** ([17], Theorem 1) Let \( q = e^{2\pi i \tau} \) and

\[
\Phi(\tau) = a_0(\Phi, v) + \sum_{n<0} a_n(\Phi, v) + \sum_{n>0} a_n(\Phi)q^n
\]

then the \( a_n \)'s are given by

\[
a_n = \begin{cases} -h_K(\log q + \log v + 2N(1, \chi_q)) & \text{if } n = 0 \\ 2\log q(\text{ord}_q(n) + 1)\rho(n) + 2\sum_{p\neq q} \log p(\text{ord}_p(n) + 1)\rho(\frac{n}{p}) & \text{if } n > 0 \\ 2\beta_1(4\pi |n|v)\rho(-n) & \text{if } n < 0 \end{cases}
\]

where \( \Lambda(s, \chi_q) \) is the completed \( L \)-function of \( \chi_q \) and \( \rho(n) := \# \{ a \subseteq \mathcal{O}_K | N_{K/\mathbb{Q}}(a) = n \} \) and \( \beta_1(t) = \int_1^{\infty} u^{-1}e^{-ut}du \).

Now they also found the same expressions for the degrees of the special cycles \( Z(n) \) for \( n > 0 \) and they also defined Green functions \( Z(t, v) \) for \( t < 0 \) and considered the Arakelov degree of them to obtain

**Theorem 1.9.** ([17], Theorem 3) (1) Let \( n > 0 \), then we have \( \text{deg } Z(n) = a_n(\Phi) \). (2) Let \( t < 0 \), then we have \( \text{deg } Z(t, v) = a_t(\Phi, v) \).

Now the constant term of the derivative of the Eisenstein series remains and they remark that Bost was able to relate it to Faltings height of the elliptic curve with CM by \( \mathcal{O}_K \) (call it \( h_{\text{Fal}}(E) \)) and prove

\[
a_0(\Phi, v) = -h_K(\log v + 4h_{\text{Fal}}(E) + \log \pi - \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} + 2\log 2\pi)
\]
where \( \Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx \) and so the interesting idea of relating the constant term to Faltings heights remained which was studied much later in [27], [28], [29], [30] which were able to prove a lot of cases of Colmez conjecture. Later Ben Howard [9] extended these results greatly (except for the constant term) to unitary Shimura varieties and was able to define special cycles and Green functions on the unitary Shimura varieties. The same setting was considered in the papers [15] and [16]. These are the results that we generalize in two ways in our thesis.

1.5 RESULTS OF THIS THESIS

In this section we are going to give an overview of the results of this thesis and state the main theorems of this thesis. This thesis has two results and in this introduction, first we talk about the result on the space of products of CM abelian varieties with a fixed CM-field and will also talk about the result of Ben Howard [9], that this result is based on. Also in the second part, we talk about the second result of the thesis which is the relation of Eisenstein series and special divisors on the CM-cycles on Rapoport-Smithling-Zhang Shimura varieties.

First Result (Equal CM-Fields)

A program envisioned in [13], [14] was to relate algebro-geometrical data, namely the degrees of some special divisors (correspondences between moduli spaces) to the coefficients of Fourier expansion of the central value of the derivative of an Eisenstein series. This has been successful in some special cases including [8] for the case of CM elliptic curves, [9] for the case of CM Abelian varieties with CM by a field containing an imaginary quadratic field. Here, we consider general CM-fields and CM abelian varieties with CM by this field and prove that degrees of some special divisors (a correspondence between the moduli space of CM abelian varieties with a CM-type and the moduli space of CM abelian varieties with a second nearby CM-type) are related to the Fourier coefficients of a Hilbert Eisenstein modular form.

Let \( K \) be a CM-field with maximal totally real subfield \( F \). Let \( O_K \) be the ring of integers of \( K \). We denote by \( \bar{K} \) the normal closure of the field \( K \) and for a prime \( p \) of \( \bar{K} \), \( \bar{k}_p \) be the residue field of \( \bar{K} \) at \( p \) and \( \bar{k}_p \) a choice of algebraic closure of it. Also suppose that \( K/Q \) has a mild ramification condition (see the main theorem below) for small primes. If \( (A, \lambda_A) \) is an abelian variety with CM by \( O_K \) with the \( O_K \)-principal polarization \( \lambda_A \), \( (B, \lambda_B) \) be an \( O_K \)-principally polarized abelian variety having action by \( O_K \)
and $O_K$-principal polarization $\lambda_B$, we make $\text{Hom}_{O_K}(A, B)$ into a Hermitian space by letting $(f, g) := \lambda_A^{-1} \circ g^\vee \circ \lambda_B \circ f$ where $g^\vee : B^\vee \to A^\vee$ is the dual of $g$. This Hermitian form is $O_K$-valued. We define a Deligne-Mumford stack $\mathcal{Z}(\alpha)$ of triples $(A, B, f)$ where $A$ is an abelian variety (over the assigned scheme) with CM by $O_K$ and $B$ is an $O_K$-principally polarized Abelian variety with CM by $O_K$ and $f \in \text{Hom}_{O_K}(A, B)$ is an $O_K$-linear homomorphism such that $(f, f) = \alpha$. We compute the Arakelov degree of this stack (as a stack over Spec$\bar{\mathcal{O}}$ (ring of integers of $\bar{K}$)) defined as follows:

$$\deg \mathcal{Z}(\alpha) = \frac{1}{[\bar{K} : Q]} \sum_{p \subseteq \bar{O}} \log N(p) \left( \sum_{\alpha \in \mathcal{Z}(\bar{O})} \frac{\text{length}(O_{\mathcal{Z}(\alpha), z})}{\# \text{Aut } z} \right)$$

where $O_{\mathcal{Z}(\alpha), z}$ is the strictly Henselian local ring at $z$. As CM points have a transitive action of a group, it follows that for a fixed prime $p$ of $\bar{O}$, above length is constant for all $z$ and the problem reduces to that of computing the length for just one point and $\# \mathcal{Z}(\alpha) / (\bar{K}_p)$. Let $p_F$ be the prime of $F$ below $p$. For an fractional ideal $I$ of $O_F$ let $\rho(I)$ be the number of fractional ideals of $O_K$ with norm equal to $I$ in $F$, we will prove:

**Theorem 1.10.** Let $\alpha \in F^{\geq 0}$, assume that $K/F$ is ramified at at least a finite prime, If $p$ is a prime such that $p_F$ is nonsplit, then

$$\# \mathcal{Z}(\alpha)(\bar{K}_p) = \sum_{(A_1, A_2, f) \in \mathcal{Z}(\alpha)(\bar{K}_p)} \frac{1}{\# \text{Aut}(A_1, A_2, f)} = \frac{|C_K|}{\omega(K)} \rho(a_F^{-\epsilon_F} O_F)$$

where $\epsilon_F$ is 1 if $K/F$ is unramified and 0 if $K/F$ is ramified.

After this theorem, to compute the lengths of the local rings $O_{\mathcal{Z}(\alpha), z}$, we relate it (using Serre-Tate theorem) to “CM $p$-divisible groups” defined over local rings and then use Grothendieck-Messing deformation theory.

**Theorem 1.11.** Let $\alpha \in F^{\geq 0}$, $p$ be a prime of $\bar{K}$ such that $p_F$ is nonsplit in $K$, then at $z \in \mathcal{Z}(\alpha)(\bar{K}_p)$:

$$\text{length}(O_{\mathcal{Z}(\alpha), z}) = \frac{1}{2} e_{p_F} (\text{ord}_{p_F}(\alpha) + 1)$$

where $e_{p_F}$ is the ramification index of $\phi^{p_F}(p_F)$ in $\bar{K}/\phi^{p_F}(F)$.

We are also able to define the “arithmetic divisors” $\mathcal{Z}(\alpha)$ for all nonzero $\alpha \in F$. The main result is the relation of this degree to the derivative of the Fourier coefficients of an $SL_2(F)$-Eisenstein series of parallel weight $(1,1,\cdots,1)$.
Theorem 1.12. Let \( \alpha \) be a nonzero element of \( F \). Suppose that the following ramification conditions are satisfied:
1) \( K / F \) is ramified at at least one finite prime.
2) For every rational prime \( l \leq \frac{[K : \mathbb{Q}]}{[K : \mathbb{Q}]} + 1 \), ramification index is less than \( l \), then we have
\[
\hat{\deg} \tilde{Z}(a) = -\frac{|C_K| \sqrt{N_F/Q(d_{K/F})}}{w(K)} b_\Phi(a, y)
\]
where \( |C_K| = |\hat{O}_F^{\times_{\geq 0}} / N_{K/F} \hat{O}_K^{\times}| h(K) \) where \( h(K) \) is the class number of \( K \), \( w(K) \) is the number of roots of unity in \( K \), \( d_{K/F} \) is the relative discriminant of \( K / F \), \( r \) is the number of places (including archimedean) ramified in \( K \), and \( b_\Phi(a, y) \) is the \( a \)th coefficient of the Fourier expansion of the derivative of a Hilbert Eisenstein series at \( s = 0 \).

Before this result, a special case was considered in [9] where \( K \) contains a quadratic imaginary field \( K_0 \). Let \( K \) be a CM-field containing a quadratic imaginary field \( K_0 \) with maximal totally real subfield \( F \) and \( \Phi \) be a CM-type of \( K \). By the signature of \( \Phi \), we mean a pair of numbers \((r, s)\) such that \( r \) is the number of elements of \( \Phi \) that restrict to identity on \( K_0 \) and \( s \) is the number of the rest. Let \( O_K, O_{K_0} \) be the ring of integers of \( K, K_0 \), respectively. We denote by \( K_\Phi \) the reflex field of \( \Phi \) and for a prime \( p \) of \( K_\Phi \), \( k_{\Phi, p} \) be the residue field of \( K_\Phi \) at \( p \) and \( \overline{k_{\Phi, p}} \) a choice of algebraic closure of it. Also suppose that \( F, K_0 \) have relatively prime odd discriminants, so in particular \( O_K = O_{K_0} \otimes O_F \). If \((A, \lambda_A)\) is an elliptic curve with CM by \( O_{K_0} \) with the principal polarization \( \lambda_A \) and \( (B, \lambda_B) \) be an \( O_K \)-principally polarized abelian variety having action by \( O_{K_0} \) and \( O_{K_0} \)-principal polarization \( \lambda_B \), we make \( \text{Hom}_{O_{K_0}}(A, B) \) into a Hermitian space by letting \( \langle f, g \rangle := \lambda_A^{-1} \circ g \circ \lambda_B \circ f \) where \( g^\vee : B^\vee \to A^\vee \) is the dual of \( g \). This Hermitian form is \( O_{K_0} \)-valued, we can also define \( \langle \cdot, \cdot \rangle_{CM} \) to be the unique \( K \)-valued Hermitian form with the property that \( \text{tr}_{K/K_0} \langle \cdot, \cdot \rangle_{CM} = \langle \cdot, \cdot \rangle \). In the aforementioned paper, Howard defines a Deligne-Mumford stack \( Z(\alpha) \) of triples \((E, A, f)\) where \( E \) is an elliptic curve (over the assigned scheme) with CM by \( O_{K_0} \) and \( A \) is an \( O_K \)-principally polarized abelian variety with CM by \( O_K \) and \( f \in \text{Hom}_{O_{K_0}}(E, A) \) is an \( O_{K_0} \)-linear homomorphism such that \( \langle f, f \rangle_{CM} = \alpha \). He computes the Arakelov degree of this stack (as a stack over \( \text{Spec} O_\Phi \) (ring of integers of \( K_\Phi \))) defined as follows:
\[
\hat{\deg} Z(\alpha) = \frac{1}{[K_\Phi : \mathbb{Q}]} \sum_{p \in O_\Phi} \log N(p) \sum_{z \in Z(\alpha)(\overline{k_{\Phi, p}})} \frac{\text{length}(O_{Z(\alpha), z})}{\#\text{Aut} z}
\]
where \( O_{Z(\alpha), z} \) is the strictly Henselian local ring at \( z \). As CM points have a transitive action of a group, it follows that for a fixed prime \( p \) of \( O_\Phi \), above
length is constant for all $z$ and the problem reduces to that of computing
the length for just one point and $\#Z(\alpha)(\overline{k}_{F,p})$.

The main result of [9] is the relation of this degree to the derivative of
the Fourier coefficients of an $SL_2(F)$-Eisenstein series of parallel weight
$(1,1,\cdots,1)$. Suppose that we have the Fourier expansion

$$E(\tau,s) = \sum_{\alpha \in F} a_\alpha(s,y) e^{2\pi i \text{Tr}_F/Q(\alpha \tau)}$$

for this Eisenstein series where $x + iy = \tau \in \mathfrak{h}^{[F:Q]}$ where $\mathfrak{h}$ is the upper-
half plane and $s \in \mathbb{C}$, then

**Theorem 1.13.** ([9], Theorem 4.2.3) For $\alpha \in F$ totally positive, we have the
following equality:

$$\deg Z(\alpha) = -\frac{h(K_0)}{w(K_0)} \frac{\sqrt{N(d_{K/F})} q'_\alpha(0,y)}{2^{r-1}[K:Q]} \quad (1.4)$$

The way to prove our main result is to compute both the coefficients and
degrees separately and see that they match. To compute the coefficients we
use Yang’s formulas [26], to compute the degrees we use Serre-Tate and
Grothendieck-Messing theory. Indeed, for a pair of CM abelian varieties,
we consider their $p$-divisible groups (so that these $p$-divisible groups have
actions by the ring of integers of a local field) and then we will compute
explicitly the condition on lifting of these $p$-divisible groups and maps
between them.

**Second Result (Relative CM-Fields)**

The second result of this thesis relates the special divisors on CM-cycles of
RSZ Shimura varieties to Eisenstein series.

In this part, we are interested in having different but included CM
fields (i.e. CM fields $K_0$ and $K$ with $K_0 \subseteq K$) and then we are going to
consider pairs of polarized abelian varieties (such that their dimensions
are relatively $[K : K_0]$ and they have action by $O_{K_0}$ (ring of integers
of $K_0$) such that the action of $O_{K_0}$ on their Lie algebras has a specific
kind. Then using the same method as in the previous result (which is a
method originally from [9]), we define special divisors and prove that their
Arakelov degrees are related to the Fourier coefficients of an Eisenstein
series. The main motivation for these kinds of results for the author is the
expected relation of the $0^{th}$ coefficient of this Eisenstein series to special
value of the derivative of L-functions.

To state our main result, we need some notations (we will repeat these
notations in the notations section below as well). Let $K_0 \subseteq K$ be CM-fields
with $F_0 \subseteq F$ being their maximal totally real subfields. Let $\Phi_0$ and $\Phi$ be some nearby CM-types (for the precise definition, see the notations section) of $K_0$ and $K$, respectively. Let $\tilde{K}$ be the reflex field of $(K, \Phi)$. For a prime $p$ of $\tilde{K}$, let $k_p$ be a choice of algebraic closure of residue field of $\tilde{K}$ at $p$. Let $A_0$ be a principally polarized abelian variety with CM by $O_{K_0}$ with polarization $\lambda_{A_0}$ and let $A$ be a polarized abelian variety with polarization $\lambda_A$ that has an action by $O_{K_0}$. We can make $\mathrm{Hom}_{O_{K_0}}(A_0, A)$ a Hermitian space by letting $\langle f, g \rangle = \lambda_{A_0}^{-1} \circ g^\vee \circ \lambda_A \circ f$ where $g^\vee : A^\vee \to A_0^\vee$ is the dual of $g$. This Hermitian form is $O_{K_0}$-valued and we define $\langle \cdot, \cdot \rangle_{CM}$ to be the unique $K$-valued Hermitian form with the property that $\mathrm{tr}_{K/K_0}(\cdot)_{CM} = \langle \cdot \rangle$.

We define the special divisors $Z(\alpha)$ and consider the following quantity (called Arakelov degree of this special divisor on specific moduli space):

$$\widehat{\deg} Z(\alpha) = \frac{1}{[\tilde{K}:\mathbb{Q}]} \sum_{p \in O_{\tilde{K}}} \log N(p) \left( \sum_{z \in Z(\alpha)(\tilde{k}_p)} \frac{\text{length}(O_{Z(\alpha), z}^{\text{et}})}{\# \text{Aut } z} \right)$$

where $O_{Z(\alpha), z}^{\text{et}}$ is the strictly Henselian local ring at $z$ and $N(p)$ is the norm of $p$ in $\mathbb{Q}$ and $O_{\tilde{K}}$ is the ring of integers of $\tilde{K}$.

We prove the same result as the first result for $Z(\alpha)$ and moduli space in the setting we wrote about above. Specifically, we find an Eisenstein series with Fourier coefficients denoted by $b_{\Phi}(\alpha, y)$ and prove the following main theorem:

**Theorem 1.14.** Let $\alpha \in F^\times$. Suppose that the following conditions are satisfied:

1. $K/F$ is ramified at at least one finite prime.
2. Relative discriminants of $K_0/F_0$ and $F/F_0$ are relatively prime.
3. The assumption in page 62 below is satisfied, then we have:

$$\widehat{\deg} Z(\alpha) = \frac{-1}{w(K_0)} \sqrt{N_{F/Q}(d_{K/F})} b_{\Phi}(\alpha, y).$$

The way to prove it is to consider CM $p$-divisible groups and find the amount of lifting of homomorphisms between CM $p$-divisible groups, and then we are going to consider the global case and define the moduli space and the special divisors as DM-stacks. Finally, we will define the Eisenstein series and prove the relation between the Fourier coefficients and the Arakelov degree which will prove the main theorem.
BACKGROUND MATERIAL

In this section, we are going to talk about some background necessary for the computations of Arakelov degree of special cycles in the next chapters. We start with the theory of $p$-divisible groups and the definition and some properties of $p$-divisible groups, then we talk about the basics of Grothendieck-Messing theory which will be used to compute the lengths of the local rings on the Shimura variety by considering the liftings of homomorphisms between $p$-divisible groups. Then in the second section of this chapter we talk about Rapoport-Smithling-Zhang Shimura varieties which is the basis for the second result (that is the relative CM-case) and we write about the definition and the main results of construction of integral models of unitary Shimura varieties constructed by Rapoport, Smithling and Zhang in [19].

2.1 $p$-DIVISIBLE GROUPS AND GROTHENDIECK-MESSING THEORY

In this section, we will first define the $p$-divisible groups and write about some properties of them and then we will start the Grothendieck-Messing theory and state the important theorems. The main references are [23], [20].

$p$-divisible Groups

Let $R$ be a ring, $m$ be a natural number and $G$ be a group scheme which is locally free of rank $m$ over $R$ (This will be called a finite group scheme of order $m$ over $R$). This means that $G$ locally over $R$ is Spec$A$ for some $R$-algebra $A$ and $\mu : A \to A \otimes_R A$ be the multiplication map (on $G$) and $\epsilon : A \to R$ the neutral element homomorphism and $i : A \to A$ be the inverse map (on $G$). A homomorphism between two finite group schemes $G$ and $G'$ over $R$ is a map $f : G \to G'$ of schemes that is locally compatible with the multiplication map, neutral element map and the inverse map. Now we define an exact sequence between finite group schemes:

**Definition 2.1.** A sequence

$$0 \to G' \xrightarrow{i} G \xrightarrow{j} G'' \to 0$$
is called exact if $i$ is a closed immersion such that the image of $G'$ in $G$ is equal to the kernel of $j$.

We have a natural exact sequence for a finite group scheme $G$ over a complete noetherian local rings $R$. This exact sequence is called the connected-étale exact sequence

$$0 \to G^0 \to G \to G^{\text{ét}} \to 0$$

where $G^0$ is connected and $G^{\text{ét}}$ is étale over $R$. Now if $G = G^0$ then $G$ is called connected and if $G = G^{\text{ét}}$, then $G$ is called étale. Let $k$ be the residue field of the local ring $R$, then the functor sending $G$ to the $\text{Gal}(\bar{k}/k)$-sets $G(\bar{k})$ (where $\bar{k}$ is the algebraic closure of $k$) is an equivalence of categories between finite étale group schemes over $R$ and finite $\text{Gal}(\bar{k}/k)$-sets (where finite here mean finite as a set). Now we define the $p$-divisible groups:

Let $p$ be prime. A $p$-divisible group $G$ is a direct system of finite group schemes $(G_n)_{n \geq 0}$ with maps $G_n \xrightarrow{i_n} G_{n+1}$ such that $G_n$ is a finite group scheme over $R$ of $p$-power order $p^{nh}$ for some $h \geq 0$ (called the height of the $p$-divisible group) and for each $n \geq 0$, the sequence

$$G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{|p^n|} G_{n+1}$$

is exact (so that $G_n = \ker [p^n](G_{n+1})$). A homomorphism between $p$-divisible groups $G \xrightarrow{f} H$ is a map of their finite parts $G_n \xrightarrow{f_n} H_n$ which are compatible with the $i_n$'s and $f_n$ is a map of group schemes.

**Examples.** (1) Let $A$ be an abelian scheme over $R$ and $d = \dim A$. Let $A[p^n] = \ker [p^n]|_A$ then $A[p^n] = (A[p^n], i_n)_{n \geq 0}$ where $i_n : A[p^n] \to A[p^{n+1}]$ is the inclusion, is a $p$-divisible group over $R$ of height $2d$.

(2) Let $G_m/R$ be the multiplicative group scheme over $R$ defined functorially by

$$G_m(A) = A^\times$$

for all $R$-algebras $A$. Then considering the kernel of $p^n$-multiplication map on $G_m$ would give a finite group scheme of order $p^n$ written as $\mu_{p^n}$ (also called the group scheme of $p^n$-roots of unity as $\mu_{p^n}(A) = \{x \in A | x^{p^n} = 1\}$ for any $R$-algebra $A$). Then the system $(\mu_{p^n}, i_n)_{n \geq 0}$ with $i_n$ defined by the inclusion $i_n : \mu_{p^n} \to \mu_{p^{n+1}}$ is a $p$-divisible group of height 1.

There is a theorem relating the homomorphisms of $p$-divisible groups to the homomorphisms of their Tate modules which we are going to state below and it will be used in some parts of the local parts of the results below.

Let $G$ be a $p$-divisible group over a complete discrete valuation ring $R$ with $\text{char}R \neq \text{char}k$ where $k$ is the residue field of $R$ (examples are $\mathbb{Z}_p$ and
$O_K$ for $K/Q_p$ a finite extension). Now we define the Tate module of the $p$-divisible group $G$:

**Definition 2.2.** Let $G$ be a $p$-divisible group as above. Then we define the Tate module of $G$ to be the inverse limit $T_G = \lim_{\leftarrow} G_n(k)$ with respect to the maps $j_n : G_{n+1} \to G_n$ where $j_n$ is the map that comes into the factorization of $[p^n]$ ($p^n$-multiplication map) in

$$0 \to G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{j_n} G_n \to 0$$

Now the main result of [23] (which will be used in the local parts below) is

**Theorem 2.3.** Let $R$ be an integrally closed noetherian domain with $\text{char} R = 0$ and $G$ and $H$ be $p$-divisible groups over $R$. Also denote the field of fraction of $R$ by $K$. Obviously $\text{Gal} (\bar{K}/K)$ acts on $TG$ and $TH$ (which are the Tate modules of $G$ and $H$ respectively). Then a homomorphism $G \otimes_R K \to H \otimes_R K$ extends uniquely to a map $G \to H$. As a corollary, we have that the map

$$\text{Hom}(G, H) \to \text{Hom}_{\text{Gal}(\bar{K}/K)}(TG, TH)$$

is an isomorphism.

**Grothendieck-Messing Theory**

In this section, we introduce the Grothendieck-Messing theory and deformation theory of $p$-divisible groups. First we define the divided power structure on ideals of commutative rings and we use that to define the crystalline site. Let $I$ be an ideal of a commutative ring $A$. Divided power structure is similar to the notion of $x^n$ when $n!$ is not necessarily invertible in $A$.

**Definition 2.4.** Divided power on $I$ means a collection of maps $\{\gamma_i : I \to A\}_{i \geq 0}$ such that these maps have the following properties:

1. $\gamma_0(x) = 1$ and $\gamma_1(x) = x$ for all $x \in I$.
2. $\gamma_i(x) \in I$ for all $i \geq 1$ and $x \in I$.
3. $\gamma_i(ax) = a^i \gamma_i(x)$ for all $a \in A$ and $x \in I$.
4. $\gamma_k(x + y) = \sum_{i+j=k} \gamma_i(x) \gamma_j(y)$ for all $x, y \in I$.
5. $\gamma_i(x) \gamma_j(x) = \frac{(i+j)!}{i!j!} \gamma_{i+j}(x)$ for all $x \in I$.
6. $\gamma_p(\gamma_q(x)) = \frac{p!}{p!(q)!^p} \gamma_{pq}(x)$ for all $x \in I$.

Then we say that $(A, I)$ has divided power structure.

A map $(A, I) \to (B, J)$ of ideals with divided power structure is a map of rings such that it is compatible with the divided power structure. We can
easily extend this definition to the ideal sheaves of sheaves on topological spaces. Now we are able to define the crystalline site of a scheme $X$ over a base scheme $S$.

Objects of the crystalline site $\text{Cris}(X/S)$ are pairs $(U \to T, \delta)$ where $U \to T$ is a closed immersion defined by an ideal sheaf $I$ and $U$ is an open subset of $X$ and $\delta$ is a divided power structure on the ideal sheaf $I$. A morphism $T \to T'$ of the crystalline site is a commutative diagram

$$
\begin{array}{ccc}
U & \longrightarrow & T \\
\downarrow & & \downarrow \\
U' & \longrightarrow & T'
\end{array}
$$

where $U \subseteq U'$ is an inclusion in the Zariski sense and $T \to T'$ is a morphism compatible with the divided power structures. We can also easily define the coverings of an object of this category.

