Explorations on Beyond Endoscopy

by

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Abstract

In this thesis we provide a description of the first paper on Beyond Endoscopy by Altuğ and explain how to generalize to totally real fields, based on a joint work of the author with Melissa Emory, Debanjana Kundu and Tian An Wong, and is a work in preparation. This part is mostly expository, and we refer the reader to the relevant paper [7]

Furthermore, we prove a conjecture of Arthur. In his original paper on Beyond Endoscopy, Langlands provides a formula for certain product of orbital integrals in $GL(2, \mathbb{Q})$, subsequently used by Altuğ to manipulate the regular elliptic part of the trace formula with the goal of isolating the contribution of the trivial representation. Arthur predicts this formula should coincide with a product of polynomials associated to zeta functions of orders constructed by Zhiwei Yun. We prove this is the case by finding the explicit polynomials and recovering the original formula from them.

We also explain how some aspects of the strategy used can be interpreted as problems of independent interest and importance of their own.

Glendower: I can call spirits from the vasty deep.

Hotspur: Why, so can I, and so can any man,
but will they come when you do call for them?

Henry IV, Act III

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Introduction

General overview

The central conjecture of the Langlands program is the Principle of Functoriality (see [11] or [5]). Loosely speaking it predicts the existence of a correspondence between the automorphic representations of any two appropriate reductive algebraic groups. Several major problems in number theory and representation theory follow from the veracity of this conjecture. Furthermore, it has been a driving force for the development of several mathematical areas and tools. In particular, the theory of the trace formula has seen huge advancements and applications due to its significant role in the Langlands Program

A trace formula for a reductive algebraic group G is an identity equating the value of two distributions built out of computing the trace of the right regular representation (with modifications to make this possible) in two distinct ways. One way uses information associated to the automorphic representations of G as, for example, the multiplicities with which they appear in the decomposition of the regular representation. This is called the *spectral side* of the trace formula. On the other side, we have geometric information related to the group under conjugation. Both sides are intricate but include within themselves two particular relevant parts that are easy to describe for $G = GL(n)$ over an algebraic number field.

On the spectral side we have the discrete part, which is given by

$$\sum_{\pi} m_\pi \mathrm{Tr}(\pi(f)), \tag{1}$$

where f is a test function in a relevant Schwartz space and $\pi(f)$ is a corresponding representation of G. Here, π runs over the representations appearing in the discrete part of the decomposition of the regular representation into discrete and continuous invariant subspaces and m_π stands for its multiplicity.

On the geometric side we have the regular elliptic part, or:

$$\sum_{[\gamma]} \mathrm{vol}(\gamma)\mathcal{O}(\gamma, f), \tag{2}$$

where $[\gamma]$ runs over the regular elliptic conjugacy classes of $GL(n, K)$, i.e. those whose corresponding characteristic polynomial over K is irreducible. Also, $\mathrm{vol}(\gamma)$ refers to the adelic volume of the centralizer G_γ of γ and

$$\mathcal{O}(\gamma, f) = \int_{G_\gamma(\mathbb{A}_{\mathbb{K}})\backslash G(\mathbb{A}_K)} f(x^{-1}\gamma x)\mathrm{d}x,$$

is the orbital integral of f.

These two parts are not equal as parts on their corresponding sides of the trace formula but there are good reasons to loosely treat them as if they did match each other.

In his article on *Beyond Endoscopy*, [10], Langlands describes a new strategy for proving the Principle of Functoriality. It relies on developing a new trace formula where the poles of $L-$functions play a prominent role. He suggests modifying the spectral side (1) to

$$\sum_{\pi} \mu_\pi m_\pi \mathrm{Tr}(\pi(f)), \tag{3}$$

where μ_π is the order of the pole at $s = 1$ for the L function $L(s, \pi, r)$ associated to π and to some irreducible representation r of $GL(n, \mathbb{C})$. That these poles exist and behave in the correct way to make this definition valid is a consequence of functoriality and as such cannot be taken for granted. Nevertheless, supposing it and working backwards, μ_π can be computed via Tauberian limits. Concretely, we get an expression

$$\mu_\pi = \lim_{n\to\infty} \frac{1}{\sharp\{p \mid p \leq n\}} \sum_{p\leq n} \log(p)\mathrm{Tr}(r(c(\pi_p))) \tag{4}$$

where $c(\pi_p)$ is a certain conjugacy class associated to π and the prime p. We can substitute this expression back into equation (3) and manipulate it into

$$\lim_{n\to\infty} \frac{1}{\sharp\{p \mid p \leq n\}} \sum_{p\leq n} \log(p) \sum_{\pi} m_\pi \mathrm{Tr}(r(c(\pi_p)))\mathrm{Tr}(\pi(f)), \tag{5}$$

and we can construct a function f^p such that the inner sum actually is

$$\lim_{n\to\infty} \frac{1}{\sharp\{p \mid p \leq n\}} \sum_{p\leq n} \log(p) \sum_{\pi} m_\pi \mathrm{Tr}(\pi(f^p)). \tag{6}$$

The sum that we have inside, running over π, is a classical discrete part of the spectral side of the trace formula for a new test function f^p.

Of course, all this has to be put solidified by actually using real parts of the trace formula and explaining carefully what we mean by the construction of f^p, but all this can be done. We refer the reader to [6]. For our motivational purposes we keep this vague description and recall that we have said this discrete spectral parts loosely correspond to the regular elliptic parts. So, we are left with

$$\lim_{n\to\infty} \frac{1}{\sharp\{p \mid p \leq n\}} \sum_{p\leq n} \log(p) \sum_{[\gamma]} \mathrm{vol}(\gamma)\mathcal{O}(\gamma, f^p). \tag{7}$$

Recall that the purpose of this is the development of a new trace formula, whose discrete part has been manipulated assuming that functoriality holds true, but now both equations (6) and (7) do not depend on functoriality at all for their definition and we may expect them to represent, respectively, our new discrete spectral side and regular elliptic geometric side of our new trace formula. The problem thus becomes: *do these limits actually converge?* The answer to this is no, since the spectral side has representations that are nontempered and as such its traces do not have good analytical properties.

For the case of GL(2), which is the case that matters for us and the one handled by Langlands and Altuğ over $\mathbb{Q}$, there is only one such representation: the trivial representation $\mathbf{1}$. If we expect this to be responsible for the failure of the spectral limit to converge, then if we take it away and write the "equality"

$$\lim_{n\to\infty}\frac{1}{\sharp\{p \mid p\leq n\}}\sum_{p\leq n}\log(p)\sum_{[\gamma]}\mathrm{vol}(\gamma)\mathcal{O}(\gamma,f^p)=\lim_{n\to\infty}\frac{1}{\sharp\{p \mid p\leq n\}}\sum_{p\leq n}\log(p)\sum_{\pi\neq\mathbf{1}}m_\pi\mathrm{Tr}(\pi(f^p))$$
$$+\lim_{n\to\infty}\frac{1}{\sharp\{p \mid p\leq n\}}\sum_{p\leq n}\log(p)\mathrm{Tr}(\mathbf{1}(f^p)),$$

pass the trivial representation to the other side and collect terms, we get the equality between

$$\lim_{n\to\infty}\frac{1}{\sharp\{p \mid p\leq n\}}\sum_{p\leq n}\log(p)\left(\sum_{[\gamma]}\mathrm{vol}(\gamma)\mathcal{O}(\gamma,f^p)-\mathrm{Tr}(\mathbf{1}(f^p))\right)$$

and

$$\lim_{n\to\infty}\frac{1}{\sharp\{p \mid p\leq n\}}\sum_{p\leq n}\log(p)\sum_{\pi\neq\mathbf{1}}m_\pi\mathrm{Tr}(\pi(f^p)).$$

We expect what remains in the discrete spectral side to converge, as we have taken away the nontempered terms, so what is on the geometric side *must* converge. Hence, the following is unavoidable: *there should be a way to manipulate the regular elliptic part that isolates the trace of the trivial representation and cancels it, such that what remains, converges*

This is exactly what Altuğ did in his series of three papers [1], [2] and [3] for a particular function f that we describe below and when r is the standard representation. In the first of these papers, what is achieved is the isolation, from the regular elliptic part, of the contribution of the trivial representation. In the second one, several estimates are done so that in the third paper the aforementioned limit can be proven to exist and correspond to the correct value. We now explain some specific details.

As mentioned above, for each prime p there is a function f^p that should become the test function to use in the regular elliptic part. Since we will not deal with the limit here we will not vary the prime p, so we will call our function f instead of f^p.

Given a number field K, we define

$$f=\prod_{q<\infty}f_q\times f_\infty,$$

as follows: at finite primes $q\neq p$ of K we define f_q as the indicator function of the maximal compact in $\mathrm{Mat}(2,K_q)$. At p we define it as

$$N_{K_p}(p)^{-k/2}\mathbf{1}\left(X\in\mathrm{Mat}(2,O_{K_p}) \mid |\det(X)|_p=N_{K_p}(p)^{-k}\right),$$

where k is a nonnegative integer.

For f_∞ we do not specify conditions beyond its orbital integral having compact support. For the explanation of why we consider these functions we refer the reader to [6]. Let us only say here that these are the functions we should consider if our original test function, before the application of the

Tauberian limit, was the indicator of the maximal compact of $\mathrm{Mat}(2, K_q)$ at each prime q, including p.

Because f is defined as a product of local factors, one per prime, the orbital integral splits accordingly and we get that the regular elliptic side of the trace formula, given by (2), becomes

$$\sum_{[\gamma]} \mathrm{vol}(\gamma) \prod_{q<\infty} \mathcal{O}(\gamma, f_q) \times \mathcal{O}(\gamma, f_\infty).$$

Langlands and Altuğ deal with the case in which $K = \mathbb{Q}$. In order to isolate the contribution of the trivial representation, the previous expression has to be manipulated. In part, this is done by explicitly computing the volume term and using germ expansions in the infinite parts.

The obstacle that remains is what to do with the finite orbital integrals. An easy fact to notice is that only regular elliptic conjugacy classes $[\gamma]$ whose characteristic polynomial is monic, has integral coefficients and the constant term is $\pm p^k$ can have a nonzero product of finite orbital integrals.

For these cases, Langlands claims in [10] the following

Proposition 1. *When $K = \mathbb{Q}$, for the function f defined above and a regular elliptic element γ that contributes to this part we have*

$$\prod_{q<\infty} \mathcal{O}(\gamma, f_q) = \sum_{d|s_\gamma} d \prod_{q|d} \left(1 - \frac{\left(\frac{D_\gamma}{q}\right)}{q}\right), \tag{8}$$

where D_γ is the fundamental discriminant of the extension $K_\gamma = \mathbb{Q}(\gamma)$ and $\tau^2 \mp 4p^k = s_\gamma^2 D_\gamma$, where the characteristic polynomial of γ is $X^2 - \tau X \pm p^k = 0$.

In here $\left(\frac{D_\gamma}{q}\right)$ is the Kronecker Symbol. He justifies this formula by noticing that this is a multiplicative function, if it were to be thought as a function of s_γ, and then he proceeds to see that for each possible value of the Kronecker symbol(that is $1, -1, 0$), he gets the corresponding value in the following

Proposition 2. *For $K = \mathbb{Q}$, the orbital integrals $\mathcal{O}(\gamma, f_q)$ of the functions f_q for contributing conjugacy classes to the regular elliptic terms, are given as follows:*

(a) If γ is split then it is q^n,

(b) If γ is not split and the associated extension is unramified then it is

$$q^n \frac{q+1}{q-1} - \frac{2}{q-1}.$$

(c) If γ is not split and the associated extension is ramified then it is

$$\frac{q^{n+1}}{q-1} - \frac{1}{q-1}.$$

At this point, we will not explain what this parameter n is precisely. In a vague sense it measures how much γ fails to generate the whole ring of integers of K_γ at q. Furthermore, in this proposition, saying γ is split, not split and ramified or not split and unramified, refers to the behaviour of the extension $\mathbb{Q}_q(\gamma)$ which is a quadratic extension of $\mathbb{Q}_q$. Notice this extension, in the split case, is not a field but a product of two fields isomorphic to $\mathbb{Q}_q$.

This formula is useful because it allows the expression for the volume and the value of the orbital integrals to be manipulated together. More concretely, the volume can be written as

$$\mathrm{vol}(\gamma) = \sqrt{|D_\gamma|}L(1, \chi_\gamma),$$

where $\chi_\gamma(\cdot) = \left(\frac{D_\gamma}{\cdot}\right)$, and this L function merges with the formula for orbital integrals. Once this is done, one wants to invoke the Poisson Summation Formula in what one obtains but this is not immediately possible because there is no clear Schwartz function that is being evaluated over some lattice.

Altuğ overcomes this obstacle by invoking a formula of a complex parameter, which in his notation is

$$L(s, \delta) = \sum_{f^2|\delta}{}' \frac{1}{f^{2s-1}} L\left(s, \left(\frac{\frac{\delta}{f^2}}{\cdot}\right)\right). \tag{9}$$

In here δ is any integer congruent to 0 or 1 modulo 4. The ' in the sum means that it is taken only over f such that δ/f^2 is itself congruent to 0 or 1. Altuğ shows that what remains after the manipulation of the volume and the finite orbital integrals is precisely $L(1, \delta)$ for a specific δ related to γ.

This allows him to define a function of a complex value, which upon evaluation at $s = 1$ recovers the value in equation 8, and to invoke the approximate functional equation with the completed $L-$function

$$\Lambda(s, \delta) = \left(\frac{|\delta|}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{z + i_\delta}{2}\right) L(s, \delta)$$

where i_δ is 0 or 1 according to whether δ is positive or negative, respectively.

Recall that the approximate functional equation is a tool to understand the behaviour of L-functions in the critical strip, where the Dirichlet series expansion is not valid. In order to carry this out, the classical functional equation for the L function must already be known. Hence, in order to do this, for the above $L-$function, it is necessary to know

$$\Lambda(s, \delta) = \Lambda(1 - s, \delta).$$

The proof of this is given at [16].

After using the Approximate Functional Equation for Λ and evaluating at $s = 1$ once more, what is obtained is an expression that can be, when merged with the infinite orbital integrals, subjected to Poisson summation.

For completeness, let us briefly state what happens after this is achieved. The approximate functional equation transforms the regular elliptic part of the trace formula into a sum of integrals, which come from the Poisson summation formula, weighted by particular character sums that resemble Kloosterman sums but are not quite the same. Altuğ manages to evaluate them precisely. Plugging in these evaluations and shifting the contour of the integrals, he obtains residues of the relevant poles. These residues include the contribution of the trivial representation, as well as the contributions of the special representations.

It is desirable to be able to repeat the whole process just described in greater generality. One of the particularly interesting aspects to understand is formula (8). One principal direction in which to address this question is to change the field, but remain in GL(2). Of course, in this case, the formula one gets can be inferred from that of $\mathbb{Q}$, and one only has to make sure the orbital integral computations, Poisson summation and Kloosterman sums evaluation, etc., make sense and proceed analogously. This is being done in [7].

Another more mysterious direction is what to do in $\mathrm{GL}(n)$ when $n > 2$. In his article [4], Arthur explains that he expects that the formula of Langlands on orbital integrals and its extension to a function of a complex parameter can be recovered from certain zeta functions of local orders constructed by Yun in [17]. In the latter paper, Yun proves that these zeta functions recover, when evaluated at $s = 1$, the value of the finite orbital integrals considered above. When multiplied over all primes of K, what one obtains is a prime candidate to use instead of formula (8), and as a function of a complex variable s instead of formula (9), except that this is given as a product and not as a concrete sum. We are thus lead to the several questions, of which we highlight the following:

Question 1: *Is the formula that Langlands proposes and Altuğ uses the same as the product of the zeta functions of local orders?* A priori, the answer to this question is not serious for the success of the whole strategy, but if the answer is affirmative then we have further evidence that using zeta functions of local orders works, as well as the possibility of using the theory of orders to further study the trace formula.

Question 2: *Can we explain formula (8)?* Its proof requires knowing its expression beforehand and then verifying that at the corresponding prime powers it recovers the values of orbital integrals. This furthermore requires us to know these values of orbital integrals which are not easy to access for higher rank groups. If we did not know this formula nor had the values of the orbital integrals to inspect and guess, could we obtain it from the local order polynomials?

Question 3: *Can we provide a different explanation of the functional equation?* As it stands $L(s, \delta)$ is introduced to extend the sum, but in order to define this function, one has to merge the L-function from the volume factor and the finite orbital integrals. Can what comes from the volume and from the orbital integrals can be extended independently to a function of s, each with its functional equation?

It is the purpose of this thesis, among other objectives, to answer these questions.

Organization and results of this thesis

This thesis is divided into four chapters, each of which discusses an specific topic related to generalizing different parts of the method we described above.

One of the main goals of the author, through his PhD work, has been to understand properly the work carried out in [1] and to be able to explain it in a way that highlights the structure behind it, as opposed to a sequence of *ad hoc* situations that might have been available for $\mathbb{Q}$ but do not generalize for other relevant situations.

The purpose of *Chapter One* is twofold. Firstly, to carefully show the aforementioned strategy and how it carries out in [1]. Secondly, to show this strategy, and the new obstacles that appear, when it is carried out in a new context. As was mentioned above, the original work happens over $\mathbb{Q}$ and it is important to extend it to general number fields K.

This generalization to number fields is being carried out in the joint work, [7], of the author with Melissa Emory, Debanjana Kundu and Tian An Wong and which is still a work in progress at the moment of writing this thesis. Nevertheless, a great part of the whole process has been done and many different isolated problems have been established as independent results.

We highlight that one of the most striking features of the general number field case is the influence of the class number of the field, which is a phenomenon that does not occur in $\mathbb{Q}$ because its class number is one. Another issue, related to number theory, is the problem of completing the regular elliptic part of the trace formula, which in the author's perspective is deeply subtle. This is related to the lack of explicit formulas for the reciprocity symbols, in any of their versions, for primes above two.

We finally conclude Chapter One with a discussion on problems pertaining to global and local methods, as apparent from the strategy once it has been completely carried out. We hope this chapter will serve as a good introduction to the work of Altuğ and the scope of future directions, as well as a good description of the work being carried out by in [7].

Chapter Two and *Chapter Three* are the main part of the thesis work and are aimed at answering the three questions posed at the end of the previous section.

Chapter two could be read independently of the rest of the thesis and its goal is to compute the polynomials related to zeta functions of orders. To be more precise, we have a local field K and a quadratic reduced $K-$algebra extension L. General theory guarantees then that one of three possibilities must occur: L is a field extension, which might be ramified or unramified, or $L = K \times K$.

In each of these three situations we study a specific sequence of orders. Recall that orders are a generalization of the rings of integers, but have lost the property of being of dimension 1 (i.e. prime ideals might not be maximal), and as a consequence, do not share the uniqueness of factorization of ideals into prime ideals. Nevertheless, several constructions of the classical maximal orders can be done in the general situation. In particular, they have associated zeta functions.

One reason to study orders is because associated to each regular elliptic class γ, there is an order R_γ whose zeta function recovers the value of the orbital integral $\mathcal{O}(\gamma, f)$ when evaluated at $s = 1$. Furthermore, these zeta functions are recovered from certain polynomials when we substitute their variable by the value q^{-s}. q stands for the cardinality of the resiude field of K. All this is explained in [17]. The quest we will pursue is to find these polynomials.

As we have said, there are three possibilities for the extension L/K and, associated to each one of them, there is a polynomial. They are given by

$$\begin{aligned}
R_n(X) &= 1 + qX^2 + q^2X^4 + ... + q^nX^{2n}, \\
U_n(X) &= (1 + X)R_{n-1}(X) + q^nX^{2n}, \\
S_n(X) &= (1 - X)R_{n-1}(X) + q^nX^{2n},
\end{aligned}$$

where R_n, U_n and S_n are the polynomials associated, respectively, to the ramified, unramified and split case.

As far as the author is aware, this is the first instance of the explicit polynomials related to zeta functions of orders as explained by Zhiwei Yun in his original article. Furthermore, the fact that the three polynomials are so closely related is the main reason that the formula of Langlands can be recovered almost immediately from them.

The way in which we will find these polynomials is by studying closely the arithmetic of a certain sequence of orders

$$\mathcal{O}_0 \supset \mathcal{O}_1 \supset \mathcal{O}_2 ...$$

with associated zeta functions $\zeta_0, \zeta_1, \zeta_2, ...$ The main result is that ideals inside these orders can be organized in a way that lends itself for a recursive strategy. The fundamental results in the arithmetic side of things, as related to orders, are Propositions 8 and 9 of Chapter Two. What they imply is that all ideals I of $\mathcal{O}_n$ can be written as

$$I = x\mathcal{O}_i,$$

for a unique $0 \leq i \leq n$ and some $x \in I$.

One of the consequences of this is manifested in a recursive equation

$$\zeta_{n+1}(s) = \zeta_{n+1}^P(s) + q^{-s}\zeta_n(s),$$

where $\zeta_{n+1}^P(s)$ is the contribution to the zeta function from the principal ideals. Once more, as far as the author is aware such a recursive equation for zeta functions of orders has not appeared before in the literature.

The challenge now shifts to computing $\zeta_{n+1}^P(s)$. We will prove principal ideals can be organized by their *type*. This is a vector associated to the ideal, constructed from its generators, but that only depend on the ideal. The contribution to the zeta function from the principal ideals can be organized by the contribution given by specific types.

We will see the ideals divide themselves into two: high ideals and low ideals, depending on their type. The high ideals have generators that contribute in all possible ways to the zeta functions, while the low ideals have generators that do not contribute in such a free way because the order where these ideals come from starts to lose the units that are necessary for a more ample contribution.

This will manifest in the rewriting of ζ_{n+1}^P as a local Dirichlet series whose coefficients can be written as indices between different groups of units $\mathcal{O}_0^*, \mathcal{O}_1^*, \mathcal{O}_2^*, ...$ We will be able to do this because of the existence of two related group actions, one for the high ideals and one for the low ideals, and whose orbits we can understand by classical results from the theory of orders, but adapted to our case. Once this is done, we can solve the recurrence and obtain the shape of the polynomials as stated above.

We highlight that all the process in this chapter works because of the very precise arithmetic information available to us, and that we prove in section 2.1. Furthermore, how this implies that the ideals can be organized in the way we do was very unexpected.

In Chapter Three we use the knowledge we gained in chapter two to prove the formula of Langlands. Firstly, we prove that in the case of a regular elliptic element γ, the polynomials associated to the orders generated by γ,

$$\mathcal{O}_{K_\mathfrak{q}}[\gamma],$$

as $\mathfrak{q}$ varies over the different primes of K, can be described uniformly with the aid of the quadratic

Hecke character, which measures the ramification of primes. Concretely, we rewrite them as

$$\left(1 - \frac{\chi_\gamma(\mathfrak{q})}{q^s}\right) R_{n-1}(s) + q^{n(1-2s)},$$

for some nonnegative integer n. Notice how, when we put $X = q^{-s}$, and choose $0, -1$ or 1 for the values of $\chi_\gamma(\mathfrak{q})$ we recover the values of the different polynomials we mentioned above.

An important stepping stone to reach the above uniform expression of the polynomials, as $\mathfrak{q}$ varies, is the proof that $\mathcal{O}_{K_\mathfrak{q}}[\gamma]$ always lands in the sequence of orders, as constructed in Chapter Two. What we prove is that all information can be recovered from a globally constructed element S_γ, which generalizes the s_γ that appeared in equation 8 above.

Concretely, what we prove is that the exponent with which $\mathfrak{q}$ divides S_γ is precisely the location of $\mathcal{O}_{K_\mathfrak{q}}[\gamma]$, within the sequence of orders of that prime $\mathfrak{q}$. This means that exponent is also the n that we should use in the corresponding evaluation of the polynomials above.

The main task, once this uniform writing has been achieved, is to glue them into a global formula, which will become Langlands' Multiplicative Formula. Once more, this uses the above property of S_γ to show that, actually, to recover the product of orbital integrals, we simply need to consider those finitely many primes that divide S_γ, for otherwise the integral has value 1 and we do not need to consider its contribution.

What this finiteness allows us to do is invoke the standard theory of multiplicative functions, in order to transform the product of local formulas into a global one. When this is done, we obtain the formula of Langlands, but this time from the knowledge of the polynomials themselves. Furthermore, since we have constructed the formula from local factors, each of which satisfies its functional equation, we deduce a functional equation for the global case.

In this way, we answer positively the three questions we mentioned above, which settle the prediction Arthur made in [4] about the polynomials of orders recovering the formula of Langlands.

Finally, in *Chapter Four* we discuss four directions in which the topic discussed in this thesis could be continued. The first one related to another generalization of the first paper of Altuğ in which what changes is not the field but the function. We highlight that the results of Chapter Three can still be applied, suitably adapted, to produce the function of a complex parameter that extends the product of the appropiate orbital integrals.

We furthermore discuss the problems associated to the application of our method when the group is $\mathrm{GL}(n)$ for $n > 2$. To put it succinctly, we discuss how we might not be able to expect so much precision in the search of polynomials as we were able to achieve here. Furthermore, we highlight that for groups that are not $\mathrm{GL}(n)$, it is not even clear which order should be the one to use.

The third problem we discuss is related to the work of Chapter Two. It asks what type of theory can be built around the notion of a "sequence of orders", as we have used it here. In it, we highlight several questions that appear immediately once we leave the quadratic case and that will have to be settled, one way or another, if this is to work analogously for higher rank situations.

Finally, the last problem discusses the limitations of both aforementioned approaches to the formula of Langlands. Ultimately, one approach is via building theory, while the other is via order theory. The question we are interested in is whether there is some sort of "dictionary" between the theory of orders and the theory of buildings.

We explore this question by discussing the results about representatives, as we explained above, and showing how with building theory alone it is not possible to prove, as far as the author can see, that there is such a neat organization of ideals as to be subjected to a recursive approach. Precisely what proves it is not possible is the introduction of the types of an ideal. This leads to the question of what should be used (and maybe such theory is already known and the author simply is unaware) to recover this theorem or similar ones from buildings alone.

Chapter 1

The first paper of Altuğ

In order to understand the work that we will develop in the following chapters, we need to know the source that motivates it. As has been mentioned in the introduction, the paper [1] manages to complete the regular elliptic part of the trace formula and find within it the contribution of the trivial representation, on the spectral side, after applying Poisson summation.

This paper occurs in a very specific situation and, as a consequence, several of the results used hide a lot of issues that appear when the process is generalized to other possible situations of interest.

