Functional transcendence in mixed Hodge theory

by

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Abstract

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Ax-Schanuel theorem is a function field analogue of the Schanuel's conjecture in transcendental number theory. Building on the works of Bakker, Gao, Klingler, Mok, Pila, Tsimerman, Ullmo and Yafaev, we extend the Ax-Schanuel theorem to mixed period mappings. Using this together with the Ax-Schanuel theorem for foliated principal bundles by Blázquez-Sanz, Casale, Freitag, and Nagloo, we furthur extend the Ax-Schanuel theorem to the derivatives of mixed period mappings. The linear subspaces in the Ax-Schanuel theorem are replaced by weak Mumford-Tate domains, which are certain group orbits of mixed Hodge structures. In particular, we prove that these domains have complex structures, and that their real-split retractions can be decomposed into semisimple and unipotent parts. We prove that the image of a mixed period mapping is contained in the weak Mumford-Tate domain that arises from the monodromy group of the variation. O-minimal geometry, namely the definable Chow theorem and the Pila-Wilkie counting theorem, are used in the proof of our extension of the Ax-Schanuel theorem.

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Contents

Chapter 1

Introduction

Transcendental number theory

In 1761, Johann Heinrich Lambert proved the irrationality of π and conjectured more generally that π cannot be a solution to a polynomial equation with rational coefficients, i.e. π is transcendental, and suggested a link between the transcendence of π and the ancient "squaring the circle" problem. More than a century later Ferdinand von Lindemann proved the following theorem, and concluded the transcendence of π from it (using Euler's identity and the transcendence of e proved earlier by Hermite in 1873).

Theorem 1.1 (Lindemann, 1882)**.** *If α is a non-zero algebraic number, then e^{α} is transcendental.*

Lindemann's theorem was generalized by Karl Weierstrass:

Theorem 1.2 (Weierstrass, 1885)**.** *If $\alpha_1, \ldots, \alpha_n$ are algebraic numbers linearly independent over $\mathbb{Q}$, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$.*

In the 1960s, Schanuel conjectured the following statement which generalizes the above theorems of Lindemann and Weierstrass.

Conjecture 1.3 (Schanuel)**.** *Let $z_1, \ldots, z_n$ be complex numbers. If*

$$\operatorname{tr.deg}_{\mathbb{Q}} \mathbb{Q}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) < n,$$

then $z_1, \ldots, z_n$ are linearly dependent over $\mathbb{Q}$.

The Ax-Schanuel theorem

In 1971, Ax proved the following function field analogue [3] of Schanuel's conjecture:

Theorem 1.4 (Ax, 1971)**.** *Let $f_1, \ldots, f_n \in \mathbb{C}[[t_1, \ldots, t_m]]$ be formal power series. If*

$$\operatorname{tr.deg}_{\mathbb{C}} \mathbb{C}(f_1, \ldots, f_n, e^{f_1}, \ldots, e^{f_n}) < n + \mathit{rank} \left(\frac{\partial f_i}{\partial t_j} \right)_{i,j},$$

then $f_1, \ldots, f_n$ are $\mathbb{Q}$-linearly dependent modulo $\mathbb{C}$.

Since the transcendental number π originates from geometry, one analogously reinterprets Ax's theorem in geometric terms and look for applications. Consider the fiber product

$$\begin{tikzcd} W \arrow[r] \arrow[d] & \mathbb{C}^n \arrow[d] \\ (\mathbb{C}^\times)^n \arrow[r, "\log"] & \mathbb{Z}^n \setminus \mathbb{C}^n. \end{tikzcd}$$

Let $p : \mathbb{C}^n \times (\mathbb{C}^\times)^n \to (\mathbb{C}^\times)^n$ be the projection onto $(\mathbb{C}^\times)^n$. Ax's theorem can restated in terms of analytic sets and their Zariski closure:

Theorem 1.5 (Ax, 1971)**.** *Let U be an irreducible analytic subset of W. Let U^{Zar} be the Zariski closure of U in $\mathbb{C}^n \times (\mathbb{C}^\times)^n$. If*

$$\dim U^{Zar} < n + \dim U,$$

then $p(U)$ is contained in a coset of a proper algebraic subtori of $(\mathbb{C}^\times)^n$.

A special case of the Ax-Schanuel theorem is the Ax-Lindemann-Weierstrass theorem, which is an analogue of the Lindemann-Weierstrass theorem in functional transcendence.

Theorem 1.6 (Ax-Lindemann-Weierstrass)**.** *Let* $\exp : \mathbb{C}^n \to (\mathbb{C}^\times)^n$ *be the componentwise exponential map. If* U *is an irreducible algebraic subvariety of* $\mathbb{C}^n$*, then the Zariski closure of* $\exp(U)$ *in* $(\mathbb{C}^\times)^n$ *is a coset of an algebraic subtori.*

Many conjectures in Diophantine geometry are on the finiteness of rational points or "special" points outside some "special" subvarieties, and they can be approached using the *Pila-Zannier method* [55]. A functional transcendence result like the Ax-Lindemann-Weierstrass theorem is very useful in reducing the Diophantine problem in question to finiteness outside proper bialgebraic subvarieties, i.e. proper algebraic subvarieties whose preimages under the uniformization map (the exponential map in the torus case) are also algebraic. This is one of the ideas in the Pila-Zannier method. In view of Lindemann's theorem in transcendental number theory, it is reasonable to expect that transcendental sets, i.e. sets that do not contain any positive dimensional semi-algebraic subsets, have "very few" rational points. The second idea in the Pila-Zannier method is to apply this guiding principle to the preimage of the subvariety in question. The third idea in the Pila-Zannier method is to reduce "very few" points to finitely many points by obtaining a lower bound for the number of Galois conjugates of a special point.

The Ax-Lindemann-Weierstrass theorem was used by Pila [49] to reprove Lang's conjecture for torsion points in subvarieties of tori. Indeed, Pila proved more generally in *loc. cit.* the André-Oort conjecture for products of elliptic curves, modular curves and tori, using the corresponding analogue of the Ax-Lindemann-Weierstrass theorem.

Functional transcendence and Hodge theory

Apart from tori, functional transcendence can be studied in a more general setting like Shimura varieties, and much more generally varieties equipped with variations of mixed Hodge structures on them.

The Ax-Lindemann-Weierstrass theorem was extended by Pila-Tsimerman [52] to uniformizations of the moduli space of principally polarized abelian varieties, and more generally to uniformizations of Shimura varieties by Klingler-Ullmo-Yafaev [37], and uniformizations of mixed Shimura varieties by Gao [28].

The Ax-Schanuel theorem was extended to the j-function and its derivatives by Pila-Tsimerman [53], uniformizations of Shimura varieties and their derivatives by Mok-Pila-Tsimerman [43], period mappings by Bakker-Tsimerman [7], and uniformizations of mixed Shimura varieties of Kuga type by Gao [29]. It was conjectured by Klingler [36] that the Ax-Schanuel theorem can be extended to mixed period mappings in general. This thesis includes the author's proof [20] of Klingler's conjecture, and the proof [21] of a more general statement involving the derivatives of mixed period mappings, see Theorem 4.4 and Theorem 5.1 respectively. In these statements, the cosets of algebraic subtori in the classical Ax-Schanuel theorem are replaced by the notion of weakly special subvarieties. A proof of Klingler's conjecture is given independently by Gao-Klingler [31].

Pila, Shankar and Tsimerman [51] proved the André-Oort conjecture using the Ax-Lindemann-Weierstrass theorem for Shimura varieties [37] and mixed Shimura varieties [28]. Analogously, the Hodge theoretic generalizations of the Ax-Schanuel theorem are used in several works [9][11][22][50] on the geometric aspects of the Zilber-Pink conjecture, which is a conjecture on atypical intersections that vastly generalizes the André-Oort conjecture, the Manin-Mumford conjecture and the Mordell-Lang conjecture in Diophantine geometry.

Lawrence-Venkatesh [39] and Lawrence-Sawin [38] used the Ax-Schanuel theorem for period mappings [7] to prove Shafarevich conjectures for hypersurfaces. Gao used the Ax-Schanuel theorem for mixed Shimura varieties of Kuga type [29] to study the generic rank of Betti map [30], which was then used by Dimitrov-Gao-Habegger [24] to prove a uniform bound for the number of rational points on curves. Hast develops a higher dimensional Chabauty-Kim method in [33] using the Ax-Schanuel theorem for mixed period mappings.

The Ax-Schanuel theorem for derivatives of j-function [53] was applied in the work of Aslanyan, Eterović, and Kirby [2] on the existential closedness problem for the j-function. The Ax-Schanuel theorem for derivatives of uniformizations of Shimura varieties [43] was applied in the work of Eterović and Zhao [27] on the same problem for uniformizations of Shimura varieties. One expects Theorem 5.1 will have applications in similar existential closedness problems for mixed period mappings, and in Zilber-Pink conjectures with derivatives for variations of mixed Hodge structures.

There are several works proving Ax-Schanuel theorems for other functions. Baldi and Ullmo [10] prove the Ax-Schanuel theorem for certain non-arithmetic ball quotients. They use Simpson's theory in addition to o-minimality and monodromy (André-Deligne). Outside Hodge theory, Blázquez-Sanz, Casale, Freitag, and Nagloo [12] prove the Ax-Schanuel theorem with derivatives for uniformizers of any Fuchsian group of the first kind and any genus. Their proof use Ax's arguments, foliated principal bundles, the Maurer-Cartan structure equation, and the model theory of differentially closed fields. Huang and Ng [34] prove the Ax-Schanuel theorem for certain meromorphic functions using Nevanlinna theory.

Chapter 2

Preliminaries

Since we are going to study functional transcendence in the setting of mixed Hodge theory, we first review the notion of variations of mixed Hodge structures. Since we are studying interactions between algebraic and analytic sets in the Ax-Schanuel statement, we need to use o-minimal geometry, which serves as a bridge between analytic geometry and algebraic geometry. We will recall two crucial theorems in o-minimal geometry, namely the definable Chow theorem and the Pila-Wilkie counting theorem, that are useful in proving algebraicity and creating semi-algebraic subets in "not wildly" analytic situations.

2.1 Pure Hodge structures

Let R be one of the rings $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. A *pure R-Hodge structure* of weight $n \in \mathbb{Z}$ is a Noetherian R-module H equipped with a bigrading

$$H_{\mathbb{C}} := H \otimes_R \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}$$

such that $H^{p,q} = \overline{H^{q,p}}$. Equivalently, a pure R-Hodge structure of weight $n \in \mathbb{Z}$ is a pair $(H, F^\bullet)$, where H is a Noetherian R-module and $F^\bullet$ is a finite decreasing filtration, called the *Hodge filtration*, of $H_{\mathbb{C}}$ such that $F^p \oplus \overline{F^{n-p+1}} = H_{\mathbb{C}}$ and $F^p \cap \overline{F^{n-p+1}} = 0$ for any p. The equivalence is given by

$$F^p = \bigoplus_{p' \geq p} H^{p',n-p'} \quad \text{and} \quad H^{p,q} = F^p \cap \overline{F^q}.$$

The Deligne torus $\mathbf{S}$ is the restriction of scalars $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbf{G}_m$. We have $\mathbf{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$. The set $\mathbf{S}(\mathbb{R})$ consists of points of the form $(z, \overline{z})$. There is a natural embedding $w : \mathbf{G}_{m,\mathbb{R}} \to \mathbf{S}$ of algebraic groups which on complex points is the diagonal embedding $z \mapsto (z, z)$.

Equivalently, a pure R-Hodge structure of weight $n \in \mathbb{Z}$ is a Noetherian R-module H equipped with a morphism of algebraic groups $\varphi : \mathbf{S} \to \mathbf{GL}(H_{\mathbb{R}})$ such that $(\varphi \circ w)(z) = z^{-n} \cdot \mathrm{Id}_H$. The equivalence with the above definition is given by

$$H^{p,q} = \{v \in H_{\mathbb{C}} : \varphi_{\mathbb{C}}(z, w) \cdot v = z^{-p} w^{-q} v \text{ for any } (z, w) \in \mathbf{S}(\mathbb{C})\}.$$

A morphism $f : (H, F^\bullet) \to (H', F'^\bullet)$ of pure R-Hodge structure is a morphism $f : H \to H'$ of R-module respecting the Hodge filtration.

For any $n \in \mathbb{Z}$, there is a unique pure R-Hodge structure $R(n)$ of weight $-2n$ on $(2\pi i)^n R$. A *polarization* for a pure R-Hodge structure $(H, F^\bullet)$ is a morphism of R-Hodge structures $Q : H \otimes H \to R(-n)$ which is $(-1)^n$-symmetric and such that the real-valued symmetric bilinear form $(2\pi i)^n Q(Cu, v)$ is positive-definite on $H_{\mathbb{R}}$, where C is the Weil operator which acts on each $H^{p,q}$ by scalar multiplication by i^{p-q}.

2.2 Mixed Hodge structures

Let $K := R \otimes_{\mathbb{Z}} \mathbb{Q}$. A *mixed R-Hodge structure* is a tuple $(H, W_\bullet, F^\bullet)$, where

- H is a Noetherian R-module,

- $W_\bullet$ is a finite increasing filtration, called the *weight filtration*, of $H_K := H \otimes_R K$ by K-vector subspaces,

- $F_\bullet$ is a finite decreasing filtration, called the *Hodge filtration*, of $H_{\mathbb{C}}$ by $\mathbb{C}$-vector subspaces,

such that for each $n \in \mathbb{Z}$,

$$(\operatorname{Gr}_n^W H_K, \operatorname{Gr}_n^W(F^\bullet))$$

is a pure K-Hodge structure of weight n.

A morphism

$$f : (H, W_\bullet, F^\bullet) \to (H', W'_\bullet, F'^\bullet)$$

of mixed R-Hodge structure is a morphism $f : H \to H'$ of R-module respecting the weight filtration and the Hodge filtration.

A *graded polarization* for a mixed R-Hodge structure $(H, W_\bullet, F^\bullet)$ is a sequence $\{Q_n\}$, where Q_n is polarization on the pure K-Hodge structure

$$(\operatorname{Gr}_n^W H_K, \operatorname{Gr}_n^W(F^\bullet))$$

for each $n \in \mathbb{Z}$.

Theorem 2.1 (Deligne [23]). *Let $(H, W_\bullet, F^\bullet)$ be a mixed $\mathbb{R}$-Hodge structure. There exists a unique bigrading, called the Deligne bigrading (or Deligne splitting),*

$$H_{\mathbb{C}} = \bigoplus_{p,q} H^{p,q}$$

such that

$$W_n = \bigoplus_{p+q \leq n} H^{p,q}, \quad F^p = \bigoplus_{r \geq p} H^{r,s},$$

and

$$H^{p,q} = \overline{H^{q,p}} \mod \bigoplus_{r<p, s<q} H^{r,s}.$$

Moreover, the splitting is functorial.

Given a mixed $\mathbb{R}$-Hodge structure $(H, W_\bullet, F^\bullet)$, its Deligne splitting on $H_\mathbb{R}$ defines a unique morphism $\varphi_\mathbb{C} : \mathbf{S}_\mathbb{C} \to \mathbf{GL}(H_\mathbb{C})$ such that

$$H^{p,q} = \{v \in H_\mathbb{C} : \varphi_\mathbb{C}(z, w) \cdot v = z^{-p} w^{-q} v \text{ for any } (z, w) \in \mathbf{S}(\mathbb{C})\}.$$

Proposition 2.2 ([57], Prop. 1.5). *Let H be a finite-dimensional $\mathbb{Q}$-vector spaces. A representation $\rho : \mathbf{S}_\mathbb{C} \to \mathbf{GL}(H_\mathbb{C})$ defines a mixed $\mathbb{Q}$-Hodge structure on H if and only if there exist a connected linear algebraic group P over $\mathbb{Q}$, a representation $\beta : P \to \mathbf{GL}(H)$ defined over $\mathbb{Q}$, and a homomorphism $\alpha : \boldsymbol{S}_\mathbb{C} \to P_\mathbb{C}$, such that $\rho = \beta_\mathbb{C} \circ \alpha$ and such that the following holds: let U be the unipotent radical of P, let $G := P/U$ and $\pi : P \to G$ be the canonical projection, then*

(1) $\pi \circ \alpha$ is already defined over $\mathbb{R}$;

(2) $\pi \circ \alpha \circ w$ is already defined over $\mathbb{Q}$, where $w : \mathbf{G}_{m,\mathbb{R}} \hookrightarrow \mathbf{S}$;

(3) the weight filtration $W_\bullet$ on $\mathrm{Lie}\, P_\mathbb{C}$, defined by composing the adjoint representation with α, satisfies $W_{-1}(\mathrm{Lie}\, P_\mathbb{C}) = \mathrm{Lie}\, U$.

A mixed R-Hodge structure $(H, W_\bullet, F^\bullet)$ is said to be of type (p, q) if $H_\mathbb{C} = H^{p,q}$ in the Deligne splitting. Elements of the Hodge structure

$$T^{m,n}(H_K) := H^{\otimes m} \otimes \mathrm{Hom}(H, R)^{\otimes n} \otimes_R \mathbb{Q}$$

of type $(0, 0)$ are called Hodge tensors. The *Mumford-Tate group* of $(H, W_\bullet, F^\bullet)$ is the largest subgroup of $\mathbf{GL}(H_K)$ which fixes any Hodge tensors in $T^{m,n}(H_K)$ for any $m, n \geq 0$. Equivalently [1, Lemma 2], the Mumford-Tate group of $(H, W_\bullet, F^\bullet)$ is the Tannakian group of the Tannakian category $\mathcal{C}$ of mixed K-Hodge structure tensorially generated by $(H, W_\bullet, F^\bullet)$ and its dual. This Tannakian formalism gives an equivalence between $\mathcal{C}$ and the category of representations of the Mumford-Tate group.

Let H be a finite-dimensional vector space over a field L of characteristic zero. Let $\{W_k\}$ be an increasing filtration on H. A *splitting* of $\{W_k\}$ is a direct sum decomposition

$$H = \bigoplus_k H_k$$

such that

$$W_\ell = \bigoplus_{k \le \ell} H_k.$$

The variety of all splitting of $\{W_k\}$ is denoted by $\mathcal{S}(W_\bullet)$. It is a smooth algebraic variety over L such that $\mathcal{S}(W_\bullet)(L')$ is the set of all splittings of $\{W_k \otimes_L L'\}$ for every field L' containing L. Let

$$W_k \operatorname{End}(H) := \{X \in \operatorname{End}(H) : X(W_\ell) \subset W_{\ell+k}\}.$$

The group $(\exp W_{-1} \operatorname{End}(H))(L')$ acts simply transitively on $\mathcal{S}(W_\bullet)(L')$ for every field L' containing L [19, Prop. 2.2].

A mixed $\mathbb{R}$-Hodge structure $(H, W_\bullet, F^\bullet)$ is said to *real-split* if it satisfies one of the following equivalent properties:

- it is a direct sum of pure real Hodge structures of different weights;
- it admits a real splitting, i.e. a bigrading $H^{p,q}$ such that $H^{p,q} = \overline{H^{q,p}}$.
- there exists $T \in \mathcal{S}(W_\bullet)(\mathbb{R})$ such that $T(F^p) \subset F^p$.

2.3 Variations of mixed Hodge structures

A *variation of mixed R-Hodge structures* on a complex manifold X with structure sheaf $\mathcal{O}_X$ is a tuple $(\mathcal{H}, \mathcal{W}_\bullet, \mathcal{F}^\bullet)$, where

- $\mathcal{H}$ is a local system of Noetherian R-modules on X,

- $\mathcal{W}_\bullet$ is a finite increasing filtration, called the *weight filtration*, of $\mathcal{H}_K := \mathcal{H} \otimes_R K$ by local subsystems of K-vector spaces,

- $\mathcal{F}^\bullet$ is a finite decreasing filtration, called the *Hodge filtration*, of $\mathcal{V} := \mathcal{H} \otimes \mathcal{O}_X$ by holomorphic vector subbundles,

such that

- for each $x \in X$, the tuple $(\mathcal{H}_x, (\mathcal{W}_\bullet)_x, (\mathcal{F}^\bullet)_x)$ is a mixed R-Hodge structure,

- the flat connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_X$ whose sheaf of horizontal sections is $\mathcal{H}_\mathbb{C}$ satisfies the *Griffiths transversality* condition, i.e. $\nabla \mathcal{F}^\bullet \subset \mathcal{F}^{\bullet-1} \otimes \Omega^1_X$.

Let $R(n)_X$ be the constant variation of pure R-Hodge structures on X of weight $-2n$ whose stalks are endowed with the pure Hodge structure $R(n)$. A *graded polarization* $\mathcal{Q}$ for $(\mathcal{H}, \mathcal{W}_\bullet, \mathcal{F}^\bullet)$ is a sequence

$$\mathcal{Q}_n : \mathrm{Gr}^W_n(\mathcal{H}_K) \otimes \mathrm{Gr}^W_n(\mathcal{H}_K) \to K(-n)_X$$

of ∇-flat bilinear form inducing graded polarizations $\mathcal{Q}_{n,x}$ on the mixed R-Hodge structure $(\mathcal{H}_x, (\mathcal{W}_\bullet)_x, (\mathcal{F}^\bullet)_x)$ for each $x \in X$.

We assume all variations of mixed R-Hodge structures are *admissible* (see [46, p. 364] for its definition).

The Mumford-Tate groups of $(\mathcal{H}_x, (\mathcal{W}_\bullet)_x, (\mathcal{F}^\bullet)_x)$ attain maximal dimension and are isomorphic to each other for all x outside a countable union (called the Hodge locus) of proper irreducible analytic varieties of X [1, Lemma 4]. We say these Mumford-Tate groups are *Hodge generic*. If x is in the Hodge locus, then the Mumford-Tate group is strictly smaller than the generic one. Points outside the Hodge locus are also called *Hodge generic* points.

2.4 O-minimal geometry

Definition 2.3. A *structure* on a non-empty set R is a sequence $\mathcal{S} = (\mathcal{S}_m)_{m\in\mathbb{N}}$ such that for each $m \geq 0$:

- $\mathcal{S}_m$ is a Boolean algebra of subsets of R^m;
- if $A \in \mathcal{S}_m$, then $R \times A$ and $A \times R$ are $\mathcal{S}_{m+1}$;
- $\{(x_1, \ldots, x_m) \in R^m : x_1 = x_m\} \in \mathcal{S}_m$;
- if $A \in \mathcal{S}_{m+1}$, then $\pi(A) \in \mathcal{S}_m$, where $\pi : R^{m+1} \to R^m$ is the projection map on the first m coordinates.

We say that a set is *definable* in $\mathcal{S}$ if it is in $\mathcal{S}_m$ for some $m \geq 0$. A function $f : X \to Y$, where $X \subset R^m$ and $Y \subset R^n$ for some $m, n \in \mathbb{N}$, is said to *definable* in $\mathcal{S}$ if the graph of f is in $\mathcal{S}_{m+n}$.

