DROPLET FORMATION IN SIMPLE NONLOCAL AGGREGATION MODELS

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ABSTRACT

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We interaction energies given by various kernels, and investigate how these kernels drive the formation of multiple flocks within a larger population. We show that for a class of kernels having a “well-barrier" shape that the energy is minimized by a sequence of indicators of finitely many balls whose supports become infinitely far apart from one another. The dichotomy case of the concentration compactness principle is a key ingredient in our proof. We also consider a toy model which forbids points in the support of an admissible density from being within a certain range of distances from one another. We show in one dimensions, that no matter the width of this range the energy is minimized by the indicator of a union of well separated intervals of length 1 and one smaller interval. Finally, we also consider weakly repulsive kernels and show that Wasserstein d_∞ local minimizers must saturate the density constraint.

CONTENTS

1

INTRODUCTION

1.1 INTRODUCTION

The study of aggregation phenomena in biology (see for example [BT13][TBL06]) and physics (see for example [Wal10][HP05]) has produced an interesting class of geometric shape optimization problems. Large scale collective behaviour, such as collaboration and formation into flocks, is shaped at least in part by social forces between individuals.

In this paper we will consider the case where the attractive and repulsive forces between individuals are determined entirely by the distances between individuals. Even in this simple case there are a wide range of complex phenomena. For instance, Carrillo, Figalli and Patacchini [CFP17] have shown that weak repulsion at short distances implies that global minimizers of the interaction energy have finite support. Lim and McCann [LM21] have shown that for certain power law kernels with mild repulsion and strong attraction, the interaction energy is uniquely minimized by measures that are uniformly distributed on the vertices of a unit simplex. For certain power law kernels, Lopes [Lop19] has shown that minimizers are unique and radially symmetric. For further examples, see [BCLR13], [CDM16], and [FL19].

We focus on the particular phenomenon of formation of multiple flocks within a large population. To this end, we consider interactions that are attractive at short distances, followed by a repulsive “barrier" at mid distances, and neutral at long distances. See Figure 1.1 for a sketch of such an interaction kernel. The short range attraction motivates individuals to flock together, while the mid range repulsion prevents the entire population from forming into a single flock. In biological applications, the aggregation problem typically involves kernels that are repulsive at short distances, to prevent collisions between individuals, for example see [MBSEK03]. In this paper, we use a density constraint, to prevent the population from concentrating to a point, and to force the population to spread out as its size increases. So, while there is no short-range repulsion built into the kernel, the density constraint provides what [FL18] refer to as a hard-core repulsion.

We show that, under certain conditions on the relative sizes of the attractive well and repulsive barrier, there are minimizing sequences of densities

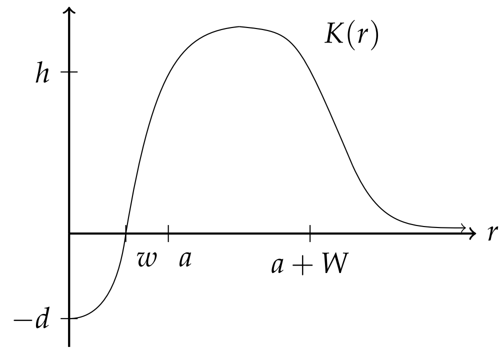

Figure 1.1: Sketch of a "well-barrier" type kernel. The parameters d and w represent the depth and width of the attractive well, and the parameters h and W represent the height and width of the repulsive barrier.

that are indicator functions of balls, whose centres get infinitely far apart from one another. To show this, we use the concentration compactness principle to show that minimizers exist in the case where distant individuals have no mutual interaction. The dichotomy case of the concentration compactness principle is our main tool for extracting a minimizer even though an arbitrary minimizing sequence is expected to split into many pieces whose supports may be far away from each other.

1.2 STATEMENT OF MAIN RESULTS

Consider the problem of minimizing the energy

$$\mathcal{E}[\rho] = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(|x-y|)\rho(x)\rho(y)\,dx\,dy$$

given by an interaction kernel K over the set

$$\mathcal{A}_m = \{\rho \in L^1(\mathbb{R}^N) \ : \ 0 \leq \rho \leq 1, \ ||\rho||_{L^1(\mathbb{R}^N)} = m\}$$

of densities having total mass m. This nonlocal interaction energy arises in the study of aggregation in biology, as outlined in [BT13]. We will also make use of the following notation to denote the interaction energy between two densities ρ and η

$$\mathcal{E}[\rho, \eta] = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(|x-y|)\rho(x)\eta(y)\,dx\,dy.$$

In chapters 2-5 we will be considering a certain family of kernels having a "well-barrier" shape. The key features of these kernels are that they are attractive at short distances, followed by a repulsive "barrier," followed by decay at infinity. More precisely, we will only consider kernels $K : [0, \infty) \to \mathbb{R} \cup \{+\infty\}$ which satisfy the following conditions:

(K1) Well. $K(0) < 0$, and K is non-decreasing on some interval $[0, a]$,

(K2) Barrier. $K(r) \geq h$ on $[a, a + W]$, for some width $W > 0$, and

(K3) Decay. $K(r) \geq 0$ on $(a + W, \infty)$, and $\lim_{r \to \infty} K(r) = 0$

It turns out, for large mass, minimizers may not exist, but we can construct minimizing sequences that consists of finitely many pieces whose supports become increasingly far apart. If we make additional assumptions on the relative sizes of the attractive well and repulsive barrier, we find that these pieces want to be indicators of balls. Let $d = -K(0)$, which represents the depth of the well, and let $w = \inf\{r > 0 : K(r) > 0\}$ be the width of the well, and consider the following two additional separation conditions:

(K4) $d < h$, and

(K5) $a + w \leq W - 2w$.

Theorem 1.1 (Existence of minimizers for compactly supported kernels)**.** *Let $K = K_b + K_\infty$ where K_b is bounded and satisfies $(\mathbf{K1}) - (\mathbf{K3})$, and K_∞ only takes values in $\{0, \infty\}$. Suppose K is zero outside of some large radius $R > 0$. Then for any $m > 0$ the infimum*

$$\inf_{\rho \in \mathcal{A}_m} \mathcal{E}[\rho]$$

is attained by some $\rho \in \mathcal{A}_m$.

The main tool in the proof of Theorem 1.1 is Lions' concentration compactness principle. We start with a minimizing sequence, and concentration compactness tells us that there is a subsequence that is vanishing, tight up to translation, or dichotomous. A common strategy when using concentration compactness is to rule out the possibility of a vanishing or dichotomous minimizing sequence. The resulting tight minimzing sequence is then used to show that a minimizer exists. In our case, we cannot rule out dichotomy, instead we use it to decompose the minimizing sequence into finitely many pieces whose supports become very far away from each other. Further, each of these pieces are tight up to translation. Then, since the kernel is zero outside of some large radius, these pieces do not actually need to spread out far away from each other in order to reach the minimal energy, allowing us to construct a minimizer. When the kernel K decays at infinity, minimizers do not generally exist, but we can still consider the minimal energy:

$$E(m) = \inf_{\rho \in \mathcal{A}_m} \mathcal{E}[\rho].$$

Theorem 1.2. *Let $K = K_b + K_\infty$ where K_b is bounded and satisfies $(\mathbf{K1}) - (\mathbf{K5})$, and K_∞ only takes values in $\{0, \infty\}$. Further, assume K is strictly increasing on $[0, a]$. Then for any $m > 0$,*

$$E(m) = \sum_{i=1}^{k} \mathcal{E}[\mathbb{1}_{B(0,r_i)}] \tag{1.1}$$

for some radii $r_i > 0$.

We prove Theorem 1.2 by first truncating the kernel K, which allows us to use the existence Theorem for kernels with no decay. A minimizing sequence is then obtained by sending the pieces of the minimizer for the truncated problem infinitely far away from each other. In 3.4 we analyze this using the notion of generalized minimizers.

In chapter **??** we investigate the size and number of the droplets in the specific example where the kernel is a power law near 0. We show in this example that generalized minimizers consist of many balls of the same size, and at most one other smaller ball. To do this we first show that the minimal energy grows linearly in m.

To better understand the underlying geometry and how the width of the barrier influences the separation into droplets, in chapter **??** we consider a toy version of this problem where the kernel K is -1 on $[0, 1]$, followed by $+\infty$ on an interval $[1, 1 + W]$, and zero elsewhere. If $W \geq 1$, then any density with finite energy can be easily decomposed into finitely many pieces whose supports have diameter at most 1. However, the geometric problem becomes more complicated when the repulsive barrier is narrower. We prove the following theorem, which says that no matter the width of the forbidden range of distances, the minimal energy in the one dimensional problem is always attained by a well separated union of intervals of length 1, plus possibly a piece of smaller mass which is supported in an interval of length one. Note that we say two densities η and ρ are well separated if $\mathcal{E}[\eta, \rho] = 0$

Theorem 1.3. *Let $N = 1$, $m > 0$, and write $m = k + a$ where $k \in \mathbb{Z}$ and $0 \leq a < 1$. Then for any $w \geq 0$,*

$$\inf_{\rho \in \mathcal{A}_m} \mathcal{E}_w[\rho] = -1 - a^2,$$

where $\mathcal{E}_w$ is the energy for the toy model with width w.

When m is an integer, we also show that the only minimizers consist of indicators of well separated intervals of length 1. However, when $w = 0$ we give an example of a minimal configuration of disjoint intervals which are not well separated.

Finally, in chapter 6 we investigate another mechanism of separation. We consider weakly repulsive kernels, that is C^2 kernels satisfying first the following condition

$$K(0) = 0, \text{ and there exists an } R > 0 \text{ such that } K(r) < 0 \text{ for } 0 < r < R, \\ \text{and } K(r) \geq 0 \text{ for } r \geq R, \tag{1.2}$$

along with the weak repulsion condition

$$\text{there exist } \alpha > 2, \text{ and } C > 0 \text{ such that } K'(r)r^{1-\alpha} \to -C/\alpha \text{ as } r \to 0. \tag{1.3}$$

In the case where one minimizes over probability measures rather than densities, in [CFP17] Carrillo, Figalli, and Patacchini proved that Wasserstein d_∞ local minimizers have discrete support. We adapt their techniques to show that when we consider the minimization problem over densities of a fixed height, d_∞ local minimizers saturate the density constraint. This reduces the problem wholly to a shape optimization problem, since minimizing densities are given by (scalar multiples of) indicators of sets.

Theorem 1.4. *Let K be a kernel satisfying (1.2) and the weak repulsion condition (6.2). If $\rho \in \mathcal{A}'_m$ is a d_∞ local minimizer of the energy $\mathcal{E}$ given by the kernel K, then the set $\{0 < \rho < 1\}$ has measure zero.*

For small mass (corresponding to high densities), one may expect that minimizers are well separated in the sense that we would expect them to be close to a probability measure minimizer, which we know has discrete support. However, as mass increases one may wonder, depending on the long range attraction of the kernel, whether the population will always form one large group or whether there is a possibility for extended configuration of distinct groups within the larger population.

2

PRELIMINARIES ON SHAPE OF MINIMIZERS

Throughout this chapter we consider kernels K which satisfy the well-barrier conditions **(K1)-(K3)**.

2.1 BALLS ARE MINIMAL FOR SMALL MASS

First we show that indicators of balls are minimal for small mass.

Lemma 2.1. *There is a mass $m_0 > 0$, such that for all $m \leq m_0$*

$$\inf_{\rho \in \mathcal{A}_m} \mathcal{E}[\rho] = \mathcal{E}[\mathbb{1}_{B(0,r)}] \tag{2.1}$$

where $B(0,r)$ is the ball of measure m.

Proof. Fix $m > 0$ and let $\rho \in \mathcal{A}_m$ and let r be the radius of a ball with measure m. First note that

$$K(r) \leq \mathbb{1}_{[0,w]}(r)K(r).$$

The right hand side, $\mathbb{1}_{[0,w]}(r)K(r)$ is non-decreasing, and so we may apply the Riesz rearrangement inequality (see [LL01] Ch.3), which says that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(|x-y|)\mathbb{1}_{|x-y|\leq w}\rho(x)\rho(y)\,dx\,dy \geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(|x-y|)\mathbb{1}_{|x-y|\leq w}\rho^*(x)\rho^*(y)\,dx\,dy$$

where ρ^* is the symmetrically decreasing rearrangement of ρ. By the bathtub principle (see [LL01] Theorem 1.14) we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(|x-y|)\mathbb{1}_{|x-y|\leq w}\rho^*(x)\rho^*(y)\,dx\,dy \geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(|x-y|)\mathbb{1}_{|x-y|\leq w}\mathbb{1}_{B(0,r)}(x)\mathbb{1}_{B(0,r)}(y)\,dx\,dy.$$

Finally, if $r \leq w$, then

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(|x-y|)\mathbb{1}_{|x-y|\leq w}\mathbb{1}_{B(0,r)}(x)\mathbb{1}_{B(0,r)}(y)\,dx\,dy = \mathcal{E}[\mathbb{1}_{B(0,r)}].$$

So, if m is small enough so that $r \leq w$, we have that $\mathcal{E}[\rho] \geq \mathcal{E}[\mathbb{1}_{B(0,r)}]$ for all $\rho \in \mathcal{A}_m$. □

2.2 FIRST ORDER CONDITION

We adapt Lemma 4.2 in [BCT18] to the case where the kernel K is allowed to take the value $+\infty$. This Lemma is used to prove that minimizing densities have compact support. It also plays a key role in showing that points in the support of a minimizing density cannot be within certain range of distances from one another.