Now we will state the theorem for the anti-equivalence of categories between $p$-divisible group schemes over a perfect field and Dieudonné modules. For this we first need to define the Dieudonné ring. Let $k$ be a perfect field of characteristic $p > 0$. First we define the Dieudonné ring. Let $W(k)$ be the ring of Witt vectors of $k$ and $\sigma : W(k) \to W(k)$ be the Frobenius map.

**Definition 2.5.** Let $D$ be the noncommutative ring $W(k)[F,V]$ with the relations below for $F$ and $V$:

1. $FV = VF = p$
2. $Fa = \sigma(a)F$ for all $a \in W(k)$.
3. $Va = \sigma^{-1}(a)V$ for all $a \in W(k)$.

Dieudonné was able to prove the following

**Theorem 2.6.** There is an anti-equivalence of categories

$$
\{p \text{- group schemes }/k\} \cong \{D \text{- modules of finite } W(k) \text{ - length}\}
$$

From this we find the anti-equivalence of categories for the $p$-divisible groups over the perfect field $k$:

**Theorem 2.7.** There is an anti-equivalence of categories

$$
\{p \text{- divisible groups }/k\} \cong \{W(k) \text{- free } D \text{- modules}\}
$$

Now we consider other base schemes for the $p$-divisible groups and we try to find an anti-equivalence of categories. Let $G/S$ be a finite flat group...
scheme. First we define the vector group over \( S \) denoted by \( V(G) \) (By a vector group over \( S \), we mean a quasi-coherent locally-free of finite rank \( O_S \)-module considered as a sheaf on the fppf site over \( S \)) associated to a finite flat group scheme: \( V(G) \) is defined universally by a map \( \alpha : G \to V(G) \) such that \( G \to M \) for any other vector group over \( S \) is factored

\[
\begin{array}{c}
M \\
\downarrow \alpha \\
G \\
\downarrow a \\
V(G)
\end{array}
\]

Now there is an extension of \( G \) (only for \( G \) with \( \text{Hom}(G, V) = 0 \) for all vector groups \( V \) over \( S \)) defined by

\[
0 \to V(G) \to E(G) \to G \to 0
\]

such that it has the universal property that for all extensions

\[
0 \to V' \to E' \to G \to 0
\]

with \( V' \) a vector group, there are unique maps \( V(G) \to V' \) and \( E(G) \to E' \) that commutes with the above exact sequences.

There’s a construction that gives a locally-free sheaf of \( O_S \)-modules \( \text{Lie}(E(G)) \) which is defined by constructing a formal Lie group from the universal extension and this is the sheaf constructed from the Lie algebra of that formal Lie group. We use this to construct a crystal \( \mathcal{D} \) for the \( p \)-divisible group \( G/S \). This crystal on a divided power immersion \( S_0 \to S \) for the group \( G_0 = G \times_S S_0 \):

\[
\mathcal{D}(G_0)_{(S_0 \to S)} = \text{Lie}(E(G))
\]

Now we are in the situation to state the Grothendieck-Messing main theorem. First we need to define the category \( C \):

**Definition 2.8.** Let \( S_0 \to S \in \text{Cris}(S) \) and let \( G_0 = G \times_S S_0 \) and \( \mathcal{D}(G_0)(S) \) be the crystal \( \mathcal{D}(G_0)_{(S_0 \to S)} \) evaluated at \( S_0 \). Then the category \( C \) is defined as follows: the objects of this category are pairs \( (G_0, \text{Fil}^1 \mathcal{D}(G_0)(S)) \) where \( G_0 \) is a \( p \)-divisible group over \( S_0 \) that can be lifted to \( S \) and \( \text{Fil}^1 \mathcal{D}(G_0)(S) \) be admissible. Admissible here means that \( \text{Fil}^1 \mathcal{D}(G_0)(S) \) is a locally-free vector subgroup with locally-free quotient such that over \( S_0 \) it restricts to \( V(G_0) \to \text{Lie}(E(G_0)) \) as above. The morphisms in this category are maps between \( p \)-divisible groups over \( S_0 \) that are compatible with the filtrations.

Now we state the
Theorem 2.9. There’s an anti-equivalence between
\[ \{ p \text{-divisible groups} / S \} \rightleftharpoons C \]
given by sending a \( p \)-divisible group \( G / S \) to \( (G \times_S S_0, V(G) \rightarrow \text{Lie}(E(G)) = D(G_0)(S)) \)

2.2 RAPPORT-SMITHLING-ZHANG SHIMURA VARIETIES

In this section, we talk about Rapoport-Smithling-Zhang Shimura varieties (abbreviated as RSZ Shimura varieties) and this section is taken from [19]. We are only interested in the definition of integral models of these Shimura varieties as Deligne-Mumford stacks (DM-stacks for short) and will not discuss the rest of [19].

First let’s fix some notations. We use the same notations as in [19]. Let \( F \) be a CM-field and \( F_0 \) its maximal totally real subfield. \( \bar{\Delta} \in \text{Gal}(F/F_0) \) is the nontrivial element of \( \text{Gal}(F/F_0) \) and \( F = F_0(\sqrt{\Delta}) \) where \( \Delta \) is a totally negative element of \( F_0 \) (meaning that it goes to negative numbers through every embedding of \( F_0 \) into \( \mathbb{R} \)). Let \( \Phi \) be a CM-type of \( F \) (also we can choose the correct \( \Delta \), and will assume that \( \Phi = \{ \phi : F \rightarrow \mathbb{C} | \phi(\sqrt{\Delta}) = ir \text{ with } r > 0 \} \)).

For a number field \( K \), let \( \mathbb{A}_K \) be the ring of adéles of \( K \). \( \mathbb{A}_K^f \) means the ring of finite adéles (outside the archimedean part) and \( \mathbb{A}_K^p \) means adéles outside the \( p \) (outside all primes above \( p \)) part.

Let \( k \) be a field and \( A \) be an étale \( k \)-algebra of degree 2. Let \( W \) be a Hermitian \( A \)-module (meaning a finite free \( A \)-module with a Hermitian form (i.e. a form that in linear in the first variable and conjugate-linear in the second variable)). We denote by \( \det W \in k^\times / \mathbb{N}_{A/K}(A^\times) \) for the determinant of \( \langle (e_i, e_j) \rangle_{i,j} \) where \( e_i \)'s are a basis for \( W / A \) and \( \langle, \rangle \) is the Hermitian form. \( -W \) is the same module as \( W \) with negative of its Hermitian form. Also for a place \( v \) of \( F_0 \) and \( W \) a Hermitian space over \( F_0 \) of dimension \( n \), let \( W_v \) be the Hermitian space over \( F_0, v \) (the completion of \( F_0 \) with respect to \( v \)) and define
\[
\text{inv}_v(W_v) = (-1)^{\frac{n(n-1)}{2}} \det W_v \in F_{0,v}^\times / \mathbb{N}_{F_v/F_0,v}(F_v^\times)
\]
If \( \text{inv}_v(W_v) = 1 \), we say that \( W_v \) is split. If \( A \) is an abelian scheme, we write \( A^\vee \) for the dual of \( A \) and \( T_p A \) for the Tate module of \( A \) and \( V_p A = T_p A \otimes \mathbb{Q} \).

Let \( \Lambda \) be an \( \mathbb{O}_{F,v} \)-lattice in some Hermitian space for \( F_v / F_{0,v} \). We call \( \Lambda \) a vertex lattice of type \( r \) (\( r \) is a natural number) if \( \Lambda \) is of co-length \( r \) in \( \Lambda^\times \) (the dual of \( \Lambda \) with respect to the Hermitian form \( \langle, \rangle \)) with \( \Lambda^\times \subseteq \pi_v^{-1} \Lambda \).
where \( \pi_v \) is the uniformizer of \( F_{0,v} \). A vertex lattice is a vertex lattice of some type \( r \). By this definition, a self-dual lattice (resp. almost self-dual lattice) is a vertex lattice of type 0 (resp. type 1). A vertex lattice is called \( \pi_v \)-modular (almost \( \pi_v \)-modular) if \( \Lambda^* = \pi_v^{-1} \Lambda \) (resp. \( \Lambda^* \) be an \( O_{F_{0,v}} \)-module of co-length 1 in \( \pi_v^{-1} \Lambda \)).

Now we define the Shimura varieties using the moduli problem. These Shimura varieties are defined for products of unitary similitude groups of \( W \). First we define the moduli problem \( \mathcal{M}_0 \). Let \( E \) be the reflex field of \( (F, \Phi) \). We define \( \mathcal{M}_0 \) over \( O_E \). So let \( S \) be an \( O_E \)-scheme, let \( \mathcal{M}_0(S) \) be the groupoid of triples \((A_0, \iota_0, \lambda_0)\) where

- \( \iota_0 \) is an action \( O_{F_0} \to \text{End} A_0 \) satisfying the Kottwitz condition of \( \text{char} (\iota_0(a) \mid_{\text{Lie}(A_0)}) (t) = \prod_{\phi \in \Phi} (t - \phi(a)) \quad \forall a \in O_F \)

- \( \lambda_0 \) is a principal polarization of \( A_0 \) such that its Rosati involution on \( O_F \) is the conjugation of \( F/F_0 \).

For a nonzero ideal \( a \) of \( O_F \), we define \( \mathcal{M}_0^a \) to be the same as above except that the polarization is not principal and has kernel \( A_0[a] \). \( \mathcal{M}_0^a \) is a DM-stack and it is finite and étale over \( \text{Spec} O_E \).

\( \mathcal{M}_0^a \) will also have a decomposition into open and closed substacks \( \mathcal{M}_0^{a,\xi} \) which we will explain in the necessary context (see section 4.3 below).

Now we can define the main moduli problem. This is the moduli problem that has a CM-cycle on it on which we are going to define the special cycles.

First, we fix \( W \) to be an \( F/F_0 \)-Hermitian space and defined this subset of places of \( F_0 \):

\[ V^W_{AT} = \{ v \mid v \text{ is inert in } F \text{ and } v \text{ is not split} \} \cup \{ v \mid v \text{ is ramified in } F \} \]

Let \( \partial^W_{AT} = \prod_{v \in V^W_{AT}} q_v \subseteq O_F \) where \( q_v \) is the unique prime in \( F \) above \( v \in V^W_{AT} \). Fix a lattice \( \Lambda \) in \( W \) such that

\[ \Lambda \subseteq \Lambda^* \subseteq (\partial^W_{AT})^{-1} \Lambda \]

and assume that

- All primes \( v \) of \( F_0 \) that are ramified over \( Q \) or divide 2 are split in \( F \).
- the pair \( (v, \Lambda_v) \) has one of the following properties (called AT-types (1) to (4)):
  1. \( v \) is inert in \( F \) and \( \Lambda_v \) is almost self-dual.
  2. \( v \) ramifies in \( F \), \( \text{dim} W \) is even and \( \Lambda_v \) is \( \pi_v \)-modular.
  3. \( v \) ramifies in \( F \), \( \text{dim} W \) is odd and \( \Lambda_v \) is almost \( \pi_v \)-modular.
(4) \( v \) ramifies in \( F \), \( \dim W = 2 \) and \( \Lambda_v \) is self-dual.

Now we define the moduli problem \( \mathcal{M} \) over \( \text{Spec} O_E \) (recall that \( E \) is the reflex field of \( (F, \Phi) \) and \( \phi_0 \in \Phi \)). For each \( O_E \)-scheme \( S \), let \( \mathcal{M}(S) \) be the groupoid of \( (A_0, \iota_0, \lambda_0, A, \iota, \lambda) \) such that \( (A_0, \iota_0, \lambda_0) \) is an object of \( \mathcal{M}_0(S) \) and

- \( A \) is an abelian scheme over \( S \) and \( \iota: O_F \to \text{End} A \) with Kottwitz condition
  
  \[
  \text{char}(\iota(a)|_{\text{Lie} A})(t) = (t - \phi_0(a))^{n-1}(t - \bar{\phi}_0(a)) \prod_{\phi \in \Phi \setminus \{\phi_0\}} (t - \phi(a))^n
  \]

- \( \lambda \) is a polarization with Rosati involution on \( O_F \) being equal to the conjugation of \( F/F_0 \).

Let \( \text{inv}_v(A_{0,s}, \iota_{0,s}, \lambda_{0,s}, A_s, \iota_s, \lambda_s) \) be the twist of sign invariant of the Hermitian space \( \text{Hom}_{O_E}(\hat{V}_p(A_0), \hat{V}_p(A)) \) (see appendix of [19]), then we impose

\[
\text{inv}_v'(A_{0,s}, \iota_{0,s}, \lambda_{0,s}, A_s, \iota_s, \lambda_s) = \text{inv}_v(-W_v)
\]

for all points \( s \) of \( S \) and all finite places \( v \) of \( F_0 \) non-split in \( F \). We also impose that the triple \( (A \otimes \mathbb{Z}_p, \iota \otimes \mathbb{Z}_p, \lambda \otimes \mathbb{Z}_p) \) has the extra conditions as in section 4 of [19] (for us they will become irrelevant as we will restrict to the case that \( A \) has CM and due to the assumption in page 62).

Now we get to the theorem we need from [19] which is:

**Theorem 2.10.** The moduli problem above is representable by a flat DM-stack over \( \text{Spec} O_E \) and

1. \( \mathcal{M} \) is smooth of relative dimension \( n-1 \) over the open subscheme of \( \text{Spec} O_E \) defined by removing finitely many primes (namely those places above \( v \) in \( \text{Spec} O_E \) such that \( (v, \Lambda_v) \) is of AT-type (4) or AT-type (1) (see above)).
2. \( \mathcal{M} \) has semistable reduction over the open subscheme of \( \text{Spec} O_E \) by removing all places \( v \) for which \( (v, \Lambda_v) \) is of AT-type (1), (4) and \( v \)'s that \( E_v/\mathbb{Q}_p \) is ramified.

We would need this theorem in our second result to define CM-cycles and be able to define special divisors on them.
EQUAL CM-FIELDS

This is the first result of the thesis. In this chapter, we consider the case of moduli space of pairs of CM abelian varieties with action by a CM-field \( K \) such that the CM-type of \( A_1 \) (denote \( \Phi_1 \)) and CM-types of \( A_2 \) (denote it by \( \Phi_2 \)) differ in exactly one embedding. Then we define the special divisors \( Z(\alpha) \) on this moduli space by specifying an \( \mathcal{O}_K \)-homomorphism \( f : A_1 \to A_2 \) with \( \langle f, f \rangle = \alpha \) (\( \langle, \rangle \) is defined as in the introduction) and compute the Arakelov degree of \( Z(\alpha) \) and also defining the arithmetic “\( Z(\alpha) \)”s and finding the arithmetic degrees of them. Finally at the end, we construct and Eisenstein series whose Fourier coefficients are related to these degrees.

3.1 LOCAL COMPUTATION

In this section, we will compute the lengths of the local rings appearing in the Arakelov degree by local considerations and use of Grothendieck-Messing theory. Here we consider the two cases of different CM-types and same CM-types, because when we change two abelian varieties by their \( p \)-divisible groups, the \( p \)-adic CM-types (to be defined below) might be equal or have exactly one difference.

\( p \)-divisible Groups with Different CM-Types

Let \( p \) be a prime, \( \mathbb{F}_p \) be the field with \( p \) elements and \( \overline{\mathbb{F}}_p \) the algebraic closure of it, \( \mathcal{W} = \mathcal{W}(\mathbb{F}_p) \) be the Witt vectors over \( \mathbb{F}_p \) (equivalently, the completion of the ring of integers of maximal unramified extension of \( \mathbb{Q}_p \)), \( \mathbb{C}_p \) the completion of the algebraic closure of \( \mathbb{Q}_p \), also in this paper by a scheme we always mean a locally noetherian scheme and an algebraic stack means a Deligne-Mumford stack, \( \mathcal{F} \) be an extension of \( \mathbb{Q}_p \) of degree \( n \), \( \mathcal{K}/\mathcal{F} \) a quadratic field extension of local fields, \( (\mathcal{O}_K,p_K),(\mathcal{O}_F,p_F) \) are their ring of integers and the nonzero prime ideals of \( \mathcal{O}_K,\mathcal{O}_F \), respectively. \( x \mapsto \bar{x} \) be the nontrivial automorphism of \( \mathcal{K} \) over \( \mathcal{F} \). If \( \phi : \mathcal{K} \hookrightarrow \mathbb{C}_p \) is an embedding, then \( \overline{\phi}(x) = \phi(\bar{x}) \).

Suppose that \( \Phi_1, \Phi_2 \) be a pair of nearby \( p \)-adic CM-types (A \( p \)-adic CM-type consists of \( n \) embeddings \( \mathcal{K} \hookrightarrow \mathbb{C}_p \) such that for an embedding \( \phi : \mathcal{K} \hookrightarrow \mathbb{C}_p \) and...
\( \mathcal{K} \hookrightarrow \mathcal{C}_p \) either \( \phi \) or \( \bar{\phi} \) is in \( \Phi \), i.e. \( \#(\Phi_1 \cap \Phi_2) = n - 1 \), so there is a unique \( \phi^{\text{op}} \in \Phi_1 \) such that \( \bar{\phi}^{\text{op}} \in \Phi_2 \). Now fix two triples \((A_1, \kappa_1, \lambda_1), (A_2, \kappa_2, \lambda_2)\) such that for \( i = 1, 2 \):

- \( A_i \) is \( p \)-divisible group over \( \mathbb{F}_p \).
- \( \kappa_i : O_{\mathcal{K}} \to \text{End} A_i \) satisfies \( \Phi_i \)-determinant condition, i.e. for \( x \in O_{\mathcal{K}} \), \( t - x \) acts on \( \text{Lie}(A_i) \) with determinant \( \prod_{\phi \in \Phi_i} (t - \phi(x)) \).
- \( \lambda_i : A_i \to A_i^\vee \) be an \( O_{\mathcal{K}} \)-linear isomorphism, i.e. for all \( x \in O_{\mathcal{K}} \), \( \lambda_i \circ x = x \circ \lambda_i \).

First of all, \( A_i \)'s are supersingular (i.e. all slopes of their Dieudonné modules are \( \frac{1}{2} \)) by proposition 2.1.1 of [9]. Now we consider the \( O_{\mathcal{K}} \)-module \( L(A_1, A_2) = \text{Hom}_{O_{\mathcal{K}}}(A_1, A_2) \) of all \( O_{\mathcal{K}} \)-linear homomorphisms from \( A_1 \) to \( A_2 \) and define a Hermitian form on it:

\[
\langle f, g \rangle = \lambda_1^{-1} \circ g^\vee \circ \lambda_2 \circ f \in \text{End}_{O_{\mathcal{K}}} A_1 = O_{\mathcal{K}}, \quad f, g \in L(A_1, A_2)
\]

Define \( S = O_{\mathcal{K}} \otimes_{\mathbb{Z}_p} \mathcal{W}, \mathcal{W}, \mathcal{W} \in \text{Aut}\mathcal{W} \) be Frobenius, then \( \mathcal{W} \) acts on \( S \) by \( x \otimes w \mapsto x \otimes \mathcal{W}^\mathcal{W} w \). Let \( \mathcal{K}^{\text{Fr}}, \mathcal{F}^{\text{Fr}} \) be the maximal unramified extensions of \( O_{\mathcal{K}} \) inside \( \mathcal{K}, \mathcal{F} \), respectively and \( O_{\mathcal{K}}^{\text{Fr}}, O_{\mathcal{F}}^{\text{Fr}} \) their rings of integers. For each \( \psi : O_{\mathcal{K}}^{\text{Fr}} \to \mathcal{W}, \) there is an idempotent \( e_{\psi} \in S \) such that \( (x \otimes 1)e_{\psi} = (1 \otimes \psi(x))e_{\psi} \) for all \( x \in O_{\mathcal{K}}^{\text{Fr}} \). They satisfy \( (e_{\psi})^{\mathcal{W}} = e_{\mathcal{W} \psi} \). Also, that \( S = \prod_{\psi} e_{\psi} S \) where \( e_{\psi} S \subseteq \mathcal{W} \mathcal{K} \) (ring of integers of the completion of maximal unramified extension of \( \mathcal{K} \)) is a DVR, so we also have the maps \( \text{ord}_{\psi} : S \to \mathbb{Z}^{\geq 0} \cup \{0\} \). Put \( m(\psi, \Phi_i) = \# \{ \phi \in \Phi_i : \phi|_{O_{\mathcal{K}}^{\text{Fr}}} = \psi \} \). Let \( D(A_i) \) be the Dieudonné module of \( A_i \). Here, we state lemma 2.3.1 of [9].

**Lemma 3.1.** For \( i = 1, 2 \), we have an isomorphism of \( S \)-modules \( D(A_i) \cong S \).

\( F, V \in \text{End} D(A_i) \) act on \( S \) by \( F = a_i \circ \mathcal{W}, V = b_i \circ \mathcal{W}^{-1} \), for some \( a_i, b_i \in S \) such that \( a_i b_i^{\mathcal{W}} = p \) and \( \text{ord}_{\psi}(b_i) = m(\psi, \Phi_i) \) for all \( \psi : O_{\mathcal{K}}^{\text{Fr}} \to \mathcal{W} \).

\[
L(A_1, A_2) \subseteq \text{Hom}_{O_{\mathcal{K}} \otimes_{\mathbb{Z}_p} \mathcal{W}}(D(A_1), D(A_2)) \cong \text{Hom}_{S}(S, S) \cong S \text{ as } \mathcal{W} \text{-modules}, \text{so by the above } L(A_1, A_2) = \{ s \in S | (b_1 s)^{\mathcal{W}} = s b_2^{\mathcal{W}} \} \text{ (these are elements of } S \text{ that are compatible with } V \text{ (and so } F))\).

**Lemma 3.2.** \( \langle \, , \, \rangle \) on \( L(A_1, A_2) \) is identified with \( \langle s_1, s_2 \rangle = \bar{\zeta} s_1 \bar{s_2} \) (with identification as a subset of \( S \)) with \( \bar{\zeta} \in S \) satisfying:

1) \( \bar{\zeta} = \zeta \)
2) \( \bar{\zeta} S = S \) (i.e. \( \bar{\zeta} \in S^\times \))
3) \( (b_1 b_2)^{\mathcal{W}} \bar{\zeta} = \zeta^{\mathcal{W}} (b_2 b_1)^{\mathcal{W}} \)

**Proof.** By polarization we have \( \mathcal{W} \)-symplectic maps \( \lambda_i : S \times S \to \mathcal{W} \) (\( i = 1, 2 \)) satisfying \( \lambda_i(sx, y) = \lambda_i(x, sy), \lambda_i(Fx, y) = \lambda_i(x,Vy)^{\mathcal{W}} \) so we can find \( \bar{\zeta_i} \in S \otimes \mathbb{Q} \) such that \( \lambda_i(s_1, s_2) = \text{tr}_{\mathcal{K}/\mathbb{Q}_p}(\bar{\zeta_i} s_1 s_2) \) such that \( \bar{\zeta} = -\zeta, p\bar{\zeta_i} = (\bar{\zeta_i} b_1 b_1)^{\mathcal{W}}, \zeta_i S = S^{-1} S \).
where $D$ is the different of $K$, so that $\bar{\zeta} = \zeta_1^{-1}\zeta_2$ has the above properties.

\[\square\]

**Proposition 3.3.** For some $\beta \in F^\times$ satisfying

$$\beta O_K = \begin{cases} p_F O_K & \text{if } K/F \text{ is unramified} \\ O_K & \text{if } K/F \text{ is ramified} \end{cases}$$

we have $L(A_1, A_2) \cong O_K$ with $\langle x, y \rangle = \beta xy$ on $O_K$.

**Proof.** First of all, as both $A_1, A_2$ are supersingular, we have an isogeny $A_2 \to A_1$, also that $\text{End} A_1 = M_n(H)$, where $H$ is the quaternion division algebra over $Q_p$. Now by Noether-Skolem, we can change this isogeny by some $h \circ f$ for $h \in M_n(H)$ so that it becomes $O_K$-linear, so:

$$L(A_1, A_2) \otimes Q_p \cong \text{Hom}_{O_K}(A_1, A_1) \otimes Q_p \cong K$$

Consequently $L(A_1, A_2)$ is free of rank 1 over $O_K$ (freeness is trivial from the definition).

Let $s$ be an $O_K$-module generator of $L(A_1, A_2)$, such that $\beta = \bar{\zeta}ss$ ($\bar{\zeta}$ is as in Lemma 3.2), so that $\bar{\zeta}S = S$.

