SOME TOPICS IN THE ARITHMETIC OF HODGE STRUCTURES AND AN AX-SCANUEL THEOREM FOR GL_n

by

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ABSTRACT

Some topics in the arithmetic of Hodge structures

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In the first part of this thesis we consider smooth projective morphisms $f : X \to S$ of K-varieties with S an open curve and K a number field. We establish upper bounds of the Weil height $h(s)$ by $[K(s) : K]$ at certain points $s \in S(\bar{K})$ that are "exceptional" with respect to the variation of Hodge structures $R^n(f^{an})_*(\mathbb{Q}_{X^{an}_{\mathbb{C}}})$, where $n = \dim X - 1$. We work under the assumption that the generic special Mumford-Tate group of this variation is $Sp(\mu, \mathbb{Q})$, the variation degenerates in a strong fashion over some fixed point s_0 of a proper curve that contains S, the Hodge conjecture holds, and that what we define as a "good arithmetic model" exists for the morphism f over the ring $\mathcal{O}_K$.

Our motivation comes from the field of unlikely intersections, where analogous bounds were used to settle unconditionally certain cases of the Zilber-Pink conjecture.

In the second part of this thesis, we prove an Ax-Schanuel type result for the exponential functions of general linear groups over $\mathbb{C}$. We prove the result first for the group of upper triangular matrices and then for the group GL_n of all $n \times n$ invertible matrices over $\mathbb{C}$. We also obtain Ax-Lindemann type results for these maps as a corollary, characterizing the bi-algebraic subsets of these maps.

Our motivation comes from the fact that Ax-Schanuel and Ax-Lindemann type results are an important tool in the theory of unlikely intersections, in the context of the Pila-Zannier method.

Στον παππού μου τον Γιωργαλή
και στον πατέρα μου.

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PUBLICATIONS

The first part of this thesis is heavily based on the author's preprint [Pap22]. The second part is practically the preprint [Pap19], parts of which have been submitted for publications.

CONTENTS

1

INTRODUCTION

In recent years there have been significant advances in so called "problems of unlikely intersections". The main goal of this thesis is to provide tools, in particular a certain height bound, that we expect will be needed in establishing some cases of the Zilber-Pink conjecture, which is the most general conjecture stated in the field of unlikely intersections. We begin this introduction by presenting a quick historical overview of the field of unlikely intersections. We then present a summary of the general strategy introduced in [PZ08] by J.Pila and U.Zannier, through which many results in the field have been resolved. After that, we proceed to summarize the work that motivated our study, namely the results in [DO21c, DO21b, DO21a], where C.Daw and M.Orr establish unconditionally some cases of the Zilber-Pink conjecture for $\mathcal{A}_g$, the moduli space of principally polarized abelian varieties. Finally, we end this introduction with an overview of this thesis.

1.1 PROBLEMS OF UNLIKELY INTERSECTIONS

One could say that the starting point of "unlikely intersections" is none other than the simple observation that if we were to consider two irreducible varieties X and Y of dimensions r and s respectively that are contained in some ambient variety Z of dimension n, then it is natural to expect that the irreducible components of the intersection $X \cap Y$ will have dimension $r+s-n$. This can be seen, for example, by "counting the conditions" that define each of X and Y as subvarieties of Z. In particular, is is reasonable to expect that looking at random X and Y as above, their intersection will be empty if $r+s-n<0$, or equivalently if the codimension $\operatorname{codim}(Y)=n-s$ of Y is strictly larger than the dimension of X.

With this observation in mind, an intersection of two such varieties X and Y as above is referred to as **atypical** whenever $X \cap Y$ has some irreducible component of dimension $> r+s-n$. Moreover, such intersections are referred to as **unlikely** whenever $X \cap Y$ has some positive dimensional irreducible component and $0 > r+s-n$.

Let us consider a classic example, by letting X and Y be two irreducible curves inside the projective plane $\mathbb{P}^2$. Then, Bezout's Theorem, see Corollary 1.8 page 71 of [Sha13], implies that $X \cap Y \neq \emptyset$. It is also easy to see that, for dimension reasons, unless the curves X and Y coincide, this intersection will in fact be a finite set. In other words, two curves in $\mathbb{P}^2$ "typically" intersect at only finitely many points.

The general context of problems of unlikely intersections is to fix a general variety X as above and allow Y to vary in some countable set $\mathcal{Y}$ of subvarieties of dimension r of the fixed ambient space Z. The sets in $\mathcal{Y}$ are usually interesting from an arithmo-theoretic point of view and are referred to as the "special" sets. It is then expected that there should not be "too many" elements $Y \in \mathcal{Y}$ for which the intersection $X \cap Y$ is unlikely unless the variety X is, in a way, "special" itself.

Lang's Conjecture

Perhaps the earliest example of such a problem is Lang's conjecture. Lang's conjecture deals with the case where the ambient variety Z is some complex torus $\mathbb{G}^n_{m,\mathbb{C}}$ and the "special subvarieties" are torsion cosets, meaning subvarieties of the form ζG where G is some irreducible algebraic subgroup of $\mathbb{G}^n_{m,\mathbb{C}}$ and ζ is a torsion point of $\mathbb{G}^n_{m,\mathbb{C}}$.

In this context Lang considered, in the above notation, intersections of a curve X in $\mathbb{G}^2_{m,\mathbb{C}}$ with the set $\mathcal{Y}$ of torsion points of this group. In our notation we also have $r + s - n = 1 + 0 - 2 = -1$, so any torsion point that is in the curve X is an unlikely intersection. Lang's conjecture, proven by Ihara, Serre, and Tate[Lan65], in this context can be rephrased as the following theorem.

Theorem 1.1 (Lang's Conjecture)**.** *Let $X \subset \mathbb{G}^2_{m,\mathbb{C}}$ be an irreducible curve. Then X contains infinitely many torsion points if and only if it is a torsion coset.*

The natural generalization of this problem to $n \geq 2$, also known as the Multiplicative Mordell-Lang conjecture, was established by Laurent in [Lau84].

Theorem 1.2 (Multiplicative Mordell-Lang)**.** *Let $X \subset \mathbb{G}^n_{m,\mathbb{C}}$ be a subvariety and let Σ_X be the set of torsion points in X. Then Σ_X is Zariski dense in X if and only if X is a finite union of torsion cosets.*

The above result also has an alternative, and equivalent, formulation, in a sense "removing" the variety X from the statement.

Theorem 1.3 (MML-Alternative formulation)**.** *Let $\Sigma \subset \mathbb{G}^n_{m,\mathbb{C}}$ be a set of torsion points in $\mathbb{G}^n_{m,\mathbb{C}}$. Then the Zariski closure of Σ is a finite union of torsion cosets.*

The study of problems of unlikely intersections where the set $\mathcal{Y}$ comprises of 0-dimensional subvarieties of the ambient space Z, commonly referred to as the "special points" of the ambient space Z, translates well to other settings as well.

The Manin-Mumford Conjecture

In the case where Z is an abelian variety the set $\mathcal{Y}$ of special points is the set of torsion points of the abelian variety in question, defined in some algebraic closure of its field of definition. In this case the natural analog of the Multiplicative Mordell-Lang conjecture is the Manin-Mumford conjecture established originally by M.Raynaud in [Ray83].

Raynaud's result may be restated, using the above vocabulary in the following form:

Theorem 1.4 (Manin-Mumford-Conjecture)**.** *Let Z be an abelian variety defined over a field K of characteristic 0 and let $\Sigma \subset Z(\bar{K})$ be a set of torsion points of Z. Then the Zariski closure of Σ is a finite union of torsion cosets of Z.*

Here once again the torsion cosets of Z play the role of "general" special subvarieties of Z and are defined as in the case of $\mathbb{G}_m^n$. Namely they are subvarieties of the form $P + B$ where P is a torsion point of Z and B is an abelian subvariety of Z.

The André-Oort Conjecture

In the case where the ambient space Z is a Shimura variety one can associate a countable collection $\mathcal{S}_Z$ of "special subvarieties", known as subvarieties of Hodge type, see [Moo98, Edi01]. The zero-dimensional ones are once again referred to as special points.

In general the description of special points is technical. In the case where the Shimura variety in question is either $\mathcal{A}_g$ or of the form $Y(1)^n$ these are easier to describe. Namely, when $Z = \mathcal{A}_g$ the special points are the $x \in Z$ that correspond to an abelian variety with complex multiplication, or CM for short. In a similar manner, when $Z = Y(1)^n$ is a cartesian product of modular curves the special points are the points $x = (x_i) \in Z$ all of whose coordinates correspond to CM elliptic curves.

In this context the analog of the above conjectures of Manin-Mumford and Mordell-Lang is the André-Oort conjecture and is a combination of conjectures made by Y.André in [And89] and F.Oort in [Oor97]. The conjecture was initially proven conditionally on the Generalized Riemann Hypothesis by B. Klingler and A.Yafaev in [KY14] and then unconditionally by J.Tsimerman in the case of $\mathcal{A}_g$ in [Tsi18] and most recently by J.Pila, A.Shankar, and J.Tsimerman for all Shimura varieties in [PST+21].

Theorem 1.5 (André-Oort conjecture)**.** *Let Z be a Shimura variety and $\Sigma \subset Z$ a set of special points of Z. Then the Zariski closure of Σ is a finite union of special subvarieties.*

The Zilber-Pink conjecture

In all of the above settings one only considers intersections of the variety X with 0-dimensional special subvarieties, the special points in each context. The Zilber-Pink conjecture, in its modern form a combination of conjectures made by B.Zilber in [Zil02] and R.Pink in [Pin05], aims to generalize all of the aforementioned conjectures by describing the expected behavior of all atypical intersections of a given variety X with special sets in the ambient space Z.

To formulate this conjecture formally we start with the following definition, following the exposition in [Pil14].

Definition 1.6. Let Z be either a torus, an abelian variety, or a mixed Shimura variety and let $\mathcal{S}_Z$ be its set of special subvarieties.

Consider X a subvariety of Z. A component A of $X \cap Y$, where $Y \in \mathcal{S}_Z$, is called an atypical subvariety of X if

$$\dim A > \dim X + \dim Y - \dim Z.$$

We also define $\operatorname{atyp}(X) := \bigcup_{Y \in \mathcal{S}_Z} A$ to be the union of all atypical subvarieties of the variety X.

A priori $\operatorname{atyp}(X)$ is a countable union of varieties. The Zilber-Pink conjecture predicts that this union is in fact finite.

Conjecture 1.7 (Zilber-Pink conjecture)**.** *Let Z be as in the previous definition and $X \subset Z$ be a subvariety of Z. Then* $\operatorname{atyp}(X)$ *is a finite union of varieties. Equivalently, the variety X contains finitely many maximal atypical subvarieties.*

While the Zilber-Pink conjecture remains largely open some partial results are known to be true, particularly in the case were X is a curve. In fact when X is a curve and the ambient space Z is a torus, the Zilber-Pink conjecture has been established by work of E.Bombieri, P.Habegger, D.Masser, U.Zannier, and G.Maurin in a series of papers [BMZ99, BMZ08, BHMZ10, Mau08].

For a more detailed introduction to problems of unlikely intersections we direct the interested reader to our primary sources for this chapter, namely [HRS+17, Pil14, Pil15, Zan12].

1.2 THE PILA-ZANNIER METHOD

The most fruitful method in tackling cases of the Zilber-Pink conjecture, and problems of unlikely intersections in general, is the strategy introduced by J.Pila and U.Zannier in [PZ08] where the authors provide with this new method a novel proof of the Manin-Mumford conjecture. We present a short summary of this method here. We keep the notation from the previous paragraph letting Z denote an ambient space such as those in the previous section and X a subvariety of Z defined over some number field K.

The starting point of this method is to consider an analytic uniformization map $\pi : \mathfrak{U} \to Z$ of the space of $\mathbb{C}$-points of the ambient space. For example, when Z is the torus $\mathbb{G}^n_{m,\mathbb{C}}$, the map π is nothing but the usual exponential function, while when Z is an n-dimensional abelian variety π is the analytic uniformization map $\mathbb{C}^n \to \mathbb{C}^n/\Lambda \simeq Z(\mathbb{C})$.

One then looks at the preimage $Y \subset \mathfrak{U}$ of X via π. In all of the examples mentioned in the previous section the preimages of the special points of the ambient space Z are either rational points or algebraic points of bounded degree. For example, when Z is either a torus or an abelian variety these preimages are rational points and when Z is $Y(1)^n$ or some $\mathcal{A}_g$ these preimages are algebraic points, meaning they are points whose coordinates are algebraic numbers.

First step: Point counting via the Pila-Wilkie theorem

The idea of the first step of the method is to then view Y as a real-analytic variety and try to show that it cannot have "too many" such points. This is achieved by restricting our attention to a fundamental domain F for the uniformizing map π. For example, when Z is an abelian variety $F = [0,1)^{2n}$ and when $Z = \mathcal{A}_g$ the fundamental domain F is the Siegel fundamental set, see [PS13]. The goal of this is to employ techniques from the theory of o-minimality to bound the number of algebraic points on the "transcendental part" of $Y \cap F$.

Definition 1.8. Let $T \subset \mathbb{R}^m$. The algebraic part of T, denoted T^{alg}, is the union of all connected semi-algebraic subsets of T of positive dimension. The transcendental part of T is then defined to be the complement $T \backslash T^{alg}$ of the algebraic part of T.

This upper bound is achieved using the powerful Pila-Wilkie counting theorem, that first appeared in [PW06], or one of its generalizations. To state this we need some notation. For $T \subset \mathbb{R}^m$ we let $N(T,h)$ be the cardinality of the set

$$\{P \in T \cap \mathbb{Q}^m : H(P) \leq h\}$$

of rational points in T with height bounded by $h > 0$.

Theorem 1.9 (Pila-Wilkie)**.** *Let $T \subset \mathbb{R}^m$ be a definable set and let $\epsilon > 0$. Then there is a constant $c(T, \epsilon) > 0$ such that*

$$N(T \backslash T^{alg}, h) \leq c(T, \epsilon) h^{\epsilon}.$$

Working with $T = Y \cap F$, in other words focusing on a fundamental domain F, allows us to relate the height $H(P)$ of the preimage of a special point $P \in T$ with natural arithmetic quantities related to the special point $\pi(P)$. For example, in the case where Z is an abelian variety $H(P)$ is the order of the torsion point $\pi(P)$, while in the case where $Z = \mathcal{A}_g$, $H(P)$ is polynomially bounded in $|\operatorname{Disc}(R_{\pi(P)})|$, the discriminant of the ring $Z(\operatorname{End}(A_{\pi(P)}))$, where $A_{\pi(P)}$ is the CM-abelian variety corresponding to the special point $\pi(P)$, see [PT13]. We will refer to this quantity as the **complexity** of the special point $s = \pi(P)$, following the exposition in [DR18], and denote it by $\Delta(s)$.

A crucial step in this process is describing the set T^{alg}. This description rests on results of functional transcendence, dubbed "Ax-Lindemann Theorems" by J.Pila. These results are remarkably surprising given that the aforementioned uniformizing maps π are highly transcendental.

Second step: Galois orbits

The second main step of the strategy consists of showing that the Galois orbits of the special points are "large" in a certain sense. Let us explicate the above statement.

The starting point here is to note that if $s \in X(\mathbb{C})$ is a special point, in all of the settings of the previous section, we have that $P \in X(\bar{K})$, in other words s is a $\bar{K}$-point of X. In that case, we get that every $g \in \operatorname{Gal}(\bar{K}/K)$ acts on $X(\bar{K})$ so that $g(s) \in X(\bar{K})$ and is a special point as well. This way we know that if one special point s is in our variety X then its entire Galois orbit $\operatorname{Gal}(\bar{K}/K) \cdot s$ is also contained in X. Note that to apply this train of thought, our assumption that the variety X is defined over a number field K is necessary.

The idea is then to try to establish a so-called "Large Galois orbits hypothesis". With our notation one could phrase this as follows:

Conjecture 1.10 (Large Galois Orbits Conjecture)**.** *Let s be a special point in the ambient space Z as above. Then there exist positive constants c and d such that*

$$|\operatorname{Gal}(\bar{K}/K) \cdot s| \geq c \Delta(s)^d.$$

We note that the above conjecture has been established in all of the settings mentioned in the previous setting. For example in the case where the ambient space Z is an abelian variety this follows from a result of D.Masser see [Mas84], while in the case where Z is a Shimura variety the result is the central theorem of [Tsi18] when $Z = \mathcal{A}_g$ and in full generality it is the central theorem of [PST+21].

Combining the bounds

Assuming that the variety X contains no proper so called "weakly special subvarieties" the Ax-Lindemann results in each case establish that the aforementioned set T^{alg} is empty. In this case the first step tells us that the number of special points in X with complexity $\leq \Delta$ is bounded above by a quantity of the form Δ^{ϵ}, where $\epsilon > 0$ is allowed to be arbitrarily small.

On the other hand, from the second step we get that if X had one special point s of complexity Δ it would contain at least Δ^d-many such points, where d is now a fixed constant. Since for large enough Δ we have

$$\Delta^d > \Delta^{\epsilon} c_0,$$

for any constant c_0, we conclude that the complexity of the special points in X is bounded! This in turn guarantees that there are only finitely many special points in X, since in all aforementioned cases there are only finitely many special points of bounded complexity in the entire ambient space Z.

Finally, to deduce the full André-Oort conjecture one then has to combine the above ideas with standard geometric arguments. See for example the remarks at the end of [PZ08] or [Tsi18].

Applications of the Pila-Zannier method

The strategy of Pila and Zannier has been extremely successful in establishing various problems of unlikely intersections. Most notably for problems of André-Oort-type, it has been used to establish the André-Oort conjecture for products of the modular curve, in [Pil11], the André-Oort conjecture for $\mathcal{A}_g$, in [Tsi18] following work in [PT13, PT14], and culminating with the proof of the André-Oort conjecture for Shimura varieties in [PST+21].

This method has had some success in establishing Zilber-Pink-type results as well. Indeed, in [HP16, BD19] the Zilber-Pink conjecture is verified when X is a curve defined over $\mathbb{C}$ and the ambient space is an abelian variety. On the front of moduli spaces, the Zilber-Pink conjecture for curves X when the ambient space $Z = Y(1)^n$ is a product of modular curves has been studied in [HP12].

When X is a curve which is not contained in a proper special subvariety of the ambient space Z, one only needs to study intersections of the curve with the set usually denoted by $\mathcal{S}^{[2]}$. This set is defined to be the union of the complex points of all special subvarieties of the ambient space Z of codimension at least 2, in other words

$$\mathcal{S}^{[2]} = \bigcup_{\operatorname{codim}(S)\geq 2} S(\mathbb{C}),$$

where the union runs over the countable set of special subvarieties of codimension ≥ 2.

In [HP12] the authors establish some cases of the Zilber-Pink conjecture. Their main result is the following theorem:

Theorem 1.11 (Theorem 1 of [HP12])**.** *Let X be an irreducible curve defined over $\bar{\mathbb{Q}}$. If X is not contained in a special subvariety of positive codimension and if X is assymetric, then the set $X(\mathbb{C}) \cap \mathcal{S}^{[2]}$ is finite.*

We note that the "assymetricity" of the curve is needed to establish a "Large Galois orbit hypothesis", in the spirit of the second step of the Pila-Zannier strategy mentioned above. We also note that establishing this "Large Galois orbit hypothesis" is the only missing step in proving the conjecture in general in this context, see Lemma 4.2 and preceding discussion in [HP12]..

A short review of Ax-Schanuel

Ax-Schanuel and Ax-Lindemann results play a major role, as already mentioned, in establishing questions in unlikely intersections through the Pila-Zannier strategy. The Ax-Schanuel Theorem for the usual exponential function of complex numbers, originally a conjecture of Schanuel, is due to J. Ax [Ax71]. Ax's proof employs techniques of differential algebra. One of the equivalent formulations of this theorem is the following

Theorem 1.12 (Ax-Schanuel)**.** *Let $y_1, \ldots, y_n \in \mathbb{C}[[t_1, \ldots, t_m]]$ have no constant terms. If the y_i are $\mathbb{Q}$-linearly independent then*

$$tr.d._{\mathbb{C}}\mathbb{C}(y_1, \ldots, y_n, e^{y_1}, \ldots, e^{y_n}) \geq n + \operatorname{rank}\left(\frac{dy_\nu}{dt_\mu}\right)_{\substack{\mu=1,\ldots,m \\ \nu=1,\ldots,n}}.$$

An immediate consequence of the above Ax-Schanuel Theorem is the characterization of all the bi-algebraic subsets of $\mathbb{C}^n$ with respect to the map $\pi : \mathbb{C}^n \to (\mathbb{C}^\times)^n$, given by $(z_1 \ldots, z_n) \mapsto (e^{z_1}, \ldots, e^{z_n})$. In other words it leads to a characterization of the subvarieties $W \subset \mathbb{C}^n$ with the property that $dim_{\mathbb{C}}(\pi(W)) = dim_{\mathbb{C}}(Zcl(\pi(W)))$.

Definition 1.13. A subvariety W of $\mathbb{C}^n$ will be called weakly special, or geodesic, if it is defined by any number $l \in \mathbb{N}$ of equations of the form

$$\sum_{i=i}^{n} q_{i,j} z_i = c_j,\ j = 1, \dots, l,$$

where $q_{i,j} \in \mathbb{Q}$ and $c_j \in \mathbb{C}$.

This characterization of bi-algebraic sets is due to the following result, dubbed Ax-Lindemann by Pila due to its resemblance to Lindemann's theorem,

Theorem 1.14 (Ax-Lindemann)**.** *Let $V \subset (\mathbb{C}^{\times})^n$ be an algebraic subvariety. Then any maximal algebraic subvariety $W \subset \pi^{-1}(V)$ is weakly special.*

Subsequent results in functional transcendence that look to achieve similar results to the above theorem for other transcendental functions have also been dubbed as "Ax-Schanuel" and "Ax-Lindemann" respectively. Ax-Schanuel results are known for affine abelian group varieties, due to J.Ax [Ax72], for semiabelian varieties, due to J. Kirby [Kir09], the j-function, due to J. Pila and J. Tsimerman [PT16], for Shimura varieties, due to N. Mok, J. Pila, and J. Tsimerman [MPT19], for mixed Shimura varieties due to Z. Gao [Gao18]. for variations of Hodge structures, due to B. Bakker and J. Tsimerman [BT17], and for mixed variations of Hodge structures independently by K.Chiu, see [Chi21a, Chi21b], and independently by Z.Gao and B.Klingler [GK21]. Finally, B. Klingler, E. Ullmo, and A. Yafaev [KUY16] have proven an Ax-Lindemann result for any Shimura variety. For more on these notions, along with a proof of Ax-Lindemann as a corollary of Ax-Schanuel, we refer to [Pil15].

1.3 A SUMMARY OF THE WORK OF DAW AND ORR

In recent work, see [DO21c, DO21b, DO21a], C.Daw and M.Orr establish some partial and some conditional cases of the Zilber-Piink conjecture for curves in $\mathcal{A}_g$, the moduli space of principally polarized g-dimensional abelian varieties, building on previous work in [DR18]. Since [DO21c] and [DO21b] were the main motivation behind much of our work we dedicate this section to summarize the results of these papers and put into context the importance of the height bounds we pursue in the main part of this text.

The goal of [DO21c, DO21b] is to establish the Zilber-Pink conjecture for curves in $\mathcal{A}_2$. Using the Pila-Zannier strategy C.Daw and J.Ren in [DR18] manage to reduce the Zilber-Pink conjecture for Shimura varieties to three essential ingredients. The first of these is the so called "hyperbolic

Ax-Schanuel conjecture", which has been settled by N.Mok, J.Pila, and J.Tsimerman in [MPT19].

The other two ingredients, are some hypotheses of arithmetic nature. The first of these hypothesis, and the second main ingredient, is a conjecture similar to the aforementioned 1.10, see Conjecture 11.1 of [DR18]. The third and final ingredient is a series of conjectures aimed at controlling the "complexity" of pre-special subvarieties, a notion made explicit in §10 of [DR18], meaning the preimages of special subvarieties under the analytic uniformization π. These can be further categorized as

1. a hypothesis on the existence of only finitely many special subvarieties of bounded complexity, see Conjecture 10.3 of loc. cit., and

2. two hypotheses, see Conjectures 12.6 and 12.7 in loc. cit, that give on the one hand upper bounds of the degrees of fields associated to special subvarieties and on the other hand upper bounds for the heights of elements of certain lattices.

We note that the first of the aforementioned hypotheses, Conjecture 10.3 in loc. cit., was established recently by D.Urbanik in [Urb21].

Unlikely intersections in $\mathcal{A}_2$

In [DO21c, DO21b] Daw and Orr reduce the validity of the Zilber-Pink conjecture for irreducible algebraic curves X in $\mathcal{A}_2$ that are not contained in any proper special subvariety of the ambient space to a large Galois orbits hypothesis, see Theorem 1.2 of [DO21c] and Theorem 1.3 in [DO21b].

To do this, they use the fact that the Zilber-Pink conjecture can be checked on a case-by-case basis in this context. Indeed, one has to only study intersections of the curve X with three types of special subvarieties of $\mathcal{A}_2$:

1. curves parameterizing abelian surfaces with quaternionic multiplication, which the authors refer to as "quaternionic curves",,

2. curves parameterizing abelian surfaces isogenous to to the square of an elliptic curve, which the authors refer to as "E^2-curves", and

3. curves parameterizing abelian surfaces isogenous to the product of two elliptic curves at least one of which has complex multiplication, which the authors have dubbed "$E \times CM$-curves".

They then proceed to establish the validity of the large Galois orbits hypothesis for certain curves for points of intersection with either $E \times CM$-curves, see Theorem 1.4 of [DO21c], or quaternionic curves, see Theorem

1.4 of [DO21b]. These results can be succinctly summarized by the following:

Theorem 1.15 ([DO21c, DO21b]). *Let X be an irreducible Hodge generic algebraic curve defined over $\bar{\mathbb{Q}}$ such that the Zariski closure of X in the Baily-Borel compactification of $\mathcal{A}_2$ intersects the 0-dimensional stratum of the boundary.*

Then there are only finitely many points of intersection of X with $E \times CM$-curves and quaternionic curves.

The conditions imposed on the curve X in this theorem are a result of the height bound of André used to prove this theorem. André's result could be rephrased as follows:

Theorem 1.16 ([And89]). *Let $f : A \to S$ be an abelian scheme over a curve S defined over a number field K. Assume that the geometric generic fiber A_η is a simple abelian variety of odd dimension $g > 1$ and that the Zariski closure of the image of $S \to \mathcal{A}_g$ intersects the 0-dimensional fiber of the Baily-Borel compactification of $\mathcal{A}_g$.*

Let h denote a Weil height on S and consider the set

$$Ш(S) := \{s \in S(\bar{\mathbb{Q}}) : \mathrm{End}^0(A_s) \not\hookrightarrow M_g(\bar{\mathbb{Q}})\}.$$

Then there exist effectively computable constants $c_1, c_2 > 0$ such that for all $s \in Ш(S)$

$$h(s) \leq c_1[K(s) : K]^{c_2}.$$

In [DO21c] the authors establish a similar height bound, see Theorem 8.1 of loc. cit., in the case when the dimension of the fibers g is even, albeit upon enforcing several assumptions about the "generic Hodge behavior" of this family of abelian varieties. Using their height bound together with the Masser-Wüstholz Isogeny theorem [MW93] and a comparison result between the stable Faltngs height and the Weil height due to Faltings, see proof of Lemma 3 in §3 of [Fal83], the authors establish the large Galois orbit hypothesis they need in the proof of 1.15.

Unlikely intersections in $\mathcal{A}_g$

In [DO21a] C.Daw and M.Orr, building on the work of [DO21c, DO21b] apply roughly the same general strategy as in the aforementioned work to investigate some cases of the Zilber-Pink conjecture in $\mathcal{A}_g$ for $g \geq 3$.

They focus on intersections of a curve $C \subset \mathcal{A}_g$ with some families of special subvarieties that the authors refer to as "special subvarieties of simple PEL type I and II". These special subvarieties essentially parameterize the set Σ of points in $\mathcal{A}_g(\mathbb{C})$ that correspond to simple abelian varieties whose endomorphism algebra is either of type I, and different from $\mathbb{Q}$, or type II in Albert's classification, see 2.24.

In this context, i.e. in the context of intersections of a curve C with the aforementioned families of special subvarieties, the authors reduce the Zilber-Pink conjecture to a "Large Galois orbits hypothesis", see Conjecture 1.5 for this hypothesis. In more detail they prove:

Theorem 1.17. *Let $g \geq 3$ and Σ be as before. If C is a Hodge generic algebraic curve in $\mathcal{A}_g$ and Conjecture 1.5 of loc.cit. holds for the curve C, then $|C \cap \Sigma| < \infty$.*

The main novel ingredient in this work is the establishment of the existence of a good parameter space for the aforementioned special subvarieties. This allows the authors to apply a Pila-Wilkie-type theorem. Using their version of André's height inequality, established in [DO21c], they then establish the aforementioned "Large Galois orbits hypothesis" in certain cases. See, Theorems 1.6 and 8.5 of [DO21a].

1.4 A ZILBER-PINK CONJECTURE FOR VARIATIONS OF HODGE STRUCTURES

In [Kli17] B.Klingler proposes some, even more, far reaching conjectures that are natural analogues of the Zilber-Pink conjecture in the setting of variations of mixed Hodge structures. Klingler's conjectures in fact imply the "classic" Zilber-Pink conjecture, which so far had been formulated for mixed Shimura varieties.

The ultimate motivation behind these conjectures is to study the Hodge locus of a variation of mixed Hodge structures $\mathbb{V}$ defined over a smooth quasi-projective complex algebraic variety S. In short, the Hodge locus $HL(S, \mathbb{V})$ is the points on S that have "more Hodge classes" than expected. This information is captured by the Mumford-Tate group.

In short, to a mixed Hodge structure one can associate a group, called the Mumford-Tate group. In that way, for each point $s \in S(\mathbb{C})$, one gets a Mumford-Tate group corresponding to the Hodge structure $\mathbb{V}_s$. One can then naturally associate, in the case where S is irreducible, a so called "generic Mumford-Tate group" to the variation that is the Mumford-Tate group of most of the points of S. The Hodge locus of the variation $(S, \mathbb{V})$ is then the points in S whose Mumford-Tate group is strictly smaller than the generic one.

Its is known by work of Cattani-Deligne-Kaplan [CDK95], in the case of variations of pure polarized Hodge structures, and Brosnan-Pearlstein-Schnell [BPS10], in the mixed admissible case, that this Hodge locus is in fact a countable union of closed irreducible algebraic subvarieties of S.

As in the case of Shimura varieties one can naturally define a set of special subvarieties of S, or more precisely special subvarieties of $(S, \mathbb{V})$, since the "special nature" of these subvarieties is tied to the information encoded by the variation $\mathbb{V}$. In analogy with the Zilber-Pink conjecture, see 1.7 and preceding discussion, one wants to focus on those special subvarieties that provide "atypical intersections". With that in mind, using Hodge-theoretic language, Klingler defines the notion of an "atypical subvariety" for a given variation of mixed Hodge structures $(S, \mathbb{V})$ and considers the atypical locus $S_{atyp}(\mathbb{V})$ of this variation, which is a subset of the aforementioned Hodge locus $HL(S, \mathbb{V})$, to be the union of all atypical subvarieties of S. His analogue of the Zilber-Pink in this setting is then:

Conjecture 1.18 (Conjecture 1.9, [Kli17]). *Let S be an irreducible smooth quasi-projective complex variety endowed with a variation of mixed Hodge structures $\mathbb{V} \to S$. Then the atypical locus $S_{atyp}(\mathbb{V})$ is a finite union of special subvarieties of $(S, \mathbb{V})$.*

1.5 OUR MAIN RESULTS

Part I: A height bound

The main result of the first part of this thesis, see 1.19, establishes analogues of André's height bound in the setting of certain pure polarized Hodge structures of arbitrary weight. The motivation behind this pursuit of ours was two-fold as can be seen by this introduction. On the one hand, establishing such height bounds seems to the author, as is evident in the work of Daw and Orr, to be the most serious obstacle in establishing a significant amount of cases of the Zilber-Pink conjecture in $\mathcal{A}_g$. Furthermore, such height bounds are, as far as the author's knowledge goes, the only tool successfully used in establishing Large Galois orbits hypotheses needed for Zilber-Pink type problems in the setting of Shimura varieties. On the other hand, Klingler's conjecture seems at the moment to be the most general version of "Zilber-Pink" type, it is the author's hope that these height bounds will play a part in establishing cases of this conjecture in the future.

Our setting: Let K be a number field and let S' be a smooth geometrically irreducible complete curve over K, $\Sigma_S \subset S'(K)$ a finite set of K-point of S', and fix s_0 an element of Σ_S. Let us consider S to be the curve $S' \backslash \Sigma_S$, X a smooth variety over K, and let $f : X \to S$ be a smooth projective morphism that is also defined over K and assume that the dimension of the fibers of f is n.

For each $i \in \{0,\dots,2n\}$ the morphism f defines variations of Hodge structures on the analytification S^{an} of S, namely the variations given by $R^i f_*^{an}\mathbb{Q}_{X_{\mathbb{C}}^{an}} \otimes \mathcal{O}_{S_{\mathbb{C}}^{an}}$. We focus on the variation with $i = n$ and set $\mathbb{V} := R^n f_*^{an}\mathbb{Q}_{X_{\mathbb{C}}^{an}}$. We furthermore assume that there exists a smooth K-scheme X' and a projective morphism $f' : X' \to S'$ such that:

1. f' is an extension of f, and
2. $Y = f^{-1}(s_0)$ is a union of transversally crossing smooth divisors Y_i entering the fiber with multiplicity 1.

Let $\Delta \subset S_{\mathbb{C}}'^{an}$ be a small disk centered at s_0 such that $\Delta^* \subset S_{\mathbb{C}}^{an}$. From work of Katz it is known that the residue at s_0 of the Gauss-Manin connection of the relative de Rham complex with logarithmic poles along Y is nilpotent if we have (2) above. From this it follows, by [Ste76] Theorem 2.21, that the local monodromy around s_0 acts unipotently on the limit Hodge structure $H^n_{\mathbb{Q}-lim}$. By the theory of the limit Hodge structure we then get the weight monodromy filtration $W_\bullet$. We let $h := \dim_{\mathbb{Q}} W_0$.

Our main result, using the above notation, is the following theorem.

Theorem 1.19. *Let S', s_0, and $f : X \to S$ be as above and all defined over a number field K. We assume that the dimension n of the fibers is odd, that the Hodge conjecture holds, and that a good arithmetic model, in the sense of 7.1, exists for the morphism f over $\mathcal{O}_K$.*

For the variation whose sheaf of flat sections is given by $\mathbb{V} := R^n f_*^{an}\mathbb{Q}_{X_{\mathbb{C}}^{an}}$ we assume the following hold true:

1. *the generic special Mumford-Tate group of the variation is $Sp(\mu,\mathbb{Q})$, where $\mu = \dim_{\mathbb{Q}} \mathbb{V}_z$ for any $z \in S^{an}$, and*
2. *$h \geq 2$.*

Let $\Sigma \subset S(\bar{\mathbb{Q}})$ be the set of points for which the decomposition $\mathbb{V}_s = V_1^{m_1} \oplus \cdots \oplus V_r^{m_r}$ of $\mathbb{V}_s$ into simple polarized sub-$\mathbb{Q}$-HS and the associated algebra $D_s := M_{m_1}(D_1) \oplus \cdots \oplus M_{m_r}(D_r)$ of Hodge endomorphisms are such that:

1. *s satisfies condition $\star$ in 8.1, and either*
2. *$h > \frac{\dim_{\mathbb{Q}} V_j}{[Z(D_j):\mathbb{Q}]}$ for some j, or*
3. *there exists at least one D_i that is of type IV in Albert's classification and $h \geq \min\{\frac{\dim_{\mathbb{Q}} V_i}{[Z(D_i):\mathbb{Q}]} : i$ such that $D_i = \mathrm{End}_{HS}(V_i)$ is of type IV $\}$.*

Then, there exist constants C_1, $C_2 > 0$ such that for all $s \in \Sigma$ we have

$$h(s) \leq C_1[K(s) : K]^{C_2},$$

where h is a Weil height on S'.

Remark 1.20. We note that CM-points of the variation will be in the set Σ of this 1.19. We can also create concrete examples of possible algebras of Hodge endomorphisms for which the conditions that guarantee $s \in \Sigma$ above can be checked fairly easily, once we have information on the weight monodromy filtration defined by the local monodromy around the point of degeneration s_0. We return to this issue in 10.1.

Part II: Ax-Schanuel for linear algebraic groups

In the second part of this thesis we study questions related to the functional transcendence of exponential functions of $n \times n$ matrices over $\mathbb{C}$. To be more precise, motivated by the exposition in [Pil15], we prove Ax-Schanuel and Ax-Lindemann type Theorems for the exponential function of upper triangular matrices, as well as the exponential function of general $n \times n$ matrices over $\mathbb{C}$. We have divided our exposition into two parts dealing with each of the cases separately.

In the most general case we will consider the exponential function $E : \mathfrak{gl}_n \to GL_n$ over $\mathbb{C}$. The strongest result that we achieve in this case is the following

Theorem 1.21 (Two-sorted Weak Ax-Schanuel for GL_n)**.** *Let $U \subset \mathfrak{gl}_n$ be a bi-algebraic subvariety that contains the origin, let $X = E(U)$, and let $V \subset U$ and $Z \subset X$ be algebraic subvarieties, such that $\vec{0} \in V$ and $I_n \in Z$. If C is a component of $V \cap E^{-1}(Z)$ with $\vec{0} \in C$, then, assuming that C is not contained in any proper weakly special subvariety of U,*

$$\dim_{\mathbb{C}} C \leq \dim_{\mathbb{C}} V + \dim_{\mathbb{C}} Z - \dim_{\mathbb{C}} X.$$

Here the term **component** of a subset $R \subset \mathfrak{gl}_n$ refers to a complex-analytically irreducible component of R, while the term **bi-algebraic** refers to a subvariety U of $\mathfrak{gl}_n$ whose image $E(U)$ under the exponential is such that its Zariski closure $Zcl(E(U))$ satisfies $\dim_{\mathbb{C}}(E(U)) = \dim_{\mathbb{C}}(Zcl(E(U)))$. The **weakly-special** subvarities will be defined later on and are characterized, as we will see, by an Ax-Lindemann-type statement. In that sense, they are naturally defined for the exponential map of matrices, mirroring the definition of weakly special subvarities for other transcendental maps, for more on those we refer to [Pil11].

Remarks 1.22. *1. The restrictions on V, C, and Z, requiring that $\vec{0} \in C$, $\vec{0} \in V$, and $I_n \in Z$ are needed to deal with the existence of positive dimensional connected components in the preimage of I_n. A phenomenon that does not appear in other transcendental maps considered so far in the literature, at least to the knowledge* *of the author. For example, in $\mathfrak{h}_2$ all matrices of the form* $\begin{pmatrix} 2k\pi i & x \\ 0 & 2l\pi i \end{pmatrix}$ *with $k \neq l$ integers are mapped to the identity matrix I_2 via the exponential. We return to this in* **??**.

2. It is worth noting that, in contrast to [Kir09] and [Ax71], our target space, the group GL_n, is no longer a commutative group and that the exponential map is no longer a group homomorphism. We are nevertheless able to extract "functional equations" satisfied by our map, those will be reflected in the weakly special subvarieties.

1.6 ORGANIZATION OF THE THESIS

Part I: Height bounds

We start by reviewing some aspects of the theory of G-functions in 2.3. The method that André uses to obtain his height bounds hinges on two results from the theory of G-functions. First is the fact that among the relative n-periods associated to the morphism $f : X \to S$ there are some that are G-functions. Namely they will be the ones that can be written as $\int_\gamma \omega$ where $\gamma_z \in Im((2\pi i N^*)^n)$ for $z \in \Delta^*$, where Δ^* and N^* are the aforementioned punctured disc and nilpotent endomorphism. The second main result we will need is a result that can be described as a "Hasse principle" for the values of G-functions. This is what will ultimately allow us to extract height bounds.

We then move on in 2.2 where we review some standard facts about the structure of the algebra of Hodge endomorphisms of a Hodge structure. After this, in 2.4 we fix some general notation with the hope of making the exposition easier.

In 3 we address some technical issues that appear later on in our exposition. Namely we consider the isomorphism between algebraic de Rham and singular cohomology for a smooth projective variety Y/k, where k is a subfield of $\bar{\mathbb{Q}}$

$$P^n : H^n_{DR}(Y/k) \otimes_k \mathbb{C} \to H^n(Y^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

The singular cohomology is endowed with a Hodge structure and we consider its algebra of Hodge endomorphisms D. Later on we will want to create splittings of both de Rham and singular cohomology with respect to actions of D on these vector spaces. To do that we will need to have an action of D on $H^n_{DR}(Y/k)$, which a priori we do not. We show that assuming the absolute Hodge conjecture we may base change Y by a finite extension L of k to obtain such an action that will be compatible with the action of D on $H^n(Y^{an}, \mathbb{Q})$ via the isomorphism P^n. We also show that this field extension may be chosen so that its degree is bounded from above only in terms of the dimension $\dim_{\mathbb{Q}} H^n(Y^{an}, \mathbb{Q})$. We believe these results are known to experts in the field, however being unable to find a reference for these arguments we include them here for the sake of completeness.

Our next goal, realized in 4.2, is to describe the trivial relations among those relative n-periods associated to the morphism f which are G-functions. This amounts to describing the polynomials defining the $\bar{\mathbb{Q}}[x]$-Zariski closure of a certain $h \times \mu$ matrix, where x here is a local parameter of S' at the point s_0. This is achieved by a monodromy argument using André's Theorem of the Fixed part.

The next part of our exposition, mainly 5.1, consists of creating relations among the values of the G-functions in question at certain exceptional points that are "non-trivial". That means that these do not come from specializing the trivial relations we described earlier.

The last part of our exposition is dedicated to showing that the relations we created are "global", see 2.3 for the term. To achieve this we need to assume the existence of certain good arithmetic models. We discuss these models in 7.1.

To achieve this we first study the relation between the algebra of Hodge endomorphisms $D_s = \mathrm{End}_{HS}(H^n(X_s^{an}, \mathbb{Q}))$ and the algebra of inertia-invariant endomorphisms of the étale cohomology group $H^n_{\acute{e}t}(\bar{X}_{s,v}, \mathbb{Q}_l)$. In 6.1 we prove that assuming the Hodge conjecture the former algebra naturally injects in the latter.

This forces an interplay between the algebra of Hodge endomorphisms and the endomorphisms of the graded quotients of the monodromy filtration of $H^n_{\acute{e}t}(\bar{X}_{s,v}, \mathbb{Q}_l)$. Taking advantage of this interplay we establish conditions in 8.1 that guarantee the impossibility of the point s being v-adically close to the degeneration s_0. Establishing this rests on the comparison between the local monodromy representation and the representation defined by inertia which follows from the theorem on the Purity of the branch locus. To employ this comparison we need to assume the existence of the arithmetic models of 7.1.

After this we put all the aforementioned ideas together in 9.1. In summary, the relations created in 5.1 are shown to be non-trivial and global, under the aforementioned conditions. Applying the "Hasse Principle" for the values of G-functions, we obtain the height bounds we want.

We finish with a section centered around examples of algebras where 1.19 applies. In particular we study the CM-points of variations satisfying the conditions of 1.19 and establish that these are in fact points of the set Σ.

We have also included an appendix on polarizations. The main result we need about polarizations in the text is a description of the relations they define among the n-periods. This description in the case where the weight of the Hodge structures is 1 already appears in [And89]. The description in the case of arbitrary odd weight is not different at all. We include it in this appendix for the sake of completeness.

Part II: Ax-Schanuel for GL_n

Our exposition is split essentially in two. We start with a short chapter where we restate essentially an immediate corollary of the original Ax-Schanuel theorem for the exponential functions to which we reduce the Ax-Schanuel results we obtain. We then move on in 12 to deal with the exponential of the algebra $\mathfrak{h}_n$ of $n \times n$ upper triangular matrices. This map is more accessible to computations. These computations form the technical part of the reduction from the Ax-Schanuel result in this case to the original Ax-Schanuel Theorem and are presented in 12.1.

We let $E : \mathfrak{h}_n \to U_n$ denote the exponential of $\mathfrak{h}_n$, U_n being the group of upper triangular invertible matrices over $\mathbb{C}$. Let A be an upper triangular matrix with entries in $\mathbb{C}[[t_1, \ldots, t_m]]$. We will denote the field extension of $\mathbb{C}$ that results from adjoining to $\mathbb{C}$ the entries of both matrices A and $E(A)$ by $\mathbb{C}(A, E(A))$. In this case our main result will be

Theorem 1.23 (Weak Ax-Schanuel for U_n)**.** *Let $f_1, \ldots, f_n, g_{i,j} \in \mathbb{C}[[t_1, \ldots, t_m]]$ be power series, where $1 \leq i < j \leq n$. We assume that the f_i do not have a constant term. Let A be the $n \times n$ upper triangular matrix with diagonal $\vec{f}$ and the (i, j) entry equal to $g_{i,j}$. Let $N = \dim_{\mathbb{Q}} \langle f_1, \ldots, f_n \rangle_{\mathbb{Q}}$, then*

$$tr.d._{\mathbb{C}}\mathbb{C}(A, E(A)) \geq N + \operatorname{rank}(J(\vec{f}, \vec{g}; \vec{t})).$$

Here $\langle f_1, \ldots, f_n \rangle_{\mathbb{Q}}$ denotes the linear span of the f_i over $\mathbb{Q}$, while $J(\vec{f}, \vec{g}; \vec{t})$ denotes the $\frac{n(n+1)}{2} \times m$ Jacobian matrix with entries of the form $\partial h_s / \partial t_j$, where h_s, with $1 \leq s \leq \frac{n(n+1)}{2}$, is an ordering of the $g_{i,j}$ and the f_i. The rank of the Jacobian is its rank over the fraction field of $\mathbb{C}[[t_1, \ldots, t_m]]$.

The main idea is that given a matrix $A \in \mathfrak{h}_n$ we are able to canonically choose a basis of generalized eigenvectors for it, based solely on the multiplicities of its eigenvalues. This basis is chosen in such a way that makes it computable in terms of the entries of the matrix A. At the same time we can determine the action of the matrix A in each of its generalized eigenspaces.

The canonical basis and its properties lead us naturally to define the notion of eigencoordinates in 12.1. These will roughly be coordinates describing the generalized eigenspaces of a matrix A along with the action of the matrix A in each of these spaces. Ultimately they will be used in reducing the Ax-Schanuel result to the original Ax-Schanuel Theorem.

After establishing the Ax-Schanuel result we turn towards characterizing the bi-algebraic subvarieties of the exponential map $E : \mathfrak{h}_n \to U_n$ that contain the origin. In the literature for other transcendental maps such subvarieties are referred to as **weakly special**.

To that end we start by defining the weakly special subvarieties of $\mathfrak{h}_n$ that contain the origin in 12.3. Roughly speaking a subvariety $V \subset \mathfrak{h}_n$ that contains the origin will be weakly special if its diagonal coordinates satisfy $\mathbb{Q}-$linear relations, while the other algebraic relations on it come from algebraic relations on the eigencoordinates, i.e. from algebraic relations between generalized eigenvectors and the actions of matrices in V on their generalized eigenspaces.

These expectations are based on two properties of the exponential of a matrix. First, that if v is an f-generalized eigenvector for the matrix A then v is also an e^f-generalized eigenvector for its exponential, the matrix $E(A)$. Secondly, the exponential is a bi-algebraic map between nilpotent and unipotent operators, the inverse being the logarithm. So the nilpotent action defined by A on a generalized eigenspace gets mapped bi-algebraically to the corresponding action of $E(A)$ on the same space.

As a corollary of our Ax-Schanuel result we obtain an Ax-Lindemann-type result. This result will imply that the weakly special sets we define will be exactly the bi-algebraic subsets of $\mathfrak{h}_n$ that contain the origin.

Having finished with the picture in U_n we deal with the same questions of functional transcendence this time for the exponential map $E : \mathfrak{gl}_n \to GL_n$ of the algebra of $n \times n$ matrices over $\mathbb{C}$ in 13.

In this case we generalize the picture we had in $\mathfrak{h}_n$. Instead of a specific canonical basis and eigencoordinates, we introduce the notion of the **data** of a matrix $A \in \mathfrak{gl}_n$. This new notion will effectively have the role that the eigencoordinates had for $\mathfrak{h}_n$.

