Three pictures of Lusztig's asymptotic Hecke algebra

by

Stefan James Dawydiak

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Department of Mathematics
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Abstract

Let $\tilde{W}$ be an extended affine Weyl group, $\mathbf{H}$ be the its Hecke algebra over the ring $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$ with standard basis $\{T_w\}_{w\in\tilde{W}}$, and J be Lusztig's asymptotic Hecke algebra, viewed as a based ring with basis $\{t_w\}_{w\in\tilde{W}}$. This thesis studies the algebra J from several perspectives, proves theorems about various incarnations of J, and provides tools to be applied for future work. We prove three types of results. In the second and third chapters, we investigate J as a subalgebra of the $(\mathbf{q}^{-1})$-adic completion of $\mathbf{H}$ via Lusztig's map ϕ. In the second chapter, we use Harish-Chandra's Plancherel formula for p-adic groups to show that the coefficient of T_x in t_w is a rational function of $\mathbf{q}$, depending only on the two-sided cell containing w, with no poles outside of a finite set of roots of unity that depends only on $\tilde{W}$. In type $\tilde{A}_n$ and type $\tilde{C}_2$, we show that the denominators all divide a power of the Poincaré polynomial of the finite Weyl group. As an application, we conjecture that these denominators encode more detailed information about the failure of the Kazhdan-Lusztig classification of $\mathbf{H}$-modules at roots of the Poincaré polynomial than is currently known. In the third chapter, we reprove the results of the second chapter without using any tools from harmonic analysis in the special case $\mathbf{G} = \mathrm{SL}_2$. In this case we also prove a positivity property for the coefficients of T_x in t_w, that we conjecture holds in general. We also produce explicit formulas for the action of J on the Iwahori invariants $\mathcal{S}^I$ of the Schwartz space of the basic affine space. In the fourth chapter, we give a triangulated monoidal category of coherent sheaves whose Grothendieck group surjects onto $J_0 \subset J$, the based ring of the lowest two sided cell of $\tilde{W}$, equipped with a monoidal functor from the category of coherent sheaves on the derived Steinberg variety. We show that this partial categorification acts on natural coherent categorifications of $\mathcal{S}^I$. In low rank cases, we construct complexes lifting the basis elements t_w of J_0 and their structure constants.

To my parents

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Contents

Chapter 1

Introduction

The representation theory of p-adic groups plays a key role in the Langlands program, and, when studied using geometric techniques, provides a bridge between harmonic analysis and algebraic geometry. Upon categorification, this bridge upgrades to a family of equivalences of categories between constructible sheaves, corresponding to harmonic analysis, and coherent sheaves, corresponding to (derived) algebraic geometry.

A key object in the representation theory of $G = \mathbf{G}(F)$, where F is a p-adic field and $\mathbf{G}$ is a connected reductive group over F—as well as in physics and knot theory—is the *affine Hecke algebra* $\mathbf{H}$ of G. This ring is the medium through which the above mentioned correspondence manifests itself, and is a $\mathscr{A} = \mathbb{Z}[\mathbf{q}^{1/2}, \mathbf{q}^{-1/2}]$-algebra deforming the group algebra of the *affine Weyl group* $\tilde{W}$ of G. Upon specializing $\mathbf{q}^{1/2}$ to $\sqrt{q}$, where q is the cardinality of the residue field of F, Iwahori and Matsumoto showed that $\mathbf{H}$ is isomorphic to the convolution algebra of certain smooth compactly-supported functions on G. Subsequently, Borel and independently Casselman showed that the category of $\mathbf{H}$-modules is equivalent to the full subcategory of smooth representations of G with nonzero fixed vectors under the *Iwahori subgroup* I, which we will term Iwahori-spherical. On the other hand, Kazhdan and Lusztig showed in 1987 that $\mathbf{H}$ is isomorphic to $K^{\mathbf{G}^\vee \times \mathbb{G}_{\mathrm{m}}}(\mathrm{St}^\vee)$, the $\mathbf{G}^\vee \times \mathbb{G}_{\mathrm{m}}$-equivariant algebraic K-theory of the *Steinberg variety* $\mathrm{St}^\vee$ of $\mathbf{G}^\vee$, the Langlands dual group of $\mathbf{G}$ taken over $\mathbb{C}$, and used this isomorphism to parameterize the simple $\mathbf{H}$-modules. This proved the local Langlands correspondence for I-spherical representations. In 2015, Bezrukavnikov upgraded this isomorphism to an equivalence of categories, lifting functions and K-theory to derived categories of constructible and coherent sheaves, respectively. Thus $\mathbf{H}$ connects harmonic analysis on G to algebraic geometry on $G^\vee$, and this connection lifts to the level of categories.

The representation theory of $\mathbf{H}$, in stark contrast to the general theory of p-adic groups, is almost insensitive to q; Kazhdan-Lusztig's parametrization holds for any $q \in \mathbb{C}^\times$ outside the roots of the *Poincaré polynomial* P_W of the finite Weyl group W. In 1987, Lusztig constructed a based ring J, the *asymptotic Hecke algebra*, with $\mathbb{Z}$-basis $\{t_w\}_{w\in\tilde{W}}$ that implements this uniformity, and equipped it with an injection $\phi\colon \mathbf{H} \hookrightarrow J \otimes_{\mathbb{Z}} \mathbb{Z}[v, v^{-1}]$. The map ϕ becomes an isomorphism upon completing both sides to allow infinite sums of basis elements, convergent in the appropriate topology. In an algebraic version of Harish-Chandra's philosophy of cusp forms, J is a direct sum of two-sided ideals J_u indexed by the unipotent conjugacy classes of $\mathbf{G}^\vee(\mathbb{C})$.

1.1 Overview and main results

This thesis studies Lusztig's asymptotic Hecke algebra J from several perspectives. It proves special cases of some of the below proposed future research directions, and provides tools to be applied to others. In this section we give an overview of its main results.

Experts should read this section directly, and then would likely be best served by reading the individual papers. The only unpublished material in this section, indeed, in this thesis, is the planned future work sketched in Section 1.2. The non-expert reader might read Section 1.3 before this one.

Let $\tilde{W}$ be an affine Weyl group. Its group algebra $\mathbb{Z}[\tilde{W}]$ is deformed by the affine Hecke algebra $\mathbf{H}$ of $\tilde{W}$. In turn, Lusztig defined the asymptotic Hecke algebra J, a based ring with basis t_w, $w \in \tilde{W}$ and structure constants determined from certain "leading terms" of the structure constants of $\mathbf{H}$. Further, he provided a morphism of algebras $\phi\colon \mathbf{H} \hookrightarrow J \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{q}^{\pm 1/2}]$ and showed it was an isomorphism after a mild completion. Thus $\mathbf{H}$ can be viewed as a subalgebra of J, and J can be viewed as a subalgebra of a completion $\mathscr{H}_{\mathrm{aff}}$ of $\mathbf{H}$. While the morphism ϕ is an essential part of Lusztig's exploration of J, until recently there have been few compelling reasons to adopt the perspective of $J \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{q}^{\pm 1/2}]$ as a subalgebra of $\mathscr{H}_{\mathrm{aff}}$. This thesis investigates this perspective, first with techniques from harmonic analysis, and then with techniques from derived algebraic geometry.

The algebra $\mathbf{H}$ appears in many areas of mathematics in many guises, but one of the most prominent relates to the representation theory of p-adic groups. Let F be a local non-archimedean field and q be the cardinality of the residue field of F. Let $\mathbf{G}$ be as above. For the purposes of harmonic analysis on $G = \mathbf{G}(F)$, it is natural to consider $\mathbf{H}|_{\mathbf{q}^{1/2}=\sqrt{q}}$, very much an algebraic object, as a subalgebra of the larger, analytically-characterized *Harish-Chandra Schwartz algebra* $\mathscr{C}(G)^I$.

In [15], Braverman and Kazhdan gave an interpretation of J in terms of harmonic analysis, casting J as an algebraic version of $\mathscr{C}(G)^I$ by defining an map $J \to \mathscr{C}(G)^I$. (They also defined a larger ring $J(G) \hookrightarrow \mathscr{C}(G)$ doing the same for the full algebra.) In Chapter 2, we show that this morphism is essentially the specialization of ϕ^{-1} at $\mathbf{q} = q$, and in particular is an injection. It is their work that motivates investigating J as a subalgebra of $\mathscr{H}_{\mathrm{aff}}$.

1.1.1 Denominators in the asymptotic Hecke algebra

In Chapter 2, we describe the asymptotic Hecke algebra J as a subalgebra of the above-mentioned completion $\mathscr{H}_{\mathrm{aff}}$ of the affine Hecke algebra $\mathbf{H}$. The injection $\phi\colon \mathbf{H} \hookrightarrow J \otimes \mathbb{Z}[\mathbf{q}^{1/2}, \mathbf{q}^{-1/2}]$ becomes an isomorphism upon replacing $\mathbf{H}$ with $\mathscr{H}_{\mathrm{aff}}$ and J with a completion $\mathscr{J}$. This means that one may write $\phi^{-1}(t_w) = \sum_x a_{x,w} T_x$ for every basis element t_w of J, where $a_{x,w}$ is a formal Laurent series in $\mathbf{q}^{-1/2}$, and the sum is convergent in the $(\mathbf{q}^{-1/2})$-adic topology. While much is known about the structure and representation theory of J, very little was known about its incarnation as a subalgebra of the completion. What was known [70] is that for finite Weyl groups, ϕ becomes an isomorphism after inverting only the Poincare polynomial P_W of W.

Thus, the coefficients $a_{x,w}$ are rational functions of $\mathbf{q}$ with denominators dividing a power of P_W. For affine Weyl groups, there are two potential types of infinity: the coefficients $a_{x,w}$ are nonzero for infinitely-many x, and are, *a priori*, formal Laurent series that need not equal rational functions.

We prove that, for affine Weyl groups, this second potential infinity does not appear. For $G = \mathrm{GL}_n(F)$, we prove that the $a_{x,w}$ are rational functions of $\mathbf{q}$ with denominators dividing a fixed power of P_W. For general reductive groups, we prove that the $a_{x,w}$ are rational functions of $\mathbf{q}$ and have poles only at finitely-many roots of unity. We conjecture that in general, the denominators of $a_{x,w}$ always divide a fixed power of P_W. The group $\tilde{W}$ is partitioned combinatorially into *two-sided cells*, and we also prove that the denominators of $a_{x,w}$ depend only on the two-sided cell containing w. Thus taking the set of poles of $a_{x,w}$ assigns to each cell a quantitative measure of its singularity in $\mathbf{q}$. By a deep theorem of Lusztig, this also gives a measure of singularity to each summand J_u of J.

In summary, the conjecture formulated in this chapter is

Conjecture 1 (Chapter 2, Conjecture 2)**.** *Let $\tilde{W}$ be an affine Weyl group, $\mathbf{H}$ its affine Hecke algebra over $\mathscr{A}$, and J its asymptotic Hecke algebra. Let $\phi\colon \mathbf{H} \hookrightarrow J \otimes_{\mathbb{Z}} \mathscr{A}$ be Lusztig's map, and let $()^\dagger$ be the Goldman involution on $\mathbf{H}$.*

1. *For all $x, w \in \tilde{W}$, $a_{x,w}$ is a rational function of $\mathbf{q}$. The denominator of $a_{x,w}$ is independent of x. As a function of w, it is constant on two-sided cells.*
2. *There exists $N_{\tilde{W}} \in \mathbb{N}$ such that upon writing*
$$()^\dagger \circ \phi^{-1}(t_w) = \sum_{x\in\tilde{W}} a_{x,w} T_x,$$
we have
$$P_W(\mathbf{q})^{N_{\tilde{W}}} a_{x,w} \in \mathscr{A}$$
for all $x, w \in \tilde{W}$.
3. *Moreover, there exists $N_{\tilde{W}} \in \mathbb{N}$ such that*
$$P_W(\mathbf{q})^{N_{\tilde{W}}} d(\omega) \in \mathscr{A}$$
for all discrete series representations ω of $\mathbf{H}$.

The main result of this chapter is

Theorem 1 (Chapter 2, Theorem 34, Theorem 36, and Theorem 19)**.** *Let $\tilde{W}$ be of type $\tilde{A}_n$ or $\tilde{C}_2$. Then Conjecture 2 is true. For general affine Weyl groups, a weaker version of the conjecture is true.*

The proof proceeds by relating Braverman-Kazhdan's action of J on the tempered representations of the p-adic group G to $\mathscr{C}(G)^I$ via Harish-Chandra's Plancherel formula, then relating the Schwartz algebra to $\mathscr{H}_{\mathrm{aff}}$.

This chapter is available as the preprint [22].

1.1.2 Positivity in the asymptotic Hecke algebra

In Chapter 3, we study the example of type $\tilde{A}_1$, for $\mathbf{G} = \mathrm{SL}_2$, where $\tilde{W}$ is the infinite dihedral group. We prove that, in this case, coefficients related to the $a_{x,w}$ display a certain positivity property, which we conjecture holds for general reductive groups $\mathbf{G}$. In particular, in this case, one may recover the main result of the first chapter by purely algebraic means without recourse to the Plancherel formula. We then use our formulas to re-prove a theorem of Braverman and Kazhdan

[15] in the case of SL_2, namely that J acts on the Schwartz space $\mathscr{S}^I$ of the basic affine space. We also obtain explicit formulas for the action of two distinguished basis elements of J; one acts by projection onto the functions invariant under the maximal compact subgroup $\mathrm{SL}_2(\mathscr{O})$, the other acts by the complementary projection for the other conjugacy class of maximal compact subgroups.

The main result of Chapter 3 is

Theorem 2 (Chapter 3, Theorem 37)**.** *1. For any w, the element $(\phi \circ j)^{-1}(t_w) \in \mathscr{H}_{\mathrm{aff}}$ has the form*

$$\sum b_{w,x} C'_x$$

where $b_{w,x}$ is a polynomial *in $\mathbf{q}^{-\frac{1}{2}}$. Moreover, $(-1)^{\ell(x)} b_{w,x}$ has nonpositive integer coefficients.*

2. For any w the element $(\phi \circ j)^{-1}(t_w) \in \mathscr{H}_{\mathrm{aff}}$ has the form

$$\sum a_{w,x} T_x$$

where $(\mathbf{q}+1)a_{w,x}$ is a polynomial in $\mathbf{q}^{-\frac{1}{2}}$.

The formulas for the action on $\mathscr{S}^I$ are summarized in Section 3.1.2.

This chapter is published as [23].

1.1.3 Categorification of J_0

Lusztig gave a categorification of J in [46] in terms of perverse sheaves on the affine flag variety $\mathscr{F}\ell$ of $\mathbf{G}$. In the spirit of the definition of J as a ring, the underlying category is again $\mathrm{Perv}(\mathscr{F}\ell)$, but with monoidal structure given by *truncated convolution* as opposed to convolution. In this chapter we define a monoidal category $\mathscr{J}_0$, admitting a functor from the coherent Hecke category, whose Grothendieck group surjects onto a large direct summand J_0 of J. This construction is compatible with the perspective of [15]. That is, we obtain a natural categorification of the action of J_0 on the unitary principal series, and produce a completely new category, a quotient of whose K-theory is J_0, as opposed to a new monoidal structure. In this chapter, to unburden notation, we exchange the roles of $\mathbf{G}$ and $\mathbf{G}^\vee$. We also categorify the map ϕ composed with projection to J_0, which we denote ϕ_0. Let $\mathscr{B}$ be the derived zero section of the Springer resolution $\tilde{\mathscr{N}} \to \mathscr{N}$, $\mathscr{B}^\heartsuit$ be its classical truncation, and recall that $J_0 \simeq K_G(\mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit)$ by [67] and [54]. Then main result in this chapter is

Theorem 3 (Chapter 4, Theorem 43)**.** *We have that*

1. the category

$$\mathscr{J}_0 := \mathrm{Coh}_{\mathbf{G}}(\mathscr{B} \times \mathscr{B})_!$$

is a triangulated subcategory of $\mathrm{Coh}_G(\mathscr{B} \times \mathscr{B})$, has a monoidal structure given by convolution, and admits a natural monoidal functor

$$\mathrm{Coh}_{\mathbf{G}\times\mathbb{G}_m}(\mathrm{St}) \to \mathrm{Coh}_{\mathbf{G}\times\mathbb{G}_m}(\mathscr{B} \times \mathscr{B})_!$$

such that

2. *the induced morphism*

$$\mathbf{H} \to K_0(\mathrm{Coh}_{\mathbf{G}\times\mathbb{G}_m}(\mathscr{B} \times \mathscr{B})_!) \twoheadrightarrow K_{\mathbf{G}\times\mathbb{G}_m}(\mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit)$$

is conjugate to ϕ_0 *and* $K_0(\mathscr{J}_0)$ *surjects onto* J_0*;*

3. *In the special case when* G *has universal cover equal to* SL_2 *or* SL_3*, there exists a family of objects* $\{t_w\}_{w\in\mathbf{c}_0}$ *in* $\mathscr{J}_0$*, such that that if* $t_w t_x = \sum_z \gamma_{w,x,z^{-1}} t_z$ *in* J_0*, then*

$$t_w \star t_x = \bigoplus_z t_z^{\oplus\gamma_{w,x,z^{-1}}}$$

in $\mathscr{J}_0$ *and such that the image in* $K_{\mathbf{G}}(\mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit)$ *of the class* $[t_w]$ *under the above morphism is* $[t_w]$*;*

4. *The category* $\mathscr{J}_0$ *acts on* $\mathrm{Coh}_{\mathbf{A}}(\mathscr{B})$ *and on* $\mathrm{Coh}_{\mathbf{G}}(\mathscr{B} \times \mathscr{B})$*.*

We emphasize that all functors and categories in this statement are derived, and refer to Section 4.3.1 for the definition of the (derived) category of coherent sheaves on a derived scheme.

This categorification has two pleasant properties relating to the map ϕ. First, under the classical identifications of $\mathbf{H}$ and J_0 with K-groups, the map ϕ is given in a non-obvious, and essentially K-theoretic, as opposed to geometric, way. In particular, the definition uses the Thom isomorphism theorem. However, on the derived level, we prove in Chapter 4 that ϕ is categorified simply by (derived) pullback of sheaves. The fact that ϕ is injective but not surjective onto J_0 is captured categorically in terms of the singular supports that can appear in the essential image of the functor lifting ϕ, as expand upon in Remark 21 and the discussion preceding it.

The algebra J is very close to being a direct sum of matrix algebras. In type A, this is the main result of Xi's monograph [69], and Bezrukavnikov-Ostrik in [10] showed this up to central extensions in general. (It is however now known [9] that this partial result is sharp; the central extensions do in fact appear in general.) Therefore the last two items are particularly relevant: J is most interesting as a based algebra admitting a morphism from $\mathbf{H}$, and can be quite simple in isolation. We hope to remove the very restrictive current hypothesis on Item 3 in a future version of this work; see Remark 19 for an explanation of why it is currently necessary.

It would be interesting to studying injectivity of the morphism

$$K_0(\mathscr{J}_{0\mathscr{A}}) \twoheadrightarrow J_{\mathbf{G}\times\mathbb{G}_m}(\mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit), \tag{1.1}$$

or a modification of the category $\mathscr{J}_{0\mathscr{A}} = \mathrm{Coh}_{\mathbf{G}\times\mathbb{G}_m}(\mathscr{B} \times \mathscr{B})$, for example, by placing additional restrictions in terms of singular supports, such that (1.1) restricts to an isomorphism. See Remark 22.

This chapter is available in the preprint [21].

1.2 Future work

Each of Chapters 2, 3, and 4 invites future work, which we now pause briefly to outline.

The Kazhdan-Lusztig classification at roots of unity

Kazhdan-Lusztig prove their classification in [37] for any $q \in \mathbb{C}^\times$ that is not a root of unity. Some decades later, Xi strengthened this result, showing the classification holds for all but finitely-many roots of unity:

Theorem 4 ([70])**.** *Let P_W be the Poincaré polynomial of the finite Weyl group of $\mathbf{G}$, and let $q \in \mathbb{C}^\times$ be such that $P_w(q) \neq 0$. Then the Kazhdan-Lusztig classification is true. Namely, in the notation of Section 2.1.3, every standard module $K(u, s, \rho, q)$ has unique simple quotient $L(u, s, \rho, q)$ and two such simple modules are isomorphic if and only if their corresponding triples are conjugate. Moreover, in the notation of Section 1.3.8 $a(L) = a(E)$, where E is the simple J-module attached to $K(u, s, \rho, q)$.*

The main result of Chapter 2 shows that the set of poles of the denominator of the coefficient of T_x in t_w is constant on two-sided cells. Hence, one obtains a measure of the singularity in $\mathbf{q}$ of each two-sided cell. In Conjecture 3 of Chapter 2, we conjecture that the Kazhdan-Lusztig conjecture continues to hold at a root q of P_W for u such that the corresponding two-sided cell $\mathbf{c}(u)$ is not singular at q. Proving this would first require strengthening Theorem 19 to the level of Theorem 34.

Categorification of J

There are several natural continuations of the work in Chapter 4, given here in ascending levels of priority.

The first future step is to show that in the case of J_0, there is a t-structure on the category $\mathscr{J}_0$, or a mild modification of it, for which the objects $\mathcal{t}_w$ lie in the heart and are simple. This would categorify J_0 not just as a ring but as a based ring. This is relevant as in many special cases, J is simply a direct sum of matrix rings, and so is most interesting together with its basis $\{t_w\}$.

One would then want to repeat the constructions of Chapter 4 for other summands J_u of J. This means defining a derived enhancement $\mathscr{B}^\vee_{u,\mathrm{der}}$ of the *Springer fibre* $\mathscr{B}^\vee_u$ over the conjugacy class u, showing that coherent sheaves on $\mathscr{B}^\vee_{u,\mathrm{der}} \times \mathscr{B}^\vee_{u,\mathrm{der}}$, again with an appropriate condition on their singular supports, are stable under convolution, and then that pullback by the natural inclusion $\mathscr{B}^\vee_u \times \mathscr{B}^\vee_u \hookrightarrow \mathrm{St}$ is monoidal. After the case $u = \{1\}$, *i.e.* J_0, most is known about the *subregular summand* of J, and this summand is a natural starting point. Afterwards, the next step would be to define a t-structure whose heart contains the sheaves lifting the basis elements t_w, in which these sheaves would moreover be simple objects. Categorifying J in terms of coherent sheaves will give clues about a prospective constructible categorification of J and any possible equivalences similar to Bezrukavnikov's.

We plan to construct an equivalence between $\mathrm{Coh}_{\mathbf{A}^\vee \times \mathbb{G}_\mathrm{m}}(\mathscr{B}^\vee_{\mathrm{der}})$ and certain constructible sheaves on F^2, the existence of which should also shed light on potential constructible categorifications of J_0. An ultimate application of such a constructible categorification for all summands of J would be to use Grothendieck's function-sheaf dictionary to prove a more general version the positivity property proved in Chapter 3 as Theorem 37.

A major impediment to this plan is that in general the Springer fibres $\mathscr{B}^\vee_u$ are singular, and so the equivariant K-theory of the classical scheme $\mathscr{B}^\vee_u \times \mathscr{B}^\vee_u$ is not necessarily a ring. Qiu and Xi propose in [56] a possible remedy of this problem in the context of J, namely restricting to the fixed-point locus of the $\mathbb{G}_\mathrm{m}$- action defined in [20], which is proven in *loc. cit.* to be smooth.

The Casselman basis

The computations in Chapter 3 also show that, for SL_2, there is a basis of $\mathscr{S}^I$ particularly adapted to the J_0-action, specializing to the *Casselman basis* in each principal series representation. Defining this "universal" Casselman basis beyond SL_2 could have relevance to the conjectures of Bump and Nakasuji [17] on the relationship of the Casselman basis to another basis defined as dual to to certain evaluation functionals defined using intertwining operators.

1.3 Background, notation, and conventions

In this section we will define the major pieces of contextual knowledge this thesis requires, but which is inappropriate for inclusion into research articles.

In addition, each chapter begins with an introduction of its own, which recalls local notation and conventions, and recapitulates points of the theory exposited below that are of particular relevance. The material in this section is for the most part entirely standard, with the exception that we are not aware of other expositions of the work of Braverman-Kazhdan in [15].

1.3.1 Conventions on reductive groups

Let k be a field and $\mathbf{G}$ be a connected reductive algebraic group over k. Let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$. We assume the reader's familiarity with these notions, but it is traditional to pause to fix notation, even though this notation is almost completely universal.

We will assume throughout, unless indicated explicitly otherwise, that $\mathbf{G}$ is split over k. Then, unless noted otherwise at the beginning of a chapter, we write $\mathbf{A}$ for a chosen split maximal torus of $\mathbf{G}$, $\mathbf{B}$ for a Borel subgroup containing $\mathbf{A}$, $\mathbf{N}$ for the unipotent radical of $\mathbf{B}$, and $\mathscr{B} = \mathbf{G}/\mathbf{B}$ for the variety of Borel subgroups (or Borel subalgebras) of $\mathbf{G}$ (or $\mathfrak{g}$). We write $(X^*, \Phi, X_*, \Phi^\vee)$ for the root datum of $\mathbf{G}$ over the algebraic closure $\bar{k}$ of k. We write $\mathbf{G}^\vee$ for the Langlands dual group of $\mathbf{G}$, defined as the connected reductive algebraic group over $\mathbb{C}$ with root datum $(X_*, \Phi^\vee, X^*, \Phi)$. We usually write $G^\vee = \mathbf{G}^\vee(\mathbb{C})$, and in general use bold face for schemes or varieties, and usual typeface for sets of points, or notions that make sense only on the level of points. The major exception to this convention is Chapter 4, where we are interested in algebraic groups as schemes, and do not take points. In that chapter we unburden notation by foregoing boldface.

Definition 1. Let $\mathbf{X}$ be a complex algebraic variety. The *Poincaré polynomial of* $\mathbf{X}$ is

$$p_{\mathbf{X}}(v) = \sum_{i=0}^{\dim_{\mathbb{C}} \mathbf{X}} v^i \dim H^i(\mathbf{X}, \mathscr{O}_{\mathbf{X}}),$$

where $\mathscr{O}_X$ is the structure sheaf of X, and v is a formal variable.

We recall the following definitions from [63].

Definition 2. Let $x \in G^\vee$ be a unipotent element. It is *subregular* if $\dim Z_{\mathbf{G}^\vee}(x) = \mathrm{rk}(\mathbf{G}) + 2$. It is *regular* if $\dim Z_{\mathbf{G}^\vee}(x) = \mathrm{rk}(G)$. If $\mathbf{G} = \mathrm{GL}_n$ or SL_n, this is equivalent to the Jordan normal form of x consisting of a single Jordan block.

We shall be concerend in almost equal measure with groups over a non-archimedean field F and over $\mathbb{C}$. In the former case we also need to talk about the group $\mathbf{G}(\mathscr{O})$ and its subgroups, where $\mathscr{O}$ is the ring of integers of F. To do so, we must fix a *model of* $\mathbf{G}$, that is, a group scheme $\mathscr{G}$ over $\mathscr{O}$ whose base-change to F is $\mathbf{G}$, and whose multiplication and inversion maps maps base-change to the given multiplication and inversion maps of $\mathbf{G}$. See, for example, the exposition in [28], especially Theorem 2.4.1 in *loc. cit.* which guarantees the existence of such models (as we consider only F-split $\mathbf{G}$). Note that this also allows us to define $\mathbf{G}(k)$, where k is the residue field. As is traditional in the literature, we will make no further comment about necessity of models, etc.

Definition 3. If $\mathbf{G}$ is a group over $\mathbb{C}$, we say that a semisimple element $s \in \mathbf{G}(\mathbb{C})$ is *compact* if s is contained in a compact subgroup of $\mathbf{G}(\mathbb{C})$ considered as a Lie group.

Other notions and notation pertaining to reductive groups will be introduced as required.

1.3.2 Weyl groups and affine Weyl groups

Let $\mathbf{G}$ be a split reductive group over F and recall its root datum $(X^*, \Phi, X_*, \Phi^\vee)$ over the algebraic closure $\bar{F}$ of F. For every root $\alpha \in \Phi$, we define a linear involution s_α of $X^* \otimes_{\mathbb{Z}} \mathbb{R}$ via

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \, \alpha.$$

The *Weyl group* W of $\mathbf{G}$ is the group generated by the s_α. It is classically known that W is a finite Coxeter group with generators

$$S = \{ s_\alpha \,|\, \alpha \in \Phi \},$$

and that it can also be realized as a subquotient of $\mathbf{G}$ as $N_{\mathbf{G}}(\mathbf{A})/\mathbf{A}$. In particular, it has a length function $\ell \colon W \to \mathbb{N}$. We write w_0 for the (unique) longest word in W.

Definition 4. The *extended affine Weyl group of* $\mathbf{G}$ is the semidirect product

$$\tilde{W} = W \ltimes X^*.$$

The *affine Weyl group of* $\mathbf{G}$ is the semidirect product

$$W_{\text{aff}} = W \ltimes \mathbb{Z}\Phi^\vee.$$

The extended affine Weyl group possesses a length function extending the length function of W that we also denote ℓ. In general, $\tilde{W}$ is not a Coxeter group, as it contains multiple elements of length zero, the automorphisms of a chosen fundamental alcove. The affine Weyl group is an infinite Coxeter group, with set of simple reflections equal to

$$S = \{ s_\alpha \,|\, \alpha \in \Phi \} \cup \{ s_0 \}.$$

We term s_0 the *affine* simple reflection; it corresponds to the added vertex in the affine Dynkin diagram containing the Dynkin diagram of W. The simple reflections contained in W we call the *finite* simple reflections.

If

$$w = s_{i_1} s_{i_2} \cdots s_{i_n} \tag{1.2}$$

is in $w \in \tilde{W}$ and $n = \ell(w)$, we say that the right hand side of (1.2) is a *reduced expression* for w.

Remark 1. In general, reduced expressions are not unique. However, the subset of nontrivial elements of $\tilde{W}$ with unique reduced expressions form a two-sided cell of $\tilde{W}$, the *subregular cell*. We will recall the definition of various types of cells in Section 1.3.7.

All of W, $\tilde{W}$, and W_{aff} are equipped with partial orders, each called the *strong Bruhat order*, which is defined as follows. We first define the order for W_{aff}, where we say that $w \leq w'$ if and only if

$$w' = s_{i_1} s_{i_2} \cdots s_{i_n}$$

for $s_{i_j} \in S$, and there are indices $i_{j_1}, \ldots, i_{j_k}$ such that

$$w = s_{i_1} \cdots s_{i_{j_1 - 1}} s_{\hat{i_{j_1}}} s_{i_{j_1+1}} \cdots s_{i_{j_2-1}} s_{\hat{i_{j_2}}} s_{i_{j_2+1}} \cdots s_{i_{j_k-1}} s_{\hat{i_{j_k}}} s_{i_{j_k+1}} \cdots s_{i_n}$$

where $\hat{\cdot}$ denotes omission. If $w, w' \in \tilde{W}$, then using the canonical isomorphism

$$\tilde{W} \simeq (\tilde{W}/W_{\text{aff}}) \ltimes W_{\text{aff}}$$

we say that $w = (w_1, w_2) \leq w' = (w_1', w_2')$ if and only if $w_1 = w_1'$ and $w_2 \leq w_2'$ in the sense just defined above.

Definition 5. If $w \in \tilde{W}$, then the *left descent set of* x is

$$\mathscr{L}(w) = \{s \in S \,|\, sw < w\} = \{s \in S \,|\, w \text{ has a reduced expression beginning with } s\}.$$

The *right descent set* $\mathscr{R}(w)$ *of* w is defined similarly.

For our purposes, it will be most natural to work by default with $\tilde{W}$, but there is no difference when $\mathbf{G}$ is simply-connected. Indeed, $\tilde{W}/W_{\text{aff}} \simeq \pi_1(\mathbf{G})$ is the fundamental group of $\mathbf{G}$. Moreover, this quotient is also isomorphic to the subgroup of length zero elements of $\tilde{W}$.

Definition 6. Let W be a either a finite Weyl group, an affine Weyl group, or an extended affine Weyl group. Then the *Poincaré series of* W is the power series

$$P_W(v) = \sum_{w \in W} v^{\ell(e)},$$

where v is a formal variable. When W is finite, we say that P_W is the *Poincaré polynomial of* W.

If $\mathbf{G}$ is a reductive algebraic group, we define the *Poincaré polynomial of* $\mathbf{G}$ to be the Poincaré polynomial of its Weyl group.

Remark 2. In parts of the literature dealing only with a single reductive group that might in the context of the representation theory of p-adic groups be denoted $\mathbf{G}^\vee$, it is common to exchange the roles of $\mathbf{G}$ and $\mathbf{G}^\vee$ and define

$$\tilde{W} = W \ltimes X^*.$$

We take this approach in Chapter 4 and include a warning to that effect in the introduction to that chapter. The groups $\mathbf{G}$ and $\mathbf{G}^\vee$ have isomorphic Weyl groups, but need not have the same extended affine Weyl groups. For example, SL_2 is simply-connected, whereas PGL_2 is of adjoint type.

1.3.3 Representations of p-adic groups

The original motivation for the study of the affine Hecke algebra is the representation theory of p-adic groups. In turn, the motivation for studying representations of p-adic groups is the theory of automorphic representations. An exposition of the theory of adèlic groups and automorphic representations being not so much beyond the scope as beside the scope of this thesis, we limit our motivation to noting that Flath's Theorem [26] explains that if F is a global field and $\mathbb{A}_F$ is its ring of adèles, then understanding automorphic representations of $\mathbf{G}(\mathbb{A}_F)$ reduces in substantial part to understanding the representations of $G(F_v)$ where v is a place of F. In fact, at almost all places, the local representations will have Iwahori-fixed vectors, and hence will be within the scope of this thesis.

Much of the most basic representation theory of p-adic groups in fact holds for arbitrary Hausdorff totally-disconnected locally- compact groups. See for example [53] for a treatment in this generality. The key definitions are the following.

Definition 7. A *td-group* is a Hausdorff, totally-disconnected and locally-compact topological group. A *smooth representation of* G is a morphism of groups $\pi\colon G \to \mathrm{GL}(V)$ for a complex vector space V, such that for all $v \in V$, the map

$$g \mapsto \pi(g)v$$

is locally constant. Equivalently, for all $v \in V$, there exists an open compact subgroup K of G such that $v \in \pi^K$ is K-fixed.

Definition 8. Let G be a td-group. A smooth representation π of G is *admissible* if the fixed-point space π^K is finite-dimensional for every open compact subgroup K of G.

Now let F be a non-archimedean local field with residue field $\mathbb{F}_q$, and let $\mathbf{G}$ be a reductive algebraic group defined over F. Then the group $G := \mathbf{G}(F)$ is td-group, with topology inherited from a closed embedding $\mathbf{G} \hookrightarrow \mathrm{GL}_n$ in the category of algebraic groups over F.

Theorem 5 (Harish-Chandra-Bernstein, [7])**.** *Let* $\mathbf{G}$ *be a reductive algebraic group defined over* F*, and* $G = \mathbf{G}(F)$*. Then every irreducible representation* π *of* G *is admissible, and in fact* $\dim \pi^K$ *is bounded by a constant that depends only on* G *and the compact open subgroup* K*.*

Remark 3. Relatively recently, this bound has been improved and sharpened in [38] by Kemarsky.

Definition 9. We write $\mathscr{M}(G)$ for the category of admissible representations of G.

Flath's theorem also explains the type of local representations that occur in an automorphic representation. Namely, their constituents are the admissible representations. Note that this definition of smooth representation requires no topology on the vector space V; in the theory of automorphic representations the vector space V will in practice be a subspace of a Hilbert space, but in general will not be closed.

1.3.4 The Hecke algebra of a p-adic group

At both the archimedean and non-archimedean places, the admissible representations are defined to be those for which there exists an equivalence between infinite-dimensional analytic problems, namely understanding the representations in the terms that they are most usually given, and collections of finite-dimensional algebraic problems, namely understanding finite-dimensional modules over some associative algebra(s). At the archimedean places, this algebra is the universal enveloping algebra of $\mathfrak{g}$, and the theory connecting the analytic the and algebraic is that of differential operators. At the non-archimedean places, the theory can be cast (following for example [53]) as that of the convolution algebra of compactly supported distributions on G. If G is a td-group and π is a smooth representation of G, then π is a module over the algebra of compactly-support distributions, with the action as follows.

Let ξ be a compactly-supported distribution on G and let $K \supset \operatorname{supp} \xi$ be a compact subset of G. Then for each $v \in V$ the natural map

$$K \to V$$

is of the form

$$k \mapsto \pi(k)v = \sum_{i=1}^{n} \chi_{K_i}(k) v_i$$

for $K_i \subset K$ compact open and $v_i \in V$. Then one may define

$$\pi(\xi)v = \sum_{i=1}^{n} \xi(\chi_{K_i}) v_i.$$

Haar's theorem holds for td-groups, and upon choosing a Haar measure on G and identifying compactly-supported distributions with the algebra $C_c^\infty(G)$, the above definition translates to

$$\pi(f)v = \int_G f(g)\pi(g)v \, \mathrm{d}g, \tag{1.3}$$

for a function $f \in C_c^\infty(G)$, and the integral reduces to a finite sum.

Definition 10. The algebra $C_c^\infty(G)$, or alternatively the algebra of compactly-supported distributions on G, is called the *(full) Hecke algebra of* G. The product is given by convolution

$$(f_1 \star f_2)(x) = \int_G f_1(g) f_2(g^{-1}x) \, \mathrm{d}g.$$

Remark 4. In Chapter 2 it will be essential to consider a class of representations which are modules over a larger algebra, the Harish-Chandra Schwartz algebra. In this case the notation (1.3) is also used, but is purely shorthand, see Section 2.2.3.

It is immediate from the definitions that the Hecke algebra is an indempotented algebra, with idempotents given by characteristic functions χ_K of compact open subgroups K. That is, we have

$$C_c^\infty(G) = \bigcup_K \chi_K \star C_c^\infty(G) \star \chi_K,$$

and $C_c^\infty(G)^{K\times K} = \chi_K \star C_c^\infty(G) \star \chi_K$ is the convolution algebra of K-biinvariant complex-valued functions.

Definition 11. A module M over an indempotented algebra $A = \bigcup_e eAe$ is *nondegenerate* if $M = \bigcup_e eM$.

For any compact open subgroup K of G, it is immediate from the definitions that one a functor

$$\mathscr{M}(G) \to C_c^\infty(G)^{K\times K} - \mathbf{Mod} \tag{1.4}$$

sending

$$\pi \mapsto \pi^K.$$

It is clear that we have an equivalence of categories from $\mathscr{M}(G)$ to the category of nondegenerate modules over $C_c^\infty(G)$, but the functors (1.4) are not in general equivalences on their own. Consider for example, that if K is a maximal compact subgroup, then for all principle series representations π, one has $\dim \pi^K \leq 1$ whether π is irreducible or not. The theory of types due to Bushnell and Kutzko [18] describes a more general version of the functors (1.4) that do give equivalences of categories. As so often happens, the functors $\pi \mapsto \pi^K$ are examples of the general story in the case of the trivial representation.

1.3.5 The Iwahori-Hecke algebra

The most striking example of when a functor of the form (1.4) is an equivalence is the case of the Iwahori subgroup I, defined to be the preimage in $K = \mathbf{G}(\mathscr{O})$ of $\mathbf{B}(\mathbb{F}_q)$, and the Iwahori-Hecke algebra $C_c^\infty(G(F))^{I\times I}$. In this case one has

Theorem 6 ([12])**.** *There is an equivalence of categories between the category of admissible representations of G with nonzero I-fixed vector, and the category of finite-dimensional modules over the Iwahori-Hecke algebra, given on objects by $\pi \mapsto \pi^I$.*

Hence it is possible to study such representations, and also all of the Local Langlands Correspondence for them, entirely in terms of $C_c^\infty(G)^{I\times I}$-modules.

In stark contrast to the general representation theory of p-adic groups, this category is not only completely insensitive to the prime power q, but is in fact nothing more than the specialization to a prime power of the entirely algebraic category $\mathbf{H} \otimes_{\mathbb{Z}} \mathbb{C} - \mathbf{Mod}$ of finitely-dimensional modules over the comlexified affine Hecke algebra. The affine Hecke algebra $\mathbf{H}$ itself is algebra over $\mathbb{Z}[\mathbf{q}^{\pm 1/2}]$ to be defined below. In particular, it makes sense to specialize $\mathbf{q}$ to any nonzero complex number, and ask how the representation theory of $\mathbf{H}|_{\mathbf{q}=q}$ depends on q. We outlined possible applications of this thesis to such questions in Section 1.2.

Decompositions

We now recall some useful decompositions of $\mathbf{G}$ as an algebraic group, and also of G.

Theorem 7 (Bruhat decomposition, [19] Theorem 3.1.9)**.** *There is an isomorphism of varieties over k*

$$\mathbf{G} = \coprod_{w\in W} \mathbf{B}\dot{w}\mathbf{B},$$

where $\dot{w}$ is a lift of w. In fact, one has

$$\mathbf{B}\dot{w}\mathbf{B} = \mathbf{B}\dot{w}\mathbf{N}.$$

(In actual fact, one can be even more precise and replace $\mathbf{N}$ *with a subgroup depending on* w*, see loc. cit.)*

Consequently, one has

$$\mathscr{B} = \coprod_{w \in W} \mathscr{B}_w = \coprod_{w \in W} \mathbf{B}\dot{w}\mathbf{B}/\mathbf{B}.$$

This is a decomposition into $\mathbf{B}$*-orbits of* $\mathscr{B}$*. Each orbit is an affine space of dimension* $\ell(w)$*. The closure relations on orbits are given by the Bruhat order on* W*.*

Note that this theorem also implies that the Poincaré polynomials of W and $\mathscr{B}$ are equal.

Now let $k = F$, and let $K = \mathbf{G}(\mathscr{O})$. Then K is a maximal compact subgroup of $G = \mathbf{G}(F)$. For p-adic groups, if λ is a coweight of $\mathbf{A}$, we write, in homage to the case of SL_2, $\varpi^\lambda = \lambda(\varpi)$. The connection between the Iwahori and affine Hecke algebras (to be defined in the next section) is the set of representatives given by the following

Theorem 8 ([35])**.** *Consider the decomposition of* G *into* I*-double cosets. Then a set of representatives is given by* $\tilde{W}$*. That is,*

$$G = \coprod_{w \in \tilde{W}} I\dot{w}I.$$

The closure relations on I*-orbits in* G/I *are given by the Bruhat order.*

Using this, two other very useful decompositions of G follow:

Theorem 9 (Cartan decomposition)**.** *We have*

$$G = KAK = \coprod_{\lambda \in X_*(\mathbb{A})^+} K\varpi^\lambda K,$$

where $X_*(\mathbf{A})^+$ *are the dominant coweights of of* $(\mathbf{G}, \mathbf{A})$*. As a consequence,* K*-orbits in* G/K *are indexed by dominant coweights.*

Note that we can realize the Weyl group inside K, whence the second equality.

The other decomposition that follows from Theorem 8 is the Iwasawa decomposition.

Theorem 10 (Iwasawa decomposition)**.** *We have*

$$G = KAN.$$

1.3.6 The affine Hecke algebra

We can now define the affine Hecke algebra, and relate it to the representation theory of p-adic groups.

Definition 12. The *affine Hecke algebra* $\mathbf{H}$ *of* $\mathbf{G}$ is the following deformation of $\mathbb{Z}[\tilde{W}]$. As a $\mathbb{Z}[\mathbf{q}^{1/2}, \mathbf{q}^{-1/2}]$-module, it is free with basis $\{T_w\}_{w \in \tilde{W}}$ (the *standard basis*), and has multiplication determined by the relations $T_w T_{w'} = T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$ and the quadratic relation $(T_s + 1)(T_s - \mathbf{q}) = 0$ for each $s \in S$.

We can now state the connection to the representation theory of p-adic groups.

Theorem 11 ([35]). *Let F be a non-archimdean local field and q the cardinality of the residue field. Let $G = \mathbf{G}(F)$. Then the map*

$$C_c^\infty(G,\mathbb{Z})^{I\times I} \to \mathbf{H}|_{\mathbf{q}^{1/2}=\sqrt{q}}$$

sending

$$\chi_{IwI} \mapsto T_w$$

is an isomorphism of algebras.

When $\mathbf{q}^{1/2}$ is specialized, we usually write H without bold typeface for the specialization.

The above is the *Coxeter presentation* of $\mathbf{H}$. There is another presentation, the *Bernstein presentation*, which we will use extensively in Chapter 4. Especially in Chapter 2, we often base-change to $\mathbb{C}[\mathbf{q}^{1/2},\mathbf{q}^{-1/2}]$ without introducing new notation.

In [36], Kazhdan and Lusztig defined two distinguished bases of $\mathbf{H}$ of great significance to representation theory. By a recursive procedure, they defined elements of $\mathbf{H}$ denoted

$$C'_w = \mathbf{q}^{-\frac{\ell(w)}{2}} \sum_{y\leq w} P_{y,w}(\mathbf{q})T_y,$$

and

$$C_w = \mathbf{q}^{\frac{\ell(w)}{2}} \sum_{y\leq w} (-1)^{\ell(w)-\ell(y)}\mathbf{q}^{-\ell(y)} P_{y,w}(\mathbf{q}^{-1})T_y,$$

where $P_{y,w}$ are certain recursively-defined polynomials with integer coefficients such that $P_{y,w} = 0$ unless $y \leq w$, $P_{w,w} = 1$, and $\deg P_{y,w} \leq \frac{1}{2}(\ell(w) - \ell(y) - 1)$. Thus the change-of-basis matrices from the standard basis to either the C_w or C'_w bases is upper triangular with respect to the Bruhat order and populated, up to appropriate signs and powers of $\mathbf{q}$, by polynomials in $\mathbf{q}$ or $\mathbf{q}^{-1}$ with integer coefficients. The same is therefore true of the inverse matrix, which is populated by the *inverse Kahzdan-Lusztig polynomials* $Q_{y,w}$, which are such that

$$T_w = \sum_{y\leq x} (-)^{\ell(x)-\ell(y)}\mathbf{q}^{\frac{\ell(y)}{2}} Q_{y,w}(\mathbf{q})C'_y,$$

or equivalently,

$$\sum_{z\leq y\leq x} (-1)^{\ell(x)-\ell(y)} Q_{y,x}(\mathbf{q})P_{z,y}(\mathbf{q}) = \delta_{zx}.$$

We shall use in Chapter 2 that $\deg Q_{y,w} \leq \frac{1}{2}(\ell(w) - \ell(y) - 1)$. See [11] for further exposition.

It is a deep theorem due independently to Beilinson-Bernstein and Brylinski-Kashiwara, and later also due to Soergel, that for finite Weyl groups, the coefficients of $P_{y,w}$ are nonnegative, and contain deep representation-theoretic information. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and let W be the Weyl group of its root system. Let $\mathfrak{h}$ be the Cartan algebra of $\mathfrak{g}$, and $\mathfrak{b} \supset \mathfrak{h}$ be a Borel subalgebra. It is well-known that, as the simple $\mathfrak{g}$-modules are indexed by weights $\lambda \in \mathfrak{h}^*$, they can be denoted $L(\lambda)$. Let λ be of the form $w \cdot 0$ and inflate the $\mathfrak{h}$-module $\mathbb{C}_\lambda$ given by λ to a $U(\mathfrak{b})$-module. Then the class of $L(w \cdot 0)$ in $K_0(\mathfrak{g} - \mathbf{Mod})$ is given as follows in terms of the classes of the *Verma modules* $M(x \cdot 0) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{x\cdot 0}$ for $w, x \in W$. Precisely, one has

Theorem 12 ([6], [16], [60])**.**

$$[L(w \cdot 0)] = \sum_{x \leq w} (-1)^{\ell(w)-\ell(x)} P_{x,w}(1)[M(x \cdot 0)].$$

This is an example of the type of representation-theoretic information one could hope is behind a positivity property of the type conjectured, and proved for SL_2, in Chapter 3.

There is a second presentation of H, due to Bernstein (and Bernstein-Zelevinskii in type A), which appears naturally in coherent descriptions of H, both on the level of K-theory and the level of categories.