A structure $\{\mathcal{S}'_m\}$ is said to be an *expansion* of $\{\mathcal{S}_m\}$ if $\mathcal{S}_m \subset \mathcal{S}'_m$ for all m.

Definition 2.4 ([25][56])**.** Let R be a dense linearly ordered non-empty set $(R, <)$ without endpoints. An *o-minimal structure* on R a structure $(\mathcal{S}_m)_{m\in\mathbb{N}}$ on R such that:

- $\{(x, y) \in R^2 : x < y\} \in \mathcal{S}_2$;
- the sets in $\mathcal{S}_1$ are exactly the finite unions of intervals and points.

The structure $\mathbb{R}_{\text{alg}}$ consisting of all semi-algebraic sets is o-minimal. The intersection of structures is a structure. Let $\mathbb{R}_{\text{an,exp}}$ be the smallest expansion of $\mathbb{R}_{\text{alg}}$ on $\mathbb{R}$ such that the graphs of the real exponential function $x \mapsto e^x$ and the graphs of all analytic functions on $[0, 1]$ are definable in it. The structure $\mathbb{R}_{\text{an,exp}}$ is o-minimal [26].

From now on, we say a subset of $\mathbb{R}^n$ is *definable* if it is definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$.

The following theorem, called the definable Chow theorem, is an analogue of the Chow theorem in the non-projective setting:

Theorem 2.5 (Peterzil-Starchenko [48]). *If S is a definable, complex analytically constructible subset of a complex algebraic variety V, then S is algebraically constructible.*

Definition 2.6. The *height* of $r \in \mathbb{Q}$ is defined to be $\max(|a|, |b|)$, where $r = a/b$ for coprime integers a, b. The height of $\alpha \in \mathbb{Q}^n$ is defined to be the maximum of the height of its components.

We will be using the following version [8, Remark 3.1.3] of the Pila-Wilkie counting theorem. Informally, it says that rational points of a definable set can be covered by subpolynomially many semi-algebraic sets.

Theorem 2.7 (Pila-Wilkie [54]). *Let $U \subset \mathbb{R}^n$ be a definable set. For any $\varepsilon > 0$, there is a finite number of definable sets $W^{(i)} \subset \mathbb{R}^n \times \mathbb{R}^{m_i}$ such that each fiber $W_y^{(i)} \subset \mathbb{R}^n$ is semi-algebraic and contained in U, and a constant $c(U, \varepsilon)$, such that all the rational points in U of height at most T are contained inside cT^ε many sets of the form $W_y^{(i)}$.*

Chapter 3

Weak Mumford-Tate domains

To formulate the Ax-Schanuel theorem for mixed period mappings, we need an analogue of the algebraic subtori. Note that algebraic subtori are images of linear subspaces under the exponential map. In the mixed Hodge setting, we replace linear subspaces by the weak Mumford-Tate domains, which will be defined as certain group orbits of mixed Hodge structures in the classifying space of graded-polarized mixed Hodge structures. We will equip these domains with complex structures. We will prove that the image of a mixed period mapping is contained in one of these domains, the one that arises from the monodromy group of the variation. We will decompose certain retraction of a weak Mumford-Tate domain into the unipotent part and the semisimple part, by first obtaining simply transitivity and transitivity results about these two parts respsectively.

3.1 Classifying spaces of graded-polarized mixed Hodge structures

We recall the definition of the classifying spaces of graded-polarized mixed Hodge structures [4][44][59].

Let H be a finite dimensional $\mathbb{R}$-vector space equipped with an increasing filtration $\{W_k\}$ and a collection of non-degenerate bilinear forms

$$q_k : \operatorname{Gr}^W_k H \otimes_{\mathbb{R}} \operatorname{Gr}^W_k H \to \mathbb{R}$$

that are $(-1)^k$-symmetric. Fix a partition of $\dim_{\mathbb{R}} H$ into non-negative integers $\{h^{p,q}\}$ such that $h^{p,q} = h^{q,p}$.

For any integer k, we let Ω_k be the Griffiths period domain parametrizing all decreasing filtrations $\{F^p_k\}_p$ on $\operatorname{Gr}^W_k H_{\mathbb{C}}$ with

$$\dim_{\mathbb{C}} F^p_k = \sum_{r \geq p} h^{r,k-r}$$

that define a real pure Hodge structure of weight k polarized by q_k. Let $\check{\Omega}_k$ be the compact dual of Ω_k parametrizing the $(q_k)_{\mathbb{C}}$-isotropic filtrations $\{F^p_k\}_p$ on $\operatorname{Gr}^W_k H_{\mathbb{C}}$ with

$$\dim_{\mathbb{C}} F^p_k = \sum_{r \geq p} h^{r,k-r}.$$

Let

$$\Omega := \prod_k \Omega_k \quad \text{and} \quad \check{\Omega} := \prod_k \check{\Omega}_k.$$

Let $\mathcal{M}$ be the corresponding classifying space of graded-polarized mixed $\mathbb{R}$-Hodge structures, i.e. the set of all decreasing filtrations $\{F^p\}$ of $H_{\mathbb{C}}$ such that $(H, W_\bullet, f^\bullet)$ is a real mixed Hodge structure graded-polarized by $\{q_k\}$ and such that

$$\dim_{\mathbb{C}}((F^p \operatorname{Gr}^W_{p+q} H_{\mathbb{C}}) \cap (\overline{F}^p \operatorname{Gr}^W_{p+q} H_{\mathbb{C}})) = h^{p,q}.$$

Let $\check{\mathcal{M}}$ be the smooth projective complex variety that parametrizes decreasing filtrations $\{F^p\}$ of $H_{\mathbb{C}}$ such that the induced filtration on the graded piece $\operatorname{Gr}^W_k H_{\mathbb{C}}$ is inside $\check{\Omega}_k$ for each k.

Let G^{Ω} be the real algebraic group $\prod_k \operatorname{Aut}(q_k)$. The group G^{Ω} acts transitively on Ω by semi-algebraic automorphisms. The complex algebraic group $G^{\Omega}(\mathbb{C})$

acts transitively on $\check{\Omega}$ by algebraic automorphisms. Let $P^{\mathcal{M}}$ be the preimage of G^{Ω} under the natural homomorphism $\mathbf{GL}(H)^W \to \mathbf{GL}(\mathrm{Gr}^W H)$. Let $P_u^{\mathcal{M}}$ be the unipotent radical of $P^{\mathcal{M}}$. The real algebraic group $P^{\mathcal{M}}(\mathbb{R})P_u^{\mathcal{M}}(\mathbb{C})$ acts transitively on $\mathcal{M}$ by semi-algebraic automorphisms [4, Prop. 3.6]. The complex algebraic group $P_u^{\mathcal{M}}(\mathbb{C})$ acts transitively on $\widecheck{\mathcal{M}}$ by algebraic automorphisms.

Proposition 3.1 ([57], Prop. 1.7)**.** *P be a connected linear algebraic group over $\mathbb{Q}$. Let U be the unipotent radical of P. Let $\mathcal{X}_P$ be a $P(\mathbb{R})U(\mathbb{C})$-conjugacy class in* $\mathrm{Hom}(\boldsymbol{S}_{\mathbb{C}}, P_{\mathbb{C}})$*. Assume that for one, and hence for all $\rho \in \mathcal{X}_P$, the conditions (1), (2) and (3) in Proposition 2.2 holds. Let $\beta : P \to \mathbf{GL}(H)$ be a faithful representation. Let Φ be the obvious map from $\mathcal{X}_P$ to the classifying space $\mathcal{M}$. There exists a unique structure, independent on the choice of the faithful representation β, on $D_P := \Phi(\mathcal{X}_P)$ as a complex manifold. This structure is $P(\mathbb{R})U(\mathbb{C})$-invariant and $U(\mathbb{C})$ acts complex analytically on D_P.*

Given P, $\mathcal{X}_P$ and D_P as in Proposition 3.1, the triple $(P, \mathcal{X}_P, D_P)$ is called a *mixed Hodge datum.* If D_P^+ is a connected component of D_P, then we say $(P, \mathcal{X}_P, D_P^+)$ is a *connected mixed Hodge datum.*

3.2 Quotient mixed Hodge datum

Let $(P, \mathcal{X}_P, D_P)$ be a mixed Hodge datum. Let $\rho_{\mathbb{C}}$ be in $\mathcal{X}_P$. Let M be a normal algebraic $\mathbb{Q}$-subgroup of P. Composing $\rho_{\mathbb{C}}$ with the quotient map $P_{\mathbb{C}} \to (P/M)_{\mathbb{C}}$ gives a representation

$$\overline{\rho}_{\mathbb{C}} : \mathbf{S}_{\mathbb{C}} \to (P/M)_{\mathbb{C}}.$$

Let $\mathcal{X}_{P/M}$ be the $(P/M)(\mathbb{R})(P/M)_u(\mathbb{C})$-conjugacy class of $\overline{\rho}_{\mathbb{C}}$. Fix an embedding of P/M into the automorphism group of some vector space.

Lemma 3.2. *The representation $\overline{\rho}_{\mathbb{C}}$ defines a mixed $\mathbb{Q}$-Hodge structure.*

Proof. The representation $\overline{\rho}_{\mathbb{C}}$ satisfies (1) and (2) of Proposition 2.2 because we have the $\mathbb{Q}$-morphism $P_u \to (P/M)_u$. Let

$$\mathrm{Ad}_1 : P \to \mathbf{GL}(\mathfrak{p})$$

and

$$\mathrm{Ad}_2 : P/M \to \mathbf{GL}(\mathfrak{p}/\mathfrak{m})$$

be the adjoint representations. By the functoriality of the adjoint representation, we have the following commutative diagram

$$\begin{tikzcd}
\mathbf{S}_{\mathbb{C}} \arrow[r] & P(\mathbb{C}) \arrow[r] \arrow[d, "\mathrm{Ad}_1"] & (P/M)(\mathbb{C}) \arrow[d, "\mathrm{Ad}_2"] \\
& \mathrm{Ad}(P(\mathbb{C})) \arrow[r] & \mathrm{Ad}((P/M)(\mathbb{C})).
\end{tikzcd}$$

Hence, the differential $\mathfrak{p}_{\mathbb{C}} \to (\mathfrak{p}/\mathfrak{m})_{\mathbb{C}}$ preserves the gradings, so $\overline{\rho}_{\mathbb{C}}$ satisfies (3) of Proposition 2.2. By Proposition 2.2, $\overline{\rho}_{\mathbb{C}}$ defines a mixed $\mathbb{Q}$-Hodge structure. $\square$

Fix Levi subgroups M_r and P_r for M and P respectively such that $M_r \subset P_r$. By [13, Corollary 14.11], P_uM/M is the unipotent radical of P/M. Let $(P/M)_r := P_rM/M$, which is a Levi subgroup of P/M. Since M is normal in P, we have $M_r = M \cap P_r$ and $M_u = M \cap P_u$ by [28, Prop. 2.13]. Hence, $(P/M)_u \simeq P_u/M_u$ and $(P/M)_r \simeq P_r/M_r$.

By Lemma 3.2 and Proposition 3.1, there is a complex manifold $D_{P/M}$ attached to $\mathcal{X}_{P/M}$. The tuple $(P/M, \mathcal{X}_{P/M}, D_{P/M})$ is a mixed Hodge datum. We also have a mixed Hodge data morphism

$$(P, \mathcal{X}_P, D_P) \to (P/M, \mathcal{X}_{P/M}, D_{P/M}),$$

where the map $D_P \to D_{P/M}$ is given by

$$\gamma_r\gamma_u \cdot h \mapsto (\gamma_r M_r(\mathbb{R}))(\gamma_u M_u(\mathbb{C})) \cdot \overline{h}$$

for any $\gamma_r \in P_r(\mathbb{R})$ and $\gamma_u \in P_u(\mathbb{C})$, and h and $\overline{h}$ are the mixed Hodge structure attached to $\rho_{\mathbb{C}}$ and $\overline{\rho}_{\mathbb{C}}$ respectively.

In the case where P is the Mumford-Tate group $\mathcal{MT}_h$ of a mixed Hodge structure h, then the quotient $\mathcal{MT}\,/\,\mathcal{MT}_u$ of $\mathcal{MT}$ by its unipotent radical is the Mumford-Tate group of $\overline{h}$, see [1, Lemma 2]. Let $D^+_{\mathcal{MT},\mathrm{Gr}} := D^+_{\mathcal{MT}\,/\,\mathcal{MT}_u}$ be the connected component of $D_{\mathcal{MT}\,/\,\mathcal{MT}_u}$ containing $\overline{h}$. The group $(\mathcal{MT}\,/\,\mathcal{MT}_u)(\mathbb{R})^+$ acts transitively on $D^+_{\mathcal{MT},\mathrm{Gr}}$, which is contained in Ω.

3.3 Complex structures on weak Mumford-Tate domains

Let X be a smooth irreducible algebraic variety over $\mathbb{C}$. Let $(\mathcal{H}, \mathcal{W}_\bullet, \mathcal{F}^\bullet, \mathcal{Q})$ be an admissible graded-polarized variation of mixed $\mathbb{Z}$-Hodge structures (GPVMHS) on X. Let η be a Hodge generic point of X. Let h_0 be the mixed $\mathbb{Z}$-Hodge structure on the stalk $\mathcal{H}_\eta$. Let $\mathcal{MT} := \mathcal{MT}_0$ be the Mumford-Tate group of h_0. Let $\mathcal{M}$ and $\breve{\mathcal{M}}$ be respectively the classifying space and the projective space defined in Section 3.1, where the fixed graded polarization and Hodge numbers are chosen to be the same as that of the mixed Hodge structures our GPVMHS is parametrizing. Let $D^+_{\mathcal{MT}}$ be the connected mixed Mumford-Tate domain, i.e. the $\mathcal{MT}_0(\mathbb{R})^+(\mathcal{MT}_0)_u(\mathbb{C})$-orbit of $h_{0\mathbb{R}}$ in $\mathcal{M}$, where $\mathcal{MT}_0(\mathbb{R})^+$ is the identity component of $\mathcal{MT}_0(\mathbb{R})^+$, and $(\mathcal{MT}_0)_u$ is the unipotent radical of $\mathcal{MT}_0$.

Definition 3.3. Let h be any mixed $\mathbb{R}$-Hodge structure in $D^+_{\mathcal{MT}}$. Let $\mathbf{M}$ be a normal algebraic $\mathbb{R}$-subgroup of the Mumford-Tate group $\mathcal{MT}_h$ of h. Let $\mathbf{M}_u$ be its unipotent radical. Let $\mathbf{M}(\mathbb{R})^+$ be the identity component of $\mathbf{M}(\mathbb{R})$. The $\mathbf{M}(\mathbb{R})^+\mathbf{M}_u(\mathbb{C})$-orbit $D(\mathbf{M})$ of h is called a *weak Mumford-Tate domain.* Denote the $\mathbf{M}(\mathbb{C})$-orbit of h by $\check{D}(\mathbf{M})$.

Theorem 3.4. *The orbit $\check{D}(\mathbf{M})$ is complex algebraic. The weak Mumford-Tate domain $D(\mathbf{M})$ is open in $\check{D}(\mathbf{M})$ in the Archimedean topology, so it has a complex analytic structure inherits from $\check{D}(\mathbf{M})$.*

Proof. Let

$$\Psi : P^{\mathcal{M}}(\mathbb{C}) \times \breve{\mathcal{M}} \to \breve{\mathcal{M}}$$

be the algebraic morphism defined by $\Psi(\gamma, h) = \gamma \cdot h$. There exists a projective compactification $(P^{\mathcal{M}}(\mathbb{C}))'$ of $P^{\mathcal{M}}(\mathbb{C})$ such that Ψ extends to a rational map

$$\Psi' : (P^{\mathcal{M}}(\mathbb{C}))' \times \breve{\mathcal{M}} \to \breve{\mathcal{M}}.$$

By the Chevalley-Remmert theorem [40, p. 291], the set

$$\breve{D}(\mathbf{M}) = \Psi'(\mathbf{M}(\mathbb{C}) \times \{h\})$$

is complex analytically constructible. By the definable Chow theorem [48], $\breve{D}(\mathbf{M})$ is complex algebraically constructible in $\breve{\mathcal{M}}$.

Since $\mathbf{M}$ is normal in $\mathcal{MT}_h$, the adjoint action of $\mathcal{MT}_h$ stabilizes $\mathfrak{m} := \operatorname{Lie} \mathbf{M}$. Hence, $\mathfrak{m}$ is equipped with a mixed $\mathbb{R}$-Hodge structure by the Tannakian formalism. Similarly, $\mathfrak{p}^{\mathcal{M}} := \operatorname{Lie} P^{\mathcal{M}}$ is equipped with a mixed $\mathbb{Q}$-Hodge structure. Let

$$\mathfrak{p}^{\mathcal{M}}_{\mathbb{C}} = \bigoplus_{r,s} \mathfrak{p}^{\mathcal{M},r,s}$$

and

$$\mathfrak{m}_{\mathbb{C}} = \bigoplus_{r,s} \mathfrak{m}^{r,s}$$

be the Deligne bigradings [23] of the mixed Hodge structures on $\mathfrak{p}^{\mathcal{M}}$ and $\mathfrak{m}$ respectively. By functoriality, $\mathfrak{m}^{r,s} = \mathfrak{m}_{\mathbb{C}} \cap \mathfrak{p}^{\mathcal{M},r,s}$. Let $\mathfrak{b}^{\mathcal{M}}$ be the Lie algebra of the $P^{\mathcal{M}}(\mathbb{C})$-stabilizer $B^{\mathcal{M}}$ of h. By (21) of [45],

$$\mathfrak{b}^{\mathcal{M}} = F^0 \mathfrak{p}^{\mathcal{M}}_{\mathbb{C}} = \bigoplus_{r \geq 0; s} \mathfrak{p}^{\mathcal{M},r,s}.$$

Let $\mathfrak{b}$ be the Lie algebra of the $\mathbf{M}(\mathbb{C})$-stabilizer B of h. Then

$$\mathfrak{b} = \mathfrak{m}_{\mathbb{C}} \cap \mathfrak{b}^{\mathcal{M}} = \mathfrak{m}_{\mathbb{C}} \cap \bigoplus_{r \geq 0; s} \mathfrak{p}^{\mathcal{M},r,s} = \bigoplus_{r \geq 0; s} \mathfrak{m}^{r,s}.$$

Since

$$\bigoplus_{r+s\leq -1} \mathfrak{m}^{r,s}$$

is a nilpotent ideal of $\mathfrak{m}_{\mathbb{C}}$, we know

$$\mathfrak{m}_{\mathbb{C}} = \mathfrak{m}_{\mathbb{C},u} + F^0\mathfrak{m}_{\mathbb{C}} + \overline{F^0\mathfrak{m}_{\mathbb{C}}},$$

where $\mathfrak{m}_{\mathbb{C},u}$ is the Lie algebra of the unipotent radical of $\mathbf{M}(\mathbb{C})$. For any $X \in \overline{F^0\mathfrak{m}_{\mathbb{C}}}$, we have

$$X = (X + \overline{X}) - \overline{X} \in \mathfrak{m}_{\mathbb{R}} + F^0\mathfrak{m}_{\mathbb{C}}.$$

Hence,

$$\mathfrak{m}_{\mathbb{C}} \subset \mathfrak{m}_{\mathbb{R}} + \mathfrak{m}_{\mathbb{C},u} + F^0\mathfrak{m}_{\mathbb{C}}.$$

Therefore, the canonical map

$$(\mathfrak{m}_{\mathbb{R}} + \mathfrak{m}_{\mathbb{C},u})/((\mathfrak{m}_{\mathbb{R}} + \mathfrak{m}_{\mathbb{C},u}) \cap \mathfrak{b}) \to \mathfrak{m}_{\mathbb{C}}/\mathfrak{b}$$

is surjective. The canonical map $D(\mathbf{M}) \to \check{D}(\mathbf{M})$ is then a submersion, and thus it is an open embedding. □

3.4 Images of period mappings

Let h_0 and $\mathcal{MT} := \mathcal{MT}_0$ be as in Section 3.3. The graded-polarized mixed Hodge structure h_0 defines a representation $\rho_{\mathbb{C}} : \mathbf{S}_{\mathbb{C}} \to \mathbf{GL}(\mathcal{H}_{\mathbb{C},\eta})$, see Section 2.2. Let $\mathcal{X}_0$ be the $\mathcal{MT}_0(\mathbb{R})(\mathcal{MT}_0)_u(\mathbb{C})$-conjugacy class of $\rho_{\mathbb{C}}$. Let $D^+_{\mathcal{MT}}$ be the mixed Mumford-Tate domain, i.e. the $(\mathcal{MT}_0)(\mathbb{R})^+(\mathcal{MT}_0)_u(\mathbb{C})$-orbit of h_0 in $\mathcal{M}$. The tuple $(\mathcal{MT}, \mathcal{X}_{\mathcal{MT}}, D^+_{\mathcal{MT}})$ is a connected mixed Hodge datum.

Let Γ be the image of the monodromy representation $\pi_1(X,\eta) \to \mathbf{GL}(\mathcal{H}_{\mathbb{Z},\eta})$ associated to the local system $\mathcal{H}$. Let $\mathbf{P}$ be the identity component of the $\mathbb{Q}$-Zariski closure of Γ in $\mathbf{GL}(\mathcal{H}_{\mathbb{Q},\eta})$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{P}$. Let D

be the $\mathbf{P}(\mathbb{R})^+\mathbf{U}(\mathbb{C})$-orbit of h_0, where $\mathbf{P}(\mathbb{R})^+$ is the identity component of $\mathbf{P}(\mathbb{R})$. Let $\check{D}$ be the $\mathbf{P}(\mathbb{C})$-orbit of h_0.