The data of a matrix A will comprise of the distinct eigenvalues of A, their multiplicities, their generalized eigenspaces, and the nilpotent operators defined by the matrix A on each such generalized eigenspace. Given the number k of distinct eigenvalues and the multiplicity m_i, $i = 1, \dots, k$, of each of them, the rest of the above information, i.e. the generalized eigenspaces and corresponding nilpotent operators, will be parametrized by an affine variety, which we will denote by $W_k(\vec{m})$. With the help of $W_k(\vec{m})$, we shall see that the Ax-Schanuel result for E is reduced to the original Ax-Schanuel Theorem.

Let A be an $n \times n$ matrix with entries in $\mathbb{C}[[t_1, \dots, t_m]]$. As before we will denote the field extension of $\mathbb{C}$ that results from adjoining to $\mathbb{C}$ the entries of the matrices A and $E(A)$ by $\mathbb{C}(A, E(A))$. The result we obtain will then be

Theorem 1.24 (Weak Ax-Schanuel for GL_n). *Let $g_{i,j} \in \mathbb{C}[[t_1, \ldots, t_m]]$ be power series with no constant term, where $1 \leq i, j \leq n$. Let f_i, where $1 \leq i \leq n$, denote the eigenvalues of the matrix $A = (g_{i,j})$. Let us also set $N = \dim_{\mathbb{Q}} \langle f_1, \ldots, f_n \rangle_{\mathbb{Q}}$, then*

$$tr.d._{\mathbb{C}} \mathbb{C}(A, E(A)) \geq N + \operatorname{rank} J((g_{i,j}); \vec{t}).$$

Again the above Ax-Schanuel result leads us to a description of the weakly special subvarieties of $\mathfrak{gl}_n$ that contain the origin. These are defined in detail in 13.1. Roughly speaking these will be subvarieties of $\mathfrak{gl}_n$ that are subject to algebraic relations of the following two types:

1. $\mathbb{Q}$-linear relations on the eigenvalues and
2. algebraic relations on the coordinates of a variety $W_k(\vec{m})$, as above, for some $k \in \mathbb{N}$ and $\vec{m} \in \mathbb{N}^k$, or in other words, relations coming from a subvariety $W \subset W_k(\vec{m})$.

In other words, the relations are either on the eigenvalues, or between the generalized eigenspaces and the corresponding nilpotent operators defined on them. All the while there can be no algebraic relations between eigenvalues and generalized eigenspaces or eigenvalues and nilpotent operators defined on those spaces. Alternatively, we require that the two types of relations considered above do not interfere with one another.

These results for the Lie algebra $\mathfrak{gl}_n$ will imply, as a corollary, Ax-Schanuel and Ax-Lindemann type results for all subalgebras $\mathfrak{g}$ of $\mathfrak{gl}_n$ and their respective exponentials.

Part I

Height bounds

2 BACKGROUND MATERIAL

In this first chapter of the first part of this thesis we have added a summary of some general background material. We start with a quick introduction to Hodge structures. We then have a short summary on the algebras of endomorphisms of Hodge structures, whose properties play a central role in 5. We then proceed with a short introduction to G-functions, reviewing some of the main results of André that we will need.

Finally, we end this first chapter with a short section fixing the notation that we will be using in this first part of the thesis.

2.1 A SHORT INTRODUCTION TO VARIATIONS OF HODGE STRUCTURES

We review here the main concepts that we will need from the theory of Hodge structures. Our main sources are [GGK12, Moo99, Voi07].

Hodge structures

Definition 2.1. A $\mathbb{Q}$-Hodge structure of weight m is a finite dimensional $\mathbb{Q}$-vector space together with a decomposition of the form

$$V_{\mathbb{C}} := V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=m} V_{\mathbb{C}}^{p,q},$$

where $V_{\mathbb{C}}^{p,q}$ are subspaces of the complex vector space $V_{\mathbb{C}}$ that satisfy

$$\overline{V_{\mathbb{C}}^{p,q}} = V_{\mathbb{C}}^{q,p}.$$

One can also define the notion of a $\mathbb{Z}$-Hodge structure, consisting of a finitely generated abelian group V and a decomposition as above of $V_{\mathbb{C}}$, or of an $\mathbb{R}$-Hodge structure.

Remark 2.2. Consider $F^p = \bigoplus_{r \geq p} V_{\mathbb{C}}^{r,s}$. The F^p define a decreasing filtration, often referred to as the Hodge filtration, of $V_{\mathbb{C}}$ and they furthermore satisfy

$$V_{\mathbb{C}} = F^p \oplus \overline{F^{m-p+1}} \text{ and } V_{\mathbb{C}}^{p,q} = F^p \cap \overline{F^{m-p}}.$$

One can alternatively define the notion of a $\mathbb{Q}$-Hodge structure of weight m as the data given by a decreasing filtration F^p of $V_{\mathbb{C}}$ that satisfies the first of the two aforementioned equalities.

The importance of this definition and the motivation behind it come from the structure of the cohomology of compact Kähler manifolds. Indeed, given a compact Kähler manifold we have for all $k \in \mathbb{N}$ a, so called, "Hodge decomposition"

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q},$$

where $\overline{H^{p,q}} = H^{q,p}$. In particular, via the isomorphism $H^k(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^k(X, \mathbb{C})$ we get a $\mathbb{Z}$-Hodge structure of weight k.

It is well known, see [Voi07] and especially Chapter 3, that projective complex manifolds are Kähler. More importantly to us, given X a projective algebraic variety over $\mathbb{C}$, the k-th cohomology group of its analytification X^{an} will carry the additional information of a $\mathbb{Q}$-Hodge structure of weight k.

Example 2.3. One of the most basic examples of a $\mathbb{Z}$-Hodge structure is the Tate structure $\mathbb{Z}(n)$. In this case V is the free $\mathbb{Z}$-module $(2\pi i)^n\mathbb{Z} \subset \mathbb{C}$ and $V_{\mathbb{C}} = V^{-n,-n}$ so that the Tate structure is in a natural way of weight $-2n$.

By tensoring with $\mathbb{Q}$ or $\mathbb{R}$ one can define in the obvious way the $\mathbb{Q}$-Hodge structure $\mathbb{Q}(n)$ or $\mathbb{R}(n)$.

Definition 2.4. Let $(V, V_{\mathbb{C}}^{p,q})$ and $(V', V_{\mathbb{C}}'^{p,q})$ be $\mathbb{Z}$-Hodge structures of the same weight m. Then a morphism of Hodge structures $f : (V, V_{\mathbb{C}}^{p,q}) \to (V', V_{\mathbb{C}}'^{p,q})$ is a homomorphism of abelian groups $f : V \to V'$ such that its base change to $\mathbb{C}$, the linear map $f_{\mathbb{C}} : V_{\mathbb{C}} \to V'_{\mathbb{C}}$, is compatible with the decompositions, i.e. $f_{\mathbb{C}}(V_{\mathbb{C}}^{p,q}) \subset V_{\mathbb{C}}'^{p.q}$.

Through the usual constructions of linear algebra one can use pre-existing Hodge structures to define new ones. Here are some examples:

Examples 2.5. Let V and V' be two $\mathbb{Z}$-Hodge structures of weights n and m respectively.

1. The group $W := \hom(V, V')$ is naturally endowed with a Hodge structure of weight $m - n$. Using the fact that $W_{\mathbb{C}} \simeq \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, V'_{\mathbb{C}})$ it is natural to define the components of the Hodge decomposition via

$$W_{\mathbb{C}}^{a,b} \bigoplus_{r-p=a, s-q=b} \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}^{p,q}, V_{\mathbb{C}}'^{r,s}).$$

From this, one can see, taking $V' = \mathbb{Z}$ as a weight 0 Hodge structure, that the dual $V^{\vee}$ can be endowed with a weight $-n$ Hodge structure.

2. The tensor product $U := V \otimes_{\mathbb{Z}} V'$ carries a Hodge structure via the decomposition with components

$$U_{\mathbb{C}}^{a,b} = \bigoplus_{p+r=a,q+s=b} V_{\mathbb{C}}^{p,q} \otimes_{\mathbb{C}} V_{\mathbb{C}}^{\prime r,s}.$$

This is a Hodge structure of weight $n+m$.

3. (Tate twists) Taking $V' = \mathbb{Z}(i)$ to be the Tate structure of weight $-2i$ the tensor product $V \otimes V'$ is denoted by $V(i)$ and is usually referred to as the Tate twist of V. This will be a Hodge structure of weight $n-2i$.

One pathology, in a way, of the category of $\mathbb{Q}$-Hodge structures is that, even though it is abelian, it is not semisimple, see [Moo99] exercises after Theorem 1.6 for an example. To get a semisimple category one looks at the subcategory of polarizable $\mathbb{Q}$-Hodge structures, In other words one looks at the $\mathbb{Q}$-Hodge structures that also come equipped with a polarization.

Definition 2.6. Let V be a $\mathbb{Q}$-Hodge structure of weight m. A polarization on V is a bilinear $(-1)^m$-symmetric form $Q : V \times V \to \mathbb{Q}$ such that its complexification $Q_{\mathbb{C}}$ satisfies the relations

1. $Q_{\mathbb{C}}(V_{\mathbb{C}}^{p,q}, V_{\mathbb{C}}^{r,s}) = 0$ for $p+r \neq m$, and
2. $i^{p-q} Q_{\mathbb{C}}(V_{\mathbb{C}}^{p,q}, \overline{V_{\mathbb{C}}^{p,q}}) > 0$ for all pairs p,q with $p+q=m$ and $V_{\mathbb{C}}^{p,q} \neq 0$.

A $\mathbb{Q}$-Hodge structure is called **polarizable** if it admits a polarization.

Polarizations appear naturally in geometry. In fact, the cohomology groups $H^k(X, \mathbb{R})$ for a X a compact Kähler manifold are polarizable. This is a consequence of one of the main results in Hodge Theory, the Hard Lefschetz theorem. See [Voi07].

An equivalent definition-The Deligne torus

One can also define Hodge structures equivalently via representations of the Deligne torus $\mathbb{S} = Res_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. This $\mathbb{S}$ is an algebraic group over $\mathbb{R}$, in fact a rank 2 torus, with $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$ and $\mathbb{S}(\mathbb{C}) \simeq \mathbb{C}^* \times \mathbb{C}^*$.

The group of characters of $\mathbb{S}$ is generated by two characters, usually denoted by z and $\bar{z}$, defined by $z : \mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C}) \to \mathbb{G}_m(\mathbb{C})$ being the identity and $\bar{z} : \mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C}) \to \mathbb{G}_m(\mathbb{C})$ coinciding with complex conjugation, under the identification $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$. Of importance is also the so called "weight co-character", i.e. the co-character w of the $\mathbb{R}$-algebraic group $\mathbb{S}$ given on the level of $\mathbb{R}$-points by the inclusion $\mathbb{G}_{m,\mathbb{R}}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{R})$.

One can then give the following definition.

Definition 2.7. A $\mathbb{Q}$-Hodge structure(resp. $\mathbb{Z}$-Hodge structure) of weight m consists of a $\mathbb{Q}$-vector space V(resp. a finitely generated abelian group V) and a homomorphism of $\mathbb{R}$-algebraic groups

$$\tilde{\phi} : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}}),$$

such that $\tilde{\phi} \circ w(r) = r^n \operatorname{id}_{V_{\mathbb{R}}}$ for all $r \in \mathbb{R}^*$.

Remark 2.8. General $\mathbb{Q}$-Hodge structures, i.e. not of fixed weight, can be defined as representations of $\mathbb{S}$ as above for which the homomorphism $\tilde{\phi} \circ w$ is defined over $\mathbb{Q}$.

Throughout our exposition we will be working with $\mathbb{Q}$-Hodge structures of a fixed weight, so called "**pure Hodge structures**", since our Hodge structures will come from geometry as cohomology groups of the analytifications of certain projective varieties.

To get from this new definition to the original definition of a Hodge structure one defines

$$V_{\mathbb{C}}^{p,q} = \{v \in V_{\mathbb{C}} : \tilde{\phi}(z)v = z^p \bar{z}^q v,\ \forall z \in \mathbb{S}(\mathbb{R})\}.$$

To recover this new definition from the original, one starts by defining an action of $\mathbb{S}_{\mathbb{C}}$ on $V_{\mathbb{C}}$, defining it first on each $V_{\mathbb{C}}^{p,q}$ via $\tilde{\phi}_{\mathbb{C}}(z_1, z_2)v = z_1^p z_2^q v$ for $v \in V_{\mathbb{C}}^{p,q}$, in other words $\phi_{\mathbb{C}}$ is such that $V_{\mathbb{C}}^{p,q}$ is the character space for the character $z_1^p z_2^q$ of $\mathbb{S}_{\mathbb{C}}$. One then needs a descent argument, which follows from the structure of the characters z and $\bar{z}$ of the group $\mathbb{S}$, and the Hodge relations $\overline{V^{p,q}} = V^{q,p}$, to show that the representation $\mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$ descends to $\mathbb{R}$. For more on this see [GGK12, Moo04].

In the case of fixed weight polarized Hodge structures it turns out to be more practical to work with representations of the algebraic subgroup of $\mathbb{S}$ defined by

$$\mathbb{U}_1 := \ker(Nm),$$

where $Nm : \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}$ is the character defined by $Nm := z\bar{z}$. Note that $\mathbb{U}_1(\mathbb{R}) \simeq S^1$ and that $\mathbb{U}_1$ is a 1-dimensional $\mathbb{R}$-algebraic torus.

Given a $\tilde{\phi}$ as above we get a representation $\phi := \tilde{\phi}|_{\mathbb{U}_1} : \mathbb{U}_1 \to \mathrm{GL}(V_{\mathbb{R}})$. For $v \in V_{\mathbb{C}}^{p,q}$, as above, we will have that $\phi(z)v = z^{p-q}v$ for $z \in \mathbb{U}_1(\mathbb{C})$ which we can identify with $\{z, z^{-1} : z \in \mathbb{C}\}$ as a subset of $\mathbb{S}(\mathbb{C}) \simeq (\mathbb{C}^*)^2$.

If one were to work with Hodge structures of a certain fixed weight m, studying representations of $\mathbb{U}_1$ turns out to be equivalent to studying representations of $\mathbb{S}$ that correspond to Hodge structures of the fixed weight m. For more on this see [GGK12] Chapter I.

We note that this new more abstract definition of Hodge structures is more flexible to work with. A lot of notions central to Hodge structures are much more easily defined via representations of the Deligne torus. One could, for example, rephrase the above definition of morphism of Hodge structures as follows:

Definition 2.9. A morphism of $\mathbb{Q}$-Hodge structures (V, ϕ) and (W, ϕ') of pure weight n is a homomorphism $f : V \to W$ such that

$$f_{\mathbb{R}}(\phi(z)v) = \phi'(z)f_{\mathbb{R}}(v), \text{ for all } v \in V_{\mathbb{R}}.$$

Variations of Hodge structures

Variations of Hodge structures are, roughly speaking, families of Hodge structures varying over a space of parameters S, which will be a complex manifold. These appear naturally in geometry as the local systems $R^k f_* \mathbb{Q}_X$ where $f : X \to S$ is a family of compact Kähler manifolds, meaning f is a proper submersion whose fibers are compact Kähler manifolds.

Definition 2.10. A variation of $\mathbb{Q}$-Hodge structures of weight m, or $\mathbb{Q}$-VHS for short, on the complex manifold S is given by a pair $(\mathbb{V}, \mathcal{F}^\bullet)$ where $\mathbb{V}$ is local system of $\mathbb{Q}$-vector spaces of finite rank on S and $\mathcal{F}^\bullet$ is a decreasing filtration of the vector bundle $\mathcal{V} = \mathbb{V} \otimes_{\mathbb{Q}} \mathcal{O}_S$ by holomorphic vector sub-bundles, such that the following hold:

1. $\forall s \in S$, the fibers $\mathcal{F}_s^\bullet$ endow the fiber $\mathbb{V}_s$ with a $\mathbb{Q}$-HS of weight m, and
2. (Griffiths' transversality) for every i we have that
$$\nabla(\mathcal{F}^i) \subset \Omega_S^1 \otimes \mathcal{F}^{i-1}.$$

Furthermore, we say that the above $\mathbb{Q}$-VHS is polarizable if it admits a polarization, meaning a bilinear form $Q : \mathbb{V} \times \mathbb{V} \to \mathbb{Q}(-m)_S$, such that for all $s \in S$ the fiber of Q on s defines a polarization of the $\mathbb{Q}$-HS on $\mathbb{V}_s$.

The connection ∇ that appears in the above definition is the Gauss-Manin connection. Its existence is based on the following:

Theorem 2.11. *Let S be a complex manifold. There is an equivalence of categories between the category of local systems of $\mathbb{C}$-vector spaces over S and holomorphic vector bundles equipped with a flat holomorphic connection.*

We sketch the related constructions. First of all, given a local system $\mathbb{V}$ of $\mathbb{C}$-vector spaces on S one gets a holomorphic vector bundle $\mathcal{V} = \mathbb{V} \otimes_{\mathbb{Q}} \mathcal{O}_S$ over S. One can then equip the bundle $\mathcal{V}$ with a flat holomorphic connection.

Definition 2.12. Let $\mathcal{V}$ be a holomorphic vector bundle on the complex manifold S. A connection on $\mathcal{V}$ is an additive morphism of sheaves

$$\mathcal{V} \to \mathcal{V} \otimes \Omega_S^1$$

that also satisfies Leibniz's rule, meaning that for all sections σ of $\mathcal{V}$ and section f of $\mathcal{O}_S$ we have that

$$\nabla(f\sigma) = f\nabla(\sigma) + \sigma \otimes df.$$

The connection ∇ is called **flat** if we have that $\nabla \circ \nabla = 0$.

In the above situation, i.e. given a local system $\mathbb{V} \subset \mathcal{V}$, one can construct connections locally on S and then glue them to obtain a global connection ∇. The idea is that the local system will be locally "killed" by the connection, so let us consider a local trivialization σ_i, $1 \leq i \leq \dim_{\mathbb{C}} \mathbb{V}_s = \mu$, of $\mathbb{V}$. Then locally a section of $\mathcal{V}$ will be of the form $\sigma = \sum_{i=1}^{\mu} f_i\sigma_i$ where f_i are local sections of $\mathcal{O}_S$. Then one sets

$$\nabla(\sigma) = \sum_{i=1}^{\mu} \sigma_i \otimes df_i.$$

Once one glues these connections to a global such object for $\mathcal{V}$ one gets a pair $(\mathcal{V}, \nabla)$. The flatness of the connection follows from the relation $d \circ d = 0$.

To invert this process one associates to a pair $(\mathcal{V}, \nabla)$ a local system $\mathbb{V}$ such that $\mathcal{V} = \mathbb{V} \otimes \mathcal{O}_S$. This is done by considering the so called "**flat sections**" of $\mathcal{V}$ with respect to the connection ∇. These will be the sections annihilated by ∇. To see that these define an actual local system one needs to argue that vector bundles equipped with a flat connection are locally trivializable. This follows essentially from our ability to to find locally solutions of ordinary differential equations with an initial value, for more on this see [CMSP17] Appendix C.4, or the discussion in Chapter 9 of [Voi07].

Remark 2.13. One can define $\mathbb{Z}$-VHS or $\mathbb{R}$-VHS in an analogous manner to our definition of $\mathbb{Z}$-HS or $\mathbb{R}$-HS.

Griffiths's transversality condition in the definition of a $\mathbb{Q}$-VHS is a condition aimed at controlling the infinitesimal behavior of the variation. Once again its inclusion to the definition is motivated from geometry, where this phenomenon was first noticed by Griffiths. See the discussion in section 4.5 of [CMSP17] for a proof of this.

Geometric variations of Hodge structures

As already noted the notions of a Hodge structure and of a variation of Hodge structures are inspired from geometry. We review here a few of the main properties of geometric variations.

Let us fix a smooth projective morphism $f : X \to S$ where S is a smooth quasi-projective variety, with f, X and S defined over some subfield K

of $\mathbb{C}$. Base changing to $\mathbb{C}$ then considering the analytications of the map $f_{\mathbb{C}}$ and the varieties $X_{\mathbb{C}}$ and $S_{\mathbb{C}}$ one gets naturally defined $\mathbb{Q}$-variations of Hodge structures on $S_{\mathbb{C}}^{an}$ constructed from the holomorphic map $f_{\mathbb{C}}^{an}$. Indeed, one has the locally constant sheaves $R^k(f_{\mathbb{C}}^{an})_*\mathbb{Q}_{X_{\mathbb{C}}^{an}}$ given by the higher direct image functors.

For these it is well known that, due to the properness of the map f, the fibers at points $z \in S_{\mathbb{C}}^{an}$ are naturally isomorphic with the singular cohomology $H^k((X_{\mathbb{C}}^{an})_z, \mathbb{Q})$ of the fiber of $f_{\mathbb{C}}^{an}$ at the point z.

More importantly for us, it turns out that the Hodge filtration on the vector bundle $\mathcal{V} := R^k(f_{\mathbb{C}}^{an})_*\mathbb{Q}_{X_{\mathbb{C}}^{an}}$ can be defined algebraically. To do this one has to instead work with relative de Rham cohomology. We paint a rough picture of these arguments in the next subsection.

de Rham and relative de Rham

Given a smooth projective n-dimensional variety Y over a field K, which for simplicity we assume is contained in $\mathbb{C}$, one has the **de Rham complex** $(\Omega_Y^\bullet, d)$, where $\Omega^1_{Y/K}$ is the sheaf of Kähler differentials and $\Omega^p_{Y/K} := \wedge^p\Omega^1_{Y/K}$ is the sheaf of differential p-forms. The relation $d \circ d = 0$ then defines a complex

$$0 \to \mathcal{O}_Y \xrightarrow{d} \Omega^1_{Y/K} \to \ldots \xrightarrow{d} \Omega^n_{Y/K} \xrightarrow{d} 0.$$

Definition 2.14. The de Rham cohomology groups of Y are defined to be the hypercohomology groups of the de Rham complex, in other words

$$H^k_{DR}(Y/K) := \mathbb{H}^k(Y, \Omega^\bullet_{Y/K}).$$

In our case, Y projective smooth, the de Rham cohomology groups are in fact K-vector spaces. Furthermore, the construction of de Rham cohomology is functorial. For example it behaves well under base changes of the form $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ where L/K is a field extension. In particular, we have that $H^i_{DR}(Y_{\mathbb{C}}/\mathbb{C})$ is naturally isomorphic with $H^i_{DR}(Y/K) \otimes_K \mathbb{C}$, where naturality here refers to a fixed inclusion $K \to \mathbb{C}$.

One can then define the so called "naive filtration". In summary one considers the cochain complex $\sigma_{\geq p}\Omega^\bullet_{Y/K}$ that is the result of taking the de Rham complex and replacing all $\Omega^i_{Y/K}$ for $i \leq p-1$ by the 0 sheaf. We note that this complex comes equipped with a natural map to the de Rham complex. One then has the following:

Definition 2.15. The Hodge filtration in de Rham cohomology is defined as

$$F^p H^k_{DR}(Y/K) := Im(\mathbb{H}^k(Y, \sigma_{\geq p}\Omega^\bullet_{Y/K}) \to H^k_{DR}(Y/K)).$$

In this context, it turns out that the $F^pH^k_{DR}(Y/K)$ are K-vector subspaces of $H^k_{DR}(Y/K)$ that define a decreasing filtration of this cohomology group.

Using Serre's GAGA, the analytic de Rham complex coincides with the analytification of the algebraic de Rham complex, after we base change everything to $\mathbb{C}$ via a fixed inclusion $K \hookrightarrow \mathbb{C}$. In other words we have isomorphisms $H^i_{DR}(Y/K) \otimes_K \mathbb{C} \to H^i_{DR}(Y^{an}_{\mathbb{C}}/\mathbb{C})$ between analytic and algebraic de Rham cohomology.

By the holomorphic Poincaré lemma, that says that the holomorphic de Rham complex $\Omega^\bullet_{Y^{an}_{\mathbb{C}}/\mathbb{C}}$ is a resolution of the constant sheaf $\mathbb{C}_{Y^{an}_{\mathbb{C}}}$, one obtains isomorphisms $H^i_{DR}(Y^{an}_{\mathbb{C}}/\mathbb{C}) \to H^i(Y^{an}_{\mathbb{C}}, \mathbb{C})$. Combining the above one gets isomorphisms

$$H^i_{DR}(Y/K) \otimes_K \mathbb{C} \to H^i(Y^{an}_{\mathbb{C}}, \mathbb{C}),$$

through which the image of the $F^pH^k_{DR}(Y/K) \otimes_K \mathbb{C}$ give rise to the Hodge filtration of $H^k(Y^{an}_{\mathbb{C}}, \mathbb{C})$.

This comparison isomorphism is known as the "algebraic de Rham to singular cohomology" or "Grothendieck's comparison isomorphism". For more on this see the very detailed [EZT14].

The relative case

The above construction has a "relative avatar", in the sense that one can instead of the smooth projective morphism $Y \to \mathrm{Spec}(K)$ study the smooth projective morphism $X \to S$, where as above for simplicity we let S be a quasiprojective variety defined over K. We present here a fairly short version of the whole story, for proofs and more on relative de Rham cohomology see [Sta22] and [Kat70, KO68].

One starts by considering the relative de Rham complex, $\Omega^\bullet_{X/S}$ which is defined analogously by letting $\Omega^k_{X/S} := \wedge^k \Omega^1_{X/S}$.

Definition 2.16. We define the relative de Rham cohomology sheaves of X/S to be

$$H^i_{DR}(X/S) := \mathbb{R}^i f_* \Omega^\bullet_{X/S}.$$

These de Rham sheaves are in fact coherent $\mathcal{O}_S$-modules. As in the non-relative case, Serre's GAGA and functoriality of the construction give isomorphisms

$$\begin{gathered} H^i_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}_{\mathbb{C}}} \simeq H^i_{DR}(X_{\mathbb{C}}/S_{\mathbb{C}}) \otimes_{\mathcal{O}_{S_{\mathbb{C}}}} \mathcal{O}_{S^{an}_{\mathbb{C}}} \simeq \\ \simeq (\mathbb{R}^i(f_C)*\Omega^\bullet_{X_{\mathbb{C}}/S_{\mathbb{C}}})^{an} \simeq \mathbb{R}^i(f^{an}_{\mathbb{C}})_*\Omega^\bullet_{X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}}. \end{gathered}$$

Similar to the non-relative case one can define the filtration algebraically.

Definition 2.17. We define the algebraic filtration sheaves of relative de Rham cohomology to be the $\mathcal{O}_S$-modules

$$\mathcal{F}^p_{alg} H^k_{DR}(X/S) := \mathbb{R}^k f_*(\sigma_{\geq p}\Omega^\bullet_{X/S}),$$

where $\sigma_{\geq p}\Omega^\bullet_{X/S}$ denotes the relative avatar of the naive filtration.

In the holomorphic world it is well known, see [Voi07] Chapter 10, that the Hodge bundles

$$\mathcal{F}^p H^k_{DR}(X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}) := \mathbb{R}^k (f^{an}_{\mathbb{C}})_*(\sigma_{\geq p}\Omega^\bullet_{X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}}),$$

actually define the Hodge filtration fiberwise, under the isomorphism

$$H^k_{DR}(X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}) \xrightarrow{\simeq} R^k (f^{an}_{\mathbb{C}})_*\mathbb{Q}_{X^{an}_{\mathbb{C}}} \otimes \mathcal{O}_{S^{an}_{\mathbb{C}}}.$$

We thus get that the analytification of the algebraic filtration sheaves $\mathcal{F}^p_{alg} H^k_{DR}(X/S)$ are in fact the Hodge bundles, due to functoriality and Serreś GAGA. As a result we get that the local systems $R^k(f^{an}_{\mathbb{C}})\mathbb{Q}$ satisfy the first part of the definition of a $\mathbb{Q}$-VHS, i.e. we have a filtration as we wanted. More importantly, this filtration is in fact defined over the field K, the field of definition of the variety S and morphism f.

What one is then left with is to establish Griffith's transversality which one can prove via arguments in differential geometry, see [Voi07] Proposition 10.12.

An important fact in this area of ideas is that the Gauss-Manin connection ∇ can in fact be defined over the field K as well. This is due to N. Katz and T. Oda, see [KO68]. In summary, they start with the exact sequence of sheaves on X

$$0 \to f^*\Omega^1_{S/K} \to \Omega^1_{X/K} \to \Omega^1_{X/S} \to 0.$$

The complex of Kähler differentials on the variety X then has a canonical decreasing filtration given by the complexes

$$L^i(\Omega^\bullet_{X/K}) := Im(\Omega^{\bullet-i}_{X/K} \otimes_{\mathcal{O}_X} f^*(\Omega^i_{S/K}) \to \Omega^\bullet_{X/K}).$$

We then end up with the short exact sequence of complexes

$$0 \to L^1/L^2 \to \Omega^\bullet_{X/K}/L^2 \to L^0/L^1 \to 0.$$

But $L^1/L^2 = f^*(\Omega^1_{S/K}) \otimes \Omega^{\bullet-1}_{X/S}$ and $L^0/L^1 = \Omega^\bullet_{X/S}$. In other words, one has

$$0 \to f^*(\Omega^1_{S/K}) \otimes \Omega^{\bullet-1}_{X/S} \to \Omega^\bullet_{X/K}/L^2 \to \Omega^\bullet_{X/S} \to 0,$$

considering the long exact sequence, that one gets from this short exact sequence via the hyper-derived functors $\mathbb{R}^q f_*$, one gets connecting homomorphisms

$$H^k_{DR}(X/S) \to \Omega^1_{S/K} \otimes_{\mathcal{O}_S} H^k_{DR}(X/S).$$

It turns out, see section 3 of [KO68], that these connecting homomorphisms coincide with the classical Gauss-Manin connection after analytification.

Hodge classes and absolute Hodge classes

As we saw earlier, the $\mathbb{Q}$-VHS that arise from algebraic geometry, i.e. from smooth projective morphisms $f : X \to S$, have structures intimately related to the geometry of the original varieties.

This relation is best captured by Grothendieck's comparison isomorphism in its relative form

$$H^k_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}_{\mathbb{C}}} \to R^k(f^{an}_{\mathbb{C}})_* \mathbb{Q}_{X^{an}_{\mathbb{C}}} \otimes_{\mathbb{Q}_{S^{an}_{\mathbb{C}}}} \mathcal{O}_{S^{an}_{\mathbb{C}}}.$$

Essentially by the functoriality of its construction, which we summarized earlier, this comparison isomorphism respects cohomology constructions between the "de Rham side" and the "Betti side".

In the non-relative setting this comparison isomorphism has another important property as a byproduct of its construction, it sends elements in the image of the cycle-class map of the algebraic de Rham cohomology to elements that are in the image of the cycle-class map of the Betti cohomology. These algebraic cycles, in the case where $k = 2p$ is an even number, are a special case of the notion of "Hodge classes".

Definition 2.18. Let (V, ϕ) be a $\mathbb{Q}$-HS of even weight $2p$. A Hodge class of this Hodge structure is an element of the set $V^{p,p}_{\mathbb{C}} \cap V$. In other words Hodge classes are vectors in V that are only of type (p, p) in the Hodge decomposition $V_{\mathbb{C}} = \oplus V^{r,s}_{\mathbb{C}}$.

We denote the set of Hodge classes by $Hdg(V)$ or by $Hdg^p(X)$, when $V = H^{2p}(X^{an}, \mathbb{Q})$ is the Hodge structure underlied by the $2p$-th singular cohomology of the analytification of a smooth projective variety X over $\mathbb{C}$.

Remark 2.19. Noting that if V is as in the definition, the Tate twist $V(p)$ is a weight 0 Hodge structure one can alternatively define Hodge classes as elements of $V(p) \cap (V(p))^{0,0}_{\mathbb{C}}$.

In the case of cohomology we get that all cycle classes in $H^{2p}(Y^{an}, \mathbb{Q})$ are in fact Hodge classes, see [CS14] for a proof, where Y is a projective variety over $\mathbb{C}$. The Hodge conjecture predicts that there are practically no other Hodge classes other than those that come from geometry.

Conjecture 2.20 (The Hodge Conjecture)**.** *Let Y be a projective variety over $\mathbb{C}$. Then for all $p \geq 0$ the subspace $Hdg^p(Y)$ of rational Hodge classes is generated by the cohomology classes of codimension p subvarieties of Y.*

The Hodge conjecture is extremely far-reaching and very few cases of it are known to hold. See [Voi16] for more details. Given the difficulty of this conjecture P. Deligne introduced the notion of "absolute Hodge classes". This notion is motivated by the good behavior of cycles of algebraic subvarieties with respect to "base changing by automorphisms of $\mathbb{C}$".

In particular, let us consider $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $Y/\mathbb{C}$ a smooth projective variety. One can then consider the, so called, conjugate variety Y^σ given by base changing by σ, i.e. we have

$$\begin{tikzcd} Y^\sigma \arrow[r, "\sigma^{-1}"] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "\sigma^*"] & \mathrm{Spec}(\mathbb{C}). \end{tikzcd}$$

The σ^{-1} are scheme isomorphisms and, as a result, induce isomorphisms, via pullback of forms, of the de Rham cohomology groups, since these were defined algebraically. In other words we have isomorphisms

$$(\sigma^{-1})^* : H^k_{DR}(Y//C) \to H^k_{DR}(Y^\sigma/\mathbb{C}),$$

which we denote by $v \mapsto v^\sigma$.

Most importantly one has that these isomorphisms commute with the formation of cycle classes of subvarieties, in other words for a subvariety Z of Y one has that $[Z]^\sigma = [Z^\sigma]$. In other words the cycle class $[Z]$ gets mapped to the cycle class of $[Z^\sigma]$ under the above isomorphism.

Definition 2.21. Let Y be a smooth projective variety over $\mathbb{C}$ and $p \geq 0$ an integer. Let $v \in H^{2p}_{DR}(Y/\mathbb{C})$. Then v is an absolute Hodge class if for all $\sigma \in \mathrm{Aut}(\mathbb{C})$ the cohomology class $v^\sigma \in H^{2p}((Y^\sigma)^{an}, \mathbb{Q})$ is a Hodge class.

From our remarks above we see that all algebraic classes are in fact absolute Hodge classes. We also note that all absolute Hodge classes are, trivially by the definition, Hodge classes. The converse of this statement is the following conjecture:

Conjecture 2.22 (Absolute Hodge conjecture)**.** *Let Y be a smooth projective variety over $\mathbb{C}$ and p a non-negative integer. Then every Hodge class in $H^{2p}(Y^{an}, \mathbb{Q})$ is absolute Hodge.*

The absolute Hodge conjecture is a theorem of P. Deligne,see [DMOS82], in the case where Y is an abelian variety. For more on the notions on Hodge classes and absolute Hodge classes one can see the original source [DMOS82], J. Milne's notes of P. Deligne's lectures on the subject, or the excellent notes [CS14] of F. Charles and C. Schnell on which our exposition is heavily based.

2.2 ENDOMORPHISM ALGEBRAS OF HODGE STRUCTURES

One of the central notions we will employ in what follows are the endomorphism algebras of polarized $\mathbb{Q}$-Hodge structures of pure weight. We present here a quick review of the main facts we will need later on about the structure of these algebras, as well as a few standard definitions and notation on Hodge-theoretic notions that we will use.

Given a $\mathbb{Q}$-Hodge structure, or $\mathbb{Q}$-HS for short, of pure weight we also get a group homomorphism $\tilde{\varphi} : \mathbb{S} \to GL(V)_{\mathbb{R}}$ of $\mathbb{R}$-algebraic groups, where $\mathbb{S}$ is the Deligne torus. Let $\mathbb{U}_1$ be the $\mathbb{R}$-subtorus of $\mathbb{S}$ with $\mathbb{U}_1(\mathbb{R}) = \{z \in \mathbb{C}^* : |z| = 1\}$ and let $\varphi := \tilde{\varphi}|_{\mathbb{U}_1}$.

Definition 2.23. Let V be a pure weight $\mathbb{Q}$-HS and $\tilde{\varphi}$ and φ be as above. The **Mumford-Tate group** of V, denoted by $G_{mt}(V)$, is defined as the $\mathbb{Q}$-Zariski closure of $\tilde{\varphi}(\mathbb{S}(\mathbb{R}))$. The **special Mumford-Tate group** of V, denoted by $G_{smt}(V)$, is defined as the $\mathbb{Q}$-Zariski closure of $\varphi(\mathbb{U}_1(\mathbb{R}))$.

Irreducible Hodge Structures: Albert's Classification

It is well known that the category of polarizable $\mathbb{Q}$-Hodge structures is semi-simple. This implies that for a polarizable $\mathbb{Q}$-HS V, its endomorphism algebra $D := \mathrm{End}(V)^{G_{smt}(V)}$ is a semi-simple $\mathbb{Q}$-algebra. If, furthermore, the polarizable $\mathbb{Q}$-HS V that we consider is simple, then D is a simple division $\mathbb{Q}$-algebra equipped with a positive involution, naturally constructed from the polarization.

Such algebras are classified by Albert's classification.

Theorem 2.24 (Albert's Classification,[Mum08]). *Let D be a simple $\mathbb{Q}$-algebra with a positive (anti-)involution ι, denoted $a \mapsto a^\dagger$. Let $F = Z(D)$, be the center of D, $F_0 = \{a \in F : a = a^\dagger\}$, $e_0 = [F_0 : \mathbb{Q}]$, $e = [F : \mathbb{Q}]$, and $d^2 = [D : F]$. Then D is of one of the following four types:*

Type I: *$D = F = F_0$ is a totally real field, so that $e = e_0$, $d = 1$, and ι is the identity.*

Type II: *D is a quaternion algebra over the totally real field $F = F_0$ that also splits at all archimedean places of F. If $a \mapsto a^* = tr_{D/F}(a) - a$ denotes the standard involution of this quaternion algebra, then there exists $x \in D$ with $x = -x^*$ such that $a^\dagger = xa^*x^{-1}$ for all $a \in D$. Finally, in this case $e = e_0$ and $d = 2$.*

Type III: *D is a totally definite[1] quaternion algebra over the totally real field $F = F_0$. In this case ι is the standard involution of this quaternion algebra and as*

1 We remind the reader that a quaternion algebra B over a number field F is called totally definite if for all archimedean places $v \in \Sigma_{F,\infty}$ we have that the algebra B is ramified at v. This requires that F is totally real so that $B \otimes_F F_v \simeq \mathbb{H}$, with $\mathbb{H}$ the standard quaternion algebra over $\mathbb{R}$, for all $v \in \Sigma_{F,\infty}$.

before $e = e_0$ and $d = 2$.

Type IV: *D is a division algebra of rank d^2 over the field F, which is a CM-field with totally real subfield F_0, i.e. $e = 2e_0$. Finally, the involution ι corresponds, under a suitable isomorphism $D \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\simeq} M_d(\mathbb{C}) \times \ldots \times M_d(\mathbb{C})$, with the involution $(A_1, \ldots, A_{e_0}) \mapsto ({}^t\bar{A}_1, \ldots, {}^t\bar{A}_{e_0})$*

Furthermore, in this case we have that for σ a generator of $\mathrm{Gal}(F/F_0)$ the following must hold:

1. *if $v \in \Sigma_{F,f}$ is such that $\sigma(v) = v$ we have that $\mathrm{inv}_v(D) = 0$, and*
2. *for all $v \in \Sigma_{F,f}$ we must have that $\mathrm{inv}_v(D) + \mathrm{inv}_{\sigma(v)}(D) = 0$.*

.

The general case

Let (V, ϕ) be a polarized $\mathbb{Q}$-HS of weight n. Then, combining the semi-simplicity of the category of polarized $\mathbb{Q}$-HS and 2.24 we get a good description of the endomorphism algebra $D = \mathrm{End}(V)^{G_{smt}(V)}$.

Indeed, we know that there exist simple polarized weight n sub-$\mathbb{Q}$-Hodge structures (V_i, φ_i) with $1 \leq i \leq r$, such that $V_i \not\simeq V_j$ for all $i \neq j$ and we have a decomposition

$$V = V_1^{m_1} \oplus \ldots \oplus V_r^{m_r}. \tag{2.1}$$

Denoting by $D_i := \mathrm{End}(V_i)^{G_{smt}(V_i)}$ the corresponding endomorphism algebras and by $F_i := Z(D_i)$ their respective centers, we then have a decomposition

$$D = M_{m_1}(D_1) \times \ldots \times M_{m_r}(D_r). \tag{2.2}$$

Finally, this implies that the center F of D is such that

$$F = F_1 \times \ldots \times F_r, \tag{2.3}$$

were each F_i is diagonally embedded into $M_{m_i}(D_i)$, and the maximal commutative semi-simple sub-algebra E of D may be written as

$$E = F_1^{m_1} \times \ldots \times F_r^{m_r}. \tag{2.4}$$

For a proof of Albert's classification see [Mum08], §21. For more on Mumford-Tate groups we direct the interested reader to our sources for this section, which are mainly [Moo99] and [GGK12].

2.3 A SHORT REVIEW OF G-FUNCTIONS IN ARITHMETIC GEOMETRY

G-functions were first introduced by Siegel in [Sie14]. We start with a short review of G-functions and we list some of their main properties.

Definition 2.25. Let K be a number field and let $y = \sum_{n=0}^{\infty} a_n x^n \in K[[x]]$. Then y is called a **G-series at the origin** if the following are true:

1. $\forall v \in \Sigma_{K,\infty}$ we have that $i_v(y) \in \mathbb{C}_v[[x]]$ defines an analytic function around 0,
2. there exists a sequence $(d_n)_{n\in\mathbb{N}}$ of natural numbers such that
   - $d_n a_m \in \mathcal{O}_K$ for all $m \leq n$,
   - there exists $C > 0$ such that $d_n \leq C^n$ for all $n \in \mathbb{N}$,
3. y satisfies a linear homogeneous differential equation with coefficients in $K(x)$.

Examples of G-series at the origin are elements of $\bar{\mathbb{Q}}(x)$ without a pole at 0, the expansion of $log(1+x)$ at 0, and any element of $\bar{\mathbb{Q}}[[x]]$ which is algebraic over $\bar{\mathbb{Q}}(x)$.

We note, see [DGS94], that we can naturally define "*G-series at ζ*", for any $\zeta \in \mathbb{C}$. We also remark that the number field K can be replaced by $\bar{\mathbb{Q}}$ without problems thanks to the third condition, which implies that the a_i are all in some finite extension of $\mathbb{Q}$. Finally, we note that the set of G-series at ζ forms a ring.

Definition 2.26. A **G-function** is a multivalued locally analytic function y on $\mathbb{C}\backslash S$, with $|S| < \infty$, such that for some $\zeta \in \mathbb{C}\backslash S$, y can be represented by a G-series at ζ.

Thanks to the Theorem of Bombieri-André and the Theorem of Chudnovsky we know that the global nature of a G-function is in fact very much dependent on the fact that it can be locally written as a G-series. That is why, essentially following [And89], we identify the two notions, especially since we will be only interested at power series centered at the origin.

For more on G-functions we direct the interested reader to the excellent introductory text [DGS94] and the more advanced [And89].

A Hasse Principle for G-functions

The main tool we will need from the theory of G-functions is a theorem of André, that generalizes work of Bombieri in [Bom81], which plays the role of a “Hasse Principle" for G-functions. First we need some definitions. For the rest of this section consider $y_0, \dots, y_{m-1}$ to be G-functions with coefficients in some number field K. We also define $Y := (y_0, \dots, y_{m-1}) \in K[[x]]^m$, we fix some homogeneous polynomial $p \in K[t_1, \dots, t_m]$, and a $\xi \in K$.

Definition 2.27. 1. We say that a relation $p(y_0(\xi), \dots, y_{m-1}(\xi)) = 0$ **holds v-adically for some place v of K** if

$$i_v(p)(i_v(y_0)(i_v(\xi)), \dots, i_v(y_{m-1})(i_v(\xi))) = 0.$$

2. A relation like that is called **non-trivial** if it does not come by specialization at ξ from a homogeneous relation of the same degree with coefficients in $K[x]$ among the y_i. Respectively, we call it **strongly non-trivial** if it does not occur as a factor of a specialization at ξ of a homogeneous irreducible relation among the y_i of possibly higher degree.

3. A relation $p(y_0(\xi), \dots, y_{m-1}(\xi)) = 0$ is called **global** if it holds v-adically for all places v of K for which $|\xi|_v < \min\{1, R_v(Y)\}$.

Theorem 2.28 (Hasse Principle for G-functions,[And89], Ch VII, §5.2)**.** *Assume that $Y \in \bar{\mathbb{Q}}[[x]]^m$ satisfies the differential system $\frac{d}{dx}Y = \Gamma Y$ where $\Gamma \in M_m(\bar{\mathbb{Q}}(x))$ and that $\sigma(Y) < \infty$. Let $Ш_\delta(Y)$, resp. $Ш'_\delta(Y)$, denote the set of ordinary points or apparent singularities $\xi \in \bar{\mathbb{Q}}^*$ where there is some non-trivial, resp. strongly non-trivial, and global homogeneous relation of degree δ.*

Then,

$$h(Ш_\delta(Y)) \le c_1(Y)\delta^{3(m-1)}(\log \delta + 1), \text{ and}$$

$$h(Ш'_\delta(Y)) \le c_2(Y)\delta^{m}(\log \delta + 1).$$

In particular any subset of $Ш_\delta(Y)$ with bounded degree over $\mathbb{Q}$ is finite.

Remark 2.29. The quantity $\sigma(Y)$ is called the size of Y. G-functions have finite size. [2]

Periods and G-functions

Our primary interest in the theory of G-functions stems from the connection between G-functions and relative periods. We give a brief review of the results in [And89] that highlight this connection together with some basic facts and definitions that we will use later on.

2 For this fact and the definition of the notion of “size" of a power series see [And89] Chapter I.

Let T be a smooth connected curve over some number field $k \subset \mathbb{C}$, $S = T\backslash\{s_0\}$, where $s_0 \in T(k)$ is some closed point, and let x be a local parameter of the curve T at s_0.

We also consider $f : X \to S$ a proper smooth morphism and we let $n = dim X - 1$. We then have the following isomorphism of $\mathcal{O}_{S^{an}_{\mathbb{C}}}$-modules

$$P^{\bullet}_{X/S} : H^{\bullet}_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}_{\mathbb{C}}} \to R^{\bullet} f^{an}_{*} \mathbb{Q}_{X^{an}_{\mathbb{C}}} \otimes_{\mathbb{Q}_{S^{an}_{\mathbb{C}}}} \mathcal{O}_{S^{an}_{\mathbb{C}}}.$$

In what follows we will be focusing on the isomorphism $P^n_{X/S}$, which from now on we will simply denote by $P_{X/S}$. We also let $\mu = \dim_{\mathbb{Q}} H^n(X^{an}_z, Q)$ where $z \in S(\mathbb{C})$.

This isomorphism is the relative version of Grothendieck's isomorphism between algebraic de Rham and Betti cohomology and it can be locally represented by a matrix. Namely, if we choose a basis ω_i of $H^n_{DR}(X/S)$ over some affine open subset $U \subset S$ and a frame γ_j of $R_n f^{an}_{*} \mathbb{Q}_{X^{an}_{\mathbb{C}}}$ over some open analytic subset V of the analytification $U^{an}_{\mathbb{C}}$, $P_{X/S}$ is represented by a matrix with entries of the form $\int_{\gamma} \omega_i$.

Definition 2.30. We define the **relative n-period matrix** (over V) to be the $\mu \times \mu$ matrix

$$\left(\tfrac{1}{(2\pi i)^n} \int_{\gamma} \omega_i \right).$$

Its entries will be called the **relative n-periods**.

A result we will need in what follows guarantees the existence of G-functions among the relative n-periods under the hypothesis that the morphism f extends over all of T. Namely, let us assume f extends to a projective morphism $f_T : X_T \to T$ with X_T a smooth k-scheme, such that $Y := f^{-1}(s_0)$ is a union of smooth transversally crossing divisors Y_i entering the fiber with multiplicity 1.

Under these assumptions we know, see [PS08] Corollary 11.19, that the local monodromy is unipotent. Let Δ be a small disk embedded in T^{an} and centered around s_0. We let $2\pi i N^*$ be the logarithm of the local monodromy acting on the sheaf $R_n(f^{an}_{\mathbb{C}})_*(\mathbb{Q})|_{\Delta^*}$.

Definition 2.31. We denote the image of the map $(2\pi i N^*)^n$ by $M_0 R_n(f^{an}_{\mathbb{C}})_*(\mathbb{Q})|_{\Delta^*}$. We call M_0-n-period any relative n-period over a cycle γ in $M_0 R_n(f^{an}_{\mathbb{C}})_*(\mathbb{Q})|_{\Delta^*}$.