Our purpose in this chapter is to show that there is nevertheless a very concrete strategy which can be broken up into five main steps. Each of these steps has very precise goals that can be readily described. The importance of this is twofold: firstly, to become familiar with its parts as problems on their own that can be studied independently (up to a point) and modified within themselves so that they adapt to different situations, as required to the context. Second, to show that this is not an *ad hoc* strategy, that occurs in the case Altuğ worked in, because of the coincidence of several good accidents, but is instead an actual defined procedure.

1.1 Description of the strategy

The goal of [1] is to manipulate the regular elliptic part of the trace formula for $\mathrm{GL}(2,\mathbb{Q})$, which looks like

$$\sum_{\gamma} \mathrm{vol}(\gamma)\mathcal{O}(\gamma, f),$$

where the sum runs over the regular elliptic classes and f is a specific test function which we will define later on more carefully.

What this manipulation must achieve is to cancel the contribution of the trivial representation in the spectral side. What this means concretely is: when we perform the manipulations, we must be able to locate explicitly the value of the trace of the trivial representation, in such a way that what remains has an explicit expression that can be used for the specific purposes of computing the required limit, as explained in the introduction.

The strategy is broken up into the following five steps which we describe in broad terms now:

Rewriting of the regular elliptic part: This part of the trace formula is written as a sum over the regular elliptic classes γ and requires the explicit evaluation of three different terms. First, the term

$$\mathrm{vol}(\gamma),$$

which is the adelic volume of the centralizer of γ. Second, the manipulation of the nonarchimidean orbital integrals, that is, the successful rewriting of

$$\prod_p \mathcal{O}(\gamma, f_p),$$

where p varies over the finite primes. What we mean by successful evaluation will become clear as we move forward. Finally, the rewriting of the archimidean orbital integrals

$$\prod_{v|\infty} \mathcal{O}(\gamma, f_v),$$

where v varies over the infinite primes. Of course, for the case of $\mathbb{Q}$ there is only one such v.

These three parts must be expressed in terms of the variables that parametrize the regular elliptic conjugacy classes. In the case of GL(2), the parameters are the trace and the determinant associated to each class. Hence, given a possible trace τ and a possible determinant δ, what must be achieved is to change the sum

$$\sum_{\gamma}$$

into a sum

$$\sum_{\delta,\tau \text{ valid}}$$

which runs only among those pairs (τ, δ) that correspond to regular elliptic classes. Then, the rewriting of the volume and the orbital integrals must be explicitly written in terms of these τ and δ in a compatible way.

Application of the Approximate Functional Equation: If the previous stage was carried out successfully, what we have is that the nonarchimidean orbital integrals and certain part of the volume term have merged to give a specific value, which can be explicitly recovered as the evaluation at $s = 1$ of a function of a complex value s. This function must behave as an $L-$function: we must be able to complete it so that we obtain an entire function and furthermore it must satisfy a functional equation.

As we know, $L-$functions usually have two versions: the Dirichlet series expansion which holds only in a certain right half plane, and the Euler product expansion, which allow us to understand global properties from the local ones. In the situation we are in, it is fundamental that the expansion we obtain is in Dirichlet series form so that we have a certain sum over ideals.

Once this is achieved, we can perform the approximate functional equation. Let us recall that the approximate functional equation is a tool that allows us to extract information from within the critical strip of the $L-$function from the knowledge of the convergence of its Dirichlet series outside this strip.

In our case, the approximate functional equation expresses the value at $s = 1$ as a weighted average of two different Dirichlet series: one associated to the right of the critical strip, the other is associated to the left of the critical strip.

These two contributions from different Dirichlet series will act as weights to the factor that comes from the archimidean orbital integrals. Recall that archimidean orbital integrals are not continuous but actually have specific types of singularities when approaching the singular set, which for GL(2) is the center. This becomes an obstacle against Poisson Summation because we cannot perform it unless we are dealing with a Schwartz function. The role of the approximate functional equation then is to combine with the archimidean orbital integral term and produce a Schwartz function to which Poisson Summation can be performed.

Completion of the Regular Elliptic Part and the performance of Poisson Summation: Once the approximate functional equation has been applied and we have obtained Schwartz functions we are left with an expression similar to the following form (we don't write the precise expression since the complete manipulation requires a lot of steps we are not getting into):

$$\sum_{\delta}\sum_{\tau \text{ valid}}\sum_{a=1}^{\infty}\sum_{d=1}^{\infty} H(\delta,\tau,a,d).$$

In here, the outer sum goes over the possible determinants, while the τ sum goes over all possible traces which furthermore satisfy that the corresponding conjugacy class is regular elliptic. The inner two sums that go over a and d are precisely what come from the Dirichlet series: one of the sums, the one over d, is given by the L function itself and is what remains of the nonarchimidean orbital integrals, while the other sum over a is what corresponds to the expansion in Dirichlet series in the approximate functional equation. Furthermore, $H(\delta,\tau,a,d)$ is the form the summand has taken after all these manipulations.

What we need is to appeal to Fubini's Theorem and swap the order of summation so that τ is inside and rewrite this as

$$\sum_{\delta}\sum_{a=1}^{\infty}\sum_{d=1}^{\infty}\sum_{\tau \text{ valid}} H(\delta,\tau,a,d).$$

Once we fix δ, a, d the inner sum is one over certain integers τ. What we expect is that

$$\sum_{\tau\in\mathbb{Z}} H(\delta,\tau,a,d),$$

is a sum that can be subjected to Poisson Summation and rewritten as

$$\sum_{\xi\in\mathbb{Z}} \widehat{H}(\delta,\xi,a,d).$$

The approximate summation part, if carried out successfully, made sure that we can indeed perform Poisson Summation here since we are now dealing with Schwartz functions.

Yet, what we really have is

$$\sum_{\delta}\sum_{a=1}^{\infty}\sum_{d=1}^{\infty}\sum_{\tau \text{ valid}} H(\delta,\tau,a,d),$$

and the inner sum lacks some integers. What we should do is add what we are missing, that is,

$$\sum_{\delta}\sum_{a=1}^{\infty}\sum_{d=1}^{\infty}\sum_{\tau \text{ not valid}} H(\delta,\tau,a,d).$$

This is precisely the problem of completion: the function $H(\delta,\tau,a,d)$ makes sense naturally by our manipulations and it has been given to us from the very beginning. On the other hand, when we add what we are missing, we get parameters δ,τ,a,d in which H might not be defined and we need to prove it *can be extended to the domain required by the completion.*

If this is achieved, Poisson summation can now be performed successfully on the τ sum.

Concrete Evaluation of Kloosterman Type Sums: We will see concretely how this takes place in the situation of Altuğ in the next section, but what occurs at this stage is the following: the function $H(\delta,\tau,a,d)$ includes constructions of number theory associated to quadratic reciprocity. When Poisson summation is performed, what we obtain are sums that include integrals coming from the Fourier transforms in the Poisson Summation. It turns out that these integrals are weighted by character sums that we want to evaluate concretely.

This is the first stage at which the main goal, which is locating within the regular elliptic part the trace of the trivial representation, enters the picture. These character sums, when evaluated explicitly, give an incredibly precise value. To appreciate this specificity, let us discuss briefly the spectral side and the trivial representation. Recall that this is the trace of the representation

$$\int_{Z^+\backslash GL(2,\mathbb{A}_K)} \mathbf{1}(f)(g)\ \mathrm{d}g = \int_{Z^+\backslash GL(2,\mathbb{A}_K)} \mathbf{1}(g)f(g)\ \mathrm{d}g,$$

where $\mathbf{1}$ is the trivial representation. This trace can be computed exactly by breaking it into its archimidean and nonarchimidean parts. For example, in the case treated by Altuğ, it becomes

$$\int_{GL(2,\mathbb{Q}_p)} f_p(g)\ \mathrm{d}g \cdot \int_{Z^+\backslash GL(2,\mathbb{R})} f_\infty(x)\ \mathrm{d}x.$$

It turns out that the nonarchimidean integral, that is,

$$\int_{GL(2,\mathbb{Q}_p)} f_p(g)\mathrm{d}g$$

is part of what appears in the very specific value the Kloosterman Sums have when evaluated. The other part of this specific value is a quotient of Dedekind zeta functions and they enter the picture in the next and last stage.

Isolation of the Trivial Representation: At this stage we have been able to evaluate the Kloosterman sums in a way thatwill return the nonarchimidean integrals associated to the trace. What we now need is to deal with the integrals that we have collected. We have two such integrals: one of them appeared at the approximate functional equation stage, while the other one appears in the Fourier Transform required to do Poisson Summation.

We want to perform standard calculus of residues to one of these integrals and leave the other fixed. When we do this and move the region of integration, we do go over poles of the integrand. These poles have two sources: some appear because of the zeta functions that were left behind in the Kloosterman sum stage, and as such we can evaluate precisely the residue they have. The others come from the approximate functional equation stage: in order to perform this approximate functional equation we need a function that allows to modulate the Dirichlet series contribution from outside the critical strip into the inside without them blowing up. These functions also introduce residues themselves.

Collecting these residues and grouping the integrals correctly, now over its new domain, we obtain several terms of a sum. One of those terms is precisely

$$\int_{Z^+\backslash GL(2,\mathbb{R})} f_\infty(x)\ \mathrm{dx},$$

after some changes of variable that are necessary, and weighted together with the values of the Kloosterman sums that is multiplying all this integral we recover the trace of the trivial representation.

This last step, then, aims to isolate the contribution of the trivial representation via the calculus of residues. We furthermore need that one of the terms that survives, after all contours have been changed and all evaluations have been done, is the archimidean part of the contribution of the trivial representation.

1.2 Altuğ's first paper broken into parts.

Now that we have described in broad terms what happens in each of the five stages of the strategy, we will show how they are specifically carried out in the case of Altuğ.

Before we throw ourselves into this, let us give the relevant assumptions. We are working with the group $\mathrm{GL}(2,\mathbb{Q})$ and we will be fixing a prime p and a nonnegative integer k. For generic primes we will use the letter q.

Our test function is given by

$$f = \prod_{q<\infty} f_q \cdot f_\infty,$$

defined as follows: for a prime $q \neq p$, f_q is the indicator of the maximal compact $\mathrm{GL}(2,\mathbb{Z}_q)$ of $\mathrm{GL}(2,\mathbb{Q}_q)$, while for the prime p we have $p^{-k/2}$ times the indicator of the set

$$\{X \in \mathrm{Mat}(2,\mathbb{Z}_p) \mid \ |\det(X)|_p = p^{-k}\},$$

in $\mathrm{GL}(2,\mathbb{Q}_p)$. Finally f_∞ is any smooth function on $\mathrm{GL}(2,\mathbb{R})$ whose orbital integrals are compactly supported.

γ will represent either a regular elliptic class or a representative of it. We usually do not distinguish betweem them and seldom will we have any problem with this. The orbital integrals that we encounter require choices of measures that we explain now: at the nonarchimedean place $\mathrm{GL}(2,\mathbb{Q}_q)$ we pick the Haar measure that gives its maximal compact subgroup $\mathrm{GL}(2,\mathbb{Z}_q)$ volume 1. Furthermore, to the centralizer $\mathrm{GL}_\gamma(2,\mathbb{Q}_p)$, which is a connected maximal torus, we also pick the Haar measure that gives its maximal compact subgroup $\mathrm{GL}_\gamma(2,\mathbb{Z}_p)$ measure one. These two Haar measures define a unique right invariant measure in the quotient $\mathrm{GL}_\gamma(2,\mathbb{Z}_p)\backslash\mathrm{GL}(2,\mathbb{Z}_p)$, so that the intgration in stages formula holds without any correcting factor. This is the measure we put in this quotient. These choices are also taken at the prime p.

At the archimidean place, we endow $\mathrm{GL}(2,\mathbb{R})$ with its standard Lebesgue measure. On the other hand, the centralizer $\mathrm{GL}_\gamma(2,\mathbb{R})$ is a connected maximal torus. It can be a split torus, conjugate to the diagonal subgroup A of matrices of the form

$$\begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_1} \end{pmatrix},$$

whose measure is $\mathrm{d}\lambda_1\mathrm{d}\lambda_2$. Or it can be a compact torus conjugate to the circle torus B of matrices

$$\begin{pmatrix} r\cos(\theta) & -r\sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{pmatrix}$$

where $\theta \in [0,2\pi)$ and $0 \leq r \leq 1$ whose measure is $\frac{\mathrm{d}r\mathrm{d}\theta}{2\pi}$. The measures put on these centralizers are precisely the pullback measures from the A or B. (We are calling these tori A and B following [15]).

Once these measures at each local place have been fixed, the adelic groups have the corresponding resricted product measure dx. In such situation the orbital integral associated to a regular elliptic element γ and our test function f is

$$\begin{aligned}\mathcal{O}(f,\gamma) &= \int_{Z^+GL_\gamma(2,\mathbb{A}_\mathbb{Q})\backslash GL(2,\mathbb{A}_\mathbb{Q})} f(x^{-1}\gamma x)dx \\ &= \prod_q \int_{GL_\gamma(2,\mathbb{Q}_q)\backslash GL(2,\mathbb{Q}_q)} f(x_q^{-1}\gamma x_q)dx_q \cdot \int_{Z^+GL_\gamma(2,\mathbb{R})\backslash GL(2,\mathbb{R})} f(x_\infty^{-1}\gamma x_\infty)dx_\infty \\ &= \prod_q \mathcal{O}(f_q,\gamma)\cdot\mathcal{O}(f_\infty,\gamma).\end{aligned}$$

This splitting of the adelic orbital integral into a product of local ones is valid because our test function is factorable itself. Finally, with these same measures we have that the volume associated to γ is

$$\mathrm{vol}(\gamma) := \int_{Z^+GL_\gamma(2,\mathbb{Q})\backslash GL_\gamma(2,\mathbb{A}_\mathbb{Q})} dx.$$

Thus, we now have that the regular elliptic part is given by

$$\sum_\gamma \mathrm{vol}(\gamma)\mathcal{O}(f,\gamma),$$

with now all the terms defined above.

On the other hand, or rather on the other side, we have the contribution of the trivial representation. Recall that this means the trace of the operator $\mathbf{1}(f)$, where $\mathbf{1}$ is the trivial representation, and f is the test function we defined above. By definition, this representation is

$$\mathbf{1}(f) = \int_{Z^+\backslash GL(2,\mathbb{A})} f(x)\mathbf{1}(x)dx,$$

which being a one dimensional representation immediately gives that

$$\begin{aligned}\mathrm{Tr}(\mathbf{1}(f)) &= \int_{Z^+\backslash GL(2,\mathbb{A})} f(x)dx \\ &= \prod_q \int_{GL(2,\mathbb{Q}_q)} f_q(x_q)dx_q \cdot \int_{Z^+\backslash GL(2,\mathbb{R})} f_\infty(x_\infty)dx_\infty,\end{aligned}$$

and again this can be broken up into parts since our test function itself is factorable. For every finite prime $q \neq p$, we have

$$\int_{GL(2,\mathbb{Q}_q)} f_q(x_q)dx_q = 1,$$

because f_q is the indicator of the maximal compact and the Haar measure was chosen to give this compact set volume 1. Thus, this reduces to

$$\mathrm{Tr}(\mathbf{1}(f)) = \int_{GL(2,\mathbb{Q}_p)} f_p(x_p)dx_p \cdot \int_{Z^+\backslash GL(2,\mathbb{R})} f_\infty(x_\infty)dx_\infty,$$

Langlands proves in his paper [10] that

$$\int_{GL(2,\mathbb{Q}_p)} f_p(x_p)dx_p \cdot = 2p^{k/2}\frac{1-p^{-(k+1)}}{1-p^{-1}},$$

and we ultimately are lead to the task of finding within the regular elliptic part, after it has been manipulated, the term

$$\mathrm{Tr}(\mathbf{1}(f)) = 2p^{k/2}\frac{1-p^{-(k+1)}}{1-p^{-1}}\int_{Z^+\backslash GL(2,\mathbb{R})} f_\infty(x_\infty)dx_\infty.$$

There are more computations to do on this remaining integral but we will discuss them below. We now content ourselves with having put in place all necessary knowledge to see how the five steps of the strategy outlined above are carried out in a way that we can understand it clearly.

1.2.1 Rewriting of the regular elliptic part

Associated to each regular elliptic matrix γ in $GL(2,\mathbb{Q})$ there is a global quadratic extension generated by its characteristic polynomial. This extension we write as $\mathbb{Q}_\gamma$, and the characteristic polynomial as $P_\gamma(X) = X^2 - \tau X + \delta$.

This quadratic extension also has associated a Dirichlet character χ_γ given by the ramification of primes of $\mathbb{Z}$. Concretely,

$$\chi_\gamma(q) = \begin{cases} 1 & \text{if } q \text{ is split} \\ -1 & \text{if } q \text{ is inert} \\ 0 & \text{if } q \text{ is ramified} \end{cases}$$

This character is primitive and has as conductor the fundamental discriminant of $\mathbb{Q}_\gamma$, which we denote in this chapter by D_γ, following Langlands and Altuğ. One of the consequences of the celebrated Quadratic Reciprocity Law is that we can write the equality

$$\chi_\gamma(n) = \left(\frac{D_\gamma}{n}\right),$$

where the symbol on the right is the Kronecker Symbol and n is any integer. It can be proved that

$$\mathrm{vol}(\gamma) = \sqrt{|D_\gamma|} \cdot L(1, \chi_\gamma),$$

where

$$L(s, \chi_\gamma) = \frac{\chi_\gamma(1)}{1^s} + \frac{\chi_\gamma(2)}{2^s} + \frac{\chi_\gamma(3)}{3^s} + \frac{\chi_\gamma(4)}{4^s} + ...$$

is the Dirichlet $L-$function associated to χ_γ. Furthermore, we can extend the $L-$function to the whole complex plane as an entire function with functional equation, once it is completed in the appropriate way. In particular, we know from Dirichlet that $L(1, \chi_\gamma) \neq 0$.

Moving on, when we consider a regular elliptic element γ, we can prove that

$$\mathcal{O}(f, \gamma) \neq 0$$

implies that $\det(\gamma) = \pm p^k$ and that τ is an integer as well. Because of this, the regular elliptic part actually only runs over conjugacy classes whose determinant is $\pm p^k$. Furthermore, we know that for $GL(2, \mathbb{Q})$ the condition of being regular elliptic is equivalent to the characteristic polynomial being irreducible over $\mathbb{Q}$. By the quadratic formula this will happen if and only if

$$\tau^2 \pm 4p^k$$

is not an integral square. This condition is written by Altuğ as

$$\tau^2 \pm 4p^k \neq \square,$$

and in this chapter we will follow this notation. Hence, the sum over regular elliptic classes γ can be parametrized as a double sum: one over $\pm$ and the other over $\tau \in \mathbb{Z}$ such that the above condition is satisfied. Given such a choice of sign and trace, there is a unique conjugacy class $\gamma = \gamma(\pm, \tau)$ corresponding to it.

We furthermore have the existence of a positive integer s_γ such that the following equality holds true

$$\tau^2 \pm 4p^k = s_\gamma^2 D_\gamma.$$

This expression allows for an expression of the sign character as

$$\chi_\gamma(n) = \left(\frac{\frac{\tau^2 \pm 4p^k}{s_\gamma^2}}{n} \right).$$

Now we come to the rewriting of the product of archimidean orbital integrals. Langlands proves the formula

$$p^{k/2} \prod_q \mathcal{O}(f_q, \gamma) = \sum_{d | s_\gamma} d \prod_{r|d} \left(1 - \frac{\chi_\gamma(r)}{r}\right),$$

where the product inside the sum runs over the different primes dividing d. In the product on the left we are including the prime p. Finally, to deal with the orbital integral at the infinite place we invoke germ expansions as used in [15]. What Altuğ proves is that there exist four functions

$g_1^+, g_1^-, g_2^+, g_2^-$ on $\mathbb{R}$, such that when using the parametrization via trace and determinant, we have

$$\mathcal{O}(f_\infty, \gamma(\tau, \pm p^k)) = g_1^\pm\left(\frac{\tau}{2p^{k/2}}\right) + \frac{1}{2}\left|\frac{\tau^2}{2p^k} - 1\right|^{-1/2} g_2^\pm\left(\frac{\tau}{2p^{k/2}}\right).$$

In this way the conjugacy classes have been parametrized by $(\pm, \tau)$ and the volume, the archimidean and nonarchimidean orbital integrals have been written using these parameters explicitly. When one substitutes all these expressions into the regular elliptic part and performs several changes of variable between terms contributing to the volume with those of the orbital integrals, one reaches the following expression:

$$\sum_{\pm} \sum_{\substack{\tau\in\mathbb{Z} \\ \tau^2-4p^k\neq\square}} \theta_\infty^\mp\left(\frac{\tau}{2p^{k/2}}\right) \sideset{}{'}\sum_{d^2|\tau^2\pm4p^k} \frac{1}{d} L\left(1, \left(\frac{(\tau^2\pm4p^k)/d^2}{\cdot}\right)\right),$$

where

$$\theta^\mp(x) = 2\left|x^2 \pm 1\right|^{1/2} g_1^\mp(x) + g_2^\mp(x),$$

is a function that appears where the volume term merges with the archimidean orbital intergral. On the other hand, the sum

$$\sideset{}{'}\sum_{d^2|\tau^2\pm4p^k} \frac{1}{d} L\left(1, \left(\frac{(\tau^2\pm4p^k)/d^2}{\cdot}\right)\right).$$

is what remains when the formula of Langlands merges with the L function coming from the volume. The $'$ that appears on top refers to a specific congruence condition of the quotient $\tau^2 \pm 4p^k/d^2$ that is necessary for this manipulation. We explain this condition in the next section. Furthermore, the Kronecker Symbol

$$\left(\frac{(\tau^2\pm4p^k)/d^2}{\cdot}\right)$$

is still a Dirichlet Character as a function of the lower entry, but is nonprimitive. Nevertheless it still has an L function and that is what is being used in here. We have skipped several details, which are not straightforward, but we can now see the kind of expression we expect to have at the end of this stage in general.

1.2.2 Application of the approximate functional equation

As we mentioned before, what we must do here is to apply the approximate functional equation to the expression that comes from the nonarchimidean orbital integral. Concretely, this term is

$$\sideset{}{'}\sum_{d^2|\tau^2\pm4p^k} \frac{1}{d} L\left(1, \left(\frac{(\tau^2\pm4p^k)/d^2}{\cdot}\right)\right).$$

The approximate functional equation requires a function of a complex parameter which satisfies a functional equation. Altuğ at this point invokes such a function, which has been used before in [16] and that is as follows: for a given integer $N \equiv 0, 1 \pmod 4$ we define

$$L(s,N) = \sum_{f^2|N}{}' \frac{1}{f^{2s-1}} L\left(s, \left(\frac{N/f^2}{\cdot}\right)\right),$$

where the $'$ means that we only consider those $f^2|N$ such that the quotient is congruent to 0 or 1 modulo 4. This is the congruence condition we mentioned above. The choice of $N \equiv 0, 1 \pmod 4$ is necessary to assure that the Kronecker symbol

$$\left(\frac{N}{\cdot}\right)$$

is a primitive character and that all the nonprimitive characters

$$\left(\frac{N/f^2}{\cdot}\right)$$

are just lifts of this primitive one. This is important because when the functional equation is used for an $L-$function of a nonprimitive character, we require the completion factors at the real and complex places that the primitive character does, as well as a factor related to the modulus of the primitive character. Because of this, we could complete all the nonprimitive $L-$ functions simultaneously since they require the same factor. Concretely, this gives

$$\Lambda(s,N) = \left(\frac{|N|}{\pi}\right)^{s/2} L_\infty\left(s, \left(\frac{N}{\cdot}\right)\right) L(s,N),$$

where L_∞ is the completion at infinity of the primitive character. Note that L_∞ depends on whether $N > 0$ or $N < 0$, because it corresponds to a quadratic extension that is totally real or totally complex.

In any case, we have that

$$\Lambda(s,N) = \Lambda(1-s,N).$$

We will discuss in the next section the proof of this functional equation. Once this has been achieved, the standard method of the approximate functional equation can be applied and its ultimate result is the following expansion

$$\begin{aligned} L(s,N) = &\sum_{f^2|N}{}' \frac{1}{f^{2s-1}} \sum_{a=1}^{\infty} \frac{1}{a^s} \left(\frac{N/f^2}{a}\right) F\left(\frac{af^2}{A}\right) \\ &+ \left(\frac{|N|}{\pi}\right)^{1/2-s} \sum_{f^2|N}{}' \frac{1}{f^{1-2s}} \sum_{a=1}^{\infty} \frac{1}{a^{1-s}} \left(\frac{N/f^2}{a}\right) H_N\left(s, \frac{af^2 A}{|N|}\right). \end{aligned}$$

There are two sums here because one of them, the first one, comes from the Dirichlet series to the right of the critical strip, while the second sum counts for the contribution of the Dirichlet Series to the left of the critical strip. In general, when applying an approximate functional equation we get two finite sums and a residue given in integral form, but in the above context the last step of the classical uses of the approximate functional equations is not applied and that is why there is no residue term appearing.

In the above expression F and H are two related functions used to mitigate the growth of the Dirichlet series as it approaches the critical strip and to ensure convergence. The F function used by Altuğ is a Bessel function of the second kind because of its oddness (that is, $F(-x) = -F(x)$) as well as the wellknown behaviour of its residues and convergence at infinity, which will be used explicitly on the last step.

This expression is evaluated at $s = 1$ and then substituted into the manipulation of the regular elliptic part. When that is done what we obtain is the following

$$\sum_{\mp} \sum_{\substack{\tau \in \mathbb{Z} \\ \tau^2 \pm 4p^k \neq \square}} \theta_\infty^\mp \left(\frac{\tau}{2p^{k/2}}\right) \sideset{}{'}\sum_{d^2 | \tau^2 \pm 4p^k} \frac{1}{d}$$
$$\times \sum_{a=1}^{\infty} \frac{1}{a} \left(\frac{(\tau^2 \pm 4p^k)/d^2}{a}\right) \left(F\left(\frac{ad^2}{A}\right) + \frac{ad^2}{\sqrt{|\tau^2 \pm 4p^k|}} H\left(\frac{ad^2 A}{|\tau^2 \pm 4p^k|}\right)\right).$$

The parameter $A > 0$ appears during the proof of the approximate functional equation and can be chosen at will. If we choose $A = |\tau^2 \pm 4p^k|^\alpha$ with $0 < \alpha < 1$, then we achieve one of the important goals toward Poisson Summation: *obtaining a Schwartz Function*. Concretely, we have that

$$\theta_\infty^\mp \left(\frac{x}{2p^{k/2}}\right) F\left(\frac{ad^2}{|x^2 \pm 4p^k|^\alpha}\right),$$

and

$$|x^2 \pm 4p^k|^{-1/2} \theta_\infty^\mp \left(\frac{x}{2p^{k/2}}\right) H\left(\frac{ad^2}{|\tau^2 \pm 4p^k|^{1-\alpha}}\right),$$

are Schwartz Functions in the variable x. In the above sum these two functions appear evaluated at τ, so it will make sense to try to perform Poisson Summation with them since they are being evaluated at (an incomplete) lattice. In this way we have achieved what we expected to do in this part.