Lemma 2.2. *Let $\rho \in \mathcal{A}_m$ be a local minimizer of the energy $\mathcal{E}$, in the L^1 topology. Then there is a $\lambda < 0$ such that for almost every x we have*

$$K * \rho(x) \begin{cases} \geq \lambda, & \text{if } \rho(x) = 0, \\ = \lambda, & \text{if } 0 < \rho(x) < 1, \\ \leq \lambda, & \text{if } \rho(x) = 1. \end{cases}$$

Proof. The proof follows the same method as in [BCT18], with some modifications to account for the kernel K taking the value $+\infty$. Let $S_0 = \{\rho = 0\}$, and $S_1 = \{\rho = 1\}$. We want to construct a perturbation of ρ, by adding to ρ on S_0, and subtracting on S_1. Fix any nonnegative, bounded, compactly supported φ and $\psi \in L^1(\mathbb{R}^d)$ with $||\varphi||_{L^1(\mathbb{R}^d)} = ||\psi||_{L^1(\mathbb{R}^d)=1}$, and $\varphi = 0$ a.e. in S_1 and $\psi = 0$ a.e. in S_0. Note that $\mathcal{E}[\psi, \rho] < \infty$, since $\psi = 0$ almost everywhere in S_0, and $\mathcal{E}[\rho] < \infty$. For now assume that $\mathcal{E}[\varphi, \rho] < \infty$. Let $\epsilon > 0$ and define

$$\varphi_\epsilon(x) = \frac{1}{||\varphi \mathbb{1}_{\rho < 1-\epsilon}||_{L^1(\mathbb{R}^N)}} \varphi(x) \mathbb{1}_{\rho < 1-\epsilon}(x),$$

$$\psi_\epsilon(x) = \frac{1}{||\psi \mathbb{1}_{\rho > \epsilon}||_{L^1(\mathbb{R}^N)}} \psi(x) \mathbb{1}_{\rho > \epsilon}(x).$$

Then $\eta_t = \rho + t(\varphi_\epsilon - \psi_\epsilon) \in \mathcal{A}_m$. For sufficiently small $t > 0$ we have $\mathcal{E}[\eta_t] \geq \mathcal{E}[\rho]$, since ρ is a local minimizer. Then,

$$\begin{aligned} 0 &\leq \lim_{t \to 0^+} \frac{\mathcal{E}[\rho + t(\varphi_\epsilon - \psi_\epsilon)] - \mathcal{E}[\rho]}{t} \\ &= \lim_{t \to 0^+} \frac{\mathcal{E}[\rho] + 2t\mathcal{E}[\rho, \varphi_\epsilon - \psi_\epsilon] + t^2 \mathcal{E}[\varphi_\epsilon - \psi_\epsilon] - \mathcal{E}[\rho]}{t} \\ &= 2 \int_{\mathbb{R}^N} K * \rho(x)(\varphi_\epsilon(x) - \psi_\epsilon(x))\, dx. \end{aligned}$$

Then taking $\epsilon \to 0$, by dominated convergence we have

$$\int_{\mathbb{R}^N} K * \rho(x)\varphi(x)\,dx \geq \int_{\mathbb{R}^N} K * \rho(x)\psi(x)\,dx. \tag{2.2}$$

The inequality (2.2) also holds for any nonnegative $\varphi, \psi \in L^1(\mathbb{R}^N)$ that satisfy $||\varphi||_{L^1(\mathbb{R}^N)} = ||\psi||_{L^1(\mathbb{R}^N)} = 1$, $\psi = 0$ a.e. in S_0, $\varphi = 0$ a.e in S_1, and $\mathcal{E}[\varphi, \rho] < \infty$, by density of bounded compactly supported functions in $L^1(\mathbb{R}^N)$. Finally, we can relax the assumption that $\mathcal{E}[\varphi, \rho] < \infty$, since (2.2) is trivial in this case. Now let

$$\lambda = \sup\left\{\int_{\mathbb{R}^N} K * \rho(x)\psi(x)\,dx \;:\; ||\psi||_{L^1(\mathbb{R}^N)} = 1,\ \psi \geq 0;\ \psi = 0 \ \text{ a.e. in } S_0\right\}. \tag{2.3}$$

Then, by (2.2)

$$\lambda \leq \inf\left\{\int_{\mathbb{R}^N} K * \rho(x)\varphi(x)\,dx \;:\; ||\varphi||_{L^1(\mathbb{R}^N)} = 1,\ \varphi \geq 0;\ \varphi = 0 \ \text{ a.e. in } S_1\right\}. \tag{2.4}$$

These two equations tell us that$K * \rho(x) \leq \lambda$ for a.e x such that $\rho(x) > 0$ and $K * p(x) \geq \lambda$ for a.e. x such that $\rho(x) < 1$, respectively.

To see that $\lambda < 0$, suppose for contradiction that $K * \rho(x) \geq 0$ on a set of positive measure $A \subset \{\rho > 0\}$. Without loss of generality, assume A has small enough diameter so that $\mathcal{E}[\rho|_A] < 0$. Then,

$$\mathcal{E}[\rho - \rho|_A] = \mathcal{E}[\rho] - 2\mathcal{E}[\rho, \rho|_A] + \mathcal{E}[\rho_A] < \mathcal{E}[\rho].$$

This says that we can improve ρ by removing a small piece of it. We can construct a competitor by adding a small piece at infinity. To make the perturbation from ρ small, we can pick A to be as small as we like. □

Again, following the general argument from [BCT18], we can now prove that local minimizers are compactly supported.

Lemma 2.3 (L^1 local minimizers have compact support)**.** *Let $\rho \in \mathcal{A}_m$ be a local minimizer of the energy $\mathcal{E}$, in the L^1 topology. Then ρ is compactly supported.*

Proof. Let $\rho \in \mathcal{A}_m$ be a local minimizer of the energy $\mathcal{E}$, in the L^1 topology. Then by Lemma 2.2, there is a $\lambda < 0$ such that, up to a set of measure zero, $\{\rho > 0\} \subseteq \{K * \rho \leq \lambda\}$ For any $\epsilon > 0$, let $R > 0$ be large enough so that

$$\int_{|y|>R} \rho(y)\,dy < \epsilon.$$

Let $r_0 > 0$ be such that $K(r) \geq 0$ for all $r > r_0$. Then, noting that $K(r)$ takes its minimum value at $r = 0$, we have

$$\begin{aligned} K * \rho(x) &\geq \int_{|x-y|<r_0} K(0)\rho(y)\,dy \\ &> K(0)\epsilon \end{aligned}$$

for all $|x| > R + r_0$. Thus

$$\liminf_{|x|\to\infty} K * \rho(x) \geq 0.$$

But, since $\lambda < 0$, this means that the set $\{K * \rho(x) \leq \lambda\}$ is bounded. □

2.3 SECOND ORDER CONDITION

Lemma 2.4. *Let $\rho \in \mathcal{A}_m$ be an L^1 local minimizer of the energy $\mathcal{E}$. Then there is an $\epsilon > 0$ such that if η is a bounded function supported on $\{0 < \rho < 1\}$ with $\int \eta = 0$, $\rho + \eta \in \mathcal{A}_m$, and $||\eta||_{L^1(\mathbb{R}^N} < \epsilon$ then*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(|x-y|)\eta(x)\eta(y)\,dx\,dy \geq 0.$$

Proof. Since ρ is an L^1 local minimizer of the energy $\mathcal{E}$, there is an $\epsilon > 0$ so that $\mathcal{E}[\rho] \leq \mathcal{E}[\rho_0]$ for any $\rho_0 \in \mathcal{A}_m$ with $||\rho - \rho_0||_{L^1(\mathbb{R}^N} < \epsilon$. Let η be as in the lemma, and let $\rho_0 = \rho + \eta$. Then

$$\begin{aligned} 0 \leq &\mathcal{E}[\rho+\eta] - \mathcal{E}[\rho] \\ = &\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(|x-y|)\eta(x)\eta(y)\,dx\,dy \\ &+ 2\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(|x-y|)\rho(x)\eta(y)\,dx\,dy. \end{aligned}$$

By lemma 2.2 there is a number λ such that $K * \rho(x) = \lambda$ for almost every $x \in \{0 < \rho < 1\}$. Since η is supported in $\{0 < \rho < 1\}$, this means

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(|x-y|)\rho(x)\eta(y)\,dx\,dy = \lambda \int_{\mathbb{R}^N} \eta(y)\,dy = 0.$$

□

2.4 SUBADDITIVITY

For $m > 0$, let

$$E(m) = \inf_{\rho\in\mathcal{A}_m} \mathcal{E}[\rho].$$

We will show that this is a subadditive function of the mass m :

Lemma 2.5 (Subadditivity)**.** *Let* $K : [0, \infty) \to \mathbb{R}$ *be a bounded kernel satisfying the "well-barrier" conditions* $(\mathbf{K1}) - (\mathbf{K3})$*. Let* $m, n > 0$*. Then,*

$$E(m+n) \leq E(m) + E(n)$$

Intuitively, considering two masses m and n one may imagine taking a minimizer for each mass and then placing them very far away from each other to obtain a minimizer for the mass $m+n$. Of course, we must actually consider minimizing sequences, but the underlying idea is the same.

Proof. Let $\{\rho_k^m\} \subset \mathcal{A}_m$ and $\{\rho_k^n\} \subset \mathcal{A}_n$ be minimizing sequences for $\inf_{\rho \in \mathcal{A}_m} \mathcal{E}[\rho]$ and $\inf_{\rho \in \mathcal{A}_n} \mathcal{E}[\rho]$, respectively. From these two sequences we will construct a sequence $\{\rho_k\} \subset \mathcal{A}_{m+n}$, for which

$$\lim_{k \to \infty} \mathcal{E}[\rho_k] = \lim_{k \to \infty} \mathcal{E}[\rho_k^m] + \lim_{k \to \infty} \mathcal{E}[\rho_k^n].$$

Our approach will be essentially to add a translated copy of ρ_k^n to ρ_k^m. The translations will be chosen so that the translated ρ_k^n has small interaction with ρ_k^m. Since these densities may not be compactly supported, we will consider their restrictions to balls of suitably large radius, and add a small corrective term to ensure the resulting density is in $\mathcal{A}_{m+n}$. For each $k \in \mathbb{N}$, there is a radius $R_k > 0$ for which

$$\int_{B(0,R_k)} \rho_k^m \geq m - \frac{1}{k} \quad \text{and} \quad \int_{B(0,R_k)} \rho_k^n \geq n - \frac{1}{k}. \tag{2.5}$$

Next, define

$$\overline{\rho}_k^m := \rho_k^m|_{B(0,R_k)} \quad \text{and} \quad \overline{\rho}_k^n := \rho_k^n|_{B(0,R_k)}.$$

Finally, set

$$\rho_k(x) := \overline{\rho}_k^m + \overline{\rho}_k^n(\cdot + x_k) + \mathbb{1}_B(x_k, r_k)$$

where $r_k \geq 0$ is chosen so that $\rho_k \in \mathcal{A}_{m+n}$, and $x_k = (k + 2R_k)e_1$. This choice for x_k means that the supports of these three pieces become arbitrarily far apart as $k \to \infty$. Now, we may expand

$$\begin{aligned} \mathcal{E}[\rho_k] =& \mathcal{E}[\overline{\rho}_k^m] + \mathcal{E}[\overline{\rho}_k^n] + \mathcal{E}[\mathbb{1}_{B(x_k,r_k)}] \\ &+ 2\mathcal{E}[\overline{\rho}_k^m, \overline{\rho}_k^n(\cdot + x_k)] + 2\mathcal{E}[\overline{\rho}_k^m, \mathbb{1}_{B(x_k,r_k)}] \\ &+ 2\mathcal{E}[\overline{\rho}_k^n(\cdot + x_k), \mathbb{1}_{B(x_k,r_k)}]. \end{aligned} \tag{2.6}$$

First, we will check that the final three terms in (2.6) approach 0 as $k \to \infty$. By our choice of x_k, $\text{dist}(B(0, R_k), B(-x_k, R_k)) \geq k$, hence

$$|\mathcal{E}[\overline{\rho}_k^m, \overline{\rho}_k^n(\cdot + x_k)]| = \left| \int_{B(0,R_k)} \int_{B(-x_k,R_k)} K(|x-y|)\rho_k^n(x+x_k)\rho_k^m(y)\, dx\, dy \right| \tag{2.7}$$

$$= \left| \int_{B(0,R_k)} \int_{B(-x_k,R_k)} K(|x-y|)\mathbb{1}_{|x-y|\geq k}\rho_k^n(x+x_k)\rho_k^m(y)\, dx\, dy \right| \tag{2.8}$$

$$\leq mn \sup_{r\geq k} K(r). \tag{2.9}$$

To conclude, use the fact that $K(r)$ tends to 0 as $r \to \infty$. Similar estimates can be done for the other two pair interaction terms.

Next, we need to check that the correction term $\mathbb{1}_{B(x_k,r_k)}$ has arbitrarily small self interaction as k gets large. By (2.5), $|B(x_k, r_k)| \leq 2/k$. Combining this with the fact that K is bounded, we see that

$$|\mathcal{E}[\mathbb{1}_{B(x_k,r_k)}]| \leq ||K||_\infty |B(0, r_k)|^2 \leq \frac{4||K||_\infty}{k^2}.$$

Finally, we need to check that

$$\lim_{k\to\infty} \mathcal{E}[\overline{\rho}_k^m] = \lim_{k\to\infty} \mathcal{E}[\rho_k^m].$$

We may write

$$\mathcal{E}[\overline{\rho}_k^m] = \mathcal{E}[\rho_k^m - (\rho_k^m - \overline{\rho}_k^m)] = \mathcal{E}[\rho_k^m] + \mathcal{E}[\rho_k^m - \overline{\rho}_k^m] - 2\mathcal{E}[\overline{\rho}_k^m, \rho_k^m - \overline{\rho}_k^m] \tag{2.10}$$

As before, using the fact that K is bounded and $||\rho_k^m - \overline{\rho}_k^m||_{L^1(\mathbb{R}^d} \leq 1/k$, it is straightforward to check that the second and third term in the right hand side of (2.10) both tend to 0 as $k \to \infty$. □

3 EXISTENCE OF MINIMIZERS

Throughout this sections we will only be considering kernel which satisfy **(K1)-(K3)**.

3.1 CONCENTRATION COMPACTNESS

In the next section, we will prove existence of minimizers for kernels which are 0 outside of some large radius. The main tool we will use is Lions' concentration compactness principle (Lemma 1.1 in [Lio84]), which will allow us to deal with the possibility that minimizing sequences can have pieces which get infinitely far away from one another.

Lemma 3.1 (Concentration Compactness)**.** *Let $\{\rho_n\}$ be a sequence in $L^1(\mathbb{R}^N)$ satisfying*

$$\rho_n \geq 0, \ \ \text{and} \ \int_{\mathbb{R}^N} \rho_n(x)\, dx = m,$$

for a fixed $m > 0$. Then there exists a subsequence $\{\rho_{n_k}\}$ that satisfies one of the following three properties:

1. *(Tightness up to translation) There exists $y_k \in \mathbb{R}^N$ such that for every $\epsilon > 0$ there exists an $R > 0$,*

$$\int_{B(y_k,R} \rho_{n_k}(x)\, dx \geq m - \epsilon.$$

2. *(Vanishing) For all $R > 0$*

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} \rho_{n_k}(x)\, dx = 0.$$

3. *(Dichotomy) There exists an $0 < \alpha < m$, such that for any $\epsilon > 0$, there exist $k_0 \geq 1$, $y_k \in \mathbb{R}^N$, and radii $R > 0$ and $R_k \to \infty$ as $k \to \infty$ such that*

$$\rho_k^1 = \rho_{n_k}|_{B(y_k,R)}, \ \ \text{and} \ \ \rho_k^2 = \rho_{n_k}|_{\mathbb{R}^N \setminus B(y_k,R_k)}$$

satisfy

$$||\rho_{n_k} - (\rho_k^1 + \rho_k^2)||_{L^1(\mathbb{R}^N)} \leq \epsilon,$$
$$||\rho_k^1||_{L^1(\mathbb{R}^N)} - \alpha \leq \epsilon, \text{ and}$$
$$||\rho_k^2||_{L^1(\mathbb{R}^N)} - (m - \alpha) < \epsilon$$

for $k \geq k_0$.

The statement for the dichotomy case here appears different from the statement in [Lio84], but the modification reflects the construction of ρ_k^1, and ρ_k^2 in [Lio84].