Definition 13. The *Bernstein presentation* of $\mathbf{H}$ is the presentation with basis $\{T_w \theta_\lambda\}_{w \in W, \lambda \in X_*}$, where

- For $w \in W$, T_w is the same basis element as in the Coxeter presentation.
- If $\lambda \in \tilde{W}$ is an antidominant character, and hence corresponds to a *geometrically* dominant cocharacter in the sense of [19], then
$$\theta_\lambda = \mathbf{q}^{-\frac{\ell(\lambda)}{2}} T_\lambda.$$
- If $\lambda \in \tilde{W}$ is a dominant character, and hence corresponds to a *geometrically* antidominant cocharacter, then
$$\theta_\lambda = \mathbf{q}^{\frac{\ell(\lambda)}{2}} T_\lambda^{-1}.$$

The sets $\{\theta_\lambda, \theta_\lambda T_{s_0}\}_{\lambda \in X_*}$ and $\{\theta_\lambda, T_s \theta_\lambda\}_{\lambda \in X_*}$ are each $\mathscr{A}$-bases. The relations are as follows:

- The same quadratic relation for T_{s_0};
- For any cocharacters λ, λ', we have $\theta_\lambda \theta_{\lambda'} = \theta_{\lambda+\lambda'}$;
- The Bernstein relation
$$\theta_{\frac{\alpha}{2}} T_{s_\alpha^\vee} - T_{s_\alpha^\vee} \theta_{-\frac{\alpha}{2}} = (\mathbf{q}-1) \frac{\theta_{\frac{\alpha}{2}} - \theta_{-\frac{\alpha}{2}}}{1 - \theta_{-\alpha}} = (\mathbf{q}-1)\theta_{\frac{\alpha}{2}}.$$

Example 1. Let $\mathbf{G}^\vee = \mathrm{PGL}_2$, so that $\tilde{W} = \mathfrak{S}_2 \ltimes X$, where X is the character lattice of $\mathbf{G} = \mathrm{SL}_2$. Write s_0 for the finite simple reflection, and s_1 for the affine simple reflection in $\tilde{W}$. Then the generators of the Bernstein subalgebra are as follows:

1. If $\lambda = -n\alpha^\vee = (s_1 s_0)^n \in \tilde{W}$, and hence corresponds to a *geometrically* dominant cocharacter in the sense of [19], then
$$\theta_{(s_1 s_0)^n} = \theta_{-n\alpha^\vee} = \mathbf{q}^{-n} T_{(s_1 s_0)^n}.$$
2. If $\lambda = n\alpha^\vee = (s_0 s_1)^n \in \tilde{W}$, and hence corresponds to a *geometrically* antidominant cocharacter, then
$$\theta_{(s_0 s_1)^n} = \theta_{n\alpha^\vee} = \mathbf{q}^n T^{-1}_{(s_1 s_0)^n}.$$

In particular, under the geometric choice of dominance, we have $\rho = -1$.

1.3.7 Cells in affine Weyl groups

In [41], Lusztig defined three partitions of $\tilde{W}$. These definitions are essential to the content of this thesis. We also refer the reader to the exposition in [49].

Recall that given two elements w and y, we write $y \prec w$ if $y < w$, $\ell(w) - \ell(y)$ is odd, and $P_{y,w}(q) = \mu(y,w)q^{(\ell(w)-\ell(y)-1)/2}$ for $\mu(y,w)$ a nonzero integer. Two elements y and w are said to be *joined* if either $y \prec w$ or $w \prec y$. We write $w - x$ for joined elements and $\tilde{\mu}(y,w)$ for $\mu(y,w)$ or $\mu(w,y)$, whichever is defined. Using this Lusztig defines a preorder $\leq_L$ by declaring that $x \leq_L y$, for $x, y \in \tilde{W}$, if there is a sequence of elements $x = x_0, x_1, x_2, \ldots, x_n = y$ in $\tilde{W}$ such that for $1 \leq i \leq n$, we have $x_{i-1} - x_i$ and $\mathscr{L}(x_{i-1}) \not\subset \mathscr{L}(x_i)$. Note that $x - x$ is never true, by length considerations, but for the sequence $x = x_0 = x_n = x$ with $n = 0$, the conditions are vacuous.

Lusztig shows in [41] that declaring $x \sim_L y$ if $x \leq_L y$ and $y \leq_L x$ and gives an equivalence relation on $\tilde{W}$.

Definition 14. The equivalence classes under $\sim_L$ are the *left cells in* $\tilde{W}$. A *right-cell in* $\tilde{W}$ is a set of the form Γ^{-1}, where Γ is a left cell.

We now define a relation on $\tilde{W}$ by saying that $x \leq_{LR} y$ if there exists a sequence $x_0 = x, x_1, x_2, \ldots, x_n = y$ such that for $1 \leq i \leq n$, we have either $x_{i-1} \leq_L x_i$ or $x_{i-1}^{-1} \leq_L x_i^{-1}$. Write $\sim_{LR}$ for the equivalence relation given by $x \sim_{LR} y$ if $x \leq_{LR} y \leq_{LR} x$.

Definition 15. The equivalence classes under $\sim_{LR}$ are the *two-sided cells of* $\tilde{W}$.

It is clear from the definition that two-sided cells are stable under inversion. Every two-sided cell is a union of one-sided cells.

Example 2.

1. The set $\{1\} \subset \tilde{W}$ is always a two-sided cell.
2. There is a two-sided cell $\mathbf{c}_0$, the *big* or *lowest two-sided cell* that is characterized by containing w_0.
3. By [40], Proposition 3.8, the set $\mathbf{c}$ of nontrivial elements of $\tilde{W}$ with unique reduced expressions is a two-sided cell, called the *subregular cell.* It is a union of the left cells

$$\Gamma_s = \{w \in \mathbf{c} \mid s \in \mathscr{R}(w)\}.$$

There is another enlightening description of cells that works as follows. We say that that $w \leq'_L x$ if there exists $h \in \mathbf{H}$ such that

$$hC_x = \sum_y a_y C_y$$

and $a_w \neq 0$. That is, C_w can be obtained by left multiplication against C_x. Likewise we define $\leq'_R$, and say that $w \leq'_{LR} x$ if there are $h, h' \in \mathbf{H}$ such that

$$hC_x h' = \sum_y a_y C_y$$

with $a_y \neq 0$. The the corresponding equivalence classes are the same as those defined above ([49], Section 8.1).

The following deep theorem of Lusztig is proved in [44]. The general proof makes essential use of the affine and asymptotic Hecke algebras, but early evidence in low rank was collected almost by hand, for example by Du in [24].

Theorem 13 (Lusztig, [44])**.** *Let $(X^*, \Phi, X_*, \Phi^\vee)$ be a root datum, and let $\tilde{W}$ be the corresponding extended affine Weyl group. Let $G^\vee = \mathbf{G}^\vee(\mathbb{C})$ be the $\mathbb{C}$-points of the corresponding Langlands dual group. There is a bijection*

$$\left\{\textit{two-sided cells of } \tilde{W}\right\} \leftrightarrow \{\textit{unipotent conjugacy classes in } G^\vee\}.$$

$$\mathbf{c} \mapsto u(\mathbf{c})$$

$$\mathbf{c}(u) \leftarrow u.$$

Definition 16. *Lusztig's a-function* $a\colon \tilde{W} \to \mathbb{N}$ is defined such that $a(w)$ is the minimal value such that $\mathbf{q}^{\frac{a(w)}{2}} h_{x,y,w} \in \mathscr{A}^+$ for all $x, y \in \tilde{W}$.

The a-function is constant of two-sided cells of $\tilde{W}$. Obviously, $a(1) = 1$, and the a-function obtains its maximum, equal to the number of positive roots, $\ell(w_0)$, on the two-sided cell containing the longest word $w_0 \in W$. In general, under the bijection between two-sided cells of $\tilde{W}$ and unipotent conjugacy classes in $G^\vee$, we have

$$a(\mathbf{c}) = \dim_{\mathbb{C}}(\mathscr{B}_u^\vee),$$

where $\mathscr{B}_u^\vee$ is the subvariety of Borel subgroups of $G^\vee$ meeting the conjugacy class u that corresponds to $\mathbf{c}$ [44]. We have $a(w) \leq \ell(w)$ for all $w \in \tilde{W}$.

Lusztig's bijection is very inexplicit; its existence is proved using the Kazhdan-Lusztig parametrization, an instance of the Local Langlands correspondence together with the inexplicit bijection in Theorem 21. However, using the a-function, one can deduce some of its properties:

Example 3.

1. $\mathbf{c}(\{1\})$ is the big cell [68]. Indeed, the value of the a-function on the big cell is $\ell(w_0) = \dim \mathscr{B}^\vee$. Hence if $u = u(\mathbf{c}_0)$, $\dim \mathscr{B}_u^\vee = \dim \mathscr{B}^\vee$. As $\mathscr{B}^\vee$ is irreducible, we have $\mathscr{B}_u^\vee = \mathscr{B}^\vee$, so $u = \{1\}$.

2. Let u be the subregular unipotent conjugacy class. Then $\mathbf{c}(u)$ is the subregular cell. Indeed, the subregular cell is the two-sided cell containing all the simple reflections. In particular, $a = 1$ on $\mathbf{c}(u)$. On the other hand, $\dim \mathscr{B}_u^\vee = 1$ is equivalent to u being subregular ([63], Section 3.8, Theorem 2 (e)).

3. Let u be the regular unipotent conjugacy class. Then $\mathbf{c}(u) = \{1\}$. Indeed, by regularity $\dim \mathscr{B}_u^\vee = 0$, and $a(w) = 0$ only for $w = 1$.

One has in general [44], that

$$a(d) \leq \ell(d) - 2\delta(d).$$

In [42], Lusztig defined a set

$$\mathscr{D} = \left\{d \in \tilde{W} \,\middle|\, a(d) = \ell(d) - 2\delta(d)\right\},$$

where $\delta(d) = \deg P_{e,d}(\mathbf{q})$.

Definition 17. The elements of $\mathscr{D}$ are the *distinguished involutions* of $\tilde{W}$.

It is shown in [42] that the distinguished involutions are in fact involutions, although this fact is not trivial.

Example 4. The identity, the longest element of the finite Weyl group, and the longest element of any finite parabolic subgroup of $\tilde{W}$, are each distinguished involutions.

Lusztig proved the following without recourse to his bijection from Theorem 13.

Theorem 14 ([42]).

1. *There are finitely many left cells, hence finitely many right and two-sided cells, in $\tilde{W}$;*
2. *The set $\mathscr{D}$ is finite; every one-sided cell contains a unique distinguished involution.*
3. *Any distinguished involution is contained in a finite parabolic subgroup of $\tilde{W}$. (*[70]*, Proof of Theorem 3.1.)*

1.3.8 Lusztig's asymptotic Hecke algebra J

In [42], Lusztig defined an associative algebra J over $\mathbb{Z}$, the *asymptotic Hecke algebra*, that implements the independence on q of the representation theory of $\mathbf{H}$. In *loc. cit.*, the algebra J was also equipped with an injection $\phi\colon \mathbf{H} \hookrightarrow J \otimes_{\mathbb{C}} \mathscr{A}$ which becomes an isomorphism after taking a certain completion of both sides, to be recalled in Section 2.4.3. As an abelian group, J is free with basis $\{t_w\}_{w\in\tilde{W}}$. The structure constants of J are obtained from those in $\mathbf{H}$ written in the $\{C_w\}_{w\in\tilde{W}}$-basis under the following procedure. Write

$$C_xC_y = \sum_{z\in W} h_{x,y,z}C_z$$

and then define the integer $\gamma_{x,y,z}$ by the condition

$$\mathbf{q}^{\frac{a(z)}{2}} h_{x,y,z^{-1}} - \gamma_{x,y,z} \in \mathbf{q}\mathscr{A}^+.$$

One then defines

$$t_xt_y = \sum_z \gamma_{x,y,z} t_{z^{-1}}.$$

The map ϕ is defined by

$$\phi(C_w) = \sum_{\substack{z\in W \\ d\in\mathscr{D}\ a(z)=a(d)}} h_{x,d,z}t_z.$$

Write ϕ_q for the specialization of this map when $\mathbf{q} = q$. It is known ([43], Proposition 1.7) that ϕ_q is injective for all $q \in \mathbb{C}^\times$.

The algebra J decomposes as a direct sum of two-sided ideals

$$J = \bigoplus_{\mathbf{c}} J_{\mathbf{c}}$$

indexed by the two-sided cells $\mathbf{c}$ of $\tilde{W}$, where

$$J_{\mathbf{c}} = \left(\sum_{d\in\mathscr{D}\cap\mathbf{c}} t_d\right) J \left(\sum_{d\in\mathscr{D}\cap\mathbf{c}} t_d\right).$$

Each left cell contains a single distinguished involution, and corresponding basis elements $\{t_d\}$ are easily seen to form a family of orthogonal idempotents, and their sum is the identity element in J. If $\mathbf{c}$ is a two-sided cell, then $J_{\mathbf{c}}$ is a unital ring, with unit

$$1_{J_{\mathbf{c}}} = \sum_{d\in\mathscr{D}\cap\mathbf{c}} t_d.$$

The fundamental result upon which this thesis' perspective on J is based is the following theorem of Lusztig.

Consider the completions

$$\boldsymbol{\mathcal{H}}_C^- := \left\{ \sum_{x\in\tilde{W}} b_x C_x \;\middle|\; b_x \to 0 \text{ as } \ell(x)\to\infty \right\}.$$

as well as

$$\boldsymbol{\mathcal{J}} = \left\{ \sum_{x\in\tilde{W}} b_x t_x \;\middle|\; b_x \to 0 \text{ as } \ell(x)\to\infty \right\}.$$

of $\mathbf{H}$ and $J\otimes_{\mathbb{Z}}\mathscr{A}$, where we say that $b_x\to 0$ as $\ell(x)\to\infty$ if for all $N>0$, $b_x\in(\mathbf{q}^{1/2})^N\check{\mathscr{A}}^-$ for all x sufficiently long.

Theorem 15 ([42])**.** *The map ϕ extends to an isomorphism of $\mathscr{A}$-algebras*

$$\phi\colon \boldsymbol{\mathcal{H}}_C^- \to \boldsymbol{\mathcal{J}},$$

so that J is realized as a subalgebra of $\boldsymbol{\mathcal{H}}_C^-$.

Deformations of the group ring

We have already remarked that, upon setting $\mathbf{q}=1$, $\mathbf{H}|_{\mathbf{q}=1}$ is isomorphic to $\mathbb{Z}[\tilde{W}]$, and so $\mathbf{H}$ is a deformation of the group algebra of $\tilde{W}$.

Let, temporarily, W be any finite Coxeter group. Then one can define its Hecke algebra $\mathbf{H}$, an algebra over $\mathbb{Z}[\mathbf{q}^{\frac{1}{2}},\mathbf{q}^{-\frac{1}{2}}]$ which deforms the group ring $\mathbb{Z}[W]$. Let $q\in\mathbb{C}^\times$. For all but finitely many values of q (all of which are roots of unity), the algebras $\mathbf{H}_{\mathbf{q}=q}$ are trivial deformations of $\mathbb{C}[W]$, and hence are all isomorphic. However, this isomorphism requires choosing a square root of q. The affine Hecke algebra provides a canonical isomorphism: away from roots of P_W, we have that $\mathbf{H}|_{\mathbf{q}=q} = \mathbf{H}\otimes_{\mathscr{A}}\mathbb{C}_q$ is isomorphic to J ([70], Proof of Theorem 3.2), c.f. [49], 20.1 (e).

Example 5. Let $W=\langle 1,s \,|\, s^2=1\rangle$ be the Weyl group of type A_1. The Kazhdan-Lusztig C_w-basis elements are $C_1=T_1$ and $C_s=\mathbf{q}^{-\frac{1}{2}}-\mathbf{q}^{\frac{1}{2}}$. There are two two-sided cells in W, and one can easily check that

$$\phi\colon C_1\mapsto t_1+t_s$$

and

$$\phi\colon C_s \mapsto -\left(\mathbf{q}^{\frac{1}{2}} + \mathbf{q}^{-\frac{1}{2}}\right) t_s.$$

Letting $\mathbf{q} = q$, we see that ϕ becomes an isomorphism whenever $\left(q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right) \neq 0$, that is, whenever $q \neq -1$.

We exposit the representation theory of $\mathbf{H}$ and J in Section 2.1.3, with emphasis on the role of J in implementing the independence on $q \in \mathbb{C}^\times$ of the theory, and also reflecting the decomposition

$$\mathbf{H} = \bigoplus_{\mathbf{c}} \left(\bigoplus_{\substack{w \\ w \leq_{LR} y}} \mathscr{A} C_w / \bigoplus_{\substack{w \\ w <_{LR} y}} \mathscr{A} C_w \right)$$

of $\mathbf{H}$ as an $\mathbf{H}-\mathbf{H}$-bimodule. Here y is arbitrary element of $\mathbf{c}$, upon which the decomposition doesn't depend ([49], 8.3).

1.3.9 K-theoretic realizations of H

At the heart of the Kazhdan-Lusztig classification is a description of J in terms of algebraic K-theory, which appeared already in [37]. There are in fact several slightly different versions of this realization. We will recall some of them in this section.

Let $\mathbf{q}^{1/2}\colon \mathbb{C}^\times \to \mathbb{C}^\times$ be the identity character. Let $\mathscr{B}^\vee$ be the flag variety of $\mathbf{G}^\vee$, and $\mathscr{N} \subset \mathfrak{g}^\vee$ the nilpotent cone.

The Steinberg variety and H

Definition 18. The *Steinberg variety of triples* is the variety

$$\mathrm{St}^\vee := \tilde{\mathscr{N}}^\vee \times_{\mathscr{N}^\vee} \tilde{\mathscr{N}}^\vee = \{(\mathfrak{b}_1, \mathfrak{b}_2, x) \mid x \in \mathfrak{b}_1 \cap \mathfrak{b}_2 \text{ is nilpotent}\} \subset T^*\mathscr{B}^\vee \times T^*\mathscr{B}^\vee.$$

It has the structure of a $\mathbf{G}^\vee \times \mathbb{C}^\times$-variety via

$$(g, z) \cdot (\mathfrak{b}_1, \mathfrak{b}_2, x) = (\mathrm{Ad}(g)(\mathfrak{b}_1), \mathrm{Ad}(g)(\mathfrak{b}_2), z^{-1}\mathrm{Ad}(g)(x)).$$

This convention on the scaling action of $\mathbb{C}^\times$ is chosen as in [19], and means in the case $\mathbf{G} = \mathrm{SL}_2$ that the fibres of $\mathbb{O}(-2)$ are scaled by $\mathbf{q}^{-1}$. Thus the fibres of $\mathbb{O}(2)$ are scaled by the character $\mathbf{q}$.

The irreducible components of St are given by conormal bundles to $\mathbf{G}^\vee$-orbits in $\mathscr{B}^\vee \times \mathscr{B}^\vee$.

Example 6. When $\mathbf{G} = \mathrm{SL}_2$, $\mathbf{G}^\vee = \mathrm{PGL}_2$, the Steinberg variety has two components, the zero section $\mathrm{St}_Y^\vee \simeq \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathrm{St}$ of the bundle projection $\mathrm{St}^\vee \to \mathbb{P}^1 \times \mathbb{P}^1$ and the diagonal,

$$\mathrm{St}_\Delta^\vee := \{(\mathfrak{b}, \mathfrak{b}, x) \mid x \in \mathfrak{b} \text{ is nilpotent}\}.$$

The diagonal component $\mathrm{St}_\Delta^\vee$ plays a large role in general. Recall that $K^{\mathbf{G}^\vee}(\mathscr{B}^\vee)$ is isomorphic to the representation ring of $\mathbf{A}^\vee$. Given a character λ of $\mathbf{A}^\vee$, let $\mathscr{O}(\lambda)$ be the corresponding sheaf on $\mathscr{B}^\vee$. Let $\Delta \subset \mathscr{B}^\vee \times \mathscr{B}^\vee \to \mathscr{B}^\vee$ be the projection from the diagonal, and let $\Delta\colon \mathrm{St}_\Delta^\vee \hookrightarrow \mathrm{St}^\vee$ be the inclusion of the diagonal component. Define sheaves $\mathscr{O}_\lambda = \Delta_* \pi_\Delta^* \mathscr{O}(\lambda)$ on $\mathrm{St}^\vee$.

For a simple reflection $s \in W$, consider the corresponding orbit $Y_s \subset \mathscr{B}^\vee \times \mathscr{B}^\vee$. Its closure in $\mathscr{B}^\vee \times \mathscr{B}^\vee$ is $\overline{Y_s} = Y_s \cup \Delta$. Let $\Omega^1_{\overline{Y_s}/\mathscr{B}^\vee}$ be the sheaf of relative 1-forms for the projection to the first coordinate. We have the projection $\pi_s \colon T^*_{\overline{Y_s}/\mathscr{B}^\vee}(\mathscr{B}^\vee \times \mathscr{B}^\vee) \to Y_s$. Define $\mathscr{Q}_s = \pi_s^* \Omega^1_{\overline{Y_s}/\mathscr{B}^\vee}$.

Definition 19. If $\mathbf{G}$ is an algebraic group and X is a $\mathbf{G}$-variety, then we write $K^{\mathbf{G}}(X)$ for the Grothendieck group of the triangulated category $D^b\mathrm{Coh}_{\mathbf{G}}(X)$ of $\mathbf{G}$-equivariant coherent sheaves on X by default. In chapter 4, we will for the most part mean the Grothendieck group of slightly different triangulated categories that arise when X is a derived scheme.

Theorem 16. *There is an isomorphism of $\mathscr{A}$-algebras*

$$\Theta \colon \mathbf{H} \to K^{\mathbf{G}^\vee \times \mathbb{C}^\times}(\mathrm{St}^\vee)$$

given in the Bernstein presentation by

$$\theta_\lambda \mapsto \mathscr{O}_{-\lambda}$$

$$C'_s \mapsto -\mathbf{q}^{1/2}\mathscr{Q}_s.$$

Example 7. When $\mathbf{G} = \mathrm{SL}_2$, the maps above are as follows. Let $\pi_Y \colon Y \to \mathbb{P}^1 \times \mathbb{P}^1$ and $\pi_\Delta \colon \Delta \to \mathbb{P}^1$ be the corresponding projections when viewing the components of St as conormal bundles. Consider the sheaf $\mathscr{O} \boxtimes \mathscr{O}(-2)$ of relative 1-forms for the projection $: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ onto the first factor, and set $\mathscr{Q} := j_* \pi_Y^*(\mathscr{O} \boxtimes \mathscr{O}(-2))$, where $j \colon \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathrm{St}$ is the inclusion of the zero section. Note that π_Y is an isomorphism. For the embedding $\Delta \colon Z_\Delta \hookrightarrow \mathrm{St}$ of the diagonal, define $\mathscr{O}_n := \Delta_* \pi_\Delta^* \mathscr{O}(n)$. The Θ is given by

$$\theta_{-n\alpha^\vee} = \theta_{(s_1 s_0)^n} \mapsto [\mathscr{O}_{-2n}], \quad C'_{s_0} \mapsto -q^{-\frac{1}{2}}[\mathbf{q}\mathscr{Q}] = -\mathbf{q}^{\frac{1}{2}} j_*(\mathscr{O} \boxtimes \mathscr{O}(-2)).$$

(In fact, [19] considers only the case when the dual group is simply-connected, but it is easy to deduce the case for an arbitrary semisimple group and its dual group from the simply-connected case, by embedding the affine Hecke algebra of $\mathbf{G}$ into the affine Hecke algebra of $\mathbf{G}^{\mathrm{ad}}$. The factor of 2 appearing while mapping the Bernstein elements arises due to this embedding.)

An isomorphism of this flavour was first defined by Kazhdan-Lusztig in [37], where it was given in the category of $\mathbf{H}$-modules. The isomorphism of algebras above is due to Chris and Ginzburg [19]; Lusztig in [48] defines a different isomorphism of algebras

$$\Theta_{\mathrm{Lus}} \colon \mathbf{H} \to K^{\mathbf{G}^\vee \times \mathbb{C}^\times}(\mathrm{St}^\vee)$$

such that when $\mathbf{G} = \mathrm{SL}_2$,

$$\theta_n \mapsto [\mathscr{O}_{-2n}] \quad C'_{s_0} \mapsto -j_*([\mathscr{O}(-1) \boxtimes \mathscr{O}(-1))].$$

As we have $[\mathscr{O} \boxtimes \mathscr{O}(2)] \neq [\mathscr{O}(-1) \boxtimes \mathscr{O}(-1)]$, which can be seen by the fact that the former has cohomology while the latter does not, these two isomorphisms are in fact different maps.

In Chapter 4 we shall identify $\mathbf{H}$ with $K^{\mathbf{G}^\vee \times \mathbb{C}^\times}(\mathrm{St}^\vee)$, but our results do not depend on a choice of isomorphism.

1.3.10 The Schwartz space of the basic affine space

Let $X = \mathbf{G}(F)/\mathbf{N}(F)$. The space X has a left G-action and a commuting right A-action.

The principal series comprise an important class of representations of G. These are representations of the form

$$\pi_\chi = \operatorname{Ind}_B^G(\chi)$$

where $\chi\colon T \to \mathbb{C}^\times$ is a character of T, which we inflate to B (by induction, we always mean normalized induction). Thus each principal series representation can be realized as a subspace of $C_c^\infty(X)$. As explained in 2.3.1, for $w \in W$, it is true for generic χ that $\pi_\chi \simeq \pi_{w\cdot\chi}$. However, as explained in 2.3.1, this is not always true.

In [14], Braverman and Kazhdan define a space $\mathscr{S}$ of functions on X that corrects this problem, in the the sense that the space of coinvariants $\mathscr{S}_\chi \simeq \mathscr{S}_{w\cdot\chi}$ for all characters χ of T and all $w \in W$, and when $\pi_\chi \simeq \pi_{w\cdot\chi}$, $\mathscr{S}_\chi \simeq \pi_\chi$. As explained in *loc. cit.*, for each $w \in W$, Gelfand-Graev define a Fourier transform Φ_w acting on $L^2(X)$, and one then puts

$$\mathscr{S} := \sum_w \Phi_w(C_c^\infty(X)) \subset L^2(X).$$

Definition 20. The space X is called the *basic affine space of* $\mathbf{G}$ (note that it is in fact only quasi-affine). The space $\mathscr{S}$ called the *Schwartz space of the basic affine space.*

Example 8. When $\mathbf{G} = \mathrm{SL}_2$, $\mathscr{S}$ has a simple description. In this case we have $G/N \simeq F^2 \setminus \{0\}$, and $\mathscr{S} = C_c^\infty(F^2)$. In this case there is a single Fourier transform, which is the usual Fourier transform on the plane: Let $\psi\colon F \to \mathbb{C}^\times$ be a nontrivial unramified character. Then for $f \in C_c^\infty(X)$, define

$$\Phi(f)(x) := \int_{F^2\setminus\{0\}} f(y)\psi(\langle y, x\rangle)\,\mathrm{d}y,$$

where

$$\langle x, y\rangle = \left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = x_1y_2 - y_1x_2.$$

We shall make use of this perspective, for $\mathscr{S}^I$, only in Chapter 3. In Chapter 4 we shall refer the following description of $\mathscr{S}^I$.

Consider the equivariant K-group $K^{A^\vee\times\mathbb{C}^\times}(\mathscr{B}^\vee) = K^{A^\vee}(\mathscr{B}^\vee)[\mathbf{q}^{1/2}, \mathbf{q}^{-1/2}]$, where $\mathbb{C}^\times$ acts trivially and $\mathbf{q}^{1/2}$ is the identity character of $\mathbb{C}^\times$.

Let us recall the $K^{\mathbf{G}^\vee\times\mathbb{C}^\times}(\mathrm{St}^\vee)$-module structure on $K^{\mathbf{A}^\vee\times\mathbb{C}^\times}(\mathscr{B}^\vee)$ following [48], Section 10.1: Viewing $\mathrm{St}^\vee$ and $T^*\mathscr{B}^\vee \times \mathscr{B}^\vee$ as closed subvarieties of $T^*\mathscr{B}^\vee \times T^*\mathscr{B}^\vee$, let $p_2\colon T^*\mathscr{B}^\vee \times \mathscr{B}^\vee \to \mathscr{B}^\vee$ be projection onto the second factor, and $p_1\colon \mathrm{St}^\vee \cap (T^*\mathscr{B}^\vee \times T^*\mathscr{B}^\vee) \to \mathscr{B}^\vee$ be the first projection. Then the cotangent bundles are smooth, p_2 is flat and p_1 is proper, so given $\mathscr{G} \in \mathrm{Coh}^{A^\vee\times\mathbb{C}^\times}(\mathscr{B}^\vee)$ and $\mathscr{F} \in \mathrm{Coh}^{G^\vee\times\mathbb{C}^\times}(\mathrm{St}^\vee)$, we may define

$$\mathscr{F} \star_0 \mathscr{G} := p_{1*}\left(\mathscr{F} \otimes_{T^*\mathscr{B}^\vee\times T^*\mathscr{B}^\vee} p_2^*\mathscr{G}\right)$$

as usual to obtain an action.

Therefore there is an H-module structure on $K^{\mathbf{A}^\vee\times\mathbb{C}^\times}(\mathscr{B}^\vee)$. In fact, there is an $H \otimes \mathbb{Z}[\tilde{W}]$-module structure, which we will now recall. We have $\mathbb{Z}[X_*(\mathbf{A})] \simeq R(\mathbf{A}^\vee) \simeq K^{\mathbf{A}^\vee}(\mathrm{pt})$, and thus

$K^{\mathbf{A}^\vee \times \mathbb{C}^\times}(\mathscr{B}^\vee)$ acquires a $\mathbb{Z}[X_*(\mathbf{A})]$-action. Obviously $\mathscr{B}^\vee$ is also a $\mathbf{G}^\vee$-variety; in such circumstances $\mathbf{A}^\vee$-equivariant K-theory acquires a W-action, as explained in [14] and [19], Section 6.1. To wit, let $\bar{w}$ be a lift of $w \in W$ to the normalizer in $\mathbf{G}^\vee$ of $\mathbf{A}^\vee$. As $\mathscr{B}^\vee$ is a $\mathbf{G}^\vee$-variety, we can define $\bar{s_0}^*\mathscr{F}$. This sheaf is again $\mathbf{A}^\vee$-equivariant for any $\mathbf{A}^\vee$-equivariant sheaf $\mathscr{F}$. This gives an action of W, and these two actions give an action of $\tilde{W}$.

This description appears already in [14] to give an alternative model for $\mathscr{S}^I$ to the function-theoretic one on G/N. In particular we have

Theorem 17 ([14], Corollary 5.7)**.** *$\mathscr{S}^I \simeq K^{\mathbf{A}^\vee \times \mathbb{C}^\times}(\mathscr{B}^\vee)|_{\mathbf{q}^{1/2}=q^{1/2}}$ as $H \otimes_{\mathbb{Z}} \mathbb{Z}[\tilde{W}]$-modules.*

It will also be relevant in Chapter 4 that

$$K^{\mathbf{G}^\vee \times \mathbb{C}^\times}(\mathscr{B}^\vee \times \mathscr{B}^\vee) \simeq K^{\mathbf{A}^\vee \times \mathbb{C}^\times}(\mathscr{B}^\vee).$$

Chapter 2

Tempered picture: Denominators in J via the Plancherel formula

2.1 Introduction

Let $\tilde{W}$ be an affine Weyl group or extended affine Weyl group, and let $\mathbf{H}$ be its associated Hecke algebra over $\mathscr{A} := \mathbb{C}[\mathbf{q}^{1/2}, \mathbf{q}^{-1/2}]$, where $\mathbf{q}$ is a formal variable. The representation theory of $\mathbf{H}$ is very well understood, behaving well and uniformly when $\mathbf{q}$ is specialized to any $q \in \mathbb{C}^\times$ that is not a root of the Poincaré polynomial P_W of the finite Weyl group $W \subset \tilde{W}$.

When $\mathbf{q}$ is specialized to a prime power q, the category of modules over the specialized algebra H is equivalent to the category of admissible representations with Iwahori-fixed vector of some p-adic group. A form of local Langlands correspondence, the Deligne-Langlands conjecture, has been established by Kazhdan and Lusztig in [37], where they classified modules over the generic algebra $\mathbf{H}$ using algebraic K-theory. A slightly different approach to this classification due to Ginzburg is explained in [19]. In both treatments, a first step is to fix a central character. In particular, one must chose a complex number $q \in \mathbb{C}^\times$ by which $\mathbf{q}$ will act. Decomposing the K-theory of certain subvarieties of Springer fibres into irreducible representations of a certain finite group yields the *standard modules.* It can happen that the standard modules are themselves simple (this is true, for example, in the case of tempered representations, which play an essential role in this chapter), but in general simple modules are obtained as a certain unique nonzero quotient of standard modules. This quotient exists when $q \in \mathbb{C}^\times$ is not a root of unity, but can be zero otherwise. Lusztig conjectured in [45] that this classification would in fact hold whenever q was not a root of P_W, and this result was proven by Xi in [70]. The failure of the classification found by Xi is related to a lack of simple $\mathbf{H}|_{\mathbf{q}=q}$-modules attached to the lowest two-sided cell. Our results in this chapter explain that the lowest two-sided cell is in a precise sense the most singular with respect to q; in Conjecture 3 below we make precise how our results might parameterize the failure of the Kazhdan-Lusztig classification.

One way Lusztig expressed the uniformity in q of the representation theory of the various algebras $\mathbf{H}|_{\mathbf{q}=q}$ is via the asymptotic Hecke algebra J. This is a $\mathbb{C}$-algebra (in fact, $\mathbb{Z}$-algebra) J with a distinguished basis $\{t_w\}_{w\in\tilde{W}}$ equipped with an injection $\phi\colon \mathbf{H} \hookrightarrow J \otimes_{\mathbb{C}} \mathscr{A}$. In this way there is a map from J-modules to $\mathbf{H}$-modules, and Lusztig has shown in [43] and [44] that when q is not a root of unity (other than 1), that the specialized map ϕ_q induces a bijection between simple $\mathbf{H}$-modules and simple J-modules, these last being defined over $\mathbb{C}$. Moreover, he showed that when $P_W(q) \neq 0$, the map ϕ_q induces an isomorphism

$$(\phi_q)_* \colon K_0(J - \mathbf{Mod}) \to K_0(\mathbf{H}|_{\mathbf{q}=q} - \mathbf{Mod})$$

of Grothendieck groups. The map ϕ becomes a bijection after completing both sides by replacing $\mathscr{A}$ with $\mathbb{C}((\mathbf{q}^{-1/2}))$ and allowing infinite sums convergent in the $(\mathbf{q}^{-1/2})$-adic topology. In this way one can write a basis element t_w as an infinite sum

$$()^\dagger \circ \phi^{-1}(t_w) = \sum_{x \in \tilde{W}} a_{x,w} T_x, \tag{2.1}$$

where each $a_{x,w}$ is a formal Laurent series in $\mathbf{q}^{-1/2}$, and $()^\dagger$ is the Goldman involution. In light of the above, it is natural to ask how $a_{x,w}$ behaves when $\mathbf{q}$ is specialized to a root of unity.

This chapter is prompted by the work of Braverman and Kazhdan, who in [15] associated a Schwartz function to an element of J, giving another way of associating to t_w an expression similar to (2.1). Along the way, we prove in Proposition 3 that their construction is essentially the map ϕ^{-1}. The following conjecture was first made by Kazhdan; we have strengthened it based on the results we obtain in the present chapter.

Conjecture 2. *Let $\tilde{W}$ be an affine Weyl group, $\mathbf{H}$ its affine Hecke algebra over $\mathscr{A}$, and J its asymptotic Hecke algebra. Let $\phi \colon \mathbf{H} \hookrightarrow J \otimes_{\mathbb{Z}} \mathscr{A}$ be Lusztig's map.*

1. *For all $x, w \in \tilde{W}$, $a_{x,w}$ is a rational function of $\mathbf{q}$. The denominator of $a_{x,w}$ is independent of x. As a function of w, it is constant on two-sided cells.*

2. *There exists $N_{\tilde{W}} \in \mathbb{N}$ such that upon writing*

$$()^\dagger \circ \phi^{-1}(t_w) = \sum_{x \in \tilde{W}} a_{x,w} T_x,$$

we have

$$P_W(\mathbf{q})^{N_{\tilde{W}}} a_{x,w} \in \mathscr{A}$$

for all $x, w \in \tilde{W}$.

3. *Moreover, there exists $N_{\tilde{W}} \in \mathbb{N}$ such that*

$$P_W(\mathbf{q})^{N_{\tilde{W}}} d(\omega) \in \mathscr{A}$$

for all discrete series representations ω of $\mathbf{H}$.

In [23], appearing here as Chapter 3, the author proved Conjecture 2 in type $\tilde{A}_1$, but with different conventions. To translate to the conventions of this chapter, the reader should replace j with the Goldman involution $()^\dagger$, and the completion of $\mathbf{H}$ with respect to the C_w basis and positive powers of $\mathbf{q}^{1/2}$ with the completion of $\mathbf{H}$ with respect to the basis $\{(-1)^{\ell(w)} C'_w\}_{w \in \tilde{W}}$ and negative powers of $\mathbf{q}^{1/2}$. Note also that we write $a_{x,w}$ instead of $a_{w,x}$ as in [23]. In [52], Neunhöffer described the coefficients $a_{x,w}$ for finite Weyl groups.

Our main result in this chapter, which appears as Theorem 34 and Theorem 36 below, is the following

Theorem 18. *Let $\tilde{W}$ be of type $\tilde{A}_n$ or $\tilde{C}_2$. Then Conjecture 2 is true.*

The asymptotic Hecke algebra can be used to organize the results of [37], as done by Lusztig in [44]. This setup is used by Xi in [70]. Let $\mathbf{c}_0$ be the lowest two-sided cell and restrict to type $\tilde{A}_n$. As noted above, in *loc. cit.*, Xi showed that the Kazhdan-Lusztig classification of representations of $\mathbf{H}$ fails whenever $P_W(q) = 0$ in the following way: Appealing to [68], Xi showed that when $P_W(q) = 0$, there are no simple $\mathbf{H}|_{\mathbf{q}=q}$-modules M such that $\mathbf{c}(M) = \mathbf{c}_0$. This shows that there is no bijection between simple $\mathbf{H}|_{\mathbf{q}=q}$-modules and triples (u, s, ρ) as in [37], as this bijection fails for $u = 0$. Examining Theorem 32, we see by Proposition 3 that in the notation of (2.11), $a_{1,d}$ has poles at every root of P_W, for all distinguished involutions $d \in \mathbf{c}_0$. On the other hand, for $d \in \mathbf{c} \neq \mathbf{c}_0$, by Theorem 32 and Proposition 1, there are roots of P_W at which $a_{1,d}$ does not have a pole. One could hope that this behaviour detects the failure of Kazhdan-Lusztig classification. We record this hope as

Conjecture 3. *Let $\tilde{W}$ be such that Conjecture 2 holds, and let $q \in \mathbb{C}^\times$ be a root of P_W. Let $\mathbf{c}$ be a two-sided cell such that if $w \in \mathbf{c}$, then $a_{x,w}$ does not have a pole at $\mathbf{q} = q$. Let $u = u(\mathbf{c})$. Then every standard module $K(u, s, \rho, q)$ has a unique simple quotient $L = L(u, s, \rho, q)$. Two such simple modules are isomorphic if and only if their corresponding triples are conjugate. We have $a(L) = a(E)$, where E is the simple J module corresponding to $K(u, s, \rho, q)$, in the notation of* [44].

Note that in type $\tilde{A}_n$, the number of two-sided cells grows as $e^{\sqrt{n}}$, whereas the number of subsets of roots of P_W is $2^{n(n+1)/2}$, but in low rank these quantities may be such that the conjecture is vacuous. For example in type $\tilde{A}_1$, there is only one root of $P_W = \mathbf{q} + 1$, but there are two two-sided cells (and both are singular at $\mathbf{q} = -1$.) The same holds for type $\tilde{C}_2$ considered in Section 2.5: all the cells are singular at all the roots of $P_{\tilde{C}_2}$. However, already in type $\tilde{A}_3$, one can see from Theorem 32 that the two-sided cell corresponding to the partition $4 = 2 + 2$ is not singular at either of the roots $\mathbf{q} = (-1)^{2/3}$ or $\mathbf{q} = -(-1)^{1/3}$ of $P_{\tilde{A}_3}(\mathbf{q}) = (1 + \mathbf{q})(1 + \mathbf{q} + \mathbf{q}^2)(1 + \mathbf{q} + \mathbf{q}^2 + \mathbf{q}^3)$.

Our main tool is Harish-Chandra's Plancherel theorem for the p-adic group G associated to $\mathbf{H}$. The proof of the theorem relies on certain cancellations taking place while computing the integrals given by the Plancherel formula. In particular, the theorem is sometimes only true thanks to cancellations involving formal degrees. For this reason, for general $\mathbf{G}$, we can presently only prove the following weaker form of Conjecture 2:

Theorem 19. *Let $\tilde{W}$ be an affine Weyl group, $\mathbf{H}$ its affine Hecke algebra over $\mathscr{A}$, and J its asymptotic Hecke algebra. Let $\phi\colon \mathbf{H} \hookrightarrow J \otimes_{\mathbb{Z}} \mathscr{A}$ be Lusztig's map.*

1. *For all $w, x \in \tilde{W}$, $a_{x,w}$ is a rational function of $\mathbf{q}$. The denominator of $a_{x,w}$ is independent of x. As a function of w, it is constant on two-sided cells.*
2. *There is a polynomial $P_{\mathbf{G}}(\mathbf{q})$ depending only on $\mathbf{G}$ such that upon writing*

$$()^\dagger \circ \phi^{-1}(t_w) = \sum_{x \in \tilde{W}} a_{x,w} T_x,$$

we have

$$P_{\mathbf{G}}(\mathbf{q}) a_{x,w} \in \mathscr{A}$$

for all $x, w \in \tilde{W}$. The roots of $P_{\mathbf{G}}(\mathbf{q})$ are all roots of unity. If d is a distinguished involution in the lowest two-sided cell, then $a_{1,d} = 1/P_W(q)$ exactly.

3. *Moreover, $P_{\mathbf{G}}(\mathbf{q})d(\omega) \in \mathscr{A}$ for all discrete series representations ω of $\mathbf{H}$.*

Remark 5. It is tempting to pose the following more precise version of Conjecture 2, (1) based on the factorization $P_M(q)P_{G/P}(q) = P_{\mathbf{G}/\mathbf{B}}(q)$: for every Levi subgroup M of G and all $\omega \in \mathscr{E}_2(M)$, the formal degree $d(\omega)$ is a rational function of q the denominator of which divides a power of $P_M(q)$. The integral over all induced twists $\mathrm{Ind}_P^G(\nu \otimes \omega)$ is a rational function of q, with denominator dividing a power of the Poincaré polynomial of the partial flag variety $(G/P)(\mathbb{C})$. For example, let $G = \mathrm{GL}_6(F)$ and $M = \mathrm{GL}_3(F) \times \mathrm{GL}_3(F)$. In this case the integral itself (omitting the factor C_M in the notation of section 2.2.6) is

$$\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{(z_1 - z_2)(z_1 - z_2)}{(z_1 - q^3 z_2)(z_1 - q^{-3} z_2)} \frac{\mathrm{d}z_1}{z_1}\frac{\mathrm{d}z_2}{z_2} = \frac{(1-q^3)^2}{(1-q^6)^2 q^3} + 1 = \frac{(1-q^3)q^{-3}}{1+q^3} + 1,$$

and by [13] Proposition 23.1, (with $q = t^2$) we have

$$P_{G/P}(q) = P_{G(3,6)}(q) = (1+q^2)(1+q+q^2+q^3+q^4)(1+q^3).$$

In examples such as the above, this does indeed happen, but only after cancellation with some terms in the numerator. In general, we will not track numerators precisely enough to show this version of Conjecture 2. We shall however see a limited demonstration of this behaviour in Corollary 1.

2.1.1 Outline of the argument

Our strategy is to determine that the Schwartz functions f_w on the p-adic group G satisfy the statements of the conjecture, and are in addition well-behaved enough that they lift to elements of a certain completion $\boldsymbol{\mathcal{H}}^-$ of $\mathbf{H}$. We therefore obtain two maps $J \subset J \otimes_{\mathbb{C}} \mathscr{A} \to \boldsymbol{\mathcal{H}}^-$: Lusztig's map, and our map induced by the construction in [15]. We prove that these maps agree, at which point our main results follow.

This chapter is organized according to our strategy for proving Theorems 34, 19, and 36. These results are each simple corollaries of computations with the Plancherel formula and some of Lusztig's results on J. The remainder of this section will introduce $\mathbf{H}$ and J precisely, and recall their basic representation theory. In Section 2.2, we introduce Harish-Chandra's Plancherel formula in detail, along with all the numerical constants that appear in it. In Section 2.2.4, we recall the results of Braverman-Kazhdan from [15]. There is no original material in the first two sections. In Section 2.3, we prepare to apply the Plancherel formula by proving that, if f_w is the Schwartz function associated by Braverman-Kazhdan to t_w, and π is a tempered representation, then $\mathrm{trace}(\pi(f_w))$ is sufficiently regular so as not to complicate the denominators of $a_{x,w}$. This section is also mostly a recollection of standard material, the only original result being Lemma 5. In Section 2.4, we prove our main results. As we are able to be more precise in type $\tilde{A}_n$, we perform each step in parallel for type $\tilde{A}_n$ and for other types: in Sections 2.4.1 and 2.4.2 we prove statements like those of Conjecture 2 for the Schwartz functions f_w. In these sections $\mathbf{q}$ is specialized to a prime power q. In Section 2.4.3 we relate the functions f_w to the basis elements t_w, turning statements that hold for all prime powers q into statements that hold for the formal variable $\mathbf{q}$. We are then able to prove our main results.

In Section 2.5, we show that Conjecture 2 also holds for type $\tilde{C}_2$.

2.1.2 The affine Hecke algebra

Let F be a non-archimedean local field, $\mathscr{O}$ its ring of integers and ϖ be a uniformizer. Let q be the cardinality of the residue field. Then $q = p^r$ is a prime power. We write $|\cdot|_F$ for the p-adic absolute value on F; when necessary, $|\cdot|_\infty$ will denote the archimedean absolute value on $\mathbb{C}$.

Let $\mathbf{G}$ be a connected reductive algebraic group defined and split over F, $\mathbf{A}$ a maximal torus of $\mathbf{G}$, and $X_* = X_*(\mathbf{A})$ the cocharacter lattice of $\mathbf{A}$. Let $\mathbf{N}$ be unipotent radical of $\mathbf{B}$, so that $\mathbf{B} = \mathbf{AN}$ is the upper-triangular Borel subgroup. Let W be the finite Weyl group of $\mathbf{G}$, and $\tilde{W} = W \ltimes X_*(\mathbf{A})$ be the affine Weyl group. Write S for the set of simple reflections in $\tilde{W}$. Let $\mathbf{G}^\vee$ be the Langlands dual group of $\mathbf{G}$, taken over $\mathbb{C}$, and write $G^\vee$ for $\mathbf{G}^\vee(\mathbb{C})$. We write $G = \mathbf{G}(F)$, $A = \mathbf{A}(F)$, etc. Let K be the maximal compact subgroup $\mathbf{G}(\mathscr{O})$. Also let I be Iwahori subgroup of G that is the preimage of a fixed Borel subgroup in $\mathbf{G}(\mathbb{F}_q)$. We sometimes write $P_{\mathbf{G}/\mathbf{B}}$ for P_W, as this polynomial is also the Poincaré polynomial of the flag variety $(\mathbf{G}/\mathbf{B})(\mathbb{C})$.

We write $\mathbf{H}$ for the affine Hecke algebra of $\tilde{W}$. It is a unital associative algebra over the ring $\mathscr{A} = \mathbb{C}[\mathbf{q}^{\frac{1}{2}}, \mathbf{q}^{-\frac{1}{2}}]$ (in fact, it is defined over $\mathbb{Z}[\mathbf{q}^{\frac{1}{2}}, \mathbf{q}^{-\frac{1}{2}}]$ but we will work over $\mathbb{C}$ to avoid having to introduce extra notation later), where $\mathbf{q}^{\frac{1}{2}}$ is a formal variable. We will think of $\mathbb{C}^\times$ as $\operatorname{Spec}\mathscr{A}$. The algebra $\mathbf{H}$ has the Coxeter presentation with standard basis $\{T_w\}_{w\in\tilde{W}}$ with $T_wT_{w'} = T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$ and quadratic relation $(T_s + 1)(T_s - \mathbf{q}) = 0$ for $s \in S$. We write θ_λ for the generators of the Bernstein subalgebra.

It is well-known that, upon extending scalars, there is an isomorphism of associative $\mathbb{C}$-algebras

$$\mathbf{H} \otimes_{\mathscr{A}} \mathbb{C} \to C_c^\infty(G)^{I\times I} =: H,$$

where $\mathbf{q}$ acts on $\mathbb{C}$ by multiplication by q (our splitness assumptions are in place to guarantee equal parameters in the definition of H).

2.1.3 Representation theory of H and J

Recall the classification given in [37]. For an extended exposition with slightly different conventions, we refer the reader to Chapters 7 and 8 of [19]. The primary difference between the setup we require and that of [19] is that we must be able to defer specializating $\mathbf{q}$ until the last possible moment, whereas specializing $\mathbf{q}$ is the first step of the construction as given in [19]. In particular, let $u \in G^\vee(\mathbb{C})$ be a unipotent element, and $s \in G^\vee(\mathbb{C})$ be a semisimple element such that $us = su$. Let ρ be an irreducible representation of the simultaneous centralizer $Z_{G^\vee}(s, u)$. The *standard H-module* $K(s, u, \rho)$ is a certain, and in general reducible, H-module defined using the geometry of the flag variety of the Langlands dual group. It may be the zero module; we say that (u, s, ρ) is *admissible* when this does not happen. Having fixed s and u, we say that ρ is admissible if (u, s, ρ) is. In this section we will define the standard modules as vector spaces (in fact, as a representation of some finite group) only; to equip it with an action of the affine- or Iwahori-Hecke algebra requires the coherent realization we recalled in .