By André [1, Proof of Theorem 1], $\mathbf{P}$ is normal in $\mathcal{MT}$. From Section 3.2, we have a quotient morphism

$$f : (\mathcal{MT}, \mathcal{X}_{\mathcal{MT}}, D^+_{\mathcal{MT}}) \to (\mathcal{MT} \,/\, \mathbf{P}, \mathcal{X}_{\mathcal{MT}\,/\,\mathbf{P}}, D^+_{\mathcal{MT}\,/\,\mathbf{P}})$$

of connected mixed Hodge datum. Let

$$\mathfrak{mt}_{\mathbb{C}} = \bigoplus_{p,q} \mathfrak{mt}^{p,q}$$

be the Deligne bigrading [23] of the mixed Hodge structure on the Lie algebra $\mathfrak{mt}$ of $\mathcal{MT}$. Let $\mathfrak{b}$ be the Lie algebra of the stabilizer B in $\mathcal{MT}(\mathbb{C})$ of h_0. By Remark 2.4 and (21) of [45], $\mathfrak{b}_{\mathcal{MT}}$ can be identified with

$$\bigoplus_{p\geq 0;q} \mathfrak{mt}^{p,q}.$$

Let $\mathfrak{v}$ be the Lie algebra of $V = \mathcal{MT}(\mathbb{R})^+\,\mathcal{MT}_u(\mathbb{C}) \cap B$. By the definition of mixed Hodge datum,

$$\mathfrak{mt}_{u,\mathbb{C}} = \bigoplus_{p+q\leq -1} \mathfrak{mt}^{p,q}.$$

We have

$$\begin{aligned}
\mathfrak{v} &= (\mathfrak{mt}_{\mathbb{R}} + \mathfrak{mt}_{u,\mathbb{C}}) \cap \mathfrak{b} \\
&= \left(\left(\bigoplus_{p+q=0} \mathfrak{mt}^{p,q} \right)_{\mathbb{R}} \oplus \left(\bigoplus_{p+q\leq -1} \mathfrak{mt}^{p,q} \right) \right) \cap \bigoplus_{p\geq 0;q} \mathfrak{mt}^{p,q} \\
&= \mathfrak{mt}^{0,0}_{\mathbb{R}} \oplus \left(\bigoplus_{p+q\leq -1;p\geq 0} \mathfrak{mt}^{p,q} \right).
\end{aligned}$$

Hence we have an identification

$$\alpha : T_{h_0} D^+_{\mathcal{MT}} \simeq \left(\bigoplus_{p\neq 0} \mathfrak{mt}^{p,-p} \right)_{\mathbb{R}} \oplus \left(\bigoplus_{p+q\leq -1;p<0} \mathfrak{mt}^{p,q} \right) =: (\mathfrak{mt}_{\mathbb{R}} + \mathfrak{mt}_{u,\mathbb{C}})^-.$$

Let β be the projection

$$\mathfrak{mt}_{\mathbb{R}} + \mathfrak{mt}_{u,\mathbb{C}} = \left(\bigoplus_{p+q=0} \mathfrak{mt}^{p,q}\right)_{\mathbb{R}} \oplus \left(\bigoplus_{p+q\leq -1} \mathfrak{mt}^{p,q}\right) \to (\mathfrak{mt}_{\mathbb{R}} + \mathfrak{mt}_{u,\mathbb{C}})^{-}.$$

Denote the kernel of β by $(\mathfrak{mt}_{\mathbb{R}} + \mathfrak{mt}_{u,\mathbb{C}})^{+}$. Let $\mathfrak{p}$ and $\mathfrak{u}$ be the Lie algebras of $\mathbf{P}$ and $\mathbf{U}$ respectively. Replacing $\mathfrak{mt}$ by $\mathfrak{mt}/\mathfrak{p}$, we also have maps γ and δ, akin to α and β respectively.

Lemma 3.5. *The connected component D_0 of $f^{-1}(f(h_0))$ that contains h_0 is contained in D.*

Proof. Define a map

$$g_1 : \mathcal{MT}(\mathbb{R})^{+}\,\mathcal{MT}_u(\mathbb{C}) \to D^{+}_{\mathcal{MT}}$$

by $m \mapsto m \cdot h_0$ for any $m \in \mathcal{MT}(\mathbb{R})^{+}\,\mathcal{MT}_u(\mathbb{C})$. Similarly, we have a map

$$g_2 : \mathbf{P}(\mathbb{R})^{+}\,\mathbf{U}(\mathbb{C}) \to D_0,$$

where $\mathbf{P}(\mathbb{R})^{+}$ is the identity component of $\mathbf{P}(\mathbb{R})$, and also a map

$$g_3 : (\mathcal{MT}\,/\,\mathbf{P})(\mathbb{R})^{+}(\mathcal{MT}\,/\,\mathbf{P})_u(\mathbb{C}) \to D^{+}_{\mathcal{MT}\,/\,\mathbf{P}}.$$

The differentials dg_1 and dg_3 of g_1 and g_3 are $\alpha^{-1} \circ \beta$ and $\gamma^{-1} \circ \delta$ respectively. Let $\mathbf{P}_r$ be a fixed Levi subgroup of $\mathbf{P}$ that is contained in $\mathcal{MT}_r$. Let

$$(\mathcal{MT}\,/\,\mathbf{P})_r := \mathcal{MT}_r\,\mathbf{P}\,/\,\mathbf{P},$$

which is a Levi subgroup of $\mathcal{MT}\,/\,\mathbf{P}$. Note that

$$\mathfrak{p}_{r,\mathbb{R}} \oplus \mathfrak{u}_{\mathbb{C}} = \mathfrak{p}_{\mathbb{R}} + \mathfrak{u}_{\mathbb{C}},$$

$$\mathfrak{mt}_{r,\mathbb{R}} \oplus \mathfrak{mt}_{u,\mathbb{C}} = \mathfrak{mt}_{\mathbb{R}} + \mathfrak{mt}_{u,\mathbb{C}}$$

and

$$(\mathfrak{mt}/\mathfrak{p})_{r,\mathbb{R}} \oplus (\mathfrak{mt}/\mathfrak{p})_{u,\mathbb{C}} = (\mathfrak{mt}/\mathfrak{p})_{\mathbb{R}} + (\mathfrak{mt}/\mathfrak{p})_{u,\mathbb{C}}.$$

We have the following commutative diagram of differentials

$$\begin{tikzcd}
T_{h_0}D_0 \arrow[r, hook] & T_{h_0}D^+_{\mathcal{MT}} \arrow[r] & T_{f(h_0)}D^+_{\mathcal{MT}/\mathbf{P}} \\
\mathfrak{p}_{r,\mathbb{R}} \oplus \mathfrak{u}_{\mathbb{C}} \arrow[r, hook] \arrow[u, "dg_2"] & \mathfrak{mt}_{r,\mathbb{R}} \oplus \mathfrak{mt}_{u,\mathbb{C}} \arrow[r, "q"] \arrow[u, "dg_1"] & (\mathfrak{mt}/\mathfrak{p})_{r,\mathbb{R}} \oplus (\mathfrak{mt}/\mathfrak{p})_{u,\mathbb{C}}, \arrow[u, "dg_3"]
\end{tikzcd}$$

where the composition of the upper horizontal maps is zero, and where q is the componentwise quotient by $\mathfrak{p}_{r,\mathbb{R}}$ and $\mathfrak{p}_{u,\mathbb{C}}$. Let $v \in T_{h_0}D_0$. Since $\mathcal{MT}(\mathbb{R})^+\,\mathcal{MT}_u(\mathbb{C})$ acts transitively on $D^+_{\mathcal{MT}}$, there exists $w \in \mathfrak{mt}_{\mathbb{R}} + \mathfrak{mt}_{u,\mathbb{C}}$ such that $(dg_1)(w) = v$. Write $w = w^- + w^+$, where $w^- \in (\mathfrak{mt}_{\mathbb{R}} + \mathfrak{mt}_{u,\mathbb{C}})^-$ and $w^+ \in (\mathfrak{mt}_{\mathbb{R}} + \mathfrak{mt}_{u,\mathbb{C}})^+$. Recall that $T_{h_0}D^+_{\mathcal{MT}} \simeq (\mathfrak{mt}_{\mathbb{R}} + \mathfrak{mt}_{u,\mathbb{C}})^-$, so

$$(dg_1)(w^-) = (dg_1)(w) - (dg_1)(w^+) = v.$$

By commutativity, we then know $q(w^-)$ is in the kernel of $dg_3 = \gamma^{-1} \circ \delta$, so

$$q(w^-) \in ((\mathfrak{mt}/\mathfrak{p})_{\mathbb{R}} + (\mathfrak{mt}/\mathfrak{p})_{u,\mathbb{C}})^+.$$

Moreover, the quotient morphism f of mixed Hodge data induces a morphism $df : \mathfrak{mt} \to \mathfrak{mt}/\mathfrak{p}$ of mixed Hodge structures, so

$$q(w^-) \in ((\mathfrak{mt}/\mathfrak{p})_{\mathbb{R}} + (\mathfrak{mt}/\mathfrak{p})_{u,\mathbb{C}})^-.$$

Hence, $q(w^-) = 0$. Since q is the componentwise quotient by $\mathfrak{p}_{r,\mathbb{R}}$ and $\mathfrak{p}_{u,\mathbb{C}}$, we thus have $w^- \in \mathfrak{p}_{\mathbb{R}} + \mathfrak{u}_{\mathbb{C}}$. Then since dg_2 is the restriction of dg_1, the differential dg_2 is surjective. Repeating the argument by replacing h_0 by any $h \in D_0$, we know g_2 is a submersion, and thus an open mapping. Repeating the argument by replacing h_0 by any $h \in D_0$, every $\mathbf{P}(\mathbb{R})^+\,\mathbf{U}(\mathbb{C})$-orbit in D_0 is open. By the connectedness of D_0, the group $\mathbf{P}(\mathbb{R})^+\,\mathbf{U}(\mathbb{C})$ acts transitively on D_0. Therefore, $D_0 \subset D$. $\square$

Lemma 3.6. *Let K be a subfield of $\mathbb{C}$. Let E be a subset of the set $\mathbb{A}^n_K(K)$ of K-points of the n-dimensional affine space. Let M be the smallest closed algebraic K-subvariety of $\mathbb{A}^n_K$ such that $E \subset M(K)$. The complex variety $M_{\mathbb{C}}$ is the smallest closed algebraic $\mathbb{C}$-subvariety of $\mathbb{A}^n_{\mathbb{C}}$ such that $E \subset M(\mathbb{C})$.*

Proof. Suppose $M'_{\mathbb{C}}$ is a closed algebraic $\mathbb{C}$-subvariety of $\mathbb{A}^n_{\mathbb{C}}$ such that $E \subset M'_{\mathbb{C}}(\mathbb{C})$. The complex algebraic variety

$$\bigcap_{\sigma \in \mathrm{Gal}(\mathbb{C}/K)} \sigma M'_{\mathbb{C}}$$

is stable under the Galois action by $\mathrm{Gal}(\mathbb{C}/K)$. so it has a closed affine model L over K by Galois descent [41, Prop. 16.1, 16.8] because K is perfect. Since $E \subset \mathbb{A}^n_K(K)$, we know $\sigma E = E$ for any $\sigma \in \mathrm{Gal}(\mathbb{C}/K)$. Hence, $E \subset L(K)$, and thus $M \subset L$. Therefore, $M_{\mathbb{C}} \subset L_{\mathbb{C}} \subset M'_{\mathbb{C}}$, as desired. □

Let $\widetilde{X}$ be the universal cover of X.

Theorem 3.7. *The domain D contains the image of the period mapping $\widetilde{X} \to D^+_{\mathcal{MT}}$. For any weak mumford Mumford-Tate domain $D(\mathbf{M})$ that contains this image,* $\dim D \leq \dim D(\mathbf{M})$.

Proof. Since $\mathbf{P}(\mathbb{Q}) \cap \Gamma$ is of finite index in Γ, the composition

$$\widetilde{X} \to D^+_{\mathcal{MT}} \to D^+_{\mathcal{MT}/\mathbf{P}}$$

descends to a period mapping on X whose associated GPVMHS has finite monodromy. Passing to a finite covering of X, we have a GPVMHS with trivial monodromy. By rigidity [17, Theorem 7.12], this GPVMHS and its associated period mapping are constant. Hence, the lifting

$$\widetilde{X} \to D^+_{\mathcal{MT}} \to D^+_{\mathcal{MT}/\mathbf{P}}$$

is constant, with value $f(h_0)$. Hence, the image of $\widetilde{X}$ in $D^+_{\mathcal{MT}}$ lies in D_0, and thus lies in D by Lemma 3.5.

Suppose $D(\mathbf{M})$ is a weak Mumford-Tate domain that contains the image of $\widetilde{X} \to D_{\mathcal{MT}}$. We have $\Gamma \cdot h \subset D(\mathbf{M})$. Since Γ is $\mathbb{Q}$-Zariski dense in $\mathbf{P}$, it is $\mathbb{C}$-Zariski dense in $\mathbf{P}_{\mathbb{C}}$ by Lemma 3.6. Hence, $\check{D} \subset \check{D}(\mathbf{M})$, and thus $\dim D \leq \dim D(\mathbf{M})$ by Lemma 3.4. □

3.5 Real-split retractions of weak Mumford-Tate domains

Let h be a mixed $\mathbb{Q}$-Hodge structure with Mumford-Tate group $\mathcal{MT}$. It induces a connected mixed Hodge datum $(\mathcal{MT}, \mathcal{X}_{\mathcal{MT}}, D^+_{\mathcal{MT}})$. By [1, Lemma 2], the group $\mathcal{MT}$ preserves the weight filtration, so it is contained in $\mathcal{P}^{\mathcal{M}}$, where $\mathcal{M}$ is the the classifying space where the fixed graded polarization and Hodge numbers are chosen to be the same as that of h. Let $\{W_\bullet\}$ be the weight filtration of h. From Section 3.2, we have a quotient morphism

$$q : (\mathcal{MT}, \mathcal{X}_{\mathcal{MT}}, D^+_{\mathcal{MT}}) \to (\mathcal{MT}\,/\,\mathcal{MT}_u, \mathcal{X}_{\mathcal{MT}\,/\,\mathcal{MT}_u}, D^+_{\mathcal{MT}\,/\,\mathcal{MT}_u})$$

of connected mixed Hodge datum. Let $\overline{h}$ be the image of h under this morphism. The group $(\mathcal{MT}\,/\,\mathcal{MT}_u)(\mathbb{R})^+$ acts transitively on $D^+_{\mathcal{MT},\mathrm{Gr}} := D^+_{\mathcal{MT}\,/\,\mathcal{MT}_u}$. We call it the Mumford-Tate domain for the associated graded $\overline{h}$. Let $\mathbf{M}$ be a normal algebraic $\mathbb{Q}$-subgroup of $\mathcal{MT}$. Let $D(\mathbf{M})$ be the corresponding weak Mumford-Tate domain, i.e. the $\mathbf{M}(\mathbb{R})^+\,\mathbf{M}_u(\mathbb{C})$-orbit of h.

Given a mixed $\mathbb{R}$-Hodge structure $(H, W_\bullet, F^\bullet)$ with Deligne bigrading $H^{p,q}$, we define a nilpotent Lie algebra

$$L^{-1,-1}_{(W,F)} := \{X \in \mathrm{End}(H_{\mathbb{C}}) : X(H^{p,q}) \subset \bigoplus_{r<p,s<q} H^{r,s}\}.$$

Proposition 3.8. [19, Prop. 2.20, 2.24] *Given a mixed $\mathbb{R}$-Hodge structure $(H, W_\bullet, F^\bullet)$, there exists a unique $\delta \in L^{-1,-1}_{(W,F)}$ such that $(H, W_\bullet, e^{-i\delta}F^\bullet)$ is a real-split mixed $\mathbb{R}$-Hodge structure. This yields a $\mathcal{P}^{\mathcal{M}}(\mathbb{R})$-equivariant smooth real semi-algebraic retraction $r : \mathcal{M} \to \mathcal{M}_{\mathbb{R}}$, where $\mathcal{M}_{\mathbb{R}}$ is the set of real-split mixed $\mathbb{R}$-Hodge structures in $\mathcal{M}$.*

For any connected mixed Hodge datum $(P, , \mathcal{X}_P, D^+_P)$, let $D^+_{P,\mathbb{R}} := r(D^+_P)$.

Lemma 3.9 ([4], Lemma 6.6)**.** *There is a $P(\mathbb{R})^+$-equivariant retraction r_P : $D_P^+ \to D_{P,\mathbb{R}}^+$ defined by restricting r to D_P^+, for every connected mixed Hodge datum $(P, , \mathcal{X}_P, D_P^+)$. These retractions are compatible with morphisms of mixed Hodge data.*

Let $D_{\mathbf{M},\mathbb{R}}^+ := r(D(\mathbf{M}))$. Let $D_{\mathbf{M},\mathrm{Gr}}^+ := q(r(D(\mathbf{M})))$. Let $D_{\mathbf{M}_u,\mathbb{R}}$ be the $\mathbf{M}(\mathbb{R})^+$-orbit in $\mathcal{S}(W_\bullet)(\mathbb{R})$ of the splitting $h_{1,s}$ of $h_1 := r(h)$.

Lemma 3.10. *The unipotent radical $\mathbf{M}_u(\mathbb{R})$ acts simply transitively on $D_{\mathbf{M}_u,\mathbb{R}}$.*

Proof. Let $\phi : \mathbf{M}(\mathbb{R})^+ \to D_{\mathbf{M}_u,\mathbb{R}}$ be the submersion defined by $\phi(g) = g \cdot h_{1,s}$ for any $g \in \mathbf{M}(\mathbb{R})^+$. This induces a surjective differential $d\phi : \mathfrak{m}_\mathbb{R} \to T_{h_{1,s}} D_{\mathbf{M}_u,\mathbb{R}}$. The real-split mixed Hodge structure $h_{1,s}$ gives a Deligne bigrading $\bigoplus_{r,s} H^{r,s}$. Let

$$B_k := \bigoplus_{r+s=k} H^{r,s}.$$

The splitting $h_{1,s} \in \mathcal{S}(W_\bullet)(\mathbb{R})$ of h_1 is given by $\{B_k\}$. Let

$$\mathfrak{m}^{p,q} := \{X \in \mathfrak{m}_\mathbb{C} : X(H^{r,s}) \subset H^{r+p,s+q} \text{ for any } r, s\}.$$

Let

$$\mathfrak{s}_k := \bigoplus_{p+q=k} \mathfrak{m}^{p,q}.$$

For any $t \in \mathbb{R}$, $X \in \mathfrak{s}_0$ and $v \in B_k$,

$$\exp(tX)(v) = v + (tX)(v) + \frac{1}{2!}(tX)((tX)(v)) + \cdots$$

is in B_k. Hence, $\exp(tX)(B_k) = B_k$ for any k. Thus, $\exp(tX) \cdot h_{1,s} = h_{1,s}$ for any $t \in \mathbb{R}$ and $X \in \mathfrak{s}_0$. We then have $(d\phi)(X) = 0$ for any $X \in \mathfrak{s}_0 \cap \mathfrak{m}_\mathbb{R}$. Therefore, $d\phi$ induces a surjection

$$\mathfrak{m}_\mathbb{R} \cap \bigoplus_{k \leq -1} \mathfrak{s}_k \to T_{h_{1,s}} D_{\mathbf{M}_u,\mathbb{R}},$$

so ϕ restricts to a surjection

$$\left(\exp \bigoplus_{k \leq -1} \mathfrak{s}_k\right)(\mathbb{R}) \to D_{\mathbf{M}_u,\mathbb{R}}.$$

By [19, Prop. 2.2], this mapping is injective. By the definition of mixed Hodge datum,

$$\mathfrak{mt}_{u,\mathbb{C}} = \bigoplus_{p+q \leq -1} \mathfrak{mt}^{p,q}.$$

By functoriality of the Deligne bigrading, $\mathfrak{m}^{p,q} = \mathfrak{m}_{\mathbb{C}} \cap \mathfrak{mt}^{p,q}$. Hence,

$$\mathfrak{m}_{u,\mathbb{C}} = \mathfrak{m}_{\mathbb{C}} \cap \mathfrak{mt}_{u,\mathbb{C}} = \bigoplus_{p+q \leq -1} \mathfrak{m}^{p,q} = \bigoplus_{k \leq -1} \mathfrak{s}_k.$$

Therefore, ϕ restricts to an isomorphism $\mathbf{M}_u(\mathbb{R}) \simeq D_{\mathbf{M}_u,\mathbb{R}}$, as desired. □

Lemma 3.11. *The group $(\mathbf{M} \,/\, \mathbf{M}_u)(\mathbb{R})^+$ acts transitively on $D^+_{\mathbf{M},\mathrm{Gr}}$.*

Proof. From Section 3.2, we have a quotient morphism

$$f : (\mathcal{MT}, \mathcal{X}_{\mathcal{MT}}, D^+_{\mathcal{MT}}) \to (\mathcal{MT} \,/\, \mathbf{M}, \mathcal{X}_{\mathcal{MT} \,/\, \mathbf{M}}, D^+_{\mathcal{MT} \,/\, \mathbf{M}})$$

between connected mixed Hodge data. Similarly, we have a map

$$\overline{q} : D^+_{\mathcal{MT} \,/\, \mathbf{M}} \to D^+_{(\mathcal{MT} \,/\, \mathbf{M})/(\mathcal{MT} \,/\, \mathbf{M})_u} =: D^+_{\mathcal{MT} \,/\, \mathbf{M},\mathrm{Gr}}.$$

Let $D^+_{\mathcal{MT} \,/\, \mathbf{M},\mathbb{R}} := r(D^+_{\mathcal{MT} \,/\, \mathbf{M}})$. By Lemma 3.9, we have an $\mathcal{MT}(\mathbb{R})^+$-equivariant retraction

$$r_{\mathcal{MT}} : D^+_{\mathcal{MT}} \to D^+_{\mathcal{MT},\mathbb{R}}$$

and an $(\mathcal{MT} \,/\, \mathbf{M})(\mathbb{R})^+$-equivariant retraction

$$r_{\mathcal{MT} \,/\, \mathbf{M}} : D^+_{\mathcal{MT} \,/\, \mathbf{M}} \to D^+_{\mathcal{MT} \,/\, \mathbf{M},\mathbb{R}},$$

satisfying $r_{\mathcal{MT} \,/\, \mathbf{M}} \circ f = f_{\mathbb{R}} \circ r$, where

$$f_{\mathbb{R}} : D^+_{\mathcal{MT},\mathbb{R}} \to D^+_{\mathcal{MT} \,/\, \mathbf{M},\mathbb{R}}$$

is the morphism induced from f by restriction. The weak Mumford-Tate domain $D(\mathbf{M})$ is in the fiber $f^{-1}(x)$ of a point $x \in D^+_{\mathcal{MT}\,/\,\mathbf{M}}$. Then

$$f_{\mathbb{R}}(r(D(\mathbf{M}))) = \{r_{\mathcal{MT}\,/\,\mathbf{M}}(x)\}.$$

From Section 3.2, we have a quotient morphism

$$g_1 : D^+_{\mathcal{MT},\mathrm{Gr}} \to D^+_{\mathcal{MT}\,/\,\mathbf{M},\mathrm{Gr}}.$$

We have the following commutative diagram:

$$\begin{tikzcd}
r(D(\mathbf{M})) \arrow[r, hook] \arrow[d, "q"] & D^+_{\mathcal{MT},\mathbb{R}} \arrow[r, "f_{\mathbb{R}}"] \arrow[d, "q"] & D^+_{\mathcal{MT}\,/\,\mathbf{M},\mathbb{R}} \arrow[d, "\overline{q}"] \\
D^+_{\mathbf{M},\mathrm{Gr}} \arrow[r, hook] & D^+_{\mathcal{MT},\mathrm{Gr}} \arrow[r, "g_1"] & D^+_{\mathcal{MT}\,/\,\mathbf{M},\mathrm{Gr}}.
\end{tikzcd}$$

The conclusion in the first paragraph that $f_{\mathbb{R}}(r(D(\mathbf{M}))) = \{r_{\mathcal{MT}\,/\,\mathbf{P}}(x)\}$ then implies that g_1 maps $D^+_{\mathbf{M},\mathrm{Gr}}$ to a point $y := \overline{q}(r(x))$.