1.2.3 Completion of the regular elliptic part and the performance of Poisson summation

We can write the above sum as

$$\sum_{\mp} \sum_{\substack{\tau \in \mathbb{Z} \\ \tau^2 - 4p^k \neq \square}} \sideset{}{'}\sum_{d^2 | \tau^2 \pm 4p^k} \sum_{a=1}^{\infty} \theta_\infty^\mp \left(\frac{\tau}{2p^{k/2}}\right) \frac{1}{ad} \left(\frac{(\tau^2 \pm 4p^k)/f^2}{a}\right) \left(F\left(\frac{ad^2}{A}\right) + \frac{ad^2}{\sqrt{|\tau^2 \pm 4p^k|}} H\left(\frac{ad^2 A}{|\tau^2 \pm 4p^k|}\right)\right).$$

for $A = |x^2 \pm 4p^k|^\alpha$ and define, for the appropriate valid parameters

$$G(\mp 1, \tau, a, d) = \theta_\infty^\mp \left(\frac{\tau}{2p^{k/2}}\right) \frac{1}{ad} \left(\frac{(\tau^2 \pm 4p^k)/f^2}{a}\right) \left(F\left(\frac{ad^2}{A}\right) + \frac{ad^2}{\sqrt{|\tau^2 \pm 4p^k|}} H\left(\frac{ad^2 A}{|\tau^2 \pm 4p^k|}\right)\right).$$

The above function makes sense for all values of $\pm 1, \tau, a, d$, regardless of whether there is a regular elliptic matrix γ that produces them or not. Hence, the next expression

$$\Sigma(\Box) := \sum_{\mp} \sum_{\substack{\tau \in \mathbb{Z} \\ \tau^2 \pm 4p^k = \Box}} \sideset{}{'}\sum_{d^2 | \tau^2 \pm 4p^k} \sum_{a=1}^{\infty} G(\mp, \tau, a, d),$$

is well defined. In this way we can add this to our original sum and get

$$\sum_{\mp} \sum_{\tau \in \mathbb{Z}} \sideset{}{'}\sum_{d^2 | \tau^2 \mp 4p^k} \sum_{a=1}^{\infty} G(\mp, \tau, a, d),$$

which is the *completed regular elliptic part.* The most important point is that the τ sum is now over all the integers, so the Schwartz Function is really going over a complete lattice. As it is, we are not ready to apply Poisson Summation because the τ sum is outside the other two sums, and we want it to be the inner sum. Due to the convergence properties of all these involved sums we can swap the order of summation and get

$$\sum_{\pm} \sum_{d=1}^{\infty} \sum_{a=1}^{\infty} \sum_{\substack{\tau \\ \frac{\tau^2 \pm 4p^k}{d^2} \equiv 0,1, \pmod 4}} G(\mp, \tau, a, d),$$

The congruence condition that appears is what replaces the $'$ that existed before the swaps between sums were made. We could argue that doing this swap has destroyed the property we tried to recover: a lattice sum over τ, because of the appearance of this congruence. Nevertheless what happens is that modulo $4ad^2$ the character

$$\left(\frac{(\tau^2 \pm 4p^k)/d^2}{a} \right)$$

is periodic in τ and as so, instead of doing the sum over τ we break it down into arithmetic sequences, each of which can be considered a complete lattice. Doing this transforms the sum into

$$\sum_{\mp} \sum_{d=1}^{\infty} \sum_{a=1}^{\infty} \frac{1}{ad} \sum_{\substack{\mu \pmod{4ad^2} \\ \mu^2 \pm 4p^k \equiv 0 \pmod{d^2} \\ \frac{\mu^2 \pm 4p^k}{d^2} \equiv 0,1, \pmod 4}} \left(\frac{(\mu^2 \pm 4p^k)/d^2}{a} \right) \times$$

$$\sum_{m \equiv \mu \pmod{4ad^2}} \theta_{\infty}^{\mp} \left(\frac{m}{2p^{k/2}} \right) \left(F\left(\frac{ad^2}{A} \right) + \frac{ad^2}{\sqrt{|m^2 \pm 4p^k|}} H\left(\frac{ad^2 A}{|m^2 \pm 4p^k|} \right) \right).$$

For concreteness we will call

$$\Phi^{\pm}(x) = \theta_{\infty}^{\mp} \left(\frac{x}{2p^{k/2}} \right) \left(F\left(\frac{ad^2}{A} \right) + \frac{ad^2}{\sqrt{|x^2 \pm 4p^k|}} H\left(\frac{ad^2 A}{|x^2 \pm 4p^k|} \right) \right).$$

With this notation we finally have a lattice in which a Schwartz function is being evaluated,

$$\sum_{m\equiv\mu \pmod{4ad^2}} \Phi^{\pm}(x),$$

and by the Poisson summation formula, this is equal to

$$\sum_{\xi\in\mathbb{Z}} \widehat{\Phi}^{\pm}(\xi),$$

as desired.

1.2.4 Concrete evaluation of Kloosterman type sums

For every Poisson summation, a character is required. Since this is the Poisson summation formula for the integers, the standard exponential is used, albeit weighted to deal with the volume of the lattice, since it is performed over arithmetic sequences and not over all integers.

Once the Poisson Summation is explicitly performed the sum becomes

$$\frac{p^{k/2}}{2}\sum_{\mp}\sum_{d=1}^{\infty}\sum_{a=1}^{\infty}\frac{1}{a^2d^3}\sum_{\xi} I(\xi,\pm,a,d)K_{a,d}(\xi,\pm p^k),$$

where $I(\xi,\pm,a,d)$ is defined as

$$\int \theta^{\mp}(x)\left(F\left(\frac{ad^2(4p^k)^{-\alpha}}{|x^2\pm 1|^{\alpha}}\right)+\frac{ad^2p^{-k/2}}{2\sqrt{|x^2\pm 1|}}H\left(\frac{ad^2(4p^k)^{\alpha-1}}{|x^2\pm 1|^{1-\alpha}}\right)\right)\exp\left(\frac{-x\xi p^{k/2}}{2ad^2}\right)\mathrm{dx},$$

and

$$K_{a,d}(\xi,\pm p^k) = \sum_{\substack{m \pmod{4ad^2}\\ m^2\pm 4p^k\equiv 0 \mod d^2\\ \frac{m^2\pm 4p^k}{d^2}\equiv 0,1, \mod 4}}\left(\frac{(m^2\pm 4p^k)/d^2}{a}\right)\exp\left(\frac{m\xi}{4ad^2}\right).$$

Several steps occurred here that we omit but let us briefly explain them: firstly, the integral that appears is the one that defines the Fourier transform. Furthermore, we know that when performing the Poisson Summation, an exponential has to appear with the corresponding weight regarding the volume associated to the lattice. The reason why part of it has gone outside the exponential is because it is independent of the variable of integration. And then, there is a change of variable to introduce the variable x in the denominators, and that brings out several factors: the $p^{k/2}/2$ outside, as well as the change from $1/ad$ to $1/a^2d^3$ are precisely due to this change of variable.

We can then swap the sums on ξ and μ and collect the terms depending on this last variable. This gives rise to the term $K_{a,d}$ which is similar to the classical Kloosterman sums, although not exactly the same.

A particularly important term in the previous expansion is the one corresponding to $\xi = 0$. In this situation the exponential factor becomes 1 and the corresponding term is

$$\frac{p^{k/2}}{2}\sum_{\mp}\sum_{d=1}^{\infty}\sum_{a=1}^{\infty}\frac{1}{a^2d^3}I(0,\pm,a,d)K_{a,d}(0,\pm p^k).$$

I is an integral and given the convergence properties of the above sum has, we can use Fubini's theorem to move the a and d sums into the integral. This leads to the expression

$$\sum_{a=1}^{\infty}\sum_{d=1}^{\infty}\frac{K_{a,d}(0,\pm p^k)}{a^2d^3},$$

and the need to evaluate it explicitly becomes apparent since, ultimately, when we try to isolate the trivial representation we do need to go through this stage of bringing the integral outside. To exploit the calculus of residues later on, it is important to not consider this as an isolated number, but once more to deal with a function of a complex variable z and define

$$D(z,\pm p^k)=\sum_{d=1}^{\infty}\frac{1}{d^{2z+1}}\sum_{a=1}^{\infty}\frac{K_{a,d}(0,\pm p^k)}{a^{z+1}},$$

where z is a complex variable. This has an *Euler Product* expansion, with Euler product at the prime q equal to

$$D_q(z,\pm p^k)=\sum_{r=0}^{\infty}\frac{1}{q^{r(2z+1)}}\sum_{v=0}^{\infty}\frac{K_{q^v,q^r}(0,\pm p^k)}{q^{v(z+1)}}.$$

This also holds at our special prime $q=p$. At odd primes, the Kloosterman-like sums have become

$$K_{q^v,q^r}(0,\pm p^k)=\sum_{\substack{m \pmod{q^{v+2r}}\\ m^2\pm 4p^k\equiv 0 \mod q^{2r}\\ \frac{m^2\pm 4p^k}{q^{2r}}\equiv 0,1, \mod 4}}\left(\frac{(m^2\pm 4p^k)/q^{2r}}{q^v}\right),$$

while for $q=2$ we have

$$K_{2^v,2^r}(0,\pm p^k)=\sum_{\substack{m \pmod{2^{2+v+2r}}\\ m^2\pm 4p^k\equiv 0 \mod 2^{2r}\\ \frac{m^2\pm 4p^k}{2^{2r}}\equiv 0,1, \mod 4}}\left(\frac{(m^2\pm 4p^k)/2^{2r}}{2^v}\right).$$

In this way we are left with the computation of these local Kloosterman-like sums, and the computation goes along the lines of counting the number of solutions of quadratic congruences modulo q. There are complications of two types. The first are those related to the case when $q=2$, because we are dealing with quadratic congruences and quadratic symbols and these do not match well together. Concretely, the relevant modulus for quadratic congruences at $q=2$ is modulo 8, which is not a field and this complicates computations. On the other hand, at odd primes q all the computations can be carried out at the field of q elements.

The other complication occurs when $q=p$, because in that situation the term $\pm 4p^k$ in the upper entry of the character becomes relevant and, according to the values of r and v, it can be either disregarded or it have an influence. Altuğ manages through a long computation to prove that the Kronecker character, which has been central to the whole manipulation, disappears and that what remains is nothing else but Euler Factors of the zeta function $\zeta_{\mathbb{Q}}$. He concretely gets that when $q\neq p$ and odd,

$$D_q(z)=\frac{1-q^{-(z+1)}}{1-q^{-z}}$$

while

$$D_2(z) = 4\frac{1-2^{-(z+1)}}{1-2^{-z}}.$$

On the other hand, when $q = p$ and odd we get

$$D_p(z) = \frac{(1-p^{-(z+1)})(1-p^{-(k+1)z})}{(1-q^{-z})(1-q^{-2z})}$$

while if $q = p = 2$ we have

$$D_2(z) = 4\frac{(1-2^{-(z+1)})(1-2^{-(k+1)z})}{(1-2^{-z})(1-2^{-2z})}.$$

Hence when we multiply all of this together, we obtain

$$D(z, \pm p^k) = 4\frac{\zeta(2z)}{\zeta(z+1)} \cdot \frac{1-p^{z(k+1)}}{1-p^{-z}},$$

where p is the fixed prime. Notice that the factor

$$\frac{1-p^{z(k+1)}}{1-p^{-z}},$$

becomes when evaluated at $z = 1$

$$\frac{1-p^{-(k+1)}}{1-p^{-1}},$$

which is one of the factors that comes from the trivial representation at the finite places, as explained above.

1.2.5 Isolation of the contribution of the trivial representation

This part is probably the most technical of the whole strategy, but the idea is quite clear. The emphasis is on the term when $\xi = 0$. This step is focused entirely on it.

We know that the term with $\xi = 0$ is

$$\frac{p^{k/2}}{2} \sum_{\mp} \sum_{d=1}^{\infty} \sum_{a=1}^{\infty} \frac{1}{a^2 d^3} \sum_{\xi} I(0, \pm, a, d) K_{a,d}(0, \pm p^k),$$

In this expression there is no complex variable. In order to introduce such variable z we use the definition of F, which was not chosen at random. As we said above, this is a Bessel function and it satisfies a Mellin Inversion Formula of the following form

$$F(y) = \frac{1}{2\pi i} \int_{\Re(z)=1} \tilde{F}(z) y^{-z} \mathrm{dz},$$

where $\tilde{F}$ is the Mellin transform of F and with z a complex number. It is via this integral that we will be able to invoke the residue theorem while using the variable u as our complex variable .

Recalling that I is defined as

$$\int \theta^{\mp}(x)\left(F\left(\frac{ad^2(4p^k)^{-\alpha}}{|x^2\pm 1|^{\alpha}}\right)+\frac{ad^2p^{-k/2}}{2\sqrt{|x^2\pm 1|}}H\left(\frac{ad^2(4p^k)^{\alpha-1}}{|x^2\pm 1|^{1-\alpha}}\right)\right)\mathrm{dx},$$

it makes sense to analyze the term associated with F and H separately. It is actually important to also separate between the $+$ and $-$ sign. As an example, let us study the one related to F and $+$; i.e.

$$\int \theta^{+}(x)F\left(\frac{ad^2(4p^k)^{-\alpha}}{|x^2+1|^{\alpha}}\right)\mathrm{dx}.$$

We have that the corresponding term is

$$\frac{p^{k/2}}{2}\sum_{-}\sum_{d=1}^{\infty}\sum_{a=1}^{\infty}\frac{1}{a^2d^3}K_{a,d}(0,+p^k)\int \theta^{+}(x)\frac{1}{2\pi i}\int_{\Re(z)=1}\tilde{F}(z)\left(\frac{ad^2(4p^k)^{-\alpha}}{|x^2+1|^{\alpha}}\right)^{-z}\mathrm{dzdx}.$$

Verifying the appropriate convergence allows us to put the d and a sums inside the integral:

$$\frac{p^{k/2}}{2}\sum_{-}\int \theta^{+}(x)\frac{1}{2\pi i}\int_{\Re(z)=1}\tilde{F}(z)\left(\frac{(4p^k)^{-\alpha}}{|x^2+1|^{\alpha}}\right)^{-z}\sum_{d=1}^{\infty}\sum_{a=1}^{\infty}\frac{K_{a,d}(0,+p^k)}{a^{(z+1)+1}d^{2(z+1)+1}}\mathrm{dzdx}.$$

Recognizing the term

$$D(z+1,+p^k)=\sum_{d=1}^{\infty}\sum_{a=1}^{\infty}\frac{K_{a,d}(0,+p^k)}{a^{(z+1)+1}d^{2(z+1)+1}}$$

and using its explicit evaluation this becomes

$$\frac{p^{k/2}}{2}\int \theta^{+}(x)\frac{1}{2\pi i}\int_{\Re(z)=1}\tilde{F}(z)\left(\frac{(4p^k)^{-\alpha}}{|x^2+1|^{\alpha}}\right)^{-z}\cdot 4\frac{\zeta(2z+2)}{\zeta(z+2)}\cdot\frac{1-p^{(z+1)(k+1)}}{1-p^{-(z+1)}}\mathrm{dzdx}.$$

Altuğ verifies that the residue theorem can be applied at this point moving the line of integration from $\Re(z)=1$ to $\Re(z)=-1$. For the above, this transforms situation the integral into

$$\frac{(4p^k)^{(1-\alpha)/2}\tilde{F}(1/2)(1-p^{-(k+1)/2})}{2\zeta(3/2)(1-p^{-1/2})}\int\frac{\theta^{+}(x)}{|x^2+1|^{\alpha/2}}\mathrm{dx}$$
$$+\frac{p^{k/2}}{2}\int \theta^{+}(x)\frac{1}{2\pi i}\int_{\Re(z)=-1}\tilde{F}(z)\left(\frac{(4p^k)^{-\alpha}}{|x^2+1|^{\alpha}}\right)^{-z}\cdot 4\frac{\zeta(2z+2)}{\zeta(z+2)}\cdot\frac{1-p^{(z+1)(k+1)}}{1-p^{-(z+1)}}\mathrm{dzdx}.$$

In the same way the other integrals, corresponding to the missing combinations of F and H with $+$ and $-$, become a residue and an integral over a new contour. Some of the terms that appear cancel in pairs due to the oddness of F and some terms with nice properties remain.

One of these terms, which is the residue at the pole at $z=0$ of the expression with H and $+$, is precisely

$$2p^{k/2}\frac{1-p^{k+1}}{1-p^{-1}}\sum_{\mp}\int \theta^{\mp}(x)\mathrm{d}x,$$

which is the contribution of the trivial representation, after the Weyl integration formula has been applied in the original computation, to make the functions $\theta^{\pm}$ appear . This alone is what was expected when all this process was carried out, but more was found. Other of the terms that remains is

$$\frac{k+1}{2}\sum_{\pm}\int \frac{\theta^{\mp}(x)}{\sqrt{|x^2 \pm 1|}}\mathrm{d}x,$$

which Langlands proved in his original article to be the contribution of the special representations. Even more so, the terms that remain, beyond the ones above, can be handled analytically in a way that is appropriate for the limit computations that have to be carried out in later stages of the whole strategy of Beyond Endoscopy that Altuğ performs in his next two papers.

This concludes our exposition of the first paper of Altuğ in a way that is broken up into different parts, each with a concise goal to be carried out.

1.3 Generalizing to totally real fields.

There are several directions in which one can try to generalize the work of [1]. Generalizing the work of Altuğ to general number fields is being carried out in [7]. We discuss below several aspects of this particular research when applied to totally real fields, which is still a work in progress but already in an advanced stage. We divide the process into the five steps discussed prior.

Our test function is constructed in the same way as in Altuğ; that is, fixing a finite prime $\mathfrak{p}$ of the base field K and defining $f_{\mathfrak{q}}$ as the indicator of the maximal compact $\mathrm{GL}(2, \mathcal{O}_{K_{\mathfrak{q}}})$ of $\mathrm{GL}(2, K_{\mathfrak{q}})$ at primes $\mathfrak{q} \neq \mathfrak{p}$. At $\mathfrak{p}$ we define it as $N_K(\mathfrak{p})^{-k/2}$ times the indicator of

$$\{X \in \mathrm{Mat}(2, \mathcal{O}_{K_{\mathfrak{p}}}) \mid \ |\det(X)|_{\mathfrak{p}} = N_K(\mathfrak{p})^{-k}\}.$$

To simplify notation several times below we will denote by p and q the norms of $\mathfrak{p}$ and $\mathfrak{q}$.

With regards to the measures put in the different relevant groups, the choices are the same as before. That is, at finite place we pick Haar measures giving the standard maximal compact measure one, while at the real places we follow the above stated conventions.

Finally, we assume K is a totally real field, although several of the facts we will discuss below are true in the general case.

1.3.1 Manipulation of the regular elliptic part

With respect to changing the base field to a general one K there are immediate challenges. One of the more striking ones is the following: it can be easily proved that

$$\mathcal{O}(\gamma, f) \neq 0$$

implies that

$$(\det(\gamma)) = \mathfrak{p}^k,$$

so that if a regular elliptic class does contribute a nonzero term to the regular elliptic part, its determinant must generate the ideal $\mathfrak{p}^k$. This last equality is automatic in $\mathbb{Q}$ because it has class number 1, but is not necessarily true for other fields if the choice of $\mathfrak{p}$ and k is not adequate, i.e. if $\mathfrak{p}^k$ is not a principal ideal.

This means two things: firstly, that if we make the "wrong" choice of prime $\mathfrak{p}$ or integer k then the regular elliptic part will be equal to 0 and the contribution of the trivial representation must be located in another part of the trace formula.

Secondly, in the case that the choice is adequate then it must be used to extract the unit: we not only fix an ideal $\mathfrak{p}$, but also fix one of the generators, say ε, of $\mathfrak{p}^k$. We write $\det(\gamma) = u\varepsilon$, for some unit $u \in \mathcal{O}_K^*$, since they generate the same ideal $\mathfrak{p}^k$. Altuğ implicitly does this when he writes $\det(\gamma) = \pm p^k$. He is picking $\varepsilon = p^k$ and u is one of 1 or -1.

With this at hand the regular elliptic part of the trace formula is written as

$$\sum_{u \in O_K^*} \sum_{\substack{\tau \in O_K \\ \tau^2 - 4u\varepsilon \neq \square}} \mathrm{vol}(\gamma)\mathcal{O}(\gamma, f),$$

where once more we have used the irreducible characteristic polynomial of γ, namely

$$P_\gamma(X) = X^2 - \tau X + u\varepsilon$$

to classify the regular elliptic conjugacy classes. This polynomial generates a quadratic field extension K_γ of K and, associated to it, we have the quadratic sign character defined by

$$\chi_\gamma(\mathfrak{q}) = \begin{cases} 1 & \text{if } q \text{ is split} \\ -1 & \text{if } q \text{ is inert} \\ 0 & \text{if } q \text{ is ramified} \end{cases}$$

Nevertheless, contrary to what occurred in the case of $\mathbb{Q}$, we do not have such an explicit expression of this character in terms of Kronecker (or related) symbols. It can be proven that the volume term is proportional to

$$\sqrt{|D_\gamma|} \cdot L(1, \chi_\gamma),$$

where D_γ is the absolute discriminant of the extension K_γ over $\mathbb{Q}$ and we want to subject it to the same treatment as was done in the case of $\mathbb{Q}$.

In order to carry out this procedure we have to discuss what happens to the orbital integrals in the regular elliptic part. Similarly to the case of the rational numbers, there exists an integer S_γ such that

$$(\tau^2 - 4u\varepsilon) = S_\gamma^2 \Delta_\gamma,$$

where Δ_γ is the relative discriminant of the extension K_γ over K. In the case of Altuğ, this coincides with the fundamental discriminant and the above equation can be taken as an equality of numbers as opposed to ideals. With this at hand, the orbital integrals at the finite primes satisfy

$$N_K(\mathfrak{p})^{k/2} \prod_{\mathfrak{q}} \mathcal{O}(\gamma, f_\mathfrak{q}) = \sum_{\mathfrak{d} | S_\gamma} N_K(\mathfrak{d}) \prod_{\mathfrak{q} | \mathfrak{d}} \left(1 - \frac{\chi_\gamma(\mathfrak{q})}{N_K(\mathfrak{q})}\right).$$

We will discuss this formula thoroughly in chapters 2 and 3. At the archimedean places we now have an expression of the form

$$\prod_{\nu | \infty} \mathcal{O}(\gamma, f_\nu),$$

where ν now varies over the infinite primes. This time we have the possible appearance of several real places. We must use for each orbital integral. For each real place this will look like

$$\mathcal{O}(\gamma, f_\nu) = \Gamma^u_{\nu,1}(\gamma) + D(\gamma)\Gamma^u_{\nu,2}(\gamma)$$

where

$$D(\gamma) = \frac{|\lambda_1\lambda_2|^{1/2}}{|\lambda_1 - \lambda_2|},$$

where λ_1, λ_2 are the eigenvalues of γ and $\Gamma^u_{\nu,1}$ and $\Gamma^u_{\nu,2}$ are certain real functions (see [15]). These are precisely the $g_1^{\pm}, g_2^{\pm}$, that Altuğ uses. The reason we can distribute the factor $\sqrt{|D_L|}$ into all the orbital integrals is because of the existence of the following identity

$$q^{k/2} = N_K(S_\gamma)^2 N_K(\Delta_\gamma) \cdot \prod_{\substack{v \\ \text{real}}} D(\lambda_1, \lambda_2),$$

which follows from several manipulations using the product formula for the field K. Using this, and some extra simplifications, we can distribute the factor of each archimidean place to its corresponding orbital integral and the factor S_γ with the formula of Langlands (8).

Similar to what occurs in the case of $\mathbb{Q}$, what comes from the archimidean factors becomes a function $\theta^u(\tau)$ evaluated at a point associated to the trace τ. To be more precise, this time the function has domain in $\mathbb{R}^n = \mathbb{R}^{r_1}$, where K has $n = r_1$ real embeddings and τ really stands for the canonical inclusion of τ in $\mathcal{O}_K$ into $\mathbb{R}^n$ via these embeddings. In this way, the regular elliptic part is roughly transformed into

$$\sum_{u \in O_K^*} \sum_{\substack{\tau \in O_K \\ \tau^2 - 4u\varepsilon \neq \square}} \theta^u(\tau) L(1, \chi_\gamma) \left(\sum_{\mathfrak{d} | S_\gamma} \frac{1}{N_K(\mathfrak{d})} \prod_{\mathfrak{q} | (S_\gamma/\mathfrak{d})} \left(1 - \frac{\chi_\gamma(\mathfrak{q})}{N_K(\mathfrak{q})}\right)\right).$$

We say *roughly* because there are several details, pertaining the sum over the units, that we are omitting and which are important, but technical in nature, and we refer the reader to the actual article to see the whole process. The final sum that we use doesn't entirely look exactly like the previous one, but has to be studied more, but the whole structure can be better explained at the level we desire here using the current form.

