# LEVI DECOMPOSABLE SUBALGEBRAS OF CLASSICAL LIE ALGEBRAS WITH REGULAR SIMPLE LEVI FACTOR

by

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# ABSTRACT

Levi Decomposable Subalgebras of Classical Lie Algebras with Regular Simple Levi Factor

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This thesis describes and characterizes a significant class of subalgebras of the classical Lie algebras, namely those which are Levi decomposable with regular and simple Levi factor, with select exceptions. Such subalgebras are entirely determined by their Levi factors and radicals. The possible Levi factors are well-established in the literature and so the contribution of this thesis is a characterization of the radicals. The radicals naturally decompose into nontrivial and trivial components. The nontrivial component is found to be fully classified by subsets of the parent root system and Weyl group. However, a classification of the trivial component requires solving the open problem of classifying solvable subalgebras of classical Lie algebras. Nonetheless, this thesis establishes a criterion on the trivial components for determining when two such subalgebras are conjugate. This thesis also briefly explores the ramifications of relaxing simplicity of the Levi factor to allow for semisimplicity.

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# CONTENTS

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# INTRODUCTION

In this thesis we study the subalgebra structure of the classical Lie algebras. We begin with an introduction to the general classification problem, stating the results currently in the literature and the major contributions of this thesis. Throughout, all Lie algebras are over an algebraically closed field $\mathbb{F}$ of characteristic zero.

## 1.1 THE GENERAL PROBLEM

The problem of classifying subalgebras of simple Lie algebras has historically been approached by noting that every subalgebra is one of three types. Explicitly, Levi's Theorem asserts that a Lie algebra is semisimple, solvable, or Levi decomposable, where the latter is defined below.

**Definition 1.1.** A Lie algebra $\mathfrak{a}$ is *Levi decomposable* if there exists a proper semisimple subalgebra $\mathfrak{l} \subsetneq \mathfrak{a}$ such that $\mathfrak{a}$ is the semidirect product of $\mathfrak{l}$ and the radical Rad $\mathfrak{a}$ of $\mathfrak{a}$. The corresponding *Levi decomposition* is expressed as $\mathfrak{a} = \mathfrak{l} \ni \text{Rad}\, \mathfrak{a}$.

There is substantial research on the semisimple subalgebras. Dynkin [Dyn00] and Minchenko [Min06] together yield a classification of the semisimple subalgebras of the exceptional Lie algebras up to inner automorphism. De Graaf [dG11] explicitly classified the semisimple subalgebras of the simple Lie algebras of rank at most eight up to linear equivalence[1]. However, far less is known in the non-semisimple setting. This thesis shall focus upon the classification of the Levi decomposable subalgebras of the classical Lie algebras.

In the literature, two distinct and complementary methodologies have been employed to tackle this classification. The first has been to provide full and explicit descriptions in particular low-rank situations. For instance, Douglas and Repka classified all Levi decomposable subalgebras of the rank two symplectic [DR15b] and special linear [DR16] Lie algebras up to inner automorphism. Mayanskiy [May16] completely described all subalgebras of the exceptional Lie algebra of type $G_2$, again up to inner automorphism.

1 Generally, a classification up to linear equivalence is weaker than a classification up to inner automorphism. For some Lie algebras, however, the two concepts coincide.

The second approach, which we adopt in this thesis, is to classify only a proper subset of the Levi decomposable subalgebras but in a rather general setting. An example of such a procedure is in Douglas and Repka's characterization of a certain class of Levi decomposable subalgebras with abelian radicals [DR15a].

This thesis is inspired by the techniques of Dynkin and, more recently, those of Douglas and Repka. In Dynkin's original classification of the semisimple subalgebras, the task was divided into two sub-problems: first a classification of the regular semisimple subalgebras and then a characterization of the remaining semisimple subalgebras. The approaches Dynkin employed in the regular setting greatly differ from those used in the non-regular case, suggesting a significant intrinsic difference between the two types of subalgebras. Therefore, it is reasonable to suppose that the undertaking of describing Levi decomposable subalgebras of classical Lie algebras is also divided into cases depending on regularity of the Levi factor. This thesis aims to examine this particular setting and we do so by expanding the techniques of Douglas and Repka. In particular, this thesis describes all Levi decomposable subalgebras of the classical Lie algebras with regular and simple Levi factor up to inner automorphism, with a few exceptions. Namely, the primary claims of this thesis apply except when the Levi factor is of sufficiently small rank, as well as in the case of Levi factor of type $A_m$ in a Lie algebra of type $D_n$. While this approach characterizes only a subset of the Levi decomposable subalgebras, it has the significant advantage of being generally applicable and is not restricted to low-rank situations. The inability to handle a few aberrant cases is unfortunate but we shall show that the theory we develop indeed breaks down in those particular settings.

## 1.2 THESIS OVERVIEW

From a computational perspective, it will be beneficial to identify the classical Lie algebras with matrix Lie algebras. In Chapter 2 we make this identification explicit. Moreover, we are interested in a classification up to inner automorphism. As such, in Chapter 2 we also state a collection of results regarding Levi decomposable subalgebras which will assist us in this endeavour.

By definition, Levi decomposable subalgebras are entirely characterized by a Levi factor and the radical. This thesis concentrates on Levi factors which are regular and simple and such subalgebras were completely described by Dynkin [Dyn00]. In Chapter 3 we present the relevant portions of Dynkin's description. Superficially, it may seem as though the regularity condition is fairly restrictive. However, it turns out that if a simple subalgebra is sufficiently large then it is necessarily regular. We prove this also in Chapter 3.

Chapter 4 focuses on a classification of the radical and is the central chapter of this thesis. In this chapter we demonstrate that the radical naturally decomposes into nontrivial and trivial components and a classification of the radical amounts to a classification of each component separately. In most instances we present a full description of the nontrivial component, but unfortunately a description of the trivial component requires a solution to the open problem of classifying all solvable subalgebras. We do not solve this problem and instead establish a criterion on the trivial component to assist in determining when two Levi decomposable subalgebras are conjugate[2].

In Chapter 5 we explore the select cases in which the results of Chapter 4 do not apply. Explicitly, we briefly discuss why our proposed description fails when the Levi factor is regular of type $A_1$, of type $A_2$ in a Lie algebra of type $B_n$, or when the Levi factor is of type $A_m$ inside a classical Lie algebra of type $D_n$. While the results of Chapter 4 do not hold in this latter situation, we shall see that they can be made to hold if we make some minor modifications.

Lastly, in Chapter 6 we discuss the effects of relaxing the simplicity condition of the Levi factor. Since the removal of this condition drastically increases the number of possibilities for the Levi factor, we only examine one special case. We show that in this particular case that the findings of Chapter 4 can be naturally extended. This suggests that the results of Chapter 4 may hold in the setting of having semisimple Levi factors if appropriate hypotheses are imposed.

2 Two subalgebras are *conjugate* if there exists an inner automorphism of the parent Lie algebra mapping one subalgebra to the other.

# 2 PRELIMINARIES

In this chapter we establish some requisite facts on Lie algebras. First, we explicitly describe our chosen identification of the classical Lie algebras with matrix Lie algebras. Afterwards, we briefly present some results on Levi decomposable Lie algebras which will assist us in our attempt to construct a classification up to inner automorphism.

## 2.1 THE CLASSICAL LIE ALGEBRAS

Several computations in this thesis will be made less arduous if we consider the classical Lie algebras as matrix Lie algebras. If $\mathfrak{g}$ is a classical Lie algebra then the type of $\mathfrak{g}$, denoted by $\mathrm{t}(\mathfrak{g})$, is $A_n$, $B_n$, $C_n$, or $D_n$ with $n$ a natural number[3]. The special linear algebra $\mathfrak{sl}_{n+1}$, consisting of traceless $(n+1)\times(n+1)$ matrices, corresponds to type $A_n$. Types $B_n$, $C_n$, and $D_n$ in turn can be identified with the special orthogonal algebra $\mathfrak{so}_{2n+1}$, the symplectic algebra $\mathfrak{sp}_{2n}$, and the special orthogonal algebra $\mathfrak{so}_{2n}$, respectively, where

$$\mathfrak{so}_{2n+1} = \{Z \in \mathfrak{gl}_{2n+1} \colon Z^\top J_B = -J_B Z\} \tag{2.1a}$$
$$\mathfrak{sp}_{2n} = \{Z \in \mathfrak{gl}_{2n} \colon Z^\top J_C = -J_C Z\} \tag{2.1b}$$
$$\mathfrak{so}_{2n} = \{Z \in \mathfrak{gl}_{2n} \colon Z^\top J_D = -J_D Z\} \tag{2.1c}$$

with $\mathfrak{gl}_k$ the Lie algebra of $k \times k$ matrices and

$$J_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}, \qquad J_C = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \qquad J_D = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \tag{2.2}$$

where $I_n$ is the $n \times n$ identity matrix[4]. Equation 2.1 can be expressed more explicitly as

$$\mathfrak{so}_{2n+1} = \left\{ \begin{pmatrix} 0 & e^\top & f^\top \\ -f & R & S \\ -e & T & -R^\top \end{pmatrix} \in \mathfrak{gl}_{2n+1} \colon e, f \in \mathbb{F}^n, R, S, T \in \mathfrak{gl}_n, \right.$$

3 We adopt the convention that the set of natural numbers $\mathbb{N}$ is the set of positive integers.
4 We omit the subscript if it is obvious from context.

$$\left. S^\top = -S, T^\top = -T \right\} \tag{2.3a}$$

$$\mathfrak{sp}_{2n} = \left\{ \begin{pmatrix} R & S \\ T & -R^\top \end{pmatrix} \in \mathfrak{gl}_{2n} \colon R, S, T \in \mathfrak{gl}_n, S^\top = S, T^\top = T \right\} \tag{2.3b}$$

$$\mathfrak{so}_{2n} = \left\{ \begin{pmatrix} R & S \\ T & -R^\top \end{pmatrix} \in \mathfrak{gl}_{2n} \colon R, S, T \in \mathfrak{gl}_n, S^\top = -S, T^\top = -T \right\} \tag{2.3c}$$

We take the standard Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ to be the subalgebra of diagonal matrices and $\Phi \subseteq \mathfrak{h}^*$ to be the corresponding root system. Let

$$\mathcal{B} = \{H_i \in \mathfrak{h} \colon i \in [n]\} \cup \{X_\alpha \in \mathfrak{g} \colon \alpha \in \Phi\} \cup \{Y_\alpha \in \mathfrak{g} \colon \alpha \in \Phi\} \tag{2.4}$$

be a Chevalley basis for $\mathfrak{g}$, where for $k \in \mathbb{N}$ we define $[k] = \{1, 2, \ldots, k\}$. If $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subseteq \Phi$ is a base for $\Phi$ we write $X_i$ and $Y_i$ instead of $X_{\alpha_i}$ and $Y_{\alpha_i}$, respectively. We can choose $\mathcal{B}$ such that for each $i \in [n]$,

$$X_i = \begin{cases} E_{i,i+1}, & \mathsf{t}(\mathfrak{g}) = A_n \\ E_{i+1,i+2} - E_{n+i+2,n+i+1}, & \mathsf{t}(\mathfrak{g}) = B_n \text{ and } i < n \\ E_{n+1,1} - E_{1,2n+1}, & \mathsf{t}(\mathfrak{g}) = B_n \text{ and } i = n \\ E_{i,i+1} - E_{n+i+1,n+i}, & \mathsf{t}(\mathfrak{g}) \in \{C_n, D_n\} \text{ and } i < n \\ E_{n,2n}, & \mathsf{t}(\mathfrak{g}) = C_n \text{ and } i = n \\ E_{n-1,2n} - E_{n,2n-1}, & \mathsf{t}(\mathfrak{g}) = D_n \text{ and } i = n \end{cases} \tag{2.5a}$$

$$Y_i = \begin{cases} 2X_i^\top, & \mathsf{t}(\mathfrak{g}) = B_n \text{ and } i = n \\ X_i^\top, & \text{else} \end{cases} \tag{2.5b}$$

where $E_{i,j}$ is the square matrix with $(i, j)$-entry equal to 1 and every other entry equal to 0.

It will also be helpful to have explicit descriptions of the classical root systems. Define $\varepsilon_1, \ldots, \varepsilon_n \in \mathfrak{h}^*$ such that for each $i \in [n]$ and $H \in \mathfrak{h}$,

$$\varepsilon_i(H) = \begin{cases} H_{i,i}, & \mathsf{t}(\mathfrak{g}) \neq B_n \\ H_{i+1,i+1}, & \mathsf{t}(\mathfrak{g}) = B_n \end{cases} \tag{2.6}$$

where $H_{k,l}$ denotes the $(k, l)$-entry of $H$. If $\mathsf{t}(\mathfrak{g}) = A_n$ we also define

$$\varepsilon_{n+1} = -\sum_{k=1}^{n} \varepsilon_k \tag{2.7}$$

Then a base $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ of $\Phi$ can be defined such that

$$\alpha_i = \begin{cases} \varepsilon_i - \varepsilon_{i+1}, & \mathrm{t}(\mathfrak{g}) = A_n \text{ or } i < n \\ \varepsilon_n, & \mathrm{t}(\mathfrak{g}) = B_n \text{ and } i = n \\ 2\varepsilon_n, & \mathrm{t}(\mathfrak{g}) = C_n \text{ and } i = n \\ \varepsilon_{n-1} + \varepsilon_n, & \mathrm{t}(\mathfrak{g}) = D_n \text{ and } i = n \end{cases} \tag{2.8}$$

It follows that

$$\Phi = \begin{cases} \{\varepsilon_i - \varepsilon_j \in \mathfrak{h}^* \colon i, j \in [n+1], i \neq j\}, & \mathrm{t}(\mathfrak{g}) = A_n \\ \{\pm\varepsilon_i \pm \varepsilon_j \in \mathfrak{h}^* \colon i, j \in [n], i \neq j\} & \\ \quad \cup \{\pm\varepsilon_i \in \mathfrak{h}^* \colon i \in [n]\}, & \mathrm{t}(\mathfrak{g}) = B_n \\ \{\pm\varepsilon_i \pm \varepsilon_j \in \mathfrak{h}^* \colon i, j \in [n], i \neq j\} & \\ \quad \cup \{\pm 2\varepsilon_i \in \mathfrak{h}^* \colon i \in [n]\}, & \mathrm{t}(\mathfrak{g}) = C_n \\ \{\pm\varepsilon_i \pm \varepsilon_j \in \mathfrak{h}^* \colon i, j \in [n], i \neq j\}, & \mathrm{t}(\mathfrak{g}) = D_n \end{cases} \tag{2.9}$$

with independent choices for the various "$\pm$". This description of $\Phi$ leads to a fairly simple characterization of the Weyl group $\mathscr{W}$. We have

$$\mathscr{W} \cong \begin{cases} S_{n+1}, & \mathrm{t}(\mathfrak{g}) = A_n \\ S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n, & \mathrm{t}(\mathfrak{g}) \in \{B_n, C_n\} \\ S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^{n-1}, & \mathrm{t}(\mathfrak{g}) = D_n \end{cases} \tag{2.10}$$

where $S_k$ is the symmetric group on $k$ letters. In Equation 2.10, the symmetric group factor acts by permuting the $\varepsilon_k$. For types $B_n$ and $C_n$, $(\mathbb{Z}/2\mathbb{Z})^n$ acts by changing the signs of the $\varepsilon_k$, while in the case of type $D_n$ the $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ factor acts by changing an even number of signs of the $\varepsilon_k$.

Since we are interested in a classification up to inner automorphism, it is sensible to interpret inner automorphisms of $\mathfrak{g}$ as conjugations by appropriate matrices. If $\mathrm{t}(\mathfrak{g}) = A_n$ then inner automorphisms can be regarded as conjugations by elements of the special linear group $SL_{n+1}$. For types $B_n$, $C_n$, and $D_n$ we may view an inner automorphism as conjugation by an element of $SO_{2n+1}$, $Sp_{2n}$, or $SO_{2n}$, respectively, where

$$SO_{2n+1} = \{P \in GL_{2n+1} \colon P^\top J_B = J_B P^{-1}\} \tag{2.11a}$$
$$Sp_{2n} = \{P \in GL_{2n} \colon P^\top J_C = J_C P^{-1}\} \tag{2.11b}$$
$$SO_{2n} = \{P \in GL_{2n} \colon P^\top J_D = J_D P^{-1}\} \tag{2.11c}$$

## 2.2 LEVI DECOMPOSABLE SUBALGEBRAS

As we are interested in a classification up to inner automorphism, we will often need to determine whether two given Levi decomposable subalgebras are conjugate. A commonly occurring situation is one in which the two Levi decomposable subalgebras have the same Levi factor but different radicals. Unfortunately, an inner automorphism mapping one such subalgebra to the other need not map the Levi factor to itself. This however is easily rectified by the following three lemmas.

**Lemma 2.1** (See Lemma 5.1 of [DR16]). *Suppose* $\psi\colon \mathfrak{l}_1 \in \mathfrak{s}_1 \to \mathfrak{l}_2 \in \mathfrak{s}_2$ *is an isomorphism, where* $\mathfrak{l}_1$ *and* $\mathfrak{l}_2$ *are semisimple and* $\mathfrak{s}_1$ *and* $\mathfrak{s}_2$ *are solvable. Then* $\psi(\mathfrak{s}_1) = \mathfrak{s}_2$.

**Lemma 2.2** (See Theorem 3.14.2 of [Var84]). *Let* $\mathfrak{a}$ *be a Levi decomposable Lie algebra such that* $\mathfrak{a} = \mathfrak{l}_1 \in \operatorname{Rad} \mathfrak{a} = \mathfrak{l}_2 \in \operatorname{Rad} \mathfrak{a}$ *for some semisimple subalgebras* $\mathfrak{l}_1, \mathfrak{l}_2 \subsetneq \mathfrak{a}$. *Then there exists* $\tau \in \operatorname{Int} \mathfrak{a}$ *such that* $\tau(\mathfrak{l}_1) = \mathfrak{l}_2$, *where* $\operatorname{Int} \mathfrak{a}$ *is the inner automorphism group of* $\mathfrak{a}$.

**Lemma 2.3.** *Let* $\mathfrak{g}$ *be a Lie algebra and let* $\mathfrak{a}_1 = \mathfrak{l} \in \operatorname{Rad} \mathfrak{a}_1$ *and* $\mathfrak{a}_2 = \mathfrak{l} \in \operatorname{Rad} \mathfrak{a}_2$ *be Levi decomposable subalgebras of* $\mathfrak{g}$ *with common Levi factor* $\mathfrak{l}$. *Suppose there exists* $\rho \in \operatorname{Int} \mathfrak{g}$ *such that* $\rho(\mathfrak{a}_1) = \mathfrak{a}_2$. *Then there exists* $\tau \in \operatorname{Int} \mathfrak{g}$ *such that* $\tau(\mathfrak{l}) = \mathfrak{l}$ *and* $\tau(\operatorname{Rad} \mathfrak{a}_1) = \operatorname{Rad} \mathfrak{a}_2$.

*Proof.* Note that $\rho(\mathfrak{a}_1) = \rho(\mathfrak{l}) \in \operatorname{Rad} \mathfrak{a}_2$ by Lemma 2.1. Thus both $\mathfrak{l}$ and $\rho(\mathfrak{l})$ are Levi factors of $\mathfrak{a}_2$. Lemma 2.2 implies there exists $\sigma \in \operatorname{Int} \mathfrak{a}_2$ such that $\sigma\rho(\mathfrak{l}) = \mathfrak{l}$. Moreover, Lemma 2.1 implies $\sigma\rho(\operatorname{Rad} \mathfrak{a}_1) = \operatorname{Rad} \mathfrak{a}_2$. By definition, $\operatorname{Int} \mathfrak{a}_2$ is generated by automorphisms of the form $\exp \operatorname{ad} Z$ for $Z \in \mathfrak{a}_2$. Regarding such elements $Z$ as elements of $\mathfrak{g}$ shows that $\sigma$ extends to some $\tilde{\sigma} \in \operatorname{Int} \mathfrak{g}$. The result now follows by setting $\tau = \tilde{\sigma}\rho$. □

In most cases in which we are concerned with conjugacy, Lemma 2.3 will suffice. However, there are occasions in which it would be beneficial for an inner automorphism to not only preserve the Levi factor, but also have it restrict to the identity on said Levi factor. This can be achieved by adding another hypothesis to Lemma 2.3.

**Corollary 2.4.** *Let* $\mathfrak{g}$, $\mathfrak{a}_1$, *and* $\mathfrak{a}_2$ *be as in Lemma 2.3. Suppose there exists* $\rho \in \operatorname{Int} \mathfrak{g}$ *such that* $\rho(\mathfrak{l}) = \mathfrak{l}$ *and* $\rho(\operatorname{Rad} \mathfrak{a}_1) = \operatorname{Rad} \mathfrak{a}_2$. *If* $\sigma = \rho|_{\mathfrak{l}}$ *is an inner automorphism of* $\mathfrak{l}$ *then there exists* $\tau \in \operatorname{Int} \mathfrak{g}$ *such that* $\tau|_{\mathfrak{l}} = \operatorname{id}$ *and* $\tau(\operatorname{Rad} \mathfrak{a}_1) = \operatorname{Rad} \mathfrak{a}_2$.

*Proof.* Since $\mathfrak{l}$ is a subalgebra of $\mathfrak{a}_2$ and $\sigma \in \operatorname{Int} \mathfrak{l}$ we conclude by an argument analogous to that in the proof of Lemma 2.3 that $\sigma^{-1}$ extends to some $\sigma_1 \in \operatorname{Int} \mathfrak{a}_2$. Since $\sigma_1$ is an automorphism of $\mathfrak{a}_2$ we deduce by

Lemma 2.1 that $\sigma_1(\text{Rad } \mathfrak{a}_2) = \text{Rad } \mathfrak{a}_2$. Further extending $\sigma_1$ to some $\sigma_2 \in \text{Int } \mathfrak{g}$ implies the existence of an inner automorphism $\sigma_2 \in \mathfrak{g}$ with $\sigma_2|_{\mathfrak{l}} = \rho^{-1}$ and $\sigma_2(\text{Rad } \mathfrak{a}_2) = \text{Rad } \mathfrak{a}_2$. The result now follows by defining $\tau = \sigma_2 \rho$. □

# 3 CLASSIFYING THE LEVI FACTOR

This chapter marks the beginning of our attempt at classifying the Levi decomposable subalgebras of the classical Lie algebras with regular simple Levi factor. This process can be separated into two stages. The first stage, in which we determine and describe all possibilities for the Levi factor, is examined in this chapter. We also remark on the seemingly limiting condition of regularity of the Levi factor and prove in the latter portion of this chapter that in many cases, sufficiently large simple subalgebras must be regular. For the remainder of this chapter $\mathfrak{g}$ will be a classical Lie algebra of rank $n$.

## 3.1 REGULAR SIMPLE SUBALGEBRAS

We begin our classification with an examination of the possible regular simple Levi factors. We first recall the definition of regularity.

**Definition 3.1.** A subspace $\mathfrak{l} \subseteq \mathfrak{g}$ is *regular* if there exists a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and corresponding root system $\Phi \subseteq \mathfrak{h}^*$ such that[5]

$$\mathfrak{l} = \mathfrak{h}_{\mathfrak{l}} \oplus \bigoplus_{\alpha \in \Phi_{\mathfrak{l}}} \mathfrak{g}_\alpha \tag{3.1}$$

for some subspace $\mathfrak{h}_{\mathfrak{l}} \subseteq \mathfrak{h}$ and subset $\Phi_{\mathfrak{l}} \subseteq \Phi$, where $\mathfrak{g}_\alpha$ is the root space of $\mathfrak{g}$ corresponding to $\alpha$. We say that $\mathfrak{l}$ *is regular relative to* $\mathfrak{h}$.

Equivalently, $\mathfrak{l}$ is regular relative to $\mathfrak{h}$ if and only if $[\mathfrak{h}, \mathfrak{l}] \subseteq \mathfrak{l}$.

*Remark* 3.2. Since all Cartan subalgebras are conjugate and we are interested in a classification up to inner automorphism, we can suppose that the Levi factors of interest are regular with respect to our chosen standard Cartan subalgebra $\mathfrak{h}$ of diagonal matrices.

Under what conditions is the regular subspace in Equation 3.1 in fact a regular simple subalgebra? It is an elementary exercise to show that such a subspace is a simple subalgebra precisely when two criteria are met.

5 Unless otherwise stated, direct sums are always interpreted as direct sums of vector spaces, not direct sums of Lie algebras.

First, $\Phi_{\mathfrak{l}}$ must be an irreducible subroot system of $\Phi$, in which case $\Phi_{\mathfrak{l}}$ is the root system of $\mathfrak{l}$. Secondly, $\mathfrak{h}_{\mathfrak{l}}$ must be the subalgebra of $\mathfrak{h}$ generated by the spaces $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ with $\alpha \in \Phi_{\mathfrak{l}}$. In this case $\mathfrak{h}_{\mathfrak{l}}$ is a Cartan subalgebra of $\mathfrak{l}$. Therefore, a complete characterization of regular simple subalgebras, and hence potential Levi factors in our classification problem, amounts to determining all possible subroot systems of $\Phi$. This was done by Dynkin and we present the relevant portions of his results in the following theorem.

**Theorem 3.3** (See Table 9 of [Dyn00]). *Let $\mathfrak{l}$ be a regular simple subalgebra of $\mathfrak{g}$. Then $t(\mathfrak{l}) \in \mathcal{T}_{\mathfrak{g}}$, where*

$$\mathcal{T}_{\mathfrak{g}} = \begin{cases} \{A_m \colon m \in [n]\}, & t(\mathfrak{g}) = A_n \\ \{A_m \colon m \in [n-1]\} \cup \{B_m \colon m \in [n]\} & \\ \cup \{D_m \colon 3 \leq m \leq n\}, & t(\mathfrak{g}) = B_n \\ \{A_m \colon m \in [n-1]\} \cup \{C_m \colon m \in [n]\}, & t(\mathfrak{g}) = C_n \\ \{A_m \colon m \in [n-1]\} \cup \{D_m \colon 3 \leq m \leq n\}, & t(\mathfrak{g}) = D_n \end{cases} \tag{3.2}$$

*In addition, for each $T \in \mathcal{T}_{\mathfrak{g}}$ there exists a regular simple subalgebra of $\mathfrak{g}$ of type $T$. Furthermore, if $\mathfrak{l}$ and $\mathfrak{l}'$ are regular simple subalgebras of $\mathfrak{g}$ with $t(\mathfrak{l}) = t(\mathfrak{l}')$ then $\mathfrak{l}$ and $\mathfrak{l}'$ are conjugate as a general rule, provided $A_1$, $B_1$, and $C_1$ are regarded as distinct types. The sole exception to this rule is when $t(\mathfrak{g}) = D_n$ with $n$ even and $t(\mathfrak{l}) = A_{n-1}$. In this case there are precisely two non-conjugate regular simple subalgebras of $\mathfrak{g}$ of type $A_{n-1}$.*

*Remark* 3.4. Notice that barring the only exception outlined in Theorem 3.3, regular simple subalgebras of classical Lie algebras are uniquely determined by type, up to inner automorphism.

*Remark* 3.5. Even though there are two non-conjugate regular simple subalgebras $\mathfrak{l}_1$ and $\mathfrak{l}_2$ of $\mathfrak{g}$ of type $A_{n-1}$ when $\mathrm{t}(\mathfrak{g}) = D_n$ with $n$ even, one can construct an outer automorphism $\tau \in \operatorname{Aut} \mathfrak{g}$ such that $\tau(\mathfrak{l}_1) = \mathfrak{l}_2$, where $\operatorname{Aut} \mathfrak{g}$ is the automorphism group of $\mathfrak{g}$.

*Remark* 3.6. Observe that we must regard types $A_1$, $B_1$, and $C_1$ as distinct despite yielding pairwise isomorphic subalgebras. To clarify, in this thesis the main result will apply to Levi decomposable subalgebras with regular simple Levi factors not of type $A_1$ and so the results of this thesis *will* apply when the Levi factor is of type $B_1$ or $C_1$. The manner in which we differentiate between these types is as follows. If $\mathfrak{l}$ is a regular simple subalgebra isomorphic to $\mathfrak{sl}_2$ with root system $\Phi_{\mathfrak{l}} \subseteq \Phi$, then $\mathfrak{l}$ is of type $B_1$ if $\mathrm{t}(\mathfrak{g}) = B_n$ and $\Phi_{\mathfrak{l}}$ consists of short roots, $\mathfrak{l}$ is of type $C_1$ if $\mathrm{t}(\mathfrak{g}) = C_n$ and $\Phi_{\mathfrak{l}}$ consists of long roots, and $\mathfrak{l}$ is of type $A_1$ otherwise.

By Theorem 3.3 we have a full description of possible Levi factors up to inner automorphism. In Chapter 4 we shall examine each Levi factor separately and determine how we can attach a solvable subalgebra to produce Levi decomposable subalgebras. A priori this runs the risk of duplication in our list. Indeed, what if $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are two conjugate Levi decomposable subalgebras of $\mathfrak{g}$ but have non-conjugate Levi factors? Then $\mathfrak{a}_1$ and $\mathfrak{a}_2$ may appear twice in our description despite actually being conjugate. This next lemma ensures we need not worry about such things.

**Lemma 3.7.** *Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be conjugate Levi decomposable subalgebras of $\mathfrak{g}$ with Levi factors $\mathfrak{l}_1$ and $\mathfrak{l}_2$, respectively. Then $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are conjugate.*

*Proof.* Suppose $\tau(\mathfrak{a}_1) = \mathfrak{a}_2$ for some $\tau \in \operatorname{Int} \mathfrak{g}$. Then $\mathfrak{a}_2 = \tau(\mathfrak{l}) \Subset \operatorname{Rad} \mathfrak{a}_2$ by Lemma 2.1. Since $\tau(\mathfrak{l}_1)$ is semisimple it follows that $\tau(\mathfrak{l}_1)$ is a Levi factor of $\mathfrak{a}_2$ and so there exists $\sigma \in \operatorname{Int} \mathfrak{a}_2$ such that $\sigma\tau(\mathfrak{l}_1) = \mathfrak{l}_2$ by Lemma 2.2. Extend $\sigma$ to some $\tilde{\sigma} \in \operatorname{Int} \mathfrak{g}$. Then $\tilde{\sigma}\tau(\mathfrak{l}_1) = \mathfrak{l}_2$. □

## 3.2 EXPLICIT DESCRIPTIONS OF THE POSSIBLE LEVI FACTORS

If $\mathfrak{a}$ is a Levi decomposable subalgebra of $\mathfrak{g}$ then Theorem 3.3 yields a list of candidates for the regular simple Levi factor $\mathfrak{l}$ of $\mathfrak{a}$. To aid us in performing the necessary calculations, it will be advantageous to have explicit descriptions of the various regular simple subalgebras. Since a regular simple subalgebra $\mathfrak{l}$ of a classical Lie algebra $\mathfrak{g}$ is uniquely defined by a subroot system, which in turn is uniquely defined by a base, providing a base $\Delta_{\mathfrak{l}} = \{\alpha_{\mathfrak{l},1}, \dots, \alpha_{\mathfrak{l},m}\}$ for $\Phi_{\mathfrak{l}}$ uniquely determines $\mathfrak{l}$.

For convenience we define

$$M = \begin{cases} n-m, & \mathfrak{t}(\mathfrak{l}) \neq A_m \\ n-m-1, & \mathfrak{t}(\mathfrak{l}) = A_m \end{cases} \tag{3.3}$$

where $m = \operatorname{rank} \mathfrak{l}$.

If $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) \in \{(B_n, B_m), (C_n, C_m), (D_n, D_m)\}$ we define

$$\Delta_{\mathfrak{l}} = \{\alpha_{M+1}, \dots, \alpha_n\} \tag{3.4}$$

If $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (B_n, B_m)$ then we obtain

$$\mathfrak{l} = \left\{ \begin{pmatrix} 0 & 0_{1,M} & e^\top & 0_{1,M} & f^\top \\ 0_{M,1} & 0_{M,M} & 0_{M,m} & 0_{M,M} & 0_{M,m} \\ -f & 0_{m,M} & R & 0_{m,M} & S \\ 0_{M,1} & 0_{M,M} & 0_{M,m} & 0_{M,M} & 0_{M,m} \\ -e & 0_{m,M} & T & 0_{m,M} & -R^\top \end{pmatrix} \in \mathfrak{so}_{2n+1} \colon e, f \in \mathbb{F}^m, \right.$$

$$R, S, T \in \mathfrak{gl}_m, \begin{pmatrix} 0 & e^\top & f^\top \\ -f & R & S \\ -e & T & -R^\top \end{pmatrix} \in \mathfrak{so}_{2m+1} \Bigg\} \tag{3.5}$$

where $0_{k,l}$ is the $k \times l$ zero matrix[6]. Otherwise, if $\mathrm{t}(\mathfrak{g}) \in \{C_n, D_n\}$ then Equation 3.4 implies

$$\mathfrak{l} = \Bigg\{ \begin{pmatrix} 0_{M,M} & 0_{M,m} & 0_{M,M} & 0_{M,m} \\ 0_{m,M} & R & 0_{m,M} & S \\ 0_{M,M} & 0_{M,m} & 0_{M,M} & 0_{M,m} \\ 0_{m,M} & T & 0_{m,M} & -R^\top \end{pmatrix} \in \mathfrak{sp}_{2n} \colon R, S, T \in \mathfrak{gl}_m,$$

$$\begin{pmatrix} R & S \\ T & -R^\top \end{pmatrix} \in \mathfrak{l}' \Bigg\} \tag{3.6}$$

where

$$\mathfrak{l}' = \begin{cases} \mathfrak{sp}_{2m}, & \mathrm{t}(\mathfrak{l}) = C_m \\ \mathfrak{so}_{2m}, & \mathrm{t}(\mathfrak{l}) = D_m \end{cases} \tag{3.7}$$

If $(\mathrm{t}(\mathfrak{g}), \mathrm{t}(\mathfrak{l})) = (B_n, D_m)$ we instead define

$$\Delta_{\mathfrak{l}} = \{\alpha_{M+1}, \ldots, \alpha_{n-1}, \alpha_{n-1} + 2\alpha_n\} \tag{3.8}$$

yielding

$$\mathfrak{l} = \Bigg\{ \begin{pmatrix} 0_{M+1,M+1} & 0_{M+1,m} & 0_{M+1,M} & 0_{M+1,m} \\ 0_{m,M+1} & R & 0_{m,M} & S \\ 0_{M,M+1} & 0_{M,m} & 0_{M,M} & 0_{M,m} \\ 0_{m,M+1} & T & 0_{m,M} & -R^\top \end{pmatrix} \in \mathfrak{so}_{2n+1} \colon R, S, T \in \mathfrak{gl}_m,$$

$$\begin{pmatrix} R & S \\ T & -R^\top \end{pmatrix} \in \mathfrak{so}_{2m} \Bigg\} \tag{3.9}$$

With this we have explicit descriptions of $\mathfrak{l}$ when $\mathfrak{l}$ is not of type $A_m$. If $\mathrm{t}(\mathfrak{l}) = A_m$ then we take

$$\Delta_{\mathfrak{l}} = \{\alpha_1, \ldots, \alpha_m\} \tag{3.10}$$

6 We omit the subscript if it is obvious from context.

resulting in

$$\mathfrak{l} = \left\{ \begin{pmatrix} R & 0_{m+1,M+1} \\ 0_{M+1,m+1} & 0_{M+1,M+1} \end{pmatrix} \in \mathfrak{sl}_{n+1} \colon R \in \mathfrak{sl}_{m+1} \right\} \tag{3.11}$$

if $\mathrm{t}(\mathfrak{g}) = A_n$. If $\mathrm{t}(\mathfrak{g}) = B_n$ then we instead obtain

$$\mathfrak{l} = \left\{ \begin{pmatrix} 0 & 0_{1,m+1} & 0_{1,M} & 0_{1,m+1} & 0_{1,M} \\ 0_{m+1,1} & R & 0_{m+1,M} & 0_{m+1,m+1} & 0_{m+1,M} \\ 0_{M,1} & 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \\ 0_{m+1,1} & 0_{m+1,m+1} & 0_{m+1,M} & -R^{\top} & 0_{m+1,M} \\ 0_{M,1} & 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \end{pmatrix} \in \mathfrak{so}_{2n+1} \colon \right.$$
$$\left. R \in \mathfrak{sl}_{m+1} \right\} \tag{3.12}$$

Finally, if $\mathfrak{g}$ is of type $C_n$ or $D_n$ then

$$\mathfrak{l} = \left\{ \begin{pmatrix} R & 0_{m+1,M} & 0_{m+1,m+1} & 0_{m+1,M} \\ 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \\ 0_{m+1,m+1} & 0_{m+1,M} & -R^{\top} & 0_{m+1,M} \\ 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \end{pmatrix} \in \mathfrak{g} \colon R \in \mathfrak{sl}_{m+1} \right\} \tag{3.13}$$

By Theorem 3.3, if $\mathrm{t}(\mathfrak{g}) = D_n$ with $n$ even and $m = n-1$ then there exists another non-conjugate regular subalgebra $\mathfrak{l}'$ of $\mathfrak{g}$ of type $A_m$. It is defined by the base

$$\Delta_{\mathfrak{l}'} = \{\alpha_1, \ldots, \alpha_{n-2}, \alpha_n\} \tag{3.14}$$

yielding

$$\mathfrak{l}' = \left\{ \begin{pmatrix} R & 0_{n-1,1} & 0_{n-1,n-1} & e \\ 0_{1,n-1} & 0_{1,1} & -e^{\top} & 0_{1,1} \\ 0_{n-1,n-1} & f & -R^{\top} & 0_{n-1,1} \\ -f^{\top} & 0_{1,1} & 0_{1,n-1} & 0_{1,1} \end{pmatrix} \in \mathfrak{so}_{2n} \colon R \in \mathfrak{sl}_{n-1}, \right.$$
$$\left. e, f \in \mathbb{F}^{n-1} \right\} \tag{3.15}$$

## 3.3 SUFFICIENTLY LARGE CLASSICAL SUBALGEBRAS ARE REGULAR

While at this point we could proceed to the heart of the thesis, namely a classification of the radical for our Levi decomposable subalgebras, we shall first address the condition of regularity. Of course, arbitrary simple subalgebras of $\mathfrak{g}$ need not be regular. For example, every simple Lie algebra can be embedded into one of type $A_n$ for sufficiently large $n$, but Theorem 3.3 asserts that the only regular such subalgebras must be of type $A_m$. Indeed, being regular implies the subalgebra's root system is contained in $\Phi$, which does appear to be a rather limiting condition. We prove in this section that regularity is not as restrictive as it initially seems by showing that classical subalgebras of sufficiently large rank are regular.

Since we are regarding our Lie algebras as matrix Lie algebras, we may view $\mathfrak{l}$ as the image of some representation. The main idea behind the proof of our claim is that if $\mathfrak{l}$ has a large enough rank, then the only permitted representations are those with regular images due to dimension considerations. As such, it is instructive to take a brief digression and briefly discuss representations. Let

$$\mathcal{B}_{\mathfrak{l}} = \{H_{\mathfrak{l},i} \in \mathfrak{h}_{\mathfrak{l}} \colon i \in [m]\} \cup \{X_{\mathfrak{l},\alpha} \in \mathfrak{l} \colon \alpha \in \Phi_{\mathfrak{l}}\} \cup \{Y_{\mathfrak{l},\alpha} \in \mathfrak{l} \colon \alpha \in \Phi_{\mathfrak{l}}\} \quad (3.16)$$

be a Chevalley basis for $\mathfrak{l}$ relative to $\mathfrak{h}_{\mathfrak{l}}$ and $\Delta_{\mathfrak{l}}$. Define the fundamental dominant weights $\lambda_1, \ldots, \lambda_m \in \mathfrak{h}_{\mathfrak{l}}^*$ such that for each $i, j \in [m]$, $\lambda_i(H_{\mathfrak{l},j}) = \delta_{i,j}$ and let

$$\delta = \sum_{k=1}^{m} \lambda_k \quad (3.17)$$

Given a dominant weight $\lambda \in \mathfrak{h}_{\mathfrak{l}}^*$ we denote the irreducible $\mathfrak{l}$-module of highest weight $\lambda$ by $V(\lambda)$. Note that our description of $\mathfrak{g}$ as a matrix Lie algebra is precisely $V(\lambda_1)$. The Weyl dimension formula asserts that

$$\dim V(\lambda) = \frac{\prod_{\alpha \in \Phi_{\mathfrak{l}}^+} \langle \delta + \lambda, \alpha \rangle}{\prod_{\alpha \in \Phi_{\mathfrak{l}}^+} \langle \delta, \alpha \rangle} \quad (3.18)$$

where $\Phi_{\mathfrak{l}}^+$ is the subset of positive roots of $\Phi_{\mathfrak{l}}$ and for all $\mu_1, \mu_2 \in \mathfrak{h}_{\mathfrak{l}}^*$ the quantity $\langle \mu_1, \mu_2 \rangle$ is defined as

$$\langle \mu_1, \mu_2 \rangle = \frac{2(\mu_1, \mu_2)}{(\mu_2, \mu_2)} \quad (3.19)$$

where $(\mu_1, \mu_2)$ is the inner product on $\mathfrak{h}_{\mathfrak{l}}^*$ induced by the Killing form. Notice that for all $i, j \in [m]$, $\langle \lambda_i, \alpha_{\mathfrak{l},j} \rangle = \delta_{i,j}$.

We previously alluded to the fact that we intend to use dimension arguments to prove that sufficiently large classical subalgebras must be regular. As such, we require some results pertaining to dimensions of representations. The first such result is an immediate consequence of Equation 3.18.

**Lemma 3.8** (See Eq. (13) of [DR15a]). *Let $\lambda = \sum_{k=1}^{m} a_k\lambda_k$ and $\mu = \sum_{k=1}^{m} b_k\lambda_k$ be dominant weights of $\mathfrak{l}$ such that $a_k \leq b_k$ for each $k \in [m]$. Then $\dim V(\lambda) \leq \dim V(\mu)$.*

In addition to Lemma 3.8, the subsequent lemma will prove useful in establishing the main claim of this section.

**Lemma 3.9.** *Let $c \in \mathbb{N}$. Then*

$$\dim V(c\lambda_1) = \begin{cases} \binom{m+c}{c} = \dim V(c\lambda_m), & t(\mathfrak{l}) = A_m \\ \frac{2m-1+2c}{2m-1}\binom{2m-2+c}{c}, & t(\mathfrak{l}) = B_m \\ \binom{2m-1+c}{c}, & t(\mathfrak{l}) = C_m \\ \frac{m-1+c}{m-1}\binom{2m-3+c}{c}, & t(\mathfrak{l}) = D_m \end{cases} \tag{3.20}$$

*Proof.* For $\alpha \in \Phi_{\mathfrak{l}}$ let $\alpha^\vee \in \Phi_{\mathfrak{l}}^\vee$ be the dual of $\alpha$, i.e.

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)} \tag{3.21}$$

If $\alpha^\vee = \sum_{k=1}^{m} c_k\alpha_{\mathfrak{l},k}^\vee$ for some $c_1, \ldots, c_m \in \mathbb{Z}$ then

$$\langle \delta, \alpha \rangle = \sum_{k=1}^{m} \frac{2(\lambda_k, \alpha)}{(\alpha, \alpha)} = \sum_{k=1}^{m} (\lambda_k, \alpha^\vee) = \sum_{k=1}^{m} c_k = \operatorname{ht} \alpha^\vee \tag{3.22}$$

where the first equality follows from Equation 3.19 and the definition of $\delta$, the second equality follows from the definition of the dual root, and $\operatorname{ht} \alpha^\vee$ denotes the height of $\alpha^\vee$. Similarly, for each $i \in [m]$ we have

$$\langle \lambda_i, \alpha \rangle = (\lambda_i, \alpha^\vee) = c_i \tag{3.23}$$

The remainder of this proof will be specific to the case of $t(\mathfrak{l}) = A_m$ since the other cases are handled analogously. In this setting, observe via Equation 2.8 and Equation 2.9 that the element in $\Phi_{\mathfrak{l}}^\vee$ with the largest height is $\sum_{k=1}^{m} \alpha_{\mathfrak{l},k}^\vee$ and so the maximum height of an element of $\Phi_{\mathfrak{l}}^\vee$ is $m$. For each $k \in [m]$ there are precisely $m - k + 1$ dual roots of $\Phi_{\mathfrak{l}}^\vee$ of height $k$. These are $\sum_{i=1}^{k} \alpha_{\mathfrak{l},i}^\vee, \sum_{i=2}^{k+1} \alpha_{\mathfrak{l},i}^\vee, \ldots, \sum_{i=m-k+1}^{m} \alpha_{\mathfrak{l},i}^\vee$. Of these, there exists exactly one dual root with $c_1 \neq 0$, namely $\sum_{i=1}^{k} \alpha_{\mathfrak{l},i}^{\vee}$, in which case we have $c_1 = 1$. Thus we conclude by Equation 3.18, Equation 3.22, and Equation 3.23 that

$$\dim V(c\lambda_1) = \frac{\prod_{k=1}^{m} k^{m-k}(k+c)}{\prod_{k=1}^{m} k^{m-k+1}} = \binom{m+c}{c} \tag{3.24}$$

A symmetric argument shows that $\dim V(c\lambda_m) = \binom{m+c}{c}$ as well. The proofs when $\mathfrak{l}$ is of the other types are similar so we omit the details. □

The next two lemmas will also be needed for our proof. The first follows from Theorems 2 and 3 of [Min06] and allows us to convert between equivalence of embeddings and equivalence of representations.

**Lemma 3.10.** *Suppose $t(\mathfrak{g}) \neq D_n$, $\pi_1\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is the representation of $\mathfrak{g}$ corresponding to $V(\lambda_1)$, $\mathfrak{l}$ is a simple Lie algebra, and $\varphi_1, \varphi_2\colon \mathfrak{l} \to \mathfrak{g}$ are two embeddings. Then $\varphi_1$ and $\varphi_2$ are equivalent*[7] *if and only if the representations $\pi_1\varphi_1$ and $\pi_1\varphi_2$ are isomorphic.*

Unfortunately, Lemma 3.10 does not apply in the case of $\mathfrak{g}$ of type $D_n$. To get around this, note that a classical Lie algebra of type $B_n$ naturally contains one of type $D_n$. Indeed, this is apparent by recalling that a classical Lie algebra of type $B_n$ is identified with $\mathfrak{so}_{2n+1}$ and one of type $D_n$ is identified with $\mathfrak{so}_{2n}$. This observation, along with the next lemma, will prove to be a suitable substitute for Lemma 3.10 when $\mathfrak{g}$ is of type $D_n$.

**Lemma 3.11** (See Lemma 4.10 of [DR15a]). *Let $\mathfrak{g}'$ be a Lie algebra of type $B_n$, let $\mathfrak{h}' \subseteq \mathfrak{g}'$ be a Cartan subalgebra, and let $\mathfrak{g}_1, \mathfrak{g}_2 \subseteq \mathfrak{g}'$ be simple subalgebras of type $D_n$. If $\mathfrak{h}' \subseteq \mathfrak{g}_1 \cap \mathfrak{g}_2$ then $\mathfrak{g}_1 = \mathfrak{g}_2$.*

We wish to prove the main claim of this section, namely that sufficiently large classical subalgebras are quite often regular. But what exactly do we mean by "quite often"? Observe from Theorem 3.3 that regular subalgebras are restricted in what type they can be. For instance, if $\mathrm{t}(\mathfrak{g}) = A_n$ then a regular simple subalgebra of $\mathfrak{g}$ must be of type $A_m$ for some $m \leq n$. Therefore, the problem of determining when a simple subalgebra is regular is only interesting for particular types of simple subalgebras. This leads to the following definition.

**Definition 3.12.** Define

$$\Omega = \{T_n\colon T \in \{A, B, C, D\}, n \in \mathbb{N}\} \tag{3.25}$$

7 We say embeddings $\varphi_1$ and $\varphi_2$ are *equivalent* if there exists an inner automorphism $\tau \in \mathrm{Int}\,\mathfrak{g}$ such that $\tau\varphi_1 = \varphi_2$

We say $(T_n, \tilde{T}_m) \in \Omega^2$ is a *regular pair* if there exists a classical Lie algebra $\mathfrak{g}$ and a regular simple subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$ such that $\mathrm{t}(\mathfrak{g}) = T_n$ and $\mathrm{t}(\mathfrak{l}) = \tilde{T}_m$.

By our previous remark, if we wish to show that sufficiently large simple subalgebras $\mathfrak{l}$ of $\mathfrak{g}$ are regular, then at the very least we must restrict our attention to the case when $(\mathrm{t}(\mathfrak{g}), \mathrm{t}(\mathfrak{l}))$ is a regular pair. With this we obtain the following result.

**Theorem 3.13.** *Let* $\mathfrak{g}, \mathfrak{l}$ *be classical Lie algebras with* $\operatorname{rank} \mathfrak{g} = n$ *and* $\operatorname{rank} \mathfrak{l} = m$ *such that* $(t(\mathfrak{g}), t(\mathfrak{l}))$ *is a regular pair. If* $\varphi \colon \mathfrak{l} \to \mathfrak{g}$ *is an embedding and* $m > c_{\mathfrak{g},\mathfrak{l}}$ *then* $\varphi(\mathfrak{l})$ *is a regular subalgebra of* $\mathfrak{g}$*, where*

$$c_{\mathfrak{g},\mathfrak{l}} = \begin{cases} \max\{\frac{1}{4}(\sqrt{16n+9}-1), \frac{\ln(2n+1)}{\ln 2}, \frac{1}{4}(2n-1)\}, & t(\mathfrak{l}) = B_m \\ \max\{\frac{1}{4}(\sqrt{16n+1}-1), \frac{1}{2}n\}, & t(\mathfrak{l}) = C_m \\ \max\{\frac{1}{4}(\sqrt{16n+9}+1), \frac{\ln(4n+2)}{\ln 2}, \frac{1}{4}(2n+1)\}, & t(\mathfrak{l}) = D_m \\ \max\{\frac{1}{2}(\sqrt{8n+9}-1), \frac{1}{2}(n-1)\}, & (t(\mathfrak{g}), t(\mathfrak{l})) = (A_n, A_m) \\ \max\{\frac{1}{2}(\sqrt{16n+1}-1), \frac{1}{3}(2n-3)\}, & (t(\mathfrak{g}), t(\mathfrak{l})) = (C_n, A_m) \\ \max\{\frac{1}{2}(\sqrt{16n+9}-1), \frac{2}{3}(n-1)\}, & \textit{else} \end{cases} \tag{3.26}$$

*Proof.* This proof will be handled on a case-by-case basis.

**Case 1: $\mathrm{t}(\mathfrak{l}) = B_m$**

By Theorem 3.3 we have $\mathrm{t}(\mathfrak{g}) = B_n$. Therefore, since $\pi_1$ is a $(2n+1)$-dimensional representation of $\mathfrak{g}$ we have that $\pi_1\varphi$ is a $(2n+1)$-dimensional-representation of $\mathfrak{l}$. By [Bou05] we have

$$\dim V(\lambda_i) = \binom{2m+1}{i}, \qquad \dim V(\lambda_m) = 2^m \tag{3.27}$$

for each $i \in [m-1]$. Since $m > c_{\mathfrak{g},\mathfrak{l}}$ we have $\binom{2m+1}{2}, 2^m > 2n+1$. It follows from Lemma 3.8 that if $\dim V(\lambda) \leq 2n+1$ then $\lambda = a_1\lambda_1$ for some $a_1 \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Observe that for $c > 1$,

$$\dim V(c\lambda_1) \geq \dim V(2\lambda_1) = \frac{2m+1}{2m-1}\binom{2m}{2} = \binom{2m+1}{2} > 2n+1$$

where the first equality is due to Lemma 3.8, the first equality is by Lemma 3.9, and the final inequality is since $m > c_{\mathfrak{g},\mathfrak{l}}$. Hence $a_1 \in \{0,1\}$. Therefore, if $\dim V(\lambda) \leq 2n+1$ we must have $\lambda = 0$, in which case $\dim V(\lambda) = 0$, or $\lambda = \lambda_1$, implying $\dim V(\lambda) = 2m+1$. Notice that having $m > c_{\mathfrak{g},\mathfrak{l}}$ implies $2(2m+1) > 2n+1$. Therefore, since $\pi_1\varphi$ is a $(2n+1)$-dimensional representation it must be that $\pi_1\varphi$ is equivalent to $(2n+1)V(0)$ or $V(\lambda_1) \oplus 2(n-m)V(0)$ as an $\mathfrak{l}$-module. The former is impossible since $\varphi$ is an embedding and thus injective and so up to equivalence there is precisely one possibility for $\pi_1\varphi$. By Lemma 3.10 there is precisely one embedding of $\mathfrak{l}$ into $\mathfrak{g}$, up to equivalence. Since Theorem 3.3 implies the existence of a regular subalgebra of $\mathfrak{g}$ of type $B_m$ it must be that $\varphi(\mathfrak{l})$ is regular.

**Case 2: $\mathrm{t}(\mathfrak{l}) = C_m$**

The arguments in this case are completely analogous to that of Case 1. The only differences are that $\mathrm{t}(\mathfrak{g}) = C_n$, $\pi_1$ is a $(2n)$-dimensional representation of $\mathfrak{g}$,

$$\dim V(\lambda_i) = \frac{m+1-i}{m+1}\binom{2m+2}{i} \tag{3.28}$$

for $1 < i \leq m$, $\dim V(\lambda_1) = 2m$, and having $m > c_{\mathfrak{g},\mathfrak{l}}$ implies $\binom{2m+1}{2} > 2n$ and $4m > 2n$. Otherwise the reasoning is identical to that of Case 1.

**Case 3: $(\mathrm{t}(\mathfrak{g}), \mathrm{t}(\mathfrak{l})) = (B_n, D_m)$** This case is also similar to that of Case 1. The primary differences are $\dim V(\lambda_i) = \binom{2m}{i}$ for $i \in [m-2]$, $\dim V(\lambda_{m-1}) = \dim V(\lambda_m) = 2^{m-1}$, and having $m > c_{\mathfrak{g},\mathfrak{l}}$ implies $\binom{2m}{2} > 2n+1$, $2^{m-1} > 2n+1$, and $4m > 2n+1$.