Let $h \in D^+_{\mathbf{M},\mathrm{Gr}}$. The map

$$a_1 : (\mathbf{M}\,/\,\mathbf{M}_u)(\mathbb{R})^+ \to D^+_{\mathbf{M},\mathrm{Gr}}$$

defined by $m \mapsto m \cdot h$ induces the differential $(\mathfrak{m}/\mathfrak{m}_u)_{\mathbb{R}} \to T_h D^+_{\mathbf{M},\mathrm{Gr}}$. Similarly, we have a map

$$a_2 : (\mathcal{MT}\,/\,\mathcal{MT}_u)(\mathbb{R})^+ \to D^+_{\mathcal{MT},\mathrm{Gr}}$$

and a map

$$a_3 : ((\mathcal{MT}\,/\,\mathbf{M})/(\mathcal{MT}\,/\,\mathbf{M})_u)(\mathbb{R})^+ \to D^+_{\mathcal{MT}\,/\,\mathbf{M},\mathrm{Gr}}.$$

There is a weight 0 Hodge structure on $\mathfrak{mt}/\mathfrak{mt}_u$. The tangent space $T_h D^+_{\mathcal{MT},\mathrm{Gr}}$ can be identified through a map α with the real points

$$(\mathfrak{mt}/\mathfrak{mt}_u)_{\mathbb{R}} \cap \bigoplus_{a \neq 0} (\mathfrak{mt}/\mathfrak{mt}_u)^{a,-a},$$

see [18, Chapter 12]. We have a projection of $(\mathfrak{mt}/\mathfrak{mt}_u)_{\mathbb{R}}$ onto $T_h D^+_{\mathcal{MT},\mathrm{Gr}}$ with kernel the real points $(\mathfrak{mt}/\mathfrak{mt}_u)_{\mathbb{R}} \cap (\mathfrak{mt}/\mathfrak{mt}_u)^{0,0}$. This projection is the differential da_2 of a_2. Let $v \in T_h D^+_{\mathbf{M},\mathrm{Gr}}$. We have the following commutative diagram of differentials

$$\begin{tikzcd}
T_h D^+_{\mathbf{M},\mathrm{Gr}} \arrow[r, hook] & T_h D^+_{\mathcal{MT},\mathrm{Gr}} \arrow[r] & T_y D^+_{\mathcal{MT}\,/\,\mathbf{M},\mathrm{Gr}} \\
(\mathfrak{m}/\mathfrak{m}_u)_{\mathbb{R}} \arrow[r, hook] \arrow[u, "da_1"] & (\mathfrak{mt}/\mathfrak{mt}_u)_{\mathbb{R}} \arrow[r, "q"] \arrow[u, "da_2"] & ((\mathfrak{mt}/\mathfrak{m})/(\mathfrak{mt}/\mathfrak{m})_u)_{\mathbb{R}}, \arrow[u, "da_3"]
\end{tikzcd}$$

where the composition of the upper horizontal maps is the zero, and where q is the quotient morphism. Since $(\mathcal{MT}\,/\,\mathcal{MT}_u)(\mathbb{R})^+$ acts transitively on $D^+_{\mathcal{MT},\mathrm{Gr}}$, there exists $w \in (\mathfrak{mt}/\mathfrak{mt}_u)_{\mathbb{R}}$ such that $(da_2)(w) = v$. Write $w = w' + w''$, where

$$w' \in (\mathfrak{mt}/\mathfrak{mt}_u)_{\mathbb{R}} \cap \bigoplus_{a \neq 0} (\mathfrak{mt}/\mathfrak{mt}_u)^{a,-a} \quad \text{and} \quad w'' \in (\mathfrak{mt}/\mathfrak{mt}_u)_{\mathbb{R}} \cap (\mathfrak{mt}/\mathfrak{mt}_u)^{0,0}.$$

Then

$$(da_2)(w') = (da_2)(w) - (da_2)(w'') = v.$$

Hence, $q(w')$ is in the kernel of da_3, so

$$q(w') \in ((\mathfrak{mt}/\mathfrak{m})/(\mathfrak{mt}/\mathfrak{m})_u)_{\mathbb{R}} \cap ((\mathfrak{mt}/\mathfrak{m})/(\mathfrak{mt}/\mathfrak{m})_u)^{0,0}.$$

Moreover, the quotient morphism g_1 of pure Hodge data of the associated graded induces a morphism

$$dg_1 : (\mathfrak{mt}/\mathfrak{mt}_u)_{\mathbb{R}} \to ((\mathfrak{mt}/\mathfrak{m})/(\mathfrak{mt}/\mathfrak{m})_u)_{\mathbb{R}}$$

of pure Hodge structures that preserves gradings, so

$$q(w') \in ((\mathfrak{mt}/\mathfrak{m})/(\mathfrak{mt}/\mathfrak{m})_u)_{\mathbb{R}} \cap \bigoplus_{a \neq 0} ((\mathfrak{mt}/\mathfrak{m})/(\mathfrak{mt}/\mathfrak{m})_u)^{a,-a}.$$

Hence, $q(w') = 0$, thus $w' \in (\mathfrak{m}/\mathfrak{m}_u)_{\mathbb{R}}$ since q is the quotient morphism. Then since da_1 is the restriction of da_2, the differential da_1 is surjective. Since the choice of h in $D^+_{\mathbf{M},\mathrm{Gr}}$ is arbitrary, the map a_1 is a submersion, and thus an open map. Again since $h \in D^+_{\mathbf{M},\mathrm{Gr}}$ is arbitrary, every $(\mathbf{M}\,/\,\mathbf{M}_u)(\mathbb{R})^+$-orbit in $D^+_{\mathbf{M},\mathrm{Gr}}$ is open. By the connectedness of $D^+_{\mathbf{M},\mathrm{Gr}}$, the group $(\mathbf{M}\,/\,\mathbf{M}_u)(\mathbb{R})^+$ acts transitively on $D^+_{\mathbf{M},\mathrm{Gr}}$. $\square$

Theorem 3.12. *The group $\mathbf{M}(\mathbb{R})^+$ acts transitively on $D^+_{\mathbf{M},\mathbb{R}} := r(D(\mathbf{M}))$. We have an $\mathbf{M}(\mathbb{R})^+$-equivariant definable isomorphism $j : D^+_{\mathbf{M},\mathbb{R}} \simeq D^+_{\mathbf{M},\mathrm{Gr}} \times D_{\mathbf{M}_u,\mathbb{R}}$ which sends h to (h_{Gr}, h_s), where $h_{\mathrm{Gr}} := q(h)$ and h_s is the splitting of h.*

Proof. The map j is injective in view of Deligne bigrading. It is clear that j is $\mathbf{M}(\mathbb{R})^+$-equivariant, and thus j is definable. Fix a Levi subgroup $\mathbf{M}_r$ of $\mathbf{M}$. It gives an isomorphism $\mathbf{M}_r \simeq \mathbf{M}\,/\,\mathbf{M}_u$. Let $(x, y) \in D^+_{\mathbf{M},\mathrm{Gr}} \times D_{\mathbf{M}_u,\mathbb{R}}$. By Lemma 3.10 and Lemma 3.11, $x = g \cdot h_{1,\mathrm{Gr}}$ and $y = u \cdot h_{1,s}$ for some $g \in \mathbf{M}_r(\mathbb{R})^+$ and $u \in \mathbf{M}_u(\mathbb{R})$. By Lemma 3.10, there exists $u' \in \mathbf{M}_u(\mathbb{R})$ such that $g^{-1}h_{1,s} = u'h_{1,s}$. Then $j(ugu'h_1) = (ugu'h_{1,\mathrm{Gr}}, ugu'h_{1,s}) = (gh_{1,\mathrm{Gr}}, uh_{1,s}) = (x, y)$ since the unipotent radical acts trivially on the associated graded. Therefore, j is surjective, and the group $\mathbf{M}(\mathbb{R})^+$ acts transitively on $D^+_{\mathbf{M},\mathbb{R}}$. $\square$

Chapter 4

Ax-Schanuel for mixed period mappings

4.1 Statements of results

Let X be a smooth irreducible algebraic variety over $\mathbb{C}$. Let $(\mathcal{H}, \mathcal{W}_\bullet, \mathcal{F}^\bullet, \mathcal{Q})$ be an admissible graded-polarized variation of mixed $\mathbb{Z}$-Hodge structures (GPVMHS) on X. Let η be a Hodge generic point of X. Let Γ be the image of the monodromy representation $\pi_1(X, \eta) \to \mathbf{GL}(\mathcal{H}_{\mathbb{Z},\eta})$ associated to the local system $\mathcal{H}$. Let $\mathbf{P}$ be the identity component of the $\mathbb{Q}$-Zariski closure of Γ in $\mathbf{GL}(\mathcal{H}_{\mathbb{Q},\eta})$. It is known that $\mathbf{P}$ is normal in the Mumford-Tate group $\mathcal{MT}$ at η [1, Proof of Theorem 1]. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{P}$. The Deligne splitting of the graded-polarized mixed Hodge structure h_0 on the stalk $\mathcal{H}_\eta$ defines a representation $\rho_0 : \mathbf{S}_\mathbb{C} \to \mathbf{GL}(\mathcal{H}_{\mathbb{C},\eta})$, see Section 2.2. Let $\mathcal{M}$ and $\breve{\mathcal{M}}$ be respectively the classifying space and the projective space defined in Section 3.1, where the fixed graded polarization and Hodge numbers are chosen to be the same as that of the mixed Hodge structures our GPVMHS is parametrizing. Let D be the $\mathbf{P}(\mathbb{R})^+\mathbf{U}(\mathbb{C})$-orbit of h_0 in $\mathcal{M}$, where $\mathbf{P}(\mathbb{R})^+$ is the identity component of $\mathbf{P}(\mathbb{R})$. Let $\check{D}$ be the $\mathbf{P}(\mathbb{C})$-orbit of h_0 in $\breve{\mathcal{M}}$. First assume $\Gamma \subset \mathbf{P}(\mathbb{Z}) \cap \mathbf{P}(\mathbb{R})^+ =: \mathbf{P}(\mathbb{Z})^+$. This is assumed everywhere outside Theorem 4.4 and its proof. Let $\psi : X \to \Gamma\backslash D$ be the period mapping. Let φ be the composition of ψ with $\Gamma\backslash D \to \mathbf{P}(\mathbb{Z})^+\backslash D$. Consider the fiber product

$$\begin{array}{ccc} W & \longrightarrow & D \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\varphi} & \mathbf{P}(\mathbb{Z})^+\backslash D. \end{array}$$

Definition 4.1. For any weak Mumford-Tate domain $D(\mathbf{M}) \subset D$, we say $\varphi^{-1}\pi(D(\mathbf{M}))$ is a *weakly special subvariety* of X.

Let $p_X : X \times \check{D} \to X$ and $p_D : X \times \check{D} \to \check{D}$ be the projections onto X and $\check{D}$ respectively. Let U be an irreducible analytic subset of W, denote by U^{Zar} the Zariski closure of U in $X \times \check{D}$. Let $V := U^{\mathrm{Zar}}$.

Theorem 4.2. *If* $\dim V - \dim U < \dim \check{D}$, *then* $p_X(U)$ *is contained in a proper weakly special subvariety.*

Remark 4.3. This theorem is equivalent to the statement with φ replaced by ψ, *cf.* Lemma 4.5. By [4, Corollary 6.7], weakly special subvarieties are indeed algebraic.

Let $\psi' : X \to \Gamma\backslash\mathcal{M}$ be the period mapping. Let $p_X : X \times \mathcal{M} \to X$ be the projection onto X.

Consider the fiber product

$$\begin{array}{ccc} W' & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow{\scriptstyle \pi'} \\ X & \xrightarrow{\psi'} & \Gamma\backslash\mathcal{M}. \end{array}$$

Let U' be an irreducible analytic subset of W'. Let U'^{Zar} be the Zariski closure of U' in $X \times \breve{\mathcal{M}}$.

Theorem 4.4. *If* $\dim U'^{\mathrm{Zar}} - \dim U' < \dim \check{D}$, *then* $p_X(U')$ *is contained in a proper weakly special subvariety.*

4.2 Sketch of the proof

Theorem 4.2 and Theorem 4.4 will be proved by induction on certain quantities involving the dimensions of X, V and U. In Section 4.3, we will prove the base cases of induction. Let $\mathbf{N}$ be the identity component of the $\mathbb{Q}$-Zariski closure of the $\mathbf{P}(\mathbb{Z})^+$-stabilizer of V. We prove that $\mathbf{N}$ is normal in $\mathbf{P}$ in Section 4.4 using the Hilbert scheme argument of Mok-Pila-Tsimerman [43]. We consider certain definable set I and its image $\overline{I}$ under the map $\mathbf{P}(\mathbb{C}) \to (\mathbf{P} / \mathbf{N})(\mathbb{C})$. We apply the Pila-Wilkie theorem on $\overline{I}$ to get a semi-algebraic curve in it that contains arbitrarily many points (see Lemma 4.16 for the precise meaning). The semi-algebraic curve can then be used to construct varieties having smaller or larger dimensions such that the induction hypothesis can be used to finish the proof.

The definable set I has to be defined in a way to facilitate the use of the induction hypothesis after the application of Pila-Wilkie on $\overline{I}$. At the first attempt, one collects all $\gamma \in \mathbf{P}(\mathbb{C})$ such that $W \cap \gamma V$ and U have the same dimension. However, since I has to be definable, one modifies the attempt by further intersecting $W \cap \gamma V$ with $X \times \Phi$, where Φ is a definable fundamental set for the action of $\mathbf{P}(\mathbb{Z})^+$ on D. The definability of $W \cap (X \times \Phi)$ follows from the definability of the mixed period mappings obtained by Bakker-Brunebarbe-Klingler-Tsimerman [4]. The definition of I is also in terms of certain conditions on weakly special subvarieties containing these intersections. The precise construction of I will be made in Section 4.5.2.

In order to use the counting theorem of Pila-Wilkie [54], we need $\overline{I}$ to contain at least polynomially many rational points with certain property. We count these points in $\overline{I}$ using the mixed point counting method in [29]. This method leads us to a trichotomy (Section 4.6), roughly as follows:

1. The projection of U to the semisimple part, modulo the stabilizing part (since we are counting modulo $\mathbf{N}(\mathbb{Q})$), has positive dimension, and the

unipotent direction grows slower than the semisimple direction. In this case, we apply the volume estimates of Griffith transverse subvarieties of a pure weak Mumford-Tate domain established by Bakker-Tsimerman [7]. This will be done in Section 4.7.

2. The unipotent direction grows faster than the semisimple direction. In this case, we prove and apply a height estimate on products of certain conjuguates of upper unitriangular matrices (upper triangular matrix with 1's on the diagonal). This will be done in Section 4.8.

3. The point count, modulo $\mathbf{N}(\mathbb{Q})$, is finite and U lies in a unipotent fiber. This case uses the definable Chow theorem [48], see Section 4.9.

The trichotomy motivates the decomposition of $r(D)$ into three parts: the unipotent part, the stabilizing semisimple part, and the non-stabilizing semisimple part. The definable fundamental domain Φ is built from the fundamental domains in each of these three parts, see Section 4.5.1. This motivates the use of the retraction r in Section 3.5 because the fundamental domain for the unipotent part has to be bounded.

In order to make sense of the comparison between the growths of the unipotent and the semisimple directions in the trichotomy, we define the height of a subset of $r(D)$ in Section 4.6.

In the proofs of the Ax-Schanuel results in preceding works [7][29][43], non-triviality of $\mathbf{N}$ was first obtained by applying the Pila-Wilkie counting theorem on I and using the induction hypothesis. The non-triviality of $\mathbf{N}$ was then used to construct a splitting of the period mappings. However, it is not known whether $\mathbf{N}$ is normal in the generic Mumford-Tate group in the mixed case in general, and the splitting may not make sense. This is the reason why we take another approach by looking at $\overline{I}$ instead of I, and thus explains why we have to look at Theorem 4.2 but not only Theorem 4.4, and therefore explains why we study the image of period mappings in Section 3.4.

4.3 Base cases of induction

We prove Theorem 4.2 and Theorem 4.4 simultaneously by induction on $\dim X$. The case when $\dim X = 0$ is trivial. Suppose $\dim X > 0$. For each $\dim X$, we prove Theorem 4.2 by induction on $(\dim V - \dim U, \dim X - \dim U)$ in lexicographical order, and deduce Theorem 4.4 from Theorem 4.2.

If $\Gamma \subset \mathbf{P}(\mathbb{Z})^+$, let $W_\Gamma := X \times_{\Gamma\backslash D} D$, and let S be the set of all distinct representatives of the cosets in $\mathbf{P}(\mathbb{Z})^+/\Gamma$, we have $W = \bigcup_{g\in S} gW_\Gamma$. Then since U is irreducible, $g^{-1}U \subset W_\Gamma$ for some $g \in S$. The following lemma then follows:

Lemma 4.5. *Theorem 4.2 is equivalent to the following: Assume $\Gamma \subset \mathbf{P}(\mathbb{Z})^+$. Let $U_\Gamma := g^{-1}U$. Let $V_\Gamma := U_\Gamma^{\mathrm{Zar}}$. If $\dim V_\Gamma - \dim U_\Gamma < \dim \check{D}$, then $p_X(U_\Gamma)$ is contained in a proper weakly special subvariety.*

Lemma 4.6. *Let k be an integer. If Theorem 4.2 holds for $\dim X = k$, then Theorem 4.4 also holds for $\dim X = k$.*

Proof. Since $\Gamma \cap \mathbf{P}(\mathbb{Q})$ is of finite index in Γ, passing to a finite covering of X, we can assume that $\Gamma \subset \mathbf{P}(\mathbb{Z})$. Similarly, we can further assume that $\Gamma \subset \mathbf{P}(\mathbb{R})^+$. Then since $W_\Gamma = W'$, Theorem 4.4 follows from Lemma 4.5. □

Lemma 4.7. *If there exists an algebraic subvariety Z of X such that $p_X(U) \subset Z \subsetneq X$, then Theorem 4.2 holds.*

Proof. By Lemma 4.5, it suffices to prove the statement about U_Γ in the lemma. Let Γ_Z be the monodromy group of the GPVMHS restricted to Z. We have a mapping $Z \to \Gamma_Z\backslash\mathcal{M}$. Let $W_Z := Z \times_{\Gamma_Z\backslash\mathcal{M}} \mathcal{M}$. Since $\dim Z < \dim X$, by induction hypothesis, Theorem 4.4 holds for $Z \to \Gamma_Z\backslash\mathcal{M}$. Let $W'_Z := Z \times_{\Gamma\backslash\mathcal{M}} \mathcal{M}$.

Let S' be the set of all distinct representatives of the cosets in Γ/Γ_Z. We have $W'_Z = \bigcup_{g \in S'} gW_Z$. For any U'_Z be an irreducible analytic subset of W'_Z, we have $g^{-1}U'_Z \in W_Z$ for some $g \in S'$. Hence, Theorem 4.4 holds for $Z \to \Gamma\backslash\mathcal{M}$. Since $g^{-1}U \subset W_\Gamma$ and $p_X(g^{-1}U) = p_X(U) \subset Z$, we know $g^{-1}U \subset W'_Z$. Therefore, the statement about U_Γ in Lemma 4.5 holds. □

Lemma 4.8. *If $p_X(U)$ is contained in an algebraic subvariety Z with $\dim p_X(U) = \dim Z$, then Theorem 4.2 holds.*

Proof. Suppose $\dim p_X(U) = \dim X$. Since $g^{-1}U \subset W_\Gamma$ and Γ acts discretely on D, we have

$$\dim g^{-1}U = \dim p_X(g^{-1}U) = \dim p_X(U) = \dim X = \dim W_\Gamma.$$

Since W_Γ is irreducible, we then have $W_\Gamma^{\mathrm{Zar}} = g^{-1}U^{\mathrm{Zar}}$. Since Γ is $\mathbb{Q}$-Zariski dense in $\mathbf{P}$, we know Γ is $\mathbb{C}$-Zariski dense in $\mathbf{P}_\mathbb{C}$ by Lemma 3.6. Then since W_Γ is invariant under Γ, we know U^{Zar} is invariant under $\mathbf{P}_\mathbb{C}$. Also, $p_X(U^{\mathrm{Zar}}) = X$. Then since $\mathbf{P}_\mathbb{C}$ acts transitively on $\check{D}$, we have $V := U^{\mathrm{Zar}} = X \times \check{D}$, and hence $\dim U > \dim X = \dim W$, which is a contradiction. Therefore, $\dim p_X(U) < \dim X$. If $p_X(U)$ is contained in an algebraic subvariety Z with $\dim p_X(U) = \dim Z$, then Theorem 4.2 holds by Lemma 4.7. □

If $\dim V = \dim U$, then since V is irreducible as an analytic set, the analytic closure $\overline{U}^{an}$ of U is V. We have

$$\dim p_X(U) = \dim \overline{p_X(U)}^{an} = \dim p_X(\overline{U}^{an}) = \dim p_X(V).$$

Then since $p_X(V)$ is algebraic, Theorem 4.2 holds by Lemma 4.8.

Since $U \subset W$, if $\dim X = \dim U$, then $\dim X = \dim U = \dim p_X(U)$, Theorem 4.2 thus holds by Lemma 4.8.

4.4 Normality of algebraic stabilizer in algebraic monodromy group

Let $\mathbf{N}$ be the identity component of the $\mathbb{Q}$-Zariski closure of

$$Stab(V) := \{\sigma \in \mathbf{P}(\mathbb{Z})^+ : \sigma V = V\}$$

in $\mathbf{P}$. Fix a Levi subgroup $\mathbf{N}_r$ of $\mathbf{N}$. Let $\mathbf{G} := \mathbf{P}_r$ be a maximal connected reductive subgroup of $\mathbf{P}$ containing $\mathbf{N}_r$. Then $\mathbf{G}$ is a Levi subgroup of $\mathbf{P}$. Similarly, we can choose Levi subgroup $\mathcal{MT}_r$ of $\mathcal{MT}$ containing $\mathbf{G}$. We have an isomorphism $\mathcal{MT}_r \simeq \mathcal{MT} / \mathcal{MT}_u$.

4.4.1 A temporary definable fundamental set Φ' for $\mathbf{P}(\mathbb{Z})^+\backslash D$

The definable fundamental set for $\mathbf{P}(\mathbb{Z})^+\backslash D$ in this section is temporary because later on when we define the definable set I in Section 4.5.2, we will switch to another fundamental set in Section 4.5.1 that depends on the normality we prove in this section.