We now recall the definition of the standard modules, following [43] and [44]. These modules were first constructed using Borel-Moore homology by Kazhdan and Lusztig in [37], and independently, although with a proof that would require modification, by Ginzburg. See [19] for a pedagogical treatment of the construction via homology. Let (u, s, ρ) be an admissible triple, and consider the Springer fibre $\mathscr{B}_u^\vee = \{\mathfrak{b} \in \mathscr{B}^\vee \mid u \in \mathfrak{b}\}$. This variety is equipped with an action of $\langle s \rangle$, the algebraic group generated by s, and we can consider $\langle s \rangle$-equivariant coherent sheaves on $\mathscr{B}_u^\vee$, as well as their Grothendieck group

$$K_{u,s} = K^{\langle s \rangle \times \mathbb{C}^\times}(\mathscr{B}_u^\vee),$$

where $\mathbb{C}^\times$ acts trivially. As explained in [37], 1.3 (j), $K_{u,s}$ has an action of $\pi_0(Z_{\mathbf{G}^\vee}(us))$. We then define the *standard module*

$$K_{u,s,\rho} = \mathrm{Hom}_{\pi_0(Z_{\mathbf{G}^\vee}(us))}(\rho, K_{u,s}).$$

Given $\lambda \in \mathbb{C}^\times$, define

$$K_{u,s,\rho}^\lambda = K_{u,s,\rho} \otimes_{\mathscr{A}} \mathbb{C}_\lambda,$$

where $\mathbf{q}^{1/2}$ acts on $\mathbb{C}_\lambda$ by multiplication by λ.

We will comment now on how to translate this parametrization into a parametrization in terms of triples (u, r, ρ), where now $sus^{-1} = u^q$ for $q > 1$, where $u^q := \exp(q \log(u))$. A similar version of this interconversion featuring the formal variable $\mathbf{q}$ will be recalled immediately after.

Let $q > 1$ be real. Let (u, r, ρ) be a triple with $u \in G^\vee$ unipotent, $sus^{-1} = u^q$, and let ρ be a representation of $\pi_0(Z_{G^\vee}(u, s))$. The following construction was given independently of Kazhdan-Lusztig by Ginzburg, but in its final correct form uses some results of of theirs from [37]: let

$$\mathscr{B}_u^{\vee s} = \{\mathfrak{b} \in \mathscr{B}^\vee \mid \mathrm{Ad}(s)(\mathfrak{b}) = \mathrm{Ad}(u)(\mathfrak{b}) = \mathfrak{b}\}.$$

and consider the Borel-Moore homology $H_*(\mathscr{B}^{\vee s}_u)$. It acquires by [19], Section 8.1, an action of $\pi_0(Z_{G^\vee}(u, s))$, and an action of the affine Hecke algebra, with central character given by (s, q). Denote this module $K_{u,s,\rho,q}$, and call ρ *admissible* if it is nonzero.

Given a triple (u, s', ρ) where u and s' commute, one can define a triple of the second kind in the following way. Let

$$\phi \colon \mathrm{SL}_2(\mathbb{C}) \to G^\vee$$

be the morphism given by the Jacobson-Morozov Theorem and u. That is, we have

$$\phi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = u.$$

Then define $s = \phi(\mathrm{diag}(q, q^{-1}))s'$. Then (u, s', ρ) is a triple of the second kind, and this procedure gives a bijection between the parameters for the two versions of the parametrization [61].

Theorem 20 (Harish-Chandra, Kazhdan-Lusztig, Ginzburg)**.** *Let $q > 1$. (For all but the last point, we may take any $q \in \mathbb{C}^\times$ not a root of unity.)*

1. *If ρ is admissible for a triple (u, s', ρ) of the first kind, then ρ is admissible for the triple (u, s, ρ) formed under the procedure of the preceding paragraph.*

2. *For $\lambda = q$, the modules $K_{u,s,\rho,q}$ and $K_{u,s,\rho}^\lambda$ are isomorphic.*

3. *If (u, s, ρ) is a triple of the second kind and $s = s'\phi(\mathrm{diag}(q, q^{-1}))$ as in the preceeding paragraph, $K_{u,s,\rho,q}$ is tempered (in the above sense of harmonic analysis) iff s' is compact.*

4. Tempered standard modules are exactly the simple tempered representations of H.

Remark 6. By [44], Theorem 9.1, $K_{u,s,\rho,q}$ has a nonzero K-fixed vector if and only if ρ is the trivial representation. We reassure the reader that when $\mathbf{G} = \mathrm{GL}_n$, and u is regular unipotent, it is easy to see that $Z_{\mathbf{G}(\mathbb{C})}(u, s)$ is never connected. This is compatible with the fact that the Steinberg representation does not have nontrivial K-fixed vectors (as evidenced, for example, by the Iwasawa decomposition).

Now we recall a version of this classification due to Lusztig, in which $\mathbf{q}$ remains a formal variable throughout. As we do not have access to the notion of absolute value of $\mathbf{q}$, the classical definitions of tempered and discrete-series representations of the corresponding p-adic group G are not available to us. However, Kazhdan-Lusztig provide the following algebraic generalization. Let $\kappa = \mathbb{C}(\mathbf{q}^{-\frac{1}{2}})$ and $\bar{\kappa}$ be the algebraic closure of κ. We write $\mathbf{H}_{\bar{\kappa}}$ for $\mathbf{H} \otimes_{\mathbb{Z}} \bar{\kappa}$.

Following [44], we choose a morphism of groups $V : \bar{\kappa}^\times \to \mathbb{R}$ such that $V(\mathbf{q}^{-\frac{1}{2}}) = 1$ and $V(a\mathbf{q}^{-\frac{1}{2}} + b) = 0$ for all $a \in \mathbb{C}$, $b \in \mathbb{C}^\times$.

Definition 21 ([37], [44])**.** Let M be any finite-dimensional $\mathbf{H}_{\bar{\kappa}}$-module. We say that M is *V-tempered* if all eigenvalues ν of θ_λ for all dominant $\lambda \in X_*(\mathbf{A})$ satisfy $V(\nu) \leq 0$. We say that M is *V-square-integrable* if $V(\nu) < 0$.

Note that when $\mathbf{q}$ is specialized a prime power $q \in \mathbb{C}^\times$ and $V(z) = \log |z|$, we recover that our *anti-tempered* (*i.e.* $V(\nu) \geq 0$) modules are *tempered* in the traditional analytic sense, and likewise for square-integrable, by a theorem of Casselman. We will recall these notions explicitly below. The reason for our choice of conventions, which match those of [43] and [44], is the presence of the Goldman involution—which exchanges these notions—in the relationship between H-modules and J-modules, see Theorem 21.

The representation theory of J is very well understood. We shall recall some notation and then state some major classification results of Lusztig, which relate the representation theory of J to certain $\mathbf{H}$-modules defined by Kazhdan-Lusztig.

Definition 22. Let E be a $J \otimes_{\mathbb{Z}} \mathbb{C}$-module. Then $E \otimes_{\mathbb{C}} \kappa$ is a $J \otimes_{\mathbb{Z}} \kappa$-module. Hence $\mathbf{H}_\kappa$ acts on E via ϕ. Denote the resulting $\mathbf{H}_\kappa$ module by ${}^\phi E$.

We need to recall a final defintion of Lusztig's from [44].

Definition 23. Let M be a $H_{\bar{\kappa}}$-module finite-dimensional over $\bar{\kappa}$. Say that $m \in M$ is an *eigenvector* if $\theta_x(m) = \chi_m(x)m$ for all dominant x in X_*. As χ_m is a character of the coweight lattice, it corresponds to an element $\sigma_m \in A^\vee(\bar{\kappa})$ in the sense that, for all cocharacters x of A, we have

$$\chi_m(x) = x(\sigma_m)$$

where x is viewed as a character of $A^\vee$. Then M is of *constant type* if there is a semisimple element $s' \in G^\vee(\mathbb{C})$ and a morphism of algebraic groups

$$\phi' : \mathrm{SL}_2(\mathbb{C}) \to Z^0_{G^\vee}(s')$$

such that for all eigenvectors m of M, the element σ_m is $G^\vee(\bar{\kappa})$-conjugate to

$$\phi'(\mathrm{diag}(\mathbf{q}^{1/2}, \mathbf{q}^{-1/2}))s',$$

where by abuse of notation we have written ϕ' again for the base-change to $\bar{\kappa}$.

The idea of the name of the definition is that $s' \in G^\vee(\mathbb{C})$ is a "constant element" not depending on $\mathbf{q}$.

Involutions on H

Definition 24. Let $j\colon \mathbf{H} \to \mathbf{H}$ be the ring (and not $\mathscr{A}$-algebra) involution of $\mathbf{H}$ defined by $j(\sum_w a_w T_w) = \sum_w \bar{a}_w (-1)^{\ell(w)} q^{-\ell(w)} T_w$.

Definition 25. Let $h \mapsto h^\dagger$ be the $\mathscr{A}$-algebra involution of $\mathbf{H}$ defined by setting

$$T_w^\dagger = (-1)^{\ell(w)} q^{\ell(w)} T_{w^{-1}}^{-1}.$$

This involution is the *Goldman* involution of $\mathbf{H}$. Given an $\mathbf{H}$ (or $\mathbf{H}_\kappa$)-module M, define ${}^\dagger M$ to be the same vector space with the $\mathbf{H}$- action twisted by this involution.

Lemma 1. *We have*

$$\overline{h^\dagger} = (\bar{h})^\dagger = j(h)$$

as $\mathscr{A}$-antilinear involutions. In particular, $C_w^\dagger = j(C_w) = (-1)^{\ell(w)} C'_w$.

Proof. We compute

$$\overline{\sum_x b_x T_x^\dagger} = \sum_x \bar{b}_x (-1)^{\ell(x)} \overline{q^{\ell(x)} \bar{T}_x} = \sum_x \bar{b}_x (-1)^{\ell(x} q^{-\ell(x)} T_x = j\left(\sum_x b_x T_x\right)$$

whereas

$$\left(\overline{\sum_x b_x T_x}\right)^\dagger = \left(\sum_x \bar{b}_x q^{-\ell(x)} (-1)^{\ell(x)} T_x^\dagger\right)^\dagger.$$

Thus we have $\bar{C}_w^\dagger = (-1)^{\ell(w)} C'_w$, whence the last claim. ☐

We now summarize the relationship between representations of $\mathbf{H}$ and of J.

Theorem 21 ([44], Prop. 2.11, Thm. 4.2, Prop. 4.4)**.** *There are bijections of sets*

$$\begin{array}{ccc}
(u,s,\rho) & & \{(u,s,\rho)\,|\,\rho \text{ admissible},\ us = su\}\,/G^\vee(\mathbb{C}) \\
\downarrow & & \downarrow \\
K(s,u,\rho)\otimes_{\mathscr{A}}\kappa & & \{M \in H_\kappa - \mathbf{Mod}\,|\,{}^\dagger M \otimes_\kappa \bar{\kappa} \text{ simple, } V-\text{tempered} \\
\| & & \\
{}^\phi E = E\otimes_{\mathbb{C}}\kappa \in H_\kappa - \mathbf{Mod} & & H_{\bar{\kappa}} - \text{module of constant type}\} \\
\uparrow & & \uparrow \\
E & & \{E \in J_{\mathbb{C}} - \mathbf{Mod}\,|\,E \text{ is simple}\}.
\end{array}$$

Moreover, for a simple $J_{\mathbb{C}}$-module E,

1. *E is finite-dimensional over $\mathbb{C}$;*

2. *There is a unique two-sided cell $\mathbf{c} = \mathbf{c}(E)$ of $\tilde{W}$ such that* $\operatorname{trace}(t_w, E) \neq 0$ *implies $w \in \mathbf{c}$.*

3. $\mathrm{trace}(t_w, E)$ *is the constant term of the polynomial*

$$(-\mathbf{q}^{1/2})^{a(\mathbf{c}(E))}\,\mathrm{trace}(C_w, M) \in \mathbb{C}[\mathbf{q}^{\frac{1}{2}}]$$

where $M \simeq {}^{\phi}E$.

In particular, $\mathrm{trace}(t_x)$ is independent of $\mathbf{q}$, and upon specializing $\mathbf{q} = q$ a prime power, will be a regular function in the inducing character in the setting of the Payley-Weiner theorem for the Iwahori-Hecke algebra of the p-adic group G, as we will explain in greater detail below.

We will comment more on the necessity of twisting by the Goldman involution in Section 2.2.4.

The affine Hecke has a filtration by two-sided ideals

$$H^{\geq i} = \mathrm{span}\,\{C_w \mid a(w) \geq i\}.$$

As such, for any simple $\mathbf{H}_{\bar{\kappa}}$-module there is an integer $a(M)$ such that

$$\mathbf{H}_{\bar{\kappa}}^{\geq i} M \neq 0$$

but

$$\mathbf{H}_{\bar{\kappa}}^{\geq i+1} M = 0.$$

Define $a(i)$ to be this integer.

One can also define

Definition 26. Define $a(E) = a(\mathbf{c}(E))$ where E and $\mathbf{c}(E)$ are as above.

Thus J linearizes the above filtration into an honest direct sum, and implements the almost-independence on $q \in \mathbb{C}^{\times}$ of the representation theory of $\mathbf{H}_{\mathbf{q}=q}$ as follows. In fact, as mentioned in Chapter 1, this holds for all but finitely-many roots of unity:

Theorem 22 ([70])**.** *1. Suppose that q is not a root of the Poincaré polynomial of $\mathbf{G}$. Then for each simple J_κ-module E, the $\mathbf{H}_\kappa$ module ${}^{\phi}E$ has a unique simple quotient L, and $a(E) = a(L)$. For all other simple subquotients L' of E, we have $a(L') < a(E)$.*

Equivalently, for all admissible triples (u, s, ρ), the representation $K(u, s, \rho, q)$ of H has a unique nonzero simple quotient $L = L(u, s, \rho, q)$ such that $a(L) = a(\mathbf{c}(u))$. That is, the Deligne-Langlands conjecture is true for $\mathbf{H}_q$.

2. If q is a root of the Poincaré polynomial of $\mathbf{G}$, then the Deligne-Langlands conjecture is false for the big cell. Namely, if $u = \{1\}$, then every simple subquotient L' of $K(u, s, \rho, q)$ has $a(L') < a(\mathbf{c}_0)$.

2.2 Harish-Chandra's Plancherel formula

We recall the notation and classical results we will need about the Plancherel formula. In this section q is a prime power (or at least a real number of absolute value strictly greater than 1). The formal variable $\mathbf{q}$ will not appear in this section.

2.2.1 Tempered and discrete series representations

Definition 27. A smooth representation π of G is *tempered* if π is unitary, of finite length, and all matrix coefficients of π belong to $L^{2+\epsilon}(G/Z(G))$ for all $\epsilon > 0$. We write $\mathscr{M}_t(G)$ for the category of tempered representations of G.

In particular, the central character χ of such a representation takes values in the circle group $\mathbb{T} \subset \mathbb{C}^\times$.

Example 9. A principal series representation $i_B^G(\chi)$ where χ is a unitary character of the maximal torus is tempered. A principal series induced from a nonunitary character will not have unitary central character, and is not tempered.

Definition 28. A representation ω of G belongs to the *discrete series* if ω is square-integrable modulo $Z(G)$. We write $\mathscr{E}_2(G)$ for the space of unitary discrete series, and $\mathscr{E}_2(G)^I$ for the space of unitary discrete series with nontrivial Iwahori-fixed vectors.

Formal degrees of discrete series representations

We will soon study the Plancherel decomposition $f = \sum_M f_M$ of the Schwartz function f determined by an element of J as explained in Section 2.2.4. As will be explained below, each function f_M is given by an integral formula that involves several constants depending on M, or are functions on the discrete series of M. These constants are rational functions of q, the most sensitive of which is the formal degree $d(\omega)$ of $\omega \in \mathscr{E}_2(M)^I$. Much is known about formal degrees for I-spherical ω, and we summarize the known formulas here, rewriting Macdonald's formulas to emphasize that all the denominators appearing divide a power of Poincaré polynomial of W.

Theorem 23 ([12])**.** *Let ω be a discrete series representation of G such that $\dim \omega^I = 1$. Then a complete list of formal degrees of such $\omega \neq \mathrm{St}_G$ is, in the notation of* [51]*:*

1. *For $\mathbf{G}$ of type B_ℓ ($\ell \geq 3$), $d(\omega)$ is equal to*

$$\tilde{W}_{B_\ell}(q^{-1}, q)^{-1} = \pm \frac{q^\ell(1+q^\ell)(1-q^{2-\ell})(1-q^{3-\ell})}{(1+q)(1+q+q^2+\cdots q^{2\ell-1})} \prod_{i=1}^{\ell-1} \frac{q^{2i-1}}{1+q+q^2+\cdots+q^{2i-1}} \prod_{i=2}^{\ell-1} \frac{q^{2i-2}(1-q^{1-\ell})(1-q^{2-\ell+i})}{1-q^{2i-2}}$$

2. *For $\mathbf{G}$ of type C_ℓ ($\ell \geq 2$), $d(\omega)$ is equal to either*

   (a)

$$\tilde{W}_{C_\ell}(q^{-1}, q^{-1}, q)^{-1} = \tilde{W}_{C_\ell}(q^{-1}, q, q^{-1})^{-1} = \prod_{i=0}^{\ell-1} \frac{(1-q^{-1})(1-q^{-\ell-i+1})}{(1-q^{-i-1})(1+q^{-i-1})(1+q^{-i+1})};$$

   or, occurring only for $\ell \geq 4$,

   (b)

$$\tilde{W}_{C_\ell}(q^{-1}, q, q)^{-1} = \prod_{i=0}^{\ell-1} \frac{(1-q^{-1})(1+q^{-\ell-i+3})}{(1-q^{-i-1})(1+q^{-i+1})(1+q^{-i+1})}.$$

3. *For* $\mathbf{G}$ *of type* F_4*,* $d(\omega)$ *is equal to*

$$\tilde{W}_{F_4}(q^{-1}, q)^{-1} = \prod_{i=0}^{3} \frac{(1-q^{-1})(1-q)}{(1-q^{i-1})(1+q^{-i+1})(1-q^i)}.$$

In each case, the formal degree of the Steinberg representation is exactly the reciprocal of the Poincaré series of $\tilde{W}$*.*

Borel also treated the case G_2, but we will use the formal degree of the unique discrete series representation ω coming from a character of H that appears below. It is well-known that $d(\mathrm{St}_G)$ is exactly the reciprocal of $P_{\mathbf{G}/\mathbf{B}}(q)$, given Theorem 23 and Bott's theorem on the Poincaré series of $\tilde{W}$ [11], Theorem. 7.1.10.

An essentially disjoint set of discrete series representations is studied by Reeder in [57]. A general formula was obtained in *loc. cit.*, but its denominator is given by a complicated expression. Thankfully, Reeder computed some examples using a computer algebra system. Reeder's obtained formulas also in the case of the Steinberg representation, but we omit them here as we included them in list Borel calculated. Reeder points out that his formulas have a geometric interpretation in terms of the complex-dual side, which gives hope that one might understand the zeros at the level of detail we require. We will not pursue this hope in the current thesis.

Theorem 24 ([57])**.** *The examples computed in* [57] *by specializing Formula A of loc. cit. give the following formal degrees of some Iwahori-spherical discrete series representations admitting a Whittaker model:*

1. *If* $\mathbf{G}$ *is of type* $\mathbf{G}_2$*, then a complete list of formal degrees of elements of* $\mathscr{E}_2(G)^I$ *is*

$$d(\tau_1) = \frac{(q^5-1)(q-1)^2}{(q^6-1)(q+1)}, \quad d(\tau_2) = \frac{1}{2}d(\tau_2)' = \frac{q(1+q^3)(q-1)^2}{6(1+q+q^2+q^3+q^4+q^5)(1+q)^2}$$

and

$$d(\tau_3) = \frac{q(q-1)^2(1+q+q^2)}{2(1+q+q^2+q^3+q^4+q^5)(1+q)}, \quad d(\tau_4) = \frac{q(q-1)^2(q+1)}{3(q^6-1)}$$

2. *If* $G = \mathrm{SO}_5(F)$*, then*

$$d(\pm\tau_2) = \frac{q(q-1)^2}{2(q^2+1)(q+1)^2}$$

3. *If* $G = \mathrm{SO}_7(F)$*, then*

$$d(\tau_2) = d(\tau_2)' = \frac{q(q-1)^3}{4(q^2+1)(q+1)^3}$$

and

$$d(\tau_3) = \frac{q(q-1)^2(q^3-1)}{4(q^3+1)(q^2+1)(q+1)}$$

4. *If* $G = \mathrm{SO}_9(F)$*, then in the case of* τ_i *regular (when reviewed as a root of the complex dual group) we have*

$$d(\pm\tau_1) = \frac{(q^4-1)(q^3-1)(q^7-1)(q^2-1)^2(q-1)^2}{2(q^8-1)(q^6-1)(q^4-1)^2(q+1)^3},$$

$$d(\pm\tau_3) = \frac{q(q^5-1)(q-1)^3}{4(q^4+1)(q^3+1)(q+1)^3}$$

and

$$d(\pm\tau_5) = \frac{q^2(q^3-1)^2(q-1)^2}{2(q^4+1)(q^2+1)(q+1)^4}$$

5. *If* $\mathbf{G}$ *is of type* $\mathbf{F}_4$*, then*

$$d(\tau) = \frac{q(q^{10}-1)(q^7-1)(q^3-1)(q-1)^3(1+q+q^2)}{2(q^{12}-1)(q^8-1)(1+q+q^2+q^3+q^4+q^5)(q+1)^2}.$$

Together, Reeder's formulas plus Borel's cover all of $\mathscr{E}_2(G)^I$ for $\mathbf{G}$ of rank at most 3.

Finally, the most general current result seems to be

Theorem 25 ([62], [25], Theorem 5.1 (b), [29] Proposition 4.1)**.** *Let G be connected reductive. Let ω be any unipotent—in particular, any Iwahori-spherical—discrete series representation. Then $d(\omega)$ is a rational function of q, the numerator and denominator of which are products of factors of the form $q^{m/2}$ with $m \in \mathbb{Z}$ and (q^n-1) with $n \in \mathbb{N}$. Moreover, there is a polynomial Δ_G depending only on G and F such that $\Delta_G d(\omega)$ is a polynomial in q.*

Remark 7. Proposition 4.1 of [29] studies not the γ-factor we are interested in, but rather its quotient by the γ-factor for the Steinberg representation. However, accounting for the the use of the Euler-Poincaré measure on G, and known formula for the formal degree of the Steinberg representation, one may recover our desired statement about formal degrees from the main theorem and equation (61) of [29].

This result is proven by first proving that the Hiraga-Ichino-Ikeda conjecture [34] holds for unipotent discrete series representations. Note that [62], [25] and [34] all use the normalization of the Haar measure on G defined in [34]. This normalization gives in our setting $\mu_{\mathrm{HII}}(K) = q^{\dim \mathbf{G}} \#\mathbf{G}(\mathbb{F}_q)$. Hence, noting that $\#\mathbf{G}(\mathbb{F}_q) = P_{\mathbf{G}/\mathbf{B}}(q) \cdot \#\mathbf{B}(\mathbb{F}_q)$ and that, as $\mathbb{F}_q$ is perfect, $\#B(\mathbb{F}_q)$ is a polynomial in q, we have

$$\mu_I = \frac{1}{q^{\dim \mathbf{G}} \#\mathbf{B}(\mathbb{F}_q)} \mu_{HII},$$

and so this question of normalization cannot affect the denominators of $d(\omega)$, for any Levi subgroup.

In the Iwahori-spherical case, Opdam showed the above result in [55], Proposition 3.27 (v), although with less control over the possible factors appearing in the numerator and denominator of $d(\omega)$. We emphasize that *loc. cit.* does not make the splitness assumption we have temporarily allowed ourselves above.

2.2.2 Harish-Chandra's canonical measure

In this section, we recall the standard coordinates used in [5], [66], [32]. Everything in this section is standard, and we have attempted to choose no eccentric notation of our own. We state the below for general G, which we shall apply to each standard Levi subgroup of GL_n.

Let $\mathfrak{a}_G := (X^*(A) \otimes_{\mathbb{Z}} \mathbb{R})^*$ be the real Lie algebra of A and let $\mathfrak{a}_{G\mathbb{C}}$ be its complexification. Let $\mathrm{Rat}(G) = \mathrm{Hom}(G(F), \mathbb{G}_{\mathrm{m}}(F))$ denote the rational characters of G defined over F. This set has a complex manifold structure under which $X(G) \simeq (\mathbb{C}^\times)^{\dim_{\mathbb{R}} \mathfrak{a}_G}$. We have a map

$$\mathrm{Rat}(G) \to \mathrm{Hom}(G, \mathbb{C}^\times)$$

given by $\chi \mapsto |\chi|_F$, where $|\chi|_F(g) = |\chi(g)|_F$. Set $G^1 = \bigcap_{\chi \in \mathrm{Rat}(G)} \ker |\chi|_F$. We have

$$\mathfrak{a}_{G\mathbb{C}}^* \longrightarrow X(G) \longrightarrow 1$$

given by

$$\chi \otimes s \mapsto (g \mapsto |\chi(g)|_F^s = q^{-s\,\mathrm{val}(\chi(g))}).$$

The kernel is all χ such that $s\,\mathrm{val}(\chi(g)) \in \frac{2\pi}{\log q}\mathbb{Z}$. Hence the kernel is $\frac{2\pi}{\log q}R$, where $R \subset X^*(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a lattice. Additionally, we have a morphism $H_G : G \to \mathfrak{a}_G$ defined by determining $\langle \chi, H_G(g) \rangle$ for all $\chi \in \mathrm{Rat}(G)$. One does this by requiring that

$$q^{-\langle \chi, H_G(g) \rangle} = |\chi|_F(g) = |\chi(g)|_F = q^{-\mathrm{val}(\chi(g))},$$

so that $R = \ker H_G$. Put $L = H_G(G)$. Then $R = L^*$ is the dual lattice, and set $\mathfrak{o}_{\mathbb{C}} := \mathfrak{a}_{G\mathbb{C}}^*/R$. Thus $\mathfrak{o}_{\mathbb{C}}$ is a $\mathbb{C}$-manifold. Let $\mathrm{d}\nu$ be the Euclidean measure on $\mathfrak{a}_G^*$ such that $\mathfrak{a}_G^*/R$ has volume 1.

This construction defines the *canonical* measure $\mathrm{d}\omega$ on the space $\mathscr{E}_2(M)$ of unitary discrete series, for each Levi subgroup M of G.

Example 10. If $G = \mathrm{SL}_2(F)$ and $M = A$ is the diagonal torus, we have $\mathfrak{a}_G^* \simeq \mathbb{R}$ and $R = \frac{2\pi}{\log q}\mathbb{Z}$ so that a fundamental domain for $\mathfrak{a}_G^*/R$ is $\left[-\frac{\pi}{\log q}, \frac{\pi}{\log q}\right)$ and $d\nu = \frac{\log q}{2\pi}\mathrm{d}x$, where $\mathrm{d}x$ is the Lebesgue measure. To obtain quasicharacters of G, we associate to $\nu \in \mathfrak{a}_{G\mathbb{C}}$ the quasicharacter $\chi_\nu(g) = q^{i\langle \nu, H_G(g) \rangle}$.

Therefore to compute the integral of a function f on $\mathscr{E}_2(G)$ supported on the unramified unitary characters of A, we compute

$$\int_{\mathscr{E}_2(A)} f(\omega)\,\mathrm{d}\omega = \int f(\chi_\nu)\,\mathrm{d}\nu = \frac{\log q}{2\pi} \int_{\frac{-\pi}{\log q}}^{\frac{\pi}{\log q}} f(e^{it\log q})\,\mathrm{d}t = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(z)}{z}\,\mathrm{d}z$$

if $t \mapsto e^{it\log q} = q^s =: z$ (here $s = it$) parameterizes the compact complex torus $\mathbb{T}$.

In general, $\mathscr{E}_2(M)$ is the disjoint union of compact tori, and the components of the Bernstein variety are quotients of these compact tori by certain symmetric groups, namely, the Weyl groups of M. For a detailed description of this complex structure, see [31], [32], and [5]. The $\mathscr{E}_2(M)^I$ is finite up to twist by unramified characters, by a result of Harish-Chandra [66].

We can now state the classification of the tempered representations of G. This is due to Harish-Chandra, but more accessibly appears as Proposition III.4.1 of [66].

Theorem 26 (Harish-Chandra)**.** *Let P be a parabolic subgroup of G with Levi subgroup M, let $\sigma \in \mathscr{E}_2(M)$, and let ν be a unitary character of M. Then $\mathrm{Ind}_P^G(\sigma \otimes \nu)$ is tempered, and is irreducible for generic ν. Even irreducible tempered representation is of this form.*

Note that the underlying vector space of $\mathrm{Ind}_P^G(\sigma \otimes \nu)$ can be taken to be independent of ν, and hence it makes sense to define endomorphisms of $\mathrm{Ind}_P^G(\sigma \otimes \nu)$ or $\mathrm{Ind}_P^G(\sigma \otimes \nu)^I$ that depend on ν. We will discuss this further in Section 2.2.5.

Remark 8. It is also possible to describe admissible representations as quotients of tempered representations, although we will not pursue this. This is the *Langlands classification* of admissible representations, and is due to Langlands for real groups. It is stated by Sillberger in [59], but with an incorrect proof. An alternative proof was given much later by Konno in [39].

2.2.3 The Harish-Chandra Schwartz algebra

Let $\mathscr{C} = \mathscr{C}(G)$ be the Harish-Chandra Schwartz algebra of G; see [31], [66], or [15] for the definition and associated notation. We will follow the formulation given in [15], which we now recall.

Write $G = KAK$ where K is a maximal compact subgroup of G and A is a maximal torus. We can write any $g \in G$ as $g = k_1\pi^{\lambda(g)}k_2$, where $k_1, k_2 \in K$ and $\lambda(g)$ is a dominant coweight depending on g Define $\Delta(g) = q^{\langle\lambda,\rho\rangle}$, where ρ is the half-sum of positive roots. The *Harish-Chandra Schwartz space* is then the space of functions $f : G \to \mathbb{C}$ such that f is bi-invariant with respect to some open compact subgroup, and such that for all polynomial functions $p : G \to F$ and $m > 0$, we have

$$\Delta(g)|f(g)| \leq \frac{C}{(\log(1 + |p(g)|))^m}$$

for some constant C depending on m and p.

In particular, we will record for future comparison with the argument in the proof of Proposition 2 that

$$q^{\frac{\ell(w)}{2}} q^{-\#W} \leq \Delta(IwI) \leq q^{\frac{\ell(w)}{2}}. \tag{2.2}$$

Let $\mathscr{C}^{I\times I}$ the subalgebra of Iwahori-biinvariant functions. As explained in [66], the Fourier transform $f \mapsto \pi(f)$ defines an endomorphism of π for every $f \in \mathscr{C}$ and every tempered representation π. We recall the precise construction now. Let π be a tempered representation and let $\tilde{\pi}$ be its smooth dual. Then for any matrix coefficient $g \mapsto \langle \tilde{v}, \pi(g)v\rangle$ of π and any Schwartz function f, one can show that the integral

$$\int_G f(g)\, \langle \tilde{v}, \pi(g)v\rangle \, \mathrm{d}g \tag{2.3}$$

is absolutely convergent, and thus that there exits a vector denoted $\pi(f)v$ in π with the property that

$$\langle \tilde{v}, \pi(f)v\rangle$$

is equal to (2.3). Moreover, it follows immediately that

$$\langle \tilde{v}, \pi(f_1 \star f_2)v\rangle = \langle \tilde{v}, \pi(f_1)\pi(f_2)v\rangle\,,$$

and so (2.3) defines an action of the Schwartz algebra.

The Plancherel formula is the statement [15] that this assignment defines an isomorphism of rings

$$\mathscr{C} \to \mathscr{E}_t(G),$$

where $\mathscr{E}_t(G)$ is the subring of the endomorphism ring of the forgetful functor $\mathscr{M}_t(G) \to \mathbf{Vect}_{\mathbb{C}}$ defined by the following conditions:

1. For all $\pi = \mathrm{Ind}_P^G(\nu \otimes \omega)$, the endomorphism $\eta_\pi = \eta_{\nu,\omega}$ is a smooth function of ν;

2. The endomorphism η_π is biinvariant with respect to some open compact subgroup of G.

For computational purposes such as ours, we require this isomorphism to be explicit. In the Iwahori-spherical case, harmonic analysis on $\mathscr{C}^{I\times I}$ can be phrased internally to H and various completions of H. In this setting Opdam gave an explicit Plancherel formula in [55]. In more general settings there are explicit formulas for $\mathrm{GL}_n(F)$ and $\mathrm{Sp}_4(F)$, which we will also make use of.

We have the obvious inclusion $\iota\colon H \hookrightarrow \mathscr{C}^{I\times I}$.

2.2.4 The algebra J as a subalgebra of the Schwartz algebra.

In [15], Braverman and Kazhdan constructed a map of $\mathbb{C}$-algebras $J \to C^{I\times I}$. We shall review this construction now. Following *loc. cit.*, let $\mathscr{E}_J^I(G)$ denote the subring of $\mathscr{E}_t(G)$ defined by the following conditions on the endomorphisms η_π:

1. For all $\pi = \operatorname{Ind}_P^G(\nu \otimes \omega)$, the endomorphism $\eta_\pi = \eta_{\nu,\omega}$ is a rational function of ν, regular on the set of non-strictly positive ν (see Definition 29).

2. The endomorphism ν_π is $I \times I$-biinvaraint.

Theorem 27 ([15], Theorem 1.8)**.** *Let G be a connected split reductive group over F. Then the following statements hold*

1. *Let (π, V) be a tempered representation of G. Then the action of H on V^I extends uniquely to J.*

2. *Let $P = MN$ be a Levi subgroup of G and ω an irreducible tempered representation of M. Let χ^{-1} be a non-strictly positive character of M and let $(\pi, V) = \operatorname{Ind}_P^G(\omega \otimes \chi)$. Then the action of $H(G, I)$ on V^I extends uniquely to an action of J.*

3. *The map $t_w \mapsto (\eta_\pi(w))_{\pi\in\mathscr{M}_t(G)}$ defines an isomorphism between J and the ring $\mathscr{E}_J^I(G)$.*

Composing this isomorphism with the inverse Fourier transform, Braverman and Kazhdan define an algebra inclusion

$$\tilde{\phi}\colon J \hookrightarrow \mathscr{C}^{I\times I}$$

sending

$$t_w \mapsto (\eta_\pi(w))_{\pi\in\mathscr{M}_t(G)} \mapsto f_w = \sum_{x\in\tilde{W}} A_{w,x} T_x \in \mathscr{C}^{I\times I},$$

where $A_{w,x} = f_w(IxI)$. By definition, $\eta_\pi(w) = \pi(f_w)$ as endomorphisms of π.

We will show later that $\tilde{\phi}$ is essentially the map ϕ^{-1}.

Remark 9. There is a gap in the proof of injectivity of the map $t_w \mapsto (\eta_\pi(w))_{\pi\in\mathscr{M}_t(G)}$ in [15]. However, we shall prove injectivity by proving injectivity of $\tilde{\phi}$ in Corollary 4, using only the fact that $\tilde{\phi}$ restricted to $\phi(H)$ is equal to the inclusion of the Iwahori-Hecke algebra into the Schwartz algebra.

Implicit in [15] is

Lemma 2. *We have a commutative diagram*

$$\begin{array}{ccc} J & \xrightarrow{\tilde{\phi}} & \mathscr{C}^{I\times I} \\ \phi_q \uparrow & & \uparrow \iota \\ H & \xleftarrow[()^\dagger]{} & H. \end{array}$$

Proof. Let π be a tempered representation of G. By [15], the H-action on it extends to a J-action, and by Corollary 2.6 of *loc. cit.*, we have $\eta_\pi(\phi_q(f^\dagger)) = \pi(f)$ for any $f \in H$. Therefore $\tilde{\phi} \circ \phi_q(f^\dagger) = f$ by Theorem 21 and the Matrix Paley-Wiener theorem. □

Remark 10. The presence of the Goldman involution is necessary for tempered representations of J to restrict to tempered representations of H. For example, in [15], Braverman and Kazhdan showed for $G = \mathrm{SL}_2(F)$, that $\mathscr{E}_J^I(G) = \mathrm{End}(\mathrm{St}^I) \oplus \mathrm{End}_{\Phi,X_*}(\mathscr{S}^I)$, matching the decomposition $J = \mathbb{C}t_1 \oplus J_0$, where Φ is Fourier transform on the Schwartz space of the basic affine space $\mathscr{S}$, and St is the Steinberg representation. In this case J has a unique one-dimensional representation, whereas H has one tempered one-dimensional representation St and one non-tempered one-dimensional representation, the trivial representation. It is easy to compute using the formulas of [23] that $\phi_q(T_s)t_1 = qt_1$, whereas $\phi_q(T_s^\dagger)t_1 = -t_1$. Thus for the Steinberg representation to restrict to the Steinberg representation, the twist is required.

2.2.5 The Braverman-Kazhdan perspective on J

We are now ready to give an exposition of the perspective on J developed in [15].

Paley-Wiener Theorems

The mapping $f \mapsto \pi(f)$ is of course a Fourier transform, which, as expected, converts between decay properties and regularity properties. Via the Plancherel formula, this allows one to define classes of functions on G by prescribing the regularity of $\pi(f)$ for π tempered as a function of the twisting character, as mentioned at the end of Section 2.2.2. As one would expect, support properties of f are converted to regularity properties of $\pi(f)$. *Payley-Wiener theorems* are statements making this conversion precise.

Endomorphism rings

Consider the forgetful functor

$$\mathrm{Forg}\colon \mathscr{M}(G) \to \mathbf{Vect}_{\mathbb{C}}$$

and its endomorphism ring

$$\tilde{\mathscr{E}} = \mathrm{End}(\mathrm{Forg})$$

Thus an element of $\tilde{\mathscr{E}}$ is the data of a linear endomorphism η_π of each (π, V) intertwining all intertwining operators. There is a $G \times G$ action on $\tilde{\mathscr{E}}$ given by $(g_1, g_2)\eta_\pi(v) = g_1\eta_\pi(g_2^{-1}v)$.

Definition 29 ([15], §1.7)**.** Let $P = M_P N_P$. A character $\chi\colon M_P \to \mathbb{C}^\times$ of M_P is *non-strictly positive* if for all root subgroups $U_\alpha \subset N_P$, we have $|\chi(\alpha^\vee(x))|_\infty \geq 1$ for $|x|_F \geq 1$.

Definition 30. We say a non-strictly positive character χ is *strictly positive* if for all root subgroups $U_\alpha \subset N_P$, we have $|\chi(\alpha^\vee(x)|_\infty > 1$ for $|x|_F \geq 1$.

Define $\mathscr{E}(G)$ to be the subring of $\tilde{\mathscr{E}}(G)$ such that

1. For all Levi subgroups $\mathbf{M}$ of $\mathbf{G}$, admissible representations σ of M, and unramified character χ, the endomorphism $\eta_{\mathrm{Ind}_P^G(\chi\otimes\sigma)}$ is a *regular* function of χ
2. There is an open compact K such that η_π is $K \times K$-invariant for all π.

One can further consider the subring of $\mathscr{E}^{I\times I}$ of I-biinvariant endomorphisms, and get endomorphisms of the forgetful functors

$$\mathrm{Forg}\colon \mathbf{Rep}(\mathbf{H}) \to \mathbf{Vect}_{\mathbb{C}}.$$

Passing to tempered representations, one can consider the forgetful functor

$$\mathrm{Forg}_t \colon \mathscr{M}_t(G) \to \mathbf{Vect}_{\mathbb{C}}$$

and ring of endomorphisms

$$\mathscr{E}_t = \left\{ \eta \in \mathrm{End}(\mathrm{Forg}_t) \,\middle|\, \eta(\mathrm{Ind}_P^G(\sigma \otimes \chi)) \text{ is a smooth fn. of } \chi,\ \eta \text{ is } K \times K \text{ invt. for some K} \right\},$$

where K is open compact and we consider unitary characters χ. Braverman and Kazhdan in [15] then defined a subring $\mathscr{E}_J \subset \mathscr{E}_t$ defined by the following condition on η_π: we require that η_π extends to a rational function $E_{\mathrm{Ind}_P^G(\sigma\otimes\chi)} \in \mathrm{End}_{\mathbb{C}}(\mathrm{Ind}_P^G(\sigma \otimes \chi))$ of χ, which is regular for χ^{-1} non-strictly positive.

Theorem 28 ([15]). *There is a commutative diagram*

$$\begin{array}{ccccc}
H & & \hookrightarrow & & \mathscr{C}^{I\times I} \\
\downarrow \sim & \overset{\phi_q}{\searrow} & & \overset{\tilde\phi}{\nearrow} & \downarrow \sim \\
 & & J & & \\
 & & \downarrow & & \\
\mathscr{E}^I & \hookrightarrow & \mathscr{E}_J^I & \hookrightarrow & \mathscr{E}_t^I,
\end{array}$$

where the outermost vertical maps are given by $f \mapsto \pi(f)$, and the uppermost inclusion is the natural one.

Moreover, if one defines $\mathscr{J}(G)$ as the preimage

$$\begin{array}{ccc}
\mathscr{J}(G) & \hookrightarrow & \mathscr{C} \\
\downarrow \sim & & \downarrow \sim \\
\mathscr{E}_J & \hookrightarrow & \mathscr{E}_t,
\end{array}$$

then there is a diagram

$$\begin{array}{ccccc}
C_c^\infty(G) & \hookrightarrow & \mathscr{J}(G) & \hookrightarrow & \mathscr{C}(G) \\
\downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
\mathscr{E} & \hookrightarrow & \mathscr{E}_J & \hookrightarrow & \mathscr{E}_t.
\end{array}$$

The isomorphisms are of the form $f \mapsto \pi(f)$.

2.2.6 The Plancherel formula for GL_n

For $G = \mathrm{GL}_n(F)$, we have access to an explicit Plancherel measure and its Bernstein decomposition, thanks to [5].

Recall that for $G = \mathrm{GL}_n(F)$, we have bijections

$$\begin{aligned}
\{\text{partitions of } n\} &\leftrightarrow \{\text{Standard Levi subgroups } M \text{ of } \mathrm{GL}_n(F)\} && (2.4)\\
&\leftrightarrow \{\text{unipotent conjugacy classes in } \mathrm{GL}_n(\mathbb{C})\} && (2.5)\\
&\leftrightarrow \mathscr{N}/\mathrm{GL}_n(\mathbb{C}) \\
&\leftrightarrow \{2-\text{sided cells } \mathbf{c} \text{ in } \tilde{W}\} && (2.6)\\
&\leftrightarrow \{\text{direct summands } J_{\mathbf{c}} \text{ of } J\}
\end{aligned}$$

where (2.4) $\leftrightarrow$ (2.5) sends a unipotent conjugacy class u to the standard Levi M such that a member of u is distinguished in M, and (2.5) $\leftrightarrow$ (2.6) is Lusztig's bijection from [44].

Let $P = M_P N_P$ be a parabolic subgroup of G and let $\mathscr{O}$ be an orbit in $\mathscr{E}_2(M_P)$ under the action of the unitary unramified characters of M_P as explained in Section 2.2.2. Write $W_{M_P} \subset W$ for the finite Weyl group of (M_P, A_P). Let $\mathrm{Stab}_{W_M}(\mathscr{O})$ be the stabilizer of $\mathscr{O}$. Recall that a parabolic subgroup of G is said to be *semistandard* if it contains A. Then the Plancherel decomposition reads

$$f = \sum_{(P=M_PN_P,\mathscr{O})/\text{association}} f_{M_P,\mathscr{O}}$$

where $f \in \mathscr{C}(G)$, the sum is taken over semistandard parabolic subgroups P up to association, and

$$f_{M_P,\mathscr{O}}(g) = c(G/M)^{-2}\gamma(G/M)^{-1}\#\mathrm{Stab}_{W_M}(\mathscr{O})^{-1}\int_{\mathscr{O}} \mu_{G/M}(\omega)d(\omega)\,\mathrm{trace}\,\pi(R_g(f))\,\mathrm{d}\omega,$$

where $R_g f(x) = f(xg)$ is the right translation of f and $\pi = \mathrm{Ind}_P^G(\nu\otimes\omega)$ is the normalized parabolic induction of the twist of ω by a unitary character ν of $Z(M)$. In [5], each term above is explicitly calculated as a rational function of q.

Lemma 3. *There is a finite set $S = \left\{(M,\omega)\,\middle|\,\omega \in \mathscr{E}_2(M)^I\right\}$ such that $\mathrm{trace}(\pi(f))$, where $\pi = \mathrm{Ind}_P^G(\chi\otimes\omega)$, is supported on π of the form $\pi = \mathrm{Ind}_P^G(\chi\otimes\omega)$ for $\omega \in S$, for all I-biinvaraiant Schwartz functions f.*

Proof. This is entirely standard. As $\mathscr{E}_2(M)^{I_M}$ is finite for every M, and there are finitely-many standard parabolics of G, we need only show that $\mathrm{Ind}_P^G(\chi\otimes\omega)^I \neq 0$ only if $\omega^{I_M} \neq 0$, where I_M is the Iwahori subgroup of the reductive group M relative to the Borel subgroup $M(\mathbb{F}_q)\cap B(\mathbb{F}_q)$ of $M(\mathbb{F}_q)$. Note that I_M is naturally a subgroup of I. For any representation σ of M, if $f \in \mathrm{Ind}_P^G(\sigma)$ is I-fixed, then for $i_M \in I_M$, we have

$$f(gi_M) = \sigma(i_M)\delta_P^{\frac{1}{2}}(i_M)f(g) = f(g),$$

and so if $\delta_P^{\frac{1}{2}}(i_M) = 1$, then it must be that $f(g) \in \sigma^{I_M}$. The Iwahori factorization $I_M = N_P^-(\pi\mathscr{O})A(\mathscr{O})N_M(\mathscr{O})$ and the fact that $\mathrm{Ad}_{\mathfrak{n}_P}$ is a representation implies that

$$\delta_P^{\frac{1}{2}}(i_M) = |\det\mathrm{Ad}_{\mathfrak{n}_P}(N_P^-(i_M)A(i_M)N_M(i_M))|_F = |\det\mathrm{Ad}_{\mathfrak{n}_P}(A(i_M))|_F = 1$$

by unipotence of the groups N_P^- and N_M and the fact that the image of i_M in $M(\mathbb{F}_q)$ lies in the Borel. □

Definition 31. Let u be a unipotent element of a reductive group M over the complex numbers. Then u is *distinguished in M* if $Z_M(u)$ contains no nontrivial torus.

Proposition 1 (c.f. [55] Proposition 8.3)**.** *The Plancherel decomposition is compatible with the decomposition $J = \bigoplus_{\mathbf{c}} J_{\mathbf{c}}$ in the sense that if $w \in \mathbf{c}$, $f = f_w$ and $u = u(\mathbf{c})$ under Lusztig's bijection, then $f_M \neq 0$ only for the unique Levi subgroup M such that u is distiguished in $[\mathbf{M}^\vee(\mathbb{C}), \mathbf{M}^\vee(\mathbb{C})]$.*

Proof. Let $\pi := \mathrm{Ind}_P^G(\nu\otimes\omega)$ be a tempered irreducible representation of G induced as usual from a standard parabolic subgroup P with Levi subgroup M. Then π^I is a tempered irreducible H-module, and is of the form $K(u,s,\rho,q)$ for $u, s \in \mathbf{G}^\vee(\mathbb{C})$ and ρ a representation of $\pi_0(Z_G(u,s))$ with

s compact. By [15], Corollary 2.6, $K(u, s, \rho, q)$ extends to a J-module. By definition of Lusztig's bijection, $\pi(f_w) \neq 0$ only if w is in the two-sided cell $\mathbf{c} = \mathbf{c}(u)$ of $\tilde{W}$ corresponding to u. On the other hand, $K(u, s, \rho, q)$ is induced from a square-integrable standard module $K_M(u, s, \tilde{\rho}, q)$ of $H(M, I_M)$. By [37], Theorem 8.3 and Lemma 4, $\mathbf{M}^\vee(\mathbb{C})$ is the unique Levi subgroup such that u is distinguished in $[\mathbf{M}^\vee(\mathbb{C}), \mathbf{M}^\vee(\mathbb{C})]$. Thus $(f_w)_M$ is the only nonzero summand in the Plancherel decomposition of f_w. □

We include the following trivial lemma for completeness, as the proof of the above proposition uses a slightly reformulated version of Theorem 8.3 of [37].

Lemma 4. *Let $\mathbf{M}$ be a reductive group and $M = \mathbf{M}(\mathbb{C})$. Let u be a unipotent element of M. Then u is distinguished in M if and only if for all semisimple elements s of M, $Z_M(s) \cap Z_M(u)$ contains no nontrivial torus.*

Proof. As we have $Z_M(s) \cap Z_M(u) \subset Z_M(u)$, one direction is obvious. For the converse, suppose that $T \subset Z_M(u)$ is a nontrivial torus. By [33], Prop. 16.4, choose $s \in T$ such that $Z_M(T) = Z_M(s)$. Then $Z_M(s) \cap Z_M(u) \supset T$, a contradiction. □

2.2.7 Plancherel measure for GL_n

We refer to [5], §5, for a summary of the Bernstein decomposition of the tempered irreducible representations of GL_n, in particular we use the description in *loc. cit.* of the Bernstein component parameterizing the I-spherical representations of G.