**Case 4: $(\mathrm{t}(\mathfrak{g}), \mathrm{t}(\mathfrak{l})) = (A_n, A_m)$**

This case is also analogous to Case 1. The only difference is that $\pi_1\varphi$ is an $(n+1)$-dimensional representation of $\mathfrak{l}$ and there are two nontrivial $(n+1)$-dimensional representations: one is equivalent to $V(\lambda_1) \oplus (n-m)V(0)$ as an $\mathfrak{l}$-module, and the other is equivalent to $V(\lambda_m) \oplus (n-m)V(0)$ as an $\mathfrak{l}$-module. Define the embeddings $\varphi_1, \varphi_2 \colon \mathfrak{l} \to \mathfrak{g}$ as

$$\varphi_1(X_{\mathfrak{l},\alpha_i}) = X_i, \qquad \varphi_1(Y_{\mathfrak{l},\alpha_i}) = Y_i \tag{3.29a}$$

$$\varphi_2(X_{\mathfrak{l},\alpha_i}) = -Y_i, \qquad \varphi_2(Y_{\mathfrak{l},\alpha_i}) = -X_i \tag{3.29b}$$

for each $i \in [m]$, where $X_{\mathfrak{l},\alpha_i}$ and $Y_{\mathfrak{l},\alpha_i}$ are Chevalley basis elements of $\mathfrak{l}$ as in Equation 3.16. One can verify that $\varphi_1$ corresponds to the $\mathfrak{l}$-module $V(\lambda_1) \oplus (n-m)V(0)$ and $\varphi_2$ corresponds to $V(\lambda_m) \oplus (n-m)V(0)$. Lemma 3.10 then implies that $\varphi$ is equivalent to $\varphi_1$ or $\varphi_2$. It is evident that $\varphi_1(\mathfrak{l}) = \varphi_2(\mathfrak{l})$ and so up to inner automorphism there is a unique subalgebra of $\mathfrak{g}$ of type $A_m$. Theorem 3.3 implies that such a subalgebra must be regular.

**Case 5: $(\mathrm{t}(\mathfrak{g}), \mathrm{t}(\mathfrak{l})) = (C_n, A_m)$**

This case is fairly similar to that of the previous four cases. Using the arguments outlined in Case 1, one can show that $\pi_1\varphi$ is a $(2n)$-dimensional representation of $\mathfrak{l}$ and that the only possibilities for such a representation are equivalent to one of the following as an $\mathfrak{l}$-module:

1. $V(\lambda_1) \oplus V(\lambda_m) \oplus 2MV(0)$
2. $2V(\lambda_1) \oplus 2MV(0)$

3. $2V(\lambda_m) \oplus 2MV(0)$
4. $V(\lambda_1) \oplus (n+M)V(0)$
5. $V(\lambda_m) \oplus (n+M)V(0)$
6. $2nV(0)$

where $M$ is as in Equation 3.3. Therefore, by Lemma 3.10 $\varphi$ is equivalent to one of these options as an $\mathfrak{l}$-module. We claim that it must be $V(\lambda_1) \oplus V(\lambda_m) \oplus 2MV(0)$.

Since $\varphi$ is an embedding and hence a nontrivial map it cannot yield an $\mathfrak{l}$-module equivalent to $2nV(0)$. Now note that not only is $\pi_1\varphi$ a $(2n)$-dimensional representation of $\mathfrak{l}$, but it is also a symplectic representation. We show explicitly that the second option is not symplectic. Similar arguments, which we omit, will show that the third, fourth, and fifth options are also not symplectic.

Define $\psi\colon \mathfrak{sl}_{m+1} \to \mathfrak{gl}_{2n}$ as

$$\psi(E_{i,j}) = E_{i,j} + E_{n+i,n+j} \tag{3.30}$$

for each $i, j \in [m+1]$ distinct. Then $\psi$ defines a $(2n)$-dimensional representation of $\mathfrak{sl}_{m+1}$ equivalent to $2V(\lambda_1) \oplus 2MV(0)$. Observe that

$$\psi(\mathfrak{sl}_{m+1}) = \left\{ \begin{pmatrix} R & 0_{m+1,M} & 0_{m+1,m+1} & 0_{m+1,M} \\ 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \\ 0_{m+1,m+1} & 0_{m+1,M} & R & 0_{m+1,M} \\ 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \end{pmatrix} \in \mathfrak{gl}_{2n} \colon R \in \mathfrak{sl}_{m+1} \right\} \tag{3.31}$$

If $\psi$ is symplectic then there exists a skew-symmetric matrix $J \in GL_{2n}$ such that for all $Z \in \mathfrak{sl}_{m+1}$, $J\psi(Z) + (\psi(Z))^\top J = 0$. Block decompose $J$ as

$$J = \begin{pmatrix} J_{1,1} & J_{1,2} & J_{1,3} & J_{1,4} \\ J_{2,1} & J_{2,2} & J_{2,3} & J_{2,4} \\ J_{3,1} & J_{3,2} & J_{3,3} & J_{3,4} \\ J_{4,1} & J_{4,2} & J_{4,3} & J_{4,4} \end{pmatrix} \tag{3.32}$$

where the block decomposition of $J$ is compatible with that of Equation 3.31. Then for all $R \in \mathfrak{sl}_{m+1}$ we have

$$0 = \begin{pmatrix} J_{1,1} & J_{1,2} & J_{1,3} & J_{1,4} \\ J_{2,1} & J_{2,2} & J_{2,3} & J_{2,4} \\ J_{3,1} & J_{3,2} & J_{3,3} & J_{3,4} \\ J_{4,1} & J_{4,2} & J_{4,3} & J_{4,4} \end{pmatrix} \begin{pmatrix} R & 0_{m+1,M} & 0_{m+1,m+1} & 0_{m+1,M} \\ 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \\ 0_{m+1,m+1} & 0_{m+1,M} & R & 0_{m+1,M} \\ 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \end{pmatrix}$$

$$+\begin{pmatrix} R^\top & 0_{m+1,M} & 0_{m+1,m+1} & 0_{m+1,M} \\ 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \\ 0_{m+1,m+1} & 0_{m+1,M} & R^\top & 0_{m+1,M} \\ 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \end{pmatrix} \begin{pmatrix} J_{1,1} & J_{1,2} & J_{1,3} & J_{1,4} \\ J_{2,1} & J_{2,2} & J_{2,3} & J_{2,4} \\ J_{3,1} & J_{3,2} & J_{3,3} & J_{3,4} \\ J_{4,1} & J_{4,2} & J_{4,3} & J_{4,4} \end{pmatrix}$$
$$= \begin{pmatrix} J_{1,1}R + R^\top J_{1,1} & R^\top J_{1,2} & J_{1,3}R + R^\top J_{1,3} & R^\top J_{1,4} \\ J_{2,1}R & 0_{M,M} & J_{2,3}R & 0_{M,M} \\ J_{3,1}R + R^\top J_{3,1} & R^\top J_{3,2} & J_{3,3}R + R^\top J_{3,3} & R^\top J_{3,4} \\ J_{4,1}R & 0_{M,M} & J_{4,3}R & 0_{M,M} \end{pmatrix} \tag{3.33}$$

Note that $J_{1,1}R + R^\top J_{1,1} = 0$ for all $R \in \mathfrak{sl}_{m+1}$. By considering the $(k,l)$-entry in the case where $R = E_{i,j}$ with $i,j \in [m+1]$ distinct, one has

$$\delta_{j,l}(J_{1,1})_{k,i} + \delta_{j,k}(J_{1,1})_{i,l} = 0 \tag{3.34}$$

for all $k,l \in [m+1]$. In particular, for each $i \in [m+1]$ let $j \in [m+1]$ be such that $j \notin \{1,i\}$. Note that this is possible since having $m > c_{\mathfrak{g},\mathfrak{l}}$ implies $m > 1$. Then by considering the $(1,j)$-entry one has $(J_{1,1})_{1,i} = 0$, i.e. the first row of $J_{1,1}$ is 0. An analogous argument shows that the first row of $J_{1,3}$ is also 0. It is evident from Equation 3.33 that $J_{1,2} = J_{1,4} = 0$ and so the first row of $J$ is 0, contradicting invertibility of $J$. It follows that $\psi$ is not symplectic and so $\varphi$ is not equivalent to $\psi$. Similar arguments show that having a symplectic $(2n)$-dimensional representation forbids the third, fourth, and fifth options and so $\varphi$ must yield $V(\lambda_1) \oplus V(\lambda_m) \oplus 2MV(0)$ as an $\mathfrak{l}$-module. Therefore, by Lemma 3.10 there is one embedding of $\mathfrak{l}$ into $\mathfrak{g}$ up to equivalence and Theorem 3.3 implies this embedding must yield a regular subalgebra.

**Case 6: $(\mathrm{t}(\mathfrak{g}), \mathrm{t}(\mathfrak{l})) = (B_n, A_m)$**

This case is nearly identical to that of Case 5. Indeed, the sole difference is showing there is precisely one nontrivial orthogonal $(2n+1)$-dimensional representation of $\mathfrak{l}$ in lieu of showing there is exactly one nontrivial symplectic $(2n)$-dimensional representation of $\mathfrak{l}$. Since the argument is identical we forgo the details.

**Case 7: $\mathrm{t}(\mathfrak{g}) = D_n$**

By Theorem 3.3 it remains to examine the case where $\mathrm{t}(\mathfrak{g}) = D_n$. Theorem 3.3 implies there exists an embedding $\psi_{reg}\colon \mathfrak{l} \to \mathfrak{g}$ such that $\psi_{reg}(\mathfrak{l})$ is a regular subalgebra of $\mathfrak{g}$. Note that for all $\rho \in \operatorname{Aut}\mathfrak{g}$ we have that $\rho\psi_{reg}(\mathfrak{l})$ is also a regular subalgebra of $\mathfrak{g}$. Indeed, this is immediate from the definition of regularity. Therefore, to prove $\varphi(\mathfrak{l})$ is regular it suffices to prove the existence of $\rho \in \operatorname{Aut}\mathfrak{g}$ such that $\psi_{reg} = \rho\varphi$.

Both $\varphi(\mathfrak{h}_\mathfrak{l})$ and $\psi_{reg}(\mathfrak{h}_\mathfrak{l})$ are contained in Cartan subalgebras of $\mathfrak{g}$. By conjugating if necessary, we may suppose without loss of generality that $\varphi(\mathfrak{h}_\mathfrak{l}), \psi_{reg}(\mathfrak{h}_\mathfrak{l}) \subseteq \mathfrak{h}$. Now let $\mathfrak{g}'$ be a simple Lie algebra of type $B_n$. By realizing $\mathfrak{g}$ as $\mathfrak{so}_{2n}$ and $\mathfrak{g}'$ as $\mathfrak{so}_{2n+1}$, we may naturally consider $\mathfrak{g}$ as a subalgebra of $\mathfrak{g}'$ via an embedding $\varphi' \colon \mathfrak{g} \to \mathfrak{g}'$. Note that $c_{\mathfrak{g}',\mathfrak{l}} = c_{\mathfrak{g},\mathfrak{l}}$ and so $m > c_{\mathfrak{g}',\mathfrak{l}}$. Thus Cases 3 and 6 imply $\varphi'\varphi(\mathfrak{l})$ is a regular simple subalgebra of $\mathfrak{g}'$. By Theorem 3.3 and Remark 3.5, $\mathfrak{g}'$ contains a unique subalgebra of type $\mathrm{t}(\mathfrak{l})$ up to automorphism[8]. Thus there exists $\tau \in \operatorname{Aut} \mathfrak{g}'$ such that

$$\tau\varphi'\varphi = \psi'_{reg} \tag{3.35}$$

where we define $\psi'_{reg} = \varphi'\psi_{reg}$. Since the case of $m = n$ is trivial we may suppose $m < n$, implying $\varphi'\varphi(\mathfrak{h}_{\mathfrak{l}}) \subsetneq \mathfrak{h}'$, where $\mathfrak{h}' = \varphi'(\mathfrak{h})$ is a Cartan subalgebra of $\mathfrak{g}'$. Hence there exist $H^1, \ldots, H^{n-m} \in \mathfrak{h}'$ such that

$$\mathfrak{h}' = \varphi'\varphi(\mathfrak{h}_{\mathfrak{l}}) \oplus \langle H^1, \ldots, H^{n-m} \rangle \tag{3.36}$$

Let $i \in [n-m]$. We have $[H^i, H] = 0$ for all $H \in \varphi'\varphi(\mathfrak{h}_{\mathfrak{l}})$ since $H^i \in \mathfrak{h}'$ and $\varphi'\varphi(\mathfrak{h}_{\mathfrak{l}}) \subsetneq \mathfrak{h}'$. Therefore, $[\tau(H^i), H] = 0$ for all $H \in \tau\varphi'\varphi(\mathfrak{h}_{\mathfrak{l}})$. By Equation 3.35 it follows that $\tau(H^i) \in C_{\mathfrak{g}'}(\psi'_{reg}(\mathfrak{h}_{\mathfrak{l}}))$, where $C_{\mathfrak{g}'}(\psi'_{reg}(\mathfrak{h}_{\mathfrak{l}}))$ is the centralizer of $\psi'_{reg}(\mathfrak{h}_{\mathfrak{l}})$ in $\mathfrak{g}'$. We claim that

$$C_{\mathfrak{g}'}(\psi'_{reg}(\mathfrak{h}_{\mathfrak{l}})) = \psi'_{reg}(\mathfrak{h}_{\mathfrak{l}}) \oplus C_{\mathfrak{g}'}(\psi'_{reg}(\mathfrak{l})) \tag{3.37}$$

First note that the sum in the right-hand side of Equation 3.37 is indeed direct since $\psi'_{reg}(\mathfrak{h}_{\mathfrak{l}})$ is a Cartan subalgebra of $\psi'_{reg}(\mathfrak{l})$. Consequently, $\psi'_{reg}(\mathfrak{h}_{\mathfrak{l}})$ is contained in $\psi'_{reg}(\mathfrak{l})$ and yet it is evident that $C_{\mathfrak{g}'}(\psi'_{reg}(\mathfrak{l}))$ contains no nonzero elements of $\psi'_{reg}(\mathfrak{l})$ since $\psi'_{reg}(\mathfrak{l})$ is simple. Now to see why Equation 3.37 holds, notice that $\psi'_{reg}(\mathfrak{h}_{\mathfrak{l}})$ is a regular subalgebra of $\mathfrak{g}'$ and centralizers of regular subalgebras are again regular. Therefore, as $C_{\mathfrak{g}'}(\psi'_{reg}(\mathfrak{h}_{\mathfrak{l}}))$ is regular one only needs to determine its decomposition as in Equation 3.1. This is a straightforward but laborious computation which yields Equation 3.37.

Since $\tau(H^i) \in C_{\mathfrak{g}'}(\psi_{reg}(\mathfrak{h}_{\mathfrak{l}}))$ we have that there exist $K^{i,1} \in \psi'_{reg}(\mathfrak{h}_{\mathfrak{l}})$ and $K^{i,2} \in C_{\mathfrak{g}'}(\psi'_{reg}(\mathfrak{l}))$ such that $\tau(H^i) = K^{i,1} + K^{i,2}$. Note that $H^i$, and thus $\tau(H^i)$, is semisimple in $\mathfrak{g}'$ since $H^i$ in $\mathfrak{h}'$. Additionally, $K^{i,1}$ is semisimple in $\mathfrak{g}'$ since $K^{i,1} \in \psi'_{reg}(\mathfrak{h}_{\mathfrak{l}}) \subseteq \mathfrak{h}'$. Further observe that $[\tau(H^i), K^{i,1}] = 0$ since $\tau(H^i) \in C_{\mathfrak{g}'}(\psi'_{reg}(\mathfrak{h}_{\mathfrak{l}}))$ and $K^{i,1} \in \psi'_{reg}(\mathfrak{h}_{\mathfrak{l}})$. It follows that $K^{i,2} = \tau(H^i) - K^{i,1}$ is semisimple in $\mathfrak{g}'$. Invariance of $C_{\mathfrak{g}'}(\psi'_{reg}(\mathfrak{l}))$ under $\operatorname{ad} K^{i,2}$ in turn implies $K^{i,2}$ is semisimple in $C_{\mathfrak{g}'}(\psi'_{reg}(\mathfrak{l}))$. Now observe that for all $i, j \in [n-m]$,

$$\begin{aligned} [K^{i,2}, K^{j,2}] &= [\tau(H^i) - K^{i,1}, \tau(H^j) - K^{j,1}] \\ &= \tau([H^i, H^j]) - [\tau(H^i), K^{j,1}] - [K^{i,1}, \tau(H^j)] + [K^{i,1}, K^{j,1}] \end{aligned} \tag{3.38}$$

8 Note that uniqueness is not guaranteed up to inner automorphism due to Remark 3.5.

The first and fourth terms in Equation 3.38 are zero because $H^i$, $H^j$, $K^{i,1}$, and $K^{j,1}$ are contained in $\mathfrak{h}'$ and the second and third terms are zero because $\tau(H^i), \tau(H^j) \in C_{\mathfrak{g}'}(\psi'_{reg}(\mathfrak{h}_{\mathfrak{l}}))$ and $K^{i,1}, K^{j,1} \in \psi'_{reg}(\mathfrak{h}_{\mathfrak{l}})$. It follows that $\langle K^{1,2}, \dots, K^{n-m,2} \rangle$ is a subalgebra of $C_{\mathfrak{g}'}(\psi'_{reg}(\mathfrak{l}))$ consisting of semisimple elements and thus there exists $\sigma_0 \in \operatorname{Int} C_{\mathfrak{g}'}(\psi'_{reg}(\mathfrak{l}))$ such that $\sigma_0(\langle K^{1,2}, \dots, K^{n-m,2} \rangle) \subseteq \mathfrak{h}'$.

We extend $\sigma_0$ to an inner automorphism $\sigma \in \operatorname{Int} \mathfrak{g}'$ such that $\sigma|_{\psi'_{reg}(\mathfrak{l})} = \mathrm{id}$. Note that this is possible since $[C_{\mathfrak{g}'}(\psi'_{reg}(\mathfrak{l})), \psi'_{reg}(\mathfrak{l})] = \{0\}$ by definition and $C_{\mathfrak{g}'}(\psi'_{reg}(\mathfrak{l})) \cap \psi'_{reg}(\mathfrak{l}) = \{0\}$. For each $i \in [n-m]$ we have $\sigma(K^{i,1}) = K^{i,1}$ since $K^{i,1} \in \psi'_{reg}(\mathfrak{h}_{\mathfrak{l}}) \subseteq \psi'_{reg}(\mathfrak{l})$. We also have $\sigma(K^{i,2}) \in \mathfrak{h}'$ by construction and thus $\sigma\tau(H^i) \in \mathfrak{h}'$. Moreover, we have $\sigma\tau\varphi'\varphi(\mathfrak{h}_{\mathfrak{l}}) = \sigma\psi'_{reg}(\mathfrak{h}_{\mathfrak{l}}) = \psi'_{reg}(\mathfrak{h}_{\mathfrak{l}}) \subseteq \mathfrak{h}'$ by Equation 3.35 and since $\psi'_{reg}(\mathfrak{h}_{\mathfrak{l}}) \subseteq \psi'_{reg}(\mathfrak{l})$. It follows from Equation 3.36 that $\sigma\tau(\mathfrak{h}') = \mathfrak{h}'$. Hence by Lemma 3.11 we conclude that $\sigma\tau(\mathfrak{g}) = \mathfrak{g}$ and thus $\sigma\tau|_{\mathfrak{g}} \in \operatorname{Aut} \mathfrak{g}$. By taking $\rho = \sigma\tau|_{\mathfrak{g}}$ we get $\rho\varphi = \psi_{reg}$, as desired. □

In a way, Theorem 3.13 is not altogether that surprising. If $\mathfrak{l}$ is a simple non-regular subalgebra of $\mathfrak{g}$ then it is not normalized by any Cartan subalgebra of $\mathfrak{g}$. As such, given a Cartan subalgebra $\tilde{\mathfrak{h}}$ of $\mathfrak{g}$ we have that $[\tilde{\mathfrak{h}}, \mathfrak{l}]$ will contain elements outside $\mathfrak{l}$. Heuristically, this requires that $\mathfrak{l}$ be "small" enough to ensure this can happen.

Note that Theorem 3.13 illustrates different behaviour depending upon whether $\mathfrak{l}$ is or is not of type $A_m$. Indeed, we find that for sufficiently large $n$, if $\mathrm{t}(\mathfrak{l}) = A_m$ and $\mathrm{t}(\mathfrak{g}) \in \{B_n, C_n, D_n\}$ then $\mathfrak{l}$ is regular if $m$ is roughly at least $\frac{2}{3}n$, but this lower bound changes to roughly $\frac{1}{2}n$ if either $\mathrm{t}(\mathfrak{l}) \neq A_m$ or $(\mathrm{t}(\mathfrak{g}), \mathrm{t}(\mathfrak{l})) = (A_n, A_m)$. This split in behaviour is more or less due to the fact that if $\mathrm{t}(\mathfrak{g}) = A_n$ then $\dim \mathfrak{g}$ is on the order of $n^2$, but if $\mathrm{t}(\mathfrak{g}) \neq A_n$ then $\dim \mathfrak{g}$ is on the order of $2n^2$. As such, subalgebras of type $A_m$ contained in classical Lie algebras not of type $A_n$ are "smaller" than their non-$A_m$ counterparts. Per our comment above it is understandable to find such a difference, especially since our proof heavily relied upon dimension arguments. However, this difference in behaviour will reappear when we classify radicals of Levi decomposable subalgebras in Chapter 4, even though our arguments there will not strongly depend on dimension considerations.

# 4 CLASSIFYING THE RADICAL

We commence an examination of the primary topic of this thesis, namely a description of nearly all Levi decomposable subalgebras of classical Lie algebras $\mathfrak{g}$ with regular simple Levi factor $\mathfrak{l}$. While the results of this chapter are fairly general, there are a few select instances in which the theory we establish breaks down. The case in which the results of this chapter do not apply is when $(\mathfrak{g}, \mathfrak{l})$ is ill-mannered as defined below.

**Definition 4.1.** Let $(\mathrm{t}(\mathfrak{g}), \mathrm{t}(\mathfrak{l}))$ be a regular pair. We say $(\mathrm{t}(\mathfrak{g}), \mathrm{t}(\mathfrak{l}))$ is *ill-mannered* if $\mathrm{t}(\mathfrak{l}) = A_1$, $(\mathrm{t}(\mathfrak{g}), \mathrm{t}(\mathfrak{l})) = (B_n, A_2)$, or $(\mathrm{t}(\mathfrak{g}), \mathrm{t}(\mathfrak{l})) = (D_n, A_m)$. If the pair $(\mathrm{t}(\mathfrak{g}), \mathrm{t}(\mathfrak{l}))$ is not ill-mannered we say it is *well-mannered*.

In Chapter 5 we shall examine precisely what goes wrong when $(\mathfrak{g}, \mathfrak{l})$ is ill-mannered. For now, we shall restrict our attention to the case of $(\mathfrak{g}, \mathfrak{l})$ well-mannered. Unless otherwise stated, for the remainder of this chapter $\mathfrak{g}$ will be a classical Lie algebra with a regular simple subalgebra $\mathfrak{l}$ such that $(\mathrm{t}(\mathfrak{g}), \mathrm{t}(\mathfrak{l}))$ is well-mannered and $\mathfrak{a}$ will be a Levi decomposable subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$. We further suppose $\mathfrak{l}$ is as in Equation 3.1. In particular, $\mathfrak{h}_{\mathfrak{l}}$ is a Cartan subalgebra of $\mathfrak{l}$ contained in $\mathfrak{h}$ and $\Phi_{\mathfrak{l}}$ is a root system of $\mathfrak{l}$ with base $\Delta_{\mathfrak{l}}$.

Theorem 3.3 yields a total description of the possible Levi factors of a Levi decomposable subalgebra of $\mathfrak{g}$. The primary goal of this chapter is to provide a complete description of the radical. We begin with a brief description of how we intend to study the radical. Our analysis will suggest splitting the radical into two components.

## 4.1 VIEWING THE RADICAL AS AN $\mathfrak{l}$-MODULE

The most obvious obstacle in attempting to classify the possibilities for the radical is the lack of a description of solvable subalgebras. There is no general theory for characterizing solvable subalgebras and such a classification is believed to be a hopeless endeavour. Indeed, a total description exists only for low-dimensional solvable Lie algebras; see [dG05, vW14]. Fortunately, we are not attempting to classify the radicals in a vacuum and we are able to invoke the simple structure of the Levi factors established in Theorem 3.3.

Since Rad $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, one can regard Rad $\mathfrak{a}$ as a $\mathfrak{g}$-module with respect to the adjoint representation. By restriction, Rad $\mathfrak{a}$ is an $\mathfrak{l}$-module with respect to the adjoint representation of $\mathfrak{l}$. Consequently, not only is Rad $\mathfrak{a}$ a solvable ideal of $\mathfrak{g}$ but it must also be an $\mathfrak{l}$-module. As $\mathfrak{l}$ is simple, Weyl's Theorem asserts that Rad $\mathfrak{a}$ decomposes into a direct sum of irreducible $\mathfrak{l}$-modules. Therefore, Rad $\mathfrak{a}$ is uniquely determined by maximal vectors and so a preliminary step is to determine these vectors. To do so, notice that $\mathfrak{g}$ is also an $\mathfrak{l}$-module with respect to the adjoint representation of $\mathfrak{l}$ and thus Rad $\mathfrak{a}$ is an $\mathfrak{l}$-submodule of $\mathfrak{g}$. It follows that any maximal vector of an irreducible component of Rad $\mathfrak{a}$ as an $\mathfrak{l}$-module is necessarily a maximal vector of an irreducible component of $\mathfrak{g}$ as an $\mathfrak{l}$-module. As such, our focus is to first decompose $\mathfrak{g}$ into irreducible $\mathfrak{l}$-submodules with respect to the adjoint representation and determine the corresponding maximal vectors. The following lemma will assist in our search for said maximal vectors.

**Lemma 4.2.** *Let $Z \in \mathfrak{g}$ be a maximal vector of weight $\lambda \in \mathfrak{h}_{\mathfrak{l}}$, where $\mathfrak{g}$ is viewed as an $\mathfrak{l}$-module with respect to the adjoint representation and*

$$Z = H + \sum_{\alpha \in \Phi} Z_\alpha \tag{4.1}$$

*where $H \in \mathfrak{h}$ and $Z_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Phi$. If $H \neq 0$ then $\lambda = 0$ and $H$ is a maximal vector of weight $\lambda$. If $Z_\alpha \neq 0$ for some $\alpha \in \Phi$ then $\lambda = \alpha|_{\mathfrak{h}_{\mathfrak{l}}}$ and $Z_\alpha$ is a maximal vector of weight $\lambda$.*

*Proof.* Since $Z$ has weight $\lambda$ we have that for all $H' \in \mathfrak{h}_{\mathfrak{l}}$,

$$\lambda(H')\left(H + \sum_{\alpha \in \Phi} Z_\alpha\right) = \lambda(H')Z = [H', Z] = \sum_{\alpha \in \Phi} \alpha(H')Z_\alpha \tag{4.2}$$

from which it follows that $\lambda(H')H = 0$ and $\lambda(H')Z_\alpha = \alpha(H')Z_\alpha$ for all $H' \in \mathfrak{h}_{\mathfrak{l}}$ and $\alpha \in \Phi$. If $H \neq 0$ then $\lambda = 0$. If $H = 0$ then there exists $\alpha \in \Phi$ such that $Z_\alpha \neq 0$, forcing $\lambda = \alpha|_{\mathfrak{h}_{\mathfrak{l}}}$.

Having $Z$ be a maximal vector implies that for all $\beta \in \Delta_{\mathfrak{l}}$ and $Z'_\beta \in \mathfrak{g}_\beta$,

$$0 = [Z, Z'_\beta] = \beta(H)Z'_\beta + \sum_{\alpha \in \Phi} [Z_\alpha, Z'_\beta] \tag{4.3}$$

By the root space decomposition of $\mathfrak{g}$ we conclude that $\beta(H) = 0$ and $[Z_\alpha, Z'_\beta] = 0$ for all $\alpha \in \Phi$ and $\beta \in \Delta_{\mathfrak{l}}$. If $H \neq 0$ then $H$ is a maximal vector of weight 0 since $[H, Z'_\beta] = \beta(H)Z'_\beta$ and $Z'_\beta \neq 0$. Similarly, if $Z_\alpha \neq 0$ for some $\alpha \in \Phi$ then $Z_\alpha$ is a maximal vector of weight $\alpha|_{\mathfrak{h}_{\mathfrak{l}}}$. $\square$

Lemma 4.2 implies that in our search for maximal vectors we need only examine elements of $\mathfrak{h}$ and root space vectors. We can facilitate future discussions by grouping the maximal vectors into two distinct categories: those with weight 0 and those with nonzero weight. Let us begin with an examination of those with weight 0.

If $Z_\alpha \in \mathfrak{g}_\alpha$ is a maximal vector then $[Z_\alpha, Z'_\beta] = 0$ for all $\beta \in \Delta_\mathfrak{l}$ and $Z'_\beta \in \mathfrak{g}_\beta$. Consequently, $\mathfrak{g}_{\alpha+\beta} = \{0\}$ for all $\beta \in \Delta_\mathfrak{l}$. In other words, $\alpha + \beta \notin \Phi_0$, where we define $\Phi_0 = \Phi \cup \{0\}$. Conversely, if $\alpha \in \Phi$ is such that $\alpha + \beta \notin \Phi_0$ for all $\beta \in \Delta_\mathfrak{l}$ then a nonzero vector $Z_\alpha \in \mathfrak{g}_\alpha$ will be a maximal vector. Therefore, to determine the root space maximal vectors one only needs to find all roots $\alpha \in \Phi$ such that $\alpha + \beta \notin \Phi_0$ for all $\beta \in \Delta_\mathfrak{l}$. This naturally leads to the following definition.

**Definition 4.3.** A root $\alpha \in \Phi$ is $\Delta_\mathfrak{l}$-*maximal* if for all $\beta \in \Delta_\mathfrak{l}$, $\alpha + \beta \notin \Phi_0$.

Now observe that for $\alpha \in \Phi$ $\Delta_\mathfrak{l}$-maximal, $\alpha|_{\mathfrak{h}_\mathfrak{l}} = 0$ if and only if $(\alpha, \beta) = 0$ for all $\beta \in \Delta_\mathfrak{l}$, which in turn is equivalent to having $(\alpha, \beta) = 0$ for all $\beta \in \Phi_\mathfrak{l}$. In other words, $\Delta_\mathfrak{l}$-maximal roots which vanish on $\mathfrak{h}_\mathfrak{l}$ are exactly the roots orthogonal to $\Phi_\mathfrak{l}$. This also naturally merits a definition.

**Definition 4.4.** The *complementary subroot system to* $\Phi_\mathfrak{l}$ is the set

$$\Phi_\mathfrak{l}^\perp = \{\alpha \in \Phi \colon \forall\, \beta \in \Delta_\mathfrak{l}, (\alpha, \beta) = 0\} \tag{4.4}$$

The subalgebra $\mathfrak{l}^\perp$ generated by the $X_\alpha$ and $Y_\alpha$ with $\alpha \in \Phi_\mathfrak{l}^\perp$ is the *complementary subalgebra to* $\mathfrak{l}$.

As the name suggests, $\Phi_\mathfrak{l}^\perp$ is a subroot system of $\Phi$ whenever $\Phi_\mathfrak{l}$ is a subroot system, provided $\Phi_\mathfrak{l}^\perp$ is nonempty. This is straightforward to verify and so we omit the details. It follows that $\mathfrak{l}^\perp$ is a regular semisimple subalgebra of $\mathfrak{g}$ when $\Phi_\mathfrak{l}^\perp$ is nonempty. The next lemma is trivial and so we do not include a proof, but it is quite important for what is to come.

**Lemma 4.5.** *The subalgebra* $\mathfrak{l}^\perp$ *satisfies* $\mathfrak{l} \cap \mathfrak{l}^\perp = [\mathfrak{l}, \mathfrak{l}^\perp] = \{0\}$.

From Lemma 4.5 we note that every nonzero element of $\mathfrak{l}^\perp$ is a maximal vector of weight 0. Also note that if $Z_\alpha \in \mathfrak{g}_\alpha$ is a maximal vector of weight 0 then $\alpha|_{\mathfrak{h}_\mathfrak{l}} = 0$ and so $Z_\alpha \in \mathfrak{l}^\perp$. In other words, the root space maximal vectors of weight 0 are precisely the root space elements of $\mathfrak{l}^\perp$.

In contrast, every maximal vector $H$ in $\mathfrak{h}$ will have weight 0 by Lemma 4.2 but $H$ may not necessarily be contained in $\mathfrak{l}^\perp$. Indeed, it is evident from Lemma 4.2 that such an element must be contained in $\bigcap_{\beta \in \Delta_\mathfrak{l}} \ker \beta$. Conversely, it is clear that every nonzero element of $\bigcap_{\beta \in \Delta_\mathfrak{l}} \ker \beta$ is a maximal vector of weight 0. It follows that the maximal vectors contained in $\mathfrak{h}$ are precisely the nonzero elements of $\bigcap_{\beta \in \Delta_{\mathfrak{l}}} \ker \beta$. Note that

$$\mathfrak{h}_{\mathfrak{l}} \cap \left( \bigcap_{\beta \in \Delta_{\mathfrak{l}}} \ker \beta \right) = \{0\} \tag{4.5}$$

Since $\dim \bigcap_{\beta \in \Delta_{\mathfrak{l}}} \ker \beta = n - m$ it follows that

$$\mathfrak{h} = \mathfrak{h}_{\mathfrak{l}} \oplus \left( \bigcap_{\beta \in \Delta_{\mathfrak{l}}} \ker \beta \right) \tag{4.6}$$

Also note that $\mathfrak{h}_{\mathfrak{l}}^{\perp} \subseteq \bigcap_{\beta \in \Delta_{\mathfrak{l}}} \ker \beta$, where $\mathfrak{h}_{\mathfrak{l}}^{\perp}$ is a Cartan subalgebra of $\mathfrak{l}^{\perp}$ contained in[9] $\mathfrak{h}$. If we define

$$\tilde{\mathfrak{h}}_{\mathfrak{l}} = \bigcap_{\beta \in \Delta_{\mathfrak{l}} \cup \Delta_{\mathfrak{l}}^{\perp}} \ker \beta \tag{4.7}$$

where $\Delta_{\mathfrak{l}}^{\perp}$ is a base for $\Phi_{\mathfrak{l}}^{\perp}$ then

$$\bigcap_{\beta \in \Delta_{\mathfrak{l}}} \ker \beta = \mathfrak{h}_{\mathfrak{l}}^{\perp} \oplus \tilde{\mathfrak{h}}_{\mathfrak{l}} \tag{4.8}$$

and $[\mathfrak{l}^{\perp}, \tilde{\mathfrak{h}}_{\mathfrak{l}}] = \{0\}$. Combined with our comments regarding the root space maximal vectors of weight 0, we obtain the following lemma.

**Lemma 4.6.** *The maximal vectors of weight* 0 *are precisely the nonzero elements of* $\mathfrak{l}^{\perp} \oplus \tilde{\mathfrak{h}}_{\mathfrak{l}}$.

We now provide a more explicit description of $\mathfrak{l}^{\perp}$.

**Lemma 4.7.** *Let* $M \in \mathbb{N}_0$ *be as in Equation 3.3. If* $M = 0$ *then* $\mathfrak{l}^{\perp} = \{0\}$. *If* $M > 0$ *then* $\mathfrak{l}^{\perp}$ *is a regular semisimple subalgebra of* $\mathfrak{g}$ *of rank* $M$ *with base*

$$\Delta_{\mathfrak{l}}^{\perp} = \begin{cases} \{\alpha_{m+2}, \dots, \alpha_n\}, & t(\mathfrak{l}) = A_m \\ \{\alpha_1, \dots, \alpha_{M-1}, \alpha_{M-1} + 2\sum_{k=M}^{n} \alpha_k\}, & (t(\mathfrak{g}), t(\mathfrak{l})) = (B_n, B_m) \\ \{\alpha_1, \dots, \alpha_{M-1}, \sum_{k=M}^{n} \alpha_k\}, & (t(\mathfrak{g}), t(\mathfrak{l})) = (B_n, D_m) \\ \left\{\alpha_1, \dots, \alpha_{M-1}, 2\sum_{k=M}^{n-1} \alpha_k + \alpha_n\right\}, & t(\mathfrak{l}) = C_m \\ \left\{\alpha_1, \dots, \alpha_{M-1}, \alpha_{M-1} + 2\sum_{k=M}^{n-2} \alpha_k \right. & \\ \quad \left. + \alpha_{n-1} + \alpha_n\right\}, & (t(\mathfrak{g}), t(\mathfrak{l})) = (D_n, D_m) \end{cases} \tag{4.9}$$

9 Since all Cartan subalgebras are conjugate we can always choose a Cartan subalgebra of $\mathfrak{l}^{\perp}$ to be contained in a fixed Cartan subalgebra of $\mathfrak{g}$.

*Furthermore,*

$$t(\mathfrak{l}^\perp) = \begin{cases} A_M, & t(\mathfrak{g}) = A_n \\ B_M, & (t(\mathfrak{g}), t(\mathfrak{l})) \in \{(B_n, A_m), (B_n, D_m)\} \\ C_M, & t(\mathfrak{g}) = C_n \\ D_M, & else \end{cases} \tag{4.10}$$

*Proof.* The proof is fairly straightforward and so we provide details only in the case of $(\mathrm{t}(\mathfrak{g}), \mathrm{t}(\mathfrak{l})) = (A_n, A_m)$. The arguments in the other cases are analogous.

If $\alpha \in \Phi_\mathfrak{l}^\perp$ then by Equation 3.10 we must have $(\alpha, \alpha_i) = 0$ for each $i \in [m]$. Equation 2.9 implies $\alpha = \varepsilon_k - \varepsilon_l$ for some $k, l \in [n+1]$ distinct and Equation 2.8 implies $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. It follows that $k \geq m+2$. Indeed, if $k \leq m$ then $(\alpha, \alpha_k) \neq 0$ and if $k = m+1$ then $(\alpha, \alpha_m) \neq 0$. A similar argument implies $l \geq m+2$ as well. Thus $m+2 \leq k, l \leq n+1$ with $k \neq l$, which requires $M > 0$. It is straightforward to verify that for all $k, l \in [n+1]$ distinct with $k, l \geq m+2$ one has $(\varepsilon_k - \varepsilon_l, \alpha_i) = 0$ for all $i \in [m]$ and so

$$\Phi_\mathfrak{l}^\perp = \{\varepsilon_k - \varepsilon_l \in \Phi \colon m+2 \leq k, l \leq n+1, k \neq l\} \tag{4.11}$$

It is now evident by Equation 2.8 that $\Phi_\mathfrak{l}^\perp$ has base $\Delta_\mathfrak{l}^\perp$ as defined in Equation 4.9. Moreover, it is easy to verify that $\Delta_\mathfrak{l}^\perp$ defines a simple subalgebra of type $A_M$. □

*Remark* 4.8. Note that as $\mathfrak{l}^\perp$ has rank $M$ we have rank $\mathfrak{l}$ + rank $\mathfrak{l}^\perp \in \{n-1, n\}$. Hence $\tilde{\mathfrak{h}}_\mathfrak{l}$ is at most one-dimensional.

We now turn our attention to finding the maximal vectors of nonzero weight. Lemma 4.2 and the ensuing remark imply we need only find root space vectors $X_\alpha \in \mathfrak{g}_\alpha$ such that $\alpha$ is $\Delta_\mathfrak{l}$-maximal and $\alpha|_{\mathfrak{h}_\mathfrak{l}} \neq 0$. Having $\alpha|_{\mathfrak{h}_\mathfrak{l}} \neq 0$ implies the existence of $\beta \in \Delta_\mathfrak{l}$ such that $(\alpha, \beta) \neq 0$. It follows that $\alpha - \beta \in \Phi_0$ and so we need only determine the $\Delta_\mathfrak{l}$-maximal elements $\alpha$ for which $\alpha - \beta \in \Phi_0$ for some $\beta \in \Delta_\mathfrak{l}$. This is easily accomplished via Equation 2.8, Equation 2.9, and Section 3.2.

Recall that we are decomposing $\mathfrak{g}$ into irreducible $\mathfrak{l}$-submodules with the ultimate goal of figuring out how the radical of a Levi decomposable subalgebra $\mathfrak{a}$ can decompose into irreducible $\mathfrak{l}$-modules. As such, we are not interested in any $\mathfrak{l}$-submodules of $\mathfrak{g}$ contained in $\mathfrak{l}$ since $\mathfrak{l} \cap \mathrm{Rad}\ \mathfrak{a} = \{0\}$. Note that the simplicity of $\mathfrak{l}$ ensures there is only one irreducible $\mathfrak{l}$-module contained in $\mathfrak{l}$, namely $\mathfrak{l}$ itself. As an $\mathfrak{l}$-module, $\mathfrak{l}$ is generated by the highest root $\xi$ of $\Phi_\mathfrak{l}$. Therefore, we are interested in the $\Delta_\mathfrak{l}$-maximal roots $\alpha$ apart from $\xi$ for which $\alpha - \beta \in \Phi_0$ for some $\beta \in \Delta_\mathfrak{l}$. This is described by the next lemma. Note that although the regular pair $(D_n, A_m)$ is ill-mannered, Lemma 4.9 will still hold in that particular setting.

**Lemma 4.9.** *Define*

$$\Phi_{\mathfrak{l}}^{\times} = \{\alpha \in \Phi \colon \alpha \text{ is } \Delta_{\mathfrak{l}}\text{-maximal}, \alpha - \beta \in \Phi_0 \text{ for some } \beta \in \Delta_{\mathfrak{l}}, \alpha \neq \xi\} \quad (4.12)$$

*where $\xi$ is the highest root of $\Phi_{\mathfrak{l}}$. If $\Delta_{\mathfrak{l}}$ is not given by Equation 3.14 then*

$$\Phi_{\mathfrak{l}}^{\times} = \begin{cases} \{\beta_k, \gamma_k \in \Phi \colon k \in [M]\} \cup \tilde{\Phi}_{\mathfrak{l}}^{\times}, & t(\mathfrak{l}) \neq A_m \\ \{\beta_k, \gamma_k \in \Phi \colon k \in [M+1]\}, & (t(\mathfrak{g}), t(\mathfrak{l})) = (A_n, A_m) \\ \{\beta_k, \gamma_k \in \Phi \colon k \in [M+1]\} & \\ \quad \cup \{\mu_k, \nu_k \in \Phi \colon k \in [M]\} \cup \{\zeta, \eta\}, & (t(\mathfrak{g}), t(\mathfrak{l})) = (B_n, A_m) \\ \{\beta_k, \gamma_k, \mu_k, \nu_k \in \Phi \colon k \in [M]\} \cup \{\zeta, \eta\}, & \text{else} \end{cases} \quad (4.13)$$

*where*

$$\beta_k = \begin{cases} \varepsilon_k + \varepsilon_{M+1}, & t(\mathfrak{l}) \neq A_m \\ \varepsilon_1, & (t(\mathfrak{g}), t(\mathfrak{l})) = (B_n, A_m) \text{ and } k = M+1 \\ \varepsilon_1 - \varepsilon_{m+k+1}, & \text{else} \end{cases} \quad (4.14a)$$

$$\gamma_k = \begin{cases} -\varepsilon_k + \varepsilon_{M+1}, & t(\mathfrak{l}) \neq A_m \\ -\varepsilon_{m+1}, & (t(\mathfrak{g}), t(\mathfrak{l})) = (B_n, A_m) \text{ and } k = M+1 \\ -\varepsilon_{m+1} + \varepsilon_{m+k+1}, & \text{else} \end{cases} \quad (4.14b)$$

$$\mu_k = \varepsilon_1 + \varepsilon_{m+k+1}, \qquad \nu_k = -\varepsilon_{m+1} - \varepsilon_{m+k+1} \quad (4.14c)$$

$$\zeta = \begin{cases} 2\varepsilon_1, & t(\mathfrak{g}) = C_n \\ \varepsilon_1 + \varepsilon_2, & \text{else} \end{cases}, \qquad \eta = \begin{cases} -2\varepsilon_{m+1}, & t(\mathfrak{g}) = C_n \\ -\varepsilon_m - \varepsilon_{m+1}, & \text{else} \end{cases} \quad (4.14d)$$

$$\tilde{\Phi}_{\mathfrak{l}}^{\times} = \begin{cases} \{\chi\}, & (t(\mathfrak{g}), t(\mathfrak{l})) = (B_n, D_m) \\ \varnothing, & \text{else} \end{cases} \quad (4.14e)$$

$$\chi = \varepsilon_{M+1} \quad (4.14f)$$

*If $\Delta_{\mathfrak{l}}$ is given by Equation 3.14 then*

$$\Phi_{\mathfrak{l}}^{\times} = \{\zeta, \eta\} \quad (4.15)$$

*where*

$$\zeta = \varepsilon_1 + \varepsilon_2, \qquad \eta = -\varepsilon_{n-1} + \varepsilon_n \tag{4.16}$$

*Proof.* We omit the details because they are fairly tedious and unilluminating, but this lemma can be easily verified using Equation 2.8, Equation 2.9, and the explicit descriptions of $\Delta_{\mathfrak{l}}$ provided in Section 3.2. □

With Lemma 4.7 and Lemma 4.9 we have a complete description of the maximal vectors and as such we can obtain a decomposition of Rad $\mathfrak{a}$ into irreducible $\mathfrak{l}$-submodules. We summarize our findings in the next lemma.

**Lemma 4.10.** *A decomposition*[10] *of* $\mathfrak{g}$ *into irreducible* $\mathfrak{l}$*-submodules is*

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{l}^{\perp} \oplus \tilde{\mathfrak{h}}_{\mathfrak{l}} \oplus \bigoplus_{\alpha \in \Phi_{\mathfrak{l}}^{\times}} V(\alpha) \tag{4.17}$$

*where* $\Phi_{\mathfrak{l}}^{\times}$ *is as in Lemma 4.9 and* $V(\alpha)$ *for* $\alpha \in \Phi_{\mathfrak{l}}^{\times}$ *is the irreducible* $\mathfrak{l}$*-module generated by* $\mathfrak{g}_\alpha$*. Additionally, by abuse of notation we write* $\mathfrak{l}^{\perp}$ *to denote the direct sum of trivial* $\mathfrak{l}$*-submodules which constitute* $\mathfrak{l}^{\perp}$*.*

**Corollary 4.11.** *The radical of* $\mathfrak{a}$ *is an* $\mathfrak{l}$*-submodule of* $\mathfrak{l}^{\perp} \oplus \tilde{\mathfrak{h}}_{\mathfrak{l}} \oplus \bigoplus_{\alpha \in \Phi_{\mathfrak{l}}^{\times}} V(\alpha)$*.*

Since irreducible $\mathfrak{g}$-modules are generated by maximal vectors, $V(\alpha)$ is uniquely defined simply by specifying $\alpha \in \Phi_{\mathfrak{l}}^{\times}$. However, this description is not terribly useful. Instead, note that $V(\alpha)$ decomposes into weight spaces, which coincide with either root spaces or subalgebras of $\mathfrak{h}$ since we are considering the adjoint representation. However, we can rule out any subalgebras of $\mathfrak{h}$ appearing as weight spaces since otherwise we would have $\alpha \in \Phi_{\mathfrak{l}}$, which is impossible since $\Phi_{\mathfrak{l}} \cap \Phi_{\mathfrak{l}}^{\times} = \varnothing$. Hence for each $\alpha \in \Phi_{\mathfrak{l}}^{\times}$, $V(\alpha)$ is a direct sum of root spaces. To see exactly which root spaces, note that if $\mathfrak{g}_\beta \subseteq V(\alpha)$ then $\beta = \alpha - \gamma$ for some $\gamma \in \Phi_{\mathfrak{l}}$ since $V(\alpha)$ is generated by $\mathfrak{l}$ acting on $\mathfrak{g}_\alpha$. Conversely, if $\alpha \in \Phi_{\mathfrak{l}}^{\times}$ and $\beta \in \Phi$ is such that $\beta = \alpha - \gamma$ for some $\gamma \in \Phi_{\mathfrak{l}}$ then $\mathfrak{g}_\beta \subseteq V(\alpha)$. We summarize our observations in the following lemma.

**Lemma 4.12.** *Let* $\alpha \in \Phi_{\mathfrak{l}}^{\times}$ *and define*

$$\alpha + \Phi_{\mathfrak{l}} = \{\beta \in \Phi \colon \beta = \alpha - \gamma \text{ for some } \gamma \in \Phi_{\mathfrak{l}}\} \tag{4.18}$$

*Then the irreducible* $\mathfrak{l}$*-module* $V(\alpha)$ *is given by*

$$V(\alpha) = \bigoplus_{\beta \in \alpha + \Phi_{\mathfrak{l}}} \mathfrak{g}_\beta \tag{4.19}$$

10 For convenience, we generally include $\tilde{\mathfrak{h}}_{\mathfrak{l}}$ in the decomposition even though it is possible that $\tilde{\mathfrak{h}}_{\mathfrak{l}} = \{0\}$ and would thus not technically be an $\mathfrak{l}$-module.

Our approach of distinguishing between maximal vectors of weight 0 and those of nonzero weight is entirely intentional. As we shall investigate in the remainder of this chapter, the $\mathfrak{l}$-modules of highest weight 0 behave markedly different from their nonzero-weight counterparts. This natural division suggests the following definitions.

**Definition 4.13.** Define

$$V^0 = \operatorname{Rad} \mathfrak{a} \cap (\mathfrak{l}^\perp \oplus \tilde{\mathfrak{h}}_\mathfrak{l}), \qquad V^\times = \operatorname{Rad} \mathfrak{a} \cap \left( \bigoplus_{\alpha \in \Phi_\mathfrak{l}^\times} V(\alpha) \right) \tag{4.20}$$

Notice that $V^0$ is a trivial $\mathfrak{l}$-module, $V^\times$ is an $\mathfrak{l}$-module such that the highest weights of the irreducible components of $V^\times$ are all nonzero, and $\operatorname{Rad} \mathfrak{a} = V^0 \oplus V^\times$. We call $V^0$ the *trivial component of* $\operatorname{Rad} \mathfrak{a}$ and $V^\times$ the *nontrivial component of* $\operatorname{Rad} \mathfrak{a}$.

*Remark* 4.14. Notice that $\mathfrak{l} \oplus \tilde{\mathfrak{h}}_\mathfrak{l}$ is the centralizer $C_\mathfrak{g}(\mathfrak{l})$ of $\mathfrak{l}$ in $\mathfrak{g}$. Since both $\operatorname{Rad} \mathfrak{a}$ and $C_\mathfrak{g}(\mathfrak{l})$ are subalgebras of $\mathfrak{g}$, $V^0$ is also a subalgebra of $\mathfrak{g}$.

Corollary 4.11 implies that if we wish to describe the possibilities for $\operatorname{Rad} \mathfrak{a}$ then we need only examine the $\mathfrak{l}$-submodules of $\mathfrak{l}^\perp \oplus \tilde{\mathfrak{h}}_\mathfrak{l} \oplus \bigoplus_{\alpha \in \Phi_\mathfrak{l}^\times} V(\alpha)$. The next lemma shows that if we are interested in a classification up to inner automorphism, then we can study the nontrivial and trivial components of $\operatorname{Rad} \mathfrak{a}$ separately.

**Lemma 4.15.** *Let* $\mathfrak{a}, \tilde{\mathfrak{a}}$ *be Levi decomposable subalgebras of* $\mathfrak{g}$ *with common Levi factor* $\mathfrak{l}$. *Let* $V^0$ *and* $\tilde{V}^0$ *be the trivial components of* $\mathfrak{a}$ *and* $\tilde{\mathfrak{a}}$, *respectively, and let* $V^\times$ *and* $\tilde{V}^\times$ *be the nontrivial components of* $\mathfrak{a}$ *and* $\tilde{\mathfrak{a}}$, *respectively. Then* $\mathfrak{a}$ *and* $\tilde{\mathfrak{a}}$ *are conjugate if and only if there exists* $\tau \in \operatorname{Int} \mathfrak{g}$ *such that* $\tau$ *preserves* $\mathfrak{l}$, *maps* $V^0$ *to* $\tilde{V}^0$, *and* $V^\times$ *to* $\tilde{V}^\times$.

*Proof.* First suppose $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$ are conjugate. By Lemma 2.3, there exists $\tau \in \operatorname{Int} \mathfrak{g}$ such that $\tau$ preserves $\mathfrak{l}$ and maps $\operatorname{Rad} \mathfrak{a}$ to $\operatorname{Rad} \tilde{\mathfrak{a}}$. Note that $[\mathfrak{l}, V^0] = \{0\}$ and $[\mathfrak{l}, V^\times] = V^\times$, implying $[\mathfrak{l}, \operatorname{Rad} \mathfrak{a}] = V^\times$. Similarly, $[\mathfrak{l}, \operatorname{Rad} \tilde{\mathfrak{a}}] = \tilde{V}^\times$. Note however that having $\tau(\mathfrak{l}) = \mathfrak{l}$ and $\tau(\operatorname{Rad} \mathfrak{a}) = \operatorname{Rad} \tilde{\mathfrak{a}}$ also implies $[\mathfrak{l}, \operatorname{Rad} \tilde{\mathfrak{a}}] = \tau(V^\times)$. Consequently, $\tau$ maps $V^\times$ to $\tilde{V}^\times$. Moreover, as $\tau$ preserves $\mathfrak{l}$ it also preserves $C_\mathfrak{g}(\mathfrak{l})$. It follows that $\tau$ maps $C_\mathfrak{g}(\mathfrak{l}) \cap \operatorname{Rad} \mathfrak{a}$ to $C_\mathfrak{g}(\mathfrak{l}) \cap \operatorname{Rad} \tilde{\mathfrak{a}}$ and so $\tau(V^0) = \tilde{V}^0$.

Conversely, if $\tau \in \operatorname{Int} \mathfrak{g}$ is such that $\tau$ preserves $\mathfrak{l}$, maps $V^0$ to $\tilde{V}^0$, and $V^\times$ to $\tilde{V}^\times$ then obviously $\tau(\mathfrak{a}) = \tilde{\mathfrak{a}}$. □

Now that we explicitly have a decomposition of $\mathfrak{g}$ into irreducible $\mathfrak{l}$-modules we can begin describing the possibilities for $\operatorname{Rad} \mathfrak{a}$. Suppose $\operatorname{Rad} \mathfrak{a}$ has nontrivial and trivial components $V^\times$ and $V^0$, respectively.

By definition, $[\mathfrak{l}, V^0] = \{0\}$. As such, $\mathfrak{l}$ and $V^0$ do not interact at all, making $V^0$ somewhat difficult to study due to the inability to utilize the simple structure of $\mathfrak{l}$. By contrast, the fact that $\mathfrak{l}$ and $V^\times$ have a nontrivial interaction allows us to better understand $V^\times$.

## 4.2 CLASSIFYING THE NONTRIVIAL COMPONENT

We commence our study of the possibilities for the radical by first analyzing the potential nontrivial components. Recall that this thesis is specifically considering *regular* Levi factors. One may naturally wonder whether regularity of the Levi factor in any way induces regularity of the radical. Since $[\mathfrak{l}, V^0] = \{0\}$ it is too hopeful to expect that we can at all guarantee regularity of $V^0$. But how about regularity of $V^\times$? Is the interaction between $\mathfrak{l}$ and $V^\times$ enough to ensure that regularity of $\mathfrak{l}$ secures regularity of $V^\times$ as well? The answer to this question is a satisfying yes and is one of the main results of this thesis. To prove this beautiful result we first establish a few definitions.