Now we have to determine $ssS$. Suppose that $d = [K^u : Q_p]$, and

$$\{\psi^0, \psi^1, \ldots, \psi^{d-1}\}$$

be the set of embeddings $O_K^u \to W$ such that $\text{Fr} \circ \psi^i = \psi^{i+1}$ (in a cyclic way). Now the relation $(b_1s)_{\text{Fr}} = b_2s^{\text{Fr}}$ implies that

$$\text{ord}_{\psi^{i+1}}(s) = \text{ord}_{\psi^i}(s) - \text{ord}_{\psi^i}(b_2) + \text{ord}_{\psi^i}(b_1) = \text{ord}_{\psi^i}(s) - m(\psi^i, \Phi_2) + m(\psi^i, \Phi_1) \tag{3.1}$$

Now we have two cases:

- If $K/F$ is ramified (same as $K^u \subseteq F$) then $x \mapsto \bar{x}$ acts trivially on $K^u$, so both $\Phi^F$ and $\Phi^F$ restrict to the same $\bar{\psi}^i$ on $O_K^u$ (call it $\psi^0$), so by the formula above, $\text{ord}_{\psi^{i+1}}(s) = \text{ord}_{\psi^i}(s)$, also $s$ is a generator, so $\text{ord}_{\psi^i}(s) = 0$ for all $i$, so that $s \in S^\times$ and $\beta S = \bar{\zeta}ssS = S$

- If $K/F$ is unramified, then there are $i \neq j$ not equal such that $\Phi^F|_{K^u} = \psi^i$, $\Phi^F|_{K^u} = \psi^j$, now $\text{Fr}^{j-i}$ gives us the conjugation $x \mapsto \bar{x}$ on $K^u$, as this is an involution, we have $j - i = \frac{d}{2} \mod d$, also as $s$ is a generator we have $\text{ord}_{\psi^i}(s) = 0$ for some $\nu$ so that we can compute $\text{ord}_{\psi^i}(s)$ for all $r$ to get the table above.

$$\text{ord}_{\psi^r}(ss) = \text{ord}_{\psi^r}(s) + \text{ord}_{\psi^{r-i}}(s) = 1$$

so $\beta S = sss = p_FS$
We know that
\[ \text{Theorem} \]

From now on we assume the following ramification condition:

Proof. Its kernel is equal to \( \text{Lemma} \)

Now suppose that \( \mathcal{K} \) is a local field containing the normal closure of \( \mathcal{K}/\mathbb{Q}_p \) and let \( \mathcal{W} \) be the ring of integers of the completion of the maximal unramified extension of \( \mathcal{K} \). Let \( m \subset \mathcal{W} \) be the maximal, \( R^{(k)} := \mathcal{W}/m^k \); there’s a unique deformation (by Grothendieck-Messing) \( (A_{i}^{(k)}, \kappa_{i}^{(k)}, \lambda_{i}^{(k)}) \) of \( (A_{i}, \kappa_{i}, \lambda_{i}) \) to \( R^{(k)} \).

\[ L^{(k)}(A_1, A_2) = \text{Im}(\text{Hom}_{\mathcal{O}_{\mathcal{K}}}(A_1^{(k)}, A_2^{(k)}) \to \text{Hom}_{\mathcal{O}_{\mathcal{K}}}(A_1, A_2)) \subseteq L(A_1, A_2) \]

From now on we assume the following ramification condition:

\( (\ast) \) If \( p \leq [\mathcal{K} : \phi^{p}(\mathcal{K})] + 1 \) then the ramification index of \( \mathcal{K}/\mathbb{Q}_p \) is less than \( p \).

**Theorem 3.4.** \( f \in L(A_1, A_2) \) with \( \langle f, f \rangle = \alpha \) can be lifted to \( L^{(k)} \) but not to \( L^{(k+1)} \), where

\[ k = \frac{1}{2} \text{ord}_{\mathcal{O}_{\mathcal{K}}}(ap_f) \]

We prove this by induction on the order of \( f \) in \( L(A_1, A_2) \cong \mathcal{O}_{\mathcal{K}} \).

First we prove a lemma about PD-thickening of \( R^{(k)} \)’s, let \( \varepsilon \) be the ramification index of \( \mathcal{K}/\phi^{p}(\mathcal{K}) \).

**Lemma 3.5.** If \( a \leq b \leq a + \varepsilon + 1 \), then \( R^{(b)} \to R^{(a)} \) is a PD-thickening.

**Proof.** Its kernel is equal to \( I = m^a/m^b \). To prove that this is a PD-thickening we have to (by definition) find some functions \( \gamma_n : m^a/m^b \to m^a/m^b \) for \( n > 0 \) that behave like \( \frac{x^n}{m^n} \) such that they are the maps \( \gamma_n : px \mapsto \frac{p^n}{m^n}x^n \) when restricted to \( \frac{(p)}{m^n} \).

We know that \( m = (\pi) \) for some uniformizer \( \pi \) of \( \mathcal{K} \). Suppose that \( (p) = (\pi)^r \) where \( r \) is the ramification index of \( \mathcal{K}/\mathbb{Q}_p \). Now we have two cases:

- \( a \geq r \) then \( I \subseteq (p)/m^b \), so that we can put canonical divided powers \( \gamma_n : px \mapsto \frac{p^n}{m^n}x^n \). Now to prove that \( \gamma_n \) is valued in \( I \), we compute the the order of its values (Suppose that \( px \in I \)):

\[ \text{ord}_{\pi}(\frac{p^n}{m^n}x^n) = n \text{ord}_{\pi}(px) - \text{ord}_{\pi}(n!) \geq na - \text{ord}_{\pi}n! = na - r\left( \frac{n}{p} \right) + \cdots = a + (n-1)a - r\left( \frac{n}{p} \right) + \cdots \geq a + r(n-1 - \frac{n-1}{p-1}) \geq a \]
• \( a < re \), By the statement of the lemma and the case we are considering, we have

\[
2re + 1 \geq re + \varepsilon + 1 > a + \varepsilon + 1 \geq b \implies 2re \geq b
\]

so that

\[
2re \geq b \iff m^{2re} / (m^b \cap m^{2re}) = 0 \iff ((p) / m^b)^2 = 0
\]

so we can still use the divided power structure \( \gamma_n : \pi x \mapsto \frac{\pi^n}{m!} x^n \), because on the ideal \((p) / m^b\) we just need \( \gamma_1 \) which is the identity and \( \gamma_i \) for \( i \geq 2 \) have to be zero maps on \((p) / m^b\) (the fact that images of \( \gamma_n \)'s lie in \( I \) uses ramification condition \((\ast)\)).

First we prove the base case:

**Proposition 3.6.** If \( f \) is an \( O_{K} \)-module generator of \( L(A_1, A_2) \) and \( \mathcal{D}_{K/F} \) be the relative different of \( K/F \).

1) If \( K/F \) is unramified, \( f \) is in \( L^{(e)}(A_1, A_2) \) but not in \( L^{(e+1)}(A_1, A_2) \).

2) If \( K/F \) is ramified, \( f \) is in \( L^{(k)}(A_1, A_2) \) but not in \( L^{(k+1)}(A_1, A_2) \) where \( k = \varepsilon \text{ord}_{O_{K}} \mathcal{D}_{K/F} \).

**Proof.** Let \( J_{\Phi} \), be the kernel of the \( \tilde{W} \)-algebra map \( O_{K} \otimes_{\mathbb{Z}_p} \tilde{W} \to \prod_{\phi \in \Phi_1} C_p(\phi) \) (sending \( x \otimes 1 \) to \( (\phi(x))_{\phi \in \Phi_1} \)) also let \( \mathcal{D}_1, \mathcal{D}_2 \) the Grothendieck-Messing crystals of \( A_1, A_2 \) respectively.

By the above lemma, the map \( R^{(e)} \to R^{(1)} \) is a PD-thickening so that:

\[
\mathcal{D}_1(R^{(e)}) = S \otimes_{\tilde{W}} R^{(e)}
\]

\[
\mathcal{D}_2(R^{(e)}) = S \otimes_{\tilde{W}} R^{(e)}
\]

by Theorem 2.1.3 of [9], Hodge filtrations of these are \( J_{\Phi_1} \mathcal{D}_1(R^{(e)}) \) and \( J_{\Phi_2} \mathcal{D}_2(R^{(e)}) \), respectively. So \( f \) lifts to \( A_1^{(e)} \to A_2^{(e)} \) iff

\[
f : J_{\Phi_1} \mathcal{D}_1(R^{(e)}) \to \mathcal{D}_2(R^{(e)}) / J_{\Phi_2} \mathcal{D}_2(R^{(e)})
\]

is trivial.

Now we prove parts 1 and 2 of the proposition:

1) Suppose that \( f \) is \( s \in S \) when writing \( L(A_1, A_2) \) as a subset of \( S \) as before, then we must prove that multiplication by \( s \) is zero mod \( m^e \) in the first map below:

\[
J_{\Phi_1}(S \otimes_{\tilde{W}} \tilde{W}) \xrightarrow{s} (S \otimes_{\tilde{W}} \tilde{W}) / J_{\Phi_2}(S \otimes_{\tilde{W}} \tilde{W}) \xrightarrow{\text{red}} \tilde{W}
\]

(3.2)
By lemma 2.1.2 of [9], we have that $J_{\Phi_1}$ is generated by

$$j(\psi, x) = e_\psi \prod_{\phi \in \Phi_1, \phi | O_K = \psi} (x \otimes 1 - 1 \otimes \phi(x))$$

for all $x \in O_K$ and $\psi : O_K^u \to W$. Now under multiplication by $s$, all $j(\psi, x)$ go to $J_{\Phi_2}(S \otimes_W \hat{W})$ except for $j(\psi_0, x)$'s where $\psi_0 = \phi^{\partial \overline{\phi}}|_{O_K}$ and $x \in O_K$, these elements go to

$$\phi^{\partial \overline{\phi}}(s) \prod_{\phi \in \Phi_1, \phi | O_K = \psi_0} (\phi^{\partial \overline{\phi}}(x) - \phi(x)) \in \hat{W}$$

after the last map. Now I claim that there’s $a \in O_K^u$ such that

$$\prod_{\phi \in \Phi_1} \phi|_{O_K} = \psi_0(\bar{a} - a)$$

is a unit in $\hat{W}$, indeed, by unramifiedness of $K/F$ (which is equivalent to $K^u \not\subseteq F$) we know that there exists $a \in O_K^u$ such that $\bar{a} \neq a$, now

$$\phi^{\partial \overline{\phi}}(\bar{a}) - \phi(a) = \psi_0(\bar{a} - a)$$

so that

$$j(\psi_0, a) = \psi_0(\bar{a} - a)^{m(\psi_0, \Phi_1)}$$

Thus $a - \bar{a}$ is a unit which implies that $j(\psi_0, a)$ is a unit.

Now we have, by the table in proposition 2.3, that $\text{ord}_{\psi_0} s = 1$, and so that the map 2.2 is zero mod $m^e$ and nonzero mod $m^{e+1}$. Now the map

$$S \otimes_W \hat{W} / J_{\Phi_2}(S \otimes_W \hat{W}) \xrightarrow{(\phi(x))_{\phi \in \Phi_2} \otimes \phi^{\partial \overline{\phi}}} C^u_p$$

is injective (because we killed the kernel), but the restriction of this map to image of multiplication by $s$ in equation 3.1 sends $x$ to $(0, \ldots, 0, \phi^{\partial \overline{\phi}}(x), 0, \ldots, 0)$, so that the image of $\phi^{\partial \overline{\phi}}$ determines everything and we can extend $f$ to $L^{(e)}$.

Also, we have that $R^{(e+1)} \to R^{(1)}$ is a PD-thickening, so that the map

$$f : J_{\Phi_1} D_1(R^{(e+1)}) \to D_2(R^{(e+1)}) / J_{\Phi_2} D_2(R^{(e+1)})$$

is just equation 3.1 mod $m^{e+1}$ which is nonzero so we are not able to extend $f$ to $L^{(e+1)}$.

2) Suppose that we have extended $f$ to $L^{(k)}$, in this case we will get that $f$ is equal to some $s \in S^\times$, by proposition 1, so $f : D(A_1) \to D(A_2)$ (these are
Dieudonné modules) is an isomorphism, If we check the \( \Phi_1 \)-determinant condition on \( D(A_1)/VD(A_1) \rightarrow D(A_2)/VD(A_2) \), we will get \( \phi^p = \bar{\phi}^p \mod m^k \), so \( k \leq \epsilon \mathrm{ord}_{O_K} D_{K/F} \).

Now if \( k \leq \epsilon \mathrm{ord}_{O_K} D_{K/F} \), then \( f : A_1 \rightarrow A_2 \) is an isomorphism of \( p \)-divisible groups with \( O_K \)-action and respecting \( \Phi_1 \)-determinant condition up to \( m^k \), now because deformations are unique and above map is an isomorphism we get an \( O_K \)-linear map

\[
f^{(k)} : A_1^{(k)} \rightarrow A_2^{(k)}
\]

that lifts \( f \), so \( f \in L^{(k)}(A_1, A_2) \) but not in \( L^{(k+1)}(A_1, A_2) \).

Now we have the induction step in the next proposition:

**Proposition 3.7.** Let \( \pi_K \) be the uniformizer of \( O_K \). If \( f \in L^{(k)} \), then \( \pi_K f \in L^{(k+\epsilon)} \), also \( \pi_K : L^{(k)}/L^{(k+1)} \rightarrow L^{(k+\epsilon)}/L^{(k+\epsilon+1)} \) is injective.

**Proof.** First of all, we have if \( j \in J_{\Phi_1} \) then

\[
(x \otimes 1 - 1 \otimes \bar{\phi}^p(x)) j \in J_{\Phi_2} = \ker(O_K \otimes \mathbb{Z}_p \bar{W} \rightarrow \prod_{\phi \in \Phi_2} \mathbb{C}_p(\phi))
\]

for all \( x \in O_K \). Henceforth, if we have an \( O_K \otimes \mathbb{Z}_p \bar{W} \)-linear map \( f : J_{\Phi_1} M \rightarrow N/J_{\Phi_2} N \) for \( O_K \otimes \mathbb{Z}_p \bar{W} \)-modules \( M, N \), then

\[
(x \otimes 1 - 1 \otimes \bar{\phi}^p(x)) \circ f = 0
\]

for all \( x \in O_K \).

Suppose that \( f \) lifts to \( f^{(k)} : A_1^{(k)} \rightarrow A_2^{(k)} \), then write \( D_1^{(k)}, D_2^{(k)} \) for Grothendieck-Messing crystals of \( A_1^{(k)}, A_2^{(k)} \), respectively. Now because we have the PD-thickening \( R^{(k+\epsilon)} \rightarrow R^{(k)} \),

\[
\begin{array}{ccc}
J_{\Phi_1} D_1^{(k)}(R^{(k+\epsilon)}) & \xrightarrow{f^{(k)}} & D_2^{(k)}(R^{(k+\epsilon)})/J_{\Phi_2}(D_2^{(k)}(R^{(k+\epsilon)})) \\
\downarrow & & \downarrow \\
J_{\Phi_1} D_1^{(k)}(R^{(k)}) & \xrightarrow{f^{(k)}} & D_2^{(k)}(R^{(k)})/J_{\Phi_2}(D_2^{(k)}(R^{(k)}))
\end{array}
\]

where the bottom row is a map, so that the top map becomes zero after \( \otimes_{R^{(k+\epsilon)}} R^{(k)} \), so its image is annihilated by \( m^\epsilon \), and so \( \pi_K f^{(k)} = \bar{\phi}^p(\pi_K) f^{(k)} \) is zero on the top row and so it can be lifted to \( L^{(k+\epsilon)} \).

Now suppose that \( f \in L^{(k)} \) but not in \( L^{(k+1)} \), we have the PD-thickening \( R^{(k+\epsilon+1)} \rightarrow R^{(k)} \), so we have a map

\[
J_{\Phi_1} D_1^{(k)}(R^{(k+\epsilon+1)}) \rightarrow D_2^{(k)}(R^{(k+\epsilon+1)})/J_{\Phi_2} D_2^{(k)}(R^{(k+\epsilon+1)})
\]
If $\pi_K f^{(k)}$ lifts to $L^{(k+\varepsilon)}$, then we must have that

$$
\pi_K f^{(k)} : I_{\Phi_1} D_1^{(k)} (R^{(k+\varepsilon)}) \to D_2^{(k)} (R^{(k+\varepsilon)}) / I_{\Phi_2} D_2^{(k)} (R^{(k+\varepsilon)})
$$

is trivial, but by above $\pi_K f^{(k)} = \overline{\phi} (\pi_K)f^{(k)}$, so $\overline{\phi} (\pi_K)f^{(k)}$ takes value in

$$m^{k+\varepsilon+1} D_2^{(k)} (R^{(k+\varepsilon)}) / I_{\Phi_2} D_2^{(k)} (R^{(k+\varepsilon)})$$

so $f^{(k)}$ takes value in $m^{k+1} D_2^{(k)} (R^{(k+\varepsilon+1)}) / I_{\Phi_2} D_2^{(k)} (R^{(k+\varepsilon+1)})$ and so the map

$$f^{(k)} : I_{\Phi_1} D_1^{(k)} (R^{(k+1)}) \to D_2^{(k)} (R^{(k+1)}) / I_{\Phi_2} D_2^{(k)} (R^{(k+1)})$$

is also trivial which means that $f$ can be lifted to $L^{(k+1)}$, which is a contradiction. $\Box$

Now we finish the proof of theorem 2:

Proof. $\beta$ be as before, $f = \pi_K^m f_0$, $f_0$ an $O_K$-module generator of $L(A_1, A_2)$. Suppose that $\mathcal{K} / \mathcal{F}$ is unramified, $f = \pi_K^m f_0$ where $f_0$ is an $O_K$-module generator of $\text{Hom}_{O_K}(A_1, A_2)$, then

$$\langle f_0, f_0 \rangle O_K = \beta O_K = p F O_K$$

Now by proposition 2, we have $f_0 \in L^{(\varepsilon)}$ but not in $L^{(\varepsilon+1)}$, so that $f \in L^{((m+1)\varepsilon)}$ but not in $L^{((m+1)\varepsilon+1)}$. Now

$$a = \langle f, f \rangle = \pi_K^{2m} \langle f_0, f_0 \rangle \implies a O_K = \pi_K^{2m} \langle f_0, f_0 \rangle O_K = \pi_K^{2m} p F O_K$$

$$\implies k = (m+1)\varepsilon = \frac{1}{2} \text{ord}_{O_K}(\alpha p F).$$

Now if $\mathcal{K} / \mathcal{F}$ is ramified, then by proposition 2 we have that $f_0$ is in $L^{((\text{ord}_{O_K} D_{\mathcal{K}/\mathcal{F}}))}$ but not in $L^{((\text{ord}_{O_K} D_{\mathcal{K}/\mathcal{F}})+1)}$, so that $f$ is in $L^{(\varepsilon (\text{ord}_{O_K} D_{\mathcal{K}/\mathcal{F}})+1)}$ and cannot be lifted more. Here we have two cases:

- $p$ is odd: In this case, $\mathcal{K} / \mathcal{F}$ is tamely ramified and so $\text{ord}_{O_K} D_{\mathcal{K}/\mathcal{F}}$ is 1 and we get that the above implies

$$a O_K = \pi_K^{2m} \langle f_0, f_0 \rangle O_K = \pi_K^{2m} O_K$$

$$\implies k = (m+1)\varepsilon = \frac{1}{2} \text{ord}_{O_K}(\alpha p F).$$

- $p = 2$, this case never happens by the ramification condition! $\Box$

$p$-divisible Groups with Similar CM-Types

Here we have the same notation as before except that $\mathcal{K}$ is just quadratic étale extension over $\mathcal{F}$. 
Now for another case of having two triples \((A_1, \kappa_1, \lambda_1), (A_2, \kappa_2, \lambda_2)\) \(p\)-divisible groups/\(\mathbb{F}_p\) with action of \(O_K\) and polarization, but now with the same CM-type \(\Phi\). Let \(\text{ART}\) be the category of local Artinian \(\mathcal{W}\)-algebras with residue field \(\mathbb{F}_p\).

**Proposition 3.8.** Let \(R\) be an object of \(\text{ART}\), \(A'_i\) be the unique deformation of \(A_i\) to \(R\), then

\[
\text{Hom}_{O_K}(A'_1, A'_2) \to \text{Hom}_{O_K}(A_1, A_2)
\]

is a bijection.

**Proof.** Let \(f : S \to R\) be a surjection in \(\text{ART}\) such that \((\text{Ker}f)^2 = 0\), so that we can define a trivial divided power structure on it, consequently if \(f \in \text{Hom}_{O_K}(A_1, A_2)\), let \(D_1, D_2\) be Grothendieck-Messing crystals of \(A_1, A_2\). As before let \(J_\Phi\) be the kernel of the \(\mathcal{W}\)-algebra map \(O_K \otimes \mathcal{W} \to \prod_{\phi \in \Phi} C_p(\phi)\) (sending \(x \otimes 1\) to \((\phi(x))_{\phi \in \Phi}\)), now we have

\[
J_\Phi(O_K \otimes \mathcal{W}) \subseteq J_\Phi
\]

(they are equal!), so that on

\[
f : D_1(S) \to D_2(S)
\]

Hodge filtrations map to each other (\(J_\Phi\)'s are Hodge filtrations by the Theorem 2.1.3 of [9]), so by Grothendieck-Messing theory we can extend \(f\) (uniquely) to a map \(A^S_1 \to A^S_2\) (\(A^S_i\) means the deformation of \(A_i\) to \(S\)), so we can do this inductively (induction on the length of \(R\)) to get the result. \(\square\)

3.2 **Global Computation**

In this section, we will define the special divisors on \(CM_{\Phi_1} \times CM_{\Phi_2}\), compute the Arakelov degree of them with putting together a computation of the number of stacky points of the special divisors over residue fields and the lengths of local rings appearing in the Arakelov degree.

Let's fix some notation first. Let \(F\) be a totally real extension of \(\mathbb{Q}\) of degree \(n\), \(K\) be a CM-field having \(F\) as its maximal totally real field, \(O_K, O_F\) be the ring of integers of \(K, F\) respectively, \(\Phi_1, \Phi_2\) be a pair of nearby \(p\)-adic CM-types of \(K\) (i.e. their intersection has \(n - 1\) elements and \(\phi^{sp} \in \Phi_1, \overline{\phi}^{sp} \in \Phi_2\)) and \(\phi^{sp} := \phi^{sp}|_F = \overline{\phi}^{sp}|_F\). \(\tilde{K}\) be the normal closure of \(K/\mathbb{Q}\) (so it can be regarded as a subfield of \(\mathbb{C}\), \(\tilde{O}\) be the ring of integers of \(\tilde{K}\). \(\chi_{K/F}: \mathbb{A}^2_F \to \{\pm 1\}\) be the quadratic character of the quadratic extension \(K/F\). Let \(p\) be a prime of \(\tilde{O}\) and \(p_K\) the prime in \(O_K\) below \(p\) by the embedding \(\phi^{sp}: K \hookrightarrow \tilde{K}\) (the use of \(\phi^{sp}\) or \(\overline{\phi}^{sp}\) does not matter in our
case because we only care about nonsplit primes of $K/F$, $\bar{k}_p$ the residue field of $\bar{K}$ at $p$ and fix an algebraic closure $\bar{k}_p$. $\mathcal{W}_p$ be the ring of integers of the completion of the maximal unramified extension of $\bar{k}_p$.

Now we define the algebraic stack of CM abelian varieties and CM-type $\Phi$: $\mathcal{CM}_\Phi$ is the algebraic stack over $\bar{O}$ such that for an $\bar{O}$-scheme $S$, $\mathcal{CM}_\Phi(S)$ is the groupoid of all triples $(A, \kappa, \lambda)$ with

- $A \to S$ is an Abelian scheme of relative dimension $n$.
- $\kappa: O_K \to \text{End} A$ satisfies the $\Phi$-determinant condition.
- $\lambda: A \to A^\vee$ is an $O_K$-linear principal polarization.

$O_K$-linear polarization means that for all $x \in O_K$, we have $\lambda \circ x = x \circ \lambda$.

Now for $(A_1, A_2) \in (\mathcal{CM}_{\Phi_1} \times_\bar{O} \mathcal{CM}_{\Phi_2})(S)$, let $L(A_1, A_2) := \text{Hom}_{O_K}(A_1, A_2)$ with the $O_K$-valued Hermitian form defined by

$$f, g \in L(A_1, A_2); \quad (f, g) := \lambda_A^{-1} \circ g \circ \lambda_{A_2} \circ f \in \text{End}_{O_K} A_1 = O_K$$

By proposition 3.1.2 of [9], for $i = 1, 2$, every $(A, \kappa, \lambda) \in \mathcal{CM}_\Phi(\bar{k}_p)$ has a unique deformation to $\mathcal{W}_p$, we call this deformation the canonical lift of $(A, \kappa, \lambda)$. Assume the ramification conditions below:

1) $K/F$ is ramified at at least one finite prime.

2) For every rational prime $l \leq |K : Q| + 1$, ramification index is less than $l$

**Proposition 3.9.** If $K/F$ is ramified at at least one finite prime, then for any CM-type $\Phi$ of $K$, $\mathcal{CM}_\Phi(\mathbb{C})$ is nonempty.