As we shall be applying the Plancherel formula only to Iwahori-biinvariant functions, it suffices to consider only irreducible tempered representations with Iwahori-fixed vectors. For GL_n, the only such representations are of the form

$$\pi = \mathrm{Ind}_P^G(\chi_1 \mathrm{St}_1 \boxtimes \cdots \boxtimes \chi_k \mathrm{St}_k)$$

where St_i is the Steinberg representation of GL_i, and $P \supset M = \mathrm{GL}_{l_1} \times \cdots \times \mathrm{GL}_{l_k}$.

These representations are parameterized as follows. Let M be a Levi subgroup corresponding to the partition $l_1 + \cdots + l_k = n$, and let $\mathbb{T}^n$ be the rank n compact complex torus. Define $\gamma \in \mathfrak{S}_n$ by $\gamma = (1 \dots l_1)(1 \dots l_2) \cdots (1 \dots l_k)$, so that the fixed-point set $(\mathbb{T}^n)^\gamma = \{(z_1, \dots, z_1, \dots, z_k, \dots, z_k)\} \simeq \mathbb{T}^k$. Then the irreducible tempered representations with Iwahori-fixed vectors induced from M are parameterized by the compact orbifold $(\mathbb{T}^n)^\gamma / Z_{\mathfrak{S}_n}(\gamma)$. For Iwahori-biinvariant Schwartz functions, the Plancherel formula is known explicitly for all connected reductive groups, and is due to Opdam in [55]. In the case of $G = \mathrm{GL}_n(F)$, we shall refer instead to [5] (where in fact the Plancherel formula is computed explicitly in its entirety for GL_n). In the case $\mathbf{G} = \mathrm{Sp}_4$, we shall refer to the unpublished work [4] of Aubert and Kim.

Theorem 29 ([5], Remark 5.6)**.** *Let $G = \mathrm{GL}_n$ and $M = \mathrm{GL}_{l_1} \times \cdots \times \mathrm{GL}_{l_k}$ be a Levi subgroup. Then the Plancherel measure of H on $X^\gamma / Z(\gamma)$ is*

$$d\nu_H(\omega) = \prod_{i=1}^{k} \frac{q^{l_i^2 - l_i}(q-1)^{l_i}}{l_i(q^{l_i} - 1)} \cdot q^{\frac{n-n^2}{2}} \cdot \prod_{(i,j,g)} \frac{q^{2g+1}(z_i - z_j q^g)(z_i - q^{-g} z_j)}{(z_i - z_j q^{-g-1})(z_i - q^{g+1} z_j)},$$

where the tuples $(i,j,g) \in \mathbb{Z} \times \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$ *are tuples such that* $0 \leq i < j \leq k$ *and* $|g_i - g_j| \leq g \leq g_i + g_j$*, where* $g_i = \frac{l_i - 1}{2}$*.*

This is the measure that we will integrate against, by successively applying the residue theorem. When carrying out explicit calculations, we will usually elide the constant

$$\prod_{i=1}^{k} \frac{q^{l_i^2 - l_i}(q-1)^{l_i}}{l_i(q^{l_i}-1)} \cdot q^{\frac{n-n^2}{2}}$$

as it depends only on M. We shall abbreviate

$$\Gamma_{i,j,g} := q^{-2g-1} \left|\Gamma_F(q^{-g}\frac{z_i}{z_j})\right|^2 = \frac{(z_i - q^g z_j)(z_i - q^{-g} z_j)}{(z_i - z_j q^{-g-1})(z_i - q^{g+1} z_j)},$$

and recall that, as noted in the proof of Theorem 5.1 in [5], the function $(z_1, \dots, z_k) \mapsto \prod_{(i,j,g)} \Gamma_{i,j,g}$ is $Z_{\mathfrak{S}_n}(\gamma)$-invariant. Hence for the purposes of integration, we may allow ourselves to integrate simply over $\mathbb{T}^k$. Moreover, there are many cancellations between the $\Gamma_{i,j,g}$ for a fixed pair $i < j$ as g varies. Indeed, putting $q_{ij} = q^{|g_i - g_j|}$, $q^{ij} = q^{g_i + g_j + 1}$, and

$$\Gamma^{ij} := \frac{(z_i - q_{ij} z_j)(z_i - (q_{ij})^{-1} z_j)}{(z_i - q^{ij} z_j)(z_i - (q^{ij})^{-1} z_j)}$$

we have

$$\prod \Gamma_{i,j,g} = \Gamma^{ij},$$

where the product is taken over all integers g appearing in triples (i,j,g) for $i < j$ fixed. We set

$$c_M := \prod_{(i,j,g)} q^{2g+1} \cdot \prod_{i=1}^{k} \frac{q^{l_i^2 - l_i}(q-1)^{l_i}}{l_i(q^{l_i}-1)} \cdot q^{\frac{n-n^2}{2}},$$

where the first product is taken over (i,j,g) such that $1 \leq i < j \leq k$ and $|g_i - g_j| \leq g \leq g_i + g_j$.

2.2.8 Beyond type A: the Plancherel formula following Opdam

Beyond type A, we still have available Opdam's explicit Plancherel formula for the Iwahori-Hecke algebra [55]. In certain cases, more precise knowledge of formal degrees is available than the general facts given in *loc. cit.*

Let $\mathbf{G}$ be a connected reductive algebraic group of Dynkin type other than type A (to avoid redundancy) and $G = \mathbf{G}(F)$. Let f be an Iwahori-biinvariant Schwartz function on G and let M be a Levi subgroup of G. Given a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, let $R_{1,+}$ and $R_{P,1,+}$ be defined as in [55], Section 2.3. Recall that the group of unramified characters of a Levi subgroup M has the structure of a complex torus, and is in fact a maximal torus of $\mathbf{M}^\vee(\mathbb{C})$. In particular, if P is a parabolic subgroup and α is a root of (M_P, A_P), then it makes sense to write $\alpha(\nu)$ for any unramified character ν of M. Then, altering Opdam's notation to match our own from Section 2.2.6, the Plancherel formula reads

Theorem 30 ([55] Thm. 4.43)**.**

$$f_{M,\mathscr{O}}(1) = \frac{q^{-\ell(w^P)}}{\#\mathrm{Stab}_{W_M}(\mathscr{O})}\int_{\mathscr{O}} d(\omega) \prod_{\alpha^\vee \in R_{1,+}\setminus R_{P,1,+}} \frac{|1-\alpha^\vee(\nu)|^2}{|1+q_\alpha^{\frac{1}{2}}\alpha^\vee(\nu)|^2|1-q_\alpha^{\frac{1}{2}}q_{2\alpha}\alpha^\vee(\nu)^{1/2}|^2}\,\mathrm{trace}(\pi(f))\,d\omega \tag{2.7}$$

where $\pi = \mathrm{Ind}_P^G(\omega\otimes\nu)$, $P \supset M$, and where q_α and $q_{2\alpha}$ are powers of q, and w^P is the longest element in the complement W^P to the parabolic subgroup W_P of W.

Note that whenever $q_{2\alpha}=1$, which holds whenever $\alpha^\vee \notin 2X_*$, the factor for α reduces to

$$\frac{(z_i - z_j^{\pm 1})^2 q_\alpha^{-1}}{(z_i - q_\alpha^{-1} z_j^{\pm 1})(z_i - q_\alpha z_j^{\pm 1})}.$$

In types A (as we have used above) and D, this simplification always occurs. In types B and C, it happens for all roots except $\alpha = 2\epsilon_i \in R_{1,+}(B_n)$ and $\alpha = 4\epsilon_i \in R_{1,+}(C_n)$, where ϵ_i is the character $\mathrm{diag}(a,\dots,a_n)\mapsto a_i$.

2.3 Regularity of the trace

In order to extract information about the expansion of the elements t_w in terms of the basis T_x via the Plancherel formula, we must pause briefly to establish a regularity property of trace $\pi(t_w)$ where π is an irreducible tempered representation of G. The needed property follows trivially from Theorem 27.

Recall the definition of non-strictly positive characters. Of course, it suffices to test this for $x = \varpi^{-1}$. The condition for χ^{-1} to be non-strictly positive is then that $|\chi(\alpha^\vee(\varpi))|_\infty \geq 1$. If ν corresponds to the vector $(z_1,\dots,z_n)\in\mathbb{C}^n$, then the condition that ν^{-1} is non-strictly positive translate to $|z_n| \geq |z_{n1}| \geq \dots \geq |z_1|$. Such conditions divide $\mathbb{C}^n$ into chambers, on which the Weyl group $\mathfrak{S}_n$ clearly acts simply-transitively. Interior points correspond to ν such that ν^{-1} is strictly positive.

2.3.1 Intertwining operators

The goal is to use the property that elements of $\mathscr{C}_J^I(G)$ commute with all intertwining operators, and regularity of the trace for unitary and non-strictly positive characters of M to deduce regularity of the trace at all characters of M.

Let ω be a discrete series representation of a Levi subgroup M of G, and let ν be any unramified character of M, not necessarily unitary. Then we may form the tempered representation $\pi = \mathrm{Ind}_P^B(\nu\otimes\omega)$ of G, where P is a parabolic with Levi factor M. We will now recall some well-known facts about the action of the Weyl group of M on such representations π. Let θ and θ' be two subsets of Δ corresponding to Levi subgroups M and M'. Let $w\in W$ be such that $w\theta=\theta'$. Then there is an intertwining operator

$$J_{P|P'}(\omega,\chi)\colon\ \mathrm{Ind}_P^B(\nu\otimes\omega)\to\mathrm{Ind}_{P'}^B(\nu\otimes\omega)$$

for each $w \in W(\theta, \theta') = \{w \in W \mid w(\theta) = \theta'\}$. This set is nonempty only if

$$M_\theta = \mathrm{GL}_{l_1} \times \cdots \times \mathrm{GL}_{l_N} \quad M_{\theta'} = \mathrm{GL}_{l'_1} \times \cdots \times \mathrm{GL}_{l'_N}$$

and $\{l_1, \ldots, l_N\} = \{l'_1 \ldots, l'_N\}$ are equal multisets. In this case , $W(\theta, \theta') \simeq \mathfrak{S}_N$ can be viewed as acting by permuting the blocks of M. It is well-known that $J_{P|P'}(\omega, \nu)$ is a meromorphic function of ν with simple poles. The poles of these operators have been studied by Shahidi in [58], and in the language of modules over the full Hecke algebra by Arthur in [2].

2.3.2 Conventions on parabolic subgroups

In the case $\mathbf{G} = \mathrm{GL}_n$, we adopt the union of the notation of [3], §5 and [58]. Let $\alpha_{ij} \colon \mathrm{diag}(t_i) \mapsto t_i t_j^{-1}$ be characters of T. The α_{ij} for $j > i$ are the positive roots of $(\mathbf{B}, \mathbf{A})$ and $\beta_i := \alpha_{ii+1}$ are our chosen simple roots.

We now recall the notation of [58]. Given a subset θ, we set $\Sigma_\theta = \mathrm{span}_{\mathbb{R}}\, \theta$, and $\Sigma_\theta^+ = \Psi^+ \cap \Sigma_\theta$ and likewise define Σ_θ^-. Let $\Sigma(\theta)$ be the roots of (P, A_P). Define the positive roots $\Sigma^+(\theta)$ to be the roots obtained by restriction of an element of $\Psi^+ \setminus \Sigma_\theta^-$.

Given two subsets $\theta, \theta' \subset \Delta$, following [58] we set

$$W(\theta, \theta') = \{w \in W \mid w(\theta) = \theta'\},$$

and then for $w \in W(\theta, \theta')$, we define

$$\Sigma(\theta, \theta', w) = \left\{[\beta] \in \Sigma^+(\theta) \,\middle|\, \beta \in \Psi^+ - \Sigma_\theta^+ \text{ and } w(\beta) \in \Psi^-\right\},$$

and then

$$\Sigma^\circ(\theta, \theta', w) := \left\{[\beta] \in \Sigma(\theta, \theta', w) \,\middle|\, w_{[\beta]} \in W(A_P)\right\}.$$

Remark 11. In [58] additional care about the relative case is taken in the notation. In our simple case this is of course unnecessary, and we omit it.

If P corresponds to the partition $n_1 + \cdots + n_p$ of n and subset $\theta \subset \Delta$, then $\Sigma(\theta) = \mathrm{span}\,\{\beta_{n_1}, \beta_{n_1+n_2}, \ldots\}$, where we view β_i as restricted to $\mathfrak{a}_P \hookrightarrow \mathfrak{a}_G$. Note that all the positive roots $\Sigma^+(\theta)$ of (P_θ, N_θ) are in N_P. Denoting by $[\alpha]$ the coset representing a root α of G restricted to P, the positive roots in N_P are the α_{ij} such that $[\alpha_{ij}] = [\beta_{n_1+\cdots+n_k}]$.

Example 11. Let $\mathbf{G} = \mathrm{GL}_6$, and P be the parabolic subgroup of block upper-triangular matrices

$$\begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ & & * & * & * & * \\ & & * & * & * & * \\ & & & & * & * \\ & & & & & * \end{pmatrix}$$

corresponding to the partition $6 = 2+2+1+1$ and $\theta = \{\alpha_{12}, \alpha_{34}\}$. We have $A_P = \{\mathrm{diag}(t_1, t_1, t_2, t_2, t_3, t_4)\}$. The positive roots are $\Sigma^+(\theta) = \{[\beta_2], [\beta_4], [\beta_5]\}$. The Weyl group $W(A_P) \simeq \mathfrak{S}_2 \times \mathfrak{S}_2$ acts by permuting the blocks. Note that the simple reflection sending $\beta_4 \mapsto -\beta_4$ does not arise by permuting the blocks (*i.e.* $\begin{pmatrix}4 & 5\end{pmatrix}$ does not send blocks to blocks) , hence $w_{[\beta_4]} \notin W(A_P)$. Hence for any $w \in W(\theta, \theta')$ we have $\Sigma^\circ(\theta, \theta', w) \subseteq \{[\beta_2], [\beta_5]\}$.

We have the following information about the poles, due, according to [58], to Harish-Chandra:

Theorem 31 ([58], Corollary 2.2.2)**.** *Let ω be an irreducible unitary representation of M. Say that ω is a subpresentation of $\operatorname{Ind}_{P_*}^{M}(\omega_*)$ for a parabolic subgroup $P_* = M_*N_*$ and ω_* is an irreducible supercuspidal representation of M_*. Let $\theta_* \subset \Delta$ be such that $P_* = P_{\theta_*}$ as parabolic subgroups of M_*.*

Then the operator

$$\prod_{\alpha \in \Sigma_r^0(\theta_*, w\theta_*, w)} (1 - \chi^2_{\omega,\nu}(h_\alpha))A(\nu, \pi, w)$$

is holomorphic on $\mathfrak{a}_{\theta\mathbb{C}^*}$. Here $\chi_{\omega,\nu}$ is the central character of the twisted representation $\omega \otimes q^{\langle \nu, H_\theta(-)\rangle}$.

In particular for the purposes of the Plancherel formula, the only relevant ω are unitary, hence have unitary central characters. Therefore $A(\nu, \omega, w)$ is holomorphic at ν if $|q^{\langle \nu, H_\theta(-)\rangle}| \neq 1$, or equivalently if

$$\Re(\langle \nu, H_\theta(-)\rangle) \neq 0.$$

In particular, there is a finite union of hyperplanes away from which each operator $A(\nu, \pi, w)$ is holomorphic, for any $w \in W$.

2.3.3 Regularity of the trace

Lemma 5. *Let M be a Levi subgroup of $G = \mathbf{G}(F)$ and let ω be a discrete series representation of M. Let $P = M_P A_P N_P$ be the standard parabolic subgroup containing $M = M_P$ and let $k = \operatorname{rk} A_P$. Let $z_1, \ldots, z_k \in (\mathbb{C}^\times)^k = X^*(A_P) \otimes_{\mathbb{Z}} \mathbb{C}$ define an unramified quasicharacter $\nu = \nu(z_1, \ldots, z_k)$ of A_P as in Section 2.2.2. Let $\pi = \operatorname{Ind}_P^G(\omega \otimes \nu)$. Let $f \in J$. Then*

$$\operatorname{trace}(\pi(f)) \in \mathbb{C}[z_1, \ldots, z_k, z_1^{-1}, \ldots, z_k^{-1}].$$

That is, the trace is a regular function on $(\mathbb{C}^\times)^k$.

Proof. We know *a priori* that

$$\operatorname{trace}(\pi(f)) \in \mathbb{C}(z_1, \ldots, z_k)$$

is a rational function of ν, as the operator $\pi(f)$ itself depends rationally on the variables z_i by Theorem 27. Therefore

$$\operatorname{trace}(\pi(f)) = \frac{p(\nu)}{h(\nu)} \in \mathbb{C}(z_1, \ldots, z_k).$$

By Theorem 31 and the discussion following it, for ν in open subset of $\mathbb{C}^k$, we have, for every each $w \in W$,

$$\frac{p(\nu)}{h(\nu)} = \frac{p(w(\nu))}{h(w(\nu))}. \tag{2.8}$$

Therefore (2.8) actually holds for all ν, *i.e.* $\operatorname{trace}(\pi(f))$ is a W-invariant rational function of ν. When ν is non-strictly positive with respect to M, by Theorem 27, $\operatorname{trace}(\pi(f))$ has poles only of the form $z_i^{n_i} = 0$. The claim now follows from the W-invariance, and thus $\operatorname{trace}(\pi(f))$ is a regular function on $(\mathbb{C}^\times)^k$. ∎

Lemma 6. *Let $d \in \tilde{W}$ be a distinguished involution and π be a tempered representation induced from the Levi corresponding to the two-sided cell containing d in the sense of Proposition 1. Then* $\operatorname{trace}(\pi(f_d))$ *is constant and a natural number.*

Proof. Let $\mathbf{c}$ be the two-sided cell that contains d. We have $\operatorname{trace}(\pi(f_d)) = \operatorname{rank}(\pi(f_d))$, and moreover we have that

$$\sum_{d' \in \mathbf{c}} \operatorname{rank}(\pi(f_{d'})) = \operatorname{rank}\left(\sum_{d' \in \mathbf{c}} \pi(f_{d'})\right) = \dim \pi,$$

where the sum is over all distinguished involutions contained in $\mathbf{c}$, and $\dim \pi$ is independent of ν. Therefore on one hand, $\operatorname{rank}(\pi(f_d))$ is lower semicontinuous in ν, and on the other hand,

$$\operatorname{rank}(\pi(f_d)) = \dim \pi - \sum_{\substack{d' \in \mathbf{c} \\ d' \neq d}} \operatorname{rank}(\pi(f_{d'}))$$

is upper semicontinuous. Therefore $\operatorname{rank}(\pi(f_d))$ is continuous in ν and the lemma follows as $\mathbb{T}$ is connected. □

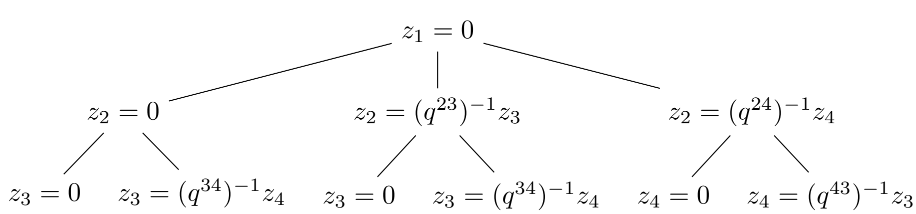
Definition 32. We write $\operatorname{rank}(f_d)$ for the constant $\operatorname{trace}(\pi(f_d))$ for π as in Lemma 6.

2.4 Proof of Conjecture 2 for GL_n, and Theorem 19 for general G

In this section, $q > 1$.

2.4.1 The functions f_w for GL_n

We will need to apply the residue theorem successively in each variable, and in doing so will we will need to sum over a certain tree tracking, for each variable, at which residues we evaluated. Upon integrating with respect to each variable z_i, we will have poles of the form $z_i = 0$ or $z_i = (q^{ij})^{-1} z_j$. For example, if we have 4 variables z_1, z_2, z_3, z_4 corresponding to a Levi subgroup $\mathrm{GL}_{l_1} \times \mathrm{GL}_{l_2} \times \mathrm{GL}_{l_3} \times \mathrm{GL}_{l_4}$, then some of the summands obtained by successively applying the residue theorem are labelled by paths on the tree



Of course, to evaluate the entire integral for M, we would also need to consider trees whose roots are decorated with $z_1 = (q^{12})^{-1} z_2$, and so on, for a total of four trees.

Definition 33. Given a Levi subgroup M with N blocks, a *bookkeeping tree* T for M is a rooted tree with $N - 1$ levels such that the vertices on the i-th level below the root each have $N - i$ child vertices, and each vertex is decorated with an equation of the form $z_i = 0$ or $z_i = (q^{ij})^{-1} z_j$, where the index

j does not appear along the path from the vertex to the root, and the parent root is decorated with an equation $z_k = (q^{ki})^{-1} z_i$ for some k. Moreover, we require that the root be decorated with an equation of the form $z_i = 0$ or $z_i = (q^{ij})^{-1} z_j$ for i minimal. A *branch* of T is a simple path in T from the root to one of the leaves.

Definition 34. Given a branch B of a bookkeeping tree, a *clump in* B is an ordered subset of indices i appearing in the decorations of successive parent-child vertices, such that no decoration of the form $z_i = 0$ occurs along the path from the closest index to the root to the farthest index from the root. We write $C \prec B$ if C is a clump of B.

Example 12. The sets of indices $\{3,4\}$, $\{2,3,4\}$, $\{2,3\}$, $\{2,3,4\}$, $\{2,4\}$, and $\{2,4,3\}$ (note the ordering) are all the clumps of the above tree. The sets $\{1,2,3\}$ and $\{1,2,4\}$ are not clumps.

Theorem 32. *Let $G = \mathrm{GL}_n(F)$ and let d be a distinguished involution such that the two-sided cell containing d corresponds to the Levi subgroup M. Let $N+1$ be the number of blocks in M such that the i-th block has size l_i. Let m_i be the number of blocks of size l_i. Define for $r \leq k$*

$$Q_{rk} = q^{i_k i_{k+1}} q^{i_{k-1} i_k} \cdots q^{i_r i_{r+1}}.$$

Then in the notation of Sections 2.2.4 and 2.3.3,

$$f_d(1) = \frac{\mathrm{rank}(f_d)}{m_1! \cdots m_{N+1}!} c_M \sum_{\text{trees } T} \sum_{\text{branches } B \text{ of } T} \prod_{\substack{C \prec B \\ C = \{i_0, \ldots, i_t\}}} \frac{(1-q^{l_{i_0}})(1-q^{l_{i_1}})}{1-q^{l_{i_0}+l_{i_1}}}$$
$$\cdot \prod_{k=1}^{t-1} \frac{(1-q^{l_{i_{k+1}}})}{(1-q^{l_{i_k}+l_{i_{k+1}}})} \prod_{r=0}^{k-1} \frac{R_{rk}}{1-Q_{rk} q^{i_r i_{k+1}}}, \quad (2.9)$$

where

$$R_{rk} = \begin{cases} 1 - Q_{rk} q^{g_{i_r} - g_{i_k}} & \text{if } k < t-1 \\ (1 - Q_{r,t-1} q^{g_{i_r} - g_{i_{t-1}}})(1 - Q_{r,t-1} q^{g_{i_k} - g_{i_r}}) & \text{if } k = t-1 \end{cases}.$$

Corollary 1. *The denominator of $f_d(1)$ divides a power of the Poincaré polynomial $P_{\mathbf{G}/\mathbf{B}}(q) = P_{\mathfrak{S}_n}(q)$ of G. Moreover, when d is in lowest two-sided cell, corresponding to $M = T$, $f_d(1) = \mathrm{rank}(\pi(f_d))/P_{\mathbf{G}/\mathbf{B}}(q)$ exactly.*

Example 13. Let $G = \mathrm{GL}_2$ and $M = T$. Then $l_1 = l_2 = 1$ and $g_1 = g_2 = 0$. Let $d = s_0$ or s_1. Then

$$\begin{aligned}
f_d(1) &= \frac{1}{2} \frac{1}{2\pi i} \frac{1}{2\pi i} q^{-1} \int_{\mathbb{T}} \int_{\mathbb{T}} q \frac{(z_1 - z_2)(z_1 - z_2)}{(z_1 - q^{-1} z_2)(z_1 - q z_2)} \frac{\mathrm{d}z_1}{z_1} \frac{\mathrm{d}z_2}{z_2} \\
&= \frac{1}{2} \frac{1}{2\pi i} \int_{\mathbb{T}} \mathrm{Res}_{z_1 = q^{-1} z_2} \frac{(z_1 - z_2)(z_1 - z_2)}{(z_1 - q^{-1} z_2)(z_1 - q z_2)} \frac{1}{z_1} + \mathrm{Res}_{z_1 = 0} \frac{(z_1 - z_2)(z_1 - z_2)}{(z_1 - q^{-1} z_2)(z_1 - q z_2)} \frac{1}{z_1} \frac{\mathrm{d}z_2}{z_2} \\
&= \frac{1}{2} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{(q^{-1} z_2 - z_2)(q^{-1} z_2 - z_2)}{(q^{-1} z_2 - q z_2) q^{-1} z_2} + \frac{z_2^2}{z_2^2} \frac{\mathrm{d}z_2}{z_2} \\
&= \frac{1}{2} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{(q^{-1} - 1)^2}{q^{-2} - 1} + 1 \frac{\mathrm{d}z_2}{z_2} \\
&= \frac{1}{2} \left(\frac{(q^{-1} - 1)^2}{q^{-2} - 1} + 1 \right) \\
&= \frac{1}{q+1}.
\end{aligned}$$

The factor $\frac{1}{2}$ reflects the fact that we integrate with the respect to the pushforward of the above $\mathfrak{S}_2$-invariant measure to the quotient $\mathbb{T} \times \mathbb{T}/\mathfrak{S}_2$.

Reassuringly, this agrees with the theorem, which instructs us to calculate $f_d(1)$ as follows: There are two trees, each of which has one vertex and no edges. The trees are $z_1 = 0$ and $z_1 = q^{-1}z_2$. Each has one branch. The first has no clumps, so the entire is product is empty. The second tree has one clump $C = \{1, 2\}$ for which $t = 1$, and we obtain

$$f_d(1) = \frac{1}{2}\left(1 + \frac{1-q}{1+q}\right) = \frac{1}{2}\frac{2}{1+q} = \frac{1}{1+q}.$$

In rank three, the computation is similar but slightly more involved.

Example 14. If $G = \mathrm{GL}_3$ and $M = T$, recalling the definition of $c_M = c_T$, we want to show that

$$\int_{\mathbb{T}}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{(z_0 - z_1)(z_0 - z_1)}{(z_0 - qz_1)(z_0 - q^{-1}z_1)} \frac{(z_0 - z_3)(z_0 - z_3)}{(z_0 - qz_3)(z_0 - q^{-1}z_3)} \frac{(z_1 - z_2)(z_1 - z_2)}{(z_1 - qz_2)(z_1 - q^{-1}z_2)} \frac{dz_1}{z_1}\frac{dz_2}{z_2}\frac{dz_3}{z_3} = 6!$$

There are three bookkeeping trees to consider, corresponding to residues at $z_0 = 0$, $z_0 = q^{-1}z_1$, and $z_0 = q^{-2}z_2$. Reusing our computation from the previous example, we have

$$\begin{aligned}
&\iiint \underbrace{\frac{(z_0 - z_1)(z_0 - z_1)}{(z_0 - qz_1)(z_0 - q^{-1}z_1)} \frac{(z_0 - z_3)(z_0 - z_3)}{(z_0 - qz_3)(z_0 - q^{-1}z_3)} \frac{(z_1 - z_2)(z_1 - z_2)}{(z_1 - qz_2)(z_1 - q^{-1}z_2)}}_{\mu(z_0,z_1,z_2)} \frac{dz_1}{z_1}\frac{dz_2}{z_2}\frac{dz_3}{z_3} \\
&= \iint \mathrm{Res}_{z_0=0}\frac{\mu(z_0,z_1,z_2)}{z_0z_1z_2} + \mathrm{Res}_{z_0=q^{-1}z_1}\frac{\mu(z_0,z_1,z_2)}{z_0z_1z_2} + \mathrm{Res}_{z_0=q^{-1}z_2}\frac{\mu(z_0,z_1,z_2)}{z_0z_1z_2} dz_0dz_1dz_2 \\
&= \iint \frac{(z_1 - z_2)(z_1 - z_2)}{(z_1 - qz_2)(z_1 - q^{-1}z_2)}\frac{1}{z_1z_2}dz_1dz_2 \\
&+ \frac{(q^{-1}-1)(q^{-1}-1)}{(q^{-1}-q)q^{-1}} \iint \frac{(q^{-1}z_1 - z_2)(q^{-1}z_1 - z_2)}{(q^{-1}z_1 - qz_2)(q^{-1}z_1 - q^{-1}z_2)} \frac{(z_1 - z_2)(z_1 - z_2)}{(z_1 - qz_2)(z_1 - q^{-1}z_2)}\frac{1}{z_1z_2}dz_1dz_2 \\
&+ \frac{(q^{-1}-1)(q^{-1}-1)}{(q^{-1}-q)q^{-1}} \iint \frac{(q^{-2}z_2 - z_1)(q^{-2}z_2 - z_1)}{(q^{-2}z_2 - qz_1)(q^{-2}z_2 - q^{-2}z_1)} \frac{(z_2 - z_1)(z_2 - z_1)}{(z_2 - qz_1)(z_2 - q^{-2}z_1)}\frac{2}{z_2z_1}dz_2dz_1 \\
&= \frac{2}{1+q} + 2\frac{1-q}{1+q}\iint \underbrace{\frac{(z_1 - qz_2)(z_1 - qz_2)}{(z_1 - q^2z_2)(z_1 - z_2)} \frac{(z_1 - z_2)(z_1 - z_2)}{(z_1 - qz_2)(z_1 - q^{-1}z_2)}}_{\mu_2(z_1,z_2)}\frac{1}{z_1z_2}dz_1dz_2 \\
&= \frac{2}{1+q} + 2\frac{1-q}{1+q}\int \mathrm{Res}_{z_1=0}\frac{\mu_2(z_1,z_2)}{z_1z_2} + \mathrm{Res}_{z_1=q^{-1}z_2}\frac{\mu_2(z_1,z_2)}{z_1z_2}dz_2 \\
&= \frac{2}{1+q} + 2\frac{1-q}{1+q}\left(1 + \frac{(1-q^2)}{1+q+q^2}\right) \\
&= 2\left(\frac{1}{1+q} + \frac{(1-q)(1+q+q^2) + (1-q)(1-q^2)}{(1+q)(1+q+q^2)}\right) \\
&= 2\left(\frac{1+q+q+^2+(1-q)(1+q+q^2) + (1+q)(1-q)^2}{(1+q)(1+q+q^2)}\right) \\
&= \frac{6!}{(1+q)(1+q+q^2)}.
\end{aligned}$$

Proof of Corollary 1. We will show that each of the three forms of denominator that appear in the conclusion of Theorem 32 divide $P_{\mathbf{G}/\mathbf{B}}(q)$, and thus that their product divides a power of $P_{\mathbf{G}/\mathbf{B}}(q)$. The denominators of c_M are all of the form $1 + q + \cdots + q^{l_i-1}$, and so divide $P_{\mathbf{G}/\mathbf{B}}(q)$ as $l_i \leq n$ for all i. Note that as $l_{i_i} + l_{i_{k+1}} \leq n$, the leftmost denominators in (2.9) satisfy the conclusion of the corollary also. Finally, $Q_{rk}q^{i_r i_{k+1}} = q^{l_{i_{k+1}}+l_{i_k}+\cdots+l_{i_r}}$, and R_{rk} is likewise always a polynomial in q (as opposed to q^{-1}) divisible by $1-q$. Again using that $l_{i_{k+1}} + l_{i_k} + \cdots + l_{i_r} \leq n$, we are done with the first statement.

We now take up the second statement, the proof of which was explained to us by A. Braverman and does not use our previous arguments at all. Recalling that the only K-spherical tempered representations of G are principal series representations [50], let μ_K be the Haar measure on G such that $\mu_K(K) = 1$, μ_I be the Haar measure such that $\mu_I(I) = 1$, and $\pi_K(f)$ denote the Fourier transform with respect to μ_K. Then we have

$$\mu_I = P_{\mathbf{G}/\mathbf{B}}(q)\mu_K$$

and,

$$\operatorname{trace}(\pi(1_K)) = P_{\mathbf{G}/\mathbf{B}}(q)\operatorname{trace}(\pi_K(1_K)) = P_{\mathbf{G}/\mathbf{B}}(q) = P_{\mathbf{G}/\mathbf{B}}(q)\frac{1}{\operatorname{rank}(\pi(f_d))}\operatorname{trace}(\pi(f_d)) \tag{2.10}$$

for any d in the lowest two-sided cell. Therefore the Plancherel formula gives that

$$f_d(1) = \frac{\operatorname{rk}(\pi(f_d))}{P_{\mathbf{G}/\mathbf{B}}(q)} = \frac{1}{P_{\mathbf{G}/\mathbf{B}}(q)}, \tag{2.11}$$

where the rank is given by [67], Proposition 5.5. (In *loc. cit.* there is the assumption of simple-connectedness, but it is easy to see that the distinguished involutions for the extended affine Weyl group $\tilde{W}(\tilde{\mathbf{G}}(F))$ of the universal cover $\tilde{\mathbf{G}}$ are distinguished involutions for $\tilde{W}$ using the definition in [42] and uniqueness of the $\{C_w\}$-basis, and that the lowest cell is just the lowest cell of $\tilde{W}(\tilde{\mathbf{G}}(F))$ intersected with $\tilde{W}$.) □

Note that (2.11) is an example of the behaviour conjectured in Remark 5.

It would be interesting to find I-biinvariant Schwartz functions f playing the role of χ_K for the other two-sided cells, namely such that $\operatorname{trace}\pi(f)$ was a constant, nonzero only for a single pair (M,ω).

Lemma 7. *Let $e_0, \ldots, e_n \in \mathbb{Z}$. Then*

$$\int_{\mathbb{T}} \cdots \int_{\mathbb{T}} z_0^{e_0} \cdots z_N^{e_N} \prod_{i<j} \Gamma^{ij} \mathrm{d}z_0 \cdots \mathrm{d}z_N = 0 \tag{2.12}$$

unless $e_0 + \cdots + e_N = -N$.

Corollary 2. *Let $w \in \tilde{W}$. Then $f_w(1)$ is a rational function of q with denominator dividing a power of the Poincaré polynomial of G. The numerator is a Laurent polynomial $p_1(q^{1/2}) + p_1(q^{-1/2})$ in $q^{1/2}$, where the degree of p_1 is bounded uniformly in terms of $\tilde{W}$. The denominator of $f_w(1)$ depends only on the two-sided cell containing w.*

Remark 12. In light of Lemma 5, Lemma 7 and Corollary 2 have the following interpretation. Let Γ be a left cell in a two-sided cell $\mathbf{c}$. Then in [69], Xi shows that all the rings $J_{\Gamma\cap\Gamma^{-1}}$ are isomorphic to the representation ring of the associated Levi subgroup $M_{\mathbf{c}}$, and $J_{\mathbf{c}}$ is a matrix algebra over $J_{\Gamma\cap\Gamma^{-1}}$.

Therefore $w \in \Gamma \cap \Gamma^{-1}$ are labelled by dominant weights of $M_{\mathbf{c}}$, and if t_λ is such an element, Xi's result show that if $\mathbf{q} = q$ and $\pi = \mathrm{Ind}_B^G(\nu)$ is an irreducible representation of $J_{\mathbf{c}}$, then

$$\mathrm{trace}(\pi(t_\lambda)) = \mathrm{trace}(\nu, V(\lambda)),$$

where we view ν as a semisimple conjugacy class in $M_{\mathbf{c}}$. Then we have that $f_\lambda(1) \neq 0$ only if λ is of height 0 with respect to the basis $\varepsilon_i \colon \mathrm{diag}(a_1, \ldots, a_n) \mapsto a_i$.

The proofs of Lemma 7 and Corollary 2 will use the notation of the proof of Theorem 32, and we defer them until after the proof of the theorem.

Proofs of Theorem 32, Lemma 7, and Corollary 2

Proof of Theorem 32. Let $n = l_0 + \cdots + l_N$. We may assume that $l_0 \leq l_2 \leq \cdots \leq l_N$. It suffices to evaluate the integral

$$\int_{\mathbb{T}} \cdots \int_{\mathbb{T}} \prod \Gamma_{i,j,g} \frac{\mathrm{d}z_0}{z_0} \cdots \frac{\mathrm{d}z_N}{z_k} = \int_{\mathbb{T}} \cdots \int_{\mathbb{T}} \prod_{i<j} \Gamma^{ij} \frac{\mathrm{d}z_0}{z_0} \cdots \frac{\mathrm{d}z_N}{z_N}.$$

We claim that the value of this integral is

$$\frac{1}{n!} c_M \sum_{\text{trees } T} \sum_{\text{branches } B \text{of } T} \prod_{\substack{C \prec B \\ C = \{i_0, \ldots, i_t\}}} \prod_{k=0}^{t-1} \frac{(1 - q^{i_k i_{k+1}} q_{i_k i_{k+1}})(1 - q^{i_k i_{k+1}} (q_{i_k i_{k+1}})^{-1})}{1 - (q^{i_k i_{k+1}})^2}$$
$$\cdot \prod_{r=0}^{k-1} \frac{(1 - Q_{rk} q_{i_r i_{k+1}})(1 - Q_{rk} (q_{i_r i_{k+1}})^{-1})}{(1 - Q_{rk} q^{i_r i_{k+1}})(1 - Q_{rk} (q^{i_r i_{k+1}})^{-1})}, \quad (2.13)$$

where the sum over trees is taken over all bookkeeping trees for the integral. When $k = 0$, we interpret the product over r as being empty.

First we explain how (2.13) simplifies to (2.9). All cancellations will take place within the same clump C of some branch B, which we now fix. We have

$$1 - Q_{rk}(q^{i_r i_{k+1}})^{-1} = 1 - q^{g_{i_k} + g_{i_{k+1}} + 1} q^{i_k i_{k+1}} \cdots q^{i_r i_{r+1}} q^{-g_{i_r} - g_{i_{k+1}} - 1} = 1 - Q_{r,k-1} q^{g_{i_k} - g_{i_r}},$$

which is one of the factors in the product $(1 - Q_{r,k-1} q_{i_r i_k})(Q_{r,k-1}(q_{i_r i_k})^{-1})$. The surviving factor in the numerator at index $(k-1, r)$ is then equal to $(1 - Q_{r,k-1} q^{g_{i_r} - g_{i_k}})$. In short, the above factors in the denominator cancel with a numerator occurring with the same r-index but k-index one lower. Such a factor occurs whenever $r < k - 1$ (note that this inequality does not hold when $k = 1$ and $r = 0$). When $r = k - 1$, we have

$$1 - Q_{k-1,k}(q^{i_k i_{i+1}})^{-1} = 1 - q^{i_k i_{k+1}} q^{i_{k-1} i_k} (q^{i_{k-1} i_{k+1}})^{-1} = q^{g_{i_k} + g_{i_{k+1}} + 1} q^{g_{i_{k-1}} - g_{i_{k+1}}} = 1 - q^{2 g_{i_k} + 1},$$

which is one of the factors in $(1 - q^{i_{k+1} i_k} q_{i_k i_{k+1}})(1 - q^{i_k i_{k+1}} (q_{i_k} i_{k+1})^{-1})$. The cancellation leaves behind the factor $1 - q^{l_{i_{k+1}}}$ in the numerator, except for $k = 0$; this term keeps both its denominators.

At this point we have shown that the factor corresponding to C in (2.13) simplifies to

$$\frac{(1-q^{l_{i_0}})(1-q^{l_{i_1}})}{1-q^{l_{i_0}+l_{i_1}}}\prod_{k=1}^{t-1}\frac{(1-q^{l_{i_{k+1}}})}{(1-q^{l_{i_k}+l_{i_{k+1}}})}\prod_{r=0}^{k-1}\frac{R_{rk}}{1-Q_{rk}q^{i_r i_{k+1}}},$$

where

$$R_{rk}=\begin{cases}1-Q_{rk}q^{g_{i_r}-g_{i_k}} & \text{if } k<t-1\\ (1-Q_{r,t-1}q^{g_{i_r}-g_{i_{t-1}}})(1-Q_{r,t-1}q^{g_{i_k}-g_{i_r}}) & \text{if } k=t-1\end{cases}.$$

This means that (2.13) simplifies to (2.9).

To prove (2.13), we will use the residue theorem for each variable consecutively, keeping track of the constant expressions in q that we extract after integrating with respect to each variable z_i. More precisely, we will track what happens in a single summand corresponding to some set of successive choices of poles to take residues at. Note that all the rational functions that will appear, namely the Γ^{ij} or the rational functions that result from substitutions into the Γ^{ij}, become equal to 1 once z_i or z_j is set to zero. Therefore poles at $z_i=0$ serve simply to remove all factors involving z_i from inside the integrand (we will see what these rational functions are below). It follows that the summand whose branch we are computing is a product over clumps in the corresponding branch, so it suffices to compute the value of a given clump for some ordered subsets $\{i_0,i_1,\dots,i_l\}$ of the indices $\{0,\dots,N\}$. As we are inside a clump, we will consider only poles occurring at nonzero complex numbers. Thus we are left only to determine what happens within a single clump.

We first integrate with respect to the variable z_{i_0}. The residue theorem gives

$$\begin{aligned}&\int_{\mathbb{T}}\cdots\int_{\mathbb{T}}\prod_{i<j}\Gamma^{ij}\frac{\mathrm{d}z_{i_0}}{z_{i_0}}\cdots\frac{\mathrm{d}z_{i_l}}{z_{i_l}}\\ &=\int_{\mathbb{T}}\cdots\int_{\mathbb{T}}\sum_{l\neq i_0}\mathrm{Res}_{z_{i_0}=(q^{i_0 l})^{-1}z_l}\frac{1}{z_{i_0}}\prod_{i<j}\Gamma^{ij}\frac{\mathrm{d}z_{i_1}}{z_{i_1}}\cdots\frac{\mathrm{d}z_{i_l}}{z_{i_l}}+\int_{\mathbb{T}}\cdots\int_{\mathbb{T}}\prod_{\substack{i<j\\ i,j\neq i_0}}\Gamma^{ij}\frac{\mathrm{d}z_{i_1}}{z_{i_1}}\cdots\frac{\mathrm{d}z_{i_l}}{z_{i_l}}.\end{aligned}$$

As noted above, the second integral belongs to a different clump; our procedure will deal with it separately, and we will now consider what happens with the first integral.

For the first integral, consider one of the summands corresponding to $z_{i_0}=(q^{i_0 i_1})^{-1}z_{i_1}$ for some i_1. We have,

$$\begin{aligned}&\int_{\mathbb{T}}\cdots\int_{\mathbb{T}}\mathrm{Res}_{z_{i_0}=(q^{i_0 i_1})^{-1}z_{i_1}}\frac{1}{z_{i_0}}\prod_{i<j}\Gamma^{ij}\frac{\mathrm{d}z_{i_1}}{z_{i_1}}\cdots\frac{\mathrm{d}z_{i_l}}{z_{i_l}}\\ &=\frac{(1-q^{i_0 i_1}q_{i_0 i_1})(1-q^{i_0 i_1}(q_{i_0 i_1})^{-1})}{1-(q^{i_0 i_1})^2}\int_{\mathbb{T}}\cdots\int_{\mathbb{T}}\prod_{j\neq i_1,i_0}\frac{(z_{i_1}-q^{i_0 i_1}q_{i_0 j}z_j)(z_{i_1}-q^{i_0 i_1}(q_{i_0 j})^{-1}z_j)}{(z_{i_1}-q^{i_0 i_1}q^{i_0 j}z_j)(z_{i_1}-q^{i_0 i_1}(q^{i_0 j})^{-1}z_j)}\\ &\prod_{\substack{i<j\\ i,j\neq i_0}}\Gamma^{ij}\frac{\mathrm{d}z_{i_1}}{z_{i_1}}\frac{\mathrm{d}z_{i_2}}{z_{i_2}}\cdots\frac{\mathrm{d}z_{i_l}}{z_{i_l}}.\end{aligned}$$

Recall that, if $i_1>2$, even though formally we have defined the symbols q_{ij} and q^{ij} only for $i<j$, the symmetry of the factors allows us to write q_{ij} even if $i>j$, thanks to the factor with $(q_{ij})^{-1}$ also present in the numerator.

Now we integrate with respect to z_{i_1}. Observe that the leftmost product over $j\neq i_1,i_0$ does not contribute poles. Indeed, the first factor each denominator does not have its zero contained in $\mathbb{T}$, and the second factor in each denominator has its zero at $z_{i_1} = q^{i_0 i_1}(q^{i_0 i_2})^{-1} z_{i_2}$. The power of q appearing is

$$g_{i_0} + g_{i_1} + 1 - (g_{i_0} + g_{i_2} + 1) = g_{i_1} - g_{i_2},$$

and so $q^{i_0 i_1}(q^{i_0 i_2})^{-1}$ is equal to $q_{i_1 i_2}$ or $(q_{i_2 i_1})^{-1}$, whichever is defined. Thus this zero cancels with a zero in the numerator of $\Gamma^{i_1 i_2}$ or $\Gamma^{i_2 i_1}$, whichever is defined. Therefore we need only consider the simple poles at $z_{i_1} = (q^{i_1 i_2})^{-1} z_{i_2}$ for $i_1 < i_2$ and $z_{i_1} = (q^{i_2 i_1})^{-1} z_{i_2}$ for $i_2 < i_1$. Observe that the residues will be the same for either inequality. In the case $i_2 < i_1$, for example, $\Gamma^{i_2 i_1}$ needs to be rewritten so that its simple pole inside $\mathbb{T}$ is in the correct format to calculate the residue by substitution:

$$\begin{aligned}
&\operatorname{Res}_{z_{i_1} = (q^{i_2 i_1})^{-1} z_{i_2}} \frac{(z_{i_2} - q_{i_2 i_1} z_{i_1})(z_{i_2} - (q_{i_2 i_1})^{-1} z_{i_1})}{(z_{i_2} - q^{i_2 i_1} z_{i_1})(z_{i_2} - (q^{i_2 i_1})^{-1} z_{i_1})} \frac{1}{z_{i_1}} \\
&= \operatorname{Res}_{z_{i_1} = (q^{i_2 i_1})^{-1} z_{i_2}} \frac{-(q^{i_2 i_1})^{-1}(z_{i_2} - q_{i_2 i_1} z_{i_1})(z_{i_2} - (q_{i_2 i_1})^{-1} z_{i_1})}{(z_{i_2} - q^{i_2 i_1} z_{i_1})(z_{i_2} - (q^{i_2 i_1})^{-1} z_{i_1})} \frac{1}{z_{i_1}} \\
&= \frac{(1 - q^{i_2 i_1} q_{i_2 i_1})(1 - q^{i_2 i_1}(q_{i_2 i_1})^{-1})}{1 - (q^{i_2 i_1})^2}.
\end{aligned}$$

Therefore after integrating within the clump at hand with respect to z_{i_0} and then z_{i_1}, we have the expression

$$\frac{(1 - q^{i_0 i_1} q_{i_0 i_1})(1 - q^{i_0 i_1}(q_{i_0 i_1})^{-1})}{1 - (q^{i_0 i_1})^2} \frac{(1 - q^{i_2 i_1} q^{i_0 i_1} q_{i_0 i_2})(1 - q^{i_1 i_2} q^{i_0 i_1}(q_{i_0 i_2})^{-1})}{(1 - q^{i_2 i_1} q^{i_0 i_1} q^{i_0 i_2})(1 - q^{i_2 i_1} q^{i_0 i_1}(q^{i_0 i_2})^{-1})} \tag{2.14}$$

$$\cdot \frac{(1 - q^{i_2 i_1} q_{i_2 i_1})(1 - q^{i_2 i_1}(q_{i_2 i_1})^{-1})}{(1 - (q^{i_2 i_1})^2)} \tag{2.15}$$

$$\cdot \int_{\mathbb{T}} \cdots \int_{\mathbb{T}} \prod_{j \neq i_0, i_1, i_2} \frac{(z_{i_2} - q^{i_1 i_2} q^{i_0 i_1} q_{i_0 j} z_j)(z_{i_2} - q^{i_1 i_2} q^{i_0 i_1}(q_{i_0 j})^{-1} z_j)}{(z_{i_2} - q^{i_1 i_2} q^{i_0 i_1} q^{i_0 j} z_j)(z_{i_2} - q^{i_1 i_2} q^{i_0 i_1}(q^{i_0 j})^{-1} z_j)} \tag{2.16}$$

$$\cdot \prod_{j \neq i_0, i_1, i_2} \frac{(z_{i_2} - q^{i_1 i_2} q_{i_1 j} z_j)(z_{i_2} - q^{i_1 i_2}(q_{i_1 j})^{-1} z_j)}{(z_{i_2} - q^{i_1 i_2} q^{i_1 j} z_j)(z_{i_2} - q^{i_1 i_2}(q^{i_1 j})^{-1} z_j)} \prod_{\substack{i<j \\ i,j \neq i_0, i_1}} \Gamma^{ij} \frac{\mathrm{d}z_{i_2}}{z_{i_2}} \cdots \frac{\mathrm{d}z_{i_k}}{z_{i_k}}. \tag{2.17}$$

Now we integrate with respect to z_{i_2}. Again, only poles from the product of Γ^{ij}'s occur with nonzero residues: in total we have simple poles contained in $\mathbb{T}$ possibly at $z_{i_2} = q^{i_1 i_2} q^{i_0 i_1}(q^{i_0 j})^{-1} z_j$, at $z_{i_2} = q^{i_1 i_2}(q^{i_1 j})^{-1} z_j$ and at $z_{i_2} = (q^{i_2, j})^{-1} z_j$ for $j \neq i_0, i_1, i_2$. It may happen that these poles are not all distinct, but all zeros in the denominator of the former two types are in fact cancelled by zeros of denominator anyway. The necessary factors occur in the product immediately adjacent on the right. Indeed, we have

$$g_{i_1} + g_{i_2} + 1 + g_{i_0} + g_{i_1} + 1 - g_{i_0} - g_j - 1 = g_{i_1} + g_{i_2} + 1 + g_{i_1} - g_j$$

and so either $q^{i_1 i_2} q^{i_0 i_1}(q_{i_0 j})^{-1} = q^{i_1 i_2} q_{i_1 j}$ or $q^{i_1 i_2} q^{i_0 i_1}(q_{i_0 j})^{-1} = q^{i_1 i_2}(q_{i_1 j})^{-1}$ (or $q_{j i_1}$ or $(q_{j i_1})^{-1}$).