Suppose for the moment that $V^\times$ is regular. Since the $V(\alpha)$ with $\alpha \in \Phi_\mathfrak{l}^\times$ are direct sums of root spaces, we have $V^\times = \bigoplus_{\alpha\in\Theta} \mathfrak{g}_\alpha$ for some $\Theta \subseteq \Phi$ satisfying $\Theta \cap \Phi_\mathfrak{l} = \Theta \cap \Phi_\mathfrak{l}^\perp = \varnothing$. Moreover, note that for all $\alpha \in \Phi_\mathfrak{l} \cup \Theta$ and $\beta \in \Theta$ that $\mathfrak{g}_\alpha \subseteq \mathfrak{a}$ and $\mathfrak{g}_\beta \subseteq \operatorname{Rad} \mathfrak{a}$, implying $\mathfrak{g}_{\alpha+\beta} \subseteq \operatorname{Rad} \mathfrak{a}$. Therefore, if $\alpha + \beta \in \Phi_0$ then $\alpha + \beta \in \Theta$ if $\mathfrak{g}_{\alpha+\beta} \subseteq V^\times$ and $\alpha + \beta \in \Phi_\mathfrak{l}^\perp$ otherwise. Consequently, we note that if $V^\times = \bigoplus_{\alpha\in\Theta_{\mathfrak{g}_\alpha}} \mathfrak{g}_\alpha$ then there are several conditions on $\Theta$. As such we make the following definition.

**Definition 4.16.** A subset $\Theta \subseteq \Phi$ is a *nontrivial $\Phi_\mathfrak{l}$-candidate* if $\Theta \cap \Phi_\mathfrak{l} = \Theta \cap \Phi_\mathfrak{l}^\perp = \varnothing$ and for all $\alpha \in \Phi_\mathfrak{l} \cup \Theta$ and $\beta \in \Theta$, if $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Theta \cup \Phi_\mathfrak{l}^\perp$.

Since we are interested in a classification up to conjugacy and we are now considering subsets of $\Phi$, it is reasonable to suspect a link to conjugacy under the Weyl group $\mathscr{W}$. Lemma 2.3 allows us to consider inner automorphisms which preserve the Levi factor, so in the Weyl group context we should consider elements which preserve $\Phi_\mathfrak{l}$. This leads to the next definition.

**Definition 4.17.** Two subsets $\Theta_1, \Theta_2 \subseteq \Phi$ are *$\Phi_\mathfrak{l}$-Weyl conjugate* if there exists $\omega \in \mathscr{W}$ such that $\omega$ preserves $\Phi_\mathfrak{l}$ and maps $\Theta_1$ to $\Theta_2$.

To prove the main claim of this section we recall a basic fact about $\mathscr{W}$.

**Lemma 4.18** (See Lemma 2.2.3 of [CM93])**.** *Define*

$$\mathcal{N}(\mathfrak{h}) = \{\tau \in \operatorname{Int} \mathfrak{g} \colon \tau(\mathfrak{h}) = \mathfrak{h}\}, \quad \mathcal{C}(\mathfrak{h}) = \{\tau \in \mathcal{N}(\mathfrak{h}) \colon \tau|_\mathfrak{h} = \operatorname{id}|_\mathfrak{h}\} \quad (4.21)$$

*Then* $\mathscr{W} \cong \mathcal{N}(\mathfrak{h})/\mathcal{C}(\mathfrak{h})$. *This isomorphism is realized via the natural action of* $\mathscr{W}$ *on* $\mathfrak{h}^*$, *transferred to* $\mathfrak{h}$ *via the contragredient representation.*

We now state one of the major results of this thesis.

**Theorem 4.19.** *Let* $V^\times$ *be the nontrivial component of* $\operatorname{Rad}\mathfrak{a}$. *Then there exists* $\Theta \subseteq \Phi$ *a nontrivial* $\Phi_\mathfrak{l}$*-candidate such that up to inner automorphism,*

$$V^\times = \bigoplus_{\alpha\in\Theta} \mathfrak{g}_\alpha \tag{4.22}$$

*Moreover, suppose* $\tilde{\mathfrak{a}}$ *is another Levi decomposable subalgebra of* $\mathfrak{g}$ *with Levi factor* $\mathfrak{l}$ *and nontrivial component of* $\operatorname{Rad}\tilde{\mathfrak{a}}$ *given by* $\tilde{V}^\times = \bigoplus_{\alpha\in\tilde{\Theta}} \mathfrak{g}_\alpha$ *for some nontrivial* $\Phi_\mathfrak{l}$*-candidate* $\tilde{\Theta} \subseteq \Phi$. *Then there exists* $\tau \in \operatorname{Int}\mathfrak{g}$ *such that* $\tau$ *preserves* $\mathfrak{l}$ *and maps* $V^\times$ *to* $\tilde{V}^\times$ *if and only if* $\Theta$ *and* $\tilde{\Theta}$ *are* $\Phi_\mathfrak{l}$*-Weyl conjugate.*

*Proof.* Suppose for the moment that there exists $\Theta \subseteq \Phi$ such that $V^\times$ is described by Equation 4.22. By our previous comments $\Theta$ is necessarily a nontrivial $\Phi_\mathfrak{l}$-candidate.

Now suppose there exists $\tau \in \operatorname{Int}\mathfrak{g}$ such that $\tau$ preserves $\mathfrak{l}$ and maps $V^\times$ to $\tilde{V}^\times$. Temporarily suppose that we can ensure that $\tau$ also preserves $\mathfrak{h}$. By Lemma 4.18, $\tau$ naturally defines an element $\omega \in \mathscr{W}$ such that $\omega$ preserves $\Phi_\mathfrak{l}$ and maps $\Theta$ to $\tilde{\Theta}$. Thus $\Theta$ and $\tilde{\Theta}$ are $\Phi_\mathfrak{l}$-conjugate.

Conversely, suppose $\Theta$ and $\tilde{\Theta}$ are $\Phi_\mathfrak{l}$-conjugate. Then there exists $\omega \in \mathscr{W}$ such that $\omega$ preserves $\Phi_\mathfrak{l}$ and maps $\Theta$ to $\tilde{\Theta}$. Lemma 4.18 then implies there exists $\tau \in \operatorname{Int}\mathfrak{g}$ such that $\tau$ preserves $\mathfrak{l}$ and maps $V$ to $\tilde{V}$, as desired.

To complete the proof it remains to establish two facts:

1. the existence of a set $\Theta$ such that up to inner automorphism, $V^\times$ is as in Equation 4.22 and
2. the ability to ensure that if there exists $\tau \in \operatorname{Int}\mathfrak{g}$ which preserves $\mathfrak{l}$ and maps $V^\times$ to $\tilde{V}^\times$ then $\tau$ can be chosen to also preserve $\mathfrak{h}$.

We dedicate the next few sections to proving these two statements. □

Before we complete the proof of Theorem 4.19 we note that this result is surprisingly clean. Despite the difficulty of classifying solvable subalgebras, Theorem 4.19 shows that imposing suitable conditions on the Levi factor, namely regularity, sufficiently restricts the behaviour of the radical so as to allow for a general description. Quite interestingly, imposing regularity on the Levi factor leads to a regularity condition on the nontrivial component of the radical. One may wonder in a future project how relaxing regularity of $\mathfrak{l}$ affects Theorem 4.19.

## 4.3 PROOF OF THEOREM 4.19 WHEN $\mathfrak{t}(\mathfrak{l}) \neq A_m$

Our proof of Theorem 4.19 will proceed via a consideration of several cases depending on $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l}))$. The first and simplest case we shall examine is when $\mathfrak{l}$ is not of type $A_m$, in which case we may suppose $\mathfrak{l}$ is described by either Equation 3.4 or Equation 3.8. Just as we saw in Theorem 3.13, the behaviour in this context is quite distinct from the case where $\mathfrak{l}$ is of type $A_m$, as we will later observe.

A consideration of certain inner automorphisms of $\mathfrak{g}$ shall prove useful in proving Theorem 4.19. Of particular significance are the following inner automorphisms.

**Lemma 4.20.** *Suppose $t(\mathfrak{l}) \neq A_m$. Then for each $i \in [M]$ there exists $\sigma_i \in \operatorname{Int}\mathfrak{g}$ such that $\sigma_i$ preserves $\mathfrak{l}$, maps $V(\beta_i)$ to $V(\gamma_i)$, $V(\gamma_i)$ to $V(\beta_i)$, and for all $k \in [M]$ distinct from $i$, $\sigma_i$ preserves $V(\beta_k)$ and $V(\gamma_k)$. If $(t(\mathfrak{g}), t(\mathfrak{l})) = (B_n, D_m)$ then $\sigma_i$ also preserves $V(\chi)$.*

*Proof.* We consider cases depending on the type of $\mathfrak{g}$.

**Case 1: $\mathfrak{t}(\mathfrak{g}) = B_n$**

Define $P_i \in GL_{2n+1}$ as

$$P_i = \begin{pmatrix} 1 & 0_{1,M} & 0_{1,m} & 0_{1,M} & 0_{1,m} \\ 0_{M,1} & I_M - E_{i,i} & 0_{M,m} & -E_{i,i} & 0_{M,m} \\ 0_{m,1} & 0_{m,M} & I_m & 0_{m,M} & 0_{m,m} \\ 0_{M,1} & -E_{i,i} & 0_{M,m} & I_M - E_{i,i} & 0_{M,m} \\ 0_{m,1} & 0_{m,M} & 0_{m,m} & 0_{m,M} & I_m \end{pmatrix} \tag{4.23}$$

It is straightforward to check that $P_i \in SO_{2n+1}$ and thus conjugation by $P_i$ defines an inner automorphism $\sigma_i \in \operatorname{Int}\mathfrak{g}$. By Equation 3.9 one has that $\sigma_i$ fixes $\mathfrak{l}$ element-wise.

For each $k \in [M]$ define

$$v_{\beta_k} = (-1)^m X_{k,\dots,n,n,n-1,\dots,M+1}, \qquad v_{\gamma_k} = (-1)^{M-k} Y_{k,\dots,M} \tag{4.24a}$$
$$= E_{k+1,n+M+2} - E_{M+2,n+k+1}, \qquad = E_{M+2,k+1} - E_{n+k+1,n+M+2} \tag{4.24b}$$

where for $c_1, \dots, c_r \in [n]$ we define

$$X_{c_1,\dots,c_r} = [[\cdots[[X_{c_1}, X_{c_2}], X_{c_3}], \cdots], X_{c_r}] \tag{4.25a}$$
$$Y_{c_1,\dots,c_r} = [[\cdots[[Y_{c_1}, Y_{c_2}], Y_{c_3}], \cdots], Y_{c_r}] \tag{4.25b}$$

Note that Equation 4.24b was obtained from Equation 4.24a via the descriptions provided in Section 2.1. If $\mathfrak{t}(\mathfrak{l}) = D_m$ we also define

$$v_\chi = -X_{M+1,\dots,n} = E_{1,n+M+2} - E_{M+2,1} \tag{4.26}$$

From Lemma 4.9 and the descriptions in Section 2.1 one has that $v_{\beta_k}$ and $v_{\gamma_k}$ are maximal vectors of $V(\beta_k)$ and $V(\gamma_k)$, respectively, and that $v_\chi$ is a maximal vector of $V(\chi)$ if $\mathfrak{t}(\mathfrak{l}) = D_m$. By simple matrix calculations we have $\tau(v_{\beta_i}) = v_{\gamma_i}$, $\tau(v_{\gamma_i}) = v_{\beta_i}$, $\tau(v_\chi) = v_\chi$, $\tau(v_{\beta_k}) = v_{\beta_k}$, and $\tau(v_{\gamma_k}) = v_{\gamma_k}$ for $k \in [M]$ distinct from $i$. The result follows.

**Case 2: $\mathfrak{t}(\mathfrak{g}) \in \{C_n, D_n\}$**

By Theorem 3.3 this is the only case left to consider. Define $P_i \in \mathfrak{gl}_{2n}$ as

$$P_i = \begin{pmatrix} I_M - E_{i,i} & 0_{M,m} & \iota E_{i,i} & 0_{M,m} \\ 0_{m,M} & I_m & 0_{m,M} & 0_{m,m} \\ \iota E_{i,i} & 0_{M,m} & I_M - E_{i,i} & 0_{M,m} \\ 0_{m,M} & 0_{m,m} & 0_{m,M} & I_m \end{pmatrix} \tag{4.27}$$

where $\iota^2 = -1$ if $\mathfrak{t}(\mathfrak{g}) = C_n$ and $\iota = -1$ if $\mathfrak{t}(\mathfrak{g}) = D_n$. Then $P_i \in Sp_{2n}$ if $\mathfrak{t}(\mathfrak{g}) = C_n$ and $P_i \in SO_{2n}$ if $\mathfrak{t}(\mathfrak{g}) = D_n$. In either case we have that conjugation by $P_i$ defines an inner automorphism $\sigma_i \in \operatorname{Int}\mathfrak{g}$.

For each $k \in [M]$ we also define

$$v_{\beta_k} = \begin{cases} (-1)^{m-1} X_{k,\dots,n,n-1,\dots,M+1}, & \mathfrak{t}(\mathfrak{g}) = C_n \\ (-1)^{m-1} X_{k,\dots,n,n-2,n-3,\dots,M+1}, & \mathfrak{t}(\mathfrak{g}) = D_n \end{cases} \tag{4.28a}$$

$$= \begin{cases} E_{k,n+M+1} + E_{M+1,n+k}, & \mathfrak{t}(\mathfrak{g}) = C_n \\ E_{k,n+M+1} - E_{M+1,n+k}, & \mathfrak{t}(\mathfrak{g}) = D_n \end{cases} \tag{4.28b}$$

$$v_{\gamma_k} = (-1)^{M-k} Y_{k,\dots,M} \tag{4.28c}$$

$$= E_{M+1,k} - E_{n+k,n+M+1} \tag{4.28d}$$

Then the $v_{\beta_k}$ and $v_{\gamma_k}$ are maximal vectors of the $V(\beta_k)$ and $V(\gamma_k)$, respectively. The result follows since $\sigma_i(v_{\beta_i}) = -\iota v_{\gamma_i}$, $\sigma_i(v_{\gamma_i}) = -\iota v_{\beta_i}$, $\sigma_i(v_{\beta_k}) = v_{\beta_k}$, and $\sigma_i(v_{\gamma_k}) = v_{\gamma_k}$ for all $k \in [M]$ distinct from $i$. □

This next result is instrumental in proving Theorem 4.19 in the case of $\mathfrak{t}(\mathfrak{l}) \neq A_m$. In fact, Lemma 4.21 will gives concrete description of $V^\times$ in this setting.

**Lemma 4.21.** *Suppose $\mathfrak{t}(\mathfrak{l}) \neq A_m$. Then up to inner automorphism, there exists $p \in [M]_0$ such that*

$$V^\times = \bigoplus_{k=1}^{p} V(\gamma_k) \tag{4.29}$$

*where we define* $[M]_0 = [M] \cup \{0\}$.

*Proof.* By Lemma 4.9 and Lemma 4.10 we note that if $V^\times \neq \{0\}$ then $V^\times$ is an $\mathfrak{l}$-submodule of $\bigoplus_{k=1}^M V(\beta_k) \oplus \bigoplus_{k=1}^M V(\gamma_k) \oplus V(\chi)$ if $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (B_n, D_m)$ and an $\mathfrak{l}$-submodule of $\bigoplus_{k=1}^M V(\beta_k) \oplus \bigoplus_{k=1}^M V(\gamma_k)$ otherwise. It is not difficult to show that $\chi|_{\mathfrak{h}_\mathfrak{l}} = \lambda_1$ when $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (B_n, D_m)$ and $\beta_k|_{\mathfrak{h}_\mathfrak{l}} = \gamma_k|_{\mathfrak{h}_\mathfrak{l}} = \lambda_1$ for all $k \in [M]$. Consequently, there exists $p \in [2M+1]_0$ if $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (B_n, D_m)$ or $p \in [2M]_0$ otherwise such that $V^\times$ consists of precisely $p$ copies of $V(\lambda_1)$. We proceed by induction on $p$ in the case where $p \leq M$.

Clearly the base case of $p = 0$ holds. Thus suppose the claim holds when $V^\times$ consists of $p$ copies of $V(\lambda_1)$ for some $p \in [M-1]_0$ and now suppose $V^\times$ consists of $p+1$ such copies. Let $w_1, \ldots, w_{p+1} \in V^\times$ be maximal vectors of these $p+1$ copies. Then for each $i \in [p+1]$ there exist $a_{i,1}, \ldots, a_{i,M}, b_{i,1}, \ldots, b_{i,M}, c_i \in \mathbb{F}$ such that

$$w_i = c_i v_\chi + \sum_{k=1}^M (a_{i,k} v_{\beta_k} + b_{i,k} v_{\gamma_k}) \tag{4.30}$$

where $c_i = 0$ if $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) \neq (B_n, D_m)$ and $v_\chi$, $v_{\beta_k}$, and $v_{\gamma_k}$ are as in Equation 4.24, Equation 4.26, and/or Equation 4.28, depending on the type of $\mathfrak{g}$. If $\mathfrak{t}(\mathfrak{g}) = C_n$ then a computation reveals that for all $i, j \in [p+1]$,

$$[w_i, w_j] = \sum_{k=1}^M (a_{j,k} b_{i,k} - a_{i,k} b_{j,k}) X_{M+1,\ldots,n,n-1,\ldots,M+1} \tag{4.31}$$

Since $X_{M+1,\ldots,n,n-1,\ldots,M+1} \in \mathfrak{l}$ we conclude that for all $i, j \in [p+1]$,

$$\sum_{k=1}^M (a_{j,k} b_{i,k} - a_{i,k} b_{j,k}) = 0 \tag{4.32}$$

If on the other hand $\mathfrak{g}$ is not of type $C_n$ then note that $Y_{M+1} \in \mathfrak{l}$. Since $w_i, w_j \in V^\times$ for all $i, j \in [p+1]$ it follows that $(-1)^m [w_i, [Y_{M+1}, w_j]] \in \operatorname{Rad} \mathfrak{a}$. Observe that if $\mathfrak{g}$ is of type $B_n$ then

$$(-1)^m [w_i, [Y_{M+1}, w_j]] = \left( c_i c_j - \sum_{k=1}^M (a_{i,k} b_{j,k} + a_{j,k} b_{i,k}) \right) X_{M+1,\ldots,n,n,\ldots,M+2} \tag{4.33}$$

whereas if $\mathfrak{t}(\mathfrak{g}) = D_n$ we have

$$(-1)^m [w_i, [Y_{M+1}, w_j]] = \sum_{k=1}^M (a_{i,k} b_{j,k} + a_{j,k} b_{i,k}) X_{M+1,\ldots,n,n-2,n-3,\ldots,M+2} \tag{4.34}$$

Since $X_{M+1,\ldots,n,n,\ldots,M+2} \in \mathfrak{l}$ when $\mathfrak{t}(\mathfrak{g}) = B_n$ and $X_{M+1,\ldots,n,n-2,n-3,\ldots,M+2} \in \mathfrak{l}$ when $\mathfrak{g}$ is of type $D_n$ we conclude from Equation 4.33 and Equation 4.34 that when $\mathfrak{t}(\mathfrak{g}) \neq C_n$,

$$c_i c_j - \sum_{k=1}^{M} (a_{i,k} b_{j,k} + a_{j,k} b_{i,k}) = 0 \tag{4.35}$$

for all $i, j \in [p+1]$.

Let $\tilde{V}^{\times}$ be the $\mathfrak{l}$-module generated by $w_1, \ldots, w_p$. Using Equation 4.32 and Equation 4.35, one can perform a tedious, albeit straightforward, series of computations to verify that $[\tilde{V}^{\times}, \tilde{V}^{\times}] \cap \left( \bigoplus_{\alpha \in \Phi_{\mathfrak{l}}^{\times}} V(\alpha) \right) \subseteq \tilde{V}^{\times}$. As such, the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{l}$ and $\tilde{V}^{\times}$ has nontrivial component of the radical given by $\tilde{V}^{\times}$. Thus by the inductive hypothesis we may suppose without loss of generality that $w_i = v_{\gamma_i}$ for each $i \in [p]$. By linear independence of $\{w_1, \ldots, w_{p+1}\}$ we may additionally suppose without loss of generality that $b_{p+1,k} = 0$ for each $k \in [p]$. By taking $j = p+1$ and considering each $i \in [p]$, we may additionally deduce that $a_{p+1,i} = 0$ by Equation 4.32 and Equation 4.35.

We construct an inner automorphism $\tau \in \operatorname{Int} \mathfrak{g}$ such that $\tau$ preserves $\mathfrak{l}$ and $\bigoplus_{k=1}^{p} V(\gamma_k)$ and maps the $\mathfrak{l}$-submodule generated by $w_{p+1}$ to $V(\gamma_{p+1})$. Define $B, D \in \mathfrak{gl}_M$ such that for each $i \in [M]$ the $i$'th columns of $B$ and $D$ are $B_i$ and $D_i$, respectively, where

$$B_i = \begin{cases} 0, & i \leq p \\ -\sum_{k=p+1}^{M} a_{p+1,k} e_k, & i = p+1 \\ \kappa \frac{a_{p+1,i-1}}{b_{p+1,l}} e_l, & p+1 < i \leq l \\ \kappa \frac{a_{p+1,i}}{b_{p+1,l}} e_l, & i > l \end{cases} \tag{4.36a}$$

$$D_i = \begin{cases} e_i, & i \leq p \text{ or } i > l \\ \sum_{k=p+1}^{M} b_{p+1,k} e_k, & i = p+1 \\ e_{i-1}, & p+1 < i \leq l \end{cases} \tag{4.36b}$$

where $\kappa = -1$ if $\mathfrak{t}(\mathfrak{g}) = C_n$, $\kappa = 1$ otherwise, and $\{e_1, \ldots, e_M\}$ is the standard basis for $\mathbb{F}^M$. Note that having $b_{p+1,l} \neq 0$ implies $\{D_1, \ldots, D_M\}$

is linearly independent, thereby ensuring invertibility of $D$. If $\mathfrak{g}$ is of type $B_n$ define $P \in \mathfrak{gl}_{2n+1}$ as

$$P = \begin{pmatrix} -1 & 0_{1,M} & 0_{1,m} & -c_{p+1}E_{1,p+1} & 0_{1,m} \\ -c_{p+1}(D^{-1})^\top E_{p+1,1} & (D^{-1})^\top & 0_{M,m} & B & 0_{M,m} \\ 0_{m,1} & 0_{m,M} & I_m & 0_{m,M} & 0_{m,m} \\ 0_{M,1} & 0_{M,M} & 0_{M,m} & D & 0_{M,m} \\ 0_{m,1} & 0_{m,M} & 0_{m,m} & 0_{m,M} & I_m \end{pmatrix} \quad (4.37)$$

If $\mathfrak{t}(\mathfrak{g}) \neq B_n$ we instead define $P \in \mathfrak{gl}_{2n}$ as

$$P = \begin{pmatrix} (D^{-1})^\top & 0_{M,m} & B & 0_{M,m} \\ 0_{m,M} & I_m & 0_{m,M} & 0_{m,m} \\ 0_{M,M} & 0_{M,m} & D & 0_{M,m} \\ 0_{m,M} & 0_{m,m} & 0_{m,M} & I_m \end{pmatrix} \quad (4.38)$$

Using Equation 4.32 and Equation 4.35 one can verify that $P \in SO_{2n+1}$ when $\mathfrak{t}(\mathfrak{g}) = B_n$, $P \in Sp_{2n}$ when $\mathfrak{t}(\mathfrak{g}) = C_n$, and $P \in SO_{2n}$ when $\mathfrak{t}(\mathfrak{g}) = D_n$. Thus conjugation by $P$ defines an inner automorphism $\tau \in \operatorname{Int}\mathfrak{g}$. One can further verify that $\tau$ preserves $\mathfrak{l}$ and maps $v_{\gamma_k}$ to $w_k$ for each $k \in [p+1]$.

We have shown that the claim holds when $V^\times$ consists of $p$ copies of $V(\lambda_1)$ with $p \leq M$, but a priori $V^\times$ may contain more than $M$ copies of $V(\lambda_1)$. To show this is impossible, suppose by contradiction that $w_1, \ldots, w_{M+1} \in V^\times$ are linearly independent maximal vectors. By what we have shown we may suppose without loss of generality that $w_k = v_{\gamma_k}$ for each $k \in [M]$. Linear independence then implies we may take $w_{M+1} = cv_\chi + \sum_{k=1}^{M} a_k v_{\beta_k}$ for some $c, a_1, \ldots, a_M \in \mathbb{F}$. Taking $j = M+1$ and considering each $i \in [M]$ separately, we deduce from Equation 4.32 and Equation 4.35 that $a_i = 0$ for each $i \in [M]$. If $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (B_n, D_m)$ then Equation 4.35 with $i = j = M+1$ yields $c = 0$. In any event we must have $w_{M+1} = 0$, which is impossible. Hence $V^\times$ contains at most $M$ copies of $V(\lambda_1)$. □

With this we have established the first necessary fact for Theorem 4.19 to hold, namely the existence of $\Theta \subseteq \Phi$ such that up to inner automorphism, $V^\times = \bigoplus_{\alpha\in\Theta} \mathfrak{g}_\alpha$. To prove Theorem 4.19 in this setting of $\mathfrak{t}(\mathfrak{l}) \neq A_m$ we must now show that if $V^\times = \bigoplus_{\alpha\in\Theta} \mathfrak{g}_\alpha$ and $\tilde{V}^\times = \bigoplus_{\alpha\in\tilde{\Theta}} \mathfrak{g}_\alpha$ for some nontrivial $\Phi_\mathfrak{l}$-candidates $\Theta$ and $\tilde{\Theta}$, then if $\tau \in \operatorname{Int}\mathfrak{g}$ preserves $\mathfrak{l}$ and maps $V^\times$ to $\tilde{V}^\times$ then we can always choose $\tau$ to also preserve $\mathfrak{h}$.

To see why this is possible, note that by Lemma 4.21 that there exist $\rho, \tilde{\rho} \in \operatorname{Int}\mathfrak{g}$ and $p \in [M]_0$ such that $\rho(V^\times) = \tilde{\rho}(\tilde{V}^\times) = \bigoplus_{k=1}^{p} V(\gamma_k)$. It is evident that $P_i$ defined in the proof of Lemma 4.20 is a generalized permutation matrix and thus the inner automorphisms of Lemma 4.20 preserve[11] $\mathfrak{h}$. Moreover, if $V^\times = \bigoplus_{\alpha \in \Theta} \mathfrak{g}_\alpha$ and $\tilde{V}^\times = \bigoplus_{\alpha \in \tilde{\Theta}} \mathfrak{g}_\alpha$ then maximal vectors of the copies of $V(\lambda_1)$ can be chosen to be root space elements. As such, $P$ defined in the proof of Lemma 4.21 is also a generalized permutation matrix and thus again defines an inner automorphism preserving $\mathfrak{h}$. Therefore, all inner automorphisms in the proof of Lemma 4.21 can be chosen to preserve $\mathfrak{l}$ and $\mathfrak{h}$ if $V^\times$ and $\tilde{V}^\times$ are regular relative to $\mathfrak{h}$.

We have argued that the inner automorphisms in Lemma 4.20 and the proof of Lemma 4.21 can be chosen to preserve $\mathfrak{h}$, but how do we know that these inner automorphisms are all that we need? Indeed, we have shown that we can always apply such inner automorphisms to get $V^\times$ in the form outlined in Equation 4.29, but it is a priori possible that $V^\times = \bigoplus_{k=1}^{p} V(\gamma_k)$ and $\tilde{V}^\times = \bigoplus_{k=1}^{\tilde{p}} V(\gamma_k)$ are conjugate for distinct $p, \tilde{p} \in [M]_0$ and that the inner automorphism taking such $V^\times$ to $\tilde{V}^\times$ preserves $\mathfrak{l}$ but cannot be made to preserve $\mathfrak{h}$. If we wish to show that such an occurrence is impossible then it will suffice to prove that for $p, \tilde{p} \in [M]_0$ distinct, $\bigoplus_{k=1}^{p} V(\gamma_k)$ and $\bigoplus_{k=1}^{\tilde{p}} V(\gamma_k)$ are not conjugate. This is the following lemma.

**Lemma 4.22.** *Let $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$ be Levi decomposable subalgebras of $\mathfrak{g}$ with common Levi factor $\mathfrak{l}$, where $\mathfrak{l}$ is not of type $A_m$. By Lemma 4.21, up to inner automorphism one may suppose that the nontrivial components $V^\times$ and $\tilde{V}^\times$ of* Rad $\mathfrak{a}$ *and* Rad $\tilde{\mathfrak{a}}$*, respectively, are $V^\times = \bigoplus_{k=1}^{p} V(\gamma_k)$ and $\tilde{V}^\times = \bigoplus_{k=1}^{\tilde{p}} V(\gamma_k)$, where $p, \tilde{p} \in [M]_0$. If there exists $\tau \in$ Int $\mathfrak{g}$ preserving $\mathfrak{l}$ and mapping $V^\times$ to $\tilde{V}^\times$ then $p = \tilde{p}$.*

*Proof.* This is immediate since $\dim V^\times = p \dim V(\lambda_1)$ and $\dim \tilde{V}^\times = \tilde{p} \dim V(\lambda_1)$. □

Lemma 4.21, Lemma 4.22, and the comments between these two lemmas prove Theorem 4.19 in the case of $\mathfrak{l}$ not of type $A_m$.

## 4.4 PROOF OF THEOREM 4.19 WHEN $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (A_n, A_m)$

It remains to examine when $\mathfrak{l}$ is of type $A_m$. Since we are restricting our attention to well-mannered pairs we may suppose a base for $\Phi_\mathfrak{l}$ is given by Equation 3.10. The first case we consider is when $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (A_n, A_m)$. We suppose $m > 1$ since $(A_n, A_m)$ is well-mannered. The general idea is quite similar to that of Section 4.3. In particular, we prove a counterpart to Lemma 4.21, ensuring the existence of a nontrivial $\Phi_\mathfrak{l}$-candidate $\Theta$. In addition, the inner automorphisms constructed in this analogous lemma will preserve $\mathfrak{h}$ in the event the maximal vectors are root space elements. We then prove that the possibilities we listed are mutually non-conjugate,

11 Recall that $\mathfrak{h}$ consists of diagonal matrices and is hence normalized by generalized permutation matrices.

which will serve as the counterpart to Lemma 4.22. This will establish the second needed assertion and thereby prove Theorem 4.19 in this setting.

We begin with the appropriate analogue to Lemma 4.21.

**Lemma 4.23.** *Suppose $(t(\mathfrak{g}), t(\mathfrak{l})) = (A_n, A_m)$. Then up to inner automorphism, there exist $p, q \in [M+1]_0$ with $p + q \leq M+1$ such that*

$$V^{\times} = \bigoplus_{k=1}^{p} V(\gamma_k) \oplus \bigoplus_{k=M+2-q}^{M+1} V(\beta_k) \tag{4.39}$$

*Proof.* By Lemma 4.9 and Lemma 4.10 we have that $V^{\times}$ is an $\mathfrak{l}$-submodule of $\bigoplus_{k=1}^{M+1} V(\gamma_k) \oplus \bigoplus_{k=1}^{M+1} V(\beta_k)$, provided that $V^{\times} \neq \{0\}$. Furthermore, $\gamma_k|_{\mathfrak{h}_{\mathfrak{l}}} = \lambda_m$ and $\beta_k|_{\mathfrak{h}_{\mathfrak{l}}} = \lambda_1$ for each $k \in [M]$. Since $(A_n, A_m)$ is well-mannered we have $m \neq 1$ and so $V^{\times}$ contains $p$ copies of $V(\lambda_m)$ and $q$ copies of $V(\lambda_1)$ for some $p, q \in [M+1]_0$.

We begin with an examination of the $p$ copies of $V(\lambda_m)$. For each $k \in [M+1]$ define $v_{\gamma_k}$ as

$$v_{\gamma_k} = (-1)^{k-1} Y_{m+1,\ldots,m+k} = E_{m+k+1,m+1} \tag{4.40}$$

Then $v_{\gamma_k}$ is a maximal vector of $V(\gamma_k)$. Let $w_1^-, \ldots, w_p^- \in V^{\times}$ be maximal vectors of the $p$ copies of $V(\lambda_m)$ in $V^{\times}$. For each $i \in [p]$ there exist $a_{i,1}, \ldots, a_{i,M+1} \in \mathbb{F}$ such that

$$w_i^- = \sum_{k=1}^{M+1} a_{i,k} v_{\gamma_k} \tag{4.41}$$

Linear independence of $\{w_1^-, \ldots, w_p^-\}$ implies linear independence of the set of coefficients $\mathcal{A} = \{(a_{i,1}, \ldots, a_{i,M+1}) \in \mathbb{F}^{M+1} : i \in [p]\}$. Therefore, there exists $A \in SL_{M+1}$ such that for each $i \in [p]$ the $i$'th column of $A$ is $(a_{i,1}, \ldots, a_{i,M})$. Define $P_1 \in SL_{n+1}$ as

$$P_1 = \begin{pmatrix} I_{m+1} & 0_{m+1,M+1} \\ 0_{M+1,m+1} & A^{-1} \end{pmatrix} \tag{4.42}$$

Conjugation by $P_1$ defines an inner automorphism $\tau_1 \in \operatorname{Int} \mathfrak{g}$. One can verify that $\tau_1$ fixes $\mathfrak{l}$ element-wise and $\tau_1(w_i^-) = v_{\gamma_i}$ for each $i \in [p]$. It follows that $\tau_1$ maps the $p$ copies of $V(\lambda_m)$ in $V^{\times}$ to $\bigoplus_{k=1}^{p} V(\gamma_k)$.

We now turn our attention to the $q$ copies of $V(\lambda_1)$ in $V^{\times}$. For each $k \in [M+1]$ define

$$v_{\beta_k} = X_{1,\ldots,m+k} = E_{1,m+k+1} \tag{4.43}$$

Then $v_{\beta_k}$ is a maximal vector of $V(\beta_k)$ for each $k \in [M+1]$. Note that since $\tau_1$ fixes $\mathfrak{l}$ element-wise $\tau_1(V^\times)$ will also contain $q$ copies of $V(\lambda_1)$. Let $w_1^+, \dots, w_q^+ \in \tau_1(V^\times)$ be maximal vectors of these $q$ copies. Then for each $i \in [q]$ there exist $b_{i,1}, \dots, b_{i,M+1} \in \mathbb{F}$ such that

$$w_i^+ = \sum_{k=1}^{M+1} b_{i,k} v_{\beta_k} \tag{4.44}$$

Like before we have that $\mathcal{B} = \{(b_{i,1}, \dots, b_{i,M+1}) \in \mathbb{F}^{M+1} \colon i \in [q]\}$ is linearly independent. Also observe that for all $i \in [p]$ and $j \in [q]$ we have $v_{\gamma_i}, w_j^+ \in \tau_1(V^\times)$ and thus $[v_{\gamma_i}, w_j^+] \in \tau_1(\mathrm{Rad}\,\mathfrak{a})$. We have

$$[v_{\gamma_i}, w_j^+] = -b_{j,i} X_{1,\dots,m} \tag{4.45}$$

Since $X_{1,\dots,m} \in \mathfrak{l} = \tau(\mathfrak{l})$ it must be that $b_{j,i} = 0$ for all $i \in [p]$ and $j \in [q]$. This observation, combined with the fact that $\mathcal{B}$ is linearly independent, forces $p \le M+1-q$, i.e. that $p+q \le M+1$. It follows that there exists $B \in SL_{M+1}$ such that for each $i \in [p]$ and $j \in [q]$ the $i$'th column of $B$ is $e_i$ and the $(M+2-j)$'th row of $B$ is $(b_{j,1}, \dots, b_{j,M+1})$. Define $P_2 \in SL_{n+1}$ as

$$P_2 = \begin{pmatrix} I_{m+1} & 0_{m+1,M+1} \\ 0_{M+1,m+1} & B \end{pmatrix} \tag{4.46}$$

Conjugation by $P_2$ defines as inner automorphism $\tau_2 \in \mathrm{Int}\,\mathfrak{g}$. Note that $\tau_2$ fixes $\mathfrak{l}$ element-wise, $\tau_2(v_{\gamma_i}) = v_{\gamma_i}$ for each $i \in [p]$, and $\tau_2(w_j^+) = v_{\beta_{M+2-j}}$ for each $j \in [q]$. The result follows by taking $\tau = \tau_2\tau_1$. □

We again note that the inner automorphisms constructed in the proof of Lemma 4.23 can be taken to be conjugation by generalized permutation matrices when the maximal vectors are root space elements. As in Section 4.3, to complete the proof of Theorem 4.19 in this setting we must show that the possibilities for $V^\times$ outlined in Lemma 4.23 are pairwise non-conjugate.

**Lemma 4.24.** *Let $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$ be Levi decomposable subalgebras of $\mathfrak{g}$ with common Levi factor $\mathfrak{l}$, where $(t(\mathfrak{g}), t(\mathfrak{l})) = (A_n, A_m)$. By Lemma 4.23, up to inner automorphism one may suppose that the nontrivial components $V^\times$ and $\tilde{V}^\times$ of* $\mathrm{Rad}\,\mathfrak{a}$ *and* $\mathrm{Rad}\,\tilde{\mathfrak{a}}$, *respectively, are $V^\times = \bigoplus_{k=1}^{p} V(\gamma_k) \oplus \bigoplus_{k=M+2-q}^{M+1} V(\beta_k)$ and $\tilde{V}^\times = \bigoplus_{k=1}^{\tilde{p}} V(\gamma_k) \oplus \bigoplus_{k=M+2-\tilde{q}}^{M+1} V(\beta_k)$, where $p, q, \tilde{p}, \tilde{q} \in [M+1]_0$ are such that $p+q \le M+1$ and $\tilde{p}+\tilde{q} \le M+1$. If there exists $\tau \in$* $\mathrm{Int}\,\mathfrak{g}$ *preserving $\mathfrak{l}$ and mapping $V^\times$ to $\tilde{V}^\times$ then $(p,q) = (\tilde{p}, \tilde{q})$.*

*Proof.* As subsets of $\mathfrak{sl}_{n+1}$ we have

$$\mathfrak{l} = \left\{ \begin{pmatrix} R & 0_{m+1,M+1} \\ 0_{M+1,m+1} & 0_{M+1,M+1} \end{pmatrix} \in \mathfrak{sl}_{n+1} \colon R \in \mathfrak{sl}_{m+1} \right\} \tag{4.47a}$$

$$\bigoplus_{k=1}^{M+1} V(\beta_k) = \left\{ \begin{pmatrix} 0_{m+1,m+1} & S \\ 0_{M+1,m+1} & 0_{M+1,M+1} \end{pmatrix} \in \mathfrak{sl}_{n+1} \colon S \text{ is } (m+1) \times (M+1) \right\} \tag{4.47b}$$

$$\bigoplus_{k=1}^{M+1} V(\gamma_k) = \left\{ \begin{pmatrix} 0_{m+1,m+1} & 0_{m+1,M+1} \\ T & 0_{M+1,M+1} \end{pmatrix} \in \mathfrak{sl}_{n+1} \colon T \text{ is } (M+1) \times (m+1) \right\} \tag{4.47c}$$

Consider $\tau$ as conjugation by some $P \in SL_{n+1}$ with block decomposition

$$P = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{4.48}$$

where $A \in \mathfrak{gl}_{m+1}$, $B$ is $(m+1) \times (M+1)$, $C$ is $(M+1) \times (m+1)$, and $D \in \mathfrak{gl}_{M+1}$. Since $\tau$ preserves $\mathfrak{l}$ it must be from Equation 4.47a that $B$ and $C$ are zero matrices. Consequently, from Equation 4.47 one can conclude that conjugation by $P$ preserves $\bigoplus_{k=1}^{M+1} V(\beta_k)$ and $\bigoplus_{k=1}^{M+1} V(\gamma_k)$. It follows that $\tau$ must map $\bigoplus_{k=1}^{p} V(\gamma_k)$ to $\bigoplus_{k=1}^{\tilde{p}} V(\gamma_k)$ and $\bigoplus_{k=M+2-q}^{M+1} V(\beta_k)$ to $\bigoplus_{k=M+2-\tilde{q}}^{M+1} V(\beta_k)$. Dimension considerations imply $(p, q) = (\tilde{p}, \tilde{q})$. □

Just as in Section 4.3, the previous two lemmas prove Theorem 4.19 in this context.

## 4.5 HELPFUL OBSERVATIONS WHEN $\mathfrak{t}(\mathfrak{g}) \neq A_n$ AND $\mathfrak{t}(\mathfrak{l}) = A_m$

To complete the proof of Theorem 4.19 we must examine the case where $\mathfrak{t}(\mathfrak{g}) \neq A_n$ and $\mathfrak{t}(\mathfrak{l}) = A_m$ with $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l}))$ well-mannered. The general idea will be the same as in the previous two sections. However, note from Lemma 4.9 that $\Phi_{\mathfrak{l}}^{\times}$ contains the most roots when $\mathfrak{t}(\mathfrak{g}) \neq A_n$ and $\mathfrak{t}(\mathfrak{l}) = A_m$. Hence one may reasonably suspect that this final case is the most troublesome to handle. This suspicion is indeed correct and so it will be beneficial to take some time in this section establishing fruitful results that will assist us in our arguments. Like in Section 4.3, certain inner automorphisms will prove useful. We take a moment to discuss these inner automorphisms and examine how they interact with certain maximal vectors. Our discussion will be divided into separate cases based on the type of $\mathfrak{g}$. Although $(D_n, A_m)$ is not well-mannered, we shall still consider that setting in this section as the remarks we make here will prove useful in Chapter 5.

**Case 1: $\mathfrak{t}(\mathfrak{g}) = B_n$**

In this context we shall consider inner automorphisms given by conjugation by $P \in SO_{2n+1}$ with $P$ of the form

$$P = \begin{pmatrix} a & 0_{1,m+1} & u^\top & 0_{1,m+1} & v^\top \\ 0_{m+1,1} & I_{m+1} & 0_{m+1,M} & 0_{m+1,m+1} & 0_{m+1,M} \\ x & 0_{M,m+1} & A & 0_{M,m+1} & B \\ 0_{m+1,1} & 0_{m+1,m+1} & 0_{m+1,M} & I_{m+1} & 0_{m+1,M} \\ y & 0_{M,m+1} & C & 0_{M,m+1} & D \end{pmatrix} \tag{4.49}$$

where $a \in \mathbb{F}$, $u, v, x, y \in \mathbb{F}^M$, and $A, B, C, D \in \mathfrak{gl}_M$ are such that the matrix $P' \in \mathfrak{gl}_{2M+1}$ defined as

$$P' = \begin{pmatrix} a & u^\top & v^\top \\ x & A & B \\ y & C & D \end{pmatrix} \tag{4.50}$$

is an element of $SO_{2M+1}$. Having $P' \in SO_{2M+1}$ forces $P$ to be an element of $SO_{2n+1}$. From Equation 3.12 we note that conjugation by $P$ fixes $\mathfrak{l}$ element-wise.

Now for each $k \in [M]$ define

$$v_{\beta_k} = X_{1,\dots,m+k} = E_{2,m+k+2} - E_{n+m+k+2,n+2} \tag{4.51a}$$

$$v_{\gamma_k} = (-1)^{k-1} Y_{m+1,\dots,m+k} = E_{m+k+2,m+2} - E_{n+m+2,n+m+k+2} \tag{4.51b}$$

$$v_{\beta_{M+1}} = -X_{1,\dots,n} = E_{1,n+2} - E_{2,1} \tag{4.51c}$$

$$v_{\gamma_{M+1}} = \tfrac{1}{2}(-1)^{M+1} Y_{m+1,\dots,n} = E_{n+m+2,1} - E_{1,m+2} \tag{4.51d}$$

$$v_{\mu_k} = (-1)^{M+1-k} X_{1,\dots,n,n,n-1,\dots,m+k+1} = E_{2,n+m+k+2} - E_{m+k+2,n+2} \tag{4.51e}$$

$$v_{\nu_k} = \tfrac{1}{4}(-1)^{M} Y_{m+1,\dots,n,n,n-1,\dots,m+k+1} = E_{n+m+k+2,m+2} - E_{n+m+2,m+k+2} \tag{4.51f}$$

$$v_{\zeta} = (-1)^{n} X_{1,\dots,n,n,n-1,\dots,2} = E_{3,n+2} - E_{2,n+3} \tag{4.51g}$$

$$v_{\eta} = \tfrac{1}{4}(-1)^{M+1} Y_{m,\dots,n,n,n-1,\dots,m+1} = E_{n+m+2,m+1} - E_{n+m+1,m+2} \tag{4.51h}$$

It is easy to check that the elements defined in Equation 4.51 are maximal vectors of their respective $\mathfrak{l}$-modules as determined by Lemma 4.9.

Relevant matrix calculations reveal that for all $k \in [M]$, if $\tau \in \operatorname{Int} \mathfrak{g}$ is conjugation by $P$ then

$$\tau(v_{\beta_k}) = -v_k v_{\beta_{M+1}} + \sum_{l=1}^{M} (D_{l,k} v_{\beta_l} + B_{l,k} v_{\mu_l}) \tag{4.52a}$$

$$\tau(v_{\gamma_k}) = -u_k v_{\gamma_{M+1}} + \sum_{l=1}^{M} (A_{l,k} v_{\gamma_l} + C_{l,k} v_{\nu_l}) \tag{4.52b}$$

$$\tau(v_{\mu_k}) = -u_k v_{\beta_{M+1}} + \sum_{l=1}^{M} (C_{l,k} v_{\beta_l} + A_{l,k} v_{\mu_l}) \tag{4.52c}$$

$$\tau(v_{\nu_k}) = -v_k v_{\gamma_{M+1}} + \sum_{l=1}^{M} (B_{l,k} v_{\gamma_l} + D_{l,k} v_{\nu_l}) \tag{4.52d}$$

$$\tau(v_{\beta_{M+1}}) = a v_{\beta_{M+1}} - \sum_{l=1}^{M} (y_l v_{\beta_l} + x_l v_{\mu_l}) \tag{4.52e}$$

$$\tau(v_{\gamma_{M+1}}) = a v_{\gamma_{M+1}} - \sum_{l=1}^{M} (x_l v_{\gamma_l} + y_l v_{\nu_l}) \tag{4.52f}$$

$$\tau(v_\zeta) = v_\zeta \tag{4.52g}$$

$$\tau(v_\eta) = v_\eta \tag{4.52h}$$

**Case 2: $\mathfrak{t}(\mathfrak{g}) \in \{C_n, D_n\}$**

In this case we regard inner automorphisms as conjugation by $P \in GL_{2n}$ with $P$ of the form

$$P = \begin{pmatrix} I_{m+1} & 0_{m+1,M} & 0_{m+1,m+1} & 0_{m+1,M} \\ 0_{M,m+1} & A & 0_{M,m+1} & B \\ 0_{m+1,m+1} & 0_{m+1,M} & I_{m+1} & 0_{m+1,M} \\ 0_{M,m+1} & C & 0_{M,m+1} & D \end{pmatrix} \tag{4.53}$$

where $A, B, C, D \in \mathfrak{gl}_M$ are such that the matrix $P' \in GL_{2M}$ defined as

$$P' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{4.54}$$

is an element of $Sp_{2M}$ if $\mathfrak{t}(\mathfrak{g}) = C_n$ and an element of $SO_{2M}$ if $\mathfrak{t}(\mathfrak{g}) = D_n$. This in turn implies $P \in Sp_{2n}$ if $\mathfrak{t}(\mathfrak{g}) = C_n$ and $P \in SO_{2n}$ if $\mathfrak{t}(\mathfrak{g}) = D_n$. Note from Equation 3.13 that conjugation by $P$ fixes $\mathfrak{l}$ element-wise.

For each $k \in [M]$ we define

$$v_{\beta_k} = X_{1,\dots,m+k} = E_{1,m+k+1} - E_{n+m+k+1,n+1} \tag{4.55a}$$

$$v_{\gamma_k} = (-1)^{k-1} Y_{m+1,\dots,m+k} = E_{m+k+1,m+1} - E_{n+m+1,n+m+k+1} \tag{4.55b}$$

$$v_{\mu_k} = \begin{cases} (-1)^{M-k-1} X_{1,\ldots,n,n-1,\ldots,m+k+1}, & \mathfrak{t}(\mathfrak{g}) = C_n \\ (-1)^{M-k} X_{1,\ldots,n,n-2,n-3,\ldots,m+k+1}, & \mathfrak{t}(\mathfrak{g}) = D_n \text{ and } k < M \\ X_{1,\ldots,n-2,n}, & \mathfrak{t}(\mathfrak{g}) = D_n \text{ and } k = M \end{cases} \tag{4.55c}$$

$$= \begin{cases} E_{1,n+m+k+1} + E_{m+k+1,n+1}, & \mathfrak{t}(\mathfrak{g}) = C_n \\ E_{1,n+m+k+1} - E_{m+k+1,n+1}, & \mathfrak{t}(\mathfrak{g}) = D_n \end{cases} \tag{4.55d}$$

$$v_{\nu_k} = \begin{cases} (-1)^{M} Y_{m+1,\ldots,n,n-1,\ldots,m+k+1}, & \mathfrak{t}(\mathfrak{g}) = C_n \\ (-1)^{M} Y_{m+1,\ldots,n,n-2,n-3,\ldots,m+k+1}, & \mathfrak{t}(\mathfrak{g}) = D_n \text{ and } k < M \\ (-1)^{M} Y_{m+1,\ldots,n-2,n}, & \mathfrak{t}(\mathfrak{g}) = D_n \text{ and } k = M \end{cases} \tag{4.55e}$$

$$= \begin{cases} E_{n+m+1,m+k+1} + E_{n+m+k+1,m+1}, & \mathfrak{t}(\mathfrak{g}) = C_n \\ E_{n+m+1,m+k+1} - E_{n+m+k+1,m+1}, & \mathfrak{t}(\mathfrak{g}) = D_n \end{cases} \tag{4.55f}$$

$$v_{\zeta} = \begin{cases} \frac{1}{2}(-1)^{n-1} X_{1,\ldots,n,n-1,\ldots,1}, & \mathfrak{t}(\mathfrak{g}) = C_n \\ (-1)^{n} X_{1,\ldots,n,n-2,n-3,\ldots,2}, & \mathfrak{t}(\mathfrak{g}) = D_n \end{cases} \tag{4.55g}$$

$$= \begin{cases} E_{1,n+1}, & \mathfrak{t}(\mathfrak{g}) = C_n \\ E_{1,n+2} - E_{2,n+1}, & \mathfrak{t}(\mathfrak{g}) = D_n \end{cases} \tag{4.55h}$$

$$v_{\eta} = \begin{cases} \frac{1}{2}(-1)^{M} Y_{m+1,\ldots,n,n-1,\ldots,m+1}, & \mathfrak{t}(\mathfrak{g}) = C_n \\ (-1)^{M} Y_{m,\ldots,n,n-2,n-3,\ldots,m+1}, & \mathfrak{t}(\mathfrak{g}) = D_n \end{cases} \tag{4.55i}$$

$$= \begin{cases} E_{n+m+1,m+1}, & \mathfrak{t}(\mathfrak{g}) = C_n \\ E_{n+m+1,m} - E_{n+m,m+1}, & \mathfrak{t}(\mathfrak{g}) = D_n \end{cases} \tag{4.55j}$$

These elements are maximal vectors of their respective $\mathfrak{l}$-modules as found in Lemma 4.9. By direct computation we have

$$\tau(v_{\beta_k}) = \sum_{l=1}^{M} (D_{l,k} v_{\beta_l} + B_{l,k} v_{\mu_l}) \tag{4.56a}$$

$$\tau(v_{\gamma_k}) = \sum_{l=1}^{M} (A_{l,k} v_{\gamma_l} + C_{l,k} v_{\nu_l}) \tag{4.56b}$$

$$\tau(v_{\mu_k}) = \sum_{l=1}^{M} (C_{l,k} v_{\beta_l} + A_{l,k} v_{\mu_l}) \tag{4.56c}$$

$$\tau(v_{\nu_k}) = \sum_{l=1}^{M} (B_{l,k} v_{\gamma_l} + D_{l,k} v_{\nu_l}) \tag{4.56d}$$

$$\tau(v_{\zeta}) = v_{\zeta} \tag{4.56e}$$

$$\tau(v_{\eta}) = v_{\eta} \tag{4.56f}$$

We briefly examined above fairly general inner automorphisms of $\mathfrak{g}$ and how they interact with maximal vectors. There are select additional inner automorphisms that we will extensively use which are not in the form of the inner automorphisms discussed above. Recall that Lemma 4.20 in Section 4.3 allowed us to swap between a fixed pair of $\mathfrak{l}$-modules $V(\beta_i)$ and $V(\gamma_i)$ while fixing the other nontrivial $\mathfrak{l}$-submodules, assisting in the arguments of Lemma 4.21. The next collection of inner automorphisms satisfies the analogous role in this setting.