By the formalism of the limit Hodge structure we have that for all $z \in \Delta^*$ the group $\pi_1(\Delta^*, z)$ acts unipotently on the fiber $(R_n(f^{an}_{\mathbb{C}})_*(\mathbb{Q})|_{\Delta^*})_z$. We also get that, letting $2\pi i N^*_z$ be the nilpotent logarithm of the image of a generator of $\pi_1(\Delta^*, z)$ via the monodromy representation, $(M_0 R_n(f^{an}_{\mathbb{C}})_*(\mathbb{Q})|_{\Delta^*})_z = Im((2\pi i N^*_z)^n)$.

Theorem 2.32 ([And89], p.185)**.** *There exists a basis of sections ω_i of $H^n_{DR}(X/S)$ over some dense open subset of S, such that for any section γ of $M_0 R_n(f^{an}_{\mathbb{C}})_*(\mathbb{Q})|_{\Delta^*}$ the Taylor expansion in x of the relative M_0-n-periods $\frac{1}{(2\pi i)^n}\int_\gamma \omega_i$ are globally bounded G-functions.*

Remark 2.33. We may assume without loss of generality that the G-functions created have coefficients in k, when $k \subset \bar{\mathbb{Q}}$. For more on this see the proof of 5.10.

2.4 THE MAIN SETTING-NOTATIONAL CONVENTIONS

Before delving into the technical parts of our argument we devote this section on describing the general setting that we will be working on in more detail. We give the definitions of the main objects and introduce the notation that we will, unless otherwise stated, keep uniform throughout our exposition.

Let S' be a smooth proper geometrically irreducible curve over some number field $K \leq \bar{\mathbb{Q}}$, let $\Sigma_S \subset S'(K)$ be a finite set of K-points and $s_0 \in \Sigma_S$ be a fixed such point. We let $S = S' \backslash \Sigma_S$ be the complement of Σ_S in S'. We also fix x a local parameter of the curve S' at s_0 and η the generic point of S.

Let us consider $f : X \to S$ a smooth projective morphism and let $n = \dim X - 1$. Assume f extends to a projective morphism $f' : X' \to S'$ with X' a smooth K-scheme and that $Y = f^{-1}(s_0)$ is a simple normal crossings divisor.

The map f defines a variation of polarized $\mathbb{Q}$-HS of weight n over $S^{an}_{\mathbb{C}}$ given by $R^n f^{an}_* \mathbb{Q}_{X^{an}_{\mathbb{C}}}$. We denote by $G_{mt,p}$, respectively by $G_{smt,p}$, the Mumford-Tate group, or respectively the special Mumford-Tate group, associated to the $\mathbb{Q}$-HS associated to the point $p \in S(\mathbb{C})$. We also let $G_{mt,\eta}$, respectively $G_{smt,\eta}$, be the generic Mumford-Tate group, or respectively the generic special Mumford-Tate group, of the variation. For each $p \in S(\mathbb{C})$ we also let $V_p = H^n(X^{an}_p, \mathbb{Q})$ be the fiber of the local system $R^n f^{an}_* \mathbb{Q}_{X^{an}_{\mathbb{C}}}$ and let $\mu = \dim_{\mathbb{Q}} V_p$.

Consider $z \in S(\mathbb{C})$ to be a Hodge generic point for the above variation of $\mathbb{Q}$-HS. The main invariant of the variation we will be interested in is the $\mathbb{Q}$-algebra

$$D := \mathrm{End}(V_z)^{G_{smt,z}} = \mathrm{End}(V_z)^{G_{smt,\eta}}.$$

Similarly, for $s \in S(\mathbb{C})$ we let

$$D_s := \mathrm{End}(V_s)^{G_{smt,s}}.$$

Definition 2.34. Let X, S, $s \in S(\mathbb{C})$, D_s, and D be as above. We call D_s the **algebra of Hodge endomorphisms at** s.

Definition 2.35. A variation of Hodge structures such as above, meaning a weight n geometric variation of $\mathbb{Q}$-HS parameterized by S whose degeneration at some $s_0 \in \Sigma_S \subset S'$ is as above, with $S = S' \backslash \{\Sigma_S\}$, with all of the above defined over some number field K, will be called **G-admissible**.

Remark 2.36. We remark that under these assumptions 2.32 applies by letting $T = S' \backslash (\Sigma_S \backslash \{s_0\})$ and f_T be the pullback of f' over T. In particular we have the existence of G-functions among the entries of the relative period matrix as described in 2.3.

Notation: We fix some notation that appears throughout the text. By Σ_K, $\Sigma_{K,f}$, $\Sigma_{K,\infty}$ we denote the set of all places of a number field K, respectively finite or infinite places of K. For $v \in \Sigma_K$ we let $i_v : K \to \mathbb{C}_v$ denote the inclusion of K into $\mathbb{C}_v$. For $y \in K[[x]]$ we let $i_v(y)$ denote the element of $\mathbb{C}_v[[x]]$ given via i_v acting coefficient-wise on y.

For a scheme Y defined over a field k we let $\bar{Y} := Y \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ and $Y_L := Y \times_{\operatorname{Spec} k} \operatorname{Spec} L$ for any extension L/k.

3

HODGE ENDOMORPHISMS AND DE RHAM COHOMOLOGY

Let K be a number field and $f : X \to S$ be a smooth projective K-morphism of K-varieties, with S a curve as above. Let us also consider a point $s \in S(L)$ for some finite extension L/K and set $Y := X_s$ which is a smooth projective variety defined over L.

In what follows we will need the existence of a natural action of the algebra of Hodge endomorphisms of $H^n(Y, \mathbb{Q})$ on both sides of the comparison isomorphism

$$P^n : H^n_{DR}(\bar{Y}/\bar{L}) \otimes_{\bar{L}} \mathbb{C} \to H^n(\bar{Y}^{an}_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}, \tag{3.1}$$

such that these actions commute with this isomorphism.

In the case of abelian varieties this is automatic from the fact that the algebra of Hodge endomorphisms is naturally realized as the algebra of endomorphisms of the abelian variety. This in turn acts naturally on both sides of the comparison isomorphism and the actions commute with the isomorphism itself. In a general variety Y we cannot hope for such a description without assuming the validity of the absolute Hodge Conjecture.

It is the author's belief that the results in this section are known to experts in the field. Since we were not able to find an exact reference of the results we needed we have dedicated this section in providing proofs for these results.

3.1 EXISTENCE OF THE ACTION

For the rest of this subsection we fix a number field L and a smooth projective n-dimensional variety Y defined over L.

Proposition 3.1. *Let Y be a smooth projective variety over the number field L of dimension n. Let $V := H^n(\bar{Y}^{an}_{\mathbb{C}}, \mathbb{Q})$ and $D := \mathrm{End}_{HS}(V)$ be the algebra of Hodge endomorphisms. Then,* ***assuming the absolute Hodge Conjecture****,* *there exists a finite Galois extension $\hat{L}$ of L such that there exists an injective homomorphism of algebras*

$$i : D \hookrightarrow \mathrm{End}_{\hat{L}}(H^n_{DR}(Y/L) \otimes_L \hat{L}).$$

Moreover, we have that $P^n(i(d)v) = d \cdot P^n(v)$ for all $d \in D$ and all $v \in H^n_{DR}(\bar{Y}/\bar{L}) \otimes_{\bar{L}} \mathbb{C}$. In other words, the action of the algebra D, that is induced by i, on de Rham cohomology coincides with the usual action of D on the Betti cohomology as endomorphisms of the Hodge structure under the comparison isomorphism P^n.

Proof. We start with some, well known, observations. First of all, the natural isomorphism

$$\alpha_0 : \mathrm{End}_{\mathbb{Q}}(V) \cong V \otimes V^*$$

is an isomorphism of $\mathbb{Q}$-HS. In particular, via α_0 the elements of D correspond to Hodge classes[1].

It is also known that the isomorphism $\alpha : H^n(\bar{Y}^{an}_{\mathbb{C}}, \mathbb{Q})^* \to H^n(\bar{Y}^{an}_{\mathbb{C}}, \mathbb{Q})(n)$, given by Poincaré duality, is an isomorphism of $\mathbb{Q}$-HS. As a consequence we get that the induced isomorphism

$$\alpha_1 : V \otimes_{\mathbb{Q}} V^* \xrightarrow{\cong} (V \otimes_{\mathbb{Q}} V)(n)$$

is also an isomorphism of $\mathbb{Q}$-HS. Moreover, it is known that the injection

$$\alpha_2 : (H^n(\bar{Y}^{an}_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^n(\bar{Y}^{an}_{\mathbb{C}}, \mathbb{Q}))(n) \hookrightarrow H^{2n}(\bar{Y}^{an}_{\mathbb{C}} \times \bar{Y}^{an}_{\mathbb{C}}, \mathbb{Q})(n),$$

given by the Künneth formula is also an injective homomorphism of $\mathbb{Q}$-HS.

Step 1: Reduction to the algebraic closure: Let us start by fixing a basis β of D over $\mathbb{Q}$. By the above remarks, for each $d \in \beta$ we get a Hodge class $\phi_d := \alpha_2 \circ \alpha_1 \circ \alpha_0(d) \in H^{2n}(\bar{Y}^{an}_{\mathbb{C}} \times \bar{Y}^{an}_{\mathbb{C}}, \mathbb{Q})(n)$.

Now, assuming the **absolute Hodge Conjecture** , from Corollary 11.3.16 of [CS14] such a class ϕ_d has to be defined over the algebraically closed field $\bar{L}$, i.e. $\phi_d = P^{2n}_{Y \times Y}(\tilde{\phi}_d)$ where $\tilde{\phi}_d \in H^{2n}_{DR}(\bar{Y} \times_{\bar{L}} \bar{Y}/\bar{L})(n)$.

Step 2: Reduction to finite extension: Let us set $Z := Y \times_L Y$. We have an L-vector space $H^{2n}(Z/L)$ and we also have an isomorphism

$$H^{2n}_{DR}(Z/L) \otimes_L F \cong H^{2n}_{DR}(Z_F/F) \tag{3.2}$$

for every extension F/L. In particular (3.2) holds for $F = \bar{L}$.

1 See Lemma 11.41 of [Voi07].

If we consider $\delta := \{\delta_1, \ldots, \delta_m\}$ the image of an L-basis of $H^{2n}_{DR}(Z/L)$ in $H^{2n}_{DR}(\bar{Z}/\bar{L})$ under the above isomorphism. For $\tilde{\phi}_d$ as above we may write

$$\tilde{\phi}_d = a_1(d)\delta_1 + \cdots + a_m(d)\delta_m. \tag{3.3}$$

Given these coefficients, we set L_d to be the field $L(a_1(d), \cdots, a_m(d))$, which is a finite extension of L. Finally, we let $\hat{L}$ be the Galois closure of the compositum of the L_d for all $d \in \beta$.

We observe that for any Galois extensions F_1, F_2 of the field L with $L \subset F_1 \subset F_2$ the diagram

$$\begin{array}{ccc} (H^{2n}_{DR}(Z/L) \otimes_L F_1) \otimes_{F_1} F_2 & \longrightarrow & H^{2n}_{DR}(Z_{F_1}/F_1) \otimes_{F_1} F_2 \\ \downarrow & & \downarrow \\ H^{2n}_{DR}(Z/L) \otimes_L F_2 & \longrightarrow & H^{2n}_{DR}(Z_{F_2}/F_2) \end{array}$$

is a commutative diagram of $\mathrm{Gal}(F_2/L)$-modules. As a consequence of this we may and do view from now on each $\tilde{\phi}_d$ as an element of $H^{2n}_{DR}(Z_{\hat{L}}/\hat{L})$.

Step 3: Back to endomorphisms: So far we have found classes $\tilde{\phi}_d \in H^{2n}_{DR}(Z_{\hat{L}}/\hat{L})$. We want to show that these naturally correspond to endomorphisms of $H^n_{DR}(Y_{\hat{L}}/\hat{L})$ and that this correspondence behaves well with respect to the comparison isomorphism of Grothendieck.

To that end, we start by noting that Grothendieck's comparison isomorphism between algebraic de Rham cohomology and Betti cohomology is compatible with the isomorphisms given by both Poincaré duality and the Künneth formula. We note that both of these, i.e. Poincaré duality and the Künneth formula, are defined for both cohomology theories in question, in fact for de Rham cohomology they are defined over L.

With that in mind we define $\alpha_{i,DR}$ mirroring the homomorphisms α_i we had earlier.

Therefore for every $d \in \beta$, viewing the class $\tilde{\phi}_d$ as an element of $H^{2n}_{DR}(Z_{\mathbb{C}}/\mathbb{C})$, due to the aforementioned compatibility, we get an element $\tilde{d} \in \mathrm{End}_{\mathbb{C}}(H^n_{DR}(\bar{Y}_{\mathbb{C}}/\mathbb{C}))$ which is such that

1. it maps to d via the comparison isomorphism, and
2. it maps to $\tilde{\phi}_d$ via the injective map $\alpha_{2,DR} \circ \alpha_{1,DR} \circ \alpha_{0,DR}$.

Property (1) above tells us that $P^n(\tilde{d}(v)) = d(P^n(v))$ for all $v \in H^n_{DR}(\bar{Y}_{\mathbb{C}}/\mathbb{C})$. Thus proving the "moreover" part of the proposition.

Since Y is defined over the field L the same is true for the $\alpha_{i,DR}$. In particular since their composition $\alpha_{DR} := \alpha_{2,DR} \circ \alpha_{1,DR} \circ \alpha_{0,DR}$ is an injective homomorphism

$$\alpha_{DR} : \mathrm{End}(H^n_{DR}(Y_{\hat{L}}/\hat{L})) \hookrightarrow H^{2n}_{DR}(Y_{\hat{L}} \times_{\hat{L}} Y_{\hat{L}}/\hat{L}),$$

we get that in fact $\tilde{d} \in \mathrm{End}(H^n_{DR}(Y_{\hat{L}}/\hat{L}))$.

Since d was a random element in a $\mathbb{Q}$-basis of D we get an injective homomorphism

$$i : D \hookrightarrow \mathrm{End}(H^n_{DR}(Y_{\hat{L}}/\hat{L})) \cong \mathrm{End}(H^n_{DR}(Y_L/L) \otimes_L \hat{L}). \tag{3.4}$$

Finally, because of the above comments, we also get that i satisfies the "moreover" part of the proposition. □

Bounds on the degree extension

Later on we want to have some control on the degree of the Galois extension $\hat{L}/L$ constructed in the proof of 3.1. In particular, we want an upper bound on the degree $[\hat{L} : L]$ that will be independent of the smooth projective variety Y/L and the field L itself. We want this bound to only depend on the dimension of Y and its n-th Betti number. In making an analogy with the case of abelian varieties, we want upper bounds akin to those achieved in [Sil92].

Proposition 3.2. *Assume the* ***absolute Hodge Conjecture*** *is true. Let Y be a smooth n-dimensional projective variety defined over the number field L. Then the field extension $\hat{L}/L$ constructed in 3.1 may be chosen so that for its degree we have*

$$[\hat{L} : L] \leq ((6.31)m^2)^{m^2},$$

where $m = \dim_{\mathbb{Q}} H^n(\bar{Y}^{an}_{\mathbb{C}}, \mathbb{Q})$ is the n-th Betti number.

Proof. Let β be a $\mathbb{Q}$-basis of D. From the proof of 3.1 we have an injective homomorphism of $\mathbb{Q}$-algebras $D \hookrightarrow \mathrm{End}_{\hat{L}}(H^n_{DR}(Y_{\hat{L}}/\hat{L}))$, given in the basis elements by $d \to \tilde{d}$ in the notation of the proof of 3.1.

By base change we have a natural action of the finite Galois group $\mathrm{Gal}(\hat{L}/L)$ on de Rham cohomology $H^n_{DR}(Y_{\hat{L}}/\hat{L})$, as an L-vector space. This induces a natural action of the same group on $\mathrm{End}_{\hat{L}}(H^n_{DR}(Y_{\hat{L}}/\hat{L}))$, viewed as an L-vector space again. We start by proving the following claim.

Claim: The above action of the Galois group induces an action on the embedding of D in $\mathrm{End}_{\hat{L}}(H^n_{DR}(Y_{\hat{L}}/\hat{L}))$. In other words for all $\sigma \in \mathrm{Gal}(\hat{L}/L)$ we have that $\sigma(D) = D$.

Proof of the claim. Assuming the absolute Hodge Conjecture, by our earlier construction, for every element d of the basis β we get an element $\tilde{d} =$

$i(d) \in \mathrm{End}_{\hat{L}}(H^n_{DR}(Y_{\hat{L}}/\hat{L}))$. By the previous proof, via Poincaré duality and the Künneth formula, we get classes $\tilde{\phi}_d \in H^{2n}_{DR}(Y_{\hat{L}} \times Y_{\hat{L}}/\hat{L})$ that map to Hodge classes $\phi_d \in H^{2n}(Y^{an} \times Y^{an}, \mathbb{Q})$. As we did in our earlier proof we let $Z := Y \times_L Y$. In the above construction we implicitly consider a fixed embedding $\sigma_0 : \hat{L} \hookrightarrow \mathbb{C}$.

By our assumption that the absolute Hodge Conjecture holds true, we get that for all embeddings $\sigma : \hat{L} \hookrightarrow \mathbb{C}$ the class $\tilde{\phi}_d \in H^{2n}(\sigma(Z_{\hat{L}})/\mathbb{C})$ is Hodge. Here $\sigma(Z_{\hat{L}})$ denotes the complex variety obtained from $Z_{\hat{L}}$ when we base change via the embedding σ to $\mathbb{C}$.

From the embedding $\sigma_0 : \hat{L} \hookrightarrow \mathbb{C}$ that we fixed earlier we get an embedding $i_0 : L \hookrightarrow \mathbb{C}$. Any embedding $\sigma : \hat{L} \hookrightarrow \mathbb{C}$ that is such that $\sigma|_L = i_0$ will correspond to an element of the Galois group $\mathrm{Gal}(\hat{L}/L)$ via the bijective map $\mathrm{Gal}(\hat{L}/L) \to \{\sigma : \hat{L} \hookrightarrow \mathbb{C} : \sigma|_L = i_0\}$ given by $\tau \mapsto \sigma_0 \circ \tau$. For notational brevity we suppress σ_0 from our notation from now on and identify $\tau \in \mathrm{Gal}(\hat{L}/L)$ with $\sigma_0 \circ \tau$, in other words we identify the elements of $\mathrm{Gal}(\hat{L}/L)$ with the corresponding embedding $\hat{L} \hookrightarrow \mathbb{C}$. With this notational convention we may and will write from now on $Y_{\mathbb{C}}$, or $Z_{\mathbb{C}}$ respectively, for the complex variety we would otherwise denote by $\sigma_0 Y_{\hat{L}}$, or $\sigma_0 Z_{\hat{L}}$ respectively.

For the above σ, since Y and hence also Z are defined over the field L, by the above remarks $H^{2n}_{DR}(\sigma Z_{\hat{L}}/\mathbb{C})$ may be identified with $H^{2n}_{DR}(Z_{\mathbb{C}}/\mathbb{C})$. Via this identification $\tilde{\phi}_d$ will get mapped to $\sigma^*(\tilde{\phi}_d) \in H^{2n}_{DR}(Z_{\hat{L}}/\hat{L})$. Here $\sigma^* : H^{2n}_{DR}(Z_{\hat{L}}/\hat{L}) \to H^{2n}_{DR}(Z_{\hat{L}}/\hat{L})$ denotes the isomorphism of L-vector spaces induced by $\sigma \in \mathrm{Gal}(\hat{L}/L)$ on cohomology.

Now, since Y and Z are both defined over the field L, both the Poincaré duality isomorphism and the Künneth formula on de Rham cohomology are defined over the field L as well. These maps, by construction, commute with the isomorphisms σ^* so we get that $\sigma^*(\tilde{d})$ maps to $\sigma^*(\tilde{\phi}_d) \in H^{2n}(Z_{\mathbb{C}}/\mathbb{C})$ via the map α_{DR} we had in the proof of 3.1.

Writing P for Grothendieck's comparison isomorphism we have that $P(\sigma^*(\tilde{d})) \in D \subset \mathrm{End}_{\mathbb{Q}} H^n(Y^{an}_{\mathbb{C}}, \mathbb{Q})$ is a Hodge endomorphism. Thus $\sigma^*(\tilde{d}) \in i(D)$ with the notation of 3.1 and the result follows. $\square$

By the claim therefore we get an action of $G := \mathrm{Gal}(\hat{L}/L)$ on the $\mathbb{Q}$-vector space D, or more precisely its image in $\mathrm{End}_{\hat{L}}(H^n_{DR}(Y_{\hat{L}}/\hat{L}))$. Let $\dim_{\mathbb{Q}} D = m_0$ and note that $m_0 \leq m^2$ trivially. We may and do assume, without loss of generality, that the field extension $\hat{L}/L$ constructed in the previous proof is minimal with the property that every cycle of the above basis $\tilde{d}$ is defined over $\hat{L}$. This implies that the corresponding group homomorphism $\mathrm{Gal}(\hat{L}/L) \to \mathrm{Aut}(D)$ is in fact injective.

Let Λ_1 be a lattice in D, and consider $\Lambda := \sum_{g \in G} g(\Lambda_0)$. This will be a lattice that is also invariant by G. From the G-invariance of Λ we get a group homomorphism

$$G \to \mathrm{GL}(\Lambda).$$

This homomorphism will be injective as well by our earlier assumption about the minimality of the extension $\hat{L}/L$.

Let $N \geq 3$. Then, we know that the kernel of the surjective map $\mathrm{GL}(\Lambda) \to \mathrm{GL}(\Lambda/N\Lambda)$ contains no element of finite order of the group $\mathrm{GL}(\Lambda)$. As a result we get $G \hookrightarrow \mathrm{GL}(\Lambda/N\Lambda)$ which implies that $|G|$ divides $|\,\mathrm{GL}(\Lambda/N\Lambda)| = |\,\mathrm{GL}_{m_0}(\mathbb{Z}/N\mathbb{Z})|$.

Following the notation of [Sil92] we let $g_r(N) := |\,\mathrm{GL}_r(\mathbb{Z}/N\mathbb{Z})|$ and $G(r) := \gcd\{g_r(N) : N \geq 3\}$. From Theorem 3.1 of [Sil92] we have that

$$G(r) < ((6.31)r)^r. \tag{3.5}$$

From the above argument we get that $|G|$ divides $G(m_0)$ and combining this with (3.5) and the fact that $m_0 \leq m^2$ we get that

$$|G| < ((6.31)m^2)^{m^2}. \tag{3.6}$$

□

4

DETERMINING THE TRIVIAL RELATIONS

Given a G-admissible variation of Hodge structures we will show that for some exceptional points $s \in S(\bar{\mathbb{Q}})$ we get so called "non-trivial" relations among the values of the relative periods at the point s. To be able to say that these relations we will create are in fact non-trivial we need to know what the trivial ones are first!

We have devoted this chapter of the thesis to determining these trivial relations in the case where the generic special Mumford-Tate group of our variation is a symplectic group.

4.1 THE ACTION OF THE LOCAL MONODROMY

We start by reviewing a key property of the local monodromy that we will need during this process. This follows the ideas in Chapter X, Lemma 2.3 of [And89].

Let Δ be a small disc embedded in $S'^{an}_{\mathbb{C}}$ centered at s_0 and such that $\Delta^* \subset S^{an}_{\mathbb{C}}$. We have already remarked in 2.3 that the logarithm of the local monodromy of $\Delta^* \subset S^{an}_{\mathbb{C}}$ acting on $R_n(f^{an}_{\mathbb{C}})_*(\mathbb{Q})|_{\Delta^*}$ defines the local subsystem $\mathcal{M}_0 := M_0 R_n(f^{an}_{\mathbb{C}})_*(\mathbb{Q})|_{\Delta^*}$. This is contained in the maximal constant subsystem of $R_n(f^{an}_{\mathbb{C}})_*(\mathbb{Q})|_{\Delta^*}$, since $2\pi i N^*$, the nilpotent logarithm associated with the action of monodromy on the limit Hodge structure, has degree of nilpotency $\leq n+1$.

We recall that, since the map $f : X \to S$ is smooth and projective, we have a bilinear form $\langle , \rangle$ on the local system $R_n f^{an}_* \mathbb{Q}$ induced by the polarizing form.

Lemma 4.1. *The local system $\mathcal{M}_0$ is a totally isotropic subsystem of $R_n f^{an}_* \mathbb{Q}|_{\Delta^*}$ with respect to the polarizing form $\langle , \rangle$.*

Proof. The skew-symmetric form $\langle , \rangle$ defines a morphism of local systems

$$R_n f^{an}_* \mathbb{Q}|_{\Delta^*} \otimes R_n f^{an}_* \mathbb{Q}|_{\Delta^*} \to \mathbb{Q}(n)|_{\Delta^*}.$$

Therefore it is invariant under the local monodromy and we conclude that for any $z \in \Delta^*$ and for all $v, w \in (R_n f_*^{an}\mathbb{Q})_z$ we have

$$\langle N_z^* v, w\rangle + \langle v, N_z^* w\rangle = 0. \tag{4.1}$$

Now let v, w be any two sections of $\mathcal{M}_0$. Then for any $z \in \Delta^*$ there exist $v_{0,z}, w_{0,z} \in (R_n f_*^{an}\mathbb{Q})_z$ such that $v_z = (2\pi i N_z^*)^n(v_{0,z})$ and $w_z = (2\pi i N_z^*)^n(w_{0,z})$. Using (4.1) we thus get

$$\begin{aligned} \langle v_z, w_z\rangle &= \langle (2\pi i N_z^*)^n(v_{0,z}), (2\pi i N_z^*)^n(w_{0,z})\rangle = \\ &= -\langle (2\pi i N_z^*)^{n-1}(v_{0,z}), (2\pi i N_z^*)^{n+1}(w_{0,z})\rangle = 0, \end{aligned}$$

where the last equality follows from the fact that N_z^* has degree of nilpotency $\leq n+1$.

Therefore we get that for all $v, w \in \mathcal{M}_0$ we have $\langle v, w\rangle = 0$. Hence $\mathcal{M}_0$ is a totally isotropic local subsystem. $\square$

4.2 TRIVIAL RELATIONS

Our setting and notations

Let $f : X \to S$ be a smooth projective morphism of k-varieties where k is a subfield of $\bar{\mathbb{Q}}$. We also fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ so that we may consider k as a subfield of $\mathbb{C}$. Assume that S is a smooth irreducible curve, that the fibers of f are n-dimensional, and let $\mu := \dim_{\mathbb{Q}} H^n(X_s^{an}, \mathbb{Q})$ for some $s \in S(\mathbb{C})$. Throughout this section we assume that n is **odd** and that $S = S' \backslash \{\Sigma_S\}$ for some finite subset Σ_S and that there exists a k-point $s_0 \in \Sigma_S$ where our VHS has a non-isotrivial degeneration.

We consider

$$P^n_{X/S} : H^n_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}_{\mathbb{C}}} \to R^n f_*^{an}\mathbb{Q}_{X^{an}_{\mathbb{C}}} \otimes_{\mathbb{Q}_{S^{an}_{\mathbb{C}}}} \mathcal{O}_{S^{an}_{\mathbb{C}}},$$

the relative period isomorphism.

Choosing bases-The Riemann relations

Let ω_i, $1 \leq i \leq \mu$, be a basis of $H^n_{DR}(X_\eta)$ over $k(S)$, where η is the generic point of S. Then there exists some dense affine open subset U of S over which these ω_i are sections of the sheaf $H^n_{DR}(X/S)$. We also fix a trivialization γ_i of $R_n f_*^{an}\mathbb{Q}_{X^{an}_{\mathbb{C}}}$, i.e. the relative homology, over an analytic open subset V of $U^{an}_{\mathbb{C}}$. Since we are interested in describing the relations among the periods archimedeanly close to the point of degeneration s_0, we may and do assume that the set V is simply connected and contained in a fixed small punctured disk Δ^* around s_0.

The matrix of $P^n_{X/S}$ with respect to this basis and trivialization will have entries in the ring $\mathcal{O}_V$. We multiply the matrix's elements by $(2\pi i)^{-n}$ and, by abuse of notation, we denote the above $\mu \times \mu$ matrix of relative n-periods by

$$P_{X/S} := ((2\pi i)^{-n} \int_{\gamma_j} \omega_i).$$

Since the morphism $f : X \to S$ is smooth, projective, and is also defined over k, it defines a polarization which will be defined over k as a form on de Rham cohomology. In particular we get, since the weight n of our variation is odd,

- a skew-symmetric form $\langle , \rangle_{DR}$ on $H^n_{DR}(X_\eta)$ with values in $k(S)$ and
- a skew-symmetric form $\langle , \rangle_B = (2\pi i)^n \langle , \rangle$ on $R_n f^{an}_* \mathbb{Q}$ with values in $\mathbb{Q}(n)$.

These two skew-symmetric forms are compatible with the isomorphism $P^n_{X/S}$, in the sense that the dual form of $\langle , \rangle_B$ coincides with the form induced by $\langle , \rangle_{DR}$ via the isomorphism $P^n_{X/S}$. The compatibility of the polarizing forms translates to relations among the periods. These relations can be described succinctly by the equality

$$ {}^t P M_{DR}^{-1} P = (2\pi i)^{-n} M_B^{-1}, \tag{4.2}$$

where M_{DR} and M_B are the matrices of $\langle , \rangle_{DR}$ and the dual of $\langle , \rangle_B$ respectively with respect to some basis and trivialization.

For more on this see A. The relations given on the periods by (4.2) are practically a direct consequence of the well known Hodge-Riemann bilinear relations defining a polarization of a Hodge structure. For this reason from now on we shall refer to (4.2) as the **Riemann relations** for brevity.

With this in mind, we may and do select the above basis ω_i and trivialization γ_j so that the following are satisfied:

- the ω_i are a symplectic basis of $H^n_{DR}(X_\eta)$ so that $\omega_1, \ldots, \omega_{\mu/2}$ constitute a basis of the maximal isotropic subspace $F^{\frac{n+1}{2}} H^n_{DR}(X_\eta)$ and the rest of the elements, i.e. $\omega_{\mu/2+1}, \ldots, \omega_\mu$ are the basis of a transverse Lagrangian of $F^{\frac{n+1}{2}} H^n_{DR}(X_\eta)$, and
- the γ_j is a symplectic trivialization of $R_n f^{an}_* \mathbb{Q}_{X^{an}_{\mathbb{C}}}|_V$, which is also such that $\gamma_1, \ldots, \gamma_h$ are a frame of the space $\mathcal{M}_0|_V$ and the $\gamma_1, \ldots, \gamma_{\mu/2}$ are a frame of a maximal totally isotropic subsystem that contains $M_0 R_n f^{an}_* \mathbb{Q}|_V$.

With these choices we may and do assume from now on that the matrices that correspond to the two aforementioned forms are $M_{DR} = M_B = J_\mu = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. With this (4.2) translates to

$$^tPJ_\mu P = (2\pi i)^{-n} J_\mu. \tag{4.3}$$

The main result

Let $y_{i,j}$ with $1 \leq i \leq \mu$ and $1 \leq j \leq h$ be the entries of the first h columns of the matrix $P_{X/S}$. The aforementioned work of André, see 2.32, guarantees that these are G-functions. This is happening with respect to a local parameter x of S' at s_0, with respect to which the $y_{i,j}$ can be written as power series.

For the remainder of this section we consider the above notation fixed. The rest of this section is dedicated to describing the generic, or "trivial", relations among the G-functions $y_{i,j}$. Indeed, we prove the following:

Proposition 4.2. *With the above notation, assume that the generic special Mumford-Tate group of the variation of $\mathbb{Q}$-HS on $S^{an}_{\mathbb{C}}$ given by $R^n f^{an}_* \mathbb{Q}_{X^{an}_{\mathbb{C}}}$ is $Sp(\mu, \mathbb{Q})$.*

Then, the Zariski closure of the $\mu \times h$ matrix $Y := (y_{i,j})$ over $\bar{\mathbb{Q}}[x]$ in $\mathbb{A}^{\mu \times h}$ is the variety whose ideal is given by the Riemann relations.

Trivial relations over $\mathbb{C}$ for the period matrix

Under the notations and assumptions of 4.2 and 4.2 we have the following:

Lemma 4.3. *Let $z \in V \subset U^{an}$ be a Hodge generic point of the $\mathbb{Q}$-VHS given by $R^n f^{an}_* \mathbb{Q}_{X^{an}_{\mathbb{C}}}$. Then the monodromy group H_z at z is $Sp(\mu, \mathbb{Q})$.*

Proof. Let $\rho_H : \pi_1(S^{an}, z) \to GL(H^n(X^{an}_z, \mathbb{Q}))$ be the monodromy representation at z. Then, by André's Theorem of the fixed part[And92] we know that H_z, which is the connected component of the the $\mathbb{Q}$-algebraic group $\rho_H(\pi_1(S^{an}, z))^{\mathbb{Q}-Zar}$, is a normal subgroup of the derived subgroup of the Mumford-Tate group $G_{mt,z}$ at z. In other words

$$H_z \trianglelefteq DG_{mt,z}.$$

On the other hand we have that $DG_{mt,z} \leq G_{smt,z}$ and trivially that $DG_{smt,z} \leq DG_{mt,z}$, where $G_{smt,z}$ is the special Mumford-Tate group at z. But, by assumption, we know that $G_{smt,z} \simeq Sp(\mu, \mathbb{Q})$, since z is Hodge generic for our variation. It is classical that $Sp(\mu, \mathbb{Q})$ satisfies $DSp(\mu, \mathbb{Q}) = Sp(\mu, \mathbb{Q})$. Hence we have $DG_{mt,z} = Sp(\mu, \mathbb{Q})$.

We thus get that $H_z \trianglelefteq Sp(\mu, \mathbb{Q})$. Finally, $Sp(\mu, \mathbb{Q})$ is a simple $\mathbb{Q}$-algebraic group, therefore $H_z = 1$ or $H_z = Sp(\mu, \mathbb{Q})$. But, if we had $H_z = 1$, then the variation of $\mathbb{Q}$-HS in question would be isotrivial, and hence extend to $T = S^{an} \cup \{s_0\}$. We get a contradiction since the local monodromy at $s_0 \in S'(\mathbb{C})$ is non-trivial by assumption. $\square$

From now on, by taking a finite étale cover of S if necessary, we may and do assume that $\rho_H(\pi_1(S^{an}, z))^{\mathbb{Q}-Zar}$ is connected, i.e. that for the Hodge generic points $z \in V$ we have $H_z = \rho_H(\pi_1(S^{an}, z))^{\mathbb{Q}-Zar}$.

The matrix of Periods and differential equations

Let us denote by M_μ the variety of $\mu \times \mu$ matrices over $\mathbb{C}$, where $\mu := \dim_{\mathbb{Q}} H^n(X^{an}_{s,\mathbb{C}}, \mathbb{Q})$ for any $s \in S(\mathbb{C})$.

The period matrix $P_{X/S}$ defines a holomorphic map

$$\phi : V \to M_\mu.$$

We let $Z \subset V \times M_\mu$ be the graph of this function. The first step in our process is determining the $\mathbb{C}$-Zariski closure of Z.

Lemma 4.4. *Let Z be as above then the $\mathbb{C}$-Zariski closure of Z is*

$$S_{\mathbb{C}} \times \{M : {}^tMJ_\mu M = (2\pi i)^{-n} J_\mu\}.$$

In order to prove this we will employ the monodromy action in an essential way. For this purpose we will need to review some further properties of the isomorphism $P^n_{X/S}$.

To this end, let us consider

$$Q^n_{X/S} : R^n f^{an}_* \otimes_{\mathbb{C}_{S^{an}}} \mathcal{O}_{S^{an}} \xrightarrow{\sim} H^n_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}},$$

the inverse of $P^n_{X/S}$.

It is known, see [Kat72] Prop.4.1.2, that this isomorphism restricts to an isomorphism of local systems

$$Q : R^n f^{an}_* \mathbb{C}_{X^{an}} \xrightarrow{\sim} \mathbb{R}^n f^{an}_* \Omega^\bullet_{X^{an}_{\mathbb{C}}/\mathbb{C}} \xrightarrow{\sim} (H^n_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}})^\nabla$$

where $(H^n_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}})^\nabla \subset \mathbb{R}^n f^{an}_* \Omega^\bullet_{X^{an}/S^{an}}$ is the local system of horizontal sections with respect to the Gauss-Manin connection.

Note that we have an inclusion of local systems $R^n f^{an}_* \mathbb{Q} \hookrightarrow R^n f^{an}_* \mathbb{C}$ on S^{an}. This leads to a commutative diagram

$$\begin{tikzcd} \pi_1(S^{an}, z) \arrow[r, "\rho_{H,\mathbb{C}}"] \arrow[d, "\rho_H"'] & \mathrm{GL}(H^n(X^{an}_z, \mathbb{C})) \\ \mathrm{GL}(H^n(X^{an}_z, \mathbb{Q})) \arrow[ur, hook] & \end{tikzcd}$$

for any point $z \in S^{an}$.

In particular, we get, under our assumptions on the connectedness of the group $(\rho_H(\pi_1(S^{an}, z)))^{\mathbb{Q}-Zar}$, that the group $G_{mono,z} := (\rho_{H,\mathbb{C}}(\pi_1(S^{an}, z)))^{\mathbb{C}-Zar}$, i.e. the $\mathbb{C}$-Zariski closure of the image of the fundamental group under $\rho_{H,\mathbb{C}}$, is such that

$$G_{mono,z} = H_z \otimes_{\mathbb{Q}} \mathbb{C}. \tag{4.4}$$

Earlier we saw that we have an isomorphism Q of local systems over S^{an}. By the equivalence of categories between local systems over S^{an} and representations of the fundamental group $\pi_1(S^{an}, z)$ we thus have that the representations

$$\rho_{H,\mathbb{C}} : \pi_1(S^{an}, z) \to \mathrm{GL}(H^n(X_z^{an}, \mathbb{C})), \text{ and}$$
$$\rho_{DR} : \pi_1(S^{an}, z) \to \mathrm{GL}((H^n_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}})^{\nabla}),$$

are conjugate. In fact, keeping in mind that all actions are on the right, we have that $\rho_{DR}(\lambda) = Q(z)^{-1}\rho_{H,\mathbb{C}}(\lambda)Q(z)$, for all $\lambda \in \pi_1(S^{an}, z)$, where $Q(z)$ is the fiber of Q at z. From this we get that

$$G_{DR,z} := (\rho_{DR}(\pi_1(S^{an}, z)))^{\mathbb{C}-Zar} = Q(z)^{-1} G_{mono,z} Q(z). \tag{4.5}$$

Let B be the matrix of the isomorphism $Q|_V$ with respect to the frame $\{\gamma_j^* : 1 \leq j \leq \mu\}$ of the trivialization of $R^n f_*^{an}\mathbb{Q}|_V \subset R^n f_*^{an}\mathbb{C}|_V$, i.e. the dual of the frame given by the γ_j on $R_n f_*^{an}\mathbb{Q}|_V$, and the basis $\{\omega_i : 1 \leq i \leq \mu\}$ chosen above. We then have that the rows b_i of B, which will correspond to $Q|_V(\gamma_i^*)$ written in the basis ω_i, will constitute a basis of the space $\Gamma(V, (H^n_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}})^{\nabla})$. In other words B is a complete solution of the differential equation $\nabla(\omega) = 0$, defined by the Gauss-Manin connection. We note that in our setting the Gauss-Manin connection is known to be defined over the field k by work of Katz and Oda. see [KO68] and [Kat70].

Let $\Gamma \in M_\mu(k(S)$ be the (local) matrix of ∇ on$U$ with respect to the basis given by the ω_i. Writing $\nabla(\omega) = d\omega + \omega\Gamma$, identifying the ω with the $1 \times n$ matrix given by the coefficients of ω in the basis given by the ω_i, we may rewrite the above equation as $d\omega = -\omega\Gamma$. The corresponding matricial differential equation then becomes

$$X' = -X\Gamma. \tag{4.6}$$

The monodromy representation ρ_{DR} defines analytic continuations of solutions at z of the differential equation 4.6. So in considering the value at the point z of the analytic continuation B^λ of the matrix B along the cycle $\lambda \in \pi_1(S^{an}, z)$, corresponding to a loop γ passing through z, all we are doing is multiplying the matrix B_z by $\rho_{DR}(\lambda)$. In other words for $\lambda \in \pi_1(S^{an}, z)$ we have that

$$(B^\lambda)_z = B_z \rho_{DR}(\lambda). \tag{4.7}$$

We apply the ideas presented in the above discussion to prove the following lemma.

Lemma 4.5. *Consider A to be the matrix of the isomorphism $P^n_{X/S}$ on the open analytic set V with respect to the basis ω_i and frame γ_j^* chosen above. Let $z \in V$ and let $\lambda \in \pi_1(S^{an}, z)$. Then the value at z of the analytic continuation A^λ of A along the loop that corresponds to λ is given by*

$$(A^\lambda)_z = A_z \rho_{H,\mathbb{C}}(\lambda)^{-1},$$

where $\rho_{H,\mathbb{C}}$ is the above representation on Betti cohomology.

Proof. We have $A \cdot B = I_\mu$ hence $A^\lambda \cdot B^\lambda = I_\mu$. Using (4.7) we get that $(A^\lambda)_z = \rho_{DR}(\lambda)^{-1} B_z^{-1} = \rho_{DR}(\lambda)^{-1} A_z$.

On the other hand, with the above notation we have that $\rho_{DR}(\lambda) = B_z^{-1} \rho_{H,\mathbb{C}}(\lambda) B_z$. This combined with the above leads to the result. □

Remark 4.6. The same relation holds for the value $P_{X/S}(z)$ at z of the matrix of relative periods $P_{X/S}$, since $P_{X/S} = (2\pi i)^{-n} A$.

We are now in the position to prove 4.4.

Proof of 4.4. Let $Z \subset V \times M_\mu \subset S_\mathbb{C} \times M_\mu$ be the graph of the isomorphism $P_{X/S}|_V$. Let $\tilde{Z}$ be the union of the graphs of all possible analytic continuations of Z. It is easy to see via analytic continuation that we have $(\tilde{Z})^{\mathbb{C}-Zar} = Z^{\mathbb{C}-Zar}$. We also note that for all $z \in V$ we have $(\tilde{Z}_z)^{\mathbb{C}-Zar} \subset (\tilde{Z}^{\mathbb{C}-Zar})_z$ for trivial reasons.

We focus on the points $z \in V$ that are Hodge generic for the variation of $\mathbb{Q}$-HS given by $R^n f_*^{an} \mathbb{Q}_{X^{an}_\mathbb{C}}|_V$. We note that the set of such z in V, which we denote by V_{Hgen}, is uncountable.

By 4.5, and the fact that the rows of the matrix B above are a complete solution of the differential system (4.6), we know that $\tilde{Z}_z = P_{X/S}(z) \rho_{H,\mathbb{C}}(\pi_1(S^{an}, z))$. From this we get that $(\tilde{Z}_z)^{\mathbb{C}-Zar} = P_{X/S}(z) G_{mono,z}$.

From (4.4) we know that $G_{mono,z} = H_z \otimes_\mathbb{Q} \mathbb{C}$ while from 4.3 we know that, since $z \in V_{Hgen}$, we have $H_z \simeq Sp(\mu, \mathbb{Q})$, hence $G_{mono,z} \simeq Sp(\mu, \mathbb{C})$. Hence we have $(\tilde{Z}_z)^{\mathbb{C}-Zar} = P_{X/S}(z) Sp(\mu, \mathbb{C})$.

Using (4.3) together with the above we arrive through elementary reasoning to

$$(\tilde{Z}_z)^{\mathbb{C}-Zar} = \{M \in \mathrm{GL}_\mu(\mathbb{C}) : {}^t M J_\mu M = (2\pi i)^{-n} J_\mu\}. \tag{4.8}$$

Applying this to the fact that for all $z \in V$ we have $(\tilde{Z}_z)^{\mathbb{C}-Zar} \subset (\tilde{Z}^{\mathbb{C}-Zar})_z = (Z^{\mathbb{C}-Zar})_z$, we get that

$$V_{Hgen} \times \{M \in \mathrm{GL}_\mu(\mathbb{C}) : {}^tMJ_\mu M = (2\pi i)^{-n} J_\mu\} \subset Z^{\mathbb{C}-Zar}. \tag{4.9}$$

Now, using the fact that V_{Hgen} is uncountable and taking Zariski closures in (4.9) we get that

$$S_{\mathbb{C}} \times \{M \in \mathrm{GL}_\mu(\mathbb{C}) : {}^tMJ_\mu M = (2\pi i)^{-n} J_\mu\} \subset Z^{\mathbb{C}-Zar}.$$

On the other hand, once again from (4.3), we know that

$$Z \subset S_{\mathbb{C}} \times \{M \in \mathrm{GL}_\mu(\mathbb{C}) : {}^tMJ_\mu M = (2\pi i)^{-n} J_\mu\}$$

which, by once again taking Zariski closures, gives the reverse inclusion. □

Trivial relations over $\mathbb{C}$ for the G-functions

As we remarked earlier, the entries of the first h columns of our matrix $P_{X/S}$ are G-functions, under our choice of basis and trivialization. Let us denote by $y_{i,j}$ these entries and by Y the respective $\mu \times h$ matrix they define. Consider the projection map $\mathrm{pr} : M_\mu \to \mathbb{A}^{\mu\times h}$ that maps a matrix $(a_{i,j}) \in M_\mu$ to the $\mu \times h$ matrix that consists of its first h columns. This maps $P_{X/S}$ to Y.

Lemma 4.7. *Let T be the subvariety of $\mathbb{A}^{\mu\times h}$ defined by the following set of polynomials*

$$\{{}^tb_iJ_\mu b_j : 1 \le i, j \le h\},$$

where b_i denotes the i-th column of a matrix of indeterminates.

Then $Y^{\mathbb{C}(S)-Zar} = T_{\mathbb{C}(S)}$.

Proof. Let $Z_Y \subset V \times M_{\mu\times h}(\mathbb{C})$ denote the graph of Y as a function $Y : V \to M_{\mu\times h}(\mathbb{C})$. It suffices to show that $Z_Y^{\mathbb{C}-Zar} = S \times T$.

The inclusion $Z_Y^{\mathbb{C}-Zar} \subset S \times T$ follows trivially from (4.3), which shows that $Z_Y \subset V \times T(\mathbb{C})$. On the other hand, we have that the map $\mathrm{id}_S \times \mathrm{pr} : S \times M_\mu \to S \times \mathbb{A}^{\mu\times h}_{\mathbb{C}}$ is topologically closed, with respect to the Zariski topology. This implies that $Z_Y^{\mathbb{C}-Zar} = (\mathrm{id} \times \mathrm{pr})(Z^{\mathbb{C}-Zar})$.

By construction we have that the columns c_i of any $\mu \times h$ matrix $C \in T(\mathbb{C})$ will be a basis that spans an isotropic subspace of dimension h with respect to the symplectic form defined by J_μ on $\mathbb{C}^\mu$. It is easy to see that we can extend this set of vectors to a basis $\{c_j : 1 \le j \le \mu\}$ of $\mathbb{C}^\mu$ that satisfies

1. ${}^tc_iJ_\mu c_j = 0$ for all i, j with $|i-j| \neq \mu/2$, and

2. ${}^t c_i J_\mu c_j = (2\pi i)^{-n}$ for $i = j + \mu/2$.

In other words, this is a symplectic basis "twisted" by a factor $(2\pi i)^{-n/2}$. The $\mu \times \mu$ matrix with columns c_i will then be such that $(s, M_C) \in Z^{\mathbb{C}-Zar}$ by 4.4 and by construction $\mathrm{pr}(M_C) = C$.

Combining the above with the fact that $Z_Y^{\mathbb{C}-Zar} = (\mathrm{id} \times \mathrm{pr})(Z^{\mathbb{C}-Zar})$ we have that $S \times T \subset Z_Y^{\mathbb{C}-Zar}$ and our result follows. $\square$

Trivial relations over $\bar{\mathbb{Q}}$

So far we have not used any arithmetic information about the $y_{i,j}$, namely the fact that they are G-functions.