Likewise we have

$$g_{i_1} + g_{i_2} + 1 - g_{i_1} - g_j - 1 = g_{i_2} - g_j$$

as happened when we integrated with respect to z_{i_0}.

Now we see the following pattern: At each stage, we integrate with respect to the variable we took a residue at in the previous step. When integrating with respect to z_{i_r}, there will be $r+1$ products of rational functions, and each rational function in the leftmost r products will contribute a pole, in addition to the pole contributed by $\Gamma^{i_r i_{r+1}}$ or $\Gamma^{i_{r+1} i_r}$, whichever is defined. However, each is nullified by having zero residue thanks to a denominator in the product immediately to the right. Integrating with respect to z_{i_r} will result in extracting $r+1$ new rational factors, each of the form claimed in the theorem. When it comes to integrating with respect to z_{i_k}, all variables z_{i_k} will cancel from the remains of the Gamma functions by homogeneity, and the factor $\frac{1}{z_{i_k}}$ will result in the final integral contributing just the remaining $k+1$ rational factors. □

Lemma 7 is a porism of the preceding proof.

Proof of Lemma 7. We will evaluate the integral in (2.12) by applying the residue theorem successively for each variable as in the proof of Theorem 32. Note that as functions of any variable z_k, the functions $\prod_{i<j} \Gamma^{ij}$ and all the other products, for example those appearing in (2.14), have numerator and denominator with equal degrees. Thus the overall sum of powers of all z_i in the integrand of (2.12) is $e_0 + \cdots + e_N$, and in general, the sum of powers of all z_i in the integrand of an expression like (2.14) is the sum of the degrees of the monomial z_{i_j} terms.

When evaluating (2.12) along a single branch, we find again that the only poles that appear are of the form $z_i = (q^{ij})^{-1} z_j$, or $z_i^{e_i} = 0$ for $e_i < 0$. We will track the effect that evaluating each successive residue has on the total degree of the integrand, and then conclude using the fact that $\int_{\mathbb{T}} z^r \, \mathrm{d}z = 2\pi i \delta_{r,-1}$. First, observe that evaluating a residue of the form $z_i = (q^{ij})^{-1} z_j$ increases the sum of all powers by 1, a factor z_j is contributed to the resulting integrand. The sum of all powers is likewise increased by 1 when evaluating the residue at a simple pole of z_i^{-1} at 0. To compute the residue at a pole of $z_i^{-e_i}$ at 0 for $e_i > 1$, consider the Taylor expansion at 0 of Γ^{ij}. We have

$$\frac{1}{z_i - q^{ij} z_j} = -\frac{1}{q^{ij} z_j} - \frac{z_i}{(q^{ij} z_j)^2} - \frac{z_i^2}{(q^{ij} z_j)^3} - \cdots$$

and

$$\frac{1}{z_i - (q^{ij})^{-1} z_j} = -\frac{q^{ij}}{z_j} - \left(\frac{q^{ij}}{z_j}\right)^2 z_i - \left(\frac{q^{ij}}{z_j}\right)^3 z_i^2 - \cdots .$$

Multiplying these series and further multiplying by the denominator $z_i^2 - (q_{ij} + q_{ij}^{-1}) z_i z_j + z_j$, it follows that the Taylor expansion of Γ^{ij} is

$$\begin{aligned} 1 &+ \frac{q^{ij} + (q^{ij})^{-1} - q_{ij} - q_{ij}^{-1}}{z_j} z_i + \frac{(q^{ij})^2 + 2 + (q^{ij})^{-2} - (q_{ij} + q_{ij}^{-1})(q^{ij} + q^{-ij})}{z_j} z_i^2 + \cdots \\ &+ \frac{-(q_{ij} + q_{ij}^{-1})((q^{ij})^{n-1} + \cdots + (q^{ij})^{-n+1}) + (q^{ij})^{n-2} + \cdots + (q^{ij})^{-n+2} + (q^{ij})^{n} + \cdots + (q^{ij})^{-n}}{z_j^n} z_i^n \\ &+ \cdots . \end{aligned} \tag{2.18}$$

It is clear that the salient point of (2.18), that z_i^n appears with a coefficient proporitional to z_j^{-n}, holds also for the all the products like those in (2.14). Thus computing a residue at the pole of $z_i^{-e_i}$ at 0 will result in a new integrand, the total degree of which has increased by $e_i - e_i + 1 = 1$. It now follows that after integrating with respect to $N-1$ variables, the final integral to be computed will be a constant times $\int_{\mathbb{T}} z_N^{e_0 + \cdots + e_N + N - 1} \mathrm{z_N}$. This is nonzero if and only if $e_0 + \cdots + e_N = -N$. Summing over all branches of a bookkeeping tree, we see that (2.12) is nonzero only if $e_0 + \cdots + e_N = -N$. □

We can now prove Corollary 2.

Proof of Corollary 2. In light of Lemmas 5 and 7, it is enough to prove that the conclusions of the present corollary hold for integrals of the form (2.12) with $e_0 + \cdots + e_N = -N$. If $e_i = -1$ for all i then the integral (2.12) is just the integral from Theorem 32. Otherwise there is some $e_{i_0} \geq 0$. We may assume that $i_0 = 0$. Then the only poles in z_0 are of the form $z_0 = (q^{0i_1})^{-1} z_{i_1}$ for indices $i_1 > 0$. Therefore we compute that (2.12) is equal to to a sum of terms of the form

$$\frac{(1 - q^{0i_1} q_{0i_1})(1 - q^{0i_1}(q_{0i_1})^{-1})}{1 - (q^{0i_1})^2} (q^{0i_1})^{-e_0 - 1} \int_{\mathbb{T}} \cdots \int_{\mathbb{T}} z_{i_1}^{e_{i_1} + 1 + e_0} z_{i_2}^{e_{i_2}} \cdots z_{i_N}^{e_{i_N}} \cdot \prod_{j \neq i_1, 0} \frac{(z_{i_1} - q^{0i_1} q_{0j} z_j)(z_{i_1} - q^{0i_1}(q_{0j})^{-1} z_j)}{(z_{i_1} - q^{0i_1} q^{0j} z_j)(z_{i_1} - q^{0i_1}(q^{0j})^{-1} z_j)} \prod_{\substack{i<j \\ i,j \neq i_0}} \Gamma^{ij} \mathrm{d}z_{i_1} \cdots \mathrm{d}z_{i_N}. \tag{2.19}$$

The total degree of the integrand is now $e_0 + \cdots e_N + 1 = -(N-1)$. Therefore either $e_{i_0} + e_{i_1} + 1 = e_{i_2} = \cdots = e_{-N} = -1$, or we may again assume without loss of generality that the exponent of some z_{i_j} is nonnegative, and proceed with evaluating (2.19) by integrating with respect to z_{i_j}. We may continue in this way, never having to deal with more than a simple pole at 0. It is now clear from the calculations in the proof of Theorem 32 that the only positive powers of $q^{1/2}$ that appear in the any numerator are those that appeared as in Theorem 32. These are controlled by the possible block sizes of M, and hence are bounded in terms of $\tilde{W}$. Throughout this procedure, the denominator has been contributed to only by the Plancherel density itself, which depends only on M_P. The last claim of the corollary now follows from Proposition 1. □

2.4.2 The functions f_w for general G

For general $\mathbf{G}$, we will follow the same plan as for $\mathbf{G} = \mathrm{GL}_n$. The only difference is that we have less control over which denominators can appear (indeed, this is true even for formal degrees). It will also be necessary to control the possible numerators of $f_w(1)$ in a different manner than for $\mathbf{G} = \mathrm{GL}_n$ in order to prove Proposition 2.

Theorem 33. *Let $w \in \tilde{W}$. Then $f_w(1)$ is a rational function of q with poles drawn from a finite set of roots of unity depending only on $\tilde{W}$. The numerator is a Laurent polynomial in $q^{1/2}$. The denominator depends only on the two-sided cell containing w.*

Proof of Theorem 33. Let $w \in \tilde{W}$ and M be the Levi subgroup corresponding to the two-sided cell containing w. Let $N = \mathrm{rk}\, A_P$. By Lemma 5, it suffices to show the conclusions of the theorem hold for integrals of the form

$$\left(\frac{1}{2\pi i}\right)^N \int_{\mathbb{T}} \cdots \int_{\mathbb{T}} \prod_{\substack{\alpha \in R_{1,+} \setminus R_{P,1,+} \\ \alpha \neq r\epsilon_i}} \frac{(z_i - z_j^{\pm 1})^2 q_\alpha^{-1}}{(z_i - q_\alpha^{-1} z_j^{\pm 1})(z_i - q_\alpha z_j^{\pm 1})} \prod_{\substack{\alpha \in R_{1,+} \setminus R_{P,1,+} \\ \alpha = r\epsilon_i}} \frac{(1 - z_i^r)^2 q_\alpha^{-1} q_{2\alpha}^{-1}}{(z_i^{r/2} + q_\alpha^{-1/2})(z_i^{r/2} + q_{\alpha^\vee}^{1/2})(z_i^{r/2} - q_\alpha^{-1/2} q_{2\alpha}^{-1})(z_i^{r/2} - q_\alpha^{1/2} q_{2\alpha})} z_1^{e_1} \cdots z_N^{e_N} \, \mathrm{d}z_1 \cdots \mathrm{d}z_N \tag{2.20}$$

where $r \in \{2,4\}$ and $e_i \in \mathbb{Z}$. Note that as every α appearing in the second product has even length as an element of $\tilde{W}$, $q_\alpha^{1/2}$ is a natural power of q by multiplicativity of the root label function. As in the case of GL_n, using the residue theorem and the quotient rule, we see that after integrating with respect to each variable z_i, the constant factors extracted are of the form

$$\frac{Q}{(1 \pm q^{f_1})^{h_1} \cdots (1 \pm q^{f_k})^{h_k}} \tag{2.21}$$

where Q is a Laurent polynomial in $q^{\frac{1}{2}}$, $f_i \in \frac{1}{2}\mathbb{N}$, and $e_i \in \mathbb{N}$. Therefore (2.20) has poles in q only at a finite number of roots of unity. Again by the quotient rule, exponents h_i and f_i depend only on M. Clearly there are only finitely-many exponents f_i that appear for any M. The theorem now follows. □

2.4.3 Relating t_w and f_w

We will now relate the Schwartz functions f_w on G to the elements $\phi^{-1}(t_w)$ of a completion $\boldsymbol{\mathcal{H}}^-$ of $\mathbf{H}$, whose definition we will now recall. In this section $\mathbf{G}$ is general.

Completions of H and $J \otimes_{\mathbb{Z}} \mathscr{A}$

Let $\hat{\mathscr{A}} = \mathbb{C}((\mathbf{q}^{-1/2}))$ and $\hat{\mathscr{A}}^- = \mathbb{C}[[\mathbf{q}^{-1/2}]]$. Write $\boldsymbol{\mathcal{H}}^-$ for the $\hat{\mathscr{A}}$-algebra

$$\boldsymbol{\mathcal{H}}^- = \left\{ \sum_{x \in \tilde{W}} b_x T_x \;\middle|\; b_x \to 0 \text{ as } \ell(x) \to \infty \right\}$$

(note the difference between C'_x and C_x), where we say that $b_x \to 0$ as $\ell(x) \to \infty$ if for all $N > 0$, $b_x \in (\mathbf{q}^{1/2})^N \check{\mathscr{A}}^-$ for all x sufficiently long.

Consider also the completions

$$\boldsymbol{\mathcal{H}}^-_{C'} := \left\{ \sum_{x \in \tilde{W}} b_x C'_x \;\middle|\; b_x \to 0 \text{ as } \ell(x) \to \infty \right\}.$$

and

$$\boldsymbol{\mathcal{H}}^-_{C} := \left\{ \sum_{x \in \tilde{W}} b_x C_x \;\middle|\; b_x \to 0 \text{ as } \ell(x) \to \infty \right\}.$$

as well as

$$\boldsymbol{\mathcal{J}} = \left\{ \sum_{x \in \tilde{W}} b_x t_x \;\middle|\; b_x \to 0 \text{ as } \ell(x) \to \infty \right\}.$$

In the same way, one can define a completion $\boldsymbol{\mathcal{J}}$ of $J \otimes_{\mathbb{C}} \mathscr{A}$. In [42], Lusztig shows that ϕ extends to an isomorphism of $\hat{\mathscr{A}}$-algebras $\boldsymbol{\mathcal{H}}^-_C \to \boldsymbol{\mathcal{J}}$. In this way the elements $t_w \in J \subset \boldsymbol{\mathcal{J}}$ may be identified with elements of $\boldsymbol{\mathcal{H}}^-$.

Lemma 8. *We have $\boldsymbol{\mathcal{H}}^-_{C'} \subset \boldsymbol{\mathcal{H}}^-$.*

Proof. Given an infinite sum $\sum_x b_x C'_x$, upon rewriting this sum in the standard basis, the coefficient of some T_y is

$$a_y := \sum_{x \geq y} b_x q^{-\frac{\ell(x)}{2}} P_{y,x}(q).$$

As $\deg P_{y,x} \leq \frac{1}{2}(\ell(x) - \ell(y) - 1)$, we have that $q^{-\frac{\ell(x)}{2}} P_{y,x}(q)$ is a polynomial in $q^{-1/2}$. Therefore the above sum defines a formal Laurent series. Moreover, as $\ell(y) \to \infty$, it is clear that $a_y \to 0$. □

The functions f_w and the basis elements t_w

We shall now explain how the map $\tilde{\phi}\colon J \to \mathscr{C}(G)^I$ induces a map of $\mathscr{A}$-algebras $\phi_1\colon J \otimes_{\mathbb{Z}} \mathscr{A} \to \boldsymbol{\mathcal{H}}^-$. The most difficult part of the proof of the proposition is showing that $a_{x,w} \to 0$ as $\ell(x) \to \infty$. To this end we have

Lemma 9. *Let $w \in \tilde{W}$. Then the degree in q of the numerator of $f_w(1)$ is bounded uniformly in w by some N depending only on $\tilde{W}$, and hence in the notation of Proposition 3 below, we have $a_{1,w} \in (\mathbf{q}^{1/2})^N \hat{\mathscr{A}}^-$ for all $w \in \tilde{W}$.*

Remark 13. In type A, the Lemma follows immediately from the order of integration used in the proof of Corollary 2. For types D and E the argument is analogous.

Proof. Let $\pi_\omega = \mathrm{Ind}_P^G(\nu \otimes \omega)$ be a tempered I-spherical representation of G arising by induction from a parabolic P such that $\operatorname{rank} A_P = k$. Let $\mathbb{T}^k$ be the compact torus parametrizing twists of ω, and let t_w be given. By Lemma 5, $\operatorname{trace}(\pi(f_w))$ is a regular function on $\mathbb{T}^k$, *i.e.* a Laurent polynomial in the coordinates $z_1, \ldots, z_k$ on $\mathbb{T}^k$. The coefficients of this Laurent polynomial are independent of q, as J and its representation theory are independent of q. As we have

$$|1 - q^r| \leq |z_i - q^r z_j| \leq |1 + q^r|$$

for all $z_i, z_j \in \mathbb{T}^k$ and r, we may bound $|\mu(z)|$ by a rational function $U(q)$. Thus we define a rational function of q by

$$\frac{h_1(q) + h_2(q^{-1})}{k_1(q) + k_2(q^{-1})} = U(q) \sum_{\omega} |d(\omega)| \max_{z \in \mathbb{T}^k} |\operatorname{trace}(\pi(f_w))|,$$

where $h_1, k_1 \in \mathbb{C}[q]$ and $h_2, k_2 \in q^{-1}\mathbb{C}[q^{-1}]$, μ is the Plancherel measure, and the sum is over the finitely many, up to unitary twist, $\omega \in \mathscr{E}_2(M)$ such that $\operatorname{trace}(\pi_\omega(f_w)) \neq 0$.

We then have

$$|f_w(1)| = \left| \int_{\mathbb{T}^k} d(\omega) \operatorname{trace}(\pi_\omega(f_w))\, \mathrm{d}\mu(z) \right| \leq \frac{h_1(q) + h_2(q^{-1})}{k_1(q) + k_2(q^{-1})}.$$

Crucially, this expression holds for all $q > 1$.

On the other hand, by Corollary 2 and Theorem 33, $f_w(1)$ is a rational function of q, and for q sufficiently large, we may write

$$|f_w(1)| = \frac{f_1(q) + f_2(q^{-1})}{d_1(q) + d_2(q^{-1})}$$

where $f_1, d_1 \in \mathbb{C}[q]$ and $f_2, d_2 \in q^{-1}\mathbb{C}[q^{-1}]$. Therefore for all $q \gg 1$ we have

$$|(f_1(q) + f_2(q^{-1}))||(k_1(q) + k_2(q^{-1}))| \leq |(d_1(q) + d_2(q^{-1})||(h_1(q) + h_1(q^{-1}))|.$$

We claim that this implies

$$|f_1(q)k_1(q)| \leq |d_1(q)h_1(q)|.$$

Indeed, let $\epsilon > 0$ be given and choose $q \gg 1$ such that

$$\big||(f_1(q) + f_2(q^{-1}))||(k_1(q) + k_2(q^{-1}))| - |f_1(q)k_1(q)|\big| \leq \epsilon$$

and

$$\big||(d_1(q) + d_2(q^{-1}))||(h_1(q) + h_2(q^{-1}))| - |d_1(q)h_1(q)|\big| \leq \epsilon.$$

Then we have

$$\begin{aligned} |f_1(q)||k_1(q)| &\leq |(f_1(q) + f_2(q^{-1}))||(k_1(q) + k_2(q^{-1}))| + \epsilon \\ &\leq |(d_1(q) + d_2(q^{-1}))||(h_1(q) + h_2(q^{-1}))| + \epsilon \\ &\leq |d_1(q)||h_1(q)| + 2\epsilon, \end{aligned}$$

which proves the claim.

Therefore

$$\deg f_1 \leq \deg f_1 + \deg k_1 \leq \deg d_1 + \deg h_1.$$

Now, the denominator of $f_1(w)$ depends only on the two-sided cell containing w, again by Corollary 2 and Theorem 33. We can also bound $\deg h_1$ uniformly in terms of $\tilde{W}$, as it depends only on the Plancherel measure and the finitely many possible formal degrees appearing in the parametrization of the I-spherical part of the tempered dual of G. This proves the lemma. □

Proposition 2. *There is a map of $\mathscr{A}$-algebras $\phi_1 \colon J \otimes_{\mathbb{Z}} \mathscr{A} \to \boldsymbol{\mathcal{H}}^-$ such that if $\phi_1(t_w) = \sum_x a_{x,w} T_x$, then $a_{x,w}(q) = f_w(x)$. Moreover, there is a constant N depending only on $\tilde{W}$ such that $a_{y,w} \in (\mathbf{q}^{1/2})^N \hat{\mathscr{A}}^-$ for all $w, y \in \tilde{W}$.*

Proof. Let $w \in \tilde{W}$. Write $\tilde{\phi}(t_w) = f_w = \sum_x A_{x,w} T_x$ as Schwartz functions on G. We must show that there is a unique element $a_{x,w} \in \hat{\mathscr{A}}$ such that $a_{x,w}(q) = A_{x,w}$ as complex numbers, where $A_{x,w} := f_w(x)$. We will then check that $a_{x,w} \to 0$ rapidly enough as $\ell(x) \to \infty$ for $\sum_x a_{x,w} T_x$ to define an element of $\boldsymbol{\mathcal{H}}^-$.

By Corollary 2 and Theorem 33 , there is a formal power series in $\hat{\mathscr{A}}^-$ with constant term equal to 1 that specializes to the denominator of $A_{1,w}$ when $\mathbf{q} = q$. Moreover, there is a unique formal Laurent series $a_{1,w} \in \hat{\mathscr{A}}$ such that $a_{1,w}(q) = A_{1,w}$ for all prime powers.

Indeed, $a_{1,w}$ is convergent for $\mathbf{q} = q$, and the difference of any two such series defines a meromorphic function of $\mathbf{q}^{-1/2}$ outside the unit disk with zeros at $q = p^r$ for every $r \in \mathbb{N}$. As these prime powers accumulate at ∞, such a meromorphic function must be identically zero.

If $f \in \mathscr{C}^{I \times I}$ is a Harish-Chandra Schwartz function, then

$$\begin{aligned} q^{-\ell(x)}(f \star T_{x^{-1}})(1) &= q^{-\ell(x)} \int_G f(g) T_{x^{-1}}(g^{-1}) \, \mathrm{d}\mu_I(g) \\ &= q^{-\ell(x)} \int_{IxI} f(g) \, \mathrm{d}\mu_I(g) = q^{-\ell(x)} \mu_I(IxI) f(x) = f(x). \end{aligned}$$

By definition, $f_w(x) = A_{x,w}$. On the other hand, according to Lemma 2, we have

$$\begin{aligned}
q^{-\ell(x)}(f_w \star T_{x^{-1}})(1) &= q^{-\ell(x)}\left(\tilde{\phi}(t_w) \star \tilde{\phi}\left(\phi_q(T^{\dagger}_{x^{-1}})\right)\right)(1) \\
&= q^{-\ell(x)}\tilde{\phi}\left(t_w\phi_q(T^{\dagger}_{x^{-1}})\right)(1) \\
&= q^{-\ell(x)}\tilde{\phi}\left(t_w\phi_q\left(\sum_{y\leq x^{-1}} q^{\frac{\ell(y)}{2}}(-1)^{\ell(x^{-1})-\ell(y)}Q_{y,x^{-1}}(q)(C'_y)^{\dagger}\right)\right)(1) \qquad (2.22)\\
&= q^{-\ell(x)}\tilde{\phi}\left(t_w\phi_q\left(\sum_{y\leq x^{-1}} q^{\frac{\ell(y)}{2}}(-1)^{\ell(x^{-1})}Q_{y,x^{-1}}(q)(C_y)\right)\right)(1) \qquad (2.23)\\
&= q^{-\ell(x)}\tilde{\phi}\left(t_w\sum_{y\leq x^{-1}} q^{\frac{\ell(y)}{2}}(-1)^{\ell(x^{-1})}Q_{y,x^{-1}}(q)\sum_{\substack{r\\ d\in\mathscr{D}\\ a(d)=a(r)}} h_{y,d,r}t_r\right)(1) \qquad (2.24)\\
&= q^{-\ell(x)}\tilde{\phi}\left(\sum_{y\leq x^{-1}} q^{\frac{\ell(y)}{2}}(-1)^{\ell(x^{-1})}Q_{y,x^{-1}}(q)\sum_{\substack{r\\ d\in\mathscr{D}\\ a(d)=a(r)=a(w)}} h_{y,d,r}t_wt_r\right)(1) \\
&= q^{-\ell(x)}\sum_{y\leq x^{-1}} q^{\frac{\ell(y)}{2}}(-1)^{\ell(x^{-1})}Q_{y,x^{-1}}(q)\sum_{\substack{r\\ a(r)=a(w)}} h_{y,d_w,r}(f_w\star f_r)(1), \qquad (2.25)
\end{aligned}$$

where d_w is the unique distinguished involution in the left cell containing w. In line (2.22), we rewrote $T_{x^{-1}}$ in terms of the C'-basis of $\mathbf{H}$, using the inverse Kazhdan-Lusztig polynomials $Q_{y,x^{-1}}$. In line (2.23), we applied the Goldman involution (see Lemma 1). In line (2.24) we applied Lusztig's map ϕ_q, and then in line (2.25), we applied the map $\tilde{\phi}$.

We use (2.25) to define $a_{x,w} \in \hat{\mathscr{A}}$. By the same arguments as above, $a_{x,w}$ is unique and defines a meromorphic function of $\mathbf{q}^{-1/2}$. It remains to show that as $\ell(x) \to \infty$, $a_{x,w} \to 0$ in the $(\mathbf{q}^{-1/2})$-adic topology. This follows in fact from (2.25). Indeed, the product $f_w \star f_r$ is an $\mathbb{N}$-linear combination of functions f_z, and the values $f_z(1)$ are rational functions of q, the numerators of which have uniformly bounded degree in q by Lemma 9. The polynomials $h_{y,d_w,r}$ have bounded degree in q (for example in terms of the a-function). Finally, the degree in q of

$$q^{-\ell(x)}q^{\frac{\ell(y)}{2}}Q_{y,x^{-1}}(q)$$

is at most

$$q^{-\ell(x)}q^{\frac{\ell(y)}{2}}q^{\frac{\ell(x^{-1})-\ell(y)-1}{2}} = q^{-\ell(x)}q^{\frac{\ell(y)}{2}}q^{\frac{\ell(x)-\ell(y)-1}{2}} = q^{\frac{-\ell(x)-1}{2}} \to 0 \qquad (2.26)$$

as $\ell(x) \to \infty$. This completes the definition of ϕ_1 as a map of $\mathscr{A}$-modules.

It is easy to see that ϕ_1 is a morphism of rings, essentially because $\tilde{\phi}$ is. Indeed, we have

$$\phi_1(t_wt_{w'}) = \sum_z \gamma_{w,w',z^{-1}}\sum_x a_{z,x}T_x \qquad (2.27)$$

while on the other hand

$$\phi_1(t_w)\phi_1(t_{w'}) = \sum_x a_{w,x}T_x \cdot \sum_y a_{w',y}T_y \qquad (2.28)$$

and when $\mathbf{q} = q$, we have that (2.28) becomes by definition

$$\tilde{\phi}(t_w) \star \tilde{\phi}(t_{w'}) = \tilde{\phi}(\sum_z \gamma_{w,w',z^{-1}} t_z) = \sum_z \sum_r \gamma_{w,w',z^{-1}} A_{z,r} T_r.$$

Hence for infinitely many prime powers we have that the specializations of (2.27) agrees with those of (2.28), and hence (2.27) is equal to (2.28) in $\boldsymbol{\mathcal{H}}^-$. A similar argument shows that ϕ_1 preserves units. $\square$

Remark 14. The proof, specifically (2.26), gives a necessary condition for an element of $\boldsymbol{\mathcal{H}}^-$ to belong to the image of ϕ_1: the coefficients must decay asymptotically at least as fast as $\mathbf{q}^{-\frac{\ell(x)}{2}}$.

Proposition 3. *There is a commutative diagram*

$$\begin{array}{ccccc} \mathbf{H} & \longrightarrow & J \otimes_{\mathbb{Z}} \mathscr{A} & \xrightarrow{\phi_1} & \boldsymbol{\mathcal{H}}^- \\ & & \big\downarrow{\scriptstyle \phi^{-1}} & & \big\uparrow \\ & & \boldsymbol{\mathcal{H}}^-_C & \xrightarrow{()^\dagger} & \boldsymbol{\mathcal{H}}^-_{C'}, \end{array}$$

and we have $\phi_1 = ()^\dagger \circ \phi^{-1}$ as morphisms of $\mathscr{A}$-algebras $J \otimes_{\mathbb{Z}} \mathscr{A} \to \boldsymbol{\mathcal{H}}^-$. In particular, $a_{x,w}$ has integer coefficients for all $x, w \in \tilde{W}$.

Proof. The second claim follows from the first if we show that ϕ_1 extends to a continuous morphism $\boldsymbol{\mathcal{J}} \to \boldsymbol{\mathcal{H}}^-$, by density of $\phi(\mathbf{H})$ in $\boldsymbol{\mathcal{J}} \simeq \boldsymbol{\mathcal{H}}^-_C$, and the third claim follows from the second.

Note that as $\tilde{\phi} \circ \phi_q = ()^\dagger$ on H for all q, we have that $\phi_1 = ()^\dagger \circ \phi^{-1}$ on $\phi(\mathbf{H})$. This says that the diagram commutes.

We now show that ϕ_1 extends to a continuous map $\boldsymbol{\mathcal{J}} \to \boldsymbol{\mathcal{H}}^-_C$. Let $\sum_w b_w t_w$ define an element of $\boldsymbol{\mathcal{J}}$ and define

$$\phi_1 \left(\sum_w b_w t_w \right) = \sum_y b'_y T_y,$$

where

$$b'_y = \sum_w b_w a_{y,w}.$$

We must first show that this infinite sum of elements of $\hat{\mathscr{A}}$ is well-defined. By Lemma 9 and (2.25), we have that there is N such that $a_{y,w} \in (\mathbf{q}^{1/2})^M \hat{\mathscr{A}}^-$ for all w, y. Therefore b'_y is well-defined, and as $a_{y,w} \to 0$ as $\ell(y) \to \infty$, we have $b'_y \to 0$ as $\ell(y) \to \infty$. Therefore ϕ_1 extends to $\boldsymbol{\mathcal{J}}$.

To show continuity, it suffices to show that if $\{\sum_x b_{x,n} t_x\}_n$ is a sequence of elements of $\boldsymbol{\mathcal{J}}$ tending to 0 as $n \to \infty$, then

$$\sum_x b_{x,n} \phi_1(t_x) = \sum_y b'_{y,n} T_y \to 0$$

as $n \to \infty$ in $\boldsymbol{\mathcal{H}}^-$, where $b'_{y,n} = \sum_w b_{w,n} a_{y,w}$. For all $R > 0$, there is $N > 0$ such that $n > N$ implies $b_{x,n} \in (\mathbf{q}^{-1/2})^R \hat{\mathscr{A}}^-$ for all x. We have seen that there is M depending only on $\tilde{W}$ such that $a_{y,w} \in (\mathbf{q}^{1/2})^M \hat{\mathscr{A}}^-$ for all w, y. Therefore $b'_{y,n} \to 0$ as $n \to \infty$, because $b_{w,n} \to 0$ as $n \to \infty$. $\square$

Remark 15. In type A, all appeals to Lemma 9 can be replaced with consequences of the integration order used in the proof of Corollary 2.

Note that Proposition 3 means in particular that $\phi_1(J) \subset \boldsymbol{\mathcal{H}}^-_C$.

Corollary 3. *We have*

$$(\phi \circ ()^{\dagger})^{-1}(t_1) = f_1(1) \sum_{w \in \tilde{W}} q^{-\ell(w)} T_w.$$

Corollary 4. *The map $\tilde{\phi}$ defined in* [15] *and recalled in Theorem 28 is injective.*

Proof. We must show that $\tilde{\phi}(j) \neq 0$. By injectivity of ϕ_1, we have $\phi_1(j) \neq 0$. By definition of the map ϕ_1, this means that there exists $u, s \in G^{\vee}$ such that $us = su$ with s compact and a representation ρ of $\pi_0(Z_{G^{\vee}}(u,s))$ such that $jK(u,s,\rho,q_0) \neq 0$. But then $jK(u,s,\rho,q) \neq 0$, $K(u,s,\rho,q)$ being a different specialization of the restriction of the same J-module $E(u,s,\rho)$ as for $K(u,s,\rho,q_0)$, and $K(u,s,\rho,q)$ is also tempered. It follows that $\tilde{\phi}(j) \neq 0$. □

2.4.4 Proof of Conjecture 2 for GL_n

Theorem 34. *Let $\tilde{W}$ be of type $\tilde{A}_n$. Then Conjecture 2 is true.*

Proof. The third statement follows from the standard formula for the formal degree of the Steinberg representation and multiplicativity of the formal degree under external tensor product.

By Corollary 2 and Propositions 2 and 3 show that $a_{1,x}$ is a rational function of $\mathbf{q}$ for all w. Then equation (2.25) implies that $a_{x,w}$, being a sum of rational functions with Laurent polynomial coefficients, is a rational function of $\mathbf{q}$ for all x. The same equation, together with the fact that $J_{\mathbf{c}}$ is a two-sided ideal for each cell $\mathbf{c}$ shows that the denominator of $a_{x,w}$ depends only on the two-sided cell containing w. This proves the first claim. The second claim now follows from the first claim and the first statement of Corollary 2 and the fact that $\tilde{W}$ has finitely-many two-sided cells. □

2.4.5 Proof of Theorem 19

Theorem 35. *Let $\tilde{W}$ be an affine Weyl group, $\mathbf{H}$ its affine Hecke algebra over $\mathscr{A}$, and J its asymptotic Hecke algebra. Let $\phi\colon \mathbf{H} \hookrightarrow J \otimes_{\mathbb{Z}} \mathscr{A}$ be Lusztig's map.*

1. *For all $w, x \in \tilde{W}$, $a_{x,w}$ is a rational function of $\mathbf{q}$. The denominator of $a_{x,w}$ is independent of x. As a function of w, it is constant on two-sided cells.*
2. *There is a polynomial $P_{\mathbf{G}}(\mathbf{q})$ depending only on $\mathbf{G}$ such that upon writing*

$$()^{\dagger} \circ \phi^{-1}(t_w) = \sum_{x \in \tilde{W}} a_{x,w} T_x,$$

we have

$$P_{\mathbf{G}}(\mathbf{q}) a_{x,w} \in \mathscr{A}$$

for all $x, w \in \tilde{W}$. The roots of $P_{\mathbf{G}}(\mathbf{q})$ are all roots of unity. If d is a distinguished involution in the lowest two-sided cell, then $a_{1,d} = 1/P_W(q)$ exactly.

3. *Moreover, $P_{\mathbf{G}}(\mathbf{q}) d(\omega) \in \mathscr{A}$ for all discrete series representations ω of $\mathbf{H}$.*

Proof. Theorem 33 together with Propositions 2 and 3 show that $a_{1,w}$ is a rational function of $\mathbf{q}$ with denominator depending only on the two-sided cell containing w. Equation (2.25) again shows that $a_{x,w}$ is a rational function of $\mathbf{q}$ with denominator depending only on the two-sided cell containing

w; up to twists the set $\mathscr{E}_2(M)$ is finite for every Levi subgroup M, so we may multiply through to include the denominators of all required formal degrees, which are in fact rational of the correct form by Theorem 25. As there are finitely-many Levi subgroups of G up to association, this defines a polynomial $P^1_{\mathbf{G}}(\mathbf{q})$ satisyfing the first two statements of the theorem. Moreover, Theorems 25 and 33 give that $a_{x,w}$ has poles occurring only in a finite set of roots of unity. The statement about distinguished involutions in the lowest two-sided cell follows from Equation (2.11). The third statement follows immediately from Theorem 25 by putting $P_{\mathbf{G}}(\mathbf{q}) = \Delta(\mathbf{q})P^1_{\mathbf{G}}(\mathbf{q})$. □

2.5 Further example: $\mathbf{G} = \mathrm{Sp}_4$

In this section we will provide some further evidence for Conjecture 2 by proving it in type $\tilde{C}_2$. In the case, the entire Plancherel formula, not just the Iwahori-spherical part, has been computed explicitly by Aubert and Kim in [4]. Set $\mathbf{G} = \mathrm{Sp}_4$ and $G = \mathbf{G}(F)$. We will use their calculations to prove a final

Theorem 36. *Conjecture 2 is true for* $\mathbf{G} = \mathrm{Sp}_4$.

We will first briefly recall some facts about Sp_4 and notation from [4]. The Poincaré polynomial of Sp_4 is

$$(1+\mathbf{q})(1+\mathbf{q}+\mathbf{q}^2+\mathbf{q}^3) = (1+\mathbf{q})^2(1+\mathbf{q}^2).$$

Denote the three proper Levi subgroups of G by writing the maximal torus $M_\emptyset \simeq \mathrm{GL}_1(F)\times\mathrm{GL}_1(F)$, and putting

$$M_s := \left\{ \begin{pmatrix} A & 0 \\ 0 & w_0{}^t A w_0^{-1} \end{pmatrix} \middle| A \in \mathrm{GL}_2(F) \right\} \simeq \mathrm{GL}_2(F)$$

where w_0 is the longest element in the finite Weyl group, and

$$M_h := \left\{ \begin{pmatrix} a & & \\ & A & \\ & & a^{-1} \end{pmatrix} \middle| A \in \mathrm{Sp}_2(F),\ a \in F^\times \right\} \simeq \mathrm{GL}_1(F)\times \mathrm{Sp}_2(F).$$

Then under the Plancherel decomposition $f = \sum_M f_M$, according to [4], p. 14, we have

$$f_{M_h}(1) = c(G/P_h)^{-2}\gamma(G/P_h)\frac{1}{\#W(M_h)}\frac{1}{2\pi i}\frac{q-1}{q+1}\int_{\mathbb{T}}\frac{q^3(z-q)(z-q^{-1})}{(z-q^{-2})(z-q^2)}\,\mathrm{trace}(\pi(f))\,\frac{\mathrm{d}z}{z}. \tag{2.29}$$

and

$$f_{M_s}(1) = c(G/P_s)^{-2}\gamma(G/P_s)\frac{1}{\#W(M_s)}\frac{q-1}{2(q+1)}\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{(z^2-1)^2}{(z+q^{-\frac12})(z-q^{-\frac32})(z-q^{\frac32})(z+q^{\frac12})}\,\mathrm{trace}(\pi(f))\,\frac{\mathrm{d}z}{z} \tag{2.30}$$

and $f_{M_\emptyset}(1)$ is equal, up to a sign, to

$$\frac{q^{11}}{1+q+q^2+q^3}\frac{1}{(2\pi i)^2}$$
$$\int_{\mathbb{T}\setminus\{\pm 1\}}\int_{\mathbb{T}\setminus\{\pm 1\}} \frac{q^4(z_2-1)^2(z_1-1)^2(z_1z_2-1)^2(z_1-z_2)^2}{(z_1-q^{-1})(z_1-q^{-1}z_2)(z_1-qz_2)(z_1-q)(z_2-q^{-1})(z_2-q)(z_1z_2-q)^2}\,\mathrm{trace}(\pi(f))\,\frac{\mathrm{d}z_1}{z_1}\frac{\mathrm{d}z_2}{z_2} \tag{2.31}$$

Proof of Theorem 36. By the same arguments using Propositions 2 and 3 as for GL_n, it suffices to check that the functions f_w are of the required form. We will check this, and along the way check that the same is true of the formal degrees we shall require.

The first step is to note that, according to [4], p. 6, all the $c(G/P)^{-1}$-factors and γ-factors that appear in equations (2.29) and (2.30) are either polynomial or have denominators dividing $P_{\mathbf{G}/\mathbf{B}}(q)$. The same is obviously true for all the formal degrees that appear. Recalling Lemma 5, it suffices in the case of (2.29) to compute that

$$\int_{\mathbb{T}} \frac{q^3(z-q)(z-q^{-1})}{(z-q^{-2})(z-q^2)} z^e\,\mathrm{d}z = \begin{cases} \frac{q^{-2e-2}(q^{-2}-q)}{1+q+q^2+q^3} & \text{if } e \geq 0 \\ \frac{q^{-2e-2}(q^{-2}-q)}{1+q+q^2+q^3} + Q & \text{if } e < 0 \end{cases},$$

and that for any $e \in \mathbb{Z}$, where Q is a Laurent polynomial in q.

In the case of (2.30), for any $e \in \mathbb{Z}$, the integral in question involves summing over residues at poles z_0, $z = q^{-3/2}$, and $z = q^{-1/2}$, and is a sum of terms of the form

$$\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{(z^2-1)^2 z^e}{(z+q^{-\frac{1}{2}})(z-q^{-\frac{3}{2}})(z-q^{\frac{3}{2}})(z+q^{\frac{1}{2}})}\,\mathrm{d}z = Q + \frac{q^{-3}q^{9/2}q^{-3e/2}}{(1+q)(1+q^2)} + \frac{q^e q^{1/2}}{(1+q)^2},$$

where Q is a Laurent polynomial in $q^{1/2}$.

As for the principal series, let $e_1, e_2 \in \mathbb{Z}$. It will be clear from the computations below that if $e_1 \geq 0$ or $e_2 \geq 0$, we obtain only a strict subset of the summands below. Therefore it suffices to assume that $e_1, e_2 < 0$ and prove that every summand below has denominators as required.

The integral (2.31) is then a sum of terms of the form

$$\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\mathbb{T}}\int_{\mathbb{T}} \underbrace{\frac{q^4(z_2-1)^2(z_1-1)^2(z_1z_2-1)^2(z_1-z_2)^2 z_1^{e_1}z_2^{e_2}}{(z_1-q^{-1})(z_1-q^{-1}z_2)(z_1-qz_2)(z_1-q)(z_2-q^{-1})(z_2-q)(z_1z_2-q)^2}}_{H(z_1,z_2)}\,\mathrm{d}z_1\mathrm{d}z_2 \\
&= \frac{1}{2\pi i}\int_{\mathbb{T}} \mathrm{Res}_{z_1=0}H(z_1,z_2) + \mathrm{Res}_{z_1=q^{-1}}H(z_1,z_2) + \mathrm{Res}_{z_1=q^{-1}z_2}H(z_1,z_2)\,\mathrm{d}z_2 \\
&= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{p(z_2)(z_2-1)^2 z_2^{e_2}}{(z_2-q^{-1})(z_2-q)} + \frac{q-1}{q+1}\frac{(z_2-1)(z_2^2-q)^2 z_2^{e_2+1}}{(z_2-q^2)(z_2+q)(z_2-q^{-1})} \\
&+ \frac{1-q}{1+q}\frac{q^{4-e_1}(1-z_2)(z_2-q)^2(z_2-q^{-1})z_2^{e_2}}{(q^{-2}-z_2)(z_2-q^2)^2}\,\mathrm{d}z_2,
\end{aligned}$$

where p is a polynomial.

The first summand has pole at $z_2 = 0$, and a simple pole at $z_2 = q^{-1}$. The residue at $z_2 = 0$ is a Laurent polynomial in q. The residue at $z_2 = q^{-1}$ is

$$\frac{p(q^{-1})}{1+q}.$$

The second summand has a possible pole at $z_2 = 0$, and a simple pole at $z_2 = q^{-1}$. The residue at $z_2 = 0$ has denominator $1 + q$. The residue at $z_2 = q^{-1}$ is

$$-\frac{(1-q)(1-q^2)(1-q^3)q^{-e_2-2}}{(1+q)(1+q+q^2+q^3)}$$

and again is as required.

The third summand has a pole at $z_2 = 0$, the residue of which has denominator $1 + q$, as well as a pole at $z_2 = q^{-1}$. The residue at $z_2 = q^{-1}$ is

$$\frac{(1+q)(1-q^3)(1+q+q^2)(q^{-2}-q^{-1})q^{-e_1-2e_2}}{(1+q)(1+q+q^2+q^3)^2}.$$

These computations show that the denominator of $f_w(1)$ divides a power of $P_{\mathbf{G}/\mathbf{B}}(q)$, for all $w \in \tilde{W}$. By the same argument as before, the denominators of $f_w(x)$ divide a power of $P_{\mathbf{G}/\mathbf{B}}(q)$ for any $x \in \tilde{W}$. The theorem follows. □

Chapter 3

Closeup: On Lusztig's asymptotic Hecke algebra for SL_2

3.1 Introduction

In this chapter we continue with the conventions from Chapter 2. Namely, let $\mathbf{G}$ be a split connected reductive algebraic group, let $\mathbf{H}$ be the corresponding affine Hecke algebra, and let J be the corresponding asymptotic Hecke algebra in the sense of Lusztig. When $\mathbf{G} = \mathrm{SL}_2$, and the parameter $\mathbf{q}$ is specialized to a prime power q, Braverman and Kazhdan showed in [15] that in this case, H has codimension two as a subalgebra of J, and described a basis for the quotient in spectral terms. It follows that in fact H has codimension two as a subalgebra of J for generic q. In this chapter we write these functions explicitly in terms of the basis $\{t_w\}$ of J, and further invert the canonical isomorphism between the completions of H and J, obtaining explicit formulas for each basis element t_w in terms of the basis $\{T_w\}$ of H. We conjecture some properties of this expansion for more general groups. We conclude by using our formulas to prove that J acts on the Schwartz space of the basic affine space of SL_2, and produce some formulas for this action.

This chapter therefore recovers the results of Chapter 2, but in much stronger form, as we obtain information not only about denominators, but also about numerators. In light of the results of Section 2.4.3, and with the aim of preserving compatibility with the published version [23] of this chapter, we will simplify notation by writing $H := \mathbf{H}$, and $\mathscr{H} := \boldsymbol{\mathcal{H}}_C^-$, and work only with the prime power q, using silently that Propositions 2 and 3 translate these into formulas with $\mathbf{q}$ replacing q.

3.1.1 The asymptotic Hecke algebra

For $\mathbf{G}$ a connected reductive algebraic group, a specialization of the affine Hecke algebra H corresponding to the affine Weyl group $\tilde{W}$ of $\mathbf{G}$ plays an important role in the representation theory of $G = \mathbf{G}(F)$ for a p-adic field F. Explicitly, given a smooth representation π of G, a function $f \in H$ yields an endomorphism $\pi(f)$ of π^I, where I is the Iwahori subgroup of G.

In [42], Lusztig defined the asymptotic Hecke algebra J, which is a $\mathbb{Z}$-algebra with basis $\{t_z\}_{z\in\tilde{W}}$

equipped with an injection $\phi\colon H \hookrightarrow J \otimes_{\mathbb{Z}} \mathscr{A}$ given by

$$\phi\left(\sum_{x\in\tilde{W}} b_x C_x\right) = \sum_{\substack{x,z\in\tilde{W}\\ d\in\mathscr{D},\ a(d)=a(z)}} b_x h_{x,d,z} t_z,$$

where $\mathscr{D}$ is the set of distinguished involutions and a is Lusztig's a-function; see Section 3.2.1 and Definition 35. Multiplication (see Remark 18) in J, and the definition of the map ϕ is given combinatorially in terms of the structure constants for H written in the $\{C_w\}$ basis. It was also shown in [42] that ϕ is an isomorphism after a certain completion, whose details we recall in Section 3.2.2.

In [15], the authors found an interpretation of J as certain $I \times I$-invariant functions on $G(F)$ and described the corresponding endomorphisms $\pi(f)$.

The purpose of this chapter is to study the map ϕ in more detail (in the case of SL_2) in order to obtain an explicit, as opposed to spectral, description of the elements of J as functions on $G(F)$. In what follows it will be convenient to twist ϕ by an involution j of H described in Section 3.2.1. Then our first main result is as follows: we give a formula for $(\phi\circ j)^{-1}(t_w)$ for all w by an explicit calculation in a self-contained way. The resulting formulas are given in Theorem 38 and Corollary 5. As a byproduct we obtain the following result:

Theorem 37. *1. For any w the element $(\phi\circ j)^{-1}(t_w) \in \mathscr{H}$ has the form*

$$\sum a_{w,x} C'_x$$

where $a_{w,x}$ is a polynomial in $q^{-\frac{1}{2}}$. Moreover, $(-1)^{\ell(x)}a_{w,x}$ has nonpositive integer coefficients.

2. For any w the element $(\phi\circ j)^{-1}(t_w) \in \mathscr{H}$ has the form

$$\sum b_{w,x} T_x$$

where $(q+1)b_{w,x}$ is a polynomial in $q^{-\frac{1}{2}}$.

Let us remark that if we work with a finite Coxeter group instead of an affine one, then while the second assertion of Theorem 37 remains true (in general $q+1$ must be replaced by the Poincaré polynomial of the corresponding flag variety), the first assertion is wrong in that case. In fact, it is clear that for finite Coxeter groups if some of the coefficients $b_{w,x}$ are genuine rational functions (i.e. not polynomials) then the same will also be true for some of the $a_{w,x}$.

We conjecture that similar statements hold more generally.