The concentration compactness principle is used to compensate for the lack of sequential compactness in $L^1(\mathbb{R}^N)$ - a sequence in $L^1(\mathbb{R}^N)$ may not have a convergent subsequence. Often concentration compactness is used by taking a minimizing sequence, and ruling out the vanishing and dichotomy cases, leaving only the possibility of tightness up to translation. In our case, we expect a minimizing sequence to break apart into many pieces, so we lean into the dichotomy case and use it to decompose our minimizing sequence.

3.2 EXISTENCE OF MINIMIZERS FOR COMPACTLY SUPPORTED KERNELS

In this section we prove Theorem 1.1. The general strategy is to use the concentration compactness principle to decompose a minimizing sequence into finitely many tight (up to translation) pieces whose supports become far away from one another. We first prove a series of lemmas related to the three cases from the concentration compactness principle. The following Lemma will be used to rule out the vanishing case.

Lemma 3.2. *Let the kernel* $K : [0, \infty) \to \mathbb{R} \cup \{+\infty\}$ *satisfy*$(\mathbf{K1}) - (\mathbf{K3})$. *Suppose* $\{\rho_k\}$ *is a sequence in* $\mathcal{A}_m$ *that satisfies the vanishing property in the concentration compactness Lemma. Then,* $\liminf_{k\to\infty} \mathcal{E}[\rho] \geq 0$.

Note in particular that a minimizing sequence cannot be vanishing, nor can it have a "vanishing part".

Proof. Fix $R > 0$ large enough so that $K(r) \geq 0$ for $r > R$. Then for any $\epsilon > 0$, using the fact that $\{\rho_k\}$ is vanishing we have for large k

$$\int_{B(y,R)} \rho(x)dx < \epsilon$$

for any $y \in \mathbb{R}^N$. We can now compute

$$\begin{aligned} \mathcal{E}[\rho_k] &\geq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(|x-y|) \mathbb{1}_{|x-y| \leq R} \rho_k(x) \rho_k(y) \, dx \, dy \\ &\geq K(0) \int_{\mathbb{R}^N} \rho_k(y) \int_{B(y,R)} \rho_k(x) \, dx \, dy \\ &\geq K(0) m \epsilon \end{aligned}$$

□

Next, the following lemma will be used to extract droplets.

Lemma 3.3. *Let the kernel* $K = K_b + K_\infty$ *where* K_b *is bounded and satisfies* $(\mathbf{K1}) - (\mathbf{K3})$*, and* K_∞ *takes values in* $\{0, \infty\}$*. Suppose* $\{\rho_k\}$ *is a sequence of densities in* $\mathcal{A}_m$ *with* $\mathcal{E}[\rho_k] < \infty$*. If* $\{\rho_k\}$ *is tight up to translation, then there is a* $\rho \in \mathcal{A}_m$ *such that*

$$\lim_{k \to \infty} \mathcal{E}[\rho_k] = \mathcal{E}[\rho].$$

Proof. This argument follows the same strategy as in [CFT15]. By tightness up to translation, there exist $y_k \in \mathbb{R}^N$ such that for any $\epsilon > 0$, there is a radius $R > 0$ such that

$$\int_{B(y_k,R)} \rho_k(y) \, dy \geq m - \epsilon \tag{3.1}$$

for all k. By the translation invariance of the energy, without loss of generality we can take $y_k = 0$ for all k. Fix any $1 < q < \infty$, then $\{\rho_k\}$ is a bounded sequence in $L^q(\mathbb{R}^N)$, so there is a $\rho \in L^q(\mathbb{R}^N)$ such that ρ_k converges to ρ weakly in $L^q(\mathbb{R}^N)$.

First, we will verify that $\rho \in \mathcal{A}_m$. By (3.1),

$$\int_{\mathbb{R}^N} \rho(y) \, dy = m.$$

To see that $\rho \geq 0$, consider the set $A = \{\rho < 0\}$. Suppose $|A| > 0$, and if $|A| = \infty$, replace A with a subset that has finite measure. Then, by weak convergence we have

$$0 \leq \lim_{k \to \infty} \int_A \rho_k(y) \, dy = \int_A \rho(y) \, dy < 0,$$

which is a contradiction. Therefore $|A| = 0$. Similarly, if we let $B = \{\rho > 1\}$, and assume $|B| > 0$, we can compute

$$|B| \geq \lim_{k \to \infty} \int_B \rho_k(y) \, dy = \int_B \rho(y) \, dy > |B|.$$

Combined, this means that $\rho \in \mathcal{A}_m$

Next, we will show that $\mathcal{E}[\rho] = \mathcal{E}_b[\rho]$ where $\mathcal{E}_b$ is the energy given by the kernel K_b. Let

$$C = \{x \in \text{supp}(\rho) : \exists y \in \text{supp}(\rho) \text{ such that } K_\infty(|x-y|) = \infty\}.$$

We need to show that C has measure zero. Suppose C has positive measure. If C has infinite measure replace it with a subset of finite measure. Since $\mathcal{E}[\rho_k] < \infty$ we must have

$$\int_C \rho_k = 0$$

for all k. But then, since ρ_k converges to ρ weakly in $L^q(\mathbb{R}^N)$ this implies that

$$\int_C \rho = 0.$$

Next, we will prove that $\lim_{k\to\infty} \mathcal{E}_b[\rho_k] = \mathcal{E}_b[\rho]$. Let

$$G_k(x) = K_b * \rho_k(x), \text{ and}$$
$$G(x) = K_b * \rho(x).$$

By (3.1), ρ_k also converges to ρ weakly in $L^1(\mathbb{R}^N)$. Then since K is bounded, this means that G_k converges to G pointwise.

Fix $\epsilon > 0$, and pick $R > 0$ such that

$$\int_{\mathbb{R}^N \setminus B(0,R)} \rho_k(y)\,dy \le \epsilon \tag{3.2}$$

for all k. Then compute

$$\begin{aligned}\mathcal{E}_b[\rho_k] - \mathcal{E}_b[\rho] &= \int_{\mathbb{R}^N} G_k(x)\rho_k(x)\,dx - \int_{\mathbb{R}^N} G(x)\rho(x)\,dx \\ &= \int_{B(0,R)} (G_k(x) - G(x))\rho_k(x)\,dx + \int_{B(0,R)} G(x)(\rho_k(x) - \rho(x))\,dx \\ &\quad + \int_{\mathbb{R}^N\setminus B(0,R)} G_k(x)\rho_k(x)\,dx. - \int_{\mathbb{R}^N\setminus B(0,R)} G(x)\rho(x)\,dx.\end{aligned} \tag{3.3}$$

The first term in (3.3) converges to 0 as $k \to \infty$ by the bounded convergence theorem. The second term also converges to 0 as $k \to \infty$ since $G|_{B(0,R)}$ is an admissible test function, noting that $||G||_{L^\infty(\mathbb{R}^N)} \le m||K_b||_{L^\infty(\mathbb{R}^N)}$ Thus, for large enough k we have

$$|\mathcal{E}_b[\rho_k] - \mathcal{E}_b[\rho]| \le 2\epsilon + 2m||K_b||_{L^\infty(\mathbb{R}^N)}\epsilon.$$

□

The next lemma deals with the case where we get a dichotomous subsequence after applying the concentration compactness principle.

Lemma 3.4. *Let the kernel $K = K_b + K_\infty$ where K_b is bounded and satisfies $(\mathbf{K1}) - (\mathbf{K3})$, and K_∞ takes values in $\{0, \infty\}$. Moreover assume that K has compact support. Let $\{\rho_k\}$ be a dichotomous sequence in $\mathcal{A}_m$ with $\mathcal{E}[\rho_k] < \infty$. Then, for some $0 < \alpha < m$ there are sequences $\{\rho_k^1\}$ and $\{\rho_k^2\}$ in $\mathcal{A}_\alpha$ and $\mathcal{A}_{m-\alpha}$, respectively, such that*

$$\lim_{k\to\infty} \left(\mathcal{E}[\rho_k] - \mathcal{E}[\rho_k^1] - \mathcal{E}[\rho_k^2] \right) = 0. \tag{3.4}$$

Moreover if ρ_k is a minimizing sequence, then

$$\liminf_{k\to\infty} \mathcal{E}[\rho_k^1] = \inf_{\rho\in\mathcal{A}_\alpha} \mathcal{E}[\rho], \tag{3.5}$$

and

$$\liminf_{k\to\infty} \mathcal{E}[\rho_k^2] = \inf_{\rho\in\mathcal{A}_{m-\alpha}} \mathcal{E}[\rho]. \tag{3.6}$$

Proof. Fix $R > 0$ such that $K(r) = 0$ for $r \geq R$. By dichotomy, after perhaps passing to a subsequence, we can find radii $R_k > 0$ and points $y_k \in \mathbb{R}^N$ so that the sequences

$$\rho_k^1 = \rho_k|_{B(y_k,R_k)}, \quad \text{and} \quad \rho_k^2 = \rho_k|_{\mathcal{R}^N \setminus B(y_k,R_k+R)} \tag{3.7}$$

satisfy

$$\lim_{k\to\infty} \int \rho_k^1 = \alpha, \quad \text{and} \tag{3.8}$$

$$\lim_{k\to\infty} \int \rho_k^2 = m - \alpha \tag{3.9}$$

for some $0 < \alpha < m$. Note that this is not exactly how the dichotomy condition is stated in lemma3.1, but this follows from applying it for a sequence $\epsilon_k \to 0$ as $k \to 0$ and passing to a further subsequence. Note that since $K(r) = 0$ for $r \geq R$,

$$\mathcal{E}[\rho_k^1, \rho_k^2] = 0.$$

So,

$$\mathcal{E}[\rho_k] = \mathcal{E}[\rho_k^1] + \mathcal{E}[\rho_k^2] + \mathcal{E}[\rho - \rho_k^1 - \rho_k^2] + 2\mathcal{E}[\rho_k^1 + \rho_k^2, \rho - \rho_k^1 - \rho_k^2] \tag{3.10}$$

Note that as $k \to \infty$ the last two terms approach 0 since

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \rho_k - \rho_k^1 - \rho_k^2 = m - \alpha - (m - \alpha) = 0$$

and $\mathcal{E}[\rho_k] < \infty$. Next, we modify $\{\rho_k^1\}$ and $\{\rho_k^2\}$ so that they have constant mass. To do so for ρ_k^1, we either decrease the radius R_k if ρ_k^1 has too much mass, or add a small piece far away if ρ_k^1 has too little mass. For each k, let

$$\overline{\rho_k^1} = \rho_k|_{B(y_k,\overline{R_k^1})} + \mathbb{1}_{B(x_k,r_k^1)},$$

where $\overline{R_k^1} \le R_k$, $r_k^1 \ge 0$, and $x_k \in \mathbb{R}^N$ are chosen so that $\overline{\rho_k^1} \in \mathcal{A}_\alpha$, and $|x_k - y_k| > R_k + r_k^1 + R$, and $r_k^1 \to 0$ as $k \to \infty$. Similarly for ρ_k^2, we either increase the radius R_k to decrease the mass, or add a small ball centered at y_k to increase the mass. That is we let

$$\overline{\rho_k^2} = \rho_k|_{\mathbb{R}^N \setminus B(y_k \overline{R_k^2}+R)} + \mathbb{1}_{B(y_k,r_k^2)},$$

where $\overline{R_k^2} \ge R_k$, and $r_k^2 \ge 0$ are chosen so that $\overline{\rho_k^2} \in \mathcal{A}_{m-\alpha}$, and $r_k^2 \to 0$ as $k \to \infty$. It is straightforward to check, using the fact that $||\rho_k^i - \overline{\rho_k^i}||_{L^1(\mathbb{R}^N)} \to 0$ as $k \to \infty$ for $i = 1, 2$ that

$$\lim_{k\to\infty} \left(\mathcal{E}[\rho_k^i] - \mathcal{E}[\overline{\rho_k^i}] \right) = 0.$$

Then, combining this with (3.10),

$$\lim_{k\to\infty} \left(\mathcal{E}[\rho_k] - \mathcal{E}[\overline{\rho_k^1}] - \mathcal{E}[\overline{\rho_k^2}] \right) = 0. \tag{3.11}$$

Now, suppose ρ_k is a minimizing sequence. By subadditivity (Lemma 2.5), we have

$$\lim_{k\to\infty} \mathcal{E}[\rho_k] = \inf_{\rho\in\mathcal{A}_m} \mathcal{E}[\rho] \le \inf_{\rho\in\mathcal{A}_\alpha} \mathcal{E}[\rho] + \inf_{\rho\in\mathcal{A}_{m-\alpha}} \mathcal{E}[\rho].$$

By (3.11)

$$\begin{aligned} \lim_{k\to\infty} \mathcal{E}[\rho_k] &= \lim_{k\to\infty} \left(\mathcal{E}[\overline{\rho_k^1}] + \mathcal{E}[\overline{\rho_k^2}] \right) \\ &\ge \liminf_{k\to\infty} \mathcal{E}[\overline{\rho_k^1}] + \liminf_{k\to\infty} \mathcal{E}[\overline{\rho_k^2}]. \end{aligned}$$

Then we are done since

$$\liminf_{k\to\infty} \mathcal{E}[\overline{\rho_k^1}] \ge \inf_{\rho\in\mathcal{A}_\alpha} \mathcal{E}[\rho], \quad \text{and} \quad \liminf_{k\to\infty} \mathcal{E}[\overline{\rho_k^1}] \ge \inf_{\rho\in\mathcal{A}_{m-\alpha}} \mathcal{E}[\rho].$$

□

Proof of Theorem 1.1. Let $\{\rho_k\}$ be a minimizing sequence.

Then by concentration compactness (Lemma 3.1), we can pass to a subsequence to a obtain a minimizing sequence which, abusing notation slightly, I will also label $\{\rho_k\}$ that is either tight up to translation, vanishing, or dichotomous. By Lemma [3.2] a minimizing sequence cannot be vanishing since we know $\inf_{\rho\in\mathcal{A}_m}\mathcal{E}[\rho] < 0$.

If the sequence is tight up to translation, we can apply Lemma 3.3, and conclude that there is a $\rho \in \mathcal{A}_m$ such that $\lim_{k\to\infty}\mathcal{E}[\rho_k] = \mathcal{E}[\rho]$, and so ρ is a minimizer and we are done.