Let $\xi \in \bar{\mathbb{Q}}$. Then a trivial polynomial relation with coefficients in $\bar{\mathbb{Q}}$ at the point ξ among the $y_{i,j} \in \bar{\mathbb{Q}}[[x]]$ is a relation that satisfies the following:

1. there exists $p(x_{i,j}) \in \bar{\mathbb{Q}}[x_{i,j}]$ such that the relation we have is of the form $p(y_{i,j}(\xi)) = 0$,

2. the relation holds v-adically for some place v of $\bar{\mathbb{Q}}$, i.e. letting $i_v : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_v$ we have that the $y_{i,j}$ converge at $i_v(\xi)$ and the above relation is an equality in $\bar{\mathbb{Q}}_v$,

3. there exists a polynomial $q(x)(x_{i,j}) \in \bar{\mathbb{Q}}[x][x_{i,j} : 1 \le i \le \mu, 1 \le j \le h]$ such that it has the same degree as p, with respect to the $x_{i,j}$, and $q(\xi)(x_{i,j}) = p(x_{i,j})$

Therefore, to describe the trivial relations among the values of our G-functions $y_{i,j}$ at some $\xi \in \bar{\mathbb{Q}}$, it is enough to determine the $\bar{\mathbb{Q}}[x]$-Zariski closure of the matrix Y. We do this in the following lemma, which is practically a more detailed rephrasing of 4.2.

Lemma 4.8. *Let Y be the $\mu \times h$ we had above. Then the $\bar{\mathbb{Q}}[x]$-Zariski closure $Y^{\bar{\mathbb{Q}}[x]-Zar}$ of Y is the subvariety of $\mathbb{A}^{\mu \times h}_{\bar{\mathbb{Q}}[x]}$ defined by the following set of polynomials*

$$\{{}^t b_i J_\mu b_j : 1 \le i, j \le h\},$$

where b_i denotes the i-th column of a matrix of indeterminates.

Proof. We let Σ be the set of polynomials above and let I_R be the ideal generated by Σ in the ring $R[x_{i,j}]$, where R will denote different fields in our proof.

In this case from 4.7 we know that $Y^{\mathbb{C}(S)-Zar}$ is equal to $V(I_{\mathbb{C}(S)})$. Note that the elements of Σ all have coefficients in $\bar{\mathbb{Q}}[x]$, in fact they have coefficients in $\bar{\mathbb{Q}}$. From this we get the result we wanted, i.e. $Y^{\bar{\mathbb{Q}}[x]-Zar} = V(I_{\bar{\mathbb{Q}}[x]})$. $\square$

Remark 4.9. Implicit in the previous proof is the fact that we have a polarization that is defined over $k \subset \bar{\mathbb{Q}}$ as a cycle in some de Rham cohomology group.

5

CONSTRUCTING NON-TRIVIAL RELATIONS

5.1 TOWARDS RELATIONS FOR EXCEPTIONAL POINTS

Let $f : X \to S$ be a G-admissible variation of $\mathbb{Q}$-HS. We start with the following definition.

Definition 5.1. Let $s \in S(\bar{\mathbb{Q}})$. Assume that in the decomposition of V_s into irreducible $\mathbb{Q}$-Hodge structures, as in (5.1), there exists at least one irreducible factor V_i whose algebra of endomorphisms D_i is of type IV in Albert's classification. We then say that the point s, or equivalently the corresponding $\mathbb{Q}$-HS, is **pseudo-CM**.

Remark 5.2. We note here that all CM-points $s \in S(\bar{\mathbb{Q}})$ of the variation will satisfy the above definition. The term "pseudo-CM" reflects the fact that the center of a type IV algebra in Albert's classification is a CM field. We note that the points considered here are far more general, at least in principle, than special points.

Notational Conventions

Let $f : X \to S$ be a G-admissible variation as above and let $s \in S(L)$ with $L \subset \bar{\mathbb{Q}}$.

First of all, note that from the semisimplicity of the category of polarized Hodge structures, we know that we may write

$$V_s = V_1^{m_1} \oplus \ldots \oplus V_r^{m_r}, \tag{5.1}$$

with (V_i, φ_i) irreducible polarized $\mathbb{Q}$-HS that are non-isomorphic to each other. Let $D_i := \mathrm{End}(V_i)^{G_{mt}(V_i)}$ be the respective endomorphism algebras so that

$$D_s = M_{m_1}(D_1) \times \ldots \times M_{m_r}(D_r).$$

From 3.1 we know that, assuming the absolute Hodge conjecture, there exists a finite extension $\hat{L}$ of L such that D_s acts on $H^n_{DR}(X_{s,\hat{L}}/\hat{L})$ and that this action is compatible with the comparison isomorphism between algebraic de Rham and singular cohomology. Again assuming the absolute Hodge conjecture, we know from 3.2 that the degree $[\hat{L} : L]$ of the extension is bounded by a bound independent of the point s. We assume from now on that $\hat{L} = L$ and return to this issue in the proof of 1.19.

We let F_i denote the center of the algebra D_i for $1 \leq i \leq r$ and note that these are number fields due to Albert's classification. We introduce the following notation

- $\hat{E}_s = F_1^{m_1} \times \ldots \times F_r^{m_r}$ the maximal commutative semi-simple algebra of D_s,
- $\hat{F}_i$ the Galois closure of the field F_i in $\mathbb{C}$,
- $\hat{F}_s$ the compositum of the fields $\hat{F}_i$ together with the field L.

Splittings in cohomology and homology

Let us assume $f : X \to S$ is a G-admissible variation as above and let $s \in S(L)$, where L/K is a finite extension. We assume that s is archimedeanly close to the point s_0 on S', with respect to a fixed inclusion $L \hookrightarrow \mathbb{C}$. In particular we assume that it is in the image of the inclusion of a punctured unit disc $\Delta^* \subset S^{an}_{\mathbb{C}}$ centered at s_0.

Under the above assumption, $L = \hat{L}$, we know that we have two splittings. Namely, on the one hand we get a splitting

$$H_n(X^{an}_{s,\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \hat{F}_s = \bigoplus_{\sigma:\hat{E}\to\mathbb{C}} \hat{W}_\sigma, \tag{5.2}$$

induced from the splitting $\hat{E}_s \otimes_{\mathbb{Q}} \hat{F}_s = \bigoplus_{\sigma:\hat{E}_s\to\mathbb{C}} \hat{F}_s^\sigma$, where $\hat{F}_s^\sigma$ denotes the field $\hat{F}_s$ viewed as an $\hat{E}_s$-module with the action of $\hat{E}_s$ being multiplication by σ. We also note that on $\hat{W}_\sigma$ the algebra $\hat{E}_s$ acts again via multiplication with its character σ.

On the other hand, we have a splitting

$$H^n_{DR}(X_s/L) \otimes_L \hat{F}_s = \bigoplus_{\sigma:\hat{E}_s\to\mathbb{C}} \hat{W}^\sigma_{DR}, \tag{5.3}$$

which once again comes from the above splitting of $\hat{E}_s \otimes_{\mathbb{Q}} \hat{F}_s$. In particular, we note that the action of $\hat{E}_s$ on $\hat{W}^\sigma_{DR}$ comes once again via σ.

Duality of the splittings

We start by highlighting how the two splittings interact with one another via the comparison isomorphism

$$P^n_{X_s} : H^n_{DR}(X_s/L) \otimes_L \mathbb{C} \to H^n(X^{an}_{s,\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

The following lemma is already noted as a property of the splittings by André, we include a short proof for the sake of completeness.

Lemma 5.3. *For all $\sigma \neq \tau$ if $\omega \in W^{\tau}_{DR}$ and $\gamma \in W_{\sigma}$ then*

$$\int_{\gamma} \omega = 0.$$

Proof. Let us fix $\sigma \neq \tau$ as above and let $\omega \in \hat{W}^{\tau}_{DR}$ and $\gamma \in \hat{W}_{\sigma}$.

For all $e \in \hat{E}_s$ we have that $P^n_{X_s}(d\omega) = P^n_{X_s}(\tau(d)\omega) = \tau(d)P^n_{X_s}(\omega) = d \cdot P^n_{X_s}(\omega)$, where the last equality follows from the moreover part of 3.1. The algebra $\hat{E}_s$, and in particular its group of invertible elements $\hat{E}_s^{\times}$, acts by definition on V_s as endomorphisms of the Hodge structure. The action of $\hat{E}_s^{\times}$ on the dual space V_s^* will thus be the dual of that of V_s.

In particular for any $\gamma \in \hat{W}_{\sigma}$, for any $e \in \hat{E}_s^{\times}$, and for any $\delta \in V_s$, we get that $(e \cdot \gamma)(e \cdot \delta) = \gamma(\delta)$. Taking $\delta = P^n_{X_s}(\omega)$ we get that for all $\gamma \in \hat{W}_{\sigma}$ and for all $e \in \hat{E}_s^{\times}$

$$\int_{\gamma} \omega = \gamma(P^n_{X_s}(\omega)) = (e \cdot \gamma)(e \cdot P^n_{X_s}(\omega)).$$

But we know that $(e \cdot \gamma)(e \cdot P^n_{X_s}(\omega)) = (\sigma(e^{-1})\gamma)(\tau(e)P^n_{X_s}(\omega))$, where we used the duality between the actions of $\hat{E}_s^{\times}$ on V_s and V_s^*. Putting everything together we get that for all $e \in \hat{E}_s^{\times}$ we will have that

$$\int_{\gamma} \omega = \sigma(e)^{-1}\tau(e) \int_{\gamma} \omega.$$

Since $\sigma \neq \tau$ we can find such an e with $\sigma(e) \neq \tau(e)$ and the lemma follows. □

Involutions and symplectic bases

In creating the relations we want we will need to construct symplectic bases with particular properties. To construct these we will need to review some facts about the involutions of the algebras of Hodge endomorphisms and see how they interact with the splittings we have.

For the weight n $\mathbb{Q}$-HS given by V_s we denote by $\langle , \rangle$ the symplectic form defined by the polarization on V_s. By duality we get a polarized $\mathbb{Q}$-HS of weight $-n$ on the dual space $V_s^* := H_n(\bar{X}^{an}_{s,\mathbb{C}}, \mathbb{Q})$, and we denote the symplectic form given by the polarization again by $\langle , \rangle$. We note that these two symplectic forms are dual.

The algebra D_s comes equipped with an involution, which we denote by $d \mapsto d^\dagger$, that is defined by the relation

$$\langle d \cdot v, w \rangle = \langle v, d^\dagger \cdot w \rangle, \tag{5.4}$$

for all $d \in D_s$ and for all $v, w \in V_s^*$, or equivalently for all $v, w \in V_s$.

In the decomposition (5.1) of V_s, or its dual V_s^*, the polarization on each V_i, or V_i^* respectively, is given by the restriction of the polarization of V_s, or its dual respectively. Therefore the involution $d \mapsto d^\dagger$ of D_s restricts to the positive involutions of the respective algebras D_i.

The algebra homomorphisms $\sigma : \hat{E}_s \to \mathbb{C}$ have a convenient description. Writing

$$\hat{E}_s = F_1^{m_1} \times \ldots \times F_r^{m_r},$$

we let $\mathrm{pr}_{j,l} : \hat{E}_s \to F_j$, where $1 \leq j \leq r$ and $1 \leq l \leq m_j$, denote the projection of $\hat{E}_s$ onto the l-th factor of $F_j^{m_j}$, which will act respectively on the l-th factor of $V_j^{m_j}$ that appears in the decomposition. Then any algebra homomorphism $\sigma : \hat{E}_s \to \mathbb{C}$ can be written as

$$\sigma = \tilde{\sigma} \circ \mathrm{pr}_{j,l} \tag{5.5}$$

for some j and l as above and some $\tilde{\sigma} : F_j \hookrightarrow \mathbb{C}$. For convenience, from now on we define the notation

$$\tilde{\sigma}_{j,l} := \tilde{\sigma} \circ \mathrm{pr}_{j,l}\,. \tag{5.6}$$

Lemma 5.4. *Consider the splitting (5.2). Then for the subspaces $\hat{W}_\sigma$ the following hold:*

1. *If $\sigma = \tilde{\sigma}_{j,l}$ then $\hat{W}_\sigma$ is contained in the l-th factor of $(V_j^*)^{m_j}$,*
2. *Let $\sigma = \tilde{\sigma}_{j,l}$ and let τ be some non-zero algebra homomorphism $\hat{E}_s \to \mathbb{C}$. Consider non-zero vectors $v \in \hat{W}_\sigma$ and $w \in \hat{W}_\tau$. If we assume that $\langle v, w \rangle \neq 0$ then one of the following cases holds*
   a) *$\sigma = \tau$ and the algebra D_j of Hodge endomorphisms is of Type I, II or III in Albert's classification, or*
   b) *$\sigma = \bar{\tau}$, where $\bar{(\cdot)}$ denotes complex conjugation, and D_j is of Type IV in Albert's classification.*

Proof. The first part of the lemma is trivial.

For the second part let $v \in \hat{W}_\sigma$ and $w \in \hat{W}_\tau$ be non-zero vectors as above with $\langle v, w\rangle \neq 0$. From the preceding discussion there exists a pair (j', l') for τ such that $\tau = \tilde{\tau} \circ \mathrm{pr}_{j',l'}$, where $\tilde{\tau} : F_{j'} \hookrightarrow \mathbb{C}$. From the first part of this lemma we also know that $\hat{W}_\tau$ is contained in the l'-th factor of $(V_{j'}^*)^{m_{j'}}$.

The subspaces V_i^* of V_s^* are symplectic, with their symplectic inner product being the restriction of that of V_s^*. This immediately implies that $(j, l) = (j', l')$.

For any $d \in \hat{E}_s$ we have that $\langle d \cdot v, d \cdot w\rangle = \langle \sigma(d)v, \tau(d)w\rangle = \sigma(d)\tau(d)\langle v, w\rangle$. On the other hand using the defining property of the involution we get

$$\langle d \cdot v, d \cdot w\rangle = \langle v, (d^\dagger d) \cdot w\rangle = \tau(d^\dagger d)\langle v, w\rangle. \tag{5.7}$$

Since, by assumption $\langle v, w\rangle \neq 0$ the above relations imply that for all $d \in \hat{E}_s$ we have

$$\sigma(d)\tau(d) = \tau(d^\dagger)\tau(d). \tag{5.8}$$

Let F_j be the center of the algebra D_j. Then (5.8) implies that for all $d \in F_j$

$$\tilde{\sigma}(d)\tilde{\tau}(d) = \tilde{\tau}(d^\dagger)\tilde{\tau}(d). \tag{5.9}$$

In particular, this implies that for all $d \in F_j$ we have that

$$\tilde{\tau}(d^\dagger) = \tilde{\sigma}(d). \tag{5.10}$$

If D_j is of Type I in Albert's classification then the involution restricts to the identity and we get trivially that $\tilde{\tau} = \tilde{\sigma}$, and hence also $\sigma = \tau$. So our result follows in this case.

If D_j is of Type II then we have that F_j is a totally real field, D_j is a quaternion algebra over F_j and there exists $a \in D_j$ such that the involution is given by $d^\dagger = a d^* a^{-1}$ on D_j, where $d^* = \mathrm{tr}_{D_j/F_j}(d) - d$. Note that for $d \in F_j = Z(D_j)$ we have that $\mathrm{tr}_{D_j/F_j}(d) = 2d$, so that $d^\dagger = d$ for all $d \in F_j$. Combining these observations with (5.10) we get that $\tau = \sigma$.

The same argument we just used for the case of Type II algebras works for the case of Type III algebras, though we do not need to introduce any element a as above since the involution in this case is equal to the canonical involution.

Finally, let us assume that D_j is of type IV in Albert's classification. In this case F_j is a CM-field. In this case the involution is known to restrict to complex conjugation on the field F_j. In other words $d^\dagger = \bar{d}$ for $d \in F_j$. This, together with (5.10), implies that $\tilde{\sigma}(d) = \tilde{\tau}(\bar{d})$. Since F_j is a CM-field this implies that $\tilde{\sigma} = \bar{\tilde{\tau}}$ and by extension $\sigma = \bar{\tau}$. $\square$

Remark 5.5. The above lemma shows that the splitting (5.2) of V_s^* is comprised of two types of mutually skew-orthogonal symplectic subspaces. On the one hand, we have the symplectic subspaces $\hat{W}_\sigma$ that are contained in some V_j that is of Type I-III, and on the other hand we have the symplectic subspaces of the form $\hat{W}_\tau \oplus \hat{W}_{\bar{\tau}}$, where $\hat{W}_\tau$ is contained in some V_j that is of Type IV. For the second type, note that we also have that $\hat{W}_\tau$ and $\hat{W}_{\bar{\tau}}$ are transverse Lagrangians of these symplectic subspaces.

Constructing relations

We return to our original G-admissible variation of $\mathbb{Q}$-HS, restricted to Δ^*. We let $s \in S(L)$ be a fixed point which we assume is in Δ^*. We then have the totally isotropic local subsystem of rank h over the ring $\mathcal{O}_{S^{an}_{\mathbb{C}}}|_{\Delta^*}$

$$\mathcal{M}_0 := M_0 R_n (f^{an}_{\mathbb{C}})_*(\mathbb{Q})|_{\Delta^*}$$

of the local system $R_n(f^{an}_{\mathbb{C}})_*(\mathbb{Q})|_{\Delta^*}$, which has rank $\mu := \dim_{\mathbb{Q}} V_s$.

We fix a basis $\{\omega_i : 1 \le i \le \mu\}$ of $H^n_{DR}(X/S)$ over some dense open subset $U \subset S$ and a trivialization $\{\gamma_j : 1 \le j \le \mu\}$ of $R_n f^{an}_* \mathbb{Q}|_V$ where V is some open analytic subset of U^{an} with $s \in V \subset \Delta^*$. We may and do choose these so that the following conditions are satisfied:

1. the matrices of the skew-symmetric forms on $H^n_{DR}(X/S)$ and $R_n f^{an}_* \mathbb{Q}$ induced by the polarization written with respect to the basis $\{\omega_i\}$ and trivialization $\{\gamma_j\}$ respectively are both equal to J_μ,

2. $\gamma_1, \dots, \gamma_h \in \mathcal{M}_0|_V$ and $\gamma_1, \dots, \gamma_{\mu/2} \in \mathcal{M}^+$,

where $\mathcal{M}^+$ is a maximal totally isotropic local subsystem of $R_n(f^{an}_{\mathbb{C}})_*(\mathbb{Q})|_V$ that contains $\mathcal{M}_0|_V$.

Let us now consider the relative comparison isomorphism

$$P^n_{X/S} : H^n_{DR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}_{\mathbb{C}}} \to R^n f^{an}_* \mathbb{Q}_{X^{an}_{\mathbb{C}}} \otimes_{\mathbb{Q}_{S^{an}_{\mathbb{C}}}} \mathcal{O}_{S^{an}_{\mathbb{C}}},$$

and restrict it over the set V. With respect to the above choices we let $P_{X/S} = \frac{1}{(2\pi i)^n} (\int_{\gamma_j} \omega_i)$ for the matrix of periods of f.

Let us write $P_{X/S} = \begin{pmatrix} \Omega_1 & \Omega_2 \\ N_1 & N_2 \end{pmatrix}$. From Theorem 2 of Chapter IX, §4 of [And89], we know that the first h columns of this matrix have entries that are G-functions. It is among their values at $\xi = x(s)$ that we want to find some relation that reflects the action of $\hat{E}_s$.

Lemma 5.6. *Assume that $h \ge 2$ and that for the point $s \in S(L)$ one of the following is true*

1. *there exists $\tau : \hat{E}_s \to \mathbb{C}$ such that $h > \dim_{\hat{F}_s} \hat{W}_\tau$, or*

2. *s is a pseudo-CM point and*

$$h \geq \min\{\dim \hat{W}_\tau : \hat{W}_\tau \subset V_{i(\tau)}, \textit{with } V_{i(\tau)} \textit{ of type IV }\}.$$

Then, there exists an algebraic relation among the values at $\xi = x(s)$ of the entries of the first h columns of the matrices Ω_1 and N_1. Moreover, this relation corresponds to some homogeneous polynomial with coefficients in $\hat{F}_s$ and degree ≤ 2.

Proof. We take cases depending on the interplay between $\mathcal{M}_{0,s} \otimes \hat{F}_s$ and the splitting (5.2). We also assume that $V_s = H^n(\bar{X}_{s,\mathbb{C}}, \mathbb{Q})$ has a decomposition as in (5.1).

Case 1: Assume there exists some $\tau : \hat{E}_s \to \mathbb{C}$ such that the following holds

$$\left(\bigoplus_{\substack{\sigma:\hat{E}_s \to \mathbb{C} \\ \sigma \neq \tau}} \hat{W}_\sigma \right) \cap (\mathcal{M}_s \otimes \hat{F}_s) \neq 0, \tag{5.11}$$

then we get, at least one relation of degree 1.

Note that for dimension reasons (5.11) is satisfied for the τ of the first condition above.

Indeed, let $\gamma \in \Gamma(V, R_n(f_{\mathbb{C}}^{an})_*(\mathbb{Q}))$ be a section such that $\gamma(s)$ belong to the non-zero space of (5.11). From 5.3 we get that for all $\omega \in \hat{W}^\tau_{DR}$ we have

$$\frac{1}{(2\pi i)^n} \int_{\gamma(s)} \omega = 0. \tag{5.12}$$

Writing γ as an $\hat{F}_s$-linear combination of the γ_j with $1 \leq j \leq h$ and ω as an $\hat{F}_s$-linear combination of the ω_i with $1 \leq i \leq \mu$, we have that (5.12) leads to a linear equation among the values of the G-functions in question at ξ.

Case 2: Assume that for all $\tau : \hat{E}_s \to \mathbb{C}$ we have

$$\left(\bigoplus_{\substack{\sigma:\hat{E}_s \to \mathbb{C} \\ \sigma \neq \tau}} \hat{W}_\sigma \right) \cap (\mathcal{M}_s \otimes \hat{F}_s) = 0, \tag{5.13}$$

then we want to show that we can create a relation of degree 2.

First of all, we may assume, which we do from now on, that $\dim_{\hat{F}_s} \hat{W}_\sigma \geq h$ for all σ, otherwise we are in case 1, for dimension reasons.

The first step in creating the relations we want is defining symplectic bases with particular properties, which we do in the following claims.

Claim 1: There exists a symplectic basis $e_1, \ldots, e_{\mu/2}, f_1, \ldots, f_{\mu/2}$ of the symplectic vector space $V_s^* \otimes_{\mathbb{Q}} \hat{F} := H_n(\bar{X}_{s,\mathbb{C}}, \mathbb{Q}) \otimes \hat{F}_s$ that satisfies the following properties

1. $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$ and $\langle e_i, f_j \rangle = \delta_{i,j}$ for all i, j.
2. $e_j = \gamma_j(s)$ for $1 \leq j \leq h$.
3. There exists $\tau : \hat{E}_s \to \mathbb{C}$ such that
   a) $\hat{W}_\tau$ is contained in $V^*_{i(\tau)} \otimes \hat{F}_s$, where $V^*_{i(\tau)}$ is some irreducible sub-Hodge Structure of V^*_s which is of Type IV, and
   b) $f_j \in \hat{W}_\tau$ for $1 \leq j \leq h$,
   c) $\dim_{\hat{F}_s} \hat{W}_\tau = h$.

Proof of Claim 1. From 4.1 we know that choosing any basis of local sections $\gamma_j(s)$ of $\mathcal{M}_{0,s}$, its vectors will satisfy $\langle \gamma_i(s), \gamma_j(s) \rangle = 0$ for all i, j. Assume that we have fixed one such basis as above and fix an indexing of the set $\{\sigma : \sigma : \hat{E}_s \to \mathbb{C}\} = \{\sigma_i : 1 \leq i \leq m(s)\}$. We can then write uniquely

$$\gamma_j(s) = w_{j,1} + \ldots + w_{j,m(s)} \tag{5.14}$$

where $1 \leq j \leq h$ and $w_{j,i} \in \hat{W}_{\sigma_i}$.

By assumption the $\mathbb{Q}$-HS V_s is pseudo-CM, therefore there exists τ such that $\hat{W}_\tau$ is as we want in the claim and the same holds for $\hat{W}_{\bar{\tau}}$. Without loss of generality assume that $\bar{\tau} = \sigma_1$. Since we are in Case 2, we also know that $\dim_{\hat{F}_s} \hat{W}_{\bar{\tau}} \geq h$ and that (5.13) holds. From (5.13) we get that the vectors $w_{j,1} \in \hat{W}_{\bar{\tau}}$ are in fact linearly independent.

By 5.4 we know that $\hat{W}_{\bar{\tau}} \oplus \hat{W}_\tau$ is a symplectic vector space with $\hat{W}_\tau$ and $\hat{W}_{\bar{\tau}}$ being transverse Lagrangians.

Let v_j with $1 \leq j \leq \dim_{\hat{F}_s} \hat{W}_{\bar{\tau}}$ be a basis of $\hat{W}_{\bar{\tau}}$ with $v_j = w_{j,1}$ for $1 \leq j \leq h$. We complete this to a symplectic basis v_i, f_j, with $1 \leq j \leq \dim_{\hat{F}_s} \hat{W}_{\bar{\tau}}$ of $\hat{W}_{\bar{\tau}} \oplus \hat{W}_\tau$ such that the f_j are a basis of $\hat{W}_\tau$. Then we have, by construction and by 5.4, that

$$\langle \gamma_i(s), f_j \rangle = \delta_{i,j} \tag{5.15}$$

for all $1 \leq i, j \leq h$.

Therefore, setting $e_i := \gamma_i(s)$ for $1 \leq i \leq h$ the result follows by extending the set of vectors $\{e_i, f_i : 1 \leq i \leq h\}$ to a symplectic basis of $V^*_s \otimes \hat{F}_s$. Finally, note that the τ was arbitrary with $\hat{W}_\tau$ being contained in a type IV sub-Hodge structure of V^*_s. Therefore, by the assumption that $h \geq \min\{\dim \hat{W}_\tau : \hat{W}_\tau \subset V^*_{i(\tau)}, \text{ with } V^*_{i(\tau)} \text{ of type IV }\}$ in our lemma and the assumption in this second case that $\dim \hat{W}_\sigma \geq h$ for all σ we get that we may find such a τ that also satisfies the last condition of our claim. □

From now on we fix the τ we found in Claim 1. Having created a symplectic basis for $H_n(\bar{X}^{an}_{s,\mathbb{C}}, \mathbb{Q}) \otimes \hat{F}_s$ we want to construct a symplectic basis of $H^n_{DR}(X_s) \otimes_L \hat{F}_s$ in a way that lets us take advantage of5.3.

Claim 2: There exists a symplectic basis $e_1^{DR}, \dots, e_{\mu/2}^{DR}, f_1^{DR}, \dots, f_{\mu/2}^{DR}$ of $H^n_{DR}(X_s) \otimes_L \hat{F}_s$ such that the following holds

1. $\forall j$ we have that $e_j^{DR} \in \hat{W}^\sigma_{DR}$ for some $\sigma \neq \tau$,
2. for $1 \leq j \leq h$ we have that $f_j^{DR} \in \hat{W}^\tau_{DR}$,
3. for $h+1 \leq h \leq \mu/2$ we have that $f_j^{DR} \in \hat{W}^\sigma_{DR}$ for some $\sigma \neq \tau$.

Proof of Claim 2. We start by noting that the results of 5.4 apply easily via duality to the splitting (5.3) via $P^n_{X_s}$, due to our assumption that $L = \hat{L}$. In particular, via duality we get that for $\sigma : \hat{E}_s \to \mathbb{C}$ the subspaces $\hat{W}^\sigma_{DR}$ are once again divided into two categories

- $\hat{W}^\sigma_{DR}$ that are symplectic subspaces, corresponding to $\hat{W}_\sigma$ that are contained in simple sub-Hodge structures of V_s^*, after these are tensored with $\hat{F}$, that are of Type I, II or III, and
- $\hat{W}^\sigma_{DR}$ that are isotropic subspaces appearing in pairs such that σ and $\bar{\sigma}$ are both algebra homomorphisms $\hat{E}_s \to \mathbb{C}$ and $\hat{W}^\sigma_{DR} \oplus \hat{W}^{\bar{\sigma}}_{DR}$ is a symplectic subspace. These correspond via duality to the $\hat{W}_\sigma$ that are contained in simple sub-Hodge structures of V_s^*, again after these are tensored with $\hat{F}$, that are of Type IV.

With that in mind, for each σ we pick vectors e_i^σ so that

- the e_i^σ are the basis of a Lagrangian subspace of $\hat{W}^\sigma_{DR}$ if we are in the first case above, so that in this case $1 \leq i \leq \frac{1}{2} \dim_{\hat{F}_s} \hat{W}^\sigma_{DR}$,
- the $e_i^{\bar{\tau}}$ are a basis of $\hat{W}^{\bar{\tau}}_{DR}$ of our fixed τ, and
- in the second case above for each $\sigma \neq \tau, \bar{\tau}$ we pick one σ for each pair $(\sigma, \bar{\sigma})$ and we let e_i^σ be a basis of $\hat{W}^\sigma_{DR}$.

Let e_j^{DR}, with $1 \leq j \leq \mu$, be any indexing of the set of all the e_i^σ above. The spanning set of these defines a Lagrangian subspace of $H^n_{DR}(X_s) \otimes_L \hat{F}_s$. In a similar manner, by the above remarks derived from 5.4, we can construct a basis of a transverse Lagrangian to the Lagrangian spanned by the e_i^{DR} with f_j^{DR} also elements of the various $\hat{W}^\sigma_{DR}$. It is also straightforward from the above that we may pick $f_1^{DR}, \dots, f_h^{DR} \in \hat{W}^\tau_{DR}$. □

Step 1: Changing bases. We note that the bases $\beta_2 := \{e_i, f_i : 1 \leq i \leq \mu/2\}$ and $\beta_2^{DR} := \{e_i^{DR}, f_i^{DR} : 1 \leq i \leq \mu/2\}$ that were created above are $\hat{F}_s$-linear combinations of the bases $\beta_1 := \{\gamma_j(s) : 1 \leq j \leq \mu\}$ and

$\beta_1^{DR} := \{\omega_j(s) : 1 \le j \le \mu\}$ respectively. Since all bases are by construction symplectic the base change matrices are all symplectic matrices. Note that for the change of base matrix $[I_\mu]_{\beta_1}^{\beta_2}$ we will have by construction of β_2 have that its first h columns will be

$$\begin{pmatrix} I_h \\ 0 \end{pmatrix}.$$

Let us consider the isomorphism $P^n_{\bar{X}_s} : H^n_{DR}(\bar{X}_s) \otimes_{\bar{\mathbb{Q}}} \mathbb{C} \to H^n(\bar{X}^{an}_{s,\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$. Let $\tilde{P}_j$ for $j = 1,2$ be the matrix[1] of this isomorphism with respect the basis β_j^{DR} and the dual of the basis β_j. We are interested in the matrices $P_j := \frac{1}{(2\pi i)^n} \tilde{P}_j$, note that P_1 is the value of the relative period matrix at $\xi = x(s)$. For the matrices P_j we have

$$P_2 = [I_{2\mu}]^{\beta_1^{DR}}_{\beta_2^{DR}} P_1 [I_{2\mu}]^{\beta_2^\vee}_{\beta_1^\vee}, \tag{5.16}$$

and all these matrices are symplectic, while the two change of base matrices will have coefficients in the field $\hat{F}_s$.

Step 2: Relations on P_2. Let us examine the matrix P_2 in more detail. Write

$$P_2 = \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \Delta_1 & \Delta_2 \end{pmatrix}, \tag{5.17}$$

where Γ_i and Δ_i are $\mu/2 \times \mu/2$ matrices. For convenience we also let $\tilde{\Gamma}_i$ and $\tilde{\Delta}_i$ for the $\mu/2 \times h$ matrices defined by the first h first columns of the matrices Γ_i and Δ_i for $i = 1,2$ respectively.

From the fact that $(2\pi i)^n P_2$ is symplectic we have the relations[2]

$${}^t\Delta_j \Gamma_j = {}^t\Gamma_j \Delta_j, \tag{5.18}$$

for $j = 1,2$ and also

$${}^t\Delta_2 \Gamma_1 - {}^t\Gamma_2 \Delta_1 = \frac{I_{\mu/2}}{(2\pi i)^n} \tag{5.19}$$

By construction of the bases in the two claims above and 5.3 we immediately get that $\tilde{\Gamma}_2 = 0$.

Let us set $\Delta_{2,h}$ to be the $h \times h$ matrix given by $(\int_{f_j} f_i^{DR})_{1 \le i,j \le h}$. Let us also set $\Delta_{1,h}$ be the $h \times h$ matrix given by $(\int_{\gamma_j} f_i^{DR})_{1 \le i,j \le h}$. In other words, $\Delta_{i,h}$ is the submatrix of Δ_i that is comprised of the entries in the first h

1 Note that in keeping with our earlier notation the matrix acts via multiplication on the right, i.e. $P^n(x) = [x]_{\beta_j^{DR}} \tilde{P}_j$.

2 This follows from the Riemann relations. See the A for more details.

columns and first h rows of Δ_i.

Claim 3: There exists an $h \times h$ matrix $T \in \mathrm{GL}_h(\hat{F}_s)$ which is such that

$$\Delta_{1,h} = \Delta_{2,h} \cdot^t T. \tag{5.20}$$

Proof of Claim 3. We have a pairing

$$\hat{W}^{\tau}_{DR} \times H_n(\bar{X}^{an}_{s,\mathbb{C}}, \mathbb{Q}) \otimes \hat{F}_s \to \mathbb{C}, \tag{5.21}$$

defined by $(\omega, \gamma) \mapsto \int_\gamma \omega$.

By 5.3 we have that this induces a perfect pairing

$$\hat{W}^{\tau}_{DR} \times \left((H_n(\bar{X}^{an}_{s,\mathbb{C}}, \mathbb{Q}) \otimes \hat{F}_s) \Big/ \Big(\bigoplus_{\substack{\sigma:\hat{E}_s \to \mathbb{C} \\ \sigma \neq \tau}} \hat{W}_\sigma \Big) \right) \to \mathbb{C},$$

On the one hand, we know that $\{f_1, \ldots, f_h\}$, the basis of $\hat{W}_\tau$, maps to a basis in the quotient $(H_n(\bar{X}^{an}_{s,\mathbb{C}}, \mathbb{Q}) \otimes \hat{F}_s) \Big/ \Big(\bigoplus_{\substack{\sigma:\hat{E}_s \to \mathbb{C} \\ \sigma \neq \tau}} \hat{W}_\sigma \Big)$. On the other hand, from the assumption (5.13), we get that the basis $\{\gamma_1, \ldots, \gamma_h\}$ also maps to a basis of the same quotient.

Let T be the transpose of the change of basis matrix from the basis induced by the γ_j on $(H_n(\bar{X}^{an}_{s,\mathbb{C}}, \mathbb{Q}) \otimes \hat{F}_s) \Big/ \Big(\bigoplus_{\substack{\sigma:\hat{E}_s \to \mathbb{C} \\ \sigma \neq \tau}} \hat{W}_\sigma \Big)$, to that induced on the same space by the f_j. Then $T \in \mathrm{GL}_h(\hat{F}_s)$ and (5.21) holds. $\square$

Let $\Gamma_{1,h}$ be, once again, the submatrix of Γ_1 that is comprised of the entries in the first h columns and first h rows of Γ_1. We have already seen that $\tilde{\Gamma}_2 = 0$ and, by the construction of Claims 1 and 2 and 5.3, that

$$\tilde{\Delta}_2 = \begin{pmatrix} \Delta_{2,h} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Using these facts we derive from (5.19) that

$${}^t\Delta_{2,h}\Gamma_{1,h} = \frac{1}{(2\pi i)^n} I_h. \tag{5.22}$$

Let us now multiply both sides of (5.22) by T. Then, using (5.20), we get

$${}^t\Delta_{1,h}\Gamma_{1,h} = \frac{T}{(2\pi i)^n}. \tag{5.23}$$

Step 3: Relations on P_1. The relation (5.23) we created on the first h columns of the matrix P_2 will translate to relations among the coefficients of the first h columns of the matrix P_1. Since these are the values of the G-functions we are interested in, this will finish the proof of this lemma.

We start with introducing some notation let $[I_\mu]^{\beta_2^\vee}_{\beta_1^\vee} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ and $[I_\mu]^{\beta_1^{DR}}_{\beta_2^{DR}} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$, were the A_i, B_i, C_i, and $D_i \in M_{\mu/2}(\hat{F}_s)$. In keeping the same notation as above, for any matrix $A \in M_{\mu/2}(\mathbb{C})$ we define $\tilde{A}$ to be the $\mu/2 \times h$ matrix defined by the first h columns of A. Note that by our construction in Claim 1 we know that $\tilde{C}_1 = 0$ and $\tilde{A}_1 = \begin{pmatrix} I_h \\ 0 \end{pmatrix}$.

With this notation (5.16) becomes

$$\begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \Delta_1 & \Delta_2 \end{pmatrix} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \begin{pmatrix} \Omega_1(s) & \Omega_2(s) \\ N_1(s) & N_2(s) \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}. \tag{5.24}$$

From this we get the following two relations

$$\Gamma_1 = A_2\Omega_1(s)A_1 + A_2\Omega_2(s)C_1 + B_2N_1(s)A_1 + B_2N_2(s)C_1, \tag{5.25}$$

and

$$\Delta_1 = C_2\Omega_1(s)A_1 + C_2\Omega_2(s)C_1 + D_2N_1(s)A_1 + D_2N_2(s)C_1. \tag{5.26}$$

Now we notice that for any matrices $A, B \in M_\mu(\mathbb{C})$ we have that $\widetilde{AB} = A\tilde{B}$ and that $A \cdot \begin{pmatrix} I_h \\ 0 \end{pmatrix} = \tilde{A}$. Using these observations on (5.25) and (5.26) we get

$$\tilde{\Gamma}_1 = A_2\tilde{\Omega}_1(s) + B_2\tilde{N}_1(s), \tag{5.27}$$

and

$$\tilde{\Delta}_1 = C_2\tilde{\Omega}_1(s) + D_2\tilde{N}_1(s) \tag{5.28}$$

Substituting these in (5.23) we get

$$^t((C_2\tilde{\Omega}_1(s) + D_2\tilde{N}_1(s))_h)(A_2\tilde{\Omega}_1(s) + B_2\tilde{N}_1(s))_h = \frac{T}{(2\pi i)^n}, \tag{5.29}$$

where, using the same notation as earlier, the subscript h signifies that we are considering the $h \times h$ submatrices that are comprised by the first h rows of these $\mu/2 \times h$ matrices.

Since we are assuming that $h \geq 2$, equation (5.29) provides relations among the values of the G-functions we want at $\xi = x(s)$ that, upon getting rid of the factor $(2\pi i)^n$, correspond to homogeneous polynomials with coefficients in $\hat{F}_s$ and degree ≤ 2. □

Some cleaning up

The technical conditions

$$\exists \tau : \hat{E}_s \to \mathbb{C} \text{ such that } h > \dim_{\hat{F}_s} \hat{W}_\tau \tag{5.30}$$

$$h \geq \min\{\dim \hat{W}_\tau : \hat{W}_\tau \subset V_{i(\tau)}, \text{ with } V_{i(\tau)} \text{ of type IV }\}. \tag{5.31}$$

that appear in 5.6 are by no means aesthetically pleasing! We have dedicated this short section to remedy this fact. In fact we prove the following lemma.

Lemma 5.7. *Condition* (5.31) *is equivalent to the condition*

$$h > \frac{\dim_{\mathbb{Q}} V_j}{[Z(D_j):\mathbb{Q}]} \text{ for some } j,$$

and condition (5.31) *is equivalent to the condition*

$$h \geq \min\{\frac{\dim_{\mathbb{Q}} V_i}{[Z(D_i):\mathbb{Q}]} : i \text{ such that } D_i = \mathrm{End}_{HS}(V_i) \text{ is of type IV }\}.$$

To prove this we work in greater generality with modules of semisimple algebras over $\mathbb{Q}$. The material in this section is definitely not new but we include it for the sake of completeness of our exposition.

Let us fix some notation. We consider a $\mathbb{Q}$-HS V with $\mu := \dim_{\mathbb{Q}} V$ that decomposes as $V = V_1^{m_1} \oplus \cdots \oplus V_r^{m_r}$. We write $D = M_{m_1}(D_1) \oplus \cdots M_{m_r}(D_r)$ for the algebra of Hodge endomorphisms of V, where D_i is the algebra of Hodge endomorphisms of V_i. For each i we let $F_i := Z(D_i)$ be the center of D_i and $f_i := [F_i : \mathbb{Q}]$. Finally, we let $\hat{F}$ be the Galois closure of the compositum of the fields F_i and $\hat{E} := F_1^{m_1} \oplus \cdots \oplus F_r^{m_r}$ be the maximal commutative semisimple algebra of D.

For the non-trivial homomorphisms of algebras $\sigma : \hat{E} \to \hat{F}$ we write $\sigma = \tilde{\sigma}_{j,l}$ as we did earlier. The above result then follows from the following lemma.

Lemma 5.8. *The $\hat{E} \otimes_{\mathbb{Q}} \hat{F}$-module $V \otimes_{\mathbb{Q}} \hat{F}$ has a decomposition as an $\hat{E} \otimes_{\mathbb{Q}} \hat{F}$-module as*

$$V \otimes_{\mathbb{Q}} \hat{F} = \bigoplus_{\sigma : \hat{E} \to \hat{F}} \hat{W}_\sigma,$$

where $\hat{W}_\sigma$ are $\hat{F}$-subspaces of $V \otimes_{\mathbb{Q}} \hat{F}$ on which $\hat{E} \otimes_{\mathbb{Q}} \hat{F}$ acts via multiplication by σ. Moreover, $\dim_{\hat{F}} W_\sigma = \frac{\dim_{\mathbb{Q}} V_{i(\sigma)}}{f_{i(\sigma)}}$ where $i(\sigma) \in \{1, \ldots, r\}$ is such that $\sigma = \tilde{\sigma}_{i(\sigma),l}$ for some l and $\tilde{\sigma}$ with our previous notation.

Proof. First of all, note that $\forall i$ we have $F_i \hookrightarrow \mathrm{End}_{\mathbb{Q}} V_i$ trivially. Therefore V_i is isomorphic, as an F_i-module, to

$$V_i \simeq F_i^{t_i} \tag{5.32}$$

for some t_i. Counting dimensions of these as $\mathbb{Q}$-vector spaces we get that $t_i = \frac{\dim_{\mathbb{Q}} V_i}{f_i}$.

Tensoring both sides of (5.32) by $\otimes_{\mathbb{Q}} \hat{F}$ we get that $V_i \otimes_{\mathbb{Q}} \hat{F} \simeq (F_i \otimes_{\mathbb{Q}} \hat{F})^{t_i}$ as F_i-modules. Now note that since $\hat{F}$ is a Galois extension that contains F_i we have that $F_i \otimes_{\mathbb{Q}} \hat{F} \simeq \bigoplus_{\tilde{\sigma}: F_i \to \hat{F}} \hat{F}^{\tilde{\sigma}}$, where $\hat{F}^{\tilde{\sigma}}$ is just $\hat{F}$ viewed as an F_i-module via the action of the embedding $\tilde{\sigma} : F_i \hookrightarrow \hat{F}$. Combining the above we get

$$V_i \otimes_{\mathbb{Q}} \hat{F} \simeq \bigoplus_{\tilde{\sigma}: F_i \to \hat{F}} \left(\hat{F}^{\tilde{\sigma}}\right)^{t_i}.$$

The result now follows trivially. $\square$

5.2 NON-TRIVIAL RELATIONS

The relations created in 5.6 were created after fixing a place $v \in \Sigma_{L,\infty}$, corresponding to an inclusion $i_v : L \to \mathbb{C}$. This is because we assumed that s is archimedeanly close to s_0, with respect to this fixed embedding $L \hookrightarrow \mathbb{C}$.

Definition 5.9. Let $s \in S(L)$ for some L/K and set $\xi = x(s) \in L$. For a place $v \in \Sigma_L$ we say that s is v-adically close to s_0 if $|\xi|_v < \min\{1, R_v(\vec{y}) := R_v(y_1, \ldots, y_{h\mu})\}$, where y_j are the G-functions we had earlier.

We want to create relations among the values of the G-functions $y_i \in K[[x]]$ at $\xi = x(s)$ for all places $v \in \Sigma_{L,\infty}$. In order to be able to create these we will need the following technical lemma, following the exposition in Ch.X, §3.1 of [And89]. We fix a priori, the matrix

$$G = \begin{pmatrix} y_1 & \cdots & y_h \\ \vdots & \ddots & \vdots \\ y_{h\mu-h+1} & \cdots & y_{h\mu} \end{pmatrix} \in M_{\mu\times h}(K[[x]]) \tag{5.33}$$

Let us now consider $\iota : K \hookrightarrow \mathbb{C}$ to be a random complex embedding of K. We then have the complex Taylor series $\iota(y_i)$. We also let G_ι be the matrix defined analogously to G with the y_i replaced by the $\iota(y_i)$.

Lemma 5.10. *For any ι as above the matrix G_ι is again the matrix that consists of the entries in the first h columns of a period matrix with respect to the same* *basis of local sections of* $H^n_{DR}(X/S)$ *and to some local frame of the local system* $R_n(f^{an}_{\iota,\mathbb{C}})_*(\mathbb{Q}_{X^{an}_{\mathbb{C}}})$.

Remark 5.11. Here by $f^{an}_{\iota,\mathbb{C}}$ we denote the analytification of the morphism $f_{\iota,\mathbb{C}}$, where $f_{\iota,\mathbb{C}}$ is the morphism induced from $f : X \to S$ via the base change given by $\iota : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec} K$.

Proof. This follows essentially from the proof of Theorem 2 in Ch.X, §4.1 of [And89], which constitutes §4.4 of the same chapter. We review the main parts we need.

Step 1: A short review of the construction. For each point $Q \in Y^{[n]}$ we can find an affine open subset U^Q of X' admitting algebraic coordinates $x_{Q,1}, \ldots, \ldots, x_{Q,n+1}$ such that $Y_i \cap U^Q = Z(x_{Q,i})$ and the local parameter x of S' at s_0 lifts to $x = x_{Q,1} \cdots x_{Q,n+1}$. To ease our notation we write simply x_i for $x_{Q,i}$. We also fix the inclusion $i_Q : U^Q \to X'$. Then loc.cit. describes a horizontal map $T_Q : H^n_{DR}(X'/S'(\log Y)) \to K[[x]]$.

This map T_Q also has an analytic description. To define it one needs some cycles $i_{Q_*}\gamma_Q$. We briefly review the definition of these cycles. For each $z \in \Delta$ we have the cycle $\gamma_{Q,z} \in H_n((U^Q_z)^{an}, \mathbb{Z})$ defined by the relations $|x_2| = \ldots = |x_{n+1}| = \epsilon$ and $x_1 x_2 \cdots x_{n+1} = x(z)$, where $\epsilon > 0$ is small. These cycles glue together to define a section $\gamma_Q \in H^0(\Delta, R_n(f'_\Delta \circ i_Q)_*\mathbb{Q})$ which we can push-forward to a cycle $i_{Q_*}\gamma_Q \in H^0(\Delta, R^n(f'_\Delta)_*\mathbb{Q})$.

We note that $(i_{Q_*}\gamma_Q)_z \in H_n(X^{an}_z, \mathbb{Q})$ is also invariant by the action of $\pi_1(\Delta^*, z)$. In fact from the exposition in loc. cit. we know that the cycles $(i_{Q_*}\gamma_Q)_z$ span the fiber $M_0 R_n(f^{an}_{\mathbb{C}})_*(\mathbb{Q})_z$ for $z \in \Delta^*$.

From the analytic description of T_Q we get that for $1 \le i \le h\mu$ there exists a point $Q \in Y^{[n]}(\bar{\mathbb{Q}})$ such that the entry $y_i|_\Delta$ is equal to

$$T_Q(\omega) = \tfrac{1}{(2\pi i)^n} \int_{i_{Q_*}\gamma_Q} \omega$$

for some $\omega \in H^n_{DR}(X'/S'(\log Y))$, where Δ is a unit disk centered at s_0.

To be able to work over K, instead of $\bar{\mathbb{Q}}$ as loc. cit. does, we assume without loss of generality that all of the above, i.e. the points Q, the algebraic coordinates, and the coefficients of the y_i are actually defined over our original field K. To achieve this we might have to a priori base change everything, i.e. $f : X \to S$ and $f' : X' \to S'$, by some fixed finite extension $\hat{K}$ of our original field K.

Step 2: Changing embeddings. Implicit in the definition of the cycles $i_{Q_*}\gamma_Q$ is the fixed embedding $K \hookrightarrow \mathbb{C}$. Shifting our point of view to the embedding $\iota : K \hookrightarrow \mathbb{C}$ we get a similar picture. Given a K-variety Z we define $Z_\iota := Z \times_{\operatorname{Spec} K, \iota} \operatorname{Spec}\mathbb{C}$ the base change of Z via $\iota : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec} K$

and similarly for the base change of a morphism $\phi : Z_1 \to Z_2$ between K-varieties. In other words we suppress reference to the original embedding $K \hookrightarrow \mathbb{C}$ but keep track of the new embeddings.