Conjecture 4. *For any split connected reductive group G and any $w \in \tilde{W}$, we have*

$$(\phi\circ j)^{-1}(t_w) = \sum a_{w,x} C'_x$$

where $a_{w,x}$ is a polynomial in $q^{-\frac{1}{2}}$ such that $(-1)^{\ell(x)}a_{w,x}$ has nonpositive coefficients. Similarly, we conjecture that

$$(\phi\circ j)^{-1}(t_w) = \sum b_{w,x} T_x$$

where $(\sum_{w\in W} q^{\ell(w)})b_{w,x}$ *is a polynomial in* $q^{-1/2}$ *(note that the sum in parentheses is over the finite Weyl group).*

Conjecture 4 (if true) is very interesting from a geometric point of view, and one can hope that the coefficients carry representation-theoretic information. More specifically, it would be extremely interesting to categorify J with its basis $\{t_w\}$. By this we mean the following: Let $\mathscr{K} = \mathbb{C}((z)), \mathscr{O} = \mathbb{C}[[z]]$, and consider the ind group-scheme $G(\mathscr{K})$. Let $\mathscr{F}\ell = G(\mathscr{K})/I$ denote the affine flag variety. Then the Iwahori-Hecke algebra H is the Grothendieck ring of the bounded derived category of mixed I-equivariant constructible sheaves on $\mathscr{F}\ell$. Under this isomorphism the elements C'_x correspond to the classes of irreducible perverse sheaves. The above conjecture suggests that the elements t_w correspond to some canonical ind-objects in the above derived category. Moreover, these objects should have the property that every simple perverse sheaf appears there, shifted according to Lusztig's a function (see Definition 35). It would be extremely interesting to find a construction of these objects.

The key simplification in type $\tilde{A}_1$ that allows the computations carried out in this note is the simple nature of the affine Weyl group and that the Kazhdan-Lusztig polynomials are all constant and equal to one, so that each C'_w is a constant function. Geometrically, this corresponds to smoothness of I-orbit closures in $\mathscr{F}\ell$. Exact formulas for the elements t_w seem to be unlikely in higher rank, when these simplifications are not present.

3.1.2 Further results

In Section 3.3 we show in an elementary way that J acts on $C_c^\infty(G/N)^I$, reproving in an elementary (in that we make make no serious use of the theory of harmonic analysis on p-adic groups, and use no algebraic geometry whatsoever) way a result of [15], and that J lies in the Harish-Chandra Schwartz space of G. These results are recorded as Propositions 7 and 8, and Theorem 40. Let $\mathscr{S}_c = C_c^\infty(G/N)$ and let $\mathscr{S}$ be the Schwartz space of the basic affine space as in [14]. In [15], it is proved that the direct summand J_0 of J corresponding to the big cell in $\tilde{W}$ is exactly the space of endomorphisms of $\mathscr{S}^I$ commuting with all Fourier transforms and all translations by cocharacters of a fixed maximal torus in G, and that $J_0 \cdot \mathscr{S}_c^I = \mathscr{S}^I$. In this way knowledge of $\mathscr{S}^I$ is equivalent to knowledge of J_0, which in the case of SL_2 is just $J_0 = \operatorname{span}\{t_w\}_{w\neq 1}$.

3.2 Formulas for the map ϕ

3.2.1 Preliminaries

Throughout, π is a uniformizer of a fixed non-archimedean local field F with ring of integers $\mathscr{O}$, and q is the cardinality of the residue field $\mathscr{O}/\pi\mathscr{O}$ (although until Section 3.3 we can also view it as an indeterminate). We shall write $G = \mathrm{SL}_2$ as algebraic groups. When there is no room for confusion, we write G for $G(F)$ as well. We fix the Borel subgroup B of upper triangular matrices, and write $I \subset G(\mathscr{O})$ for the corresponding Iwahori subgroup. Put $\tilde{W}$ for the affine Weyl group of G ,with length function ℓ and set S of simple reflections. Let H be the Iwahori-Hecke algebra of G, over the ring $\mathscr{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. We recall that H has a basis $\{T_w\}_{w\in\tilde{W}}$, where multiplication is defined by relations $T_wT_{w'} = T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$ and quadratic relation $(T_s + 1)(T_s - q) = 0$ for

$s \in S$. Additionally, we have the Kazhdan-Lusztig basis

$$C_w = \sum_{y \leq w} (-1)^{\ell(w)-\ell(y)} q^{\frac{\ell(W)-\ell(y)}{2}} P_{y,w}(q^{-1}) q^{-\frac{\ell(y)}{2}} T_y$$

and the basis $\{C'_w\}_{w\in\tilde{W}}$, which we recall is related to the $\{C_w\}_{w\in\tilde{W}}$ basis by $C'_w = (-1)^{\ell(w)} j(C_w)$. Here j is the algebra involution on H defined in [36] by $j(\sum a_w T_w) = \sum \bar{a_w}(-1)^{\ell(w)} q^{-\ell(w)} T_w$, where $\bar{(\,)}\colon \mathscr{A} \to \mathscr{A}$ is the involution defined by $\overline{q^{\frac{1}{2}}} = q^{-\frac{1}{2}}$. The bar involution of $\mathscr{A}$ extends to the bar involution of H, and we have $\bar{C_w} = C_w$ and $\bar{C'_w} = C'_w$ for all w. Several definitions will be given in terms of the structure constants of H in the basis $\{C_w\}$, and we write $h_{x,y,z}$ to mean those elements of $\mathscr{A}$ such that $C_x C_y = \sum_z h_{x,y,z} C_z$.

Let $\alpha\colon \mathrm{diag}(a, a^{-1}) \mapsto a^2$ be the positive root of SL$_2$, and $\alpha^\vee$ the corresponding coroot. Write $X_*(A)$ for the cocharacter group of the maximal torus A of diagonal matrices. From now on, $\tilde{W} = W \ltimes X_*(A) = W \ltimes \mathbb{Z}\langle\alpha^\vee\rangle$ is the affine Weyl group for $G = \mathrm{SL}_2$, with fixed presentation $\tilde{W} = \langle s_0, s_1 \,|\, s_0^2 = s_1^2 = 1\rangle$. We write $S = \{s_0, s_1\}$, with s_1 the affine reflection, so that $W = \langle s_0 \rangle$ is the finite Weyl group. When working with this presentation, all the words we write down will be reduced. The identification between this presentation and the semidirect product realization of $\tilde{W}$ sends s_0 to the simple reflection s_α corresponding to α, and s_1 corresponds to $s_\alpha \pi$, where $\pi = \pi^{\alpha^\vee}$. Our convention is that α is dominant, so that dominant coweights correspond to positive integers, with $\pi^n = \pi^{n\alpha^\vee} = (s_0 s_1)^n$ being dominant,and $\pi^{-n} = (s_1 s_0)^n$ being antidominant. The distinguished involutions in $\tilde{W}$ are $\mathscr{D} = \{1, s_0, s_1\}$. We remark that as an abstract group, $\tilde{W}$ is the infinite dihedral group, with s_0 and s_1 playing symmetric roles. However, as seen above, under the identification we have fixed, the finite and affine simple reflections play different roles. There is however an automorphism of H exchanging T_{s_0} and T_{s_1}, see Section 3.3.2. In our special case, we have

$$C'_w = q^{-\frac{\ell(w)}{2}} \sum_{y \leq w} T_y,$$

where $\leq$ is the strong Bruhat order *i.e.* $y \leq w$ if and only if after writing a reduced word for w and deleting some letters, we obtain a word for y.

Example 15. We have $C'_e = 1 = T_e$ is the unit in H, where e is the unit element in $\tilde{W}$, and

$$C'_{s_0 s_1 s_0} = q^{-\frac{3}{2}} \left(T_{s_0 s_1 s_0} + T_{s_1 s_0} + T_{s_0 s_1} + T_{s_0} + T_{s_1} + 1\right).$$

3.2.2 The map ϕ

Proposition 4 ([42], Section 2.4)**.** *The map $\phi\colon H \to J \otimes_{\mathbb{Z}} \mathscr{A}$ defined in Section 3.1.1 is a morphism of algebras.*

We now recall the details of the completion mentioned above. Let $\hat{\mathscr{A}}$ be the ring of formal Laurent series in $q^{\frac{1}{2}}$, and let $\hat{\mathscr{A}}^+$ be the ring of formal power series in $q^{\frac{1}{2}}$. We obtain a completion $\mathscr{H}$ of H whose elements are (possibly infinite) $\hat{\mathscr{A}}$-linear combinations $\sum_x b_x C_x$ such that $b_x \to 0$ in the (q)-adic topology on $\hat{\mathscr{A}}^+$ *i.e.* such that for any $N > 0$, $b_x \in (q^{\frac{1}{2}})^N \hat{\mathscr{A}}^+$ for $\ell(x)$ sufficiently large. When working with the basis $\{C'_w\}_{w\in\tilde{W}}$, we complete with respect to the negative powers of q. The involution j naturally extends to a homeomorphism between these different completions. In the same way, we obtain a completion $\mathscr{J}$ of $J \otimes_{\mathbb{Z}} \mathscr{A}$. The definition of ϕ (see Proposition 4) carries over verbatim, yielding an isomorphism $\phi\colon \mathscr{H} \xrightarrow{\sim} \mathscr{J}$.

Over the course of the next three lemmas, we shall see that the definition of this map simplifies considerably in our case. We first recall two special cases of results of Lusztig. We refer to the exposition in [49] for this material. There Lusztig writes T_w for our $q^{-\frac{\ell(w)}{2}}T_w$, c_w for our C'_w, and in our case $p_{y,w} = q^{\frac{-\ell(w)+\ell(y)}{2}}$. We write $\mathscr{R}(w) = \{s \in S \mid ws < w\}$. If $w = rs_i$ is nontrivial, $\mathscr{R}(w) = \{s_i\}$ is a singleton.

Lemma 10 ([49], Corollary 6.7)**.** *Let $w \in \tilde{W}$ and $s = s_i$. Then*

$$C_wC_s = \begin{cases} -\left(q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right)C_w & \text{if } s \in \mathscr{R}(w) \\ \sum_{\substack{|\ell(w)-\ell(y)|=1 \\ ys<y}} C_y & \text{if } s \notin \mathscr{R}(w) \end{cases}.$$

Definition 35 (Lusztig's a function.)**.** For $w \in \tilde{W}$, define $a(w)$ to be the smallest integer such that $(-q)^{\frac{a(w)}{2}}h_{x,y,w} \in \mathscr{A}^+$ for all $x, y \in \tilde{W}$.

Lemma 11 ([49], Section 13.4, Lemma 13.5, Proposition 13.7)**.** *Let $w \in \tilde{W}$. If $w = 1$, then $a(w) = 0$. Otherwise $a(w) = 1$.*

Assembling Lemmas 10 and 11, we can describe ϕ explicitly.

Lemma 12. *Let $i \neq j$ and $i, j \in \{0, 1\}$. Then*

$$\phi(C_{s_i}) = -\left(q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right)t_{s_i} + t_{s_is_j}.$$

More generally, if $\ell(w) \geq 2$ and $w = rs_i$, then

$$\phi(C_w) = -\left(q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right)t_{rs_i} + t_r + t_{rs_is_j}.$$

Proof. We need only note that the condition $ys_j < y$ from Lemma 10 implies y ends in s_j. □

Recall that the unit in J is $1_J = t_{s_0} + t_{s_1} + t_1$, the sum of the basis elements corresponding to distinguished involutions. As ϕ preserves units, we have $\phi(C_1) = t_1 + t_{s_1} + t_{s_0}$.

Definition 36. If w and y are elements in $\tilde{W}$, we say that w *starts with* y if we have reduced expressions $y = s_{i_1} \cdots s_{i_n}$ and $w = s_{i_1} \cdots s_{i_n} s_{i_{n+1}} \cdots s_{i_{n+m}}$ for some $m \geq 0$.

Lemma 13. *We have*

$$\phi\left(\sum_{\substack{w\in\tilde{W} \\ w \text{ starts with } s_0}} q^{\frac{\ell(w)}{2}}C_w\right) = -t_{s_0},$$

and likewise

$$\phi\left(\sum_{\substack{w\in\tilde{W} \\ w \text{ starts with } s_1}} q^{\frac{\ell(w)}{2}}C_w\right) = -t_{s_1}.$$

Proof. Under ϕ, the infinite sum $\sum_{\substack{w\in\tilde{W}\\ w \text{ starts with } s_0}} q^{\frac{\ell(w)}{2}} C_w$ is sent to

$$q^{\frac{1}{2}}\left(-\left(q^{\frac{1}{2}}+q^{-\frac{1}{2}}\right)t_{s_0}+t_{s_0s_1}\right) \tag{3.1}$$
$$+q\left(-\left(q^{\frac{1}{2}}+q^{-\frac{1}{2}}\right)t_{s_0s_1}+t_{s_0}+t_{s_0s_1s_0}\right) \tag{3.2}$$
$$+q^{\frac{3}{2}}\left(-\left(q^{\frac{1}{2}}+q^{-\frac{1}{2}}\right)t_{s_0s_1s_0}+t_{s_0s_1}+t_{s_0s_1s_0s_1}\right) \tag{3.3}$$
$$+\cdots.$$

By Lemma 12, cancellation of terms appearing in $\phi(C_w)$ with $\ell(w)=n$ can occur only against terms appearing in $\phi(C_m)$ with $|n-m|=1$, and we see that after cancellations between the terms on lines (3.1) through (3.3), corresponding to lengths at most 3, the sum stands as

$$-t_{s_0}-q^2 t_{s_0s_1s_0}+t_{s_0s_1s_0s_0s_1}+\text{terms from longer words.}$$

Further, if r starts with s_0 and $w=rs_0$, the term $-q^{\frac{\ell(w)-1}{2}}q^{\frac{1}{2}}t_r$ from $\phi(C_r)$ cancels with the term $q^{\frac{\ell(w)}{2}}t_r$ coming from $\phi(C_w)$, and the term $q^{\frac{\ell(w)-1}{2}}t_w$ from $\phi(C_r)$ cancels with the term $-q^{\frac{\ell(w)}{2}}q^{-\frac{1}{2}}t_w$ in $\phi(C_w)$. Likewise the terms $-q^{\frac{\ell(w)}{2}}q^{\frac{1}{2}}t_w$ cancels with a term from $\phi(C_{ws_1})$ and $q^{\frac{\ell(w)}{2}}t_{ws_1}$ cancels with the term $-q^{\frac{\ell(w)+1}{2}}q^{-\frac{1}{2}}t_{ws_1}$ from $\phi(C_{ws_1})$. The case for w ending in s_1 is identical, and cancellations happen between terms from two words ending both in s_0. The calculation for t_{s_1} is identical. □

The formula for ϕ^{-1} is implicit in the proof Lemma 13. Indeed, the lemma upgrades to

Lemma 14. *Let $y=s_{i_1}\cdots s_{i_n}$, and let $i=i_n$. Then*

$$\phi\left(\sum_{\substack{w\in\tilde{W}\\ w \text{ starts with } y}} q^{\frac{\ell(w)}{2}}C_w\right)=-q^{\frac{\ell(y)-1}{2}}t_y+q^{\frac{\ell(y)}{2}}t_{ys_i}.$$

Proof. Direct calculation as in Lemma 13. Let s_j be the generator that is not s_i. Then the first terms are

$$q^{\frac{\ell(y)}{2}}\left(-\left(q^{\frac{1}{2}}+q^{-\frac{1}{2}}\right)t_y+t_y s_i+t_{ys_j}\right)+q^{\frac{\ell(y)+1}{2}}\left(-\left(q^{\frac{1}{2}}+q^{-\frac{1}{2}}\right)t_{ys_j}+t_y+t_{ys_js_i}\right)+\cdots,$$

and the cancellations in the proof of Lemma 13 pick up from this point, leaving only $-q^{\frac{\ell(y)-1}{2}}t_y+q^{\frac{\ell(y)}{2}}t_{ys_i}$. □

We can therefore calculate $\phi^{-1}(t_y)$ up to an error term of length $\ell(ys_i)<\ell(y)$. Given that we can calculate $\phi(t_{s_i})$, we can cancel the error terms inductively, yielding a formula for ϕ^{-1}.

Theorem 38. *Let $y=s_{i_1}s_{i_2}\cdots s_{i_n}$ so that $\ell(y)=n>0$, and for $k\leq n$, write $y_k=s_{i_1}\cdots s_{i_k}$. Then*

$$-q^{\frac{n-1}{2}}\phi^{-1}(t_y)=\sum_{k=1}^{n}q^{n-k}\sum_{\substack{w\in\tilde{W}\\ w \text{ starts with } y_k}} q^{\frac{\ell(w)}{2}}C_w$$

Proof. It suffices to prove that the images of the left-hand side and of the right-hand side under ϕ are equal. To do this, apply Lemma 14 to the last $\ell(y)-1$ summands and Lemma 13 to the first. □

Example 16. We calculate $\phi^{-1}(t_{s_0s_1s_0s_1})$, where $n=4$. Under ϕ,

$$\begin{aligned}
&q^2 C_{s_0s_1s_0s_1} + q^{\frac{5}{2}} C_{s_0s_1s_0s_1s_0} + q^3 C_{s_0s_1s_0s_1s_0s_1} + \cdots \\
&+ q\left(q^{\frac{3}{2}} C_{s_0s_1s_0} + q^2 C_{s_0s_1s_0s_1} + q^{\frac{5}{2}} C_{s_0s_1s_0s_1s_0} + q^3 C_{s_0s_1s_0s_1s_0s_1} + \cdots\right) \\
&+ q^2\left(q^{C}_{s_0s_1} + q^{\frac{3}{2}} C_{s_0s_1s_0} + q^2 C_{s_0s_1s_0s_1} + q^{\frac{5}{2}} + C_{s_0s_1s_0s_1s_0} + q^3 C_{s_0s_1s_0s_1s_0s_1} + \cdots\right) \\
&+ q^3 \sum_{\substack{w\in\tilde{W} \\ w \text{ starts with } s_0}} q^{\frac{\ell(w)}{2}} C_w
\end{aligned}$$

is sent to

$$q^2 t_{s_0s_1s_0} - q^{\frac{3}{2}} t_{s_0s_1s_0s_1} + q^{\frac{5}{2}} t_{s_0s_1} - q^2 t_{s_0s_1s_0} + q^3 t_{s_0} - q^{\frac{5}{2}} t_{s_0s_1} - q^3 t_{s_0} = -q^{\frac{3}{2}} t_{s_0s_1s_0s_1}.$$

Corollary 5. *If y is as above, we have*

$$-q^{\frac{1-n}{2}}(\phi\circ j)^{-1}(t_y) = \sum_{k=1}^{n} q^{k-n}\left(\sum_{\substack{w\in\tilde{W} \\ w \text{ starts with } y_k}} \frac{(-1)^{\ell(w)}q^{-\ell(w)+1}}{1+q} T_w + \sum_{\substack{w\in\tilde{W} \\ w \text{ does not start with } y_k \\ \ell(w)\geq k}} \frac{(-1)^{\ell(w)+1}q^{-\ell(w)}}{1+q} T_w \right.$$
$$\left. + \frac{(-1)^k q^{-k+1}}{1+q} \sum_{\substack{w\in\tilde{W} \\ w \text{ does not start with } y_k \\ \ell(w)<k}} T_w \right).$$

The constant factor $q(1+q)^{-1}$ in each summand appears as $\sum_{n=0}^{\infty}(-1)^n q^{-n}$.

3.2.3 The functions f and g

In [15], Braverman and Kazhdan gave a spectral definition of two functions f and g on G, which, viewed as elements in J, span J/H when q is specialized to a prime power.

They are

$$f = T_1 + T_{s_0} + \sum_{n=1}^{\infty} q^{-2n}\left(T_{(s_1s_0)^n} + T_{s_0(s_1s_0)^n} - q\left(T_{(s_0s_1)^n} + T_{s_1(s_0s_1)^n}\right)\right)$$

and

$$g = \sum_{w\in\tilde{W}} (-1)^{\ell(w)} q^{-\ell(w)} T_w$$

We find their images under ϕ and show they lie in J by explicit calculation in Theorem 39.

By [15] equation 4.1, we have $J = \mathrm{End}(\mathrm{St}^I) \oplus J_0$, where St is the Steinberg representation of SL_2, and J_0 is the algebra of endomorphisms of $C_c^\infty(F^2)^I$ that commute with translation and Fourier transform, see Section 3.3.1. The function g is the matrix coefficient of St^I and induces an integral operator spanning $\mathrm{End}(\mathrm{St}^I)$. The function f does not have such a nice description, but the closely-related function $\tilde{f}$ (see equation (3.6)) is defined to be constant on I-orbits on $G(\mathscr{O})\backslash G(F)$

by putting $\tilde{f}\restriction_X = (-q)^{-\dim X - 1}$ for I-orbits X. We conjecture that $\tilde{f}$ thus defined lies in J for any connected reductive group G.

Remark 16. The function f is defined in [15] directly as a function on $\mathrm{SL}_2(\mathscr{O})\backslash\mathrm{SL}_2(F)/I$. Our definition is equivalent, as can be seen by writing

$$\mathrm{SL}_2(\mathscr{O})\cdot\mathrm{diag}(t^n, t^{-n})\cdot I = I\cdot\mathrm{diag}(\pi^n,\pi^{-n})\cdot I \coprod I\cdot\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\mathrm{diag}(\pi^{-n},\pi^n)\cdot I.$$

It is easy to rewrite elements given in the T_w basis to elements given in the C'_w basis; the change of basis is "upper-triangular with monomial entries." Precisely, we have the following

Proposition 5. *We have*

$$T_w = \sum_{y\le w} q^{\frac{\ell(y)}{2}}(-1)^{\ell(w)-\ell(y)}C'_y.$$

Proof. Clearly the proposition is true for $\ell(w)=0$, and for $\ell(w)=1$. Now write $w = s_i r s_j$, so that

$$C'_w = q^{-\frac{\ell(w)}{2}}\left(T_w + T_{rs_j} + T_{s_i r} + \cdots\right) = q^{-\frac{\ell(w)}{2}}\left(T_w + T_{rs_j} + q^{\frac{\ell(s_i r)}{2}}C'_{s_i r}\right)$$

whence

$$q^{\frac{\ell(w)}{2}}C'_w - q^{\frac{\ell(s_i r)}{2}}C'_{s_i r} = T_w + T_{rs_j}.$$

The claim follows by induction on $\ell(w)$. □

We can now rewrite the functions f and g in the C'_w basis, in preparation for applying $\phi\circ j$ to them. In the case of g, we have

$$g = \sum_{w\in\tilde{W}}(-1)^{\ell(w)}q^{-\ell(w)}T_w = \sum_{w\in\tilde{W}}(-1)^{\ell(w)}q^{-\ell(w)}\left(\sum_{y\le w} q^{\frac{\ell(y)}{2}}(-1)^{\ell(w)-\ell(y)}C'_y\right),$$

and we see that the coefficient b_w of C'_w is a power series in $q^{-\frac{1}{2}}$ of order $q^{\frac{\ell(w)}{2}}$. Indeed, C'_w will appear once in the expansion of T_w, and then twice for each length greater than $\ell(w)$, and thus

$$b_w = (-1)^{\ell(w)}q^{-\ell(w)}q^{\frac{\ell(w)}{2}} + 2\left(\sum_{n=\ell(w)+1}^{\infty}(-1)^n(-1)^{n-\ell(w)}q^{\frac{\ell(w)}{2}}q^{-n}\right).$$

For $z\in\tilde{W}$ such that $\ell(z) = n \ge \ell(w)$, $(-1)^n q^{-n}$ is the coefficient of T_z in rewriting g, and $(-1)^{n-\ell(w)}q^{\frac{\ell(w)}{2}}$ is the coefficient of C_w in the expansion of T_z according to Proposition 5. Therefore

$$b_w = (-1)^{\ell(w)}q^{-\frac{\ell(w)}{2}}\left(1 + 2\frac{q^{-1}}{1-q^{-1}}\right),$$

and so

$$g = \left(1 + 2\frac{q^{-1}}{1-q^{-1}}\right)\sum_{w\in\tilde{W}}(-1)^{\ell(w)}q^{-\frac{\ell(w)}{2}}C'_w. \tag{3.4}$$

We note that $1 + 2\frac{q^{-1}}{1-q^{-1}} = 1 + 2q^{-1} + 2q^{-2} + \cdots = \sum_{w\in\tilde{W}} q^{-\ell(w)}$ is a unit in $\mathbb{Z}[\![q^{-\frac{1}{2}}]\!]$.

Rewriting the function f is simpler, in the sense that no infinite series coefficients appear. In order to simplify the eventual calculation, we will work with a related function

$$\tilde{f} = f - T_1 - T_{s_0} = \sum_{m=1}^{\infty} q^{-2m} \left(\underbrace{T_{s_0(s_1s_0)^m}}_{A} + \underbrace{T_{(s_1s_0)^m}}_{B} - q \left(\underbrace{T_{(s_0s_1)^m}}_{C} + \underbrace{T_{s_1(s_0s_1)^m}}_{D} \right) \right). \tag{3.5}$$

The first thing is again to calculate the coefficients b_w such that $\tilde{f} = \sum_{w \in \tilde{W}} b_w C'_w$. For coefficients $b_{s_0s_1}$, we see that instances of C'_w are contributed by the C- and D-type terms starting from $m = n$, and that, for length reasons, almost all the contributions cancel, leaving just $-qq^{-n}$. The type A terms contribute starting from $m = n$, and the type B terms, from $m = n+1$. For the same reason, only the first instance of $C'_{(s_0s_1)^n}$ coming from $T_{(s_0s_1)^n}$ fails to cancel, so that $b_{(s_0s_1)^n} = q^n(-1-q)$.

No terms $C'_{(s_1s_0)^n}$ appear. Indeed, A- and B-type terms both begin contributing at $m = n$, but have contributions with opposite signs. The same goes for C- and D-type terms, which both start contributing from $m = n+1$. For exactly the same reasons (except the A and B-type terms start to contribute at $m = n+1$ as well), no terms $C'_{s_1(s_0s_n)^n}$ appear.

For $b_{s_0(s_1s_0)^n}$, the A-type terms contribute from $m = n$ onwards, and the B-type terms, from $m = n+1$. All contributions except the first cancel, leaving $q^{-n+\frac{1}{2}}$. The type C and D terms contribute from $m = n+1$ and $= n+2$, respectively, with opposite signs as usual. Their contribution simplifies to $qq^{-n-\frac{3}{2}}$, making $b_{s_0(s_1s_0)^n} = q^{-n}(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$.

Therefore

$$\tilde{f} = \sum_{n=1}^{\infty} q^{-n}(-1-q)C'_{(s_0s_1)^n} + q^{-n}\left(q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right)C'_{s_0(s_1s_0)^n}. \tag{3.6}$$

Recall from Section 3.2.3 the functions f and g defined in [15] that form a basis of J/H.

Theorem 39. *We have*

1. $\phi(j(g)) = \left(1 + 2\frac{q}{1-q}\right) t_1$;
2. $\phi(j(\tilde{f})) = \left(q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right) t_{s_0s_1} - (q+1)t_{s_0}$.

Proof. Applying j to equation (3.4), we get $j(g) = \left(1 + 2\frac{q}{1-q}\right) \sum_{w \in \tilde{W}} q^{\frac{\ell(w)}{2}} C_w$. We conclude by adding the results of Lemma 13 together and recalling that ϕ preserves units.

Applying j to expression (3.6), we obtain

$$j(\tilde{f}) = (1 - q^{-1}) \sum_{n=1}^{\infty} q^n C_{(s_0s_1)^n} + q^{n+\frac{1}{2}} C_{s_0(s_1s_0)^n},$$

to which we apply Lemma 14. □

3.3 The elements t_w as functions on G

3.3.1 The Harish-Chandra Schwartz space

From now on, we write t_w for $(\phi \circ j)^{-1}(t_w)$ and we view q as the cardinality of the residue field of F.

Recall that we can interpret H as the convolution algebra $C_c^\infty(I\backslash G/I)$. Using Corollary 5, we can see in an elementary way that the functions t_y lie in the Harish-Chandra Schwartz space $\mathscr{C}(G)$, whose definition we recalled in Section 2.2.3. Note that in our case, we identify coweights with nonnegative integers.

Proposition 6. *The functions defined in Corollary 5 all lie in $\mathscr{C}(G)$.*

Proof. Clearly the t_y are all bi-invariant with respect to the Iwahori subgroup, which is open and closed in the compact subgroup K, as it is the preimage of the discrete group $B(\mathbb{F}_q)$, hence is open compact. Fix y and let $f = t_y$.

Let $g \in K\pi^\lambda K = I\pi^\lambda I \sqcup Is_0\pi^\lambda I \sqcup I\pi^\lambda s_0 I \sqcup I\pi^{-\lambda} I$ for $\lambda = \lambda(g) = n > 0$. Thus g lies in an Iwahori double coset corresponding to an element of $\tilde{W}$ of length $2n \pm 1$. Here π^λ is $(s_0 s_1)^n$. In our case, $\Delta(g) = q^{\lambda(g)}$, and so by Corollary 5, up to a multiplicative scalar depending on f we have $\Delta(g)|f(g)| \leq q^{-n+2}$ if λ is identified with n. We must therefore bound $q^{2-n}(\log(1+|p(g)|))^m$ uniformly in n. If $\lambda(g) = 0$, then $\Delta(g)|f(g)| \leq q^2$ up to the same scalar. Let p and m be given. Then

$$p(g) = p(k_1 a k_2) = \sum_{i=-N_1}^{N_2} (\pi^\lambda)^i p_i(k_1, k_2)$$

where the p_i are polynomials in the eight entries of k_1 and k_2, and $N_1, N_2 \in \mathbb{N}$. Therefore

$$|p(g)| \leq \max_i \left|(\pi^\lambda)^i p_i(k_1, k_2)\right| \leq \max_i |\pi^{ni}| C_p \leq q^{nM_p} C_p$$

for $C_p > 0$ and $M_p \in \mathbb{N}$ depending on p. Then

$$\begin{aligned}
\log(1+|p(g)|) &\leq \log(q^{nM_p} + q^{nM_p} C_p) \\
&= \log(q^{nM_p}(1+C_P)) \\
&= nM_p \log(q(1+C_p)^{1/nM_p}) \\
&\leq nM_p \log(q(1+C_p)) \\
&= nM_p D_p
\end{aligned}$$

with $D_p > 0$. Therefore $M_p^m D_p^m (\log(1+|p(g)|)^{-m} \geq n^{-m}$. By elementary calculus, there is $F_m > 0$ such that $n^m \leq F_m q^n$ for all $n \in \mathbb{N}$. It follows that

$$\frac{1}{q^{n+2}} \leq \frac{1}{q^{n-1}} \leq \frac{q^2 F_m M_p^m D_p^m}{(\log(1+|p(g)|))^m}$$

as required. □

3.3.2 Action on functions on the plane

The plane

Let $N = N(F)$ be the subgroup of upper triangular matrices with 1s on the diagonal, and recall that $G/N = F^2 \setminus \{0\}$. Recalling the Iwasawa decomposition $G = KAN$, where $K = \mathrm{SL}_2(\mathscr{O})$ and A is the maximal torus of diagonal matrices, we see that K-orbits in $F^2 \setminus \{0\}$ are labelled by $\mathbb{Z} = X_*(A)$, and are of the form

$$K\pi^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \pi^n e \\ \pi^n g \end{pmatrix}.$$

if elements of K are written $k = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Note that we cannot have both e and g divisible by π, and therefore K-orbits are precisely of the form $\pi^n \mathscr{O}^2 \setminus \pi^{n+1}\mathscr{O}^2$. Indeed, e and g are not both in $\pi\mathscr{O}$, so one is a unit. If e is a unit, then we may chose $k = \begin{pmatrix} e & 0 \\ g & e^{-1} \end{pmatrix}$. If g is a unit, we may chose $k = \begin{pmatrix} e & -g^{-1} \\ g & 0 \end{pmatrix}$.

Each K-orbit decomposes into two I-orbits. The two cases that partition the points $k\pi^n(1,0)^T$ are $k \in I$ and $k \notin I$. If $k \in I$, then the I-orbit consists of points of the form

$$\begin{pmatrix} \pi^n e \\ \pi^{n+1} g \end{pmatrix} \in \begin{pmatrix} \pi^n \mathscr{O}^\times \\ \pi^{n+1}\mathscr{O} \end{pmatrix} \subset \pi^n\mathscr{O}^2 \setminus \pi^{n+1}\mathscr{O}^2.$$

We denote the characteristic functions of such orbits by ψ_n. The remaining orbit consists of points of the form

$$\begin{pmatrix} \pi^n e \\ \pi^{n+1} g \end{pmatrix} \in \begin{pmatrix} \pi^n \mathscr{O} \\ \pi^{n}\mathscr{O}^\times \end{pmatrix} \subset \pi^n\mathscr{O}^2 \setminus \pi^{n+1}\mathscr{O}^2.$$

We denote the characteristic functions of such orbits by φ_n. The characteristic functions of the closures of these orbits are

$$\bar{\varphi}_n := \sum_{k=n}^{\infty} \varphi_k + \psi_k$$

and

$$\bar{\psi}_n := \sum_{k=n}^{\infty} \psi_k + \varphi_{k+1}.$$

The Iwahori subgroup acts on functions on G/N by translation as $(g \cdot f)(x) = f(g^{-1}x)$, and the functions $\bar{\varphi}_n$ and $\bar{\psi}_n$ give a basis for $C_c^\infty(F^2)^I$. Note that we have, for example, $\varphi_0 = \bar{\varphi}_0 - \bar{\psi}_0$. The functions $\bar{\varphi}_n$ give a basis for $C_c^\infty(F^2)^K$.

Recall also that $I\backslash G/NA(\mathscr{O}) \simeq \tilde{W}$, hence I-invariant functions (which are automatically $A(\mathscr{O})$-invariant) on $F^2 \setminus \{0\}$ are the same as functions on the set of alcoves; in our case, intervals in $\mathbb{R}$ with integer endpoints. A basis for $C_c^\infty(F^2)^I$ is then given under this identification by half lines with integer boundary points, corresponding to semi-infinite orbit closures. For the general construction with a different normalization, see [14]. We now fix some relevant notation and identifications for alcoves. We identify the alcove corresponding to φ_0 with the interval $[-1, 0]$ and the alcove corresponding ψ_0 with the interval $[0, 1]$, so that *e.g.* φ_2 corresponds to $[3, 4]$.

Convolutions

We can now describe how the affine Hecke algebra acts on functions on the plane. The content of the following lemmas is well known; for a general combinatorial description of them with different normalizations, see [47]. It will be useful to observe that the convolution action commutes with the right action of $2\mathbb{Z}$ on the set of alcoves, and that the functions φ_n, ψ_n are periodic in the sense that

$(m\alpha^\vee)\cdot\varphi_n = \varphi_{n+m}$ and likewise for ψ_n.

We view the convolution action as follows: given T_w and the characteristic function χ_X of an I-orbit X, we have a multiplication map

$$IwI \times X \to G/N,$$

which descends to the quotient of the left-hand side by the equivalence relation $(g,x)\sim(gi,i^{-1}x)$ for $i\in I$, yielding a map

$$IwI \underset{I}{\times} X \to G/N.$$

The image of this map is finitely-many I-orbits, and the coefficient of the characteristic function of each orbit is the number of points in the fibre over any point in that orbit.

It will be useful to note that T_{s_0} and T_{s_1} are related by the following automorphism Φ of G. Let Θ be the automorphism given by inverse-transpose, Ψ be conjugation by $\mathrm{diag}(1,\pi)\in\mathrm{GL}_2(F)$, and then $\Phi=\Psi\circ\Theta$. Observe that Φ preserves I, and therefore induces an automorphism of H, which exchanges T_{s_0} and T_{s_1}. In particular, T_{s_1} can be realized as the characteristic function of $K'\setminus I$, where K' is the maximal compact subgroup

$$\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\,\middle|\, a,d\in\mathscr{O},\ c\in\pi\mathscr{O},\ b\in\pi^{-1}\mathscr{O}\right\}.$$

The complement of I is then the subset of such matrices with $b\in\pi^{-1}\mathscr{O}^\times$.

Lemma 15. *We have*

1. $T_{s_0}\star\psi_n=\varphi_n$;
2. $T_{s_0}\star\varphi_n=(q-1)\varphi_n+q\psi_n$;
3. $T_{s_1}\star\varphi_n=\psi_{n-1}$;
4. $T_{s_1}\star\psi_n=(q-1)\psi_n+q\varphi_{n+1}$.

Proof. By periodicity of φ_n and ψ_n and the fact that the action of H commutes with translation, it suffices to prove the formulas in the case $n=0$. To prove the first formula, let $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in K\setminus I := Y$ *i.e.* with $c\in\mathscr{O}^\times$ and let $\mathbf{x}$ be an element in the orbit X corresponding to ψ_0. Then $\mathbf{x}=(x,y)$ with $x\in\mathscr{O}^\times$ and $y\in\pi\mathscr{O}$, and

$$gx=\begin{pmatrix} ax+by\\ cx+dy\end{pmatrix}$$

so that $cx+dy\in\mathscr{O}^\times$, and $ax+by$ is obviously integral. Thus $T_{s_0}\star\psi_0$ is proportional to φ_0. To prove the formula it remains to show that all fibres have size one. Without loss of generality, we may assume that the situation is $g_1(1,0)=g_2(1,0)$ *i.e.* the first columns of g_1 and g_2 agree. It follows that $g_2^{-1}g_1\in N^+(\mathscr{O})$, which stabilizes $(1,0)$ in $N^+\cap I$. Therefore all fibres have size one.

To prove the second formula, let g be as above and let $\mathbf{x}=(x,y)\in\mathscr{O}^2$ with $y\in\mathscr{O}^\times$. Then $g\mathbf{x}$ is an integral vector, and does not lie in $\pi\mathscr{O}^2$ as $\mathbf{x}$ is nonzero modulo π, and g is invertible modulo π.

Therefore $T_{s_0} \star \varphi_0$ is a linear combination of φ_0 and ψ_0. Consider the map

$$\xi\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{a}{c} \mod \pi$$

into $\mathbb{F}_q$, which descends to the quotient Y/I. Therefore the fibre over any point (x, y) in either orbit injects into $\mathbb{F}_q$. In the case where $y \in \mathscr{O}^\times$, taking the fibre over $\mathbf{x} = (0, -1)$ we see that $a \in \mathscr{O}^\times$, so that ξ is into $\mathbb{F}_q^\times$ in this case. If $a \in \mathbb{F}_q^\times$, then

$$\begin{pmatrix} a & 0 \\ 1 & a^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \begin{pmatrix} \mathscr{O} \\ \mathscr{O}^\times \end{pmatrix}$$

is a product of a matrix in $K \setminus I$ with a vector in the orbit corresponding to φ_0. This shows that the coefficient of φ_0 is $q - 1$. For any $a \in \mathbb{F}_q$, we have

$$\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \begin{pmatrix} \mathscr{O}^\times \\ \pi\mathscr{O} \end{pmatrix}.$$

Therefore the coefficient of ψ_0 is q.

The case for the third formula is similar: if the matrices with entries a_i, b_i, c_i, d_i are in I, then

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} & \pi^{-1} \\ -\pi & \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \pi^{-1} a_1 d_1 - \pi b_1 b_2 \\ \pi^{-1} c_1 d_2 - \pi b_2 d_1 \end{pmatrix} \tag{3.7}$$

has top entry in $\pi^{-1}\mathscr{O}^\times$ and bottom entry in $\mathscr{O}$. Indeed, $\pi \nmid a_1$ and $\pi \nmid d_2$, and $\pi \mid c_1$, so the bottom row of (3.7) is integral. Therefore $T_{s_1} \star \varphi_0$ is proportional to ψ_{-1}. To show the fibres all have size one, we can again calculate that any two matrices of the above form whose right columns agree are in the same $N^-(\mathscr{O}) \cap I = \mathrm{Stab}_I((0, 1))$ coset.

For the fourth formula, the fact that we have

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} & \pi^{-1} \\ -\pi & \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 c_2 \pi^{-1} - a_2 b_1 \pi \\ c_1 c_2 \pi^{-1} - a_2 d_1 \pi \end{pmatrix} \in \begin{pmatrix} \mathscr{O} \\ \pi\mathscr{O} \end{pmatrix} \tag{3.8}$$

is clear. We want to see that these products lie in

$$\begin{pmatrix} \mathscr{O}^\times \\ \pi\mathscr{O} \end{pmatrix} \coprod \begin{pmatrix} \pi\mathscr{O} \\ \pi\mathscr{O}^\times \end{pmatrix} \subset \begin{pmatrix} \mathscr{O} \\ \pi\mathscr{O} \end{pmatrix}.$$

The complement of the disjoint union in $(\mathscr{O}, \pi\mathscr{O})^T$ is $(\pi\mathscr{O}, \pi^2\mathscr{O})^T$. Any matrix in K' with its left column in the complement would have determinant in $\pi\mathscr{O}$, and so the products all lie in the disjoint union. Therefore $T_{s_1} \star \phi_0$ is a linear combination of ψ_0 and φ_1. To count points in the fibre, we will use that $T_{s_1} = \chi_{K' \setminus I}$. Define $\xi'\colon K' \setminus I \to \mathbb{F}_q$ by

$$\xi'\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{d}{\pi b} \mod \pi,$$

and note this function is right I-invariant. For any $d \in \mathbb{F}_q$, we have that

$$\begin{pmatrix} 0 & \pi^{-1} \\ -\pi & d \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \pi \end{pmatrix} \in \begin{pmatrix} \pi\mathscr{O} \\ \pi\mathscr{O}^\times \end{pmatrix}$$

is the product of a matrix in $K' \setminus I$ and a vector in X. Therefore the coefficient of φ_1 is q. Taking the fibre over $(1, 0)$, we see that $d \in \mathscr{O}^\times$, so that ξ' is into $\mathbb{F}_q^\times$ in this case. If $d \in \mathbb{F}_q^\times$, then

$$\begin{pmatrix} d^{-1} & \pi^{-1} \\ 0 & d \end{pmatrix} \begin{pmatrix} d \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \begin{pmatrix} \mathscr{O}^\times \\ \pi\mathscr{O} \end{pmatrix}$$

shows that the coefficient of ψ_0 is $q - 1$. □

Assembling the formulas from Lemma 15 and the definitions of $\bar{\varphi}_n$ and $\bar{\psi}_n$ recovers the following fact.

Corollary 6. *The Iwahori-Hecke algebra H acts on $C_c^\infty(F^2)$. We have*

1. $T_{s_0} \star \bar{\varphi}_n = q\bar{\varphi}_n$;
2. $T_{s_1} \star \bar{\psi}_n = q\bar{\psi}_n$;
3. $T_{s_0} \star \bar{\psi}_n = \bar{\varphi}_n - \bar{\psi}_n + q\bar{\varphi}_{n+1}$;
4. $T_{s_1} \star \bar{\varphi}_n = \bar{\psi}_{n-1} - \bar{\varphi}_n + q\bar{\psi}_n$.

Lemma 16. *We have*

1. $T_{(s_1 s_0)^n} \star \psi_m = \psi_{m-n}$;
2. $T_{s_0(s_1 s_0)^n} \star \psi_m = \varphi_{m-n}$;
3. $T_{(s_0 s_1)^n} \star \varphi_m = \varphi_{m-n}$;
4. $T_{s_1(s_0 s_1)^n} \star \varphi_m = \psi_{m-n-1}$;
5. $$T_{(s_1 s_0)^n} \star \varphi_m = q^{2n}\varphi_{m+n} + (q-1)\sum_{k=1}^{2n} q^{2n-k}\psi_{m+n-k};$$
6. $$T_{s_0(s_1 s_0)^n} \star \varphi_m = q^{2n+1}\varphi_{m+n} + (q-1)\sum_{k=0}^{2n} q^{2n-k}\varphi_{m+n-k};$$
7. $$T_{(s_0 s_1)^n} \star \psi_m = q^{2n}\psi_{m+n} + (q-1)\sum_{k=1}^{2n} q^{2n-k}\varphi_{m+n+1-k};$$
8. $$T_{s_1(s_0 s_1)^n} \star \psi_m = q^{2n+1}\varphi_{m+n+1} + (q-1)\sum_{k=0}^{2n} q^{2n-k}\psi_{m+n-k}.$$

Proof. Formulas 1–4 follow directly from Lemma 15, and the remaining formulas follow from 1–4 and another application of the lemma. For example, to prove formula 1, write $T_{(s_1s_0)^n} = T_{s_1}T_{s_0}\cdots T_{s_1}T_{s_0}$ and successively apply formulas 1 and 3 from Lemma 15. Formula 5 is proved by induction on n, the base case being

$$T_{s_1s_0}\star\varphi_m = T_{s_1}T_{s_0}\star\varphi_m = q^2\varphi_{m+1} + (q-1)(q\psi_m + \psi_{m-1}),$$

which again follows from Lemma 15, formulas 2, 3, and 4. Then by induction we have

$$\begin{aligned}T_{s_1s_0}T_{(s_1s_0)^n}\star\varphi_m &= T_{s_1s_0}\star q^{2n}\varphi_{m+n} + (q-1)\sum_{k=1}^{2n} q^{2n-k}\psi_{m+n-k}\\ &= q^{2n+2}\varphi_{m+n+1} + (q-1)q^{2n}\left(q\psi_{m+n}+\psi_{m+n-1}\right) + (q-1)\sum_{k=1}^{2n} q^{2n-k}\psi_{m+n-1-k}\\ &= q^{2n+2}\varphi_{m+n+1} + (q-1)\left(q^{2n+1}\psi_{m+n} + q^{2n}\psi_{m+n-1} + \sum_{k=3}^{2n+2} q^{2n+2-k}\psi_{m+n+1-k}\right),\end{aligned}$$

where between the first and second line we used the base case and formula 1 of this lemma. □

Remark 17. Observe that the formulas in Lemma 16 recover those of Lemma 15 upon specifying n, provided that sums with decreasing indices are interpreted as empty.

We can now describe the action of J on functions on the plane. To begin with, we present an elementary proof of the result from the discussion following equation 4.1 in [15], namely that t_1 acts trivially.

Proposition 7. *We have $t_1\star\psi_m = t_1\star\varphi_m = 0$ for all m.*

Proof. It suffices to check that g (identified with a scalar multiple of t_1 by theorem 39) acts trivially, and for this it suffices to check that $g\star\varphi_0 = g\star\psi_0 = 0$. Now, g sends ψ_0 to

$$\begin{aligned}\psi_0 - q^{-1}(q-1)(\varphi_0 + q\varphi_1 + (q-1)\psi_0) &+ q^{-2}\left(q^2\psi_1 + (q-1)\left(q\varphi_1+\varphi_0\right) + \psi_{-1}\right)\\ &- q^{-3}\left(\varphi_{-1} + q^3\varphi_2 + (q-1)\left(q^2\psi_1 + q\psi_0 + \psi_{-1}\right)\right)\\ &+ q^{-4}\left(\psi_{-2} + q^4\psi_2 + (q-1)\left(q^3\varphi_2 + q^2\varphi_1 + q\varphi_0 + \varphi_{-1}\right)\right)\\ &- q^{-5}\left(\varphi_{-2} + q^5\varphi_3 + (q-1)\left(q^4\psi_2 + q^3\psi_1 + q^2\psi_0 + q\psi_{-1} + \psi_{-2}\right)\right)\\ &+\cdots\end{aligned}$$

and after cancellations between these terms we are left with

$$-q^4\left(q^3\varphi_2 + q^2\varphi_1 + q\varphi_0 + \varphi_{-1}\right) - q^{-5}\left(\varphi_{-2} + q^5\varphi_3 - \left(q^4\psi_2 + q^3\psi_1 + q^2\psi_0 + q\psi_{-1} + \psi_{-2}\right)\right) + \cdots$$

Further, all cancellation of terms corresponding to elements of length l occurs between terms corresponding to lengths $l \pm 2$, and proceeds as follows. We have

$$- q^{-2n+1}\left(\varphi_{-n+1} + q^{2n-1}\varphi_n + (q-1)\sum_{k=0}^{2n-2} q^{2n-2-k}\psi_{n-1-k}\right) \tag{3.9}$$

$$+ q^{-2n}\left(\psi_{-n} + q^{2n}\psi_n + (q+1)\sum_{k=1}^{2n} q^{2n-k}\varphi_{n+1-k}\right) \tag{3.10}$$

$$- q^{-2n-1}\left(\varphi_{-n} + q^{2n+1}\varphi_{n+1} + (q-1)\sum_{k=0}^{2n} q^{2n-k}\psi_{n-k}\right) \tag{3.11}$$

$$+ q^{-2n-2}\left(\psi_{-n-1} + q^{2n+2}\psi_{n+1} + (q-1)\sum_{k=1}^{2n+2} q^{2n+2-k}\varphi_{n+2-k}\right) \tag{3.12}$$

$$- q^{-2n-3}\left(\varphi_{-n-1} + q^{2n+3}\varphi_{n+2} + (q-1)\sum_{k=0}^{2n+2} q^{2n+2-k}\psi_{n+1-k}\right), \tag{3.13}$$

where line (3.9) corresponds to $T_{s_0(s_1s_0)^{n-1}}\star\psi_0+T_{s_1(s_0s_1)^{n-1}}\star\psi_0$, line (3.10) corresponds to $T_{(s_1s_0)^n}\star\psi_0 + T_{(s_0s_1)^n}\star\psi_0$ and so on up to line (3.13) corresponding to $T_{s_0(s_1s_0)^{n+1}}\star\psi_0 + T_{s_1(s_0s_1)^{n+1}}\star\psi_0$.

We will explain the cancellation of the terms in line (3.11); the cancellation of terms in odd-numbered lines follows the same pattern. The lead term in line (3.11) cancels with the final term in q times the sum in line (3.12), and the second cancels with the first term in q times the sum. The first and last terms in q times the sum in line (3.11) cancel with the leading terms of line (3.10), and the middle terms cancel with -1 times the sum in line (3.9). The terms in -1 times the sum in line (3.11) cancel with the middle terms of q times the sum in line (3.13).