1.3.2 Application of the approximate functional equation:

The approximate functional equation requires us to find an entire function of a complex parameter which satisfies a functional equation such that at $s = 1$ we recover

$$L(1, \chi_\gamma) \left(\sum_{\mathfrak{d} | S_\gamma} \frac{1}{N_K(\mathfrak{d})} \prod_{\mathfrak{q} | (S_\gamma/\mathfrak{d})} \left(1 - \frac{\chi_\gamma(\mathfrak{q})}{N_K(\mathfrak{q})}\right)\right).$$

In this case, to find this function we do not merge the $L-$function and the sum over the divisors of S_γ. We define

$$\mathbb{L}(s, \chi_\gamma) = \sum_{\mathfrak{d} | S_\gamma} \frac{1}{N_K(\mathfrak{d})^{2s-1}} \prod_{\mathfrak{q} | (S_\gamma/\mathfrak{d})} \left(1 - \frac{\chi_\gamma(\mathfrak{q})}{N_K(\mathfrak{q})^s}\right).$$

Using this we can complete to

$$\mathbf{\Lambda}(s,\chi_\gamma) = N_K(S_\gamma)^s \Lambda(s,\chi_\gamma)\mathbb{L}(s,\chi_\gamma).$$

Here Λ is the standard completion of the quadratic sign character, which we know is an entire function with a functional equation. Actually, it can be proven that

$$\mathbf{\Lambda}(s,\chi_\gamma) = \mathbf{\Lambda}(1-s,\chi_\gamma).$$

Since Λ already has a functional equation, what must be verified is that

$$N_K(S_\gamma)^s\mathbb{L}(s,\chi_\gamma)$$

also does. At first sight this follows from a clever algebraic manipulation, namely, that we can write

$$N_K(S_\gamma)^s\mathbb{L}(s,\chi_\gamma) = N_K(S_\gamma)^{1/2}\sum_{a|S_\gamma}\frac{\chi_\gamma(a)\mu(a)}{\sqrt{N_K(a)}}N_K\left(\frac{S_\gamma}{a}\right)^{s-1/2}\sum_{b|(S_\gamma/a)}\frac{1}{N_K(b)^{2s-1}},$$

and verify this is invariant under the change s goes to $1-s$. Here μ is the Mobius function. The previous expansion briefly appears, only for $\mathbb{Q}$, in the paper [16] when they actually need to prove the functional equation. In [7] a full explanation for this is given.

This algebraic trick, as useful as it is in our case, is definitely not something we can expect or desire to generalize. Fortunately, there is another way to prove this functional equation via local methods, as we will study in chapter 2 and 3. Nevertheless, if we meditate for one moment on why we are able to have such an approach as the one above, it is clear we do because we know the explicit Langlands Formula (8). This is to say, we know the Dirichlet character version of the function we are working with, as opposed to only having its Euler factors (as in the approach in chapter 2 and 3) where we could potentially lose the information of the coefficients in the Dirichlet expansion.

In any case, we can perform the approximate functional equation, which once more introduces two functions F and H. Again, F is a certain type of Bessel Function which is odd and whose residues we understand precisely. H is closely related to this function and as such we know its poles and residues. We are not getting into these details, but at least the following comment must be said: precisely because there are possible several real places, all the L functions at infinity must be used. These amount to several Gamma factors evaluated at different values, according to the nature of the completion and the quadratic character χ_γ. All the estimates that are needed, which are more complicated in nature, but that are necessary for the exchange of integrals and the application of the calculus of residues, can be carried out successfully.

In any case, the expansion we get from the approximate functional equation, as a function of a complex variable s, is

$$\sum_{\mathfrak{d}|S_\gamma}\frac{1}{N(\mathfrak{d})^{2s-1}}\sum_{\mathfrak{a}}\frac{\chi_{\mathfrak{d}}(\mathfrak{a})}{N(\mathfrak{a})^s}F\left(\frac{N(\mathfrak{d})^2N(\mathfrak{a})}{A}\right)$$
$$+\left(N(S_\gamma)|D_K|^{1/2}N(\Delta_\gamma)^{1/2}\right)^{1-2s}\sum_{\mathfrak{d}|S_\gamma}\frac{1}{N(\mathfrak{d})^{1-2s}}\sum_{\mathfrak{a}}\frac{\chi_{\mathfrak{d}}(\mathfrak{a})}{N(\mathfrak{a})^{1-s}}H\left(s,\frac{N(\mathfrak{d})^2N(\mathfrak{a})A}{N(S_\gamma)^2|D_K|N(\Delta_\gamma)}\right).$$

We have written N instead of N_K to simplify notation and once more $A > 0$ is a real parameter that we are free to introduce in whichever way is convenient to us. The most important difference of what is appearing here as opposed to what Altuğ has is the appearance of $\chi_{\mathfrak{d}}(\mathfrak{a})$. To understand what has happened let us compare this to what we had at this point over the rationals, which is

$$\sum_{d^2|N}' \frac{1}{d^{2s-1}} \sum_{a=1}^{\infty} \frac{1}{a^s} \left(\frac{N/d^2}{a}\right) F\left(\frac{ad^2}{A}\right) + \left(\frac{|N|}{\pi}\right)^{1/2-s} \sum_{d^2|N}' \frac{1}{d^{1-2s}} \sum_{a=1}^{\infty} \frac{1}{a^{1-s}} \left(\frac{N/d^2}{a}\right) H_N\left(s, \frac{ad^2A}{|N|}\right),$$

and where $N = \tau^2 \pm 4p^k$. A quick comparison convinces us that they are basically the same, and the differences are related to the way certain L functions at infinity are being displayed notationally. The most important difference between them is that in the former expansion we have $\chi_{\mathfrak{d}}(\mathfrak{a})$ while in the latter one we have

$$\left(\frac{N/d^2}{a}\right).$$

Theoretically these two objects are the same, but are written at different levels of explicitness and constructed with different tools. As we have said before, the quadratic reciprocity law allows us to write for $\mathbb{Q}$ the equality

$$\chi_\gamma(m) = \left(\frac{(\tau^2 \pm 4p^k)/s_\gamma^2}{m}\right),$$

for any integer m. Recall that on the right hand side we have a Kronecker Symbol and that it makes sense in general. In particular, if both entries of the Kronecker Symbol are not relatively prime then it has the value 0. The division by f^2 that appears in the upper entry has the function to change where does the symbols gives the value 0 or ± 1, because it can happen that the f^2 cancels entirely a prime that divides m and so avoids the situation in which the entries are not relatively prime.

The reason why this has to be done is because of the merging of the L- function of the volume and the formula of Langlands. In this merging, the Euler factor expansion of the L-function,

$$\prod_{\mathfrak{q}} \left(1 - \frac{\chi_\gamma(\mathfrak{q})}{N_K(\mathfrak{q})^s}\right)^{-1},$$

multiplies the factor

$$\prod_{\mathfrak{q}|(S_\gamma/\mathfrak{d})} \left(1 - \frac{\chi_\gamma(\mathfrak{q})}{N_K(\mathfrak{q})^s}\right),$$

from the Langlands formula. This multiplication cancels the factors associated to primes $\mathfrak{p}|(S_\gamma/\mathfrak{d})$, and so what remains is the Euler product expansion of another character associated to the sign character χ_γ but not quite the same *because it vanishes at more primes.* Concretely, the character becomes non-primitive and the conductor grows via a multiplication factor. Thanks to a change of variable that occurs when changing $\mathfrak{d}$ with $S_\gamma/\mathfrak{d}$ this multiplication is expressed as a division. We can explicitly express this when we have the Kronecker symbol available because we simply write the new conductor on the upper entry but when we do not have such an expansion, we must leave the change of conductor implicit. This is what $\chi_{\mathfrak{d}}$ means! It is the nonprimitive character, associated to the χ_γ, whose conductor Δ_γ has been multiplied by $S_\gamma/\mathfrak{d}$.

Now that we understand what this is, we can evaluate at $s = 1$ and substitute into the regular elliptic part. This gives

$$\sum_{u\in\mathcal{O}_K^*}\sum_{\substack{\tau\in O_K\\ \tau^2-4u\varepsilon\neq\square}}\theta^u(\tau)\left(\sum_{\mathfrak{d}|S_\gamma}\frac{1}{N(\mathfrak{d})}\sum_{\mathfrak{a}}\frac{\chi_{\mathfrak{d}}(\mathfrak{a})}{N(\mathfrak{a})}F\left(\frac{N(\mathfrak{d})^2N(\mathfrak{a})}{A}\right)\right)$$
$$+\sum_{u\in\mathcal{O}_K^*}\sum_{\substack{\tau\in O_K\\ \tau^2-4u\varepsilon\neq\square}}\theta^u(\tau)\frac{1}{N(\partial)^{1/2}|D_K|^{1/2}}\sum_{\mathfrak{d}|S_\gamma}\frac{1}{N(\mathfrak{d})^{-1}}\sum_{\mathfrak{a}}\chi_{\mathfrak{d}}(\mathfrak{a})H\left(1,\frac{N(\mathfrak{d})^2N(\mathfrak{a})A}{N(\partial)|D_K|}\right).$$

Picking $A=N_K(\tau^2-4u\varepsilon)|D_K|^\alpha$, for any $0<\alpha<1$ and defining

$$\mathcal{P}_u(x)=\prod_i\left(\frac{|x_i^2-4u_i\varepsilon_i|}{|u_i\varepsilon_i|^{\frac{1}{2}}}\right),$$

where the i varies over all archimidean primes, we can prove that the functions

$$\Phi_u(x):=\theta^u(x)F\left(\frac{N(\mathfrak{d})^2N(\mathfrak{a})q^{-k\alpha}}{|D_K|^\alpha}\mathcal{P}_u(x)^{2\alpha}\right)$$

and

$$\Psi_u(x):=\theta^u(x)\mathcal{P}_u(x)|D_K|^{-1/2}\,q^{-k/2}H\left(\frac{N(\mathfrak{d})^2N(\mathfrak{a})q^{-k(1-\alpha)}}{|D_K|^{1-\alpha}}\mathcal{P}_u(x)^{2(1-\alpha)}\right),$$

are Schwartz Functions. Furthermore, a calculation proves that $\Phi_u(\tau)$ and $\Psi_u(\tau)$ recover what appears in the previous expansion of the completed regular elliptic part. More concretely, the completed regular elliptic sum becomes

$$\sum_{u\in\mathcal{O}_K^*}\sum_{\substack{\tau\in O_K\\ \tau^2-4u\varepsilon\neq\square}}\sum_{\mathfrak{d}|S_\gamma}\sum_{\mathfrak{a}}\chi_{\mathfrak{d}}(\mathfrak{a})\left(\frac{\Phi_u(\tau)}{N(\mathfrak{d})N(\mathfrak{a})}+N(\mathfrak{d})\Psi_u(\tau)\right)$$

1.3.3 Completion of the regular elliptic part and the performance of the Poisson Summation

At this stage we can change the order of summation to obtain

$$\sum_{u\in\mathcal{O}_K^*}\sum_{\mathfrak{d}|S_\gamma}\sum_{\mathfrak{a}}\sum_{\substack{\tau\in O_K\\ \tau^2-4u\varepsilon\neq\square}}\chi_{\mathfrak{d}}(\mathfrak{a})\left(\frac{\Phi_u(\tau)}{N(\mathfrak{d})N(\mathfrak{a})}+N(\mathfrak{d})\Psi_u(\tau)\right)$$

When we discussed the case of $\mathbb{Q}$ we had at this point

$$\sum_{\pm}\sum_{\substack{\tau\in\mathbb{Z}\\ \tau^2-4p^k\neq\square}}\sideset{}{'}\sum_{d^2|\tau^2\pm4p^k}\sum_{a=1}^{\infty}\theta_\infty^\pm\left(\frac{\tau}{2p^{k/2}}\right)\frac{1}{ad}\left(\frac{(\tau^2\pm4p^k)/f^2}{a}\right)\left(F\left(\frac{ad^2}{A}\right)+\frac{ad^2}{\sqrt{|\tau^2\pm4p^k|}}H\left(\frac{ad^2A}{|\tau^2\pm4p^k|}\right)\right),$$

for certain value of A, and we defined

$$G(\pm 1, \tau, a, d) = \theta_\infty^\pm \left(\frac{\tau}{2p^{k/2}}\right) \frac{1}{ad} \left(\frac{(\tau^2 \pm 4p^k)/f^2}{a}\right) \left(F\left(\frac{ad^2}{A}\right) + \frac{ad^2}{\sqrt{|\tau^2 \pm 4p^k|}} H\left(\frac{ad^2 A}{|\tau^2 \pm 4p^k|}\right)\right).$$

The main difference between what we have, as has been said above, is the character. In the case of $\mathbb{Q}$ we can simply define $G(\pm 1, \tau, a, d)$ by exactly the same expression that we have already from the manipulations. On the other hand, for the general number field, we have to prove such a G exists. Because the problem is located concretely in the character, and not in the Schwartz function or the norms, we localize the problem there. Concretely, we have to settle the following issue:

(E) Existence of the Completion Function: *Define a function $G(u, \tau, \mathfrak{a}, \mathfrak{d})$, where u is any unit, τ any integer, and $\mathfrak{a}, \mathfrak{d}$ any integral ideals. It must satisfy that whenever there exists a regular elliptic matrix γ that generates u (corresponding to the determinant times ε), τ to the trace, $\mathfrak{d}|S_\gamma$ and $\mathfrak{a}$ is arbitrary we have*

$$G(u, \tau, \mathfrak{a}, \mathfrak{d}) = \chi_{\mathfrak{d}}(\mathfrak{a}),$$

where χ is the quadratic sign character associated to the extension given by the characteristic polynomial of γ.

The reason why such existence is needed is because in that way we can define concretely what we are missing, which is

$$\sum_{u \in \mathcal{O}_K^*} \sum_{\mathfrak{d}} \sum_{\mathfrak{a}} \sum_{\substack{\tau \in O_K \\ \tau^2 - 4u\varepsilon = \square}} G(u, \tau, \mathfrak{a}, \mathfrak{d}) \left(\frac{\Phi_u(\tau)}{N(\mathfrak{d})N(\mathfrak{a})} + N(\mathfrak{d})\Psi_u(\tau)\right)$$

Notice that it is important to insist in the condition

$$G(u, \tau, \mathfrak{a}, \mathfrak{d}) = \chi_{\mathfrak{d}}(\mathfrak{a}).$$

for those values that come from a given γ. Otherwise we could define an arbitrary completion in any way whatsoever for the terms we are missing but we wouldn't be able to do the next step we need: in order to carry out the Poisson Summation formula we need a Schwartz function summed over a lattice, and the character cannot be part of the Schwartz function so it has to be taken out of the τ sum. Altuğ does this by invoking periodicity of the character. If we persist in completing in a random way, the fact that the τ sum remains separated into two parts - the completing part $\tau^2 - 4u\varepsilon = \square$ and the one that comes directly for the trace formula $\tau^2 - 4u\varepsilon \neq \square$ - prevents us from guaranteeing complete lattices and the process of completion would be useless.

The existence of such a G can be settled for any number field. Let us assume we have such a function G. As we said above, we also need to be able to take out from the τ sum the character, otherwise we do not have a Schwartz function. The way to do this is by invoking periodicity in the variable τ. We demand the following:

(P) Periodicity of G: *Prove there exists a function $G(u, \tau, \mathfrak{a}, \mathfrak{d})$, satisfying (E), such that for* *given u, $\mathfrak{a}$ and $\mathfrak{d}$,*

$$G(u,\tau_1,\mathfrak{a},\mathfrak{d}) = G(u,\tau_2,\mathfrak{a},\mathfrak{d})$$

whenever

$$\tau_1 \equiv \tau_2 \pmod{4\mathfrak{a}\mathfrak{d}^2}.$$

Notice that for Altuğ, with our new understanding of G as extending the character exclusively , we have

$$G(\pm 1,\tau,a,d) = \left(\frac{(\tau^2 \pm 4p^k)/f^2}{a}\right),$$

which indeed has period $4af^2$. Once more, the existence of such a G can be guaranteed for all number fields and, by breaking the τ sum via the modulus condition, we have that the completed sum can be written up as

$$\sum_{u\in\mathcal{O}_K^*}\sum_{\mathfrak{d}}\sum_{\mathfrak{a}}\sum_{m\in\mathcal{O}_K/4\mathfrak{a}\mathfrak{d}^2} G(u,m,,\mathfrak{a},\mathfrak{d})\sum_{\tau\equiv m}\left(\frac{\Phi_u(\tau)}{N(\mathfrak{d})N(\mathfrak{a})} + N(\mathfrak{d})\Psi_u(\tau)\right)$$

The m sum here varies over a fixed set of representatives of the quotient $\mathcal{O}_K/4\mathfrak{a}\mathfrak{d}^2$. The sum over $\tau \equiv m \pmod{4\mathfrak{a}\mathfrak{d}^2}$ is a lattice sum and what is being added are Schwartz functions (the norms that appear can be taken out of the sums so they do not really matter). Performing standard Poisson Summation, using an appropriate exponential character e, we have that this sum can be rewritten as

$$\sum_{u\in\mathcal{O}_K^*}\sum_{\mathfrak{d}}\sum_{\mathfrak{a}}\frac{1}{N(\mathfrak{d})^3N(\mathfrak{a})^2}\left(\sum_{\xi}\widehat{\Phi}_u(\xi) + N(\mathfrak{d})^2N(\mathfrak{a})\widehat{\Psi}_u(\xi)\right)K_{\mathfrak{a},\mathfrak{d}}(\xi,u;G),$$

where we define the *Kloosterman Type Sums* as

$$K_{\mathfrak{a},\mathfrak{d}}(\xi,u;G) = \sum_{m\in\mathcal{O}_K/4\mathfrak{a}\mathfrak{d}^2} G(u,m,\mathfrak{a},\mathfrak{d})e(\xi\cdot m).$$

In here ξ is the variable of the dual Poisson summation sum and it has to run over the correct lattice. We will not go into the specifics details of this, but simply say that this Poisson summation is occurring inside $\mathbb{R}^n$ after the integers $\mathcal{O}_K$ are embedded, and it is there where the dual lattice is obtained.

1.3.4 The concrete evaluation of Kloosterman type sums

Since we will only deal with the case $\xi = 0$ from now on, we will simplify $K_{\mathfrak{a},\mathfrak{d}}(0,u;G)$ to $K_{\mathfrak{a},\mathfrak{d}}(u;G)$. Just as in the case of $\mathbb{Q}$ we expect to find the contribution of the trivial representation when $\xi = 0$. In such a situation we are led, exactly for the same reasons as over the rationals, to the function

$$D(s,u;G) = \sum_{\mathfrak{d}}\frac{1}{N(\mathfrak{d})^{2s+1}}\sum_{\mathfrak{a}}\frac{K_{\mathfrak{a},\mathfrak{d}}(0,u;G)}{N(\mathfrak{a})^{s+1}},$$

where s is a complex variable.

At this point, in the rational case, we use the Fundamental Theorem of Arithmetic as well as the local nature of the Kronecker symbol, to factor the above sum into its local factors. Since we are asking G to be what generalizes the characters, and these are local, it makes sense to expect a factorization of G into local factors.

We have to be careful when writing down what we need: usually, when we want to say that a global character that gets evaluated at ideals admits a factorization into local factors, what we mean is that on the completions there is a character such that the product of all local characters give the global one. Here we have two ideals, $\mathfrak{a}$ and $\mathfrak{d}$. The factorization and evaluation in local factors should be applied to $\mathfrak{a}$, since it is the argument of χ_γ. What we need to answer then is: *how does $\mathfrak{d}$ contribute?*

As we know, a property of the Kronecker symbol is that

$$\left(\frac{a}{b}\right) = 0,$$

whenever there is a prime p such that $p|a$ and $p|b$. The role that $\mathfrak{d}$ plays is to prevent this from occurring excessively. Over the rationals, the way in which it occurs is precisely by the division by d^2 and it is relevant when the completion happens. In general, if the completion is done carelessly and this division, or a similar process, is not done, then in the completion step we added many terms where the above situation is happening and so we are really just adding a lot of zeros. To understand this specific aspect of how $\mathfrak{d}$ plays a role was a major challenge against the successful understanding of what we need, technically speaking, to complete the regular elliptic part. With this in mind, we now write what we expect:

(L) Factorability of** G**: *Define for each prime $\mathfrak{q}$ a function*

$$G_{\mathfrak{q}}(u', m', v, r)$$

where $u' \in O_{K^*_{\mathfrak{q}}}$, $m' \in O_{K_{\mathfrak{q}}}$ and v, r are nonnegative integers such that the product

$$G(u, m, \mathfrak{a}, \mathfrak{d}) := \prod_{\mathfrak{q}|\mathfrak{a}} G_{\mathfrak{q}}(u, m, val_q(\mathfrak{a}), val_q(\mathfrak{d})),$$

satisfies (E) and (P) and where $\mathfrak{a}$ and $\mathfrak{d}$ are arbitrary ideals of O_K and $u \in O_{K^*}$, $m \in O_K$.

Notice that in the above product we can consider each $u \in O_{K^*}$, $m \in O_K$ as elements of the corresponding completions. Once more we can guarantee such a construction for any number field, and when this is assumed we have that

$$D(s, u; G) = \prod_{\mathfrak{q}} D_{\mathfrak{q}}(s, u; G_{\mathfrak{q}})$$

where

$$D_{\mathfrak{q}}(s, u; G) = \sum_{r=0}^{\infty} \frac{1}{q^{r(2s+1)}} \sum_{v=0}^{\infty} \frac{K_{v,r}(0, u, \mathfrak{q}; G_{\mathfrak{q}})}{q^{v(s+1)}}.$$

Here, when $\mathfrak{q}$ is a prime not above 2 we have

$$K_{v,r}(0, u, \mathfrak{q}; G_{\mathfrak{q}}) = \sum_{x \mod \mathfrak{q}^{v+2r}} G_{\mathfrak{q}}(u, x, v, r).$$

while for $\mathfrak{q}$ above 2 we have

$$K_{v,r}(0,u,\mathfrak{q};G_{\mathfrak{q}}) = \sum_{x \mod \mathfrak{q}^{v+2r+2e_{\mathfrak{q}}}} G_{\mathfrak{q}}(u,x,v,r),$$

where $e_{\mathfrak{q}}$ is the ramification degree of $\mathfrak{q}$ over 2. Recall that we need the explicit evaluation of these sums because it is from them that we will get certain residues, via the zeta function, as well as the contribution of the trivial representation via the evaluations when $\mathfrak{q}=\mathfrak{p}$, our fixed special prime. Concretely then, we get the last property we need:

(K) Kloosterman Summable: *Define for each prime $\mathfrak{q}$ a function*

$$G_{\mathfrak{q}}(u',m',v,r)$$

where $u' \in O_{K_{\mathfrak{q}}^*}$, $m' \in O_{K_{\mathfrak{q}}}$ and v,r are nonnegative integers such that they satisfy (L). Furthermore, it must satisfy for $\mathfrak{q}\neq\mathfrak{p}$ not above 2

$$D_{\mathfrak{q}}(s,u;G_{\mathfrak{q}}) = \frac{1-q^{-(s+1)}}{1-q^{-s}},$$

and for primes $\mathfrak{q}\neq\mathfrak{p}$ above 2

$$D_{\mathfrak{q}}(s,u;G_{\mathfrak{q}}) = q^{2e_{\mathfrak{q}}}\frac{1-q^{-(s+1)}}{1-q^{-s}}.$$

While, if $\mathfrak{p}$ is not above 2, we have

$$D_{\mathfrak{p}}(s,u;G_{\mathfrak{q}}) = \frac{(1-p^{-(s+1)})(1-p^{-(k+1)s})}{(1-p^{-s})(1-p^{-2s})},$$

and if it is above 2 then

$$D_{\mathfrak{p}}(s,u;G_{\mathfrak{q}}) = p^{2e_{\mathfrak{p}}}\frac{(1-p^{-(s+1)})(1-p^{-(k+1)s})}{(1-p^{-s})(1-p^{-2s})}.$$

We have been very specific when defining the successive set of properties (E), (P), (L) and now (K) to express concretely how deep it is to do the correct completion. In a sense, each property puts extra constraints on a possible completing function G, and as we have seen in the case of $\mathbb{Q}$, we need (K) to hold. At the moment of this writing, we can prove such functions $G_{\mathfrak{q}}$ satisfying (L) exist and in a sense are very natural, but we do not know why those satisfying (K) do exist.

We have been very mysterious about the construction of these functions $G_{\mathfrak{q}}$, so let us discuss it now in more detail. When $\mathfrak{q}$ is a prime that is not above the prime 2 we can define

$$G_{\mathfrak{q}}(u,m,v,r) = \left(\frac{(m^2-4u\varepsilon)\pi_{\mathfrak{q}}^{-2r}}{\mathfrak{q}}\right)^{v},$$

where $(\cdot/\mathfrak{q})$ is the quadratic power residue symbol in $K_{\mathfrak{q}}$ and $\pi_{\mathfrak{q}}$ is a uniformizer. Recall that for units $u \in O_{K_{\mathfrak{q}}}^*$ we have that

$$(u/\mathfrak{q}) = \begin{cases} 1 & u \text{ is a square modulo } \mathfrak{q} \\ -1 & u \text{ is not a square modulo } \mathfrak{q} \end{cases}$$

and it is extended to have the value 0 in the rest of $K_{\mathfrak{q}}$. When we use this $G_{\mathfrak{q}}$ the corresponding parts of (K) that deal with odd primes hold.

The serious issue is when $\mathfrak{q}$ is an even prime. In this situation, the power residue symbol is not defined as above, just as the Legendre symbol $(\cdot/2)$ is not defined, and when we speak of the Kronecker symbol $(\cdot/2)$ we mean an extension such that the appropriate reciprocity law (or product law if we were thinking of Hilbert symbols) is satisfied.

Notice, finally, that despite not knowing at this moment that condition (K) holds for any number field this process is very important since we can now locate what we want as the existence of a particular function in a local field. Hence, we assume such a completion satisfying (K) exists and under this condition we have

$$D(s, u; G) = 4^n \frac{\zeta_K(2s)}{\zeta_K(s+1)} \cdot \frac{1 - p^{s(k+1)}}{1 - p^{-s}},$$

where $p = N(\mathfrak{p})$ and $n = [K : \mathbb{Q}]$.

1.3.5 Isolation of the contribution of the trivial representation

We will not go into the greater details of this part, since at the time of writing this thesis the process is still being carried out. We refer the reader to the paper for the final version of these computations.

We will content ourselves with mentioning the following: when computing directly the trace of the trivial representation, we find it is

$$\mathrm{tr}(\mathbf{1}(f)) = \frac{1 - q^{-(k+1)}}{1 - q^{-1}} \mathcal{O}(\gamma, f_\infty),$$

which is exactly the same as the case over $\mathbb{Q}$. The main difference is that the orbital integral actually breaks up into local orbital integrals

$$\mathcal{O}(\gamma, f_\infty) = \prod_{\nu} \mathcal{O}(\gamma, f_\nu),$$

and we can apply *exactly* the same process to each one of them, as in the rational case, to obtain a specific form. Concretely, we apply Weyl integration formula to express each one as

$$\mathcal{O}(\gamma, f_\nu) = \sum_{\pm} \int \theta_\nu^{\pm}(x) \mathrm{dx}.$$

This $\theta_\nu^{\pm}$ are normalized versions of the $\Gamma_{\nu,1}^u$ and $\Gamma_{\nu,2}^u$ in which the contribution of the unit has been uniformized via a specific change of variables. To understand the reason behind this we compare the two sums that we obtain from both sides: the geometric side, we are getting a potentially infinite sum if the number of units on O_K is infinite. On the other hand, the trivial representation does not bring forth an infinite sum, rather just finite sums that come from the Weyl integration formula.