**Proof.** Consider $\mathbb{C}^n$ constructed from the $\phi \in \Phi$. Let $\zeta \in K^\times$ satisfy $\bar{\zeta} = -\zeta$, also (using weak approximation theorem) assume that $\phi(\zeta)i > 0$ for all $\phi \in \Phi$. Now $\lambda(x, y) = \text{tr}_{K/Q}(\bar{\zeta}xy)$ is an $\mathbb{R}$-symplectic form on $K_\mathbb{R} \cong \mathbb{C}^n$ and $\lambda(ix, x)$ is positive definite. By class field theory and the ramification condition in the hypothesis, the norm map from ideal class group of $K$ to narrow ideal class group of $F$ is surjective, so there is a fractional $O_K$-ideal $I$ and $u \in F^{\times}$ such that $uI = \zeta^{-1}D^{-1}$ where $D$ is the different of $K/Q$. So by replacing $\zeta$ with $\zeta u^{-1}$ we have $\zeta I \bar{I} = \mathcal{D}^{-1}$, so that

$$I = \{x \in K_\mathbb{R} | \lambda(x, I) \subseteq \mathbb{Z}\}$$

Now $\lambda$ is a Riemann form on $K_\mathbb{R}/I$, so it is an Abelian variety with the given CM-type. \qed

**Proposition 3.10.** Suppose that $k$ is an algebraically closed field and $(A_1, A_2) \in (\mathcal{CM}_{\Phi_1} \times_\bar{O} \mathcal{CM}_{\Phi_2})(k)$, if $f \in V(A_1, A_2) := L(A_1, A_2) \otimes \mathbb{Q}$ is such that $\langle f, f \rangle \neq 0$, then $k$ has nonzero characteristic and $A_1, A_2$ are $O_K$-isogenous.
Proof. Suppose that \( f \in L(A_1, A_2) \) and let \( T_i(A_i) \) be the Tate module at a prime \( l \neq \text{char} k \) and \( T_i^p(A_i) := T_i(A_i) \otimes \mathbb{Q} \), consider the Weil pairing induced by polarizations:

\[
i = 1, 2; \quad \Lambda_i : T_i^p(A_i) \times T_i^p(A_i) \to \mathbb{Q}_l(1)
\]

By existence of \( f \), we get a map

\[
f_i : T_i^p(A_1) \to T_i^p(A_2)
\]

Consider the adjoint of \( f_i \):

\[
\Lambda_1(x, f_i^* y) = \Lambda_2(f_i x, y)
\]

and we get that \( f_i^* \circ f_i = \langle f, f \rangle \) as elements in \( F \subseteq \text{End}(T_i^p A_i) \otimes \mathbb{Q} \), so that \( f_i \) is injective and an isogeny, so we have \( O_K \)-linear isogeny \( A_1 \sim A_2 \). Now, because CM-types of \( A_1, A_2 \) are different, \( \text{char } k > 0 \). \( \square \)

From now on, we fix some \((A_1, A_2) \in (CM_{\Phi_1} \times O CM_{\Phi_2})(\overline{k}_p)\).

Remark 3.11. If \( (CM_{\Phi_1} \times O CM_{\Phi_2})(\overline{k}_p) \) is nonempty with an \( f \) as above, then \( p_F \) is non-split in \( K \). Indeed, On the level of Lie algebras, using the isogeny \( f \), we must have \( \phi^{sp} = \overline{\phi^{sp}} \), so that \( x = x \mod p \) for all \( x \in K \), so \( p = p \), so that \( p_K = \overline{p_K} \) and \( p_F \) is non-split in \( K \).

Proposition 3.12. Let \( k \) be an algebraically closed field and \( \text{char } k > 0 \) with \( f \in L(A_1, A_2) \) such that \( \langle f, f \rangle \neq 0 \), then \( L(A_1, A_2) \) is a projective \( O_K \)-module of rank 1. Let \( q \) be a rational prime and \( q \) above it in \( F \), then the map

\[
L(A_1, A_2) \otimes_{O_F} O_{F, q} \to \text{Hom}_{O_K}(A_1[q^\infty], A_2[q^\infty])
\]

is an isomorphism.

Proof. By the last proposition, we get an \( O_K \)-linear isogeny \( A_2 \to A_1 \), so we get

\[
\text{Hom}_{O_K}(A_1, A_2) \to \text{Hom}_{O_K}(A_1, A_1) \cong O_K
\]

which is injective and has finite cokernel, so \( \text{Hom}_{O_K}(A_1, A_2) \) is a (nonzero) fractional ideal of \( O_K \) (as \( 0 \neq f \in \text{Hom}_{O_K}(A_1, A_2) \)). Also, the map \( A_2[q^\infty] \to A_1[q^\infty] \) will give us the injective map

\[
\text{Hom}_{O_K}(A_1[q^\infty], A_2[q^\infty]) \to \text{Hom}_{O_K}(A_1[q^\infty], A_1[q^\infty]) = O_K \otimes \mathbb{Z}_q
\]

with finite cokernel, so that \( \text{Hom}_{O_K}(A_1[q^\infty], A_2[q^\infty]) \subseteq O_K \otimes \mathbb{Z}_q \) is a projective \( O_K \otimes \mathbb{Z}_q \)-module of rank 1.
The map

$$\text{Hom}_{O_K}(A_1, A_2) \otimes \mathbb{Z} \mathcal{Z}_q \rightarrow \text{Hom}_{O_K}(A_1[q^\infty], A_2[q^\infty])$$

is injective with $\mathcal{Z}_q$-torsion free cokernel and as the $\mathcal{Z}_q$-ranks of domain and codomain are equal (by the fact just mentioned), this is an isomorphism.

So after taking $q$-parts, $L(A_1, A_2) \otimes_{O_F} O_{F,q} \rightarrow \text{Hom}_{O_K}(A_1[q^\infty], A_2[q^\infty])$ is an isomorphism. \qed

Now consider the group $C_K = \mathcal{I}_K / \mathcal{P}_K$ where $\mathcal{I}_K$ consists of pairs $(I, \zeta)$ such that $I$ is a fractional $O_K$-ideal and $\zeta \in F^\times$ that $\zeta I = O_K$, multiplication is given by $((I_1, \zeta_1), (I_2, \zeta_2)) = (I_1 I_2, \zeta_1 \zeta_2)$ and $\mathcal{P}_K = \{(z^{-1} O_K, z \mathcal{Z}) | z \in K^\times \}$ is a subgroup of it. Now $C_K$ acts on the set of all $(L, H)$ (where $L$ is an $O_K$-fractional ideal and $H$ is an $O_K$-valued $O_K$-Hermitian form on $L$) as follows: If $(I, \zeta) \in \mathcal{I}_K$, change this element by an element of $\mathcal{P}_K$ so that $\zeta \in O_K$, and define the action by

$$(I, \zeta).(L, H) = (IL, \zeta H)$$

Now for an Abelian scheme $A \in \mathcal{C}\mathcal{M}_{\Phi_1}(S)$ and for $I$ a fractional ideal of $O_K$, write $A^I$ for the Serre construction $\mathcal{I} \otimes_{O_K} A \in \mathcal{C}\mathcal{M}_{\Phi_1}(S)$.

For a ring $R$, denote $R \otimes \hat{\mathcal{O}}$ by $\hat{R}$ and for an $O_K$-module $M$, we denote $M \otimes_{O_K} \hat{O}_K$ by $\hat{M}$ (so that $\hat{L}(A_1, A_2) = L(A_1, A_2) \otimes_{O_K} \hat{O}_K$). Now we have

**Proposition 3.13**. Let $S$ be a connected $\hat{O}$-scheme and $(A_1, A_2) \in (\mathcal{C}\mathcal{M}_{\Phi_1} \times_{\hat{O}} \mathcal{C}\mathcal{M}_{\Phi_2})(S)$, then for any $(I, \zeta) \in \mathcal{I}_K$, we have an isomorphism of $\hat{O}_K$-modules $\hat{L}(A_1, A_2) \cong \hat{L}(A_1, A'_2)$ such that if $\langle , \rangle$ is the Hermitian form of $\hat{L}(A_1, A'_2)$, we have

$$(\langle , \rangle)^I = \zeta z \langle , \rangle$$

where $z$ is a finite idele of $K$ such that $z O_K = I$.

Now for $(A_1, A_2) \in (\mathcal{C}\mathcal{M}_{\Phi_1} \times_{\hat{O}} \mathcal{C}\mathcal{M}_{\Phi_2})(S)$, define $L_{\text{Betti}}(A_1, A_2) = \text{Hom}_{O_K}(H_1(A_1, C), H_1(A_2, C))$.

**Proposition 3.14**. Let $(A_1, A_2) \in (\mathcal{C}\mathcal{M}_{\Phi_1} \times_{\hat{O}} \mathcal{C}\mathcal{M}_{\Phi_2})(S)$, then there exists $\beta \in \hat{O}_F$ and an isomorphism

$$(\hat{L}_{\text{Betti}}(A_1, A_2), \langle , \rangle) \cong (\hat{O}_K, \beta xy)$$

Also, $\langle , \rangle$ is negative definite at $\infty^p$ and positive definite at others.

**Proof.** We give a sketch below:

We get the polarizations $\lambda_1(x, y) = \text{tr}_{K/Q}(\zeta xy)$, $\lambda_2(x, y) = \text{tr}_{K/Q}(\zeta^{-1} xy)$, then $\zeta^{-1} O_K = O_K$. $L_B(A_1, A_2)$ has Hermitian form $\zeta^{-1} xy$. So that $\phi(\zeta^{-1}) > 0$ for $\phi \neq \phi^p, \phi^{\infty}$ and $\phi(\zeta^{-1}) < 0, \phi^{\infty}(\zeta^{-1}) < 0$. \qed
Now consider all pairs \((L, H)\) such that
- \(L\) is a fractional ideal of \(O_K\).
- \(H\) is an \(O_K\)-valued \(O_K\)-Hermitian form on \(L\).
- \((\tilde{L}, H) \cong (O_K, \beta x \bar{y})\) with \(\beta \in \tilde{\mathcal{O}}_K^\times\).
- \((L, H)\) has signature \(-1\) at \(\infty^{sp}\) and \(+1\) at other embeddings.

The set of these Hermitian spaces has a \(C\) isomorphism \(\tilde{O}\).

**Proof.** There exists a unique lift of \(\beta\) respecting Hermitian forms.

Now we have the \(\beta\) below it, then there are isomorphisms of Hermitian \(\beta\) inverse image of maximal ideal of \(\mathcal{O}\) for \(q\) \(A\) modules of \(3\) \(CM\) \(1\).

\[ \beta = \left\{ \begin{array}{ll}
1 & \text{if } K/F \text{ is unramified} \\
0 & \text{if } K/F \text{ is ramified}
\end{array} \right. \]

**Theorem 3.15.** Let \(p\) be a prime of \(\tilde{K}\) such that \(p_F\) is nonsplit and \((A_1, A_2) \in (CM_{\Phi_1} \times CM_{\Phi_2})(\tilde{K}_p)\) and there exists \(f \in L(A_1, A_2)\) such that \((f, f) \in F^\times\), then

\[ (\tilde{L}(A_1, A_2), (\cdot, \cdot)) \cong (\tilde{O}_K, \beta x \bar{y}) \]

where \(\beta \in \tilde{F}^\times\) and \(\beta O_F = p_F^{\epsilon_p}\) where

\[ \epsilon_p = \left\{ \begin{array}{ll}
1 & \text{if } K/F \text{ is unramified} \\
0 & \text{if } K/F \text{ is ramified}
\end{array} \right. \]

**Proof.** There exists a unique lift of \((A_1, A_2)\) to \(C_p\) and then the isomorphism of \(\tilde{K}\)-algebras \(C_p \cong C\), we may view the unique lift as a pair \((A_1', A_2')\) in \((CM_{\Phi_1} \times CM_{\Phi_2})(C)\). Let \(q \subseteq O_F\) be a prime with the rational prime \(q\) below it, then there are isomorphisms of Hermitian \(O_{K,q}\)-modules:

\[ L_{\text{Betti}}(A_1', A_2') \otimes_{O_K} O_{K,q} \cong \text{Hom}_{O_K}(A_1'[q^\infty], A_2'[q^\infty]) \]

\[ L(A_1, A_2) \otimes_{O_K} O_{K,q} \cong \text{Hom}_{O_K}(A_1[q^\infty], A_2[q^\infty]) \]

Now we have the

**Lemma 3.16.** With the notation as above, let \(q \subseteq O_F\) be a prime \(\neq p_F\), we have \(O_K\)-linear isomorphisms

\[ \text{Hom}_{O_K}(A_1'[q^\infty], A_2'[q^\infty]) \cong \text{Hom}_{O_K}(A_1[q^\infty], A_2[q^\infty]) \]

respecting Hermitian forms.

**Proof.** Let \(q\) be the rational prime below \(q\), If \(q \neq p\), then the \(q\)-adic Tate modules of \(A_i', A_i\) are isomorphic, so that it is true.

For \(q = p\) and \(i = 1, 2\), consider \(\Phi_i(q)\) to be all of \(\phi \in \Phi_i\) such that the inverse image of maximal ideal of \(O_{C_p}\) in \(K \hookrightarrow \tilde{C}_p\) is \(q\). Now we have that \(\Phi_1(q) = \Phi_2(q)\) because \(\Phi^{sp}\) cannot be in \(\Phi_i(q)\) and similarly \(\Phi^{sp}\) cannot be
in $\Phi_2(q)$ (as $q$ is not $p_F$ by hypothesis). So by passing to $p$-divisible groups $A_1[q]\cap$ with the action of $O_{K,\mathfrak{A}}$, we are in the situation of proposition 2.8 and letting $A_1^{\text{can}}$ be the canonical lifting of $A_1$ to $\tilde{W}_p$, we have

$$\text{Hom}_{O_K}(A_1^{\text{can}}[p^{\infty}], A_2^{\text{can}}[q^{\infty}]) \to \text{Hom}_{O_K}(A_1[p^{\infty}], A_2[q^{\infty}])$$

is an isomorphism, now by the base change $\tilde{W}_p \to C_p$ we get a map

$$\text{Hom}_{O_K}(A_1^{\text{can}}[p^{\infty}], A_2^{\text{can}}[q^{\infty}]) \to \text{Hom}_{O_K}(A_1'[p^{\infty}], A_2'[q^{\infty}]) \quad (3.3)$$

Now we have Tate’s theorem that says for two $p$-divisible groups $G, H$ with Tate modules $TG, TH$ (over specific types of rings $R$, which includes $\tilde{W}_p$ and $C_p$ with $E = \text{Frac}(R)$) the map

$$\text{Hom}(G, H) \to \text{Hom}_{\text{Gal}(E/E)}(TG, TH)$$

is an isomorphism. So that the map in equation 3.3 is injective with image the submodule of $\text{Aut}(C_p/\text{Frac}(\tilde{W}_p))$-invariants of $\text{Hom}_{O_K}(A_1'[p^{\infty}], A_2'[q^{\infty}])$, so that its cokernel is $\mathbb{Z}_p$-torsion free. Now both sides of the composit map $\text{Hom}_{O_K}(A_1[p^{\infty}], A_2[q^{\infty}]) \to \text{Hom}_{O_K}(A_1'[p^{\infty}], A_2'[q^{\infty}])$ are free of rank 1 over $O_{K,\mathfrak{A}}$, so that it is an isomorphism. It is obvious that the isomorphism respects Hermitian forms.

Now to prove the theorem, we just collect the propositions we proved to get: If $q \neq p_F$, then

$$L(A_1, A_2) \otimes_{O_F} O_{F,\mathfrak{A}} \cong O_{K,\mathfrak{A}}$$

with Hermitian form $\beta_q x\bar{y}$ with $\beta_q \in O_{F,\mathfrak{A}}^\times$.

If $q = p_F$, then we use prop 2.3 to get $\beta_q O_{F,\mathfrak{A}} = p_F^{e_{p_F}}$.

Now set $\beta = \prod_q \beta_q$, so that the ideal of $L(A_1, A_2)$ is $\beta O_F = p_F^{e_{p_F}}$. So we have that $L(A_1, A_2)$ isomorphic to $\tilde{O}_K$ with Hermitian form given by $\beta x\bar{y}$. Now to prove $\chi_{K/F}(\beta) = 1$, we have that $L(A_1, A_2) \otimes_{O_K} K$ is $K$ with the Hermitian form given by $c x\bar{y}$ with $c \in F^{\gg 0}$ and $\beta$ differs by norm with $c$ at each place of $K$, so that

$$\chi_{K/F}(\beta) = \chi_{K/F}(c) = 1$$

Let $S$ be an $\tilde{O}$-scheme, then to each $(A_1, A_2) \in (C_{M_{\Phi_1}} \times_{\tilde{O}} C_{M_{\Phi_2}})(S)$, we have $(L(A_1, A_2), \langle \cdot, \cdot \rangle)$.

Let $\mathcal{Z}(\alpha)$ (for $\alpha \in F^\times$ totally positive) be the algebraic stack that for an
\(\tilde{O}\)-scheme \(S\) assigns the groupoid of triples \((A_1, A_2, f)\), where \((A_1, A_2) \in (CM_{\Phi_1} \times CM_{\Phi_2})(S)\) and \(f\) is an element of \(L(A_1, A_2)\) with \(\langle f, f \rangle = \alpha\).

**Proposition 3.17.** Let \(\alpha \in F^\times\),

1) Suppose that \(\alpha \gg 0\), then the stack \(Z(\alpha)\) has dimension 0 and is supported in nonzero characteristics.
2) If \(p\) is a prime in \(K\) for which \(Z(\alpha)(\tilde{k}_p)\) is nonempty, then \(p_F\) is nonsplit.

**Proof.** 1) Second part of the statement comes from proposition 3.2. For the first part, we have that the map of stacks \(Z(\alpha) \to CM_{\Phi_1} \times \tilde{O}CM_{\Phi_2}\) sending \((A_1, A_2, f)\) to \((A_1, A_2)\) is unramified, so that each of \(\tilde{O}_{Z(\alpha), \tilde{A}}\) is a quotient of \(\tilde{W}_p\), but it can not be the whole \(\tilde{W}_p\) because \(Z(\alpha)\) does not have points in characteristic 0, so it has to be of dimension 0, and so \(Z(\alpha)\) has dimension zero.

2) See Remark 3.3. \(\square\)

Now we count the number of \(\tilde{k}_p\) points of the stack \(Z(\alpha)\). For an fractional ideal \(I\) of \(O_F\) let \(\rho(I)\) be the number of fractional ideals of \(O_K\) with norm equal to \(I\) in \(F\), then

**Theorem 3.18.** Let \(\alpha \in F^{>0}\), assume that \(K/F\) is ramified at at least a finite prime, If \(p\) is a prime such that \(p_F\) is nonsplit, then

\[
\#Z(\alpha)(\tilde{k}_p) = \sum_{(A_1, A_2, f) \in Z(\alpha)(\tilde{k}_p)} \frac{1}{\#Aut(A_1, A_2, f)} = \frac{|C_K|}{\omega(K)} \rho(\alpha p^{-\epsilon_p} O_F)
\]

**Proof.** First we define a subgroup of \(C_K\) denoted \(C^\circ_K\): Let \(H\) be the algebraic group over \(F\) defined for an \(F\)-algebra \(R\) to be \(H(R) = \ker(N_{K/F} \otimes \text{id} : (K \otimes F R) \times \to (F \otimes F R) \times)\) and a compact open subgroup of \(\tilde{F}\)-points of it, \(U = \ker(N_{K/F} : \tilde{O}_K^\times \to \tilde{O}_L^\times) \subseteq H(\tilde{F})\). Let \(C^\circ_K\) be the double quotient (it is a finite group) \(H(\tilde{F}) \setminus H(\tilde{F})/U\). Also let \(\eta : \tilde{O}_K^\times / N_{K/F} \tilde{O}_K^\times \to \{\pm 1\}\) be the restriction of \(\chi_{K/F}\) to \(\tilde{O}_K^\times\).

Now we have the exact sequence

\[
1 \to C^\circ_K \to C_K \to \tilde{O}_F^\times / \text{Norm}\tilde{O}_K^\times \xrightarrow{\eta} \{\pm 1\}
\]  

(3.4)
For a set $S$, let $1_S$ be the characteristic function of $S$, we compute

$$
\sum_{f \in L(A_1, A_2^1)} \sum_{x \in L(A_1, A_2) \otimes \mathbb{Q}} 1_{L(A_1, A_2)}(x) = \sum_{f \in L(A_1, A_2^1)} \sum_{x \in L(A_1, A_2) \otimes \mathbb{Q}} 1_{L(A_1, A_2)}(h^{-1}x)
$$

Let $\mu(K)$ be the roots of unity in $K$, then for $z \in \mathbb{Z}(a)(\mathbb{F}_p)$, we have $Aut A \cong \mu(K)$, so that $Aut(A_1, A_2^1) \cong \mu(K)^2$, also we have $H(F) \cap U = \mu(K)$. Thus, we get

$$
\sum_{f \in L(A_1, A_2^1)} \sum_{x \in L(A_1, A_2) \otimes \mathbb{Q}} \frac{w(K)}{\#Aut(A_1, A_2)} = \sum_{h \in H(F)/U} \sum_{x \in H(F) \setminus L(A_1, A_2) \otimes \mathbb{Q}} 1_{L(A_1, A_2)}(h^{-1}x)
$$

If there exists an $x \in H(F) \setminus L(A_1, A_2) \otimes \mathbb{Q}$ satisfying $\langle x, x \rangle = \alpha$, then we get a simply transitive action of $H(F)$ on all such $x$ and so the last sum can be removed with fixing an $x \in L(A_1, A_2) \otimes \mathbb{Q}$ with the property $\langle x, x \rangle = \alpha$:

$$
\sum_{f \in L(A_1, A_2^1)} \sum_{x \in L(A_1, A_2) \otimes \mathbb{Q}} \frac{w(K)}{\#Aut(A_1, A_2)} = \frac{1}{w(K)} \sum_{h \in H(F)/U} 1_{L(A_1, A_2)}(h^{-1}x)
$$

Summing above over $I \in C_K/C_K^0$ and using exactness of equation (3.4), we have

$$
\sum_{f \in L(A_1, A_2^1)} \sum_{x \in L(A_1, A_2) \otimes \mathbb{Q}} \frac{1}{\#Aut(A_1, A_2^1)} = \frac{1}{w(K)} \sum_{\xi \in \ker \eta} \sum_{h \in H(F)/U} 1_{\hat{L}(A_1, A_2)}(h^{-1}x_\xi)
$$

where $x_\xi$ is an element of $L(A_1, A_2) \otimes \mathbb{Q}$ such that $\langle x_\xi, x_\xi \rangle = \xi \alpha$. Now let’s fix $x \in \hat{K}$ such that $\alpha = \beta x \bar{x}$ (recall that $\hat{L}(A_1, A_2)$ has Hermitian form $\beta x \bar{y}$) and compute the orbital integral

$$
O_\alpha(A_1, A_2) = \sum_{h \in H(F)/U} 1_{\hat{L}(A_1, A_2)}(h^{-1}x)
$$
Also define the local orbital integrals
\[ O_{\alpha,v}(A_1, A_2) = \sum_{h \in H(F_v)/U_v} 1_{O_{K,v}}(h^{-1}x_v) \]

Thus we can factor \( O_{\alpha}(A_1, A_2) \) as
\[ O_{\alpha}(A_1, A_2) = \prod_v O_{\alpha,v}(A_1, A_2) \]

where the product is over the finite places of \( F \). If \( v \) is nonsplit in \( K \), then
\[ O_{\alpha,v}(A_1, A_2) = \left\{ \begin{array}{cl} 1 & \text{if } \alpha \beta^{-1} \in O_{F,v} \\ 0 & \text{otherwise} \end{array} \right. \]

and if \( v \) is split in \( K \), then
\[ K_v \cong F_v \otimes F_v \] and by fixing a uniformizer \( \pi \in F_v \), we find that \( H(F_v)/U_v \) is the cyclic group generated by \((\pi, \pi^{-1}) \in F_v^* \times F_v^*\) and
\[ O_{\alpha,v}(A_1, A_2) = \left\{ \begin{array}{cl} 1 + \ord_v(\alpha \pi^{-1}) & \text{if } \alpha \beta^{-1} \in O_{F,v} \\ 0 & \text{otherwise} \end{array} \right. \]

So that \( O_{\alpha,v}(A_1, A_2) \) is the number of ideals \( a \subseteq O_{K,v} \) such that
\[ \beta_v a_v a_{\bar{v}} = a O_{F,v} \]

and therefore
\[ O_{\alpha}(A_1, A_2) = \rho(\alpha \beta^{-1} O_F). \]

Now in the RHS of equation (3.5) we have that there is a unique \( \xi \in \ker \eta \) such that the sum is not zero (because we must have that \( \rho(\alpha \beta^{-1} O_F) \neq 0 \) and this means that there exists an ideal \( a \subseteq O_K \) such that \( a O_F = \beta a \bar{a} \) and so this means that there’s a unique \( \xi \in \hat{O}_F^* / \text{Norm}(\hat{O}_K^*) \) that \( \xi \bar{a} \) is represented by \( \beta \bar{x} \bar{a} \), also as \( \chi_{K/F}(\beta) = 1 \), we get \( \xi \in \ker \eta \). Thus, equation (3.5) is
\[ \sum_{\xi \in \ker \eta} O_{\xi, a}(A_1, A_2) = \rho(\alpha \beta^{-1} O_F) = \rho(\alpha p^{-\varepsilon_{PF}}) \]

Therefore as \( C_K \) acts simply transitively on \( CM_{\Phi}(\bar{k}_p) \) for a CM-type \( \Phi \) of \( K \), we get
\[ \sum_{A_1 \in CM_{A_1}(\bar{k}_p)} \sum_{f \in L(A_1, A_2)} \frac{1}{\#Aut(A_1, A_2)} = \frac{|C_K|}{w(K)} \rho(\alpha p^{-\varepsilon_{PF}}) \]

which completes the proof. \( \square \)

Now we have
Theorem 3.19. Let $\alpha \in F^{\neq 0}$, $p$ be a prime of $\bar{K}$ such that $p_F$ is nonsplit in $K$, then at $z \in Z(\alpha)(k_p)$:

$$\text{length}(O_{Z(\alpha), z}^e) = \frac{1}{2} e_{p_F}(\text{ord}_{p_F}(\alpha) + 1)$$

where $e_{p_F}$ is the ramification index of $\varphi^{p_F}(p_F)$ in $\bar{K}/F$.