The algebraic coordinates $x_1, \ldots, x_{n+1}$ on U^Q pullback to algebraic coordinates $\iota^* x_1, \ldots, \iota^* x_{n+1}$ on U_ι^Q. We write $x_{i,\iota}$ for $\iota^* x_i$ and also consider a unit disk $\Delta_\iota \subset (S'_\iota)^{an}$ centered at s_0.

Once again we have that $x_{1,\iota} \cdots x_{n+1,\iota} = x_\iota$. We define the cycles $\gamma_{Q,\iota}$ similarly:

for $z \in \Delta_\iota$ we let $\gamma_{Q,\iota,z} \in H_n((U_\iota^Q)^{an}, \mathbb{Z})$ be defined by $|x_{2,\iota}| = \ldots = |x_{n+1,\iota}| = \epsilon$ and $x_{1,\iota} x_{2,\iota} \cdots x_{n+1,\iota} = x_\iota(z)$. Once again these glue together to give cycles $i_{Q,\iota_*} \gamma_{Q,\iota} \in H^0(\Delta_\iota, R_n(f'_{\iota,\Delta_\iota})_*^{an} \mathbb{Q})$.

The cycles $(i_{Q_*} \gamma_{Q,\iota})_z$, for Q varying in the set $Y^{[n]}$, will span the fiber of the local system $M_0 R_n (f_\iota^{an})_* (\mathbb{Q}_{X_{\iota,\mathbb{C}}^{an}})_z$ for $z \in \Delta_\iota^*$. This follows from the exposition in loc.cit. since the proof does not depend on the embedding $K \hookrightarrow \mathbb{C}$.

Among these we may choose a frame of $M_0 R_n (f_\iota^{an})_* (\mathbb{Q}_{X_{\iota,\mathbb{C}}^{an}})|_V$ and then extend that to a frame of $R_n (f_\iota^{an})_* (\mathbb{Q}_{X_{\iota,\mathbb{C}}^{an}})|_V$, where $V \subset \Delta_\iota^*$ is some open subset of Δ_ι^*. We thus get a relative period matrix P_1 of the morphism f. Finally, Deligne's trick, see Remark 1 page 21 of [And89] together with the exposition in the aforementioned proof show that in fact $G_1 = G_\iota$, where G_1 is the matrix that consists of the first h columns of P_1, and the result follows. $\square$

Construction of the actual relations

Let $s \in S(L)$ be a point of the variation satisfying either of the conditions of 5.6. We assume that s is v_0-adically close for some fixed $v_0 \in \Sigma_{L,\infty}$. Considering the embedding $i_{v_0} : L \to \mathbb{C}$, which we drop from notation from now on writing just $L \hookrightarrow \mathbb{C}$, the construction of 5.6 goes through.

We consider now G, as in (5.33) above, to be the matrix of G-functions created with respect to that embedding. For any other place $v \in \Sigma_{L,\infty}$ such that s is v-adically close to s_0 we repeat the process of 5.6, this time we replace K by $i_v(K)$, L by $i_v(L)$, and X_s by $X_s \times_L i_v(L)$. Thanks to 5.10 we may choose trivializations so that the corresponding $\mu \times h$ matrix of G-functions we are interested in is G_{ι_v}.

As a result for any such archimedean place v we get a polynomial q_v with coefficients in L such that $i_v(q_v)(i_v(y_1)(\xi), \ldots, i_v(y_{h\mu})(i_v(\xi))) = 0$. We let

$$q = \prod_{\substack{v \in \Sigma_{L,\infty} \\ |\xi|_v < \min\{1, R_v(\vec{y})\}}} q_v. \tag{5.34}$$

The relation we were looking for is the one coming from this polynomial. This relation holds v-adically for all archimedean places for which $|\xi|_v < \min\{1, R_v(\vec{y})\}$, by construction.

Later on, we describe conditions that guarantee that s cannot be v-adically close to s_0 for $v \in \Sigma_{L,f}$. This will, effectively, make (5.34), or to be more precise the relation it induces at ξ, global in that case.

Leaving the proof of globality for later we note that the relation induced from (5.34) satisfies the other key property we want. Namely we have the following lemma.

Lemma 5.12. *The relations created above are non-trivial, assuming that the generic special Mumford-Tate group of our variation is $Sp(\mu, \mathbb{Q})$.*

Proof. This follows by comparing the relations in (5.12) and (5.29) with the polynomials described in 4.8. For Case 2 above, it is easier to see the non-triviality of the relations in question by looking at (5.23) instead of (5.29). $\square$

6

GROTHENDIECK'S MONODROMY THEOREM AND HODGE ENDOMORPHISMS

6.1 REVIEW ON THE ACTION OF THE INERTIA GROUP

Here we review some notions about the action of the inertia group on étale cohomology of varieties over local fields. We then apply these results to our case of interest that of G-admissible variations of $\mathbb{Q}$-HS.

Grothendieck's monodromy Theorem

Let X/K be a smooth projective variety with K a local field whose residue field has characteristic p. We let $G_K := Gal(K_s/K)$ be the absolute Galois group of the field K, $I_K \leq G_K$ be its inertia subgroup, and we also let $X_{K_s} = X \times_K K_s$. We have a natural action of these groups, which we denote by

$$\rho_l : G_K \to Aut(H^i_{ét}(X_{K_s}, \mathbb{Q}_l)),$$

on the étale cohomology groups of X_{K_s} for $l \neq p$. This action for the inertia group is described by the following classical theorem of Grothendieck.

Theorem 6.1. *[[ST68]] Let X be a smooth projective variety over K. Then the inertia group I acts quasi-unipotently on the étale cohomology group $H^i_{ét}(X_{K_s}, \mathbb{Q}_l)$.*

From this we get that there exists a finite field extension L/K for which the inertia group I_L acts unipotently on $H^i_{ét}(X_{K_s}, \mathbb{Q}_l)$. The unipotency of this action can be described more explicitly and provides an ascending filtration, called the **monodromy filtration**, $M_\bullet$ of $H^i_{ét}(X_{K_s}, \mathbb{Q}_l)$. We present a short review of these facts here.

Let us choose a uniformizer $\varpi \in \mathcal{O}_L$ and consider, for each $n \geq 0$ the map

$$t_{l,n} : I_L \to \mu_{l^n}$$

which is defined by $\sigma(\varpi^{\frac{1}{l^n}}) = t_{l,n}(\sigma)\varpi^{\frac{1}{l^n}}$. We then define the map

$$t_l : I_L \to \mathbb{Z}_l(1),$$

as the inverse limit of the maps $t_{l,n}$.

Then there exists a nilpotent map, called the monodromy operator, $N : H^i_{ét}(X_{K_s}, \mathbb{Q}_l)(1) \to H^i_{ét}(X_{K_s}, \mathbb{Q}_l)$ such that for all $\sigma \in I_L$ we have $\rho_l(\sigma) = exp(Nt_l(\sigma))$.

The monodromy filtration

From the above operator N we can construct the monodromy filtration on $H^i_{ét}(X_{K_s}, \mathbb{Q}_l)$ written as

$$0 = M_{-i-1} \subset M_{-i} \subset \ldots \subset M_{i-1} \subset M_i = H^i_{ét}(X_{K_s}, \mathbb{Q}_l)$$

we also define the j-th graded quotient of the filtration to be $Gr^M_j := M_j/M_{j-1}$.

We record the main properties of the monodromy filtration in the following lemma.

Lemma 6.2. *Let $M_\bullet$ be the above monodromy filtration. Then the following hold*

1. $NM_j(1) \subset M_{j-2}$ *for all j,*
2. *the map N^j induces an isomorphism $Gr^M_j(j) \xrightarrow{\cong} Gr^M_{-j}$ for all j,*
3. *the monodromy filtration is the unique ascending filtration satisfying the above two properties, and*
4. *the inertia group I_L acts trivially on Gr^M_j and $M^I \subset M_0$.*

Proof. This follows from [Del80], Prop. 1.6.1 and the above discussion. □

Filtrations and endomorphisms

In what follows we will need the following lemma.

Lemma 6.3. *Let k be a field of characteristic 0 and let $A = A_1 \oplus \ldots \oplus A_r$ be a semisimple algebra over k, where A_i are its simple summands, and assume that $A \hookrightarrow (\mathrm{End}(V))^N$, where N is a nilpotent endomorphism of the finite dimensional k-vector space V.*

Let $W_\bullet$ be the ascending filtration of V defined by N and let $h_i := \dim_k \mathrm{Gr}^{W_\bullet}_i$. Then, for each $1 \leq i \leq r$ we have that there exists $j(i)$ with

$$A_i \hookrightarrow \mathrm{End}(\mathrm{Gr}^{W_\bullet}_{j(i)}) \simeq M_{h_{j(i)}}(k).$$

Proof. We assume without loss of generality that $i = 1$ and proceed by induction on the degree $n + 1$ of nilpotency of N, and hence the length $2n$ of the filtration $W_\bullet$,

$$0 \subset W_{-n} \subset W_{-n+1} \subset \ldots \subset W_{n-1} \subset W_n = V.$$

Let $\psi_1 : \mathrm{End}(V)^N \to \mathrm{End}(W_{-n})$ be the homomorphism of k-algebras defined by $F \mapsto F|_{W_{-n}}$. Since the algebra A_i is simple, either one of two things will happen

1. $\ker \psi_1 \cap A_1 = \{0\}$, in which case we are done, or
2. $A_1 \subset \ker \psi_1$.

From now on we assume that we are in the second case.

Let N_1 be the nilpotent endomorphism induced by N on the quotient W_{n-1}/W_{-n}. We note that the degree of nilpotency of N_1 is n, i.e. it has dropped by 1. We than make the following

Claim. *There is a natural embedding of k-algebras*

$$A_1 \hookrightarrow \mathrm{End}(W_{n-1}/W_{-n})^{N_1}.$$

Proof of the Claim. First of all, note that we have the map

$$\phi_1 : \ker \psi_1 \to \mathrm{End}(W_{n-1}/W_{-n})^{N_1},$$

given by $F \mapsto F|_{W_{n-1}} \mod W_{-n}$.

Since $A_1 \subset \ker \psi_1$ is simple we get that, once again, either $A_1 \subset \ker \phi_1$ or $A_1 \cap \ker \phi_1 = \{0\}$. We clearly cannot have the first case since all elements of $\ker \phi_1$ are nilpotent, in fact $F^3 = 0$ for all such F. Therefore, $A_1 \cap \ker \phi_1 = \{0\}$ and the algebra homomorphism ϕ_1 restricts to an embedding of A_1 into $\mathrm{End}(W_{n-1}/W_{-n})^{N_1}$ as we wanted. □

By induction our result follows, noting that the case $n = 1$ is trivially dealt with by the above argument. □

Inertia and endomorphisms

We believe the results in this subsection are broadly known to experts. Unable to find a reference for these we include them here for the sake of completeness.

Let us consider $f : X \to S$ a G-admissible variation of $\mathbb{Q}$-HS defined over the fixed number field K as usual. We also fix a point $s \in S(L)$ that is v-adically close to s_0 for some $v \in \Sigma_{L,f}$. Our main goal in this section is a brief study of the relation between the algebra of Hodge endomorphisms at $s \in S(L)$ and its relation with the endomorphisms of $H^n_{ét}(\bar{X}_{s,v}, \mathbb{Q}_l)$ where $l \neq p(v)$ and $\bar{X}_{s,v} = (X_s \times_L L_v) \times_{L_v} \bar{L}_v$.

This relation is captured by the following proposition.

Proposition 6.4. *Assume the Hodge conjecture is true for X_s and that the action of the inertia group I_{L_v} on $H^n_{ét}(\bar{X}_{s,v}, \mathbb{Q}_l)$ is unipotent. Then*

$$D_s \otimes \mathbb{Q}_l := (\mathrm{End}(H^n(\bar{X}^{an}_{s,\mathbb{C}}, \mathbb{Q})))^{M_s} \otimes_{\mathbb{Q}} \mathbb{Q}_l \hookrightarrow (\mathrm{End}(H^n_{ét}(\bar{X}_{s,v}, \mathbb{Q}_l)))^{I_{L_v}}.$$

In order to prove this we follow the same strategy of proof as that of Theorem 1 of [Rib75]. We first start with the following lemma.

Lemma 6.5. *There is a natural endomorphism of $\mathbb{Q}_l$ algebras*

$$(\mathrm{End}(H^n(\bar{X}^{an}_{s,\mathbb{C}}, \mathbb{Q})))^{G_{mt,s}} \otimes_{\mathbb{Q}} \mathbb{Q}_l \hookrightarrow \mathrm{End}(H^n_{ét}(\bar{X}_{s,v}, \mathbb{Q}_l)).$$

Proof. As a corollary[1] of the Smooth base change Theorem for lisse l-adic sheaves we have that

$$H^n_{ét}(\bar{X}_{s,v}, \mathbb{Q}_l) \cong H^n_{ét}(\bar{X}_s, \mathbb{Q}_l) \cong H^n_{ét}(\bar{X}_{s,\mathbb{C}}, \mathbb{Q}_l), \tag{6.1}$$

where $\bar{X}_s = X_s \times_L \bar{L}$ and $\bar{X}_{s,\mathbb{C}} = \bar{X}_s \times_{\bar{L}} \mathbb{C}$.

Applying Artin's comparison theorem for lisse l-adic sheaves, we get

$$H^n_{ét}(\bar{X}_{s,\mathbb{C}}, \mathbb{Q}_l) \cong H^n(\bar{X}^{an}_{s,\mathbb{C}}, \mathbb{Q}_l). \tag{6.2}$$

Combining (6.1) with (6.2) we get

$$H^n_{ét}(\bar{X}_{s,v}, \mathbb{Q}_l) \cong H^n(\bar{X}^{an}_{s,\mathbb{C}}, \mathbb{Q}_l),$$

and the inclusion map we want follows. □

Proof of 6.4. It suffices to show that given $\sigma \in I_{L_v}$ and $f \in (\mathrm{End}(H^n(\bar{X}^{an}_{s,\mathbb{C}}, \mathbb{Q})))^{G_{mt,s}}$ the corresponding elements of $\mathrm{End}(H^n_{ét}(\bar{X}_{s,v}, \mathbb{Q}_l))$ commute with each other.

Since $f \in (\mathrm{End}(H^n(\bar{X}^{an}_{s,\mathbb{C}}, \mathbb{Q})))^{G_{mt,s}}$ the Hodge conjecture implies that f is defined over some finite extension $\tilde{L}/L$. Therefore, by compatibility of the cycle class maps in étale and singular cohomology with Artin comparison, f commutes with σ^k for some $k \geq 1$. The result then follows from Lemma 1.2 of [Rib75] since the action in question is unipotent. □

1 See Corollary 4.3, Ch. VI of [Mil80] applied to the field extension $\bar{L}_v/\bar{L}$ and $\mathbb{C}/\bar{L}$.

7

GOOD MODELS AND LOCAL MONODROMY

7.1 GOOD MODELS AND v-ADIC PROXIMITY

We return from now on to the notation of 2.4. So let us fix for the remainder a G-admissible variation of $\mathbb{Q}$-HS given by the map $f : X \to S$ with all the relevant notions of 2.4 defined over some fixed number field K. Let us also fix a local parameter $x \in K(S')$ of S' at the point s_0.

We start by considering a regular proper model $\tilde{S}$ of the curve S' over the ring of integers $\mathcal{O}_K$. We also consider a point $s \in S(L)$, where L/K is some finite extension. By the valuative criterion of properness s_0 and s give sections, which we denote by $\tilde{s}_0$ and $\tilde{s}$, of the arithmetic pencil

$$\tilde{S} \underset{\mathcal{O}_K}{\times} \mathcal{O}_L \to \mathcal{O}_L.$$

Lemma 7.1. *The basis ω_i of $H^n_{DR}(X/S)$ in 2.32 may be chosen so that the following property holds:*

For any L/K and any $s \in S(L)$ we have the if s is v-adically close to s_0 for some $v \in \Sigma_{L,f}$ then $\tilde{s}$ and $\tilde{s}_0$ have the same image in $\tilde{S}(\mathbb{F}_{q(v)})$.

Proof. This follows from the discussion on page 209 of [And89]. In essence we might need to multiply a given basis of sections of $H^n_{DR}(X/S)$ by a factor of the form $\frac{\zeta}{\zeta - x}$, for an appropriately chosen $\zeta \in K^\times$. This amounts to multiplying the G-functions by the same factor.

We still get G-functions but these will have possibly smaller local radii for a finite set of finite places to ensure the property above. □

Good arithmetic models

We aim to apply the results of §4 of [PST+21]. To apply these results we need to assume the existence of good arithmetic models over $\mathcal{O}_K$ both for the curve S and the morphism f.

Definition 7.2. We say that S has a **good arithmetic model** over $\mathcal{O}_K$ if for the triple $(S', S, \{s_0\})$ there exists a model $(\tilde{S}, \mathcal{C}, D)$ over $\mathcal{O}_K$ such that $\tilde{S}$ is smooth and proper and D is a normal crossings divisor.

The smooth proper K-morphism $f : X \to S$ provides us, as we have seen, with a weight $n = (\dim X - 1)$ variation of Hodge structures on the $\mathbb{C}$-manifold $S^{an}_{\mathbb{C}}$. We are also provided with a $\mathbb{Z}$-local system $\mathbb{V} := R^n f^{an}_* \mathbb{Z}_{X^{an}_{\mathbb{C}}}$, contained in the local system of flat sections of the variation of Hodge structures we study.

On the other hand, we know that for any prime l the morphism f defines an associated l-adic étale local system (lisse l-adic sheaf) over S which we denote by $\mathbb{V}_l := \varprojlim R^n f_*(\mathbb{Z}/l^m\mathbb{Z})$. We note that the analytification of $\mathbb{V}_l$ is nothing but the l-adic completion of the local system $\mathbb{V}$.

From the proper base change theorem in étale cohomology and Artin's comparison theorem we know that for each $s \in S(L)$ we have an isomorphism

$$H^n_{\text{ét}}(\bar{X}_s, \mathbb{Z}_l) = (\mathbb{V}_l)_{\bar{s}} \simeq (\mathbb{V}_l)^{an}_{\bar{s}} = H^n(X^{an}_s, \mathbb{Z}_l). \tag{7.1}$$

Extending the étale local system

Let us fix some notation. For a finite place v of an extension L/K, let $p = p(v)$ be the characteristic of the residue field $\mathcal{O}_{L_v}/m_{L_v}$, and $l \neq p(v)$ a prime. We also let $M := L^{ur}_v$ and $\mathcal{O}_M$ be its ring of integers. We consider $(\tilde{S}, \mathcal{C}, D)$ to be as in 7.2, and let $\mathbb{V}$ and $\mathbb{V}_l$ be as in the previous section.

For our argument to work, we need to have the analogue of Lemma 4.3 in [PST+21]. In other words we would like to be able to extend the l-adic étale local system $\mathbb{V}_l$ to some l-adic étale local system $\widetilde{\mathbb{V}}_l$ on $\mathcal{C}_{\mathcal{O}_M} := \mathcal{C} \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(\mathcal{O}_M)$.

The argument in [PST+21] assumes that there is at least one CM point that is integral. This is achieved by a standard spreading out argument. Since we want to be able to deal with all of the finite places v of the field L, we cannot employ a similar tactic.

With that in mind we start with the following definition.

Definition 7.3 (Arithmetic models for f)**.** Let $f : X \to S$ be projective smooth morphism of K-varieties with S a curve, as above. We say f has **a good arithmetic model over** $\mathcal{O}_K$ if there exists a good arithmetic model for the triple $(S', S, \{s_0\})$ over $\mathcal{O}_K$, such that for each L/K finite and each $v \in \Sigma_{L,f}$ we have that

1. there exists an $\mathcal{O}_M$-scheme $\mathcal{X}_v$ such that $(\mathcal{X}_v)_{L^{ur}_v} = X_{L^{ur}_v}$, and
2. there exists a smooth proper morphism $\tilde{f}_v : \mathcal{X}_v \to \mathcal{C}_{\mathcal{O}_M}$ of $\mathcal{O}_M$-schemes whose generic fiber is the morphism f_v(see below).

Let us fix a finite extension L/K and a place $v \in \Sigma_{L,f}$. Assume the existence of such a pair $(\mathcal{X}_v, \tilde{f}_v)$ and let f_v be the base change of f via the morphism Spec $L_v^{ur} \to$ Spec K. We then define $\widetilde{\mathbb{V}}_l$ to be the l-adic étale sheaf on $\mathcal{C}_{\mathcal{O}_M}$ given by

$$R^n(\tilde{f}_v)_*(\mathbb{Z}_l) = \varprojlim R^n(\tilde{f}_v)_*(\mathbb{Z}/l^m\mathbb{Z}).$$

Lemma 7.4. *Let $f : X \to S$ over K be as above. Assume that there exists a good arithmetic model $(\mathcal{X}, \tilde{f})$ for the morphism f over $\mathcal{O}_K$. Then the l-adic étale sheaf $\widetilde{\mathbb{V}}_l := R^n(\tilde{f}_v)_*(\mathbb{Z}_l)$ on $\mathcal{C}_{\mathcal{O}_M}$ is an l-adic étale local system that extends the l-adic étale local system $R^n(f_v)_*(\mathbb{Z}_l)$ on S_M.*

Proof. From the proper base change theorem for lisse l-adic sheaves we have that $R^n(\tilde{f}_v)_*(\mathbb{Z}_l)$ is an extension of the sheaf $R^n(f_v)_*(\mathbb{Z}_l)$.

The fact that $\widetilde{\mathbb{V}}_l$ is also an l-adic étale local system on $\mathcal{C}_{\mathcal{O}_M}$ follows from the smooth proper base change theorem for lisse l-adic sheaves. □

7.2 A COMPARISON OF MONODROMY OPERATORS

The crux of our argument rests on a comparison between the action of inertia and that of the local monodromy around the degeneration s_0. To be able to establish this connection we need to assume the existence of the aforementioned good arithmetic models.

Review on the monodromy weight filtration

We present a short review of the two monodromy operators we want to compare.

The local monodromy operator

Let $f : X \to S$ be a G-admissible variation defined over the number field K, $\Delta \subset S'^{an}_{\mathbb{C}}$ a unit disk, relative to a fixed embedding $K \hookrightarrow \mathbb{C}$, centered at the point of degeneration $s_0 \in S'(K)$.

We have by our assumptions that the local monodromy around s_0 on $\mathbb{V}_z := H^n(X_z^{an}, \mathbb{Q})$ is in fact unipotent for any $z \in \Delta^*$. We denote by $N_z : \mathbb{V}_z \to \mathbb{V}_z$ the corresponding nilpotent endomorphism that is the logarithm of the unipotent endomorphism that defines this action.

We know that N_z has degree of nilpotency $\leq n+1$ and hence, by (6.4) of [Sch73], we get an ascending filtration

$$0 \subset W_0^B \subset W_1^B \subset \ldots \subset W_{2n-1}^B \subset W_{2n}^B = \mathbb{V}_z. \tag{7.2}$$

The filtration $W_\bullet^B$ is called the weight monodromy filtration. By Lemma (6.4) of [Sch73], we know that this is the unique ascending filtration characterized by the properties

1. $N_z(W_i^B) \subset W_{i-2}^B$ for all i, and
2. for all $1 \leq i \leq n$ the endomorphism N_z^i defines an isomorphism $\mathrm{Gr}^{W_\bullet^B}_{n+i} \to \mathrm{Gr}^{W_\bullet^B}_{n-i}$.

We define from now on $h_i^B := \dim_{\mathbb{Q}} \mathrm{Gr}_i^{W_\bullet^B}$ and let $h_{\max}^B := \max\{h_j : 0 \leq j \leq 2n\}$. We note that the number h, the dimension of the local system $M_0 R_n (f_{\mathbb{C}}^{an})_* \mathbb{Q}_{X_{\mathbb{C}}^{an}}|_{\Delta^*}$, is equal to h_0^B.

We note that while the matrix N_z depends on the point z the numbers h_j^B do not.

The monodromy operator from inertia

Let $s \in S(L)$ with L/K finite, be a point of the G-admissible variation of $\mathbb{Q}$-HS given by the morphism $f : X \to S$.

Let $v \in \Sigma_{L,f}$ be a finite place of L. We then have the nilpotent endomorphism N_v, given by the action of some subgroup of finite index of the inertia group I_{L_v}, acting on the étale cohomology group $H^n_{\text{ét}}(\bar{X}_{s,v}, \mathbb{Q}_l)$, where $l \neq p(v)$.

Let $W_\bullet^{\text{ét}}$ be the monodromy filtration defined on $H^n_{\text{ét}}(\bar{X}_{s,v}, \mathbb{Q}_l)$ via the action of the above N_v. We also let $h_i^{\text{ét}} := \dim_{\mathbb{Q}_l} \mathrm{Gr}_i^{W_\bullet^{\text{ét}}}$.

v-adic proximity and comparison of operators

Consider $f : X \to S$ some G-admissible variation of Hodge structures. Throughout this subsection we consider $s \in S(L)$ a fixed point, where L/K is some finite extension, and $v \in \Sigma_{L,f}$ some fixed finite place of L. We let $p = p(v)$ be the characteristic of the finite field $\mathcal{O}_{L_v}/m_v$ and we fix some $l \neq p$.

We assume that a good arithmetic model, in the sense of 7.1, $(\tilde{S}, \mathcal{C}, D)$ over $\mathcal{O}_K$ exists for S and that a good arithmetic model for the morphism f exists over $\mathcal{O}_K$ with respect to this triple.

Motivated by the exposition in §4 of [PST+21] we prove the following lemma.

Lemma 7.5. *Under the above assumptions, we have that if s is v-adically close to s_0 we have $h_i^{\text{ét}} = h_{i+n}^B$ for all i, where h_j^B and $h_j^{\text{ét}}$ are as in 7.2.*

Proof. We start with a bit of notation following the exposition in §4 of [PST+21]. First of all, for any group G we will denote by $G_{(l)}$ its maximal pro-l quotient. As always we fix a punctured unit disk $\Delta^* \subset S^{an}$ centered at s_0.

We let y and y_0 be the sections of the arithmetic pencil $\tilde{S} \times_{\mathcal{O}_K} \mathcal{O}_L \to \mathcal{O}_L$ whose generic fibers are the points s and s_0 respectively. By 7.1 we may assume without loss of generality that $y_{\bar{\mathbb{F}}_p} = y_{0,\bar{\mathbb{F}}_p} \in D(\bar{\mathbb{F}}_p)$. We also let t be such that it cuts out D in $\tilde{S}$.

We have, by our assumptions, that the local monodromy $\pi_1(\Delta^*, z)$ acts unipotently on the fiber $(R^n f_*^{an}\mathbb{Z})_z$ for all $z \in \Delta^*$. Letting $\gamma_0 \in \pi_1(\Delta^*, z)$ be a generator, we denote by $U_0 \in \mathrm{GL}(H^n(X_z^{an}, \mathbb{Z}))$ the unipotent endomorphism it maps to via the local monodromy representation. We define N_z to be the nilpotent logarithm of U_0.

By Grothendieck's quasi-unipotent action theorem we know that there exists a finite extension F/L_v such that the Galois representation $\rho_l : G_{L_v} \to \mathrm{GL}(H^n_{\acute{e}t}(\bar{X}_{s,v}, \mathbb{Z}_l))$ restricts to a unipotent representation of the inertia group I_F. In other words $\rho_l|_{I_F}$ is given by $\sigma \mapsto \exp(N_v t_l(\sigma))$ with N_v nilpotent with degree of nilpotency $\leq n+1$.

We let $M := F^{ur}$ and $\mathcal{O}_M$ be its ring of integers. We consider the rings $R_1 := \mathcal{O}_M[[t]]\left[\frac{1}{t}\right]$, $R_2 := M[[t]]\left[\frac{1}{t}\right]$, and $R_3 := \mathbb{C}[[t]]\left[\frac{1}{t}\right]$. We note that, after fixing an inclusion $\mathcal{O}_M \hookrightarrow \mathbb{C}$, these define the following commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} R_3 \arrow[r] \arrow[d, "g_3"'] & \operatorname{Spec} R_2 \arrow[r] \arrow[d, "g_2"] & \operatorname{Spec} R_1 \arrow[d, "g_1"] \\
S_{\mathbb{C}} \arrow[r] & S_M \arrow[r] & \mathcal{C}_{\mathcal{O}_M}
\end{tikzcd}$$

with g_1, g_2 and g_3 being étale. In fact, note that the $\operatorname{Spec} R_i$ are étale neighborhoods of the geometric point $\bar{y}_M = \bar{s}$.

Since $y_{\bar{\mathbb{F}}_p} \in D$, we get that t pulls back to an element t_M of the maximal ideal of $\mathcal{O}_M$ via $\operatorname{Spec}(\mathcal{O}_M) \xrightarrow{y_{\mathcal{O}_M}} \tilde{S}_{\mathcal{O}_M}$. From this we get a morphism $f_1 : \mathcal{O}_M[[t]]\left[\frac{1}{t}\right] \to M$, since $v_M(t_M) > 0$.

By Lemma 4.2 of [PST+21] we get an induced map

$$\phi_1 : G_M^{(p)} \to \pi_1^{\acute{e}t}(\operatorname{Spec} R_1, \bar{y}_M)^{(p)},$$

between the prime-to-p Galois groups, whose image is completely determined by $v_M(t_M) > 0$. In particular we get a map

$$\phi_{1,l} : (G_M)_{(l)} \to \pi_1^{\acute{e}t}(\operatorname{Spec} R_1, \bar{y}_M)_{(l)}.$$

With respect to the above embedding $\mathcal{O}_M \hookrightarrow \mathbb{C}$ we get a map $f_2 : R_1 \to R_3$ which induces an isomorphism of prime-to-p Galois groups, and hence of their maximal pro-l quotients, which we denote by

$$\phi_{2,l} : \pi_1^{\acute{e}t}(\operatorname{Spec} R_3, \bar{y}_M)_{(l)} \to \pi_1^{\acute{e}t}(\operatorname{Spec} R_1, \bar{y}_M)_{(l)}.$$

We also consider the composition $\phi_l := \phi_{2,l}^{-1} \circ \phi_{1,l} : (G_M)_{(l)} \to \pi_1^{ét}(\operatorname{Spec} R_3, \bar{y}_M)_{(l)}$.

Finally, we let $F : R_3 \to R_3$ be the map defined by $t \mapsto t^{v_M(t_M)}$, by abuse of notation we also let $F : \operatorname{Spec}(R_3) \to \operatorname{Spec}(R_3)$ be the étale cover induced from F. Letting $\bar{y}_1$ be any geometric point in the fiber of $\bar{y}_M = \bar{s}$ over F, we then get from F an induced morphism

$$\psi_l : \pi_1^{ét}(\operatorname{Spec} R_3, \bar{y}_1)_{(l)} \to \pi_1^{ét}(\operatorname{Spec} R_3, \bar{y}_M)_{(l)}.$$

We also note that by construction we have that ψ_l has the same image as ϕ_l.

On S_M we have the lisse l-adic sheaf $\mathbb{V}_l := (\underset{\leftarrow}{\lim} R^n(f_v)_*(\mathbb{Z}/l^i\mathbb{Z}))$. Its analytification $\mathbb{V}_l^{an}$ is nothing but the local system $R^n f_*^{an}\mathbb{Z}_{X_{\mathbb{C}}^{an}} \otimes_{\mathbb{Z}} \mathbb{Z}_l$ on $S_{\mathbb{C}}^{an}$, which is the l-adic completion of the $\mathbb{Z}$-local system that underlies the variation of $\mathbb{Z}$-HS we are studying. The assumption that good arithmetic models for the morphism f exist over $\mathcal{O}_K$ then implies, see 7.4, that we have that $\mathbb{V}_l$ extends to a lisse l-adic sheaf $\widetilde{\mathbb{V}}_l$ on $\mathcal{C}_{\mathcal{O}_M}$. We let $\widetilde{\mathbb{V}}_{l,1}$ be the pullback of this sheaf via g_1. Note that the pullback of $\mathbb{V}_l$ via the morphism $\operatorname{Spec}(R_2) \xrightarrow{g_2} S_M$, which we will denote by $\mathbb{V}_{l,1}$, is nothing but the generic fiber of the lisse l-adic sheaf $\widetilde{\mathbb{V}}_{l,1}$. By abuse of notation we also denote by $\mathbb{V}_{l,1}$ the pullback of this last sheaf via the morphism $\operatorname{Spec}(R_3) \to \operatorname{Spec}(R_2)$ above. Finally, we denote by $\mathbb{V}_{l,2}$ the pullback of $\mathbb{V}_{l,1}$ via the map F.

For any $z \in \Delta^*$ we have the local monodromy representation given by $\rho : \pi_1(\Delta^*, z) \to \mathrm{GL}((\mathbb{V}_l^{an})_z)$. By our assumptions this action is unipotent given by $\gamma_0 \mapsto U_0$, where γ_0 a generator of $\pi_1(\Delta^*, z)$. From the tower of étale covers $(\operatorname{Spec} R_3)^{an} \xrightarrow{F^{an}} (\operatorname{Spec} R_3)^{an} \xrightarrow{g_3^{an}} S_{\mathbb{C}}^{an}$ we get an inclusion $\pi_1(\operatorname{Spec}(R_3)^{an}, z_2) \hookrightarrow \pi_1(\operatorname{Spec}(R_3)^{an}, z_1) \hookrightarrow \pi_1(\Delta^*, z)$ where $z_1 \in (g_3^{an})^{-1}(z)$ and $z_2 \in (F^{an})^{-1}(z_1)$. Letting $\gamma_i \in \pi_1(\operatorname{Spec}(R_3)^{an}, z_i)$ for $i = 1,2$ be generators of these groups we can identify these with $\gamma_0^{a_i}$, where $a_i \in \mathbb{Z}$ and $a_1 | a_2$. Without loss of generality we may and do assume that $a_i > 0$.

By construction we then have the following commutative diagrams for $i = 1,2$

$$\begin{array}{ccc} \pi_1(\operatorname{Spec}(R_3)^{an}, z_i) & \xrightarrow{\rho_i} & \mathrm{GL}((\mathbb{V}_{l,i})_{z_i}^{an}) \\ \downarrow & & \downarrow \\ \pi_1(\Delta^*, z) & \xrightarrow{\rho} & \mathrm{GL}((\mathbb{V}_l)_z) \end{array}$$

from which we see that γ_i maps to $U_{i,z} := U_0^{a_i}$. In particular, the nilpotent logarithm $N_{i,z}$ of $U_{i,z}$ is $a_i N_z$.

Now, let γ_3 be a generator of the group $\pi_1(\operatorname{Spec}(R_3)^{an}, \bar{y}_M) \simeq \mathbb{Z}$. Considering the representation $\pi_1(\operatorname{Spec}(R_3)^{an}, \bar{y}_M) \to \mathrm{GL}((\mathbb{V}_{l,1}^{an})_{\bar{y}_M})$ with $\gamma_3 \mapsto U_3$.

We get that U_3 is conjugate to $U_{1,z}$, i.e. there exists some P with $U_3 = PU_{1,z}P^{-1}$. So that U_3 is also unipotent and we may write $N_3 := PN_{1,z}P^{-1}$ for its nilpotent logarithm.

Putting everything together we get that

$$N_3 = a_1 P N_z P^{-1}. \tag{7.3}$$

We also have a commutative diagram

$$\begin{tikzcd} \pi_1(\mathrm{Spec}(R_3)^{an}, \bar{y}_M) \arrow[r, "\rho_1"] \arrow[d] & \mathrm{GL}((\mathbb{V}_{l,1})^{an}_{\bar{y}_M}) \arrow[d] \\ \pi_1^{\text{ét}}(\mathrm{Spec}(R_3), \bar{y}_M) \arrow[r, "\rho_1^{\text{ét}}"] & \mathrm{GL}((\mathbb{V}_{l,1})_{\bar{y}_M}) \end{tikzcd}$$

We note that $\pi_1(\mathrm{Spec}(R_3)^{an}, \bar{y}_M) \simeq \mathbb{Z}$ and that it is dense in the étale fundamental group $\pi_1^{\text{ét}}(\mathrm{Spec}\, R_3, \bar{y}_M)$. Thus, the continuity of $\rho_1^{ét}$ implies that the image of $\rho_1^{ét}$ is $P_1 U_3^{\mathbb{Z}_l} P_1^{-1} := \{P_1 \exp(\alpha N_3) P_1^{-1} : \alpha \in \mathbb{Z}_l\}$. This is a pro-$l$ group and we therefore get that $\rho_1^{ét}$ factors through $\pi_1^{\text{ét}}(\mathrm{Spec}(R_3), \bar{y}_M)_{(l)}$.

We know turn our attention to the action of inertia. We know that the following diagram is commutative

$$\begin{tikzcd} G_M \arrow[r] \arrow[rr, bend right, "\rho_l"'] & \pi_1^{ét}(\mathrm{Spec}\, R_3, \bar{y}_M) \arrow[r, "c"] & \mathrm{GL}(H^n_{\text{ét}}(\bar{X}_{s,v}, \mathbb{Z}_l)), \end{tikzcd}$$

where c is the homomorphism induced by the composition of the identification of $(\mathbb{V}_{l,1})_{\bar{y}_M} \simeq H^n_{\text{ét}}(\bar{X}_{s,v}, \mathbb{Z}_l)$ and the action of $\pi_1^{ét}(\mathrm{Spec}\, R_3, \bar{y}_M)$ on $(\mathbb{V}_{l,1})_{\bar{y}_M}$.

By construction, the image of $\rho_l : G_M \to \mathrm{GL}(H^n_{\text{ét}}(\bar{X}_{s,v}, \mathbb{Z}_l))$ above is a unipotent group. This implies that $Im(\rho_l)$ is in fact a pro-l group. In more detail, we have $Im(\rho_l) = \{\exp(\alpha N_v) : \alpha \in \mathbb{Z}_l\}$ which is a pro-l group.

From our earlier comments we get that the image of c is a pro-l group as well. We thus have that the above diagram induces a commutative diagram

$$\begin{tikzcd} (G_M)_{(l)} \arrow[r, "\phi_l"] \arrow[rr, bend right, "\bar{\rho}_l"'] & \pi_1^{ét}(\mathrm{Spec}\, R_3, \bar{y}_M)_{(l)} \arrow[r, "c"] & \mathrm{GL}(H^n_{\text{ét}}(\bar{X}_{s,v}, \mathbb{Z}_l)), \end{tikzcd}$$

where ϕ_l is as above.

In particular we get that $\exp(N_v) \in Im(c)$. In more detail this implies that $\exp(N_v) = QP_1 \exp(\alpha N_3) P_1^{-1} Q^{-1}$ for some $\alpha \in \mathbb{Z}_l$. Taking logarithms and combining this with (7.3) this translates to

$$N_v = \alpha a_1 P_0 N_z P_0^{-1}, \tag{7.4}$$

where $P_0 \in \mathrm{GL}_\mu(Z_l)$, upon considering $\mathbb{Z}_l$-bases of the fibers of the various lisse l-adic sheaves.

Note that $\alpha a_1 \neq 0$ since $Im(\phi_l) = Im(\psi_l)$. Indeed, in this case we would have that $Im(\psi_l) = 0$ and, hence, that the monodromy representation $\rho_2 : \pi_1(\mathrm{Spec}(R_3), z_2) \to \mathrm{GL}((\mathbb{V}_{l,2}^{an})_{z_2})$ is trivial. This would imply that $N_{2,z} = 0$ and hence that $N_z = 0$ which is impossible since we have by assumption of having a G-admissible variation a non-trivial singularity at s_0.

Combining this with 7.6, our result follows. □

Lemma 7.6. *Let k be a field with $char(k) = 0$ and let N_1 and N_2 be two nilpotent elements of of $\mathrm{End}(V)$, where V is some μ-dimensional k-vector space. Assume that N_1 is conjugate to aN_2 for some $a \in k^\times$. Let $W^1_\bullet$ and $W^2_\bullet$ be the ascending filtrations of V associated to N_1 and N_2 respectively, and let $h_i^j := \dim_k \mathrm{Gr}_i^{W^j_\bullet}$ for $j = 1, 2$. Then $h_i^1 = h_i^2$ for all i.*

Proof. The easiest way to see this is via the explicit formulas for the filtration $W_\bullet$ associated to a nilpotent endomorphism N, see Remark (2.3) of [SZ85]. The fact that $N_1 = PaN_1P^{-1}$ for some $P \in \mathrm{GL}(V)$ and some $a \in k^\times$ defines, using the aforementioned formulas, isomorphisms $\mathrm{Gr}_i^{W^1_\bullet} \simeq \mathrm{Gr}_i^{W^2_\bullet}$ for all i and the result follows.

□

8

THE ISSUE OF v-ADIC PROXIMITY

8.1 CONDITIONS FOR v-ADIC PROXIMITY

Motivated by 6.3 we make the following definition.

Definition 8.1. Let $f : X \to S$ be a G-admissible variation defined over the number field K. Let $s \in S(\bar{\mathbb{Q}})$ be a point of the variation whose associated algebra of Hodge endomorphism is $D_s = M_{m_1}(D_1) \oplus \ldots \oplus M_{m_r}(D_r)$. Let us also define $F_i := Z(D_i)$, $d_i^2 := [D_i : F_i]$, and $f_i := [F_i : \mathbb{Q}]$.

We say that the point s satisfies condition $\star$ if it satisfies any of the following

$\star_1$ if we have that

$$\sum_{i=1}^{r} m_i f_i > \mu - \dim_{\mathbb{Q}} Im(N^B). \qquad (\star_1)$$

$\star_2$ if there exists i such that for the set

$$\Pi^i_{D_s} := \{l \in \Sigma_{\mathbb{Q},f} : \exists w \in \Sigma_{F_i,f}, \text{ with } w|l \text{ and } [F_{i,w} : \mathbb{Q}_l] > \tfrac{h^B_{\max}}{m_i}\}$$

we have that

$$|\Pi^i_{D_s}| \geq 2. \qquad (\star_2)$$

$\star_3$ if we have that $\exists i \in \{1, \ldots, r\}$ such that

a) $d_i m_i \geq h^B_{\max}$, and

b) for the sets $P^i_{D_s} := \{l \in \Sigma_{\mathbb{Q},f} : l \text{ is totally split in } F_i\}$ and $Q^i_{D_s} := \{l \in \Sigma_{\mathbb{Q},f} : \exists w \in \Sigma_{F_i,f}, w|l \text{ and } \mathrm{inv}_w(D_i) \notin \mathbb{Z}\}$ we have that $|P^i_{D_s} \cap Q^i_{D_s}| \geq 2$.

$\star_4$ if for some i we have that D_i is a quaternion algebra, and letting $R^i_{D_s} := \{l \in \Sigma_{\mathbb{Q},f} : \exists w \in \Sigma_{F_i,f}, w|l, \mathrm{inv}_w(D_i) \notin \mathbb{Z}, \text{and } m_i 4[F_{i,w}, \mathbb{Q}_l] \ / \ |h^B_j \ \forall j\}$, we have that $|R^i_{D_s}| \geq 2$.

$\star_5$ if for some i we have that D_i is a quaternion algebra, and letting $S^i_{D_s} := \{l \in \Sigma_{\mathbb{Q},f} : \exists w \in \Sigma_{F_i,f}, w|l, \mathrm{inv}_w(D_i) \in \mathbb{Z}, \text{and } m_i 2[F_{i,w}, \mathbb{Q}_l] \ / \ |h^B_j \ \forall j\}$, we have that $|S^i_{D_s}| \geq 2$.

$\star_6$ if for some i we have that D_i is a quaternion algebra and for the above sets $|R^i_{D_s} \cup S^i_{D_s}| \geq 2$.

$\star_7$ if for some i we have that D_i is of Type IV with $d_i^2 = \dim_{F_i}[D_i : F_i]$ and for the set

$$T^i_{D_s} := \{l \in \Sigma_{\mathbb{Q},f} : \exists w \in \Sigma_{F_i,f},\ w|l \text{ and } m_i d_i [F_{i,w}, \mathbb{Q}_l] \nmid h^B_j\ \forall j\},$$

we have that $|T^i_{D_s}| \geq 2$.

Remark 8.2. 1. We remark that all conditions above only depend on data coming from the original G-admissible variation $f : X \to S$. These should be thought of as demanding that we have "a lot of endomorphisms". This "a lot" here should be contrasted with the fact that the generic algebra of Hodge endomorphisms is just $\mathbb{Q}$.

2. The necessity for demanding that several of the sets of primes in these conditions have at least two elements arises from 7.5. As we will see in the proof of 8.6, once we fix the place v, and by extension the prime $p \in \mathbb{Q}$ with $v|p$, we need some condition as those in $\star_2$-$\star_7$ to hold for some prime $l \neq p$ to be able to apply the aforementioned result.

3. We note that the sets in conditions $\star_2$, $\star_5$, $\star_6$, and $\star_7$ will potentially have infinitely many elements, while the sets in $\star_3$ and $\star_4$ will have at most finitely many elements. This rests on the fact that $\mathrm{inv}_w(D) \notin \mathbb{Z}$ only for finitely many primes.

4. We note that, assuming that the extension $F_i/\mathbb{Q}$ is cyclic, condition $\star_5$ is equivalent to the existence of some i for which $m_i 2[F_i : \mathbb{Q}] \nmid h^B_j$ for all j.

5. The different treatment needed for general type IV algebras, addressed in $\star_7$, stems from the fact that even if we know that that the restriction of the division algebra at some place w is not of the form $M_t(F_{i,w})$ for some, it might still be of the form $M_t(D')$ with D' a central division algebra D' with $[D' : F_{i,w}] > 1$ and $t > 1$. This issue does not appear in the case of quaternion algebras since the only two possibilities are $D_i \otimes_{F_i} F_{i,w} \simeq M_2(F_{i,w})$ or $D_i \otimes_{F_i} F_{i,w}$ is a quaternion algebra over $F_{i,w}$.

8.2 SOME LINEAR ALGEBRAIC LEMMAS

To prove the result we want, first we will need some elementary lemmas from linear algebra and the theory of linear algebraic groups.

Lemma 8.3. *Let $N \in M_n(k)$, where k is a field with $char(k) = 0$, be a non-zero nilpotent upper-triangular matrix with $\dim_k Im(N) = r$. Then, there exist* *unipotent upper triangular matrices* $Q_L, Q_R \in GL_n(k)$ *such that the matrix* $N_{red} := Q_L N Q_R$ *is strictly upper triangular with at most one non-zero entry in each row and column.*

Proof. We employ row and column reduction together with induction. □

Remark 8.4. Another way of phrasing the above lemma is that $N_{red} = (\epsilon_{i,j})$ such that there exist exactly r entries $\epsilon_{i,j} \neq 0$, which we can take without loss of generality to be equal to 1. These entries are all in distinct rows and columns, i.e. if $\epsilon_{i,j}, \epsilon_{i',j'} \neq 0$ then $i \neq i'$ and $j \neq j'$.

Lemma 8.5. *Let* V *be an* n*-dimensional vector space over an algebraically closed field* k *of characteristic* 0 *and let* $N \in \mathrm{End}(V)$ *be a nilpotent linear operator with* $r := \dim_k Im(N) > 0$*. Let* T *be a sub-torus of the algebraic group* $\mathrm{GL}(V)^N$ *of automorphisms of* V *commuting with* N*. Then* $\dim_k T \leq n - r$*.*

Proof. We may assume that the sub-torus T is maximal in $G_N := \mathrm{GL}(V)^N$. The torus T will be contained in a maximal sub-torus T_m of the group $\mathrm{GL}(V)$. From the theory of linear algebraic groups[1] all maximal tori of $GL(V)$ are conjugate. Hence, there exists $P \in \mathrm{GL}(V)(k)$ such that PT_mP^{-1} is the torus of diagonal matrices in $\mathrm{GL}(V)$, with respect to a fixed basis $\{\vec{e}_i : 1 \leq i \leq n\}$.

We notice that, setting $N_P := PNP^{-1}$, we have that $P\,\mathrm{GL}(V)^N P^{-1} = \mathrm{GL}(V)^{N_P}$, and the sub-torus T will be isomorphic to the sub-torus PTP^{-1}. For the element N_P we will have that it is nilpotent and that $r = \dim_k Im(N) = \dim_k Im(N_P)$. We may thus assume from now on, replacing T by PTP^{-1} and N by N_P, that T is comprised of diagonal matrices and is contained in the subgroup of diagonal matrices $\mathbb{G}_m^n \subset \mathrm{GL}_n$. Let $A \in T(k)$ be generic and write $A = \mathrm{diag}(a_1, \ldots, a_n)$. Since A and N commute they are simultaneously upper triangularizable, so there exists $P_1 \in GL_n(k)$ such that $A_1 = P_1AP_1^{-1}$ and $N_1 := P_1NP_1^{-1}$ are both upper triangular and $A_1N_1 = N_1A_1$.