With these we conclude our description of generalizing Altuğ's strategy to certain number fields and see how it follows the same strategy than the case of $\mathbb{Q}$. We will have the opportunity to connect and discuss these stages, for certain other possible generalizations, in the following chapters.

1.4 Highlight of some problems

At this point we could discuss more the individual problems of each part, but there is already plenty of literature on that regard, so we refer the reader to the bibliography. Instead, to conclude this chapter, we prefer to briefly discuss two of the problems arising from this process that seem to require further investigation and that relate to what we will do in later chapters.

The first issue concerns *global - local methods and computations*. If we were to describe how the computations occurred, we would first note the following: each term is local because the orbital integrals can be computed locally and the volume can also be computed locally. All this is a consequence of the factorability of the function.

The multiplicative formula of Langlands manages to make the orbital integrals a global object, and we perform the approximate functional equation to this global object. Once this is completed and Poisson summation is performed, we break up again into local factors and compute locally. We then multiply together and get a global object (the zeta function) whose poles we understand. Finally we perform the isolation, which occurs to this global object.

The highlights here is the following process: Langlands' multiplicative formula is a gluing process, which goes from local to global; while finding the Euler factors of $D(s, u, G)$ is a global to local process. The question would then be: *is it possible to perform an equivalent process that remains in the local setting?*

As we will discuss in chapter 3, it is possible to already extend the *local* orbital integrals to functions of a complex variable satisfying functional equation. It is possible to perform approximate functional equation to each of these local functions. The real task is to deduce the Kloosterman Sum values from these local manipulations in a way that several of the character sums that were involved in the manipulations are not required anymore.

To be more concrete, when the computations are carried out, several simplifications require us to know what certain character sums are. These sums are concrete and have precise values, but the their evaluation requires one to study the number of solutions of certain curves over fields (or rings in the case of the primes above 2). Yet, in the end, all the contributions of the character disappear since they add up to zero and we end up with the zeta function alone.

The character is relevant for the multiplicative formula of Langlands (8) as well as for the volume, which allows us to pass to the global picture to perform the approximate functional equation. If this can be carried out locally, we might be able to mitigate the influence of the character and avoid those character sums.

The second problem precisely concerns the multiplicative formula of Langlands (8). As we will see in Chapter three, this formula can be proven from local methods entirely. When using these local methods, the local factors arise naturally, at least theoretically. We can write down an explicit formula because those local factors can be deduced precisely, but maybe this will not be the case in general. It is thus important to know, if this local passage mentioned above is going to work, what are the properties of the local factors needed to carry out the analogous application of the approximate functional equation.

These properties should be concrete conditions on the coefficients of the Dirichlet version of the functions that recover the local orbital integrals, so that from them we can deduce the vanishing of the contribution of the characters as well as growth properties that are important for the exchange of integrals.

Chapter 2

Zeta functions of orders in the quadratic local case

We have talked in the introduction and in Chapter One about the importance of the multiplicative formula of Langlands for the product of finite orbital integrals (for our specific function). This formula is only explicitly known for GL(2). It is relevant to investigate how we could find similar formulas that can serve the analogous purpose for higher rank groups.

One important property of this formula is that it is multiplicative, as its name says. This means that it is a product of similar formulas evaluated at prime powers. Furthermore, we need these expressions to extend to functions of a complex variable s in a way that this multiplicativity still holds. We will study this in detail in Chapter three. For the moment let us just say that these factors, if normalized appropiately, are polynomials in q^{-s}.

In [4] a conjecture is made in this regard. Explicitly, it is stated that these polynomials are constructed as zeta functions of certain orders inside reduced algebra quadratic extensions following the work presented in [17]. The content in this chapter, which could be read completely independently of the rest of the thesis, is to compute explicitly the polynomials of certain orders in the quadratic case.

2.1 Orders and their arithmetic

2.1.1 Orders

We will discuss orders in the relevant context. In this subsection K will be a local number field and L a finite dimensional reduced K-algebra. $\mathcal{O}_K$ denotes the ring of integers of K and $\mathcal{O}_L$ the integral closure of $\mathcal{O}_K$ in L. In this context, and following [17], we have the following

Definition 1. *An **order** $\mathcal{O} \subseteq L$ is a finitely generated $\mathcal{O}_K$-module such that*

$$\mathcal{O} \otimes_{\mathcal{O}_K} K = L.$$

Let us briefly discuss this definition. By standard localization theory we have

$$\mathcal{O} \otimes_{\mathcal{O}_K} K = (\mathcal{O}_K^*)^{-1}[\mathcal{O}].$$

In the classical context of orders, L is also a field, in which case we know that $\mathcal{O}$ itself has no zero divisors and so it has its own field of fractions, which, up to isomorphism, is the smallest field containing $\mathcal{O}$. In particular, that implies

$$(\mathcal{O}_K*)^{-1}[\mathcal{O}] = (\mathcal{O}^*)^{-1}[\mathcal{O}],$$

since both sides are fields, and the left side is always inside the right side. Hence, we arrive at the conclusion that the field of fractions of $\mathcal{O}$ is L itself. This last property is what clasically defined orders; that is, a subring of the integral elements such that it itself has the same field of fractions.

In this way we understand the definition given above: it recovers that one of the standard case, but furthermore, it extends to situations where we deal with nonintegral domains. As we will see below, this generality is important to consider because it does arise.

Notice that any subring $\mathcal{O}_K[\Delta]$, with Δ an element of degree equal to $[L : K]$ is an order. Among all possible orders we give a name to this ones:

Definition 2. *In the context above, an order $\mathcal{O}$ is **monogenic** if there exists an element $\Delta \in L$ such that*

$$\mathcal{O} = \mathcal{O}_K[\Delta].$$

$\mathcal{O}$ is a subring of $\mathcal{O}_L$ and as such it has its own ideals. Some of those are also ideals in $\mathcal{O}_L$. The largest of them is particularly important and is defined as follows:

Definition 3. *If $\mathcal{O} \subseteq \mathcal{O}_L$ is an order, we define its **conductor** as*

$$(\mathcal{O} : \mathcal{O}_L) = \{x \in \mathcal{O} \mid x\mathcal{O}_L \subseteq \mathcal{O}\}.$$

It is easy to prove that the conductor is an ideal of both $\mathcal{O}$ and $\mathcal{O}_L$ and a characterizing property of it is that it is the largest one with this property. This ideal will play an important role in the theory below.

With this we conclude the facts we need from the theory of orders and we are ready to study their zeta functions in the monogenic case. Nevertheless let us emphasize that orders have a very rich theory and very interesting properties, despite us not actually needing them here. One of their remarkable facts, among other things, is that despite them losing properties of the maximal order (i.e. the ring of integers), they do it in a controlled way. Concretely, we know that the ring of integers in a number field is a Dedekind domain and consequently its ideals have uniqueness of factorization. This is no longer true in general orders, but is true among those ideals relatively prime to the conductor. In the same way, the theory of extensions of primes above and below, which is so important in number theory, remains true among ideals relatively prime to the conductor. We hope that this brief commentary motivates these definitions as objects of interest in and of themselves despite our admittedly succinct description of them.

2.1.2 Arithmetic of the orders O_n

Let K be a local algebraic number field and L be a reduced $K-$algebra of dimension 2 over K. This means that L is one of the following two possibilities: a quadratic extension of K or $K \times K$. We call these cases, respectively, the *nonsplit* and the *split* cases. We will need, later on, to distinguish further between the unramified nonsplit case and the ramified nonsplit case, which we will simply call *unramified case* and *ramified case*.

We make the following conventions about K:

- p is a uniformizer of K and q is the cardinality of its residue field.
- The ring of integers of K will be denoted by O_K.
- The valuation of K will be denoted by val_p and the norm of p with respect to K by N_K.
- When $L = K \times K$, we always consider K embedded diagonally in L.

The purpose of this section is to prove a theorem that allows us to relate ideals of different orders in a very specific way. The theorem is the same in both situations, nonsplit or split, but certain results require more care in the split case than in the nonsplit one. The following conventions about L aim at giving a uniform way to treat these cases.

- The integer closure of K inside L is denoted by O_L. In the nonsplit case this is the ring of integers of L while in the split case it coincides with $O_K \times O_K$.
- Since O_K is a discrete valuation ring and a principal ideal domain, every $K-$algebra of finite dimension is generated by a single element. We denote by $\Delta \in O_L$ an element such that

$$O_L = O_K[\Delta].$$

  In the split case, we furthermore put $\Delta = (\Delta_1, \Delta_2)$ and we assume without loss of generality that Δ_1, Δ_2 are nonzero. Notice that they must be different.

- Δ satisfies a quadratic equation and we suppose it is given by

$$\Delta^2 = \tau_\Delta \Delta - \delta_\Delta,$$

  with $\tau_\Delta, \delta_\Delta \in O_K$. This equation exists because $1, \Delta$ form an O_K basis of O_L, by the choice of Δ.

- In the nonsplit case we denote by π a uniformizer of O_L, by val_π the corresponding valuation of L, write Q for the cardinality of the residue field and N_L for the norm on O_L. Furthermore, we define the ramification and inertia degrees by

$$Q = q^f$$
$$pO_L = (\pi O_L)^e.$$

  Notice e, f satisfy $ef = 2$.

- For the split case we will denote $\mathbf{1} = (1, 1)$, which is the unit of the ring to distinguish it from 1 the unit of K. Of course, by the diagonal embedding we mentioned above, they are identified but it will be useful to distinguish them at times.

We will always be referring to L and K as in the above fashion. When we do not make a distinction between split and nonsplit cases is because the definitions or arguments make sense and are the same in both cases.

With these conventions at hand we can define the sequence of orders we will be interested in:

Definition 4. *We define the* ***main sequence of orders*** *by*

$$O_n = O_K[p^n\Delta],\ n \geq 1.$$

More specifically, $O_n \subseteq O_L$ consists of the elements of O_L that can be written as

$$x + yp^n\Delta,$$

for some $x, y \in O_K$.

Notice that $x + y\Delta \in O_L$ belongs to O_n if and only if $p^n|y$. Also, we write here, once and for all, the product in these coordinates, since it will be particularly important to refer to it several times later. We have

$$(a + b\Delta)(x + y\Delta) = (ax - by\delta_\Delta) + (ay + bx + by\tau_\Delta)\Delta. \tag{2.1}$$

If in this equation we write bp^n, instead of b, we get

$$(a + bp^n\Delta)(x + y\Delta) = (ax - bp^n y\delta_\Delta) + (ay + bp^n x + bp^n y\tau_\Delta)\Delta, \tag{2.2}$$

which immediately implies that O_i is an O_n-module for $0 \leq i \leq n$. Our first goal is to prove there are no other ones.

Proposition 3. *For each $n \geq 1$ define $\psi_n : O_n \longrightarrow O_K/p^nO_K$ by*

$$\psi_n(x + yp^n\Delta) = x \pmod{p^nO_K}.$$

Then ψ_n is a surjective ring homomorphism with kernel p^nO_0 which induces an O_n-module structure on O_K/p^nO_K given by

$$(x + yp^n\Delta) \cdot \overline{b} = \psi_n(x + yp^n\Delta)\overline{b} = \overline{xb},$$

where the overline means class in O_K/p^nO_K.

Proof. That ψ_n is a surjective additive group homomorphism is straightforward. For the product, we have

$$(a + bp^n\Delta)(x + yp^n\Delta) = (ax - bp^{2n}y\delta_\Delta) + (ayp^n + bp^n x + bp^{2n}y\tau_\Delta)\Delta.$$

Hence

$$\psi_n((a + bp^n\Delta)(x + yp^n\Delta)) \equiv ax - bp^{2n}y\delta_\Delta \equiv ax \pmod{p^nO_K}.$$

This proves it is a homomorphishm. The kernel of ψ_n consists of those $x + yp^n\Delta$ with $p^n|x$, that is,

$$\ker \psi_n = p^n O_0.$$

This implies that

$$\frac{O_n}{p^n O_0} = \frac{O_K}{p^n O_K}.$$

As we have mentioned above, O_0 is an O_n module, and so $p^n O_0$ is also one with the same scalar multiplication (i.e. the product in O_L). This means that $\frac{O_n}{p^n O_0}$ has an O_n- module structure inherited in the standard way to the quotient space. Since the quotient map is clearly ψ_n, we get that the module structure is exactly the one claimed in the proposition. This concludes the proof. □

The following was part of the previous proof but we will specifically need it later so we isolate it here now:

Corollary 1. *For $n \geq 1$ we have*

$$\frac{O_n}{p^n O_0} = \frac{O_K}{p^n O_K}.$$

Notice how the right hand side depends only on K and not on the extension L. Now we can list all the O_n-modules:

Proposition 4. *For $n \geq 0$, the only O_n-modules M with*

$$O_n \subseteq M \subseteq O_0$$

are precisely $O_0, O_1, ..., O_n$.

Proof. By the correspondence theorem, and using the previous corollary, all possible modules M as listed in the proposition correspond to the O_n modules in $\dfrac{O_K}{p^n O_K}$

The module structure, induced by ψ_n, is

$$(x + yp^n) \cdot \overline{b} = \overline{xb}.$$

This only depends on $x \pmod{p^n O_K}$. Hence the action descends to one of $\dfrac{O_K}{p^n O_K}$ on itself by

$$\overline{x} \cdot \overline{b} = \overline{xb},$$

and every submodule on the former action corresponds to a submodule on the latter one. But the latter action submodules are just the ideals of $O_K/p^n O_K$, and we know there are $n + 1$ of them. Hence, the list $O_0, ..., O_n$ is the complete list of O_n modules between O_n and O_0. □

We also have the following fact stating who is the conductor between different orders.

Proposition 5. *The conductor ideal*

$$(O_{n+m} : O_n) := \{z \in O_{n+m} \mid zO_n \subseteq O_{n+m}\},$$

is given by

$$(O_{n+m} : O_n) = p^m O_n.$$

In particular, $(O_n : O_0) = p^n O_0$.

Proof. For $a + bp^{n+m}\Delta$ to satisfy

$$(a + bp^{n+m}\Delta)(x + yp^n\Delta) \in O_{n+m},$$

for all $x, y \in O_K$, it is enough that it does it for $x = 0$ and $y = 1$. In this case we have

$$(a + bp^{n+m}\Delta)(p^n\Delta) = -bp^{2n+m}\delta_\Delta + (ap^n + bp^{n+m}\tau_\Delta)p^n\Delta.$$

This implies $p^{n+m}|ap^n$, which is equivalent to $p^m|a$. Hence

$$a + bp^{n+m}\Delta = p^n(a_0 + bp^m\Delta) \in p^n O_m,$$

as desired. Furthermore, if the previous inclusion holds then $a + bp^{n+m}\Delta$ is in the conductor. This concludes the proof. □

We now study the units of O_n. Let us recall that in a local ring, if $x + y$ is a unit, then one of x or y is a unit. Equivalently: if x is a unit and y is not, then $x + y$ is a unit.

In the nonsplit case, both O_K and O_L are local rings and we can use use this fact. On the other hand, in the split case $O_L = O_K \times O_K$ is not a local ring and this fact doesn't hold. What we will do is to prove that, even though in general it is not true, something weaker is true but that is enough for our purposes:

Proposition 6. *Let* $n \geq 1$. *Then* $a + bp^n\Delta \in O_n$ *is a unit of* O_L *if and only if* a *is a unit of* O_K.

Proof. Notice that the equation in O_L

$$1 = (a + bp^n\Delta)(x + y\Delta) = (ax - bp^n y\delta_\Delta) + (ay + bp^n x + bp^n y\tau_\Delta)\Delta, \tag{2.3}$$

is equivalent to the system of equations in O_K

$$\begin{aligned} 1 &= ax - bp^n y\delta_\Delta \\ 0 &= ay + bp^n x + bp^n y\tau_\Delta. \end{aligned}$$

Firstly, let $a + bp^n\Delta$ be a unit of O_L and $x + y\Delta$ the inverse. We get from the first equation of the system, since $n \geq 1$, that $ax \in O_K^*$, which implies a is a unit as well.

On the other hand, suppose $a + bp^n\Delta$ has $a \in O_K^*$. We must show the system above is solvable in O_K. The matrix representing this system of equations is

$$\begin{pmatrix} a & -bp^n\delta_\Delta \\ bp^n & a + bp^n\tau_\Delta \end{pmatrix},$$

whose determinant is

$$a(a + bp^n\tau_\Delta) + b^2p^{2n}\delta_\Delta = a^2 + abp^n\tau_\Delta + b^2p^{2n}\delta_\Delta$$

which is a unit in O_K because a^2 is and the sum of the other terms is a multiple of p^n. We conclude the system can be solved, via Cramer's Rule for instance, within O_K as desired. □

With this proposition at hand we can prove the following results describing the units of these orders.

Proposition 7. *We have* $O_n^* = O_L^* \cap O_n$.

Proof. For $n = 0$ this is true because $O_0 = O_L$.

For $n \geq 1$, we clearly have $O_n^* \subseteq O_L^* \cap O_n$. So what we need to prove is that if $a + bp^n\Delta$ is a unit of O_L, then its inverse also lies in O_n. By the previous proposition a is a unit of O_K. If $x + y\Delta$ is the inverse, then from the product equation (2.3) we deduce

$$ay + bp^n x + bp^n y\tau_\Delta = 0,$$

hence $p^n | ay$. We deduce that $p^n | y$, since a is a unit. That is, $x + y\Delta \in O_n$, as desired. □

Now we understand the units of O_n for every n. Using the previous result, we immediately get the following criterion which we will use repeatedly:

Theorem 1. *Let* $n \geq 1$. *Then* $a + bp^n\Delta \in O_n$ *is a unit if and only if* a *is a unit of* O_K.

The following definition will become useful to have later.

Definition 5. *Let* $x \in O_0 = O_L$ *be any element. We define its* ***type*** *by*

$$\varepsilon(x) = \begin{cases} val_\pi(x) & \text{in nonsplit case} \\ (val_p(x_1), val_p(x_2)) & \text{in split case} \end{cases}$$

where in the split case we have $x = (x_1, x_2)$. *Furthermore, given an ideal* $I \subseteq O_n$*, we say* $x \in I$ *is a* ***representative*** *of* I *if* $\varepsilon(x) = val_\pi(x)$ *is minimal among* $x \in I$ *in the nonsplit case, and if*

$$x_1 = \min_{(x,y)\in I}(val_p(x))$$
$$x_2 = \min_{(x,y)\in I}(val_p(y))$$

in the split case.

It might seem this definition is different for each case, but it can be described in a homogeneous way: the type is registering the valuation at each coordinate, and a representative of an ideal is just an element of the ideal that has minimum possible valuation at each coordinate *simultaneously*. For the nonsplit case there is only one coordinate, while for the split case there are two, corresponding to the two copies of K in $K \times K$.

We will be saying repeatedly "rank 2 ideals". The rank we refer to is as an O_K module. All the ideals considered are submodules of some O_n, and as such are themselves free modules of some rank. Notice that only if they have rank 2 they will have finite index.

Proposition 8. *Let* $n \geq 0$ *and* $I \subseteq O_n$ *a rank* 2 *ideal. Then* I *has a representative. Furthermore, all its representatives have nonzero entries.*

Proof. This is obvious in the nonsplit case since there is only one variable over which to minimize the valuation. We now address the split case.

Define $(x_1, y_1) \in I$ as an element such that

$$x_1 = \min_{(x,y)\in I}(\mathrm{val}_p(x)),$$

and $(x_2, y_2) \in I$ as an element such that

$$y_2 = \min_{(x,y)\in I}(\mathrm{val}_p(y)).$$

Both of these elements exists since, along a single coordinate, minima of the valuations exist.

If either of these elements has the other coordinate also a minimum then we are done. Otherwise we must have

$$\mathrm{val}_p(x_2) > \mathrm{val}_p(x_1), \mathrm{val}_p(y_1) > \mathrm{val}_p(y_2).$$

In this situation, define $(x_0, y_0) = (x_1 + x_2, y_1 + y_2)$. Since I is closed under addition, we have $(x_0, y_0) \in I$. Furthermore,

$$\mathrm{val}_p(x_0) = \mathrm{val}_p(x_1 + x_2) = \min(\mathrm{val}_p(x_1), \mathrm{val}_p(x_2)) = \mathrm{val}_p(x_1) = \min_{(x,y)\in I}(\mathrm{val}_p(x)).$$

Notice we have used that the equality of the triangle inequality for valuations happens if the terms involved have different valuations, as is our case. Analogously,

$$\mathrm{val}_p(y_0) = \min_{(x,y)\in I}(\mathrm{val}_p(y)),$$

and this proves (x_0, y_0) satisfies what we are looking for.

Furthermore, these valuations are nonnegative integers (as opposed to ∞). Otherwise, the ideal would have all of its elements with 0's in the same coordinate which results in it not having rank 2.

□

Now that we know representatives of ideals exist, we are ready to justify the name "representative":

Proposition 9. *Let $n \geq 0$. For every rank 2 ideal $I \subseteq O_n$ and every representative $x \in I$, there exists $0 \leq i \leq n$ such that*

$$I = xO_i.$$

Furthermore, i is independent of the representative and only depends on I.

Proof. The argument is the same for both split and nonsplit cases but we only write it for the split case, since we prefer to show how we handle the two coordinates. The argument for the nonsplit case is obtained analogously.

For the ideal I, in the split case, using the previous proposition, there is a representative element

$(x_0, y_0) \in I$ such that

$$x_0 = \min_{(x,y)\in I}(\mathrm{val}_p(x)),$$
$$y_0 = \min_{(x,y)\in I}(\mathrm{val}_p(y)).$$

For any element $(x, y) \in I$ consider

$$(z_1, z_2) = (xx_0^{-1}, yy_0^{-1}).$$

Taking valuations we get

$$\mathrm{val}_p(z_1) \geq \mathrm{val}_p(x) - \mathrm{val}_p(x_0) \geq 0,$$
$$\mathrm{val}_p(z_2) \geq \mathrm{val}_p(y) - \mathrm{val}_p(y_0) \geq 0.$$

by definition of (x_0, y_0). This implies

$$(z_1, z_2) \in O_0,$$

which is the same as saying

$$I \subseteq (x_0, y_0)O_0.$$

Of course we furthermore have, because I is an ideal, that

$$(x_0, y_0)O_n \subseteq I.$$

Hence,

$$O_n \subseteq (x_0^{-1}, y_0^{-1})I \subseteq O_0.$$

Notice we can take inverses since the entries are nonzero. Since we know the complete list of O_n-modules between O_n and O_0, we deduce that there exists a unique i, with respect to (x_0, y_0), such that

$$(x_0^{-1}, y_0^{-1})I = O_i,$$

that is, $I = (x_0, y_0)O_i$.

Suppose (x_1, y_1) and (x_2, y_2) are two representatives for the same ideal with

$$I = (x_1, y_1)O_i = (x_2, y_2)O_j.$$

Without loss of generality, let $i \leq j$. There exists $o_i \in O_i$ and $o_j \in O_j$ with

$$(x_1, y_1) = (x_2, y_2)o_j$$
$$(x_2, y_2) = (x_1, y_1)o_i$$

This implies $o_i o_j = \mathbf{1}$. Since $O_j \subseteq O_i$ we have $o_j \in O_i$, which means that o_j and o_i are units of O_i. Hence

$$I = (x_1, y_1)O_i = (x_2, y_2)o_j O_i = (x_2, y_2)O_i.$$

This implies $i = j$ for otherwise it contradicts the unicity of the index for the representative (x_2, y_2). $\square$

We are now in position to prove the main structural result regarding ideals in the main sequence of orders:

Proposition 10. *Let $n \geq 1$ and $I \subseteq O_n$ be a nonprincipal ideal that has rank 2 as an O_K-module. Then there exists a unique ideal $J \subseteq O_{n-1}$ such that $I = pJ$.*

Proof. For the ideal I, using the previous proposition, write $I = xO_i$ for some $x \in I$ and a unique $0 \leq i \leq n$. Because I is not principal, we know $i < n$. In particular, because

$$xO_i \subseteq O_n,$$

x lies in the corresponding conductor $(O_n : O_i) = p^{n-i}O_i$. In particular, $x = p^{n-i}y$, for some $y \in O_i$. Writing this for our expression of I we get

$$I = xO_i = p(p^{n-i-1}yO_i).$$

Every ideal of O_i that lies in O_{n-1} is also an ideal of O_{n-1}. This implies $J = (p^{n-i-1}yO_i)$ is an O_{n-1} ideal. This concludes the proof of the existence of such a J. To notice it is unique just realize that $J = p^{-1}I$, which as a set is uniquely determined by I. $\square$

We finally have

Theorem 2. *For $n \geq 0$, let $\mathcal{I}_n$ be the set of ideals $I \subseteq O_n$ of rank-2 as O_K-modules. Define the map*

$$T_n : \mathcal{I}_n \longrightarrow \mathcal{I}_{n+1},$$

by

$$T_n(J) = pJ.$$

Then no principal ideal of O_{n+1} is in the image of T_n. Furthermore, T_n is a bijection onto its image, which consists of the nonprincipal ideals of O_{n+1} that have rank 2 as O_K-modules.

Proof. We have basically proved everything we need, so we just collect the facts:

- A nonzero rank 2 principal ideal of O_n, say x_nO_n, cannot be in the image of this map since otherwise we could write

  $$x_nO_n = pJ = py_iO_i,$$

  for some $y_i \in J$ and $0 \leq i \leq n-1$. This would contradict the uniqueness of proposition (9) when applied to x_nO_n.

- The previous proposition establishes precisely that the nonprincipal ideals form the image.

- Finally, this is a bijection onto the image since $pJ_1 = pJ_2$ clearly implies $J_1 = J_2$.

These three facts conclude the proof. $\square$

Definition 6. *The map T_n from Theorem 2 will be called* ***the traveling map****.*

2.2 The zeta function of orders

2.3 A general construction

Let L/K be as in the previous section. We have said there are two possibilities: either L is itself a field extension, or $L = K \times K$. In section 2.2. of [17], the definition of the zeta function for orders is given. In order to construct it in general, the correct module has to be chosen, for otherwise the final result will not hold. What we will discuss now is a simplification that allows us to change the zeta function as defined in [17] for a simpler one.