Proof. We have the decomposition

$$A[p_F^\infty] = \prod_{q \subseteq O_F} A[q^\infty]$$

Now we have a map $f : A_1 \to A_2$ and by Serre-Tate, the deformation functor of $(A_1, A_2, f)$ is the same as the deformation functor of $(A_1[p_F^\infty], A_2[p_F^\infty], f[p_F^\infty] : A_1[p_F^\infty] \to A_2[p_F^\infty])$, now we have the decompositions

$$f[q^\infty] : A_1[q^\infty] \to A_2[q^\infty]$$

Case 1) $q \neq p_F$, then we can use proposition 3.8 to prove that the points $(A_1[q^\infty], A_2[q^\infty], f[q^\infty])$ can always be deformed to objects of ART.

Case 2) $q = p_F$, we get that the deformation functor of $(A_1[p_F^\infty], A_2[p_F^\infty], f[p_F^\infty])$ is pro-represented by $\hat{W}_p/m^k$ ($m$ is the maximal of $\hat{W}_p$) where $k = \frac{1}{2}\text{ord}_p(ap_F) = \frac{1}{2} e_{p_F}(\text{ord}_{p_F}(\alpha) + 1)$ by Theorem 3.4.

Collecting everything together we get the concluding theorem:

Theorem 3.20. We have

$$\widehat{\text{deg}} Z(\alpha) = \frac{|C_K|}{w(K)} \sum_{p \subseteq O_F} \frac{\log N(p)}{[K : Q]} (\text{ord}_p(\alpha) + 1) \rho(\alpha p^{-\epsilon_p})$$

where $p$ goes over the nonsplit primes of $F$. 
Proof. Let $f_{p/p_F}$ be the residue degree of $p$ over $p_F$. Now by theorem 3.10 and 3.11, we have

$$
\deg_Z(\alpha) = \frac{1}{[\bar{K} : Q]} \sum_{p \subset \bar{O}} \log N(p) \sum_{z \in Z(\alpha)(\bar{F})} \frac{\text{length}(O^{et}_{Z(\alpha), z})}{\# \text{Aut } z}
$$

$$
= \frac{1}{[\bar{K} : Q]} \sum_{p \subset \bar{O}} \log N(p) e_{p_F}(\text{ord}_{p_F}(\alpha) + 1) |C_K| \rho(\alpha p_F^{-e_{p_F} O_F})
$$

$$
= \frac{|C_K|}{2w(K)[\bar{K} : Q]} \sum_{p \subset \bar{O}_F} \log N(p) [\bar{K} : F](\text{ord}_p(\alpha) + 1) \rho(\alpha p^{-e_p} O_F)
$$

where prime over sigma means that we are considering primes $p$ in $\bar{K}$ such that $p_F$ is nonsplit in $K$ in the first two sums, and primes in $F$ that are nonsplit in $K$ in the last two sums. This finishes the proof of the theorem. \(\square\)

### 3.3 Arithmetic Chow Group

An arithmetic divisor on $CM_{\Phi_1} \times_{\bar{O}} CM_{\Phi_2}$ is a pair $(Z, \text{Gr})$ where $Z$ is a divisor and $\text{Gr}$ a Green function for $Z$. As $Z(\alpha)$ does not have characteristic zero points, the Green function for $Z(\alpha)$ can be any complex-valued function on the finite set

$$
\coprod_{\sigma : \bar{K} \to C} (CM_{\Phi_1} \times_{\bar{O}} CM_{\Phi_2})^\sigma(C)
$$

Recall that for a scheme $S \to \text{Spec } \bar{K}$ and a map $\sigma : \bar{K} \to C$, $S^\sigma$ is obtained by the base change along the map $\text{Spec } C \to \text{Spec } \bar{K}$. We define the Green’s function for $Z(\alpha)$ first for $(CM_{\Phi_1} \times_{\bar{O}} CM_{\Phi_2})(C)$ at $(A_1, A_2) = z \in (CM_{\Phi_1} \times_{\bar{O}} CM_{\Phi_2})(C)$ to be

$$
\text{Gr}_\alpha(y, z) = \sum_{f \in L_{\text{et}}(A_1, A_2)} \beta_1(4\pi |y_\alpha|_{\text{et}})
$$
where $\beta_1 : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ is $\beta_1(t) = \int_{1}^{\infty} e^{-tu}du$. Also, to define $\text{Gr}_\alpha$ on $z \in (CM_{\Phi_1} \times CM_{\Phi_2})^r(C)$, first we extend $\sigma : \hat{K} \rightarrow C$ to $\tilde{\sigma} \in \text{Aut}(C)$ and we obtain two new CM-types $\tilde{\sigma} \circ \Phi_1$, $\tilde{\sigma} \circ \Phi_2$ and then

$$(CM_{\Phi_1} \times CM_{\Phi_2})^r(C) = (CM_{\tilde{\sigma} \circ \Phi_1} \times CM_{\tilde{\sigma} \circ \Phi_2})$$

and then we take the $\text{Gr}_\alpha(y,z)$ for $(A_1, A_2) = z \in (CM_{\Phi_1} \times CM_{\Phi_2})^r(C)$ to be

$$\sum_{f \in L_{Betti}(A_1, A_2)} \beta_1(4\pi |y\alpha|_{\tilde{\sigma} \circ \infty})$$

Now as $\langle \cdot, \cdot \rangle$ is negative definite at $\infty$ and positive definite at other places, we have $(\text{Gr})_\alpha = 0$ unless $\alpha$ is negative definite at exactly one place. So, we define

$$\tilde{Z}(\alpha) = (Z(\alpha), \text{Gr}_\alpha) \in \widehat{\text{CH}^1}(CM_{\Phi_1} \times CM_{\Phi_2})$$

where for $\alpha$ not totally positive, $Z(\alpha) = 0$ in $\text{CH}^1(CM_{\Phi_1} \times CM_{\Phi_2})$.

Now the degree of an arithmetic divisor is defined by the composition of the following maps:

$$\widehat{\deg} : \widehat{\text{CH}^1}(CM_{\Phi_1} \times CM_{\Phi_2}) \rightarrow \widehat{\text{CH}^1}(\text{Spec } \hat{O}) \rightarrow \mathbb{R}$$

where left hand side map is the push forward of $(CM_{\Phi_1} \times CM_{\Phi_2}) \rightarrow \text{Spec } \hat{O}$. Then, we would have

$$\widehat{\deg}(Z, \text{Gr}) = \frac{1}{[K : Q]} \left( \sum_{p \in O \in Z(\tilde{K})} \frac{\log N(p)}{\#\text{Aut} z} + \sum_{c : K \rightarrow C} \sum_{z \in (CM_{\Phi_1} \times CM_{\Phi_2})^r(C)} \frac{\text{Gr}(z)}{\#\text{Aut} z} \right)$$

Then by Theorem 3.20 and the above discussion, we have

$$\widehat{\deg} \tilde{Z}(\alpha) = \begin{cases} \frac{|C_k|}{w(K)} \log N(p) (\text{ord}_p(\alpha) + 1)\rho(\alpha p^{-v}) & \text{if } \alpha \gg 0 \\
\frac{|C_k|}{w(K)} \rho(\alpha) \beta_1(4\pi |y\alpha|_v) & \alpha \text{ is negative definite at exactly one place } v \\
0 & \text{otherwise} \end{cases}$$

3.4 EISENSTEIN SERIES

In this section, we will construct an Eisenstein series for $SL_2(F)$ such that the degree of the divisors and arithmetic divisors, as computed in the preceding section, is up to an explicit multiplicative constant equal to the Fourier expansion of the central value of the derivative of this Eisenstein series.
Let $v$ be a place of $F$, $\psi_F = \psi_Q \circ \text{tr}_{F/Q}$ an additive character of $\mathbb{A}_F / F$, where $\psi_Q$ is the canonical additive character of $\mathbb{A}_Q / Q$, $\psi_F = \psi_F|_{F_v}$, $\mathbb{H}_F = \{ x + iy \in F_C | y \gg 0 \}$, for $s \in \mathbb{C}$, $g \in SL_2(F_v)$, $x \in F_v^\times$, let $\Phi_{\alpha, \psi_F}(g, s) \in I(\chi_v, s)$ be the same section defined in [9], and $\Phi_{c, \psi_F} = \otimes_v \Phi_{c, \psi_F}$, $\psi$ a (general) additive character of $F_v$, for simplicity, $\chi$ be the character of $\mathbb{A}_F^\times$ associated to the quadratic extension $K/F$ and $\chi_v$ the character of $F_v^\times$ associated to $K_v/F_v$, $L(s, \chi) = \prod_w L(s, \chi_w)$ where the product is over all places (including archimedean ones) $w$ of $F$ (If $w$ is archimedean, let $L(s, \chi_w) = \pi^{-\frac{w+1}{2}} \Gamma(s+\frac{1}{2})$, $\pi_w$ be a uniformizer of $F_v$ and $p_v$ the prime ideal of $v$ in $F$, for $I$ an ideal of $O_F$, $N(I) = \#(O_F/I)$ as usual.

For $c \in \mathbb{A}_F^\times$, $g \in SL_2(F_v)$, We construct an Eisenstein series

$$E(g, s, c, \psi_F) := \sum_{\gamma \in B(F) \cap SL_2(F)} \Phi_{c, \psi_F} (\gamma g, s)$$

where $B$ is the Borel subgroup of SL. For $\tau = x + iy \in \mathbb{H}_F$, take

$$g_{\tau} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{bmatrix}$$

at the archimedean places and identity at finite places.

Now define

$$E(\tau, s, c, \psi_F) = \frac{L(s + 1, \chi)}{N_{F/Q}(y)^{\frac{1}{2}}} E(g_{\tau}, s, c, \psi_F)$$

Also, we define the local Whittaker function at $v$ to be $(\alpha \in F_v^\times, g \in SL_2(F_v))$:

$$W_\alpha (g, s, c, \psi) = \int_{F_v} \Phi_{c, \psi}(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g, s) \psi(-ax) d x$$

where $dx$ is the Haar measure on $F_v$ with respect to $\psi$.

If $\delta \in F_v^\times$, we set $(\delta \psi)(x) = \psi(\delta x)$, then $\Phi_{\alpha, \delta \psi} = \Phi_{\delta \alpha, \psi}$, then

$$W_\alpha (g, s, c, \delta \psi) = |\delta|^{\frac{1}{2}} W_{\delta \alpha} (g, s, \delta c, \psi) \quad (3.6)$$

Now $E(\tau, s, c, \psi_F)$ is a Hilbert modular form and has a Fourier expansion

$$E(\tau, s, c, \psi_F) = \sum_{\alpha \in F} E_\alpha (\tau, s, c, \psi_F)$$

Now suppose that $c$ is a unit at finite places and $c_v = 1$ for all archimedean places and $\chi(c) = -1$, then $E$ has parallel weight $1$ and is incoherent, so
\( E(\tau, 0, c, \psi_F) = 0. \)
Now define the final Eisenstein series to be the following sum
\[
E_\Phi(\tau, s) = \sum_{c \in \Xi} E(\tau, s, c, \psi_F)
\]

Where \( \Xi \) is the \( N_{K/F}(\hat{O}_K) \)-orbits of \( c \) satisfying:
1) \( c \) is a unit at finite places.
2) \( c_v = 1 \) for all archimedean places.
3) \( \chi(c) = -1. \)

Define the difference set \( \text{Diff}(\alpha, c) = \{ v \mid \chi_v(\alpha c) = -1 \} \) (By the condition \( \chi(c) = -1 \) and the product formula, this set has odd cardinality, also every \( v \in \text{Diff}(\alpha, c) \) is nonsplit in \( K \)), also as before set
\[
\rho_v(I) = \#\{ J \subseteq O_{K,v} | JI = IO_{K,v} \}
\]
We have that \( \rho \) is a multiplicative function.
Now we compute the Fourier coefficients at \( s = 0 \) of the derivative of \( E_\alpha. \)

**Proposition 3.21.** Suppose that the following ramification conditions are satisfied:
1) \( K/F \) is ramified at at least one finite prime.
2) For every rational prime \( l \leq \frac{|K/Q|}{|F/Q|} + 1, \) ramification index is less than \( l. \)
3) \( \alpha \in F^\times, d_{K/F} \) be the relative discriminant of \( K/F, r \) the number of places of \( F \) (including archimedean places) ramified in \( K \) and \( c \in \Xi, \) then:
   1) \( \text{ord}_s=0 E_\alpha(\tau, s, c, \psi_F) \geq \#\text{Diff}(\alpha, c) \) so if \( \#\text{Diff}(\alpha, c) \geq 2, \) then \( \frac{d}{ds} E_\alpha(\tau, s, c, \psi_F)|_{s=0} = 0. \)
   2) If \( \text{Diff}(\alpha, c) = \{ p \} \) with \( p \) finite, then
\[
\frac{d}{ds} E_\alpha(\tau, s, c, \psi_F)|_{s=0} = \frac{-2^{r-1}}{N(d_{K/F})^2} \rho(\alpha p^{-e_p}) (\text{ord}_p \alpha + 1) \log N(p) q^\alpha
\]
where \( q^\alpha \) means \( e^{2\pi iry/F(\alpha x)} \), and \( e_p \) is, as before, \( 0 \) if \( p \) is unramified in \( K \) and is \( 1 \) otherwise.
3) If \( \text{Diff}(\alpha, c) = \{ v \} \) for \( v \) archimedean place of \( F, \) then
\[
\frac{d}{ds} E_\alpha(\tau, s, c, \psi_F)|_{s=0} = \frac{-2^{r-1}}{N(d_{K/F})^2} \rho((\alpha)) \beta_1(4\pi |y\alpha|_v) q^\alpha
\]
where \( \beta_1 \) is defined in section 4.

**Proof.** For \( g \in \text{SL}_2(F_v), \) consider the normalized local Whittaker function
\[
W_{\alpha_v}^*(g_v, s, c_v, \psi_{F_v}) = L(s + 1, \chi_v) W_{\alpha_v}(g_v, s, c_v, \psi_{F_v})
\]
Now we have the factorization of Fourier coefficients of $\mathcal{E}$:

$$E_a(\tau, s, c, \psi_F) = N_{F/Q}(y)^{-\frac{1}{2}} \prod_v W^*_v(g_{\tau, p}, s, c, \psi_{F_v})$$

So by equation (3), we have

$$E_a(\tau, s, c, \psi_F) = N_{F/Q}(y)^{-\frac{1}{2}} \prod_v W^*_v(g_{\tau, p}, s, 1, c, \psi_{F_v})$$

where $(c\psi_F)(x) = \psi_F(cx)$ is an unramified character of $\mathcal{A}_F$ and $(c, \psi_{F_v})(x) = \psi_{F_v}(cx)$.

Now by the formulas of Yang, stated in prop. 2.1, 2.2, 2.3, 2.4 of [26], we have the following:

Suppose that $v$ is a finite prime of $F$, then

$$\chi_v(ac) = -1 \iff W^*_v(g_{\tau, p}, 0, 1, c, \psi_{F_v}) = 0$$

(Recall that for a finite prime $v$, we have $g_{\tau, p} = I \in SL_2(F_v)$), so, we have the two cases:

1) $\chi_v(ac) = 1$, then

$$W^*_v(g_{\tau, p}, 0, 1, c, \psi_{F_v}) =$$

$$\chi_v(-1)e\left(\frac{1}{2}, \chi_v, c, \psi_{F_v}\right)\rho_v(\alpha O_{F_v}) \left\{ \begin{array}{ll}
2N(\pi_{F_v})^{-\frac{1}{2}\ord_v(d_{K/F})} & \text{if } v \text{ ramified in } K/F \\
1 & \text{if } v \text{ unramified in } K/F
\end{array} \right.$$  

2) $\chi_v(ac) = -1$, then

$$\frac{d}{ds} W^*_v(g_{\tau, p}, s, 1, c, \psi_{F_v})|_{s=0} =$$

$$= \chi_v(-1)e\left(\frac{1}{2}, \chi_v, c, \psi_{F_v}\right)\rho_v(\alpha O_{F_v}) \left\{ \begin{array}{ll}
2N(\pi_{F_v})^{-\frac{1}{2}\ord_v(d_{K/F})} & \text{if } v \text{ ramified in } K/F \\
\rho_v(\alpha p_{\psi_v}^{-1}) & \text{if } v \text{ unramified in } K/F
\end{array} \right.$$  

The point here is that:

1) If $v$ is ramified, $c^{-1}_v \in O_{F_v}$ if $\rho_v(\alpha_v O_{F_v}) = 1$, so $\rho_v(\alpha_v O_{F_v}) = 1_{O_{F_v}}(c^{-1}_v)$.  
2) $f = \ord_v d_{K/F} = 1$ (in Yang’s formula prop. 2.3 (2), if $p \neq 2$ which is the case in here by our ramification condition) and also if $K_v/F_v$ is unramified, $\chi_v(\alpha_v c^{-1}_v) = -1 \iff K_v/F_v$ is inert and $\ord_v \alpha$ is odd (so we need $p_{\psi_v}^{-1}$ to make $\alpha p_{\psi_v}^{-1}$ have even order to get $\rho_v(\alpha p_{\psi_v}^{-1}) = 1_{O_{F_v}}(\alpha_v c^{-1}_v)$).
3) $\chi_v(-1)e\left(\frac{1}{2}, \chi_v, c, \psi_{F_v}\right) = 1$ for $v$ unramified by prop 2.8 of [14].
4) If $v$ is split, then $\rho_v(\alpha_v O_{F_v}) = \ord_v \alpha + 1$ and if $v$ is inert, then $\rho_v(\alpha_v O_{F_v}) =$
Also for an archimedean place \( v \) of \( F \), we have

\[
W^*_{c\alpha}(g, \tau, v, 0, 1, c\psi_F) = 2\chi_v(-1)e\left(\frac{1}{2} \mathcal{A}_v \epsilon \right) y_v^{\frac{1}{2}} e^{2\pi i \epsilon \tau_v}
\]

If \( \chi_v(\alpha c) = -1 \) then \( W^*_{c\alpha}(g, \tau, v, 0, 1, c\psi_F) = 0 \) and

\[
\frac{d}{ds} W^*_{c\alpha}(g, \tau, v, 0, 1, c\psi_F) |_{s=0} = \chi_v(-1)e\left(\frac{1}{2} \mathcal{A}_v \epsilon \right) y_v^{\frac{1}{2}} e^{2\pi i \epsilon \tau_v} \beta_1(4\pi |y\alpha|_v)
\]

Now if \( \text{Diff}(\alpha, c) = \{ w \} \), then

\[
\frac{d}{ds} \mathcal{E}_{\alpha}(\tau, s, c, \psi_F) |_{s=0} = N_{F/Q}(y)^{-\frac{1}{2}} \frac{d}{ds} W^*_{c\alpha}(g, \tau, w, s, 1, c\psi_F) |_{s=0} \prod_{v \neq w} W^*_{c\alpha}(g, \tau, v, 0, 1, c\psi_F)
\]

So by the aforementioned formulas, we obtain the formulas stated in the theorem by using the fact that

\[
\prod_v e\left(\frac{1}{2} \mathcal{A}_v \epsilon \right) = \chi(c) \prod_v e\left(\frac{1}{2} \mathcal{A}_v \epsilon \right) = (-1)(1) = -1
\]

Now we have the Fourier expansion

\[
\mathcal{E}_\Phi = \sum_{\alpha \in F} \mathcal{E}_{\Phi, \alpha}
\]

for \( \mathcal{E}_\Phi(\tau, s) = \sum_{c \in \Xi} \mathcal{E}_{\alpha}(\tau, s, c, \psi_F) \), so that

\[
\frac{d}{ds} \mathcal{E}_{\Phi, \alpha} |_{s=0} = \sum_{c \in \Xi} \mathcal{E}_{\alpha}'(\tau, s, c, \psi_F) \frac{-2r-1}{N(d_K/F)^{\frac{1}{2}}} \sum_{p \subseteq O_F \text{ nonsplit}} (\text{ord}_p \alpha + 1) (\rho(a p^{-\epsilon_\tau})) (\log N(p)) q^a =: b_\Phi(\alpha, y)
\]

So we get to the main theorem:

**Theorem 3.22.** Suppose that conditions of proposition 3.21 are satisfied, then we have

\[
\hat{\text{deg}} \tilde{Z}(\alpha) = -\frac{|C_K| \sqrt{N_{F/Q}(d_K/F)}}{w(K)^{2r-1} |K : Q|} b_\Phi(\alpha, y)
\]
RELATIVE CM-FIELDS

This is the second result of thesis. In this chapter, we fix a CM-field $K_0$ and we consider the moduli space of pairs of abelian varieties $(A_0, A)$ such that $A_0$ is a CM abelian variety and this CM is given by $O_{K_0}$ and $A$ is a higher dimensional abelian variety with an action of $O_{K_0}$ (not necessarily CM). Now we consider the integral models of this moduli space using the Rapoport-Smithling-Zhang Shimura variety. Then we consider CM-cycles on this integral model and define special divisors on this CM-cycle. Then we compute the Arakelov degrees of these special divisors (also extended version to arithmetic divisor) and we relate these degrees to the central derivative of Fourier coefficients of some specific Hilbert Eisenstein series computed in the last section.

4.1 NOTATIONS

Let $p$ be a prime number. Let $K_0 \subseteq K$ be CM fields with $F_0 \subseteq F$ their maximal totally real subfields. Let $K_0 = F_0(\sqrt{\Delta})$ for a totally negative element $\Delta \in F_0$. For a local or global field $L$, let $O_L$ be the ring of integers or valuation ring of $L$. For a global field $L$, let $O_L(\varphi)$ be the localisation of ring of integers of $L$ at $\varphi$ and for $\varphi$ a prime of a number field $L$, $L_\varphi$ is the completion of $L$ at $\varphi$ and $O_{L,\varphi}$ the valuation ring of $L_\varphi$. Let $\mathbb{F}_p$ be an algebraic closure of $\mathbb{F}_p$, the field of $p$ elements. $\bar{L}$ is a choice of algebraic closure of $L$. For a local field $L$, $p_L$ denotes the maximal ideal of $O_L$. For a prime $q$ of $F$, let $\epsilon_q$ be 0 if $q$ is ramified in $K$ and 1 if $q$ is unramified in $K$. For a prime $\varphi$ of $K$, $v_\varphi$ a prime of $K_0$ with $\varphi | v_\varphi | p$. Let $v, v_0, v_1, v_2$ be primes in $F, F_0$ respectively below $v, v_0$. Let $n = [K : K_0]$ and $2d = [K_0 : \mathbb{Q}]$. Let $\chi : \mathbb{A}_F^\times \to \{\pm 1\}$ be the quadratic character associated to $K/F$. By a CM-type $\Phi_L$ of a CM-field $L$ with complex conjugation $(\cdot)$, we mean a subset of $\text{Hom}(L, \mathbb{C})$ such that $\Phi_L \sqcup \overline{\Phi}_L = \text{Hom}(L, \mathbb{C})$, where $\overline{\Phi}_L = \{\bar{\phi} | \phi \in \Phi_L\}$ ($\bar{\phi}(x) = \phi(\bar{x})$ for $x \in L$). For a finite extension $L/\mathbb{Q}_p$ and subfield $L_0 \subseteq L$ of index 2 and $\text{Gal}(L/L_0) = \langle (\cdot) \rangle$, a $p$-adic CM-type $\Phi_L$ is a subset of $\text{Hom}_{\mathbb{Q}_p}(L, \mathbb{C}_p)$ with $\Phi_L \sqcup \overline{\Phi}_L = \text{Hom}_{\mathbb{Q}_p}(L, \mathbb{C}_p)$ where $\overline{\Phi}_L = \{\bar{\phi} | \phi \in \Phi_L\}$ ($\bar{\phi}(x) = \phi(\bar{x})$ for $x \in L$). Let $\Phi$ (resp. $\Phi_0$) be a CM-type of $K$ (resp. $K_0$) with

$$\Phi = \{\phi_1^1, \phi_1^2, \ldots, \phi_1^n, \phi_2^1, \ldots, \phi_2^n, \ldots, \phi_d^1, \ldots, \phi_d^n\}$$
\( \Phi_0 = \{ \phi_1, \phi_2, \cdots, \phi_d \} \)

with \( \phi_i|_{K_0} = \phi_i \) if \( (i,j) \neq (1,1) \) and \( \phi_1|_{K_0} = \overline{\phi_1} \). Also let \( \Phi_0 \) be the CM-type of \( K \) induced by \( \Phi_0 \). Fix \( \iota : C \cong \mathbb{C} \). Let \( \Phi_0 \) (resp. \( \Phi_{p_0} \)) be \( p \)-adic CM-type of \( K_0 \) relative to \( F_0 \) (resp. \( K_{0,p_0} \) relative to \( F_{0,p_0} \)) consisting of all \( \phi \in \Phi \) (resp. \( \phi_0 \in \Phi_0 \)) with the property that \( (i \circ \phi)^{-1}(p_{C_p}) = v \) (resp. \( (i \circ \phi_0)^{-1}(p_{C_p}) = v_0 \)). Let \( \bar{K} \subseteq C \) (resp. \( \bar{K}_p \subseteq C_p \) with the abuse of notation) be a large enough Galois extension of \( Q \) (resp. \( Q_p \)) such that for \( \sigma \in \text{Aut}(C/\bar{K}) \) (resp. \( \text{Aut}(C_p/\bar{K}_p) \)), we have \( \Phi^\sigma = \Phi \) (resp. \( \Phi^\sigma_0 = \Phi_0 \)) and \( \Phi_0^\sigma = \Phi_0 \) (resp. \( \Phi_0^{\sigma_0} = \Phi_{0,p_0} \)) where \( \Phi^\sigma = \{ \sigma \circ \phi | \phi \in \Phi \} \) and similarly for \( \Phi_0 \). For example, we can take \( \bar{K} \) (resp. \( \bar{K}_p \)) to be the Galois closure of \( K \) over \( Q \) (resp. \( K_0 \) over \( Q_p \)). Let \( \bar{k}_p \) be the residue field of \( \bar{K} \) at \( p \). Let \( \bar{k}_p \) be an algebraic closure of \( \bar{k}_p \). \( \bar{W} \) be the valuation ring of the maximal unramified extension of \( \bar{k}_p \) and \( m \) be its maximal ideal. ART be the category of local Artinian \( \bar{W} \)-algebras with residue field \( \mathbb{F}_p \). For a \( p \)-divisible group \( A \) defined over \( R \in \text{obj}(\text{ART}) \) with an action \( \kappa : O_K \rightarrow \text{End}(A) \), we say it has \( \Phi \)-determinant condition if determinant of the action of \( \sum_{i=1}^r f_i x_i \) (\( x_i \)'s \( \in \) \( O_K \) and \( f_i \)'s variables) on \( \text{Lie} A \) is given by the image of \( \prod_{\phi \in \Phi} (\sum_{i=1}^r f_i \phi(x_i)) \) in \( R[t_1, \cdots, t_r] \). For \( R \in \text{obj}(\text{ART}) \), we let \( m_R \) be the maximal ideal of \( R \), then we denote \( R^{(i)} = R/m_R^i \). Let \( J_{\phi_0} \) (resp. \( J_{0,\phi_{0,p_0}} \)) be the kernel of the \( \bar{W} \)-algebra map

\[
O_{K_0} \otimes_{\mathbb{Z}_p} \bar{W} \rightarrow \prod_{\phi \in \Phi_0} C_p
\]

(resp. \( O_{K_{0,p_0}} \otimes_{\mathbb{Z}_p} \bar{W} \rightarrow \prod_{\phi \in \Phi_{0,p_0}} C_p \)) given by \( x \otimes 1 \mapsto (\phi(x))_{x \in \phi_0} \) (resp. \( x \otimes 1 \mapsto (\phi(x))_{x \in \phi_{0,p_0}} \)). \( \mathcal{D}_0 \) and \( \mathcal{D}_{0,p} \) be differentials of \( K_0/Q_p \) and \( K_{0,p_0}/Q_p \) respectively. Let \( W \) be ring of integers of maximal unramified extension of \( K_0 \) if \( v \) is known in the context.