**Lemma 4.25.** *Suppose $t(\mathfrak{g}) \neq A_n$ and $t(\mathfrak{l}) = A_m$. Then for each $i \in [M]$ there exists $\sigma_i \in \operatorname{Int}\mathfrak{g}$ such that $\sigma_i$ preserves $\mathfrak{l}$, maps $V(\beta_i)$ to $V(\mu_i)$, $V(\mu_i)$ to $V(\beta_i)$, $V(\gamma_i)$ to $V(\nu_i)$, and $V(\nu_i)$ to $V(\gamma_i)$ while preserving $V(\beta_k)$, $V(\mu_k)$, $V(\gamma_k)$, $V(\nu_k)$, $V(\zeta)$, and $V(\eta)$ for all $k \in [M]$ distinct from $i$. Moreover, if $t(\mathfrak{g}) = B_n$ then $\sigma_i$ also preserves $V(\beta_{M+1})$ and $V(\gamma_{M+1})$.*

*Proof.* Define $A, B, C, D \in \mathfrak{gl}_M$ as

$$A = D = I - E_{i,i}, \qquad B = C = \iota E_{i,i} \tag{4.57}$$

where $\iota^2 = -1$ if $\mathfrak{t}(\mathfrak{g}) = C_n$ and $\iota = 1$ otherwise. If $\mathfrak{g}$ is of type $B_n$ then take $P$ to be as in Equation 4.49, where $a = 1$ and $u = v = x = y = 0$. Else take $P$ to be as defined in Equation 4.53. In both cases the result follows by taking $\sigma_i$ to be conjugation by $P$. □

**Lemma 4.26.** *Suppose $t(\mathfrak{g}) \neq A_n$ and $t(\mathfrak{l}) = A_m$. Then there exists $\rho \in \operatorname{Int}\mathfrak{g}$ such that $\rho$ preserves $l$ and for all $k \in [M]$, $\rho$ maps $V(\beta_k)$ to $V(\nu_k)$, $V(\nu_k)$ to $V(\beta_k)$, $V(\mu_k)$ to $V(\gamma_k)$, $V(\gamma_k)$ to $V(\mu_k)$, $V(\zeta)$ to $V(\eta)$, and $V(\eta)$ to $V(\zeta)$. If $t(\mathfrak{g}) = B_n$ then $\rho$ maps $V(\beta_{M+1})$ to $V(\gamma_{M+1})$ and $V(\gamma_{M+1})$ to $V(\beta_{M+1})$ as well.*

*Proof.* If $\mathfrak{t}(\mathfrak{g}) = B_n$ define $P \in SO_{2n+1}$ as

$$P = \begin{pmatrix} 1 & 0_{1,m+1} & 0_{1,M} & 0_{1,m+1} & 0_{1,M} \\ 0_{m+1,1} & 0_{m+1,m+1} & 0_{m+1,M} & I_{m+1} & 0_{m+1,M} \\ 0_{M,1} & 0_{M,m+1} & I_M & 0_{M,m+1} & 0_{M,M} \\ 0_{m+1,1} & I_{m+1} & 0_{m+1,M} & 0_{m+1,m+1} & 0_{m+1,M} \\ 0_{M,1} & 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & I_M \end{pmatrix} \tag{4.58}$$

If $\mathfrak{g}$ is not of type $B_n$ we instead define $P \in GL_{2n}$ as

$$P = \begin{pmatrix} 0_{m+1,m+1} & 0_{m+1,M} & \iota I & 0_{m+1,M} \\ 0_{M,m+1} & I & 0_{M,m+1} & 0_{m+1,M} \\ \iota I & 0_{m+1,M} & 0_{m+1,m+1} & 0_{m+1,M} \\ 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & I \end{pmatrix} \tag{4.59}$$

where $\iota^2 = -1$ if $\mathfrak{t}(\mathfrak{g}) = C_n$ and $\iota = 1$ if $\mathfrak{t}(\mathfrak{g}) = D_n$. A simple verification reveals that $P \in Sp_{2n}$ when $\mathfrak{t}(\mathfrak{g}) = C_n$ and $P \in SO_{2n}$ when $\mathfrak{t}(\mathfrak{g}) = D_n$.

Let $\rho \in \operatorname{Int} \mathfrak{g}$ be conjugation by $P$. Then $\rho$ preserves $\mathfrak{l}$. Note that $P$ is a generalized permutation matrix and thus $\rho$ preserves $\mathfrak{h}$. Hence $\rho$ maps root spaces to root spaces. In particular, by recalling that weight spaces of $V(\alpha)$ are root spaces of $\mathfrak{g}$ for each $\alpha \in \Phi_{\mathfrak{l}}^{\times}$, we observe that $\rho$ maps maximal vectors to root space vectors. Thus for each $\alpha \in \Phi_{\mathfrak{l}}^{\times}$ let $\sigma(\alpha) \in \Phi$ be such that $\rho$ maps $v_\alpha$ into $\mathfrak{g}_{\sigma(\alpha)}$. Then for each $k \in [M]$, $\sigma(\beta_k) \in \nu_k + \Phi_{\mathfrak{l}}$, $\sigma(\nu_k) \in \beta_k + \Phi_{\mathfrak{l}}$, $\sigma(\mu_k) \in \gamma_k + \Phi_{\mathfrak{l}}$, $\sigma(\gamma_k) \in \mu_k + \Phi_{\mathfrak{l}}$, $\sigma(\zeta) \in \eta + \Phi_{\mathfrak{l}}$, and $\sigma(\eta) \in \zeta + \Phi_{\mathfrak{l}}$, where $\alpha + \Phi_{\mathfrak{l}}$ is defined as in Equation 4.18. If $\mathsf{t}(\mathfrak{g}) = B_n$ then $\sigma(\beta_{M+1}) \in \gamma_{M+1} + \Phi_{\mathfrak{l}}$ and $\sigma(\gamma_{M+1}) \in \beta_{M+1} + \Phi_{\mathfrak{l}}$ as well. □

One may note that all of the inner automorphisms we have constructed in this section have roughly the same form. Indeed, they have all amounted to conjugation by a matrix which decomposes simply into blocks, most of which are zero matrices. If we hope to prove the equivalents of Lemma 4.22 and Lemma 4.24, we will need to consider all inner automorphisms which preserve $\mathfrak{l}$. While it would greatly facilitate our arguments if all inner automorphisms preserving $\mathfrak{l}$ could be chosen to be conjugation by simple block matrices, it is not a prioiri guaranteed that this is always possible. While this next lemma may not appear to be related to the discussion at hand, it will indeed prove quite instrumental in later sections.

**Lemma 4.27.** *Let $m > 1$, $\kappa \in \{-1, 1\}$, and $A, B, C, D \in \mathfrak{gl}_{m+1}$ be such that for all $Z \in \mathfrak{sl}_{m+1}$,*

$$AZB^\top - \kappa BZ^\top A^\top = 0 \tag{4.60a}$$

$$CZD^\top - \kappa DZ^\top C^\top = 0 \tag{4.60b}$$

$$A^\top D + \kappa C^\top B = I \tag{4.60c}$$

*If $D \neq 0$ then $B = 0$.*

*Proof.* Suppose by contradiction that $B \neq 0$. Then there exist $r, s \in [m+1]$ such that $B_{r,s} \neq 0$. For each $i, j \in [m+1]$ distinct we may take $Z = E_{i,j}$ in Equation 4.60a. By considering the $(k,l)$-entry for each $k, l \in [m+1]$ we conclude that

$$A_{k,i} B_{l,j} = \kappa B_{k,j} A_{l,i} \tag{4.61}$$

In particular, taking $l = r$ and $j = s$ in Equation 4.61 yields

$$A_{k,i} = \frac{\kappa A_{r,i}}{B_{r,s}} B_{k,s} \tag{4.62}$$

Consequently, for each $i \in [m+1]$ distinct from $s$ we have that the $i$'th column of $A$ is a scalar multiple of the $s$'th column of $B$. Thus $A$ has rank at most 2. Moreover, Equation 4.62 allows us to replace $A_{k,i}$ and $A_{l,i}$ in Equation 4.61, yielding

$$0 = A_{r,i}(\kappa B_{k,j} B_{l,s} - B_{k,s} B_{l,j}) \tag{4.63}$$

for all $i, j, k, l \in [m+1]$ with $i \neq j, s$.

Suppose by contradiction that $A$ has rank 2. Then there exists $g \in [m+1]$ distinct from $s$ such that the $g$'th column of $A$ is nonzero. As $g \neq s$ we have that the $g$'th column of $A$ is a scalar multiple of the $s$'th column of $B$. Since $B_{r,s} \neq 0$ it must be that $A_{r,g} \neq 0$. By taking $i = g$ and $k = r$ in Equation 4.63 we conclude that for all $j \in [m+1]$ distinct from $g$,

$$B_{l,j} = \frac{\kappa B_{r,j}}{B_{r,s}} B_{l,s} \tag{4.64}$$

Hence for all $j \in [m+1]$ distinct from $g$ we have that the $j$'th column of $B$ is a scalar multiple of the $s$'th column of $B$.

Since $m > 1$ there exists $j \in [m+1]$ distinct from $g$ and $s$. By taking $Z = E_{g,g} - E_{j,j}$ in Equation 4.60a and considering the $(r, l)$-entry with $l \in [m+1]$, we conclude by Equation 4.62, Equation 4.64, and the fact that $A_{r,g} \neq 0$ that

$$B_{l,g} = \frac{B_{r,g}}{A_{r,g}} A_{l,g} \tag{4.65}$$

for all $l \in [m+1]$. Therefore, the $g$'th column of $B$ is a scalar multiple of the $g$'th column of $A$. However, recall that since $g \neq s$ we have that the $g$'th column of $A$ is a scalar multiple of the $s$'th column of $B$. It follows that every column of $B$ is a scalar multiple of the $s$'th column of $B$.

By considering $Z = E_{s,s} - E_{j,j}$ in Equation 4.60a with $j \in [m+1]$ distinct from $g$ and $s$ and examining the $(k, r)$-entry for each $k \in [m+1]$, we obtain by Equation 4.62 and Equation 4.64 that

$$A_{k,s} = \frac{A_{r,s}}{B_{r,s}} B_{k,s} \tag{4.66}$$

Hence the $s$'th column of $A$ is a scalar multiple of the $s$'th column of $B$. However, we previously showed that for all $i \in [m+1]$ distinct from $s$ that the $i$'th column of $A$ is a scalar multiple of the $s$'th column of $B$. It follows that rank $A = 1$, contradicting our assumption. Hence rank $A \leq 1$.

Since $D \neq 0$ we can apply the exact same argument to imply rank $C \leq 1$ by using Equation 4.60b in lieu of Equation 4.60a. Since rank $A$, rank $C \leq 1$ we have rank $A^\top D$, rank $C^\top B \leq 1$. Consequently, rank $(A^\top D + \kappa C^\top B) \leq$

2, which contradicts Equation 4.60c since $m > 1$, forcing $\operatorname{rank} I = m + 1 > 2$. Hence it must be that $B = 0$. □

Lemma 4.27 is the first lemma which will aid us in examining $\mathfrak{l}$-preserving inner automorphisms of $\mathfrak{g}$. The main use of this lemma will be in proving analogues to Lemma 4.22 and Lemma 4.24. This next result will also be useful in this endeavour.

**Lemma 4.28.** *Suppose $\mathsf{t}(\mathfrak{g}) \neq A_n$ and $\mathsf{t}(\mathfrak{l}) = A_m$. Consider the subspace $U \subseteq \mathfrak{g}$ defined either as*

$$U = \operatorname{span}\{v_{\nu_1}, \dots, v_{\nu_p}, v_{\gamma_1}, \dots, v_{\gamma_q}\} \tag{4.67a}$$

$$\text{or } U = \operatorname{span}\{v_{\nu_1}, \dots, v_{\nu_p}, v_{\gamma_1}, \dots, v_{\gamma_q}, v_{\gamma_{M+1}}\} \tag{4.67b}$$

*for some $p, q \in [M]_0$ with $q \leq p$, where Equation 4.67b is only possible if $\mathfrak{g}$ is of type $B_n$. If $U$ contains a subspace $U'$ such that for all $u_1, u_2 \in U'$, $[u_1, [Y_m, u_2]] = 0$ then $\dim U' \leq p$.*

*Proof.* Let $d' = \dim U'$ and $\{u_1, \dots, u_{d'}\}$ be a basis for $U'$. Since $U' \subseteq U$ we have that for each $i \in [d']$ there exist $a_{i,1}, \dots, a_{i,M}, b_{i,1}, \dots, b_{i,M}, c_i \in \mathbb{F}$ such that

$$u_i = c_i v_{\gamma_{M+1}} + \sum_{k=1}^{M} (a_{i,k} v_{\gamma_k} + b_{i,k} v_{\nu_k}) \tag{4.68}$$

where $a_{i,k} = b_{i,l} = 0$ for $q < k \leq M$ and $p < l \leq M$ and $c_i = 0$ if $U$ is as in Equation 4.67a. For each $i \in [d']$ define $w_i \in \mathbb{F}^d$ as

$$w_i = \sum_{k=1}^{p} b_{i,k} e_k + \sum_{k=p+1}^{p+q} a_{i,k-p} e_k + c_i e_d \tag{4.69}$$

where $d = \dim U$. Note that linear independence of $\{u_1, \dots, u_{d'}\}$ implies linear independence of $\{w_1, \dots, w_{d'}\}$. Now let $Q$ be the $d' \times d$ matrix with rows $w_1^\top, \dots, w_{d'}^\top$. Without loss of generality we may choose $u_1, \dots, u_{d'}$ such that $Q$ is in reduced row echelon form. Consequently, there exists a minimal index $l \in [p]$ such that for all $i \in \{l+1, \dots, d'\}$ we have $b_{i,1} = \cdots = b_{i,M} = 0$. Now observe that for each $i, j \in [d']$ we have $(-1)^M [w_i, [Y_m, w_j]] = 0$, implying

$$0 = \begin{cases} -\frac{1}{4}\left(c_i c_j + \sum_{k=1}^{M} (a_{i,k} b_{j,k} + a_{j,k} b_{i,k})\right) Y_{m,\dots,n,n,n-1,\dots,m+1}, & \mathsf{t}(\mathfrak{g}) = B_n \\ \left(\sum_{k=1}^{M} (a_{i,k} b_{j,k} - a_{j,k} b_{i,k})\right) Y_{m,\dots,n,n-1,\dots,m+1}, & \mathsf{t}(\mathfrak{g}) = C_n \\ -\left(\sum_{k=1}^{M} (a_{i,k} b_{j,k} + a_{j,k} b_{i,k})\right) Y_{m,\dots,n,n-2,n-3,\dots,m+1}, & \mathsf{t}(\mathfrak{g}) = D_n \end{cases} \tag{4.70}$$

Hence

$$c_i c_j + \sum_{k=1}^{M} (a_{i,k} b_{j,k} + \kappa a_{j,k} b_{i,k}) = 0 \tag{4.71}$$

for all $i, j \in [d']$, where $\kappa = -1$ if $\mathsf{t}(\mathfrak{g}) = C_n$ and $\kappa = 1$ otherwise. Since $b_{i,1} = \cdots = b_{i,M} = 0$ for all $i \in \{l+1, \ldots, d'\}$ we conclude by taking $j = i$ in Equation 4.71 that $c_i = 0$ for all $i \in \{l+1, \ldots, d'\}$.

For each $i \in \{l+1, \ldots, d'\}$ and $j \in [l]$ define $x_i, y_j \in \mathbb{F}^p$ as

$$x_i = \sum_{k=1}^{p} a_{i,k} e_k, \qquad y_j = \sum_{k=1}^{p} b_{j,k} e_k \tag{4.72}$$

Since $Q$ is in reduced row echelon form, $\{w_1, \ldots, w_{d'}\}$ is linearly independent, and $c_i = \cdots = c_{d'} = 0$, we conclude by the definition of $l$ that $\{x_{l+1}, \ldots, x_{d'}\}$ and $\{y_1, \ldots, y_l\}$ are linearly independent.

Now observe that Equation 4.71 in particular holds for all $i \in \{l+1, \ldots, d'\}$ and $j \in [l]$. Since $b_{i,1} = \cdots = b_{i,M} = 0$ and $c_i = 0$ for all $i \in \{l+1, \ldots, d'\}$ and $a_{k,p+1} = \cdots = a_{k,M} = b_{k,p+1} = \cdots = b_{k,M} = 0$ for all $k \in [d']$ it follows from Equation 4.71 that $\sum_{k=1}^{p} a_{i,k} b_{j,k} = 0$ for all $i \in \{l+1, \ldots, d'\}$ and $j \in [l]$. Since $\{y_1, \ldots, y_l\}$ is linearly independent we conclude that the dimension of the subspace spanned by $\{x_{l+1}, \ldots, x_{d'}\}$ is bounded by $p - l$. Linear independence of $\{x_{l+1}, \ldots, x_{d'}\}$ thus implies $d' - l \leq p - l$, i.e. $d' \leq p$. □

The final three lemmas of this section are unrelated to the issue of conjugacy. Nonetheless, they are short, simple, and will prove quite helpful in completing the proof of Theorem 4.19.

**Lemma 4.29.** *Suppose $\mathsf{t}(\mathfrak{g}) \neq A_n$ and $\mathsf{t}(\mathfrak{l}) = A_m$. Then the $\mathfrak{l}$-module $V(\zeta) \oplus V(\eta)$ is not contained in $V^\times$.*

*Proof.* Suppose by contradiction that $V(\zeta) \oplus V(\eta) \subseteq V^\times$. Define $\zeta' \in \Phi$ as

$$\zeta' = \begin{cases} \zeta - \sum_{k=1}^{m} \alpha_k, & \mathsf{t}(\mathfrak{g}) = C_n \\ \zeta - \sum_{k=2}^{m} \alpha_k, & \text{else} \end{cases} \tag{4.73a}$$

$$= \varepsilon_1 + \varepsilon_{m+1} \tag{4.73b}$$

By construction, $\zeta' \in \zeta + \Phi_{\mathfrak{l}}$ and so $\mathfrak{g}_{\zeta'} \subseteq V(\eta)$. Since $V(\zeta) \oplus V(\eta) \subseteq \operatorname{Rad} \mathfrak{a}$ it follows that $[\mathfrak{g}_{\zeta'}, \mathfrak{g}_\eta] = \mathfrak{g}_{\zeta'+\eta} \subseteq \operatorname{Rad} \mathfrak{a}$. If $\mathsf{t}(\mathfrak{g}) \neq C_n$ then $\zeta' + \eta = \varepsilon_1 - \varepsilon_m = \sum_{k=1}^{m-1} \alpha_k$ whereas if $\mathsf{t}(\mathfrak{g}) = C_n$ then $\zeta' + \eta = \varepsilon_1 - \varepsilon_{m+1} = \sum_{k=1}^{m} \alpha_k$. In any event we have $\zeta' + \eta \in \Phi_{\mathfrak{l}}$, implying $\mathfrak{g}_{\zeta'+\eta} \subseteq \mathfrak{l}$. This is a contradiction. □

**Lemma 4.30.** *Suppose $t(\mathfrak{g}) \neq A_n$ and $t(\mathfrak{l}) = A_m$. If $V(\gamma_k) \oplus V(\nu_k) \subseteq V^\times$ for some $k \in [M]$ then $V(\eta) \subseteq V^\times$. If $t(\mathfrak{g}) = B_n$ and $V(\gamma_{M+1}) \subseteq V^\times$ then $V(\eta) \subseteq V^\times$.*

*Proof.* Define $\tilde{\nu}_k \in \Phi$ as

$$\tilde{\nu}_k = \begin{cases} \nu_k, & \mathrm{t}(\mathfrak{g}) = C_n \\ \nu_k - \alpha_m, & \text{else} \end{cases} \tag{4.74}$$

Then $\gamma_k \in \gamma_k + \Phi_{\mathfrak{l}}$ and $\tilde{\nu}_k \in \nu_k + \Phi_{\mathfrak{l}}$. Therefore, if $V(\gamma_k) \oplus V(\nu_k) \subseteq V^\times$ then $[\mathfrak{g}_{\gamma_k}, \mathfrak{g}_{\tilde{\nu}_k}] = \mathfrak{g}_\eta \subseteq \operatorname{Rad} \mathfrak{a}$, implying $V(\eta) \subseteq V^\times$. Similarly, if $\mathrm{t}(\mathfrak{g}) = B_n$ then observe that $\gamma_{M+1}, \gamma_{M+1} - \alpha_m \in \gamma_{M+1} + \Phi_{\mathfrak{l}}$. Since $[\mathfrak{g}_{\gamma_{M+1}}, \mathfrak{g}_{\gamma_{M+1}-\alpha_m}] = \mathfrak{g}_\eta$ we similarly conclude that $V(\eta) \subseteq V^\times$ if $V(\gamma_{M+1}) \subseteq V^\times$. □

**Lemma 4.31.** *Suppose $t(\mathfrak{g}) \in \{B_n, D_n\}$ and $t(\mathfrak{l}) = A_3$. Define*

$$w = av_\zeta + bv_\eta \tag{4.75}$$

*where $a, b \in \mathbb{F}$. If $w \in V^\times$ then $ab = 0$.*

*Proof.* We have

$$[[Y_2, w], w] = 2ab\kappa X_{1,2,3} \tag{4.76}$$

where $\kappa = 1$ if $\mathrm{t}(\mathfrak{g}) = B_n$ and $\kappa = -1$ if $\mathrm{t}(\mathfrak{g}) = D_n$. As $Y_2 \in \mathfrak{l}$ and $w \in \operatorname{Rad} \mathfrak{a}$ we have $[[Y_2, w], w] \in \operatorname{Rad} \mathfrak{a}$. Since $X_{1,2,3} \in \mathfrak{l}$ the result follows. □

## 4.6 PROOF OF THEOREM 4.19 WHEN $(\mathrm{t}(\mathfrak{g}), \mathrm{t}(\mathfrak{l})) = (B_n, A_m)$

This section aims to complete the proof of Theorem 4.19 in the case of $\mathfrak{g}$ of type $B_n$ and $\mathfrak{l}$ of type $A_m$. As in the other sections, this will be partially achieved by proving a lemma that yields a concrete description of $V^\times$ as we did in Lemma 4.21 and Lemma 4.23. Since $(\mathfrak{g}, \mathfrak{l})$ is ill-mannered if $m \leq 2$ we may suppose $m > 2$.

**Lemma 4.32.** *Suppose $(t(\mathfrak{g}), t(\mathfrak{l})) = (B_n, A_m)$. Then up to inner automorphism there exist $p, q, c, d \in [M]_0$ with $q \leq p$, $d \leq p - q$, and $c \leq M - p$ such that $V^\times$ is one of the following:*

$$V^\times = V(\eta) \oplus \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k) \oplus \bigoplus_{k=p-d+1}^{p} V(\beta_k) \tag{4.77a}$$

$$\text{or } V^\times = \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=M-c+1}^{M} V(\beta_k) \oplus \bigoplus_{k=p-d+1}^{p} V(\beta_k) \tag{4.77b}$$

$$\text{or } V^\times = V(\gamma_{M+1}) \oplus V(\eta) \oplus \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k) \oplus \bigoplus_{k=p-d+1}^{p} V(\beta_k) \tag{4.77c}$$

*Moreover, if $V^\times$ is as in Equation 4.77b then we may further suppose $c + d \leq p$.*

*Proof.* Lemma 4.9 and Lemma 4.10 imply that if $V^\times \neq \{0\}$ then $V^\times$ is an $\mathfrak{l}$-submodule of $\bigoplus_{k=1}^{M+1} V(\beta_k) \oplus \bigoplus_{k=1}^{M+1} V(\gamma_k) \oplus \bigoplus_{k=1}^{M} V(\mu_k) \oplus \bigoplus_{k=1}^{M} V(\nu_k) \oplus V(\zeta) \oplus V(\eta)$. For each $k \in [M]$, $\beta_k|_{\mathfrak{h}_\mathfrak{l}} = \beta_{M+1}|_{\mathfrak{h}_\mathfrak{l}} = \mu_k|_{\mathfrak{h}_\mathfrak{l}} = \lambda_1$, $\gamma_k|_{\mathfrak{h}_\mathfrak{l}} = \gamma_{M+1}|_{\mathfrak{h}_\mathfrak{l}} = \nu_k|_{\mathfrak{h}_\mathfrak{l}} = \lambda_m$, $\zeta|_{\mathfrak{h}_\mathfrak{l}} = \lambda_2$, and $\eta|_{\mathfrak{h}_\mathfrak{l}} = \lambda_{m-1}$.

Since $(B_n, A_m)$ is well-mannered we have $m > 2$. As such, $\lambda_1$, $\lambda_2$, $\lambda_{m-1}$, and $\lambda_m$ are all pairwise distinct weights, unless $m = 3$. It follows that if $m \neq 3$ then there exist $r^+, r^- \in [2M+1]_0$ and $s^+, s^- \in \{0, 1\}$ such that $V^\times$ contains $r^+$ copies of $V(\lambda_1)$, $r^-$ copies of $V(\lambda_m)$, $s^+$ copies of $V(\lambda_2)$, and $s^-$ copies of $V(\lambda_{m-1})$. Lemma 4.29 implies $s^+s^- = 0$. Moreover, if one of $s^+$ or $s^-$ is nonzero then we may suppose without loss of generality due to Lemma 4.26 that $s^- = 1$ and $s^+ = 0$. If however $s^- = s^+ = 0$ then we again use Lemma 4.26 to suppose $r^- \geq r^+$ without loss of generality. This allows us to assume $c + d \leq p$ in Equation 4.77b, provided we can show Equation 4.77b holds.

Notice that if $m = 3$ then the above assertions still hold. Indeed, the only potential issue when $m = 3$ is that $\lambda_2 = \lambda_{m-1}$. However, Lemma 4.29 ensures that $V^\times$ can contain at most one copy of $V(\lambda_2)$ in this case, and Lemma 4.26 and Lemma 4.31 together imply that up to inner automorphism we may suppose such a copy is given by $V(\eta)$. Hence in what follows we may suppose $m \geq 3$.

We proceed via induction on $r^-$ to show that there exist $p, q \in [M]_0$ with $q \leq p$ such that the $r^-$ copies of $V(\lambda_m)$ in $V^\times$ are, up to inner automorphism, given by either $\bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$ with $p + q = r^-$ or $V(\gamma_{M+1}) \oplus \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$ with $p + q = r^- - 1$. The base case of $r^- = 0$ trivially holds. Thus suppose the assertion holds when $\operatorname{Rad} \mathfrak{a}$ contains $r^-$ copies of $V(\lambda_m)$ and now suppose $V^\times$ contains $r^- + 1$ such copies. Let $w_1^-, \ldots, w_{r^-+1}^- \in V^\times$ be maximal vectors of these $r^- + 1$ copies. For each $i \in [r^- + 1]$ there exist $a_{i,1}, \ldots, a_{i,M}, b_{i,1}, \ldots, b_{i,M}, c_i \in \mathbb{F}$ such that

$$w_i^- = c_i v_{\gamma_{M+1}} + \sum_{k=1}^{M} (a_{i,k} v_{\gamma_k} + b_{i,k} v_{\nu_k}) \tag{4.78}$$

where $v_{\gamma_{M+1}}$, $v_{\gamma_k}$, and $v_{\nu_k}$ are as in Equation 4.51.

Let $\tilde{V}_m^\times$ be the $\mathfrak{l}$-module generated by $w_1^-, \ldots, w_{r^-}^-$. One can verify that $[\tilde{V}_m^\times, \tilde{V}_m^\times] \cap \left( \bigoplus_{k=1}^{M+1} V(\gamma_k) \oplus \bigoplus_{k=1}^{M} V(\nu_k) \right) \subseteq \tilde{V}_m^\times$. Consequently, the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{l}$ and $\tilde{V}_m^\times$ is a Levi decomposable subalgebra such that the nontrivial component of the radical has precisely $r^-$ copies of $V(\lambda_m)$ given by $\tilde{V}_m^\times$. By the inductive hypothesis, we may therefore suppose without loss of generality that there exist $p, q \in [M]_0$ with $q \leq p$ such that $\tilde{V}_m^\times = \bigoplus_{k=1}^p V(\nu_k) \oplus \bigoplus_{k=1}^q V(\gamma_k)$ or $\tilde{V}_m^\times = V(\gamma_{M+1}) \oplus \bigoplus_{k=1}^p V(\nu_k) \oplus \bigoplus_{k=1}^q V(\gamma_k)$. Linear independence of $\{w_1^-, \ldots, w_{r^-+1}^-\}$ implies that we may suppose $a_{r^-+1,1} = \cdots = a_{r^-+1,q} = b_{r^-+1,1} = \cdots = b_{r^-+1,p} = 0$. We proceed via a systematic consideration of all possible cases.

**Case 1: $\tilde{V}_m^\times = V(\gamma_{M+1}) \oplus \bigoplus_{k=1}^p V(\nu_k) \oplus \bigoplus_{k=1}^q V(\gamma_k)$ and there exists $l \in \{q+1, \ldots, p\}$ such that $a_{r^-+1,l} \neq 0$**

Note that this case requires $q < p$. Since $v_{\gamma_{M+1}} \in \tilde{V}_m^\times$ we can add an appropriate scalar multiple of $v_{\gamma_{M+1}}$ to $w_{r^-}^-$ to suppose without loss of generality that

$$c_{r^-+1}^2 + 2\sum_{k=1}^M a_{r^-+1,k} b_{r^-+1,k} = 0 \tag{4.79}$$

Define $A, C \in \mathfrak{gl}_M$ such that for each $i \in [M]$ the $i$'th column of $A$ and $C$ is $A_i$ and $C_i$, respectively, where

$$A_i = \begin{cases} e_i, & i \leq q \text{ or } i > l \\ \sum_{k=1}^M a_{r^-+1,k} e_k, & i = q+1 \\ e_{i-1}, & q+1 < i \leq l \end{cases} \tag{4.80a}$$

$$C_i = \begin{cases} -\frac{b_{r^-+1,i}}{a_{r^-+1,l}} e_l, & i \leq q \text{ or } i > l \\ \sum_{k=1}^M b_{r^-+1,k} e_k, & i = q+1 \\ -\frac{b_{r^-+1,i-1}}{a_{r^-+1,l}} e_l, & q+1 < i \leq l \end{cases} \tag{4.80b}$$

Invertibility of $A$ is ensured by linear independence of $\{A_1, \ldots, A_M\}$ and so we define $B, D \in \mathfrak{gl}_M$ as $B = 0$ and $D = (A^{-1})^\top$. Also define $a \in \mathbb{F}$ and $u, v, x, y \in \mathbb{F}^M$ as $a = 1$, $u = -c_{r^-+1} e_{q+1}$, $v = x = 0$, and $y = a_{r^-+1} D e_{q+1}$. Since $q < p$ and Equation 4.79 holds we have that $P'$ as defined in Equation 4.50 is an element of $SO_{2M+1}$. Consequently, $P$ as given by Equation 4.49 is an element of $SO_{2n+1}$ and hence defines an $\mathfrak{l}$-preserving inner automorphism $\tau \in \operatorname{Int} \mathfrak{g}$. By Equation 4.52 we have that $\tau$ maps $v_{\gamma_{q+1}}$ to $w_{r^-+1}^-$ and $\bigoplus_{k=1}^q V(\gamma_k)$ to itself. From Equation 4.52 we also observe that to show that $\tau$ maps $V(\gamma_{M+1}) \oplus \bigoplus_{k=1}^p V(\nu_k)$ to itself it suffices to verify that the first $p$ columns of $D$ are contained in $\operatorname{span}\{e_1, \ldots, e_p\}$. This can be immediately deduced from Equation 4.80a and the fact that $A^\top D = I$. Therefore, $\tau^{-1}$ maps the $r^- + 1$ copies of $V(\lambda_m)$ to $V(\gamma_{M+1}) \oplus \bigoplus_{k=1}^p V(\nu_k) \oplus \bigoplus_{k=1}^{q+1} V(\gamma_k)$ with $q+1 \leq p$ in this case.

**Case 2:** $\tilde{V}_m^\times = V(\gamma_{M+1}) \oplus \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$ **and** $a_{r^-+1,k} = 0$ **for all** $k \in \{q+1, \dots, p\}$

Note that this case implies $p < M$ and that there exists $l \in \{p+1, \dots, M\}$ such that $a_{r^-+1,l} \neq 0$ or $b_{r^-+1,l} \neq 0$. Otherwise $w_{r^-+1}$ is a scalar multiple of $v_{\gamma_{M+1}}$, contradicting the fact that $V(\gamma_{M+1}) \subseteq \tilde{V}_m^\times$. By Lemma 4.25 and since $q \leq p$ we may suppose $b_{r^-+1,l} \neq 0$. Define $B, D \in \mathfrak{gl}_M$ such that the $i$'th column of $B$ and $D$ is $B_i$ and $D_i$, respectively, for each $i \in [M]$, where

$$B_i = \begin{cases} -\frac{a_{r^-+1,i}}{b_{r^-+1,l}} e_l, & i \leq p \text{ or } i > l \\ \sum_{k=1}^{M} a_{r^-+1,k} e_k, & i = p+1 \\ -\frac{a_{r^-+1,i-1}}{b_{r^-+1,l}} e_l, & p+1 < i \leq l \end{cases} \tag{4.81a}$$

$$D_i = \begin{cases} e_i, & i \leq p \text{ or } i > l \\ \sum_{k=1}^{M} b_{r^-+1,k} e_k, & i = p+1 \\ e_{i-1}, & p+1 < i \leq l \end{cases} \tag{4.81b}$$

Note that $\{D_1, \dots, D_M\}$ is linearly independent so $D$ is invertible. Define $A, C \in \mathfrak{gl}_M$, $a \in \mathbb{F}$, and $u, v, x, y \in \mathbb{F}^M$ as $A = (D^{-1})^\top$, $C = 0$, $a = 1$, $u = y = 0$, $v = -c_{r^-+1} e_{p+1}$, and $x = c_{r^-+1} A e_{p+1}$. As in Case 1 this defines an inner automorphism $\tau \in \operatorname{Int} \mathfrak{g}$ preserving $\mathfrak{l}$ via conjugation by $P$ as in Equation 4.49. Note from Equation 4.52 that $\tau$ maps $v_{\nu_{p+1}}$ to $w_{r^-+1}^-$ and $\bigoplus_{k=1}^{p} V(\nu_k)$ to itself. By verifying that the first $q$ columns of $A$ are contained in $\operatorname{span}\{e_1, \dots, e_q\}$ we also have that $\tau$ preserves $\bigoplus_{k=1}^{q} V(\gamma_k)$. Therefore, $\tau^{-1}$ maps $w_{r^-+1}^-$ to $v_{\nu_{p+1}}$, $\bigoplus_{k=1}^{p} V(\nu_k)$ to itself, and $\bigoplus_{k=1}^{q} V(\gamma_k)$ to itself. It remains to determine how $\tau^{-1}$ acts on $V(\gamma_{M+1})$.

A straightforward calculation reveals that

$$\tau^{-1}(v_{\gamma_{M+1}}) = v_{\gamma_{M+1}} + c_{r^-+1} v_{\gamma_{p+1}} \tag{4.82}$$

Therefore, by what we have shown so far we may note that this case is equivalent to the case where

$$\{w_1^-, \dots, w_{r^-+1}^-\} = \{v_{\gamma_1}, \dots, v_{\gamma_q}, v_{\nu_1}, \dots, v_{\nu_{p+1}}, v_{\gamma_{M+1}} + c v_{\gamma_{p+1}}\} \tag{4.83}$$

for some $c \in \mathbb{F}$. There is nothing to do if $c = 0$ so suppose $c \neq 0$. Since $v_{\nu_{p+1}} \in \operatorname{Rad} \mathfrak{a}$ we can replace $w_{r^-+1}^- = v_{\gamma_{M+1}} + c v_{\gamma_{p+1}}$ with

$$\tilde{w}_{r^-+1}^- = v_{\gamma_{M+1}} + c v_{\gamma_{p+1}} - \frac{1}{2c} v_{\nu_{p+1}} \tag{4.84}$$

Define $A, C \in \mathfrak{gl}_M$ such that the $i$'th column for each $i \in [M]$ is $A_i$ and $C_i$, respectively, where

$$A_i = \begin{cases} e_i, & i \le q \text{ or } i > p+1 \\ ce_{p+1}, & i = q+1 \\ e_{i-1}, & q+1 < i \le p+1 \end{cases}, \quad C_i = \begin{cases} -\frac{1}{2c}e_{p+1}, & i = q+1 \\ 0, & i \neq q+1 \end{cases} \tag{4.85}$$

It is evident that $A$ is invertible so define $B, D \in \mathfrak{gl}_M$ as $B = 0$ and $D = (A^{-1})^\top$. Also define $a \in \mathbb{F}$ and $u, v, x, y \in \mathbb{F}^M$ as $a = 1$, $u = -e_{q+1}$, $v = x = 0$, and $y = De_{q+1}$. Then $P$ as in Equation 4.49 is in $SO_{2n+1}$ and hence defines an inner automorphism $\tau \in \operatorname{Int}\mathfrak{g}$ by conjugation. Such $\tau$ preserves $\mathfrak{l}$. Additionally, from Equation 4.52 we have that $\tau$ maps $v_{\gamma_{q+1}}$ to $\tilde{w}^-_{r^-+1}$ and $\bigoplus_{k=1}^q V(\gamma_k)$ to itself. By verifying that the first $p+1$ columns of $D$ are contained in $\operatorname{span}\{e_1, \dots, e_{p+1}\}$ we also see that $\tau$ preserves $\bigoplus_{k=1}^{p+1} V(\nu_k)$. Thus we have constructed an inner automorphism that allows us to suppose the $r^- + 1$ copies of $V(\lambda_m)$ in $V^\times$ are precisely $\bigoplus_{k=1}^{p+1} V(\nu_k) \oplus \bigoplus_{k=1}^{q+1} V(\gamma_k)$.

**Case 3: $\tilde{V}_m^\times = \bigoplus_{k=1}^p V(\nu_k) \oplus \bigoplus_{k=1}^q V(\gamma_k)$, Equation 4.79 holds, and there exists $l \in \{q+1, \dots, p\}$ such that $a_{r^-+1,l} \neq 0$**

In Case 1, the only time we used the fact that $\tilde{V}_m^\times$ contains $V(\gamma_{M+1})$ was to add an appropriate scalar multiple of $v_{\gamma_{M+1}}$ to $w^-_{r^-+1}$ to ensure that Equation 4.79 holds. Since this is already true in this case, the argument here is identical to that of Case 1.

**Case 4: $\tilde{V}_m^\times = \bigoplus_{k=1}^p V(\nu_k) \oplus \bigoplus_{k=1}^q V(\gamma_k)$, Equation 4.79 holds, and $a_{r^-+1,q+1} = \cdots = a_{r^-+1,p} = 0$**

This implies there exists $l \in \{p+1, \dots, M\}$ such that $a_{r^-+1,l} \neq 0$ or $b_{r^-+1,l} \neq 0$. Indeed, otherwise $w^-_{r^-+1}$ would be a scalar multiple of $v_{\gamma_{M+1}}$, which is impossible since Equation 4.79 would then not hold. The argument is then identical to that of Case 2.

**Case 5: $\tilde{V}_m^\times = \bigoplus_{k=1}^p V(\nu_k) \oplus \bigoplus_{k=1}^q V(\gamma_k)$ and Equation 4.79 does not hold**

If $w^-_{r^-+1}$ is a scalar multiple of $v_{\gamma_{M+1}}$ we are done. Otherwise define

$$\tilde{w}^- = w^-_{r^-+1} + \kappa v_{\gamma_{M+1}} \tag{4.86}$$

where $\kappa \in \mathbb{F}$ is chosen such that

$$(c_{r^-+1} + \kappa)^2 + 2\sum_{k=1}^{M} a_{r^-+1,k} b_{r^-+1,k} = 0 \tag{4.87}$$

Since $w^-_{r^-+1}$ is not a scalar multiple of $v_{\gamma_{M+1}}$ we have that $\tilde{w}^- \neq 0$ and thus $\tilde{w}^-$ is a maximal vector of weight $\lambda_m$. By Cases 3 and 4 there exists an inner automorphism $\tau \in \operatorname{Int} \mathfrak{g}$ such that $\tau$ preserves $\mathfrak{l}$ and maps $\bigoplus_{k=1}^{p} V(\nu_k)$ to itself, $\bigoplus_{k=1}^{q} V(\gamma_k)$ to itself, and either maps $\tilde{w}^-$ to $v_{\gamma_{q+1}}$ as in Case 3 or $\tilde{w}^-$ to $v_{\nu_{p+1}}$ as in Case 4. Recall from these two cases that mapping to $v_{\gamma_{q+1}}$ was only possible if $q < p$ and mapping to $v_{\nu_{p+1}}$ was only possible if $p < M$. By reviewing the inner automorphisms described in Cases 3 and 4, we can determine where $\tau$ maps $w^-_{r^-+1}$.

If $\tau(\tilde{w}^-) = v_{\gamma_{q+1}}$ with $q < p$ then we observe that

$$\tau(v_{\gamma_{M+1}}) = v_{\gamma_{M+1}} + (c_{r^-+1} + \kappa) v_{\nu_{q+1}} \tag{4.88}$$

It follows that

$$\tau(w^-_{r^-+1}) = \tau(\tilde{w}^-) - \kappa \tau(v_{\gamma_{M+1}}) = v_{\gamma_{q+1}} - \kappa(v_{\gamma_{M+1}} + (c_{r^-+1} + \kappa) v_{\nu_{q+1}}) \tag{4.89}$$

If instead $\tau(\tilde{w}^-) = v_{\nu_{p+1}}$ with $p < M$ then

$$\tau(v_{\gamma_{M+1}}) = v_{\gamma_{M+1}} + (c_{r^-+1} + \kappa) v_{\gamma_{p+1}} \tag{4.90}$$

implying

$$\tau(w^-_{r^-+1}) = \tau(\tilde{w}^-) - \kappa \tau(v_{\gamma_{M+1}}) = v_{\nu_{p+1}} - \kappa(v_{\gamma_{M+1}} + (c_{r^-+1} + \kappa) v_{\gamma_{p+1}}) \tag{4.91}$$

In any event, we conclude from Equation 4.89 and Equation 4.91 that this case is equivalent to having $\tilde{V}_m^\times = \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$ and either

$$w^-_{r^-+1} = c_{r^-+1} v_{\gamma_{M+1}} + a_{r^-+1,q+1} v_{\gamma_{q+1}} + b_{r^-+1,q+1} v_{\nu_{q+1}} \tag{4.92}$$

with $q < p$ or

$$w^-_{r^-+1} = c_{r^-+1} v_{\gamma_{M+1}} + a_{r^-+1,p+1} v_{\gamma_{p+1}} + b_{r^-+1,p+1} v_{\nu_{p+1}} \tag{4.93}$$

with $p < M$. In either case Equation 4.79 does not hold. Also note from Equation 4.89 and Equation 4.91 that $a_{r^-+1,q+1} \neq 0$ if $w^-_{r^-+1}$ is as in Equation 4.92 and $b_{r^-+1,p+1} \neq 0$ if $w^-_{r^-+1}$ is as in Equation 4.93.

If $w^-_{r^-+1}$ is given by Equation 4.92 we have $q < p$ and so $v_{\nu_{q+1}} \in \tilde{V}_m^\times$. Therefore, by adding an appropriate scalar multiple of $v_{\nu_{q+1}}$ to $w^-_{r^-+1}$ we may in fact suppose Equation 4.79 holds. Since $a_{r^-+1,q+1} \neq 0$ the argument is now identical to that of Case 3.

If $w^-_{r^-+1}$ is instead given by Equation 4.93 we have $p < M$ and $b_{r^-+1,p+1} \neq 0$. Since Equation 4.79 does not hold, by rescaling we may suppose without loss of generality that

$$c^2_{r^-+1} + 2a_{r^-+1,p+1}b_{r^+1,p+1} = 1 \tag{4.94}$$

Define $A, B, C, D \in \mathfrak{gl}_M$ as

$$A = I + \left(\frac{(c_{r^-+1} - 1)^2}{2b_{r^-+1,p+1}} - 1\right) E_{p+1,p+1}, \quad B = -\frac{(c_{r^-+1} + 1)^2}{4b_{r^-+1,p+1}} E_{p+1,p+1} \tag{4.95a}$$

$$C = -b_{r^-+1,p+1}E_{p+1,p+1}, \qquad D = I + \left(\frac{b_{r^-+1,p+1}}{2} - 1\right) E_{p+1,p+1} \tag{4.95b}$$

Also define $a \in \mathbb{F}$ and $u, v, x, y \in \mathbb{F}^M$ as $a = c_{r^-+1}$ and

$$u = (c_{r^-+1} - 1)e_{p+1}, \qquad v = -\tfrac{1}{2}(c_{r^-+1} + 1)e_{p+1} \tag{4.96a}$$

$$x = -a_{r^-+1,p+1}e_{p+1}, \qquad y = -b_{r^-+1,p+1}e_{p+1} \tag{4.96b}$$

Due to Equation 4.94 we have that $P'$ as defined in Equation 4.50 is an element of $SO_{2M+1}$, in turn implying $P$ as in Equation 4.49 is in $SO_{2n+1}$. If $\tau \in \operatorname{Int}\mathfrak{g}$ is conjugation by such $P$ then $\tau$ preserves $\mathfrak{l}$ and by Equation 4.52 we also have that $\tau$ maps $v_{\gamma_{M+1}}$ to $w^-_{r^-+1}$, $\bigoplus_{k=1}^{p} V(\nu_k)$ to itself, and $\bigoplus_{k=1}^{q} V(\gamma_k)$ to itself. Thus up to inner automorphism we have that the $r^- + 1$ copies of $V(\lambda_m)$ are given by $V(\gamma_{M+1}) \oplus \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$. This concludes this inductive portion of the proof.

In summary, we have shown that if $V^\times$ contains a $V(\lambda_2)$ component or a $V(\lambda_{m-2})$ component then it must contain $V(\eta)$ and not $V(\zeta)$. We have also shown that the $r^-$ copies of $V(\lambda_m)$ in $V^\times$, which we shall denote by $V^\times_m$, are either $V(\gamma_{M+1}) \oplus \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$ with $p + q + 1 = r^-$ or $\bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$ with $p + q = r^-$ for some $p, q \in [M]_0$. To complete the proof, we must turn our attention to the $r^+$ copies of $V(\lambda_1)$ in $V^\times$. We again proceed by induction, this time on $r^+$. Explicitly, we shall show that up to inner automorphism there exist $c, d \in [M]_0$ with $c + d = r^+$, $c \leq M - p$, and $d \leq p - q$ such that the $r^+$ copies of $V(\lambda_1)$ in $V^\times$ are $\bigoplus_{k=M-c+1}^{M} V(\beta_k) \oplus \bigoplus_{k=p-d+1}^{p} V(\beta_k)$, where $c = 0$ if $V(\eta) \subseteq V^\times$ or if $q > 0$, provided that $V^\times_m$ is as described above.

The base case of $r^+ = 0$ is immediate. Thus suppose the assertion holds when $\operatorname{Rad}\mathfrak{a}$ contains $r^+$ copies of $V(\lambda_1)$ and now suppose $\operatorname{Rad}\mathfrak{a}$ contains $r^+ + 1$ such copies. Let $w^+_1, \ldots, w^+_{r^++1} \in V^\times$ be maximal vectors of these

$r^+ + 1$ copies. Then for each $i \in [r^+ + 1]$ there exist $a_{i,1}, \ldots, a_{i,M}, b_{i,1}, \ldots, b_{i,M}, c_i \in \mathbb{F}$ such that

$$w_i^+ = c_i v_{\beta_{M+1}} + \sum_{k=1}^{M} (a_{i,k} v_{\beta_k} + b_{i,k} v_{\mu_k}) \tag{4.97}$$

where $v_{\beta_{M+1}}$, $v_{\beta_k}$, and $v_{\mu_k}$ are given by Equation 4.51. We make a few observations.

Firstly, for each $i, j \in [r^+ + 1]$ note that having $w_i^+, w_j^+ \in V^\times$ and $Y_1 \in \mathfrak{l}$ implies $[w_i^+, [Y_1, w_j^+]] \in \operatorname{Rad} \mathfrak{a}$. Observe that

$$[w_i^+, [Y_1, w_j^+]] = (-1)^n \left( c_i c_j + \sum_{k=1}^{M} (a_{i,k} b_{j,k} + a_{j,k} b_{i,k}) \right) X_{1,\ldots,n,n,n-1,\ldots,2} \tag{4.98}$$

Since $X_{1,\ldots,n,n,n-1,\ldots,2} \in V(\zeta)$ and we previously argued that we can ensure that $V(\zeta) \not\subseteq V^\times$, we conclude that for all $i, j \in [r^+ + 1]$ we must have

$$0 = c_i c_j + \sum_{k=1}^{M} (a_{i,k} b_{j,k} + a_{j,k} b_{i,k}) \tag{4.99}$$

Secondly, for each $i \in [r^+ + 1]$ and $j \in [q]$ we have $w_i^+, v_{\gamma_j} \in V^\times$ and thus $[w_i^+, v_{\gamma_j}] \in \operatorname{Rad} \mathfrak{a}$. As

$$[w_i^+, v_{\gamma_j}] = a_{i,j} X_{1,\ldots,m} \tag{4.100}$$

and $X_{1,\ldots,m} \in \mathfrak{l}$ it must be that $a_{i,j} = 0$ for all $i \in [r^+ + 1]$ and $j \in [q]$. By similarly considering $[w_i^+, v_{\nu_j}]$ for $j \in [p]$ and $[w_i^+, v_{\gamma_{M+1}}]$ we conclude that $b_{i,j} = 0$ for all $i \in [r^+ + 1]$ and $j \in [p]$ and that $c_i = 0$ for all $i \in [r^+ + 1]$ if $V(\gamma_{M+1}) \subseteq V^\times$.

Lastly, note that for each $i \in [r^+ + 1]$ we have

$$[[Y_{1,\ldots,m}, w_i^+], v_\eta] = (-1)^{m-1} \left( c_i v_{\gamma_{M+1}} + \sum_{k=1}^{M} (a_{i,k} v_{\nu_k} + b_{i,k} v_{\gamma_k}) \right) \tag{4.101}$$

If $V(\eta) \subseteq V^\times$ then for each $i \in [r^+ + 1]$ we must have $[[Y_{1,\ldots,m}, w_i^+], v_\eta] \in \operatorname{Rad} \mathfrak{a}$. Since either $V_m^\times = V(\gamma_{M+1}) \oplus \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$ or $V_m^\times = \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$ it must be that for all $i \in [r^+ + 1]$, if $V(\eta) \subseteq V^\times$ then $a_{i,p+1} = \cdots = a_{i,M} = 0$ if $p < M$ and $b_{i,q+1} = \cdots = b_{i,M} = 0$ if $q < M$. Putting everything together we see that for all $i \in [r^+ + 1]$, $a_{i,k} = 0$ for $k \in [q]$ and $b_{i,l} = 0$ for $l \in [p]$. If $V(\eta) \subseteq V^\times$ we also have that $a_{i,k} = 0$ for $p < k \leq M$, $b_{i,l} = 0$ for $q < l \leq M$, and $c_i = 0$.

Let $\tilde{V}_1^\times$ be the $\mathfrak{l}$-module generated by $w_1^+, \dots, w_{r^+}^+$. As usual, by noting that $[\tilde{V}_1^\times, \tilde{V}_1^\times] \cap \left(\bigoplus_{k=1}^{M+1} V(\beta_k) \oplus \bigoplus_{k=1}^{M} V(\nu_k)\right) \subseteq \tilde{V}_1^\times$ we have that the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{l}$, $V_m^\times$, and $\tilde{V}_1^\times$ is Levi decomposable with Levi factor $\mathfrak{l}$ such that the copies of $V(\lambda_m)$ in the nontrivial component of the radical are given by $V_m^\times$ and the copies of $V(\lambda_1)$ are given by $\tilde{V}_1^\times$. Therefore, by the inductive hypothesis we may suppose without loss of generality that there exist $c, d \in [M]_0$ with $c + d = r^+$, $c \le M - p$, and $d \le p - q$ such that

$$\tilde{V}_1^\times = \bigoplus_{k=M-c+1}^{M} V(\beta_k) \oplus \bigoplus_{k=p-d+1}^{p} V(\beta_k) \tag{4.102}$$

with $c = 0$ if $V(\eta) \subseteq V^\times$ or $q > 0$. Hence we may further suppose without loss of generality that $a_{r^++1,M-c+1} = \cdots = a_{r^++1,M} = 0$ if $c > 0$ and $a_{r^++1,p-d+1} = \cdots = a_{r^++1,p} = 0$ if $d > 0$ by linear independence of $\{w_1^+, \dots, w_{r^++1}^+\}$. We proceed by considering various cases.

**Case (i): $V_m^\times = V(\gamma_{M+1}) \oplus \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$**

In this case $c = 0$ by Lemma 4.30. The comments following Equation 4.101, combined with linear independence of $\{w_1^+, \dots, w_{r^++1}^+\}$ and the fact that $q \le p$, imply we can take

$$w_{r^++1}^+ = \sum_{k=q+1}^{p-d} a_{r^++1,k} v_{\beta_k} \tag{4.103}$$

Hence there exists $l \in \{q+1, \dots, p-d\}$ such that $a_{r^++1,l} \neq 0$. Define $D \in \mathfrak{gl}_M$ such that for each $i \in [M]$ the $i$'th column of $D$ is $D_i$, where

$$D_i = \begin{cases} e_i, & i < l \text{ or } i > p - d \\ e_{i+1}, & l \le i < p - d \\ \sum_{k=q+1}^{p-d} a_{r^++1,k} e_k, & i = p - d \end{cases} \tag{4.104}$$

It is evident that $D$ is invertible. Define $A, B, C \in \mathfrak{gl}_M$, $a \in \mathbb{F}$, and $u, v, x, y \in \mathbb{F}^M$ as $A = (D^{-1})^\top$, $B = C = 0$, $a = 1$, and $u = v = x = y = 0$. Then $P$ as in Equation 4.49 is an element of $SO_{2n+1}$ defining an inner automorphism $\tau \in \operatorname{Int} \mathfrak{g}$ which preserves $\mathfrak{l}$. From Equation 4.52 we have that $\tau$ maps $v_{\beta_{p-d}}$ to $w_{r^++1}^+$, $\bigoplus_{k=p-d+1}^{p} V(\beta_k)$ to itself, $V(\gamma_{M+1})$ to itself, and $\bigoplus_{k=1}^{p} V(\nu_k)$ to itself. To see that $\tau$ preserves $\bigoplus_{k=1}^{q} V(\gamma_k)$ one need only verify that the first $q$ columns of $A$ are contained in $\operatorname{span}\{e_1, \dots, e_q\}$. This is immediate from noting that $A^\top D = I$. Therefore, in this case we have that up to inner automorphism the $r^+ + 1$ copies of $V(\lambda_1)$ in $V^\times$ are $\bigoplus_{k=p-d}^{p} V(\beta_k)$.

**Case (ii): $V_m^\times = \bigoplus_{k=1}^p V(\nu_k) \oplus \bigoplus_{k=1}^q V(\gamma_k)$ and there exists $l \in \{p+1, \dots, M-c\}$ such that $a_{r^++1,l} \neq 0$ or $b_{r^++1,l} \neq 0$**

Note that this requires $q = 0$ and $V(\eta) \not\subseteq V^\times$. Indeed, if $q \neq 0$ then $p \neq 0$ since $q \leq p$. Lemma 4.30 then implies $V(\eta) \subseteq V^\times$, which in turn forces $w_{r^++1}^+$ to be given by Equation 4.103. As such $q = 0$. Therefore, by Lemma 4.25 we may suppose $a_{r^++1,l} \neq 0$. Define $B, D \in \mathfrak{gl}_M$ such that the $i$'th column of $B$ and $D$ is $B_i$ and $D_i$, respectively, for each $i \in [M]$, where

$$B_i = \begin{cases} -\frac{b_{r^++1,i}}{a_{r^++1,l}} e_l, & i < l \text{ or } i > M-c \\ -\frac{b_{r^++1,i+1}}{a_{r^++1,l}} e_l, & l \leq i < M-c \\ \sum_{k=1}^M b_{r^++1,k} e_k, & i = M-c \end{cases} \tag{4.105a}$$

$$D_i = \begin{cases} e_i, & i < l \text{ or } i > M-c \\ e_{i+1}, & l \leq i < M-c \\ \sum_{k=1}^M a_{r^++1,k} e_k, & i = M-c \end{cases} \tag{4.105b}$$

Linear independence of $\{D_1, \dots, D_M\}$ ensures $D$ is invertible. Define $P$ as in Equation 4.49 with $A = (D^{-1})^\top$, $C = 0$, $a = 1$, $u = y = 0$, $v = -c_{r^++1} e_{M-c}$, and $x = c_{r^++1}(D^{-1})^\top e_{M-c}$. Since Equation 4.99 holds in particular when $i = j = r^+ + 1$ one can verify that $P$ is an element of $SO_{2n+1}$. Consequently, $P$ defines via conjugation an inner automorphism $\tau \in \operatorname{Int} \mathfrak{g}$ preserving $\mathfrak{l}$. Such $\tau$ maps $v_{\beta_{M-c}}$ to $w_{r^++1}$, $\bigoplus_{k=M-c+1}^M V(\beta_k)$ to itself, $\bigoplus_{k=p-d+1}^p V(\beta_k)$ to itself, and $\bigoplus_{k=1}^p V(\nu_k)$ to itself. Hence the $r^+ + 1$ copies of $V(\lambda_1)$ in $V^\times$ are given by $\bigoplus_{k=M-c}^M V(\beta_k) \oplus \bigoplus_{k=p-d+1}^p V(\beta_k)$ in this case.