Definition 4.9 ([4])**.** Let Y be a definable locally compact subset in $\mathbb{R}^n$ and Γ a group acting on Y by definable homeomorphism. A subset F in Y is a **fundamental set** for $\Gamma\backslash Y$ if $\Gamma \cdot F = Y$ and the set $\{g \in \Gamma : F \cap gF \neq \varnothing\}$ is finite.

The image Λ of $\mathbf{P}(\mathbb{Z})^+$ in $\mathbf{G}(\mathbb{Q})$ is an arithmetic subgroup containing $\mathbf{G}(\mathbb{Z})^+ := \mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^+$. Let $\Phi_{\mathbf{G}}$ be a definable open fundamental set for the action of Λ on D_{Gr} [6, Theorem 1.1]. Let $\Phi_{\mathbf{U}}$ be a bounded definable open fundamental set for the cocompact action of $\mathbf{U}(\mathbb{Z})$ on $D_{\mathbf{U},\mathbb{R}}$. Let $D_{\mathrm{Gr}} := D^+_{\mathbf{P},\mathrm{Gr}}$. Let $D_{\mathbf{U},\mathbb{R}}$ be the $\mathbf{P}(\mathbb{R})^+$-orbit in $\mathcal{S}(W_\bullet)(\mathbb{R})$ of the splitting h_{1s} of $h_1 := r(h_0)$. Recall the isomorphism

$$j : r(D) \simeq D_{\mathrm{Gr}} \times D_{\mathbf{U},\mathbb{R}}$$

obtained by taking $\mathbf{M} = \mathbf{P}$ in Theorem 3.12.

Lemma 4.10. *The set* $\Phi'_{\mathbb{R}} := j^{-1}(\Phi_{\mathbf{G}} \times \Phi_{\mathbf{U}})$ *is a definable open fundamental set for the action of* $\mathbf{P}(\mathbb{Z})^+$ *on* $r(D)$*. Hence* $\Phi' := r^{-1}(\Phi'_{\mathbb{R}})$ *is a definable open fundamental set for the action of* $\mathbf{P}(\mathbb{Z})^+$ *on* D*.*

Proof. Let $(gh_{1\mathrm{Gr}}, uh_{1s})$ be in $D_{\mathrm{Gr}} \times D_{\mathbf{U},\mathbb{R}}$, where $g \in \mathbf{G}(\mathbb{R})^+$ and $u \in \mathbf{U}(\mathbb{R})$. There exists $\gamma_{\mathbf{G}} \in \Lambda$ such that $\gamma_{\mathbf{G}}^{-1} gh_{1\mathrm{Gr}} \in \Phi_{\mathbf{G}}$. There exists $\gamma_{\mathbf{U}} \in \mathbf{U}(\mathbb{Q})$ such that $\gamma_{\mathbf{G}}\gamma_{\mathbf{U}} \in \mathbf{P}(\mathbb{Z})^+$. By Lemma 3.10, $\gamma_{\mathbf{U}}^{-1}\gamma_{\mathbf{G}}^{-1} uh_{1s} = u'h_{1s}$ for some unique $u' \in \mathbf{U}(\mathbb{R})$. By the definition of $\Phi_{\mathbf{U}}$, there exists $\gamma'_{\mathbf{U}} \in \mathbf{U}(\mathbb{Z})$ such that $\gamma_{\mathbf{U}}'^{-1} u'h_{1s} \in \Phi_{\mathbf{U}}$, which implies $\gamma_{\mathbf{U}}'^{-1}\gamma_{\mathbf{U}}^{-1}\gamma_{\mathbf{G}}^{-1} uh_{1s} \in \Phi_{\mathbf{U}}$. By the triviality of the $\mathbf{U}(\mathbb{R})$ action on the associated graded,

$$\gamma_{\mathbf{G}}\gamma_{\mathbf{U}}\gamma'_{\mathbf{U}} \cdot (\gamma_{\mathbf{G}}^{-1} gh_{1\mathrm{Gr}}, \gamma_{\mathbf{U}}'^{-1}\gamma_{\mathbf{U}}^{-1}\gamma_{\mathbf{G}}^{-1} uh_{1s}) = (gh_{1\mathrm{Gr}}, uh_{1s}).$$

Therefore, $\mathbf{P}(\mathbb{Z})^+ \cdot (\Phi_{\mathbf{G}} \times \Phi_{\mathbf{U}}) = D_{\mathrm{Gr}} \times D_{\mathbf{U},\mathbb{R}}$.

Suppose $\gamma \in \mathbf{P}(\mathbb{Z})^+$ such that $\gamma(\Phi_{\mathbf{G}} \times \Phi_{\mathbf{U}}) \cap (\Phi_{\mathbf{G}} \times \Phi_{\mathbf{U}}) \neq \varnothing$. Write $\gamma = \gamma_{\mathbf{G}}\gamma_{\mathbf{U}}$, where $\gamma_{\mathbf{G}} \in \Lambda$ and $\gamma_{\mathbf{U}} \in \mathbf{U}(\mathbb{Q})$. Then $\gamma_{\mathbf{G}}\Phi_{\mathbf{G}} \cap \Phi_{\mathbf{G}} \neq \varnothing$, thus $\gamma_{\mathbf{G}}$ has only finitely many choices. Since $\gamma_{\mathbf{U}}\Phi_{\mathbf{U}} \cap \gamma_{\mathbf{G}}^{-1}\Phi_{\mathbf{U}} \neq \varnothing$ and since $\gamma_{\mathbf{G}}^{-1}\Phi_{\mathbf{U}}$ overlaps with only finitely many translates of $\Phi_{\mathbf{U}}$, we then know $\gamma_{\mathbf{U}}$ has only finitely many choices. □

4.4.2 Normality of algebraic stabilizer in algebraic monodromy group

We apply the Hilbert scheme argument to prove that $\mathbf{N}$ is normal in $\mathbf{P}$. This argument was used in [7], [29], and [43].

Let $(X \times \check{D})'$ be a projective compactification of $X \times \check{D}$. Let M be the Hilbert scheme of all subvarieties of $(X \times \check{D})'$ with the same Hilbert polynomial as V', where V' is the Zariski closure of V in $(X \times \check{D})'$. Let $\mathcal{V} \to M$ be the universal family over M, with a natural embedding $\mathcal{V} \hookrightarrow (X \times \check{D})' \times M$. Let $\mathcal{V}_W$ be the base change to $W \times M$. Each $m \in M$ corresponds to a subvariety called V_m. Write $m = [V_m]$.

Let T be the set of all pairs $(p, m) \in W \times M$, such that $V_m \cap W$ has dimension at least $\dim U$ around p. The set T is closed and analytic in $\mathcal{V}_W$, see proof of [47, Lemma 8.2]. Let T_0 be the irreducible component containing $(p, [V])$ for some and hence any $p \in U$.

The action of $\mathbf{P}(\mathbb{Z})^+$ on $X \times D$, defined by $\gamma \cdot (x, h) = (x, \gamma \cdot h)$, lifts to $\mathcal{V}_W$. There is also an action of $\mathbf{P}(\mathbb{Z})^+$ on T. Let $Y := \mathbf{P}(\mathbb{Z})^+ \backslash T_0$ be the image of T_0 in $\mathbf{P}(\mathbb{Z})^+ \backslash T$.

Lemma 4.11. *The period mapping $\varphi : X \to \mathbf{P}(\mathbb{Z})^+ \setminus D$ is definable.*

Proof. We follow Bakker-Brunebarbe-Klingler-Tsimerman [4, §5] and make suitable modifications (recall that $\mathbf{P}$ is the connected algebraic monodromy group). By [4, Lemma 4.1], by passing X to finite étale covering if necessary, X is the union of finitely many punctured polydisks such that the GPVMHS has unipotent monodromy over each such polydisk. It suffices to prove that the period map $\varphi|_{(\Delta^*)^n}$ restricted to each such polydisk, say $(\Delta^*)^n$, is definable. By [4, Prop. 5.2], the restriction to any vertical strip E of the lifting $\widetilde{\varphi}$ of $\varphi|_{(\Delta^*)^n}$ is definable, so it suffices to prove that the image $\widetilde{\varphi}(E)$ lies in a finite union of definable fundamental sets of $\mathbf{P}(\mathbb{Z})^+ \setminus D$. By [16, Cor. 2.34], the composition of $\widetilde{\varphi}$ with $D \to D_{\mathbf{U},\mathbb{R}}$ is bounded on any vertical strip. It suffices to prove that the image of $\widetilde{\varphi}(E)$ in D_{Gr} lies in a finite union of Siegel sets. By [1, Cor. 2], $\mathbf{G}$ is semisimple. Then by [14, 7.5 and 7.7], it suffices to prove that $\widetilde{\varphi}(E)$ lies in a finite union of Siegel sets in $D^+_{\mathcal{MT},\mathrm{Gr}}$. This holds by [6, Theorem 1.5]. □

Hence, the set

$$W \cap (X \times \Phi') = \{(x, F^\bullet) \in X \times \Phi' : \varphi(x) = \pi|_{\Phi'}(F^\bullet)\}$$

is definable.

Since the Hilbert scheme M is proper, the composition

$$T \hookrightarrow W \times M \to W$$

is proper, so

$$\mathbf{P}(\mathbb{Z})^+ \backslash T \to \mathbf{P}(\mathbb{Z})^+ \backslash W = X$$

is proper, and thus the induced map $q : Y \to X$ is proper. The intersection

$$\mathcal{V} \cap ((W \cap (X \times \Phi')) \times M)$$

is a definable fundamental set for the action of $\mathbf{P}(\mathbb{Z})^+$ on $\mathcal{V}_W$. By [4, Proposition 2.3], $\mathbf{P}(\mathbb{Z})^+ \backslash \mathcal{V}_W$ and similarly Y have definable structures, so the projection q is definable. Then $q(Y)$ is closed, complex analytic and definable in X, and therefore algebraic by definable Chow [48].

Since $q(Y) \supset p_X(U)$, by Lemma 4.7, we can assume $q(Y) = X$. Let $\mathcal{F}$ be the family of algebraic varieties parametrized by the projection of T_0 in M. The family $\mathcal{F}$ is stable under the image Γ_Y of $\pi_1(Y) \to \pi_1(X) \to \Gamma$. Let $\Gamma_{\mathcal{F}} \subset \Gamma_Y$ be the subgroup of elements γ such that every fiber in $\mathcal{F}$ is invariant under γ. For any $\mu \in \Gamma_Y - \Gamma_{\mathcal{F}}$, define E_μ to be the image in M of the union of all fibers which are invariant under μ. We have $E_\mu \subsetneq M$. Hence, the Γ_Y-stabilizer of a very general fiber in $\mathcal{F}$, i.e. a fiber outside a countable union of proper subvarieties of $\mathcal{F}$, is $\Gamma_{\mathcal{F}}$. The point $[V]$ is a limit point of a set of points in M that corresponds to very general fibers. Recall that $\dim V = \dim V_m$ and $\dim V_m \cap W \geq \dim U$ for any m in image of T_0 in M. Hence, if the Ax-Schanuel holds for all very general fibers, then it also holds for V, so we can assume V is very general.

Theorem 4.12. *The subgroup* $\mathbf{N}$ *is normal in* $\mathbf{P}$*.*

Proof. Since q is definable, each fiber of q has only finitely many components. Then Γ_Y is of finite index in Γ. Since Γ is Zariski-dense in the connected group

$\mathbf{P}$, it follows that Γ_Y is Zariski-dense in $\mathbf{P}$. Every element $\gamma \in \Gamma_Y$ sends a very general fiber of $\mathcal{F}$ to a very general fiber, so

$$Stab(V) = \Gamma_{\mathcal{F}} = Stab(\gamma V) = \gamma Stab(V)\gamma^{-1}.$$

Since Γ_Y is Zariski-dense in $\mathbf{P}$, $\mathbf{N}$ is then normal in $\mathbf{P}$. $\square$

4.5 Definable quotient $\overline{I}$

4.5.1 Definable fundamental set Φ for $\mathbf{P}(\mathbb{Z})^+\backslash D$

Let $\mathbf{N}_u$ be the unipotent radical of $\mathbf{N}$. Recall that we fixed Levi subgroup $\mathbf{N}_r$ in $\mathbf{N}$ and Levi subgroup $\mathbf{G}$ in $\mathbf{P}$ such that $\mathbf{N}_r \subset \mathbf{G}$. By Theorem 4.12, $\mathbf{N}$ is normal in $\mathbf{P}$, so $\mathbf{N}_u = \mathbf{N} \cap \mathbf{U}$ and $\mathbf{N}_r = \mathbf{N} \cap \mathbf{G}$ by [28, Prop. 2.13]. Thus the $\mathbb{Q}$-group $\mathbf{N}_r$ is normal in $\mathbf{G}$. Similarly, $\mathbf{G}$ is normal in $\mathcal{MT}_r$. By [1, Cor. 2], $\mathbf{G}$ is semisimple, so there exists a connected normal subgroup $\mathbf{L}$ of $\mathbf{G}$ such that the map $\mathbf{N}_r \times \mathbf{L} \to \mathbf{G}$ defined by $(g_1, g_2) \mapsto g_1 g_2$ is an isogeny [42, Theorem 21.51]. This induces an isogeny $\beta : \mathbf{L} \to \mathbf{G} / \mathbf{N}_r$. Let $D_{\mathbf{N}_r}$ and $D_{\mathbf{L}}$ be respectively the $\mathbf{N}_r(\mathbb{R})^+$-orbit and the $\mathbf{L}(\mathbb{R})^+$-orbit of the pure Hodge structure $h_{1\mathrm{Gr}}$. By [32], we have an isomorphism $D_{\mathbf{N}_r} \times D_{\mathbf{L}} \simeq D_{\mathrm{Gr}}$. Combining this with the isomorphism in Theorem 3.12, we have an isomorphism

$$j : r(D) \simeq D_{\mathbf{N}_r} \times D_{\mathbf{L}} \times D_{\mathbf{U},\mathbb{R}}.$$

The unipotent radical acts trivially on the associated graded, so for any $u \in \mathbf{U}(\mathbb{R})$, $g_1 \in \mathbf{G}_r(\mathbb{R})^+$, and $g_2 \in \mathbf{L}(\mathbb{R})^+$, we have

$$j(ug_1g_2h_1) = (g_1 h_{1\mathrm{Gr}}, g_2 h_{1\mathrm{Gr}}, ug_1g_2h_{1s}).$$

Let $\mathbf{N}_r(\mathbb{Z})^+ := \mathbf{N}_r(\mathbb{Z}) \cap \mathbf{N}_r(\mathbb{R})^+$ and $\mathbf{L}(\mathbb{Z})^+ := \mathbf{L}(\mathbb{Z}) \cap \mathbf{L}(\mathbb{R})^+$.

Lemma 4.13. *We have*

$$\mathbf{P}(\mathbb{Z})^+ = \bigcup_{i=1}^{k} \mathbf{U}(\mathbb{Z})\,\mathbf{N}_r(\mathbb{Z})^+\,\mathbf{L}(\mathbb{Z})^+\rho_i$$

for some $\rho_1, \ldots, \rho_k \in \mathbf{P}(\mathbb{Z})^+$, where one of the ρ_i is the identity.

Proof. By [58, p. 173, Cor. 2], $\mathbf{U}(\mathbb{Z})\,\mathbf{G}(\mathbb{Z})$ is of finite index in $\mathbf{P}(\mathbb{Z})$, so $\mathbf{U}(\mathbb{Z})\,\mathbf{G}(\mathbb{Z})^+$ is of finite index in $\mathbf{P}(\mathbb{Z})^+$. Consider the isogeny $\mathbf{N}_r \times \mathbf{L} \to \mathbf{G}$. By [58, Theorem 4.1], $\mathbf{N}_r(\mathbb{Z})\,\mathbf{L}(\mathbb{Z})$ is of finite index in $\mathbf{G}(\mathbb{Z})$, so $\mathbf{N}_r(\mathbb{Z})^+\,\mathbf{L}(\mathbb{Z})^+$ is of finite index in $\mathbf{G}(\mathbb{Z})^+$. □

Recall the fundamental set $\Phi_{\mathbf{U}}$ for the $\mathbf{U}(\mathbb{Z})$-action on $D_{\mathbf{U},\mathbb{R}}$ and recall the arithmetic subgroup Λ of $\mathbf{G}(\mathbb{Q})$ in Section 4.4.1. Let $\Phi_{\mathbf{L}}$ and $\Phi_{\mathbf{N}_r}$ be definable open fundamental sets for the actions of $\mathbf{L}(\mathbb{Z})^+$ and $\mathbf{N}_r(\mathbb{Z})^+$ on $D_{\mathbf{L}}$ and $D_{\mathbf{N}_r}$ respectively [6, Theorem 1.1]. We can assume that $\Phi_{\mathbf{L}}$ and $\Phi_{\mathbf{N}_r}$ contain $h_{1\mathrm{Gr}}$. Since $\mathbf{N}_r(\mathbb{Z})^+\,\mathbf{L}(\mathbb{Z})^+$ is of finite index in Λ [58, Theorem 4.1], the image of $\Phi_{\mathbf{L}} \times \Phi_{\mathbf{N}_r}$ in D_{Gr} is a definable open fundamental set for the action of Λ on D_{Gr}. By Lemma 4.10 with $\Phi_{\mathbf{G}}$ replaced by $\Phi_{\mathbf{N}_r} \times \Phi_{\mathbf{L}}$, the set

$$j^{-1}(\Phi_{\mathbf{N}_r} \times \Phi_{\mathbf{L}} \times \Phi_{\mathbf{U}})$$

is a definable open fundamental set for the action of $\mathbf{P}(\mathbb{Z})^+$ on $r(D)$, so

$$\Phi_{\mathbb{R}} := \bigcup_{i=1}^{k} \rho_i \cdot j^{-1}(\Phi_{\mathbf{N}_r} \times \Phi_{\mathbf{L}} \times \Phi_{\mathbf{U}})$$

is also a definable open fundamental set for the action of $\mathbf{P}(\mathbb{Z})^+$ on $r(D)$. Then $\Phi := r^{-1}(\Phi_{\mathbb{R}})$ is a definable open fundamental set for the action of $\mathbf{P}(\mathbb{Z})^+$ on D.

4.5.2 Definable subset I of $\mathbf{P}(\mathbb{C})$

Definition 4.14. For an irreducible analytic set $E \subset X \times \check{D}$, let $\mathbf{P}_E$ be the connected algebraic monodromy group of the GPVMHS restricted to $p_X(E)^{\mathrm{Zar}}$.

Let $E^{ws} := \varphi^{-1}\pi(D(\mathbf{P}_E))$. For an analytic set $E \subset X \times \check{D}$ and a weakly special subvariety $B \subset X$, we define $S_d(E, B)$ to be the set of points $x \in E$ around which x is regular and of dimension d, and such that the irreducible analytic component E_c containing x satisfies $E_c^{ws} = B$.

Let

$$I := \{\gamma \in \mathbf{P}(\mathbb{C}) : S_{\dim U}(\gamma^{-1}V \cap W \cap (X \times \Phi), U^{ws}) \neq \varnothing\}.$$

where γ acts on V by acting on the $\check{D}$-coordinates. By [4, Proposition 2.3], $\mathbf{P}(\mathbb{Z})^+\backslash D$ has a definable structure such that the canonical map $\Phi \to \mathbf{P}(\mathbb{Z})^+\backslash D$ is definable. By Lemma 4.11, the set

$$W \cap (X \times \Phi) = \{(x, F^\bullet) \in X \times \Phi : \varphi(x) = \pi|_\Phi(F^\bullet)\}$$

is then definable. Then since regular condition and dimension condition can be expressed in terms of derivatives, I is definable. Similarly, $W \cap (X \times \gamma\Phi)$ is definable for any $\gamma \in \mathbf{P}(\mathbb{C})$.

4.5.3 Definable quotient $\overline{I}$

Recall that $\mathbf{N}$ is the identity component of the $\mathbb{Q}$-Zariski closure of

$$Stab(V) := \{\sigma \in \mathbf{P}(\mathbb{Z})^+ : \sigma V = V\}$$

in $\mathbf{P}$, so $\mathbf{N}_\mathbb{C}$ is the identity component of the $\mathbb{C}$-Zariski closure of $Stab(V)$ by Lemma 3.6. Moreover, V is algebraic and invariant under $Stab(V)$, so V is invariant under $\mathbf{N}_\mathbb{C}$. Let $\overline{I}$ be the definable image of I under the map $\mathbf{P}(\mathbb{C}) \to (\mathbf{P}/\mathbf{N})(\mathbb{C})$, i.e.

$$\overline{I} := \{[\gamma] : \gamma \in \mathbf{P}(\mathbb{C}) \text{ and } S_{\dim U}(\gamma^{-1}V \cap W \cap (X \times \Phi), U^{ws}) \neq \varnothing\}.$$

Let $\mathbf{W}$ be the unipotent radical of $\mathbf{P}/\mathbf{N}$. By [13, Corollary 14.11], $\mathbf{W} = \mathbf{U}\mathbf{N}/\mathbf{N}$. The group $\mathbf{H} := \mathbf{G}\mathbf{N}/\mathbf{N}$ is a Levi subgroup of $\mathbf{P}/\mathbf{N}$. We have $\mathbf{H} \cong \mathbf{G}/(\mathbf{G} \cap \mathbf{N})$.

The image $I_{\mathbf{W}}$ of $\overline{I}$ under the definable projection

$$\pi : (\mathbf{P} / \mathbf{N})(\mathbb{C}) \simeq \mathbf{W}(\mathbb{C}) \rtimes \mathbf{H}(\mathbb{C}) \to \mathbf{W}(\mathbb{C})$$

is definable. We have

$$\begin{aligned} I_{\mathbf{W}} = \{[\gamma] : & \gamma \in \mathbf{U}(\mathbb{C}) \text{ and } S_{\dim U}((\gamma\eta)^{-1} V \cap W \cap (X \times \Phi), U^{ws}) \neq \varnothing \\ & \text{for some } \eta \in \mathbf{L}(\mathbb{C})\}. \end{aligned}$$

Let $\mathbf{N}(\mathbb{Z})^+ := \mathbf{N}(\mathbb{Z}) \cap \mathbf{P}(\mathbb{R})^+$.

Lemma 4.15. *If γ is in $\mathbf{P}(\mathbb{Z})^+$ such that*

$$U \cap (X \times \gamma \bigcup_{\sigma \in \mathbf{N}(\mathbb{Z})^+} \sigma\Phi) \neq \varnothing,$$

then $S_{\dim U}(\gamma^{-1} V \cap W \cap (X \times \Phi), U^{ws}) \neq \varnothing$.