We note that we can write

$$A_1 = \mathrm{diag}(a'_1, \ldots, a'_n) + A' \tag{8.1}$$

where A' is some strictly upper triangular matrix. We note that $(a'_1, \ldots, a'_n) = (a_{(r_1)}, \ldots, a_{(r_n)})$ where $i \mapsto r_i$ defines a permutation of the set $\{1, \ldots n\}$.

We apply 8.3 to the matrix N_1 and let Q_L, Q_R be unipotent matrices as in the lemma. We then have

$$(Q_LN_1Q_R)(Q_R^{-1}A_1Q_R) = (Q_LA_1Q_L^{-1})(Q_LN_1Q_R). \tag{8.2}$$

1 The results we used from the theory of Linear algebraic groups can be found in Chapter 6 of [Spr98].

Let $B = (b_{i,j}) := (Q_L N_1 Q_R)(Q_R^{-1} A_1 Q_R)$ and $Q_L N_1 Q_R = (\epsilon_{i,j}) = N_{red}$. Note that, since the matrices Q_L and Q_R are unipotent and upper triangular, the matrices $(Q_R^{-1} A_1 Q_R)$, $(Q_L^{-1} A_1 Q_L)$, and A_1 are all upper triangular matrices that have the same diagonal.

Let $I := \{(i_t, j_t) : 1 \leq t \leq r\}$ denote the set of indices with $\epsilon_{i,j} = 1$, note that the rest of the entries of the matrix N_{red} are zero. For each pair $(i_t, j_t) \in I$ we compute via the left hand side of (8.2)

$$b_{i_t, j_t} = a'_{j_t}, \tag{8.3}$$

while computing these using the right hand side of (8.2) we have

$$b_{i_t, j_t} = a'_{i_t}, \tag{8.4}$$

Combining the above we find that for $1 \leq t \leq r$ we have

$$a'_{i_t} = a'_{j_t}. \tag{8.5}$$

These imply that the Zariski closure of $\operatorname{diag}(a'_1, \dots, a'_n)$ in $\mathbb{G}_m^n$ is contained in the subvariety defined by the ideal

$$J_T := \langle X_{i_t} - X_{j_t} : (i_t, j_t) \in I \rangle \tag{8.6}$$

of $k[X_1, \dots, X_n]$.

Let $f_t = X_{i_t} - X_{j_t}$ be the above polynomials generating the ideal J_T. Note that each indeterminant X_i appears at most twice among the polynomials f_t, at most once as one of the X_{i_t} and at most once as one of the X_{j_t}. We also note that by construction we have $i_t < j_t$ for all t.

We start by proving the following claim:

Claim: The polynomials f_t are linearly independent over k.

Proof of Claim. We proceed by induction on r. The cases $r = 1, 2$, are trivial. So let us assume that $r = 3$.

Assume, without loss of generality, that

$$f_3 = \lambda_1 f_1 + \lambda f_2. \tag{8.7}$$

Since X_{i_3} appears on the left it must appear on the right, so either $i_3 = j_1$ or $i_3 = j_2$, due to the above remarks. Assuming without loss of generality that $i_3 = j_1$ we get that $\lambda_1 = -1$. Similarly we get that $j_3 = i_1$ or $j_3 = i_2$. If $j_3 = i_1$ it would contradict the fact that $i_t < j_t$ for all t, so $j_3 = i_2$.

This implies $\lambda_2 = -1$ and (8.7) becomes

$$X_{i_3} - X_{j_3} = -(X_{i_1} - X_{j_1}) - (X_{i_2} - X_{j_2}). \tag{8.8}$$

Applying the above we get $X_{j_2} - X_{i_1} = 0$ which implies $i_1 = j_2$.

Putting everything together we get

$$i_3 < j_3 = i_2 < j_2 = i_1 < j_1 = i_3, \tag{8.9}$$

which is obviously a contradiction.

Now assume that $r \geq 4$. To reach a contradiction we assume that there exist $\lambda_i \in k$ such that

$$X_{i_r} - X_{j_r} = \lambda_1 f_1 + \ldots + \lambda_{r-1} f_{r-1}. \tag{8.10}$$

Again, since j_r appears on the left there exists a unique i_t such that $i_t = j_r$. We assume, without loss of generality that $t = r-1$ so that $\lambda_{r-1} = -1$. Similarly without loss of generality we may assume that $i_r = j_{r-2}$ and $\lambda_{r-2} = -1$.

Using the above with (8.10) we get

$$X_{i_r} - X_{j_r} = -(X_{j_r} - X_{j_{r-1}}) - (X_{i_{r-2}} - X_{i_r}) + \lambda_1(X_{i_1} - X_{j_1}) + \ldots + \lambda_{r-3}(X_{i_{r-3}} - X_{j_{r-3}}), \tag{8.11}$$

which, after canceling, becomes

$$X_{i_{r-2}} - X_{j_{r-1}} = \lambda_1(X_{i_1} - X_{j_1}) + \ldots + \lambda_{r-3}(X_{i_{r-3}} - X_{j_{r-3}}). \tag{8.12}$$

This gives a contradiction by the inductive step as follows:

First, note that $i_{r-2} < j_{r-2} = i_r < j_r = i_{r-1} < j_{r-1}$. Now, consider the following set of pairs of indices

$$I' = \{(i_1, j_1), \ldots (i_{r-3}, j_{r-3}), (i_{r-2}, j_{r-1})\}.$$

Notice that this satisfies all the properties of the original set of pairs of indices, namely $i_t < j_t$ for all t, where we can "rename" j_{r-2} as j_{r-1}, each i_t appears at most once in the first coordinate of the pairs and the same hold for the j_t in the second coordinate of the pairs.

□

Since the f_t are linearly independent linear polynomials it follows that $Z(J_T) \subset \mathbb{A}_k^n$ has dimension $n - r$ and the result follows. □

8.3 RULING OUT v-ADIC PROXIMITY

With an eye towards "globality" of relations among the values of G-functions at a certain point, we establish the following proposition.

Proposition 8.6. *Let $f : X \to S$ be a G-admissible variation of $\mathbb{Q}$-HS defined over some number field K. Let $s \in S(L)$, where L/K is some finite extension, and let $v \in \Sigma_{f,L}$ be a finite place of L.*

If s satisfies condition $\star$ then, assuming the Hodge conjecture holds and that a good arithmetic model exists for the morphism f, the point s cannot be v-adically close to the degeneration s_0.

Proof. **Step 1:** Assume that s is v-adically close to s_0. We then get by 7.5 that $h_i^{\text{ét}} = h_{i+n}^B$, assuming the existence of a good arithmetic model for f over $\mathcal{O}_K$.

From 6.4, assuming the validity of the Hodge conjecture, we get that

$$D_s \otimes_{\mathbb{Q}} \mathbb{Q}_l \hookrightarrow \text{End}(H^n_{\text{ét}}(\bar{X}_{s,v}, \mathbb{Q}_l))^{N_v}, \tag{8.13}$$

where $l \neq p(v)$. We are thus in a position to use 6.3 for the algebra $D_s \otimes \mathbb{Q}_l$ and the nilpotent endomorphism N_v on the space $H^n_{\text{ét}}(\bar{X}_{s,v}, \mathbb{Q}_l)$.

From this we get that we must have that for all $1 \leq i \leq r$ and all $w \in \Sigma_{F_j,f}$ with $w|l$ there exists $j(i,w)$ such that

$$M_{m_i}(D_i \otimes_{F_i} F_{i,w}) \hookrightarrow M_{h^B_{j(i,w)}}(\mathbb{Q}_l). \tag{8.14}$$

Step 2: We start by ruling out points that satisfy $\star_1$. This follows from 8.5. Indeed the dimension of the maximal subtorus of $(D_s \otimes \bar{\mathbb{Q}}_l)^\times$ is $\sum_{i=1}^{r} m_i f_i$. On the other hand, the maximal subtorus of $\text{GL}(H^n_{\text{ét}}(\bar{X}_{s,v}, \mathbb{Q}_l))^{N_v}$ has dimension $\leq \mu - \dim_{\mathbb{Q}_l} N_v$. The result then follows from 7.5, which implies that $\dim_{\mathbb{Q}_l} Im(N_v) = \dim_{\mathbb{Q}} Im(N_B) > 0$.

Assume now that s satisfies condition $\star_2$. We then get that there exists i, a prime $l \neq p(v)$, and a place $w \in \Sigma_{F_i,f}$ with $w|l$ for which $[F_{i,w} : \mathbb{Q}_l]m_i > h^B_{\max}$. This contradicts the validity of (8.14). Indeed the maximal commutative semisimple algebra of $M_{m_i}(D_j \otimes_{F_i} F_{i,w})$ has dimension $\geq [F_{i,w} : \mathbb{Q}_l]m_i$ over $\mathbb{Q}_l$, while that of $M_{h^B_{\max}}(\mathbb{Q}_l)$ has dimension $h^B_{\max}$ over $\mathbb{Q}_l$.

Assume that s satisfies condition $\star_3$ and choose i as in $\star_3$. Then we have that there exists $l \in \Sigma_{\mathbb{Q},f}$ that is totally split in F_i with $l \neq p(v)$. Therefore we have that $F_{i,w} = \mathbb{Q}_l$ for all $w \in \Sigma_{F_i,f}$ with $w|l$.

Also by $\star_3$ we know that we can find $w \in \Sigma_{F_i,f}$ with $w|l$ such that $\mathrm{inv}_v(D_i) \notin \mathbb{Z}$. Since once again by assumption we have $(m_i d_i) \geq h^B_{\max}$ we get that (8.14) is an isomorphism in this case, with $h^B_{j(i,w)} = h^B_{\max}$. This would imply that $\mathrm{inv}_w(D_i) = 0 \in \mathbb{Q}/\mathbb{Z}$ contradicting the above assumption.

Assume that s satisfies condition $\star_4$. Then by assumption there exists a prime $l \neq p(v)$ such that there exists $w \in \Sigma_{F_i,f}$ for which $D_{i,w} = D_i \otimes_{F_i} F_{i,w}$ is a quaternion algebra over $F_{i,w}$. If a $j(i,w)$ satisfying (8.14) existed for the simple summand $M_{m_i}(D_{i,w})$ of $D_s \otimes_{\mathbb{Q}} \mathbb{Q}_l$, we would have $m_i \dim_{\mathbb{Q}_l} D_{i,w} | h^B_{j(i,w)}$ which contradicts our assumptions. Indeed, such an embedding would imply an isomorphism of $M_{m_i}(D_{i,w})$-modules $\mathbb{Q}_l^{h^B_{j(i,w)}} \cong (D^{m_i}_{i,w})^r$ for some r. Comparing $\mathbb{Q}_l$-dimensions the contradiction follows.

The argument for $\star_5$ and $\star_6$ is practically identical to that of condition $\star_4$.

Finally, the proof in the case where s satisfies condition $\star_7$ is practically identical to that of $\star_4$ but has a small twist. Let us assume that $|T^i_{D_s}| \geq 2$ and let $l \in T^i_{D_s}$ and $w|l$ be such that $l \neq p(v)$ and $m_i d_i [F_{i,w} : \mathbb{Q}_l] \nmid h^B_j$ for all j.

If $\mathrm{inv}_w(D) \neq 0$ we have that $D_i \otimes_{F_i} F_{i,w} \simeq M_r(D')$ with D' a division algebra with center $F_{i,w}$. Let $d' := \sqrt{[D' : F_{i,w}]}$. If an index $j(i,w)$ satisfying (8.14) existed, by the same argument as above, we would have that $m_i r \dim_{\mathbb{Q}_l}(D') | h^B_{j(i,w)}$. Since $rd' = d_i$ and $\dim_{\mathbb{Q}_l}(D') = d'^2 [F_{i,w} : \mathbb{Q}_l]$ we get the contradiction we wanted.

The case $\mathrm{inv}_w(D) = 0$ follows from the same argument above, though we do not have to introduce a new division algebras D' since $D_i \otimes_{F_i} F_{i,w} \simeq M_{d_i}(F_{i,w})$. □

Remark 8.7. The proof in the case where condition $\star_7$ is satisfied shows that $\star_7$ is weaker than the strongest condition we can actually impose on these algebras.

In fact the proof shows that we would need to guarantee the impossibility of the occurrence of

$$rd'^2 \cdot [F_{i,w} : \mathbb{Q}_l] \mid h^B_j \text{ for some } j,$$

where r and d' are as in the proof of $\star_7$.

9
THE HEIGHT BOUND

9.1 PROOF OF OUR RESULT

Finally, we combine all parts of our exposition to prove 1.19.

Proof of 1.19: Let $s \in S(L)$ be a point satisfying the conditions in 1.19 for the variation $\mathbb{V} = R^n f_*^{an} \mathbb{Q}_{X_{\mathbb{C}}^{an}}$ where L/K is some finite extension. We let $\xi := x(s)$, where x is the local parameter of S' at s_0 with respect to which the y_i are written as power series.

By 3.1 there exists a finite extension $\hat{L}/L$ such that D_s acts on $H^n_{DR}(X_s \times_L \hat{L}/\hat{L})$. From 3.2 we also know that $\hat{L}$ may be chosen so that $[\hat{L} : L]$ is bounded only in terms of $\mu := \dim_{\mathbb{Q}} H^n(X_s^{an}, \mathbb{Q})$.

Let $y_1, \ldots, y_{h\mu}$ be the G-functions that comprise the first h columns of the relative n-period matrix associated to the morphism f. We then have the polynomials (5.34) with coefficients in $\hat{F}_s$ and degree $\leq [\hat{L} : \mathbb{Q}]$. By 5.12 we know that these polynomials define relations among the values of the G-functions in question at ξ that are non-trivial, as long as there exists at least one archimedean place $v \in \Sigma_{\hat{L},\infty}$ for which s is v-adically close to s_0.

Consider now $v \in \Sigma_{\hat{L},f}$ to be any finite place of $\hat{L}$. By 8.6 we know that s cannot be v-adically close to s_0, in other words that $|\xi|_v \geq \min\{1, R_v(\vec{y})\}$, where $R_v(\vec{y})$ is the local radius of convergence of the y_i.

Now we split into two cases.

Case 1: For all archimedean places $v \in \Sigma_{\hat{L},\infty}$ we have that $|\xi|_v \geq \min\{1, R_v(\vec{y})\}$.

Combining this assumption with the above result we get that

$$h(\xi^{-1}) \leq \rho(\vec{y}),$$

where $\rho(\vec{y})$ is the global radius of the collection of power series y_i. Combining Lemma 2 of I.§2.2 of [And89] with the Corollary of VI.§5 of loc.cit., we get that $h(s) = h(\xi) = h(\xi^{-1}) \leq \rho(\vec{y}) < \infty$.

This concludes this case.

Case 2: There exists at least one archimedean place $v \in \Sigma_{\hat{L},\infty}$ for which s is v-adically close to s_0.

In this case the relation defined by the polynomials in (5.34) among the values at ξ of the y_i is by construction global, since there are no other places $v \in \Sigma_{\hat{L},f}$ for which s is v-adically close to s_0.

Since we know that the relation (5.34) is both non-trivial and global we get from 2.28 that

$$h(\xi) \leq c_1(\vec{y})\delta^{3\mu h-1}(\log \delta + 1), \tag{9.1}$$

where δ is the degree of the polynomial (5.34) in $\bar{\mathbb{Q}}[x_1, \dots x_{h\mu}]$.

By construction of (5.34) we know that $\delta \leq [\hat{L} : \mathbb{Q}] = [\hat{L} : L] \cdot [L : \mathbb{Q}]$. On the other hand 3.2 gives the bound $[\hat{L} : L] \leq ((6.31)\mu^2)^{\mu^2}$. Combining these remarks with (9.1) we get that there exist positive constants C_1, C_2, independent of the point s, such that

$$h(\xi) \leq C_1([L : \mathbb{Q}] + 1)^{C_2}, \tag{9.2}$$

as we wanted.

The result follows by combining the above two cases, or simply by replacing C_1 in (9.2) by $\max\{C_1, \rho(\vec{y})\}$. $\square$

10 EXAMPLES: CM AND OTHER EXCEPTIONAL POINTS

We construct examples of possible algebras of endomorphisms of Hodge structures where the conditions for points s to be in the set Σ of 1.19 are easy to check. We start with the case of CM-Hodge structures and then construct infinite families of possible such algebras of all possible types, i.e. type I-IV in Albert's classification 2.24.

10.1 COMPLEX MULTIPLICATION

Hodge Structures with complex multiplication

Hodge structures with complex multiplication, or simply CM Hodge structures, play the same role for Hodge structures with weight ≥ 2 that abelian varieties with complex multiplication play for the weight 1 Hodge structures. For an introduction to these we point the interested reader to [GGK12] and [Moo99].

The main ingredient we will need about CM Hodge structures is the following lemma that describes their algebra of Hodge endomorphisms. Following [GGK12] we write "CMpHS" as an abbreviation of the term "polarized Hodge structure with complex multiplication".

Lemma 10.1. *Let V be a CMpHS then there is a unique decomposition of V into simple Strongly CMpHS's $V = V_1^{\oplus m_1} \oplus \cdots \oplus V_l^{\oplus m_r}$. In particular for the algebra D of Hodge endomorphisms of V we have that*

$$D \cong \bigoplus_{i=1}^{r} M_{m_i}(K_i),$$

where $K_i \cong D_i$, the algebra of Hodge endomorphisms of V_i, is a CM field with $[K_i : \mathbb{Q}] = \dim_{\mathbb{Q}} V_i$.

Proof. See [GGK12] Ch.V and especially the facts on page 195. For a proof see [Moo99] where the necessary machinery is developed. □

Motivated by this we make the following definition.

Definition 10.2. Let $\mathbb{V}$ be a polarized variation of $\mathbb{Q}$-HS over some base T. We say that the point $t \in T(\mathbb{C})$ is a **CM point of the variation**, or a **special point of the variation**, if the Hodge structure $\mathbb{V}_t$ is a CMpHS.

Remarks 10.3. *1. Consider a CMpHS (V, ϕ) as in 10.1. Let E be the maximal commutative semi-simple algebra of the algebra D. We have by 10.1 that*

$$E = K_1^{m_1} \times \cdots \times K_r^{m_r}. \tag{10.1}$$

We also have that $\dim_{\mathbb{Q}} E = \sum_{j=1}^{r} m_j [K_j : \mathbb{Q}]$. Noting that $\dim_{\mathbb{Q}} V = \sum_{j=1}^{r} m_j \dim_{\mathbb{Q}} V_j$ and that $\dim_{\mathbb{Q}} V_j = [K_j : \mathbb{Q}]$ we get that

$$\dim_{\mathbb{Q}} V = \dim_{\mathbb{Q}} E. \tag{10.2}$$

2. The above property guarantees that CM-points satisfy the conditions of 5.6. Indeed, from the above and 5.7 we have that CM-points satisfy (5.30) and so we get non-trivial relations among the values of the G-functions we study at $\xi = x(s)$.

The globality of these relations follows from the fact that CM-points satisfy condition $\star_2$.

CM-points have potentially good reduction

Given a CM abelian variety A defined over a number field K it is well known, see [ST68], that A will have potentially good reduction at each finite place of K. The linear algebraic lemma 8.5 has an interesting consequence. Namely, it shows that a similar picture holds true for smooth projective varieties with CM polarized Hodge structure defined over a number field K. Even though the term "good reduction" or "potentially good reduction" cannot be expected to hold in general, as in the existence of Néron models, the term still makes sense from the point of view of Galois representations.

Proposition 10.4. *Let $f : X \to S$ be a G-admissible variation of $\mathbb{Q}$-HS defined over the number field K. Let L/K be a finite extension, let $s \in S(L)$ be a CM point for this variation, and let $v \in \Sigma_{L,f}$ be a finite place of L.*

Assume the Hodge conjecture holds. Then the l-adic Galois representation of G_{K_v} on $H^n_{ét}(\bar{X}_{s,v}, \mathbb{Q}_l)$ has potentially good reduction for all $l \neq p(v)$.

Proof. Let D_s be the algebra of Hodge endomorphisms at s. We set $V_s := H^n(\bar{X}^{an}_{s,\mathbb{C}}, \mathbb{Q})$ to be the $\mathbb{Q}$-HS corresponding to s, with $\dim_{\mathbb{Q}} V_s = 2\mu$.

From 10.1 we know that V_s decomposes as $V_s = \bigoplus_{i=1}^{r} V_i^{\oplus m_i}$, so that $D_s \cong \bigoplus_{i=1}^{r} M_{m_i}(K_i)$, where V_i are irreducible CM polarized $\mathbb{Q}$-HS with algebras of Hodge endomorphisms K_i being CM fields with $n_i := [K_i : \mathbb{Q}] = \dim_{\mathbb{Q}} V_i$. We also note that, trivially from the above, we have that $\mu = \sum_{i=1}^{r} m_i n_i$, where $\mu := \dim_{\mathbb{Q}} H^n(X_s^{an}, \mathbb{Q})$.

Let $p = p(v)$ be the characteristic of the residue field $\kappa(v) := \mathcal{O}_{L_v}/m_{L_v}$. We fix $l \in \mathbb{N}$ a prime with $l \neq p$. We then notice that the $\bar{\mathbb{Q}}_l$-algebra $\bar{D}_{s,l} := D_s \times_{\mathbb{Q}} \bar{\mathbb{Q}}_l \cong (D_s \times_{\mathbb{Q}} \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \bar{\mathbb{Q}}_l$ is such that $\bar{D}_{s,l}^{\times}$ contains all of the closed $\bar{\mathbb{Q}}_l$-points of a μ-dimensional torus.

From 6.1 we know that the inertia group I_{L_v} acts quasi-unipotently on $H^n_{\text{ét}}(\bar{X}_{s,v}, \mathbb{Q}_l)$. Therefore up to a finite extension of L_v we may and do assume that it in fact acts unipotently. In this case we get an associated nilpotent endomorphism N_v, as we saw in our earlier discussion, whose exponential determines the action of the inertia group.

In this case, from 6.4, we have that

$$\bar{D}_{s,l}^{\times} \hookrightarrow \mathrm{GL}(H^n_{\text{ét}}(\bar{X}_{s,v}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \bar{\mathbb{Q}}_l)^{N_v},$$

where N_v is the above nilpotent matrix associated to the action of the inertia group. The result now follows from 8.5. Indeed, if $N_v \neq 0$ we get that $\dim_{\bar{\mathbb{Q}}_l} \bar{D}_{s,l}^{\times} \leq \mu - \mathrm{rank}(N_v) < \mu$ which contradicts the above. $\square$

10.2 SOME MORE COMPLICATED EXAMPLES

We start with highlighting some examples where 1.19 applies. We fix for the remainder some general notation as in the main part of our exposition.

We consider $f : X \to S$ a G-admissible variation of Hodge structures with fibers of odd dimension n and generic special Mumford-Tate group $Sp(\mu, \mathbb{Q})$, where $\mu := \dim_{\mathbb{Q}} H^n(X_z^{an}, \mathbb{Q})$ for any $z \in S^{an}$ and let $h := \dim_{\mathbb{Q}} Im((N^*)^n)$.

As before, for a point $s \in S(\bar{\mathbb{Q}})$ we let $V_s := H^n(X_s^{an}, \mathbb{Q})$ and assume that the decomposition of V_s into simple polarized sub-$\mathbb{Q}$-HS is given by $V_1^{m_1} \oplus \ldots \oplus V_r^{m_r}$. We also let $D_s = M_{m_1}(D_1) \oplus \ldots \oplus M_{m_r}(D_r)$ be its algebra of Hodge endomorphisms.

For our process to kick in we need to have that

$$h > \frac{\dim_{\mathbb{Q}} V_j}{[Z(D_j) : \mathbb{Q}]} \text{ for some } j \text{, or that} \tag{10.3}$$

$$h \geq \min\{\frac{\dim_{\mathbb{Q}} V_i}{[Z(D_i):\mathbb{Q}]} : i \text{ such that } D_i = \mathrm{End}_{HS}(V_i) \text{ is of type IV }\}. \quad (10.4)$$

Some examples of CM-fields and totally real fields

The fields $F_{0,n}$ and F_n: Let $n \in \mathbb{N}$ and let p be a prime with $p \equiv 1 \mod 2n$. It is well known that the field $\mathbb{Q}(\zeta_p)$ is a cyclic extension of $\mathbb{Q}$ with totally real subfield $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$. We can therefore find a subfield $F_{0,n}$ of $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$ with $[F_{0,n} : \mathbb{Q}] = n$. Note that by construction this extension is also cyclic and totally real.

By Frobenius density it follows[1]that the set

$$In(F_{0,n}) := \{l \in \Sigma_{\mathbb{Q},f} : l \text{ is inert in } F_{0.n}\}$$

is infinite. Fix $l_1, l_2 \in In(F_n)$ with $l_1 \neq l_2$. Then from quadratic reciprocity we can find a prime $q = q(l_1, l_2) \in \mathbb{Q}$ such that in the extension $F_n := F_{0,n}(i\sqrt{q})$ the prime ideals $l_j\mathcal{O}_{F_{0,n}}$, $j = 1, 2$, split in F_n.

In conclusion we get that for the field F_n constructed above there are distinct primes l_1, $l_2 \in \mathbb{Q}$ whose splitting in F_n is given by

$$l_j\mathcal{O}_{F_n} = w_{j,1} \cdot w_{j,2}.$$

In particular we get that $[(F_n)_{w_{j,i}} : \mathbb{Q}_{l_j}] = n$ for all i, j and that, trivially by construction, if σ denotes complex conjugation in $F_n/\mathbb{Q}$, we have $\sigma w_{j,1} = w_{j,2}$.

Note that the family of CM-fields constructed above is infinite, for fixed n, by varying either the infinitely many pairs (l_1, l_2) or the infinite choices $q(l_1, l_2)$.

Examples of algebras of Type IV

Consider F_n a field as those constructed above. Let D be a division algebra over F_n and let $d^2 = [D : F_n]$. For such an algebra to be of type IV in Albert's classification it needs to satisfy the following conditions:

1. for any finite place $v \in \Sigma_{F_n,f}$ we have that $\mathrm{inv}_v(D) + \mathrm{inv}_{\sigma(v)}(D) = 0$, and
2. $\mathrm{inv}_v(D) = 0$ for all such places that satisfy $\sigma(v) = v$.

By construction of our fields F_n we get places $w_{j,i}$, $1 \leq i, j \leq 2$, that come in pairs with $\sigma(w_{j,1}) = w_{j,2}$ for $j = 1, 2$. We want, in view of the conditions

1 See [Jan73] Ch. IV, Corollary 5.4.

$\star$ introduced in 8.1, to work with algebras D that ramify at these finite places. For that reason we consider the following set of central division algebras, or CDA for short, with center F_n

$$\mathcal{D}_{IV}(n,d) := \{D : CDA/F_n,\ [D : F_n] = d^2,\ \mathrm{inv}_{w_{j,i}}(D) \notin \mathbb{Z}\}.$$

We note that the set $\mathcal{D}_{IV}(n,d)$ actually depends on the numbers n, l_j, $q(l_1,l_2)$, and d so it would be perhaps more accurate to denote this by $\mathcal{D}_{IV}(d,n,l_1,l_2,q(l_1,l_2))$, though we avoid this for notational brevity.

We note that for all choices of n, l_j, and $q(l_1,l_2)$ as above the set $\mathcal{D}_{IV}(n,2)$, i.e. the set of quaternion algebras satisfying the above conditions is non-empty. This follows from the classification theorem of quaternion algebras over global fields, see [Vig80] Ch. III, Theoreme 3.1.

Let $f : X \to S$ be as above and let $s \in S(\bar{\mathbb{Q}})$ and assume that one of the algebras D_k that appear in the decomposition of the algebra of Hodge endomorphisms D_s at the point s is an element of $\mathcal{D}_{IV}(n,d)$ for some choice of $(n,d,l_1,l_2,q(l_1,l_2))$. Then we can find simple conditions to check whether s is in the set Σ of 1.19.

Indeed, assuming that

$$2n \geq \frac{\dim_{\mathbb{Q}} V_k}{h} \tag{10.5}$$

is enough to guarantee the validity of (10.4). After this we just need to check the validity of at least one of the conditions in 8.1. In our case, by construction of the fields F_n, conditions $\star_4$ and $\star_7$ translate into easy to check conditions, mainly thanks to the fact that $[(F_n)_{w_{j,i}} : \mathbb{Q}_{l_j}] = n$.

In more detail, by our construction in the case $d = 2$ condition $\star_4$ follows from

$$4nm_k \nmid h_j^B \text{ for all } j, \tag{10.6}$$

where h_j^B are as in 7.2. Condition $\star_7$, only applicable in our case when $d = d_k \geq 3$, follows from

$$m_k dn \nmid h_j^B \text{ for all } j. \tag{10.7}$$

Finally, for the case $d = 1$ so that $D_k = F_n$, or in other words the case where V_k is a CM-HS while the other V_t are arbitrary polarized sub-HS of V_s, it is easy to create a condition analogous to the conditions $\star$ created in 8.1. Indeed, it is easy to check, using arguments as in 8.6, that the the condition

$$m_k n \nmid h_j^B \text{ for all } j, \tag{10.8}$$

is enough to guarantee the impossibility of v-adic proximity for all finite places $v \in \Sigma_L$, where $L = K(s)$ with our usual notation.

With these observations in mind we define $\Sigma_{n,IV} \subset S(\bar{\mathbb{Q}})$ to be the set that consists of the points $s \in S(\bar{\mathbb{Q}})$ whose corresponding algebra of Hodge endomorphisms satisfies the above hypothesis, i.e. for some k we have that $D_k \in \mathcal{D}_{IV}(n,d)$ for some choice of $(n,d,l_1,l_2,q(l_1,l_2))$, condition (10.5) holds, and condition (10.6) holds if $d_k = 2$, or (10.7) holds if $d_k \geq 3$, or (10.8) holds if $d_k = 1$.

For the points in $\Sigma_{n,IV}$ we get that they satisfy the conditions needed so that they are in the set Σ of 1.19. We thus have the following corollary.

Corollary 10.5. Let $f : X \to S$ be a morphism over K defining a G-admissible variation of $\mathbb{Q}$-HS satisfying the conditions of 1.19. Let $\Sigma_{n,IV}$ be the above set of points.

Then, there exist constants $C_1, C_2 > 0$ such that for all $s \in \Sigma_{n,IV}$ we have

$$h(s) \leq C_1[K(s) : K]^{C_2},$$

where h is a Weil height on S'.

Algebras of types I-III

The totally real fields $F_{0,n}$ with $[F_{0,n} : \mathbb{Q}] = n$ that we created above, help us construct convenient examples to check the conditions in 8.1, mainly since they are cyclic. In fact, every such field constitutes a type I algebra in Albert's classification.

By the aforementioned classification of quaternion algebras over number fields, see [Vig80] or [Voi21], we have a bijection between the set

$$\{\text{Quaternion algebras over } F_{0,n} \text{ up to isomorphism}\}$$

and the set[2]

$$\{P \subset \Sigma_F : |P| \equiv 0 \mod 2\}$$

given by $B \mapsto \mathrm{Ram}(B)$, with $\mathrm{Ram}(B)$ the set of places over which the quaternion algebra B ramifies.

With this in mind we define, in parallel to the type IV case above, the following sets of quaternion algebras $D/F_{0,n}$:

$$\mathcal{D}_{II}(n) := \{D : \mathrm{Ram}(D) \cap \Sigma_{F,\infty} = \emptyset,\ w_{j,i} \in \mathrm{Ram}(D) \text{ for all } j,i\}$$

$$\mathcal{D}_{III}(n) := \{D : \Sigma_{F,\infty} \subset \mathrm{Ram}(D),\ w_{j,i} \in \mathrm{Ram}(D) \text{ for all } j,i\}.$$

2 Normally we would have to make sure that the the subsets P in question do not have any complex archimedean places. Since our fields are totally real this condition is simply vacuous.

Remark 10.6. We note that by the aforementioned classification theorem it follows that these sets are in fact infinite, for fixed n and pair of primes (l_1, l_2).

Note that for these algebras we will have by construction the following:

1. $[(F_{0,n})_{w_{j,i}} : \mathbb{Q}_{l_j}] = [F_{0,n} : \mathbb{Q}] = n$ for all $1 \leq i, j \leq 2$, and
2. for $D \in \mathcal{D}_{III}(n) \cup \mathcal{D}_{II}(n)$ we will have that $\mathrm{inv}_{w_{j,i}}(D) \neq 0$.

Given these it is much easier to check for the validity of the conditions for a point $s \in S(\bar{\mathbb{Q}})$ to be in the set Σ of 1.19, assuming that one of the algebras D_k appearing in the decomposition of the algebra of Hodge endomorphisms D_s is an element of $\mathcal{D}_{III}(n) \cup \mathcal{D}_{II}(n)$, or even that $D_k = F_{0,n}$ for some n.

Indeed, assume that for some k we have $D_k \in \mathcal{D}_{III}(n) \cup \mathcal{D}_{II}(n)$. Then, condition (10.3) translates to simply checking

$$n > \frac{\dim_{\mathbb{Q}} V_k}{h}. \tag{10.9}$$

On the other hand, checking $\star_4$, which is the strongest out of the conditions in 8.1, becomes straightforward. Indeed, letting h_j^B be the dimensions of the quotients resulting from the weight monodromy filtration as in 7.2, $\star_4$ in this case follows from

$$4m_k n \not| h_j^B \text{ for all } j. \tag{10.10}$$

In the case where $D_k = F_{0,n}$ it is easy to check, with the same arguments as in the proof of 8.6, that the the condition

$$m_k n \not| h_j^B \text{ for all } j, \tag{10.11}$$

is enough to guarantee the impossibility of v-adic proximity for all finite places $v \in \Sigma_L$, where $L = K(s)$ with our usual notation.

With the above in mind, for our fixed morphism $f : X \to S$, we consider the set $\Sigma_n \subset S(\bar{\mathbb{Q}})$ that consists of the points $s \in S(\bar{\mathbb{Q}})$ that are such that, if the corresponding algebra of Hodge endomorphisms is given by $D_s = M_{m_1}(D_1) \oplus \ldots \oplus M_{m_r}(D_r)$, we have that there exists $1 \leq k \leq r$ and primes $l_1, l_2 \in \mathbb{Q}$ such that (10.9) holds and either $D_k \in \mathcal{D}_{III}(n) \cup \mathcal{D}_{II}(n)$ and (10.10) holds, or $D_k = F_{0,n}$ and (10.11) holds.

In this case, 1.19 applies to such points and we have the following corollary.

Corollary 10.7. Let $f : X \to S$ be a morphism over K defining a G-admissible variation of $\mathbb{Q}$-HS satisfying the conditions of 1.19. Let Σ_n be the above set of points.

Then, there exist constants $C_1, C_2 > 0$ such that for all $s \in \Sigma_n$ we have

$$h(s) \leq C_1[K(s) : K]^{C_2},$$

where h is a Weil height on S'.

Remarks 10.8. *1. We note that in both 10.5 and 10.7 the conditions imposed on the points s revolve around only one of the division algebras D_k that appear in the decomposition of the algebra of Hodge endomorphisms D_s. The rest of the algebras D_t with $t \neq k$ could have arbitrary properties.*

2. We could simplify the situation by considering points s for which the algebra D_s is equal to one of the algebras constructed in the above examples, i.e. we are in the case where $r = m_1 = 1$ and D_s is a central simple algebra itself.

Part II

An Ax-Schanuel theorem for the matrix exponential

11

AX-SCHANUEL IN FAMILIES

In each of the two cases we deal with, we start by approaching the Ax-Schanuel result from a functional standpoint. We then use properties of the exponential maps in question to reduce to the following corollary of the classic Ax-Schanuel Theorem

Proposition 11.1 (Weak Ax-Schanuel in families)**.** *Let $f_i, g_j \in \mathbb{C}[[t_1, \ldots, t_m]]$, where $1 \leq i \leq n$, $1 \leq j \leq k$, be power series. Then, assuming that the f_i are $\mathbb{Q}-$linearly independent modulo $\mathbb{C}$,*

$$tr.d._{\mathbb{C}}\mathbb{C}(\{f_i, g_j, e^{f_i} : 1 \leq i \leq n, 1 \leq j \leq k\}) \geq n + \operatorname{rank}(J(\vec{f}, \vec{g}; \vec{t})).$$

Proof. [1]Without loss of generality, we may assume that the set

$$\{f_i, g_j : 1 \leq i \leq n, 1 \leq j \leq k\}$$

is a $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$ subset of $\mathbb{C}[[t_1, \ldots, t_m]]$ and that the g_j for $1 \leq j \leq k$ are algebraically independent over $\mathbb{C}$. Consider the fields $K_1 = \mathbb{C}(\{f_i, g_j, e^{f_i}, e^{g_j} : 1 \leq i \leq n, 1 \leq j \leq k\})$ and $K_2 = \mathbb{C}(\{f_i, g_j, e^{f_i} : 1 \leq i \leq n, 1 \leq j \leq k\})$.

Then by the classic Ax-Schanuel Theorem we have

$$tr.d._{\mathbb{C}}K_1 \geq n + k + \operatorname{rank}(J(\vec{f}, \vec{g}; \vec{t})).$$

On the other hand

$$tr.d._{\mathbb{C}}\mathbb{C}(\{e^{g_j} : 1 \leq j \leq k\}) + tr.d._{\mathbb{C}}K_2 \geq tr.d._{\mathbb{C}}K_1, \text{ and}$$
$$tr.d_{\mathbb{C}}\mathbb{C}(\{e^{g_j} : 1 \leq j \leq k\}) \leq k.$$

Combining all three of these the result follows. □

Remarks 11.2. *1. We note that 11.1 is probably known as a result in the field. However, since we couldn't find a reference, we have dedicated this short section to its proof.*
2. Following the ideas in [Tsi15] we can obtain the Full Ax-Schanuel Theorem for families along with a few corollaries. For these the interested readers are referred to the Appendix.

1 The author thanks the anonymous referee for this proof that helped shorten this section significantly.

We finally note, that the same proof gives, by reduction to the classic Ax-Schanuel Theorem[2] as above, the following more abstract variant of the above proposition:

Proposition 11.3. *Let $\mathbb{Q} \subset C \subset F$ be a tower of fields and Δ a set of derivations of the field F with $\bigcap_{D\in\Delta} kerD = C$. Let $f_1, \ldots, f_m \in F$ and $z_1, \ldots, z_m \in F^*$ be such that for all $D \in \Delta$ and $1 \leq i \leq m$, $Df_i = \frac{Dz_i}{z_i}$. Let us also assume that for some $n \leq m$ the f_i for $1 \leq i \leq n$ are $\mathbb{Q}$-linearly independent modulo C. Then,*

$$tr.d._C C(\{f_i, z_j : 1 \leq i \leq m, 1 \leq j \leq n\}) \geq n + rank(Df_i)_{\substack{1\leq i\leq m \\ D\in\Delta}}.$$

2 See Theorem 3 in [Ax71].

12 UPPER TRIANGULAR MATRICES

Notation: We will denote by $U = U_n$ the algebraic group of $n \times n$ upper triangular invertible matrices over the field $\mathbb{C}$, and by $\mathfrak{h} = \mathfrak{h}_n$ its Lie algebra, i.e. the algebra of $n \times n$ upper triangular matrices. Also we denote the corresponding exponential map $\mathfrak{h}_n \to U_n$ by $E = E_n$ and its non-diagonal entries by $E_{i,j}$, $1 \le i < j \le n$.

We will mainly concern ourselves with transcendence degrees over the field $\mathbb{C}$ of extensions of the form $\mathbb{C}(\Sigma)$ with Σ a finite subset of some ring of power series, or a finite subset of regular functions on some variety over the field $\mathbb{C}$. Of particular interest will be the case were Σ is the set of entries of a matrix A, or a matrix A and its exponential $E(A)$. In that case we denote the field extension $\mathbb{C}(\Sigma)$ over $\mathbb{C}$ by $\mathbb{C}(A)$ and $\mathbb{C}(A, E(A))$ respectively.

Consider elements $f_i, g_{i,j} \in \mathbb{C}[[t_1, \ldots, t_m]]$, with $1 \le i < j \le n$. As in the introduction, we will denote by $\langle f_1, \ldots, f_n \rangle_{\mathbb{Q}}$ the linear span of the $f_i \in \mathbb{C}[[t_1, \ldots, t_m]]$ over $\mathbb{Q}$. Also, as in the introduction, $J(\vec{f}, \vec{g}; \vec{t})$ denotes the $\frac{n(n+1)}{2} \times m$ Jacobian matrix with entries of the form $\partial h_s / \partial t_j$, where h_s with $1 \le s \le \frac{n(n+1)}{2}$ is an ordering of the $g_{i,j}$ and the f_i. The rank of the Jacobian is its rank over the fraction field of $\mathbb{C}[[t_1, \ldots, t_m]]$.

We also introduce some rings that will be needed in some of the proofs that follow. First consider z_i, $1 \le i \le n$, and $x_{i,j}$, $1 \le i < j \le n$, to be independent variables over $\mathbb{C}$. Let $M_0 := \mathbb{C}(\{z_i, x_{i,j} : 1 \le i < j \le n\})$. Then for $1 \le i < j \le n$, a subset $I \subset \{(s,t) : 1 \le s < t \le n\}$, and a subset $J \subset \{1, \ldots, n\}$ we define the subrings

$$R_{i,j}(I) := \mathbb{Q}[x_{s,t}, \tfrac{1}{z_l - z_j} : \min(i,j) \le s < t \le \max(s,t), (l,j) \notin I, (s,t) \neq (i,j)], \text{ and}$$

$$D_{i,j}(J) := \mathbb{Q}(z_j)[\tfrac{1}{z_l - z_j}, x_{s,t} : \min(i,j) \le s < t \le \max(i,j), i < l < j, l \notin J]$$

of the field M_0.

We note that elements of M_0 can be naturally viewed as rational functions on $\mathfrak{h}_n$, with the $z_i(A)$ and $x_{s,t}(A)$ the corresponding entries of $A \in \mathfrak{h}_n$. Likewise, since they are subrings, the same is true for elements of the rings

$D_{i,j}(J)$ and $R_{i,j}(I)$ defined above.

Our goal here is to state an Ax-Schanuel-type Theorem and reduce its proof to 11.1. The first step towards that will be to find a lower bound for the transcendence degree

$$tr.d._{\mathbb{C}}\mathbb{C}(A, E(A)) = tr.d._{\mathbb{C}}\mathbb{C}(\{f_i, g_{i,j}, E_{i,j}(A), e^{f_i} : 1 \leq i < j \leq n\})$$

where $f_i, g_{i,j} \in \mathbb{C}[[t_1, \ldots, t_m]]$, with $1 \leq i < j \leq n$, and $A = \mathrm{diag}(\vec{f}) + (g_{i,j})$.

We start with the case of U_n, instead of jumping straight to the case of GL_n, for a few reasons. First of all, the case of U_n, as we will see, is open to more calculations and, because of this, examples and notions, such as weakly special sets, can be more easily formulated in this setting. Furthermore, while the technical difficulties that appear in dealing with the exponential map seem to be of a similar nature in both of these linear groups, they are easier to deal with in the case of U_n, mainly again thanks to us being able to adopt a more computational approach. Finally, we believe that, in future work, the restrictions imposed by our method will be easier to lift in the case of U_n first, so as to gain insight in the more technical case of GL_n. We return to this in **??**.

12.1 EIGENCOORDINATES

In this section we consider fixed $f_i, g_{i,j} \in \mathbb{C}[[t_1, \ldots, t_m]]$ such that the f_i are without constant terms, with $1 \leq i < j \leq n$, and let $A = \mathrm{diag}(\vec{f}) + (g_{i,j})$ be the matrix they define.

The main idea is that eigenvectors and generalized eigenvectors for a matrix A will remain as such for the matrix $E(A)$. As we will see shortly the other information, that will naturally appear, and that we will have to keep track of, are the nilpotent operators defined by A and $E(A)$ on their respective generalized eigenspaces.

In this section we define a canonically chosen basis of generalized eigenvectors for a given matrix $A \in \mathfrak{h}_n$ that will only depend on the multiplicities of the eigenvalues of A. To this basis we can assign coordinates, which will be rational functions on the entries of A. At the same time we achieve a canonical description of the respective nilpotent operators defined by A and $E(A)$ on each of their generalized eigenspaces. To each such operator we will be able to naturally assign certain rational functions of the entries of A. The combination of the above rational functions, both those describing the basis and those describing the nilpotent operators, will be what we will refer to as **eigencoordinates**.

These new notions have a distinct advantage, as we will see, in our setting, when dealing with questions surrounding transcendence properties. Namely they will allow us to:

1. replace the $g_{i,j}$ by the eigencoordinates of A, when dealing with transcendence questions, and

2. capture the essence of the map E, as far as transcendence is concerned, and replace the $E_{i,j}$ by the eigencoordinates of A, again in questions concerning transcendence.

The scope of this section is to state and prove the main lemmas that we will need concerning these new notions. As a motivation we first deal with the case where all of the eigenvalues f_i of our matrix A are distinct. After that we proceed with dealing with the general case.

Distinct Eigenvalues

Let A, f_i, and $g_{i,j}$ be as defined above, we assume that all the eigenvalues of our matrix A are distinct. This is equivalent to the eigenvalues being distinct for both A and $E(A)$, since the f_i have no constant term. Among all the possible bases of eigenvectors for A we choose one in a canonical way.

Let K be an algebraic closure of the field $\mathbb{C}(\{f_i, g_{i,j} : 1 \leq i < j \leq n\})$, and let $t_{i,j} \in K$, with $1 \leq i < j \leq n$ be such that the vector

$$\vec{v}_i = (-t_{1,i}, \ldots, -t_{i-1,i}, 1, 0, \ldots, 0)$$

is an f_i-eigenvector for A. In this case $\vec{v}_i$ will also be an $e^{f_i}-$eigenvector for $E(A)$. We leave the proof of the existence of this canonical basis for A, chosen as above, to 12.3.

For the above, we will have the following

Lemma 12.1. *Let A and $t_{i,j}$ be as above. Then*

$$tr.d._{\mathbb{C}}\mathbb{C}(A) = tr.d._{\mathbb{C}}\mathbb{C}(\{f_i, t_{i,j} : 1 \leq i < j \leq n\}).$$

Proof. The condition $A\vec{v}_j = f_j\vec{v}_j$ translates to the following system of equations

$$\begin{aligned} \frac{g_{i,j} - g_{i,j-1}t_{j-1,j} - \dots - g_{i,i+1}t_{i+1,j}}{f_i - f_j} &= t_{i,j} \\ &\vdots \\ \frac{g_{j-1,j}}{f_{j-1} - f_j} &= t_{j-1,j} \end{aligned} \tag{12.1}$$

Since all of the f_i are distinct we can write the $t_{i,j}$ as rational functions on the entries of A by solving the above system of equations. In particular we get that for all $i < j$:

a. $t_{i,j} \in \mathbb{Z}[\frac{1}{f_s - f_j}, g_{s,r} : \min(i,j) \leq s < r \leq \max(i,j)]$, and

b. Let $I_{i,j} = \{(s,j) : s \leq i \text{ or } s \geq j\}$. There exists $Q_{i,j} \in R_{i,j}(I_{i,j})$ such that

$$t_{i,j}(A) = \tfrac{g_{i,j} - Q_{i,j}(A)}{f_i - f_j}.$$

Our result now follows trivially from the above remarks. □

Since the matrices A and $E(A)$ have the same eigenvectors, in the case where both A and $E(A)$ have distinct eigenvalues, we get that, for all $1 \leq i < j \leq n$, we will have $t_{i,j}(A) = t_{i,j}(E(A))$. This remark, together with the proof of 12.1, implies

Lemma 12.2. *Let A and $t_{i,j}$ be as defined above. Then*

$$tr.d._{\mathbb{C}}\mathbb{C}(E(A)) = tr.d._{\mathbb{C}}\mathbb{C}(\{e^{f_i}, t_{i,j}(A) : 1 \leq i < j \leq n\}), \text{ and}$$

$$tr.d._{\mathbb{C}}\mathbb{C}(A, E(A)) = tr.d._{\mathbb{C}}\mathbb{C}(\{f_i, e^{f_i}, t_{i,j}(A) : 1 \leq i < j \leq n\}).$$

Proof. Let $\vec{v}_i$ be the canonically chosen basis for A. Then, for all $1 \leq i < j \leq n$, by combining the proof of 12.1 and the equality $t_{i,j}(A) = t_{i,j}(E(A))$, there exists $Q_{i,j}$ as in the proof of 12.1 such that

$$E_{i,j}(A) = (g_{i,j} - Q_{i,j}(A))\tfrac{e^{f_i} - e^{f_j}}{f_i - f_j} + Q_{i,j}(E(A)).$$

The result then follows trivially from these remarks. □

Repeating eigenvalues

In the case where we have eigenvalues with multiplicity greater than 1 we have to alter our approach. The idea is to generalize the approach of the previous subsection. In other words, we wish to find a canonically defined basis, which will allow us to define coordinates that characterize our original matrix A uniquely. Furthermore we wish to describe those coordinates as algebraic functions of the coordinates of A. Our ultimate goal is to obtain results about transcendence degrees similar to those we proved in the previous case.