If the sequence is dichotomous, then by Lemma 3.4 there are $\rho_k \in \mathcal{A}_{\alpha_1}$, and $\rho_k \in \mathcal{A}_{\alpha_2}$, with $\alpha_1 + \alpha_2 = m$, and

$$\lim_{k\to\infty}\mathcal{E}[\rho_k] = \lim_{k\to\infty}\left(\mathcal{E}[\rho_k^1] + \mathcal{E}[\rho_k^2]\right).$$

We then apply concentration compactness principle to $\{\rho_k^1\}$ and $\{\rho_k^2\}$, noting that neither can have a vanishing subsequence. Applying Lemmas 3.3 and 3.4, and then concentration again iteratively, we eventually obtain (after passing to subsequences, and some relabelling) sequences $\rho_k^1, \rho_k^2, \ldots \rho_k^l$, which are tight up to translation with $\rho_k^i \in \mathcal{A}_{\alpha_i}$, $\alpha_1 + \cdots + \alpha_l = m$, and

$$\lim_{k\to\infty}\mathcal{E}[\rho_k] = \lim_{k\to\infty}\left(\mathcal{E}[\rho_k^1] + \cdots + \mathcal{E}[\rho_k^l]\right).$$

Note that this process of applying concentration compactness iteratively must end after finitely many steps, because the α_i's cannot become arbitrarily small. This is because by Lemma 2.1 $\inf_{\rho\in\mathcal{A}_\alpha}\mathcal{E}[\rho]$ is attained by the indicator of a ball if $\alpha \le m_0$. So for any $\alpha \le m_0$, and any dichotomous sequence $\{\eta_k\}$ in $\mathcal{A}_\alpha$, we would have

$$\liminf_{k\to\infty}\mathcal{E}[\eta_k] > \inf_{\rho\in\mathcal{A}_\alpha}\mathcal{E}[\rho],$$

which contradicts Lemma 3.4. To conclude, by Lemma 3.3 there exist $\rho^1 \in \mathcal{A}_{\alpha_1}, \ldots \rho^l \in \mathcal{A}_{\alpha_l}$ such that

$$\lim_{k\to\infty}\mathcal{E}[\rho_k] = \mathcal{E}[\rho^1] + \cdots + \mathcal{E}[\rho^l].$$

Each ρ^i is a minimizer of $\mathcal{E}$ in $\mathcal{A}_{\alpha_i}$, and so by Lemma 2.3, they each have compact support. So, we can construct a minimizer for the truncated problem by letting $\rho = \rho(z_1 + \cdot) + \ldots \rho^l(z_l + \cdot)$ for suitably chosen $z_1, \ldots, z_l \in \mathbb{R}^N$.

Finally, assume additionally that K is strictly increasing on $[0, a]$, and $a + w \le W - 2w$. Let ρ be any minimizing density. By Lemma 3.5, any two points x, y points in the support of ρ satisfy the distance condition $|x - y| \notin [a + w, a + W - w]$. Then since $a + w \le W - 2w = (a + W -$

$w) - (a+w)$, suppρ can be decomposed into pieces each with diameter at most $a+w$. On each of these pieces, the energy is uniquely minimized (up to translations and modifications on sets of measure zero) by the indicator of a ball. □

3.3 SEPARATION LEMMA

So far we have shown existence of minimizers for compactly supported kernels, to understand their shape in this section we will prove a separation lemma, which applies as long as the repulsive barrier is high and wide enough relative to the attractive well. More precisely, the next lemma will allow us to conclude that minimizers of the energy $\mathcal{E}$ cannot have points in their support within a certain range of “forbidden" distances. As long as this range of forbidden distances is wide enough, this will force the support of a minimizing energy to be separated into disjoint pieces which do not interact with one another.

Lemma 3.5. *Let $\rho \in \mathcal{A}_m$. Suppose $x_1, x_2 \in \mathbb{R}^N$ satisfy*

1. $K * \rho(x_1), K * \rho(x_2) \leq 0$, *and*
2. $a + w \leq |x_1 - x_2| \leq a + W - w$.

Then, $x_1, x_2 \notin \text{supp}(\rho)$.

Proof. For $i = 1, 2$ we have

$$\begin{aligned} 0 \geq K * \rho(x_i) &= \int_{|x_i - y| \leq w} K(|x_i - y|)\rho(y)\, dy + \int_{|x_i - y| > w!} K(|x_i - y|)\rho(y)\, dy \\ &\geq -d \int_{|x_i - y| \leq w} \rho(y)\, dy + h \int_{a \leq |x_i - y| \leq a+W} \rho(y)\, dy. \end{aligned}$$

Rearranging, we obtain

$$\int_{a \leq |x_i - y| \leq a+W} \rho(y)\, dy \leq \frac{d}{h} \int_{|x_i - y| \leq w} \rho(y)\, dy \tag{3.12}$$

This estimate says that ρ cannot have too much mass in the annulus ...... relative to its mass near x_i. Also note that for $i \neq j$

$$\int_{|x_i - y| \leq w} \rho(y)\, dy \leq \int_{a \leq |x_j - y| \leq a+W} \rho(y)\, dy \tag{3.13}$$

since $a + w \leq |x_1 - x_2| \leq a + W - w$. Then, for $i \neq j$ we alternate between using (3.12) and (3.13) to obtain

$$\begin{aligned}\int_{|x_i - y| \leq w} \rho(y)\, dy &\leq \int_{a \leq |x_j - y| \leq a+W} \rho(y)\, dy \\ &\leq \frac{d}{h} \int_{|x_j - y| \leq w} \rho(y)\, dy \\ &\leq \frac{d}{h} \int_{a \leq |x_i - y| \leq a+W} \rho(y)\, dy \\ &\leq \left(\frac{d}{h}\right)^2 \int_{|x_i - y| \leq w} \rho(y)\, dy.\end{aligned}$$

Then since $\rho \geq 0$ and $d < h$, we must have

$$\int_{|x_i - y| \leq w} \rho(y)\, dy = 0.$$

□

3.4 GENERALIZED MINIMIZERS CONSIST OF TUPLES OF INDICATORS OF BALLS

In this section we will prove Theorem 1.2. To do this, we first prove the existence of "generalized minimizers," and then use the separation lemma (lemma 3.5) to conclude that in fact these generalized minimizers consist of a tuple of indicators of balls. The following notion of a generalized minimizer comes from [KMN16]. It allows us to deal with the fact that minimizing sequences may have many pieces that become infinitely far apart.

Definition 3.6. A **generalized minimizer** of $\mathcal{E}$ in $\mathcal{A}_m$ is a collection of densities $(\rho_1, \ldots, \rho_M)$, for some $M \in \mathbb{N}$, and each $\rho_i \in \mathcal{A}_{m_i}$ is a minimizer for $\mathcal{E}$ in $\mathcal{A}_{m_i}$. Additionally, $\sum_{i=1}^{M} m_i = m$, and

$$\inf_{\rho \in \mathcal{A}_m} \mathcal{E}[\rho] = \sum_{i=1}^{M} \mathcal{E}[\rho_i]. \tag{3.14}$$

Let $\overline{K} = K \cdot \mathbb{1}_{[0,a+W]}$, and denote

$$\overline{\mathcal{E}}[\rho] = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \overline{K}(|x-y|)\rho(x)\rho(y)\, dx\, dy. \tag{3.15}$$

Theorem 3.7 (Existence of Generalized Minimizers)**.** *Let* $K : [0, \infty) \to \mathbb{R}$ *be a bounded kernel satisfying* $(\mathbf{K1}) - (\mathbf{K5})$*. Further, assume K is strictly increasing* *on* $[0, a]$ *and* $a + w \leq W - 2w$. *then, generalized minimizers exist and up to sets of measure zero any generalized minimizer has the form*

$$\rho = (\mathbb{1}_{B(x_i, r_i)})_{i=1}^{k} \tag{3.16}$$

for some centres $x_i \in \mathbb{R}^N$ *and radii* $r_i > 0$ *such that* $\mathcal{E}[\mathbb{1}_{B(x_i,r_i)}, \mathbb{1}_{B(x_j,r_j)}] = 0$ *for* $i \neq j$.

Proof. Let $\rho \in \mathcal{A}_m$ be a minimizer for the truncated problem. Note that $\bar{K} \leq K$, so $\bar{\mathcal{E}}[\eta] \leq \mathcal{E}[\eta]$, for any $\eta \in \mathcal{A}_m$.

By (**??**), we may write

$$\rho = \sum_{i=1}^{k} \mathbb{1}_{B(x_i, r_i)}, \tag{3.17}$$

for some centres $x_i \in \mathbb{R}^N$ and radii $r_i > 0$ such that $\mathcal{E}[\mathbb{1}_{B(x_i,r_i)}, \mathbb{1}_{B(x_j,r_j)}] = 0$ for $i \neq j$. From this, we may construct a minimizing sequence for $\mathcal{E}$, by picking new centers $x_{i,l}$ so that $|x_{i,l} - x_{j,l}| \to \infty$ as $k \to \infty$. So,

$$\inf_{\rho \in \mathcal{A}_m} \mathcal{E}[\rho] = \inf_{\rho \in \mathcal{A}_m} \bar{\mathcal{E}}[\rho].$$

So, by taking $\rho_i = \mathbb{1}_{B(x_i,r_i)}$, we obtain a generalized minimizer for ρ □

4

SIZE OF DROPLETS

4.1 LINEAR GROWTH

We will show that the minimal energy

$$E(m) = \inf_{\rho \in \mathcal{A}_m} \mathcal{E}[\rho]$$

grows linearly in m.

Theorem 4.1. *Let $K : [0, \infty) \to \mathbb{R}$ be a bounded kernel satisfying $(\mathbf{K1}) - (\mathbf{K5})$. Further, assume K is strictly increasing on $[0, a]$ and $a + w \leq W - 2w$. For each $m > 0$, let $E(m) = \inf_{\rho \in \mathcal{A}_m} \mathcal{E}[\rho]$ and $g(m)$ be the energy of the ball with mass m. Then*

$$\lim_{m \to \infty} \frac{E(m)}{m} = \min_{m > 0} \frac{g(m)}{m}. \tag{4.1}$$

Proof. The first step is to show that the limit in the left hand side of (4.1) exists. By Fekete's subadditive Lemma (see [Kuc09] Theorem 16.2.9) we have

$$\lim_{m \to \infty} \frac{E(m)}{m} = \inf_{m > 0} \frac{E(m)}{m}.$$

Next, by Theorem 1.2 given any $m > 0$ we may write $E(m) = g(m_1) + g(m_2) + \cdots + g(m_k)$ for some positive numbers $m_1, \ldots m_k$ such that $m_1 + \cdots + m_k = m$. Then,

$$\frac{E(m)}{m} = \frac{g(m_1) + \cdots + g(m_k)}{m} \geq \frac{(m_1 + \cdots + m_k) \inf_{m>0} \frac{g(m)}{m}}{m} = \inf_{m>0} \frac{g(m)}{m}$$

On the other hand, $E(m) \leq g(m)$ for all m, so

$$\inf_{m>0} \frac{E(m)}{m} = \inf_{m>0} \frac{g(m)}{m}.$$

To conclude we need to show that the infimum on the right is actually a minimum. Note that g is continuous on $(0, \infty)$, and is positive for large values of m. Also, since $K(r) \geq K(0)$ for all $r > 0$,

$$g(m) \geq K(0) m^2.$$

Then, using the squeeze theorem, and the fact that g is negative for small values of m,

$$\lim_{m \to 0^+} \frac{g(m)}{m} = 0.$$

□

4.2 POWER LAW KERNELS

Theorem 3.7 says that generalized minimizers are tuples of indicators of balls. The linear growth estimate (4.1) suggests that when m is larger enough, the droplets should all be relatively close in size. In this section we will consider kernels that are power laws near 0, to further explore the question of droplet size. We will see that generalized minimizers for such kernels consist of many indicators of balls of one size, and possibly one ball of a smaller size.

Let

$$K(r) = \begin{cases} r^p - d, & \text{if } 0 \leq r \leq a, \\ f(r), & \text{if } r > a, \end{cases} \tag{4.2}$$

where $f : (a, \infty) \to \mathbb{R}$ is a non-negative function such that $f(r) \geq a^p + d$ for all $a < r \leq w + a$ and such that $\lim_{r \to \infty} f(r) = 0$. To understand generalized minimizers for such a kernel, we first compare the indicator of one ball versus two balls.

Lemma 4.2. *Let $g(m) = \mathcal{E}[\mathbb{1}_{ball}]$, and set $f(t) = g(tm) + g((1-t)m)$ for $0 \leq t \leq 1/2$. Then, there are numbers $0 < m_0 < m_1$ such that*

1. *If $0 < m \leq m_0$, then f is minimized at $t = 0$*
2. *If $m_0 < m < m_1$ f is minimized at some $0 < t_0 < 1/2$*
3. *If $m \geq m_1$, then f is minimized at $t = 1/2$.*

Moreover,

$$m_0 = \left(\frac{2d}{C_{n,p}(2 + p/n)} \right)^{n/p}, \text{ and} \tag{4.3}$$

$$m_1 = \left(\frac{2^{1+p/n} d}{C_{n,p}(2 + p/n)(1 + p/n)} \right)^{n/p}, \tag{4.4}$$

where

$$C_{n,p} = \frac{1}{|B_1|^{2+p/n}} \int_{B_1} \int_{B_1} |x - y|^p \, dx \, dy.$$

Proof. Let $m > 0$ and $r > 0$ such that $m = |B_r|$. Then,

$$\begin{aligned} g(m) = \mathcal{E}[\mathbb{1}_{B_r}] &= \int_{B_r}\int_{B_r} \left(|x-y|^p - d\right)\, dx\, dy \\ &= r^{2n+p}\int_{B_1}\int_{B_1} |x-y|^p\, dx\, dy\; - dm^2 \\ &= C_{n,p} m^{2+p/n} - dm^2. \end{aligned}$$

We seek to minimize $f(t) = g(tm) + g((1-t)m)$ for $t \in [0, 1/2]$. Compute

$$\begin{aligned} f'(t) =& m^2 \left(C_{n,p}(2+p/n) m^{p/n} (t^{1+p/n} - (1-t)^{1+p/n}) - 2d(2t-1) \right) \\ f''(t) =& m^2 \left(C_{n,p}(2+p/n)(1+p/n) m^{p/n} (t^{p/n} + (1-t)^{p/n}) - 4d \right) \end{aligned}$$

It is straightforward to check that $t \mapsto t^{p/n} + (1-t)^{p/n}$ is decreasing on $[0, 1/2]$ if $p/n > 1$. So, f'' is decreasing, which means f' is concave. Note that $f'(1/2) = 0$, so f can have at most one critical value in the interval (0,1/2). To determine whether such a critical value exists, we need to look at the signs of $f'(0)$ and $f''(1/2)$.

$$\begin{aligned} f'(0) =& m^2 C_{n,p}(2+p/n)(m_0^{p/n} - m^{p/n}), \\ f''(1/2) =& 2^{1-p/n} m^2 C_{n,p}(2+p/n)(1+p/n)(m^{p/n} - m_1^{p/n}). \end{aligned}$$

Let m_0 and m_1 be as defined in Lemma 4.2. Then,

1. If $0 < m \leq m_0$, then $f'(0) \geq 0$, and so f has no critical values in $(0, 1/2)$. Moreover, $f'(t) \geq 0$ for all $t \in [0, 1/2]$. So, f has its minimum at $t = 0$.

2. If $m_0 < m < m_1$, then $f'(0) < 0$ and $f''(1/2) < 0$. Then there is a point $0 < t_0 < 1/2$ where $f(t_0) = 0$. Moreover, $f'(t) < 0$ on $(0, t_0)$ and $f'(t) > 0$ on $(t_0, 1/2)$. So, f has its minimum at $t = t_0$.