We begin by recalling the definition of the *different ideal*. This ideal is defined as

$$\mathfrak{d}_{L/K} = \{x \in L \mid x\mathcal{O}_L^* \subset \mathcal{O}_L\},$$

where

$$\mathcal{O}_L^* = \{x \in L \mid \operatorname{Tr}(x\mathcal{O}_L) \subseteq O_K\}.$$

The different is an ideal of L and, being a principal ideal ring, it has a generator which we call c. If we modify the trace pairing Tr to

$$(x, y) := \operatorname{Tr}(c^{-1}xy)$$

we can define for any order $\mathcal{O} \subset \mathcal{O}_L$ its dual

$$\check{\mathcal{O}} := \{x \in L \mid (x, \mathcal{O}) \subseteq \mathcal{O}_K\}.$$

In the case where L is a number field, we have from the general theory of commutative algebra, that because $\mathcal{O}_n$ is a monogenic order, it is a Gorenstein Ring and so counting $\mathcal{O}_n$-fractional ideals in $\check{\mathcal{O}}_n$ is equivalent to counting $\mathcal{O}_n$-fractional ideals in $\mathcal{O}_n$, which are precisely the ideals. (See for example [8]).

On the other hand, we have the split case in which we cannot invoke the results in the same way, but we still need to change the module $\check{\mathcal{O}}_n$ to $\mathcal{O}_n$ to make our computations easier. We thus have the following

Proposition 11. *$\mathcal{O}_n$ and $\check{\mathcal{O}}_n$ are isomorphic as $\mathcal{O}_n$-modules.*

Proof. Since $\mathcal{O}_n$ is a free $\mathcal{O}_n$-module of rank 1 over itself, what we must prove is that $\check{\mathcal{O}}_n$ is also free of rank 1 over $\mathcal{O}_K$. To that end, notice that in this case $\operatorname{Tr}(x_1, x_2) = x_1 + x_2$. Hence, from

$$\mathcal{O}_L^* = \{x \in L \mid \operatorname{Tr}(x\mathcal{O}_L) \subseteq O_K\}.$$

we deduce

$$\operatorname{Tr}((x_1, x_2)(1, 0)) \in \mathcal{O}_K, \operatorname{Tr}((x_1, x_2)(0, 1)) \in \mathcal{O}_K,$$

that is, x_1, x_2 are in $\mathcal{O}_K$ and this clearly is a sufficient condition since $\mathcal{O}_L = \mathcal{O}_K \times \mathcal{O}_K$. We deduce

$$\mathcal{O}_L^* = \mathcal{O}_L.$$

With this, the definition of the different becomes

$$\mathfrak{d}_{L/K} = \{x \in L \ \mid x\mathcal{O}_L \subset \mathcal{O}_L\},$$

which immediately implies $\mathfrak{d}_{L/K} = O_L$ as well. Thus, we can put $c = (1,1)$, the unit of L. In this way, we can write

$$\check{\mathcal{O}}_n = \{x \in L \ \mid \ \mathrm{Tr}(x\mathcal{O}_n) \subseteq \mathcal{O}_K\},$$

and this is equivalent to

$$\mathrm{Tr}(x \cdot (1,1)) \in \mathcal{O}_K, \ \mathrm{Tr}(x \cdot (p^n\Delta_1, p^n\Delta_2)) \in \mathcal{O}_K.$$

In coordinates $x = (x_1, x_2)$ this is

$$\begin{aligned} x_1 + x_2 &= a \\ x_1\Delta_1 + x_2\Delta_2 &= p^{-n}b, \end{aligned}$$

for some integers a, b in $\mathcal{O}_K$. We can solve for this system and we get

$$x = \begin{pmatrix} 1 & 1 \\ \Delta_1 & \Delta_2 \end{pmatrix}^{-1} \begin{pmatrix} a \\ p^{-n}b \end{pmatrix} = \frac{1}{\Delta_2 - \Delta_1} \begin{pmatrix} \Delta_2 & -1 \\ -\Delta_1 & 1 \end{pmatrix} \begin{pmatrix} a \\ p^{-n}b \end{pmatrix}$$

We deduce that

$$\check{\mathcal{O}} = \mathcal{O}_K \cdot v + \mathcal{O}_K \cdot w,$$

where

$$v = \frac{1}{\Delta_2 - \Delta_1} \begin{pmatrix} \Delta_2 \\ -\Delta_1 \end{pmatrix}, \ w = \frac{p^{-n}}{\Delta_2 - \Delta_1} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

We now define the following O_k-linear map given by $R : \check{\mathcal{O}}_n \longrightarrow \mathcal{O}_n$ by

$$\begin{aligned} R(v) &= p^{2n}\delta \\ R(w) &= p^n\Delta \end{aligned}$$

The $\mathcal{O}_n$-module structure on both $\mathcal{O}_n$ and $\check{\mathcal{O}}_n$ is by entrywise multiplication in $K \times K$. What we must prove is that it respects the scalar product structure, and for that is enough to verify it with $p^n\Delta$ (since the other generator is the unit). We have, in $\mathcal{O}_n$, that

$$\begin{aligned} (p^n\Delta) \cdot 1 &= p^n\Delta \\ (p^n\Delta) \cdot (p^n\Delta) &= (p^n\tau) \cdot p^n\Delta - (p^{2n}\delta) \cdot 1. \end{aligned}$$

On the other hand, in $\check{\mathcal{O}}_n$, we have

$$\begin{aligned} (p^n\Delta) \cdot v &= (p^{2n}\delta)w \\ (p^n\Delta) \cdot w &= (p^n\tau)w - v \end{aligned}$$

Both of these systems are obtained by performing the entrywise multiplication in coordinates and using that $\Delta^2 = \tau\Delta - \delta \cdot 1$. Comparing them, we realize that it means

$$R(p^n\Delta \cdot v) = p^n\Delta \cdot R(v)$$
$$R(p^n\Delta \cdot w) = p^n\Delta \cdot R(w),$$

so that R is an $\mathcal{O}_n-$module morphism.

We claim $\{w\}$ is a basis for $\mathcal{O}_n$. Indeed, given arbitrary a, b in $\mathcal{O}_K$ we have

$$\begin{aligned}((b + ap^n\tau) - ap^n\Delta) \cdot w &= (b + ap^n\tau)w - a(p^n\Delta) \cdot w \\ &= (b + ap^n\tau)w - a((p^n\tau)w - v) \\ &= av + bw,\end{aligned}$$

showing that we can generate any element of $\check{\mathcal{O}}_n$. Now suppose $x \in \mathcal{O}_n$ is such that

$$x \cdot w = 0,$$

then applying the map R above to this equation we have

$$x \cdot p^n\Delta = 0,$$

which means in coordinates

$$(x_1, x_2) \cdot (\Delta_1, \Delta_2) = 0.$$

Since Δ_1, Δ_2 are both nonzero, we conclude that $x = 0$. Hence, $\check{\mathcal{O}}_n$ is free of rank 1 as desired. □

In this way, counting fractional $\mathcal{O}_n$ ideals in $\check{\mathcal{O}}_n$ is the same as counting them in $\mathcal{O}_n$, but in the latter case, it is just counting the ideals, which is what we want.

2.4 The zeta function in our case

In the paper [17] a zeta function for general orders of number fields is defined and, furthermore, a local version of such zeta function is also defined for orders of reduced K- algebras where K is a local field. This general version is necessary to account for the diversity of algebras that appear when considering extensions by polynomials since an irreducible polynomial over a global number field may not remain irreducible over the completion at a prime.

Definition 7. *Let K be a local number field and $\mathcal{O} \subseteq O_K$ be an order, where O_K is the ring of integers of K. We define its zeta function as*

$$\zeta_{\mathcal{O}}(s) = \sum_{I \subseteq \mathcal{O}} \frac{1}{[\mathcal{O} : I]^s},$$

where the sum runs over all ideals $I \subseteq \mathcal{O}$ of finite index.

In [17] a more general zeta function is defined, but it is not the one we have defined above. For general orders a more careful construction is needed in order to make sure the results proven hold true. Concretely, the results of [17] are for the zeta function

$$\zeta_{\check{\mathcal{O}}}(s) = \sum_{I \subseteq \check{\mathcal{O}}} \frac{1}{[\check{\mathcal{O}} : I]^s},$$

where the sum runs over $\mathcal{O}$-fractional ideals inside of $\check{\mathcal{O}}$. Nevertheless, thanks to the results discussed in the previous section, we know that for the quadratic cases the above zeta function simplifies to definition 7, which is what we will use.

The main result of [17] is the existence of a functional equation for the zeta functions of orders. The proof goes by studying them locally, and verifying that the local versions satisfy the functional equation when they are properly modified.

Let us restrict to the local case, where we have a local field K and a finite dimensional reduced $K-$algebra L which can be written as product of g fields

$$L = L_1 \times ... \times L_g.$$

The residue field of each of these fields has cardinality $Q_i = q^{f_i}$ and the prime ideal $\pi_i O_{L_i}$ in O_{L_i} satisfies

$$(\pi_i O_{L_i})^{e_i} = pO_{L_i},$$

where π_i is a uniformizer of O_{L_i} and p is one of O_K. With this at hand, we define the following function of a complex parameter s

$$V(s) = \prod_{i=1}^{g} (1 - q^{-f_i s}),$$

Considering now an order $\mathcal{O}$ of L, define

$$n = \text{length}_{O_K}(O_L/\mathcal{O}) = \log_q([O_L : \mathcal{O}]).$$

Notice that $V(s)$ does not depend on the order $\mathcal{O}$ selected, and there is a good reason for this: as will be made clear later, this function is capturing the behaviour of ideals in O_L. Now we can state the following result, which is Theorem 2.5 in [17].

Proposition 12. *Let K be a local field, L be a finite dimensional reduced K-algebra and $\mathcal{O} \subseteq O_L$. In the context of the previous discussion, define the function*

$$\tilde{J}(s) = q^{ns} V(s) \zeta_{\mathcal{O}}(s).$$

Then there exists a polynomial $P(x) \in 1 + x\mathbb{Z}[x]$ of degree $2n$ such that

$$V(s)\zeta_{\mathcal{O}}(s) = P(q^{-s}).$$

Furhtermore, $\tilde{J}$ satisfies the functional equation

$$\tilde{J}(s) = \tilde{J}(1 - s),$$

or equivalently, the polynomial P satisfies

$$(qx^2)^n P\left(\frac{1}{qx}\right) = P(x).$$

2.5 Study of the principal part

We now return to our main sequence of orders in the quadratic case, for which we have the

Definition 8. *For $n \geq 0$ we define the **zeta function** of the order O_n, following definition 7 in page 52, by*

$$\zeta_n(s) = \sum_{I \subseteq O_n} \frac{1}{[O_n : I]^s},$$

*where the sum runs over all ideals $I \subseteq O_n$ of finite index. We will call these ideals **contributing ideals to** O_n or simply **contributing ideals**.*

Furthermore, we denote by $\mathfrak{P}_n$ the set of contributing principal ideals of O_n and define

$$\zeta_n^P(s) = \sum_{I \in \mathfrak{P}_n} \frac{1}{[O_n : I]^s},$$

*which we call **the principal part** of $\zeta_n(s)$.*

The purpose of the traveling map is to isolate the principal part of ζ_n away from the zeta function ζ_{n-1}. More precisely, we have

Theorem 3. *For $n \geq 1$, the zeta functions of the orders O_n satisfy the recurrence relation*

$$\zeta_n(s) = \zeta_n^P(s) + q^{-s}\zeta_{n-1}(s).$$

Proof. It is clear that

$$\zeta_n(s) = \zeta_n^P(s) + \zeta_n^i(s),$$

where we define

$$\zeta_n^i(s) = \sum_{I \notin \mathfrak{P}_n} \frac{1}{[O_n : I]^s}.$$

If $J \subseteq O_{n-1}$ then $pJ \subseteq O_n \subseteq O_{n-1}$ and $pJ \subseteq J \subseteq O_{n-1}$. Using the tower theorem for indices we get

$$[O_n : pJ] = \frac{[O_{n-1} : pJ]}{[O_{n-1} : O_n]} = \frac{[O_{n-1} : J][J : pJ]}{[O_{n-1} : O_n]}$$

We know that $[J : pJ] = q^2$ and $[O_{n-1} : O_n] = q$. Both of these equalities follow from elementary divisor theory, since for J and pJ the elementary divisors are p, p while for O_{n-1} and O_n they are $1, p$, by construction. The previous equality thus becomes

$$[O_n : pJ] = q[O_{n-1} : J].$$

We then deduce

$$\begin{aligned}
\zeta_n^i(s) &= \sum_{I \notin \mathfrak{P}_n} \frac{1}{[O_n : I]^s} \\
&= \sum_{J \subseteq O_{n-1}} \frac{1}{[O_n : pJ]^s} \\
&= \sum_{J \subseteq O_{n-1}} \frac{1}{q^s [O_{n-1} : J]^s} \\
&= \sum_{J \subseteq O_{n-1}} \frac{1}{q^s [O_{n-1} : J]^s} \\
&= q^{-s} \zeta_{n-1}(s),
\end{aligned}$$

where we used the traveling map in the second equality. □

With this at hand the challenge becomes to find a usable expression for the principal part. In this direction we introduce the following

Definition 9. *For $I \in \mathfrak{P}_n$ we denote by*

$$R(I) := \{\alpha \in O_n \mid \alpha O_n = I\},$$

its set of representatives.

We will organize ideals by the type their representatives have. In order to do this successfully we prove the following

Proposition 13. *For $I \in \mathfrak{P}$ and $\alpha \in R(I)$, the type of $\varepsilon(\alpha)$ is independent of α, that is, it only depends on the ideal I.*

Proof. For the nonsplit case, if α_1, α_2 belong to $R(I)$ then $\alpha_1 = u\alpha_2$ for some unit $u \in O_n^* \subseteq O_L^*$. Hence have

$$\mathrm{val}_\pi(\alpha_1) = \mathrm{val}_\pi(u\alpha_2) = \mathrm{val}_\pi(\alpha_2),$$

since u is a unit.

Analogously, for the split case, let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ be two representatives. There exists a unit $u = (u_1, u_2) \in O_n^* \subseteq O_L^*$ such that $\alpha = \beta u$. Hence

$$\mathrm{val}_p(\alpha_i) = \mathrm{val}_p(\beta_i u_i) = \mathrm{val}_p(\beta_i),$$

for $i = 1, 2$, as desired. □

We now make a division of principal ideals into two different classes, in which the contribution to the zeta function is similar but not quite the same. More precisely we have

Definition 10. *Let $I \in \mathfrak{P}_n$. We define the* ***type of*** *I by*

$$\varepsilon(I) := \varepsilon(x),$$

for any $x \in R(I)$. We also define for every $x \in O_L$

$$\eta(x) = \min\{val_p(a) \mid a \text{ is an entry of } \varepsilon(x)\},$$

and write $\eta(I) = \eta(x)$ for any $x \in R(I)$. Notice the previous proposition implies this is well defined. We will say that I is a ***low ideal*** *if*

$$\eta(I) < \begin{cases} ne & \text{in the nonsplit case,} \\ n & \text{in the split case.} \end{cases}$$

Otherwise, we say I is a ***high ideal****. We will refer to ne and n, respectively in the nonsplit and split cases, as the* ***ideal threshold*** *and denote if uniformly by t_n.*

The role of low and high ideals will become clearer as we move forward. For the moment, let us only say that high ideals contribute in all possible ways available, while low ideals have a more restricted contribution to the principal part of the zeta function.

Several times in what follows we will speak about the *possible types* of an ideal in $\mathfrak{P}_n$. With this we just mean an element of the image of η, which at least potentially is $\eta(I)$ for some ideal $I \in \mathfrak{P}_n$

We begin by showing what is the contribution of each ideal of $\mathfrak{P}_n$. We have the following

Definition 11. *Let ω be a possible type of an ideal in $\mathfrak{P}_n$. We define* ***the type contribution*** *as*

$$c(\omega) := \begin{cases} f\omega & \text{in the nonsplit case,} \\ \omega_1 + \omega_2 & \text{in the split case.} \end{cases}$$

In here we are denoting $\omega = (\omega_1, \omega_2)$ in the split case, where the possible types are pairs of nonnegative integers.

Proposition 14. *Let $I \in \mathfrak{P}_n$. The index $[O_n : I]$ is given by*

$$[O_n : I] = q^{-c(\varepsilon(I))}.$$

Proof. Pick $\alpha \in R(I)$. We have two chains of inclusions:

$$\alpha O_n \subseteq \alpha O_0 \subseteq O_0,$$

and also

$$\alpha O_n \subseteq O_n \subseteq O_0.$$

By the tower theorem of indices we have

$$[O_0 : \alpha O_0][\alpha O_0 : \alpha O_n] = [O_0 : O_n][O_n : \alpha O_n].$$

Of course, since α is nonzero, we have

$$[\alpha O_0 : \alpha O_n] = [O_0 : O_n],$$

so that

$$[O_n : \alpha O_n] = [O_0 : \alpha O_0].$$

For the nonsplit case, by definition $\alpha = \pi^{\eta(I)}u$, for some unit u. We conclude

$$[O_n : I] = [O_n : \alpha O_n] = N_L(\alpha) = Q^{-\eta(I)} = q^{-f\eta(I)} = q^{-c(\varepsilon(I))},$$

as desired. For the split case we deduce

$$[O_n : I] = [O_0 : \alpha O_0] = q^{-(l+m)} = q^{-c(\varepsilon(I))}.$$

We used for the second equality that α is a generator of $p^l \times p^m$ if the type of I is (l, m). This concludes the proof. □

Given a possible type ω, which in the nonsplit case is a nonnegative integer, and in the split one a pair of nonnegative integers, define $X_\omega := \varepsilon^{-1}(\omega)$. Each ideal in X_ω contributes the same to the zeta function and grouping those terms together we deduce the principal part of the zeta function can be written as

$$\zeta_n^P(s) = \sum_{\omega} \frac{|X_\omega|}{q^{c(\omega)}},$$

and the success of our analysis depends on whether we are able to compute the $|X_\omega|$ or not.

The first step is to deal with the high ideals. The following explains why this distinction is important.

Proposition 15. *Let $x \in O_0 = O_L$ be any element with $\eta(x) \geq t_n$, then $x \in O_n$.*

Proof. The point to realize is that whenever $x \in O_0$ with $\eta(x) \geq t_n$ we get that $\varepsilon(x)$ is a multiple of p^n in O_0.

For example, in the nonsplit case, $\eta(x) = \varepsilon(x) = \mathrm{val}_\pi(x) \geq ne$. This means

$$x = \pi^{\mathrm{val}_\pi(x)}u = \pi^{ne}\pi^{\mathrm{val}_\pi(x)-ne}u = p^n\left(\pi^{\mathrm{val}_\pi(x)-ne}u\right)$$

While for the split case

$$x = (x_1, x_2) = (p^{\mathrm{val}_p(x_1)}u_1, p^{\mathrm{val}_p(x_2)}u_2) = p^n(p^{\mathrm{val}_p(x_1)-n}u_1, p^{\mathrm{val}_p(x_2)-n}u_2).$$

In any case, we conclude

$$x \in p^n O_0 = (O_0 : O_n) \subseteq O_n,$$

proving that $x \in O_n$. □

With this at hand we now define, for every possible type ω,

$$O_n^{[\omega]} = \bigsqcup_{\varepsilon(I)=\omega} R(I),$$

that is, the set of all representatives of principal ideals in $\mathfrak{P}_n$ that contribute $q^{-c(\omega)}$ to the principal part of the zeta function.

What the previous proposition is really saying then is that

$$O_n^{[\omega]} = \left\{x \in O_0 \mid \eta(x) = \omega\right\}.$$

as long as $\eta(\omega) \geq t_n$. We can now state:

Proposition 16. *Let ω be a possible type for an ideal $I \in \mathfrak{P}_n$ with $\eta(\omega) \geq t_n$. Define the map $\Psi_\omega : O_n^{[\omega]} \longrightarrow O_L^*$ given by*

$$\Psi_\omega(x) = \pi^{-\omega} x$$

in the nonsplit case and by

$$\Psi_\omega(x_1, x_2) = (p^{-\omega_1} x_1, p^{-\omega_2} x_2)$$

in the split case. Then Ψ_ω is an O_n^*-equivariant isomorphism.

In here, $O_n^{[\omega]}$ and O_L^* are equipped with the natural actions of O_n^* by left multiplication. As a consequence, the orbits of one are in correspondence with the orbits of the other, and so

$$|X_\omega| = [O_0^* : O_n^*].$$

Proof. The equivariance is straightforward and the previous proposition implies this is surjective, and since it clearly is injective, it is an isomorphism. Finally, notice that the orbits of $O_n^{[\omega]}$ are in correspondence with the ideals that satisfy $\eta(I) = \omega$, since any two representatives of the same ideal differ by a unit of O_n. □

This solidifies the contribution of the high ideals. We now have to deal with the low ideas. The analysis is similar, but we have to find how to deal with the fact that Proposition 15 is no longer true. The first step to do this is the following:

Proposition 17. *Let $I \in \mathfrak{P}_n$ be a low ideal and $x + yp^n\Delta \in R(I)$ a representative. Then $val_p(x)$ is independent of the representative of I chosen and, furthermore, its value is smaller than n and is equal to $val_p(a)$ for every entry a of $\varepsilon(I)$.*

Proof. Realize there are no low ideals for $n = 0$, so we may assume $n \geq 1$.

For the nonsplit case define $t = \mathrm{val}_p(x)$. Suppose that $t \geq n$ and write $x = p^t u$ for some unit $u \in O_K^*$. Then

$$\begin{aligned}
\eta(I) &= \mathrm{val}_\pi(p^t u + yp^n\Delta) \\
&= \mathrm{val}_\pi(p^n(p^{t-n}u + y\Delta)) \\
&= \mathrm{val}_\pi(p^n) + \mathrm{val}_\pi(p^{t-n}u + y\Delta) \\
&\geq n\mathrm{val}_\pi(p),
\end{aligned}$$

where for the inequality we have used that $p^{t-n}u + y\Delta \in O_L$. Since $\mathrm{val}_\pi(p) = e$, we conclude that

$$\eta(I) \geq ne,$$

which contradicts that I is a low ideal. Hence, $t < n$.

Now take any other representative, say $\beta = w + zp^n\Delta \in R(I)$. Because α and β represent the same ideal, there exists a unit $u = a + bp^n\Delta$ such that

$$\begin{aligned}
x + yp^n\Delta &= (w + zp^n\Delta)(a + bp^n\Delta) \\
&= wa - bzp^{2n}\delta_\Delta + (wb + az + bzp^n\tau_\Delta)p^n\Delta,
\end{aligned}$$

whence $x + bzp^{2n}\delta_\Delta = wa$. Taking valuations, we get

$$\begin{aligned}\mathrm{val}_p(w) &= \mathrm{val}_p(aw)\\ &= \mathrm{val}_p(x + bzp^{2n}\delta_\Delta)\\ &= \min\{\mathrm{val}_p(x), \mathrm{val}_p(bzp^{2n}\delta_\Delta)\}\\ &= \mathrm{val}_p(x),\end{aligned}$$

since

$$\mathrm{val}_p(bzp^{2n}\delta_\Delta) \geq 2n > n = \mathrm{val}_p(x).$$

Notice we also used that a has to be a unit of O_K in order for u to be a unit of O_n. This proves the independence on representatives, as well as that $\mathrm{val}_p(x) < n$. The assertion about all entries being the same is obvious here since there is only one.

For the nonsplit case pick $\alpha \in R(I)$, say $\alpha = (\alpha_1, \alpha_2)$, and put $\varepsilon(I) = (l, m)$. Since I is low one of l or m is smaller than n. Without loss of generality suppose $l < n$.

Write $\alpha = a \cdot \mathbf{1} + b \cdot (p^n\Delta)$, for some $a, b \in O_K$ which in coordinates becomes

$$\begin{aligned}\alpha_1 &= a + bp^n\Delta_1,\\ \alpha_2 &= a + bp^n\Delta_2.\end{aligned}$$

Suppose $\mathrm{val}_p(a) \geq n$. Then

$$\begin{aligned}l &= \mathrm{val}_p(\alpha_1)\\ &= \mathrm{val}_p(a + bp^n\Delta_1)\\ &\geq \min(\mathrm{val}_p(a), \mathrm{val}_p(bp^n\Delta_1))\\ &= \min(\mathrm{val}_p(a), n + \mathrm{val}_p(b\Delta_1))\\ &\geq n,\end{aligned}$$

which is a contradiction. We conclude $t := \mathrm{val}_p(a) < n$, say $a = p^t u$ for some unit $u \in O_K$. Then, for $i = 1, 2$,

$$\alpha_i = p^t u + bp^n\Delta_i = p^t(u + bp^{n-t}\Delta_i),$$

and because $n - t > 0$ we have $bp^{n-t}\Delta_i$ is not a unit of O_K, and since u is, then $u + bp^{n-t}\Delta_i$ is also a unit. We conclude $\mathrm{val}_p(\alpha_i) = t = \mathrm{val}_p(a)$, which proves independence, equality of entries and its value smaller than n, as desired.

□

Due to this proposition we can make the following

Definition 12. *Let $I \in \mathfrak{P}_n$ a low ideal. The common value of $\mathrm{val}_p(x)$ for all the $x + yp^n\Delta \in R(I)$ is denoted by $\eta_p(I)$.*

Contrary to what happened in the high ideals, where we obtained in proposition 16 that all possible *high* types do contribute, we have that not all possible *low* types do. More precisely, we have

Proposition 18. *Let $I \in \mathfrak{P}_n$ be a low ideal. Then, in the nonsplit case, we have*

$$\eta(I) = e\eta_p(I),$$

and so the only possible types for low ideals are $0, e, 2e, ..., (n-1)e$. On the other hand, for the split case

$$\varepsilon(I) = (\eta_p(I), \eta_p(I)),$$

and so the only possible types for low ideals are $(0,0), (1,1), ..., (n-1, n-1)$.

Proof. For the split case we have for $x + yp^n\Delta \in R(I)$ with $x = p^t u$ and $u \in O_K^*$ that

$$\eta(I) = \mathrm{val}_\pi(x + yp^n\Delta) = \mathrm{val}_\pi(p^t(u + yp^{n-t}\Delta)) = t\mathrm{val}_\pi(p) = te,$$

as desired. Notice that for the last equality we have used that $t < n$, by the previous proposition, and hence that $u + yp^{n-t}\Delta$ is a unit of O_L. This implies that the possible types of low ideals, which are precisely the values of $\varepsilon(I) = \eta(I)$, can only be the multiples of e from 0 to $(n-1)e$.

For the split case, the previous proposition implies the entries of $\varepsilon(I)$ are equal and smaller than n. Hence, the possible types are $(0,0), ..., (n-1, n-1)$, as claimed.