We assume that the matrix A has eigenvalues with multiplicities possibly greater than 1. By our assumption that the f_i have no constant term, each eigenvalue e^{f_i} of $E(A)$ has the same multiplicity as the respective eigenvalue f_i of A.

The Canonical Basis

Our first objective will be to describe and prove the existence of a certain canonical basis for A. We begin by describing the canonical basis of each eigenspace, then we combine these to create the basis we want.

Just as before we let $A = \operatorname{diag}(\vec{f}) + (g_{i,j})$ be an upper triangular matrix with entries in $\mathbb{C}[[t_1, \ldots, t_m]]$ and K be an algebraic closure of the field $\mathbb{C}(\{f_i, g_{i,j} : 1 \leq i < j \leq n\})$.

Lemma 12.3. *Let $z = f_{i_1} = \ldots = f_{i_{k-1}} = f_{i_k}$ with $i_1 < \ldots < i_k$. We also assume that $f_i \neq z$ for all $i \neq i_j$. Let $M(z)$ be the generalized eigenspace for the eigenvalue z. Then there exists a unique basis $\mathcal{B}_z$ of $M(z)$ consisting of vectors $\vec{v}_{i_j}$, $1 \leq j \leq k$, such that*

1. *$\vec{v}_{i_j}$ is of the form*
$$\vec{v}_{i_j} = (-t_{1,i_j}, \ldots, -t_{i_j-1,i_j}, 1, 0, \ldots, 0), \text{ and}$$
2. *$t_{l,i_j} = 0$ for $l = i_r$ where $1 \leq r \leq j-1$, and*
3. *there exist $s_{i_l,i_r} \in K$ for $1 \leq l < r \leq k$ such that*
$$A\vec{v}_{i_j} = z\vec{v}_{i_j} + s_{i_1,i_j}\vec{v}_{i_1} + \cdots + s_{i_{j-1},i_j}\vec{v}_{i_{j-1}}.$$

Proof. We proceed by induction on $k = \dim_{\mathbb{C}}(M(z))$. For $k = 1$ the uniqueness follows from the unique solution to the $t_{i,j}$ described by the equations (12.1).

Assume that $k = 2$. Then applying the system (12.1) we can determine the vector $\vec{v}_{i_1}$ which will be an eigenvector for A. Since $\vec{v}_{i_2}$ will in general be a generalized eigenvector then we will have
$$(A - zI_n)\vec{v}_{i_2} = s_{i_1,i_2}\vec{v}_{i_1},$$
for some $s_{i_1,i_2} \in K$. We can therefore assume without loss of generality that $t_{i_1,i_2} = 0$.

This relation will describe the coefficients of $\vec{v}_{i_2}$ uniquely thanks to the following series of equations:

1. In the range $i_1 < j < i_2$ we get

$$\begin{aligned} 0 &= -(f_{i_2-1} - f_{i_2})t_{i_2-1,i_2} + g_{i_2-1,i_2} \\ &\vdots \\ 0 &= -(f_{i_1+1} - f_{i_2})t_{i_1+1,i_2} - \ldots - g_{i_1+1,i_2-1}t_{i_2-1,i_2} + g_{i_1+1,i_2} \end{aligned} \tag{12.2}$$

2. For $j = i_1$

$$-g_{i_1,i_1+1}t_{i_1+1,i_2} - \ldots - g_{i_1,i_2-1}t_{i_2-1,i_2} + g_{i_1,i_2} = s_{i_1,i_2} \tag{12.3}$$

3. In the range $1 \leq j < i_1$

$$-(f_j - f_{i_2})t_{j,i_2} - \ldots - g_{j,i_2-1}t_{i_2-1,i_2} + g_{j,i_2} = s_{i_1,i_2}t_{j,i_1}. \tag{12.4}$$

Solving the above system, starting from the first equation of (12.2) and moving to the final equation described in the system (12.4) provides a unique solution for t_{j,i_2} in terms of the f_s and $g_{s,t}$.

Assume the result holds for $\dim_{\mathbb{C}} M(z) = k$. Then in order to prove the inductive step we can create a similar system of equations with unique solution for the t_{i_s,i_t} in terms of the coefficient of the matrix A.

We force relations on the canonical basis to be chosen so that

$$A\vec{v}_{i_j} = z\vec{v}_{i_j} + s_{i_1,i_j}\vec{v}_{i_1} + \cdots + s_{i_{j-1},i_j}\vec{v}_{i_{j-1}}. \tag{12.5}$$

Then by induction it is enough to determine the $t_{s,i_{k+1}}$, since the rest of the vectors will constitute a basis for the respective eigenspace of a smaller diagonal submatrix of A. To do this we just translate (12.5) for $j = k+1$ to a system of equations similar to the systems (12.2), (12.3), and (12.4). □

Combining all of the canonical bases of the eigenspaces we get a basis

$$\vec{v}_i = (-t_{1,i}, \ldots, -t_{i-1,i}, 1, 0, \ldots, 0),$$

with $1 \leq i \leq n$, $t_{i,j} \in K$ for $1 \leq i < j \leq n$, and such that $t_{i,j} = 0$ if $f_i = f_j$. This will be the **canonical basis** of A.

Basic Lemmas

In 12.3 we introduced the coefficients s_{i_l,i_r} for $l < r$. These coefficients determine uniquely the nilpotent operator defined by A on the generalized eigenspace $M(z)$, i.e. they determine the nilpotent operator $(A - zI_n)|_{M(z)}$.

In particular if $z = f_{i_1} = \ldots = f_{i_k}$ then we have $s_{i_l,i_r} \in K$, where K is an algebraic closure of $\mathbb{C}(A)$, and they are such that for $1 \le l < r \le k$

$$A\vec{v}_{i_j} = z\vec{v}_{i_j} + s_{i_1,i_j}\vec{v}_{i_1} + \cdots + s_{i_{j-1},i_j}\vec{v}_{i_{j-1}}. \tag{12.6}$$

From the proof of 12.3 both the $t_{i,j}$ and the $s_{i,j}$ are rational functions of the f_i and $g_{i,j}$. Their combined information turns out to be exactly what we will need in what follows.

Definition 12.4. Let $A = \operatorname{diag}(\vec{f}) + (g_{i,j})$ be an upper triangular matrix with entries in $\mathbb{C}[[t_1, \ldots, t_m]]$, such the diagonal entries f_i have no constant term. Let $\vec{v}_i = (-t_{1,i}, \ldots, -t_{i-1,i}, 1, 0, \ldots, 0)$ be a canonical basis for A. Also we consider the $s_{i,j}$ that satisfy the equations in (12.6) for those eigenvalues of A with multiplicity greater than 1. We define the **eigencoordinates** of A to be

$$T_{i,j}(A) = \begin{cases} t_{i,j}, & \text{if } f_i \neq f_j \\ s_{i,j}, & \text{if } f_i = f_j \end{cases}.$$

At this point we want to replicate the results of 12.1 and 12.2. We start with the following

Lemma 12.5. *Let A be as above and let $T_{i,j}(A)$ be its eigencoordinates. Then*

$$tr.d._{\mathbb{C}}\mathbb{C}(A) = tr.d._{\mathbb{C}}\mathbb{C}(\{f_i, T_{i,j}(A) : 1 \le i < j \le n\}).$$

Proof. Consider the set $I_{i,j}(A) = \{(s,j) : s \le i \text{ or } s \ge j, \text{ and } f_s \neq f_j\}$. For notational convenience let us define the ring

$$R_{i,j}(A) := R_{i,j}(I_{i,j}(A)).$$

From the proof of 12.3 it follows that the $T_{i,j}$ are rational functions on the coordinates of the matrix such that

(a) if $f_i \neq f_j$ then there exists a $Q_{i,j} \in R_{i,j}(A)$ such that

$$T_{i,j}(A) = \tfrac{g_{i,j} - Q_{i,j}(A)}{f_i - f_j}.$$

(b) if $f_i = f_j$ then there exists $Q_{i,j} \in R_{i,j}(A)$ such that

$$T_{i,j}(A) = g_{i,j} - Q_{i,j}(A).$$

In other words the map $A \mapsto \operatorname{diag}(f_1, \ldots, f_n) + (T_{i,j}(A))$ is bijective and the coordinates are rational functions on the entries of A with only factors of the form $f_i - f_j$ appearing in the denominator.

The equality of the transcendence degrees in question then follows easily from the above remarks. $\square$

Remark 12.6. The proof of the above lemma actually gives us more. Namely from the proof it follows that

$$\mathbb{C}(A) = \mathbb{C}(\{f_i, T_{i,j}(A) : 1 \leq i < j \leq n\}).$$

The next step here is to study the effects of the exponential function on the eigencoordinates. We record the main such results we will need in the following

Proposition 12.7. *Let A be as above and let $T_{i,j}(A)$ be its eigencoordinates. Then*

$$tr.d._{\mathbb{C}}\mathbb{C}(E(A)) = tr.d._{\mathbb{C}}\mathbb{C}(\{e^{f_i}, T_{i,j}(A) : 1 \leq i < j \leq n\}).$$

From this and 12.5 we conclude that

$$tr.d._{\mathbb{C}}\mathbb{C}(A, E(A)) = tr.d._{\mathbb{C}}\mathbb{C}(\{f_i, e^{f_i}, T_{i,j}(A) : 1 \leq i < j \leq n\}).$$

Proof. We start by fixing some notation. We let $R_{i,j}(A)$ be the rings defined in the proof of 12.5.

Since we have already dealt with the case where all eigenvalues are distinct, we only have to study the behavior of the exponential with respect to the generalized eigenspaces of dimension greater than 1.

We assume that $z = f_{i_1} = \ldots f_{i_k}$ with $f_i \neq z$ if $i \notin \{i_1, \ldots, i_k\}$ so that the system (12.6) actually describes the eigencoordinates of A. We start by looking at the effect of the exponential on the equations of (12.6).

By induction and the definition of the exponential we get the following relation for the exponential matrix $E(A)$

$$\begin{aligned} E(A)\vec{v}_{i_j} = e^z\vec{v}_{i_j} + e^z[s_{i_1,i_j}\vec{v}_{i_1} + \cdots + s_{i_{j-1},i_j}\vec{v}_{i_{j-1}}] + \\ + e^z[S_{i_1,i_j}\vec{v}_{i_1} + \cdots + S_{i_{j-2},i_j}\vec{v}_{i_{j-2}}] \end{aligned} \tag{12.7}$$

where $S_{i_t,i_j} := \sum_{t<l_1<\ldots<l_m<j} \frac{1}{m!} s_{i_t,i_{l_1}} \cdots s_{i_{l_{k-1}},i_{l_k}}$.

Most importantly (12.7) implies that

$$T_{i,j}(E(A)) = \begin{cases} e^z[T_{i,j}(A) + S_{i,j}(A)] & \text{if } f_i = f_j \\ T_{i,j}(A) & \text{if } f_i \neq f_j \end{cases}, \tag{12.8}$$

where the $S_{i,j}$ will again be elements of the ring $R_{i,j}$, due to the proof of 12.5, and can therefore be considered as functions on A.

Assuming that $f_i = f_j$ we define $J_{i,j}(A) = \{i_1, \ldots, i_k\}$ and we also define the ring

$$D_{i,j}(A) := D_{i,j}(J_{i,j}(A)).$$

Claim: Let i, j be such that $f_i = f_j$. Then there exists $P_{i,j} \in D_{i,j}(A)$ such that

$$T_{i,j}(E(A)) = e^z(T_{i,j}(A)) + P_{i,j}(E(A)). \tag{12.9}$$

Assuming this claim we go about proving that the transcendence degrees in question are in fact equal. First we define the following fields:

$$K_1 = \mathbb{C}(\{e^{f_i}, T_{i,j}(A) : 1 \le i < j \le n\}),\ K_0 = \mathbb{C}(E(A)), \text{ and}$$
$$L = \mathbb{C}(\{E(A), T_{i,j}(E(A)), T_{i,j}(A) : 1 \le i < j \le n\}).$$

We then have from the equations (12.8), for the case $f_i \ne f_j$, and (12.9), for the case $f_i = f_j$, that the extension $L/K_0(\{T_{i,j}(E(A)) : 1 \le i < j \le n\})$ is algebraic. From the proof of 12.5, applied to the matrix $E(A)$, we get that $K_0(\{T_{i,j}(E(A)) : 1 \le i < j \le n\}) = K_0$. So $tr.d._{\mathbb{C}} L = tr.d._{\mathbb{C}} K_0$.

On the other hand, again from the proof of 12.5, we have that $L = K_1(\{T_{i,j}(E(A)) : 1 \le i < j \le n\})$. While (12.8), together with the definition of $S_{i,j}$, tells us that the extension $K_1(\{T_{i,j}(E(A)) : 1 \le i < j \le n\})/K_1$ is algebraic. So that $tr.d._{\mathbb{C}} L = tr.d._{\mathbb{C}} K_1$ and the result follows.

Proof of the Claim. We assume we are in the same situation as above. Namely we assume that $z = f_{i_1} = \ldots f_{i_k}$ with $f_i \ne z$ if $i \notin \{i_1, \ldots, i_k\}$, so (12.7) holds for all $1 \le j \le k$. In particular, we need to show that, for all pairs $1 \le t < j \le k$, there exists $P_{i_t,i_j} \in D_{i_t,i_j}(A)$ such that $T_{i_t,i_j}(E(A)) = e^z(T_{i_t,i_j}(A)) + P_{i_t,i_j}(E(A))$. By (12.8) it suffices to show the existence of a $P_{i_t,i_j} \in D_{i_t,i_j}(A)$ such that $S_{i_t,i_j} = P_{i_t,i_j}(E(A))$. From the definition of $S_{i,j}$ it suffices to prove that for all pairs $1 \le t < j \le k$, there exists $F_{i_t,i_j} \in D_{i_t,i_j}(A)$ such that $s_{i_t,i_j}(A) = F_{i_t,i_j}(E(A))$. We prove this last assertion by induction on $j - t$.

Let us start with $j - t = 1$. As a consequence of (12.7) we have that $T_{i_{j-1},i_j}(E(A)) = e^z T_{i_{j-1},i_j}(A)$ for all j. We can rewrite this as

$$s_{i_{j-1},i_j}(A) = e^{-z} s_{i_{j-1},i_j}(E(A)). \tag{12.10}$$

The assertion now follows from the proof of 12.5 applied to the matrix $E(A)$ that shows $s_{i_{j-1},i_j}(E(A)) = E(A)_{i_{j-1},i_j} - Q_{i_{j-1},i_j}(E(A))$, where $Q_{i_{j-1},i_j} \in R_{i_{j-1},i_j}(A)$.

For $t = j - 2$ the definition of $S_{i,j}$ and (12.8) imply

$$s_{i_{j-2},i_j}(E(A)) = e^z s_{i_{j-2},i_j}(A) + e^z s_{i_{j-2},i_{j-1}}(A) s_{i_{j-1},i_j}(A).$$

This together with (12.10) imply

$$s_{i_{j-2},i_j}(A) = e^{-z}(s_{i_{j-2},i_j}(E(A))) - e^{-2z} s_{i_{j-2},i_{j-1}}(E(A)) s_{i_{j-1},i_j}(E(A)). \tag{12.11}$$

Once again the assertion follows as above from the proof of 12.5.

Assume the assertion for all pairs (i_x, i_y) with $y - x \leq m$. Let $j = t + m + 1$. Then, by definition of $S_{i,j}$ and the inductive hypothesis, we get that there exists $P_{i_t,i_j} \in Di_t, i_j(A)$ such that $S_{i_t,i_j} = P_{i_t,i_j}(E(A))$.

From (12.8) we get that $s_{i_t,i_j}(A) = F_{i_t,i_j}(E(A)) = e^{-z}(s_{i_t,i_j}(E(A)) - P_{i_t,i_j}(E(A)))$. Once again by the proof of 12.5 applied to $E(A)$ as above the assertion follows and the claim has been proven. □

12.2 AX-SCHANUEL FOR UPPER TRIANGULAR MATRICES

At this point we are able to state and prove a Weak Ax-Schanuel-type result for the map $E : \mathfrak{h}_n \to U_n$. We also record a corollary of our result, as well as an alternate geometric view in the spirit of [Pil15].

As we have been doing so far, we let $E_{i,j}(A)$ be the corresponding entry of the matrix $E(A)$ for $1 \leq i < j \leq n$.

Theorem 12.8 (Weak Ax-Schanuel for U_n)**.** *Let $f_1, \dots, f_n, g_{i,j} \in \mathbb{C}[[t_1, \dots, t_m]]$ be power series, where $1 \leq i < j \leq n$. We assume that the f_i do not have a constant term. Let A be the $n \times n$ upper triangular matrix with diagonal $\vec{f}$ and the (i, j) entry equal to $g_{i,j}$. Then, assuming that the f_i are $\mathbb{Q}-$linearly independent,*

$$tr.d._{\mathbb{C}}\mathbb{C}(A, E(A)) \geq n + \operatorname{rank}(J(\vec{f}, \vec{g}; \vec{t})).$$

Proof. From 12.2 we may replace the left hand side of the above inequality by $tr.d._{\mathbb{C}}\mathbb{C}(\{f_i, g_{i,j}, e^{f_i} : 1 \leq i < j \leq n\})$. This reduces the proof to 11.1, by giving the $g_{i,j}$ a new indexing g_k, $1 \leq k \leq \frac{n(n-1)}{2}$. □

Replacing 12.2 with 12.7 in the above proof yields the following

Corollary 12.9. Let $f_1, \dots, f_n, g_{i,j} \in \mathbb{C}[[t_1, \dots, t_m]]$ be power series, where $1 \leq i < j \leq n$. We assume that the f_i do not have a constant term. Let A be the $n \times n$ upper triangular matrix with diagonal $\vec{f}$ and the (i, j) entry equal to $g_{i,j}$. Let also $N = \dim_{\mathbb{Q}}\langle f_1, \dots, f_n\rangle_{\mathbb{Q}}$, then

$$tr.d._{\mathbb{C}}\mathbb{C}(A, E(A)) \geq N + \operatorname{rank}(J(\vec{f}, \vec{g}; \vec{t})).$$

An Alternate Formulation

In the spirit of [Pil15] we can give an alternate form of 12.8. This time the background is slightly changed. We let $V \subset \mathfrak{h}_n$ be an open subset and $X \subset V$ an irreducible complex analytic subvariety of V with $dim_{\mathbb{C}}(X) = m$ such that X contains the origin, and locally at $\vec{0}$ the coordinate functions, $f_1, \dots, f_n$ and $g_{s,t}$ for $s < t$, are meromorphic functions on X.

For reasons of convenience, and in keeping a similar notation to the previous version, we let $A = (\vec{f}) + (g_{s,t})$ denote the matrix corresponding to the coordinates f_i and $g_{i,j}$.

Theorem 12.10 (Weak Ax-Schanuel-Alternate Formulation). *In the above context, if the f_i are $\mathbb{Q}-$linearly independent modulo $\mathbb{C}$, then*

$$tr.d._{\mathbb{C}}\mathbb{C}(A, E(A)) \geq n + \dim(X).$$

Proof. We choose $t_1, \ldots, t_m$ that are independent holomorphic coordinates on X locally at $\vec{0}$ so that $\mathrm{rank}(J(\vec{f}, \vec{g}; \vec{t})) = dim_{\mathbb{C}}(X)$. Then this reduces to 12.8. □

Also similarly to above we can translate 12.9 in this context.

Corollary 12.11. In the above context, if the f_i are are such that $N = \dim_{\mathbb{Q}}\langle f_1, \ldots, f_n\rangle_{\mathbb{Q}}$, then

$$tr.d._{\mathbb{C}}\mathbb{C}(A, E(A)) \geq N + \dim(X).$$

This form of the Ax-Schanuel result is the one we will use in what follows.

12.3 WEAKLY SPECIAL SUBVARIETIES

We turn our attention to describing the weakly special subvarities of $\mathfrak{h}_n$ that contain the origin, i.e. the zero matrix. Geometrically 12.7 gives us a significant amount of motivation. We can expect that the weakly special subvarieties will be determined by the following information:

1. A system Σ of $\mathbb{Q}$-linear equations on the diagonal coordinates of the matrices, i.e. the eigenvalues, and
2. A system of equations on the eigen-coordinates of a generic matrix in the subvariety of $\mathfrak{h}_n$ defined by the system Σ.

Conditions on Eigenvalues

We start by making this idea more explicit. Let us assume that we have a $\mathbb{Q}$-linearly independent set Σ of $\mathbb{Q}$-linear polynomials on the diagonal coordinates of $\mathfrak{h}_n$, i.e. polynomials of the form

$$F(\vec{f}) = \sum_{i=1}^{n} q_i f_i,$$

where $q_i \in \mathbb{Q}$. Let $Z(\Sigma) \subset \mathfrak{h}_n$ be the algebraic subvariety of $\mathfrak{h}_n$ defined by Σ.

Picking a generic matrix $A \in Z(\Sigma)$ the multiplicities of the eigenvalues will be determined by Σ. More specifically, depending on Σ we have fixed multiplicities of eigenvalues on a dense open subset of $Z(\Sigma)$, which we denote by U_Σ. To define this latter subset we start by considering the set

$$I_\Sigma = \{(i,j) : 1 \leq i < j \leq n, Z(\Sigma) \subset Z(f_i - f_j)\}.$$

Then we take

$$U_\Sigma = Z(\Sigma) \backslash (\bigcup_{(i,j) \notin I_\Sigma} Z(f_i - f_j)).$$

Passing to the eigencoordinates

Let us restrict our attention to U_Σ. On this dense open subset we can define, by 12.3 a canonical basis for any matrix $A \in U_\Sigma$. The eigen-coordinates of A will be well defined regular functions on U_Σ, thanks again to the proof of 12.3.

The relations proven during the proof of 12.7 reinterpreted geometrically show that any algebraic relation satisfied by the $T_{i,j}(A)$ will translate to an algebraic relation for the $T_{i,j}(E(A))$ and vice versa. In order to translate this into a geometric language we must first find a more convenient description for $Z(\Sigma)$ and U_Σ.

We start by noting that we have an isomorphism

$$Z(\Sigma) \cong L \times \mathbb{A}_{\mathbb{C}}^{\frac{n(n-1)}{2}},$$

where L is a $\mathbb{Q}$-linear subspace of $\mathbb{C}^k$ where k is the number of generically distinct eigenvalues of $Z(\Sigma)$. Let us also consider the dense open subset $D_\Sigma \subset \mathbb{C}^k$ with

$$D_\Sigma = \mathbb{C}^k \backslash (\bigcup_{1 \leq i < j \leq k} Z(x_i - x_j)),$$

where x_i denote the coordinates of $\mathbb{C}^k$.

Combining the above, we consider $L' = L \cap D_\Sigma$ under this identification. We may then take the following isomorphism

$$U_\Sigma \cong L' \times \mathbb{A}_{\mathbb{C}}^{\frac{n(n-1)}{2}}. \tag{12.12}$$

At this point we apply 12.5, which shows that we can change coordinates on the $\mathbb{A}_{\mathbb{C}}^{\frac{n(n-1)}{2}}$ part of the right hand side of the above isomorphism from $g_{i,j}$ to $T_{i,j}$. In other words, we have an isomorphism

$$T_\Sigma : U_\Sigma \to L' \times \mathbb{A}_{\mathbb{C}}^{\frac{n(n-1)}{2}}, \tag{12.13}$$

where the $\mathbb{A}_{\mathbb{C}}^{\frac{n(n-1)}{2}}$ on the right signifies the affine space of the eigencoordinates $T_{i,j}$.

Conditions on the eigencoordinates

Let $V \subset \mathbb{A}_{\mathbb{C}}^{\frac{n(n-1)}{2}}$ be an irreducible subvariety of $\mathbb{A}_{\mathbb{C}}^{\frac{n(n-1)}{2}}$ that contains the origin, where the latter is considered as the space of the eigencoordinates. Then if we consider $W = L' \times V$ this will be an irreducible subvariety of $L' \times \mathbb{A}_{\mathbb{C}}^{\frac{n(n-1)}{2}}$. We now consider its inverse under the isomorphism T_Σ of (12.13). Finally we consider the Zariski closure of the resulting set in $\mathfrak{h}_n$, which we will denote by $X(\Sigma, V)$.

Notice that $X(\Sigma, V)$ satisfies exactly what we wanted, the diagonal coordinates are only subject to $\mathbb{Q}$-linear equations, any relation on the strictly upper triangular part comes from relations on the eigencoordinates, it is irreducible and it contains the origin.

Definition 12.12 (Weakly Special Subvarieties)**.** An irreducible subvariety X of $\mathfrak{h}_n$ that contains the origin will be called **weakly special** if there exist:

1. a system Σ of $\mathbb{Q}$-linear equations on the diagonal entries, and
2. an irreducible subvariety V defined as above,

such that $X = X(\Sigma, V)$, where the latter is as defined in the above discussion.

Some examples

We present two examples of weakly special subvarieties as a motivation for the above definition.

Example 12.13. Let $V_1 \subset \mathfrak{h}_3$ be the set of all upper triangular 3×3 matrices A satisfying the following conditions:

(C_1) A has diagonal (f_1, f_2, f_3) with $f_i \neq f_j$ for all $i \neq j$,

(C_2) $\vec{v}_2 = (1, 1, 0)$ is an f_2-eigenvector, and

(C_3) there exists $s \in \mathbb{C}$ such that $\vec{v}_3 = (s, s^2, 1)$ is an f_3-eigenvector.

Notice that all upper triangular matrices will have $\vec{v}_1 = (1, 0, 0)$ as an eigenvector for the eigenvalue f_1. We also note that here the choice of "s^2" is arbitrary and can be replaced by any algebraic function of s.

If we set $W_1 = Zcl(V_1)$ to be the Zariski closure of V_1 in $\mathfrak{h}_n$, then W_1 will be a weakly special subvariety of $\mathfrak{h}_n$.

To see that this is natural to expect, consider the set $X_1 \subset U_3$ of all invertible 3×3 upper triangular matrices A satisfying:

(D_1) A has diagonal (f_1, f_2, f_3) with $f_i \neq f_j$ for all $i \neq j$,

(D_2) $\vec{v}_2 = (1,1,0)$ is an f_2-eigenvector, and

(D_3) there exists $s \in \mathbb{C}$ such that $\vec{v}_3 = (s, s^2, 1)$ is an f_3-eigenvector.

This set will be contained in $E(V_1)$, by the remarks above. In fact, X_1 is a Zariski open subset of $E(V_1)$ and $dim_{\mathbb{C}}(X_1) = dim_{\mathbb{C}}(E(V_1)) = dim_{\mathbb{C}}(Zcl(E(V_1))) = 4$.

Example 12.14. Once again let us denote by $\vec{v}_1$ the vector $(1,0,0)$. We consider $V_2 \subset \mathfrak{h}_3$ to be the set of all upper triangular matrices A satisfying the following conditions:

(C_1) A has diagonal (f_1, f_2, f_1) wtih $f_1 \neq f_2$,

(C_2) there exists $s \in \mathbb{C}$ such that $\vec{v}_2 = (s, 1, 0)$ is an f_2-eigenvector,

(C_3) for the same s as above, the vector $\vec{v}_3 = (0, s^2, 1)$ is a generalized eigenvector for the eigenvalue f_1, and

(C_4) for the same s as above, we have that
$$A\vec{v}_3 = f_1\vec{v}_3 + s^5\vec{v}_1.$$

Here again the choices of "s^2" and "s^5" in C_3 and C_4 are arbitrary and can be replaced by any algebraic function in s.

Setting $W_2 = Zcl(V_2)$ we get a weakly special subvariety of $\mathfrak{h}_3$.

We argue as before, by considering the set $X_2 \subset U_3$ of all invertible 3×3 upper triangular matrices A satisfying:

(D_1) A has diagonal (f_1, f_2, f_1) wtih $f_1 \neq f_2$,

(D_2) there exists $s \in \mathbb{C}$ such that $\vec{v}_2 = (s, 1, 0)$ is an f_2-eigenvector,

(D_3) for the same s as above, the vector $\vec{v}_3 = (0, s^2, 1)$ is a generalized eigenvector for the eigenvalue f_1, and

(D_4) for the same s as above, we have that
$$A\vec{v}_3 = f_1\vec{v}_3 + f_1 s^5\vec{v}_1.$$

The change in D_4 from s^5 to $f_1 s^5$ is related to the action of the exponential on the nilpotent operator on the f_1-generalized eigenspace. It turns out that X_2 is a Zariski open subset of $E(V_2)$ and $dim_{\mathbb{C}}(X_2) = dim_{\mathbb{C}}(E(V_2)) = dim_{\mathbb{C}}(Zcl(E(V_2))) = 3$.

12.4 AX-LINDEMANN AND OTHER COROLLARIES

Here we record some corollaries of our Ax-Schanuel result. We start with a "two-sorted version"[1] of 12.11 and then use that to prove the Ax-Lindemann result. The latter allows us to characterize the bi-algebraic subsets for the map E that contain the origin. The exposition follows in the spirit of [Pil15].

We start by defining the notion of a component.

Definition 12.15. Let $W \subset \mathfrak{h}_n$ and $V \subset U_n$ be algebraic subvarieties. Then a **component** X of $W \cap E^{-1}(V)$ will be a complex-analytically irreducible component of $W \cap E^{-1}(V)$.

The context in which we will be using our Ax-Schanuel result is the one described in 12.10 and the discussion leading up to it.

Theorem 12.16 (Two-sorted Weak Ax-Schanuel for U_n)**.** *Let $U \subset \mathfrak{h}_n$ be a weakly special subvariety, containing the origin, and set $X = E(U)$. Let $W \subset U$ and $V \subset X$ be algebraic subvarieties, with $0_n \in W$ and $I_n \in V$. If the component C of $W \cap E^{-1}(V)$ that contains the origin is not contained in any proper weakly special subvariety of U then*

$$\dim_{\mathbb{C}} C \leq \dim_{\mathbb{C}} V + \dim_{\mathbb{C}} W - \dim_{\mathbb{C}} X.$$

Proof. Following the discussion of the previous section, we can associate to the subvariety U a system Σ of $\mathbb{Q}-$linear equations on the diagonal entries, as well as the corresponding $\mathbb{Q}-$linear subspace L of $\mathbb{C}^k$ where k is the number of generically distinct eigenvalues, and a subvariety Z of $\mathbb{A}_{\mathbb{C}}^{\frac{n(n-1)}{2}}$. In other words with the notation of the previous section $U = X(\Sigma, Z)$. We also denote by U_Σ the corresponding dense open subset we had in the discussion of the previous section.

At this point we let $B = U_\Sigma \cap C$, then B is again a complex analytically irreducible subset that is dense in C and it is not contained in a proper weakly special subvariety of U. In particular we will have $\dim_{\mathbb{C}} B = \dim_{\mathbb{C}} C$.

We denote by f_i the diagonal coordinates of a matrix as functions on B and similarly for the coordinates $g_{i,j}$. Likewise we denote the diagonal coordinates of the exponential map by E_i and the strictly upper triangular by $E_{i,j}$ and we consider them as functions on B as well, keeping in mind that $E_i(A) = e^{f_i}$. For reasons of convenience we let $A = \mathrm{diag}(\vec{f}) + (g_{i,j})$ denote the matrix of the corresponding coordinates.

1 We are borrowing this term from the relative discussion in [Pil15].

We start with some simple remarks concerning our setting. First of all, we will have

$$\dim_{\mathbb{C}} W \geq tr.d._{\mathbb{C}}\mathbb{C}(A), \text{ and} \tag{12.14}$$

$$\dim_{\mathbb{C}} V \geq tr.d._{\mathbb{C}}\mathbb{C}(E(A)). \tag{12.15}$$

Next, we employ 12.11, to get that, if $N = \dim_{\mathbb{Q}}\langle f_1, \ldots, f_n\rangle_{\mathbb{Q}}$, then

$$\dim_{\mathbb{C}} B + N \leq tr.d._{\mathbb{C}}\mathbb{C}(A, E(A)). \tag{12.16}$$

We also set

$$m = tr.d._{\mathbb{C}}\mathbb{C}(A, E(A)), \text{ and}$$

$$l = tr.d._{\mathbb{C}}\mathbb{C}(\{T_{i,j}(A) : 1 \leq i < j \leq n\}),$$

with $T_{i,j}$ denoting once again the eigencoordinates of a matrix. From this point on for convenience we will denote simply by $T_{i,j}$ the elements $T_{i,j}(A)$.

At this point we turn our attention to 12.5, 12.7, along with equations (12.8) and (12.9). On B the eigencoordinates $T_{i,j}$ are well defined as functions on B. From the aforementioned lemmas we also get

$$tr.d._{\mathbb{C}}\mathbb{C}(A) = tr.d._{\mathbb{C}}\mathbb{C}(\{T_{i,j}, f_i : 1 \leq i < j \leq n\}), \text{ and} \tag{12.17}$$

$$tr.d._{\mathbb{C}}\mathbb{C}(E(A)) = tr.d._{\mathbb{C}}\mathbb{C}(\{T_{i,j}, E_i : 1 \leq i < j \leq n\}). \tag{12.18}$$

By the definition of weakly special subvarieties, and the minimality of U in containing C, we see that

$$\dim_{\mathbb{C}} U = \dim_{\mathbb{C}} E(U) = \dim_{\mathbb{C}} X = N + l. \tag{12.19}$$

On the other hand, we get that, if $K = \mathbb{C}(\{T_{i,j} : 1 \leq i < j \leq n\})$,

$$m = l + tr.d._K K(\{E_i, f_i : 1 \leq i \leq n\})$$

We also have that

$$tr.d._{\mathbb{C}}\mathbb{C}(A) = tr.d._{\mathbb{C}}K + tr.d._K K(f_1, \ldots, f_n),$$

and likewise that

$$tr.d._{\mathbb{C}}\mathbb{C}(E(A)) = tr.d._{\mathbb{C}}K + tr.d._K K(E_1, \ldots, E_n).$$

Combining the above equalities implies that

$$m \leq tr.d._{\mathbb{C}}\mathbb{C}(E(A)) + tr.d._{\mathbb{C}}\mathbb{C}(A) - l. \tag{12.20}$$

Using (12.20) along with (12.15) and (12.14) yields

$$m \leq \dim_{\mathbb{C}} W + \dim_{\mathbb{C}} V - l.$$

Together with (12.16), (12.19), and the fact that $\dim_{\mathbb{C}} B = \dim_{\mathbb{C}} C$, this finishes the proof. $\square$

Corollary 12.17 (Ax-Lindemann for U_n)**.** Let $V \subset U_n$ be an algebraic subvariety with $I_n \in V$. If $W \subset E^{-1}(V)$ is a maximal irreducible subvariety that contains the origin, then W is a weakly special subvariety.

Proof. Let U be the minimal weakly special subvariety that contains W, $X = E(U)$, and let $V' = V \cap X$. We use 12.16 for $C = W$ to get

$$\dim_{\mathbb{C}} W \leq \dim_{\mathbb{C}} W + \dim_{\mathbb{C}} V' - \dim_{\mathbb{C}} X.$$

This implies $\dim_{\mathbb{C}} X \leq \dim_{\mathbb{C}} V'$, and since $V' \subset X$ we get that $X \subset V$ and that $W \subset U \subset E^{-1}(V)$. Maximality of W then implies that $W = U$ is weakly special. $\square$

13

GENERAL LINEAR GROUP

Having studied the exponential of $\mathfrak{h}_n$ we can expect to achieve similar Ax-Schanuel and Ax-Lindemann results for the case of general matrices. Once again the key role will be played by the eigenvalues of our matrix.

We start with considering certain subsets of $\mathfrak{gl}_n$ that will assist us in formulating the Ax-Schanuel and Ax-Lindemann results. We then proceed in a similar fashion to the upper triangular case. Namely we start by stating the Ax-Schanuel result and then reduce its proof to 11.1. Finally, we conclude with some corollaries of our result.

Notation: For the remainder we will denote the Lie algebra of $n \times n$ matrices over $\mathbb{C}$ by $\mathfrak{gl}_n$ and the respective exponential function by

$$E : \mathfrak{gl}_n \to GL_n.$$

13.1 DATA OF A MATRIX AND THE EXPONENTIAL

We begin our study by defining the **data** of a matrix A, a notion that will generalize the eigencoordinates we had in the upper triangular case. With the help of this new notion we can define, as we will see, the weakly special subvarieties and achieve a simpler description of the exponential.

As we did in the case of the upper triangular matrices, throughout this section we present as lemmas the equalities of transcendence degrees that we will need in the proofs of our main results.

The Data of a matrix

Let V be a $\mathbb{C}-$linear space with $\dim_{\mathbb{C}} V = n$. Let also $A \in \mathrm{Hom}(V, V) = \mathfrak{gl}_n$ then A is uniquely characterized by the following data:

1. A number of distinct complex numbers $f_1, \ldots, f_k$, the eigenvalues of A,

2. for each eigenvalue f_i an $m_i \in \mathbb{N}$, the multiplicity of that eigenvalue, such that $\sum_{i=1}^{k} m_i = n$,

3. for each f_i as above, a subspace $V_i \leq V$, with $\dim V_i = m_i$, such that $V = \bigoplus_{i=1}^{k} V_i$, i.e. to every eigenvalue a corresponding generalized eigenspace, and

4. for each f_i as above, a nilpotent operator $N_i \in \mathrm{Hom}(V_i, V_i)$, i.e. $N_i = (A - f_i I_n)|_{V_i}$.

The above picture also holds over an arbitrary algebraically closed field.

Definition 13.1. Let A be a matrix as above. Then we define the **data of the matrix** to be the data

$$((f_1, \ldots, f_k), (m_1, \ldots, m_k), (V_1, \ldots, V_k), (N_1, \ldots, N_k)).$$

The information of the generalized eigenspaces V_i and nilpotent operators N_i of a matrix A with k distinct eigenvalues, each with respective multiplicity m_i, is parametrized by a variety which we will denote by $W_k(\vec{m})$. We also let $w_{k,\vec{m}} = \dim W_k(\vec{m})$. In what follows we will need to consider a set of coordinates on such a variety, which we will denote by T_j with $1 \leq j \leq w_{k,\vec{m}}$.

These T_j will play the role of the eigencoordinates of Part I. We digress here to properly define these varieties and make the above ideas more rigorous. We do this over $\mathbb{C}$, though the same construction clearly works over any algebraically closed field.

Some auxiliary varieties

We fix an $n-$dimensional vector space V over $\mathbb{C}$, we also fix $k \in \mathbb{N}$ and $m_1 \leq \ldots \leq m_k \in \mathbb{N}$ such that $\sum_{i=1}^{k} m_i = n$. We need a space parametrizing all pairs of $k-$tuples of the form $((V_1, \ldots, V_k), (N_1, \ldots, N_k))$ where V_i is an m_i-dimensional subspace of V, N_i is a nilpotent operator on V_i, and the V_i are such that $V = \bigoplus_{i=1}^{k} V_i$.

To this end, consider the product of Grassmannians

$$G(\vec{m}, V) := Gr_{m_1}(V) \times \ldots \times Gr_{m_k}(V).$$

On this space we consider the trivial bundle $S := V \times G(\vec{m}, V)$ and for $1 \leq i \leq k$ the subbudle $\mathbb{V}_i$ of S that is the pullback of the tautological bundle of the Grassmannian $Gr_{m_i}(V)$ on $G(\vec{m}, V)$.

Now consider the morphism of vector bundles over $G(\vec{m}, V)$

$$\phi_{(\vec{m},V)} : \bigwedge^{m_1} \mathbb{V}_1 \wedge \ldots \wedge \bigwedge^{m_k} \mathbb{V}_k \to \bigwedge^{n} S.$$

The set $A(\vec{m}, V) := \{P \in G(\vec{m}, V) : \phi_{(\vec{m},V),P} \neq 0\}$ is an open subvariety of $G(\vec{m}, V)$.

Definition 13.2. Let X be a topological space and F a finite dimensional vector bundle on X. Then we define $Nil(F)$ to be the vector bundle over X whose fiber at $x \in X$ is the vector space $Nil(F_x)$ of nilpotent operators on F_x.

Let us now consider the vector bundle $E(\vec{m}) = Nil(\mathbb{V}_1) \times \ldots \times Nil(\mathbb{V}_k)$ over $G(\vec{m}, V)$. Then the restriction $E(\vec{m})|_{A(\vec{m},V)}$ of this vector bundle on $A(\vec{m}, V)$ is exactly the space we want. So we define $W_k(\vec{m}) := E(\vec{m})|_{A(\vec{m},V)}$.

Some auxiliary maps

In what follows we will also need to consider a group action on $X_k(\vec{m}) = \mathbb{A}^k_{\mathbb{C}} \times W_k(\vec{m})$. Consider the equivalence relation on $\{1, \ldots, k\}$ given by $i \sim j$ if and only if $m_i = m_j$. Let $i_1, \ldots, i_r$ be representatives for the equivalence classes of this equivalence relation, and for $1 \leq j \leq r$ we let

$$n_j := |\{s : 1 \leq s \leq k, s \sim i_j\}|$$

and note that $\sum_{j=1}^{r} n_j = k$.

Let $S_{\vec{m}} := S_{n_1} \times \ldots \times S_{n_r}$ be the direct product of the symmetric groups S_{n_j}. Each group S_{n_j} acts naturally as permutations on $\mathbb{A}^{n_j}_{\mathbb{C}}$ and again as permutations of the factors $Gr_{m_s}(V)$ with $s \sim i_j$ of $G(\vec{m}, V)$. As a result we get a natural action of each S_{n_j} on $E(\vec{m})$ and by restriction on $W_k(\vec{m})$.

Putting all of these actions together we get an action of $S_{\vec{m}}$ on $\mathbb{A}^k_{\mathbb{C}}$ and one on $W_k(\vec{m})$. Because of our convention that $m_1 \leq \ldots \leq m_k$, we may assume that the $S_{n_1}-$factor of $S_{\vec{m}}$ acts on the first n_1 coordinates of $\mathbb{A}^k_{\mathbb{C}}$, the $S_{n_2}-$factor on the next n_2 coordinates and so on. These two actions of $S_{\vec{m}}$ combine to give a diagonal action on $X_k(\vec{m})$.

If we let $f_1, \ldots, f_k$ be coordinates on $\mathbb{A}^k_{\mathbb{C}}$ we define $\Gamma_k := \mathbb{A}^k_{\mathbb{C}} \backslash \bigcup_{i<j} Z(f_i - f_j)$ and define $Y_k(\vec{m}) := (\Gamma_k \times W_k(\vec{m}))/S_{\vec{m}}$. We also have a finite surjective morphism[1] that we denote by

$$\pi_{k,\vec{m}} : \Gamma_k \times W_k(\vec{m}) \to Y_k(\vec{m}).$$

For coordinates f_i, $1 \leq i \leq k$, of $\mathbb{A}^k_{\mathbb{C}}$ and coordinates T_j, $1 \leq j \leq w_{k,\vec{m}}$ of $W_k(\vec{m})$ we denote $[(\overrightarrow{(f_i(P))}, \overrightarrow{(T_j(P))})] \in Y_k(\vec{m})$ the image of the point $P = (\overrightarrow{(f_i(P))}, \overrightarrow{(T_j(P))}) \in X_k(\vec{m})$. We also define the map

$$\Psi_{k,\vec{m}} : Y_k(\vec{m}) \to \mathfrak{gl}_n$$

1 See Theorem 1, pages 104-105 of [Mum08].

such that $\{[(\overrightarrow{(f_i(P))}, \overrightarrow{(T_j(P))})]\} \mapsto QJQ^{-1}$, with J being a block diagonal matrix, with its blocks being Jordan blocks, where we allow elements of the superdiagonal to be either 0 or 1, and Q being the transition matrix that is defined by the subspaces V_i parameterized by the coordinates T_j.

We also define

$$\Phi_{k,\vec{m}} := \Psi_{k,\vec{m}} \circ \pi_{k,\vec{m}} : \Gamma_k \times W_k(\vec{m}) \to \mathfrak{gl}_n.$$

Remarks 13.3. *1. The image of $\Psi_{k,\vec{m}}$ will be exactly the set of all matrices with k distinct eigenvalues whose multiplicities are given by the entries of the vector $\vec{m}$. This is true since the Jordan canonical form of a matrix is uniquely determined, up to permutation, by the Jordan blocks. Permuting these blocks also results in respective permutations of the columns of the transition matrix P, which are parameterized by the T_j.*

2. We note that while $\Psi_{k,\vec{m}}$ is injective, $\Phi_{k,\vec{m}}$ is not. Nevertheless, it is a quasi-finite morphism of varieties, since all of its fibers are finite.

3. The map $\pi_{k,\vec{m}}$ is étale, since the action of $S_{\vec{m}}$ is free. Since $\Phi_{k,\vec{m}}$ is an open immersion onto its image, $\Psi_{k,\vec{m}}$ and $\Phi_{k,\vec{m}}$ are also étale onto their image.

4. The group action that we defined above reflects a new level of geometric complexity to the case of $\mathfrak{gl}_n$ compared to that for $\mathfrak{h}_n$. This stems from the fact that in $\mathfrak{gl}_n$ there is no a priori order to the eigenvalues of a matrix, in contrast to what happens in $\mathfrak{h}_n$, where they are naturally ordered in the diagonal.

Changing coordinates

What is most important in our setting is that the passage from a matrix to its data preserves the transcendence degree. As in the upper triangular case, we start by considering elements in some ring of formal power series.

Let $g_{i,j} \in \mathbb{C}[[t_1, \ldots, t_l]]$, $1 \le i, j \le n$ and write $A = (g_{i,j})$. Then the eigenvalues of A are elements of the integral closure of $\mathbb{C}[[t_1, \ldots, t_l]]$. By the Newton-Puiseux Theorem we know that this is contained in the field

$$\mathbb{L} = \bigcup_{\vec{r} \in \mathbb{N}^l} \mathbb{C}((t_1^{\frac{1}{r_1}})) \cdots ((t_l^{\frac{1}{r_l}})).$$

Assume that there are exactly k distinct such eigenvalues f_i of A, and that they have corresponding multiplicities m_i. Then the coordinates $T_j(A)$ of the point of $W_k(\vec{m})$ are also elements of the field $\mathbb{L}$.

We start by making rigorous the fact that changing from coordinates of $\mathfrak{gl}_n$ to coordinates of the data does not affect the transcendence degree.

Lemma 13.4. *Let $g_{i,j} \in \mathbb{C}[[t_1, \ldots, t_l]]$, $1 \le i, j \le n$. Let K be an algebraic closure of the field $\mathbb{C}(A)$ and assume that the matrix $A = (g_{i,j})$ has exactly k distinct eigenvalues $f_1, \ldots, f_k \in K$ with respective multiplicities m_i. Let also*

$T_j = T_j(A)$, $1 \leq j \leq w_{k,\vec{m}}$, be the coordinates of the point in the variety $W_k(\vec{m})$ parameterizing the rest of the corresponding data of A. Then

$$tr.d._{\mathbb{C}}\mathbb{C}(A) = tr.d._{\mathbb{C}}\mathbb{C}(\{f_i, T_j : 1 \leq i \leq k, 1 \leq j \leq w_{k,\vec{m}}\}).$$

Proof. Let $K_0 = \mathbb{C}(\{f_i, T_j : 1 \leq i \leq k, 1 \leq j \leq w_{k,\vec{m}}\})$ and

$$K_1 = \mathbb{C}(\{g_{i,j}, T_r, f_s : 1 \leq i, j \leq n, 1 \leq s \leq k, 1 \leq r \leq w_{k,\vec{m}}\}).$$

The T_j and f_i are algebraic over $\mathbb{C}(A)$. So $tr.d._{\mathbb{C}}\mathbb{C}(A) = tr.d._{\mathbb{C}}K_1$.

On the other hand the T_j and f_i determine the $g_{i,j}$ via an algebraic process, in particular by determining the Jordan canonical form and the transition matrix Q above. So we get that $tr.d._{\mathbb{C}}K_0 = tr.d._{\mathbb{C}}K_1$.