3. If $m \geq m_1$, then $f''(1/2) \geq 0$, which means $f'(t) \leq 0$ for all $t \in [0, 1/2]$. Thus f has its minimum at $t = 1/2$

Note, it is straightforward to check that $m_0 < m_1$

□

Lemma 4.3. *Let $m > 0$ and $(\rho_i)_{i=1}^k$ be a generalized minimizer. Then there exist radii $r_1, r_2 > 0$ such that each $\rho_i = \mathbb{1}_{B_{r_1}}$ or $\mathbb{1}_{B_{r_2}}$.*

Proof. Let $m > 0$ $(\rho_i)_{i=1}^k$ be a generalized minimizer. Consider the minimization problem

$$\min_{\substack{t_1+\cdots+t_{k-1}<1\\ t_i>0}} g(t_1 m) + \dots g(t_{k-1}m) + g((1-t_1-\cdots-t_{k-1})m). \tag{4.5}$$

Then, $(\rho_i)_{i=1}^k$ corresponds to a minimizer $(t_1, \dots, t_{k-1})$ of (4.5). Computing the gradient yields

$$mg'(t_i m) - mg'((1-t_1-\cdots-t_{k-1})m) = 0$$

for each $i = 1, \dots, k-1$. In particular this means that

$$g'(t_i m) = g'(t_j m)$$

for each $i \neq j$. Note that g' strictly decreasing then increasing on $[0, \infty)$, so for any $y \in \mathbb{R}$ the equation $g'(x) = y$ has at most two solutions. □

Note that the only thing we used was that g is concave then convex so this argument applies more generally than to just the power law kernels.

Lemma 4.4. *Generalized minimizers have the form* $(\mathbb{1}_{B_s}, \mathbb{1}_{B_r}, \dots \mathbb{1}_{B_r})$*, for some* $s \leq r$.

Proof. In the previous Lemma we saw that generalized minimizers will consist of indicators of balls having at most two different sizes.

Let m_0 and m_1 be the thresholds from Lemma 4.2.

The claim follows by induction on the number of balls in a generalized minimizer. First, for a given $m > 0$ assume that a generalized minimizer consists of four balls, two of which have mass M, and the other two have size N. Then, by Lemma 4.2 $2M \geq m_1$, $2N \geq m_1$, and $M + N \leq m_1$. So,

$$2m_1 \leq 2M + 2N \leq 2m_1,$$

which means $M = N$.

For the inductive step, assume the claim is true for generalized minimizers that consist of k balls. Then, let $(\rho_i)_{i=1}^{k+1}$ be a generalized minimizer that consists of $k+1$ indicators balls, $k_M > 1$ of mass M and k_N of mass N. Then, removing on of the balls of mass M will give a generalized minimizer for the problem with with smaller mass, and so $k_N = 1$.

To see that the single ball must have smaller mass, consider a generalized minimizer $(\mathbb{1}_{B_s}, \mathbb{1}_{B_r}, \dots \mathbb{1}_{B_r})$, for some $s \neq r$. By Lemma 4.2, $|B_s| + |B_r| < m_1$ and $2|B_r| \geq m_1$, so $|B_s| < |B_r|$. □

5 TOY MODEL

In this chapter we analyze a toy model which emphasizes one way that the supports of densities may be forced to break apart into well separated pieces. In these models there is a range for distances that are forbidden between points in the support of an admissible density. We will prove Theorem 1.3.

In the case of kernels satisfying the well-barrier conditions **(K1)-(K5)**, the separation into distinct droplets was driven by the same mechanism that forces separation in the toy model. If the repulsive barrier was high and wide enough relative to the attractive well, then they were a range of distances which were disallowed between points in the support of a minimizing density.

For $w > 0$, define the kernel $K_w : [0, \infty) \to \mathbb{R} \cup \{\infty\}$ by

$$K_w(r) = \begin{cases} -1, & \text{if } 0 \leq r \leq 1, \\ +\infty, & \text{if } 1 < r < 1 + w, \\ 0, & \text{if } r \geq 1 + w, \end{cases}$$

Denote the corresponding interaction energy by $\mathcal{E}_w$. This kernel is attractive for distances between 0 and 1, followed by a range of strictly forbidden distances. Understanding the geometry of the supports of densities for which $\mathcal{E}_w[\rho]$ is finite will be important in our analysis for a family of more general kernels. Later, we will see in fact that suitable assumptions on the attractive well and repulsive barrier in fact create a forbidden region associated to the support of minimizers.

The parameter w represents the width of the range of forbidden distances. If w is large, it turns out that admissible densities decompose into pieces that are separated by large distances. If w is small, admissible densities can have many configurations, so the first step in minimizing the energy is to understand the geometry of admissible densities. In dimension 1 we show that if the mass m is an integer then an optimal density consists of intervals of diameter 1. The key detail is to observe that non-separated configurations cannot allow mass to concentrate too much.

The most extreme situation occurs when $w = 0$. For $w = 0$, the interaction energy is given by

$$\mathcal{E}_0[\rho] = \begin{cases} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \mathbb{1}_{[0,1)}(|x-y|)\, dx\, dy, & |x-y| \neq 1 \text{ whenever } \rho(x), \rho(y) > 0 \\ +\infty, & \text{else .} \end{cases}$$

Again in the 1-dimensional case, if the mass m is an integer then an optimal density consists of intervals of diameter 1. For non-integer mass, there are examples of configurations having the same energy as many intervals of length 1, and one smaller interval, although it is not clear yet whether this is the minimal energy.

5.1 LARGE RANGE OF UNFAVOURABLE DISTANCES

Theorem 5.1. *Assume $w \geq 1$. Let $m > 0$, and write $m = n|B(0,1/2)| + \alpha$ for some natural number n and $0 \leq \alpha < |B(0,1/2)|$. Then,*

$$\inf_{\rho \in \mathcal{A}_m} \mathcal{E}_w[\rho] = -n|B(0,1/2)|^2 - \alpha^2.$$

Lemma 5.2. *Assume $w \geq 1$. If $m \leq |B(0,1/2)|$, then*

$$\inf_{\rho \in \mathcal{A}_m} \mathcal{E}_w[\rho] = -m^2,$$

and the infimum is attained by the indicator of a ball, or any density which is supported on a set of diameter no larger than 1. These are the only minimizers

Proof. Fix any $\rho \in \mathcal{A}_m$. Note that $K_w(r) \geq -1$ for all r. So,

$$\mathcal{E}_w[\rho] \geq -\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \rho(x)\rho(y)\; dx\; dy = -m^2.$$

It is obvious that any $\rho \in \mathcal{A}_m$ that is supported on a set of diameter no greater than 1 will attain this. □

The key ingredient in the proof of Theorem 5.1 is the following decomposition Lemma. We write ρ as a sum of densities whose supports are of distance at least 2 from each other, and which each have small mass.

Lemma 5.3 (Decomposition Lemma for $w \geq 1$)**.** *Let $w \geq 1$. Let $\rho \in \mathcal{A}_m$ have finite energy $\mathcal{E}_w[\rho]$. Then we may decompose ρ into a finite sum of densities*

$$\rho = \sum_{i=1}^{k} \rho_i$$

satisfying the following:

$$||\rho_i||_{L^1(\mathbb{R}^d)} \leq |B(0,1/2)| \text{ for each } i, \text{and} \tag{5.1}$$

$$\text{dist}(\text{supp}(\rho_i), \text{supp}(\rho_j)) \geq 1 + w, \text{ for } i \neq j \tag{5.2}$$

Proof. Let $\rho \in \mathcal{A}_m$ have finite energy. We will first assume that ρ is compactly supported. First observe that we may define an equivalence relation on $\text{supp}(\rho)$ by $x \sim y$ if and only if $|x - y| \leq 1$. It is obvious that this relation is reflexive and symmetric. To see that it is transitive, let $x, y, z \in \text{supp}(\rho)$ be such that $|x - y| \leq 1$ and $|y - z| \leq 1$. By the triangle inequality $|x - z| \leq 2 \leq 1 + w$. Then $|x - z| \leq 1$ since $\mathcal{E}_w[\rho] < \infty$. Thus, we may decompose $\text{supp}(\rho)$ into sets of diameter at most 1, noting that each of these must be separated by a distance of at least $1 + w$.

If ρ is not compactly supported, then there is a radius $R > 0$ so that

$$\int_{B(0,R)} \rho(x)dx \geq m - |B(0,1/2)|.$$

Let

$$A = \left(\cup_{x \in \text{supp}(\rho) \cap B(0,R)} B(x,1)\right) \cup B(0,R).$$

Set $\rho_0 = \rho|_{\mathbb{R}^d \setminus A}$. Our choice of A guarantees that the supports of ρ_0 and $\rho - \rho_0$ are at a distance of at least $1 + w$ from each other. Also, $||\rho_0||_{L^1(\mathbb{R}^d)} \leq |B(0,1/2)|$ and $\rho - \rho_0$ is compactly supported, so the above argument can be used to decompose $\rho - \rho_0$.

□

Proof of Theorem 5.1. Let $w \geq 1$. Let $m = n|B(0,1/2)| + \alpha$. Let $\rho \in \mathcal{A}_m$ have finite energy. Decompose ρ as in Lemma 5.3. Then,

$$\mathcal{E}_w[\rho] = \sum_{i=1}^{k} \mathcal{E}_w[\rho_i] + \sum_{i \neq j} \mathcal{E}_w[\rho_i, \rho_j].$$

The second sum is 0 by (5.2), and we may apply Lemma 5.2 to the first sum since each ρ_i has mass at most $|B(0,1/2)|$. Thus,

$$\begin{aligned}\mathcal{E}_w[\rho] &\geq \sum_{i=1}^{k} \mathcal{E}[\rho_i] \\ &\geq -\sum_{i=1}^{k} ||\rho_i||^2_{L^1(\mathbb{R}^d)}.\end{aligned}$$

Let $m_i = ||\rho_i||_{L^1(\mathbb{R}^d)}$. We seek to minimize $-\sum_{i=1}^k m_i^2$ with the constraints that

$$\sum_{i=1}^k m_i = m, \text{ and } 0 \le m_i \le |B(0,1/2)| \text{ for } i = 1, \dots k.$$

This amounts to minimizing a concave function over a polytope, and it is now straightforward to check that

$$\mathcal{E}_w[\rho] \ge -n|B(0,1/2)|^2 - \alpha^2.$$

Finally, to conclude that

$$\inf_{\rho \in \mathcal{A}_m} \mathcal{E}_w[\rho] = -n|B(0,1/2)|^2 - \alpha^2,$$

set

$$\rho = \sum_{i=1}^n \mathbb{1}_{B(x_i,1/2)} + \mathbb{1}_{B(x_0,r)},$$

where r is chosen so $\alpha = |B(0,r)|$, and the points $x_i \in \mathbb{R}^d$ are chosen so that $|x_i - x_j| > 2 + w$. □

5.2 SMALLER RANGE OF UNFAVOURABLE DISTANCES

We now consider the case where $0 < w < 1$, and dimension $N = 1$. In this case we show that for integer valued mass m, the interaction energy is minimal only for indicators of unions of separated intervals of length 1.

Theorem 5.4. *Let $0 < w < 1$, $N = 1$, and $m > 0$ be an integer. Then*

$$\inf_{\rho \in \mathcal{A}_m} \mathcal{E}_w[\rho] = -m.$$

Moreover, $\rho \in \mathcal{A}_m$ is a minimizer if and only if $\rho = \mathbb{1}_{I_1 \cup \dots \cup I_m}$ for some intervals $I_1, \dots, I_k$ each of length 1 such that $\operatorname{dist}(I_i, I_j) > 1$ for $i \neq j$.

To prove this, we need the following Lemma, which says that admissible densities ρ cannot fill more than half of interval of length 2. Moreover, it says that if ρ is spread out in an interval of length 2, then it fills strictly less than half of the interval.

Lemma 5.5. *Let $0 < w < 1$, and $N = 1$. Suppose ρ has finite energy $\mathcal{E}_w[\rho]$. If*

$$\operatorname{diam}(\operatorname{supp}(\rho) \cap [x-1, x+1]) > 1,$$

then

$$|\operatorname{supp}(\rho) \cap [x-1, x+1]| \le 1 - w$$

Proof. Let $0 < w < 1$, and $N = 1$. Suppose ρ has finite energy $\mathcal{E}_w[\rho]$. Let x be such that

$$\operatorname{diam}(\operatorname{supp}(\rho) \cap [x-1, x+1]) > 1.$$

First, consider the case that $\operatorname{supp}(\rho) \cap [x-1, x+1])$ is a finite union of disjoint intervals $I_1, \ldots, I_k$. First note that $(I_j \pm 1) \cap \operatorname{supp}(\rho) = \emptyset$ for each j. In particular, $|\operatorname{supp}(\rho) \cap [x-1, x]| \leq 1 - |\operatorname{supp}(\rho) \cap [x, x+1]|$. From the assumption $\operatorname{diam}(\operatorname{supp}(\rho) \cap [x-1, x+1]) > 1$, there are two intervals, say $I_1 \subseteq \operatorname{supp}(\rho) \cap [x-1, x]$ and $I_2 \subseteq \operatorname{supp}(\rho) \cap [x, x+1]$, that are at distance at least $1 + w$ from each other. Since $\operatorname{supp}(\rho)$ consists of finitely many intervals we may also assume without loss of generality that I_2 is the interval closest to $I_1 + 1 + w$ from the right. See figure 5.1 for an illustration of of I_1, I_2, and their associated "forbidden regions", $I_1 + 1$ and $I_2 - 1$ which are drawn in red. Note that the interval $[\inf(I_2 - 1) - w, \inf(I_2 - 1)]$ is also "forbidden" since it contains numbers whose distance from I_2 are between 1 and $1 + w$. This interval is drawn in yellow in figure 5.1.