□

Notice how the behaviour of high and low ideals is very different. For the high ideals, every possible type appears in the principal part of the zeta function since all the possible representatives exist within O_n because they are elements of the conductor $(O_n : O_0)$. On the other end, with the low ideals, not all possible types can contribute because O_n simply does not have elements with the correct valuations in the corresponding order. In both cases we have a *linear behaviour* for the possible low types which is a remarkable fact for the split case since, a priori, there were several possible other types that end up never appearing.

To emphasize this, we find instructive to visualize the types of rank 2 ideals of O_n, as compared to those of O_0 as some sort of linear shift. We will do it for $n = 3$. The code convention for the types will be as follows

Low ideal type	⦿
High ideal type	●
Non appearing type	○

Table 2.1: Label convention for ideal types

In the nonsplit unramified case, we have $e = 1$, and the possible types are numbers $0, 1, 2, 3, ...$ that we visualize on the number line. In the left we have the types of $\mathcal{O}_0$, which are all high types. On the right we have the types of $\mathcal{O}_3$. $0, 1, 2$ are low types, while $3, 4, 5, ...,$ are high types. In this situation all numbers actually appear as the type of some ideal. The figure is as follows

Figure 2.1: Nonsplit unramified types for $\mathcal{O}_0$ and $\mathcal{O}_3$.

For the ramified case, in which $e = 2$, the types of $\mathcal{O}_0$ behave in exactly the same fashion. Nevertheless, the types of $\mathcal{O}_3$ do not: this time there are non appearing types, that is, numbers which do not appear as the type of any ideal. In this case those are $1, 3, 5$. The low ideals are $0, 2, 4$, and the high ideals are $6, 7, 8, 9, \ldots$ The picture is

Figure 2.2: Nonsplit ramified types for $\mathcal{O}_0$ and $\mathcal{O}_3$.

For the split case the types are pairs of nonnegative integers. On the left we have those for $\mathcal{O}_0$, which all occur and all are high ideal types. On the other hand, for $\mathcal{O}_3$, we have three low types: $(0,0), (1,1)$ and $(2,2)$. The high ideals are all pairs with both entries at least 3. Furthermore, this time there are an infinite number of nonnapearing types: all of those pairs with unequal entries, at least one of which is smaller than 3. The picture looks like

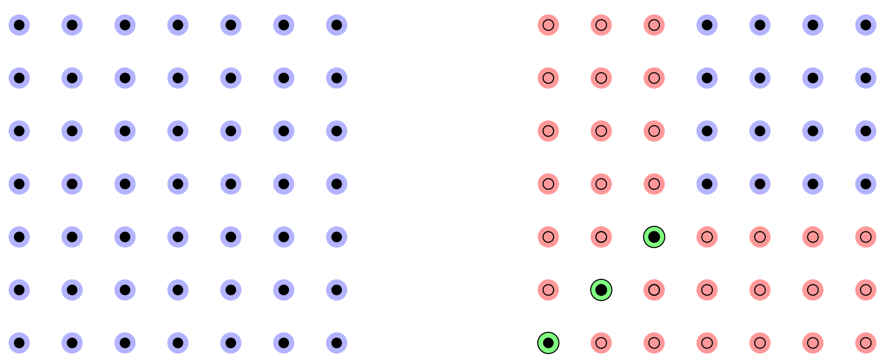

Figure 2.3: Split ramified types for $\mathcal{O}_0$ and $\mathcal{O}_3$.

For each case, what we see then is that in O_0 (which is the figure on the left) all possible types come from high ideals, while for those of O_3(which is the figure on the right) the high ones behave the same way as in O_0, creating an infinite grid, but it is shifted in a linear fashion, possibly producing in this displacement non appearing types.

We now study the contribution of each type to the principal part of the zeta function. We have

Proposition 19. *Let ω be one of the types of the previous proposition (i.e. the possible ones for low ideals). Define $\Psi_\omega : O_n^{[\omega]} \to O_{n-d}^*$ by*

$$\Psi_\omega(x) = p^{-d}x.$$

where $d = \eta(\omega)/e$ in the nonsplit case and $d = \eta(\omega)$ in the split case. Then Ψ is a O_n^*-equivariant isomorphism between the O_n^*-spaces $O_n^{[\omega]}$ and O_{n-d}^*, where in both spaces the action is by left multi-

plication. As a consequence, we have

$$|X_\omega| = [O^*_{n-d} : O^*_n] = \frac{[O^*_0 : O^*_n]}{[O^*_0 : O^*_{n-d}]}$$

Proof. For the nonsplit case, an ideal I whose representatives have $\eta(I) = \omega$ satisfies

$$\eta(I) = de, \eta_p(I) = d.$$

On the other hand, for the split case what we have is $\varepsilon(x) = (d, d)$. Hence, in both cases, we can write any representative as

$$p^d u + yp^n \Delta,$$

for some unit $u \in O^*_K$. Since $d < n$ we can factor the p^d and get

$$p^d(u + yp^{n-d}\Delta),$$

and since u is a unit of O_K, then $u + yp^{n-d}$ is a unit of O_{n-d}. Since all of these steps are reversible, we get that $\Psi_d(O_n^{[\omega]}) = O^*_{n-d}$.

That it is equivariant is immediate and that all orbits count follows from the fact that the representatives are nondivisors of 0 and we can appeal to Proposition (14).

□

Now we are ready to prove the main result of this section, which is the following

Proposition 20. *The principal part of the zeta functions of the orders O_n satisfy*

$$\zeta_n^P(s) = \sum_{d=0}^{n-1} \frac{[O^*_0 : O^*_n]}{[O^*_0 : O^*_{n-d}]} \cdot \frac{1}{q^{2ds}} + \frac{[O^*_0 : O^*_n]}{q^{2ns} V(s)}.$$

In here $V(s)$ is the factor appearing in Proposition 12 on page 53.

Proof. Collecting the contributions of both high and low ideals, we will obtain this result. For the nonsplit case, the high ideals contribute

$$\begin{aligned} \sum_{j \geq 0} \frac{|X_{ne+j}|}{q^{f(ne+j)s}} &= \sum_{j \geq 0} \frac{[O^*_0 : O^*_n]}{q^{2ns+fjs}} \\ &= \frac{[O^*_0 : O^*_n]}{q^{2ns}} \sum_{j \geq 0} \frac{1}{q^{fjs}} \\ &= \frac{[O^*_0 : O^*_n]}{q^{2ns}(1 - q^{-fs})} \end{aligned}$$

For the low ideals we only get contributions for $0, e, ..., (n-1)e$, and for each one of them we have

$|X_{de}|$ such terms. Hence, we obtain that the low ideals contribute

$$\begin{aligned}\sum_{d=0}^{n-1} \frac{|X_{de}|}{q^{f(de)s}} &= \sum_{d=0}^{n-1} \frac{|X_{de}|}{q^{2ds}} \\ &= \sum_{d=0}^{n-1} \frac{[O_0^* : O_n^*]}{[O_0^* : O_{n-d}^*]} \cdot \frac{1}{q^{2ds}}\end{aligned}$$

Notice that we have used $fe = 2$. On the other hand, for the high ideals we get

$$\begin{aligned}\sum_{\eta(I)\geq n} \frac{1}{[O_n : I]^s} &= \sum_{l,m\geq n} \sum_{\varepsilon(I)=(l,m)} \frac{1}{[O_n : I]^s} \\ &= \sum_{l,m\geq n} \frac{[O_L^* : O_n^*]}{[O_n : I]^s} \\ &= \frac{[O_L^* : O_n^*]}{q^{2ns}} \sum_{i,j\geq 0} \frac{1}{q^{(i+j)s}} \\ &= \frac{[O_L^* : O_n^*]}{q^{2ns}} \cdot \frac{1}{(1-q^{-s})^2}.\end{aligned}$$

For the low ideals we have the possible types are only $(0,0), ..., (n-1, n-1)$. Hence, the contribution is

$$\sum_{d=0}^{n-1} \frac{|X_{(d,d)}|}{q^{(d+d)s}} = \sum_{d=0}^{n-1} \frac{|X_{(d,d)}|}{q^{2ds}} = \sum_{d=0}^{n-1} \frac{[O_0^* : O_n^*]}{[O_0^* : O_{n-d}^*]} \cdot \frac{1}{q^{2ds}}.$$

Finally, by inspection of the two cases, we see $V(s)$ coincides with $(1-q^{-fs})$ in the nonsplit case and with $(1-q^{-s})^2$ in the split one. $\square$

2.6 Solution of the recurrence

We have seen that the principal part depends on the indices $[O_0^* : O_n^*]$. We now compute the values of these indices. Recall that we have computed before that the conductor is $(O_0 : O_n) = p^n O_0$ and this is the largest ideal of both O_0 and O_n.

The following formula is classical in the general theory of orders (see for example [14]) but as we are in a particularly simple case, since we do not have to deal with class numbers, we give a proof:

Proposition 21. *The following formula holds*

$$[O_0^* : O_n^*] = \frac{\left|\left({}^{O_0}/{}_{p^n O_0}\right)^*\right|}{\left|\left({}^{O_n}/{}_{p^n O_0}\right)^*\right|}$$

Proof. Let us explain the result for the nonsplit case first. Later we discuss the slight modifications for the split case. The result holds because of the existence of an exact sequence

$$1 \longrightarrow \left({}^{O_n}/{}_{p^n O_0}\right)^* \xrightarrow{\alpha} \left({}^{O_0}/{}_{p^n O_0}\right)^* \xrightarrow{\beta} {}^{O_0^*}/{}_{O_n^*} \longrightarrow 1$$

We now describe each map and argue the exactness. Notice that once this is done the isomorphism theorem implies exactly what we want. We begin by describing the maps:

Description of α: Let $\bar{x} \in \left({}^{O_n}/_{p^n O_0}\right)^*$ represented by $x \in O_n$, and furthermore, let $y \in O_n$ represent its inverse. That is:

$$\bar{x}\bar{y} \equiv 1 \pmod{p^n O_0}.$$

This implies, $xy = 1 + p^n o$, for some $o \in O_0$ and as such, xy belongs to the higher unit groups $U_L^{(n)}$. In particular, both x and y are units of O_0. We thus define $\alpha(x)$ as the class of x in ${}^{O_0}/_{p^n O_0}$, which represents an invertible element there, whose inverse is represented in O_0 by y.

We will use $\bar{\cdot}$ and $\tilde{\cdot}$ to represent the classes of elements in $\left({}^{O_n}/_{p^n O_0}\right)^*$ and $\left({}^{O_0}/_{p^n O_0}\right)^*$, respectively. With this notation, our definition is

$$\alpha(\bar{x}) = \tilde{x}.$$

Description of β: Pick $\tilde{x} \in \left({}^{O_0}/_{p^n O_0}\right)^*$. We can pick $y \in O_0$ to represent its inverse and just as before $xy \in U_L^{(n)}$. Hence, x is a unit of O_0^* and we define β to be its class modulo O_n^*. More concretely, if we denote these last classes by $\hat{\cdot}$, then our definition is

$$\beta(\tilde{x}) = \hat{x}.$$

Notice that each map is a group homomorphism since canonical maps from groups to quotient groups are morphisms, and in these definitions what is happening is only to change which quotient map is applied to the representative element. Now we proceed to explain the exactness of the sequence:

Injectivity of α: If $\bar{x}$ is such that $\tilde{x} = \tilde{1}$, then $x = 1 + p^n o$ for some $o \in O_n$, but then this equation also holds in O_n and taking quotient we have $\bar{x} = \bar{1}$, as desired.

Exactness at the middle: Notice that $\beta(\alpha(\bar{x})) = \hat{x}$. As an element x, representing $\bar{x}$, we can suppose it lies in O_n and our description of the maps above proves that in such case it is a unit in O_0^*, hence it is in O_n^*. This means that, by definition, $\hat{x} = 1$.

Furthermore, if $\beta(\tilde{x}) = \hat{x} = \hat{1}$, then we have that the representative $x \in O_0$ is a unit that actually lies in O_n. That means $x \in O_n^*$ represent an invertible class $\bar{x}$ with $\alpha(\bar{x}) = \tilde{x}$. This proves exactness in the middle.

Surjectivity of β: Pick any class $\hat{x}$ represented by a unit $x \in O_n^*$. Then x, as an element of O_0, represents an invertible element $\tilde{x}$ and we have, by definition, $\beta(\tilde{x}) = \hat{x}$, proving surjectivity.

This concludes the proof in the nonsplit case.

For the split case the proof again relies in the existence of the same exact sequence. All the arguments of its construction and exactness are the same as before, and the only point we need to be careful about is the claim that we repeatedly used that an invertible class in any of the quotients is represented by a unit. For instance, if we pick $\bar{x} \in \left({}^{O_n}/_{p^n O_0}\right)^*$, we get there exists $y \in O_n$ representing its inverse and as such

$$xy - \mathbf{1} \in p^n O_0,$$

which coordinatewise means

$$x_i y_i - 1 \in p^n O_K, \text{ for } i = 1, 2.$$

This in turn implies x_1, x_2 are units of O_K, and as such $x = (x_1, x_2)$ is a unit of O_0. That is, invertible classes have representatives that are units themselves in O_0. This holds in exactly the same way for the other two quotients in this sequence. With this at hand now the proof follows as in the nonsplit case. □

We have now reached the point where it is desirable to make the distinction between the two possible nonsplit cases.

Proposition 22. *The indices of the units subgroups satisfy:*

Ramified Case, i.e. $f = 1$: *For every $n \geq 0$, we have*

$$[O_0^* : O_n^*] = q^n$$

Unramified Case, i.e. $f = 2$: *For every $n \geq 1$ we have*

$$[O_0^* : O_n^*] = (q+1)q^{n-1},$$

and, of course, $[O_0^* : O_0^*] = 1$.

Split Case: *For every $n \geq 1$ we have*

$$[O_0^* : O_n^*] = (q-1)q^{n-1},$$

and, of course, $[O_0^* : O_0^*] = 1$.

Proof. We begin with the nonsplit cases, which can be treated mostly the same as long as we pay careful attention to the appeareances of e and f. Recall that $ef = 2$. We have that ${}^{O_0}/_{p^n O_0} = {}^{O_L}/_{\pi^{ne} O_L}$, and we have the well known isomorphisms (see for example [12])

$$\begin{aligned} \left({}^{O_L}/_{\pi^{ne} O_L}\right)^* &\cong {}^{O_L^*}/_{U_L^{(ne)}}, \\ {}^{U_L^{(m)}}/_{U_L^{(m+1)}} &\cong {}^{O_L}/_{\pi O_L}, m \geq 1. \end{aligned}$$

We also know that $O_L^* = \mu_{Q-1} \times U_L^{(1)}$, where μ_{Q-1} is the group of $(Q-1)$ roots of unity. From this we also have

$${}^{O_L^*}/_{U_L^{(1)}} \cong \mu_{Q-1}.$$

Now we can iterate and obtain

$$\begin{aligned} \left|\left({}^{O_L}/_{\pi^{ne} O_L}\right)^*\right| &= \left|{}^{O_L^*}/_{U_L^{(ne)}}\right| \\ &= \left|{}^{O_L^*}/_{U_L^{(1)}}\right| \left|{}^{U_L^{(1)}}/_{U_L^{(2)}}\right| \cdots \left|{}^{U_L^{(ne-1)}}/_{U_L^{(ne)}}\right| \\ &= \left|\mu_{Q-1}\right| \left|{}^{O_L}/_{\pi O_L}\right|^{ne-1} \\ &= (Q-1)Q^{ne-1}. \end{aligned}$$

Using the second part of proposition 1 on page 44, we have

$${}^{O_n}/_{p^n O_0} \cong {}^{O_K}/_{p^n O_K},$$

and so

$$\left({}^{O_n}/{}_{p^n O_0}\right)^* \cong \left({}^{O_K}/{}_{p^n O_K}\right)^*,$$

Hence by the same iterative argument that

$$\left|\left({}^{O_n}/{}_{p^n O_n}\right)^*\right| = (q-1)q^{n-1}.$$

Now we specialize to each of our cases:

Ramified, $f = 1$**:** Then $e = 2$ and, using the previous proposition result, and recalling $Q = q^f = q$, we get

$$[O_0^* : O_n^*] = \frac{(Q-1)Q^{2n-1}}{(q-1)q^{n-1}} = \frac{(q-1)q^{2n-1}}{(q-1)q^{n-1}} = q^n,$$

and notice that this same formula works if $n = 0$.

Unamified, $f = 2$**:** In this situation $Q = q^2$ and $e = 1$, then

$$[O_0^* : O_n^*] = \frac{(Q-1)Q^{n-1}}{(q-1)q^{n-1}} = \frac{(q^2-1)q^{2(n-1)}}{(q-1)q^{n-1}} = (q+1)q^{n-1}.$$

This concludes the proof of the nonsplit cases. For the split case we have

$$O_0^* = O_K^* \times O_K^*,$$

and as such

$${}^{O_0^*}/{}_{p^n O_0} = {}^{O_K^* \times O_K^*}/{}_{p^n O_K \times p^n O_K} = \left({}^{O_K^*}/{}_{p^n O_K}\right) \times \left({}^{O_K^*}/{}_{p^n O_K}\right).$$

Using the corresponding iterative process we get

$$\left|{}^{O_0^*}/{}_{p^n O_0}\right| = \left|{}^{O_K^*}/{}_{p^n O_K}\right|^2 = (q-1)^2 q^{2(n-1)}.$$

Again by proposition 1,

$${}^{O_n}/{}_{p^n O_n} \cong {}^{O_K}/{}_{p^n O_K}.$$

With this, and once again by the iterative argument, we get

$$\left|\left({}^{O_n}/{}_{p^n O_0}\right)^*\right| = (q-1)q^{n-1}.$$

We conclude

$$[O_0^* : O_n^*] = \frac{(q-1)^2 q^{2(n-1)}}{(q-1)q^{n-1}} = (q-1)q^{n-1},$$

which finishes the proof. □

We have found before that the recurrence relation in each case is

$$\zeta_n(s) = \zeta_n^P(s) + q^{-s}\zeta_{n-1}(s).$$

We are now ready to solve this equations explicitly. We begin by finding the initial condition:

Proposition 23. *The zeta function of the order $O_0 = O_L$ is in the nonsplit case*

$$\zeta_0(s) = \frac{1}{1 - q^{-fs}},$$

and

$$\zeta_0(s) = \frac{1}{(1 - q^{-s})^2}.$$

in the split case. In particular, in each case, $\zeta_0(s) = V(s)^{-1}$, where $V(s)$ is the factor appearing in Proposition 12 at page 53.

Proof. We know that for $n = 0$ all ideals are high. We have computed before, in the proof of Proposition 20 on page 62, that the high ideals contribute $\frac{[O_0^* : O_n^*]}{q^{2ns}V(s)}$. In our present case this means

$$\zeta_0(s) = \zeta_0^P(s) = \frac{1}{V(s)},$$

and $V(s)$ is $1 - q^{-fs}$ for the nonsplit case and $(1 - q^{-s})^2$ for the split one. □

We finally get

Theorem 4. *For each $n \geq 0$ define the following polynomials:*

$$R_n(X) = 1 + qX^2 + q^2X^4 + ... + q^nX^{2n},$$

and for $n \geq 1$ define

$$U_n(X) = (1 + X)R_{n-1}(X) + q^nX^{2n},$$
$$S_n(X) = (1 - X)R_{n-1}(X) + q^nX^{2n}.$$

Finally, also put $U_0(X) = S_0(X) = 1$. Then we have:

Ramified Case: *The solution of the ramified case recurrence relationship satisfies*

$$(1 - q^{-s})\zeta_n(s) = R_n(q^{-s}).$$

Unramified Case: *The solution of the unramified case recurrence relationship satisfies*

$$(1 - q^{-2s})\zeta_n(s) = U_n(q^{-s}).$$

Split Case: *The solution of the split case recurrence relationship satisfies*

$$(1 - q^{-s})^2\zeta_n(s) = S_n(q^{-s}).$$

Proof. Since the polynomials of the ramified case are related to the other two cases as well, we begin by solving its recurrence. By plugging in the values of the indices we find for $n \geq 1$

$$\begin{aligned}\zeta_n^P(s) &= \sum_{d=0}^{n-1} \frac{q^n}{q^{n-d}} \cdot \frac{1}{q^{2ds}} + \frac{q^n}{q^{2ns}} \cdot \frac{1}{1-q^{-s}} \\ &= \sum_{d=0}^{n-1} q^d \cdot q^{-2sd} + \frac{q^{(1-2s)n}}{1-q^{-s}}\end{aligned}$$

Notice that for $n = 0$ we have

$$\zeta_0(s) = \zeta_0^P(s) = \frac{1}{1-q^{-s}},$$

and so we get

$$\begin{aligned}\zeta_1(s) &= \zeta_1^P(s) + q^{-s}\zeta_0(s) \\ &= 1 + \frac{q^{1-2s}}{1-q^{-s}} + \frac{q^{-s}}{1-q^{-s}}.\end{aligned}$$

If we now multiply by $V(s) = (1 - q^{-s})$ we obtain

$$\begin{aligned}(1-q^{-s})\zeta_0(s) &= 1 \\ (1-q^{-s})\zeta_1(s) &= 1 - q^{-s} + q \cdot q^{-2s} + q^{-s} \\ &= 1 + q \cdot q^{-2s}.\end{aligned}$$

These values are, respectively, $R_0(q^{-s})$ and $R_1(q^{-s})$ as expected. Let us suppose now this holds up to $n - 1$, we have

$$(1-q^{-s})\zeta_n(s) = (1-q^{-s}) \sum_{d=0}^{n-1} q^d \cdot q^{-2sd} + q^n \cdot q^{-2sn} + q^{-s}(1-q^{-s})\zeta_{n-1}(s).$$

Based on the previous expression, let us define the polynomial $P(x)$ by

$$P(x) = (1-x) \sum_{d=0}^{n-1} q^d x^{2d} + q^n x^{2n} + xR_{n-1}(x).$$

Notice that $P(q^{-s}) = (1 - q^{-s})\zeta_n(s)$, where we used induction in the last term. Our job now is to prove $P(x) = R_n(x)$. This is straighforward from

$$\begin{aligned}P(x) &= (1-x) \sum_{d=0}^{n-1} q^d x^{2d} + q^n x^{2n} + xR_{n-1}(x) \\ &= (1-x)R_{n-1}(x) + q^n x^{2n} + xR_{n-1}(x) \\ &= R_{n-1}(x) + q^n x^{2n} \\ &= R_n(x).\end{aligned}$$

Let us move now onto the unramified case. We have

$$\zeta_0(s) = \zeta_0^P(s) = \frac{1}{1-q^{-2s}},$$

and plugging in the values of the indices of units for these cases we get

$$\zeta_n^P(s) = \sum_{d=0}^{n-1} q^d \cdot q^{-2ds} + \frac{(q+1)q^{n-1}q^{-2ns}}{1-q^{-2s}}.$$

For $n = 0$ and $n = 1$ we get, using the recurrence relationship and simplifying, that

$$\begin{aligned}(1-q^{-2s})\zeta_0(s) &= 1\\ (1-q^{-2s})\zeta_1(s) &= 1 + q^{-s} + q \cdot q^{-2s},\end{aligned}$$

which are, respectively, $U_0(q^{-s})$ and $U_1(q^{-s})$. Proceeding by induction we find

$$\begin{aligned}(1-q^{-2s})\zeta_n(s) &= (1-q^{-2s})\sum_{d=0}^{n-1} q^d \cdot q^{-2ds} + (q+1)q^{n-1}q^{-2ns} + q^{-s}(1-q^{-2s})\zeta_{n-1}(s)\\ &= (1-q^{-2s})R_{n-1}(q^{-s}) + (q+1)q^{n-1}q^{-2ns} + q^{-s}U_{n-1}(q^{-s}),\end{aligned}$$

This clearly is a polynomial evaluated at q^{-s}. If we change q^{-s} for x, to simplify notation, we find that

$$\begin{aligned}&(1-x^2)R_{n-1}(x) + (q+1)q^{n-1}x^{2n} + xU_{n-1}(x)\\ &= (1-x^2)R_{n-1}(x) + q^n x^{2n} + q^{n-1}x^{2n} + x((1+x)R_{n-2}(x) + q^{n-1}x^{2(n-1)})\\ &= (1-x^2)R_{n-1}(x) + q^n x^{2n} + q^{n-1}x^{2n} + x(1+x)R_{n-2}(x) + q^{n-1}x^{2n-1}\\ &= (1-x^2)R_{n-1}(x) + x(1+x)(R_{n-1}(x) - q^{n-1}x^{2(n-1)}) + q^n x^{2n} + q^{n-1}x^{2n} + q^{n-1}x^{2n-1}\\ &= (1+x)R_{n-1}(x) + q^n x^{2n}\\ &= U_n(x),\end{aligned}$$

as desired. Finally, let us settle the split case. Once more, we have for $n \geq 1$

$$\zeta_n^P(s) = \sum_{d=0}^{n-1} q^d \cdot q^{-2ds} + \frac{(q-1)q^{n-1}q^{-2ns}}{(1-q^{-s})^2}.$$

For the first two values we do recover $S_0(q^{-s})$ and $S_1(q^{-s})$ since we get

$$\begin{aligned}(1-q^{-s})^2\zeta_0(s) &= 1\\ (1-q^{-s})^2\zeta_1(s) &= 1 - q^{-s} + q \cdot q^{-2s}.\end{aligned}$$

Hence, just as in the unramified case, we have

$$\begin{aligned}(1-q^s)^2\zeta_n(s) &= (1-q^s)^2\sum_{d=0}^{n-1} q^d\cdot q^{-2ds} + (q-1)q^{n-1}q^{-2ns} + q^{-s}(1-q^s)^2\zeta_{n-1}(s)\\ &= (1-q^s)^2R_{n-1}(s) + (q-1)q^{n-1}q^{-2ns} + q^{-s}S_{n-1}(q^{-s})\end{aligned}$$

This is a polynomial evaluated at q^{-s} and, putting x instead of q^{-s}, we get

$$\begin{aligned}&(1-x)^2R_{n-1}(x) + (q-1)q^{n-1}x^{2n} + xS_{n-1}(x)\\ &= (1-x)^2R_{n-1}(x) + q^nx^{2n} + q^{n-1}x^{2n} + x((1-x)R_{n-2}(x) + q^{n-1}x^{2(n-1)})\\ &= (1-x)^2R_{n-1}(x) + q^nx^{2n} + q^{n-1}x^{2n} + x(1-x)R_{n-2}(x) + q^{n-1}x^{2n-1}\\ &= (1-x)^2R_{n-1}(x) + x(1-x)(R_{n-1}(x) - q^{n-1}x^{2(n-1)}) + q^nx^{2n} + q^{n-1}x^{2n} + q^{n-1}x^{2n-1}\\ &= (1-x)R_{n-1}(x) + q^nx^{2n}\\ &= S_n(x),\end{aligned}$$

concluding this last case. □

We close this chapter by exemplifying the property

$$(qX^2)^nP\left(\frac{1}{qX}\right) = P(X),$$

that each of these polynomials satisfy, for the appropriate n. We verify it for

$$S_4(X) = 1 - X + qX^2 - qX^3 + q^2X^4 - q^2X^5 + q^3X^6 - q^3X^7 + q^4X^8.$$

Indeed, we have

$$\begin{aligned}(qX^2)^4S_4\left(\frac{1}{qX}\right) &= q^4X^8\left(1 - \frac{1}{qX} + \frac{q}{q^2X^2} - \frac{q}{q^3X^3} + \frac{q^2}{q^4X^4} - \frac{q^2}{q^5X^5} + \frac{q^3}{q^6X^6} - \frac{q^3}{q^7X^7} + \frac{q^4}{q^8X^8}\right)\\ &= q^4X^8 - q^3X^7 + q^3X^6 - q^2X^5 + q^2X^4 - qX^3 + qX^2 - X + 1\\ &= S_4(X),\end{aligned}$$

as desired. A table for the first five polynomials of each case is shown below.