**Case (iii): $V_m^\times = \bigoplus_{k=1}^p V(\nu_k) \oplus \bigoplus_{k=1}^q V(\gamma_k)$ and $a_{r^++1,l} = b_{r^++1,l} = 0$ for all $l \in \{p+1, \dots, M-c\}$**

In this case we have $w_{r^++1} = c_{r^++1} v_{\beta_{M+1}} + \sum_{k=q+1}^p a_{r^++1,k} v_{\beta_k}$ by the remarks following Equation 4.101. Taking $i = j = r^+ + 1$ in Equation 4.99 implies $c_{r^++1} = 0$. The inner automorphism constructed in Case (i) will then suffice.

With this we complete this second inductive portion and thus complete the proof. □

As in Section 4.3 and Section 4.4, all inner automorphisms in the proof of Lemma 4.32 can be chosen to preserve $\mathfrak{h}$ when the maximal vectors are selected to be root space elements. To complete the proof of Theorem 4.19 in this setting, we need only show that no additional inner automorphisms beyond those considered in the proof of Lemma 4.32 are necessary. We show this by proving that the descriptions for $V^\times$ in Lemma 4.32 are pairwise non-conjugate.

**Lemma 4.33.** *Let $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$ be Levi decomposable subalgebras of $\mathfrak{g}$ with common Levi factor $\mathfrak{l}$, where $(t(\mathfrak{g}), t(\mathfrak{l})) = (B_n, A_m)$. By Lemma 4.32, up to inner automorphism one may suppose that the nontrivial components $V^\times$ and $\tilde{V}^\times$ of* $\operatorname{Rad}\mathfrak{a}$ *and* $\operatorname{Rad}\tilde{\mathfrak{a}}$*, respectively, are as in Equation 4.77, with $p, q, c, d \in [M]_0$ corresponding to $V^\times$ and $\tilde{p}, \tilde{q}, \tilde{c}, \tilde{d} \in [M]_0$ corresponding to $\tilde{V}^\times$. If there exists $\tau \in \operatorname{Int}\mathfrak{g}$ preserving $\mathfrak{l}$ and mapping $V^\times$ to $\tilde{V}^\times$ then $(p, q, c, d) = (\tilde{p}, \tilde{q}, \tilde{c}, \tilde{d})$ and $V^\times$ and $\tilde{V}^\times$ are both given by Equation 4.77a, Equation 4.77b, or Equation 4.77c.*

*Proof.* Regard $\tau$ as conjugation by some $P \in SO_{2n+1}$. Since $\tau$ preserves $\mathfrak{l}$, one can show via Equation 3.12 and basic matrix computations that

$$P = \begin{pmatrix} a & 0_{1,m+1} & u^\top & 0_{1,m+1} & v^\top \\ 0_{m+1,1} & \tilde{A} & 0_{m+1,M} & \tilde{B} & 0_{m+1,M} \\ x & 0_{M,m+1} & A & 0_{M,m+1} & B \\ 0_{m+1,1} & \tilde{C} & 0_{m+1,M} & \tilde{D} & 0_{m+1,M} \\ y & 0_{M,m+1} & C & 0_{M,m+1} & D \end{pmatrix} \tag{4.106}$$

for some $a \in \mathbb{F}$, $u, v, x, y \in \mathbb{F}^M$, $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \in \mathfrak{gl}_{m+1}$, and $A, B, C, D \in \mathfrak{gl}_M$ such that $P_\mathfrak{l} \in SO_{2(m+1)}$ and $P_{\mathfrak{l}^\perp} \in SO_{2M+1}$, where

$$P_\mathfrak{l} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}, \qquad P_{\mathfrak{l}^\perp} = \begin{pmatrix} a & u^\top & v^\top \\ x & A & B \\ y & C & D \end{pmatrix} \tag{4.107}$$

For all $R \in \mathfrak{sl}_{m+1}$,

$$\begin{aligned} &\tau\left(\begin{pmatrix} 0 & 0_{1,m+1} & 0_{1,M} & 0_{1,m+1} & 0_{1,M} \\ 0_{m+1,1} & R & 0_{m+1,M} & 0_{m+1,m+1} & 0_{m+1,M} \\ 0_{M,1} & 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \\ 0_{m+1,1} & 0_{m+1,m+1} & 0_{m+1,M} & -R^\top & 0_{m+1,M} \\ 0_{M,1} & 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \end{pmatrix}\right) \\ &= \begin{pmatrix} 0 & 0_{1,m+1} & 0_{1,M} & 0_{1,m+1} & 0_{1,M} \\ 0_{m+1,1} & \tilde{A}R\tilde{D}^\top - \tilde{B}R^\top\tilde{C}^\top & 0_{m+1,M} & \tilde{A}R\tilde{B}^\top - \tilde{B}R^\top\tilde{A}^\top & 0_{m+1,M} \\ 0_{M,1} & 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \\ 0_{m+1,1} & \tilde{C}R\tilde{D}^\top - \tilde{D}R^\top\tilde{C}^\top & 0_{m+1,M} & \tilde{C}R\tilde{B}^\top - \tilde{D}R^\top\tilde{A}^\top & 0_{m+1,M} \\ 0_{M,1} & 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \end{pmatrix} \end{aligned} \tag{4.108}$$

By Equation 3.12 we have $\tilde{A}R\tilde{B}^\top - \tilde{B}R^\top\tilde{A}^\top = 0$ and $\tilde{C}R\tilde{D}^\top - \tilde{D}R^\top\tilde{C}^\top = 0$ for all $R \in \mathfrak{sl}_{m+1}$. Since having $P_\mathfrak{l} \in SO_{2(m+1)}$ forces $\tilde{A}^\top\tilde{D} + \tilde{C}^\top\tilde{B} = I$, we conclude by Lemma 4.27 that either $\tilde{D} = 0$ or $\tilde{B} = 0$.

Define $U \subseteq V^\times$ and $\tilde{U} \subseteq \tilde{V}^\times$ to be the subspaces spanned by the maximal vectors of $V^\times$ and $\tilde{V}^\times$, respectively, of highest weight $\lambda_m$. We consider cases depending on $V^\times$.

**Case 1: $V^\times$ is as in Equation 4.77a**

We first show that $\tilde{V}^\times$ must also be described by Equation 4.77a. Firstly, note that $V(\eta) \subseteq V^\times$ must be mapped to $V(\eta)$ or $V(\zeta)$ by dimension considerations since $\tau$ preserves $\mathfrak{l}$. As such, $\tilde{V}^\times$ is not given by Equation 4.77b. Suppose by contradiction that $\tilde{V}^\times$ is instead given by Equation 4.77c. Then $V(\eta)$ is mapped to $V(\eta)$. By Equation 4.51h, appropriate matrix computations reveal that if $\tilde{D} = 0$ then this is impossible. Consequently, $\tilde{D} \neq 0$ and so $\tilde{B} = 0$. Since $\tilde{A}^\top \tilde{D} + \tilde{C}^\top \tilde{B} = I$ we conclude that $\tilde{D}^\top = \tilde{A}^{-1}$. Equation 4.108 then implies $\tau|_\mathfrak{l} \in \operatorname{Int} \mathfrak{l}$ and so by Corollary 2.4 we may suppose without loss of generality that $\tau|_\mathfrak{l} = \mathrm{id}$. It follows that $\tau$ maps weight vectors to other weight vectors of the same weight. As such, $\tau$ maps $\bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$ to $V(\gamma_{M+1}) \oplus \bigoplus_{k=1}^{\tilde{p}} V(\nu_k) \oplus \bigoplus_{k=1}^{\tilde{q}} V(\gamma_k)$ and $\bigoplus_{k=p-d+1}^{p} V(\beta_k)$ to $\bigoplus_{k=\tilde{p}-\tilde{d}+1}^{\tilde{p}} V(\beta_k)$. Dimension considerations in turn imply $p + q = \tilde{p} + \tilde{q} + 1$ and $d = \tilde{d}$.

Since $\tau|_\mathfrak{l} = \mathrm{id}$ we conclude that $\tau$ maps $U$ to $\tilde{U}$. Note that there exists a subspace $U'$ of $U$ such that $\dim U' = p$ and for all $u, u' \in U'$ we have $[u, [Y_m, u']] = 0$. Indeed, $U' = \operatorname{span}\{v_{\nu_1}, \dots, v_{\nu_p}\}$. Since $\tau|_\mathfrak{l} = \mathrm{id}$ the subspace $\tilde{U}' = \tau(U')$ of $\tilde{U}$ must also satisfy the condition that $[\tilde{u}, [Y_m, [\tilde{u}']] = 0$ for all $\tilde{u}, \tilde{u}' \in \tilde{U}'$. Lemma 4.28 implies $p \leq \tilde{p}$. Conversely, there exists a $\tilde{p}$-dimensional subspace $\tilde{U}''$ of $\tilde{U}$ such that for all $\tilde{u}, \tilde{u}' \in \tilde{U}''$ we have $[\tilde{u}, [Y_m, \tilde{u}']] = 0$, namely $\tilde{U}'' = \operatorname{span}\{v_{\nu_1}, \dots, v_{\nu_{\tilde{p}}}\}$. By consideration of $\tau^{-1}(\tilde{U}'')$ we conclude by Lemma 4.28 that $\tilde{p} \leq p$. Hence $p = \tilde{p}$, implying $q = \tilde{q} + 1$. Note this implies $q > 0$.

To obtain our desired contradiction, let $W$ and $\tilde{W}$ be the subalgebras of $\mathfrak{g}$ generated by the irreducible $\mathfrak{l}$-submodules of $V^\times$ and $\tilde{V}^\times$, respectively, of highest weight $\lambda_m$. In other words, $W$ is the subalgebra of $\mathfrak{g}$ generated by $\bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$ and $\tilde{W}$ is the subalgebra of $\mathfrak{g}$ generated by $V(\gamma_{M+1}) \oplus \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q-1} V(\gamma_k)$. If $\tau$ maps $V^\times$ to $\tilde{V}^\times$ and $\tau|_\mathfrak{l} = \mathrm{id}$ then $\tau$ maps $W$ to $\tilde{W}$. Hence $\tau$ maps $N_\mathfrak{g}(W)$ to $N_\mathfrak{g}(\tilde{W})$, where $N_\mathfrak{g}(W)$ is the normalizer of $W$ in $\mathfrak{g}$. We shall show this is impossible by proving $\dim N_\mathfrak{g}(W) \neq \dim N_\mathfrak{g}(\tilde{W})$.

As $\bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$ and $V(\gamma_{M+1}) \oplus \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q-1} V(\gamma_k)$ are regular relative to $\mathfrak{h}$ the Jacobi identity implies $N_\mathfrak{g}(W)$ and $N_\mathfrak{g}(\tilde{W})$ are also regular relative to $\mathfrak{h}$. Due to the explicit description of $\Phi$ in Equation 2.9 we can easily determine both $N_\mathfrak{g}(W)$ and $N_\mathfrak{g}(\tilde{W})$. We find

$$N_\mathfrak{g}(W) = \mathfrak{g} \oplus V(\eta) \oplus \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k) \oplus \mathfrak{h}_{\mathfrak{l}^\perp} \oplus \tilde{\mathfrak{h}}_\mathfrak{l} \oplus \bigoplus_{\alpha \in \Phi_W} \mathfrak{g}_\alpha \quad (4.109a)$$

$$N_{\mathfrak{g}}(\tilde{W}) = \mathfrak{g} \oplus V(\eta) \oplus V(\gamma_{M+1}) \oplus \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q-1} V(\gamma_k) \oplus \mathfrak{h}_{\mathfrak{l}^\perp} \oplus \tilde{\mathfrak{h}}_{\mathfrak{l}} \oplus \bigoplus_{\alpha \in \Phi_{\tilde{W}}} \mathfrak{g}_\alpha \tag{4.109b}$$

where

$$\begin{aligned}\Phi_W = &\{\varepsilon_{i+m+1} - \varepsilon_{j+m+1} \in \Phi \colon p < i < j \leq M \text{ or } 1 \leq i < j \leq p\} \\ &\cup \{-\varepsilon_{i+m+1} + \varepsilon_{j+m+1} \in \Phi \colon q < i < j \leq M \text{ or } 1 \leq i < j \leq q\} \\ &\cup \{\varepsilon_{i+m+1} + \varepsilon_{j+m+1} \in \Phi \colon p < i < j \leq M \text{ or } 1 \leq i < j \leq q\} \\ &\cup \{-\varepsilon_{i+m+1} - \varepsilon_{j+m+1} \in \Phi \colon q < i < j \leq M \text{ or } (1 \leq i < j \leq p \text{ and } \\ &i \in [q])\} \cup \{\varepsilon_{i+m+1} \in \Phi \colon p < i \leq M\} \cup \{-\varepsilon_{i+m+1} \in \Phi \colon q < i \leq M\}\end{aligned} \tag{4.110a}$$

$$\begin{aligned}\Phi_{\tilde{W}} = &\{\varepsilon_{i+m+1} - \varepsilon_{j+m+1} \in \Phi \colon p < i < j \leq M \text{ or } 1 \leq i < j \leq p\} \\ &\cup \{-\varepsilon_{i+m+1} + \varepsilon_{j+m+1} \in \Phi \colon q \leq i < j \leq M \text{ or } 1 \leq i < j < q\} \\ &\cup \{\varepsilon_{i+m+1} + \varepsilon_{j+m+1} \in \Phi \colon p < i < j \leq M \text{ or } 1 \leq i < j < q\} \\ &\cup \{-\varepsilon_{i+m+1} - \varepsilon_{j+m+1} \in \Phi \colon q \leq i < j \leq M \text{ or } (1 \leq i < j \leq p \text{ and } \\ &i \in [q-1]\} \cup \{\varepsilon_{i+m+1} \in \Phi \colon 1 \leq i < q\} \cup \{-\varepsilon_{i+m+1} \in \Phi \colon 1 \leq i \leq p\}\end{aligned} \tag{4.110b}$$

If $\dim N_{\mathfrak{g}}(W) = \dim N_{\mathfrak{g}}(\tilde{W})$ then $|\Phi_W| = |\Phi_{\tilde{W}}|$. We have

$$|\Phi_W| = 2M^2 - 2(p+q)M + \tfrac{3}{2}p^2 + \tfrac{3}{2}q^2 - \tfrac{1}{2}p - \tfrac{3}{2}q + pq \tag{4.111a}$$
$$|\Phi_{\tilde{W}}| = 2M^2 - 2(p+q)M + \tfrac{3}{2}p^2 + \tfrac{3}{2}q^2 + \tfrac{1}{2}p - \tfrac{5}{2}q + pq + 1 \tag{4.111b}$$

Consequently, if $|\Phi_W| = |\Phi_{\tilde{W}}|$ then $p = q - 1$, which is a contradiction since $p \geq q$. Therefore, $\tilde{V}^\times$ must not be given by Equation 4.77c.

By the above contradiction it must be that Equation 4.77a describes $\tilde{V}^\times$. As before we note that $\tau$ must preserve $V(\eta)$. Since this is impossible when $\tilde{D} = 0$ it must be that $\tilde{B} = 0$. Hence $\tau|_{\mathfrak{l}} = \mathrm{id}$. It follows that $\tau$ maps $\bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$ to $\bigoplus_{k=1}^{\tilde{p}} V(\nu_k) \oplus \bigoplus_{k=1}^{\tilde{q}} V(\gamma_k)$ and $\bigoplus_{k=p-d+1}^{p} V(\beta_k)$ to $\bigoplus_{k=\tilde{p}-\tilde{d}+1}^{\tilde{p}} V(\beta_k)$. Dimension considerations then imply $p + q = \tilde{p} + \tilde{q}$ and $d = \tilde{d}$. The existence of $U' = \mathrm{span}\{v_{\nu_1}, \ldots, v_{\nu_p}\} \subseteq V^\times$ implies $p \leq \tilde{p}$ by Lemma 4.28 since $\tau|_{\mathfrak{l}} = \mathrm{id}$. In a similar manner, considering $\tilde{U}'' = \mathrm{span}\{v_{\nu_1}, \ldots, v_{\nu_{\tilde{p}}}\} \subseteq \tilde{V}^\times$ implies $\tilde{p} \leq p$ by Lemma 4.28. Hence $p = \tilde{p}$, implying $q = \tilde{q}$, as desired.

**Case 2: $V^\times$ is as in Equation 4.77b**

By Case 1 $\tilde{V}^\times$ is not given by Equation 4.77a. Moreover, as $V(\eta), V(\zeta) \not\subseteq V^\times$ it must be that Equation 4.77c does not describe $\tilde{V}^\times$ and so $\tilde{V}^\times$ is given by Equation 4.77b. Dimension considerations imply $p + c + d = \tilde{p} + \tilde{c} + \tilde{d}$.

Suppose by contradiction that $p \neq \tilde{p}$. Then without loss of generality we may suppose $p > \tilde{p}$ and in particular $p \geq 1$. If $\tilde{D} = 0$ then $\tau$ maps $\bigoplus_{k=1}^{p} V(\nu_k)$ to $\bigoplus_{k=M-\tilde{c}+1}^{M} V(\beta_k) \oplus \bigoplus_{k=\tilde{p}-\tilde{d}+1}^{\tilde{p}} V(\beta_k)$, implying $p = \tilde{c} + \tilde{d}$ by dimension arguments. This however is a contradiction since $p > \tilde{p} \geq \tilde{c} + \tilde{d}$. Hence $\tilde{D} \neq 0$ and so $\tilde{B} = 0$. However, this also leads to a contradiction as this forces $\tau$ to map $\bigoplus_{k=1}^{p} V(\nu_k)$ to $\bigoplus_{k=1}^{\tilde{p}} V(\nu_k)$, requiring $p = \tilde{p}$. Hence we conclude by contradiction that $p = \tilde{p}$ and so $c + d = \tilde{c} + \tilde{d}$.

Let $W$ and $\tilde{W}$ be the subalgebras of $\mathfrak{g}$ generated by $\mathfrak{l} \oplus V^\times$ and $\mathfrak{l} \oplus \tilde{V}^\times$, respectively. If $\tau$ maps $V^\times$ to $\tilde{V}^\times$ and preserves $\mathfrak{l}$ then it must also map $W$ to $\tilde{W}$. As both $\mathfrak{l} \oplus V^\times$ and $\mathfrak{l} \oplus \tilde{V}^\times$ are regular relative to $\mathfrak{h}$ we have that $W$ and $\tilde{W}$ are also regular relative to $\mathfrak{h}$ by the Jacobi identity. Direct computation reveals that

$$W = \mathfrak{l} \oplus V^\times \oplus \bigoplus_{\alpha \in \Phi_W} \mathfrak{g}_\alpha, \qquad \tilde{W} = \mathfrak{l} \oplus \tilde{V}^\times \oplus \bigoplus_{\alpha \in \Phi_{\tilde{W}}} \mathfrak{g}_\alpha \tag{4.112}$$

where

$$\Phi_W = \{-\varepsilon_{m+i+1} - \varepsilon_{m+j+1} \in \Phi \colon i \in \{p-d+1, \ldots, p\} \cup \{M-c+1, \ldots, M\}, j \in [p], i \neq j\} \tag{4.113a}$$

$$\Phi_{\tilde{W}} = \{-\varepsilon_{m+i+1} - \varepsilon_{m+j+1} \in \Phi \colon i \in \{p-\tilde{d}+1, \ldots, p\} \cup \{M-\tilde{c}+1, \ldots, M\}, j \in [p], i \neq j\} \tag{4.113b}$$

Since $\tau$ mapping $W$ to $\tilde{W}$ requires $\dim W = \dim \tilde{W}$ we have $|\Phi_W| = |\Phi_{\tilde{W}}|$. Observe that

$$|\Phi_W| = p(c+d) - \tfrac{1}{2}d^2 - \tfrac{1}{2}d, \qquad |\Phi_{\tilde{W}}| = p(\tilde{c}+\tilde{d}) - \tfrac{1}{2}\tilde{d}^2 - \tfrac{1}{2}\tilde{d} \tag{4.114}$$

Therefore, as $|\Phi_W| = |\Phi_{\tilde{W}}|$ and $c + d = \tilde{c} + \tilde{d}$ we conclude that

$$0 = \tfrac{1}{2}(d^2 + d - \tilde{d}^2 - \tilde{d}) = \tfrac{1}{2}(d - \tilde{d})(d + \tilde{d} + 1) \tag{4.115}$$

from which it follows that $d = \tilde{d}$. Hence $c = \tilde{c}$, as desired.

**Case 3: $V^\times$ is as in Equation 4.77c**

By Cases 1 and 2 we have that $\tilde{V}^\times$ must also be described by Equation 4.77c. Since $V(\eta) \subseteq V^\times, \tilde{V}^\times$ we must have that $\tilde{D} \neq 0$ and so $\tilde{B} = 0$. As such, $\tau$ maps $V(\gamma_{M+1}) \oplus \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$ to $V(\gamma_{M+1}) \oplus \bigoplus_{k=1}^{\tilde{p}} V(\nu_k) \oplus \bigoplus_{k=1}^{\tilde{q}} V(\gamma_k)$ and $\bigoplus_{k=p-d+1}^{p} V(\beta_k)$ to $\bigoplus_{k=\tilde{p}-\tilde{d}+1}^{\tilde{p}} V(\beta_k)$. Dimension arguments then imply $p + q = \tilde{p} + \tilde{q}$ and $d = \tilde{d}$.

Consideration of $U' = \operatorname{span}\{v_{\nu_1}, \ldots, v_{\nu_p}\} \subseteq V^\times$ implies $p \leq \tilde{p}$ by Lemma 4.28. Similarly, existence of $\tilde{U}'' = \operatorname{span}\{v_{\nu_1}, \ldots, v_{\nu_{\tilde{p}}}\} \subseteq \tilde{V}^\times$ implies $\tilde{p} \leq p$, again by Lemma 4.28. Hence $p = \tilde{p}$ and so $q = \tilde{q}$, as desired. □

As in Section 4.3 and Section 4.4, the proof of Theorem 4.19 in the case of $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (B_n, A_m)$ follows from Lemma 4.32 and Lemma 4.33.

## 4.7 PROOF OF THEOREM 4.19 WHEN $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (C_n, A_m)$

As we are only concerned with well-mannered pairs in this chapter the only case left to consider is when $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (C_n, A_m)$ with $m > 1$. The proof of Theorem 4.19 in this final setting proceeds in nearly the same manner as in Section 4.6, albeit with a somewhat simpler argument. Indeed, part of the tedium in the proof of Lemma 4.32 was due to the existence of roots $\gamma_{M+1}$ and $\beta_{M+1}$ in $\Phi_{\mathfrak{l}}^{\times}$. These roots were an artifact of the odd parity of $\mathfrak{so}_{2n+1}$. In contrast, the even parity of $\mathfrak{sp}_{2n}$ ensures a lack of these roots, allowing for a cleaner argument. While $\mathfrak{so}_{2n}$ is also an even-dimensional representation, we shall see in Chapter 5 that $\gamma_{M+1}$ and $\beta_{M+1}$ are needed in an orthogonal setting, which is why $(\mathfrak{g}, \mathfrak{l})$ is ill-mannered when $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (D_n, A_m)$. For the moment however we are getting ahead of ourselves and we should first establish the veracity of Theorem 4.19 in the final remaining case of $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (C_n, A_m)$. This proof will be an immediate consequence of this next two lemmas.

**Lemma 4.34.** *Suppose $(t(\mathfrak{g}), t(\mathfrak{l})) = (C_n, A_m)$. Then up to inner automorphism there exist $p, q, c, d \in [M]_0$ with $q \leq p$, $d \leq p - q$, and $c \leq M - p$ such that $V^{\times}$ is one of the following:*

$$V^{\times} = V(\eta) \oplus \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k) \oplus \bigoplus_{k=p-d+1}^{p} V(\beta_k) \tag{4.116a}$$

$$\text{or } V^{\times} = \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=M-c+1}^{M} V(\beta_k) \oplus \bigoplus_{k=p-d+1}^{p} V(\beta_k) \tag{4.116b}$$

*Moreover, if $V^{\times}$ is as in Equation 4.116b then we may further suppose $c + d \leq p$.*

*Proof.* By Lemma 4.9 and Lemma 4.10 we have that if $V^{\times} \neq \{0\}$ then $V^{\times}$ is an $\mathfrak{l}$-submodule of $\bigoplus_{k=1}^{M} V(\beta_k) \oplus \bigoplus_{k=1}^{M} V(\gamma_k) \oplus \bigoplus_{k=1}^{M} V(\mu_k) \oplus \bigoplus_{k=1}^{M} V(\nu_k) \oplus V(\zeta) \oplus V(\eta)$. For each $k \in [M]$, one can verify that $\beta_k|_{\mathfrak{h}_{\mathfrak{l}}} = \mu_k|_{\mathfrak{h}_{\mathfrak{l}}} = \lambda_1$, $\gamma_k|_{\mathfrak{h}_{\mathfrak{l}}} = \nu_k|_{\mathfrak{h}_{\mathfrak{l}}} = \lambda_m$, $\zeta|_{\mathfrak{h}_{\mathfrak{l}}} = 2\lambda_1$, and $\eta|_{\mathfrak{h}_{\mathfrak{l}}} = 2\lambda_m$.

Having $(\mathfrak{g}, \mathfrak{l})$ well-mannered implies $m > 1$. Consequently, there exist $r^+, r^- \in [2M]_0$ and $s^+, s^- \in \{0, 1\}$ such that $V^{\times}$ contains $r^+$ copies of $V(\lambda_1)$, $r^-$ copies of $V(\lambda_m)$, $s^+$ copies of $V(2\lambda_1)$, and $s^-$ copies of $V(2\lambda_m)$. By the same reasoning as in the proof of Lemma 4.32 we may suppose without loss of generality that $s^+ = 0$ and if $s^- = 0$ as well then $r^+ \leq r^-$. This allows us to suppose $c + d \leq p$ in Equation 4.116b, provided that Equation 4.116b holds.

As in the proof of Lemma 4.32 we first proceed by induction on $r^-$ to show there exist $p, q \in [M]_0$ with $q \le p$ such that the $r^-$ copies of $V(\lambda_m)$ in $V^\times$ are, up to inner automorphism, given by $\bigoplus_{k=1}^p V(\nu_k) \oplus \bigoplus_{k=1}^q V(\gamma_k)$ with $p + q = r^-$. The base case of $r^- = 0$ is certainly true so suppose our assertion holds when Rad $\mathfrak{a}$ contains $r^-$ copies of $V(\lambda_m)$. Suppose now that Rad $\mathfrak{a}$ contains $r^- + 1$ such copies and let $w_1^-, \ldots, w_{r^-+1}^- \in V^\times$ be maximal vectors of these $r^- + 1$ copies. Then for each $i \in [r^- + 1]$ there exist $a_{i,1}, \ldots, a_{i,M}, b_{i,1}, \ldots, b_{i,M} \in \mathbb{F}$ such that

$$w_i^- = \sum_{k=1}^{M} (a_{i,k} v_{\gamma_k} + b_{i,k} v_{\nu_k}) \tag{4.117}$$

where $v_{\gamma_k}$ and $v_{\nu_k}$ are as in Equation 4.55.

Let $\tilde{V}_m^\times$ be the $\mathfrak{l}$-module generated by $w_1^-, \ldots, w_{r^-}^-$. By verifying that $[\tilde{V}_m^\times, \tilde{V}_m^\times] \cap \left( \bigoplus_{k=1}^M V(\gamma_k) \oplus \bigoplus_{k=1}^M V(\nu_k) \right) \subseteq \tilde{V}_m^\times$ we see that $\tilde{V}^\times$ could be the $r^-$ copies of $V(\lambda_m)$ appearing in the nontrivial component of some Levi decomposable subalgebra $\tilde{\mathfrak{a}}$ with Levi factor $\mathfrak{l}$. Therefore, by the inductive hypothesis we may suppose without loss of generality that $\tilde{V}_m^\times = \bigoplus_{k=1}^p V(\nu_k) \oplus \bigoplus_{k=1}^q V(\gamma_k)$ for some $p, q \in [M]_0$ with $q \le p$. By linear independence of $\{w_1^-, \ldots, w_{r^-+1}^-\}$ we may suppose $a_{r^-+1,1} = \cdots = a_{r^-+1,q} = b_{r^-+1,1} = \cdots = b_{r^-+1,p} = 0$. We proceed by cases.

**Case 1: There exists $l \in \{q+1, \ldots, p\}$ such that $a_{r^-+1,l} \neq 0$**

Note that this case requires $q < p$. Define $A, C \in \mathfrak{gl}_M$ such that the $i$'th column of $A$ and $C$ is $A_i$ and $C_i$ for each $i \in [M]$, where

$$A_i = \begin{cases} e_i, & i \le q \text{ or } i > l \\ \sum_{k=1}^M a_{r^-+1,k} e_k, & i = q+1 \\ e_{i-1}, & q+1 < i \le l \end{cases} \tag{4.118a}$$

$$C_i = \begin{cases} \frac{b_{r^-+1,i}}{a_{r^-+1,l}} e_l, & i \le q \text{ or } i > l \\ \sum_{k=1}^M b_{r^-+1,k} e_k, & i = q+1 \\ \frac{b_{r^-+1,i-1}}{a_{r^-+1,l}} e_l, & q+1 < i \le l \end{cases} \tag{4.118b}$$

We have that $A$ is invertible since $\{A_1, \ldots, A_M\}$ is linearly independent. Further define $B, D \in \mathfrak{gl}_M$ such that $B = 0$ and $D = (A^{-1})^\top$. Then $P'$ in Equation 4.54 is an element of $Sp_{2M}$ and so $P$ in Equation 4.53 is in $Sp_{2n}$. Taking $\tau \in \operatorname{Int} \mathfrak{g}$ to be conjugation by $P$ we have that $\tau$ preserves $\mathfrak{l}$ and $\bigoplus_{k=1}^q V(\gamma_k)$ while also mapping $v_{\gamma_{q+1}}$ to $w_{r^-+1}^-$. Since the first $p$ columns of $D$ are contained in $\operatorname{span}\{e_1, \ldots, e_p\}$ we have that $\tau$ also preserves $\bigoplus_{k=1}^p V(\nu_k)$. Therefore, $\tau^{-1}$ maps the $r^- + 1$ copies of $V(\lambda_m)$ in $V^\times$ to $\bigoplus_{k=1}^p V(\nu_k) \oplus \bigoplus_{k=1}^{q+1} V(\gamma_k)$ with $q + 1 \le p$.

**Case 2: $a_{r^-+1,q+1} = \cdots = a_{r^-+1,p} = 0$**

Note that this case requires $p < M$ as otherwise $w^-_{r^-+1} = 0$. Additionally, there must exist $l \in \{p+1, \ldots, M\}$ such that $a_{r^-+1,l} \neq 0$ or $b_{r^-+1,l} \neq 0$. By Lemma 4.25 we may suppose without loss of generality that $b_{r^-+1,l} \neq 0$. Define $B, D \in \mathfrak{gl}_M$ such that the $i$'th column of $B$ and $D$ for each $i \in [M]$ is $B_i$ and $D_i$, respectively, where

$$B_i = \begin{cases} \frac{a_{r^-+1,i}}{b_{r^-+1,l}} e_l, & i \leq p \text{ or } i > l \\ \sum_{k=1}^{M} a_{r^-+1,k} e_k, & i = p+1 \\ \frac{a_{r^-+1,i-1}}{b_{r^-+1,l}} e_l, & p+1 < i \leq l \end{cases} \tag{4.119a}$$

$$D_i = \begin{cases} e_i, & i \leq p \text{ or } i > l \\ \sum_{k=1}^{M} b_{r^-+1,k} e_k, & i = p+1 \\ e_{i-1}, & p+1 < i \leq l \end{cases} \tag{4.119b}$$

As per usual, $D$ is invertible so we define $A, C \in \mathfrak{gl}_M$ as $A = (D^{-1})^\top$ and $C = 0$. Then $P$ in Equation 4.53 is an element of $Sp_{2n}$ and defines $\tau \in \operatorname{Int} \mathfrak{g}$ which preserves $\mathfrak{l}$ and $\bigoplus_{k=1}^{p} V(\nu_k)$ while mapping $v_{\nu_{p+1}}$ to $w_{r^-+1}$. By checking that the first $q$ columns of $A$ are in $\operatorname{span}\{e_1, \ldots, e_q\}$ we conclude that $\tau$ also preserves $\bigoplus_{k=1}^{q} V(\gamma_k)$. Hence $\tau^{-1}$ preserves $\mathfrak{l}$ and maps the $r^- + 1$ copies of $V(\lambda_m)$ in $V^\times$ to $\bigoplus_{k=1}^{p+1} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$ in this case.

With this we conclude the first inductive portion of this proof in which we showed that up to inner automorphism there exist $p, q \in [M]$ such that the $r^-$ copies of $V(\lambda_m)$ in $V^\times$ are $\bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$. It remains to consider the $r^+$ copies of $V(\lambda_1)$ in $V^\times$. By another inductive proof we show that up to inner automorphism, we can additionally suppose that the $r^+$ copies of $V(\lambda_1)$ in $V^\times$ can be assumed to be $\bigoplus_{k=M-c+1}^{M} V(\beta_k) \oplus \bigoplus_{k=p-d+1}^{p} V(\beta_k)$ for some $c, d \in [M]_0$ with $d \leq p - q$, $c \leq M - p$, and $c = 0$ if $q > 0$, provided that the copies of $V(\lambda_m)$ in $V^\times$ are given by $\bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$. The base case is again trivial so suppose the claim holds when $V^\times$ contains $r^+$ copies of $V(\lambda_1)$. Now suppose $\operatorname{Rad} \mathfrak{a}$ contains $r^+ + 1$ copies of $V(\lambda_1)$. Taking the maximal vectors to be $w_1^+, \ldots, w^+_{r^++1}$ we have that there exist $a_{i,1}, \ldots, a_{i,M}, b_{i,1}, \ldots, b_{i,M} \in \mathbb{F}$ for each $i \in [r^+ + 1]$ such that

$$w_i^+ = \sum_{k=1}^{M} (a_{i,k} v_{\beta_k} + b_{i,k} v_{\mu_k}) \tag{4.120}$$

with $v_{\beta_k}$ and $v_{\mu_k}$ as in Equation 4.55. Observe that for each $i \in [r^+ + 1]$, $j \in [q]$, and $k \in [p]$ we have

$$[w_i^+, v_{\gamma_j}] = a_{i,j} X_{1,\dots,m}, \qquad [w_i^+, v_{\nu_k}] = b_{i,k} X_{1,\dots,m} \tag{4.121}$$

As $X_{1,\dots,m} \in \mathfrak{l}$ it must be that $a_{i,j} = b_{i,k} = 0$. In addition, for each $i, j \in [r^+ + 1]$ we have

$$[w_i^+, w_j^+] = 2\sum_{k=1}^{M} (a_{i,k} b_{j,k} - a_{j,k} b_{i,k}) v_\zeta \tag{4.122}$$

where $v_\zeta$ is as in Equation 4.55. Since $V(\zeta) \not\subseteq V^\times$ it must be that for all $i, j \in [r^+ + 1]$,

$$\sum_{k=1}^{M} (a_{i,k} b_{j,k} - a_{j,k} b_{i,k}) = 0 \tag{4.123}$$

Also note that for each $i \in [r^+ + 1]$,

$$[[Y_{1,\dots,m}, w_i^+], v_\eta] = (-1)^m \sum_{k=1}^{M} (a_{i,k} v_{\nu_k} + b_{i,k} v_{\gamma_k}) \tag{4.124}$$

Therefore, since we may suppose that the copies of $V(\lambda_1)$ appearing in $V^\times$ are given by $\bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$ we conclude that if $V(\eta) \subseteq V^\times$ then for each $i \in [r^+ + 1]$, $a_{i,p+1} = \cdots = a_{i,M} = b_{i,q+1} = \cdots = b_{i,M} = 0$.

If $\tilde{V}_1^\times$ is the $\mathfrak{l}$-module generated by $w_1^+, \dots, w_{r^+}^+$ then one can easily produce a Levi decomposable subalgebra $\tilde{\mathfrak{a}}$ of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ such that Rad $\tilde{\mathfrak{a}}$ has $r^+$ copies of $V(\lambda_1)$ given by $\tilde{V}_1^\times$ and the copies of $V(\lambda_m)$ in Rad $\mathfrak{a}$ are $\bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$. By the inductive hypothesis, we may thus suppose

$$\tilde{V}_1^\times = \bigoplus_{k=M-c+1}^{M} V(\beta_k) \oplus \bigoplus_{k=p-d+1}^{p} V(\beta_k) \tag{4.125}$$

with $d \leq p - q$, $c \leq M - p$, and $c = 0$ if $q > 0$. By linear independence of $\{w_1^+, \dots, w_{r^++1}^+\}$ we may additionally suppose $a_{r^++1,M-c+1} = \cdots = a_{r^++1,M} = 0$ and $a_{r^++1,p-d+1} = \cdots = a_{r^++1,p} = 0$. We consider cases.

**Case (i): There exists $l \in \{p+1, \dots, M-c\}$ such that $a_{r^++1,l} \neq 0$ or $b_{r^++1,l} \neq 0$**

Observe that this case requires $c < M - p$. Since having $V(\eta) \subseteq V^\times$ implies $a_{r^++1,p+1} = \cdots = a_{r^++1,M} = b_{r^++1,q+1} = \cdots = b_{r^++1,M} = 0$, we conclude by Lemma 4.30 that this case requires $q = 0$ and $V(\eta) \not\subseteq V^\times$. As such, we may suppose without loss of generality due to Lemma 4.26 that

$a_{r^++1,l} \neq 0$. Define $B, D \in \mathfrak{gl}_M$ such that for each $i \in [M]$ the $i$'th columns of $B$ and $D$ are $B_i$ and $D_i$, respectively, where

$$B_i = \begin{cases} \frac{b_{r^++1,i}}{a_{r^++1,l}} e_l, & i < l \text{ or } i > M - c \\ \frac{b_{r^++1,i+1}}{a_{r^++1,l}} e_l, & l \leq i < M - c \\ \sum_{k=1}^{M} b_{r^++1,k} e_k, & i = M - c \end{cases} \tag{4.126a}$$

$$D_i = \begin{cases} e_i, & i < l \text{ or } i > M - c \\ e_{i+1}, & l \leq i < M - c \\ \sum_{k=1}^{M} a_{r^++1,k} e_k, & i = M - c \end{cases} \tag{4.126b}$$

Further define $A, C \in \mathfrak{gl}_M$ such that $A = (D^{-1})^\top$ and $C = 0$, where invertibility of $D$ is assured by linear independence of $\{D_1, \ldots, D_M\}$. Then $P$ as defined in Equation 4.53 is an element of $Sp_{2n}$. The resulting inner automorphism $\tau$ obtained via conjugation by $P$ is such that $\tau \in \operatorname{Int} \mathfrak{g}$ preserves $\mathfrak{l}$, $\bigoplus_{k=M-c+1}^{M} V(\beta_k) \oplus \bigoplus_{k=p-d+1}^{p} V(\beta_k)$, and $\bigoplus_{k=1}^{p} V(\nu_k)$ while mapping $v_{\beta_{M-c}}$ to $w_{r^++1}^+$.

**Case (ii): $a_{r^++1,l} = b_{r^++1,l} = 0$ for all $l \in \{p+1, \ldots, M-c\}$**

Since $w_{r^++1}^+ \neq 0$ it must be that $d < p - q$ and there exists $l \in \{q + 1, \ldots, p - d\}$ such that $a_{r^++1,l} \neq 0$. Define $D \in \mathfrak{gl}_M$ such that the $i$'th column of $D$ is $D_i$ for each $i \in [M]$, where

$$D_i = \begin{cases} e_i, & i < l \text{ or } i > p - d \\ e_{i+1}, & l \leq i < p - d \\ \sum_{k=1}^{M} a_{r^++1,k} e_k, & i = p - d \end{cases} \tag{4.127}$$

Note that $D$ is invertible. By defining $A, B, C \in \mathfrak{gl}_M$ as $A = (D^{-1})^\top$ and $B = C = 0$ we get that $P$ as in Equation 4.53 is in $Sp_{2n}$. Observe that conjugation by $P$ defines an inner automorphism $\tau \in \operatorname{Int} \mathfrak{g}$ preserving $\mathfrak{l}$, $\bigoplus_{k=M-c+1}^{M} V(\beta_k) \oplus \bigoplus_{k=p-d+1}^{p} V(\beta_k)$, and $\bigoplus_{k=1}^{p} V(\nu_k)$ while mapping $v_{\beta_{p-d}}$ to $w_{r^++1}^+$. By further noting that the first $q$ columns of $A$ are contained in $\operatorname{span}\{e_1, \ldots, e_q\}$ we conclude that $\tau$ also preserves $\bigoplus_{k=1}^{q} V(\gamma_k)$. This concludes this second inductive portion and thus the proof. □

Lemma 4.34 establishes existence of the nontrivial $\Phi_{\mathfrak{l}}$-candidate needed in Theorem 4.19. As usual, the inner automorphisms employed in the proof of Lemma 4.34 preserve $\mathfrak{h}$ when the maximal vectors are root space elements. To finally complete the proof of Theorem 4.19 we now demonstrate that the possibilities listed for $V^\times$ in Lemma 4.34 are mutually non-conjugate.

**Lemma 4.35.** *Let $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$ be Levi decomposable subalgebras of $\mathfrak{g}$ with common Levi factor $\mathfrak{l}$, where $(t(\mathfrak{g}), t(\mathfrak{l})) = (C_n, A_m)$. By Lemma 4.34, up to inner automorphism one may suppose that the nontrivial components $V^\times$ and $\tilde{V}^\times$ of* Rad $\mathfrak{a}$ *and $\tilde{\mathfrak{a}}$, respectively, are as in Equation 4.116 with $p, q, c, d \in [M]_0$ corresponding to $V^\times$ and $\tilde{p}, \tilde{q}, \tilde{c}, \tilde{d} \in [M]_0$ corresponding to $\tilde{V}^\times$. If there exists $\tau \in$ Int $\mathfrak{g}$ preserving $\mathfrak{l}$ and mapping $V^\times$ to $\tilde{V}^\times$ then $(p, q, c, d) = (\tilde{p}, \tilde{q}, \tilde{c}, \tilde{d})$ and $V^\times$ and $\tilde{V}^\times$ are both given by either Equation 4.116a or Equation 4.116b.*

*Proof.* Considering $\tau$ as conjugation by $P \in Sp_{2n}$, we conclude by Equation 3.13 and the fact that $\tau$ preserves $\mathfrak{l}$ that

$$P = \begin{pmatrix} \tilde{A} & 0_{m+1,M} & \tilde{B} & 0_{m+1,M} \\ 0_{M,m+1} & A & 0_{M,m+1} & B \\ \tilde{C} & 0_{m+1,M} & \tilde{D} & 0_{m+1,M} \\ 0_{M,m+1} & C & 0_{M,m+1} & D \end{pmatrix} \tag{4.128}$$

for some $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \in \mathfrak{gl}_{m+1}$ and $A, B, C, D \in \mathfrak{gl}_M$ such that $P_\mathfrak{l} \in Sp_{2(m+1)}$ and $P_{\mathfrak{l}^\perp} \in Sp_{2M}$, where

$$P_\mathfrak{l} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}, \qquad P_{\mathfrak{l}^\perp} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{4.129}$$

For all $R \in \mathfrak{sl}_{m+1}$,

$$\begin{aligned} &\tau\left(\begin{pmatrix} R & 0_{m+1,M} & 0_{m+1,m+1} & 0_{m+1,M} \\ 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \\ 0_{m+1,m+1} & 0_{m+1,M} & -R^\top & 0_{m+1,M} \\ 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \end{pmatrix}\right) \\ &= \begin{pmatrix} \tilde{A}R\tilde{D}^\top + \tilde{B}R^\top\tilde{C}^\top & 0_{m+1,M} & -\tilde{A}R\tilde{B}^\top - \tilde{B}R^\top\tilde{A}^\top & 0_{m+1,M} \\ 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \\ \tilde{C}R\tilde{D}^\top + \tilde{D}R^\top\tilde{C}^\top & 0_{m+1,M} & -\tilde{C}R\tilde{B}^\top - \tilde{D}R^\top\tilde{A}^\top & 0_{m+1,M} \\ 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \end{pmatrix} \end{aligned} \tag{4.130}$$

Thus Equation 3.13 implies $\tilde{A}R\tilde{B}^\top + \tilde{B}R^\top\tilde{A}^\top = 0$ and $\tilde{C}R\tilde{D}^\top + \tilde{D}R^\top\tilde{C}^\top = 0$ for all $R \in \mathfrak{sl}_{m+1}$. Having $P_\mathfrak{l} \in Sp_{2(m+1)}$ implies $\tilde{A}^\top\tilde{D} - \tilde{C}^\top\tilde{B} = I$ and so either $\tilde{B} = 0$ or $\tilde{D} = 0$ by Lemma 4.27.

If $V^\times$ is as in Equation 4.116a then the same argument as in Case 1 of the proof of Lemma 4.33 implies $\tilde{V}^\times$ is also described by Equation 4.116a. Then by the same argument as in that case $(p, q, d) = (\tilde{p}, \tilde{q}, \tilde{d})$.

Similarly, if $V^\times$ is as in Equation 4.116b then the reasoning provided in Case 2 of the proof of Lemma 4.33 forces $\tilde{V}^\times$ to also be as in Equation 4.116b. Dimension arguments imply $p + c + d = \tilde{p} + \tilde{c} + \tilde{d}$. Once again, as in that case we find $p = \tilde{p}$ and so $c + d = \tilde{c} + \tilde{d}$.

Let $W$ and $\tilde{W}$ be the subalgebras of $\mathfrak{g}$ generated by $\mathfrak{l} \oplus V^\times$ and $\mathfrak{l} \oplus \tilde{V}^\times$, respectively. As $\tau$ preserves $\mathfrak{l}$ and maps $V^\times$ to $\tilde{V}^\times$ we have that $\tau$ maps $W$ to $\tilde{W}$. One can verify that $W$ and $\tilde{W}$ are given by Equation 4.112 with $\Phi_W$ and $\Phi_{\tilde{W}}$ as in Equation 4.113. As such, the same argument as in Case 2 of the proof of Lemma 4.33 implies $d = \tilde{d}$ and consequently $c = \tilde{c}$. □

With Lemma 4.34 and Lemma 4.35 we finally conclude the segment of the thesis dedicated to Theorem 4.19. The ability to ensure that the nontrivial component can always be taken to be regular relative to $\mathfrak{h}$ in the well-mannered setting is quite nice and allows for an explicit and elegant description of said nontrivial component as given by Lemma 4.22, Lemma 4.24, Lemma 4.33, and Lemma 4.35. Moreover, the issue of conjugacy is typically quite difficult to solve as there is no general procedure for determining when two subalgebras are conjugate. With Theorem 4.19 however we have transformed this difficult problem into the far simpler task of determining $\Phi_{\mathfrak{l}}$-Weyl conjugacy. The classical root systems and Weyl groups are well-known and straightforward to work with, thereby allowing us to easily establish conjugacy in the well-mannered context.

## 4.8 THE TRIVIAL COMPONENT OF THE RADICAL

In this section we turn our attention to an analysis of the trivial component of the radical. In contrast to the nontrivial component, the trivial component is not nearly as well-behaved. Indeed, let $\mathfrak{s} \subseteq \mathfrak{l}^\perp$ be a solvable subalgebra. Then $\mathfrak{a} = \mathfrak{l} \Subset \mathfrak{s}$ is a Levi decomposable subalgebra with Levi factor $\mathfrak{l}$ and radical $\mathfrak{s}$ and it is evident that the trivial component of $\mathfrak{s}$ is $\mathfrak{s}$ itself. As such, every solvable subalgebra of $\mathfrak{l}^\perp$ gives rise to a Levi decomposable subalgebra of $\mathfrak{g}$. Provided that $\mathfrak{l}^\perp \neq \{0\}$, we have by Lemma 4.7 that $\mathfrak{l}^\perp$ is simple and so we would require a classification of the solvable subalgebras of simple Lie algebras to obtain an adequate classification of the trivial component of the radical. As this is a hopelessly difficult open problem, we are not able to provide a description of the trivial component of the radical.

With that said, we can still establish a fairly nice result pertaining to the trivial component. Suppose $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are Levi decomposable subalgebras of $\mathfrak{g}$ with common Levi factor $\mathfrak{l}$. Let $V_1^\times, V_1^0 \subseteq \operatorname{Rad} \mathfrak{a}_1$ and $V_2^\times, V_2^0 \subseteq \operatorname{Rad} \mathfrak{a}_2$ be the nontrivial and trivial components of $\operatorname{Rad} \mathfrak{a}_1$ and $\operatorname{Rad} \mathfrak{a}_2$, respectively. How can we determine whether $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are conjugate? By Lemma 4.15, this is equivalent to asking whether there exists $\tau \in \operatorname{Int} \mathfrak{g}$ which preserves $\mathfrak{l}$ and maps $V_1^\times$ to $V_2^\times$ and $V_1^0$ to $V_2^0$. Theorem 4.19 provides criteria for when it is possible to preserve $\mathfrak{l}$ and map $V_1^\times$ to $V_2^\times$. If Theorem 4.19 asserts that no inner automorphism can preserve $\mathfrak{l}$ and map $V_1^\times$ to $V_2^\times$ then there is nothing to do, but what if such an inner automorphism exists? Then without loss of generality we may suppose $V_1^\times = V_2^\times = V^\times = \bigoplus_{\alpha\in\Theta} \mathfrak{g}_\alpha$ for some nontrivial $\Phi_\mathfrak{l}$-candidate $\Theta \subseteq \Phi$. The task now becomes deciding whether it is possible to find an inner automorphism which preserves $\mathfrak{l}$ and $V^\times$ and maps $V_1^0$ to $V_2^0$. This section focuses on solving this problem.

In what follows we will often be interested in inner automorphisms $\tau \in \operatorname{Int}\mathfrak{g}$ which preserve $\mathfrak{l}$. Having $\tau$ preserve $\mathfrak{l}$ implies $\tau$ necessarily preserves $C_\mathfrak{g}(\mathfrak{l}) = \mathfrak{l}^\perp$ and as such, $\tau$ restricts to an automorphism of $C_\mathfrak{g}(\mathfrak{l})$. This restriction is in fact an inner automorphism of $C_\mathfrak{g}(\mathfrak{l})$ as shown below.

**Lemma 4.36.** *Suppose $\tau \in \operatorname{Int}\mathfrak{g}$ preserves $\mathfrak{l}$. Then $\tau|_{C_\mathfrak{g}(\mathfrak{l})} \in \operatorname{Int} C_\mathfrak{g}(\mathfrak{l})$.*

*Proof.* We first claim that $\tau$ fixes $\tilde{\mathfrak{h}}_\mathfrak{l}$ element-wise. Indeed, if $\mathfrak{t}(\mathfrak{l}) \neq A_m$ then $\tilde{\mathfrak{h}}_\mathfrak{l} = \{0\}$. If $\mathfrak{t}(\mathfrak{l}) = A_m$ then

$$\tilde{\mathfrak{h}}_\mathfrak{l} = \operatorname{span}\{\tilde{H}\} \tag{4.131}$$

where

$$\tilde{H} = \begin{cases} \begin{pmatrix} (M+1)I_{m+1} & 0_{m+1,M+1} \\ 0_{M+1,m+1} & -(m+1)I_{M+1} \end{pmatrix}, & \mathfrak{t}(\mathfrak{g}) = A_n \\ \begin{pmatrix} 0 & 0_{1,m+1} & 0_{1,M} & 0_{1,m+1} & 0_{1,M} \\ 0_{m+1,1} & I_{m+1} & 0_{m+1,M} & 0_{m+1,m+1} & 0_{m+1,M} \\ 0_{M,1} & 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \\ 0_{m+1,1} & 0_{m+1,m+1} & 0_{m+1,M} & -I_{m+1} & 0_{m+1,M} \\ 0_{M,1} & 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \end{pmatrix}, & \mathfrak{t}(\mathfrak{g}) = B_n \\ \begin{pmatrix} I_{m+1} & 0_{m+1,M} & 0_{m+1,m+1} & 0_{m+1,M} \\ 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \\ 0_{m+1,m+1} & 0_{m+1,M} & -I_{m+1} & 0_{m+1,M} \\ 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & 0_{M,M} \end{pmatrix}, & \text{else} \end{cases} \tag{4.132}$$

As $\tau$ preserves $\mathfrak{l}$ we have from Section 3.2 that $\tau$ preserves $\tilde{H}$ and thus fixes $\tilde{\mathfrak{h}}_\mathfrak{l}$ element-wise.

We claim that $\tau$ also restricts to an inner automorphism of $\mathfrak{l}^\perp$. This can be proved by examining each possibility for $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l}))$ separately and using the descriptions of $\mathfrak{l}$ provided in Section 3.2 and the description of $\mathfrak{l}^\perp$ given by Equation 4.9. The relevant computations are rather straightforward but somewhat tedious and lengthy and so we provide the argument for just one prototypical case.

Suppose $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (C_n, A_m)$. In the proof of Lemma 4.35 we noted that any inner automorphism of $\mathfrak{g}$ which preserves $\mathfrak{l}$ must be conjugation by some $P \in Sp_{2n}$ with $P$ as in Equation 4.128, $P_{\mathfrak{l}} \in Sp_{2(m+1)}$, and $P_{\mathfrak{l}^\perp} \in Sp_{2M}$, where $P_{\mathfrak{l}}$ and $P_{\mathfrak{l}^\perp}$ are as in Equation 4.129. We have

$$\mathfrak{l}^\perp = \left\{ \begin{pmatrix} 0_{m+1,m+1} & 0_{m+1,M} & 0_{m+1,m+1} & 0_{m+1,M} \\ 0_{M,m+1} & R & 0_{M,m+1} & S \\ 0_{m+1,m+1} & 0_{m+1,M} & 0_{m+1,m+1} & 0_{m+1,M} \\ 0_{M,m+1} & T & 0_{M,m+1} & -R^\top \end{pmatrix} \in \mathfrak{sp}_{2n} \colon R, S, T \in \mathfrak{gl}_M, \right.$$
$$\left. \begin{pmatrix} R & S \\ T & -R^\top \end{pmatrix} \in \mathfrak{sp}_{2M} \right\} \tag{4.133}$$

from which our claim is immediate. The arguments for other possible regular pairs are nearly identical.