Proof. We have $\dim U \cap (X \times \gamma\sigma\Phi) = \dim U$ for some $\sigma \in \mathbf{N}(\mathbb{Z})^+$. Pick

$$x \in U \cap (X \times \gamma\sigma\Phi)$$

around which, $V \cap W \cap (X \times \gamma\sigma\Phi)$ is regular and of dimension $\dim U$. Then around $\sigma^{-1}\gamma^{-1}x$, the analytic set

$$(\sigma^{-1}\gamma^{-1}V) \cap W \cap (X \times \Phi) = \sigma^{-1}\gamma^{-1}(V \cap W \cap (X \times \gamma\sigma\Phi))$$

is regular and of dimension $\dim U$.

Let $(U \cap (X \times \gamma\sigma\Phi))_c$ be the irreducible component of $U \cap (X \times \gamma\sigma\Phi)$ containing x. The component $(U \cap (X \times \gamma\sigma\Phi))_c$ contains an open subset of U, so

$$p_X((U \cap (X \times \gamma\sigma\Phi))_c)$$

contains an open subset of $p_X(U)$, hence $(U \cap (X \times \gamma\sigma\Phi))_c^{ws}$ contains an open subset of $p_X(U)$. Since U is irreducible, $p_X(U)$ is irreducible, so

$$(U \cap (X \times \gamma\sigma\Phi))_c^{ws}$$

contains $p_X(U)$. Therefore, by Theorem 3.7,

$$(U \cap (X \times \gamma\sigma\Phi))_c^{ws} = U^{ws}.$$

The irreducible analytic component of

$$\sigma^{-1}\gamma^{-1}(V \cap W \cap (X \times \gamma\sigma\Phi))$$

containing $\sigma^{-1}\gamma^{-1}x$ is the irreducible analytic component

$$(\sigma^{-1}\gamma^{-1}(U \cap (X \times \gamma\sigma\Phi)))_c$$

of $\sigma^{-1}\gamma^{-1}(U \cap (X \times \gamma\sigma\Phi))$ containing $\sigma^{-1}\gamma^{-1}x$. Moreover,

$$(\sigma^{-1}\gamma^{-1}(U \cap (X \times \gamma\sigma\Phi)))_c^{ws} = (U \cap (X \times \gamma\sigma\Phi))_c^{ws} = U^{ws}.$$

Therefore, $S_{\dim U}(\gamma^{-1}V \cap W \cap (X \times \Phi), U^{ws}) \neq \varnothing$. $\square$

Lemma 4.16. *If for any positive integer p, the set $\overline{I}$ contains a semialgebraic curve containing at least p points of the form $[\gamma]$, where $\gamma \in \mathbf{P}(\mathbb{Z})^+$, then Theorem 4.2 holds. Similarly, if for any integer p, the set $I_{\mathbf{W}}$ contains a semialgebraic curve containing at least p points of the form $[\gamma]$, where $\gamma \in \mathbf{U}(\mathbb{Z})$, then Theorem 4.2 holds.*

Proof. Since $\mathbf{N}(\mathbb{C})$ is of finite index q in the $\mathbb{C}$-Zariski closure of $Stab(V)$, we can choose $p > q$. Let $C_{\mathbb{R}}$ be a semialgebraic curve in $\overline{I}$ that contains at least p points of the form $[\gamma]$, where $[\gamma] \in \mathbf{P}(\mathbb{Z})^+$. Let C be a complex algebraic curve containing $C_{\mathbb{R}}$. By definition of $\overline{I}$, for each $[c] \in C_{\mathbb{R}}$, there exists an irreducible analytic component U_c of $c^{-1}V \cap W \cap (X \times \Phi)$ of dimension $\dim U$ such that $U_c^{ws} = U^{ws}$. Let V' be the smallest algebraic variety containing $C^{-1}V$. If $V' \cap W$ contains infinitely many pairwise distinct U_c, then there exists an irreducible analytic component U' of $V' \cap W$ which contains infinitely many pairwise distinct U_c. Then U' is of dimension at least $\dim U + 1$. Since the curve $C_{\mathbb{R}}$ contains at least p points of the form $[\gamma]$, where $\gamma \in \mathbf{P}(\mathbb{Z})^+$, and since $p > q$, we have $C^{-1}V \neq V$, so $\dim V' = \dim V + 1$. If $p_X(U')$ is contained in a proper weakly special subvariety, then by Theorem 3.7, $U'^{ws} \neq X$, so $U^{ws} = U_c^{ws} \neq X$, which implies that $p_X(U)$ is contained in a proper weakly special subvariety. We can therefore replace V and U by U'^{Zar} and U' and induct over $\dim X - \dim U$.

Otherwise $V' \cap W$ contains finitely many pairwise distinct U_c. Hence, there is a component $U'' = U_c$ contained in infinitely many translates $c^{-1}V$, thus by analyticity contained in all such translates by $c \in C$. For some $\gamma \in \mathbf{P}(\mathbb{Z})^+$, we have $\gamma V \neq V$ and $[\gamma] \in C$, so

$$\dim U'' \leq \dim(\bigcap_{c \in C} c^{-1}V) \leq \dim V - 1.$$

If $p_X(U'')$ is contained in a proper weakly special subvariety, then since $U^{ws} = U''^{ws}$, $p_X(U)$ is also contained in a proper weakly special subvariety by Theorem 3.7. Thus, we can replace V and U by U''^{Zar} and U'' and induct over $\dim V - \dim U$. $\square$

4.6 Heights and trichotomy

We will define the height of a subset of $r(D)$. After that, we can then apply Gao's mixed point counting method [29, Theorem 5.2] to get a trichotomy.

Fix an embedding $\overline{\phi} : \mathbf{P} / \mathbf{N} \hookrightarrow \mathbf{GL}_m$ for some m. By conjugation, we can assume $\mathbf{W}$ is mapped by $\overline{\phi}$ into the $\mathbb{Q}$-group $\mathbb{U}_m$ of upper unitriangular $m \times m$ matrices. Let $\phi : \mathbf{P} \hookrightarrow \mathbf{GL}(\mathcal{H}_{\mathbb{Q},\eta}) \cong \mathbf{GL}_\ell$ be the inclusion followed by an isomorphism, where $\ell := \dim \mathcal{H}_{\mathbb{Q},\eta}$.

Definition 4.17. For any rational square matrix A, define the *height* $\mathbf{ht}\, A$ of A to be the maximum of the naive heights of the entries. For any $[\gamma] \in \mathbf{P}(\mathbb{Q})/\mathbf{N}(\mathbb{Q})$, define the *height* $\mathbf{ht}[\gamma]$ of $[\gamma]$ to be $\mathbf{ht}\, \overline{\phi}([\gamma])$. For any $\gamma \in \mathbf{P}(\mathbb{Q})$, define the *height* $\mathbf{ht}\, \gamma$ of γ to be $\mathbf{ht}\, \phi(\gamma)$.

Recall the isogeny $\beta : \mathbf{L} \to \mathbf{G} / \mathbf{N}_r$ in Section 4.5.1. Recall that $\mathbf{N}_r = \mathbf{N} \cap \mathbf{G}$. The map $\alpha : \mathbf{G} / \mathbf{N}_r \to \mathbf{G}\,\mathbf{N} / \mathbf{N} =: \mathbf{H}$ defined by $g\,\mathbf{N_G} \mapsto g\,\mathbf{N}$ is an isomorphism. The map $\tau := \alpha \circ \beta$ is an isogeny.

Lemma 4.18. *There exist constants $k_1, k_2 > 0$ such that $\mathbf{ht}\,\tau(\gamma) \leq k_1(\mathbf{ht}\,\gamma)^{k_2}$ for any $\gamma \in \mathbf{L}(\mathbb{Q})$.*

Proof. The isogeny τ, and the embeddings $\overline{\phi}, \phi$ in the definitions of heights, are algebraic. □

For any nonempty $E \subset r(D)$, define

$$\mathbf{ht_W}\,E := \max_{\gamma \in \mathbf{U}(\mathbb{Z})} \{\mathbf{ht}[\gamma] : E \cap \gamma \bigcup_{\eta \in \mathbf{L}(\mathbb{Z})^+, \sigma \in \mathbf{N}_r(\mathbb{Z})^+} \sigma\eta\Phi_{\mathbb{R}} \neq \varnothing\}.$$

Consider the projection $p_{\mathbf{L}} : r(D) \to D_{\mathbf{L}}$. Let $Z := r(p_D(U))$. Fix h in $Z \cap \Phi_{\mathbb{R}}$. Denote the radius T ball in $D_{\mathbf{L}}$ centered at a point $p_{\mathbf{L}}(h)$ by $B_{p_{\mathbf{L}}(h)}(T)$. Let $Z(T)$ be the irreducible analytic component of $Z \cap p_{\mathbf{L}}^{-1}(B_{p_{\mathbf{L}}(h)}(T))$ which contains h.

Lemma 4.19. *Let k be a positive integer. For any $k \times k$ integer invertible matrix A, we have $\mathbf{ht}\,A^{-1} \leq (k-1)!(\mathbf{ht}\,A)^{k-1}$.*

Proof. Denote the ij-minor by M_{ij}. Since $\det A = \pm 1$, $\mathbf{ht}\,A^{-1} = \mathbf{ht}\,\mathbf{adj}A = \max_{i.j} |\det M_{ij}| \leq (k-1)!(\mathbf{ht}\,A)^{k-1}$. □

Lemma 4.20. *There exists a constant $c_1 > 0$ such that for any $T \gg 0$, if $\eta\Phi_{\mathbf{L}} \cap B_{p_{\mathbf{L}}(h)}(T) \neq \varnothing$ for some $\eta \in \mathbf{L}(\mathbb{Z})^+$, then $\mathbf{ht}\,\eta \leq e^{c_1 T}$.*

Proof. This follows from [7, Theorem 4.2] and Lemma 4.19. □

Trichotomy Fix a number $\lambda > 2k_2c_1 2^m m$. We are in one of the following three cases:

1. We have $\dim p_{\mathbf{L}}(Z) > 0$, and for some sequence $\{T_i \in \mathbb{R}\}_{i\in\mathbb{N}}$ such that $T_i \to \infty$, we have $\mathbf{ht_W}\, Z(T_i) \leq e^{\lambda T_i}$ for all i.

2. We have $\mathbf{ht_W}\, Z(T) > e^{\lambda T}$ for all $T \gg 0$. (This includes the case where $p_{\mathbf{L}}(Z) = 0$ and $\mathbf{ht_W}\, Z$ is infinite, because if $p_{\mathbf{L}}(Z) = 0$, then

$$Z \subset p_{\mathbf{L}}^{-1}(B_{p_{\mathbf{L}}(h)}(T))$$

for all T. Thus, for all T, $\mathbf{ht_W}\, Z(T) = \mathbf{ht_W}\, Z$, which is infinite.)

3. We have $\dim p_{\mathbf{L}}(Z) = 0$ and $\mathbf{ht_W}\, Z$ is finite.

4.7 Proof of case (1)

The main idea, borrowed from Gao [29], of the proof of the following theorem is to use the volume estimates established by Bakker-Tsimerman [7] to produce enough semisimple integer points, and attach to each of these points a unipotent integer point of comparable (or smaller) heights using the assumption of case (1).

Theorem 4.21. *Suppose we are in case (1), described in the previous section. There exist constants $c_3, c_4 > 0$ such that for any $i \gg 0$, there exists at least $c_3 T_i^{c_4}$ rational points $[\rho] \in (\mathbf{P} / \mathbf{N})(\mathbb{Q})$ of heights at most T_i such that $U \cap (X \times \rho'\Phi) \neq \varnothing$ for some $\rho' \in [\rho] \cap \mathbf{P}(\mathbb{Z})^+$.*

Proof. Suppose we have an element η in $\mathbf{L}(\mathbb{Z})^+$ such that $p_{\mathbf{L}}(Z(T_i)) \cap \eta\Phi_{\mathbf{L}} \neq \varnothing$. Let $z_{\mathbf{L}} \in p_{\mathbf{L}}(Z(T_i)) \cap \eta\Phi_{\mathbf{L}}$. Write $p_{\mathbf{L}}(z) = z_{\mathbf{L}}$ for some $z \in Z(T_i)$. Let $(z_{\mathbf{N}_r}, z_{\mathbf{L}}, z_{\mathbf{U}})$ be the image of z under the isomorphism

$$j : r(D) \simeq D_{\mathbf{N}_r} \times D_{\mathbf{L}} \times D_{\mathbf{U},\mathbb{R}}.$$

There exists $\sigma \in \mathbf{N}_r(\mathbb{Z})^+$ such that $z_{\mathbf{N}_r} \in \sigma\Phi_{\mathbf{N}_r}$. Recall that $\mathbf{P}(\mathbb{R})^+$ acts on $D_{\mathbf{U},\mathbb{R}}$ *a priori*. There exists $\gamma \in \mathbf{U}(\mathbb{Z})$ such that $z_{\mathbf{U}} \in \gamma\sigma\eta\Phi_{\mathbf{U}}$. Recall that we fixed h

in $Z \cap \Phi_{\mathbb{R}}$ in Section 4.6. For some $g_1 \in \mathbf{N}_r(\mathbb{R})^+$, $g_2 \in \mathbf{L}(\mathbb{R})^+$ and $u \in \mathbf{U}(\mathbb{R})$, we have $g_1 h_{\mathrm{Gr}} \in \Phi_{\mathbf{N}_r}$, $g_2 h_{\mathrm{Gr}} \in \Phi_{\mathbf{L}}$, $uh_s \in \Phi_{\mathbf{U}}$, and

$$(z_{\mathbf{N}_r}, z_{\mathbf{L}}, z_{\mathbf{U}}) = (\sigma g_1 h_{\mathrm{Gr}}, \eta g_2 h_{\mathrm{Gr}}, \gamma\sigma\eta u h_s).$$

Let $g := g_1 g_2$. By Lemma 3.10, there exists $u' \in \mathbf{U}(\mathbb{R})$ such that $g^{-1}h_s = u'h_s$. Then since $\mathbf{U}(\mathbb{R})$ acts trivially on the associated graded, we have

$$ugu'h = j^{-1}((g_1 h_{\mathrm{Gr}}, g_2 h_{\mathrm{Gr}}, ugu'h_s)) \in \Phi_{\mathbb{R}},$$

and thus (recall that $\mathbf{N}_r \times \mathbf{L} \to \mathbf{G}$ in Section 4.5.1 is an isogeny, and hence a homomorphism)

$$z = j^{-1}((z_{\mathbf{N}_r}, z_{\mathbf{L}}, z_{\mathbf{U}})) = \gamma\sigma\eta ugu'h \in \gamma\sigma\eta\Phi_{\mathbb{R}}.$$

Then $z \in Z(T_i) \cap \gamma\sigma\eta\Phi_{\mathbb{R}}$. By Lemma 4.18 and Lemma 4.20,

$$\mathbf{ht}[\eta] = \mathbf{ht}\,\tau(\eta) \le k_1(\mathbf{ht}\,\eta)^{k_2} \le k_1 e^{k_2 c_1 T_i}.$$

By $\mathbf{N}_r \subset \mathbf{N}$ and assumption, $\mathbf{ht}[\gamma\sigma] = \mathbf{ht}[\gamma] \le e^{\lambda T_i}$. It follows that

$$\mathbf{ht}[\gamma\sigma\eta] \le (\mathbf{ht}[\gamma\sigma]\,\mathbf{ht}[\eta])^m = O(e^{m(\lambda + k_2 c_1)T_i}).$$

Since $\dim p_{\mathbf{L}}(Z) > 0$, by [7, Theorem 1.2] and [7, Proposition 3.2], there exists a constant $c > 0$ such that for any $T > 0$, there exist at least e^{cT} integer points η in $\mathbf{L}(\mathbb{Z})^+$ of heights at most $e^{c_1 T}$ such that $p_{\mathbf{L}}(Z(T)) \cap \eta\Phi_{\mathbf{L}} \neq \varnothing$.

Combining with what we have proved above, taking into consideration that τ is an isogeny, we know there exist constants $c_3, c_4 > 0$ such that, for any $i \ge 0$, there exist at least $c_3 T_i^{c_4}$ points $[\rho]$ in $(\mathbf{P}\,/\,\mathbf{N})(\mathbb{Q})$ of heights at most T_i such that $Z(T_i) \cap \rho'\Phi_{\mathbb{R}} \neq \varnothing$ for some $\rho' \in [\rho] \cap \mathbf{P}(\mathbb{Z})^+$. Since $Z = r(p_D(U))$ and r is $\mathbf{P}(\mathbb{Z})^+$-equivariant, the theorem follows. $\square$

Theorem 4.22. *Theorem 4.2 holds in case (1).*

Proof. By Theorem 4.21 and Lemma 4.15, for any $i \gg 0$, $\overline{I}$ contains at least $c_3 T_i^{c_4}$ rational points $[\rho]$ of heights at most T_i, where $\rho \in \mathbf{P}(\mathbb{Z})^+$. Recall the assumption that $T_i \to \infty$. Since $\overline{I}$ is definable, by Theorem 2.7 and pigeonhole principle, for any positive integer p, $\overline{I}$ contains a semi-algebraic set containing at least p points of the form $[\gamma]$, where $\gamma \in \mathbf{P}(\mathbb{Z})^+$. By intersecting this semi-algebraic set with other semi-algebraic sets interpolating these points, we know $\overline{I}$ contains a semi-algebraic curve containing at least p points of the form $[\gamma]$, where $\gamma \in \mathbf{P}(\mathbb{Z})^+$. By Lemma 4.16, the theorem follows. $\square$

4.8 Proof of case (2)

The idea of the proof of this case is to first define a family of connected graphs encoding how Z is intersecting the translates of the fundamental sets. Next we understand how walking along a path in the graph is the same as multiplying conjugates of unitriangular matrices. Then once we get an upper estimate of the height of the product of these conjugates, we can get enough points in the graphs. We then have enough points in the unipotent projection $I_{\mathbf{W}}$ of $\overline{I}$.

For any $T > 0$, let Q_T be a graph with vertex set and edge set as follows:

$$V(Q_T) := \{[\gamma] : \gamma \in \mathbf{U}(\mathbb{Z}), Z(T) \cap \gamma \bigcup_{\eta \in \mathbf{L}(\mathbb{Z})^+, \sigma \in \mathbf{N}_r(\mathbb{Z})^+} \sigma\eta\Phi_{\mathbb{R}} \neq \varnothing\},$$

$$\begin{aligned} E(Q_T) := \{([\gamma_1], [\gamma_2]) &: \gamma_1, \gamma_2 \in \mathbf{U}(\mathbb{Z}), \\ Z(T) \cap (\gamma_1 \bigcup_{\eta \in \mathbf{L}(\mathbb{Z})^+, \sigma \in \mathbf{N}_r(\mathbb{Z})^+} \sigma\eta\Phi_{\mathbb{R}}) &\cap (\gamma_2 \bigcup_{\eta \in \mathbf{L}(\mathbb{Z})^+, \sigma \in \mathbf{N}_r(\mathbb{Z})^+} \sigma\eta\Phi_{\mathbb{R}}) \neq \varnothing\}. \end{aligned}$$

Lemma 4.23. *The graph Q_T is connected.*

Proof. Pick any vertices $[\gamma_1], [\gamma_2]$ of Q_T. For each $j = 1, 2$, choose

$$x_j \in Z(T) \cap \gamma_j \bigcup_{\eta \in \mathbf{L}(\mathbb{Z})^+, \sigma \in \mathbf{N}_r(\mathbb{Z})^+} \sigma\eta\Phi_{\mathbb{R}}.$$

By Lemma 4.13 and the definition of $\Phi_{\mathbb{R}}$,

$$r(D) = \bigcup_{\rho \in \mathbf{P}(\mathbb{Z})^+} \rho \cdot j^{-1}(\Phi_{\mathbf{N}_r} \times \Phi_{\mathbf{L}} \times \Phi_{\mathbf{U}}) = \bigcup_{\gamma \in \mathbf{U}(\mathbb{Z}), \eta \in \mathbf{L}(\mathbb{Z})^+, \sigma \in \mathbf{N}_r(\mathbb{Z})^+} \gamma\sigma\eta\Phi_{\mathbb{R}}.$$

Since $Z(T)$ is path-connected, there exists a path in

$$Z(T) = \bigcup_{\{\gamma \in \mathbf{U}(\mathbb{Z}) : [\gamma] \in V(Q_T)\}} \bigcup_{\eta \in \mathbf{L}(\mathbb{Z})^+, \sigma \in \mathbf{N}_r(\mathbb{Z})^+} Z(T) \cap \gamma\sigma\eta\Phi_{\mathbb{R}}$$

joining x_1 and x_2. This induces a path in the graph Q_T joining $[\gamma_1]$ and $[\gamma_2]$. It follows that Q_T is connected. $\square$

Let $S := \{\delta \in \mathbf{P}(\mathbb{Z})^+ : \delta\Phi_{\mathbb{R}} \cap \Phi_{\mathbb{R}} \neq \varnothing\}$, which is a finite set by the definition of a fundamental set. For any $\delta \in \mathbf{P}(\mathbb{Z})^+$, write $[\delta] = [\delta]^{\mathbf{W}}[\delta]^{\mathbf{H}}$, where $[\delta]^{\mathbf{W}} \in \mathbf{W}(\mathbb{Q})$ and $[\delta]^{\mathbf{H}} \in \mathbf{H}(\mathbb{Q})$.

Lemma 4.24. *When $T \gg 0$, the following holds: Suppose $[\gamma_1]$ and $[\gamma_2]$ are adjacent vertices in Q_T. Then $[\gamma_2] = [\gamma_1][\eta_1][\delta]^{\mathbf{W}}[\eta_1]^{-1}$ for some $\delta \in S$ and some $\eta_1 \in \mathbf{L}(\mathbb{Z})^+$ satisfying $\mathbf{ht}[\eta_1] \leq e^{2k_2c_1T}$, where k_2 is in Lemma 4.18 and c_1 is in Lemma 4.20, both independent of T.*

Proof. There exist $\sigma_1, \sigma_2 \in \mathbf{N}_r(\mathbb{Z})^+$ and $\eta_1, \eta_2 \in \mathbf{L}(\mathbb{Z})^+$ such that

$$Z(T) \cap \gamma_1\sigma_1\eta_1\Phi_{\mathbb{R}} \cap \gamma_2\sigma_2\eta_2\Phi_{\mathbb{R}} \neq \varnothing.$$

Then

$$\Phi_{\mathbb{R}} \cap ((\gamma_1\sigma_1\eta_1)^{-1}\gamma_2\sigma_2\eta_2)\Phi_{\mathbb{R}} \neq \varnothing,$$

so $\gamma_2\sigma_2\eta_2 = \gamma_1\sigma_1\eta_1\delta$ for some $\delta \in S$. We then have

$$[\gamma_2][\eta_2] = [\gamma_1][\eta_1][\delta] = [\gamma_1][\eta_1][\delta]^{\mathbf{W}}[\eta_1]^{-1}[\eta_1][\delta]^{\mathbf{H}}.$$

Since $\mathbf{P}/\mathbf{N} = \mathbf{W} \rtimes \mathbf{H}$, we have $[\gamma_2] = [\gamma_1][\eta_1][\delta]^{\mathbf{W}}[\eta_1]^{-1}$.