\( \mathcal{D}_0, \mathcal{D} \) be the differentials of \( K_0/Q \) and \( K/Q \) respectively. For two number fields \( L_1 \subseteq L_2 \), \( \partial_{L_2/L_1} \) be the relative different of \( L_2 \) over \( L_1 \). We assume that \( K_0/F_0 \) is ramified at at least one finite prime and the relative discriminants of \( K_0/F_0 \) and \( F/F_0 \) are relatively prime (this is to ensure the existence of CM abelian varieties with \( O_K \)-action and \( O_{K_0} \)-action). Also for two abelian varieties \( A_0 \) and \( A \) with CM by \( O_{K_0} \), let \( L(A_0, A) \) be \( \text{Hom}_{O_{K_0}}(A_0, A) \) \( (O_{K_0} \)-linear mappings from \( A_0 \) to \( A) \).

\[4.2 \text{ LOCAL PART}\]

**Lifting of Homomorphisms**

We assume \( O_{K_0} \otimes \mathbb{Z} O_F = O_K \), also we assume the following ramification condition:
If \( p \leq \left\lfloor \frac{\kappa_0, \rho_0}{\kappa_0, \rho_0} \right\rfloor + 1 \), then ramification index of \( \tilde{K}_p / \mathbb{Q}_p \) is less than \( p \).

Let \( \nu \) (resp. \( \nu_0 \)) be a prime of \( K \) (resp. \( K_0 \)) over \( p \) such that \( \nu | \nu_0 \) and \( A \) (resp. \( A_0 \)) be a \( p \)-divisible group over \( \mathbb{F}_p \) with an action by \( O_{K_0} \) (resp. \( O_{K_0, \nu_0} \)) given by \( \kappa : O_K \to \text{End} \ A \) (resp. \( \kappa_0 : O_{K_0, \nu_0} \to \text{End} \ A_0 \)) having \( \Phi_\nu \)-determinant (resp. \( \Phi_{\nu, \nu_0} \)-determinant) condition. Also we assume that they have an \( O_{K_0, \nu_0} \)-linear (resp. \( O_{K_0, \nu_0} \)-linear) polarization \( \lambda : A \to A^\vee \) with kernel \( A[a] \) where \( a \) is an ideal of \( O_K \) (resp. principal polarization \( \lambda_0 : A_0 \to A_0^\vee \)). Now we consider these two cases:

1. All elements of \( \Phi_\nu \) restricted to \( K_{0, \nu_0} \) become elements of \( \Phi_{0, \nu_0} \).
2. There’s exactly one element of \( \Phi_\nu \) such that when restricted to \( K_{0, \nu_0} \) becomes conjugate of an element of \( \Phi_{0, \nu_0} \) and \( K_0 \neq F_{\nu_0} \).

First we assume we have case 1:

**Proposition 4.1.** Let \( T \in \text{obj}(\text{ART}) \) and \( (A'_0, \kappa'_0, \lambda'_0), (A', \kappa', \lambda') \) be the unique deformations of \( (A_0, \kappa_0, \lambda_0) \) and \( (A, \kappa, \lambda) \) to \( T \) (which exist by theorem 2.1.3 of [9]). The reduction map

\[
\text{Hom}_{O_{K_0, \nu_0}}(A'_0, A) \to \text{Hom}_{O_{K_0, \nu_0}}(A_0, A)
\]

is a bijection.

**Proof.** If \( g : S \to R \) is a surjection in ART with kernel \( \ker g \) having property \( (\ker g)^2 = 0 \). Denote by \( (A^R, \kappa^R, \lambda^R) \) the deformation of \( (A, \kappa, \lambda) \) to \( R \) and similarly for \( (A_0, \kappa_0, \lambda_0) \). Now assume that we have \( f \in \text{Hom}_{O_{K_0, \nu_0}}(A_0^R, A^R) \) and let \( D^R_{A^0} \) and \( D_{A^0} \) be the Grothendieck-Messing crystals of \( A_0^R, A^R \) respectively. \( f \) induces a map \( f : D^R_{A^0}(S) \to D_{A^0}(S) \). Now as we are in case 1, we have

\[
J_{0, \Phi_{0, \nu_0}}(O_K \otimes \mathbb{Z}_p \tilde{W}) \subseteq J_{\Phi_\nu}
\]

and so

\[
f(J_{0, \nu_0} D^R_{A^0}(S)) = J_{0, \nu_0} f(1) D_{A^0}(S) \subseteq J_{\nu} D_{A^0}(S).
\]

By the proof of theorem 2.1.3 in [9], Hodge filtrations of the deformations to \( S \) correspond to

\[
J_{0, \Phi_{0, \nu_0}} D_{A^0}(S) \subseteq D_{A^0}(S)
\]

and

\[
J_{\Phi_\nu} D_{A^0}(S) \subseteq D_{A^0}(S)
\]

and as \( f \) preserves this filtration by above, \( f \) can be uniquely lifted to a map in \( \text{Hom}_{O_{K_0, \nu_0}}(A_0^S, A^S) \) where \( A_0^S \) and \( A^S \) are unique lifts of \( A_0^R \) and \( A^R \) to \( S \), respectively. Now using induction on \( n \) and using \( \cdots \subseteq R/m^n R \subseteq \cdots \subseteq R/m_R = \mathbb{F}_p \), we get the proposition.
Now we consider case 2. Consider the $O_{K_v}$-module
$L(A_0, A) = \text{Hom}_{O_{K_0,v_0}}(A_0, A)$ with the Hermitian form defined by $\langle f, g \rangle = \lambda_0^{-1} \circ g^\vee \circ \lambda \circ f$ so that for all $x \in O_{K_v}$ we have
$$\langle xf, g \rangle = \langle f, xg \rangle$$

Now using the above property we can find a unique $O_{K_v}$-valued $O_{K_v}$-Hermitian form $\langle , \rangle_{CM}$ on $L(A_0, A)$ satisfying $\langle f, g \rangle = \text{tr}_{K_v/K_0, v_0} \langle f, g \rangle_{CM}$ by a standard argument.

Let $S = O_{K_v} \otimes_{\mathbb{Z}_p} W$, $\text{Fr} \in \text{Aut} W$ be the Frobenius automorphism, then on $S$ we have the induced automorphism $(x \otimes w)^{\text{Fr}} = x \otimes w^\text{Fr}$. For each $\psi : O_{K_v}^u \to W$, there exists an idempotent $e_\psi \in S$ satisfying $(x \otimes 1)e_\psi = (1 \otimes \psi(x))e_\psi$ for all $x \in O_{K_v}^u$. They satisfy $e_\psi^{\text{Fr}} = e_{\text{Fr} \circ \psi}$, $S = \prod_{\psi : O_{K_v}^u \to W} e_\psi S$ and $e_\psi S \cong O_{K_v}$, where $\tilde{K}_v$ is the maximal unramified extension of $K_v$. Let $m(\psi, \Phi_v) = \# \{ \phi \in \Phi_v | \phi|_{O_{K_v}}^u = \psi \}$. Let $S_0 = O_{K_0,v_0} \otimes_{\mathbb{Z}_p} W$, then do the same as above for $S_0$. By Lemma 2.3.1 of [9], we have that there exist $b \in S, b_0 \in S_0$ such that
$$L(A_0, A) \cong \{ s \in S | (b_0 s)^{\text{Fr}} = b^{\text{Fr}} s \}$$

**Proposition 4.2.** For some $\beta \in F_{\psi_v}^\times$ satisfying
$$\beta O_K = \begin{cases} \alpha p_{F_v} D_{v_0} D_v^{-1} O_{K_v} & \text{if } K_v / F_v \text{ is unramified} \\ \alpha D_{v_0} D_v^{-1} O_{K_v} & \text{if } K_v / F_v \text{ is ramified} \end{cases}$$
we have $L(A_0, A) \cong O_{K_v}$ as an $O_{K_v}$-module with $\langle x, y \rangle_{CM} = \beta x \bar{y}$ on $O_{K_v}$.

**Proof.** In the same way as in lemma 2.3.2 of [9], $L(A_0, A)$ is a free $O_{K_v}$-module of rank 1, let $s$ be $L(A_0, A) = s O_{K_v}$. Again as in lemma 2.3.2 of [9], we get $\zeta \in S \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfying $\langle a, b \rangle = \zeta a \bar{b}$ for $a, b \in L(A_0, A) \subseteq S$ and $\zeta S = a D_{v_0} D_v^{-1} S$. Now we want to compute $s \otimes s$. Let $\{ \psi_0^i, \psi_1^i, \ldots, \psi_{f-1}^i \}$ be the set of embeddings $O_{K_v}^u \to W$ and $\psi_0^i$ be the restriction of $\psi_i$ to $O_{K_0,v_0}^u (0 \leq i \leq f - 1)$. Now by above we have $(b_0 s)^{\text{Fr}} = b^{\text{Fr}} s$, so we get
$$\text{ord}_{\psi^{i+1}}(s) = \text{ord}_{\psi^i}(s) - \text{ord}_{\psi^i}(b_0) + \text{ord}_{\psi^i}(b_0) =$$
$$\text{ord}_{\psi^i}(s) - m(\psi^i, \Phi_v) + e(K_v / K_0, v_0) m(\psi_0^i, \Phi_{0,v_0})$$.
Assuming $K_v/F_v$ is unramified, an easy computation shows

$$m(\psi^i, \Phi_v) - e(K_v/K_0, v_0)m(\psi_0^i, \Phi_{0, v_0}) = \begin{cases} 0 & \text{if } \phi^1|_{\mathcal{O}_{K_v}} \neq \phi^i, \phi|_{\mathcal{O}_{K_0, v_0}} = \psi_0^i \\ -1 & \text{if } \phi^1|_{\mathcal{O}_{K_v}} = \phi^i, \phi|_{\mathcal{O}_{K_0, v_0}} = \psi_0^i \\ 1 & \text{if } \phi^1|_{\mathcal{O}_{K_v}} = \phi^i, \phi|_{\mathcal{O}_{K_0, v_0}} = \psi_0^i \\ 0 & \text{if } \phi^1|_{\mathcal{O}_{K_v}} \neq \phi^i, \phi|_{\mathcal{O}_{K_0, v_0}} = \psi_0^i \end{cases}$$

so the sequence $(\text{ord} \psi^0(s), \text{ord} \psi^1(s), \ldots, \text{ord} \psi^{i-1}(s))$ has the form $(0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, \ldots, 0)$ with the same number (say $j = \frac{1}{2}$) of 0’s and 1’s where $\psi^j$ is the restriction of conjugation (nontrivial automorphism of $\text{Gal}(K_v/F_v)$) to $K_v^0$, and we then get

$$\text{ord}_{\psi^j}(s) + \text{ord}_{\psi^{j+1}}(s) = 1$$

for all $i$ and so $ss = p_{F_v}S$.

Now assuming $K_v/F_v$ ramified, we get

$$m(\psi^i, \Phi_v) = e(K_v/K_0, v_0)m(\psi_0^i, \Phi_{0, v_0})$$

so

$$\text{ord}_{\psi^{i+1}}(s) = \text{ord}_{\psi^i}(s)$$

for all $i$ and $ssS = S$. Let $\epsilon$ be the ramification index of $\bar{K}_F/K_v$. \qed

**Proposition 4.3.** Suppose that $f$ is an $O_{K_v}$-module generator of $L(A_0, A)$, then one can lift $f$ to $L^{(k)}(1)$, $L^{(k+1)}$ with $k = e_{ord_{K_0, v_0}}D_v$ if $K_v/F_v$ is ramified (resp. $k = \epsilon$ if $K_v/F_v$ is unramified).

Let $D_v, D_v^0$ be Grothendieck-Messing crystals of $A_0$, $A$. Now $\ker(W^{(2)} \rightarrow \bar{W}^{(1)} = \overline{F}_p)$ has a divided power structure compatible with $p\bar{W}^2$ (either the trivial divided power structure if $\bar{W}/W$ is ramified and the canonical divided powers on $p\bar{W}^2$ otherwise), now we have by [9],

$$D_v^0(\bar{W}^{(2)}) \cong S_0 \otimes_W \bar{W}^{(2)}$$

$$D_v(\bar{W}^{(2)}) \cong S \otimes_W \bar{W}^{(2)}$$

Hodge filtrations are $J_{\Phi_0, v_0}D_v^0(\bar{W}^{(2)})$ and $J_{\Phi, v}D_v(\bar{W}^{(2)})$ and $f$ lifts to a map $A_0^{(2)} \rightarrow A^{(2)}$ (where $A_0^{(2)}$ and $A^{(2)}$ are unique deformations of $A_0$ and $A$ to $\bar{W}^{(2)}$) iff

$$f : J_{\Phi_0, v_0}D_v^0(\bar{W}^{(2)}) \rightarrow D_v(\bar{W}^{(2)})/J_{\Phi, v}D_v(\bar{W}^{(2)})$$
is trivial. If \( f \in \text{Hom}_{O_{k_{0\nu_0}}} (A_0, A) \subseteq S \) corresponds to \( s \in S \), then we consider the multiplication by \( s \)

\[
J_{O_{k_{0\nu_0}}} (S \otimes_W \tilde{W}) \rightarrow (S \otimes_W \tilde{W}) / J_{\nu} (S \otimes_W \tilde{W}).
\]

Now by mapping

\[
(S \otimes_W \tilde{W}) / J_{\nu} (S \otimes_W \tilde{W}) \xrightarrow{(\phi_1^1, \phi_1^2, \ldots, \phi_1^n)} C_p^{hd}
\]

Firstly, assuming \( K_{\nu} / F_{\nu} \) is unramified, \( \phi_f^i(s) = 0 \) for all \((i, j) \neq (1, 1)\) and for \( \Phi_1^1 \), it goes to

\[
\hat{\Phi}_1^1(s) \prod_{\phi_1^1|_{O_{k_{0\nu_0}}} = \phi_1^1|_{O_{k_{0\nu_0}}}} (\phi_1(s) - \phi(s))
\]

where the product is over the \( \phi : k_{0\nu_0} \rightarrow C_p \) with the aforementioned property. Now as \( \phi_1^1|_{O_{k_{0\nu_0}}} \neq \phi_1|_{O_{k_{0\nu_0}}} \) all components of product above are units except for \( \phi_1^1(s) \) which has valuation \( 1 \) in \( W \), so valuation \( \epsilon \) in \( \tilde{W} \). So using a similar idea for \( \tilde{W}^{(1)}, \tilde{W}^{(2)}, \ldots, \tilde{W}^{(\epsilon)} \) we can lift \( f \) to them, but not to \( \tilde{W}^{(\epsilon + 1)} \).

Secondly, if \( K_{\nu} / F_{\nu} \) is ramified, suppose that we extended \( f \) to \( \tilde{W}^{(k)} \), then by prop. 2.3.3 of [9], \( s \in S^\times \) so the induced map on Dieudonne modules \( D(A_0) \otimes_{O_{k_{0\nu_0}F_0}} O_{F_{\nu}} \rightarrow D(A) \) is an isomorphism. So \( f \) induces an isomorphism of Lie algebras

\[
\text{Lie}(A_0) \otimes_{O_{k_{0\nu_0}F_0}} O_{F_{\nu}} \cong \text{Lie}(A)
\]

Now as we assumed \( f \) extends to \( \tilde{W}^{(k)} \), Nakayama’s lemma implies that the induced map

\[
\text{Lie}(A_0^{(k)}) \otimes_{O_{k_{0\nu_0}F_0}} O_{F_{\nu}} \cong \text{Lie}(A^{(k)})
\]

where \( A_0^{(k)} \) and \( A^{(k)} \) are deformations of \( A_0 \) and \( A \) to \( \tilde{W}^{(k)} \). So in \( \tilde{W}^{(k)}[t] \), we have

\[
\prod_{\phi \in \Phi_0} (t - \phi(x)) = \prod_{\phi \in \Phi_0} (t - \phi(x))
\]

where \( \Phi_0 \) is \( \{ \phi_1^1, \phi_1^2, \ldots, \phi_1^n, \phi_2^1, \ldots \} \). So we get that \( \Phi_1^1 = \Phi_1^1 (\text{mod } m^k) \) which implies that \( k \leq \text{cord}_{k_{0\nu_0}} D_{\nu_0} \).

Now suppose that \( k \leq \text{cord}_{k_{0\nu_0}} D_{\nu_0} \), then the \( O_{k_{\nu}} \)-action on \( A_0^{(k)} \otimes_{O_{k_{0\nu_0}F_0}} O_{F_{\nu}} \) satisfies \( \Phi_\nu \)-determinant condition and so \( f : A_0 \otimes_{O_{k_{0\nu_0}F_0}} O_{F_{\nu}} \rightarrow A \) is
an isomorphism of \( p \)-divisible groups and so one can see \( A_0^{(k)} \otimes_{O_{\mathbb{F}_0^{(k)}}} O_{F^k} \) as a deformation of \( A \) to \( \tilde{W}(k) \). By uniqueness of such deformations, there exists an \( O_{K_0} \)-linear isomorphism

\[
A_0^{(k)} \otimes_{O_{\mathbb{F}_0^{(k)}}} O_{F^k} \rightarrow A^{(k)}
\]

lifting \( f \), so by composing with \( A_0^{(k)} \hookrightarrow A_0^{(k)} \otimes_{O_{\mathbb{F}_0^{(k)}}} O_{F^k} \), we get the lift \( A_0^{(k)} \rightarrow A^{(k)} \) of \( f \).

**Proposition 4.4.** Let \( \pi_{K_0} \) be a uniformizer of \( O_{K_0} \). If \( f \in L^{(k)} \), then \( \pi_{K_0} f \in L^{(k+\varepsilon)} \) and the multiplication map by \( \pi_{K_0} \) map induces an injective map \( L^{(k)} \setminus L^{(k+1)} \rightarrow L^{(k+\varepsilon)} \setminus L^{(k+\varepsilon+1)} \).

**Proof.** Let \( D_0^{(k)}, D^{(k)} \) be Grothendieck-Messing crystals of \( A_0^{(k)} \) and \( A^{(k)} \), now \( \tilde{W}^{(k+\varepsilon)} \rightarrow \tilde{W}^{(k)} \) is a PD-thickening

\[
\begin{array}{ccc}
J_{0,\Phi_{0;0}} D_0^{(k)}(\tilde{W}^{(k+\varepsilon)}) & \xrightarrow{f^{(k)}} & D^{(k)}(\tilde{W}^{(k+\varepsilon)}) / J_{\Phi_\epsilon}(D^{(k)}(\tilde{W}^{(k+\varepsilon)})) \\
\downarrow & & \downarrow \\
J_{0,\Phi_{0;0}} D_0^{(k)}(\tilde{W}^{(k)}) & \xrightarrow{f^{(k)}} & D^{(k)}(\tilde{W}^{(k)}) / J_{\Phi_\epsilon}(D^{(k)}(\tilde{W}^{(k)}))
\end{array}
\]

bottom row is the zero map, so the top row becomes zero after \( \otimes_{\tilde{W}^{(k+\varepsilon)}} \tilde{W}^{(k)} \), so its image annihilated by \( m^\varepsilon \), and so \( \pi_{K_0} f^{(k)} = \phi_1^{(1)}(\pi_{K_0}) f^{(k)} \) is zero on top row and so can be lifted to \( L^{(k+\varepsilon)} \). Now suppose that \( f \in L^{(k)} \), we have the PD-thickening \( \tilde{W}^{(k+\varepsilon+1)} \rightarrow \tilde{W}^{(k)} \), so we get a diagram

\[
\begin{array}{ccc}
J_{0,\Phi_{0;0}} D_0^{(k)}(\tilde{W}^{(k+\varepsilon+1)}) & \xrightarrow{f^{(k)}} & D^{(k)}(\tilde{W}^{(k+\varepsilon+1)}) / J_{\Phi_\epsilon}(D^{(k)}(\tilde{W}^{(k+\varepsilon+1)})) \\
\downarrow & & \downarrow \\
J_{0,\Phi_{0;0}} D_0^{(k)}(\tilde{W}^{(k+\varepsilon)}) & \xrightarrow{f^{(k)}} & D^{(k)}(\tilde{W}^{(k+\varepsilon)}) / J_{\Phi_\epsilon}(D^{(k)}(\tilde{W}^{(k+\varepsilon)})) \\
\downarrow & & \downarrow \\
J_{0,\Phi_{0;0}} D_0^{(k)}(\tilde{W}^{(k+1)}) & \xrightarrow{f^{(k)}} & D^{(k)}(\tilde{W}^{(k+1)}) / J_{\Phi_\epsilon}(D^{(k)}(\tilde{W}^{(k+1)})) \\
\downarrow & & \downarrow \\
J_{0,\Phi_{0;0}} D_0^{(k)}(\tilde{W}^{(k)}) & \xrightarrow{f^{(k)}} & D^{(k)}(\tilde{W}^{(k)}) / J_{\Phi_\epsilon}(D^{(k)}(\tilde{W}^{(k)}))
\end{array}
\]

now assume that \( \pi_{K_0} f^{(k)} \) can be lifted to \( L^{(k+\varepsilon+1)} \), so in the top row of the diagram above \( f^{(k)} \) has image inside \( m^{k+1} \), so the map \( J_{0,\Phi_{0;0}} D_0^{(k)}(\tilde{W}^{(k+1)}) \rightarrow \)
$D^{(k)}(\tilde{W}^{(k+1)})/f_\Phi, D^{(k)}(\tilde{W}^{(k+1)})$ gotten by the PD-thickening $\tilde{W}^{(k+1)} \to \tilde{W}^{(k)}$ is zero and $f^{(k)}$ can be lifted to $L^{(k+1)}$.