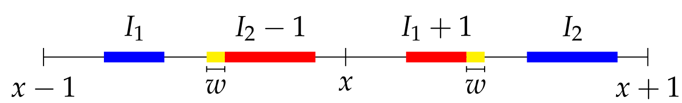

Figure 5.1: Two intervals I_1 and I_2 and their associated "forbidden regions"

Suppose $\operatorname{supp}(\rho) \cap [x, x+1] = \cup_{i=2}^{l} I_i$. Then the sets $I_i - 1$ for $i = 2, \ldots, l$ and $[\inf(I_2 - 1) - w, \inf(I_2 - 1)]$ are all disjoint and

$$|\operatorname{supp}(\rho) \cap [x-1, x]| \leq 1 - w - \sum_{i=2}^{l} |I_i| = 1 - w + |\operatorname{supp}(\rho) \cap [x, x+1]|,$$

and so

$$|\operatorname{supp}(\rho) \cap [x-1, x+1]| \leq 1 - w.$$

□

We are now ready to prove Theorem. 5.4

Proof of Theorem 5.4. Let $0 < w < 1$, and $m > 0$ be an integer. Let $\rho \in \mathcal{A}_m$ have finite energy $\mathcal{E}_w[\rho]$. Then by Lemma 5.5, $|\operatorname{supp}(\rho) \cap [x-1, x+1]| \leq 1$ for every $x \in \mathbb{R}$. Then, also using the fact that $0 \leq \rho \leq 1$, we have

$$\begin{aligned} K_w * \rho(x) &= -\int_{x-1}^{x+1} \rho(y)\, dy \\ &\geq -1. \end{aligned}$$

Then, we may compute the following lower bound for the energy

$$\begin{aligned}\mathcal{E}_w[\rho] &= \int_{\mathbb{R}} K_w * \rho(x)\rho(x)\, dx \\ &\geq -\int_{\mathbb{R}} \rho(x)\, dx. \\ &= -m.\end{aligned}$$

This value is obtained for any $\rho = \mathbb{1}_{I_1 \cup \cdots \cup I_m}$ where $I_1, \ldots, I_k$ are intervals of length 1 such that $\operatorname{dist}(I_i, I_j) > 1$ for $i \neq j$. On the other hand, if ρ does not take the form described above, then either

$$\operatorname{diam}(\operatorname{supp}(\rho) \cap [x-1, x+1] < 1 \text{ for all } x \in \operatorname{supp}(\rho)$$

or there is an $x \in \operatorname{supp}(\rho)$ such that

$$\operatorname{diam}(\operatorname{supp}(\rho) \cap [x-1, x+1] > 1.$$

In the first case, $|\operatorname{supp}(\rho) \cap [x-1, x+1]| < 1$ for all x, and so the inequalities in our earlier calculations become strict and $\mathcal{E}_w[\rho] > -m$. In the second case, we may apply Lemma 5.5 to conclude that there is an $x \in \operatorname{supp}(\rho)$ such that $|\operatorname{supp}(\rho)[x-1, x+1]| \leq 1 - w$. Then, there is a set of positive measure containing x on which $K_w * \rho > -1$, and so $\mathcal{E}_w[\rho] > -m$. □

5.3 SINGLE FORBIDDEN DISTANCE

We now consider the 1-dimensional problem with distance 1 forbidden. That is, we consider the energy

$$\mathcal{E}_0[\rho] = \begin{cases} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{[0,1)}(|x-y|)\rho(x)\rho(y)\, dx\, dy, & \text{if } |x-y| \neq 1 \text{ whenever } \rho(x), \rho(y) > 0 \\ +\infty, & \text{else.} \end{cases}$$

As before, if $\mathcal{E}_0[\rho] < \infty$, then we have for any $x \in \mathbb{R}$

$$|\operatorname{supp}(\rho) \cap [x-1, x+1]| \leq 1.$$

Note that we have not proved existence of minimizers for this energy. The strategy is the same as the proof of Theorem (ref), we consider a minimizing sequences of densities and apply concentration compactness iteratively. We use the following lemma to rule out the possibility of a minimizing sequence that vanishes.

Lemma 5.6. *Let $\{\rho_k\}$ be a sequence in $\mathcal{A}_m$ that is vanishing. Then*

$$\liminf_{k\to\infty} \mathcal{E}_0[\rho_k] \geq 0.$$

Proof. Without loss of generality, we may assume $\mathcal{E}_0[\rho_k] < \infty$, since we are proving a lower bound on $\liminf_{k\to\infty} \mathcal{E}_0[\rho_k]$. But now, for those densities with finite energy, the energy is given by the kernel $K(r) = \mathbb{1}_{[0,1)}(r)$, which is bounded, so lemma 3.2 applies. □

Lemma 5.7. *Let $\{\rho_k\}$ be a tight up to translation sequence in $\mathcal{A}_m$ such that $\mathcal{E}_0[\rho_k] < \infty$. Then there is a $\rho \in \mathcal{A}_m$ such that*

$$\lim_{k\to\infty} \mathcal{E}_0[\rho] = \mathcal{E}_0[\rho].$$

Proof. Since each ρ_k has finite energy, its energy is given by the bounded interaction kernel $K(r) = \mathbb{1}_{[0,1)}(r)$. Lemma 3.3 then says that there is a $\rho \in \mathcal{A}_m$ such that

$$\lim_{k\to\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{[0,1)}(|x-y|)\rho_k(x)\rho_k(y)\,dx\,dy = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{[0,1)}(|x-y|)\rho(x)\rho(y)\,dx\,dy.$$

Moreover, from the proof of lemma 3.3 we know that ρ_k converges to ρ in $L^q(\mathbb{R})$ for some $1 < q < \infty$. It might be the case that $\mathcal{E}_0[\rho] = \infty$. Let

$$A = \{x \in \{\rho > 0\} \;:\; \exists y \in \{\rho > 0\} \text{ such that } |x-y| = 1\}$$

be the set of points in $\{\rho > 0\}$ which are distance 1 from at least one other point of $\{\rho > 0\}$. We will show that A has positive measure. If $|A| = \infty$, replace it with a subset of finite measure. By the weak convergence of ρ_k to ρ in $L^q(\mathbb{R})$,

$$\lim_{k\to\infty} \int_A \rho_k = \int_A \rho.$$

But since each ρ_k has finite energy, $\int_A \rho_k = 0$, and so $\int_A \rho = 0$. The only way this is possible is if $|A| = 0$. Let $\overline{\rho} = \rho|_{\mathbb{R}\setminus A}$. Then, $\overline{\rho} \in \mathcal{A}_m$, $\mathcal{E}_0[\overline{\rho}] < \infty$ and

$$\lim_{k\to\infty} \mathcal{E}_0[\rho_k] = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{[0,1)}(|x-y|)\rho(x)\rho(y)\,dx\,dy = \mathcal{E}_0[\overline{\rho}].$$

□

Lemma 5.8. *$\{\rho_k\}$ be a dichotomous sequence in $\mathcal{A}_m$ such that $\mathcal{E}_0[\rho_k] < \infty$. Then, for some $0 < \alpha < m$ there are sequences $\{\rho_k^1\}$ and $\{\rho_k^2\}$ in $\mathcal{A}_\alpha$ and $\mathcal{A}_{m-\alpha}$, respectively, such that*

$$\lim_{k\to\infty} \left(\mathcal{E}_0[\rho_k] - \mathcal{E}_0[\rho_k^1] - \mathcal{E}_0[\rho_k^2] \right) = 0. \tag{5.3}$$

Moreover if ρ_k is a minimizing sequence, then

$$\liminf_{k\to\infty} \mathcal{E}_0[\rho_k^1] = \inf_{\rho\in\mathcal{A}_\alpha} \mathcal{E}_0[\rho], \tag{5.4}$$

and

$$\liminf_{k\to\infty} \mathcal{E}_0[\rho_k^2] = \inf_{\rho\in\mathcal{A}_{m-\alpha}} \mathcal{E}_0[\rho]. \tag{5.5}$$

Proof. Since each ρ_k has finite energy, its energy is given by the bounded, compactly supported interaction kernel $K(r) = \mathbb{1}_{[0,1)}(r)$. So by lemma 3.4, we obtain the sequence $\{\rho_k^1\}$ and $\{\rho_k^2\}$. We need to check that $\mathcal{E}_0[\rho_k^1], \mathcal{E}_0[\rho_k^2] < \infty$. From the proof of lemma 3.4, we know that

$$\rho_k^1 = \rho_k|_{(y_k-R_k, y_k+R_k)} + \mathbb{1}_{(x_k-r_{1,k}, x_k+r_{1,k})}$$

for some sequences $\{y_k\}$, $\{x_k\}$ of real numbers, and radii $R_k, r_{1,k}$ such that $|x_k - y_k| \geq R_k + r_{1,k} + 1$ and $r_{i,k} \to 0$, and $R_k \to infty$ as $k \to \infty$. This guarantees that $\mathcal{E}_0[\rho_k^1] < \infty$ at least for large enough k so that $r_{1,k} < 1/2$. We also know that

$$\rho_k^2 = \rho_k|_{\mathbb{R}\setminus(y_k-R_k-1, y_k+R_k+1)} + \mathbb{1}_{(y_k-r_{2,k}, y_k+r_{2,k})},$$

for some radii so that $r_{2,k} \to 0$ as $k \to \infty$. Again, as long as k is large enough so that $r_{2,k} < 1/2$ then $mcE_0[\rho_k^2] < \infty$ as needed. □

Now that we have the three analogous lemmas, to prove existence of minimizers, we may then follow the same exact strategy as before by taking a minimizing sequence and iteratively applying the concentration compactness principle.

Theorem 5.9. *Let $m > 0$, and write $m = k + a$ where $k \in \mathbb{Z}$ and $0 \leq a < 1$. Then,*

$$\inf_{\rho\in\mathcal{A}_m} \mathcal{E}_0[\rho] = -k - a^2. \tag{5.6}$$

To prove this, we will need three lemmas. The first says that we can find a minimizing density that saturates the density constraint, that is there is a set A such that $\mathbb{1}_A$ is a minimizer. Note that it is not true that minimizers must always saturate the density constraint. If m is not an integer, then there are minimizers so that the set $\{0 < \rho < 1\}$ has positive measure. The second lemma says that moreover, this set A can be assumed to be "saturated from the left", which we define by:

Definition 5.10. A bounded set A is *saturated from the left* if

$$|[\inf(A), \inf(A) + 2] \cap A| = 1.$$

That is, if the first interval of length two is half filled by the mass of A.

The third lemma tells us that a set A that is saturated from the left can be rearranged into the union of an interval of length one and another set which are well separated so that the rearranged set has less or equal energy.

Lemma 5.11. *Let $m > 0$. If $\rho \in \mathcal{A}_m$ is a minimizer for the energy $\mathcal{E}_0$, then there is a $\overline{\rho} \in \mathcal{A}_m$ such that the set $\{0 < \overline{\rho} < 1\}$ has measure zero, $\{\overline{\rho} > 0\} \subseteq \{\rho > 0\}$, and $\mathcal{E}_0[\overline{\rho}] = \mathcal{E}_0[\rho]$.*

Proof. Assume that the set $\{0 < \rho < 1\}$ has positive measure, since otherwise, we may take $\overline{\rho} = \rho$. We first assume that $\operatorname{diam}(\{0 < \rho < 1\}) \leq 1$. Let A be a subset of $\{0 < \rho < 1\}$ such that

$$|A| = \int_{\{0<\rho<1\}} \rho. \tag{5.7}$$

We will alter ρ by saturating the density on the set A, and removing all mass in $\{0 < \rho < 1\} \setminus A$. That is, set $B = \{0 < \rho < 1\} \setminus A$ and let

$$\overline{\rho} = \rho + (\mathbb{1}_A - \rho|_A) - \rho|_B. \tag{5.8}$$

By lemma, **??** there is a number λ such that $K * \rho(x) = \lambda$ for almost every $x \in \{0 < \rho < 1\}$. In particular, this means that

$$\begin{aligned}\mathcal{E}_0[\rho, \mathbb{1}_A - \rho|_A] - \mathcal{E}_0[\rho, \rho|_B] =& \int_{\mathbb{R}} K * \rho(x)(\mathbb{1}_A(x) - \rho|_A(x) - \rho|_B(x))\, dx \\ =& \lambda \int_{\mathbb{R}} (\mathbb{1}_A(x) - \rho|_{\{0<\rho<1\}}(x))\, dx \\ =& 0.\end{aligned}$$

We then compute

$$\begin{aligned}\mathcal{E}_0[\overline{\rho}] =& \mathcal{E}_0[\rho] + \mathcal{E}_0[\mathbb{1}_A - \rho|_A] + \mathcal{E}_0[\rho|_B] \\ &+ 2\mathcal{E}_0[\rho, \mathbb{1}_A - \rho|_A] - 2\mathcal{E}_0[\rho, \rho|_B] - 2\mathcal{E}_0[\mathbb{1}_A - \rho|_A, \rho|_B] \\ =& \mathcal{E}_0[\rho] + \mathcal{E}_0[\mathbb{1}_A - \rho|_A] + \mathcal{E}_0[\rho|_B] - 2\mathcal{E}_0[\mathbb{1}_A - \rho|_A, \rho|_B] \\ =& \mathcal{E}_0[\rho] - ||\mathbb{1}_A - \rho|_A||^2_{L^1(\mathbb{R})} - ||\rho|_B||^2_{L^1(\mathbb{R})} + 2||\rho|_B||_{L^1(\mathbb{R})}||\mathbb{1}_A - \rho|_A||_{L^1(\mathbb{R})} \\ =& \mathcal{E}_[\rho],\end{aligned}$$

where the second to last equality follows from the fact that $\operatorname{diam}(A \cup B) \leq 1$.

Now, if $\operatorname{diam}(\{0 < \rho < 1\}) > 1$, noting that ρ must be compactly supported, we may partition the set $\{0 < \rho < 1\}$ into finitely many disjoint subsets each of diameter no greater than one. Iteratively following the procedure above, we obtain $\overline{\rho}$.

□

Lemma 5.12. *Let $m > 1$ and $\mathbb{1}_A \in \mathcal{A}_m$ be a minimizer for $\mathcal{E}_0$. Then, there is a set $\overline{A}$ with measure m that is saturated from the left such that $\mathcal{E}_0[\mathbb{1}_A] \geq \mathcal{E}_0[\mathbb{1}_{\overline{A}}]$.*

Proof. Assume that A is not saturated from the left, since otherwise we are done. Let $a = \inf A$. Consider the continuous function defined by

$$F(x) = \int_a^x \mathbb{1}_A \, dx.$$

Since A is not saturated from the left, $F(a+2) < 1$. Since $|A| = m > 1$, this means for large enough x $F(x) > 1$. By the Intermediate Value Theorem, there is some $b > a+2$ such that $F(b) = 1$. Let

$$A_L = (a, b-1) \cap A, \; A_R = (b-1, b) \cap A, \text{ and } A_+ = (b, \infty) \cap A,$$

We will construct $\overline{A}$ by moving the mass of A_L to fill as much as possible in the interval $(b-2, b-1)$. Let

$$\overline{A_L} = (b-1, b) \setminus A - 1,$$

and set

$$\overline{A} = \overline{A_L} \cup A_R \cup A_+.$$

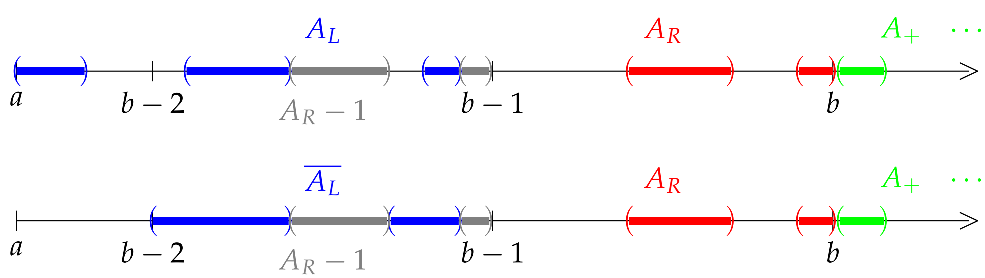

Figure 5.2: Illustration of construction of $\overline{A}$ for a particular A. Note that no mass can be placed in the set $A_R - 1$, and $\overline{A_L}$ fills the interval $[b-2, b-1]$ as much as possible.