Combining the two equalities of transcendence degrees the result follows. □

The Exponential Map

Let $A = (g_{i,j})$ be a matrix with $g_{i,j} \in \mathbb{C}[[t_1, \ldots, t_l]]$, where the $g_{i,j}$ have no constant term. Let us also assume that A has data given by

$$((f_1, \ldots, f_k), (m_1, \ldots, m_k), (V_1, \ldots, V_k), (N_1, \ldots, N_k)).$$

We are able to consider such data working over an algebraic closure K of the field $\mathbb{C}(A)$. We would like to extract from this a simpler way of computing the effect of the exponential on A.

Since the $g_{i,j}$ have no constant term, it is easy to see that the distinct f_i cannot differ by an integral multiple of $2\pi i$, and hence the corresponding data for the matrix $E(A)$ will be:

1. the eigenvalues will be the distinct elements $e^{f_1}, \ldots, e^{f_k} \in \mathbb{L}$,
2. the multiplicities m_i will be the same,
3. the generalized eigenspaces V_i will remain as such, and
4. the nilpotent operator corresponding to e^{f_i} is
$$N_i' = e^{f_i}(E(N_i) - id_{V_i}).$$

Let $E_{i,j}(A)$ denote the (i,j) entry of the exponential matrix $E(A)$. Then we will have the following

Proposition 13.5. *Let $g_{i,j} \in \mathbb{C}[[t_1, \ldots, t_l]]$, $1 \leq i, j \leq n$, be such that the $g_{i,j}$ have no constant term. We assume that $A = (g_{i,j})$ has exactly k distinct eigenvalues $f_1, \ldots, f_k$ with respective multiplicities $m_1, \ldots, m_k$. Let K be an* *algebraic closure of the field* $\mathbb{C}(A)$ *and* $T_j = T_j(A)$, $1 \le j \le w_{k,\vec{m}}$, *be coordinates for the variety* $W_k(\vec{m})$ *parameterizing the corresponding data of* A. *Then*

$$tr.d._{\mathbb{C}}\mathbb{C}(E(A)) = tr.d._{\mathbb{C}}\mathbb{C}(\{e^{f_i}, T_j : 1 \le i \le k, 1 \le j \le w_{k,\vec{m}}\}),\ \textit{and}$$

$$tr.d._{\mathbb{C}}\mathbb{C}(A, E(A)) = tr.d._{\mathbb{C}}\mathbb{C}(\{f_i, e^{f_i}, T_j : 1 \le i \le k, 1 \le j \le w_{k,\vec{m}}\}).$$

Proof. Let $\bar{T}_j = T_j(E(A))$, $1 \le j \le w_{k,\vec{m}}$, be the coordinates in $W_k(\vec{m})$ parameterizing the corresponding data of $E(A)$. From 13.4 applied to the matrix $E(A)$ we get

$$tr.d._{\mathbb{C}}\mathbb{C}(E(A)) = tr.d._{\mathbb{C}}\mathbb{C}(\{e^{f_i}, \bar{T}_j : 1 \le i \le k, 1 \le j \le w_{k,\vec{m}}\}).$$

Therefore we are left with proving the following equality

$$tr.d._{\mathbb{C}}\mathbb{C}(\{e^{f_i}, \bar{T}_j : 1 \le i \le k, 1 \le j \le w_{k,\vec{m}}\}) =$$
$$= tr.d._{\mathbb{C}}\mathbb{C}(\{e^{f_i}, T_j : 1 \le i \le k, 1 \le j \le w_{k,\vec{m}}\}).$$

By the remarks above though the T_j and $\bar{T}_j$ will parameterize the same V_i, so that their only difference is located in those $\bar{T}_j$ and T_j that parameterize the nilpotent operators. For the latter we know that we will have

$$N_i' = e^{f_i}(E(N_i) - id_{V_i}).$$

Claim: The map $f : Nil(m) \to Nil(m)$ given by $N \mapsto E(N) - id$ is a bialgebraic map, where $Nil(m)$ denotes the space of nilpotent operators on an $m-$dimensional $\mathbb{C}-$vector space.

Assuming this claim holds, if T_j^i, $\bar{T}_j^i$, $j \in J_i$ denote the elements among the T_j, and $\bar{T}_j$ respectively, that parameterize the information of the nilpotent operators N_i' and N_i, then the above shows that

$$\mathbb{C}(\{e^{f_i}, T_j^i : j \in J_i\}) = \mathbb{C}(\{e^{f_i}, \bar{T}_j^i : j \in J_i\}),$$

for all $i = 1, \dots, k$. Combining this with the fact that $T_j = \bar{T}_j$ for all of the rest, i.e. those parameterizing the V_i, the result follows trivially.

The above argument shows that in fact $\mathbb{C}(\{e^{f_i}, T_j : 1 \le j \le w_{k,\vec{m}}\}) = \mathbb{C}(\{e^{f_i}, \bar{T}_j : 1 \le j \le w_{k,\vec{m}}\})$. Combining this with the remark at the end of the proof of 13.4 we get that the field $F_1 = \mathbb{C}(\{A, e^{f_i}, T_j : 1 \le i \le k, 1 \le j \le w_{k,\vec{m}}\})$ is a finite algebraic extension of the field $\mathbb{C}(A, E(A))$. Similarly, $F_2 = \mathbb{C}(\{f_i, e^{f_i}, T_j : 1 \le i \le k, 1 \le j \le w_{k,\vec{m}}\})$ is a finite algebraic extension of F_1 which finishes the proof of the second equality.

Proof of the Claim. Let N be a nilpotent operator on an $m-$dimensional vector space and let $k \in \mathbb{N}$ be such that $N^k \neq 0$ and $N^{k+1} = 0$. Then $E(N) = \sum_{r=0}^{k} \frac{N^r}{r!}$ so that $N \mapsto E(N) - id$ is obviously algebraic.

On the other hand, define $L : Nil(m) \to Nil(m)$ given by

$$N \mapsto \log(id + N) = \sum_{r=1}^{\infty} \frac{(-1)^{r+1} N^r}{r}.$$

Since our operators are nilpotent this sum is finite and the map is algebraic, similarly to the above argument. The two functions are inverse of each other, which proves the claim. □

Remark 13.6. This shows that the T_j are the natural generalization of the notion of "eigencoordinates" we saw in the case of $\mathfrak{h}_n$. We note that in the case of $\mathfrak{h}_n$ instead of the more involved spaces $Y_k(\vec{m})$ we had $\mathbb{A}^k_{\mathbb{C}} \times \mathbb{A}_{\mathbb{C}}^{\frac{n(n-1)}{2}}$. The role of $\mathbb{A}_{\mathbb{C}}^{\frac{n(n-1)}{2}}$, the space of the eigencoordinates, is now played by $W_k(\vec{m})$.

13.2 AX-SCHANUEL FOR THE GENERAL LINEAR GROUP

We continue with our study of $E : \mathfrak{gl}_n \to GL_n$ by stating the Ax-Schanuel result and proving it by reducing to 11.3.

We will denote the coordinates of the map E by $E_{i,j}$. We start with stating the theorem in the functional point of view.

Theorem 13.7 (Weak Ax-Schanuel for GL_n)**.** *Let $g_{i,j} \in \mathbb{C}[[t_1, \ldots, t_l]]$ be power series with no constant term, where $1 \leq i, j \leq n$. Let f_i, where $1 \leq i \leq n$, denote the eigenvalues of the matrix $A = (g_{i,j})$. Let us also set $N = \dim_{\mathbb{Q}} \langle f_1, \ldots, f_n \rangle_{\mathbb{Q}}$, then*

$$tr.d._{\mathbb{C}}\mathbb{C}(A, E(A)) \geq N + \operatorname{rank} J((g_{i,j}); \vec{t}).$$

Proof. Let $\mathbb{L}$ be the field of Puiseux series defined above. Assume $A = (g_{i,j}) \in \mathfrak{gl}_n(\mathbb{C}[[t_1, \ldots, t_l]])$ has exactly k distinct eigenvalues. Let us also assume that the data of the matrix A is given by

$$(f_1, \ldots, f_k), (m_1, \ldots, m_k),\ (V_1, \ldots, V_k), \text{ and } (N_1, \ldots, N_k).$$

Let $T_j = T_j(A)$, $1 \leq j \leq w_{k,\vec{m}}$ be the coordinates of the point in $W_k(\vec{m})(\mathbb{L})$ describing the above data of A.

We are therefore in a position to apply 13.5 to get that

$$tr.d._{\mathbb{C}}\mathbb{C}(A, E(A)) = tr.d._{\mathbb{C}}\mathbb{C}(\{f_i, e^{f_i}, T_j : 1 \leq i \leq k, 1 \leq j \leq w_{k,\vec{m}}\}).$$

Using this together with 11.3, applied to the field $\mathbb{L}$, we get that

$$tr.d._{\mathbb{C}}\mathbb{C}(A, E(A)) \geq N + rank(J(\vec{f}, \vec{T}; \vec{t})).$$

Following the remarks at the end of 13.1, the map $\Phi_{k,\vec{m}}$ is étale so

$$\text{rank}\, J((g_{i,j}); \vec{t}) = rank(J(\vec{f}, \vec{T}; \vec{t})),$$

and the result follows. □

An Alternate Formulation

Similar to the alternate formulation 12.10 for the Ax-Schanuel we had for U_n we can give an alternate form of 13.7, again we have to change the background accordingly.

We let $V \subset \mathfrak{gl}_n$ be an open subset and $X \subset V$ an irreducible complex analytic subvariety of V containing the origin such that locally at $\vec{0}$ the functions $g_{i,j}$ for $1 \leq i, j \leq n$, are meromorphic functions on X.

Once again, for reasons of convenience, and notational coherence, we let $A = (g_{i,j})$ denote the matrix corresponding to the coordinates $g_{i,j}$.

Theorem 13.8 (Weak Ax-Schanuel-Alternate Formulation)**.** *In the above context, if the eigenvalues f_i of the matrix $A = (g_{i,j})$ are such that $N = \dim_{\mathbb{Q}}\langle f_1, \ldots, f_n\rangle_{\mathbb{Q}}$, then*

$$tr.d._{\mathbb{C}}\mathbb{C}(A, E(A)) \geq N + \dim_{\mathbb{C}} X.$$

Proof. We choose $t_1, \ldots, t_l$ that are independent holomorphic coordinates on X locally at $\vec{0}$ so that $\text{rank}(J((g_{i,j}; \vec{t})) = dim_{\mathbb{C}}(X)$. This reduces the proof to 13.7 □

This version of the Ax-Schanuel result is the one most useful when extracting geometric corollaries, as we have already seen.

13.3 THE WEAKLY SPECIAL SUBVARIETIES

We return once more to $\mathfrak{gl}_n$ and proceed towards defining the weakly special subvarieties. The results we had so far lead us in a natural way to consider some specific subsets of $\mathfrak{gl}_n$.

Consider a vector space V over $\mathbb{C}$ with $\dim_{\mathbb{C}} V = n$, some fixed $k \in \{1, \ldots, n\}$, some fixed $m_i \in \mathbb{N}$ for $1 \leq i \leq k$ such that $m_1 + \ldots + m_k = n$ and $W_k(\vec{m})$ the algebraic variety over $\mathbb{C}$ we defined earlier.

Relations on Eigenvalues

We expect that the only algebraic relations that will be allowable on the eigenvalues will be $\mathbb{Q}-$linear relations. Since we have already accounted for the number of distinct eigenvalues we also require that these relations do not force any more eigevalues to be equal.

With that in mind, we let Σ be a finite set of $\mathbb{Q}-$linear polynomials of the form $F(\vec{f}) = \sum_{i=1}^{k} q_i f_i$ on the f_i, and let I be the ideal generated by Σ in $\mathbb{C}[f_1, \ldots, f_k]$. We also assume that $\nexists i, j$ such that $f_i - f_j \in I$, i.e. in $Z(\Sigma)$ none of the previously distinct f_i coincide, where $Z(\Sigma)$ is the algebraic subvariety of $\mathbb{A}^k_{\mathbb{C}}$ defined by Σ. Finally, we also let

$$\Gamma_\Sigma = \Gamma_k \cap Z(\Sigma) \subset \mathbb{A}^k,$$

where Γ_k is the Zariski open subset of $\mathbb{A}^k_{\mathbb{C}}$ defined earlier.

Other Relations

For the rest of the data of the matrix we allow any algebraic relation that does not depend on the eigenvalues. So we consider $W \subset W_k(\vec{m})$ to be a subvariety of $W_k(\vec{m})$. Then if we are given a set Σ as above and a subvariety $W \subset W_k(\vec{m})$ we let

$$X(k, \vec{m}, \Sigma, W) = \Gamma_\Sigma \times W.$$

All of the above lead us naturally to the following definition.

Definition 13.9. An irreducible subvariety $U \subset \mathfrak{gl}_n$ containing the origin will be called **weakly special** if there exist a natural number $1 \leq k \leq n$, a vector $\vec{m} = (m_1, \ldots, m_k) \in \mathbb{N}^k$ such that $\sum_{i=1}^{k} m_i = n$, a set Σ of $\mathbb{Q}-$linear polynomials, and a subvariety $W \subset W_k(\vec{m})$, all defined as above, such that

$$U = Zcl(\Phi_{k,\vec{m}}(X(k, \vec{m}, \Sigma, W)),$$

where $Zcl(R)$ denotes the Zariski closure in $\mathfrak{gl}_n$ of a subset $R \subset \mathfrak{gl}_n$.

13.4 AX-LINDEMANN AND OTHER COROLLARIES

We approach this in the same way as we did for the corresponding result in 12.4. We start with defining components in this setting. After that we prove a two-sorted version of 13.8, similar to 12.16, and then, just as in 12.4, use this to infer our Ax-Lindemann result.

Definition 13.10. Let $W \subset \mathfrak{gl}_n$ and $V \subset GL_n$ be algebraic subvarieties. Then a **component** C of $W \cap E^{-1}(V)$ will be a complex-analytically irreducible component of $W \cap E^{-1}(V)$.

Theorem 13.11 (Two-sorted Weak Ax-Schanuel for GL_n). *Let $U \subset \mathfrak{gl}_n$ be a weakly special subvariety that contains the origin, let $X = E(U)$, and let $V \subset U$ and $Z \subset X$ be algebraic subvarieties, such that $\vec{0} \in V$ and $I_n \in Z$. If C is a component of $V \cap E^{-1}(Z)$ with $\vec{0} \in C$, then, assuming that C is not contained in any proper weakly special subvariety of U,*

$$\dim_{\mathbb{C}} C \leq \dim_{\mathbb{C}} V + \dim_{\mathbb{C}} Z - \dim_{\mathbb{C}} X.$$

Proof. Let $U = Zcl(\Phi_{k,\vec{m}}(X(k, \vec{m}, \Sigma, W)))$, for brevity we let $B_k := X(k, \vec{m}, \Sigma, W)$, and $B = \Phi_{k,\vec{m}}(B_k) \cap C$. We will have $\dim_{\mathbb{C}} B = \dim_{\mathbb{C}} C$ and on B the $g_{i,j}$ are well defined meromorphic functions. We let K be the algebraic closure of the field $\mathbb{C}((g_{i,j}))$. Let f_i and T_j denote the coordinates of the point in $T_\Sigma(K)$ and $W \subset W_k(\vec{m})(K)$ respectively with $\Phi_{k,\vec{m}}((\vec{f}, \vec{T})) = g_{i,j}$. We also let $N = \dim_{\mathbb{Q}} \langle f_1, \ldots, f_k \rangle_{\mathbb{Q}}$.

Then we may use 13.8 to deduce that

$$tr.d._{\mathbb{C}}\mathbb{C}(A, E(A)) \geq N + \dim_{\mathbb{C}} C. \tag{13.1}$$

On the other hand, we have the following inequalities:

$$\dim_{\mathbb{C}} Z \geq tr.d._{\mathbb{C}}\mathbb{C}(E(A)), \text{ and}$$

$$\dim_{\mathbb{C}} V \geq tr.d._{\mathbb{C}}\mathbb{C}(A).$$

Combining these with 13.4 and 13.5 we conclude that

$$\dim_{\mathbb{C}} Z \geq tr.d._{\mathbb{C}}\mathbb{C}(\{e^{f_i}, T_j : 1 \leq i \leq k, 1 \leq j \leq w_{k,\vec{m}}\}), \text{ and} \tag{13.2}$$

$$\dim_{\mathbb{C}} V \geq tr.d._{\mathbb{C}}\mathbb{C}(\{f_i, T_j : 1 \leq i \leq k, 1 \leq j \leq w_{k,\vec{m}}\}). \tag{13.3}$$

On the other hand, if we set $L = \mathbb{C}(\{T_j : 1 \leq j \leq w_{k,\vec{m}}\})$, and let $M = tr.d._{\mathbb{C}}L$, we get that

$$\dim_{\mathbb{C}} X = N + M, \tag{13.4}$$

by the minimality of U, in containing C, and hence B.

On the other hand we have that

$$\begin{aligned} tr.d._{\mathbb{C}}\mathbb{C}(\{f_i, T_j : 1 \leq i \leq k, 1 \leq j \leq w_{k,\vec{m}}\}) &= \\ = M + tr.d._{L}L(\{f_i : 1 \leq i \leq k\}), & \end{aligned} \tag{13.5}$$

$$\begin{aligned} tr.d._{\mathbb{C}}\mathbb{C}(\{e^{f_i}, T_j : 1 \leq i \leq k, 1 \leq j \leq w_{k,\vec{m}}\}) &= \\ = M + tr.d._{L}L(\{e^{f_i} : 1 \leq i \leq k\}), \text{ and} & \end{aligned} \tag{13.6}$$

$$tr.d._{\mathbb{C}}\mathbb{C}(A, E(A)) = M + tr.d._{L}L(\{f_i, e^{f_i} : 1 \leq i \leq k\}). \tag{13.7}$$

Combining these with (13.2) and (13.3) we get that

$$\dim_{\mathbb{C}} V \geq M + tr.d._L L(\{f_i : 1 \leq i \leq k\}), \text{ and}$$

$$\dim_{\mathbb{C}} Z \geq M + tr.d._L L(\{e^{f_i} : 1 \leq i \leq k\}).$$

The rest of the proof follows similarly to that of 12.16. □

As we did in 12.4, we conclude with the characterization of bi-algebraic sets that contain the origin.

Corollary 13.12 (Ax-Lindemann for GL_n)**.** Let $Z \subset GL_n$ be an algebraic subvariety with $I_n \in Z$. If $V \subset E^{-1}(Z)$ is a maximal irreducible subvariety that contains the origin, then V is a weakly special subvariety.

Proof. Let U be the minimal weakly special subvariety that contains V, $X = E(U)$ and let $Z' = Z \cap E(U)$. We use 13.11 for $C = V$ to get

$$\dim_{\mathbb{C}} V \leq \dim_{\mathbb{C}} V + \dim_{\mathbb{C}} Z' - \dim_{\mathbb{C}} X.$$

This implies $\dim_{\mathbb{C}} X \leq \dim_{\mathbb{C}} Z'$, and since $Z' \subset X$ we get that $X \subset Z$ and that $V \subset U \subset E^{-1}(Z)$. Maximality of V then implies that $V = U$ is weakly special. □

Remarks 13.13. *1. We note that the above results imply, as a direct corollary, Weak Ax-Schanuel and therefore also Ax-Lindemann results for all linear algebraic groups.*

2. In mimicking the classical Ax-Schanuel statement, we can extract Weak Ax-Schanuel and Ax-Lindemann type statements for Cartesian powers of the exponential map of a Lie algebra from our results.

Even more generally, we can infer such results for the Cartesian products of exponentials $E_i : \mathfrak{g}_i \to G_i$ of Lie algebras $\mathfrak{g}_i$, $1 \leq i \leq r$. We achieve this by noticing that the exponential of the Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \times \ldots \times \mathfrak{g}_r$ is the Cartesian product of the E_i.

A

SOME NOTES ON POLARIZATIONS

A.1 THE NON-RELATIVE CASE

Notation: Let X/k be a smooth projective variety over a subfield k of $\mathbb{C}$ and let $n = \dim_k X$.

Short review on polarizing forms

For all $d \in \mathbb{N}$ there exist non-degenerate bilinear polarizing forms

$$\langle , \rangle_{DR} : H^d_{DR}(X/k) \otimes_k H^d_{DR}(X/k) \to k, \text{ and}$$

$$\langle , \rangle_B : H^d(X^{an}_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^d(X^{an}_{\mathbb{C}}, \mathbb{Q}) \to (2\pi i)^{-d}\mathbb{Q} = \mathbb{Q}(-d),$$

on de Rham cohomology and Betti cohomology respectively. We also write $\langle , \rangle_B = (2\pi i)^{-d} \langle , \rangle$, where $\langle , \rangle$ has values in $\mathbb{Q}$ and is of the same type, i.e. symmetric or skew-symmetric, as $\langle , \rangle_B$.

These two are the polarizing forms of the corresponding cohomology group. Their existence follows from the fact that X is projective and smooth and they are constructed via a very ample line bundle [Del71].

We also have that, via the two embeddings $k \hookrightarrow \mathbb{C}$ and $(2\pi i)^{-d}\mathbb{Q} \hookrightarrow \mathbb{C}$, and the comparison isomorphism

$$P^d_X : H^d(X/k) \otimes_k \mathbb{C} \to H^d(X^{an}_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

the two bilinear forms $\langle , \rangle_{DR}$ and $\langle , \rangle_B$ are compatible under P^d_X, meaning that

$$\langle v, w \rangle_{DR} = \langle P^d_X(v), P^d_X(w) \rangle_B, \forall v, w \in H^d_{DR}(X/k) \otimes_k \mathbb{C}. \tag{A.1}$$

Relations on periods-Notation

From now on we assume that $d = n = \dim_k X$. We can and do consider from now on the above polarizing forms $\langle , \rangle_{DR}$, $\langle , \rangle_B$, and the form $\langle , \rangle$ as vectors in the spaces $H^n_{DR}(X/k)^* \otimes_k H^n_{DR}(X/k)^*$, $(H^n(X^{an}_{\mathbb{C}}, \mathbb{Q})^* \otimes_{\mathbb{Q}} H^n(X^{an}_{\mathbb{C}}, \mathbb{Q})^*)(-n)$, and $H^n(X^{an}_{\mathbb{C}}, \mathbb{Q})^* \otimes_{\mathbb{Q}} H^n(X^{an}_{\mathbb{C}}, \mathbb{Q})^*$ respectively.

In this case, i.e. $d = n$, via Poincaré duality, these forms will correspond to elements $t_{DR} \in H^n_{DR}(X/k) \otimes_k H^n_{DR}(X/k)$, $t_B \in (H^n(X^{an}_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^n(X^{an}_{\mathbb{C}}, \mathbb{Q}))(n)$, and $t \in H^n(X^{an}_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} H^n(X^{an}_{\mathbb{C}}, \mathbb{Q})$, respectively.

The compatibility of the comparison isomorphism P^n_X with Poincaré duality implies that $P^n_X \otimes P^n_X(t_{DR}) = t_B = (2\pi i)^n t$. In particular t_{DR} is a Hodge class defined over the field k. For cycles such as this it is known[1] that they impose polynomial relations among the n-periods with coefficients in the field $k((2\pi i)^n)$.

In what follows we show that the relations constructed by t_{DR} are in fact the Riemann-relations, i.e. they are the equations imposed on the n-periods by (A.1). This is used without proof by André in, essentially, the case were $n = 1$. The author is sure that this part is known to experts in the field and includes it only for the sake of completeness of the exposition.

Notation: We consider from now on a fixed basis $\{\gamma_i : 1 \leq i \leq \mu := \dim_{\mathbb{Q}} H^n(X^{an}_{\mathbb{C}}, \mathbb{Q})\}$ of $H_n(X^{an}_{\mathbb{C}}, \mathbb{Q})$ and we let γ_i^* be the elements of its dual basis, which constitutes a basis of $H^n(X^{an}_{\mathbb{C}}, \mathbb{Q})$. We also consider ω_i, $1 \leq i \leq \mu$, a fixed k-basis of $H^n_{DR}(X/k)$.

With respect to these choices the isomorphism P^n_X corresponds to the matrix $(\int_{\gamma_j} \omega_i)$. We denote this matrix also by P^n_X so that the isomorphism is nothing but $P^n_X(v) = {}^t v P^n_X$, were on the right we have the matrix acting on the right. Vectors in the various spaces will be considered as column vectors in the various bases. Finally, we denote the matrix of the n-periods by $P := (2\pi i)^{-n} P^n_X$.

With the above notation fixed we let $\langle \omega_i, \omega_j \rangle_{DR} = d_{i,j}$ and let $M_{DR} = (d_{i,j}) \in \mathrm{GL}_\mu(k)$, which will be the matrix corresponding to the form $\langle , \rangle_{DR}$, i.e.

$$\langle v, w \rangle_{DR} = {}^t v M_{DR} w.$$

Considering, alternatively as above, $\langle , \rangle_{DR}$ as an element of the space $H^n_{DR}(X/k)^* \otimes_k H^n_{DR}(X/k)$, the above are equivalent to

$$\langle , \rangle_{DR} = \sum_{i,j=1}^{\mu} d_{i,j} \omega_i^* \otimes \omega_j^*.$$

Similarly we let $q_{i,j} = \langle \gamma_i^*, \gamma_j^* \rangle \in \mathbb{Q}$ and set $M_B = (q_{i,j}) \in \mathrm{GL}_\mu(\mathbb{Q})$. This implies that $\langle \gamma_i^*, \gamma_j^* \rangle = (2\pi i)^{-n} q_{i,j}$. Same as above these relations can be rewritten as

$$\langle v, w \rangle = {}^t v M_B w \text{ and } \langle v, w \rangle_B = {}^t v((2\pi i)^{-n} M_B) w,$$

1 See page 169 of [And89].

for all $v, w \in H^n(X_{\mathbb{C}}^{an}, \mathbb{C})$. Alternatively, if we were to consider $\langle , \rangle$ and $\langle , \rangle_B$ as elements of $H^n(X_{\mathbb{C}}^{an}, \mathbb{C})^* \otimes_{\mathbb{C}} H^n(X_{\mathbb{C}}^{an}, \mathbb{C})^*$ we can write these as $\langle , \rangle = \sum_{i,j=1}^{\mu} q_{i,j}\gamma_i \otimes \gamma_j$, and $\langle , \rangle_B = \sum_{i,j=1}^{\mu} (2\pi i)^{-n} q_{i,j}\gamma_i \otimes \gamma_j$ respectively.

We now consider the Poincaré duality isomorphisms $\Pi_{DR} : H^n_{DR}(X/k) \to H^n_{DR}(X/k)^*$ and $\Pi_B : H^n(X_{\mathbb{C}}^{an}, \mathbb{Q}) \to H^n(X_{\mathbb{C}}^{an}, \mathbb{Q})$, which we have since $\dim_k X = n$. With respect to the bases $\{\omega_i\}$ and $\{\omega_i^*\}$ the isomorphism Π_{DR} corresponds to an invertible matrix which we denote by $A_{DR} \in \mathrm{GL}_{\mu}(k)$. Similarly, with respect to the bases $\{\gamma_i\}$ and $\{\gamma_i^*\}$ we get the invertible matrix A_B corresponding to Π_B.

Finally, let us write $t_{DR} = \sum_{i,j=1}^{\mu} \lambda_{i,j}\omega_i \otimes \omega_j$, $t = \sum_{i,j=1}^{\mu} \tau_{i,j}\gamma_i^* \otimes \gamma_j^*$ and $t_B = \sum_{i,j=1}^{\mu} (2\pi i)^n \tau_{i,j}\gamma_i^* \otimes \gamma_j^*$, where $\lambda_{i,j} \in k$ and $\tau_{i,j} \in \mathbb{Q}$. We also define $\Lambda_{DR} = (\lambda_{i,j})$ and $\Lambda_{\mathbb{Q}} = (\tau_{i,j})$.

Classes and forms

With the above notation fixed from now on we turn to describing the relation between the classes t_{DR}, and t and the respective forms.

By definition we have $\Pi_{DR}^{\otimes 2}(t_{DR}) = \langle , \rangle_{DR}$. This implies that

$$\sum_{i,j=1}^{\mu} \lambda_{i,j}\Pi_{DR}(\omega_i) \otimes \Pi_{DR}(\omega_j) = \sum_{i,j=1}^{\mu} d_{i,j}\omega_i^* \otimes \omega_j^*. \tag{A.2}$$

We know that $\Pi_{DR}(\omega_i) = \Sigma a_{i,j}\omega_j^*$, with $A_{DR} = (a_{i,j})$. Applying this to (A.2) its is easy to see, with a few trivial computations, that ${}^tA_{DR}\Lambda_{DR}A_{DR} = M_{DR}$, or equivalently we get the equality

$$\Lambda_{DR} = {}^tA_{DR}^{-1}M_{DR}A_{DR}^{-1}. \tag{A.3}$$

Similarly for the pair t and $\langle , \rangle$ we find that

$$\Lambda_{\mathbb{Q}} = {}^tA_B^{-1}M_BA_B^{-1}, \tag{A.4}$$

coming from the equality $\Pi_B^{\otimes 2}(t) = \langle , \rangle$.

The relation given by t_{DR}.

We review how a relation on the n-periods is constructed from t_{DR}. We start from the equality $(2\pi i)^{-n}(P_X^n)^{\otimes 2}(t_{DR}) = t$. This in turn implies that for all l, m, with the notation as above, we have

$$\sum_{i,j=1}^{\mu} \lambda_{i,j}((2\pi i)^{-n}\int_{\gamma_l}\omega_i)((2\pi i)^{-n}\int_{\gamma_m}\omega_j) = (2\pi i)^{-n}\tau_{l,m}. \tag{A.5}$$

These equations are the relations between n-periods that we eluded to earlier. Putting them altogether the previous relation is equivalent to the equality

$$^{t}P\Lambda_{DR}P = (2\pi i)^{-n}\Lambda_{\mathbb{Q}}. \tag{A.6}$$

Comparing the matrices A_B and A_{DR}.

Earlier on we had the matrices A_{DR} and A_B corresponding to the respective Poincaré duality isomorphisms. We saw in (A.3) and (A.4) how these matrices relate the "Λ-matrices" and "M-matrices". We would like to replace the "Λ-matrices in (A.6) by the corresponding "M-matrices", showing thus that the relations created are nothing but the Riemann-relations[2] . The first step is to describe how the matrices A_{DR} and A_B relate to one another.

We had the isomorphisms Π_{DR} and Π_B and the matrices A_{DR} and A_B that represented these with respect to the bases we have chosen. We know that the comparison isomorphism P_X^n respects Poincaré duality, meaning that the following diagram commutes:

$$\begin{tikzcd}
H^n_{DR}(X/k)\otimes_k \mathbb{C} \arrow[r, "P_X^n"] \arrow[d, "\Pi_{DR}\otimes_k\mathbb{C}"'] & H^n(X^{an}_{\mathbb{C}},\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C} \arrow[d, "\Pi_B\otimes_{\mathbb{Q}}\mathbb{C}"] \\
H^n_{DR}(X/k)^*\otimes_k \mathbb{C} & H^n(X^{an}_{\mathbb{C}},\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C} \arrow[l, "(P_X^n)^\vee"]
\end{tikzcd}$$

where $(P_X^n)^\vee(f) = f\circ P_X^n$ for all $f\in H^n(X^{an}_{\mathbb{C}},\mathbb{Q})^*\otimes_{\mathbb{Q}}\mathbb{C}$.

Looking at what the relation of the above diagram, i.e. $\Pi_{DR}\otimes_k\mathbb{C} = (P_X^n)^\vee\circ(\Pi_B\otimes_{\mathbb{Q}}\mathbb{C})\circ P_X^n$, does to the basis $\{\omega_i\}$, and using the fact that with respect to the bases $\{\gamma_j\}$ and $\{\omega_i^*\}$ the matrix representing $(P_X^n)^\vee$ will be the matrix $^{t}(\int_{\gamma_j}\omega_i)$, i.e. the transpose of P_X^n, we conclude that

$$A_{DR} = (\int_{\gamma_j}\omega_i)\cdot A_B\cdot {}^{t}(\int_{\gamma_j}\omega_i). \tag{A.7}$$

Conclusions

2 See 4.2 for a definition and the reason of why we needed these.

Combining (A.6) with (A.3) and (A.4) we get

$$ {}^tP({}^tA_{DR}^{-1}M_{DR}A_{DR}^{-1})P = (2\pi i)^{-n}({}^tA_B^{-1}M_BA_B^{-1}). \tag{A.8}$$

From (A.7) we get

$$ {}^tP\,{}^tA_{DR}^{-1} = \frac{1}{(2\pi i)^n}\,{}^tA_B^{-1}P, \text{ and} \tag{A.9}$$

$$ A_{DR}^{-1}P = \frac{1}{(2\pi i)^n}\,{}^tP^{-1}A_B^{-1}. \tag{A.10}$$

Using (A.9) and (A.10) together with (A.8) we get

$$ PM_B\,{}^tP = (2\pi i)^{-n}M_{DR}. \tag{A.11}$$

But, this is the relation we get between the above matrices by looking at the equation (A.1) and translating it in terms of matrices. Indeed, (A.1) translates to

$$ {}^tvP_X^n((2\pi i)^{-n}M_B)\,{}^t({}^twP_X^n) = {}^tvM_{DR}w \text{ for all } v,w \in H_{DR}^n(X/k)\otimes_k \mathbb{C}.$$

From this we recover (A.11) by multiplying on both sides by $(2\pi i)^{-n}$ and noting that $P = (2\pi i)^{-n}P_X^n$.

What is actually of use to us is not exactly (A.11) but rather the same relation for the transpose of P. To obtain this, first from (A.11) we get trivially

$$ (2\pi i)^nM_B = P^{-1}M_{DR}\,{}^tP^{-1},$$

then taking inverses on both sides we get

$$ {}^tPM_{DR}^{-1}P = (2\pi i)^{-n}M_B^{-1}. \tag{A.12}$$

A.2 THE RELATIVE CASE

Setting: We consider $f : X \to S$ a smooth projective morphism of k-varieties itself defined over the same subfield k of $\mathbb{C}$. We also assume that S is a smooth connected curve which is not necessarily complete over k and the dimension of the fibers of f is n.

We then have, for all $d \in \mathbb{N}$, the relative version of the comparison isomorphism between the algebraic de Rham and the Betti cohomology

$$ P_{X/S}^d : H_{DR}^d(X/S)\otimes_{\mathcal{O}_S}\mathcal{O}_{S_{\mathbb{C}}^{an}} \to R^df_*^{an}\mathbb{Q}_{X_{\mathbb{C}}^{an}}\otimes_{\mathbb{Q}_{S_{\mathbb{C}}^{an}}}\mathcal{O}_{S_{\mathbb{C}}^{an}}. \tag{A.13}$$

Once again we let, in parallel to the non-relative case we studied earlier, μ denote the rank of these sheaves.

We once again have the same picture, as far as polarizing forms are concerned, as in the non-relative case. In other words we have $\langle,\rangle_{DR}$ a polarizing form of the de Rham cohomology sheaves $H^d_{DR}(X/S)$ which is defined over the field k, and a polarizing form $\langle,\rangle_B = (2\pi i)^{-n}\langle,\rangle$ of the sheaves on the right of (A.13). These two forms will be compatible with the relative isomorphism (A.13), meaning that we have

$$\langle P^d_{X/S}(v), P^d_{X/S}(w)\rangle_B = \langle v, w\rangle_{DR}, \tag{A.14}$$

holds for all sections v, w of the sheaf on the right of (A.13).

From now on we focus on the case $d = n$. We choose $U \subset S$ a non-empty affine open subset. Then the form $\langle,\rangle_{DR}|_U$ will map, via the relative version of the Poincaré duality isomorphism, to a class $t_{DR} \in H^n_{DR}(X/S) \otimes_{\mathcal{O}_S} H^n_{DR}(X/S)|_U$.

Similarly we repeat this process for the forms $\langle,\rangle$ and $\langle,\rangle_B$, over the analytification $U^{an}_{\mathbb{C}}$, to get elements $t \in (R^n f_*\mathbb{Q} \otimes_{\mathbb{Q}_{S^{an}_{\mathbb{C}}}} R^n f_*\mathbb{Q})|_{U^{an}_{\mathbb{C}}}$ and $t_B \in (R^n f_*\mathbb{Q} \otimes_{\mathbb{Q}_{S^{an}_{\mathbb{C}}}} R^n f_*\mathbb{Q})(n)|_{U^{an}_{\mathbb{C}}}$ with $t_B = (2\pi i)^n t$.

Compatibility of Poincaré duality with the relative comparison isomorphism shows that $P^n_{X/S} \otimes P^n_{X/S}|_U(t_{DR}) = t_B$. In other words the class t_{DR} is a relative Hodge class thus defining polynomial relations among the relative n-periods.

Now we can repeat the arguments we made in the non-relative case. First, we choose $\{\omega_i\}$ a basis of section of $H^n_{DR}(X/S)$ over the affine open subset $U \subset S$ and $\{\gamma_j^*\}$ a frame of $R^n f^{an}_* \mathbb{Q}_{X^{an}_{\mathbb{C}}}|_V$, or equivalently a frame $\{\gamma_j\}$ of the relative homology $R_n f^{an}_* \mathbb{Q}_{X^{an}_{\mathbb{C}}}|_V$, where $V \subset U^{an}_{\mathbb{C}}$ is some open analytic subset. We get that the matrix $P_{X/S} = ((2\pi i)^{-n} \int_{\gamma_j} \omega_i)$ satisfies

$$P_{X/S} M_B \, {}^t P_{X/S} = (2\pi i)^{-n} M_{DR}, \tag{A.15}$$

where M_B and M_{DR} are the matrices of the forms $\langle,\rangle$ and $\langle,\rangle_{DR}$ with respect to the basis of section and the frame chosen above.

The same process as before shows us that (A.15) is equivalent to the validity of the polynomial relations on the relative n-periods defined by the relative Hodge class t_{DR}. Finally, the same elementary argument as before shows the validity of the relative analogue of relation (A.12), i.e. the Riemann relations that we use in 4.2.

B

THE FULL AX-SCHANUEL THEOREM IN FAMILIES

In this appendix, following the argument in [Tsi15], we prove the "Full Ax-Schanuel" analog of 11.1. As a consequence we also obtain a slightly more general result that could be dubbed "Full Ax-Schanuel in affine families". We believe the results of this section are known to experts in the field, however since we couldn't find a reference for them, and we expect that they will play a role in subsequent progress towards a "Full Ax-Schanuel for GL_n", we include them in this appendix.

We consider the uniformizing map $\pi_k : \mathbb{C}^n \times \mathbb{C}^k \to (\mathbb{C}^\times)^n$, which is given by

$$(x_1, \ldots, x_n, y_1, \ldots, y_k) \mapsto (e^{x_1}, \ldots, e^{x_n}).$$

We define $D_k = \Gamma(\pi_k)$, i.e. as a subset of $\mathbb{C}^n \times \mathbb{C}^k \times (\mathbb{C}^\times)^n$

$$D_k = \{(\vec{u}, \vec{v}) : \pi_k(\vec{u}) = \vec{v}\}.$$

Furthermore, let π_a be the projection on the first n coordinates of the space $\mathbb{C}^n \times \mathbb{C}^k \times (\mathbb{C}^\times)^n$, and π_m be the projection on the last n coordinates of the same space.

Following the proof of the Full Ax-Schanuel Theorem in [Tsi15] we prove:

Theorem B.1 (Full Ax-Schanuel in families)**.** *Let $V \subset \mathbb{C}^n \times \mathbb{C}^k \times (\mathbb{C}^\times)^n$ be an irreducible algebraic subvariety, and U a connected complex-analytic irreducible component of $V \cap D_k$. Assuming that $\pi_m(U)$ is not contained in the coset of a proper subtorus of $(\mathbb{C}^\times)^n$, then*

$$\dim_{\mathbb{C}} V \geq n + \dim_{\mathbb{C}} U.$$

Proof. We employ induction on $k \geq 0$. For $k = 0$ this is a consequence of the Ax-Schanuel Theorem[1].

Assume that $k \geq 1$ and that the result holds for $k - 1$. Then we consider the projection

1 See Theorem 1.3 in [Tsi15].

$$p_0 : \mathbb{C}^n \times \mathbb{C}^k \times (\mathbb{C}^\times)^n \to \mathbb{C},$$

of our space to the $(n+k)-$th coordinate, i.e.

$$p_0(x_1, \ldots, x_n, y_1, \ldots, y_k, z_1, \ldots, z_n) = y_k.$$

Let also $V_0 = p_0(V)$ and, for $y \in V_0$, we consider the fibre V_y of V over y. Similarly we consider the corresponding fibre U_y of U over y. With this notation we get $V = \bigcup\limits_{y \in V_0} \{y\} \times V_y$.

Since V is irreducible, if $\dim(V_0) = 0$ then $V_0 = \{y_0\}$ will be a single point. This implies that $V = V' \times \{y_0\}$ is isomorphic to an irreducible algebraic subvariety $V' \subset \mathbb{C}^n \times \mathbb{C}^{k-1} \times (\mathbb{C}^\times)^n$. In this case, $U \subset V \cap D_k$ is isomorphic to a connected complex-analytic irreducible component of $V' \cap D_{k-1}$ and the result follows by induction.

We may therefore assume that $\dim V_0 = 1$. This tells us that V_0 contains a non-empty affine open subset of $\mathbb{C}$ and that for $y \in V_0$ generic we get

$$\dim V = \dim V_y + 1.$$

The rest of the proof comprises of considering the only two possible cases for the generic behaviour of the fibres U_y.

First Case: Suppose that $\pi_a(U_y)$ is generically[2] not contained in the translate of a proper $\mathbb{Q}-$linear subspace of $\mathbb{C}^n$.

We have that $V_y \subset \mathbb{C}^n \times \mathbb{C}^{k-1} \times (\mathbb{C}^\times)^n$ is an irreducible algebraic subvariety, and U_y is a connected complex-analytic irreducible component of $V_y \cap D_{k-1}$. Therefore by the previous assumption and the inductive hypothesis we get that for $y \in V_0$ generic

$$\dim V_y \geq n + \dim U_y.$$

This in turn implies that $\dim V \geq n + (1 + \dim U_y)$ and, since $1 + \dim U_y \geq \dim U$, the result follows.

Second Case: If the assumption of the previous case does not hold, then for $y \in V_0$ chosen generically, $\pi_a(U_y) \subset \mathbb{C}^n$ will be contained in the translate of some proper $\mathbb{Q}-$linear subspace of $\mathbb{C}^n$. In other words $U_y \subset Z(f_y)$, where $f_y = c(y) + \sum\limits_{i=1}^{n} q_i(y) x_i \in \mathbb{C}[x_1, \ldots, x_n]$, is a linear polynomial with the coefficients $q_i(y) \in \mathbb{Q}$ and $c(y) \in \mathbb{C}$ depending on y.[3]

At this point we consider another projection, namely we let

$$p_1 : \mathbb{C}^n \times \mathbb{C}^k \times (\mathbb{C}^\times)^n \to \mathbb{C}^n \times \mathbb{C}^{k-1} \times (\mathbb{C}^\times)^n$$

2 Generically here refers to $y \in V_0$.

3 Here $Z(f_y) = \{(x_1, \ldots, x_n) : f_y(x_1, \ldots, x_n) = 0\}$, is just the set of solutions of $f_y = 0$ in $\mathbb{C}^n$.

be the projection given by

$$p_1(x_1, \ldots, x_n, y_1, \ldots, y_k, z_1, \ldots, z_n) = (x_1, \ldots, x_n, y_1, \ldots, y_{k-1}, z_1, \ldots, z_n).$$

We also let $V_1 = p_1(V)$, $U_1 = p_1(U)$, $V_1' = Zcl(V_1)$, the Zariski closure of V_1, and $U_1' = \bar{U}_1$, the closure of U_1 with respect to the standard topology on $\mathbb{C}^n \times \mathbb{C}^{k-1} \times (\mathbb{C}^\times)^n$.

For these new sets we get that V_1' is an irreducible subvariety of $\mathbb{C}^n \times C^{k-1} \times (\mathbb{C}^\times)^n$ and U_1' is a connected irreducible complex-analytic component of $V_1' \cap D_{k-1}$. We also get that $\pi_a(U) = \pi_a(U_1)$ and hence, by the initial assumption on U, $\pi_a(U_1')$ is not contained in the translate of a $\mathbb{Q}-$linear subspace of $\mathbb{C}^n$. Therefore we may apply the inductive hypothesis to get

$$\dim V_1' \geq n + \dim U_1'.$$

From the preceding discussion we get that $\dim V \geq \dim V_1 = \dim V_1'$. On the other hand, since, by assumption, $\pi_a(U)$ is not contained in the translate of a $\mathbb{Q}-$linear subspace of $\mathbb{C}^n$ then the $Z(f_y)$, and hence the f_y, will vary with y. This in turn implies[4] that $\dim U = \dim U_1 = \dim U_1'$.

Combining all of the above we reach the conclusion. □

By the same arguments as in [Tsi15], the above theorem implies the following

Corollary B.2. Let D_k and π_k be as above and $U \subset D_k$ be an irreducible complex analytic subspace such that $\pi_m(U)$ is not contained in a coset of a proper subtorus of $(\mathbb{C}^\times)^n$. Then

$$\dim Zcl(U) \geq n + \dim_{\mathbb{C}} U.$$

A generalization-Affine families

As a corollary of the above proof we are able to extract Ax-Schanuel results for a larger family of spaces. The idea is that we are able to replace $\mathbb{C}^k$ by a random affine variety. We approach this in a geometric setting similar to the previous subsection.

Let W be an affine variety over $\mathbb{C}$ and let $\pi_n : \mathbb{C}^n \to (\mathbb{C}^\times)^n$ be the map given by

$$(x_1, \ldots, x_n) \mapsto (e^{x_1}, \ldots, e^{x_n}).$$

4 The coordinate function y_k restricted to U will depend on the rest of the coordinates of U.

We consider the uniformizing map $\pi_n \times id_W : \mathbb{C}^n \times W \to (\mathbb{C}^\times)^n \times W$, the product of π_n and the identity morphism id_W of W. Let also $p_1 : (\mathbb{C}^\times)^n \times W \to (\mathbb{C}^\times)^n$ be the projection on $(\mathbb{C}^\times)^n$ and let $\phi : \mathbb{C}^n \times W \to (\mathbb{C}^\times)^n$ be its composition with $\pi_n \times id_W$.

We also define $D_k(W) = \Gamma(\phi)$, i.e. as a subset of $\mathbb{C}^n \times W \times (\mathbb{C}^\times)^n$

$$D_k(W) = \{(\vec{u}, \vec{v}) : \phi(\vec{u}) = \vec{v}\}.$$

Corollary B.3 (Full Ax-Schanuel in Affine families)**.** Let $V \subset \mathbb{C}^n \times W \times (\mathbb{C}^\times)^n$ be an irreducible algebraic subvariety, and U a connected complex-analytic irreducible component of $V \cap D_k(W)$. Assuming that $\pi_m(U)$ is not contained in the coset of a proper subtorus of $(\mathbb{C}^\times)^n$, then

$$\dim_{\mathbb{C}} V \geq n + \dim_{\mathbb{C}} U.$$

Proof. By Noether's Normalization Lemma there exists a finite surjective morphism $f : W \to \mathbb{A}^d_{\mathbb{C}}$ where $d = \dim W$. The product of this morphism with the identity of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$ in turn gives a finite morphism

$$F : \mathbb{C}^n \times W \times (\mathbb{C}^\times)^n \to \mathbb{C}^n \times \mathbb{A}^d_{\mathbb{C}} \times (\mathbb{C}^\times)^n.$$

Indeed a morphism of affine varieties is finite if and only if it is proper[5], since F is proper as the product of two such morphisms it will also be finite. The image of the irreducible subvariety V under this map will be an irreducible subvariety V' of $\mathbb{C}^n \times \mathbb{A}^d_{\mathbb{C}} \times (\mathbb{C}^\times)^n$, since finite morphisms are closed.

We also note that F maps the set $D_k(W)$ to the set $D_k(\mathbb{A}^d_{\mathbb{C}})$. So that the closure $U' = cl(F(U))$ of the image of U with respect to the Euclidean topology will be a component of $V' \cap D_k(\mathbb{A}^d_{\mathbb{C}})$.

Since F is finite we get that $\dim_{\mathbb{C}} U = \dim_{\mathbb{C}} U'$, $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} V'$, and by the construction of F it follows that $\pi_m(U')$ is not contained in the coset of a proper subtorus of $(\mathbb{C}^\times)^n$, since this is true for $\pi_m(U)$. Therefore the result follows from B.1. $\square$

5 See Exercises II.4.1 and II.4.6 in [Har77].

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Colophon

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