Let $z \in Z(T) \cap \gamma_1\sigma_1\eta_1\Phi_{\mathbb{R}}$. Write $z = \gamma_1\sigma_1\eta_1 ph$ for some $p \in \mathbf{P}(\mathbb{R})^+$. Write $p = ug_1g_2$ for some $u \in \mathbf{U}(\mathbb{R})$, $g_1 \in \mathbf{N}_r(\mathbb{R})^+$ and $g_2 \in \mathbf{L}(\mathbb{R})^+$. Since the unipotent radical acts trivially on the associated graded, $g_2h_{\mathrm{Gr}} \in \Phi_{\mathbf{L}}$. Thus $p_{\mathbf{L}}(z) = \eta_1 g_2 h_{\mathrm{Gr}} \in \eta_1\Phi_{\mathbf{L}}$ (recall that $\mathbf{N}_r \times \mathbf{L} \to \mathbf{G}$ in Section 4.5.1 is an isogeny, and hence a homomorphism), so $\eta_1\Phi_{\mathbf{L}} \cap p_{\mathbf{L}}(Z(T)) \neq \varnothing$. By Lemma 4.20, $\mathbf{ht}\,\eta_1 \leq e^{c_1T}$. Hence by Lemma 4.18,

$$\mathbf{ht}[\eta_1] = \mathbf{ht}\,\tau(\eta_1) \leq k_1(\mathbf{ht}\,\eta_1)^{k_2} \leq e^{2k_2c_1T}$$

when $T \gg 0$. $\square$

Lemma 4.25. *Let $\mathcal{Y}$ be a subset of the group $\mathbb{U}_m(\mathbb{Q})$ of upper unitriangular (upper triangular with 1's on the diagonal) $m \times m$ rational matrices such that the heights of matrices in $\mathcal{Y}$ are bounded by a constant. Let*

$$s_1 := \min\{d \in \mathbb{Z}^+ : dY \in \mathbf{GL}_m(\mathbb{Z}) \text{ for all } Y \in \mathcal{Y}\}.$$

For any positive integer t, and any $Y_1, \ldots, Y_t \in \mathcal{Y}$, we have

$$s_1^{m-1} Y_1 \cdots Y_t \in \mathbf{GL}_m(\mathbb{Z}).$$

Proof. We prove by induction on m. By induction assumption,

$$s_1^{m-2} \cdot (Y_1 \cdots Y_t)_{ij} \in \mathbb{Z}$$

for any $(i,j) \neq (1,m)$. Moreover,

$$s_1^{m-1} \cdot (Y_1 \cdots Y_t)_{1,m} = \sum_{i=1}^{t} s_1^{m-1} \cdot (Y_i)_{1,m} + \sum_{k=2}^{t}\sum_{i=2}^{m-1} s_1^{m-2} \cdot (Y_1 \cdots Y_{k-1})_{1,i} \cdot s_1 \cdot (Y_k)_{i,m}$$

is an integer. $\square$

Lemma 4.26. *Let $\mathcal{S}$ be a finite subset of the group $\mathbb{U}_m(\mathbb{Q})$ of upper unitriangular $m \times m$ rational matrices such that the heights of matrices in $\mathcal{S}$ are bounded by a constant s_0. There exists a constant $C > 0$, depending only on $\mathcal{S}, m$ and s_0, such that the following holds:*

Let f be a positive integer. Let $l_0 > 0$. Let $\mathcal{L} \subset \mathbf{GL}_m(\mathbb{Q})$ such that $fB \in \mathbf{GL}_m(\mathbb{Z})$ and $\mathbf{ht}\, B \leq l_0$ for all $B \in \mathcal{L}$. Let $A_1, \ldots, A_r \in \mathcal{S}$ and $B_1, \ldots, B_r \in \mathcal{L}$ satisfying $B_j A_j B_j^{-1} \in \mathbb{U}_m(\mathbb{Q})$ for all j. Then

$$\mathbf{ht}\left(\prod_{j=1}^{r} B_j A_j B_j^{-1}\right) \leq C \cdot l_0^{2^m m} \cdot r^{m-1}.$$

Proof. We prove by induction on m. The case when $m = 1$ is trivial. Let $\mathcal{S}'$ (resp. $\mathcal{S}''$) be the set of all $(m-1) \times (m-1)$ upper unitriangular matrices obtained by deleting the last (resp. first) row and the last (resp. first) column of matrices in $\mathcal{S}$. By induction assumption, we have a constant $C' > 0$ (resp. $C'' > 0$) such that the lemma holds with m and $\mathcal{S}$ being replaced by $m-1$ and $\mathcal{S}'$ (resp. $\mathcal{S}''$). Let $\mathcal{L}'$ (resp. $\mathcal{L}''$) be the set of all $(m-1) \times (m-1)$ matrices obtained by deleting the last (resp. first) row and the last (resp. first) column of matrices in $\mathcal{L}$. Write $Y_j := B_j A_j B_j^{-1} \in \mathbb{U}_m(\mathbb{Q})$. Define

$$s_1 := \min\{s \in \mathbb{Z}^+ : sA \in \mathbf{GL}_m(\mathbb{Z}) \text{ for all } A \in \mathcal{S}\}.$$

By Lemma 4.19, for all j,

$$\mathbf{ht}\, Y_j \leq s_1\, \mathbf{ht}(fB_j)(s_1 A_j)(fB_j)^{-1} \leq s_1 m^2 (m-1)!\, \mathbf{ht}(s_1 A_j)(\mathbf{ht}(fB_j))^m = O(l_0^m).$$

We know $s_1 Y_j \in \mathbf{GL}_m(\mathbb{Z})$ for all $j = 1, \ldots r$. By Lemma 4.25 with $\mathcal{Y}$ being the set of matrices obtained by deleting the last row and column of some Y_j, and by the induction hypothesis, we have

$$\begin{aligned}
\mathbf{ht}\left(\prod_{j=1}^{r} Y_j\right)_{1,m} &\leq s_1^{m-1}\,\mathbf{ht}\left(\sum_{i=1}^{r} s_1^{m-1}(Y_i)_{1,m} + \sum_{k=2}^{r}\sum_{i=2}^{m-1} s_1^{m-2}\left(\prod_{j=1}^{k-1} Y_j\right)_{1,i} s_1(Y_k)_{i,m}\right) \\
&\leq s_1^{m-1}\left(\sum_{i=1}^{r} |s_1^{m-1}(Y_i)_{1,m}| + \sum_{k=2}^{r}\sum_{i=2}^{m-1} \left| s_1^{m-2}\left(\prod_{j=1}^{k-1} Y_j\right)_{1,i}\right| |s_1(Y_k)_{i,m}|\right) \\
&\leq O(l_0^m \cdot r) + \sum_{k=2}^{r}\sum_{i=2}^{m-1} O(l_0^{2^{m-1}(m-1)}(k-1)^{m-2}\cdot l_0^m) \\
&\leq O(l_0^m \cdot r) + \sum_{k=2}^{r} O(l_0^{2^m m}(r-1)^{m-2}) \\
&\leq O(l_0^{2^m m}\cdot r^{m-1})
\end{aligned}$$

By induction assumption, the heights of the other entries are less than

$$\max\{C', C''\}\cdot l_0^{2^{m-1}(m-1)}\cdot r^{m-2}.$$

The lemma follows. $\square$

Theorem 4.27. *There are constants $c_7, c_8 > 0$ such that when $T \gg 0$, the graph $Q_{\log T^{1/\lambda}}$ has at least $c_7 T^{c_8}$ vertices of heights at most T.*

Proof. The assumption in case (2) says $\mathbf{ht}_{\mathbf{W}}\, Z(T) > e^{\lambda T}$ for all $T \gg 0$. Fix such large T, such that $e^{\lambda T} > 1 = \mathbf{ht}[id]$. Since $h \in \Phi_{\mathbb{R}}$, the identity $[id]$ is in $V(Q_T)$ for all $T > 0$. Then by Lemma 4.23, there exist $[\gamma] \in V(Q_T)$ for which $\mathbf{ht}[\gamma] > e^{\lambda T}$ and a path in the graph Q_T joining $[id]$ and $[\gamma] =: [\gamma_r]$ with intermediate vertices $[\gamma_1], \ldots, [\gamma_{r-1}]$ of heights less than $e^{\lambda T}$, where $[\gamma_i]$ and $[\gamma_{i+1}]$ are adjacent. By Lemma 4.24, $[\gamma] = \prod_{j=0}^{r-1}[\eta_j][\delta_j]^{\mathbf{W}}[\eta_j]^{-1}$ for some $\delta_j \in S$ and some $\eta_j \in \mathbf{L}(\mathbb{Z})^+$ such that $\mathbf{ht}[\eta_j] \leq e^{2k_2 c_1 T}$. The quotient map $\mathbf{P}(\mathbb{Q}) \to (\mathbf{P}\,/\,\mathbf{N})(\mathbb{Q})$ is defined by finitely many rational polynomials. Let $f \in \mathbb{Z}^+$ such that $f[\eta] \in (\mathbf{P}\,/\,\mathbf{N})(\mathbb{Z})$ for all $\eta \in \mathbf{L}(\mathbb{Z})^+$. Let $\mathcal{S} = \{[\delta]^{\mathbf{W}} : \delta \in S\}$, $\mathcal{L} = \{[\eta_0], \ldots, [\eta_{r-1}]\}$, and $l_0 = e^{2k_2 c_1 T}$.

By Lemma 4.26 with such $\mathcal{S}, f, \mathcal{L}$ and l_0, we have

$$e^{\lambda T} < \mathbf{ht}[\gamma] = \mathbf{ht} \prod_{j=0}^{r-1} [\eta_j][\delta_j]^{\mathbf{W}}[\eta_j]^{-1} = O(e^{2k_2 c_1 T 2^m m} r^{m-1}).$$

Hence, there are constants $c_5, c_6 > 0$ ($c_6 > 0$ because $\lambda > 2k_2 c_1 2^m m$ by assumption in Section 4.6) independent of T such that Q_T has at least $c_5 e^{c_6 T}$ vertices of heights at most $e^{\lambda T}$. The theorem follows. $\square$

Theorem 4.28. *Theorem 4.2 holds in case (2).*

Proof. By Lemma 4.27, when $T \gg 0$, there are at least $c_7 T^{c_8}$ points $[\gamma]$ in $\mathbf{W}(\mathbb{Q})$ of heights at most T such that for some $\gamma' \in [\gamma] \cap \mathbf{U}(\mathbb{Z})$,

$$Z \cap \gamma' \bigcup_{\eta \in \mathbf{L}(\mathbb{Z})^+, \sigma \in \mathbf{N}_r(\mathbb{Z})^+} \sigma\eta\Phi_{\mathbb{R}} \neq \varnothing.$$

Since $Z = r(p_D(U))$ and r is $\mathbf{P}(\mathbb{Z})^+$-equivariant, this condition implies that

$$U \cap (X \times \gamma' \bigcup_{\eta \in \mathbf{L}(\mathbb{Z})^+, \sigma \in \mathbf{N}_r(\mathbb{Z})^+} \sigma\eta\Phi) \neq \varnothing.$$

By Lemma 4.15, when $T \gg 0$, $I_{\mathbf{W}}$ contains at least $c_7 T^{c_8}$ points $[\gamma]$ in $\mathbf{W}(\mathbb{Q})$ of heights at most T such that $\gamma \in \mathbf{U}(\mathbb{Z})$. By Theorem 2.7 and pigeonhole principle, for any positive integer p, $I_{\mathbf{W}}$ contains a semi-algebraic set containing at least p points of the form $[\gamma]$, where $\gamma \in \mathbf{U}(\mathbb{Z})$. By intersecting this semi-algebraic set with other semi-algebraic sets interpolating these points, we know $I_{\mathbf{W}}$ contains a semi-algebraic curve containing at least p points of the form $[\gamma]$, where $\gamma \in \mathbf{U}(\mathbb{Z})$. By Lemma 4.16, the theorem follows. $\square$

4.9 Proof of case (3)

Since $\mathbf{ht}_{\mathbf{W}} Z$ is finite, the set

$$\mathcal{T} := \{[\gamma] : \gamma \in \mathbf{U}(\mathbb{Z}), Z \cap \gamma \bigcup_{\sigma \in \mathbf{N}_r(\mathbb{Z})^+} \sigma\Phi_{\mathbb{R}} \neq \varnothing\}$$

is finite. Write $\mathcal{T} = \{[\gamma_1], \dots, [\gamma_n]\}$, where γ_i satisfy the conditions in $\mathcal{T}$. Also, Z is contained in a fiber of $p_{\mathbf{L}}$. We have $p_{\mathbf{L}}(Z) \in \Phi_{\mathbf{L}}$. Recall that we let $\mathbf{N}(\mathbb{Z})^+ := \mathbf{N}(\mathbb{Z}) \cap \mathbf{P}(\mathbb{R})^+$.

Lemma 4.29. *We have*

$$Z \subset \bigcup_{i=1}^{n} \bigcup_{\sigma \in \mathbf{N}(\mathbb{Z})^+} \gamma_i \sigma \Phi_{\mathbb{R}}.$$

Proof. Suppose $z \in Z$. Let $(z_{\mathbf{N}_r}, z_{\mathbf{L}}, z_{\mathbf{U}})$ be the image of z under the isomorphism

$$r(D) \simeq D_{\mathbf{N}_r} \times D_{\mathbf{L}} \times D_{\mathbf{U},\mathbb{R}}.$$

There exists $\sigma \in \mathbf{N}_r(\mathbb{Z})^+$ such that $z_{\mathbf{N}_r} \in \sigma \Phi_{\mathbf{N}_r}$. Recall that $\mathbf{P}(\mathbb{R})^+$ acts on $D_{\mathbf{U},\mathbb{R}}$ *a priori*. There exists $\gamma \in \mathbf{U}(\mathbb{Z})$ such that $z_{\mathbf{U}} \in \gamma\sigma\Phi_{\mathbf{U}}$. For some $g_1 \in \mathbf{N}_r(\mathbb{R})^+$, $g_2 \in \mathbf{L}(\mathbb{R})^+$ and $u \in \mathbf{U}(\mathbb{R})$, we have $g_1 h_{\mathrm{Gr}} \in \Phi_{\mathbf{N}_r}$, $g_2 h_{\mathrm{Gr}} \in \Phi_{\mathbf{L}}$, $u h_s \in \Phi_{\mathbf{U}}$, and

$$(z_{\mathbf{N}_r}, z_{\mathbf{L}}, z_{\mathbf{U}}) = (\sigma g_1 h_{\mathrm{Gr}}, g_2 h_{\mathrm{Gr}}, \gamma\sigma u h_s).$$

Let $g := g_1 g_2$. There exists $u' \in \mathbf{U}(\mathbb{R})$ such that $g^{-1} h_s = u' h_s$. Then since $\mathbf{U}(\mathbb{R})$ acts trivially on the associated graded, we have

$$ugu'h = j^{-1}((g_1 h_{\mathrm{Gr}}, g_2 h_{\mathrm{Gr}}, ugu'h_s)) \in \Phi_{\mathbb{R}},$$

and thus

$$z = j^{-1}((z_{\mathbf{N}_r}, z_{\mathbf{L}}, z_{\mathbf{U}})) = \gamma\sigma ugu'h \in \gamma\sigma\Phi_{\mathbb{R}}.$$

The inclusion follows from the definition of $\mathcal{T}$. $\square$

Theorem 4.30. *Theorem 4.2 holds in case (3).*

Proof. By Lemma 4.29, by the definition of Z and that r is $\mathbf{P}(\mathbb{Z})^+$-equivariant,

$$p_X(U) = \bigcup_{i=1}^{n} \bigcup_{\sigma \in \mathbf{N}(\mathbb{Z})^+} p_X\left(U \cap (X \times \gamma_i \sigma \Phi)\right) = \bigcup_{i=1}^{n} \bigcup_{\sigma \in \mathbf{N}(\mathbb{Z})^+} p_X\left(\gamma_i^{-1}\sigma^{-1} U \cap (X \times \Phi)\right);$$

here σ^{-1} and γ_i^{-1} can be switched because $\mathbf{N}(\mathbb{Z})^+ := \mathbf{N}(\mathbb{Z}) \cap \mathbf{P}(\mathbb{R})^+$ is normal in $\mathbf{P}(\mathbb{Z})^+$. Since V is algebraic and invariant under $Stab(V)$, and since $\mathbf{N}_{\mathbb{C}}$ is the identity component of the $\mathbb{C}$-Zariski closure of $Stab(V)$, V is invariant under $\mathbf{N}_{\mathbb{C}}$. We know $\gamma_i^{-1}\sigma^{-1}U \cap (X \times \Phi)$ is a finite union of components of

$$\gamma_i^{-1}\sigma^{-1}(V \cap W) \cap (X \times \Phi) = \gamma_i^{-1}V \cap W \cap (X \times \Phi).$$

Since $\gamma_i^{-1}V \cap W \cap (X \times \Phi)$ has only finitely many components, $\gamma_i^{-1}\sigma^{-1}U \cap (X \times \Phi)$ are equal to finitely many possible sets as σ varies, each of which is definable. The set $p_X(U)$ is then definable and complex analytic. By the definable Chow theorem [48] (see also [43, Cor. 2.3]), $p_X(U)$ is algebraically constructible. By Lemma 4.8, Theorem 4.2 holds. $\square$

Chapter 5

Ax-Schanuel with derivatives for mixed period mappings

5.1 Statements of results

Let X, $(\mathcal{H}, \mathcal{W}_\bullet, \mathcal{F}^\bullet, \mathcal{Q})$, η, Γ, $\mathbf{P}$, $\mathbf{U}$, D, and $\check{D}$ be as in Section 4.1. First assume $\Gamma \subset \mathbf{P}(\mathbb{Z}) \cap \mathbf{P}(\mathbb{R})^+ =: \mathbf{P}(\mathbb{Z})^+$. This is assumed everywhere outside Corollary 5.2. Let $\psi : X \to \Gamma\backslash D$ be the period mapping. Let φ be the composition of ψ with $\Gamma\backslash D \to \mathbf{P}(\mathbb{Z})^+\backslash D$. Let $q : D \to \mathbf{P}(\mathbb{Z})^+\backslash D$ and $q' : D \to \Gamma\backslash D$ be the quotient maps.

Let k be a non-negative integer. Let $J_k(X, \check{D})$ be the set of all k-jets of germs of holomorphic mappings between open subsets of X and $\check{D}$. Since X and $\check{D}$ are algebraic, $J_k(X, \check{D})$ can be given an algebraic structure. Let $\pi_X : J_k(X, \check{D}) \to X$ be the mapping defined by projecting the k-jet of a germ to the center of the germ. Let W_k (resp. $W_{k,\Gamma}$) be the analytic subset of $J_k(X, \check{D})$ consists of all k-jets of germs of local liftings of the period mapping φ (resp. ψ). For any irreducible analytic subset U of W_k, denote by U^{Zar} its Zariski closure in $J_k(X, \check{D})$.

Theorem 5.1. *Let U be an irreducible analytic subset of W_k. If*

$$\dim U^{\mathrm{Zar}} - \dim U < \dim W_k^{\mathrm{Zar}} - \dim W_k,$$

then $\pi_X(U)$ is contained in a proper weakly special subvariety of X.

By passing to finite covering, we have the following corollary without the assumption that $\Gamma \subset \mathbf{P}(\mathbb{Z}) \cap \mathbf{P}(\mathbb{R})^+$.

Corollary 5.2. *Let U be an irreducible analytic subset of $W_{k,\Gamma}$. If $\dim U^{\mathrm{Zar}} - \dim U < \dim W_{k,\Gamma}^{\mathrm{Zar}} - \dim W_{k,\Gamma}$, then $\pi_X(U)$ is contained in a proper weakly special subvariety of X.*

Let Δ be the open unit disk. We have the following version of mixed Ax-Schanuel in terms of transcendence degree and derivatives.

Corollary 5.3. *Let $\widetilde{\psi}$ be a local lifting of the period mapping ψ on an open subset B. Let $v : \Delta^{\dim \check{D}} \to \check{D}$ and $u : \Delta^{\dim X} \to B$ be open embeddings such that $(\widetilde{\psi} \circ u)(\Delta^{\dim X}) \subset v(\Delta^{\dim \check{D}})$. Let $f : \Delta^m \to B$ be a holomorphic mapping such that $f(\Delta^m) \subset u(\Delta^{\dim X})$. Write $z = (z_1, \ldots, z_m)$, where z_i are the coordinates of Δ^m. If*

$$\operatorname{tr.deg}_{\mathbb{C}} \mathbb{C}((u^{-1} \circ f)(z), \partial^\alpha (v^{-1} \circ \widetilde{\psi} \circ f)(z) : |\alpha| \le k) < \operatorname{rank}(f) + \dim W_{k,\Gamma}^{\mathrm{Zar}} - \dim W_{k,\Gamma},$$

then $f(\Delta^m)$ is contained in a proper weakly special subvariety of X.

5.2 Sketch of the proof

Blázquez-Sanz, Casale, Freitag, and Nagloo established in [12] the Ax-Schanuel theorem for analytically foliated complex algebraic principal bundles. They proved that if the algebraic group acting on the bundle is sparse (a notion introduced in their paper concerning the analytic subgroups), and if the dimension of an algebraic subvariety of the bundle does not drop too much after intersection with a leaf, then the projection of the intersection under the bundle map is contained in a ∇-special subvariety, which was also introduced in their paper.

To use their result, we prove in Section 5.3 that when $k \gg 0$, the set $P := \mathbf{P}(\mathbb{C}) \cdot W_k$ is an algebraic principal bundle over X, and that there is a foliation on P where each leaf is of the form $g \cdot W_{k,\Gamma}$ for some $g \in \mathbf{P}(\mathbb{C})$. In particular, the algebraicity is proved in Lemma 5.4 using the definable Chow theorem of Peterzil-Starchenko [48] and the definable fundamental set for the action of $\mathbf{P}(\mathbb{Z})^+$ on D constructed in Section 4.5.1. The freeness of the group action on the fibers is proved in Lemma 5.5 using Theorem 4.2, the definable Chow theorem, and the Griffiths conjecture proved by Bakker-Brunebarbe-Tsimerman [5].

In Section 5.4, we use André-Deligne [1] to prove that any ∇-special subvariety of X is contained in a proper weakly special subvariety. We use the semisimple-unipotent Levi decomposition of $\mathbf{P}(\mathbb{C})$ in [1] to prove that $\mathbf{P}(\mathbb{C})$ is sparse. Then in Section 5.5, we prove our main theorems for all $k \geq 0$ by applying the aforementioned Ax-Schanuel theorem for principal bundles [12] followed by projection to lower order jet spaces.