**Theorem 4.5.** Suppose that the ramification condition is satisfied, then for any nonzero $f \in L(A_0, A)$ with $(f, f) = a$, we have $f \in L^{(k)} \setminus L^{(k+1)}$ where

$$k = \frac{1}{2} \text{ord}_p(aa^{-1}p_{F, F_v} D^{-1}D_v).$$

**Proof.** Let $f = f_0 \pi^{w}_{K_0}$ for an $O_{K_0}$-module generator $f_0$ of $L(A_0, A)$. By previous proposition, we know that we can lift $f$ to $L^{(n+1)} \setminus L^{(n+1)+1}$. In order to compute $k = (n+1)e$ in terms of $a$, we have

- If $K_0/F_{F_v}$ is unramified,

  $$aO_{K_0} = (f, f)O_{K_0} = p_{K_0}^{2n+1} vD_{v_0}D_{v}^{-1}O_{K_0},$$

  so

  $$n = \frac{\text{ord}_{K_0}(aa^{-1}D_{v_0}^{-1}D_v) - 1}{2} \Rightarrow (n+1)e = \frac{\text{ord}_{K_0}(aa^{-1}D_{v_0}^{-1}D_v p_{F_v})}{2}.$$

- If $K_0/F_{F_v}$ is ramified, $aO_{K_0} = (f, f)O_{K_0} = p_{K_0}^{2n} vD_{v_0}D_{v}^{-1}O_{K_0}$ so

  $$n = \frac{1}{2} \text{ord}_{K_0}(aa^{-1}D_{v_0}^{-1}D_v) \Rightarrow (n+1)e = \frac{\text{ord}_{K_0}(aa^{-1}D_{v_0}^{-1}D_v p_{F_v})}{2}.$$

$\square$

### 4.3 Global Part

*Rapoport-Smithling-Zhang Shimura Varieties*

Here we shall recall some notions from [19] that we are going to use later. We use notations from notations section freely. Also if $p$ is a prime in $K$, we assume that $K_0$ in the notations section is $K_p$. We first define the Deligne-Mumford stack $M_0$ over $O_K$. This is the Deligne-Mumford stack that for a scheme $S$ over $O_K$, gives the groupoid of tuples $(A_0, i_0, \lambda_0)$ with $A_0$ an abelian scheme over $S$ with $O_{K_0}$-action $i : O_{K_0} \to \text{End}A_0$ with $\Phi_0$-Kottwitz condition:

$$\text{charpol}(i_0(a))|_{\text{Lie}(A_0)}(t) = \prod_{\phi \in \Phi_0} (t - \phi(a))$$

for all $a \in O_{K_0}$ and $\lambda : A_0 \to A_0^\vee$ is a principal polarization such that its Rosati involution on $O_{K_0}$ by $K_0$ is the nontrivial conjugation of $K_0/F_0$. Now
consider $L_{\Phi_0}$ to be the set of isomorphism classes of pairs $(\Lambda_0, \langle \cdot, \cdot \rangle_0)$ where $\Lambda_0$ is a locally free $O_{K_0}$-module of rank 1 with a nondegenerate alternating pairs $\langle \cdot, \cdot \rangle_0 : \Lambda_0 \times \Lambda_0 \to \mathbb{Z}$ with $\langle ax, y \rangle = \langle x, ay \rangle$ for all $x, y \in O_{K_0}$ such that $x \to \langle \sqrt{Ax}, x \rangle_0$ is negative definite quadratic form on $\Lambda_0$ and that for some $\xi$ (Hermitian form) for a prime $v_0$ one has $\xi(\bigotimes_{v}O_v)$ flat over $K_0$. Fix a free $O_{K_0}$-module $W$ of rank $d$ equipped with a nondegenerate $K_0/F_0$ Hermitian form. For a prime $v_0$ of $K_0$, let $W_{v_0}$ be the completion of $W$ at $v_0$. Let $M$ be the Deligne-Mumford stack that for each $O_{K_0}$-scheme $S$ gives the groupoid of tuples $(A_0, i_0, \lambda_0, A, \iota, \lambda)$ where $(A_0, i_0, \lambda_0) \in M_{\Phi_0}(S)$ for some $\xi \in L_{\Phi_0} \equiv M_{\Phi_0}$ and $A/S$ is an abelian scheme with $O_{K_0}$-action $\iota : O_{K_0} \to \text{End} A$ with Kottwitz condition:

$$\text{charpol}(\iota(a)|_{\text{Lie}(A)})(t) = (t - \phi_1(a))^{n-1}(t - \hat{\phi}_1(a)) \prod_{\phi \in \Phi_0 \setminus \{\phi_1\}} (t - \phi(a))^n$$

and $\lambda$ is a principal polarization whose Rosati involution on $O_{K_0}$ by $\iota$ gives the nontrivial conjugation $K_0/F_0$. Also impose the sign condition:

$$\text{inv}_0^\iota(A_0, i_0, \lambda_0, A, \iota, \lambda) = \text{inv}_0(-W_0)$$

(See appendix A of [19] for the definition of $\text{inv}_0^\iota$) for any $s \in S$ and $v$ a finite place of $F_0$ nonsplit in $K_0$, also we assume that for any place $p$ of $\mathbb{K}$ with $p$ its residue characteristic, the triple $(A \otimes \mathbb{Z}_p, \iota \otimes \mathbb{Z}_p, \lambda \otimes \mathbb{Z}_p)$ over $S \times_{\text{Spec} O_{\mathbb{K}, p}} O_{\mathbb{K}, p}$ satisfies the conditions in section 4 of [19]. One of the results in [19] is the following:

**Theorem 4.7.** (Theorem 5.2 of [19]) $M$ over $O_{\mathbb{K}}$ is representable by a Deligne-Mumford stack. $M$ is flat over $O_\mathbb{K}$ and smooth of relative dimension $d - 1$ over $O_\mathbb{K}$ after removing all $p \in \text{Spec} O_{\mathbb{K}}$ with AT-type (1) or (4) (refer to section 4.4 of [19] for the definition of AT-type).
Stacks $\mathcal{Z}(\alpha)$ and $\mathcal{X}$

First we define the Deligne-Mumford stack $\mathcal{CM}_0^\Phi$:

**Definition 4.8.** Let $\mathcal{CM}_0^\Phi$ be the Deligne-Mumford stack over $O_K$ such that for each $S$ over $O_K$, we get $\mathcal{CM}_0^\Phi(S)$ is the groupoid of $(A, t, \lambda)$ with:

- $A/S$ is an abelian scheme of relative dimension $dn$.
- $t : O_K \to \text{End} A$ satisfies $\Phi$-Kottwitz condition:
  \[ \text{charpol}(t(a)|_{\text{Lie}(A)})(t) = \prod_{\phi \in \Phi} (t - \phi(a)) \quad \forall a \in O_K \]
- $\lambda : A \to A^\vee$ is a polarization with kernel $A[\alpha]$ whose Rosati involution on $O_K \subseteq \text{End} A$ gives the nontrivial involution of $K/F$.

Fix $\xi \in L_{\Phi}/\cong$ from now on. Now we define the algebraic stack $\mathcal{X}$ to be the substack of $\mathcal{M}_0^\xi \times_{O_K} \mathcal{CM}_0^\Phi$ whose $S$-points (for an $O_K$-scheme $S$) consists of $(A_0, t_0, \lambda_0, A, t, \lambda)$ with $\text{inv}_\nu(A_0s, t_0s, \lambda_0s, A_s, t_s, \lambda_s) = \text{inv}_\nu(-W_\nu)$ and $(A \otimes Z_{(p)}^{\nu}, t \otimes Z_{(p)}^{\nu}, \lambda \otimes Z_{(p)}^{\nu})$ satisfies the conditions of section 4 of [19], then we have a forgetful map $\mathcal{X} \to \mathcal{M}$ by sending $(A_0, t_0, \lambda_0, A, t, \lambda)$ in $\mathcal{X}(S)$ to $(A_0, t_0, \lambda_0, A, t|_{O_K}, \lambda)$ in $\mathcal{M}(S)$. Now it follows from [9] prop. 3.1.2 that $\mathcal{X} \to \mathcal{M}$ is étale and proper. By the same proposition, for all $(A, t, \lambda) \in \mathcal{CM}_0^\Phi(\bar{k}_p)$ we have a unique canonical lift $(A_{\text{can}}, t_{\text{can}}, \lambda_{\text{can}})$ to $\mathcal{CM}_0^\Phi(\bar{W})$. Also for all $(A_0, t_0, \lambda_0) \in \mathcal{M}_0^\xi(\bar{k}_p)$, we have a unique canonical lift $(A_{0\text{can}}, t_{0\text{can}}, \lambda_{0\text{can}}) \in \mathcal{M}_0^\xi(\bar{W})$

**Proposition 4.9.** We have $D_0D^{-1} = \partial_{F/F_2}^{-1} O_K$.

**Proof.** An easy calculation using the ramification condition introduced in notations section. 

Now fix a sextuple $(A_0, t_0, \lambda_0, A, t, \lambda) \in \mathcal{X}(S)$ for some $O_K$-scheme $S$. We can define a Hermitian space as follows: Consider $\text{Hom}_{O_K}(A_0, A)$, this has a normal $O_K$-valued Hermitian form given by:

$$\langle f, g \rangle = \lambda_0^{-1} \circ g^\vee \circ \lambda \circ f$$

As $A$ has $O_K$-action, we can change the Hermitian form and define $\langle \cdot, \cdot \rangle_{CM}$ to be the unique $K$-valued Hermitian form satisfying

$$\langle f, g \rangle = \text{tr}_{K/K_0} \langle f, g \rangle_{CM}$$
**Proposition 4.10.** Suppose $k$ is an algebraically closed field and that $(A_0, t_0, \lambda_0, A, t, \lambda) \in \mathcal{X}(k)$, if there is $f \in \text{Hom}_{\mathcal{O}_F}(A_0, A) \otimes \mathbb{Q}$ with $\langle f, f \rangle_{\text{CM}} \neq 0$, then $\text{char} \ k \neq 0$ and $A_0 \otimes_{\mathcal{O}_F} O_F$ and $A$ are $O_K$-isogenous.

**Proof.** $f : A_0 \to A$ induces the $\mathcal{O}_F$-linear map $\tilde{f} : A_0 \otimes_{\mathcal{O}_F} O_F \to A$ (where $A_0 \otimes_{\mathcal{O}_F} O_F$ is the abelian variety over $K$ defined by Serre construction and having action by $\mathcal{O}_K \otimes_{\mathcal{O}_F} O_F = O_K$). Now for $l \mid \text{char} \ k$, let $T_l(A)$, $T_l(A_0)$ be Tate modules of $A$ and $A_0$ respectively and $T_l^0(A_0) = T_l(A_0) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ and $T_l^0(A) = T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. The polarization $\lambda_0$ gives $\mathbb{Q}_l$-linear map $T_l^0(A_0) \times T_l^0(A_0) \to \mathbb{Q}_l(1)$ so that by tensoring $\otimes_{\mathcal{O}_F} F_l$ gives

$$\Lambda_0 : T_l^0(A_0 \otimes_{\mathcal{O}_F} O_F) \times T_l^0(A_0 \otimes_{\mathcal{O}_F} O_F) \to F_l(1)$$

Also polarization $\lambda$ gives $\mathbb{Q}_l$-linear map

$$\Lambda : T_l^0(A) \times T_l^0(A) \to \mathbb{Q}_l(1)$$

so that $\Lambda$ can be written uniquely as $\text{tr}_{F_l/\mathbb{Q}_l} \tilde{\Lambda}$ for some

$$\tilde{\Lambda} : T_l^0(A) \times T_l^0(A) \to F_l(1)$$

Now $\tilde{f}$ gives us a $\mathbb{Q}_l$-linear map $\tilde{f}_1 : T_l^0(A_0 \otimes_{\mathcal{O}_F} O_F) \to T_l^0(A)$ and we call the adjoint of $\tilde{f}_1$ by $\tilde{f}_1^\dagger$ (which is the unique $\mathbb{Q}_l$-linear map $\tilde{f}_1^\dagger : T_l^0(A) \to T_l^0(A_0 \otimes_{\mathcal{O}_F} O_F)$ for which $\Lambda_0(x, \tilde{f}_1^\dagger(y)) = \tilde{\Lambda}(\tilde{f}_1(x), y))$ for all $x \in T_l^0(A_0 \otimes_{\mathcal{O}_F} O_F)$ and $y \in T_l^0(A)$. Now we have $\langle f, f \rangle_{\text{CM}} = \tilde{f}_1^\dagger \circ \tilde{f}_1$ as elements of $F_l \subseteq \text{End}_{\mathbb{Q}_l}(T_l^0(A_0 \otimes_{\mathcal{O}_F} O_F))$, so that by $\langle f, f \rangle_{\text{CM}} \neq 0$, we have that $\tilde{f}_1$ is injective, so $\tilde{f} : A_0 \otimes_{\mathcal{O}_F} O_F \to A$ is an $O_K$-isogeny, so $A$ is isogenous to $A_0 \otimes_{\mathcal{O}_F} O_F \cong A_0 \times A_0 \times \cdots A_0$ (d times). This isogeny cannot happen if $\text{char} \ k = 0$ as the signatures of $A$ and $A_0 \otimes_{\mathcal{O}_F} O_F$ are different. 

Now we have the theorem relating the local parts of $L(A_0, A) = \text{Hom}_{\mathcal{O}_F}(A_0, A)$ to $\text{Hom}_{\mathcal{O}_F}(A_0[q_0^\infty], A[q^\infty])$:

**Proposition 4.11.** Suppose that $k$ is an algebraically closed field of $\text{char} \ k > 0$. Suppose that $(A_0, t_0, \lambda_0, A, t, \lambda) \in \mathcal{X}(k)$ is such that $A_0 \otimes_{\mathcal{O}_F} O_F$ and $A$ are $O_K$-isogenous, then $L(A_0, A)$ is a projective $O_K$-module of rank 1 and letting $q$ be a prime of $F$ over the prime $q_0$ of $F_0$ over the rational prime $q$, the map

$$L(A_0, A) \otimes_{O_F} O_{F,q} \to \text{Hom}_{\mathcal{O}_F}(A_0[q_0^\infty], A[q^\infty])$$

is an isomorphism.
Proof. We have an $O_K$-isogeny $A \rightarrow A_0 \otimes_{O_{F_0}} O_F$, so this induces a map

$$\text{Hom}_{O_K}(A_0, A) \rightarrow \text{Hom}_{O_K}(A_0, A_0 \otimes_{O_{F_0}} O_F) = \text{Hom}_{O_K}(A_0, A_0^d) \cong O_{K_0}^d$$

which is an injection with finite cokernel, so that $\text{Hom}_{O_K}(A_0, A)$ is a projective $O_{K_0}$-module of rank $d$. Also for $p$-divisible group, we have

$$\text{Hom}_{O_K}(A_0[q_0^\infty], A[q_0^\infty]) \rightarrow \text{Hom}_{O_K}(A_0[q_0^\infty], (A_0 \otimes_{O_{F_0}} O_F)[q_0^\infty])$$

$$= \text{Hom}_{O_K}(A_0[q_0^\infty], A_0[q_0^\infty]^d) = O_{K_0, A_0}^d$$

is injective with finite kernel, so $\text{Hom}_{O_K}(A_0[q_0^\infty], A[q_0^\infty])$ is a projective $O_{K_0, A_0}$-module of rank $d$. Also the map

$$\text{Hom}_{O_K}(A_0, A) \otimes_{O_{F_0}} O_{F_0, A_0} \rightarrow \text{Hom}_{O_K}(A_0[q_0^\infty], A[q_0^\infty])$$

is injective with $\mathbb{Z}_q$-torsion free cokernel and also the $\mathbb{Z}_q$-rank of domain and codomain are equal by above, so it is an isomorphism. Also $L(A_0, A)$ is a projective $O_K$-module a fortiori of rank 1 by being a projective $O_{K_0}$-module of rank $d$. Taking $q$-parts from both sides we get the statement of proposition.

We define the groups $C_K$ and $C_0^K$ in the same way as in [9]. For an ideal $I$ of $O_K$, let $A^I = I \otimes_{O_K} A$ be the abelian variety constructed by the Serre construction.

**Proposition 4.12.** Suppose that $S$ is a connected $O_K$-scheme and $(A_0, \iota_0, \lambda_0, A, \iota, \lambda) \in \mathcal{X}(S)$. For each $(I, \zeta) \in C_K$, we have an $\hat{O}_K$-linear isomorphism

$$\hat{L}(A_0, A^I) \cong \hat{L}(A_0, A)$$

where the Hermitian form $\langle \cdot \rangle^I_{CM}$ on left is $\text{gen}(I) \langle \cdot \rangle_{CM}$ on the right (the map gen is defined in [9]).

**Proof.** Same as prop 3.3.1 of [9].

For a pair of abelian varieties $(A_0, A) \in \mathcal{X}(C)$, define

$$L_{\text{Betti}}(A_0, A) = \text{Hom}_{O_K}(H_1(A_0, C), H_1(A, C))$$

Now we have the following structure theorem for $L_{\text{Betti}}$:

**Proposition 4.13.** There is $\beta \in \hat{F}^\times$ with $\beta O_F = \partial_{F/F_0}^{-1} a$ with an isomorphism

$$\hat{L}(A_0, A), \langle \cdot \rangle_{CM} \cong (\hat{O}_K, \beta x \bar{y})$$
Also showing $⟨x, y⟩_{CM}$ at archimedean places by $βx y$ as well, we get that $β$ is negative definite at $∞^p = φ_p^q$ and positive definite at other archimedean places of $F$.

**Theorem 4.14.** Let $p$ be a prime of $\overline{K}$ with $p_F$ nonsplit in $K$, and let

$$(A_0, A) \in X(\overline{k}_p)$$

then there is an isomorphism $(\hat{L}(A_0, A), ⟨, ⟩_{CM}) \cong (\hat{O}_K, βx y)$ with $β \in \hat{F}^×$ such that $βO_F = a\hat{F}_1^{-1} p_F^{c_p}$. Also we have $χ(β^∞) = 1$ ($β^∞$ is the element of $A_F^∞$ that has trivial archimedean components and at finite places, it is the same as $β \in \hat{F}^×$).

Let $A'_0$ (resp. $A'$) be the unique lift of $A_0$ (resp. $A$) to $C_P$ and by fixing $\overline{K}$-linear isomorphism $C \cong C_p$, we see $A'_0$ and $A'$ as abelian varieties in $C$. Now for a prime $q$ of $F$ with $q_0$ below it in $F_0$, there are isomorphisms of $O_{K,q}$-Hermitian spaces

$$L_{\text{Betti}}(A'_0, A') \otimes_{O_K} O_{K,q} \cong \text{Hom}_{O_{K,q}}(A'_0[q^∞], A'[q^∞])$$

because of the fact that $A'_0[q^∞]$ and $A'[q^∞]$ are constant $p$-divisible groups.

Also by proposition 4.6 there is an isomorphism

$$\text{Hom}_{O_{K,q}}(A_0, A) \otimes_{O_K} O_{K,q} \cong \text{Hom}_{O_{K,q}}(A_0[q^∞], A[q^∞])$$

Now we have the following lemma:

**Lemma 4.15.** If $q$ is not $p_F$, then there’s an $O_{K}$-linear isomorphism of Hermitian spaces

$$\text{Hom}_{O_{K,q}}(A'_0[q^∞], A'[q^∞]) \cong \text{Hom}_{O_{K,q}}(A_0[q^∞], A[q^∞])$$

**Proof:** Recall that $p$ is the characteristic of $\overline{k}_p$ and $A_0$ and $A$ are defined over $\overline{k}_p$. If the rational prime below $q$ is not $p$ (and call it $q$), then the Tate modules of $A_0$ and $A'_0$, and also the Tate modules of $A$ and $A'$ are going to be canonically isomorphic, so

$$\text{Hom}_{O_{K,q}}(A'_0[q^∞], A'[q^∞]) \cong \text{Hom}_{O_{K,q}}(A_0[q^∞], A[q^∞])$$

taking $q$-parts we get the wanted isomorphism. It is clear that it respects the Hermitian forms. Now suppose that the rational prime below $q$ is $p$, now because $q \neq p_F$ by hypothesis, we have that the set $Φ(q)$ of all embeddings $φ : K \to C_p$ with $q = φ^{-1}(pO_{C_p})$ satisfies $φ_1^q \notin Φ(q)$ (because of the fact that $p_F$ is the prime below $p$ using the inclusion $\hat{F}_1(F) \subseteq \hat{F}_1(K) \subseteq \overline{K}$). Similarly let $Φ_0(q_0)$ be all embeddings $φ : K_0 \to C_p$ with $q_0 = φ^{-1}(pO_{C_p})$. Then we have the following relation between $Φ(q)$ and $Φ_0(q_0)$:

$$Φ(q) = \{φ : K \to C_p|φ|_{K_0} \in Φ_0(q_0)\}$$
Also that \( A[q^\infty] \) (resp. \( A_0[q_0^\infty] \)) is a CM \( p \)-divisible group with action \( O_{K,\mathfrak{q}} \) (resp. \( O_{K_0,\mathfrak{q}_0} \)) and \( \Phi(q) \) (resp. \( \Phi_0(q_0) \))-determinant condition. Letting \( A_0^{can} \) and \( A^{can} \) be the unique lifts of \( A_0 \) and \( A \) respectively to \( \tilde{W} \). We have that

\[
\text{Hom}_{O_{K_0}}(A_0^{can}[q_0^\infty], A^{can}[q^\infty]) \to \text{Hom}_{O_{K_0}}(A_0[q_0^\infty], A[q^\infty])
\]

is an isomorphism. Now base change \( \tilde{W} \equiv \mathbb{C}_p \) defines an injection

\[
F : \text{Hom}_{O_{K_0}}(A_0^{can}[q_0^\infty], A^{can}[q^\infty]) \to \text{Hom}_{O_{K_0}}(A_0[q_0^\infty], A'[q^\infty])
\]

Now we have Tate’s theorem which says for two \( p \)-divisible groups \( G, H \) with Tate modules \( TG, TH \) respectively (over specific types of rings \( R \) including \( \tilde{W} \) and \( \mathbb{C}_p \) with \( E = \text{Frac}(R) \)) the map

\[
\text{Hom}(G, H) \to \text{Hom}_{\text{Gal}(\bar{E}/E)}(TG, TH)
\]

is an isomorphism. So the image of \( F \) is \( \text{Aut}(\mathbb{C}_p/\text{Frac}(\tilde{W})) \)-invariants of \( \text{Hom}_{O_{K_0}}(A_0'[q_0^\infty], A'[q^\infty]) \) so that the map has \( \mathbb{Z}_p \)-torsion-free cokernel. Now propositions 4.6 and 4.8 and isomorphisms

\[
L_{\text{Betti}}(A_0', A') \otimes_{O_K} O_{K,\mathfrak{q}} \cong \text{Hom}_{O_{K_0}}(A_0'[q_0^\infty], A'[q^\infty])
\]

\[
\text{Hom}_{O_{K_0}}(A_0, A) \otimes_{O_K} O_{K,\mathfrak{q}} \cong \text{Hom}_{O_{K_0}}(A_0[q_0^\infty], A[q^\infty])
\]

imply that both domain and codomain of \( F \) are free of rank 1 over \( O_{K,\mathfrak{q}} \), so that \( F \) is an isomorphism (clearly also an isomorphism of Hermitian spaces).

Now we prove the theorem using the lemma: First if \( q \) is not \( \mathfrak{p}_F \), then by lemma we have

\[
L_{\text{Betti}}(A_0', A') \otimes_{O_F} O_{F,\mathfrak{q}} \cong L(A_0, A) \otimes_{O_F} O_{F,\mathfrak{q}}
\]

so that by proposition 4.8, we have that \( L(A_0, A) \otimes_{O_F} O_{F,\mathfrak{q}} \cong O_{K,\mathfrak{q}} \) with the Hermitian form given by \( \beta_q x \bar{y} \) with \( \beta_q \in F_\mathfrak{q}^\times \) with \( \beta_q O_{F,\mathfrak{q}} = \partial_{F/F_0}^{-1} O_{F,\mathfrak{q}}. \) Now suppose that \( q = \mathfrak{p}_F \), then considering \( \Phi(q) \) and \( \Phi_0(q_0) \) as before, by proposition 3.2 gives us that \( L(A_0, A) \otimes_{O_F} O_{F,\mathfrak{q}} \cong \text{Hom}_{O_{K_0}}(A_0[q_0^\infty], A[q^\infty]) \cong O_{K,\mathfrak{q}} \) with Hermitian form given by \( \beta_q x \bar{y} \) with \( \beta_q \in F_\mathfrak{q}^\times \) with \( \beta_q O_{F,\mathfrak{q}} = \partial_{F/F_0}^{-1} \mathfrak{p}_F \mathfrak{p}_F^{-1} O_{F,\mathfrak{q}}. \)

So we have the required isomorphism as in the statement of the theorem. Now as \( L(A_0, A) \otimes_{O_F} K \) is a \( K \)-Hermitian space, we have that \( \beta \) differs from some \( \beta^* \in F_\mathfrak{q}^\times \) by a norm at each place, so that \( \chi(\beta) = \chi(\beta^*) = 1 \) which proves the theorem.
Now we are going to define \( Z(\alpha) \) for \( \alpha \in F^{\gg 0} \). It is the Deligne-Mumford stack over \( \mathcal{O}_K \) such that for an \( \mathcal{O}_K \)-scheme \( S \) it gives us the groupoid of \( (A_0, A, f) \) with \( (A_0, A) \in \mathcal{X}(S) \) and \( f \in L(A_0, A) \) with \( (f, f)_{CM} = \alpha \).

**Proposition 4.16.** (1) Let \( \alpha \in F^{\gg 0} \), the stack \( Z(\alpha) \) has dimension 0 and it is supported in nonzero characteristic.

(2) If \( p \) is a prime of \( \bar{K} \) with \( Z(\alpha)(\bar{k}_p) \) nonempty, then \( p_F \) is nonsplit.

**Proof.** (1) The forget map \( Z(\alpha) \to \mathcal{X} \) is unramified, so induces a surjection on the completed strictly Henselian local rings, so that if \( z \in Z(\alpha)(\bar{k}_p) \) is a point, then \( \hat{O}_{Z(\alpha), z} \) is a quotient of \( \hat{W} \), so because \( Z(\alpha) \) does not have a point in characteristic 0 (due to the fact that signatures of \( (A_0, A) \) have to be different) and has dimension 0.