The cancellations in $g\star\varphi_0$ follow the same pattern. □

Lemma 17. *We have (note that none of the sums below contains a T_1 term)*

1.
$$\sum_{\substack{w\in\tilde{W}\\ w \text{ starts with } s_0}} (-1)^{\ell(w)}q^{-\ell(w)}T_w\star\varphi_m = -\bar{\varphi}_m;$$

2.
$$\sum_{\substack{w\in\tilde{W}\\ w \text{ starts with } s_1}} (-1)^{\ell(w)}q^{-\ell(w)}T_w\star\varphi_m = \bar{\psi}_m;$$

3.
$$\sum_{\substack{w\in\tilde{W}\\ w \text{ starts with } s_0}} (-1)^{\ell(w)}q^{-\ell(w)}T_w\star\psi_m = \bar{\varphi}_{m+1};$$

4.
$$\sum_{\substack{w\in\tilde{W}\\ w \text{ starts with } s_1}} (-1)^{\ell(w)}q^{-\ell(w)}T_w\star\psi_n = -\bar{\psi}_n.$$

Proof. It suffices by periodicity of φ_m, ψ_m to prove the lemma for $m = 0$. We evaluate each convolution term-by-term, and then explain the cancellations that occur between adjacent terms. After accounting for the contributions of the first few terms, this gives the results of the lemma.

In the case of formula 1, we have adjacent terms of the form

$$- q^{-2n+1}\left(\underbrace{\overbrace{q^{2n+1}\psi_{n-1}}^{D}+(q-\overbrace{1}^{A})\sum_{k=0}^{2n-2}q^{2n-2-k}\varphi_{n-1-k}+}_{T_{s_0(s_1s_0)^{n-1}}}\right)+q^{-2n}\underbrace{\overbrace{\varphi_{n-1}}^{C}}_{T_{(s_0s_1)^n}}$$

$$- q^{-2n-1}\left(q^{2n+1}\psi_n+(\overbrace{q}^{B}-1)\sum_{k=0}^{2n}q^{2n-k}\varphi_{n-k}\right).$$

Adding the contributions $A+B+C+D$ gives $-(\varphi_n+\psi_n)$. The other terms cancel out similarly by induction. Starting this procedure from $n=1$ captures the contributions of all terms starting from T_{s_0}, although we must add the contribution of the first D- and B-type terms. Thus formula 1 is proved.

In the case of formula 2, we have adjacent terms of the form

$$q^{-2n+2}\left(\underbrace{\overbrace{q^{2n-2}\varphi_{n-1}}^{H}+(q-\overbrace{1}^{E})\sum_{k=1}^{2n-2}q^{2n-2-k}\psi_{n-1-k}}_{T_{(s_1s_0)^{n-1}}}\right)-q^{-2n+1}\underbrace{\overbrace{\psi_{-n}}^{L}\,{}_{T_{s_1(s_0s_1)^{n-1}}}}$$

$$+ q^{-2n}\left(+q^{2n}\varphi_n+(\overbrace{q}^{F}-1)\sum_{k=1}^{2n}q^{2n-k}\psi_{n-k}\right).$$

Adding terms $E+F+L+H$ gives $\varphi_{n-1}+\psi_{n-1}$. We can start this cancellation from $n=2$, adding the contributions of the first type L and F terms. This proves formula 2.

The remaining formulas follow the same pattern. □

Proposition 8. *For all m:*

1. *We have $t_{s_0}\star\bar{\varphi}_m=\bar{\varphi}_m$, and $t_{s_0}\star\bar{\psi}_m=0$. Thus t_{s_0} acts by a projector*

$$C_c^\infty(F^2)^I\twoheadrightarrow C_c^\infty(F^2)^K.$$

2. *We have: $t_{s_1}\star\bar{\psi}_m=\psi_m$, and $t_{s_1}\star\bar{\varphi}_m=0$. Therefore t_{s_1} acts as $\mathrm{id}-t_{s_0}$.*

Proof. It is enough to prove the proposition for $m=0$. We first calculate $t_{s_0}\star(\varphi_0+\psi_0)$, then using periodicity we will obtain formulas for $t_{s_0}\star(\varphi_n+\psi_n)$. The last step will be to take

$$t_{s_0}\star\bar{\varphi}_0=\sum_{n=0}^{\infty}t_{s_0}\star(\varphi_n+\psi_n).$$

Indeed, it follows from Corollary 5 and Lemma 17 that

$$-q^{-1}(1+q)(t_{s_0}\star(\varphi_0+\psi_0))=-(1+q^{-1})(\varphi_0+\psi_0)$$

so that $t_{s_0} \star \bar{\varphi}_0 = \bar{\varphi}_0$. The first statement follows. Again using periodicity to calculate $t_{s_0} \star (\psi_n + \varphi_{n+1})$, we get that $t_{s_0} \star \bar{\psi}_n = 0$. Therefore t_{s_0} kills all basis functions that are not K-invariant.

The calculation for t_{s_1} is similar. □

Remark 18. In this chapter, we have made essentially no use of the ring structure on J, but we note that these formulas reflect the fact that t_{s_0} and t_{s_1} are idempotents whose sum is the identity element in J_0, the based ring of the lowest two-sided cell (see Section 4.2.2).

Theorem 40. *The algebra J acts on $C_c^\infty(F^2)^I$.*

Proof. The last sentence of Proposition 8 says that the identity of J acts on $C_c^\infty(F^2)^I$ by the identity endomorphism; recall that we have shown t_1 acts trivially in Proposition 7. By Corollary 6, the action of H on $C_c^\infty(F^2)^I$ is well-defined. By Proposition 8, both t_{s_0} and t_{s_1} have well-defined actions. Now, using the first formula of Lemma 12, we see that $t_{s_i s_j}$ has a well-defined action. Then, using the second formula of that lemma, we see that $t_{s_i s_j s_i}$ has a well-defined action, and so on. □

Chapter 4

Coherent picture: A categorification of J_0

4.1 Introduction

In this chapter, we give a triangulated monoidal category of coherent sheaves whose Grothendieck group surjects onto J_0, the based ring of the lowest two sided cell of an affine Weyl group. Our category is equipped with a monoidal functor from the category of coherent sheaves on the derived Steinberg variety. We show that our almost-categorification acts on natural coherent categorifications of the Iwahori invariants of the Schwartz space of the basic affine space. In low-rank cases, we construct complexes that lift the basis elements t_w of J_0 and their structure constants.

This chapter deals only with the coherent side of Bezrukavnikov's equivalence [8], and is the first step towards a putative version of it for J. In particular, in the conventions of the preceeding chapters, only objects associated to $\mathbf{G}^\vee$ are discussed in this chapter. To unburden notation, in this chapter only, we exchange the roles of $\mathbf{G}$ and $\mathbf{G}^\vee$ and phrase our results as being about a connected simply-connected reductive group $\mathbf{G}$ defined over $\mathbb{C}$. As well, we will write $G = \mathbf{G}$ as algebraic groups, H for $\mathbf{H}$, and generally forgo boldface everywhere except in Section 4.4.2.

4.2 Functions and algebras

In this section we will recall the various algebras whose categorifications we will discuss in Section 4.3. There is no new material in this section, although we could not find a recollection of all the relationships below in one place in the existing literature.

4.2.1 The affine Hecke algebra

Let G be a connected simply-connected reductive group defined over $\mathbb{C}$, with chosen Borel subgroup B and maximal torus $T \subset B$. Write $\mathscr{B}^\heartsuit = G/B$ for the classical flag variety of G. Let X^* be the character lattice of T and let $\tilde{W} = W \ltimes X^*$, where W is the finite Weyl group of G. Let H be the corresponding affine Hecke algebra over $\mathscr{A} = \mathbb{Z}[\mathbf{q}^{1/2}, \mathbf{q}^{-1/2}]$ with standard basis $\{T_w\}_{w \in \tilde{W}}$. The multiplication in this, the Coxeter presentation, of H is determined by $T_w T_{w'} = T_{ww'}$ when

$\ell(ww') = \ell(w) + \ell(w')$ and the quadratic relation $(T_s + 1)(T_s - \mathbf{q}) = 0$ for all $s \in S$, where $S \subset \tilde{W}$ is the set of simple reflections. The Coxeter presentation is well-suited to studying the action of H on admissible representations, and to the constructible categorification of H.

4.2.2 The based ring of the lowest two-sided cell

Recall the definition of the asymptotic Hecke algebra from Chapter 2. In this chapter, we discuss only the two-sided ideal corresponding to the lowest two-sided cell. This summand is also referred to in the literature as the *based ring of the lowest two-sided cell*.

Definition 37. *Lusztig's a-function* $a\colon \tilde{W} \to \mathbb{N}$ is defined such that $a(w)$ is the minimal value such that $q^{\frac{a(w)}{2}} h_{x,y,w} \in \mathscr{A}^+$ for all $x, y \in \tilde{W}$.

It is known that a is constant on two-sided cells of $\tilde{W}$ and that

$$a(\mathbf{c}) = \dim \mathscr{B}_u^{\heartsuit}$$

where u is the unipotent conjugacy class in G corresponding to $\mathbf{c}$ under Lusztig's bijection. It is also known that $a(w) \leq \ell(w)$ for all $w \in \tilde{W}$.

In [42] Lusztig defined an associative algebra J over $\mathbb{Z}$ equipped with an injection $\phi\colon H \hookrightarrow J \otimes_{\mathbb{Z}} \mathscr{A}$ which becomes an isomorphism after taking a certain completion of both sides. As an abelian group, J has a basis $\{t_w\}_{w \in W}$. Recalling the Kazhdan-Lusztig basis elements

$$C_w = \sum_{y \leq w} (-1)^{\ell(w)-\ell(y)} \mathbf{q}^{\frac{\ell(w)}{2} - \ell(y)} P_{y,w}(\mathbf{q}^{-1}) T_y,$$

the structure constants of J are obtained from those in H written in the $\{C_w\}_{w \in W}$-basis under the following procedure. Using the structure constants

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z$$

for $h_{x,y,z} \in \mathscr{A}$, Lusztig then defines the integer $\gamma_{x,y,z}$ by the condition

$$q^{\frac{a(z)}{2}} h_{x,y,z^{-1}} - \gamma_{x,y,z} \in q\mathscr{A}^+.$$

The product in J is then defined as

$$t_x t_y = \sum_z \gamma_{x,y,z} t_{z^{-1}}.$$

One then defines

$$\phi(C_w) = \sum_{\substack{z \in W,\ d \in \mathscr{D} \\ a(z) = a(d)}} h_{x,d,z} t_z,$$

where $\mathscr{D} \subset \tilde{W}$ is the set of distinguished involutions. The elements t_d for distinguished involutions d are orthogonal idempotents. Moreover, $J = \bigoplus_{\mathbf{c}} J_{\mathbf{c}}$ is a direct sum of two-sided ideals indexed by two-sided cells $\mathbf{c} \subset \tilde{W}$. The unit element in each summand is $\sum_{\mathscr{D} \cap \mathbf{c}} t_d$, and the unit element of J is $\sum_{d \in \mathscr{D}} t_d$.

The lowest two-sided cell

Let $\mathbf{c}_0$ be the lowest two-sided cell, also called the "big cell." It can be characterized as the two-sided cell containing the longest element of W, and we have $a(\mathbf{c}_0) = \ell(w_0)$. The summand $J_0 := J_{\mathbf{c}_0}$ is particularly well-understood, and has historically been the first summand for which any structure-theoretic result has been achieved (consider, for example, the progression [67], [69], [10]). Write ϕ_0 for ϕ composed with the projection $J \otimes_{\mathbb{Z}} \mathscr{A} \to J_0 \otimes_{\mathbb{Z}} \mathscr{A}$.

By [67] and [54], we have the following description of $\mathbf{c}_0 \subset \tilde{W}$. Let $\tilde{\mathbf{c}_0}$ be the lowest cell of the affine Weyl group of the universal covering group $\tilde{G}$ of G with maximal torus $\tilde{T}$. Then $\mathbf{c}_0 = \tilde{\mathbf{c}}_0 \cap \tilde{W}$, and

$$\tilde{\mathbf{c}_0} = \left\{ f^{-1} w_0 \chi g \,\middle|\, f, g \in \Sigma, \chi \in X^*(\tilde{T})^+ \right\},$$

where $\Sigma = \{w x_w \,|\, x \in W\} \subset \tilde{W}(\tilde{G})$, where

$$x_w = w^{-1} \left(\prod_{\substack{\alpha \in \Delta \\ w^{-1}(\alpha) < 0}} \varpi_\alpha \right) \in X^*(\tilde{T}), \tag{4.1}$$

where ϖ_α is the fundamental dominant weight corresponding to α. We have

$$t_{f^{-1} w_0 \lambda g} t_{(f')^{-1} w_0 \nu g'} = 0$$

if $g \neq f'$, and

$$t_{f^{-1} w_0 \lambda g} t_{g^{-1} w_0 \nu g'} = \sum_\mu m^\mu_{\lambda,\nu} t_{f^{-1} w_0 \mu g'},$$

where $m^\mu_{\lambda,\nu}$ is the multiplicity of $V(\mu)$ in $V(\lambda) \otimes V(\nu)$.

On a theorem of Steinberg

Steinberg showed in [64] that $K_{\tilde{T}}(\mathrm{pt})$ is a free $K_{\tilde{G}}(\mathrm{pt})$ module with basis $\{x_w\}_{w \in W}$. Under the isomorphism $K_T(\mathrm{pt}) \simeq K_{\tilde{G}}(\mathscr{B}^\heartsuit)$, the x_w define an $K_{\tilde{G}}(\mathrm{pt})$- basis $\{\mathscr{F}_w\}_w$ of the latter ring, where $\mathscr{F}_w = \mathscr{O}_{\mathscr{B}^\heartsuit}(x_w)$. In [37], Kazhdan and Lusztig show that the natural pairing

$$\langle -, - \rangle : K_{\tilde{G}}(\mathscr{B}^\heartsuit) \otimes_{K_{\tilde{G}}(\mathrm{pt})} K_{\tilde{G}}(\mathscr{B}^\heartsuit) \to K_{\tilde{G}}(\mathrm{pt})$$

is nondegenerate. While the dual basis is often employed in the literature starting from *loc. cit.*, we are not aware of an explicit description of it. We provide one here in very low rank cases in type A. The lack of a description in other cases of the dual basis elements as classes in K-theory of some natural objects of $\mathrm{Coh}_{\tilde{G}}(\mathscr{B}^\heartsuit)$ is the only obstruction to proving Proposition 13 in greater generality.

Lemma 18. *Let $\tilde{G} = \mathrm{SL}_2$ or SL_3. The collection $\mathscr{G}_w = \mathscr{O}(y_w)[\ell(w)]$, where*

$$y_w = \left(w^{-1} \prod_{\substack{\alpha \in \Delta \\ w^{-1}(\alpha) > 0}} \varpi_\alpha \right) \rho^{-1}$$

defines the basis dual to Steinberg's basis of $K_{\tilde{G}}(\mathscr{B}^{\heartsuit})$ under the above pairing.

Example 17. In type A_1 and additive notation, we have $x_1 = 0$ and $x_{s_\alpha} = s_\alpha(\varpi_\alpha) = 1 - 2 = -1$. In this case the Steinberg basis is self-dual, with $y_1 = \varpi_\alpha - \rho = 1 - 1 = 0$ and $y_{s_\alpha} = s_\alpha(0) - \rho = -1$.

The lemma can be proved by direct computation, for example by computer.

Remark 19. The classes $[\mathscr{G}_w]$ cease to pair correctly with the Steinberg basis classes starting for $G = \mathrm{SL}_4$, in a way apparently governed by singularities of Schubert cells. For example, for SL_4, one has

$$\langle [\mathscr{F}_w], [\mathscr{G}_1] \rangle = \mathrm{triv}_{\mathrm{SL}_4}$$

where $w = 1$, or when $w = \sigma$ is the product of the two permutations in $\mathfrak{S}_4$ that index singular Schubert varities. In this case, the element dual to $[\mathscr{F}_1]$ is $[\mathscr{G}_1] + [\mathscr{G}_\sigma]$. We hope to produce natural complexes in future work that will lift these sums and pair correctly.

4.3 Sheaves and categories

In this section we will recall the absolute minimal amount of derived algebraic geometry required to state and prove our main results. We will need only a very modest version of the theory. Sections 4.3.1 and 4.3.3 recall the necessary material to define the category $\mathscr{J}_0$ and contain no new material.

4.3.1 Derived schemes

Definition 38. A *commutative differential graded algebra* or *cdga* is an associative graded-commutative algebra $A^\bullet = \bigoplus_i A^i$ endowed with a differential $d\colon A^\bullet \to A^\bullet$ such that $d \circ d = 0$, and

$$d(ab) = (da)b + (-1)^k a(db),$$

where $a, b \in A^k$ are homogeneous elements of degree k. A *morphism* from one cdga to another is a morphism of complexes that respects the multiplication, or, equivalently, a morphism of graded algebras that respects differentials.

Definition 39. A cdga $A^\bullet$ is *Noetherian* if $H^0(A^\bullet)$ is Noetherian and $H^i(A^\bullet)$ is a finitely-generated $H^0(A^\bullet)$-module for all i.

The grading and differential means that one way to view a dg-algebra $A = \bigoplus A^i$ is as a cochain complex

$$\cdots \xrightarrow{d} A^i \xrightarrow{d} A^{i+1} \xrightarrow{d} A^{i+2} \xrightarrow{d} \cdots$$

of abelian groups. The algebra structure and its compatibility with the differential mean that this cochain complex is a monoid object in the category of cochain complexes of abelian groups: the multiplication $A \otimes A \to A$ is a chain map, where $\otimes$ is the tensor product of cochain complexes.

Definition 40. The *homotopy groups* of A are defined to be $\pi_n(A) = H^{-n}(A)$.

Definition 41. A dg-algebra A is *connective* if $\pi_n(A) = 0$ for $n < 0$. It is *eventually connective* if there exists N such that $\pi_n(A) = 0$ for $n \geq N$.

Definition 42. If A is a dg-algebra, a *dg-A-module* M is a graded A-module with differential d_M such that the action map $A \otimes M \to M$ is a morphism of cochains, where we think of $M = \bigoplus_i M^i$ as a cochain complex of A^0-modules. In particular, $\pi_i(M)$ becomes an $\pi_0(A)$-module for all i.

Definition 43. The category of *affine derived schemes* is the opposite category to the category of connective commutative differential-graded algebras. We say that $\operatorname{Spec} A^\bullet$ is *Noetherian* if A is. If $X = \operatorname{Spec} A^\bullet$, then $\operatorname{QCoh}(X)$ is the category of dg-A-modules, and $\operatorname{Coh}(X)$ is the category of quasi-coherent sheaves on X whose homotopy sheaves are coherent sheaves on the classical truncation and have bounded cohomological amplitude.

All schemes, derived or otherwise, that we will encounter will be Noetherian.

Let $(X, \mathscr{O}_X)$ be a scheme, and let $\mathscr{E}^\bullet$ be a connective complex of $\mathscr{O}_X$-modules with a multiplication map

$$\mathscr{E}^i \otimes_{\mathscr{O}_X} \mathscr{E}^j \to \mathscr{E}^{i+j}$$

that respects the differential, such that $\pi_0(\mathscr{E}^\bullet) = \mathscr{O}_X$ and $\pi_i(\mathscr{E}^\bullet)$ is a quasi-coherent $\pi_0(\mathscr{E}^\bullet)$-module for all i. Then there is a derived scheme

$$\mathbf{Spec}\,(\mathscr{E}^\bullet).$$

We say that its *classical truncation* is

$$\mathbf{Spec}\,(\mathscr{E}^\bullet)^\heartsuit = \mathbf{Spec}\,\big(H^0(\mathscr{E})\big) = X.$$

Definition 44. We say $\mathbf{Spec}\,\mathscr{E}^\bullet$ is *Noetherian* if $\pi_i(\mathscr{E}^\bullet)$ is a coherent $\mathscr{O}_X$-module for all i.

A morphism

$$\mathbf{Spec}\,\mathscr{E}^\bullet \to \mathbf{Spec}\,\mathscr{F}^\bullet$$

is the data of a morphism of $\mathscr{F}^\bullet \to \mathscr{E}^\bullet$ of complexes of $\mathscr{O}_X$-modules that respects the multiplication. There are morphisms

$$\mathbf{Spec}\,\mathscr{E}^\bullet \to X$$

and

$$X \to \mathbf{Spec}\,\mathscr{E}^\bullet.$$

We will refer to them as the bundle projection and inclusion of the zero-section, respectively. From now on, we will also drop the explicit bullet notation.

One should think that all the geometric content of a derived scheme X is found in the underlying scheme, and that the higher homotopy groups are to X as nilpotent elements of a ring A are to $\operatorname{Spec}(A)$.

This is a very incomplete account of derived algebraic geometry, but it is almost sufficient for our purposes. The only morphisms that we will see below not constructed in the above manner arise by fibre products, which exist in the category of derived schemes.

In fact, all of the derived schemes we will encounter will appear as derived fibre products of classical schemes. Indeed, if X, Y, Z are schemes, then the fibre product $X \times_Z Y$ has structure sheaf equal to $\mathscr{O}_X \otimes_{\mathscr{O}_Z} \mathscr{O}_Y = \mathscr{T}or^0_{\mathscr{O}_Z}(\mathscr{O}_Z, \mathscr{O}_Y)$. Its derived enhancement is

$$X \times^L_Z Y := (X \times_Z Y, \mathscr{T}or^{\bullet}_{\mathscr{O}_Z}(\mathscr{O}_X, \mathscr{O}_Y)).$$

Observe that if either $X \to Z$ or $Y \to Z$ is flat, then the derived fibre product becomes a classical scheme again.

The following two examples cover, in spirit, all the cases we will see below.

Example 18 (Self-intersection of the origin). Let V be a finite-dimensional vector space, and let $\pi\colon W = \{0\} \to V$ be inclusion of the origin. Of course, as classical schemes, $\pi^{-1}(0) = \{0\}$. The derived fibre, however, remembers more information.

We wish to compute the Tor sheaf. Recall that there is an exact complex

$$\cdots \to \Lambda^3 V^* \otimes_{\mathbb{C}} \operatorname{Sym}(V^*) \to \Lambda^2 V^* \otimes_{\mathbb{C}} \operatorname{Sym}(V^*) \to \Lambda^1 V^* \otimes_{\mathbb{C}} \operatorname{Sym}(V^*) \to \operatorname{Sym} V^* \to \mathbb{C} \to 0,$$

where we view $\mathbb{C}$ as being in degree 0.

Remark 20. If V is a representation of G, this complex is G-equivariant.

Thus when we apply the functor $- \otimes_{\operatorname{Sym} V^*} \mathbb{C}$ (as $\mathscr{O}_0$ is the skyscraper sheaf $\mathbb{C}$ at zero) then we obtain the complex

$$\cdots \to \Lambda^3 V^* \to \Lambda^2 V^* \to \Lambda^1 V^* \to \Lambda^0 V^* = \mathbb{C} \to \mathbb{C} \otimes_{\operatorname{Sym} V^*} \mathbb{C} = \pi^{-1}(0) \to 0.$$

Note that every map in this complex is now zero.

We have replaced $\mathbb{C}$ with a quasi-isomorphic complex of flat, in fact, free, $\operatorname{Sym} V^*$-modules and applied the abelian tensor product functor to that complex. Therefore $\operatorname{Sym}^{\bullet}(V^*[1])$ computes the derived tensor product $\mathbb{C} \otimes_{\operatorname{Sym} V^*} \mathbb{C}$. Note that $\pi_0(\operatorname{Sym}^{\bullet}(V^*[1])) = \mathbb{C}$ and $\operatorname{Spec} \mathbb{C}$ is the underlying classical scheme of the self-intersection. Note that, as expected, this fibre product of Noetherian schemes is a Noetherian derived scheme.

Example 19 (Intersection of two subspaces, both nontrivial). Let $W_1, W_2 \subset V$ be two nontrivial, non-transverse subspaces of V. For example, take the x-axis and the x, y plane in $V = \mathbb{C}^3$. The ring of functions on W_1 is $\mathbb{C}[x]$ and functions on W_2 are $\operatorname{Sym} W_2^* = \mathbb{C}[x, y]$ as modules over $\mathbb{C}[x, y, z]$.

We wish to calculate the derived tensor product

$$\mathbb{C}[x] \otimes^L_{\mathbb{C}[x,y,z]} \mathbb{C}[x, y],$$

and claim that

$$\mathbb{C}[x] \otimes^L_{\mathbb{C}[x,y,z]} \mathbb{C}[x, y] \simeq \operatorname{Sym}^{\bullet}((\mathbb{C}^3/W_2)^*[1]) \otimes_{\mathbb{C}} \mathbb{C}[x].$$

We expect its zero cohomology to give us the classical intersection and its first cohomology to tell us about failure of W_1 and W_2 to be transverse.

We will resolve $\mathbb{C}[x, y]$ by free $\mathbb{C}[x, y, z]$-modules. We have in fact a short exact sequence

$$0 \longrightarrow \mathbb{C}z \otimes_{\mathbb{C}} \mathbb{C}[x, y, z] \longrightarrow \mathbb{C}[x, y, z] \longrightarrow \mathbb{C}[x, y] \longrightarrow 0$$

where the first map is given by $\alpha z \otimes p(x, y, z) \mapsto \alpha z p(x, y, z)$ for $\alpha \in \mathbb{C}$, and the second map is given by $p(x, y, z) \mapsto p(x, y, 0)$.

Applying the functor $- \otimes_{\mathbb{C}[x,y,z]} \mathbb{C}[x]$, we get

$$\mathbb{C}z \otimes_{\mathbb{C}} \mathbb{C}[x] \simeq \mathbb{C}z \otimes_{\mathbb{C}} \mathbb{C}[x,y,z] \otimes_{\mathbb{C}[x,y,z]} \mathbb{C}[x] \xrightarrow{0}$$

$$\mathbb{C}[x,y,z] \otimes_{\mathbb{C}[x,y,z]} \mathbb{C}[x] \simeq \mathbb{C}[x] \xrightarrow{f(x)\mapsto 1\otimes f(x)} C[x,y] \otimes_{\mathbb{C}[x,y,z]} \mathbb{C}[x] \longrightarrow 0$$

The first map is zero because it is given by $p(x,y,z) \mapsto q(x) \mapsto zp(x,y,z) \otimes q(x) = 0$. The second map is as described above because it is given by

$$p(x,y,z) \otimes q(x) \mapsto p(x,y,0) \otimes q(x) = 1 \otimes p(x,0,0)q(x)$$

and $p(x,y,z) \otimes q(x) = 1 \otimes p(x,0,0)q(x)$. Thus when we take homology of

$$0 \longrightarrow \mathbb{C}z \otimes_{\mathbb{C}} \mathbb{C}[x] \simeq \mathbb{C}z \otimes_{\mathbb{C}} \mathbb{C}[x,y,z] \otimes_{\mathbb{C}[x,y,z]} \mathbb{C}[x] \xrightarrow{0} \mathbb{C}[x,y,z] \otimes_{\mathbb{C}[x,y,z]} \mathbb{C}[x] \simeq \mathbb{C}[x] \longrightarrow 0$$

we get $\mathbb{C}[x]$ in degree zero, the classical intersection, which is just the x-axis. The first homology is

$$\mathbb{C}z \otimes_{\mathbb{C}} \mathbb{C}[x]$$

with $\mathbb{C}z \simeq \Lambda^1\mathbb{C}$ recording that the intersection $W_1 \cap W_2$ is not transverse, and the direction it fails to point in is the z-direction.

If X is classical, then $\mathrm{Coh}(X)$ and $\mathrm{Coh}_G(X)$ are the usual bounded derived categories. We write $\mathbf{Rep}(G) := \mathrm{Coh}_G(\mathrm{pt})$ and $R(G) = K_0(\mathbf{Rep}(G))$. We will often use silently the fact that if $f\colon X \to Y$ is a morphism of smooth locally-Noetherian schemes, then the pullback functor f^* preserves coherence. The classical schemes we work with will of course be exclusively locally-Noetherian, and the flag variety and bundles over it are smooth.

As derived schemes will appear below with approximately the same frequency as their truncations, there is no notational savings to be had by adopting either the convention that all schemes are derived unless otherwise indicated, or the opposite convention. Therefore from now on all categories, functors, and schemes are derived unless indicated otherwise. We emphasize especially that all fibre products are derived (although frequently this consideration will have no effect). In particular, in any case where a derived scheme and its classical truncation appears, the derived scheme will be without decoration, as will all classical schemes that appear without any derived enhancement.

We now state some results that we will use repeatedly below.

Definition 45 ([1])**.** A derived scheme X is *quasi-smooth* if it is Zariski-locally a fibre product

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{A}^n \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathbb{A}^m \end{array}$$

taken in the category of derived schemes.

Definition 46. A morphism of derived schemes is *quasi-compact* if the corresponding morphism of classical truncations is quasi-compact.

Proposition 9 ([27], Proposition 2.2.2 (b) and Lemma 3.2.4)**.** *Let $f\colon X \to Y$ be a quasi-compact morphism of derived schemes.*

1. *Then for any Cartesian diagram*

$$\begin{array}{ccc} X \times^L_Y X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Z, \end{array}$$

the base-change property holds.

2. *Let* $\mathscr{F} \in \mathrm{QCoh}(X)$ *and* $\mathscr{G} \in \mathrm{QCoh}(Y)$. *Then there is an isomorphism*

$$\mathscr{G} \otimes f_*\mathscr{F} \to f_*\left(f^*\mathscr{G} \otimes \mathscr{F}\right)$$

in $\mathrm{QCoh}(Y)$. *That is, the projection formula is true.*

Unless otherwise indicated, by "to apply base-change" we mean "to apply this proposition," and likewise for "apply the projection formula."

4.3.2 The scheme of singularities and singular support

Given a coherent sheaf $\mathscr{F}$ on a scheme X, Arinkin and Gaitsgory in [1] define a classical scheme $\mathrm{SingSupp}(\mathscr{F})$, the *singular support* of $\mathscr{F}$. The singular support serves in particular to measure the extent to which an object of $\mathrm{Coh}(X)$ fails to lie in $\mathrm{Perf}(X)$. We will require only very special cases of the theory of singluar support. All definitions and results recalled in this section are due to Arinkin-Gaitsgory in [1].

Let X be a quasi-smooth derived scheme. The classical scheme $\mathrm{Sing}(X)$ measures how far from being smooth X is. Let $T^*(X)$ be the cotangent complex of X and $T(X)$ its dual. Then one defines

$$\mathrm{Sing}(X) := \mathbf{Spec}\left(\mathrm{Sym}_{\mathscr{O}_{X^\heartsuit}} H^1(T(X))\right) \to X^\heartsuit.$$

The scheme of singularities is affine over $X^\heartsuit$, but is not in general a vector bundle. The singular support will be a conical subset of $\mathrm{Sing}(X)$. In general, if a morphism

$$f\colon X \to Y$$

exhibits $f^{-1}(\mathrm{pt})$ as quasi-smooth, then given $x \in X$,

$$\mathrm{Sing}(X)_x = \mathrm{coker}(df_x)^*.$$

Note that if $f\colon V \to W$ is a linear map between vector spaces, then the dg-algebra of functions on the derived scheme $f^{-1}(0)$ is

$$\mathscr{O}_{\ker f} \otimes \mathrm{Sym}\left(\mathrm{coker}(f)^*[1]\right).$$

Let $f\colon Z_1 \to Z_2$ be a morphism of derived schemes. Then f induces a morphism $T(Z_1) \to f^*T(Z_2)$ as usual. Define

$$\mathrm{Sing}(Z_2)_{Z_1} := \left(\mathrm{Sing}(Z_2) \times_{Z_2} Z_1\right)^\heartsuit = \mathbf{Spec}\left(\mathrm{Sym}_{\mathscr{O}_{Z_1}}\left(f^*T(Z_2)[1]\right)\right)^\heartsuit.$$

Definition 47. The *singular codifferential of* f is the induced morphism of classical schemes

$$\mathrm{Sing}(f)\colon \mathrm{Sing}(Z_2)_{Z_1} \to \mathrm{Sing}(Z_1).$$

As we state above, we need the theory of singular support to guarantee that certain functors will preserve coherence. We record the precise results we will use as

Proposition 10 ([1], Proposition 7.1.3 (b) and 7.2.2 (b))**.** *1. Let* $f\colon Z_1 \to Z_2$ *be quasi-compact, and let* $Y_i \subset \mathrm{Sing}(Z_i)$ *be conical Zariski-closed subsets. If*

$$\mathrm{Sing}(f)^{-1}(Y_1) \subset Y_2 \times_{Z_2} Z_1,$$

then f_* *preserves coherence.*

2. Let Z *be a derived scheme and* $\mathscr{F}, \mathscr{F}' \in \mathrm{Coh}(Z)$*. Suppose that*

$$\mathrm{SingSupp}(\mathscr{F}) \cap \mathrm{SingSupp}(\mathscr{F}') \subset \{0\}$$

in $\mathrm{Sing}(Z)$*. Then* $\mathscr{F} \otimes \mathscr{F}' \in \mathrm{Coh}(Z)$*.*

4.3.3 Categorification of H and Bezrukavnikov's equivalence

We now recall a derived setup upgrading the contents of the last section to the level of categories.

Write $\tilde{\mathscr{N}} = T^*(\mathscr{B}^\heartsuit)$, and denote the Steinberg variety by $\mathrm{St} = \tilde{\mathscr{N}} \times^L_{\mathfrak{g}} \tilde{\mathscr{N}}$. It is naturally a derived scheme. The category $\mathrm{Coh}_{G\times\mathbb{G}_\mathrm{m}}(\mathrm{St})$ is monoidal under convolution of sheaves. Moreover, this categorifies the affine Hecke algebra, and is one half of an equivalence of categories due to Bezrukavnikov [8] that upgrades the discussion in Section 2.1.3 (note that in keeping with the conventions of this chapter, we reverse the roles of **G** and its Langlands dual from those in [8]):

Theorem 41 (Bezrukavnikov, [8])**.** *Let* $F = \overline{\mathbb{F}_q}((t))$*. Let* $G^\vee = \mathbf{G}^\vee(F)$*, where* $\mathbf{G}$ *is as usual. Let* $I^\vee$ *be an Iwahori subgroup of* $G^\vee$*. Then there is an equivalence of categories*

$$D_{I^\vee I^\vee} := D^b_{I^\vee}(G^\vee/I^\vee) \to \mathrm{Coh}^G(\mathrm{St}),$$

where the left hand side is the Bernstein-Lunts equivariant derived category. Morevover, it is possible to realize $\mathbf{G}^\vee(F)/I^\vee$ *as the* $\bar{\mathbb{F}}_q$*-points of an ind-group-scheme over* $\bar{\mathbb{F}}_q$ *such that the equivalence intertwines the automorphism of the left hand side induced by the Frobenius automorphism with pullback by the automorphism of the derived Steinberg variety induced by*

$$(X, \mathfrak{b}_1, \mathfrak{b}_2) \mapsto (qX, \mathfrak{b}_1, \mathfrak{b}_2).$$

The Grothendieck groups of both categories are isomorphic as $\mathscr{A}$*-algebras to* H*.*

The category $\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\mathrm{St})$ *is equipped with a functor to* $D_{I^\vee I^\vee}$*. There is a* t*-structure on* $\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\mathrm{St})$ *such that there is a functor from its heart to the category of Weil sheaves on* $G^\vee/I^\vee$*.*

It would be desirable to state and prove a similar constructible-coherent equivalence for J. The main result of this chapter is a first step towards the definition of the putative coherent side.

We now explain more about the derived Steinberg variety. Defining the composite morphism

$$\tilde{\mathscr{N}} \times \tilde{\mathscr{N}} \xrightarrow{i} \mathfrak{g} \oplus \mathfrak{g} \xrightarrow{f} \mathfrak{g}$$

$$(x, \mathfrak{b}, y, \mathfrak{b}') \longmapsto (x, y) \longmapsto x - y$$

we see that St fits into the pullback diagram

$$\begin{array}{ccc} \mathrm{St} & \longrightarrow & \tilde{\mathscr{N}} \times \tilde{\mathscr{N}} \\ \downarrow & & \downarrow{\scriptstyle f\circ i} \\ \mathrm{pt} & \longrightarrow & \mathfrak{g}. \end{array}$$

Therefore St is quasi-smooth, and Sing(St) is defined. We will compute its fibres over $\mathrm{St}^{\heartsuit}$. Recall that for any variety X, if ξ is a cotangent vector at $x \in X$, then

$$T_{(x,\xi)}(T^*X) \simeq T_xX \oplus T^*_xX.$$

Thus

$$T_{(\mathfrak{b},x)}(\tilde{\mathscr{N}}) = \mathfrak{g}/\mathfrak{b} \oplus \mathfrak{n}.$$

The morphism $f \circ i$ induces the differential

$$(\mathfrak{g}/\mathfrak{b}_1 \oplus \mathfrak{n}_1) \oplus (\mathfrak{g}/\mathfrak{b}_2 \oplus \mathfrak{n}_2) \to \mathfrak{g}$$

$$((x_1, y_1), (x_2, y_2)) \mapsto y_1 - y_2.$$

Therefore

$$\mathrm{coker}\left(d_{((x_1,y_1),(x_2,y_2))}\right)(f \circ i)) = \mathfrak{g}/(\mathfrak{n}_1 \oplus \mathfrak{n}_2).$$

Generically this quotient is the Cartan subalgebra $\mathfrak{h}$. Over the diagonal component of St, it is the opposite Borel subalgebra. Thus

$$\mathrm{Sing}(\mathrm{St})_{(X,\mathfrak{b}_1,\mathfrak{b}_2)} = (\mathfrak{g}/(\mathfrak{n}_1 \oplus \mathfrak{n}_2))^*.$$

4.3.4 Categorification of J_0

Derived enhancement of the flag variety

Let $|\mathscr{E}|$ be the total space of the quotient

$$0 \to \tilde{\mathscr{N}} \to \mathscr{B}^{\heartsuit} \times \mathfrak{g} \to |\mathscr{E}| \to 0$$

and define

$$\mathscr{B} = \mathbf{Spec}(\mathrm{Sym}_{\mathscr{O}_{\mathscr{B}^{\heartsuit}}} \mathscr{E}[1]) = \mathbf{Spec}\left(\mathrm{Sym}_{\mathscr{O}_{\mathscr{B}^{\heartsuit}}} \left(\mathscr{B}^{\heartsuit} \times \mathfrak{g}/\tilde{\mathscr{N}}\right)^* [1]\right). \tag{4.2}$$

Then $\mathscr{B}$ is naturally a derived scheme with classical truncation $\mathscr{B}^{\heartsuit}$, equipped with a morphism $i\colon \mathscr{B} \to \tilde{\mathscr{N}}$, and hence with a morphism

$$i_{\mathrm{der}}\colon \mathscr{B} \times \mathscr{B} \to \mathrm{St}.$$

By construction, we have a pullback diagram

$$\begin{array}{ccc} \mathscr{B} & \longrightarrow & \tilde{\mathscr{N}} \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathscr{B}^{\heartsuit} \times \mathfrak{g}, \end{array} \tag{4.3}$$

where $\mathscr{B}^{\heartsuit} \times \mathfrak{g} \to \mathscr{B}^{\heartsuit}$ is the trivial bundle with fibre $\mathfrak{g}$ and $\{0\}$ is its zero-section. Therefore $\mathscr{B}$ is a quasi-smooth DG-scheme in the sense of [1]. The description of $\mathscr{B}$ as a fibre product yields a similar description of $\mathscr{B} \times \mathscr{B}$. Indeed, the diagram

$$\begin{array}{ccccc} \tilde{\mathscr{N}} & \longrightarrow & \mathfrak{g} & \longleftarrow & \tilde{\mathscr{N}} \\ \downarrow & & \downarrow \scriptstyle{\mathrm{id}} & & \downarrow \\ \mathscr{B}^{\heartsuit} \times \mathfrak{g} & \longrightarrow & \mathfrak{g} & \longleftarrow & \mathscr{B}^{\heartsuit} \times \mathfrak{g} \\ \uparrow & & \uparrow & & \uparrow \\ \{0\} & \longrightarrow & \mathrm{pt} & \longleftarrow & \{0\} \end{array} \tag{4.4}$$

gives immediately the description

$$\begin{array}{ccc} \mathscr{B} \times \mathscr{B} & \xrightarrow{i_{\mathrm{der}}} & \mathrm{St} \\ \downarrow & & \downarrow \scriptstyle{p_{\mathrm{St}}} \\ \{0\} & \xrightarrow{i_{\{0\}}} & \mathscr{B}^{\heartsuit} \times \mathscr{B}^{\heartsuit} \times \mathfrak{g}, \end{array} \tag{4.5}$$

where $\{0\}$ now means the zero-section of the trivial bundle $\mathscr{B}^{\heartsuit} \times \mathscr{B}^{\heartsuit} \times \mathfrak{g}$.

The category $\mathrm{Coh}_G(\mathscr{B} \times \mathscr{B})$ is a module category over the monoidal category $\mathbf{Rep}(G)$, via

$$V \cdot \mathscr{F} = \pi^* V \otimes_{\mathscr{O}_{\mathscr{B} \times \mathscr{B}}} \mathscr{F},$$

for $V \in \mathbf{Rep}(G)$, where $\pi \colon \mathscr{B} \times \mathscr{B} \to \mathrm{pt}$. The same procedure makes $\mathscr{J}_0$ into a module category over $\mathbf{Rep}(G)$.

Lemma 19. *If* $\mathscr{F} \in \mathrm{Coh}_{G \times \mathbb{G}_{\mathrm{m}}}(\mathrm{St})$*, then*

$$\mathrm{SingSupp}(i_{\mathrm{der}}^* \mathscr{F}) \subseteq \Delta \tilde{\mathfrak{g}} \subset \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}.$$

Proof. We seek to apply Proposition 7.1.3 of [1]. We have

$$\begin{array}{ccc} \mathbf{Spec}\, \mathrm{Sym}_{\mathscr{O}_{\mathscr{B}^{\heartsuit} \times \mathscr{B}^{\heartsuit}}} (i_{\mathrm{der}}^* T(\mathrm{St})[1]) & \xrightarrow{\mathrm{Sing}(i_{\mathrm{der}})} & \mathrm{Sing}(\mathscr{B} \times \mathscr{B}) \\ & \searrow \quad \swarrow & \\ & \mathrm{St}^{\heartsuit}. & \end{array}$$

Fibrewise, the singular codifferential is the linear map

$$\mathrm{Sing}(\mathrm{St})_{(0, \mathfrak{b}_1, \mathfrak{b}_2)} = (\mathfrak{g}/(\mathfrak{n}_1 \oplus \mathfrak{n}_2))^* \to (\mathfrak{g}/\mathfrak{n}_1)^* \oplus (\mathfrak{g}/\mathfrak{n}_2)^* = \mathfrak{b}_1 \oplus \mathfrak{b}_2 = \mathrm{Sing}(\mathscr{B} \times \mathscr{B})_{(\mathfrak{b}_1, \mathfrak{b}_2)} \tag{4.6}$$

induced by the direct sum of projections

$$\mathfrak{g}/\mathfrak{n}_1 \oplus \mathfrak{g}/\mathfrak{n}_2 \twoheadrightarrow \mathfrak{g}/(\mathfrak{n}_1 \oplus \mathfrak{n}_2).$$

This implies that (4.6) is the simply the diagonal embedding, and the lemma follows. □

Definition of the category $\mathscr{J}_0$

We now define the category $\mathscr{J}_0$, the main point being the condition we impose on the singular supports of its objects. In our case

$$\mathrm{Sing}(\mathscr{B}) = \mathbf{Spec}\left(\mathrm{Sym}_{\mathscr{O}_{\mathscr{B}^\heartsuit}} \mathscr{E}[2]\right) \to \mathscr{B}^\heartsuit.$$

Koszul duality gives an equivalence

$$\mathrm{KD}\colon \mathrm{Coh}(\mathscr{B}) \to \mathrm{Sym}_{\mathscr{O}_{\mathscr{B}^\heartsuit}} \mathscr{E}[2] - \mathbf{Mod}^{\text{f.g.}},$$

and following [1], we set

$$\mathrm{SingSupp}(\mathscr{F}) = \mathrm{supp}(\mathrm{KD}(\mathscr{F})) \subset |\mathscr{E}|.$$

Thus for $\mathscr{F} = \mathscr{F}_1 \boxtimes \mathscr{F}_2 \in \mathrm{Coh}(\mathscr{B} \times \mathscr{B})$, $\mathrm{SingSupp}(\mathscr{F})$ is just the usual support of some other sheaf on the total space of the bundle $\mathrm{Sing}(\mathscr{B}) \times \mathrm{Sing}(\mathscr{B})$.

Lemma 20. $\mathrm{Sing}(\mathscr{B})$ *is none other than the bundle* $\tilde{\mathfrak{g}}$.

Proof. Indeed, the fibres of $\mathscr{B}$ are

$$\mathrm{Spec}\,\mathrm{Sym}_{\mathbb{C}}((\mathfrak{g}/\mathfrak{n})^*[1]) = \mathrm{Spec}\,\mathrm{Sym}_{\mathbb{C}}(\mathfrak{b}[1]),$$

and Koszul duality identifies

$$\mathrm{Sym}_{\mathbb{C}}(\mathfrak{b}[1]) - \mathbf{Mod} \simeq \mathrm{Sym}_{\mathbb{C}}(\mathfrak{b}^*[2]) - \mathbf{Mod}. \tag{4.7}$$

□

It makes sense to take the support of a module on the right- hand side of (4.7) on the scheme $\mathfrak{b}$, by defining the support to be the support of the cohomology over the classical ring $\mathrm{Sym}_{\mathbb{C}} \mathfrak{b}^*$.

We define $\mathscr{J}_0$ to be the full subcategory of $\mathrm{Coh}_G(\mathscr{B} \times \mathscr{B})$ with objects $\mathscr{F}$ such that the projection

$$\mathrm{SingSupp}(\mathscr{F}) \to \mathrm{Sing}(\mathscr{B})$$

onto the first factor is a proper morphism. We write

$$\mathscr{J}_0 := \mathrm{Coh}_G(\mathscr{B} \times \mathscr{B})_!$$

and

$$\mathscr{J}_{0\mathscr{A}} := \mathrm{Coh}_{G \times \mathbb{G}_\mathrm{m}}(\mathscr{B} \times \mathscr{B})_!,$$

where $\mathbb{G}_\mathrm{m}$ acts trivially on $\mathscr{B}$.

There are two obvious ways that the projection onto the first factor can be proper: either $\mathrm{KD}(\mathscr{F})$ is of form $\Delta_*\mathscr{F}'$, where Δ is the diagonal, or $\mathrm{KD}(\mathscr{F}) = \mathscr{F}_1' \boxtimes \mathscr{F}_2'$ with $\mathrm{supp}(\mathscr{F}_2)$ contained in the zero section. This latter case arises precisely from sheaves $\mathscr{F}_1 \boxtimes \mathscr{F}_2 \in \mathscr{J}_0$ such that $\mathscr{F}_2$ is perfect. These are essentially the only examples that we will encounter. The image of i^*_{der} consists of sheaves of the first type (this is especially easy to see for those sheaves whose images in K-theory are contained in $Z(J_0) = \phi_0(Z(\mathbf{H}))$), and the sheaves t_w that we define in Section 4.3.4 are all examples of the second kind.

Remark 21. This fact, together with the second statement of the main theorem, can be viewed as a categorification of the fact that ϕ_0 is injective but not surjective.

Consider the pairing operation defined by

$$\langle \mathscr{F}, \mathscr{G} \rangle = \pi_*(\mathscr{F} \otimes \mathscr{G})$$

where $\pi \colon \mathscr{B} \to \mathrm{pt}$. In general, this operation does not define a functor

$$\langle -, - \rangle : \mathrm{Coh}_G(\mathscr{B}) \times \mathrm{Coh}_G(\mathscr{B}) \to \mathbf{Rep}(G),$$

but will do so when it comes to convolution of objects of $\mathscr{J}_0$. We recall

Theorem 42 ([67], [54])**.** *There is an isomorphism of based rings*

$$\sigma \colon J_0 \to K_G(\mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit).$$

We can now state the main result of this chapter.