Then,

$$\begin{aligned}\mathcal{E}_0[\mathbb{1}_{\overline{A}}] - \mathcal{E}_0[\mathbb{1}_A] =& \mathcal{E}_0[\mathbb{1}_{\overline{A_L}}] - \mathcal{E}_0[\mathbb{1}_{A_L}] \\ &+ 2(\mathcal{E}_0[\mathbb{1}_{\overline{A_L}}, \mathbb{1}_{A_R}] - \mathcal{E}_0[\mathbb{1}_{A_L}, \mathbb{1}_{A_R}]).\end{aligned}$$

First $\mathcal{E}_0[\mathbb{1}_{\overline{A_L}}] - \mathcal{E}_0[\mathbb{1}_{A_L}] \le 0$ since $|A_L| = |\overline{A_L}|$ and $\operatorname{diam}(\overline{A_L}) \le 1$. Also note that $\mathbb{1}_{\overline{A_L}} \ge \mathbb{1}_{A_L}$, which means that $\mathcal{E}_0[\mathbb{1}_{\overline{A_L}}, \mathbb{1}_{A_R}] - \mathcal{E}_0[\mathbb{1}_{A_L}, \mathbb{1}_{A_R}] \le 0$. □

Lemma 5.13. *Let $m \ge 1$ and $A \subseteq \mathbb{R}$ be saturated from the left. Let $a = \inf A$. Then,*

$$\mathcal{E}_0[\mathbb{1}_A] \ge \mathcal{E}_0[\mathbb{1}_{\overline{A}}],$$

where $\overline{A} = (a, a+1) \cup ((a+2, \infty) \cap A)$.

Proof. Denote

$$A_L = (a, a+1) \cap A,\ A_R = (a+1, a+2) \cap A,\ \text{and}\ A_+ = (a+2, \infty) \cap A.$$

Let

$$\overline{A} = A_L \cup (A_R - 1) \cup A_+.$$

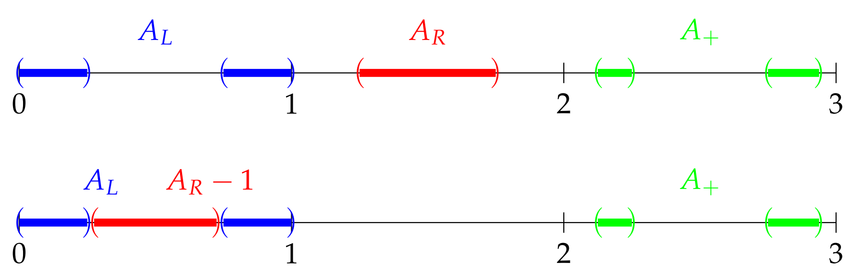

Figure 5.3: Illustration of construction of $\overline{A}$ for a particular A.

Then,

$$\begin{aligned}\mathcal{E}_0[\mathbb{1}_{\overline{A}}] - \mathcal{E}_0[\mathbb{1}_A] =& 2\mathcal{E}_0[\mathbb{1}_{A_L}, \mathbb{1}_{A_R - 1}] - 2\mathcal{E}_0[\mathbb{1}_{A_L}, \mathbb{1}_{A_R}] - 2\mathcal{E}_0[\mathbb{1}_{A_R}, \mathbb{1}_{A_+}] \\ =& -2|A_L||A_R| - 2\mathcal{E}_0[\mathbb{1}_{A_R}, \mathbb{1}_{A_L} + \mathbb{1}_{A_+}].\end{aligned}$$

So, we are left to show that $\mathcal{E}_0[\mathbb{1}_{A_R}, \mathbb{1}_{A_L} + \mathbb{1}_{A_+}] \ge -|A_L||A_R|$. Note that $A_+ \cap [a+2, a+3] \subseteq ([a+1, a+2] \setminus A_R + 1)$. But, since A is saturated from the left, up to a set of zero measure $A_L = [a+1, a+2] \setminus A_R - 1$. So, up to a set of zero measure,

$$A_+ \cap [a+2, a+3] \subseteq A_L + 2.$$

Then, for $x \in A_R$,

$$\begin{aligned}K * (\mathbb{1}_{A_L} + \mathbb{1}_{A_+})(x) =& -|A_L \cap [x-1, a+1]| - |A_+ \cap [a+2, x+1]| \\ \ge& -|A_L \cap [x-1, a+1]| - |(A_L + 2) \cap [a+2, x+1]| \\ =& -|A_L|,\end{aligned}$$

which implies that $\mathcal{E}_0[\mathbb{1}_{A_R}, \mathbb{1}_{A_L} + \mathbb{1}_{A_+}] \ge -|A_L||A_R|$, as needed. □

We now have all the ingredients needed to prove Theorem 5.9.

Proof of Theorem 5.9. Fix $m > 0$, and write $m = k + a$ where $k \in \mathbb{Z}$ and $0 \leq a < 1$. The case $k = 0$ has already been shown. Assume $k \geq 1$. Let A be a set of mass m such that $\mathbb{1}_A$ minimizes the energy $\mathcal{E}_0$. By lemma 5.12 we may assume that A is saturated from the left. Then, by lemma 5.13, there is some $b \in \mathbb{R}$ such that

$$\mathcal{E}_0[\mathbb{1}_A] = -1 + \mathcal{E}_0[\mathbb{1}_{A\cap[b,\infty)}]$$

and $|A \cap [b, \infty)| = (k-1) + a$. This means that

$$\inf_{\rho \in \mathcal{A}_{k+a}} \mathcal{E}_0[\rho] = -1 + \inf_{\rho \in \mathcal{A}_{k-1+a}} \mathcal{E}_0[\rho].$$

We may repeat the same argument for $m = k - 1 + a, m = k - 2 + a, \ldots,$ eventually yielding

$$\inf_{\rho \in \mathcal{A}_{k+a}} \mathcal{E}_0[\rho] = -k + \inf_{\rho \in \mathcal{A}_a} \mathcal{E}_0[\rho] = -k - a^2.$$

□

5.4 CHARACTERIZATION OF MINIMIZERS

In the previous section, we found the minimal energy for the toy model with distance 1 forbidden. In this section, we will explore what minimizers can look like. In the case where the mass is an integer, minimizing densities must be indicators of unions of well separated intervals of length one. When the mass is not an integer, there is a wider class of minimizing densities. Minimizers need not saturate the density constraint everywhere, nor must they be supported on a union of well separated intervals.

Integer mass

Recall that for any density ρ with $\mathcal{E}_0[\rho] < \infty$, we have that $K * \rho(x) \geq -1$ for all x. If m is an integer, then

$$\inf_{\rho \in \mathcal{A}_m} \mathcal{E}_0[\rho] = -m.$$

Suppose $\rho \in \mathcal{A}_m$ is a minimizer, then ρ is compactly supported. Let $a = \inf(\text{supp}(\rho))$. If $\rho < 1$ on a set of positive measure in the interval $[a, a+1]$, then $K * \rho > -1$ on a set of positive measure in $\{\rho > 0\}$, which in turn would mean $\mathcal{E}_0[\rho] > -m$, which cannot be true if ρ is a minimizer.

So, $\rho = 1$ almost everywhere in $[a, a+1]$. Then, $\rho = 0$ almost everywhere in $[a+1, a+2]$. We may repeat this process to see that up to a set of measure zero, ρ is the indicator of a union of intervals of length one.

Non-integer mass

Suppose $m = k + a$ for some $k \in \mathbb{Z}$ and $0 < a < 1$. One natural type of minimizer to consider is $\rho = \mathbb{1}_{I_1 \cup \cdots \cup I_l} + \eta$ where $I_1, \ldots I_l$ are intervals of length one, and $\eta \in \mathcal{A}_a$ is a supported in an interval of length one which is well separated from $I_1, \ldots I_l$. Notably, η need not saturate the density constraint.

There are other minimizers, for example if $m = 1 + a$ consider $\rho = \mathbb{1}_A$ where

$$A = (0, a) \cup \left(\frac{1+a}{2}, 1\right) \cup \left(1+a, \frac{3+a}{2}\right) \cup (2, 2+a)\,.$$

The set A is illustrated in figure 5.4.

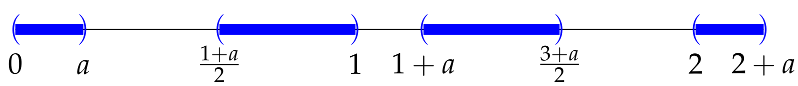

Figure 5.4: Example of a set A with $|A| = m = 1 + a$ with $\mathcal{E}[\mathbb{1}_A] = -1 - a^2$

6

WEAKLY REPULSIVE KERNELS

6.1 INTRO

In the previous chapters, we saw one class of kernels which force separation of populations. This chapter will consider kernels which are weakly repulsive. As in [CFP17], we consider C^2 kernels which satisfy

$$K(0) = 0, \text{ and there exists an } R > 0 \text{ such that } K(r) < 0 \text{ for } 0 < r < R, \\ \text{and } K(r) \geq 0 \text{ for } r \geq R, \tag{6.1}$$

along with the weak repulsion condition

$$\text{there exist } \alpha > 2, \text{ and } C > 0 \text{ such that } K'(r)r^{1-\alpha} \to -C/\alpha \text{ as } r \to 0. \tag{6.2}$$

Throughout the following, we will consider densities in

$$\mathcal{A}'_m = \left\{ \rho \in L^1(\mathbb{R}^N) \,:\, 0 \leq \rho \leq \frac{1}{m}, \text{ and } ||\rho||_{L^1(\mathbb{R}^N)} = 1 \right\}.$$

Note that the problem of minimizing the energy $\mathcal{E}$ over the set $\mathcal{A}'_m$ is equivalent to minimizing it over the set $\mathcal{A}_m$ since $\rho \in \mathcal{A}'_m$ if and only if $m\rho \in \mathcal{A}_m$. Furthermore,

$$\inf_{\rho \in \mathcal{A}'_m} \mathcal{E}[\rho] = \frac{1}{m^2} \inf_{\rho \in \mathcal{A}_m} \mathcal{E}[\rho].$$

In [CFP17], the authors show the support of a d_∞ local minimizer consists only of isolated points. We consider the minimization problem with a density constraint, and adapt their argument to show that d_∞ local minimizers saturate the density constraint.

6.2 PRELIMINARIES ON WASSERSTEIN d_∞ DISTANCES

In this section, we state the definition of the Wasserstein d_∞ distance. One may see [San14] or [GS84] for a further reference. We also mention an important property, which forms the basis of our analysis. Namely, that if two measures differ on a set of small diameter, then they are close in d_∞. First, define for probability measures μ and ν on $\mathbb{R}^N$ the set of transport plans between μ and ν

$$\Pi(\mu,\nu) = \{\pi \in \mathcal{P}(\mathbb{R}^N \times \mathbb{R}^N) : \pi(A \times \mathbb{R}^N) = \mu(A), \pi(\mathbb{R}^N \times A) = \nu(A) \text{ for all } A \in \mathcal{B}(\mathbb{R}^N)\}.$$

The Wasserstein d_∞ distance between μ and ν is then given by

$$d_\infty(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} \sup_{(x,y) \in \text{supp}(\pi)} |x-y|. \tag{6.3}$$

Throughout this chapter, we will use the following lemma, which says if two densities differ on a set of small diameter, then they must be close in d_∞.

Lemma 6.1. *Let $\rho, \eta \in L^1(\mathbb{R}^N)$ be bounded non-negative functions with $||\rho||_{L^1(\mathbb{R}^N)} = ||\eta||_{L^1(\mathbb{R}^N)} = 1$. If the set $\{x \in \mathbb{R}^N : \rho(x) \neq \eta(x)\}$ has diameter ϵ, then $d_\infty(\rho,\eta) \leq 2\epsilon$.*

This is a corollary of lemma 1 from [BCLR13], which we state here:

Lemma 6.2. *Assume that the probability measures μ and ν are convex combinations $\mu = m_0\mu_0 + m_1\mu_1$ and $\nu = m_0\nu_0 + m_1\mu_1$ where μ_0 and ν_0 are supported in the ball $B(x_0,\epsilon)$ for some $x_0 \in \mathbb{R}^N$ and $\epsilon > 0$. Then, $d_\infty(\mu,\nu) \leq 2\epsilon$.*

To see how lemma 6.1 follows from 6.2, let $A = \{x \in \mathbb{R}^N : \rho(x) \neq \eta(x)\}$ and let

$$a = ||\rho|_A||_{L^1(\mathbb{R}^N)} = ||\eta|_A||_{L^1(\mathbb{R}^N)},$$

and write

$$\rho = a\frac{\rho|_A}{a} + (1-a)\frac{\rho|_{\mathbb{R}^N\setminus A}}{1-a}, \text{ and } \eta = a\frac{\eta|_A}{a} + (1-a)\frac{\rho|_{\mathbb{R}^N\setminus A}}{1-a}.$$

6.3 FIRST AND SECOND ORDER CONDITIONS FOR d_∞ LOCAL MINIMIZERS

Lemma 6.3. *Euler Lagrange conditions for d_∞ local minimizers Let $\rho \in \mathcal{A}'_m$ be a d_∞ local minimizer for the energy $\mathcal{E}$. Then there is an $\epsilon > 0$ such that for any $x_0 \in \mathbb{R}^N$ there is a $\lambda \in \mathbb{R}$ such that*

$$K * \rho(x) \begin{cases} \leq \lambda & \text{if } \rho(x) = 1/m \\ = \lambda & \text{if } 0 < \rho(x) < 1 \\ \geq \lambda & \text{if } \rho(x) = 0, \end{cases} \tag{6.4}$$

for almost every $x \in B(x_0, \epsilon)$

Proof. The proof follows the same method as the proof of lemma 2.2. The key difference is that we vary ρ in a small ball. Let $\rho \in \mathcal{A}'_m$ be a d_∞ local minimizer for the energy $\mathcal{E}$, and fix $x_0 \in \mathbb{R}^N$. Then there is an $\epsilon > 0$ such that $\mathcal{E}[\rho] \leq \mathcal{E}[\eta]$ for all densities $\eta \in \mathcal{A}'_m$ with $d_\infty(\rho, \eta) < \epsilon$. In the cases where $\rho = 0$ almost everywhere in $B(x_0, \epsilon)$ or $\rho = 1$ almost everywhere in $B(x_0, \epsilon)$, the result follows from the boundedness of $K * \rho$. Assume that $0 < \rho < 1$ on a set of positive Lebesgue measure in $B(x_0, \epsilon)$ Let $S_0 = \{x \in B(x_0, \epsilon) : \rho(x) = 0\}$, and $S_1 = \{x \in B(x_0, \epsilon) : \rho(x) = 1\}$. We need to construct a variation which takes mass from S_1 and move it to S_0. Since we only alter the density within the ball $B(x_0, \epsilon)$, then the resulting density will be close to ρ in d_∞. Let $\phi, \psi \in L^1(\mathbb{R}^{\mathbb{N}})$ be bounded non-negative functions such that $||\phi \mathbb{1}_{B(x_0,\epsilon)}||_{L^1(\mathbb{R}^{\mathbb{N}}} = ||\phi \mathbb{1}_{B(x_0,\epsilon)}||_{L^1(\mathbb{R}^{\mathbb{N}}} = 1$, ϕ is 0 a.e. in S_1 and ψ is 0 a.e. in S_0. Let $\delta > 0$ be small , and consider

$$\phi_\delta = \frac{\phi \mathbb{1}_{B(x_0,\epsilon) \cap \{\rho < 1/m - \delta\}}}{||\phi \mathbb{1}_{B(x_0,\epsilon) \cap \{\rho < 1/m - \delta\}}||_{L^1(\mathbb{R}^{\mathbb{N}}}}, \tag{6.5}$$

and

$$\psi_\delta = \frac{\psi \mathbb{1}_{B(x_0,\epsilon) \cap \{\rho < \delta\}}}{||\psi \mathbb{1}_{B(x_0,\epsilon) \cap \{\rho < 1/m - \delta\}}||_{L^1(\mathbb{R}^{\mathbb{N}}}}. \tag{6.6}$$