Since $[\mathfrak{l}^\perp, \tilde{\mathfrak{h}}_{\mathfrak{l}}] = \{0\}$, every inner automorphism of $\mathfrak{l}^\perp$ extends to one of $\mathfrak{l}^\perp \oplus \tilde{\mathfrak{h}}_{\mathfrak{l}}$ by defining the extension to fix $\tilde{\mathfrak{h}}_{\mathfrak{l}}$ element-wise. As $\tau|_{\tilde{\mathfrak{h}}_{\mathfrak{l}}} = \mathrm{id}$ and $C_{\mathfrak{g}}(\mathfrak{l}) = \mathfrak{l}^\perp \oplus \tilde{\mathfrak{h}}_{\mathfrak{l}}$ the result follows. □

In the next two sections we provide a necessary and sufficient criterion on trivial components to solve the problem of determining when two Levi decomposable subalgebras with the same Levi factor, same nontrivial component of the radical, but different trivial components are conjugate. To do this we will make use of certain invariants, namely normalizers and centralizers. Clearly, if two subalgebras are conjugate then their normalizers and centralizers are conjugate. Generally however two subalgebras can have conjugate normalizers without themselves being conjugate, and an analogous statement holds true when regarding centralizers. Fortunately for us, it turns out that when dealing with trivial components of radicals that in some sense we can use conjugacy of normalizers and centralizers to determine conjugacy of the subalgebras. If $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) \neq (C_n, A_m)$ then only the normalizer will be needed, whereas if $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (C_n, A_m)$ then both the normalizer and centralizer will be required.

## 4.9 THE TRIVIAL COMPONENT WHEN $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) \neq (C_n, A_m)$

One might hope to tackle the problem of finding an inner automorphism $\tau \in \mathrm{Int}\, \mathfrak{g}$ preserving $\mathfrak{l}$ and $V^\times$ and mapping $V_1^0$ to $V_2^0$ in the following manner. Suppose that there exists an inner automorphism $\rho \in \mathrm{Int}\, C_{\mathfrak{g}}(\mathfrak{l})$ such that $\rho$ maps $V_1^0$ to $V_2^0$. Since $[C_{\mathfrak{g}}(\mathfrak{l}), \mathfrak{l}] = \{0\}$, one can then extend $\rho$ to some $\tilde{\rho} \in \mathrm{Int}\, \mathfrak{g}$ by asserting that $\tilde{\rho}|_{\mathfrak{l}} = \mathrm{id}$. Unfortunately, the issue here is that although $C_{\mathfrak{g}}(\mathfrak{l})$ and $\mathfrak{l}$ interact trivially via the Lie bracket, $[C_{\mathfrak{g}}(\mathfrak{l}), V^\times]$

need not necessarily be trivial and as such there is no guarantee that $\tilde{\rho}$ preserves $V^\times$. In short, this naive approach requires some tweaking.

Suppose $\tilde{\rho}$ indeed preserves $V^\times$. Inspired by invariants, and especially normalizers, we define

$$\mathfrak{n}(V^\times) = N_\mathfrak{g}(V^\times) \cap C_\mathfrak{g}(\mathfrak{l}) \tag{4.134}$$

Observe that

$$\rho(\mathfrak{n}(V^\times)) = \tilde{\rho}(N_\mathfrak{g}(V^\times)) \cap C_\mathfrak{g}(\mathfrak{l}) = N_\mathfrak{g}(\tilde{\rho}(V^\times)) \cap C_\mathfrak{g}(\mathfrak{l}) = \mathfrak{n}(V^\times) \tag{4.135}$$

Therefore, if $\tilde{\rho}$ indeed preserves $V^\times$ then $\rho$ must also preserve $\mathfrak{n}(V^\times)$. It turns out that if $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) \neq (C_n, A_m)$ then this condition is also sufficient.

**Theorem 4.37.** *Suppose $(t(\mathfrak{g}), t(\mathfrak{l})) \neq (C_n, A_m)$ and let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be Levi decomposable subalgebras of $\mathfrak{g}$ with common Levi factor $\mathfrak{l}$. Further suppose the radicals of $\mathfrak{a}_1$ and $\mathfrak{a}_2$ have the same nontrivial component $V^\times$ and let $V_1^0$ and $V_2^0$ be the trivial components of* Rad $\mathfrak{a}_1$ *and* Rad $\mathfrak{a}_2$*, respectively. Then $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are conjugate if and only if there exists $\rho \in \text{Int}\, C_\mathfrak{g}(\mathfrak{l})$ such that $\rho$ preserves $\mathfrak{n}(V^\times)$ and maps $V_1^0$ to $V_2^0$.*

*Proof.* Without loss of generality we may suppose by Theorem 4.19 that $V^\times$ is given by Equation 4.22 for some nontrivial $\Phi_\mathfrak{l}$-candidate $\Theta \subseteq \Phi$.

First suppose $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are conjugate. By Lemma 4.15 there exists $\tau \in \text{Int}\, \mathfrak{g}$ that maps $V_1^0$ to $V_2^0$ and preserves both $\mathfrak{l}$ and $V^\times$. Lemma 4.36 implies $\rho = \tau|_{C_\mathfrak{g}(\mathfrak{l})}$ is an inner automorphism of $C_\mathfrak{g}(\mathfrak{l})$. Clearly $\rho$ maps $V_1^0$ to $V_2^0$. Moreover, by Equation 4.135 $\rho$ preserves $\mathfrak{n}(V^\times)$, as desired.

Conversely, suppose there exists $\rho \in \text{Int}\, C_\mathfrak{g}(\mathfrak{l})$ such that $\rho$ preserves $\mathfrak{n}(V^\times)$ and maps $V_1^0$ to $V_2^0$. Extend $\rho$ to some $\tilde{\rho} \in \text{Int}\, \mathfrak{g}$ by requiring $\tilde{\rho}|_\mathfrak{l} = \text{id}$. By construction, $\tilde{\rho}$ maps $V_1^0$ to $V_2^0$ and preserves $\mathfrak{l}$. To show that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are conjugate it suffices to show that $\tilde{\rho}$ preserves $V^\times$.

Observe that

$$\mathfrak{n}(\tilde{\rho}(V^\times)) = N_\mathfrak{g}(\tilde{\rho}(V^\times)) \cap C_\mathfrak{g}(\mathfrak{l}) = \tilde{\rho}(N_\mathfrak{g}(V^\times)) \cap C_\mathfrak{g}(\mathfrak{l}) = \tilde{\rho}(\mathfrak{n}(V^\times)) \tag{4.136}$$

Since $\rho$ preserves $\mathfrak{n}(V^\times)$ it follows that

$$\mathfrak{n}(\tilde{\rho}(V^\times)) = \mathfrak{n}(V^\times) \tag{4.137}$$

Also note that since $V^\times$ is an $\mathfrak{l}$-module and $\tilde{\rho}$ preserves $\mathfrak{l}$ we have that $\tilde{\rho}(V^\times)$ is also an $\mathfrak{l}$-module. As such, $\mathfrak{h}_\mathfrak{l} \subseteq \mathfrak{n}(\tau(V^\times))$. Moreover, since $V^\times$ is as in Equation 4.22 we note that $\mathfrak{h}_\mathfrak{l}^\perp, \tilde{\mathfrak{h}}_\mathfrak{l} \subseteq \mathfrak{n}(V^\times) = \mathfrak{n}(\tilde{\rho}(V^\times))$. Since

$\mathfrak{h} = \mathfrak{h}_\mathfrak{l} \oplus \mathfrak{h}_\mathfrak{l}^\perp \oplus \tilde{\mathfrak{h}}_\mathfrak{l}$ it follows that $\mathfrak{h} \subseteq \mathfrak{n}(\tau(V^\times))$ and so $\tau(V^\times)$ is regular relative to $\mathfrak{h}$. We complete the proof via an examination of various cases.

**Case 1: $\mathfrak{t}(\mathfrak{l}) \neq A_m$**

By Lemma 4.21 there exists $p \in [M]_0$ such that $V^\times$ is described by Equation 4.29. To show that $\tilde{\rho}$ preserves $V^\times$, it suffices to show by dimension considerations that $v_{\beta_i} \notin \tilde{\rho}(V^\times)$ for each $i \in [M]$, $v_{\gamma_j} \notin \tilde{\rho}(V^\times)$ for each $j \in \{p+1, \ldots, M\}$, and $v_\chi \notin \tau(V)$ if $\mathfrak{g}$ is of type $B_n$ since $\tilde{\rho}(V^\times)$ is regular relative to $\mathfrak{h}$, where $v_{\beta_i}$, $v_{\gamma_j}$, and $v_\chi$ are as in Equation 4.24, Equation 4.26, and/or Equation 4.28 depending on the type of $\mathfrak{g}$.

Firstly, $v_\chi \notin \tau(V)$ as having otherwise would violate Equation 4.35.

Now let $i \in [p]$ and suppose by contradiction that $v_{\beta_i} \in \tau(V^\times)$. If $i \neq 1$ then one can verify that $X_1, \ldots, X_{i-1} \in \mathfrak{n}(V^\times)$. By Equation 4.137 it follows that $[X_{i-1}, v_{\beta_i}] \in \tilde{\rho}(V^\times)$. Since $\alpha_{i-1} + \beta_i \in \beta_{i-1} + \Phi_\mathfrak{l}$ we conclude by an inductive argument that this forces $v_{\beta_1}, \ldots, v_{\beta_i} \in \tilde{\rho}(V^\times)$. Moreover, one can check that $Y_1, \ldots, Y_{M-1} \in \mathfrak{n}(V^\times)$. Since $-\alpha_i + \beta_i \in \beta_{i+1} + \Phi_\mathfrak{l}$ for $i < M$ we conclude that $v_{\beta_{i+1}}, \ldots, v_{\beta_M} \in \tilde{\rho}(V^\times)$. Thus it must be that $p = M$ and $\tilde{\rho}(V^\times) = \bigoplus_{k=1}^M V(\beta_k)$. Now define

$$\epsilon = \begin{cases} \alpha_{M-1} + 2\sum_{k=M}^n \alpha_k, & (\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (B_n, B_m) \\ \sum_{k=M}^n \alpha_k, & (\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (B_n, D_m) \\ 2\sum_{k=M}^{n-1} \alpha_k + \alpha_n, & \mathfrak{t}(\mathfrak{g}) = C_n \\ \alpha_{M-1} + 2\sum_{k=M}^{n-2} \alpha_k + \alpha_{n-1} + \alpha_n, & \mathfrak{t}(\mathfrak{g}) = D_n \end{cases} \tag{4.138}$$

Since $\tilde{\rho}(V^\times) = \bigoplus_{k=1}^M V(\beta_k)$ and $\epsilon + \alpha \in \bigcup_{k=1}^M (\beta_k + \Phi_\mathfrak{l})$ whenever $\epsilon + \alpha \in \Phi_0$ for all $\alpha \in \bigcup_{k=1}^M (\beta_k + \Phi_\mathfrak{l})$, we conclude that $\mathfrak{g}_\epsilon \subseteq \mathfrak{n}(\tilde{\rho}(V^\times))$. Equation 4.137 then implies $\mathfrak{g}_\epsilon \subseteq \mathfrak{n}(V^\times)$. Since $p = M$ we have $v_{\gamma_M} \in V^\times$, implying $\epsilon + \gamma_M \in \bigcup_{k=1}^M (\gamma_k + \Phi_\mathfrak{l})$ if $\epsilon + \gamma_M \in \Phi_0$. However, direct computation reveals this is not true and so we reach a contradiction. Hence $v_{\beta_i} \notin \tilde{\rho}(V^\times)$ for each $i \in [p]$.

Next suppose by contradiction that $v_{\beta_i} \in \tilde{\rho}(V^\times)$ for some $i \in \{p+1, \ldots, M\}$. Note this requires $p < M$. If $i \neq M$ then $Y_i, \ldots, Y_{M-1} \in \mathfrak{n}(V^\times) = \mathfrak{n}(\tilde{\rho}(V^\times))$. As previously noted, $-\alpha_i + \beta_i \in \beta_{i+1} + \Phi_\mathfrak{l}$. As such, an inductive argument reveals that $v_{\beta_{i+1}}, \ldots, v_{\beta_M} \in \tilde{\rho}(V^\times)$. It is easy to check that $\mathfrak{g}_{-\epsilon} \subseteq \mathfrak{n}(V^\times)$. Thus $\mathfrak{g}_{-\epsilon} \subseteq \mathfrak{n}(\tilde{\rho}(V^\times))$, thereby implying $[\mathfrak{g}_{-\epsilon}, \langle v_{\beta_M} \rangle] \subseteq \tilde{\rho}(V^\times)$.

If $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (B_n, D_m)$ then $-\epsilon + \beta_M \in \chi + \Phi_\mathfrak{l}$, implying $v_\chi \in \tilde{\rho}(V^\times)$. This is a contradiction as we previously showed that $v_\chi \notin \tilde{\rho}(V^\times)$. If $\mathfrak{t}(\mathfrak{g}) = C_n$ then $-\epsilon + \beta_M \in \gamma_M + \Phi_\mathfrak{l}$, implying $v_{\gamma_M} \in \tilde{\rho}(V^\times)$. Having both $v_{\beta_M}$ and $v_{\gamma_M}$ in $\tilde{\rho}(V^\times)$ contradicts Equation 4.32. Lastly, if $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (B_n, B_m)$ or if $\mathfrak{t}(\mathfrak{g}) = D_n$ then $-\epsilon + \beta_M \in \gamma_{M-1} + \Phi_\mathfrak{l}$. Thus $v_{\gamma_{M-1}} \in \tilde{\rho}(V^\times)$. Since $v_{\beta_{i+1}}, \ldots, v_{\beta_M} \in \tilde{\rho}(V^\times)$ it must be that $i = M$ as otherwise we contradict Equation 4.35. Since $Y_1, \ldots, Y_{M-2} \in \mathfrak{n}(V^\times) = \mathfrak{n}(\tilde{\rho}(V^\times))$ and $-\alpha_k + \gamma_{k+1} \in \gamma_k + \Phi_\mathfrak{l}$ for each $k \in [M-2]$ we conclude by induction that $v_{\gamma_1}, \ldots, v_{\gamma_{M-1}} \in \tilde{\rho}(V^\times)$. As $v_{\beta_M} \in \tilde{\rho}(V^\times)$ as well we have $p = M$, contradicting the initial assumption that $p < M$. It follows that $v_{\beta_i} \notin \tilde{\rho}(V^\times)$ for each $i \in \{p+1, \ldots, M\}$ and thus also for each $i \in [M]$.

To conclude this case it remains to prove that $v_{\gamma_j} \notin \tilde{\rho}(V^\times)$ for each $j \in \{p+1, \ldots, M\}$ with $p < M$ so suppose by contradiction this is not the case. We have $Y_1, \ldots, Y_{j-1} \in \mathfrak{n}(V^\times) = \mathfrak{n}(\tilde{\rho}(V^\times))$. Since $-\alpha_{j-1} + \gamma_j \in \gamma_{j-1} + \Phi_\mathfrak{l}$ we conclude by induction that $v_{\gamma_1}, \ldots, v_{\gamma_j} \in \tilde{\rho}(V^\times)$. Since $j > p$ and $\tilde{\rho}|_\mathfrak{l} = \mathrm{id}$ this contradicts the fact that $V^\times$, and thus $\tilde{\rho}(V^\times)$, can have at most $p$ linearly independent maximal vectors.

**Case 2: $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (A_n, A_m)$**

Lemma 4.23 allows us to suppose there exist $p, q \in [M+1]_0$ with $p + q \leq M + 1$ such that $V^\times$ is as in Equation 4.39. To show that $\tilde{\rho}(V^\times) = V^\times$ we need only verify that for each $i \in [M]$, if $v_{\beta_i} \in \tilde{\rho}(V^\times)$ then $i > M + 1 - q$ and if $v_{\gamma_i} \in \tilde{\rho}(V^\times)$ then $i \leq p$ due to dimension arguments and regularity of $\tilde{\rho}(V^\times)$ relative to $\mathfrak{h}$.

First suppose $v_{\beta_i} \in \tilde{\rho}(V^\times)$ for some $i \in [M]$. Since $\tilde{\rho}|_\mathfrak{l} = \mathrm{id}$ the existence of such $v_{\beta_i}$ implies $q > 0$ and so $i \leq M$. We thus have $X_{m+i+1}, \ldots, X_n \in \mathfrak{n}(V^\times) = \mathfrak{n}(\tilde{\rho}(V^\times))$ by direct verification. Since $\alpha_{m+i+1} + \beta_i \in \beta_{i+1} + \Phi_\mathfrak{l}$ we conclude by induction that $v_{\beta_i}, \ldots, v_{\beta_{M+1}} \in \tilde{\rho}(V^\times)$. However, having $\tilde{\rho}$ preserve $\bigoplus_{k=1}^M V(\beta_k)$ implies $\tilde{\rho}(V^\times)$ has at most $q$ linearly independent maximal vectors of weight $\lambda_1$. Consequently, $i > M + 1 - q$.

Next suppose $v_{\gamma_i} \in \tilde{\rho}(V^\times)$ for some $i \in [M+1]$. Once again, preservation of $\bigoplus_{k=1}^M V(\beta_k)$ and $\bigoplus_{k=1}^M V(\gamma_k)$ under $\tilde{\rho}$ implies $p \geq 1$. Thus we have $X_{m+2}, \ldots, X_{m+i} \in \mathfrak{n}(V^\times) = \mathfrak{n}(\tilde{\rho}(V^\times))$. By noting that $\alpha_{m+i} + \gamma_i \in \gamma_{i-1} + \Phi_\mathfrak{l}$ we conclude by induction that $v_{\gamma_1}, \ldots, v_{\gamma_i} \in \tilde{\rho}(V^\times)$. Since $\tilde{\rho}(V^\times)$ can have at most $p$ linearly independent maximal vectors of weight $\lambda_m$ we conclude that $i \leq p$.

**Case 3: $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (B_n, A_m)$**

Since we are considering the case where $(\mathfrak{g}, \mathfrak{l})$ is well-mannered and $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) \neq (C_n, A_m)$ by assumption we note that this is the last case to analyze. By Lemma 4.32 there exist $p, q, c, d \in [M]_0$ with $q \leq p$, $d \leq p - q$, and $c \leq M - p$ such that $V^\times$ is given by either Equation 4.77a, Equation 4.77b, or Equation 4.77c. Additionally, $c + d \leq p$ if Equation 4.77b describes $V^\times$. Since $\tilde{\rho}|_\mathfrak{l} = \mathrm{id}$ we have that $\tilde{\rho}$ preserves $\bigoplus_{k=1}^{M+1} V(\beta_k) \oplus \bigoplus_{k=1}^M V(\mu_k)$, $\bigoplus_{k=1}^{M+1} V(\gamma_k) \oplus \bigoplus_{k=1}^M V(\nu_k)$, and $V(\eta)$. We now proceed by an examination of subcases.

**Case 3a: Equation 4.77a describes $V^\times$**

Since $v_\eta$ is the only maximal vector, modulo scalar multiples, of weight $\lambda_{m-1}$ and $\tilde{\rho}|_\mathfrak{l} = \mathrm{id}$ we have that $\tilde{\rho}$ preserves $V(\eta)$. We now show that $\tilde{\rho}$ preserves $\bigoplus_{k=1}^p V(\nu_k) \oplus \bigoplus_{k=1}^q V(\gamma_k)$. Since $\tilde{\rho}|_\mathfrak{l} = \mathrm{id}$ it suffices to show by dimension considerations and regularity of $\tilde{\rho}(V^\times)$ that for all $i \in [M]$, if $v_{\gamma_i} \in \tilde{\rho}(V^\times)$ then $i \leq q$ and if $v_{\nu_i} \in \tilde{\rho}(V^\times)$ then $i \leq p$.

Suppose by contradiction that $v_{\gamma_i} \in \tilde{\rho}(V^\times)$ for some $i \in \{q+1, \ldots, M\}$. As $Y_{m+i+1,\ldots,n} \in \mathfrak{n}(V^\times) = \mathfrak{n}(\tilde{\rho}(V^\times))$ and $-\sum_{k=m+i+1}^{n} \alpha_k + \gamma_i \in \gamma_{M+1} + \Phi_{\mathfrak{l}}$ we conclude that $v_{\gamma_{M+1}} \in \tilde{\rho}(V^\times)$. Since $v_{\gamma_{M+1}} \notin V^\times$ this contradicts Lemma 4.33. Similarly, suppose by contradiction that $v_{\nu_i} \in \tilde{\rho}(V^\times)$ for some $i \in \{p+1, \ldots, M\}$. We have that $X_{m+i+1,\ldots,n} \in \mathfrak{n}(V^\times)$ and so $X_{m+i+1,\ldots,n} \in \mathfrak{n}(\tilde{\rho}(V^\times))$ by Equation 4.137. Since $\sum_{k=m+i+1}^{n} \alpha_k + \nu_i \in \gamma_{M+1} + \Phi_{\mathfrak{l}}$ we again reach a contradiction. Therefore $\tilde{\rho}$ preserves $\bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$.

We now show that $\tilde{\rho}$ preserves $\bigoplus_{k=p-d+1}^{p} V(\beta_k)$. This amounts to proving that $v_{\beta_i} \notin \tilde{\rho}(V^\times)$ for all $i \in [p-d] \cup \{p+1, \ldots, M\}$ and $v_{\mu_j} \notin \tilde{\rho}(V^\times)$ for all $j \in [M]$. If $d = 0$ this is trivial so assume $d > 0$.

First suppose by contradiction that $v_{\beta_i} \in \tilde{\rho}(V^\times)$ for some $i \in [p-d] \cup \{p+1, \ldots, M\}$. Since $V(\eta) \subseteq V^\times$ and $\tilde{\rho}|_{\mathfrak{l}} = \mathrm{id}$ we have that $V(\eta) \subseteq \tilde{\rho}(V^\times)$. Consequently, as $v_{\beta_i}, v_\eta \in \tilde{\rho}(V^\times)$, Equation 4.101 implies $v_{\nu_i} \in \tilde{\rho}(V^\times)$. Since we have already shown that $\tilde{\rho}$ preserves $\bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$ it must be that $i \leq p$ and so $i \in [p-d]$. For each $k \in \{p-d+1, \ldots, p\}$ we have $X_{m+i+1,\ldots,m+k} \in \mathfrak{n}(V^\times) = \mathfrak{n}(\tilde{\rho}(V^\times))$, in turn implying $v_{\beta_k} \in \tilde{\rho}(V^\times)$ since $\sum_{l=m+i+1}^{m+k} \alpha_l + \beta_i \in \beta_k + \Phi_{\mathfrak{l}}$. Therefore, $v_{\beta_i} \in \tilde{\rho}(V^\times)$ and $v_{\beta_{p-d+1}}, \ldots, v_{\beta_p} \in \tilde{\rho}(V^\times)$ which is impossible since $V^\times$, and hence $\tilde{\rho}(V^\times)$, contains at most $d$ linearly independent maximal vectors of weight $\lambda_1$.

It remains to prove that $v_{\mu_j} \notin \tilde{\rho}(V^\times)$ for all $j \in [M]$ so suppose there exists $j \in [M]$ with $v_{\mu_j} \in \tilde{\rho}(V^\times)$. Since $v_\eta \in \tilde{\rho}(V^\times)$ we deduce from Equation 4.101 that $v_{\gamma_j} \in \tilde{\rho}(V^\times)$, forcing $j \leq q$. Since $d > 0$ this forces $j < p$ from which one can verify that $Y_{m+j+1,\ldots,n,n,n-1,\ldots,m+p+1} \in \mathfrak{n}(V^\times) = \mathfrak{n}(\tilde{\rho}(V^\times))$. This then implies $v_{\beta_p} \in \tilde{\rho}(V^\times)$ since $-\sum_{k=m+j+1}^{m+p} \alpha_k - 2\sum_{k=m+p+1}^{n} \alpha_k + \mu_j \in \beta_p + \Phi_{\mathfrak{l}}$. However, for each $k \in \{p-d+1, \ldots, p-1\}$ we have $Y_{m+k+1,\ldots,m+p} \in \mathfrak{n}(V^\times) = \mathfrak{n}(\tilde{\rho}(V^\times))$ which implies $v_{\beta_k} \in \tilde{\rho}(V^\times)$ since $-\sum_{l=m+k+1}^{m+p} \alpha_l + \beta_p \in \beta_k + \Phi_{\mathfrak{l}}$. This contradicts the fact that $\tilde{\rho}(V^\times)$ contains at most $d$ linearly independent maximal vectors of weight $\lambda_1$ since $v_{\mu_j} \in \tilde{\rho}(V^\times)$ and $v_{\beta_{p-d+1}}, \ldots, v_{\beta_p} \in \tilde{\rho}(V^\times)$. Hence $\tilde{\rho}$ preserves $V^\times$ in this subcase.

**Case 3b: Equation 4.77b describes $V^\times$**

We first show that for all $i \in [M]$, $v_{\gamma_i}, v_{\mu_i} \notin \tilde{\rho}(V^\times)$. If $v_{\gamma_i} \in \tilde{\rho}(V^\times)$ for some $i \in [M]$ then having $Y_{m+i+1,\ldots,n} \in \mathfrak{n}(V^\times) = \mathfrak{n}(\tilde{\rho}(V^\times))$ implies $v_{\gamma_{M+1}} \in \tilde{\rho}(V^\times)$ since $-\sum_{k=m+i+1}^{n} \alpha_k + \gamma_i \in \gamma_{M+1} + \Phi_{\mathfrak{l}}$. This contradicts Lemma 4.33 since $v_{\gamma_{M+1}} \notin V^\times$. Similarly, if $v_{\mu_i} \in \tilde{\rho}(V^\times)$ then having $Y_{m+i+1,\ldots,n} \in \mathfrak{n}(\tilde{\rho}(V^\times))$ and $-\sum_{k=m+i+1}^{n} \alpha_k + \mu_i \in \beta_{M+1} + \Phi_{\mathfrak{l}}$ implies $v_{\beta_{M+1}} \in \tilde{\rho}$. By Equation 4.98 this implies $V(\zeta) \subseteq \tilde{\rho}(V^\times)$, contradicting Lemma 4.29.

Since $\tilde{\rho}|_{\mathfrak{l}} = \mathrm{id}$ we have that there exist $a_1, \ldots, a_p \in [M]$ distinct and $b_1, \ldots, b_{c+d} \in [M]$ distinct such that

$$\tilde{\rho}(V^{\times}) = \bigoplus_{k=1}^{p} V(\nu_{a_k}) \oplus \bigoplus_{k=1}^{c+d} V(\beta_{b_k}) \tag{4.139}$$

We claim that $a_1, \ldots, a_p, b_1, \ldots, b_{c+d} \in [p] \cup \{M-c+1, \ldots, M\}$. Indeed, if $a_i \in \{p+1, \ldots, M-c\}$ for some $i \in [p]$ then $X_{m+a_i+1,\ldots,n} \in \mathfrak{n}(V^{\times}) = \mathfrak{n}(\tilde{\rho}(V^{\times}))$, implying $v_{\gamma_{M+1}} \in \tilde{\rho}(V^{\times})$ since $\sum_{k=m+a_i+1}^{n} \alpha_k + \nu_{a_k} \in \gamma_{M+1} + \Phi_{\mathfrak{l}}$. This is a contradiction. In a similar vein, if $b_i \in \{p+1, \ldots, M-c\}$ for some $i \in [c+d]$ then having $X_{m+b_i+1,\ldots,n} \in \mathfrak{n}(\tilde{\rho}(V^{\times}))$ and $\sum_{k=m+b_i+1}^{n} \alpha_k + \beta_{b_i} \in \beta_{M+1} + \Phi_{\mathfrak{l}}$ implies $v_{\beta_{M+1}} \in \tilde{\rho}(V^{\times})$, which is again a contradiction. We now consider two further subcases.

**Case 3b.1: There exists $l \in [p]$ such that $a_l \in [p-d]$**

For each $i \in [a_l - 1]$ we have $Y_{m+i+1,\ldots,m+a_l} \in \mathfrak{n}(V^{\times}) = \mathfrak{n}(\tilde{\rho}(V^{\times}))$. Since $-\sum_{k=m+i+1}^{m+a_l} \alpha_k + \nu_{a_l} \in \nu_i + \Phi_{\mathfrak{l}}$ we have $v_{\nu_i} \in \tilde{\rho}(V^{\times})$. Similarly, for each $j \in \{a_l+1, \ldots, p\}$ we have $X_{m+a_l+1,\ldots,m+j} \in \mathfrak{n}(V^{\times}) = \mathfrak{n}(\tilde{\rho}(V^{\times}))$. Then $v_{\nu_j} \in \tilde{\rho}(V^{\times})$ since $\sum_{k=m+a_l+1}^{m+j} \alpha_k + \nu_{a_l} \in \nu_j + \Phi_{\mathfrak{l}}$. As such, $v_{\nu_1}, \ldots, v_{\nu_p} \in \tilde{\rho}(V^{\times})$. As $b_1, \ldots, b_{c+d} \in [p] \cup \{M-c+1, \ldots, M\}$ we may suppose without loss of generality that $b_1, \ldots, b_d \in [p]$ and $b_{d+1}, \ldots, b_{c+d} \in \{M-c+1, \ldots, M\}$. Indeed, we may do so because of Theorem 4.19 and the fact that $V^{\times} = \bigoplus_{\alpha \in \Theta} \mathfrak{g}_{\alpha}$, where

$$\begin{aligned}\Theta = \{&= \varepsilon_i - \varepsilon_{m+k+1} \in \mathbb{F}^n : i \in [m+1], k \in [p]\} \\ &\cup \{\varepsilon_i - \varepsilon_{m+k+1} \in \mathbb{F}^n : i \in [m+1], k \in \{p-d+1, \ldots, p\} \\ &\cup \{M-c+1, \ldots, M\}\}\end{aligned} \tag{4.140}$$

Hence $\{b_{d+1}, \ldots, b_{c+d}\} = \{M-c+1, \ldots, M\}$ and so it remains to prove that $\{b_1, \ldots, b_d\} = \{p-d+1, \ldots, p\}$. If this is not the case then there exists $i \in [d]$ such that $b_i \in [p-d]$. Then for each $j \in \{p-d+1, \ldots, p\}$ we have $X_{m+b_i+1,\ldots,m+j} \in \mathfrak{n}(V^{\times}) = \mathfrak{n}(\tilde{\rho}(V^{\times}))$. As $\sum_{k=m+b_i+1}^{m+j} \alpha_k + \beta_{b_i} \in \beta_j + \Phi_{\mathfrak{l}}$ it follows that $v_{\beta_j} \in \tilde{\rho}(V^{\times})$. Hence $v_{\beta_l} \in \tilde{\rho}(V^{\times})$ and $v_{\beta_{p-d+1}}, \ldots, v_{\beta_p} \in \tilde{\rho}(V^{\times})$, which is clearly impossible by consideration of the cardinality of $\{b_1, \ldots, b_d\}$. Hence $\{b_1, \ldots, b_d\} = \{p-d+1, \ldots, p\}$ from which it follows that $\tilde{\rho}$ preserves $V^{\times}$.

**Case 3b.2: $a_1, \ldots, a_p \notin [p-d]$**

Since $a_1, \ldots, a_p \in [p] \cup \{M-c+1, \ldots, M\}$ it must be that $a_1, \ldots, a_p \in \{p-d+1, \ldots, p\} \cup \{M-c+1, \ldots, M\}$. Recall that $p \geq c+d$ and so in this subcase it must be that $p = c+d$ and $\{a_1, \ldots, a_p\} = \{p-d+1, \ldots, p\} \cup \{M-c+1, \ldots, M\}$. Since $V^{\times} = \bigoplus_{\alpha \in \Theta} \mathfrak{g}_{\alpha}$ with $\Theta$ as in Equation 4.140 we may suppose without loss of generality due to Theorem 4.19 that $b_1, \ldots, b_d \in \{p-d+1, \ldots, p\} \cup \{M-c+1, \ldots, M\}$ and $b_{d+1}, \ldots, b_{c+d} \in$

$[p-d]$ since $b_1, \ldots, b_{c+d} \in [p] \cup \{M-c+1, \ldots, M\}$. Moreover, as $p = c+d$ we have $\{b_{d+1}, \ldots, b_{c+d}\} = [p-d]$. We claim that $\{b_1, \ldots, b_d\} = \{p-d+1, \ldots, p\}$. Indeed, if there exists $i \in [d]$ such that $b_l \in \{M-c+1, \ldots, M\}$ then $Y_{m+j+1,\ldots,m+b_i} \in \mathfrak{n}(V^\times) = \mathfrak{n}(\tilde{\rho}(V^\times))$ for each $j \in \{p-d+1, \ldots, p\}$, implying $v_{\beta_j} \in \tilde{\rho}(V^\times)$ since $-\sum_{k=m+j+1}^{m+b_i} \alpha_k + \beta_{b_i} \in \beta_j + \Phi_{\mathfrak{l}}$. Consequently, $v_{\beta_{b_l}} \in \tilde{\rho}(V^\times)$ and $v_{\beta_{p-d+1}}, \ldots, v_{\beta_p} \in \tilde{\rho}(V^\times)$, which is impossible by cardinality of $\{b_1, \ldots, b_d\}$. It follows that $\{b_1, \ldots, b_d\} = \{p-d+1, \ldots, p\}$. Therefore, in this subcase we have $p = c+d$ and

$$\tilde{\rho}(V^\times) = \bigoplus_{k=p-d+1}^{p} V(\nu_k) \oplus \bigoplus_{k=M-c+1}^{M} V(\nu_k) \oplus \bigoplus_{k=1}^{b} V(\beta_k) \tag{4.141}$$

In particular, note that $\tilde{\rho}$ does not preserve $V^\times$. However, let $\tau \in \operatorname{Int} \mathfrak{g}$ be the inner automorphism constructed in Lemma 4.26 as in Equation 4.58. Note that $\tau$ fixes $C_{\mathfrak{g}}(\mathfrak{l})$ element-wise, preserves $\mathfrak{l}$, and maps $\tilde{\rho}(V^\times)$ to $V^\times$. Thus $\tau\tilde{\rho}$ maps $\mathfrak{a}_1$ to $\mathfrak{a}_2$.

**Case 3c: Equation 4.77c describes $V^\times$**

Since $v_\eta$ is the only maximal vector (up to scalar multiples) of weight $\lambda_{m-1}$ and $\tilde{\rho}|_{\mathfrak{l}} = \mathrm{id}$ we conclude by Lemma 4.33 that $V(\eta), V(\gamma_{M+1}) \subseteq \tilde{\rho}(V^\times)$. As $v_{\gamma_{M+1}} \in \tilde{\rho}(V^\times)$ and $X_{m+i+1,\ldots,n} \in \mathfrak{n}(V^\times)$ for each $i \in [q]$ we have $X_{m+i+1,\ldots,n} \in \mathfrak{n}(\tilde{\rho}(V^\times))$ for each $i \in [q]$ by Equation 4.137. Since $\sum_{k=m+i+1}^{n} \alpha_k + \gamma_{M+1} \in \gamma_i + \Phi_{\mathfrak{l}}$ we conclude that $v_{\gamma_1}, \ldots, v_{\gamma_q} \in \tilde{\rho}(V^\times)$. Therefore, since $\tilde{\rho}$ preserves $\bigoplus_{k=1}^{M+1} V(\gamma_k) \oplus \bigoplus_{k=1}^{M} V(\nu_k)$ we conclude that $\tilde{\rho}$ preserves $V(\gamma_{M+1}) \oplus \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$.

Now suppose by contradiction that $v_{\mu_i} \in \tilde{\rho}(V^\times)$ for some $i \in [M]$. Since $v_\eta \in \tilde{\rho}(V^\times)$ we have by Equation 4.101 that $v_{\gamma_i} \in \tilde{\rho}(V^\times)$, forcing $i \leq q$. However, this implies $Y_{m+i+1,\ldots,n} \in \mathfrak{n}(V^\times) = \mathfrak{n}(\tilde{\rho}(V^\times))$, which in turn implies $v_{\beta_{M+1}} \in \tilde{\rho}(V^\times)$ since $-\sum_{k=m+i+1}^{n} \alpha_k + \mu_i \in \beta_{M+1} + \Phi_{\mathfrak{l}}$. By Equation 4.98 we have $V(\zeta) \subseteq \tilde{\rho}(V^\times)$, contradicting Lemma 4.29. Hence $v_{\mu_1}, \ldots, v_{\mu_M} \notin \tilde{\rho}(V^\times)$.

Now let $i \in [M]$ be such that $v_{\beta_i} \in \tilde{\rho}(V^\times)$. Equation 4.101 implies $v_{\nu_i} \in \tilde{\rho}$ and so $i \leq p$. Suppose by contradiction that $i \in [p-d]$. Then for each $j \in \{p-d+1, \ldots, p\}$ we have $X_{m+i+1,\ldots,m+j} \in \mathfrak{n}(V^\times) = \mathfrak{n}(\tilde{\rho}(V^\times))$. Since $\sum_{k=m+i+1}^{m+j} + \beta_i \in \beta_j + \Phi_{\mathfrak{l}}$ we conclude that $v_{\beta_j} \in \tilde{\rho}(V^\times)$. Hence $v_{\beta_i} \in \tilde{\rho}(V^\times)$ and $v_{\beta_{p-d+1}}, \ldots, v_{\beta_p} \in \tilde{\rho}(V^\times)$, which is impossible since $\tilde{\rho}(V^\times)$ contains at most $d$ linearly independent maximal vectors of weight $\lambda_1$. It follows that $i \in \{p-d+1, \ldots, p\}$ and so $\tilde{\rho}$ preserves $V^\times$. □

With Theorem 4.37 we have a rather elegant description of conjugacy of Levi decomposable subalgebras when $(\mathrm{t}(\mathfrak{g}), \mathrm{t}(\mathfrak{l})) \neq (C_n, A_m)$. Indeed, Theorem 4.37 allows us to pass the question of conjugacy in $\mathfrak{g}$ to determining conjugacy in $C_{\mathfrak{g}}(\mathfrak{l})$. Thus we are still faced with the arduous task of establishing conjugacy but at the very least we can tackle this issue in the smaller subalgebra $C_{\mathfrak{g}}(\mathfrak{l})$.

## 4.10 THE TRIVIAL COMPONENT WHEN $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (C_n, A_m)$

Of course, we must address the fact that Theorem 4.37 awkwardly excludes the situation where $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (C_n, A_m)$. What goes wrong in this particular setting? Unfortunately, it turns out that having an inner automorphism $\rho \in \operatorname{Int} C_{\mathfrak{g}}(\mathfrak{l})$ mapping $V_1^0$ to $V_2^0$ and preserving $\mathfrak{n}(V^\times)$ is not sufficient for ensuring that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are conjugate. Indeed, consider the case where $V^\times = V(\eta) \oplus \bigoplus_{k=1}^{M-1} V(\nu_k) \oplus V(\gamma_1)$, $V_1^0 = \langle X_{m+2,\dots,n,n-1,\dots,m+2} \rangle$, and $V_2^0 = \langle X_n \rangle$. It is straightforward to verify that $\mathfrak{a}_1 = \mathfrak{l} \Subset (V^\times \oplus V_1^0)$ and $\mathfrak{a}_2 = \mathfrak{l} \Subset (V^\times \oplus V_2^0)$ are indeed Levi decomposable subalgebras. We claim there exists $\rho \in \operatorname{Int} C_{\mathfrak{g}}(\mathfrak{l})$ mapping $V_1^0$ to $V_2^0$ which preserves $\mathfrak{n}(V^\times)$ even though $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are not conjugate.

By direct computation, as well as by using the fact that $\mathfrak{n}(V^\times)$ is regular relative to $\mathfrak{h}$ since $V^\times$ is regular to $\mathfrak{h}$, we find

$$\mathfrak{n}(V^\times) = \bigoplus_{\alpha \in \Phi_{V^\times}} \mathfrak{g}_\alpha \tag{4.142}$$

where

$$\begin{aligned}\Phi_{V^\times} = &\{\varepsilon_{m+i+1} - \varepsilon_{m+j+1} \in \mathbb{F}^n : 1 \le i < j < M\} \\ &\cup \{-\varepsilon_{m+i+1} + \varepsilon_{m+j+1} \in \mathbb{F}^n : 1 < i < j \le M\} \cup \{2\varepsilon_{m+2}\} \cup \{2\varepsilon_n\} \\ &\cup \{\varepsilon_{m+i+1} + \varepsilon_{m+j+1} \in \mathbb{F}^n : 1 \le i \le j \le M, (i,j) \neq (1, M)\}\end{aligned} \tag{4.143}$$

Define $P \in GL_{2n}$ as

$$P = \begin{pmatrix} I & 0_{m+1,M} & 0_{m+1,m+1} & 0_{m+1,M} \\ 0_{M,m+1} & Q & 0_{M,m+1} & 0_{M,M} \\ 0_{m+1,m+1} & 0_{m+1,M} & I & 0_{m+1,M} \\ 0_{M,m+1} & 0_{M,M} & 0_{M,m+1} & Q \end{pmatrix} \tag{4.144}$$

where $Q \in \mathfrak{gl}_M$ is defined as

$$Q = \sum_{k=1}^{M} E_{M-k+1,k} \tag{4.145}$$

One can verify that $P \in Sp_{2n}$, implying conjugation by $P$ defines an inner automorphism $\tau \in \operatorname{Int}\mathfrak{g}$. Appropriate matrix calculations reveal that for all $i, j, k, l \in [M]$ with $i < j$ and $k \leq l$,

$$\tau(\mathfrak{g}_{\varepsilon_{m+i+1}-\varepsilon_{m+j+1}}) = \mathfrak{g}_{-\varepsilon_{n-j+1}+\varepsilon_{n-i+1}}, \quad \tau(\mathfrak{g}_{-\varepsilon_{m+i+1}+\varepsilon_{m+j+1}}) = \mathfrak{g}_{\varepsilon_{n-j+1}-\varepsilon_{n-i+1}} \tag{4.146a}$$

$$\tau(\mathfrak{g}_{\varepsilon_{m+k+1}+\varepsilon_{m+l+1}}) = \mathfrak{g}_{\varepsilon_{n-l+1}+\varepsilon_{n-k+1}}, \quad \tau(\mathfrak{g}_{-\varepsilon_{m+k+1}-\varepsilon_{m+l+1}}) = \mathfrak{g}_{-\varepsilon_{n-l+1}-\varepsilon_{n-k+1}} \tag{4.146b}$$

It follows from Equation 4.142 and Equation 4.143 that $\tau$ preserves $\mathfrak{n}(V^\times)$ and maps $V_1^0$ to $V_2^0$. By taking $\rho = \tau|_{C_\mathfrak{g}(\mathfrak{l})}$ we have by Lemma 4.36 the existence of an inner automorphism of $C_\mathfrak{g}(\mathfrak{l})$ which preserves $\mathfrak{n}(V^\times)$ and maps $V_1^0$ to $V_2^0$.

Now suppose by contradiction that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are conjugate via some $\sigma \in \operatorname{Int}\mathfrak{g}$. By Lemma 4.15 we may suppose without loss of generality that $\sigma$ preserves $\mathfrak{l}$ and $V^\times$ while mapping $V_1^0$ to $V_2^0$. Having $\sigma$ preserve $\mathfrak{l}$ implies $\sigma$ is conjugation by some $P \in Sp_{2n}$ as in Equation 4.128 with $P_\mathfrak{l} \in Sp_{2(m+1)}$ and $P_{\mathfrak{l}^\perp} \in Sp_{2M}$, where $P_\mathfrak{l}$ and $P_{\mathfrak{l}^\perp}$ are given by Equation 4.129.

Since $V(\eta)$ is the only irreducible $\mathfrak{l}$-submodule of $V^\times$ with dimension $\binom{m+1}{2}$ it follows that $\sigma$ must preserve $V(\eta)$. Via appropriate matrix calculations and Lemma 4.27 we conclude that $\tilde{B} = 0$. Consequently, the restriction of $\sigma$ to $\mathfrak{l}$ is an inner automorphism. Thus by Corollary 2.4 we may suppose without loss of generality that $\sigma$ restricts to the identity on $\mathfrak{l}$. Hence $\tilde{A}$ and $\tilde{D}$ are nonzero scalar multiples of the identity matrix.

We find

$$\sigma(w_1) = (-1)^M \begin{pmatrix} 0_{m+1,m+1} & 0_{m+1,M} & 0_{m+1,m+1} & 0_{m+1,M} \\ 0_{M,m+1} & 2AE_{1,1}C^\top & 0_{M,m+1} & -2AE_{1,1}A^\top \\ 0_{m+1,m+1} & 0_{m+1,M} & 0_{m+1,m+1} & 0_{m+1,M} \\ 0_{M,m+1} & 2CE_{1,1}C^\top & 0_{M,m+1} & -2CE_{1,1}A^\top \end{pmatrix} \tag{4.147}$$

where we have defined $w_1 = X_{m+2,\dots,n,n-1,\dots,m+2}$. Since $\sigma$ maps $V_1^0$ to $V_2^0$ it must be that $\sigma(w_1) = \kappa X_n$ for some nonzero $\kappa \in \mathbb{F}$. Equation 2.5a implies $-2AE_{1,1}A^\top = \kappa E_{M,M}$ and in particular $A_{M,1} \neq 0$. In a manner similar to how Equation 4.56b was obtained we find that

$$\sigma(v_{\gamma_1}) = \sum_{l=1}^{M} (a_{l,1} v_{\gamma_l} + c_{l,1} v_{\nu_l}) \tag{4.148}$$

with $a_{M,1} \neq 0$ since $A_{M,1} \neq 0$ and both $\tilde{A}$ and $\tilde{D}$ are nonzero scalar multiples of the identity matrix. Consequently, $\sigma(v_{\gamma_1}) \notin V^\times$ even though

$v_{\gamma_1} \in V^\times$, contradicting the assumption that $\sigma$ preserves $V^\times$. As such $\mathfrak{a}_1$ and $\mathfrak{a}_2$ can not possibly be conjugate.

The above counterexample illustrates the failure of Theorem 4.37 when $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (C_n, A_m)$. Fortunately, there is an elementary solution to this potential problem. In Section 4.9 we noted that if $\rho \in \text{Int } C_{\mathfrak{g}}(\mathfrak{l})$ could be extended to an inner automorphism $\tilde{\rho} \in \text{Int } \mathfrak{g}$ such that $\tilde{\rho}$ preserves $\mathfrak{l}$ and $V^\times$ then $\rho$ must also preserve $\mathfrak{n}(V^\times)$, which was defined using the normalizer. Of course we may also consider the centralizer and so we define a subalgebra $\mathfrak{c}(V^\times)$ as

$$\mathfrak{c}(V^\times) = C_{\mathfrak{g}}(V^\times) \cap C_{\mathfrak{g}}(\mathfrak{l}) \tag{4.149}$$

In precisely the same manner one can argue that such an inner automorphism $\rho$ must also preserve $\mathfrak{c}(V^\times)$. The inclusion of this additional assumption remedies the difficulty we previously faced, as shown in this next theorem.

**Theorem 4.38.** *Suppose $(t(\mathfrak{g}), t(\mathfrak{l})) = (C_n, A_m)$ and let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be Levi decomposable subalgebras of $\mathfrak{g}$ with common Levi factor $\mathfrak{l}$. Further suppose the radicals of $\mathfrak{a}_1$ and $\mathfrak{a}_2$ have the same nontrivial component $V^\times$ and let $V_1^0$ and $V_2^0$ be the trivial components of* Rad $\mathfrak{a}_1$ *and* Rad $\mathfrak{a}_2$*, respectively. Then $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are conjugate if and only if there exists $\rho \in \text{Int } C_{\mathfrak{g}}(\mathfrak{l})$ such that $\rho$ preserves both $\mathfrak{n}(V^\times)$ and $\mathfrak{c}(V^\times)$ and maps $V_1^0$ to $V_2^0$.*

*Proof.* Without loss of generality we may suppose by Lemma 4.34 that $V^\times$ is given by Equation 4.116a or Equation 4.116b. If $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are conjugate then the obvious modification of the argument in the proof of Theorem 4.37 implies the existence of $\rho \in \text{Int } C_{\mathfrak{g}}(\mathfrak{l})$ mapping $V_1^0$ to $V_2^0$ and preserving both $\mathfrak{n}(V^\times)$ and $\mathfrak{c}(V^\times)$.

Conversely, suppose there exists $\rho \in \text{Int } C_{\mathfrak{g}}(\mathfrak{l})$ such that $\rho$ preserves $\mathfrak{n}(V^\times)$ and $\mathfrak{c}(V^\times)$. Extend $\rho$ to some $\tilde{\rho} \in \text{Int } \mathfrak{g}$ by stipulating that $\tilde{\rho}|_{\mathfrak{l}} = \text{id}$. As $\tilde{\rho}$ maps $V_1^0$ to $V_2^0$ and preserves $\mathfrak{l}$ by construction we need only show that $\tilde{\rho}$ preserves $V^\times$. By Equation 4.137 and its appropriate analogue we have that

$$\mathfrak{n}(\tilde{\rho}(V^\times)) = \mathfrak{n}(V^\times), \qquad \mathfrak{c}(\tilde{\rho}(V^\times)) = \mathfrak{c}(V^\times) \tag{4.150}$$

The same reasoning as in the proof of Theorem 4.37 implies $\tilde{\rho}(V^\times)$ is regular relative to $\mathfrak{h}$. We proceed via a consideration of two cases.

**Case 1: Equation 4.116a describes $V^\times$**

Since $\tilde{\rho}|_{\mathfrak{l}} = \text{id}$ it is immediate that $\tilde{\rho}$ preserves $V(\eta)$. By dimension arguments and regularity of $\tilde{\rho}(V^\times)$ relative to $\mathfrak{h}$, to show that $\tilde{\rho}$ preserves $\bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$ it suffices to show that for all $i, j \in [M]$, if $v_{\gamma_i} \in \tilde{\rho}(V^\times)$ then $i \leq q$ and if $v_{\nu_j} \in \tilde{\rho}(V^\times)$ then $i \leq p$.

If $v_{\gamma_i} \in \tilde{\rho}(V^\times)$ for some $i \in \{q+1, \ldots, M\}$ then one can verify that $Y_{m+i+1,\ldots,n,n-1,\ldots,m+i+1} \in \mathfrak{c}(V^\times)$. As such, $Y_{m+i+1,\ldots,n,n-1,\ldots,m+i+1} \in \mathfrak{c}(\tilde{\rho}(V^\times))$ by Equation 4.150. However, $-2\sum_{k=m+i+1}^{n-1} \alpha_k - \alpha_n + v_{\gamma_i} \in \nu_i + \Phi_\mathfrak{l}$, implying $[Y_{m+i+1,\ldots,n,n-1,\ldots,m+i+1}, v_{\gamma_i}]$ is a nonzero element of $\mathfrak{g}_{\nu_i}$, contradicting the fact that $Y_{m+i+1,\ldots,n,n-1,\ldots,m+i+1} \in \mathfrak{c}(\tilde{\rho}(V^\times))$. Similarly, suppose $j \in \{p+1, \ldots, M\}$ is such that $v_{\nu_j} \in \tilde{\rho}(V^\times)$. Since $X_{m+j+1,\ldots,n,n-1,\ldots,m+j+1} \in \mathfrak{c}(V^\times) = \mathfrak{c}(\tilde{\rho}(V^\times))$ and $2\sum_{k=m+j+1}^{n-1} \alpha_k + \alpha_n + \nu_j \in \gamma_j + \Phi_\mathfrak{l}$ we reach a contradiction. It follows that $\tilde{\rho}$ preserves $\bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$.

Now let $i \in [M]$ be such that $v_{\beta_i} \in \tilde{\rho}(V^\times)$. Equation 4.124 implies $v_{\nu_i} \in \tilde{\rho}(V^\times)$ and thus $i \leq p$. If $i \leq p-d$ then for each $j \in \{p-d+1, \ldots, p\}$ we have $X_{m+i+1,\ldots,m+j} \in \mathfrak{n}(V^\times) = \mathfrak{n}(\tilde{\rho}(V^\times))$. Since $\sum_{k=m+i+1}^{m+j} \alpha_k + \beta_i \in \beta_j + \Phi_\mathfrak{l}$ we note that $v_{\beta_j} \in \tilde{\rho}(V^\times)$. As such, $v_{\beta_i} \in \tilde{\rho}(V^\times)$ and $v_{\beta_{p-d+1}}, \ldots, v_{\beta_p} \in \tilde{\rho}(V^\times)$, contradicting dimension considerations. It follows that if $v_{\beta_i} \in \tilde{\rho}(V^\times)$ then $i \in \{p-d+1, \ldots, p\}$.

Lastly, suppose by contradiction that there exists $j \in [M]$ with $v_{\gamma_i} \in \tilde{\rho}(V^\times)$. From Equation 4.124 we conclude that $v_{\gamma_j} \in \tilde{\rho}(V^\times)$ as well, implying $j \leq q$. However, as $j \leq q \leq p$ we have $v_{\nu_j} \in \tilde{\rho}(V^\times)$ as well. This is a contradiction because $\mu_j + \nu_j \in \Phi_\mathfrak{l}$.

**Case 2: Equation 4.116b describes $V^\times$**

One can verify that for all $i \in [M]$, $Y_{m+i+1,\ldots,n,n-1,\ldots,m+i+1} \in \mathfrak{c}(V^\times) = \mathfrak{c}(\tilde{\rho}(V^\times)$. Since $-2\sum_{k=m+i+1}^{n-1} \alpha_k - \alpha_n + \gamma_i \in \nu_i + \Phi_\mathfrak{l}$ and $-2\sum_{k=m+i+1}^{n-1} \alpha_k - \alpha_n + \mu_i \in \beta_i + \Phi_\mathfrak{l}$, it follows that $v_{\gamma_i}, v_{\mu_i} \notin \tilde{\rho}(V^\times)$ for all $i \in [M]$.