Theorem 43. *Let $\mathscr{B}$ be the derived zero section of the Springer resolution $\tilde{\mathscr{N}} \to \mathscr{N}$. Then*

1. *the category*
$$\mathscr{J}_0 := \mathrm{Coh}_G(\mathscr{B} \times \mathscr{B})_!$$
is a triangulated subcategory of $\mathrm{Coh}_G(\mathscr{B} \times \mathscr{B})$, has a monoidal structure given by convolution, and admits a natural monoidal functor
$$\mathrm{Coh}_{G \times \mathbb{C}^\times}(\mathrm{St}) \to \mathrm{Coh}_{G \times \mathbb{G}_\mathrm{m}}(\mathscr{B} \times \mathscr{B})_!$$
such that

2. *the induced morphism*
$$H \to K_0(\mathrm{Coh}_{G \times \mathbb{G}_\mathrm{m}}(\mathscr{B} \times \mathscr{B})_!) \to K_{G \times \mathbb{C}^\times}(\mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit)$$
is conjugate to ϕ_0 and $K_0(\mathscr{J}_0)$ surjects onto J_0;

3. *In the special case when G has universal cover equal to SL_2 or SL_3, there exists a family of objects $\{t_w\}_{w \in \mathbf{c}_0}$ in $\mathscr{J}_0$, such that that, if $t_w t_x = \sum_z \gamma_{w,x,z} t_{z^{-1}}$ in J_0, then*
$$t_w \star t_x = \bigoplus_z t_{z^{-1}}^{\oplus \gamma_{w,x,z}}$$

in $\mathscr{J}_0$ and the image in $K_G(\mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit)$ of the class $[t_w]$ under the above morphism is $[t_w]$;

4. *The category $\mathscr{J}_0$ acts on $\mathrm{Coh}_T(\mathscr{B})$ and on $\mathrm{Coh}_G(\mathscr{B} \times \mathscr{B})$.*

Proof. Each of Propositions 11, 12, 13, and 14 proves one statement of the theorem. □

In the introduction, we noted two pleasant properties of this categorification relating to the map ϕ. We repeat this here.

First, it is classical that H and J_0 can both be realized as equivariant algebraic K-theory of certain complex varieties, but the map ϕ in this picture is given in a non-obvious, and essentially K-theoretic (as opposed to geometric; the definition of ϕ uses the Thom isomorphism theorem) way. However, on the derived level, we prove in this chapter that ϕ is categorified simply by (derived) pullback of sheaves. The fact that ϕ is injective but not surjective onto J_0 is reflected categorically in terms of the singular supports that can appear in the essential image of the functor lifting ϕ.

Remark 22. As remarked after the statement of this theorem in Section 1.1.3, it would be interesting to study injectivity of the morphism

$$K_0(\mathscr{J}_{0\mathscr{A}}) \twoheadrightarrow J_{G\times\times\mathbb{G}_m}(\mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit),$$

or subcategories of $\mathscr{J}_{0\mathscr{A}} = \mathrm{Coh}_{G\times\mathbb{G}_m}(\mathscr{B} \times \mathscr{B})_!$ for which the above surjection would restrict to an isomorphism.

We will take this question up in future work.

Proposition 11. *The category $\mathscr{J}_0$ is a monoidal category under convolution of sheaves, and the pullback i^*_{der} defines a monoidal functor*

$$i^*_{\mathrm{der}} \colon \mathrm{Coh}_{G\times\mathbb{G}_m}(\mathrm{St}) \to \mathscr{J}_{0\mathscr{A}}$$

such that

$$\mathrm{SingSupp}(i^*_{\mathrm{der}}\mathscr{F}) \subset \Delta\tilde{\mathfrak{g}}$$

for all $\mathscr{F}$. The category $\mathscr{J}_0$ is a triangulated subcategory of $\mathrm{Coh}_G(\mathscr{B} \times \mathscr{B})$.

Additionally,

1. *If $\mathscr{F}_1 \boxtimes \mathscr{G}$ and $\mathscr{G}' \boxtimes \mathscr{F}_2$ are in $\mathscr{J}_0$, then $\langle \mathscr{G}, \mathscr{G}' \rangle \in \mathbf{Rep}(G)$ and*

$$\mathscr{F}_1 \boxtimes \mathscr{G} \star \mathscr{G}' \boxtimes \mathscr{F}_2 = \langle \mathscr{G}, \mathscr{G}' \rangle\, \mathscr{F}_1 \boxtimes \mathscr{F}_2;$$

2. *If $V_1, V_2 \in \mathbf{Rep}(G)$, then*

$$(V_1 \cdot \mathscr{F}) \star (V_2 \cdot \mathscr{G}) = (V_1 \otimes_{\mathbb{C}} V_2) \cdot \mathscr{F} \star \mathscr{G}$$

for $\mathscr{F}, \mathscr{G} \in \mathscr{J}_0$.

Remark 23. The category $\mathrm{Coh}_G(\mathscr{B} \times \mathscr{B})$ is not monoidal.

Proof. Let

$$\mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to$$

be a distinguished triangle in $\mathrm{Coh}_G(\mathscr{B}\times\mathscr{B})$ such that $\mathscr{F}_1, \mathscr{F}_2 \in \mathscr{J}_0$. Applying Koszul duality, we get

$$\mathrm{KD}(\mathscr{F}_1) \to \mathrm{KD}(\mathscr{F}_2) \to \mathrm{KD}(\mathscr{F}_3) \to$$

in $\mathrm{Coh}_G(\tilde{\mathfrak{g}}\times\tilde{\mathfrak{g}})$. Then

$$\operatorname{supp}\mathrm{KD}(\mathscr{F}_3) \subset \operatorname{supp}\mathrm{KD}(\mathscr{F}_1) \cup \operatorname{supp}\mathrm{KD}(\mathscr{F}_2),$$

and we see that projection $\operatorname{supp}\mathrm{KD}(\mathscr{F}_3) \to \tilde{\mathfrak{g}}$ onto the first factor is proper.

Let $\mathscr{F} \in \mathscr{J}_0$ and let $\mathscr{G} \in \mathrm{Coh}(\mathscr{B}\times\mathscr{B})$. Then their convolution will be coherent if

$$(\mathscr{F}_{12} \boxtimes \mathscr{O}_{\mathscr{B}}) \otimes (\mathscr{O}_{\mathscr{B}} \boxtimes \mathscr{G}_{23}),$$

is coherent, where the subscripts ij indicate which factors inside $\mathscr{B}\times\mathscr{B}\times\mathscr{B}$ a given sheaf sits on. Noting that $\mathrm{SingSupp}(\mathscr{O}_{\mathscr{B}}) = \{0\}$, we have

$$\mathrm{SingSupp}(\mathscr{F}_{12} \boxtimes \mathscr{O}_{\mathscr{B}}) \cap \mathrm{SingSupp}(\mathscr{O}_{\mathscr{B}} \boxtimes \mathscr{G}_{23}) \subset \{0\} \times V \times \{0\}. \tag{4.8}$$

We claim that this intersection is in fact contained in the zero section of $V\times V\times V$. First, projection from $\mathrm{SingSupp}(\mathscr{F}_{12}) \times \{0\}$ to the first coordinate is a proper morphism, and so the same is true for projection from the intersection; $\{0\} \times \mathrm{SingSupp}(\mathscr{G}_{23})$ is closed. As $\mathrm{SingSupp}(\mathscr{F}_{12})$ is a conical subset, it now follows that the intersection is contained in $\{0\}\times\{0\}\times\{0\}$. The claim now follows from Proposition 10 (1).

Now suppose that projection to the first factor from $\mathrm{SingSupp}(\mathscr{G}_{23})$ is also proper. We have

$$\mathrm{Sing}(p_{13})\colon \tilde{\mathfrak{g}}\times\tilde{\mathfrak{g}}\times\mathscr{B} \to \tilde{\mathfrak{g}}\times\tilde{\mathfrak{g}}\times\tilde{\mathfrak{g}}$$

is the inclusion of the zero section into the second coordinate, and is the identity on the other coordinates.

Define

$$Y_2 = \{(x_1,x_3) \,|\, (x_1,z) \in \mathrm{SingSupp}(\mathscr{F}_{12}),\ (z,x_3) \in \mathrm{SingSupp}(\mathscr{G}_{23}) \text{ for some } z \in \{0\}\}.$$

Then

$$\begin{aligned}
&\mathrm{Sing}(p_{13})^{-1}\left(\{(x_1,x_2,x_3) \,|\, (x_1,x_2) \in \mathrm{SingSupp}(\mathscr{F}_{12})\ (x_2,x_3) \in \mathrm{SingSupp}(\mathscr{G}_{23})\}\right)\\
&= \{(x_1,z,x_3) \,|\, (x_1,z) \in \mathrm{SingSupp}(\mathscr{F}_{12})\ (x_2,z) \in \mathrm{SingSupp}(\mathscr{G}_{23}),\ z \in \{0\}\}\\
&= Y_2 \times_{\mathscr{B}\times\mathscr{B}} \mathscr{B}\times\mathscr{B}\times\mathscr{B}.
\end{aligned}$$

It follows from Proposition 10 (1) that $\mathrm{SingSupp}(\mathscr{F}\star\mathscr{G}) \subset Y_2$. It therefore suffices to show that the projection $Y_2 \to \tilde{\mathfrak{g}}$ is proper. Indeed, we have

$$p_1^{-1}(K) \subset K \times p_{2,\mathscr{G}}\left(p_{1,\mathscr{G}}^{-1}(\{0\})\right),$$

for any K, where $p_{i,\mathscr{G}}$ is the projection $\mathrm{SingSupp}(\mathscr{G}) \to \tilde{\mathfrak{g}}$ onto the i-th factor.

Therefore $\mathscr{J}_0$ is a monoidal category. The same of course goes for $\mathscr{J}_{0\mathscr{A}}$.

Now we show the first formula. It is easy to see that if $\mathscr{F}\boxtimes\mathscr{G} \in \mathscr{J}_0$, then $\mathrm{SingSupp}(\mathscr{G})$ must be contained in the zero section, *i.e.* that $\mathscr{G}$ must be perfect. Then, $\langle \mathscr{G}, \mathscr{G}' \rangle$ is coherent because $\mathscr{G} \otimes \mathscr{G}'$ is and the map to pt is proper. Its pullback to $\mathscr{B} \times \mathscr{B}$ is then perfect. The remainder of the formula is obtained by carrying out the calculations in Lemma 5.2.28 of [19]. The required projection formula and base-change are provided by Proposition 9.

We next claim that if $\mathscr{F} \in \mathrm{Coh}_{G \times \mathbb{G}_m}(\mathrm{St})$, then $i_{\mathrm{der}}^* \mathscr{F}$ is coherent, and the projection

$$p \colon \mathrm{SingSupp}(i_{\mathrm{der}}^* \mathscr{F}) \subset \mathscr{B} \times \mathscr{B} \to \mathscr{B}$$

onto the first factor is proper. Coherence follows again from (4.4). Indeed, we need only show that the pushforward of $i_{\mathrm{der}}^* \mathscr{F}$ to $\mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit$ is coherent, and base-change says that this equals $i_{\{0\}}^* p_{\mathrm{St}*} \mathscr{F}$. By hypothesis $p_{\mathrm{St}*}\mathscr{F}$ is coherent, and hence by smoothness of $\mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit \times \mathfrak{g}$ the pullback is also coherent. We must check that $i_{\mathrm{der}}^* \mathscr{F} \in \mathscr{J}_0$. Indeed, this follows immediately from Lemma 19, which says that $\mathrm{SingSupp}(i_{\mathrm{der}}^* \mathscr{F}) \subset \Delta V$.

We now check that i_{der}^* is monoidal. Diagrams 4.3 and

$$\begin{array}{ccccc}
\tilde{\mathscr{N}} & \longrightarrow & \mathfrak{g} & \longleftarrow & \mathfrak{g} \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{B}^\heartsuit & \longrightarrow & \mathrm{pt} & \longleftarrow & \mathfrak{g} \\
\uparrow & & \uparrow & & \uparrow \\
\mathscr{B}^\heartsuit & \longrightarrow & \mathrm{pt} & \longleftarrow & \mathrm{pt}
\end{array}$$

imply that

$$\tilde{\mathscr{N}} \times_{\mathscr{B}^\heartsuit \times \mathfrak{g}} \mathscr{B}^\heartsuit \simeq \tilde{\mathscr{N}} \times_{\mathfrak{g}} \mathrm{pt} \simeq \mathscr{B}.$$

Thus

$$\begin{aligned}
\mathscr{B} \times \mathscr{B} \times_{\mathrm{St}} \tilde{\mathscr{N}} \times_{\mathfrak{g}} \tilde{\mathscr{N}} \times_{\mathfrak{g}} \tilde{\mathscr{N}} &\simeq \mathscr{B} \times \mathscr{B} \times_{\mathfrak{g}} \tilde{\mathscr{N}} \\
&\simeq \mathscr{B} \times \mathscr{B} \times \mathrm{pt} \times_{\mathfrak{g}} \tilde{\mathscr{N}} \\
&\simeq \mathscr{B} \times \mathscr{B} \times \mathscr{B},
\end{aligned}$$

and we can apply base-change to the pullback diagram

$$\begin{array}{ccc}
\mathscr{B} \times \mathscr{B} \times \mathscr{B} & \xrightarrow{i \times i \times i} & \tilde{\mathscr{N}} \times_{\mathfrak{g}} \tilde{\mathscr{N}} \times_{\mathfrak{g}} \tilde{\mathscr{N}} \\
\big\downarrow{\scriptstyle \pi_{ij}} & & \big\downarrow{\scriptstyle p_{ij}} \\
\mathscr{B} \times \mathscr{B} & \xrightarrow{\quad i_{\mathrm{der}} \quad} & \mathrm{St},
\end{array} \tag{4.9}$$

where π_{ij} and p_{ij} are the projections.

With diagram (4.9) in hand, the remainder is entirely formal. Indeed, according to the definition of convolution on St, we compute as follows: Let $\mathscr{F}, \mathscr{G} \in \mathrm{Coh}_{G \times \mathbb{G}_m}(\mathrm{St})$. Then

$$\begin{aligned}
i_{\mathrm{der}}^*(\mathscr{F} \star \mathscr{G}) &= i_{\mathrm{der}}^* p_{13*}\left(p_{12}^* \mathscr{F} \otimes p_{23}^* \mathscr{G}\right) && (4.10) \\
&\simeq \pi_{13*}(i \times i \times i)^* \left(p_{12}^* \mathscr{F} \otimes p_{23}^* \mathscr{G}\right) && (4.11) \\
&\simeq \pi_{13*}\left(\pi_{12}^* i_{\mathrm{der}}^* \mathscr{F} \otimes \pi_{23}^* i_{\mathrm{der}}^* \mathscr{G}\right) && (4.12) \\
&= i_{\mathrm{der}}^* \mathscr{F} \star i_{\mathrm{der}}^* \mathscr{G}.
\end{aligned}$$

We used base-change for the diagram (4.9) with $ij = 13$ between lines (4.10) and (4.11), and just commutativity of (4.9) for $ij = 12$ and $ij = 23$ on line (4.12). □

Recall that $a(\mathbf{c}_0) = \dim \mathscr{B}$, any quasicoherent sheaf on $\mathscr{B}^\heartsuit$ has cohomology only in degrees at most $a(\mathbf{c}_0)$. On the level of K-theory, this reflects the influence of the a-function on the multiplication in J.

The sheaves t_w

Xi, in [67] for G simply-connected, and Nie in [54] in general gave a description in K-theory of the elements t_w for $w \in \mathbf{c}_0$. We recall this construction below in Section 4.2.2; here we follow it on the level of categories in the special case $G = \mathrm{SL}_2$ or SL_3, where it can be carried out almost verbatim. As remarked above, we hope to move beyond these two special cases in future work.

Recalling the equivalence

$$\mathrm{Ind}_B^G\colon \mathrm{Coh}_B(\mathrm{pt}) \to \mathrm{Coh}_G(\mathscr{B}^\heartsuit),$$

we define

$$\mathscr{F}_w = \mathrm{Ind}_B^G \mathrm{Infl}_T^B(x_w),$$

where $x_w \in \mathrm{Coh}_T(\mathrm{pt})$ is as in (4.1) and

$$\mathrm{Infl}_T^B\colon \mathrm{Coh}_T(\mathrm{pt}) \to \mathrm{Coh}_B(\mathrm{pt})$$

is inflation. Likewise, define

$$\mathscr{G}_w = \mathrm{Ind}_B^G \mathrm{Infl}_T^B(y_w)$$

where y_w is the dual basis from Lemma 18.

Now if $w = fw_0g^{-1}$, define

$$t_w = \mathscr{F}_f \boxtimes p^*\mathscr{G}_g,$$

and if $w = fw_0\chi g^{-1}$, define

$$t_w = V(\chi)t_{fw_0g^{-1}},$$

where we view $\mathscr{F}_f$ as pushed forward under the inclusion of the zero section of $\mathscr{B}$, and $p\colon \mathscr{B} \to \mathscr{B}^\heartsuit$. Clearly $t_w \in \mathrm{Coh}_G(\mathscr{B} \times \mathscr{B})$. Moreover, as $\mathscr{B}^\heartsuit$ is smooth, $\mathscr{G}_g$ is perfect, and hence $p^*\mathscr{G}_g$ is perfect. Therefore $\mathrm{SingSupp}(\mathscr{G})$ is contained in the zero section of $\mathrm{Sing}(\mathscr{B})$, and $t_w \in \mathscr{J}_0$.

By Proposition 11 (or using Proposition 10 (2) directly), $\langle \mathscr{G}_g, \mathscr{F}_f \rangle$ is defined for all g, f and takes values in $\mathbf{Rep}(G)$. In fact, it agrees with the pairing $\langle -, - \rangle_\heartsuit$ on the classical truncation given the by the same procedure:

$$\langle p^*\mathscr{G}_g, \mathscr{F}_f \rangle = \pi_*(p^*\mathscr{G}_g \otimes \mathscr{F}_f) = \pi^\heartsuit_* p_*(p^*\mathscr{G}_g \otimes \mathscr{F}_f) = \pi^\heartsuit_* (\mathscr{G}_g \otimes p_*\mathscr{F}_f) = \langle \mathscr{G}_g, \mathscr{F}_f \rangle_\heartsuit$$

where

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{\quad p \quad} & \mathscr{B}^\heartsuit \\ & \searrow^{\pi} \quad \swarrow_{\pi^\heartsuit} & \\ & \mathrm{pt}. & \end{array}$$

Therefore by the Borel-Weil-Bott theorem, we have

$$\mathscr{F}_f \boxtimes p^*\mathscr{G}_g \star \mathscr{F}_{f'} \boxtimes p^*\mathscr{G}_{g'} = \begin{cases} \mathscr{F}_f \boxtimes \mathscr{G}_{g'} & \text{if } g = f' \\ 0 & \text{otherwise} \end{cases}.$$

Moreover, as G is reductive, we have again by Proposition 11 that

$$(V(\lambda)\mathscr{F}_f \boxtimes p^*\mathscr{G}_g) \star (V(\nu)\mathscr{F}_g \boxtimes p^*\mathscr{G}_{g'}) = (V(\lambda) \otimes V(\nu))\, \mathscr{F}_f \boxtimes p^*\mathscr{G}_{g'} = \bigoplus_\mu V(\mu)(\mathscr{F}_f \boxtimes p^*\mathscr{G}_g)^{\oplus m^\mu_{\lambda,\nu}},$$

where $m^\mu_{\lambda,\nu}$ is the multiplicity of $V(\mu)$ in $V(\lambda) \otimes V(\nu)$.

4.4 K-theory

In this section we show that the functor i^*_{der} categorifies Lusztig's homomorphism ϕ_0. By K-theory we shall always mean simply the Grothendieck group. Recall the coherent description of the affine Hecke algebra from Section 2.1.3.

4.4.1 K-theory of classical schemes and Lusztig's homomorphism

We first relate Lusztig's homomorphism to a construction in K-theory of classical schemes given in [19]. There is no new material in this section; when G is simply-connected the relationship is given by [71], and the analogous result in general follows from [10]. In order to perform calculations, though, we must devote significant space to fixing conventions.

Recall that the $K_{G\times\mathbb{G}_\mathrm{m}}(\mathrm{St}) \simeq H$ as $\mathscr{A}$-algebras. We will use the explicit isomorphism given by Chriss and Ginzburg in [19], Theorem 7.2.5.

Recall that by [10], $J_0 \simeq K_G(\mathbf{Y}\times\mathbf{Y})$ for a centrally-extended set $\mathbf{Y}$ of cardinality $\#W$. Moreover, by 5.5 (a) of *loc. cit.*, the stabilizer of every $y \in \mathbf{Y}$ is G. When G is simply-connected, it has no nontrivial central extensions, and hence in this case $J_0 \simeq \mathrm{Mat}_{\#W}(R(G))$ is a matrix ring, as first shown in [67]. Combining these results, we obtain an injection $\varphi_1 \colon J_0 \hookrightarrow \mathrm{Mat}_{\#W}(R(\tilde{G}))$, where $\tilde{G} \twoheadrightarrow G$ is a simply-connected. Write $H(\tilde{W}(\tilde{G}))$ for the affine Hecke algebra of $\tilde{G}$.

In parallel, when G is simply-connected, the external tensor product gives an isomorphism $K_G(\mathscr{B}\times\mathscr{B}) \simeq K_G(\mathscr{B}) \otimes_{R(G)} K_G(\mathscr{B}) \simeq \mathrm{Mat}_{\#W}(R(G))$, by [19], Theorem 6.2.4. This theorem does not hold when G is not simply-connected. Indeed, the trivial and Steinberg representations of H give two one-dimensional representations of J_0 for $\mathbf{G} = \mathrm{SL}_2$. On the other hand, the external tensor product still gives an inclusion

$$\psi_1 \colon K^G(\mathscr{B}\times\mathscr{B}) \hookrightarrow K^{\tilde{G}}(\mathscr{B}\times\mathscr{B}) \simeq \mathrm{Mat}_{\#W}(R(\tilde{G})).$$

(Note that G and $\tilde{G}$ have canonically isomorphic flag varieties and Weyl groups.)

By [54], we have an isomorphism $\sigma \colon J_0 \to K_G(\mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit)$ regardless of whether G is simply-connected or not.

Lemma 21 ([71])**.** *The following diagram of $\mathscr{A}$-algebras*

$$\begin{array}{ccccccc}
K_{G\times\mathbb{G}_{\mathrm{m}}}(\mathrm{St}) & \longrightarrow & K_{G\times\mathbb{G}_{\mathrm{m}}}(\tilde{\mathscr{N}}\times\tilde{\mathscr{N}}) & \xrightarrow{\bar{\iota}^*\circ\bar{p}_*} & K_{G\times\mathbb{G}_{\mathrm{m}}}(\mathscr{B}\times\mathscr{B}) & \overset{\psi_1}{\hookrightarrow} & \mathrm{Mat}_{\#W\times\#W}(R(\tilde{G}\times\mathbb{G}_{\mathrm{m}})) \\
\sim\uparrow & & & & & & \mathrm{Ad}(A)\uparrow \\
\mathbf{H} & & \xrightarrow{\qquad\qquad\phi_0\qquad\qquad} & & J_0\otimes_{\mathbb{Z}}\mathscr{A} & \overset{\varphi_1}{\hookrightarrow} & \mathrm{Mat}_{\#W\times\#W}(R(\tilde{G}\times\mathbb{G}_{\mathrm{m}}))
\end{array}$$

commutes, where A is the change-of-basis matrix from the the $Z(H(\tilde{W}(\tilde{G}))$-basis $\{\theta_{e_w}C \mid w\in W\}$ of $H(\tilde{W}(\tilde{G}))$ to the $Z(H(\tilde{W}(\tilde{G})))$-basis $\{C_{d_w w_0}\mid w\in W\}$, where e_w and d_w are as in [71].

K-theory of derived schemes and Lusztig's homomorphism

Koszul duality identifies $\mathrm{Coh}_{G\times\mathbb{G}_{\mathrm{m}}}(\mathscr{B}_{\mathrm{der}})$ with $\mathrm{Coh}_{G\times\mathbb{G}_{\mathrm{m}}}(\tilde{\mathfrak{g}}[2])$.

We now establish the relationship in K-theory between the monoidal functor i^*_{der} from Section 4.3.4 and the morphism ϕ_0. Write

$$K_{G\times\mathbb{G}_{\mathrm{m}}}(\mathscr{B}\times\mathscr{B}) := K_0\left(\mathrm{Coh}_{G\times\mathbb{G}_{\mathrm{m}}}(\mathscr{B}\times\mathscr{B})_!\right)$$

and

$$K_{G\times\mathbb{G}_{\mathrm{m}}}(\mathrm{St}) := K_0\left(\mathrm{Coh}_{G\times\mathbb{G}_{\mathrm{m}}}(\mathrm{St})\right).$$

and write $K(X^\heartsuit) := K_0(\mathrm{Coh}(X^\heartsuit))$ whenever $X^\heartsuit$ is a classical scheme, and similarly for equivariant K-theory.

If X is a derived scheme with classical truncation $X^\heartsuit$, we may define a morphism

$$K(X)\to K(X^\heartsuit)$$

by

$$[\mathscr{F}]\mapsto\sum_i(-1)^i[\pi_i(\mathscr{F})], \tag{4.13}$$

where $\pi_i(F)$ is viewed as a $\pi_0(\mathscr{O}_X)$-module.

Recalling that the derived structure on X is to be thought of as "higher nilpotents," this morphism is identical in spirit to identifying $K_0(\mathrm{Coh}(\operatorname{Spec}A))$ and $K_0(\mathrm{Coh}(\operatorname{Spec}A_{\mathrm{red}}))$, where A is Noetherian and A_{red} is reduced. Indeed, the map (4.13) is also an isomorphism of abelian groups, and both isomorphisms are consequences of dévissage; see e.g. [65].

Lemma 22. *Pushforward by bundle projection $p\colon \mathrm{St}\to\mathrm{St}^\heartsuit$ induces the map* (4.13) *on $G\times\mathbb{G}_{\mathrm{m}}$-equivariant K-theory. This map is an isomorphism of rings.*

Proof. By the remarks preceding the lemma, it suffices to show that p_* respects convolution in K-theory. By definition, we have

$$p_*([\mathscr{F}])\star p_*([\mathscr{G}]) = \sum_{i,j}(-1)^{i+j}\pi_i(\mathscr{F})\star\pi_j(\mathscr{G}), \tag{4.14}$$

whereas

$$p_*([\mathscr{F}]\star[\mathscr{G}]) = p_*p_{13*}(p_{12}^*\mathscr{F}\otimes_{q_2^*\mathscr{O}_{\mathrm{St}}}p_{23}^*\mathscr{G}) = p^\heartsuit_{13*}p_{3*}(p_{12}^*\mathscr{F}\otimes_{q_2^*\mathscr{O}_{\mathrm{St}}}p_{23}^*\mathscr{G})$$

by commutativity of the diagram

$$\begin{array}{ccc} \left(\tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathcal{N}}\right) \times_{\tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathcal{N}}} \left(\tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathcal{N}}\right) & \xrightarrow{p_{13}} & \mathrm{St} \\ \downarrow{\scriptstyle p_3} & & \downarrow{\scriptstyle p_{\mathrm{St}}} \\ \tilde{\mathcal{N}} \times_{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathcal{N}} & \xrightarrow{p_{13}^{\heartsuit}} & \mathrm{St}^{\heartsuit}. \end{array}$$

We have

$$p_{13*}^{\heartsuit} p_{3*}(p_{12}^* \mathscr{F} \otimes_{q_2^* \mathscr{O}_{\mathrm{St}}} p_{23}^* \mathscr{G}) = p_{13*}^{\heartsuit} \sum_n (-1)^n \pi_n \left(p_{12}^* \mathscr{F} \otimes_{q_2^* \mathscr{O}_{\mathrm{St}}} p_{23}^* \mathscr{G}\right), \tag{4.15}$$

and so for (4.14) to equal (4.15), we need

$$\pi_n \left(p_{12}^* \mathscr{F} \otimes_{q_2^* \mathscr{O}_{\mathrm{St}}} p_{23}^* \mathscr{G}\right) = \sum_{i+j=n} p_{12}^{\heartsuit *} \pi_i(\mathscr{F}) \otimes_{\tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \times \tilde{\mathcal{N}}} p_{23}^{\heartsuit *} \pi_j(\mathscr{G}),$$

which follows from the Künneth formula [30], Théorème 6.7.8.

□

In the case of $K_{G \times \mathbb{G}_{\mathrm{m}}}(\mathscr{B} \times \mathscr{B})$, we can define another map to the K-theory of the truncation. Recall $p\colon \mathscr{B} \to \mathscr{B}^{\heartsuit}$ is the bundle projection morphism, and let $i\colon \mathscr{B}^{\heartsuit} \to \mathscr{B}$ be the inclusion of the zero section. Then define Φ to be the composite

$$\Phi\colon K(\mathscr{J}_0) \overset{\mathrm{id} \times i^*}{\to} K_G(\mathscr{B} \times \mathscr{B}^{\heartsuit}) \overset{p_* \times \mathrm{id}}{\to} K_G(\mathscr{B}^{\heartsuit} \times \mathscr{B}^{\heartsuit}).$$

That is, if $\mathscr{F} \boxtimes \mathscr{G} \in \mathscr{J}_0$, then

$$\Phi([\mathscr{F} \boxtimes \mathscr{G}]) = [p_* \mathscr{F}] \boxtimes [i^* \mathscr{G}].$$

Remark 24. It is necessary that the source of $(\mathrm{id} \times i)^*$ (which we will show makes sense a functor) is $\mathscr{J}_0$ and not all of $\mathrm{Coh}_G(\mathscr{B} \times \mathscr{B})$; the functor

$$i^*\colon \mathrm{QCoh}(\mathscr{B}) \to \mathrm{QCoh}(\mathscr{B}^{\heartsuit})$$

does not preserve coherence in general.

Lemma 23. *The morphism Φ is well-defined and is a surjective morphism of $R(G \times \mathbb{G}_{\mathrm{m}})$-algebras.*

Proof. By the Künneth formula, we have $\mathrm{Sing}(\mathscr{B} \times \mathscr{B}^{\heartsuit}) \simeq \tilde{\mathfrak{g}} \times \tilde{\mathcal{N}}^*$. Then

$$\mathrm{Sing}(\mathrm{id} \times i)\colon \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}} \times \tilde{\mathcal{N}}^*$$

is given by the identity in the first coordinate, and then the zero map

$$(X, \mathfrak{b}) \mapsto ((Y, \mathfrak{b}) \mapsto \kappa(X, Y) = 0),$$

in the second coordinate, where κ is the Killing form (although one may of course also take the trace form).

Therefore $\ker \mathrm{Sing}(\mathrm{id} \times i) = \{0\} \times \tilde{\mathfrak{g}}$. Now let $\mathscr{F} \in \mathscr{J}_0$. The argument is essentially the same as in the proof of Proposition 11. The set-theoretic intersection

$$\left(\mathrm{SingSupp}(\mathscr{F}) \times_{\mathscr{B} \times \mathscr{B}} \mathscr{B} \times \mathscr{B}^{\heartsuit}\right) \cap \ker \mathrm{Sing}(\mathrm{id} \times i) = \mathrm{SingSupp}(\mathscr{F}) \cap (\{0\} \times \tilde{\mathfrak{g}})$$

is contained in the zero-section of $\operatorname{Sing}(\mathscr{B} \times \mathscr{B})$. Indeed, $\mathscr{F} \in \mathscr{J}_0$ and $\mathscr{B}^{\heartsuit}$ is compact, so the above intersection must be compact. As $\operatorname{SingSupp}(\mathscr{F})$ is conical, we see the intersection must be contained in the zero-section. We conclude by [1], Proposition 7.2.2 (d) that $(\mathrm{id} \times i)^*$ is well-defined. Obviously $p_* \times \mathrm{id}$ is well-defined, and hence Φ is well-defined. As each of $(\mathrm{id} \times i)^*$ and $(p \times \mathrm{id})_*$ are $R(G)$-linear (the latter by the projection formula), so is Φ.

Finally, we show that Φ is a morphism of rings. Using linearity, we compute

$$\Phi([\mathscr{F}_1]\boxtimes[\mathscr{G}])\star\Phi([\mathscr{G}']\boxtimes[\mathscr{F}_2]) = p_*[\mathscr{F}_1]\boxtimes i^*_{\mathrm{der}}[\mathscr{G}]\star p_*[\mathscr{G}']\boxtimes i^*_{\mathrm{der}}[\mathscr{F}_2] = \langle i^*_{\mathrm{der}}[\mathscr{G}], p_*[\mathscr{G}']\rangle_{\mathscr{B}^{\heartsuit}} \cdot p_*[\mathscr{F}_1]\boxtimes i^*_{\mathrm{der}}[\mathscr{F}_2],$$

whereas

$$\Phi([\mathscr{F}_1] \boxtimes [\mathscr{G}] \star [\mathscr{G}'] \boxtimes [\mathscr{F}_2]) = \langle [\mathscr{G}], [\mathscr{G}'] \rangle_{\mathscr{B}} \cdot p_*[\mathscr{F}_1] \boxtimes i^*_{\mathrm{der}}[\mathscr{F}_2].$$

Write $\pi \colon \mathscr{B}^{\heartsuit} \to \operatorname{Spec}\mathbb{C}$. Then

$$\langle i^*_{\mathrm{der}}[\mathscr{G}], p_*[\mathscr{G}']\rangle_{\mathscr{B}^{\heartsuit}} = \pi_*(i^*_{\mathrm{der}}[\mathscr{G}] \otimes_{\mathscr{O}_{\mathscr{B}^{\heartsuit}}} p_*[\mathscr{G}']) = \pi_* p_* \left(p^* i^*_{\mathrm{der}}[\mathscr{G}] \otimes_{\mathscr{O}_{\mathscr{B}}} [\mathscr{G}']\right) = \langle [\mathscr{G}], [\mathscr{G}'] \rangle_{\mathscr{B}}$$

by the formulation of the projection formula, Proposition 9 (2). Surjectivity follows as Φ has a section Ψ defined $[\mathscr{F}] \boxtimes [\mathscr{G}] \mapsto i_*[\mathscr{F}] \boxtimes p^*[\mathscr{G}]$. This completes the proof. □

Lemma 24. *We have* $\Phi(\mathscr{O}_{\Delta\mathscr{B}}(\lambda)) = \mathscr{O}_{\Delta\mathscr{B}^{\heartsuit}}(\lambda)$

Proof. This is a local computation that amounts to the map

$$\mathbb{C}[x, \epsilon] \otimes \mathbb{C}[y, \delta]/(x - y, \epsilon - \delta) \to \mathbb{C}[x, y]/(x - y)$$

quotienting by δ and leaving the first factor untouched, where $|x| = |y| = 0$ and $|\epsilon| = |\delta| = -1$. One sees that quotienting by δ also kills ϵ. □

Proposition 12. *The following diagram of $R(\mathbb{G}_{\mathrm{m}})$-algebras*

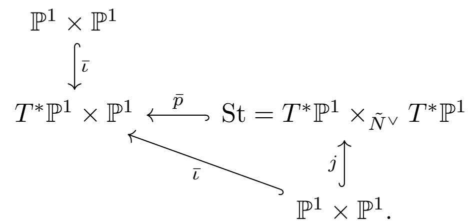
$$\begin{array}{ccccccc}
K_{G\times\mathbb{G}_{\mathrm{m}}}(\mathrm{St}_{\mathrm{der}}) & & \xrightarrow{\quad i^*_{\mathrm{der}} \quad} & & K_{G\times\mathbb{G}_{\mathrm{m}}}(\mathscr{B}\times\mathscr{B}) & & \\
\downarrow{\scriptstyle p_{\mathrm{St}*}} & & & & \downarrow{\scriptstyle \Phi} & & \\
K_{G\times\mathbb{G}_{\mathrm{m}}}(\mathrm{St}) & \longrightarrow & K_{G\times\mathbb{G}_{\mathrm{m}}}(\tilde{\mathscr{N}}\times\tilde{\mathscr{N}}) & \xrightarrow{\bar{\iota}^*\circ\bar{p}_*} & K_{G\times\mathbb{G}_{\mathrm{m}}}(\mathscr{B}\times\mathscr{B}) & \xhookrightarrow{\psi_1\otimes\mathrm{id}_{\mathscr{A}}} & \mathrm{Mat}_{\#W\times\#W}(R(\tilde{G}\times\mathbb{G}_{\mathrm{m}})) \\
\sim\uparrow & & & & & & \uparrow{\scriptstyle \mathrm{Ad}(A)} \\
\mathbf{H} & & \xrightarrow{\quad \phi_0 \quad} & & J_0\otimes_{\mathbb{Z}}\mathscr{A} & \xhookrightarrow{\phi_1\otimes\mathrm{id}_{\mathscr{A}}} & \mathrm{Mat}_{\#W\times\#W}(R(\tilde{G}\times\mathbb{G}_{\mathrm{m}}))
\end{array}$$

commutes.

We will first describe the middle morphism on the K-theory of the classical schemes. Consider the diagrams

$$\begin{array}{ccc}
\tilde{\mathscr{N}} & \xrightarrow{\pi_\Delta} & \mathscr{B}^{\heartsuit}_{\Delta} \\
\big\downarrow{\scriptstyle \iota_\Delta} & & \\
\mathrm{St}^{\heartsuit} & & \big\downarrow{\scriptstyle \Delta} \\
\big\downarrow{\scriptstyle \bar{p}} & & \\
\mathscr{B}^{\heartsuit}\times\tilde{\mathscr{N}} & \xrightarrow{\mathrm{id}\times\pi} & \mathscr{B}\times\mathscr{B},
\end{array} \tag{4.16}$$

which is Cartesian, and the diagram



(4.17)

Let

$$\mathscr{O}_\lambda := [\iota_{\Delta *}\pi_\Delta^* \mathscr{O}_{\mathscr{B}^\heartsuit}(\lambda)].$$

Lemma 25. *We have*

$$\bar{\iota}^* \circ \bar{p}_*(\mathscr{O}_\lambda) = \Delta_* \mathscr{O}_{\mathscr{B}^\heartsuit}(\lambda),$$

and in the case when $G = \mathrm{SL}_2$*, we have*

$$\bar{\iota}^* \circ \bar{p}_*(-q^{1/2} j_* \mathscr{O}(0,-2)) = -q^{\frac{1}{2}}\mathscr{O}(0,-2) + q^{-\frac{1}{2}}\mathscr{O}(0,0).$$

Proof. By diagram (4.17)

$$\bar{\iota}^* \circ \bar{p}_*(-q^{1/2} j_* \mathscr{O}(0,-2)) = -q^{\frac{1}{2}}\bar{\iota}^*\bar{\iota}_*\mathscr{O}(0,-2) = -q^{\frac{1}{2}}\lambda \otimes \mathscr{O}(0,-2)$$

by [19], Lemma 5.4.9, where $\lambda = [\mathrm{Sym}(\mathbb{P}^1 \times T\mathbb{P}^1[1])] = [\mathscr{O}(0,0)] - q^{-1}[\mathscr{O}(0,2)]$. Here we have multiplied $\mathscr{O}(0,2)$ by the character q^{-1}, giving its fibres the trivial $\mathbb{C}^\times$-action, which restores equivariance of the complex defining the class λ. (When confronted with a linear map $V \to W$ where W has trivial $\mathbb{C}^\times$-action and V is scaled by a character, one restores equivariance by tensoring V with the inverse character.) Thus we have

$$\bar{\iota}^* \circ \bar{p}_*(-q^{1/2} j_* \mathscr{O}(0,-2)) = -q^{\frac{1}{2}}\left(\mathscr{O}(0,0) - q^{-1}\mathscr{O}(0,2)\right) \otimes \mathscr{O}(0,-2) = -q^{\frac{1}{2}}\mathscr{O}(0,-2) + q^{-\frac{1}{2}}\mathscr{O}(0,0).$$

To prove the second formula, apply base-change diagram (4.16) and use that, according to the Thom isomorphism theorem, $((\mathrm{id} \times \pi)^*)^{-1} = \bar{\iota}^*$. Then we have

$$\bar{\iota}^* \circ \bar{p}_*(\mathscr{O}_\lambda) = \bar{\iota}^*(\bar{p} \circ \iota_\Delta)_* \pi_\Delta^* \mathscr{O}(\lambda)$$

by base-change we have $(\mathrm{id} \times \pi)^*\Delta_* = (\bar{p} \circ \iota_\Delta)_*\pi_\Delta^*$, hence $\Delta_* = ((\mathrm{id} \times \pi)^*)^{-1}(\bar{p} \circ \iota_\Delta)_*\pi_\Delta^*$. □

Proof of Proposition 12. That the bottom square commutes is the combination of the main results of [71] and [54].

By Proposition 11, the morphism i^*_{der} induces a morphism on K-theory as above. Hence by Lemma 23 and the above discussion, all the morphisms in the diagram are well-defined morphisms of algebras.

We first show the diagram commutes on the Bernstein subalgebra. Recalling from Section 4.3.3 that the diagonal component of the Steinberg variety is a classical rather than a derived scheme, and so $\mathscr{O}_\lambda$ is naturally an element of $K_{G \times \mathbb{G}_\mathrm{m}}(\mathrm{St})$ for which $p_{\mathrm{St}*}\mathscr{O}_\lambda = \mathscr{O}_\lambda$ with the right-hand side regarded as an object of $K_{\mathbf{G} \times \mathbb{G}_\mathrm{m}}(\mathrm{St}^\heartsuit)$. Then by Lemma 25, it suffices to show that $\Phi([i^*\mathscr{O}_\lambda]) = [\Delta_*\mathscr{O}_{\mathscr{B}^\heartsuit}(\lambda)]$.

Indeed, though, we have

$$i^*(\mathscr{O}_\lambda) = [\mathscr{O}_{\Delta\mathscr{B}}(\lambda)],$$

as the structure sheaf of the diagonal pulls back to the structure sheaf of the diagonal. By Lemma 24 we have $\Phi([\mathscr{O}_{\Delta\mathscr{B}}]) = [\mathscr{O}_{\Delta\mathscr{B}^\heartsuit}]$, and likewise for the twists. Thus the diagram commutes for the Bernstein subalgebra.

Consider the diagram

$$\begin{array}{ccc} \mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit \times \operatorname{Spec}(\operatorname{Sym}(\mathfrak{g}[1])) & \longrightarrow & \mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit \\ \downarrow r & & \downarrow j \\ \mathscr{B} \times \mathscr{B} & \xrightarrow{i_{\mathrm{der}}} & \mathrm{St} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathfrak{g}. \end{array} \tag{4.18}$$

Note that, by (4.5), the bottom square is Cartesian. Moreover, we have

$$\mathscr{B} \times \mathscr{B} \times_{\mathrm{St}} \mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit = (\mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit \times \{0\}) \times_{\mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit \times \mathfrak{g}} (\mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit \times \{0\}) = \mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit (\times \{0\} \times_{\mathfrak{g}} \{0\}),$$

where we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form. Therefore the large square is also Cartesian, and so the upper square is also Cartesian.

We will use this to explain how to compute $i_{\mathrm{der}}^* j_* \mathscr{F}$ for any coherent sheaf $\mathscr{F}$ on $\mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit$. Pullback by the top map sends $\mathscr{F} \mapsto \mathscr{F} \otimes \operatorname{Sym}(\mathfrak{g}[1])$. By (4.2), the map p is given by the identity on classical truncations, together with the map of cdgas which is pointwise the obvious map $\operatorname{Sym}(\mathfrak{b}[1]) \to \operatorname{Sym}(\mathfrak{g}[1])$ given by including a Borel subalgebra $\mathfrak{b} \hookrightarrow \mathfrak{g}$. Pushforward by r corresponds pointwise to equipping the module structure given by

$$\operatorname{Sym}(\mathfrak{b}_1[1]) \otimes \operatorname{Sym}(\mathfrak{b}_2[1]) \to \operatorname{Sym}(\mathfrak{g}[1]).$$

Appying the functor $(p \times \mathrm{id})_*$ forgets the $\operatorname{Sym}[\mathfrak{b}_1[1]]$-action, and then applying $(\mathrm{id} \times i)^*$ quotients by the remaining $\operatorname{Sym}[\mathfrak{b}_2[1]]$-action. That is, fibrewise we have

$$\mathbb{C} \otimes_{\operatorname{Sym}(\mathfrak{b}_2[1])} \operatorname{Sym}(\mathfrak{g}[1]) \otimes \mathscr{F}_{\mathfrak{b}_1,\mathfrak{b}_2} = \operatorname{Sym}(\mathfrak{g}/\mathfrak{b}_2[1]) \otimes \mathscr{F} = \operatorname{Sym}(\mathfrak{n}_2^*[1]) \otimes \mathscr{F}_{\mathfrak{b}_1,\mathfrak{b}_2},$$

where $\mathfrak{n}_2$ is the radical of $\mathfrak{b}_2$. Comparing with Lemma 25 and the first sentence of its proof, this says precisely that the diagram commutes when $G = \mathrm{SL}_2$.

Returning to general G, let s be a simple reflection. Let $\bar{Y}_s \subset \mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit$ be the closure of the G-orbit labelled by s, and let

$$\pi_s \colon T^*_{\bar{Y}_s}(\mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit) \to \bar{Y}_s$$

be the conormal bundle. Put $\mathscr{Q}_s = \pi_s^* \Omega^1_{\bar{Y}_s/\mathscr{B}^\heartsuit}$. Then by equation 7.6.34 in [19], it suffices to show that

$$\Phi\left(i_{\mathrm{der}}^* i_{\mathrm{St}*} \mathscr{Q}_s\right) = \mathscr{O}_{\mathscr{B}^\heartsuit} \boxtimes i_{X_s*}(q[\mathscr{O}(2)] - [\mathscr{O}]),$$

where i_{X_s} is the inclusion of the Schubert variety $X_s \simeq \mathbb{P}^1$ into $\mathscr{B}^\heartsuit$. That is, the image of $\mathscr{Q}_s$ is just the pushforward of the answer in the SL_2 case. But it is clear that this is indeed the case.

□

We have now nearly proved

Proposition 13. *For $\tilde{G} = \mathrm{SL}_2$ or SL_3, there exists a family of objects $\{t_w\}_{w\in\mathbf{c}_0}$ in $\mathscr{J}_0$, such that that if $t_w t_x = \sum_z \gamma_{w,x,z^{-1}} t_z$ in J_0, then*

$$t_w \star t_x = \bigoplus_z t_{z^{-1}}^{\oplus \gamma_{w,x,z}}$$

in $\mathscr{J}_0$ and such that $\Phi([t_w]) = [t_w]$.

Proof. The discussion in Section 4.3.4 proves all but the last statement of the proposition. Finally, by Proposition 12, if $w = f w_0 g^{-1}$ we have

$$\Phi([t_w]) = \Phi([\mathscr{F}_f] \boxtimes [p^*\mathscr{G}_g]) = [\mathscr{F}_f] \boxtimes i^*p^*[\mathscr{G}_g] = [\mathscr{F}_f] \boxtimes [\mathscr{G}_g].$$

The general claim follows by linearity over $K^G(\mathrm{pt})$ and the parameterization in Section 4.3.4. □

4.4.2 The Schwartz space of the basic affine space

The Schwartz space of the basic affine space $\mathcal{S}$ was defined by Braverman-Kazhdan in [14] to organize the principal series representations of $\mathbf{G}^\vee(F)$ in a way insensitive to the poles of intertwining operators. In [15], Braverman and Kazhdan gave the following description of J_0 in terms of the Iwahori-invariants $\mathcal{S}^I$ of $\mathcal{S}$. In *loc. cit.* it was shown that $\mathcal{S}^I$ is isomorphic to $K_{T\times\mathbb{G}_\mathrm{m}}(\mathscr{B}^\heartsuit)$ as an $H \otimes \mathbb{C}[\tilde{W}]$-module, and in [15], it was proven that $J_0 \simeq \mathrm{End}_{\tilde{W}}(\mathcal{S}^I)$, where the action of $\tilde{W}$ is as defined in *loc. cit.*

Example 20. Let $\mathbf{G} = \mathrm{SL}_2$, with $\tilde{W} = \left\langle s_0, s_1 \,\middle|\, s_0^2 = s_1^2 = 1 \right\rangle$ and s_0 the finite simple reflection. Then we have that $K_{T\times\mathbb{G}_\mathrm{m}}(\mathbb{P}^1)$ has basis $\{[\mathscr{O}_{\mathbb{P}^1}], [\mathscr{O}_{\mathbb{P}^1}(-1)]\}$, and we have $t_{s_0} = [\mathscr{O}_{\mathbb{P}^1}] \boxtimes [\mathscr{O}_{\mathbb{P}^1}]$ and $t_{s_1} = [\mathscr{O}(-1)] \boxtimes [\mathscr{O}(-1)][1]$ under the identification in Lemma 21. The basis elements corresponding to the two distinguished involutions in $\mathbf{c}_0$ act by projectors, with t_{s_0} preserving $\mathscr{O}_{\mathbb{P}^1}$ and killing $\mathscr{O}_{\mathbb{P}^1}(-1)$, and vice-versa for t_{s_1}.

Recalling that $K_{G\times\mathbb{G}_\mathrm{m}}(\mathscr{B}^\heartsuit \times \mathscr{B}^\heartsuit) \simeq K_{T\times\mathbb{G}_\mathrm{m}}(\mathscr{B}^\heartsuit)$, we see that we have two natural coherent categorifications of $\mathcal{S}^I$, and that $\mathscr{J}_0$ acts on both of them:

Proposition 14. *The category $\mathscr{J}_0$ acts on $\mathrm{Coh}_{G\times\mathbb{G}_\mathrm{m}}(\mathscr{B}\times\mathscr{B})$ and on $\mathrm{Coh}_{T\times\mathbb{G}_\mathrm{m}}(\mathscr{B})$.*

Proof. This is a porism of Proposition 11. Indeed, the proof that if $\mathscr{F}, \mathscr{G} \in \mathscr{J}_0$ then $\mathscr{F} \star \mathscr{G} \in \mathrm{Coh}_G(\mathscr{B}\times\mathscr{B})$ used nothing about $\mathrm{SingSupp}(\mathscr{G})$. The proof for $\mathrm{Coh}_T(\mathscr{B})$ is entirely similar. □

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