Let $\eta_t = \rho + t(\phi_\delta - \psi_\delta)$. If t is small enough, then $\eta_t \in \mathcal{A}'_m$, and $d_\infty(\rho, \eta_t) < \epsilon$ by construction. Then

$$\begin{aligned} 0 \leq \lim_{t \to 0^+} \frac{\mathcal{E}[\eta_t] - \mathcal{E}[\rho]}{t} &= \lim_{t \to 0^+} \frac{2t\mathcal{E}[\rho, \phi_\delta - \psi_\delta] + t^2 \mathcal{E}[\phi_\delta - \psi_\delta}{t}] \\ &= 2\mathcal{E}[\rho, \phi_\delta - \psi_\delta] \\ &= 2 \int_{\mathbb{R}^{\mathbb{N}}} \int_{\mathbb{R}^{\mathbb{N}}} K(|x-y|)\rho(y)(\phi_\delta(x) - \psi_\delta(x))\, dx\, dy. \end{aligned}$$

Then, letting $\delta \to 0^+$, by dominated convergence we have

$$\int_{B(x_0,\epsilon)} K * \rho(x)\phi(x)\,dx \geq \int_{B(x_0,\epsilon)} K * \rho(x)\psi(x)\,dx.$$

To conclude as in the proof of lemma 2.2, let

$$\lambda = \sup \left\{ \int_{B(x_0,\epsilon)} K * \rho(x)\psi(x)\,dx \;:\; ||\psi \mathbb{1}_{B(x_0,\epsilon)}||_{L^1(\mathbb{R}^N)} = 1,\ \psi \geq 0;\ \psi = 0 \text{ a.e. in } S_0 \right\}.$$

□

Lemma 6.4. *Let $\rho \in \mathcal{A}'_m$ be a d_∞ local minimizer of the energy $\mathcal{E}$. Then there is an $\epsilon > 0$ such that for any bounded function η supported in $B(x_0,\epsilon) \cap \{0 < \rho < 1\}$ with $\rho + \eta \in \mathcal{A}'_m$ we have*

$$\int\int K(|x-y|)\eta(x)\eta(y)\,dy\,dy \geq 0. \tag{6.7}$$

Proof. Let $\rho \in \mathcal{A}'_m$ be a d_∞ local minimizer of the energy $\mathcal{E}$. Let $\epsilon > 0$ be given by lemma 6.1. Let η be a bounded function supported in $B(x_0,\epsilon) \cap \{0 < \rho < 1\}$ for some $x_0 \in \mathbb{R}^N$. Then

$$0 \leq \mathcal{E}[\rho + \eta] - \mathcal{E}[\rho] = 2\mathcal{E}[\rho,\eta] + \mathcal{E}[\eta].$$

Then,

$$\begin{aligned}
\mathcal{E}[\eta] \geq& -2\mathcal{E}[\rho,\eta] \\
=& -2\int K * \rho(x)\eta(x)\,dx \\
=& -2\int_{B(x_0,\epsilon)\cap\{0<\rho<1\}} K * \rho(x)\eta(x)\,dx \\
=& -2\lambda \int \eta(x)\,dx \\
=&0
\end{aligned}$$

□

6.4 MINIMIZERS SATURATE DENSITY CONSTRAINT

Define for $p > 0$ and $r \geq 0$

$$K_p(r) = -K(pr)p^{-\alpha}. \tag{6.8}$$

Let A_ρ be the support of the function $f = \mathbb{1}_{\{0<\rho<1\}}$. That is, $x \in A_\rho$ if and only if for every $\delta > 0$ the set $B(x,\delta) \cap \{0 < \rho < 1\}$ has positive measure.

In this section we will show that if $\rho \in \mathcal{A}_m'$ is a d_∞ local minimizer, then the set A_ρ is empty. In [CFP17], the authors prove a geometric constraint on the support of a minimizing probability measure. We adapt their proof to obtain the same geometric constraint on the set A_ρ. The following lemma corresponds to step 1 in the proof of Theorem 1.1 in [CFP17].

Lemma 6.5 (Geometric constraint on A_ρ)**.** *Suppose $\rho \in \mathcal{A}_m'$ is a d_∞ local minimizer of the energy $\mathcal{E}$. Then, for sufficiently small $p > 0$, if $x_1, x_2 \in A_\rho$ with $|x_1 - x_2| = p$ we have*

$$A_\rho \cap S_p(x_1, x_2) = \emptyset$$

where $S_p(x_1, x_2)$ is the set of all $u \in \mathbb{R}^N$ such that

$$\mathrm{proj}_{(x_1,x_2)} u \in [x_1, x_2], \text{ and } H_p\left(\frac{|\mathrm{proj}_{(x_1,x_2)} u - x_1|}{|x_1 - x_2|}, \frac{\mathrm{proj}_{(x_1,x_2)} u - u}{|x_1 - x_2|}\right) < 0, \tag{6.9}$$

and the function H_p is defined for any $0 \leq t \leq 1$ and $y \in \mathbb{R}^N$ by

$$H_p(t, y) = \sqrt{K_p(t + |y|)} + \sqrt{K_p(1 - t + |y|)} - \sqrt{K_p(1)}.$$

Proof. Let $\rho \in \mathcal{A}_m'$ be a d_∞ local minimizer and take $\epsilon > 0$ smaller than the one in lemma 6.4. Let $x_0, x_1, x_2 \in A_\rho$ all lie within some ball $B(x, \epsilon)$. We will construct a variation which is supported within this ball as follows. Let $\delta > 0$ be small enough so that the balls $B(x_0, \delta), B(x_1, \delta), B(x_2, \delta)$ are all disjoint and lie within $B(x, \epsilon)$. Further pick $\delta' \leq \delta$ so that

$$\int_{B(x_0,\delta')} \rho \leq \min\left(\int_{B(x_1,\delta)} \left(\frac{1}{m} - \rho\right), \int_{B(x_2,\delta)} \left(\frac{1}{m} - \rho\right)\right).$$

We construct a variation which moves the mass of ρ from $B(x_0, \delta')$ to both $B(x_1, \delta)$ and $B(x_2, \delta)$. Let $\alpha = \int_{B(x_0,\delta')} \rho$, and let $0 < \beta, \gamma \leq 1$ satisfy

$$\beta \int_{B(x_1,\delta)} \left(\frac{1}{m} - \rho\right) = \gamma \int_{B(x_2,\delta)} \left(\frac{1}{m} - \rho\right) = \alpha.$$

We then let for any $0 \leq \lambda \leq 1$

$$\eta_\lambda = -\rho_0 + \lambda\rho_1 + (1 - \lambda)\rho_2,$$

where

$$\rho_0 = \rho|_{B(x,\delta')}, \text{ and } \rho_1 = \beta\left(\frac{1}{m} - \rho\right)\bigg|_{B(x_1,\delta)}, \text{ and } \rho_2 = \gamma\left(\frac{1}{m} - \rho\right)\bigg|_{B(x_2,\delta)}.$$

Note that as long as ϵ is small enough then $\mathcal{E}[\rho_i] \leq 0$ for $i = 0, 1, 2$. Additionally, $\rho + \eta_\lambda \in \mathcal{A}'_m$, and so by lemma 6.4

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(|x-y|)\eta_\lambda(x)\eta_\lambda(y)\,dx\,dy \geq 0. \tag{6.10}$$

Expanding, and using the fact that $\mathcal{E}[\rho_i] \leq 0$ for $i = 0, 1, 2$ we obtain

$$\lambda\mathcal{E}[\rho_0, \rho_1] + (1-\lambda)\mathcal{E}[\rho_0, \rho_2] \geq \lambda(1-\lambda)\mathcal{E}[\rho_1, \rho_2]. \tag{6.11}$$

If ϵ is sufficiently small, then K is decreasing on $[0, 2\epsilon]$, and so in particular

$$\begin{aligned}
K(|x-y|) \leq& K(|x_0 - x_1| - \delta - \delta') \text{ for } x \in B(x_0, \delta') \text{ and } y \in B(x_1, \delta),\\
K(|x-y|) \leq& K(|x_0 - x_2| - \delta - \delta') \text{ for } x \in B(x_0, \delta') \text{ and } y \in B(x_2, \delta), \text{ and}\\
K(|x-y|) \geq& K(|x_1 - x_2| + 2\delta) \text{ for } x \in B(x_1, \delta) \text{ and } y \in B(x_2, \delta).
\end{aligned}$$

Then, combining with (6.11), we arrive at the inequality

$$\lambda K(|x_0 - x_1| - \delta - \delta') + (1-\lambda)K(|x_0 - x_2| - \delta - \delta') \leq \lambda(1-\lambda)K(|x_1 - x_2| + 2\delta).$$

Taking $\delta \to 0^+$, noting that $\delta' \leq \delta$, by the continuity of K, we arrive at the inequality

$$\lambda K(|x_0 - x_1|) + (1-\lambda)K(|x_0 - x_2|) \leq \lambda(1-\lambda)K(|x_1 - x_2|). \tag{6.12}$$

Let

$$a = \frac{K(|x_0 - x_1|)}{K(|x_1 - x_2|)}, \text{ and } b = \frac{K(|x_0 - x_2|)}{K(|x_1 - x_2|)}.$$

Note that $a, b > 0$. We obtain

$$\lambda(1-\lambda) \leq \lambda a + (1-\lambda)b \tag{6.13}$$

for all $0 \leq \lambda \leq 1$. This is the same equation as equation (3.1) in [CFP17]. As shown in [CFP17], by optimizing λ this implies that

$$\sqrt{a} + \sqrt{b} \geq 1,$$

which by plugging back in the values of a and b yields

$$\sqrt{-K(|x_0 - x_1|)} + \sqrt{-K(|x_0 - x_2|)} \geq \sqrt{-K(|x_1 - x_2|)}.$$

Let $p = |x_1 - x_2|$ and $z = \text{proj}_{(x_1, x_2)} x_0$. Then

$$|x_0 - x_1| \leq |x_0 - z| + |z - x_1|$$

and

$$|x_0 - x_2| \le |x_0 - z| + |z - x_2| = |x_0 - z| + p - |z - x_1|.$$

Note by the weak repulsion assumption $\sqrt{-K(r)}$ is non-decreasing for small r, so

$$\sqrt{-K(|x_0 - z| + |z - x_1|)} + \sqrt{-K(|x_0 - z| + p - |z - x_1|} \ge \sqrt{-K(p)}.$$

Dividing both sides of the inequality by $p^{\alpha/2}$ then yields

$$\sqrt{K_p\left(\frac{|x_0 - z|}{p} + \frac{|z - x_1|}{p}\right)} + \sqrt{K_p\left(\frac{|x_0 - z|}{p} + 1 - \frac{|z - x_1|}{p}\right)} \ge \sqrt{K_p(1)}.$$

Hence $x_0 \notin S_p(x_1, x_2)$, which is what we needed to show. □

The following lemma corresponds to step 2 in the proof of Theorem 1.1 in [CFP17], and it follows from their same argument, which we omit. Essentially, it follows from lemma 6.5 by taking $p \to 0$.

Lemma 6.6 (Asymptotic geometric constraint on A_ρ)**.** *Suppose $\rho \in \mathcal{A}'_m$ is a d_∞ local minimizer of the energy $\mathcal{E}$. Then, $x_1, x_2 \in A_\rho$ are asymptotically close we have*

$$A_\rho \cap S_\alpha(x_1, x_2) = \emptyset$$

where $S_\alpha(x_1, x_2)$ is the set of all $u \in \mathbb{R}^N$ such that

$$\text{proj}_{(x_1,x_2)} u \in [x_1, x_2], \text{ and } H_\alpha\left(\frac{|\text{proj}_{(x_1,x_2)} u - x_1|}{|x_1 - x_2|}, \frac{\text{proj}_{(x_1,x_2)} u - u}{|x_1 - x_2|}\right) < 0, \tag{6.14}$$

and the function H_α is defined for any $0 \le t \le 1$ and $y \in \mathbb{R}^N$ by

$$H_\alpha(t, y) = (t + |y|)^{\alpha/2} + (1 - t + |y|)^{\alpha/2} - 1.$$

Now that we have the asymptotic geometric constraint we are ready to prove Theorem **??**.

Proof of Theorem **??**. Let $\rho \in \mathcal{A}_m$ be a d_∞ local minimizer of the energy $\mathcal{E}$. Suppose for the sake of contradiction that the set A_rho has positive measure. Then there is an $x \in A_\rho$ which is a Lebesgue point of $f = \mathbb{1}_{A_\rho}$ and a sequence $x_k \in A_\rho$ which converges to x. Without loss of generality by the translation invariance of the energy $\mathcal{E}$ we may take $x = 0$. Then by the asymptotic geometric constraint lemma 6.6, for large k we must have $A_\rho \cap S_\alpha(0, x_k) = \emptyset$. Let $p_k = |x_k|$. Then noting that $S_\alpha(0, x_k) \subset B(0, p_k)$, we have

$$|B(0, p_k) \cap A_\rho| \le |B(0, p_k)| - |S_\alpha(0, x_k)|.$$

By Lebesgue Differentiation Theorem (see for reference Theorem 3.21 in [Fol13]),

$$1 = \mathbb{1}_{A_\rho}(0) = \lim_{k\to\infty} \frac{1}{|B(0,p_k)|} \int_{B(0,p_k)} \mathbb{1}_{A_\rho} \leq 1 - \frac{|S_\alpha(0,x_k)|}{|B(0,p_k)|}$$

Let v be any unit vector. Then, it is straightforward to check that $\lambda S_\alpha(0,u) = S_\alpha(0,\lambda u)$ for all $\lambda > 0$. In particular, noting that $|x_k| = p_k$, $\frac{|S_\alpha(0,x_k)|}{|B(0,p_k)|}$ is a constant which doesn't depend on k. To argue to a contradiction we also need to check that $|S_\alpha(0,v)| > 0$. To see this, we show that for small enough $r > 0$, $S_\alpha(0,v)$ contains the ball $B(v/2,r)$. Let $u \in (v/2,r)$, then $1/2 - r \leq |\text{proj}_{(0,v)} u| \leq 1/2 + r$ and $|\text{proj}_{(0,v)} u - u| \leq r$. Thus,

$$H_\alpha(|\text{proj}_{(0,v)} u|, \text{proj}_{(0,v)} u - u) \leq 2\left(\frac{1}{2} + 2r\right)^{\alpha/2} - 1.$$

Since $\alpha > 2$, for small enough r this is negative, and hence $u \in S_\alpha(0,v)$. $\square$

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