Since $\tilde{\rho}|_\mathfrak{l} = \mathrm{id}$, $\tilde{\rho}(V^\times)$ is regular relative to $\mathfrak{h}$, and $V^\times$ is a sum of $p$ irreducible $\mathfrak{l}$-submodules of highest weight $\lambda_m$ and $c+d$ irreducible $\mathfrak{l}$-submodules of highest weight $\lambda_1$, it must be that there exist $a_1, \ldots, a_p \in [M]$ distinct and $b_1, \ldots, b_{c+d} \in [M]$ distinct such that $\tilde{\rho}(V^\times)$ is as in Equation 4.139. Suppose by contradiction there exists $l \in [p]$ with $p < a_l \leq M - c$. Note that we may suppose $p > 0$ as otherwise there is nothing to show. In this case, we have $Y_{m+a_l} \in \mathfrak{c}(V^\times) = \mathfrak{c}(\tilde{\rho}(V^\times))$, but this is impossible since $-\alpha_{m+a_l} + \nu_{a_l} \in \nu_{a_l - 1}$. We obtain a similar contradiction if we suppose there exist $l \in [c+d]$ such that $p < b_l \leq M - c$ since $-\alpha_{m+b_l} + \beta_{b_l} \in \beta_{b_l - 1} + \Phi_\mathfrak{l}$. Therefore, $a_1, \ldots, a_p, b_1, \ldots, b_{c+d} \in [p] \cup \{M-c+1, \ldots, M\}$. We consider two cases: the first in which there exists $l \in [p]$ such that $a_l \in [p-d]$ and the other in which $a_1, \ldots, a_p \in \{p-d+1, \ldots, p\}$. The arguments in these cases are identical to those of Case 3b.1 and Case 3b.2, respectively, in the proof of Theorem 4.37. □

Like Theorem 4.37, Theorem 4.38 allows us to study the problem of conjugacy in a much smaller subalgebra. While it also requires finding an inner automorphism which preserves both $\mathfrak{n}(V^\times)$ and $\mathfrak{c}(V^\times)$, this downside is quite minimal compared to the advantage of establishing conjugacy in a smaller, and hence simpler, setting. Indeed, there is no general algorithm for determining whether two subalgebras are conjugate. Thus the smaller the Lie algebra the easier it is to examine and tinker with. The cost of restricting our focus to this smaller subalgebra is requiring knowledge of $\mathfrak{n}(V^\times)$ and $\mathfrak{c}(V^\times)$. However, this is a small price to pay since we can take advantage of the regularity of $\mathfrak{n}(V^\times)$ and $\mathfrak{c}(V^\times)$ to easily compute these subalgebras, and preservation of these subalgebras typically leads to useful constraints on the potential inner automorphisms. Hence in some sense preserving $\mathfrak{n}(V^\times)$ and $\mathfrak{c}(V^\times)$ is an advantage as it places constraints on any possible inner automorphism, making it easier for one to either find such an inner automorphism or disprove its existence.

Note that Theorem 4.37 implies Theorem 4.38 if we allow $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) \neq (C_n, A_m)$. Hence by considering both $\mathfrak{n}(V^\times)$ and $\mathfrak{c}(V^\times)$ we can merge Theorem 4.37 and Theorem 4.38 into a single general theorem. While such a presentation is sleeker, the author has opted to separate this result into two theorems since $\mathfrak{c}(V^\times)$ is only needed in one specific context. As such, the author views Theorem 4.37 as the general statement with Theorem 4.38 as one specific exception.

Theorem 4.19 yields a compact description of the nontrivial component of the radical of a Levi decomposable subalgebra, whereas Theorem 4.37 and Theorem 4.38 provide necessary and sufficient criteria for establishing conjugacy of such subalgebras. These results comprise the core findings of this thesis, allowing us to quite generally describe the Levi decomposable subalgebras of the classical Lie algebras with regular simple Levi factor. Of course, we must recall that our description is not entirely general since we have restricted our attention solely to well-mannered pairs. For completeness, we begin a brief discussion of ill-mannered pairs.

# 5

# ILL-MANNERED PAIRS

While Theorem 4.19, Theorem 4.37, and Theorem 4.38 provide substantial information in the setting of well-mannered pairs, we have yet to discuss what happens when $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l}))$ is ill-mannered. From the definition we note that the collection of all ill-mannered pairs naturally divides into two groups: those which are ill-mannered because $\mathfrak{l}$ is of low rank, and those which are ill-mannered because they are of the form $(D_n, A_m)$. We shall separately investigate these two cases and try to determine why the results of Chapter 4 do not apply. In both cases, we shall see in particular that Theorem 4.19 fails for ill-mannered pairs.

## 5.1 ILL-MANNERED PAIRS WITH $\mathfrak{l}$ OF LOW RANK

The first type of ill-mannered pairs we shall consider are those for which $\mathfrak{t}(\mathfrak{l}) = A_m$ with $m$ sufficiently small. In particular, if $\mathfrak{g}$ is a classical Lie algebra and $\mathfrak{l}$ is a simple subalgebra of $\mathfrak{g}$ then we first examine the case where either $\mathfrak{t}(\mathfrak{l}) = A_1$ or $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) \in \{(B_n, A_2), (D_n, A_2)\}$. It is not all that shocking that $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l}))$ is ill-mannered in this context. Indeed, as a general rule exceptions are expected in low-rank situations. For instance, the classification of simple Lie algebras can be elegantly described by four families, barring some exceptional low-rank cases. Similarly, Theorem 3.13 implies that if $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l}))$ is a regular pair then $\mathfrak{l}$ is a regular simple subalgebra of $\mathfrak{g}$, provided that the rank of $\mathfrak{l}$ is sufficiently large. With these examples in mind it is thus expected that Theorem 4.19 fails in certain low-rank environments, especially if the rank of the parent Lie algebra is not also small.

Recall the structure of the proofs of Lemma 4.23, Lemma 4.32, and Lemma 4.34. The idea in those proofs was to handle the components of $V^\times$ corresponding to copies of $V(\lambda_m)$ first, after which we analyze the structure of the copies of $V(\lambda_1)$. Note that in all cases, the elements of the $V(\lambda_m)$ components correspond to negative roots while the elements of the $V(\lambda_1)$ components are positive root vectors. In these proofs, having $m > 1$ allowed us to separate the positive and negative roots. The fact that we cannot uncouple positive and negative roots when $m = 1$ is precisely the reason why we have ill-mannered pairs, as we shall now demonstrate.

**Lemma 5.1.** *Let $\mathfrak{g}$ be a classical Lie algebra of rank at least 4 and let $\mathfrak{l}$ be a regular simple subalgebra of $\mathfrak{g}$ of type $A_1$. Then there exists a Levi decomposable subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ and nontrivial component of the radical $V^\times \subseteq \operatorname{Rad}\mathfrak{a}$ such that for all $\tau \in \operatorname{Int}\mathfrak{g}$ preserving $\mathfrak{l}$, there is no nontrivial $\Phi_\mathfrak{l}$-candidate $\Theta$ such that*

$$\tau(V^\times) = \bigoplus_{\alpha\in\Theta} \mathfrak{g}_\alpha \tag{5.1}$$

*Proof.* Since the result holds when applied to subalgebras conjugate to $\mathfrak{a}$ we may suppose without loss of generality that $\mathfrak{l}$ is described by Equation 3.10. Since $n \geq 4$ we have $M \geq 2$. As such we may define

$$w = v_{\gamma_1} + v_{\beta_2} = Y_2 + X_{1,2,3} \tag{5.2}$$

where $v_{\gamma_1}$ and $v_{\beta_1}$ are as in Equation 4.40, Equation 4.43, Equation 4.51, and/or Equation 4.55. Then $w$ is a maximal vector of weight $\lambda_1$. It is straightforward to verify that $[Y_1, w] = Y_{1,2} + X_{2,3}$ and that $[w, [Y_1, w]] = 2X_3$. One can then check that $\mathfrak{a} = \mathfrak{l} \inplus (V^\times \oplus V^0)$ with $V^\times = \langle w, [Y_1, w]\rangle$ and $V^0 = \langle X_3 \rangle$ defines a Levi decomposable subalgebra of $\mathfrak{g}$ with nontrivial and trivial components of the radical $V^\times$ and $V^0$, respectively. Note that $V^\times$ is an irreducible $\mathfrak{l}$-module of highest weight $\lambda_1$. Therefore, if there exists $\tau \in \operatorname{Int}\mathfrak{g}$ preserving $\mathfrak{l}$ and a nontrivial $\Phi_\mathfrak{l}$-candidate such that $\tau(V^\times)$ is as in Equation 5.1 then it must be that $\bigoplus_{\alpha\in\Theta}\mathfrak{g}_\alpha$ is also an irreducible $\mathfrak{l}$-module of highest weight $\lambda_1$ due to Lemma 4.10 and the fact that $\beta_k|_{\mathfrak{h}_\mathfrak{l}} = \gamma_k|_{\mathfrak{h}_\mathfrak{l}} = \lambda_1$ for each $k \in [M]$.

Clearly $V^\times$ is not an abelian subalgebra of $\mathfrak{g}$ since $[w, [Y_1, w]] \neq 0$. Since $V(\beta_i)$ and $V(\gamma_i)$ are abelian subalgebras of $\mathfrak{g}$ for all $i \in [M]$ the result follows by Lemma 4.9 and Lemma 4.10 if $\mathfrak{g}$ is of type $A_n$. If $\mathfrak{g}$ is not of type $A_n$ then we note that $V(\mu_i)$ and $V(\nu_i)$ are also abelian subalgebras of $\mathfrak{g}$ for all $i \in [M]$. The claim thus also holds when $\mathfrak{g}$ is of type $C_n$ or $D_n$. However, both $V(\beta_{M+1})$ and $V(\gamma_{M+1})$ are not abelian subalgebras of $\mathfrak{g}$ in the case of $\mathfrak{g}$ of type $B_n$. Therefore, if there exists $\tau \in \operatorname{Int}\mathfrak{g}$ preserving $\mathfrak{l}$ and a nontrivial $\Phi_\mathfrak{l}$ candidate $\Theta$ such that $\tau(V^\times)$ is as in Equation 5.1, then it must be that $\tau(V^\times) = V(\beta_{M+1})$ or $\tau(V^\times) = V(\gamma_{M+1})$. We prove both scenarios are impossible.

First suppose by contradiction that $\tau(V^\times) = V(\gamma_{M+1})$. We may regard $\tau$ as conjugation by some $P \in SO_{2n+1}$. Since $\tau$ preserves $\mathfrak{l}$, a straightforward matrix argument reveals that $P$ is as in Equation 4.106 with $P_\mathfrak{l}$ and $P_{\mathfrak{l}^\perp}$ as in

Equation 4.107 satisfying $P_{\mathfrak{l}} \in SO_4$ and $P_{\mathfrak{l}^\perp} \in SO_{2M+1}$. By the description of $v_{\gamma_{M+1}}$ given in Equation 4.51 we find

$$\tau^{-1}(v_{\gamma_{M+1}}) = \begin{pmatrix} 0 & -ae_2^\top \tilde{A} & 0_{1,M} & -ae_2^\top \tilde{B} & 0_{1,M} \\ a\tilde{B}^\top e_2 & 0_{2,2} & \tilde{B}^\top e_2 u^\top & 0_{2,2} & \tilde{B}^\top e_2 v^\top \\ 0_{M,1} & -ve_2^\top \tilde{A} & 0_{M,M} & -ve_2^\top \tilde{B} & 0_{M,M} \\ a\tilde{A}^\top e_2 & 0_{2,2} & \tilde{A}^\top e_2 u^\top & 0_{2,2} & \tilde{A}^\top e_2 v^\top \\ 0_{M,1} & -ue_2^\top \tilde{A} & 0_{M,M} & -ue_2^\top \tilde{B} & 0_{M,M} \end{pmatrix} \tag{5.3}$$

Every automorphism of $\mathfrak{l}$ is inner and so by Corollary 2.4 we may suppose without loss of generality that $\tau|_{\mathfrak{l}} = \mathrm{id}$. In particular, $\tau$ maps maximal vectors to maximal vectors and so $\tau^{-1}(v_{\gamma_{M+1}}) = \kappa w$ for some $\kappa \in \mathbb{F}$ nonzero. Since $w = E_{4,3} - E_{n+3,n+4} + E_{2,5} - E_{n+5,n+2}$ we have

$$\kappa E_{1,2} = -ve_2^\top \tilde{A}, \qquad -\kappa E_{2,1} = -ue_2^\top \tilde{B}, \qquad 0 = -ue_2^\top \tilde{A} \tag{5.4}$$

Equation 5.4 implies the $(1,2)$-entry of $-ve_2^\top \tilde{A}$ is $\kappa$. Consequently $\tilde{A}_{2,2} \neq 0$. In a completely analogous manner, Equation 5.4 implies the $(2,1)$-entry of $-ue_2^\top \tilde{B}$ is $-\kappa$. Hence $u_2 \neq 0$. However, by Equation 5.4 we have that the $(2,2)$-entry of $(-ue_2^\top \tilde{A})$ is 0, implying $0 = -u_2\tilde{A}_{2,2}$. As this is a contradiction we conclude that no such $\tau$ exists.

It remains to examine the case where $\tau(V^\times) = V(\beta_{M+1})$. However, we can immediately note by Lemma 4.26 that this would imply the existence of $\rho \in \mathrm{Int}\,\mathfrak{g}$ preserving $\mathfrak{l}$ satisfying $\rho(V^\times) = V(\gamma_{M+1})$, which we have already shown to be impossible. □

Lemma 5.1 confirms our suspicion that the description of Levi decomposable subalgebras with regular simple Levi factor need not behave so elegantly when the Levi factor has a sufficiently small rank. From this we can see why we chose to consider the regular pairs $(\mathfrak{t}(\mathfrak{g}), A_1)$ as ill-mannered, excluding such pairs from consideration in Chapter 4. Of note is that Lemma 5.1 operated under the assumption that the rank of $\mathfrak{g}$ was at least 4. The interested reader can examine what happens when rank $\mathfrak{g} < 4$ but we shall not pursue this question here. Indeed, such an endeavour is unnecessary with respect to this thesis since the purpose of Lemma 5.1 was to demonstrate that as a general rule, regular simple subalgebras of type $A_1$ lead to ill-mannered pairs.

Continuing our investigation of ill-mannered pairs with $\mathfrak{l}$ of low rank, we note that having $\mathfrak{l}$ of type $A_2$ and $\mathfrak{g}$ of type either $B_n$ or $D_n$ is also regarded as ill-mannered. If $\mathfrak{g}$ is of type $B_n$ or $D_n$ and $m \geq 2$ then $\zeta|_{\mathfrak{h}_\mathfrak{l}} = \lambda_2$ and $\eta|_{\mathfrak{h}_\mathfrak{l}} = \lambda_{m-1}$. Since $\beta_i|_{\mathfrak{h}_\mathfrak{l}} = \mu_i|_{\mathfrak{h}_\mathfrak{l}} = \lambda_1$ and $\gamma_i|_{\mathfrak{h}_\mathfrak{l}} = \nu_i|_{\mathfrak{h}_\mathfrak{l}} = \lambda_m$ for all $i \in [M]$ we saw, just as in Lemma 4.32, that there is no issue when $m > 2$ but when $m = 2$ we again are unable to untangle the negative roots from the positive roots. Consequently, we encounter the same issue as when $\mathfrak{l}$ was of type $A_1$. We now present the analogous statement of Lemma 5.1 in the context of $\mathfrak{t}(\mathfrak{l}) = A_2$.

**Lemma 5.2.** *Let $\mathfrak{g}$ be a simple Lie algebra of type $B_n$ or $D_n$ with $n \geq 4$ and let $\mathfrak{l}$ be a regular simple subalgebra of $\mathfrak{g}$ of type $A_2$. Then there exists a Levi decomposable subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ and nontrivial component of the radical $V^\times$ such that for all $\tau \in \operatorname{Int}\mathfrak{g}$ preserving $\mathfrak{l}$, there is no nontrivial $\Phi_\mathfrak{l}$-candidate such that $\tau(V^\times)$ is as in Equation 5.1.*

*Proof.* Like in the proof of Lemma 5.1 we may suppose without loss of generality that $\mathfrak{l}$ is described by Equation 3.10. Since $n \geq 4$ we have $M \geq 1$ and so we may define

$$w = v_{\nu_1} + v_\zeta \tag{5.5}$$

where $v_{\nu_1}$ and $v_\zeta$ are as in Equation 4.51 or Equation 4.55, depending on the type of $\mathfrak{g}$. Then $w$ is a maximal vector of weight $\lambda_2$. If we denote the irreducible $\mathfrak{l}$-module generated by $w$ as $W$, then $W$ is the linear span of $w$, $[Y_2, w]$, and $[Y_1, [Y_2, w]]$. A simple matrix calculation reveals that $[w, [Y_2, w]] = -2v_{\beta_1}$ and so $W$ is not an abelian subalgebra of $\mathfrak{g}$.

Define $\mathfrak{a}$ to be the Levi decomposable subalgebra of $\mathfrak{a}$ with Levi decomposition $\mathfrak{a} = \mathfrak{l} \inplus V^\times$, where $\operatorname{Rad}\mathfrak{a} = V^\times = W \oplus V(\beta_1)$. One can verify that this indeed defines a Levi decomposable subalgebra of $\mathfrak{g}$. We have that $W$ has highest weight $\lambda_2$ and $V(\beta_1)$ has highest weight $\lambda_1$ and so $V^\times$ is a sum of two irreducible $\mathfrak{l}$-modules of different highest weights.

If there exists $\tau \in \operatorname{Int}\mathfrak{g}$ preserving $\mathfrak{l}$ and a nontrivial $\Phi_\mathfrak{l}$-candidate $\Theta$ such that $\tau(V^\times)$ is given by Equation 5.1 then it must be that $\bigoplus_{\alpha\in\Theta}\mathfrak{g}$ is a sum of two irreducible $\mathfrak{l}$-modules since the same is true of $V^\times$. Since $W$ is not an abelian subalgebra of $\mathfrak{g}$ but $\bigoplus_{k=1}^M V(\gamma_k) \oplus \bigoplus_{k=1}^M V(\nu_k)$ and $\bigoplus_{k=1}^M V(\beta_k) \oplus \bigoplus_{k=1}^M V(\mu_k)$ are abelian subalgebras of $\mathfrak{g}$, the result follows from Lemma 4.9, Lemma 4.10, and Lemma 4.29 if $\mathfrak{t}(\mathfrak{g}) = D_n$.

If $\mathfrak{g}$ is of type $B_n$ then the above implies either $V(\gamma_{M+1}) \subsetneq \tau(V^\times)$ or $V(\beta_{M+1}) \subsetneq \tau(V^\times)$. From Lemma 4.9, Lemma 4.10, Lemma 4.26, and Lemma 4.30 we deduce that $\tau(V^\times)$ is either $V(\gamma_{M+1}) \oplus V(\eta)$ or $V(\beta_{M+1}) \oplus V(\zeta)$. We prove both cases are impossible.

Define

$$V_1 = \bigoplus_{k=1}^{M+1} V(\beta_k) \oplus \bigoplus_{k=1}^{M} V(\mu_k) \oplus V(\eta), \quad V_m = \bigoplus_{k=1}^{M+1} V(\gamma_k) \oplus \bigoplus_{k=1}^{M} V(\nu_k) \oplus V(\zeta) \tag{5.6}$$

By construction, $V_1$ and $V_m$ consist of all $\mathfrak{l}$-submodules of $\mathfrak{g}$ of highest weight $\lambda_1$ and $\lambda_m$, respectively. Since $\tau$ preserves $\mathfrak{l}$, we note that if two $\mathfrak{l}$-

modules $M_1$ and $M_2$ have the same highest weight then $\tau(M_1)$ and $\tau(M_2)$ will also have the same highest weight. Consequently, $\tau$ either preserves or interchanges $V_1$ and $V_m$. Without loss of generality we can suppose by Lemma 4.26 that $\tau$ preserves $V_1$ and $V_m$.

If $\tau$ maps $V^\times$ to $V(\gamma_{M+1}) \oplus V(\eta)$ or $V(\beta_{M+1}) \oplus V(\zeta)$ then $\tau$ must also map $C_\mathfrak{g}(V^\times) \cap V_m$ to either $C_\mathfrak{g}(V(\gamma_{M+1}) \oplus V(\eta)) \cap V_m$ or $C_\mathfrak{g}(V(\beta_{M+1}) \oplus V(\zeta)) \cap V_m$ since $\tau$ preserves $V_m$. We shall show both of these cases are impossible. To do so, we first determine $C_\mathfrak{g}(V^\times) \cap V_m$.

Since $V(\beta_1) \subseteq V^\times$ we have $C_\mathfrak{g}(V^\times) \cap V_m \subseteq C_\mathfrak{g}(V(\beta_1)) \cap V_m$. Also note that $C_\mathfrak{g}(V(\beta_1))$ is regular since $V(\beta_1)$ is regular. Since $\alpha_3 \in \beta_1 + \Phi_\mathfrak{l}$, $\gamma_k \in \gamma_k + \Phi_\mathfrak{l}$, and $\alpha_3 + \gamma_k \in \Phi_0$ for all $k \in [M]$ we have $\gamma_1, \ldots, \gamma_{M+1} \notin C_\mathfrak{g}(V(\beta_1)) \cap V_m$. Similarly, for each $k \in \{2, \ldots, M\}$ we have $\nu_k \in \nu_k + \Phi_\mathfrak{l}$ and $\alpha_3 + \nu_k \in \Phi_0$, implying $v_{\nu_2}, \ldots, v_{\nu_M} \notin C_\mathfrak{g}(V(\beta_1)) \cap V_m$. Consequently, regularity implies $C_\mathfrak{g}(V(\beta_1)) \cap V_m \subseteq V(\nu_1) \oplus V(\zeta)$ and so $C_\mathfrak{g}(V^\times) \cap V_m \subseteq V(\nu_1) \oplus V(\zeta)$.

Direct calculations reveal that $v_{\nu_1} - v_\zeta \in C_\mathfrak{g}(V^\times) \cap V_m$. In particular, $C_\mathfrak{g}(V^\times) \cap V_m \neq \{0\}$. Therefore, the Jacobi identity implies $C_\mathfrak{g}(V^\times)$, and thus $C_\mathfrak{g}(V^\times) \cap V_m$, is an $\mathfrak{l}$-submodule of $V(\nu_1) \oplus V(\zeta)$. It is a proper $\mathfrak{l}$-submodule since $[[Y_2, w], v_{\nu_1}] = X_{1,2,3} \neq 0$ and so by Weyl's Theorem it must be an irreducible $\mathfrak{l}$-module. As $\tilde{w} = v_{\nu_1} - v_\zeta$ is a maximal vector it follows that $C_\mathfrak{g}(V^\times) \cap V_m = \tilde{W}$, where $\tilde{W}$ is the irreducible $\mathfrak{l}$-module generated by $\tilde{w}$.

Note that $C_\mathfrak{g}(V(\gamma_{M+1}) \oplus V(\eta)) \cap V_m$ is not an irreducible $\mathfrak{l}$-module. Indeed, straightforward calculations reveal that $V(\nu_k) \subseteq C_\mathfrak{g}(V(\gamma_{M+1}) \oplus V(\zeta)) \cap V_m$ for each $k \in [M]$ and $V(\eta) \subseteq C_\mathfrak{g}(V(\gamma_{M+1}) \oplus V(\zeta)) \cap V_m$. Thus it must be that $\tau$ maps $V^\times$ to $V(\beta_{M+1}) \oplus V(\zeta)$ and $C_\mathfrak{g}(V^\times) \cap V_m$ to $C_\mathfrak{g}(V(\beta_{M+1}) \oplus V(\zeta)) \cap V_m$.

Regularity of $V(\beta_{M+1}) \oplus V(\zeta)$ implies $C_\mathfrak{g}(V(\beta_{M+1}) \oplus V(\zeta)) \cap V_m$ is also regular. For each $k \in [M+1]$ we have $\zeta - \alpha_2 \in \zeta + \Phi_\mathfrak{l}$, $\gamma_k \in \gamma_k + \Phi_\mathfrak{l}$, and $\zeta - \alpha_2 + \gamma_k \in \Phi_0$. As such $v_{\gamma_1}, \ldots, v_{\gamma_{M+1}} \notin C_\mathfrak{g}(V(\beta_{M+1}) \oplus V(\zeta)) \cap V_m$. Similarly, for each $k \in [M]$ we have $\nu_k \in \nu_k + \Phi_\mathfrak{l}$ and $\zeta - \alpha_2 + \nu_k \in \Phi_0$, implying $v_{\nu_1}, \ldots, v_{\nu_M} \notin C_\mathfrak{g}(V(\beta_{M+1}) \oplus V(\zeta)) \cap V_m$. Since $[V(\zeta), V(\beta_{M+1}) \oplus V(\zeta)] = \{0\}$ it follows that $C_\mathfrak{g}(V(\beta_{M+1}) \oplus V(\zeta)) \cap V_m = V(\zeta)$. Consequently, $\tau$ maps $\tilde{W}$ to $V(\zeta)$. However, in much the same way we showed that $W$ is not an abelian subalgebra of $\mathfrak{g}$ one can prove that $\tilde{W}$ is also not an abelian subalgebra of $\mathfrak{g}$. This is a contradiction since $V(\zeta)$ is abelian. □

Lemma 5.1 and Lemma 5.2 illustrate the importance of being able to distinguish between positive and negative roots. When the rank of $\mathfrak{l}$ is too small, there is not enough structure in $\mathfrak{l}$ to effectively separate positive and negative roots, at least from a representation theoretical perspective. If we are unable to even disentangle negative roots from positive ones then what hope is there to obtain a result as elegant as Theorem 4.19?

## 5.2 ILL-MANNERED PAIRS OF THE FORM $(D_n, A_m)$

While ill-mannered pairs with $\mathfrak{l}$ of low rank are anticipated, it is highly surprising that those of the form $(D_n, A_m)$ behave poorly in the context of Chapter 4. Why is this the only infinite family of ill-mannered pairs? What differentiates this setting from all others? To answer these questions it is first instructive to examine an explicit subalgebra which does not align with Theorem 4.19. Since we have already examined the case where $\mathfrak{t}(\mathfrak{l}) = A_m$ with $m \leq 2$, we shall suppose in this section that $m > 2$.

**Lemma 5.3.** *Let $\mathfrak{g}$ be a simple Lie algebra of type $D_n$ and let $\mathfrak{l}$ be a simple subalgebra of $\mathfrak{g}$ of type $A_m$, where $2 \leq m \leq n-2$. Then there exists a Levi decomposable subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ and nontrivial component of the radical $V^\times$ such that for all $\tau \in \operatorname{Int}\mathfrak{g}$ preserving $\mathfrak{l}$, there is no nontrivial $\Phi_\mathfrak{l}$-candidate such that $\tau(V^\times)$ is as in Equation 5.1.*

*Proof.* We may suppose without loss of generality that $\mathfrak{l}$ is prescribed by Equation 3.10. Since $m \leq n-2$ we have $M \geq 1$ and so we may define

$$w = v_{\gamma_1} + v_{\nu_1} = Y_{m+1} + (-1)^M Y_{m+1,\dots,n,n-2,n-3,\dots,m+2} \tag{5.7}$$

with $v_{\gamma_1}$ and $v_{\nu_1}$ as in Equation 4.55. We have that $w$ is a maximal vector of weight $\lambda_m$ so let $W$ be the corresponding irreducible $\mathfrak{l}$-module generated by $w$. Then $W$ is the linear span of $w, [Y_m, w], [Y_{m-1,m}, w], \dots, [Y_{1,\dots,m}, w]$. An inductive argument shows that for all $k \in [m]$,

$$[Y_{k,\dots,m}, w] = Y_{k,\dots,m+1} + (-1)^{n-k} Y_{k,\dots,n,n-2,n-3,\dots,m+2} \tag{5.8}$$

From this we can conclude $[[Y_m, w], w] = 2v_\eta$ and so $W$ is not an abelian subalgebra of $\mathfrak{g}$.

Define $\mathfrak{a}$ to be the Levi decomposable subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ and radical $W \oplus V(\eta)$. Then $V^\times = \operatorname{Rad}\mathfrak{a}$ is the nontrivial component of the radical and $\{0\}$ is the trivial component of the radical. Suppose by contradiction that there exists $\tau \in \operatorname{Int}\mathfrak{g}$ preserving $\mathfrak{l}$ and $\Theta$ a nontrivial $\Phi_\mathfrak{l}$-candidate such that $\tau(V^\times)$ is described by Equation 5.1. Since $W$ has highest weight $\lambda_m$ and $V(\eta)$ has highest weight $\lambda_{m-1}$ we note that $\tau(V^\times)$ consists of two irreducible $\mathfrak{l}$-modules of different highest weight. Therefore, if $\tau(V^\times)$ is to be as in Equation 5.1 we conclude by Lemma 4.9, Lemma 4.10, and dimension considerations that $\tau$ maps $W$ to $V(\beta_i)$, $V(\mu_i)$, $V(\gamma_i)$, or $V(\nu_i)$ for some $i \in [M]$. This is, however, not possible since the

$V(\beta_i)$, $V(\mu_i)$, $V(\gamma_i)$, and $V(\nu_i)$ are all abelian subalgebras of $\mathfrak{g}$, whereas $W$ is not. □

*Remark* 5.4. Note that there is nothing special about how $w$ was defined in Equation 5.7. Indeed, we can define

$$w = \sum_{k=1}^{M} (a_k v_{\gamma_k} + b_k v_{\nu_k}) \tag{5.9}$$

where $a_1, \ldots, a_M, b_1, \ldots, b_M \in \mathbb{F}$ are such that $\sum_{k=1}^{M} a_k b_k \neq 0$. Then $w$ is a maximal vector of highest weight $\lambda_m$ and

$$[[\Upsilon_m, w], w] = \left(2 \sum_{k=1}^{M} a_k b_k\right) v_\eta \tag{5.10}$$

Consequently, the irreducible $\mathfrak{l}$-module generated by $w$ is not an abelian subalgebra of $\mathfrak{g}$. Therefore, the easily satisfied condition of $\sum_{k=1}^{M} a_k b_k \neq 0$ is what causes problems.

With Lemma 5.3 we can attempt to examine what exactly makes $(D_n, A_m)$ ill-mannered. Unlike the ill-mannered pairs in Section 5.1, the issue is *not* the inability to separate negative roots from positive roots. Instead, the issue boils down to two things:

1. the existence of an irreducible $\mathfrak{l}$-module which is not an abelian subalgebra of $\mathfrak{g}$ and

2. having $V(\alpha)$ be an abelian subalgebra of $\mathfrak{g}$ for all $\alpha \in \Phi_{\mathfrak{l}}^{\times}$, where $\Phi_{\mathfrak{l}}^{\times}$ is as in Equation 4.13.

Having both of these facts hold is precisely what leads to being ill-mannered. For instance, recall that $(B_n, A_m)$ with $m > 2$ is well-mannered despite the fact that there exists an irreducible $\mathfrak{l}$-module which is not an irreducible $\mathfrak{l}$-module. However, this is fine since any such $\mathfrak{l}$-module could be mapped to $V(\gamma_{M+1})$, which is also not abelian. Indeed, a closer look at the proof of Lemma 4.32 will reveal that this is precisely what we did. In some sense, the difference in parity of $\mathfrak{so}_{2n+1}$ versus $\mathfrak{so}_{2n}$ allows for the existence of additional roots $\gamma_{M+1}$ and $\beta_{M+1}$ when $\mathfrak{g} \cong \mathfrak{so}_{2n+1}$, solving our problem.

Now why are the pairs $(C_n, A_m)$ with $m > 1$ well-mannered? Indeed, $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n}$ are even-dimensional representations and neither $\gamma_{M+1}$ nor $\beta_{M+1}$ exist when $\mathfrak{t}(\mathfrak{g}) = C_n$. Moreover, in this case we do have that $V(\alpha)$ is an abelian subalgebra of $\mathfrak{g}$ for all $\alpha \in \Phi_{\mathfrak{l}}^{\times}$ and so one may wonder why $(C_n, A_m)$ is well-mannered. It turns out, and we omit the easily verified details, that every irreducible $\mathfrak{l}$-module in that setting is also an abelian subalgebra of $\mathfrak{g}$. As such, there are no potentially troublesome $\mathfrak{l}$-modules when $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (C_n, A_m)$.

There is another way in which one can intuit the difference between $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (C_n, A_m)$ and $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (D_n, A_m)$. Consider the matrices $A, C \in \mathfrak{gl}_M$ defined as in Equation 4.118. Recall that these matrices were used to define a matrix $P \in Sp_{2n}$. One can check that one of the conditions on $A$ and $C$ to ensure $P$ is an element of $Sp_{2n}$ is that $A^\top C = C^\top A$. Note that this condition yields no restrictions on the dot product[12] of the $k$'th column of $A$ with the $k$'th column of $C$ for all $k \in [M]$. In particular, there is no condition on the dot product of the $(q+1)$'th column of $A$ with the $(q+1)$'th column of $C$. By contrast, if we wanted $P$ to be an element of $SO_{2n}$ then we would instead require $A^\top C = -C^\top A$, which *does* place restrictions on the dot product of the $(q+1)$'th column of $A$ and the $(q+1)$'th column of $C$. By noting what the $(q+1)$'th columns of $A$ and $C$ are, we see why we may encounter issues when $\mathfrak{t}(\mathfrak{g}) = D_n$.

Of course, the above remarks regarding $A$ and $C$ do not prove anything beyond the fact that if $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (D_n, A_m)$ were well-mannered then a proof different from that of Lemma 4.34 would be required. This in itself however does illustrate a fundamental difference, and unfortunately we observe from Lemma 5.3 that this difference assures that Theorem 4.19 cannot hold when $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (D_n, A_m)$. However, the cause for being ill-mannered is not as severe as in Section 5.1. Indeed, if we want a result akin to Theorem 4.19 to hold then at the very least $\mathfrak{l}$ should be able to distinguish between positive and negative roots, which we can do when $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (D_n, A_m)$. As such, there is hope that Theorem 4.19 can be sufficiently modified to yield a meaningful statement in this case.

Let us try to adapt Theorem 4.19 to permit regular pairs of the form $(D_n, A_m)$. Since maximal vectors as in Equation 5.9 with $\sum_{k=1}^M a_k b_k \neq 0$ are troublesome we shall temporarily exclude these possibilities. If we ignore these potentially ruinous maximal vectors, then we indeed obtain an analogue to Lemma 4.32 and Lemma 4.34.

**Lemma 5.5.** *Suppose $\mathfrak{g}$ is a simple Lie algebra of type $D_n$ and let $\mathfrak{l}$ be a simple subalgebra of $\mathfrak{g}$ of type $A_m$, where $m > 2$. Further suppose $\mathfrak{a}$ is a Levi decomposable subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ such that the nontrivial component of the radical $V^\times$ can be decomposed into a direct sum[13] of irreducible $\mathfrak{l}$-submodules, each of which is also an abelian subalgebra. Then there exist $p, q, c, d \in [M]_0$ with $q \leq p$, $d \leq p - q$, and $c \leq M - p$ such that $V^\times$ is described by either Equation 4.116a or Equation 4.116b. If $V^\times$ is as in Equation 4.116b then we may further suppose $c + d \leq p$.*

12 If $v = \sum_{k=1}^M a_k e_k$ and $w = \sum_{k=1}^M b_k e_k$, by abuse of notation we call the quantity $\sum_{k=1}^M a_k b_k$ the *dot product of $v$ and $w$*.

13 A direct sum as $\mathfrak{l}$-modules.

*Proof.* If $m = n-1$ and $n$ is even then it is possible that $\mathfrak{l}$ is given by Equation 3.14. In this case, Lemma 4.9 and Lemma 4.10 imply Rad $\mathfrak{a} \subseteq V(\zeta) \oplus V(\eta)$. Since $\zeta|_{\mathfrak{h}_\mathfrak{l}} = \lambda_2$ and $\eta|_{\mathfrak{h}_\mathfrak{l}} = \lambda_{m-1}$ we have that the result is immediate if $m > 3$ by Lemma 4.29. If $m = 3$ the result is obtained by Equation 4.75. This concludes the case where Equation 3.14 describes $\mathfrak{l}$.

If $\mathfrak{l}$ is not given by Equation 3.14 then without loss of generality we may suppose Equation 3.10 describes $\mathfrak{l}$. We shall exclude many of the details since the proof is virtually identical to that of Lemma 4.32. By Lemma 4.9 and Lemma 4.10 we have that if $V^\times \neq \{0\}$ then $V^\times$ is an $\mathfrak{l}$-submodule of $\bigoplus_{k=1}^M V(\beta_k) \oplus \bigoplus_{k=1}^M V(\gamma_k) \oplus \bigoplus_{k=1}^M V(\mu_k) \oplus \bigoplus_{k=1}^M V(\nu_k) \oplus V(\zeta) \oplus V(\eta)$. For each $k \in [M]$ we have $\beta_k|_{\mathfrak{h}_\mathfrak{l}} = \mu_k|_{\mathfrak{h}_\mathfrak{l}} = \lambda_1$, $\gamma_k|_{\mathfrak{h}_\mathfrak{l}} = \nu_k|_{\mathfrak{h}_\mathfrak{l}} = \lambda_m$, $\zeta|_{\mathfrak{h}_\mathfrak{l}} = \lambda_2$, and $\eta_{\mathfrak{h}_\mathfrak{l}} = \lambda_{m-1}$. In precisely the same manner as in the proof of Lemma 4.32, there exist $r^+, r^- \in [2M]_0$ and $s^- \in \{0,1\}$ such that $V^\times$ contains $r^+$ copies of $V(\lambda_1)$, $r^-$ copies of $V(\lambda_m)$, and $s^-$ copies of $V(\lambda_{m-1})$, while also supposing $r^- \geq r^+$ if $s^- = 0$.

We first proceed via induction on $r^-$ to show there exist $p, q \in [M]_0$ with $q \leq p$ such that the $r^-$ copies of $V(\lambda_m)$ in $V^\times$ are, up to inner automorphism, given by $\bigoplus_{k=1}^p V(\nu_k) \oplus \bigoplus_{k=1}^q V(\gamma_k)$ with $p + q = r^-$. The base case of $r^- = 0$ clearly holds. As such, we shall suppose the assertion holds when $V^\times$ contains $r^-$ copies of $V(\lambda_m)$ and that $V^\times$ now contains $r^- + 1$ such copies. By assumption, we may choose these copies of $V(\lambda_m)$ to be abelian subalgebras of $\mathfrak{g}$. Let $w_1^-, \ldots, w_{r^-+1}^-$ be maximal vectors of these $r^- + 1$ copies. Then for each $i \in [r^- + 1]$ there exist $a_{i,1}, \ldots, a_{i,M}, b_{i,1}, \ldots, b_{i,M} \in \mathbb{F}$ such that

$$w_i^- = \sum_{k=1}^{M} (a_{i,k} v_{\gamma_k} + b_{i,k} v_{\nu_k}) \tag{5.11}$$

where the $v_{\gamma_k}$ and $v_{\nu_k}$ are given by Equation 4.55. Since the irreducible $\mathfrak{l}$-modules generated by the $w_i^-$ are abelian subalgebras of $\mathfrak{g}$, we conclude by Equation 5.10 that for all $i \in [r^- + 1]$,

$$\sum_{k=1}^{M} a_{i,k} b_{i,k} = 0 \tag{5.12}$$

Let $\tilde{V}_m^\times$ be the $\mathfrak{l}$-module generated by $w_1^-, \ldots, w_{r^-}^-$. Then $\tilde{V}_m^\times$ is an abelian subalgebra of $\mathfrak{g}$ and so there exists a Levi decomposable subalgebra $\tilde{\mathfrak{a}}$ of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ and radical $\tilde{V}_m^\times$. By the inductive hypothesis we may thus suppose without loss of generality that there exist $p, q \in [M]_0$ with $q \leq p$ such that $\tilde{V}_m^\times = \bigoplus_{k=1}^p V(\nu_k) \oplus \bigoplus_{k=1}^q V(\gamma_k)$. Moreover, if the irreducible $\mathfrak{l}$-module generated by $w_{r^-+1}^-$ is an abelian subalgebra of $\mathfrak{g}$, then it is evident that the irreducible $\mathfrak{l}$-module generated by $\tau(w_{r^-+1}^-)$ is also an abelian subalgebra of $\mathfrak{g}$ for all $\tau \in \text{Int}\,\mathfrak{g}$ preserving $\mathfrak{l}$. Hence we may continue assuming Equation 5.12 holds.

If there exists $l \in \{q+1, \ldots, p\}$ such that $a_{r^-+1,l} \neq 0$ then the argument is nearly identical to that of Case 3 in the proof of Lemma 4.32. Otherwise, the argument is essentially that of Case 4 in the proof of Lemma 4.32. These two cases conclude this initial inductive portion of the proof and so up to inner automorphism the copies of $V(\lambda_m)$ in $V^\times$ can be assumed to be $\bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k)$ for some $p, q \in [M]_0$ with $q \leq p$. We now examine the $r^+$ copies of $V(\lambda_1)$ in $V^\times$. We prove by induction on $r^+$ that up to inner automorphism, there exist $c, d \in [M]_0$ with $c + d = r^+$ such that the $r^+$ copies of $V(\lambda_1)$ in $V^\times$ are $\bigoplus_{k=M-c+1}^{M} V(\beta_k) \oplus \bigoplus_{k=p-d+1}^{p} V(\beta_k)$, where $c = 0$ if $V(\eta) \subseteq V^\times$ or if $q > 0$.

The base case of $r^+ = 0$ clearly holds. Now suppose the assertion holds when $\text{Rad}\,\mathfrak{a}$ contains $r^+$ copies of $V(\lambda_1)$ and further suppose $\text{Rad}\,\mathfrak{a}$ contains $r^+ + 1$ such copies. Let $w_1^+, \ldots, w_{r^++1}^+ \in V^\times$ be maximal vectors by these $r^+ + 1$ copies. For each $i \in [r^+ + 1]$ there exist $a_{i,1}, \ldots, a_{i,M}, b_{i,1}, \ldots, b_{i,M} \in \mathbb{F}$ such that

$$w_i^+ = \sum_{k=1}^{M} (a_{i,k} v_{\beta_k} + b_{i,k} v_{\mu_k}) \tag{5.13}$$

where $v_{\beta_k}$ and $v_{\mu_k}$ are as in Equation 4.55. Now observe that for all $i, j \in [r^+ + 1]$,

$$[w_i^+, [Y_1, w_j^+]] = -\sum_{k=1}^{M} (a_{i,k} b_{j,k} + a_{j,k} b_{i,k}) v_\zeta \tag{5.14}$$

It follows that for all $i, j \in [r^+ + 1]$,

$$0 = \sum_{k=1}^{M} (a_{i,k} b_{j,k} + a_{j,k} b_{i,k}) \tag{5.15}$$

Moreover, for each $i \in [r^+ + 1]$ and $j \in [q]$ we have $w_i^+, v_{\gamma_j} \in V^\times$ and thus $[w_i^+, v_{\gamma_j}] \in \text{Rad}\,\mathfrak{a}$. By performing calculations similar to that of Equation 4.100 we conclude that for all $i \in [r^+ + 1]$, $j \in [q]$, and $k \in [p]$ that $a_{i,j} = b_{i,k} = 0$. The equivalent to Equation 4.101 in this setting also implies that if $V(\eta) \subseteq V^\times$ then for each $i \in [r^+ + 1]$ we have $a_{i,p+1} = \cdots = a_{i,M} = 0$ and $b_{i,q+1} = \cdots = b_{i,M} = 0$. These observations, along with the fact that the $\mathfrak{l}$-module generated by $w_1^+, \ldots, w_{r^++1}^+$ is an abelian subalgebra of $\mathfrak{g}$, imply there exists a Levi decomposable subalgebra $\tilde{\mathfrak{a}}$ of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ and nontrivial component of the radical $\bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k) \oplus \tilde{V}_1^\times$ or $V(\eta) \oplus \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k) \oplus \tilde{V}_1^\times$, where $\tilde{V}_1^\times$

is the $\mathfrak{l}$-module generated by $w_1^+, \ldots, w_{r^+}^+$. Therefore, by the inductive hypothesis we may suppose without loss of generality that there exist $c, d \in [M]_0$ with $c + d = r^+$, $c \leq M - p$, and $d \leq p - q$ such that

$$\tilde{V}_1^\times = \bigoplus_{k=M-c+1}^{M} V(\beta_k) \oplus \bigoplus_{k=p-d+1}^{p} V(\beta_k) \tag{5.16}$$

with $c = 0$ if $V(\eta) \subseteq V^\times$ or $q > 0$.

If there exists $l \in \{p+1, \ldots, M-c\}$ such that $a_{r^++1,l} \neq 0$ or $b_{r^++1,l} \neq 0$ then the argument is virtually identical to that in Case (ii) in the proof of Lemma 4.32. Otherwise the argument in Case (iii) of the proof of Lemma 4.32 applies. This concludes this second inductive component of the proof. The assertion now follows from consideration of Lemma 4.30 and Lemma 4.31 if $m = 3$. □

From our discussion following Lemma 5.3 we suspected that the pairs of the form $(D_n, A_m)$ are ill-mannered due to the existence of maximal vectors $w$ as in Equation 5.9 with $\sum_{k=1}^{M} a_k b_k \neq 0$. Lemma 5.5 shows that such maximal vectors are indeed the sole cause of being ill-mannered. While Lemma 5.5 yields an elegant result, it applies to the fairly specific setting in which $V^\times$ decomposes in a very particular manner. As we saw in Lemma 5.3, general nontrivial components of the radical need not decompose in this way. Fortunately, it is not too difficult to augment Lemma 5.5 to allow for greater generality. Before stating this more general result, we first require a simple lemma.

**Lemma 5.6.** *Let $\mathfrak{g}$ be a simple Lie algebra of type $D_n$ and let $\mathfrak{l}$ be the simple subalgebra of type $A_m$ defined by Equation 3.10 with $m > 2$. For each $k \in [M]$ define $v_{\beta_k}$, $v_{\gamma_k}$, $v_{\mu_k}$, and $v_{\nu_k}$ as in Equation 4.55 and further define $w^-, w^+ \in \mathfrak{g}$ as*

$$w^- = \sum_{k=1}^{M} (a_k v_{\gamma_k} + b_k v_{\nu_k}), \qquad w^+ = \sum_{k=1}^{M} (c_k v_{\beta_k} + d_k v_{\mu_k}) \tag{5.17}$$

*where for each $k \in [M]$, $a_k, b_k, c_k, d_k \in \mathbb{F}$. Let $W^-, W^+ \subseteq \mathfrak{g}$ be the $\mathfrak{l}$-modules generated by $w^-$ and $w^+$, respectively. Then $W^-$ is an abelian subalgebra of $\mathfrak{g}$ if and only if $\sum_{k=1}^{M} a_k b_k = 0$. Similarly, $W^+$ is an abelian subalgebra of $\mathfrak{g}$ if and only if $\sum_{k=1}^{M} c_k d_k = 0$.*

*Proof.* Since $\gamma_k|_{\mathfrak{h}_\mathfrak{l}} = \nu_k|_{\mathfrak{h}_\mathfrak{l}} = \lambda_m$ and $\beta_k|_{\mathfrak{h}_\mathfrak{l}} = \mu_k|_{\mathfrak{h}_\mathfrak{l}} = \lambda_1$ we note that $w^-$ and $w^+$ are maximal vectors of weight $\lambda_m$ and $\lambda_1$, respectively. Therefore, $W^-$

and $W^+$ are irreducible $\mathfrak{l}$-modules. It is well-known that $W^-$ is spanned by $w^-, [Y_m, w^-], \ldots, [Y_{1,\ldots,m}, w^-]$. For each $i \in [M]$ we have

$$[Y_{i,\ldots,m}, w^-] = \sum_{k=1}^{M} (a_k(-1)^{k-1} Y_{i,\ldots,m+k} + b_k(-1)^M Y_{i,\ldots,n,n-2,n-3,\ldots,m+k+1}) \tag{5.18}$$

If we abuse notation and define $[Y_{i,\ldots,m}, w^-]$ to be $w^-$ when $i = m+1$ then for all $i, j \in [m+1]$ with $i < j$,

$$[[Y_{i,\ldots,m}, w^-], [Y_{j,\ldots,m}, w^-]] = \left(2(-1)^{i+j} \sum_{k=1}^{M} a_k b_k\right) Y_{i,\ldots,n,n-2,n-3,\ldots,j} \tag{5.19}$$

from which the claim follows for $W^-$. The assertion also holds for $W^+$ by Lemma 4.26. □

With Lemma 5.6 we can now prove that although Theorem 4.19 does not hold for regular pairs of the form $(D_n, A_m)$, we still do have a rather simple description of the nontrivial component in this setting by generalizing Lemma 5.5.

**Lemma 5.7.** *Suppose $\mathfrak{g}$ is a simple Lie algebra of type $D_n$, $\mathfrak{l}$ is a regular simple subalgebra of type $A_m$ with $m > 2$, and $\mathfrak{a}$ is a Levi decomposable subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$. Then there exusts $p, q, c, d \in [M]_0$ with $q \leq p$, $d \leq p - q$, and $c \leq M - p$ such that the nontrivial component $V^\times$ of* Rad $\mathfrak{a}$ *is one of the following:*

$$V^\times = V_\eta \oplus \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=1}^{q} V(\gamma_k) \oplus \bigoplus_{k=p-d+1}^{p} V(\beta_k) \tag{5.20a}$$

$$\text{or } V^\times = \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=M-c+1}^{M} V(\beta_k) \oplus \bigoplus_{k=p-d+1}^{p} V(\beta_k) \tag{5.20b}$$

$$\text{or } V^\times = V_\eta \oplus \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=p-d+1}^{p} V(\beta_k) \oplus W_p \tag{5.20c}$$

$$\text{or } V^\times = \bigoplus_{k=1}^{p} V(\nu_k) \oplus \bigoplus_{k=M-c+1}^{M} V(\beta_k) \oplus \bigoplus_{k=p-d+1}^{p} V(\beta_k) \oplus W_p \tag{5.20d}$$

*where $W_p$ is the $\mathfrak{l}$-module generated by $v_{\gamma_{p+1}} + v_{\nu_{p+1}}$. Moreover, if $V^\times$ is given by Equation 5.20b or Equation 5.20d then $c + d \leq p$ and if $V^\times$ is given by Equation 5.20c or Equation 5.20d then $p < M$ and $c < M - p$.*

*Proof.* If $\mathfrak{l}$ is given by Equation 3.14 then the same argument as in the proof of Lemma 5.5 applies. Thus we may instead suppose Equation 3.10

describes $\mathfrak{l}$. Lemma 4.9 and Lemma 4.10 imply $V^\times$ is an $\mathfrak{l}$-submodule of $\bigoplus_{k=1}^M V(\beta_k) \oplus \bigoplus_{k=1}^M V(\gamma_k) \oplus \bigoplus_{k=1}^M V(\mu_k) \oplus \bigoplus_{k=1}^M V(\nu_k) \oplus V(\zeta) \oplus V(\eta)$ if $V^\times \neq \{0\}$. For each $k \in [M]$ note that $\beta_l|_{\mathfrak{h}_\mathfrak{l}} = \mu_k|_{\mathfrak{h}_\mathfrak{l}} = \lambda_1$, $\gamma_k|_{\mathfrak{h}_\mathfrak{l}} = \nu_k|_{\mathfrak{h}_\mathfrak{l}} = \lambda_m$, $\zeta|_{\mathfrak{h}_\mathfrak{l}} = \lambda_2$, and $\eta|_{\mathfrak{h}_\mathfrak{l}} = \lambda_{m-1}$.

As per usual let $r^+, r^- \in [2M]_0$ and $s^- \in [1]_0$ be the number of copies of $V(\lambda_1)$, $V(\lambda_m)$, and $V(\lambda_{m-1})$ in $V^\times$, respectively. The same arguments as usual imply we may suppose $r^- \geq r^+$ if $s^- = 0$, which is why we may suppose $c + d \leq p$ in Equation 5.20b and Equation 5.20d.

Let $w_1^-, \ldots, w_{r^-}^- \in V^\times$ and $w_1^+, \ldots, w_{r^+}^+ \in V^\times$ be maximal vectors of the $r^-$ copies of $V(\lambda_m)$ and $r^+$ copies of $V(\lambda_1)$ in $V^\times$, respectively. For each $i \in [r^-]$ and $j \in [r^+]$ we have

$$w_i^- = \sum_{k=1}^{M} (a_{i,k} v_{\gamma_k} + b_{i,k} v_{\nu_k}), \qquad w_j^+ = \sum_{k=1}^{M} (c_{j,k} v_{\beta_k} + d_{j,k} v_{\mu_k}) \tag{5.21}$$

where $a_{i,k}, b_{i,k}, c_{j,k}, d_{j,k} \in \mathbb{F}$ for each $k \in [M]$. We have $V(\zeta) \not\subseteq V^\times$ by Lemma 4.26 and Lemma 4.29. Therefore, by the natural equivalent of Equation 4.101 we have $\sum_{k=1}^M c_{j,k} d_{j,k} = 0$ for each $j \in [r^+]$.

If it is possible to choose $w_1^-, \ldots, w_{r^-}^-$ such that for each $i \in [r^-]$, $\sum_{k=1}^M a_{i,k} b_{i,k} = 0$ then the result would be immediate by Lemma 5.5 and Lemma 5.6. Therefore we need only examine the case where without loss of generality we have $\sum_{k=1}^M a_{r^-,k} b_{r^-,k} \neq 0$. Notice that by adding appropriate scalar multiples of $w_{r^-}^-$ to $w_1^-, \ldots, w_{r^--1}^-$ we may suppose that for each $i \in [r^- - 1]$ Equation 5.12 holds. As such, by Lemma 5.6 the $\mathfrak{l}$-module $\tilde{V}^\times$ generated by $w_1^-, \ldots, w_{r^--1}^-, w_1^+, \ldots, w_{r^+}^+$ is such that $\tilde{V}^\times$ can be decomposed as a direct sum of irreducible $\mathfrak{l}$-submodules, each of which is also an abelian subalgebra. Since we also observe that the inner automorphisms in the proof of Lemma 5.5 preserve $\mathfrak{l}$ we may therefore suppose by Lemma 5.5 that there exist $p, q, c, d \in [M]_0$ with $q \leq p$, $d \leq p - q$, and $c \leq M - p$ such that $\tilde{V}^\times$ is as in Equation 4.116a or Equation 4.116b with $c + d \leq p$ if $\tilde{V}^\times$ is in the latter situation.

Notice that $q = 0$. Indeed, if $q \neq 0$ then $p \neq 0$ and so $v_{\gamma_1}, v_{\nu_1} \in \tilde{V}^\times \subseteq V^\times$. As such, we could add an appropriate scalar multiple of $v_{\gamma_1} + v_{\nu_1}$ to $w_{r^-}^-$ to form a new maximal vector $\tilde{w}_{r^-}^- = \sum_{k=1}^M (\tilde{a}_{r^-,k} v_{\gamma_k} + \tilde{b}_{r^-,k} v_{\nu_k})$ satisfying $\sum_{k=1}^M \tilde{a}_{r^-,k} \tilde{b}_{r^-,k} = 0$. Then $\tilde{W}^-$ is an abelian subalgebra by Lemma 5.6. As this would contradict our original assumption that we cannot decompose $V^\times$ into such a sum it must be that $q = 0$.