5.3 Foliated jet bundle attached to the mixed period mapping

Let H be the kernel of homomorphism $\mathbf{P}_{\mathbb{C}} \to \mathrm{Aut}(\check{D})$ induced by the $\mathbf{P}_{\mathbb{C}}$-action on $\check{D}$. The group $\mathbf{P}_{\mathbb{C}}$ acts on $J_k(X, \check{D})$ by postcomposition.

Let $\mathbf{P}(\mathbb{Z})^+ := \mathbf{P}(\mathbb{Z}) \cap \mathbf{P}(\mathbb{R})^+$. From Section 4.5.1, we have a definable open fundamental set $F := \Phi$ for the action of $\mathbf{P}(\mathbb{Z})^+$ on D. By [4, Prop. 2.3], $q|_F$ is definable. Let $W_{k,F}$ be the analytic set of all k-jets of germs of local liftings into F of the period mapping φ. Then $W_{k,F}$ is definable by [4, Prop. 5.2].

Lemma 5.4. *The set $P := \mathbf{P}(\mathbb{C}) \cdot W_k$ is an algebraically constructible subvariety of $J_k(X, \check{D})$.*

Proof. Define the algebraic morphism

$$\Psi : \mathbf{P}(\mathbb{C}) \times J_k(X, \check{D}) \to J_k(X, \check{D})$$

by postcomposition. There exist projective compactifications $\mathbf{P}(\mathbb{C})'$ and $J_k(X, \check{D})$ of $\mathbf{P}(\mathbb{C})$ and $J_k(X, \check{D})$ respectively, such that Ψ extends to a rational map

$$\Psi' : \mathbf{P}(\mathbb{C})' \times J_k(X, \check{D})' \to J_k(X, \check{D})'.$$

By the Chevalley-Remmert theorem [40, p. 291], the set P, which is the image under Ψ' of an analytically constructible set, is analytically constructible. Moreover,

$$P := \mathbf{P}(\mathbb{C}) \cdot W_k = \mathbf{P}(\mathbb{C}) \cdot W_{k,F}$$

is definable. By the definable Chow theorem of Peterzil-Starchenko [48] (see also [43, Cor. 2.3]), P is algebraically constructible. □

We explain the idea of the proof of the following lemma. We first use the Griffiths conjecture proved by Bakker-Brunebarbe-Tsimerman [5] to reduce to the case where the liftings are submersions onto its image. Hence if $g \in \mathbf{P}(\mathbb{C})$ stabilizes the germ a local lifting, then g fixes the image of the lifting. By Theorem 4.2, this will imply that $g \in H$. By Noether's chain condition, similar statement holds when the germ is truncated at some finite order. We then make this order independent of the local lifting using the definable Chow theorem [48] and the chain condition the second time.

Lemma 5.5. *There exists an integer $k_0 > 0$ such that H is the $\mathbf{P}_{\mathbb{C}}$-stabilizer of any jet in W_k for any $k \geq k_0$.*

Proof. Since H is normal in $\mathbf{P}_{\mathbb{C}}$, it suffices to show that there exists an integer $k_0 > 0$ such that H is the $\mathbf{P}_{\mathbb{C}}$-stabilizer of any jet in $W_{k,F}$ for any $k \geq k_0$.

Let j be the germ of a local lifting into F of the period mapping ψ. Let j_k be the k-jet of j. Let $S_{j,k}$ be the $\mathbf{P}_{\mathbb{C}}$-stabilizer of j_k. Consider analytic fiber product

$$\begin{array}{ccc} W_0 & \longrightarrow & D \\ \downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{\varphi} & \mathbf{P}(\mathbb{Z})^+\backslash D. \end{array}$$

Let H' be the pointwise $\mathbf{P}(\mathbb{C})$-stabilizer of the image of $W_0 \to D$. By the Griffiths conjecture for mixed period mappings proved in [5, Cor. 2.11], there exists an algebraic variety Y such that the period mapping is the composition of a dominant algebraic morphism $f : X \to Y$ and a closed immersion $\iota : Y \to \mathbf{P}(\mathbb{Z})^+\backslash D$. By restricting the GPVMHS to a Zariski open subset of X, we can assume that f is smooth, and thus surjective on tangent spaces. Therefore,

$$H' = \bigcap_{k \geq 0} S_{j,k}$$

by the identity theorem.

Let $h \in H'$. Let D_h be the subset of elements in D that are fixed by h. Let $V_0 := X \times D_h$. We have $W_0 \subset V_0$. The projection of W_0 to X is equal to X. By Theorem 4.2,

$$\dim V_0 \geq \dim W_0 + \dim \check{D} = \dim X + \dim \check{D},$$

so $D_h = D$. Therefore, $H' = H$.

The sequence $\{S_{j,k}\}_{k \geq 0}$ of subgroups of $\mathbf{P}_{\mathbb{C}}$ is decreasing. Since $\mathbf{P}_{\mathbb{C}}$ is Noetherian, there exists $k_j > 0$ such that $H = S_{j,k}$ for all $k \geq k_j$.

For any $k \geq 0$, let X_k be the definable analytic subset of points in X for which the k-jets of the germs of local liftings, centered at these points, into F have $\mathbf{P}_{\mathbb{C}}$-stabilizer equal to H. By Peterzil-Starchenko [48], X_k is algebraically constructible. From above,

$$X = \bigcup_{k \geq 0} X_k.$$

The sequence $\{X_k\}$ is increasing. Hence, there exists $k_0 > 0$ such that $X = X_k$ for all $k \geq k_0$. The claim follows. $\square$

Theorem 5.6. *Let $k \geq k_0$. The map $\pi_X|_P$ makes P a principal $\mathbf{P}_{\mathbb{C}}/H$-bundle over X. There is a foliation on P where each leaf is of the form $g \cdot W_{k,\Gamma}$ for some $g \in \mathbf{P}(\mathbb{C})$, and vice versa. The leaves are transverse to the fibers of the bundle.*

Proof. Let $\lambda : B \to D$ be a local lifting of $\psi : X \to \Gamma \backslash D$, such that B is an open subset and that $\lambda(B)$ does not intersect any other $\mathbf{P}(\mathbb{Z})^+$-translate of it. Let $W_{k,\lambda}$ be the analytic set of all k-jets of germs of λ. For any $x \in B$, let $J_{k,x}\lambda$ be the k-jet of the germ of λ at x. By Lemma 5.5, the map

$$\kappa : \mathbf{P}(\mathbb{C})/H \times B \to (\mathbf{P}(\mathbb{C})/H) \cdot W_{k,\lambda}$$

defined by $(gH, x) \mapsto g \cdot J^d_{k,x}\lambda$ is a biholomorphism. We then have

$$\pi_X|_P^{-1}(B) = (\mathbf{P}(\mathbb{C})/H) \cdot W_{k,\lambda} \simeq (\mathbf{P}(\mathbb{C})/H) \times B.$$

Moreover, $\mathbf{P}_{\mathbb{C}}/H$ acts transitively and freely (by Lemma 5.5) on the fibers of $\pi_X|_P$.

Suppose $\lambda_1 : B_1 \to D$ is another such local lifting on an open subset B_1 which overlaps with B. Similarly, we have

$$\pi_X|_P^{-1}(B_1) \simeq (\mathbf{P}(\mathbb{C})/H) \times B_1.$$

By restricting λ and λ_1 to $B \cap B_1$, we have an automorphism on

$$(\mathbf{P}(\mathbb{C})/H) \times (B \cap B_1),$$

which is a product of an automorphism of $\mathbf{P}(\mathbb{C})/H$ and the identity on $B \cap B_1$. The cocycle condition can also be checked.

Let $\mathcal{L}$ be the set of all local liftings λ satisfying the condition as above. Let

$$\mathcal{S} := \{g \cdot W_{k,\lambda} : g \in \mathbf{P}(\mathbb{C}), \lambda \in \mathcal{L}\}.$$

Define an equivalence relation $\sim$ on $\mathcal{S}$ as follows:

$$g_0 \cdot W_{k,\lambda_0} \sim g_k \cdot W_{k,\lambda_\ell}$$

in $\mathcal{S}$ if and only if there exist $g_i \cdot W_{k,\lambda_i} \in \mathcal{S}$ for each $0 < i < \ell$, such that

$$g_{i-1} \cdot W_{k,\lambda_{i-1}} \cap g_i \cdot W_{k,\lambda_i} \neq \varnothing$$

for all $1 \leq i \leq \ell$. Then we have a foliation on P where each leaf has the same dimension as X and is of the form

$$\bigcup_{g \cdot W_{k,\lambda} \sim g_0 \cdot W_{k,\lambda_0}} g \cdot W_{k,\lambda} \quad \text{for some } g_0 \cdot W_{k,\lambda_0} \in \mathcal{S},$$

and vice versa. Hence each leaf is of the form $g_0 \cdot W_{k,\Gamma}$ for some $g_0 \in \mathbf{P}(\mathbb{C})$, and vice versa. The transversality follows from that κ is a biholomorphism. $\square$

5.4 Ax-Schanuel for foliated principal bundles

We recall the definitions of ∇-special subvarieties and sparse groups, and the Ax-Schanuel theorem for foliated principal bundles proved by Blázquez-Sanz, Casale, Freitag, and Nagloo [12]. Then we prove that any ∇-special subvariety of X is contained in a proper weakly special subvariety, and that $\mathbf{P}(\mathbb{C})/H$ is sparse.

Let G be a complex algebraic group. Let ∇ be a flat principal G-connection on a principal G-bundle P over a complex algebraic variety X. The Galois group $\mathrm{Gal}(\nabla)$ of ∇ is the algebraic group

$$\{g \in G : g \cdot M = M\}$$

for any minimal ∇-invariant subvariety M of P. A subvariety Z of X is **∇-special** [12] if for each irreducible component Z_i with smooth locus Z_i^*, the group $\mathrm{Gal}(\nabla|_{Z_i^*})$ is a proper subgroup of G.

A Lie subalgebra of the Lie algebra $\mathfrak{g}$ of G is said to be algebraic if it is the Lie algebra of an algebraic subgroup of G. The algebraic envelop $\overline{\mathfrak{h}}$ of a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is the smallest algebraic Lie subalgebra containing $\mathfrak{h}$. An algebraic group G is said to be **sparse** [12] if for any proper Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the algebraic envelop $\overline{\mathfrak{h}}$ is a proper Lie subalgebra of $\mathfrak{g}$.

Theorem 5.7 ([12]). *Let G be a sparse complex algebraic group. Let ∇ be a flat principal G-connection on the principal G-bundle P over a complex algebraic variety X. Assume that the Galois group $\mathrm{Gal}(\nabla) = G$. Let V be an algebraic subvariety of P and L an horizontal leaf. If*

$$\dim V < \dim(V \cap L) + \dim G,$$

then the projection of $V \cap L$ in X is contained in a ∇-special subvariety.

Theorem 5.8. *Let P be the foliated principal $\mathbf{P}_{\mathbb{C}}/H$-bundle over X in Theorem 5.6. Let ∇ be the flat principal $\mathbf{P}_{\mathbb{C}}/H$-connection on P induced by the foliation. If Z is an irreducible ∇-special subvariety in X, then Z is contained in a proper weakly special subvariety. Moreover, $\mathbf{P}_{\mathbb{C}}/H = \mathrm{Gal}(\nabla)$.*

Proof. Let ℓ be a horizontal leaf over the smooth locus Z^* of Z. By Theorem 5.6, each leaf in P is of the form $g \cdot W_{k,\Gamma}$ for some $g \in \mathbf{P}(\mathbb{C})$. Let Γ_1 be the monodromy group of Z^*. The group $g\Gamma_1 g^{-1}$ stabilizes ℓ. Therefore, the algebraic group $g\overline{\Gamma_1}g^{-1}$ stabilizes the Zariski closure $\overline{\ell}$ of ℓ. By definition and [12, Lemma 2.2],

$$\mathrm{Gal}(\nabla|_{Z^*}) = \{gH \in \mathbf{P}_{\mathbb{C}}/H : g \cdot \overline{\ell} = \overline{\ell}\},$$

so

$$(g\overline{\Gamma_1}g^{-1})H/H \subset \mathrm{Gal}(\nabla|_{Z^*}).$$

Then since Z is ∇-special, $(g\overline{\Gamma_1}g^{-1})H/H$ is a proper subgroup of $\mathbf{P}_{\mathbb{C}}/H$, so $\overline{\Gamma_1}$ is a proper subgroup of $\mathbf{P}_{\mathbb{C}}$. By André-Deligne [1] and the algebraicity [4, Corollary 6.7] of weakly special subvariety, Z is contained in a proper weakly special subvariety. Similarly, $\overline{\Gamma}/H \subset \mathrm{Gal}(\nabla)$. Since $\Gamma \subset \mathbf{P}(\mathbb{Z})^+$, the $\mathbb{Q}$-closure of Γ is $\mathbf{P}$. By Lemma 3.6, the $\mathbb{C}$-Zariski closure of Γ is $\mathbf{P}_{\mathbb{C}}$. Therefore,

$$\mathbf{P}_{\mathbb{C}}/H = \overline{\Gamma}/H \subset \mathrm{Gal}(\nabla),$$

so $\mathbf{P}_{\mathbb{C}}/H = \mathrm{Gal}(\nabla)$. □

Lemma 5.9. *If G is an algebraic group whose quotient by its unipotent radical G_u is semisimple, then G is sparse.*

Proof. Let $\mathfrak{g}, \mathfrak{g}_s$ and $\mathfrak{g}_u$ be the Lie algebras of G, G_s and G_u respectively, where G_s is a Levi subgroup of G. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g} = \mathfrak{g}_s \ltimes \mathfrak{g}_u$. It is a general fact that $\mathfrak{h}$ is an ideal in its algebraic envelope $\overline{\mathfrak{h}}$ [12, Example 3.5]. Suppose $\overline{\mathfrak{h}} = \mathfrak{g}$. By [15, §6, no. 8, Cor. 4], $\mathfrak{h} \cap \mathfrak{g}_u$ is the radical of $\mathfrak{h}$ and $\mathfrak{h} \cap \mathfrak{g}_s$ is a Levi subalgebra of $\mathfrak{h}$, so $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{g}_s) \ltimes (\mathfrak{h} \cap \mathfrak{g}_u)$. The ideal $\mathfrak{h} \cap \mathfrak{g}_s$ of the semisimple Lie algebra $\mathfrak{g}_s$ is semisimple, so there exists an algebraic subgroup H_1 of G_s whose Lie algebra is $\mathfrak{h} \cap \mathfrak{g}_s$. Moreover, the exponential map gives an algebraic variety isomorphism [41, Prop. 14.32] between the unipotent group G_u and its Lie algebra, so there exists an algebraic subgroup H_2 of G_u whose Lie algebra is $\mathfrak{h} \cap \mathfrak{g}_u$. The Lie algebra of the algebraic subgroup $H_1 \ltimes H_2$ of G is thus $\mathfrak{h}$, so $\mathfrak{h} = \overline{\mathfrak{h}} = \mathfrak{g}$. Therefore, if $\mathfrak{h}$ is proper, then $\overline{\mathfrak{h}}$ is also proper. □

Corollary 5.10. *The algebraic monodromy group $\mathbf{P}_{\mathbb{C}}$ and the quotient $\mathbf{P}_{\mathbb{C}}/H$ are sparse.*

Proof. Formations of the radical and the unipotent radical commute with field extensions in characteristic 0, so by André [1, Corollary 2], we can write $\mathbf{P}_{\mathbb{C}} = \mathbf{P}_s \ltimes \mathbf{P}_u$, where $\mathbf{P}_s$ is a semisimple Levi subgroup, while $\mathbf{P}_u$ is the unipotent radical and the radical. By Lemma 5.9, $\mathbf{P}_{\mathbb{C}}$ and $\mathbf{P}_{\mathbb{C}}/H$ are sparse. □

5.5 Proofs of Theorem 5.1 and Corollary 5.3

We now prove Theorem 5.1 and Corollary 5.3, which are restated as Theorem 5.11 and Corollary 5.12 below.

Theorem 5.11. *Let U be an irreducible analytic subset of W_k. If*

$$\dim U^{\mathrm{Zar}} - \dim U < \dim W_k^{\mathrm{Zar}} - \dim W_k,$$

then $\pi_X(U)$ is contained in a proper weakly special subvariety of X.

Proof. Let S be the set of all distinct representatives of the cosets in $\mathbf{P}(\mathbb{Z})^+/\Gamma$. We have

$$W_k = \bigcup_{g \in S} W_{k,\Gamma}$$

and

$$\dim W_k = \dim X = \dim W_{k,\Gamma}.$$

Since $\Gamma \subset \mathbf{P}(\mathbb{Z})^+$, the $\mathbb{Q}$-closure of Γ is $\mathbf{P}$. By Lemma 3.6, the $\mathbb{C}$-Zariski closure of Γ is $\mathbf{P}_{\mathbb{C}}$. Then since $\Gamma \cdot W_{k,\Gamma} = W_{k,\Gamma}$, we have $\mathbf{P}_{\mathbb{C}} \cdot W_{k,\Gamma}^{\mathrm{Zar}} = W_{k,\Gamma}^{\mathrm{Zar}}$. By Lemma 5.4, $\mathbf{P}(\mathbb{C}) \cdot W_{k,\Gamma} = \mathbf{P}(\mathbb{C}) \cdot W_k =: P$ is algebraic, so $P = W_{k,\Gamma}^{\mathrm{Zar}} = W_k^{\mathrm{Zar}}$.

First assume $k \geq k_0$. Since U is irreducible, $g^{-1}U \in W_{k,\Gamma}$ for some $g \in S$. By Theorem 5.6, $W_{k,\Gamma}$ is a leaf in P and $\dim P - \dim W_{k,\Gamma} = \dim(\mathbf{P}_{\mathbb{C}}/H)$. Then

$$\begin{aligned}
\dim(g^{-1}U)^{\mathrm{Zar}} &= \dim U^{\mathrm{Zar}} \\
&< \dim U + \dim W_k^{\mathrm{Zar}} - \dim W_k \\
&= \dim g^{-1}U + \dim W_{k,\Gamma}^{\mathrm{Zar}} - \dim W_{k,\Gamma} \\
&\leq \dim((g^{-1}U)^{\mathrm{Zar}} \cap W_{k,\Gamma}) + \dim(\mathbf{P}_{\mathbb{C}}/H).
\end{aligned}$$

We have

$$\pi_X(U) = \pi_X(g^{-1}U) \subset \pi_X((g^{-1}U)^{\mathrm{Zar}} \cap W_{k,\Gamma}).$$

By Lemma 5.4, $(g^{-1}U)^{\mathrm{Zar}} \subset P$. Then by Corollary 5.10, Theorem 5.7 and Theorem 5.8, $\pi_X(U)$ is contained in a proper weakly special subvariety of X.

We now prove the theorem for $1 \leq k < k_0$. Let W_{k_0} be the analytic subset of $J_{k_0}(X, \check{D})$ consists of all k_0-jets of germs of local liftings of the period mapping ψ. Let $P_{k_0} := \mathbf{P}(\mathbb{C}) \cdot W_{k_0}$. Let $\rho : P_{k_0} \to P$ be the projection defined by lowering the order of jets. Let $U_{k_0} := W_{k_0} \cap \rho^{-1}(U)$, which implies that $U_{k_0}^{\mathrm{Zar}} \subset \rho^{-1}(U^{\mathrm{Zar}})$. We have

$$\dim \rho^{-1}(U^{\mathrm{Zar}}) - \dim U^{\mathrm{Zar}} \leq \dim P_{k_0} - \dim P.$$

Moreover, since $\rho|_{W_{k_0}}$ is equidimensional, we have

$$\dim W_{k_0} - \dim W_k = \dim U_{k_0} - \dim U.$$

Hence,

$$\begin{aligned}
\dim U_{k_0}^{\mathrm{Zar}} - \dim U_{k_0} &\leq \dim \rho^{-1}(U^{\mathrm{Zar}}) - \dim U_{k_0} \\
&\leq \dim P_{k_0} - \dim P + \dim U^{\mathrm{Zar}} - \dim U_{k_0} \\
&< \dim P_{k_0} + \dim U - \dim W_k - \dim U_{k_0} \\
&= \dim W_{k_0}^{\mathrm{Zar}} - \dim W_{k_0}.
\end{aligned}$$

By the case for $k = k_0$, $\pi_X(U_{k_0})$ is contained in a proper weakly special subvariety of X. We are done since $\pi_X(U) \subset \pi_X(U_{k_0})$. $\square$

Let Δ be the open unit disk.

Corollary 5.12. *Let $\widetilde{\psi}$ be a local lifting of the period mapping ψ on an open subset B. Let $v : \Delta^{\dim \check{D}} \to \check{D}$ and $u : \Delta^{\dim X} \to B$ be open embeddings such that $(\widetilde{\psi} \circ u)(\Delta^{\dim X}) \subset v(\Delta^{\dim \check{D}})$. Let $f : \Delta^m \to B$ be a holomorphic mapping such that $f(\Delta^m) \subset u(\Delta^{\dim X})$. Write $z = (z_1, \ldots, z_m)$, where z_i are the coordinates of Δ^m. If*

$$\operatorname{tr.deg}_{\mathbb{C}} \mathbb{C}((u^{-1} \circ f)(z), \partial^\alpha (v^{-1} \circ \widetilde{\psi} \circ f)(z) : |\alpha| \leq k) < \operatorname{rank}(f) + \dim W_{k,\Gamma}^{\mathrm{Zar}} - \dim W_{k,\Gamma},$$

then $f(\Delta^m)$ is contained in a proper weakly special subvariety of X.

Proof. We have a map $\sigma : \Delta^m \to W_{k,\Gamma}$ defined by $\sigma(z) = J_{k,f(z)}\widetilde{\psi}$, where $J_{k,f(z)}\widetilde{\psi}$ is the k-jet of $\widetilde{\psi}$ at $f(z)$. Let U be the image of σ. Using the coordinate charts u and v, the map σ can be expressed as a tuple of functions, including $(u^{-1} \circ f)(z)$ and $\partial^\alpha(v^{-1} \circ \widetilde{\psi} \circ f)(z)$, where $|\alpha| \leq k$. Then

$$\operatorname{rank}(f) \leq \operatorname{rank}(\sigma) = \dim U.$$

We also have

$$\dim U^{\mathrm{Zar}} = \operatorname{tr.deg}_{\mathbb{C}} \mathbb{C}((u^{-1} \circ f)(z), \partial^\alpha(v^{-1} \circ \widetilde{\psi} \circ f)(z) : |\alpha| \leq k).$$

Then by assumption,

$$\dim U^{\mathrm{Zar}} < \dim U + \dim W_{k,\Gamma}^{\mathrm{Zar}} - \dim W_{k,\Gamma},$$

so $f(\Delta^m) \subset \pi_X(U)$ is contained in a proper weakly special subvariety of X by Corollary 5.2. $\square$

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