(2) If \( p \) is a prime that \( Z(\alpha)(\bar{k}_p) \) is nonempty, then by the signatures of \( A_0, A \) we have that \( \phi_1 \hat{=} \phi_1 \), so that \( x = \hat{x} \mod p \) for that \( x \in K \) and so \( p = \hat{p} \) and so \( p_F \) is nonsplit \( K \). \( \square \)

Now let \( C^0_K \subseteq C_K \) be the subgroup defined by the exact sequence

\[
1 \to C^0_K \to C_K \xrightarrow{gen} \hat{O}_F / N_{K/F}(\hat{O}_K) \xrightarrow{\eta} \{ \pm 1 \}
\]

where the map is the restriction of the character \( \chi \) where \( gen(I, \xi) = \xi z^z \) where \( z \in \hat{K}^\times \) has the property \( zO_K = I \). Now we have the following assumption for the rest of our manuscript.

**Assumption.** We assume that \( [K : K_0] \) is even and that for all primes \( p \) of \( \hat{K} \)

with residue characteristic \( p \) the CM abelian varieties \( (A, I, \lambda) \) appearing in \( (A_0, A) \in \mathcal{CM}_\Phi(\bar{k}_p), (A \otimes \mathbb{Z}_p, I \otimes \mathbb{Z}_p, \lambda \otimes \mathbb{Z}_p) \) satisfies the conditions in chapter 4 of [19] (it is sufficient to assume this for primes \( p \) such that \( p_F \)

is nonsplit in \( K \) and only the conditions happening in section 4.4 of [19] as the other conditions are satisfied).

Assuming the assumption above, for each place \( v \) of \( F_0 \), choose \( W_v \) in a way that there exists at least one \( (A_0, A) \) in each of \( \mathcal{X}(\bar{k}_p) \). We see that

we have exactly one \( C^0_K \)-orbit in each \( \mathcal{X}(\bar{k}_p) \) (because the sign conditions of \( \mathcal{X} \) implies that there’s exactly one genus of Hermitian spaces in each fiber of \( \mathcal{M}^\alpha_0 \otimes \mathcal{O}_K \mathcal{M}^\alpha_0 \) which by [9] page 1137, \( C^0_K \) acts simply transitively on. Now we compute the number of stacky points of \( Z(\alpha)(\bar{k}_p) \). Let \( w(K), w(K_0) \) be the number of roots of unity in \( K, K_0 \), respectively.

**Theorem 4.17.** Suppose that \( \alpha \in F \) and \( \alpha \gg 0 \). Also let \( \beta \) be the one appearing in Theorem 4.9. If \( p \) is a prime of \( \bar{K} \) with \( p_F \) nonsplit in \( K \), then

\[
\sum_{(A_0, A, f) \in Z(\alpha)(\bar{k}_p)} \frac{1}{\#Aut(A_0, A, f)} = \frac{1}{w(K_0)} \rho(\alpha \partial_{F/F_0}^{-1} p_F^{-\epsilon_F})
\]
if \( \alpha \beta \in N_{K/F}(\hat{K}^\times) \) and 0 if not.

Proof: We have

\[
\sum_{l \in C^0_K} \# \{ f \in L(A_0, A^l) \mid \langle f, f \rangle_{CM}^l = \alpha \} = \sum_{l \in C^0_K} \sum_{x \in L(A_0, A) \cap \mathbb{Q}} 1_{l, L(A_0, A)}(x)
\]

where \( 1_A \) is the characteristic function of \( A \). Now using the presentation \( C^0_K = H(F) \setminus H(\hat{F}) / U \) using the algebraic group \( H \) and \( U \) defined in [9]. The sum above is equal to

\[
\sum_{h \in H(F) \setminus H(\hat{F}) / U} \sum_{x \in V(A_0, A) \setminus \{ x \} \cap \mathbb{Q}} 1_{\hat{L}(A_0, A)}(h^{-1}x) =
\]

\[
\#(H(F) \cap U) \sum_{h \in H(\hat{F}) / U} \sum_{x \in H(F) \setminus V(A_0, A) \setminus \{ x \} \cap \mathbb{Q}} 1_{\hat{L}(A_0, A)}(h^{-1}x)
\]

Let \( \mu(K_0) \), \( \mu(K) \) be the group of roots of unity of \( K_0, K \) respectively. Now we have that \( H(F) \cap U = \mu(K) \) and also \( Aut(A_0, A) \cong \mu(K_0) \times \mu(K) \). So we get that

\[
\sum_{l \in C^0_K} \sum_{f \in L(A_0, A^l)} \frac{\#(Aut(A_0, A^l))}{\langle f, f \rangle_{CM}^l = \alpha} \] 

\[
= \sum_{h \in H(\hat{F}) / U} \sum_{x \in H(F) \setminus V(A_0, A) \setminus \{ x \} \cap \mathbb{Q}} 1_{L(A_0, A)}(h^{-1}x)
\]

(4.1)

Now there are two cases, either there is an \( x \in V(A_0, A) \) with \( \langle x, x \rangle_{CM} = \alpha \) or there is no \( x \) with \( \langle x, x \rangle_{CM} = \alpha \). In the latter case, the RHS is zero and in the former case \( H(F) \) acts simply transitively on them and so the RHS is

\[
\frac{1}{\#(K_0)} \sum_{h \in H(\hat{F}) / U} 1_{L(A_0, A)}(h^{-1}x)
\]

Now we define the orbital integral for \( \alpha \in F^\times \) by

\[
O_{\alpha}(A_0, A) = \sum_{h \in H(F) / U} 1_{L(A_0, A)}(h^{-1}x)
\]

where \( x \in \hat{V}(A_0, A) \) has the property \( \langle x, x \rangle_{CM} = \alpha \). If such an \( x \) does not exist, \( O_{\alpha}(A_0, A) \) is defined to be zero. Now the RHS of equation 4.1 is \( \frac{1}{\#(K_0)} O_{\alpha}(A_0, A) \). Now let \( \beta \) be the element of \( \hat{F}^\times \) such that the Hermitian form on \( \hat{L}(A_0, A) \) is \( \beta x \bar{y} \). We break the orbital integral into local parts:

\[
O_{\alpha}(A_0, A) = \prod_{\nu} O_{\alpha, \nu}(A_0, A)
\]
where $O_{a,v}(A_0, A) = \sum_{h \in H(F_v)} 1_{O_{h,v}}(h^{-1}x_v)$. We now have two cases, either $v$ is nonsplit in $K$ and we will get

$$O_{a,v}(A_0, A) = \begin{cases} 1 & \text{if } a\beta^{-1} \in O_{F,v} \\ 0 & \text{otherwise} \end{cases} \quad (4.2)$$

or in the split case we see that (in the same way as in [9]):

$$O_{a,v}(A_0, A) = \begin{cases} 1 + \text{ord}_v(a\beta^{-1}) & \text{if } a\beta^{-1} \in O_{F,v} \\ 0 & \text{otherwise} \end{cases} \quad (4.3)$$

so the product above is going to be $\rho(a\beta^{-1}O_F) = \# \{ J \triangleleft O_K | N_{K/F}J = a\beta^{-1}O_F \}$ and now if $\rho(a\beta^{-1}O_F) \neq 0$ then $a\beta^{-1} \in N_{K/F}(\hat{K}^\times)$ and using the fact that we have the ideal of $\beta O_F$, if $a\beta^{-1} \not\in N_{K/F}(\hat{K}^\times)$, then $\rho(a\beta^{-1}O_F) = 0$ and we get $O_n(A_0, A) = 0$, So we finally get the statement of the theorem. \qed

Now we need a theorem about lengths of strictly henselian local rings:

**Theorem 4.18.** Let $\alpha \in F^\times$ and $p$ a prime of $\hat{K}$ such that $p_F$ is nonsplit in $K$. Then at a point $z \in Z(\alpha)(\hat{K}_p)$, we have

$$\text{length}(O_{s,h, T(a),z}) = \frac{1}{2} \text{ord}_{\hat{K}_p}(a \beta^{-1} p^{-1} \partial_{F/F_0})$$

**Proof.** Consider $(A_0, A, f) \in Z(\alpha)(\hat{K}_p)$ be the triple corresponding to $z$, then the completed strictly henselian ring $\hat{O}_{s,h, T(a),z}$ pro-represents the deformations of $(A_0, A, f)$ to objects of ART which by Serre-Tate, is in turn the same as deformations of $(A_0[p], A[p^\infty], f[p^\infty])$ to the objects of ART. Now we have the decomposition $A_0[p^\infty] = \prod_{q \nmid p} A_0[q^\infty]$ and $A[p^\infty] = \prod_{q \nmid p} A[q^\infty]$, so that the map $f[p^\infty]$ is decomposed into

$f_{q_0,q} : A_0[q_0^\infty] \rightarrow A[q^\infty]$ for different $q_0 \nmid O_{F_0}$ and $q \nmid O_F$ above the prime $p$, so we have to analyze the liftings of $f[q_0^\infty]$ to higher Artin rings (i.e. to higher powers $k$ in $W/m^k$). Now we have two cases:

1. $q \neq p_F$, in this case, the $p$-adic CM-types of $A_0[q_0^\infty]$ and $A[q^\infty]$ are compatible (i.e. the $p$-adic CM-type of $A[q^\infty]$ is exactly the embeddings whose restriction to $F_0,q_0$ induces the embeddings in the $p$-adic CM-types of $A_0[q_0^\infty]$).

2. $q = p_F$, in this case, the $p$-adic CM-types of $A_0[q_0^\infty]$ and $A[q^\infty]$ are incompatible and there’s exactly one embedding in the $p$-adic CM-type of $A[q^\infty]$ such that restriction to $F_0,q_0$ is the conjugation of one embedding of $p$-adic CM-type of $A_0[q_0^\infty]$, so we are in the situation of theorem 3.5.
in section 3 and so the deformations of \((A_0[q^\infty], A[q^\infty], f_{q_0, q})\) to objects of \(\text{ART}\) is pro-represented by \(W/m^k\) where \(k = \frac{1}{2}\ord_{k_p} (aa^{-1}p_F \partial_{F/F_0}) = \frac{1}{2}\ord_{k_p} (aa^{-1}p_F \partial_{F/F_0})\) So we get

\[
\text{length}(O_{\mathcal{Z}(h),Z(a,z)}) = \text{length}(O_{\mathcal{Z}(h),Z(a,z)}) = \frac{1}{2}\ord_{k_p} (aa^{-1}p_F \partial_{F/F_0})
\]

\(\square\)

Now we collect everything from theorem 4.12 and 4.13 and we get the main result of this section:

**Theorem 4.19.** If \(\alpha \in F^\times 0\) and for \(p\) a prime of \(\hat{K}\), let \(\beta_p\) be the \(\beta\) appearing in Theorem 4.9, then

\[
\widehat{\deg} \mathcal{Z}(\alpha) = \frac{1}{2w(K_0)} \sum_{p \in \mathcal{Z}(h),Z(a,z)} \rho(aa^{-1}d_{F/F_0})_{ord_{\hat{k}_p} (aa^{-1}p_F \partial_{F/F_0})} \log N_{\hat{k}/Q}(p) = \frac{1}{w(K_0)} \sum_{q \in \mathcal{Z}(h),Z(a,z)} \rho(aa^{-1}d_{F/F_0},q^{-\epsilon_0})_{ord_{\hat{k}_p} (aa^{-1}q \partial_{F/F_0})} \log N_{F/Q}(q) \]

where \(q\) changes over the primes of \(O_F\) nonsplit in \(K\) and \(p\) appearing in \(\beta_p\) in the second sum is a choice of prime \(p\) of \(K\) above \(q\).

**Proof.** This results from theorem 4.12 and 4.13. \(\square\)

### 4.4 Arithmetic Chow Group

In this section, we are going to define the arithmetic divisors as elements of \(\widehat{\text{CH}}^1(X')\) and find their degrees. These degrees will in turn be related to not positive definite coefficients of the Eisenstein series that we are going to define later (see section 4.5).

An arithmetic divisor of \(X = \mathcal{M}_0^L \times_{O_K} C \mathcal{M}_0^L\) is a pair \((Z, \text{Gr})\) such that \(Z\) is a Weil divisor on \(X\) and \(\text{Gr}\) is a Green function for \(Z\). We are going to define the arithmetic divisors \(\mathcal{Z}(\alpha)\) for \(0 \neq \alpha \in F^\times\) that are not necessarily totally positive. If \(\alpha \gg 0\), we want to get \(\mathcal{Z}(\alpha) = (\mathcal{Z}(\alpha), 0)\), and in the other cases we want to get \(\mathcal{Z}(\alpha) = (0, \text{Gr}_\alpha)\) for some Green function \(\text{Gr}_\alpha\).

As \(\text{Gr}_\alpha\) is a Green function for \(\mathcal{Z}(\alpha)\) and \(\mathcal{Z}(\alpha)\) does not have any characteristic zero points, we have that \(\text{Gr}_\alpha\) can be any complex-valued function on the finite set

\[
\prod_{\sigma: \hat{K} \to \mathbb{C}, \sigma|_{K_0} \in \Phi_0} ((\mathcal{M}_0^L) \times_{\hat{K}} C \mathcal{M}_0^L)^\sigma(C)
\]
Now we define the Green functions $Gr_{\alpha}$ on the point $z \in (\mathcal{M}_0^s \times_k \mathcal{M}_\Phi^s)^{\sigma}(\mathbb{C})$ corresponding to $(A_0, A)$ to be

$$Gr_{\alpha}(y, \alpha) = \sum_{f \in L_{Betti}(A_0, A)} \beta_1(4\pi|y\alpha|_{\mathbb{C}^{\sigma_{Betti}}})$$

where $\sigma \in Aut(\mathbb{C})$ is an extension of $\sigma : \mathbb{C} \to \mathbb{C}$ and $\beta_1(a) = \int_{1}^{\infty} e^{-tu} du$. We have that $\tilde{Z}(\alpha) = (Z(\alpha), Gr_{\alpha})$ is an element of the first Chow group $\tilde{CH}^1(\mathcal{X})$ and on this Chow group, we have a degree function that maps:

$$\tilde{deg} : \tilde{CH}^1(\mathcal{X}) \to \tilde{CH}^1(Spec \ O_{\mathbb{K}}) \to \mathbb{R}$$

and for an arithmetic divisor $(Z, Gr)$, it is defined to be

$$\tilde{deg}(Z, Gr) = \frac{1}{[K: \mathbb{Q}]} \left( \sum_{p \in \mathcal{O}_K} \sum_{z \in \mathbb{Z}(F_p)} \log N(p) \#Aut z + \sum_{\sigma: \mathbb{K} \to \mathbb{C}} \sum_{\sigma: \mathbb{K} \to \mathbb{C}} \frac{Gr(z)}{\#Aut z} \right)$$

Now an easy computation for $\alpha$ not positive definite shows that

$$\tilde{deg}(\tilde{Z}(\alpha)) = \begin{cases} \frac{1}{\varepsilon(K_0)|K_0: \mathbb{Q}|} \beta_1(4\pi|y\alpha|_{\mathbb{C}}) \rho(a \partial F/F_{\alpha} a^{-1}) & \text{if } \alpha \text{ is negative definite at exactly one place} \\ 0 & \text{otherwise} \end{cases}$$

4.5 EISENSTEIN SERIES

For completeness we define the Eisenstein series in this section. These are Eisenstein series with the property that the Fourier coefficients of this Eisenstein series are related to the degree of divisors considered in previous sections.

Let the notations be as in the notations section. Fix a place $v$ of $K$ and let $v_F$ be the prime below $v$ in $F$ and some $c \in F_{v_F}^\times$, $\chi_{v_F} : F_{v_F}^\times \to \mathbb{C}^\times$ be character of $K_0/F_{v_F}$. $\psi$ be an additive character $F_{v_F} \to \mathbb{C}^\times$. Now there’s a space of Schwartz functions $G(K_0)$ and $L(\chi_{v}, s)$ the space of induced representation of $\chi_v(x)|x|^s$. These two spaces have actions of $SL_2(F_{v_F})$ and we have an operator

$$\lambda_{c, \psi} : G(K_0) \to L(\chi_{v}, 0)$$

$$\lambda_{c, \psi}(\chi)(g) = (\omega_{c, \psi}(g)\phi)(0)$$

There’s a unique section $\Phi_{c, \psi}(g, s) \in I(\chi_{v}, s)$ with $\Phi(., 0) = \lambda_{c, \psi}(1_{O_{K_0}})$ ($1_{O_{K_0}}$ is the characteristic function of $O_{K_0}$ in $K_0$) and $\Phi(g, s)$ is independent of $s$ for $g$ in maximal compact subgroup of $SL_2(F_{v_F})$ if $v$ is nonarchimedean.
\[ \Phi(\cdot, 0) = \lambda_{c, \psi}(e^{-2\pi |cx|_v}) \text{ if } v \text{ is archimedean. For } \alpha \in F_{\nu_v}, \text{ define the local Whittaker function} \]
\[
W_\alpha(g, s, \Phi_{c, \psi}, \psi) = \int_{F_{\nu_v}} \Phi_{c, \psi}([0 & -1; 1 & 0] \chi^s [x; 1]) (\alpha, g, s) \psi_v(-ax) dx
\]

Now I want to define the setup for global situation. Let \( \psi_Q : \mathbb{A}_Q / \mathbb{Q} \rightarrow \mathbb{C}^\times \) be the additive character with \( \psi_Q(x) = e^{2\pi i x} \) for \( x \in \mathbb{R} \) and unramified nonarchimedean components (i.e. \( \psi_Q(\mathbb{Z}_p) = 1 \) for all \( p \) where \( \mathbb{Z}_p \) is \( \mathbb{Z}_p \mathbb{Q} / \mathbb{Q} \subseteq \mathbb{A}_Q / \mathbb{Q} \)). Let \( \psi_F(x) = \psi_Q(tr_F / Q(x)) \) and \( \chi : \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times \) be the character of \( K / F \). For \( c \in \mathbb{A}_F^\times \), let \( \Phi_{c, \psi_F} = \otimes_v \Phi_{c, \psi_v} \) and define an Eisenstein series

\[ E(g, s, c, \psi_F) = \sum_{\gamma \in B(F) \setminus SL_2(F)} \Phi_{c, \psi_F}(\gamma g, s) \]

(\( B \) is the Borel subgroup of \( SL_2 \) of upper-triangular matrices). For normalizing the above Eisenstein series, let \( H_F = \{ x + iy \in F \otimes \mathbb{Q} C \mid x, y \in F \otimes \mathbb{Q} \mathbb{R}, y \gg 0 \} \). For \( \tau = x + iy \), let \( g_\tau \in SL_2(A_F) \) have archimedean components

\[
\begin{bmatrix}
1 & x \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
y^\frac{1}{2} & 0 \\
0 & y^{-\frac{1}{2}}
\end{bmatrix}
\in SL_2(F \otimes \mathbb{Q} \mathbb{R})
\]

and trivial components. Now let the normalized Eisenstein series be (by abuse of notation):

\[ E(\tau, s, c, \psi_F) = N_{F/Q}(\partial_{F/F_0})^{1/2} \frac{L(s + 1, \chi)}{N_{F/Q}(y)^2} E(g_\tau, s, c, \psi_F) \]

where \( L(s, \chi) \) is the Dirichlet function of \( \chi \). \( E \) has a Fourier expansion

\[ E(\tau, s, c, \psi_F) = \sum_{\alpha \in F} E_\alpha(\tau, s, c, \psi_F) \]

with

\[ E_\alpha(\tau, s, c, \psi_F) = N_{F/Q}(y)^{-1/2} \int_{F\mathbb{A}_F} E([1 & b; 0 & 1], g_\tau, s, c, \psi_F(-bu)) db \]

Now let \( c \) has the property \( c\mathbb{O}_F = \partial_{F/F_0}^{-1} a \) and \( c \) has trivial archimedean components with \( \chi(c) = -1 \). \( \chi(c) = -1 \) implies that the Eisenstein series is incoherent and so \( E(\tau, 0, c, \psi_F) = 0 \).

We finally define the Eisenstein series to be

\[ \mathcal{E}(\tau, s) = E(\tau, s, c, \psi_F) \]
where \( c \in \mathbb{A}_F^\times \) is taking \( \beta \) appearing in prop 4.8 and replacing the component at \( \infty^p \) with a positive definite element. This Eisenstein series has Fourier coefficients \( \mathcal{E}(\tau, s) = \sum_{\alpha \in F} \mathcal{E}_\alpha(\tau, s) \). The theorem below computes the Fourier coefficients of the derivative of \( \mathcal{E}(\tau, s) \) at \( s = 0 \) and also the order of Fourier coefficients of \( \mathcal{E}(\tau, s) \) at \( s = 0 \). Let \( \text{Diff}(\alpha, c) = \{ v | \chi_v(\alpha c) = -1 \} \) be a finite subset of places of \( F \). This set is easily seen to have odd cardinality by \( \chi(c) = -1 \).

**Theorem 4.20.** Let \( \alpha \) be nonzero \( F \) and \( d_{K/F} \) be the relative discriminant of \( K/F \), \( r \) be the number of places of \( F \) ramified in \( K \) (including the archimedean places). Then

- \( \#\text{Diff}(\alpha, c) > 1 \Rightarrow \text{ord}_{s=0}(E_\alpha(\tau, s, c, \psi_F)) > 1 \)
- If \( \text{Diff}(\alpha, c) = \{ p \} \) with \( p \) a finite prime of \( F \), then

\[
E'_\alpha(\tau, 0, c, \psi_F) = \frac{-2^{r-1}}{N_{F/Q}(d_{K/F})^{1/2}} \rho(\alpha \partial_{F/F_0} a^{-1} p^{-e_p}) \text{ord}_p(\alpha \partial_{F/F_0} a^{-1} p) \log(N_{F/Q}(p)) e^{2\pi i tr(F/Q(\alpha \tau))}
\]

- If \( \text{Diff}(\alpha, c) = \{ w \} \) where \( w \) is an archimedean place, then

\[
E'_\alpha(\tau, 0, c, \psi_F) = \frac{2^{-(r-1)}}{N(d_{K/F})^{1/2}} \rho(\alpha \partial_{F/F_0} a^{-1}) \beta_1(4\pi |y\alpha|_w) q^a
\]

**Proof.** For \( g \in SL_2(F_v) \), consider the normalized local Whittaker function

\[
W^*_{\alpha_c}(g_v, s, c_v, \psi_{F_v}) = L(s + 1, \chi_v) W_{\alpha_c}(g_v, s, c_v, \psi_{F_v})
\]

Now we have the factorization

\[
E_\alpha(\tau, s, c, \psi_F) = N_{F/Q}(y)^{-1/2} \prod_v W^*_{\alpha_c}(g_{\tau, v}, s, c_v, \psi_{F_v})
\]

so we have

\[
E_\alpha(\tau, s, c, \psi_F) = N_{F/Q}(y)^{-1/2} \prod_v W^*_{c_{v, 1} \alpha_c}(g_{\tau, v}, s, 1, c_v \psi_{F_v})
\]

where \( (c \psi_F)(x) = \psi_F(cx) \) is an unramified character of \( \mathbb{A}_F^\times \) and also \( (c_v \psi_{F_v})(x) = \psi_{F_v}(c_v x) \). By Yang’s formula

\[
\chi_v(\alpha c) = -1 \iff W^*_{c_{v, 1} \alpha_c}(g_{\tau, v}, 0, 1, c_v \psi_{F_v}) = 0
\]
If \( v \) is nonarchimedean then by Yang’s formulas [26], we get:

(1) If \( \chi_v(\alpha c) = 1 \), then

\[ W^*_{c_0,1} (g_{\tau,v}, 0, 1, c_v \psi_{F_v}) = \chi_v(-1) e\left( \frac{1}{2} \chi_v(c_v \psi_{F_v}) \rho(\alpha d_{F/F_0} a^{-1}) \times \right. \]
\[ \left. \times \begin{cases} 2N(\pi_{F_v})^{-\frac{\ord_v(d_{K/F})}{2}} & \text{if } v \text{ is ramified in } K/F \\ 1 & \text{if } v \text{ is unramified in } K/F \end{cases} \right) \]

Now if Diff \( (\tau, c_0) = \{w\} \), then

\[ \frac{d}{ds} E_a(\tau, s, c, \psi_F) |_{s=0} = N_{F/Q}(y)^{\frac{1}{2}} \frac{d}{ds} W^*_{c_0,1} (g_{\tau,v}, 0, 1, c_v \psi_{F_v}) |_{s=0} \prod_{v \neq w} W^*_{c_0,1} (g_{\tau,v}, 0, 1, c_v \psi_{F_v}) \]

and we get the formulas stated in the statement of theorem.

So we get that for \( \alpha \gg 0 \) in \( F \):

\[ E_a'(\tau, 0) = E_a'(\tau, 0, c, \psi_F) = \]
\[ = \frac{-2^{r-1}}{\sqrt{N_{F/Q}(d_{K/F})}} \rho(\alpha d_{F/F_0} a^{-1} p^{-e_p}) \ord_p(\alpha d_{F/F_0} a^{-1}) \log(N(p)) e^{2\pi i \tau/\epsilon} =: b_\Phi(\alpha, y) \]

for \( p \subseteq O_F \) nonsplit with Diff \( (\alpha, c) = \{p\} \). Now using the above, one can derive the main result:
Theorem 4.21. Let $\alpha$ be nonzero in $F$. Suppose that the ramification condition in the introduction is satisfied and the assumption on page 62 is satisfied, then

$$\deg \hat{Z}(\alpha) = \frac{-1}{w(K_0)} \frac{N_{F/Q}(d_{K/F})}{2^{r-1}[K : Q]} b_{\Phi}(\alpha, y).$$