Linear independence of $\{w_1^-, \ldots, w_{r^-}^-\}$ implies we may suppose that $b_{r^-,1}, \ldots, b_{r^-,p} = 0$. Moreover, we may suppose $a_{r^-,1}, \ldots, a_{r^-,p} = 0$. Indeed, if $a_{r^-,l} \neq 0$ for some $l \in [p]$ then we could add some scalar multiple of $v_{\nu_l}$ to $w_{r^-}^-$, allowing us to replace the $\mathfrak{l}$-module generated by $w_{r^-}^-$ with one which is an abelian subalgebra, contradicting our original assumption. Therefore,

$b_{r^-,1} = \cdots = b_{r^-,p} = 0$ and $a_{r^-,1} = \cdots = a_{r^-,p} = 0$. Since $w^-_{r^-} \neq 0$ it must be that $p < M$. Also note that for each $i \in \{M-c+1, \ldots, M\}$,

$$[w^-_{r^-}, v_{\beta_i}] = a_{r^-,i} X_{1,\ldots,m} \tag{5.22}$$

Since $X_{1,\ldots,m} \in \mathfrak{l}$ we have $a_{r^-,i} = 0$. Therefore, there must exist $l \in \{p+1, \ldots, M-c\}$ such that $a_{r^-,l} b_{r^-,l} \neq 0$. Indeed, this is immediate since $\sum_{k=1}^{M} a_{r^-,k} b_{r^-,k} \neq 0$, $a_{r^-,1} = \cdots = a_{r^-,p} = 0$, and $a_{r^-,M-c+1} = \cdots = a_{r^-,M} = 0$. Note that this implies $c < M - p$.

We have $w^-_{r^-} = \sum_{k=p+1}^{M} (a_{r^-,k} v_{\gamma_k} + b_{r^-,k} v_{\nu_k})$ with $a_{r^-,M-c+1} = \cdots = a_{r^-,M} = 0$. By rescaling if necessary we may suppose $\sum_{k=p+1}^{M} a_{r^-,k} b_{r^-,k} = 1$. Define $\tilde{A}_1, \tilde{D}_1 \in \mathbb{F}^{M-p}$ as

$$\tilde{A}_1 = \sum_{k=1}^{M-p} a_{r^-,p+k} e_k, \qquad \tilde{D}_1 = \sum_{k=1}^{M-p} b_{r^-,p+k} e_k \tag{5.23}$$

Since $a_{r^-,l} \neq 0$, the set $U = \{u \in \mathbb{F}^{M-p} : u^\top \tilde{A}_1 = 0\}$ is an $(M-p-1)$-dimensional subspace of $\mathbb{F}^{M-p}$. Since $a_{r^-,M-c+1} = \cdots = a_{r^-,M} = 0$ we have $e_{M-p-c+1}, \ldots, e_{M-p} \in U$. Since $c < M - p$ we can define $\tilde{D}_i = e_i$ for each $i \in \{M-p-c+1, \ldots, M-p\}$. Then $\{\tilde{D}_{M-p-c+1}, \ldots, \tilde{D}_{M-p}\}$ is a linearly independent subset of $U$ and so we extend it to a basis $\{\tilde{D}_2, \ldots, \tilde{D}_{M-p}\}$ for $U$. As $\tilde{D}_1 \notin U$ we have that $\{\tilde{D}_1, \ldots, \tilde{D}_{M-p}\}$ is a basis for $\mathbb{F}^{M-p}$. In particular, the matrix $\tilde{D} \in GL_{M-p}$ defined such that the $i$'th column of $\tilde{D}$ is $\tilde{D}_i$ for each $i \in [M-p]$ is invertible. Now for each $j \in \{2, \ldots, M-p\}$ define $\tilde{A}_j = (\tilde{D}^{-1})^\top e_j$ and let $\tilde{A} \in \mathfrak{gl}_{M-p}$ be such that for each $i \in [M-p]$, the $i$'th column of $\tilde{A}$ is $\tilde{A}_i$. Note that $\tilde{D}^\top \tilde{A} = I$. Now define $A, D \in \mathfrak{gl}_M$ as

$$A = \begin{pmatrix} I_p & 0_{p,M-p} \\ 0_{M-p,p} & \tilde{A} \end{pmatrix}, \qquad D = \begin{pmatrix} I_p & 0_{p,M-p} \\ 0_{M-p,p} & \tilde{D} \end{pmatrix} \tag{5.24}$$

If we further define $B, C \in \mathfrak{gl}_M$ as $B = C = 0$ then $P'$ as in Equation 4.54 is an element of $SO_{2M}$. Therefore, $P$ as in Equation 4.53 is an element of $SO_{2n}$ and thus defines an inner automorphism $\tau \in \operatorname{Int} \mathfrak{g}$. From Equation 4.56 we conclude that $\tau$ preserves $\mathfrak{l}$, $V(\eta)$, $\bigoplus_{k=1}^{p} V(\nu_k)$, and $\bigoplus_{k=M-c+1}^{M} V(\beta_k) \oplus \bigoplus_{k=p-d+1}^{p} V(\beta_k)$ while also mapping $v_{\gamma_{p+1}} + v_{\nu_{p+1}}$ to $w^-_{r^-}$. The result follows. □

Although Theorem 4.19 does not hold in our setting of $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (D_n, A_m)$, Lemma 5.7 is the next best description. In some sense Lemma 5.7 allows us to ensure that most of the nontrivial component of Rad $\mathfrak{a}$ can be made to be regular relative to $\mathfrak{h}$. Only one irreducible component is not regular relative to $\mathfrak{h}$ but at least we have an explicit description of what this irreducible component is.

Lemma 5.7 is only a partial extension of Theorem 4.19. Indeed, Theorem 4.19 also relates conjugacy to $\Phi_{\mathfrak{l}}$-Weyl conjugacy. Since we can not assume $V^{\times}$ is regular relative to $\mathfrak{h}$ there is no meaningful sense of $\Phi_{\mathfrak{l}}$-Weyl conjugacy in this context.

One might wonder whether we can construct a counterpart to Lemma 5.7 in the context of $\mathfrak{l}$ of sufficiently low rank as in Section 5.1. It is certainly something to consider but one should not anticipate such a statement to be as clean as that of Lemma 5.7. Indeed, recall that the problem we faced in Section 5.1 was much more fundamental as we could not use $\mathfrak{l}$ to separate positive and negative roots. As such, a fruitful future project would be to more deeply examine the cases in Section 5.1 but we shall not further pursue this topic in this thesis.

After having suitably adapted Theorem 4.19 in the case of $(\mathfrak{t}(\mathfrak{g}), \mathfrak{t}(\mathfrak{l})) = (D_n, A_m)$ one may wonder at the possibility of successfully extending either Theorem 4.37 or Theorem 4.38 as well. This is another natural subject to consider but as of yet the author of this thesis has not succeeded in doing so. The prospect of slowly expanding the results of Chapter 4 to allow for more general cases is exciting, but we will have to leave such endeavours for future projects.

# 6 RELAXING SIMPLICITY OF THE LEVI FACTOR

Thus far this thesis has been concerned with Levi decomposable subalgebras where the Levi factor is simple as opposed to the more general semisimple possibility. However, a survey of the contents of this thesis will reveal that we hardly utilized simplicity of the Levi factor. Indeed, the only instance in which simplicity was used was when we stated Dynkin's classification of the regular simple subalgebras in Theorem 3.3. As such, simplicity was used merely as a tool to classify the possibilities for the Levi factor, nothing more. Because of this one may ask whether we can extend the notion of well-mannered pairs to include pairs of classical Lie algebras and regular semisimple subalgebras, thereby broadening the results of Chapter 4. This is a substantial task well-suited for a future project and we shall not discuss much here. With that said, in this chapter we shall inspect just one particular example of a Levi decomposable with regular semisimple Levi factor. We will see that adequately modified versions of the results of Chapter 4 will still apply, thus giving hope that a subsequent venture can successfully incorporate semisimple Levi factors.

## 6.1 REGULAR SEMISIMPLE SUBALGEBRAS

Let $\mathfrak{g}$ be a classical Lie algebra with semisimple subalgebra $\mathfrak{l}$. Then $\mathfrak{l}$ is a direct sum of simple ideals $\mathfrak{l}_1, \ldots, \mathfrak{l}_s \subseteq \mathfrak{l}$. These simple ideals are unique up to automorphism and so we denote the type of $\mathfrak{l}$ as $\mathsf{t}(\mathfrak{l}) = \sum_{i=1}^{s} \mathsf{t}(\mathfrak{l}_i)$.

In [Dyn00], Dynkin classified all regular semisimple subalgebras of the classical Lie algebras, not just those which are simple. Since Theorem 3.3 is the only place where we required simplicity, one may wonder why did we restrict our attention to just the regular simple subalgebras? If a full classification of the regular semisimple subalgebras is known, why not work through Chapter 4 in that far more general setting? The reason for limiting the scope of this project is purely for convenience. Recall that many of the proofs in Chapter 4 operated on a case-by-case basis. Due to the existence of ill-mannered pairs we see that a consideration of cases is a necessity. However, if we allow for semisimple Levi factors the number of possible cases to consider explodes. This is illustrated in the semisimple counterpart to Theorem 3.3 given below.

**Theorem 6.1** (See [Dyn00]). *Let $\mathfrak{g}$ be a classical Lie algebra of rank $n$ and let $\mathfrak{l} \subseteq \mathfrak{g}$ be a regular semisimple subalgebra. Then $\mathfrak{l} \in \mathcal{S}_{\mathfrak{g}}$, where*

$$\mathcal{S}_{\mathfrak{g}} = \begin{cases} \{\sum_{i=1}^{s} A_{m_i} \colon s, m_1, \ldots, m_n \in \mathbb{N}, \sum_{i=1}^{s}(m_i+1) \leq n+1\}, & t(\mathfrak{g}) = A_n \\ \{\sum_{i=1}^{s} A_{m_i} + \sum_{i=1}^{t} D_{l_i} + B_k \colon s, t, k, m_1, \ldots, m_s \in \mathbb{N}, & \\ \quad l_1, \ldots, l_t > 1, \sum_{i=1}^{s}(m_i+1) + \sum_{i=1}^{t} l_i + k \leq n\}, & t(\mathfrak{g}) = B_n \\ \{\sum_{i=1}^{s} A_{m_i} + \sum_{i=1}^{t} C_{l_i} \colon s, t, m_1, \ldots, m_s, l_1, \ldots, l_t \in \mathbb{N}, & \\ \quad \sum_{i=1}^{s}(m_i+1) + \sum_{i=1}^{t} l_i \leq n\}, & t(\mathfrak{g}) = C_n \\ \{\sum_{i=1}^{s} A_{m_i} + \sum_{i=1}^{t} D_{l_i} \colon s, t, m_1, \ldots, m_t \in \mathbb{N}, & \\ \quad l_1, \ldots, l_t > 1, \sum_{i=1}^{s}(m_i+1) + \sum_{i=1}^{t} l_i \leq n\}, & t(\mathfrak{g}) = D_n \end{cases} \tag{6.1}$$

*In addition, for each $T \in \mathcal{S}_{\mathfrak{g}}$ there exists a regular semisimple subalgebra of $\mathfrak{g}$ of type $T$. Furthermore, if $\mathfrak{l}$ and $\mathfrak{l}'$ are regular semisimple subalgebras of $\mathfrak{g}$ with $t(\mathfrak{l}) = t(\mathfrak{l}')$ then $\mathfrak{l}$ and $\mathfrak{l}'$ are conjugate as a general rule, provided $A_1$, $B_1$, $C_1$ are regarded as distinct types, $A_1 + A_1$ and $D_2$ are distinct, and $A_3$ and $D_3$ are distinct. The sole exception to this rule is when $t(\mathfrak{g}) = D_n$ with $n$ even and $t(\mathfrak{l}) = \sum_{i=1}^{s} A_{m_i}$ with $m_1, \ldots, m_s$ odd and $\sum_{i=1}^{s} m_i = n$. In this case there are precisely two non-conjugate regular semisimple subalgebras of $\mathfrak{g}$ of type $\sum_{i=1}^{s} A_{m_i}$.*

In comparison to Theorem 3.3 we greatly increase the number of possible Levi factors if we relax the simplicity condition, which is why this thesis (up until now) has been solely focused on simple Levi factors.

## 6.2 AN EXAMPLE: $t(\mathfrak{g}) = A_n$ AND $t(\mathfrak{l}) = \sum_{i=1}^{s} A_{m_i}$ WITH $\sum_{i=1}^{s}(m_i+1) = n+1$

Considering all possible regular semisimple subalgebras outlined in Theorem 6.1 is a daunting task. To motivate the possibility of extending Chapter 4 to semisimple Levi factors, we only examine one specific example. More explicitly, in the remainder of this chapter we consider the case where $\mathfrak{g}$ is a simple Lie algebra of type $A_n$ and $\mathfrak{l}$ is a semisimple subalgebra of type $\sum_{i=1}^{s} A_{m_i}$ with $\sum_{i=1}^{s}(m_i+1) = n+1$. By Theorem 6.1 such a subalgebra is unique up to inner automorphism and so we can choose an explicit realization of $\mathfrak{l}$. For each $i \in [s]$ define $\Delta_i \subseteq \Delta$ as

$$\Delta_i = \{\alpha_j \in \Delta \colon M_{i-1} < j < M_i\} \tag{6.2}$$

where

$$M_0 = 0, \qquad M_i = \sum_{j=1}^{i}(m_j+1) \tag{6.3}$$

Then $\Delta_i$ is a base for a root system of type $A_{m_i}$. Since $\Delta_i \cap \Delta_j = \varnothing$ for $i, j \in [s]$ distinct we conclude that

$$\Delta_{\mathfrak{l}} = \bigcup_{i=1}^{s} \Delta_i \tag{6.4}$$

is a base for $\Phi_{\mathfrak{l}}$. Viewing $\mathfrak{g}$ as $\mathfrak{sl}_{n+1}$ yields

$$\mathfrak{l} = \left\{ \begin{pmatrix} Z_1 & 0_{m_1+1,m_2+1} & \cdots & 0_{m_1+1,m_s+1} \\ 0_{m_2+1,m_1+1} & Z_2 & \cdots & 0_{m_2+1,m_s+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{m_s+1,m_1+1} & 0_{m_s+1,m_2+1} & \cdots & Z_s \end{pmatrix} \in \mathfrak{sl}_{n+1} \colon Z_i \in \mathfrak{sl}_{m_i+1} \right\} \tag{6.5}$$

The reason we are considering the case where $M_s = n+1$ is because $\Phi_{\mathfrak{l}}^{\perp} = \varnothing$, implying $\mathfrak{l}^{\perp} = \{0\}$. Having a trivial complementary subalgebra to $\mathfrak{l}$ will greatly simplify future arguments.

Suppose $\mathfrak{a}$ is a Levi decomposable subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$. To determine all the possibilities for Rad $\mathfrak{a}$ we shall proceed in the same manner as in Chapter 4. First we recall from Section 4.1 that it will be beneficial to first decompose $\mathfrak{g}$ into irreducible $\mathfrak{l}$-modules with respect to the adjoint representation of $\mathfrak{l}$.

As in Section 4.1, all maximal vectors of the $\mathfrak{l}$-modules appearing in the decomposition of $\mathfrak{g}$ can be taken to be root vectors or elements of $\mathfrak{h}$. Those which are elements of $\mathfrak{h}$ are given by $\tilde{\mathfrak{h}}_{\mathfrak{l}}$ as defined in Equation 4.7. Those which are root vectors correspond to $\Delta_{\mathfrak{l}}$-maximal roots. Since we are interested in a description of the radical we may ignore the $\Delta_{\mathfrak{l}}$-maximal roots appearing in $\Phi_{\mathfrak{l}}$. Indeed, these roots will instead generate the $s$ ideals appearing in the decomposition of $\mathfrak{l}$ into simple ideals. As such, we define

$$\Phi_{\mathfrak{l}}^{\times} = \{\alpha \in \Phi \colon \alpha \text{ is } \Delta_{\mathfrak{l}}\text{-maximal}, \alpha \notin \Phi_{\mathfrak{l}}\} \tag{6.6}$$

With Equation 2.9, straightforward calculations reveal that

$$\Phi_{\mathfrak{l}}^{\times} = \{\chi_{i,j} \in \Phi \colon i, j \in [s], i \neq j\} \tag{6.7}$$

where

$$\chi_{i,j} = \varepsilon_{M_{i-1}+1} - \varepsilon_{M_j} \tag{6.8}$$

With this, combined with the observation that $\mathfrak{l}^{\perp} = \{0\}$, we obtain the decomposition of $\mathfrak{g}$ into irreducible $\mathfrak{l}$-submodules.

**Lemma 6.2.** *The decomposition of $\mathfrak{g}$ into irreducible $\mathfrak{l}$-submodules with respect to the the adjoint representation of $\mathfrak{l}$ is*

$$\mathfrak{g} = \bigoplus_{i=1}^{s} \mathfrak{l}_i \oplus \tilde{\mathfrak{h}}_{\mathfrak{l}} \oplus \bigoplus_{\alpha \in \Phi_{\mathfrak{l}}^{\times}} V(\alpha) \tag{6.9}$$

*where $V(\alpha)$ for $\alpha \in \Phi_{\mathfrak{l}}^{\times}$ is the irreducible $\mathfrak{l}$-module generated by $\mathfrak{g}_\alpha$ and $\bigoplus_{i=1}^{s} \mathfrak{l}_i = \mathfrak{l}$ is the decomposition of $\mathfrak{l}$ into simple ideals $\mathfrak{l}_i$ such that each $\mathfrak{l}_i$ has base $\Delta_i$ as in Equation 6.2. Additionally, by abuse of notation we write $\tilde{\mathfrak{h}}_{\mathfrak{l}}$ to denote the direct sum of trivial $\mathfrak{l}$-submodules which constitute $\tilde{\mathfrak{h}}_{\mathfrak{l}}$.*

*Proof.* Clearly $\mathfrak{l}$ is itself an $\mathfrak{l}$-module with respect to the adjoint representation and thus $\mathfrak{l}$ decomposes into a sum of irreducible $\mathfrak{l}$-submodules. Since we are considering the adjoint, such a decomposition is equivalent to a decomposition into simple ideals. As $\mathfrak{l}^{\perp} = \{0\}$ the result follows by the arguments outlined in Section 4.1. □

Note that $V(\alpha)$ for $\alpha \in \Phi_{\mathfrak{l}}^{\times}$ is given by Equation 4.19, just as it was in the simple Levi factor setting. We also have the following corollary, which is closely related to Corollary 4.11.

**Corollary 6.3.** *If $\mathfrak{a}$ is a Levi decomposable subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ then* Rad $\mathfrak{a}$ *is an $\mathfrak{l}$-submodule of $\tilde{\mathfrak{h}}_{\mathfrak{l}} \oplus \bigoplus_{\alpha \in \Phi_{\mathfrak{l}}^{\times}} V(\alpha)$.*

In what follows $\mathfrak{a}$ is a Levi decomposable subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$, trivial component of the radical $V^0$, and nontrivial component of the radical $V^{\times}$ as in Equation 4.20. Explicitly,

$$V^0 = \text{Rad } \mathfrak{a} \cap \tilde{\mathfrak{h}}_{\mathfrak{l}}, \qquad V^{\times} = \text{Rad } \mathfrak{a} \cap \left( \bigoplus_{\alpha \in \Phi_{\mathfrak{l}}^{\times}} V(\alpha) \right) \tag{6.10}$$

Note also that Lemma 4.15 still holds.

If we are attempting to generalize the results of Chapter 4 to this chapter, we should aim to prove two results:

1. Up to inner automorphism, $V^{\times}$ is a direct sum of root spaces corresponding to a nontrivial $\Phi_{\mathfrak{l}}$-candidate. Additionally, conjugacy of

nontrivial components is related to $\Phi_{\mathfrak{l}}$-Weyl conjugacy. This is the analogue to Theorem 4.19.

2. There exists some fairly simple condition on the trivial component of the radical that determines conjugacy. This is the counterpart to Theorem 4.37/Theorem 4.38.

We dedicate the remainder of this chapter to establishing these two facts. To prove the former we first require some notation. For each $i \in [s]$ and $j \in [m_i]$ define

$$H_{i,j} = H_{M_{i-1}+j} \tag{6.11}$$

Then the set of all such $H_{i,j}$ is a basis for $\mathfrak{h}_{\mathfrak{l}}$. Define the fundamental dominant weights $\lambda_{i,j} \in \mathfrak{h}_{\mathfrak{l}}^*$ for each $i \in [s]$ and $j \in [m_i]$ such that for each $k \in [s]$ and $l \in [m_k]$, $\lambda_{i,j}(H_{k,l}) = \delta_{i,k}\delta_{j,l}$. Also note that

$$\chi_{i,j} + \Phi_{\mathfrak{l}} = \{\varepsilon_k - \varepsilon_l \in \Phi \colon M_{i-1} < k \leq M_i \text{ and } M_{j-1} < l \leq M_j\} \tag{6.12}$$

for each $i, j \in [s]$ distinct. In our identification of $\mathfrak{g}$ with $\mathfrak{sl}_{n+1}$,

$$V(\chi_{i,j}) = \begin{cases} \langle X_{r,\ldots,s} \in \mathfrak{g} \colon M_{i-1} < r \leq M_i \leq M_{j-1} \leq s < M_j \rangle, & i < j \\ \langle Y_{r,\ldots,s} \in \mathfrak{g} \colon M_{j-1} < r \leq M_j \leq M_{i-1} \leq s < M_i \rangle, & j < i \end{cases} \tag{6.13a}$$

$$= \langle E_{r,s} \in \mathfrak{sl}_{n+1} \colon M_{i-1} < r \leq M_i \text{ and } M_{j-1} < s \leq M_j \rangle \tag{6.13b}$$

We now work on proving the equivalent of Theorem 4.19. Throughout, we assume $\mathfrak{a}$ is a Levi decomposable subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ with nontrivial component of Rad $\mathfrak{a}$ given by $V^{\times}$ and trivial component of Rad $\mathfrak{a}$ given by $V^0$.

**Lemma 6.4.** *There exists $\Theta \subseteq \Phi$ a nontrivial $\Phi_{\mathfrak{l}}$-candidate such that up to inner automorphism, $V^{\times}$ is given by Equation 4.22.*

*Proof.* As in the proof of Theorem 4.19, if there exists a subset $\Theta \subseteq \Phi$ such that $V^{\times}$ is as in Equation 4.22 then $\Theta$ is necessarily a nontrivial $\Phi_{\mathfrak{l}}$-candidate. By Equation 6.7 and Corollary 6.3, the nontrivial component of the radical is an $\mathfrak{l}$-submodule of $\bigoplus_{1 \leq i,j \leq s, i \neq j} V(\chi_{i,j})$. For each $i, j \in [s]$ distinct, one can verify that $\chi_{i,j}|_{\mathfrak{h}_{\mathfrak{l}}} = \lambda_{i,1} + \lambda_{j,m_j}$ and so $V(\chi_{i,j})$ has highest weight $\lambda_{i,1} + \lambda_{j,m_j}$. Notice that if $V^{\times}$ contains an irreducible $\mathfrak{l}$-module of highest weight $\lambda_{i,1} + \lambda_{j,m_j}$ then if $(m_i, m_j) \neq (1,1)$ then it must contain $V(\chi_{i,j})$ since that is the only irreducible $\mathfrak{l}$-submodule of $\mathfrak{g}$ with that highest weight. If however $(m_i, m_j) = (1,1)$ then an irreducible $\mathfrak{l}$-module of highest weight $\lambda_{i,1} + \lambda_{j,1}$ will have maximal vector $w \in V^\times$ defined as

$$w = av_{i,j} + bv_{j,i} \tag{6.14}$$

for some $a, b \in \mathbb{F}$, where $v_{i,j}$ and $v_{j,i}$ are maximal vectors of $V(\chi_{i,j})$ and $V(\chi_{j,i})$, respectively. Note that for $k, l \in [s]$ distinct we can take

$$v_{k,l} = \begin{cases} X_{M_{k-1}+1,\dots,M_l-1}, & k < l \\ (-1)^{M_{l-1}-M_k} Y_{M_k,\dots,M_{l-1}}, & k > l \end{cases} = E_{M_{k-1}+1,M_l} \tag{6.15}$$

Since $w \in \operatorname{Rad}\mathfrak{a}$ and $Y_{M_{i-1}+1} \in \mathfrak{l}_i \subseteq \mathfrak{l}$ we have $[w, [Y_{M_{i-1}+1}, w]] \in \operatorname{Rad}\mathfrak{a}$. By direct computation,

$$[w, [Y_{M_{i-1}+1}, w]] = 2abX_{M_j-1} \tag{6.16}$$

Since $X_{M_j-1} \in \mathfrak{l}_j \subseteq \mathfrak{l}$ it must be that $ab = 0$, implying $V^\times$ contains $V(\chi_{i,j})$ or $V(\chi_{j,i})$. Therefore, independent of $(m_i, m_j)$ we observe that $V^\times$ can be taken to be a sum of the $V(\chi_{i,j})$. Lemma 4.12 implies the result. □

*Remark* 6.5. Notice from the above proof that if $V \subseteq \operatorname{Rad}\mathfrak{a}$ is an irreducible $\mathfrak{l}$-module of highest weight $\lambda \neq 0$ then $\lambda = \lambda_{i,1} + \lambda_{j,m_j}$ for some $i, j \in [s]$ distinct. If $(m_i, m_j) \neq (1,1)$ then $V$ must be $V(\chi_{i,j})$, whereas if $(m_i, m_j) = (1,1)$ then $V$ is either $V(\chi_{i,j})$ or $V(\chi_{j,i})$. This rigidity in the possibilities for the irreducible $\mathfrak{l}$-submodules greatly simplifies future arguments.

The similarity between Lemma 6.4 and Theorem 4.19 gives hope to the possibility that we can augment the notion of being well-mannered to include pairs involving a semisimple subalgebra. Note however that we specifically chose to consider the case of $\mathfrak{t}(\mathfrak{g}) = A_n$ and $\mathfrak{t}(\mathfrak{l}) = \sum_{i=1}^{s} A_{m_i}$ with $\sum_{i=1}^{s}(m_i+1) = n+1$ since this case appears to be the most tractable with respect to the decomposition of $\mathfrak{g}$ into irreducible $\mathfrak{l}$-modules. If we instead had $\sum_{i=1}^{s}(m_i+1) \leq n$ then not only would $\mathfrak{l}^\perp$ be nontrivial, but there would be significantly more $\mathfrak{l}$-modules which can appear in the nontrivial component of the radical. A similar situation arises if $\mathfrak{g}$ were not of type $A_n$. In the setting of this chapter, we were rather fortunate in having the irreducible $\mathfrak{l}$-modules nearly all of different highest weight but such coincidences need not occur in general. Hence although there is some anticipation that well-mannered pairs need not specifically apply to cases involving simple Levi factors, a broader setting would require a far more involved proof.

Lemma 6.4 is only a partial analogue to Theorem 4.19. Indeed, if we wanted a true analogue then we would need to show that conjugacy of the nontrivial components is related to $\Phi_{\mathfrak{l}}$-Weyl conjugacy of the corresponding nontrivial $\Phi_{\mathfrak{l}}$-candidates. To facilitate later arguments we make the following definition.

**Definition 6.6.** A subset $\Gamma \subseteq [s]^2$ is *admissible* if for all $i, j, k \in [s]$, if $(i, j) \in \Gamma$ then $(j, i) \notin \Gamma$, and if $(i, j), (j, k) \in \Gamma$ then $(i, k) \in \Gamma$.

From Lemma 6.4 we have that $V^\times = \bigoplus_{(i,j)\in\Gamma} V(\chi_{i,j})$ for some $\Gamma \in [s]^2$. It turns out that $\Gamma$ is necessarily admissible, as we now prove.

**Lemma 6.7.** *Let $V^\times = \bigoplus_{(i,j)\in\Gamma} V(\chi_{i,j})$ for some $\Gamma \in [s]^2$. Then $\Gamma$ is admissible.*

*Proof.* Let $(i, j) \in \Gamma$. If $(j, i) \in \Gamma$ then $V(\chi_{i,j}), V(\chi_{j,i}) \subseteq V^\times$. As $E_{M_{i-1}+1,M_j} \in V(\chi_{i,j})$ and $E_{M_j,M_i} \in V(\chi_{j,i})$ we have that $[E_{M_{i-1}+1,M_j}, E_{M_j,M_i}] \in V^\times$. However, $[E_{M_{i-1}+1,M_j}, E_{M_j,M_i}] = X_{M_{i-1}+1,\ldots,M_{i-1}+m_i} \in \mathfrak{l}$, yielding a contradiction.

Now suppose $(i, j), (j, k) \in \Gamma$. Then having $E_{M_i,M_j} \in V(\chi_{i,j})$ and $E_{M_j,M_k} \in V(\chi_{j,k})$ implies $[E_{M_i,M_j}, E_{M_j,M_k}] = E_{M_i,M_k} \in V^\times$. Since $E_{M_i,M_k} \in V(\chi_{i,k})$ it follows that $(i, k) \in \Gamma$. □

We now tackle the question of conjugacy. We hope for a result similar to Theorem 4.19, namely that conjugacy of the nontrivial component of the radical is equivalent to $\Phi_{\mathfrak{l}}$-Weyl conjugacy of the corresponding $\Phi_{\mathfrak{l}}$-candidates. To prove this, it suffices just as in the proof of Theorem 4.19 to prove that all inner automorphisms that we require can be chosen to preserve $\mathfrak{h}$. To show this it will be beneficial to consider a particular block decomposition of an element of $\mathfrak{gl}_{n+1}$.

**Definition 6.8.** Let $P \in \mathfrak{gl}_{n+1}$. The *block decomposition of $P$ relative to $\mathfrak{l}$* is the decomposition of $P$ into block matrices as in

$$P = \begin{pmatrix} P^{1,1} & P^{1,2} & \cdots & P^{1,s} \\ P^{2,1} & P^{2,2} & \cdots & P^{2,s} \\ \vdots & \vdots & \ddots & \vdots \\ P^{s,1} & P^{s,2} & \cdots & P^{s,s} \end{pmatrix} \tag{6.17}$$

where for each $k, l \in [s]$, $P^{k,l}$ is an $(m_k + 1) \times (m_l + 1)$ matrix. We write $P = (P^{k,l})$ to denote this decomposition.

With this definition, it is easy to verify that for all $i, j \in [s]$ distinct,

$$V(\chi_{i,j}) = \{(P^{k,l}) \in \mathfrak{gl}_{n+1} \colon P^{k,l} = 0 \text{ for } (k, l) \neq (i, j)\} \tag{6.18}$$

and for each $t \in [s]$,

$$\mathfrak{l}_t = \{(P^{k,l}) \in \mathfrak{gl}_{n+1} \colon P^{k,l} = 0 \text{ for } (k, l) \neq (t, t), P^{t,t} \in \mathfrak{sl}_{m_t+1}\} \tag{6.19}$$

The relative simplicity of Equation 6.18 and Equation 6.19 illustrates why we are interested in such a block decomposition.

This next result will aid us in proving that our desired inner automorphisms can be ensured to preserve $\mathfrak{h}$.

**Lemma 6.9.** *Let $P \in SL_{n+1}$ have block decomposition $P = (P^{k,l})$ relative to $\mathfrak{l}$. Let $\tau \in \operatorname{Int}\mathfrak{g}$ be conjugation by $P$ and suppose $\tau$ maps $V(\chi_{i,j})$ to $V(\chi_{i',j'})$. Then for all $k \in [s]$ distinct from $i'$ and for all $l \in [s]$ distinct from $j$, $P^{k,i} = 0$ and $P^{j',l} = 0$. Moreover, $m_{i'} = m_i$, $m_{j'} = m_j$, and both $P^{i',i}$ and $P^{j',j}$ are invertible.*

*Proof.* Let $Z = (Z^{k,l}) \in V(\chi_{i,j})$. By Equation 6.18 there exists an $(m_i+1) \times (m_j+1)$ matrix $R$ such that $Z^{i,j} = R$ and $Z^{k,l} = 0$ for all $k, l \in [s]$ with $(k,l) \neq (i,j)$. Since $\tau$ maps $V(\chi_{i,j})$ to $V(\chi_{i',j'})$ there exists an $(m_{i'}+1) \times (m_{j'}+1)$ matrix $S$ such that $\tilde{Z} = \tau(Z)$ has block decomposition $\tilde{Z} = (\tilde{Z}^{k,l})$ relative to $\mathfrak{l}$, where $\tilde{Z}^{i',j'} = S$ and $\tilde{Z}^{k,l} = 0$ for all $k, l \in [s]$ with $(k,l) \neq (i',j')$.

Define $A = PZ$ and $B = \tilde{Z}P$. Since $\tau(Z) = \tilde{Z}$ we have $A = B$. In particular, for each $k \in [s]$ distinct from $i'$ we have $A^{k,j} = B^{k,j}$. It is evident that $A^{j,k} = P^{k,i}R$ and $B^{j,k} = 0$ since $k \neq i'$. Therefore, as $A^{j,k} = B^{j,k}$ independent of $R$ it must be that $P^{k,i} = 0$ for all $k \in [s]$ distinct from $i'$. Similarly, $A^{i',l} = B^{i',l}$ for all $l \in [s]$ distinct from $j$. We have $A^{i',l} = 0$ since $l \neq j$ and $B^{i',l} = SP^{j',l}$. Since this must hold for all $(m_{i'}+1) \times (m_{j'}+1)$ matrices $S$ we conclude that $P^{j',l} = 0$ for all $l \in [s]$ distinct from $j$.

Since $P$ is invertible, having $P^{k,i} = 0$ for all $k \in [s]$ distinct from $i'$ implies $P^{i',i}$ has rank $m_i + 1$. Hence $m_{i'} + 1 \geq m_i + 1$. In an entirely similar manner, having $P^{j',l} = 0$ for all $l \in [s]$ distinct from $j$ implies $P^{j',j}$ has rank $m_{j'} + 1$. Thus $m_j + 1 \geq m_{j'} + 1$. Therefore, $m_{i'} \geq m_i$ and $m_j \geq m_{j'}$. However, note that $\tau^{-1}$ maps $V(\chi_{i',j'})$ to $V(\chi_{i,j})$. Thus applying the same argument as above to $\tau^{-1}$ yields $m_i \geq m_{i'}$ and $m_{j'} \geq m_j$. Hence $m_{i'} = m_i$ and $m_{j'} = m_j$, from which it follows that $P^{i',i}$ and $P^{j',j}$ are invertible. □

We now determine the conjugacy relations among the various possibilities for the nontrivial component of the radical outlined in Lemma 6.4.

**Lemma 6.10.** *Let $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$ be Levi decomposable subalgebras of $\mathfrak{g}$ with common Levi factor $\mathfrak{l}$. By Lemma 6.4 and Lemma 6.7 there exist $\Gamma, \tilde{\Gamma} \subseteq [s]^2$ admissible such that the nontrivial components $V^\times$ and $\tilde{V}^\times$ of $\operatorname{Rad}\mathfrak{a}$ and $\operatorname{Rad}\tilde{\mathfrak{a}}$, respectively, are*

$$V^\times = \bigoplus_{(i,j)\in\Gamma} V(\chi_{i,j}), \qquad \tilde{V}^\times = \bigoplus_{(i,j)\in\tilde{\Gamma}} V(\chi_{i,j}) \tag{6.20}$$

*Then there exists $\tau \in \operatorname{Int} \mathfrak{g}$ preserving $\mathfrak{l}$ and mapping $V^\times$ to $\tilde{V}^\times$ if and only if there exists $\sigma \in S_s$ such that for all $i \in [s]$, $m_{\sigma(i)} = m_i$ and*

$$\tilde{\Gamma} = \{(\sigma(i), \sigma(j)) \in [s]^2 \colon (i, j) \in \Gamma\} \tag{6.21}$$

*Proof.* Suppose $\tau \in \operatorname{Int} \mathfrak{g}$ preserves $\mathfrak{l}$ and maps $V^\times$ to $\tilde{V}^\times$. Since $\tau$ preserves $\mathfrak{l}$ we have that $\tau$ maps irreducible $\mathfrak{l}$-modules to irreducible $\mathfrak{l}$-modules. By Remark 6.5 there thus exists a bijection $\rho \colon [s]^2 \to [s]^2$ such that

$$\tilde{\Gamma} = \{\rho(i, j) \in [s]^2 \colon (i, j) \in \Gamma\} \tag{6.22}$$

We claim there exists $\sigma \in S_s$ such that for all $i, j \in [s]$, $\rho(i, j) = (\sigma(i), \sigma(j))$. To prove this, it suffices to verify that if $\rho(i, j) = (i', j')$, then for each $k, l \in [s]$ with $k \neq j$ and $l \neq i$, if $\rho(k, j) = (\tilde{k}, \tilde{j})$ then $\tilde{j} = j'$ and if $\rho(i, l) = (\tilde{i}, \tilde{l})$ then $\tilde{i} = i'$.

Regard $\tau$ as conjugation by some $P \in SL_{n+1}$, where $P$ has block decomposition $P = (P^{k,l})$ relative to $\mathfrak{l}$. By Lemma 6.9, having $\rho(i, j) = (i', j')$ implies $P^{j',y} = 0$ for all $y \in [s]$ distinct from $j$ and that $P^{j',j}$ is invertible. Given $k \in [s]$ distinct from $j$, if $\rho(k, j) = (\tilde{k}, \tilde{j})$ then Lemma 6.9 analogously implies $P^{\tilde{j},y} = 0$ for all $y \in [s]$ distinct from $j$ and that $P^{\tilde{j},j}$ is invertible. Hence $\tilde{j} = j'$. Similarly, given $l \in [s]$ distinct from $i$, if $\rho(i, l) = (\tilde{i}, \tilde{l})$ then Lemma 6.9 implies $P^{x,i} = 0$ for all $x \in [s]$ distinct from $\tilde{i}$ and that $P^{\tilde{i},i}$ is invertible. Consequently, since Lemma 6.9 also implies $P^{x,i} = 0$ for all $x \in [s]$ distinct from $i'$ and that $P^{i',i}$ is invertible we conclude that $\tilde{i} = i'$. From Lemma 6.9 we also have that $m_{i'} = m_i$ and $m_{j'} = m_j$. Therefore, by defining $\sigma \colon [s] \to [s]$ as $\sigma(i) = i'$ we conclude the existence of $\sigma \in S_s$ with the desired properties.

Conversely, suppose there exists $\sigma \in S_s$ such $\tilde{\Gamma}$ is as in Equation 6.21 and for all $i \in [s]$, $m_{\sigma(i)} = m_i$. Define $P = (P^{k,l}) \in \mathfrak{gl}_{n+1}$ such that for each $k, l \in [s]$,

$$P^{k,l} = \begin{cases} cI & k = \sigma(l) \\ 0_{m_k+1, m_l+1}, & k \neq \sigma(l) \end{cases} \tag{6.23}$$

where $c \in \mathbb{F}$. Note that if $k = \sigma(l)$ then $P^{k,l}$ is $(m_l + 1) \times (m_l + 1)$ and so $P$ is well-defined. Evidently $P$ is a generalized permutation matrix and is therefore invertible. Consequently, we can choose $c$ such that $P \in SL_{n+1}$. As such the inner automorphism $\tau \in \operatorname{Int} \mathfrak{g}$ defined as conjugation by $P$ preserves $\mathfrak{h}$. Moreover, it is evident by Equation 6.19 that $\tau$ maps $\mathfrak{l}_k$ to $\mathfrak{l}_{\sigma(k)}$ for each $k \in [s]$ and so $\tau$ preserves $\mathfrak{l}$. One can also verify via Equation 6.18 that for all $(i, j) \in \Gamma$, $\tau$ maps $V(\chi_{i,j})$ to $V(\chi_{\sigma(i),\sigma(j)})$. Since $\tilde{\Gamma}$ is given by Equation 6.21 we conclude that $\tau$ maps $V^\times$ to $\tilde{V}^\times$, as desired. □

Lemma 6.4 shows the existence of a subset $\Theta \subseteq \Phi$ such that up to inner automorphism $V^\times$ is given by Equation 4.22. In addition, Lemma 6.10 establishes necessary and sufficient conditions regarding conjugacy of such nontrivial components. It can be observed that the inner automorphisms employed in the proof of Lemma 6.10 preserve $\mathfrak{h}$. Therefore, by employing the same argument as in Section 4.2 we arrive at the following result.

**Theorem 6.11.** *Let $\mathfrak{g}$ and $\mathfrak{l} \subseteq \mathfrak{g}$ be as in this chapter. If $\mathfrak{a}$ is a Levi decomposable subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$, then there exists $\Theta \subseteq \Phi$ a nontrivial $\Phi_\mathfrak{l}$-candidate such that up to inner automorphism, the nontrivial component $V^\times$ of* $\operatorname{Rad} \mathfrak{a}$ *is given by Equation 4.22. Moreover, suppose $\tilde{\mathfrak{a}}$ is another Levi decomposable subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ and nontrivial component of* $\operatorname{Rad} \tilde{\mathfrak{a}}$ *given by $\tilde{V}^\times = \bigoplus_{\alpha \in \tilde{\Theta}} \mathfrak{g}_\alpha$ for some nontrivial $\Phi_\mathfrak{l}$-candidate $\tilde{\Theta} \subseteq \Phi$. Then there exists $\tau \in$* Int $\mathfrak{g}$ *such that $\tau$ preserves $\mathfrak{l}$ and maps $V^\times$ to $\tilde{V}^\times$ if and only if $\Theta$ and $\tilde{\Theta}$ are $\Phi_\mathfrak{l}$-Weyl conjugate.*

The obvious similarities between Theorem 4.19 and Theorem 6.11 are inspiring. We are admittedly only considering one isolated case in this chapter, but Theorem 6.11 provides some assurance that there is likely a way to extend the notion of well-behaved pairs to include semisimple subalgebras. Extending Theorem 6.11 to other settings in which the Levi factor is semisimple is a natural candidate for a subsequent project and is an enterprise that the author plans to pursue in the future.

We see that the hypotheses of Theorem 4.19 require essentially no changes if we wish to adapt to the particular setting of this chapter, suggesting in some sense that the nontrivial component of the radical is fairly well-behaved. What about the trivial component of the radical? Can we hope for a result similar to Theorem 4.37 or Theorem 4.38? Firstly, note from Corollary 6.3 that the trivial component of the radical of a Levi decomposable subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ will be contained in $C_\mathfrak{g}(\mathfrak{l}) = \tilde{\mathfrak{h}}_\mathfrak{l}$. Since $\tilde{\mathfrak{h}}_\mathfrak{l}$ is abelian, every subspace of $\tilde{\mathfrak{h}}_\mathfrak{l}$ is also a subalgebra. Moreover, since $\tilde{\mathfrak{h}}_\mathfrak{l} \subseteq \mathfrak{h}$ and $\bigoplus_{(i,j)\in\Gamma} V(\chi_{i,j})$ is regular relative to $\mathfrak{h}$ for every admissible subset $\Gamma \subseteq [s]^2$, it follows that every subspace of $\tilde{\mathfrak{h}}_\mathfrak{l}$ is a possible trivial component of the radical. Hence unlike in Chapter 4, the various possibilities for the trivial component of the radical are completely known. Of course, this is precisely due to the precisely chosen context we are examining in this chapter and one should not at all anticipate such a straightforward classification in a more general environment.

Now comes the question of determining conjugacy between the potential trivial components. In Theorem 4.37 and Theorem 4.38, conjugacy in $\mathfrak{g}$ of the trivial components was related to conjugacy in $C_\mathfrak{g}(\mathfrak{l})$. However, in our current setting conjugacy in $C_\mathfrak{g}(\mathfrak{l})$ is incredibly restrictive. Indeed, the only inner automorphism of $C_\mathfrak{g}(\mathfrak{l}) = \tilde{\mathfrak{h}}_\mathfrak{l}$ is the identity since $C_\mathfrak{g}(\mathfrak{l})$ is abelian. As such, we expect that the hypotheses of Theorem 4.37 and Theorem 4.38 require some tweaking if we expect a similar result to hold in this chapter. What such tweaks should we anticipate?

In some sense, Lemma 6.10 shows that inner automorphisms of $\mathfrak{g}$ preserving $\mathfrak{l}$ can be thought of as permuting the $\mathfrak{l}_k$. Any such permutation of the $\mathfrak{l}_k$ will naturally yield a permutation on $\tilde{\mathfrak{h}}_\mathfrak{l}$ and so perhaps in this semisimple setting we should look at inner automorphisms of $C_\mathfrak{g}(\mathfrak{l})$, modulo permuting the simple factors of $\mathfrak{l}$. As it turns out, this is exactly the additional assumption needed, at least in the context of this chapter. We make this more precise.

Given $i, j \in [s]$ distinct define $H(i,j) \in \mathfrak{g}$ such that the block decomposition $H(i,j) = (H(i,j)^{k,l})$ of $H(i,j)$ relative to $\mathfrak{l}$ is

$$H(i,j)^{k,l} = \begin{cases} (m_j+1)I, & (k,l) = (i,i) \\ -(m_i+1)I, & (k,l) = (j,j) \\ 0_{m_k+1,m_l+1}, & \text{else} \end{cases} \tag{6.24}$$

for all $k, l \in [s]$. It is easy to verify that for all $i, j \in [s]$ distinct we have $H(i,j) \in \tilde{\mathfrak{h}}_\mathfrak{l}$ and that $\{H(1,2), H(2,3), \ldots, H(s-1,s)\}$ is a basis for $\tilde{\mathfrak{h}}_\mathfrak{l}$. Note the similarity to $\tilde{H}$ in Equation 4.132.

Suppose $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are Levi decomposable subalgebras of $\mathfrak{g}$ with common Levi factor $\mathfrak{l}$. If $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are conjugate then Lemma 4.15 implies the existence of $\tau \in \operatorname{Int}\mathfrak{g}$ such that $\tau$ preserves $\mathfrak{l}$, maps the nontrivial component $V_1^\times$ of Rad $\mathfrak{a}_1$ to the nontrivial component $V_2^\times$ of Rad $\mathfrak{a}_2$, and maps the trivial component $V_1^0$ of Rad $\mathfrak{a}_1$ to the trivial component $V_2^0$ of Rad $\mathfrak{a}_2$. Hence if $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are conjugate then without loss of generality we may suppose $V_1^\times = V_2^\times = V^\times = \bigoplus_{(i,j)\in S} V(\chi_{i,j})$ for some admissible subset $S \subseteq [s]^2$ by Lemma 6.4. If we were to take inspiration from Theorem 4.37 and Theorem 4.38 we would look at inner automorphisms of $C_\mathfrak{g}(\mathfrak{l})$ which preserve $\mathfrak{n}(V^\times)$ or $\mathfrak{c}(V^\times)$ if we wished to determine conjugacy, where $\mathfrak{n}(V^\times)$ and $\mathfrak{c}(V^\times)$ are defined by Equation 4.134 and Equation 4.149, respectively. Since $C_\mathfrak{g}(\mathfrak{l})$ has trivial inner automorphism group we need not consider such a condition. Instead, unlike in the case where the Levi factor is simple we now have the ability to permute the simple factors of $\mathfrak{l}$. Note that a permutation of $[s]$ naturally acts on the basis $\{H(1,2), H(2,3), \ldots, H(s,s-1)\}$ of $C_\mathfrak{g}(\mathfrak{l})$. This leads to the following.

**Theorem 6.12.** *Let $\mathfrak{g}$ and $\mathfrak{l} \subseteq \mathfrak{g}$ be as in this chapter. Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be Levi decomposable subalgebras of $\mathfrak{g}$ with common Levi factor $\mathfrak{l}$ and suppose the radicals of $\mathfrak{a}_1$ and $\mathfrak{a}_2$ have the same nontrivial component of the radical $V^\times$. Let $V_1^0$ and $V_2^0$*

*be the trivial components of* Rad $\mathfrak{a}_1$ *and* Rad $\mathfrak{a}_2$*, respectively. From* Lemma 6.4 *and* Lemma 6.7 *there exists an admissible subset* $\Gamma \subseteq [s]^2$ *such that*

$$V^\times = \bigoplus_{(i,j)\in\Gamma} V(\chi_{i,j}) \tag{6.25}$$

*Moreover,*

$$V_1^0 = \operatorname{span}\left\{\sum_{k=1}^{s-1} a_{i,k} H(k,k+1) \in \tilde{\mathfrak{h}}_{\mathfrak{l}} \colon i \in [d]\right\} \tag{6.26}$$

*for some* $a_{i,k} \in \mathbb{F}$*, where* $d = \dim V_1^0$*. Then* $\mathfrak{a}_1$ *and* $\mathfrak{a}_2$ *are conjugate if and only if there exists* $\sigma \in S_s$ *such that for all* $i \in [s]$*,* $m_{\sigma(i)} = m_i$ *and*

$$\Gamma = \{(\sigma(i),\sigma(j)) \in [s]^2 \colon (i,j) \in \Gamma\} \tag{6.27}$$

$$V_2^0 = \operatorname{span}\left\{\sum_{k=1}^{s-1} a_{i,k} H(\sigma(k),\sigma(k+1)) \in \tilde{\mathfrak{h}}_{\mathfrak{l}} \colon i \in [d]\right\} \tag{6.28}$$

*Proof.* First suppose $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are conjugate. Lemma 4.15 implies there exists $\tau \in \operatorname{Int}\mathfrak{g}$ such that $\tau$ preserves $\mathfrak{l}$ and $V^\times$ while mapping $V_1^0$ to $V_2^0$. By Lemma 6.10 and Remark 6.5, there exists $\sigma \in S_s$ such that for all $i \in [s]$, $m_{\sigma(i)} = m_i$ and $\Gamma$ satisfies Equation 6.27. Indeed, $\tau(V(\chi_{i,j})) = V(\chi_{\sigma(i),\sigma(j)})$ for each $(i,j) \in \Gamma$.

Regard $\tau$ as conjugation by some $P \in SL_{n+1}$ with block decomposition $P = (P^{k,l})$ relative to $\mathfrak{l}$. Lemma 6.9 implies that for each $i \in [s]$, $P^{\sigma(i),i}$ is invertible and if $k \in [s]$ is distinct from $\sigma(i)$ then $P^{k,i} = 0$. By construction, $V_2^0 = \tau(V_1^0)$. Since $PH(i,j)P^{-1} = H(\sigma(i),\sigma(j))$ for all $i,j \in [s]$ distinct the result follows.

Conversely, suppose there exists $\sigma \in S_s$ such that for all $i \in [s]$, $m_{\sigma(i)} = m_i$ and both Equation 6.27 and Equation 6.28 hold. Let $P = (P^{k,l}) \in SL_{n+1}$ be as in Equation 6.23 and define $\tau \in \operatorname{Int}\mathfrak{g}$ as conjugation by $P$. As in the proof of Lemma 6.10 we have that $\tau$ preserves $\mathfrak{l}$ and $V^\times$. Moreover, $\tau(H(k,k+1)) = H(\sigma(k),\sigma(k+1))$ and so $\tau$ maps $V_1^0$ to $V_2^0$, as desired. □

Theorem 6.12 serves as this chapter's counterpart to Theorem 4.37 and Theorem 4.38. While an analysis of one particular case does not indicate general behaviour, it is not unreasonable to suspect that Theorem 6.12 can be extended to a broader context. Our exploration in this chapter highlights at least one key difference between the simple and semisimple settings, namely the ability to permute the simple components of the Levi factor. This observation, along with the form of Theorem 6.12, suggests that the results of Chapter 4 which pertain to the trivial component of the radical can be suitably extended if we account for this ability to permute.

# 7 CONCLUSION

Overall, our exploration into the structure of Levi decomposable subalgebras of classical Lie algebras has been rather fruitful. With the basic groundwork established in Chapter 2 and Chapter 3, we were able to illuminate much of the structure of such subalgebras in Chapter 4. While our results were for well-mannered pairs specifically, this restriction is quite minimal since all regular pairs, apart from one infinite family of such pairs, is well-mannered with only finitely many exceptions.

Dynkin's classification of regular simple subalgebras was portrayed in Chapter 3. While regularity appears to be a rather restrictive condition, we proved in Theorem 3.13 that sufficiently large classical subalgebras of appropriate type are necessarily regular. This theorem illustrates that regularity is not as prohibitive of a demand as it may seem. With the Levi factor handled, our task in Chapter 4 was to examine the possibilities for the radical. We found that the radical naturally decomposed into a nontrivial and trivial component. The nontrivial component has a surprisingly elegant description as given by Theorem 4.19. Indeed, these components were found to be entirely characterized by certain subsets of the root system and conjugacy as subalgebras was determined to be equivalent to conjugacy in the Weyl group. Root systems are incredibly simple to work with and so the ability to ensure that the nontrivial component of the radical can always to given by Equation 4.22 greatly facilitates dealing with such Levi decomposable subalgebras.

By contrast, a classification of the trivial component of the radical could not be obtained due to the sheer difficulty of classifying solvable subalgebras of classical Lie algebras. Moreover, triviality assured by definition that we were unable to utilize the structure of the Levi factor to glean data on the trivial component. Nonetheless, we were still able to gather some information on the trivial component, particularly with respect to conjugacy. By noting that the trivial component is always contained in the centralizer of the Levi factor, we were able to find conditions, as outlined in Theorem 4.37 and Theorem 4.38, on when inner automorphisms of this centralizer could be extended to an inner automorphism of the Lie algebra as a whole to detect conjugacy of two Levi decomposable subalgebras. While a full classification of the trivial component would be a considerable achievement, the conjugacy conditions we established are certainly significant. The general difficulty of ascertaining when two subalgebras are conjugate cannot be overstated and so any tools that can help solve this problem are greatly beneficial.

Unfortunately, we did remark that not all regular pairs are well-mannered. In Chapter 5 we briefly examined the regular pairs for which the results of Chapter 4 did not apply. Aside from one infinite family, the ill-mannered pairs arise when the Levi factor is of sufficiently small rank. This is quite unsurprising. Indeed, the structure of the Levi factor was used to extract information on the radical. If the Levi factor has small rank then in some sense it is too simple and thus does not impose enough restrictions on the radical. This observation complements Theorem 3.13, in which we found that sufficiently small simple subalgebras did not have enough structure to force regularity.

Of particular interest is the infinite family of ill-mannered pairs $(D_n, A_m)$. While we were able to explain why these pairs are ill-mannered, the justification is somewhat unsatisfactory. The author hopes to discover a deeper reason as to why these particular pairs are ill-mannered in the near future. Despite this, we found that these pairs are not terribly ill-mannered. Indeed, we were able to slightly modify Theorem 4.19 in this setting in order to characterize the nontrivial components of the radical.

The vast majority of this thesis was dedicated to studying the case where the Levi factor is simple. In Chapter 6, this condition was relaxed as we examined one particular case where the Levi factor was not simple. While it is certainly possible that this one example we studied is a fluke, we found that we were able to naturally extend the results of Chapter 4. This leaves the task of establishing when it is possible to extend our results in the general semisimple context, and we leave this to a prospective venture.

A natural direction in which this thesis can be taken is to allow the possibility that the parent Lie algebra is exceptional instead of classical. On the one hand, there is no reason a priori that the findings of this thesis do not apply to exceptional Lie algebras. However, we did find some ill-mannered pairs in the classical setting and the increased difficulty in working with the exceptional Lie algebras lends credit to the risk that the exceptional Lie algebras will yield ill-mannered pairs. Even so, exploring this problem in detail is a worthwhile undertaking.

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## COLOPHON

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