Table 2.2: Polynomials for the local quadratic case

Unramified Case	Ramified Case	Split case
1	1	1
$1+X+qX^2$	$1+qX^2$	$1-X+qX^2$
$1+X+qX^2+qX^3+q^2X^4$	$1+qX^2+q^2X^4$	$1-X+qX^2-qX^3+q^2X^4$
$1+X+qX^2+qX^3+q^2X^4+q^2X^5+q^3X^6$	$1+qX^2+q^2X^4+q^3X^6$	$1-X+qX^2-qX^3+q^2X^4-q^2X^5+q^3X^6$
$1+X+qX^2+qX^3+q^2X^4+q^2X^5+q^3X^6+q^3X^7+q^4X^8$	$1+qX^2+q^2X^4+q^3X^6+q^4X^8$	$1-X+qX^2-qX^3+q^2X^4-q^2X^5+q^3X^6-q^3X^7+q^4X^8$

Chapter 3

The multiplicative formula of Langlands

In this chapter we will use the results of the previous chapter to prove the validity of the formula of Langlands for the product of orbital integrals, that is, the formula

$$\prod_{\mathfrak{q}} \mathcal{O}(\gamma, f_{\mathfrak{q}}) = \sum_{\mathfrak{d}|S_\gamma} N_K(\mathfrak{d}) \prod_{\mathfrak{q}|\mathfrak{d}} \left(1 - \frac{\chi_\gamma(\mathfrak{q})}{N_K(\mathfrak{q})}\right), \tag{3.1}$$

as well as of the function of the complex parameter s, whose value at $s = 1$ recovers this value. As we know from Chapter One the function is

$$N_K(S_\gamma)^s \sum_{\mathfrak{d}|S_\gamma} \frac{1}{N_K(\mathfrak{d})^{2s-1}} \prod_{\mathfrak{q}|(S_\gamma/\mathfrak{d})} \left(1 - \frac{\chi_\gamma(\mathfrak{q})}{N_K(\mathfrak{q})^s}\right). \tag{3.2}$$

Concretely, our goal is to answer the following two questions:

Question 1: *How can we produce such formulas?* As we ave mentioned in Chapter One, the available proofs of the former formula depend on us already knowing the values of orbital integrals and then being able to produce out of them a general formula. This will not generalize unless we already know the values of the integrals, which is not an easy task. Approaching the problem via the zeta functions of orders might provide a different perspective with which to progress in this direction.

Question 2: *Can we simplify the proof of the functional equation?* We mentioned before that the function of a complex parameter is invariant when changing s to $1 - s$. As we highlighted in Chapter One, this follows from an algebraic manipulation on the completed function. We will see how with the zeta functions of orders, which we already know satisfy this invariance property, we can recover the functional equation in a more systematic way.

The answer to these two questions fulfill what we promised in the introduction, which was to confirm what Arthur predicts in [4]. Namely, that within the correct context (as explained below) the zeta functions indeed lead to the formula of Langlands for the product of orbital integrals, over any number field, and that the functional equation needed for the approximate functional equation can be deduced from that of the zeta functions.

3.1 Orbital integrals and zeta functions of orders

3.1.1 The setup in GL$(2, K)$

In this chapter, we will consider K a global number field and we will denote by $\mathcal{O}_K$ its ring of integers. We are in the context of studying the regular elliptic part of the trace formula, as was done in Chapter One, and so we will denote by f the function defined by

$$f = \prod_{\mathfrak{q}} f_{\mathfrak{q}},$$

where $\mathfrak{q}$ is the indicator of the maximal compact $\mathrm{GL}(2, \mathcal{O}_{K_\mathfrak{q}})$ of $\mathrm{GL}(2, K_\mathfrak{q})$. At the prime $\mathfrak{p}$ we define $f_\mathfrak{p}$ as the indicator of

$$\{X \in \mathrm{Mat}(2, \mathcal{O}_{K_\mathfrak{p}}) \mid \ |\det(X)|_\mathfrak{p} = N_K(\mathfrak{p})^{-k}\}.$$

Notice that we are not putting the factor $N_K(\mathfrak{p})^{-k/2}$ this time.

As we have explained in the first chapter, as part of our manipulations of the regular elliptic part of the trace formula, we have to deal with the product of local orbital integrals

$$\mathcal{O}(\gamma, f_\mathfrak{q}) = \int_{G_\gamma(K_\mathfrak{q})\backslash G(K_\mathfrak{q})} f_\mathfrak{q}(x^{-1}\gamma x) \mathrm{dx}_\mathfrak{q},$$

where $\mathrm{dx}_\mathfrak{q}$ is the measure in the quotient $G_\gamma(K_\mathfrak{q})\backslash G(K_\mathfrak{q})$ obtained via the integration in stages formula when the Haar measures on both $\mathrm{GL}(2, K_\mathfrak{q})$ and $\mathrm{GL}_\gamma(2, K_\mathfrak{q})$ give their maximal compact subgroups volume one.

We fix a regular elliptic element $\gamma \in \mathrm{GL}(2, K)$, whose characteristic polynomial is

$$P_\gamma(X) = X^2 - \tau X + \delta,$$

and such that

$$\prod_{\mathfrak{q}} \mathcal{O}(\gamma, f_\mathfrak{q}) \neq 0.$$

This implies that

$$(\det(\gamma)) = \mathfrak{p}^k$$

and that τ and δ are elements of $\mathcal{O}_K$.

3.1.2 Orders associated to γ

Associated to this polynomial is the quadratic field extension generated by γ,

$$K_\gamma = \frac{K[X]}{(P_\gamma(X))},$$

and we identiy γ with the coset $X + (P_\gamma(X))$. Notice that this γ is an integral element of K_γ. At any prime $\mathfrak{q}$ we can localize K and extend using the same polynomial P_γ, that is, define

$$K_\mathfrak{q}(\gamma) := \frac{K_\mathfrak{q}[X]}{(P_\gamma(X))},$$

which is a reduced quadratic $K_\mathfrak{q}-$algebra. This puts us into the context of the previous chapter, since $K_\mathfrak{q}(\gamma)/K_\mathfrak{q}$ is the kind of extension that we studied there. In particular, associated to γ we have a canonical choice for a local order, named

$$\mathcal{O}_{K_\mathfrak{q}}[\gamma],$$

which is the smallest ring (which happens to be an order) containing $\mathcal{O}_{K_\mathfrak{q}}$ and γ.

Using the notation of the previous chapter as well as the set up just explained, a result that is proven in [17], as Theorem 2.5, is the following

Theorem 5. *For a regular elliptic element $\gamma \in GL(2, K)$ we have the equality*

$$\tilde{J}_{\mathcal{O}_{K_\mathfrak{q}}[\gamma]}(1) = \mathcal{O}(\gamma, f_\mathfrak{q}).$$

Recall that $\tilde{J}_{\mathcal{O}_{K_\mathfrak{q}}[\gamma]}$ is the completed zeta function for the order $\mathcal{O}_{K_\mathfrak{q}[\gamma]}$. The proof of this result comes from regrouping the definition of the completed zeta function and proving that one obtains an expression which coincides with the computation obtained from counting lattices in the building of $\mathrm{SL}(2, K_\mathfrak{q})$.

3.1.3 The element S_γ

In order to relate local and global information we need an extra tool from algebraic number theory. We begin with a general fact that can be reviewed in [9].

Proposition 24. *Let E be a global algebraic number field, O_E its ring of integers, F a finite extension of E and O_F the integral closure of O_E in F. For $\alpha \in O_F$ and p a prime of E, if p does not divide*

$$\frac{Disc_{F/E}(\alpha)}{Disc(F/E)}$$

then $O_{F_p} = O_{E_p}[\alpha]$. Here, $Disc(F/E)$ is the relative discriminant of the extension F/E while $Disc_{F/E}(\alpha)$ is the discriminant of the element α.

Recall that for an element α in a quadratic extension F/E we have

$$\mathrm{Disc}(\alpha) = (\alpha - \sigma(\alpha))^2$$

where $\sigma : F \to F$ is the nontrivial Galois isomorphism. In the case of a field extension, it is the nontrivial element of the Galois group, while for a split case, is the exchange of coordinates. In particular, if α is an algebraic integer that generates the quadratic extension then

$$\mathrm{Disc}(\alpha) = a^2 - 4b,$$

where the minimal polynomial of α is

$$P_\alpha(X) = X^2 - aX + b.$$

Furthermore, the general theory of free modules and discriminants in number fields implies that the quotient of discriminants in definition (24) is actually the square of some integer S.

Returning to our discussion with the regular elliptic element γ and the quadratic extension K_γ/K it generates we have the following

Definition 13. *For the regular elliptic element γ define*

$$S_\gamma^2 = \frac{Disc_{L/K}(\gamma)}{\Delta_\gamma},$$

where Δ_γ is the relative discriminant of the extension K_γ/K.

Notice that another way of writing this is

$$(\tau^2 - 4\delta) = S_\gamma^2 \Delta_\gamma,$$

which over $\mathbb{Z}$ is precisely the formula we had in Chapter One, namely

$$\tau^2 - 4\delta = s_\gamma^2 D_\gamma,$$

since the relative discriminant is generated by the fundamental discriminant, and the signs are taken in such a way that the above equality is one of numbers as opposed to ideals.

3.2 The multiplicative formula of Langlands

Now we are ready to discuss the *multiplicative formula of Langlands* and its version as a complex valued function.

As we have said, given our regular elliptic element $\gamma \in \mathrm{GL}(2, K)$ and a prime $\mathfrak{q}$ of $\mathcal{O}_K$ we can consider the local quadratic extension $K_\mathfrak{q}(\gamma)/K_\mathfrak{q}$ and the order $O_{K_\mathfrak{q}}[\gamma]$. According to the splitting behaviour of $\mathfrak{q}$ in the extension K_γ we know that $K_\mathfrak{q}(\gamma)$ is a field or $K_\mathfrak{q} \times K_\mathfrak{q}$. In any case, we can apply the results of the previous chapter to this order and obtain its zeta function or the associated polynomial. What is striking is that we can put order in these extensions in a way that we can read everything off the global behaviour of γ directly. We explain this now.

We know that there exists an element $\Delta_\mathfrak{q}$ such that $\mathcal{O}_{K_\mathfrak{q}}[\Delta_\mathfrak{q}]$ is the ring of integers of $K_\mathfrak{q}(\gamma)$ and, using the notation of the previous chapter, we have its associated main sequence of orders:

$$\mathcal{O}_0 \supset \mathcal{O}_1 \supset \mathcal{O}_2 \supset \mathcal{O}_3 \supset \ldots$$

where

$$\mathcal{O}_n = O_{K_\mathfrak{q}}[\pi_\mathfrak{q}^n \Delta_\mathfrak{q}],$$

and $\pi_\mathfrak{q}$ is a uniformizer of $K_\mathfrak{q}$. We are omitting the dependence on $\mathfrak{q}$ from the main sequence of orders. We begin by proving

Proposition 25. *For a fixed prime* $\mathfrak{q}$*, the main sequence of orders*

$$\mathcal{O}_0 \supset \mathcal{O}_1 \supset \mathcal{O}_2 \supset \mathcal{O}_3 \supset ...$$

is independent from $\Delta_{\mathfrak{q}}$.

Proof. If $\Delta'_{\mathfrak{q}}$ is another generator of $O_{K_{\mathfrak{q}}(\gamma)}$ over $O_{K_{\mathfrak{q}}}$, then we must have

$$\Delta_{\mathfrak{q}} = x + y\Delta'_{\mathfrak{q}},$$

for some $x, y \in O_{K_{\mathfrak{q}}}$. Furthermore, for the $O_{K_{\mathfrak{q}}}$ bases $\{1, \Delta_{\mathfrak{q}}\}$ and $\{1, \Delta'_{\mathfrak{q}}\}$, the elementary divisors are $1, y$. Since both bases generate $O_{K_{\mathfrak{q}}(\gamma)}$, y is a unit. Hence, we get from the previous equation

$$\Delta'_{\mathfrak{q}} = y^{-1}(\Delta_{\mathfrak{q}} - x).$$

We conclude that if $\alpha = a + b\pi_{\mathfrak{q}}^n\Delta'_{\mathfrak{q}}$, for some $a, b \in O_{K_{\mathfrak{q}}}$ and some integer $n \geq 0$, then

$$\alpha = a + b\pi_{\mathfrak{q}}^n(y^{-1}(\Delta_{\mathfrak{q}} - x)) = a - x\pi_{\mathfrak{q}}^n by^{-1} + (by^{-1})\pi_{\mathfrak{q}}^n\Delta_{\mathfrak{q}},$$

proving that

$$O_{K_{\mathfrak{q}}}[\pi_{\mathfrak{q}}^n\Delta'_{\mathfrak{q}}] \subseteq O_{K_{\mathfrak{q}}}[\pi_{\mathfrak{q}}^n\Delta_{\mathfrak{q}}].$$

Changing the roles of $\Delta_{\mathfrak{q}}$ and $\Delta'_{\mathfrak{q}}$ we conclude

$$O_{K_{\mathfrak{q}}}[\pi_{\mathfrak{q}}^n\Delta_{\mathfrak{q}}] = O_{K_{\mathfrak{q}}}[\pi_{\mathfrak{q}}^n\Delta'_{\mathfrak{q}}],$$

proving the desired result. □

The previous proposition leads naturally to the following two questions: (1) *is the order* $O_{K_{\mathfrak{q}}}[\gamma]$ *part of the main sequence?* and if so, (2) *which position it has in the main sequence?* The first part of the amazing answer to these questions is given by the following

Proposition 26. *The order* $O_{K_{\mathfrak{q}}}[\gamma]$ *is always part of the main sequence of orders. Furthermore, if we write*

$$\gamma = a + b\Delta_{\mathfrak{q}},$$

for some $a, b \in O_{K_{\mathfrak{q}}}$ *then*

$$O_{K_{\mathfrak{q}}}[\gamma] = O_{K_{\mathfrak{q}}}[\pi_{\mathfrak{q}}^n\Delta_{\mathfrak{q}}] = \mathcal{O}_n,$$

where $n = val_{\mathfrak{q}}(b)$.

Proof. Let $\gamma = a + b\Delta_{\mathfrak{q}} = a + u(\pi_{\mathfrak{q}}^n\Delta_{\mathfrak{q}}) \in O_{K_p}[\pi_{\mathfrak{q}}^n\Delta_p]$, where u is a unit. We then have

$$O_{K_{\mathfrak{q}}}[\gamma] \subseteq O_{K_{\mathfrak{q}}}[\mathfrak{q}^n\Delta_{\mathfrak{q}}].$$

On the other hand, since $u \in O_K$ is a unit we have

$$\pi_{\mathfrak{q}}^n\Delta_{\mathfrak{q}} = -u^{-1}a + u^{-1}\gamma \in O_{K_{\mathfrak{q}}}[\gamma],$$

proving

$$O_{K_\mathfrak{q}}[\pi_\mathfrak{q}^n \Delta_\mathfrak{q}] \subseteq O_{K_\mathfrak{q}}[\gamma].$$

By double inclusion we thus get

$$O_{K_\mathfrak{q}}[\pi_\mathfrak{q}^n \Delta_\mathfrak{q}] = O_{K_\mathfrak{q}}[\gamma],$$

and since $n = \mathrm{val}_p(b)$, by definition of u, we obtain the result. □

Up to this point we know that the main sequence is relevant since it contains all the local orders associated to our regular elliptic elements. Unfortunately, as it stands we need the valuation of b in the expansion

$$\gamma = a + b\Delta_\mathfrak{q},$$

to get its location. Since the polynomial *does* depend on the location within the sequence it is important to know exactly this number from information readily available to us from γ without the extra burden of finding a generator $\Delta_\mathfrak{q}$ and the corresponding b. To solve this issue we have the following

Proposition 27. *For the regular elliptic matrix γ consider the number S_γ as we have defined it in the previous section. Then we have*

$$val_\mathfrak{q}(S_\gamma) = val_\mathfrak{q}(b),$$

where $\gamma = a + b\Delta_\mathfrak{q}$.

Proof. Because discriminants are local objects, we can localize the definition of S_γ at the prime $\mathfrak{q}$ to get

$$\pi_\mathfrak{q}^{2n_\mathfrak{q}} = \frac{\mathrm{Disc}_{K_\mathfrak{q}(\gamma)/K_\mathfrak{q}}(\gamma)}{\mathrm{Disc}(K_\mathfrak{q}(\gamma)/K_\mathfrak{q})},$$

where $n_\mathfrak{q} = \mathrm{val}_\mathfrak{q}(S_\gamma)$. In the local picture we have an $\mathcal{O}_{K_\mathfrak{q}}$-integral basis for $K_\mathfrak{q}(\gamma)$, precisely $1, \Delta_\mathfrak{q}$, and so the discriminant can be computed with the standard formula using the Galois Conjugates. If we denote by $\overline{\Delta_\mathfrak{q}}$ the conjugate of $\Delta_\mathfrak{q}$ we get

$$\mathrm{Disc}(K_\mathfrak{q}(\gamma)/K_\mathfrak{q}) = (\Delta_\mathfrak{q} - \overline{\Delta_\mathfrak{q}})^2.$$

Analogously, for the discriminant of γ we get

$$\mathrm{Disc}_{K_\mathfrak{q}(\gamma)/K_\mathfrak{q}}(\gamma) = (\gamma - \overline{\gamma})^2.$$

Finally, taking conjugate to $\gamma = a + b\Delta_\mathfrak{q}$, we get $\overline{\gamma} = a + b\overline{\Delta_\mathfrak{q}}$, which upon substraction leads to

$$(\gamma - \overline{\gamma}) = b(\Delta_\mathfrak{q} - \overline{\Delta_\mathfrak{q}}).$$

Squaring and taking valuations we conclude $\mathrm{val}_p(b) = n_\mathfrak{q}$, as desired. □

The reason why we say this is amazing is because, once we know the factorization of S_γ which is a global object immediately available (from the theoretical point of view at least) from γ itself, we can know the location of its local orders within the main sequence and hence the polynomials associated to them. This settles the first part of our quest: we have an object that tells us precisely what are the orders and hence the polynomials associated to them.

Nevertheless, the polynomials are different according to whether the extension is split, ramified or unramified. Recall that theorem (4), in page 67, give us precisely that the form of these polynomials is, respectively,

$$\begin{aligned} R_n(X) &= 1 + qX^2 + q^2X^4 + ... + q^nX^{2n}, \\ U_n(x) &= (1+X)R_{n-1}(X) + q^nX^{2n}, \\ S_n(X) &= (1-X)R_{n-1}(X) + q^nX^{2n}. \end{aligned} \tag{3.3}$$

These three families of polynomials cannot be described as a single object when we are considering arbitrary reduced quadratic $K_{\mathfrak{q}}-$algebras, but in our situation the quadratic algebras come directly from a global quadratic field constructed by a matrix and via localizing at a prime $\mathfrak{q}$. The nature of the extension is equivalent to the splitting behaviour of the prime $\mathfrak{q}$ in the quadratic extension K_γ, which is precisely measured by the quadratric sign Hecke character. To put this concretely, we have

Proposition 28. *Let χ_γ be the associated quadratic sign Hecke character associated to the extension K_γ/K. Then we have*

$$\tilde{J}_{\mathcal{O}_{K_{\mathfrak{q}}}[\gamma]}(s) = q^{ns}\left(\left(1 - \frac{\chi_\gamma(\mathfrak{q})}{q^s}\right)R_{n-1}(s) + q^{n(1-2s)}\right),$$

where $n = val_{\mathfrak{q}}(S_\gamma)$. In here $R_{-1} = 0$.

Proof. For $n > 0$ this follows from substituting the definition of the character χ_γ, namely

$$\chi_\gamma(\mathfrak{q}) = \begin{cases} 1 & \text{if } \mathfrak{q} \text{ splits,} \\ -1 & \text{if } \mathfrak{q} \text{ is inert,} \\ 0 & \text{if } \mathfrak{q} \text{ ramifies,} \end{cases}$$

into

$$\tilde{J}_{\mathcal{O}_{K_{\mathfrak{q}}}[\gamma]}(s) = q^{ns}\left(\left(1 - \frac{\chi_\gamma(\mathfrak{q})}{q^s}\right)R_{n-1}(s) + q^{n(1-2s)}\right),$$

and comparing with the polynomials (3.3). For $n = 0$ we use proposition (23) which precisely implies

$$\tilde{J}_0(s) \equiv 1.$$

□

What we find is that the quadratic character χ_γ allows us to give a uniform description of the orders associated to a given regular elliptic matrix. One important corollary of all the previous facts is the following

Proposition 29. *Let $\mathfrak{q}$ be a prime that **does not** divide S_γ, then*

$$\tilde{J}_{\mathcal{O}_{K_{\mathfrak{q}}}[\gamma]}(s) \equiv 1.$$

Also, upon evaluation at $s = 1$, we have that

$$\mathcal{O}(\gamma, f_{\mathfrak{q}}) = 1.$$

Proof. If $\mathfrak{q}$ doesn't divide S_γ, by proposition (24) we know we can take $\Delta_{\mathfrak{q}} = \gamma$. In that case,

$$\mathcal{O}_0 = \mathcal{O}_{K_{\mathfrak{q}}(\gamma)},$$

and so we can put $n = 0$ in the previous proposition obtaining the first part of the proposition. Upon evaluation at $s = 1$, and using theorem (5) in page 73, we obtain the second equality. □

We now define

Definition 14. *For a given regular elliptic element define*

$$\mathcal{O}(s, \gamma) = \prod_{\mathfrak{q}} \tilde{J}_{\mathcal{O}_{K_{\mathfrak{q}}}[\gamma]}(s).$$

The previous propositions are not as innocent as they might look. They imply two things: firstly, the product that defines $\mathcal{O}(s, \gamma)$ is finite and hence the theory of arithmetic functions will let us *glue* the product into a sum over the divisors of some ideal given by

$$\prod_{\mathfrak{q}|S_\gamma} \mathfrak{q}^{n_{\mathfrak{q}}},$$

and *a priori* this might not be S_γ. By itself this could be a minor point, but in actuality it is an important issue to address. It regards the manipulation of the regular elliptic part of the trace formula, as explained in Chapter One since the volume term $\mathrm{vol}(\gamma)$ involves S_γ specifically for the manipulation required between the volume and the formula of Langlands. If we are to do the computations as desired we need these two objects to be compatible. The previous proposition assures this, because we know $\mathfrak{q}^{n_{\mathfrak{q}}}$ is the exact power of $\mathfrak{q}$ that divides S_γ. This fact plays a fundamental role in the proof of the global formula because it is what allows an *a priori* infinite product to become a finite one where the theory of multiplicative functions can be applied. We are now ready to glue all the parts together.

Theorem 6. *$\mathcal{O}(s, \gamma)$ is an entire function that satisfies the functional equation*

$$\mathcal{O}(s, \gamma) = \mathcal{O}(1 - s, \gamma).$$

We furthermore have the expansion

$$\mathcal{O}(s, \gamma) = N_K(S_\gamma)^s \sum_{d|S_\gamma} N_K(d)^{1-2s} \prod_{\mathfrak{q}|d'} \left(1 - \frac{\chi_\gamma(\mathfrak{q})}{N_K(\mathfrak{q})^s}\right),$$

where $d' = S_\gamma/d$.

Proof. Using the previous proposition, we have

$$\mathcal{O}(s, \gamma) = \prod_{\mathfrak{q}|S_\gamma} \tilde{J}_{\mathcal{O}_{K_{\mathfrak{q}}}[\gamma]}(s).$$

This proves $\mathcal{O}$ is an entire function that satisfies the functional equation, since each of the *finite* number of factors $\tilde{J}_{\mathcal{O}_{K_{\mathfrak{q}}}[\gamma]}(s)$ does. Here we are using proposition (12) on page 12.

For a prime $\mathfrak{q}$, let

$$n_{\mathfrak{q}} = \mathrm{val}_{\mathfrak{q}}(S_\gamma).$$

Using proposition (28) and noticing that the ideal divisors of $\mathfrak{q}^{n_{\mathfrak{q}}}$ are precisely $1, \mathfrak{q}, ..., \mathfrak{q}^{n_{\mathfrak{q}}}$, we can write the following equation

$$\tilde{J}_{O_{K_{\mathfrak{q}}}[\gamma]}(s) = N_K(\mathfrak{q})^{n_{\mathfrak{q}}s} \sum_{d|\mathfrak{q}^{n_{\mathfrak{q}}}} N_K(d)^{(1-2s)} \prod_{r|d'} \left(1 - \frac{\chi_\gamma(r)}{N_K(r)^s}\right), \tag{3.4}$$

where $d' = \mathfrak{q}^{n_{\mathfrak{q}}}/d$ and where r runs over prime divisors of d'.

Invoking the standard theory of multiplicative arithmetic functions we get from equation (3.4), now that we have a finite product, that

$$\begin{aligned}
\mathcal{O}(s,\gamma) &= \prod_{\mathfrak{q}|S_\gamma} \tilde{J}_{O_{K_{\mathfrak{q}}}[\gamma]}(s) \\
&= \prod_{\mathfrak{q}|S_\gamma} \left(N_K(\mathfrak{q})^{n_{\mathfrak{q}}s} \sum_{d|\mathfrak{q}^{n_{\mathfrak{q}}}} N_K(d)^{(1-2s)} \prod_{r|(\mathfrak{q}^{n_{\mathfrak{q}}}/d)} \left(1 - \frac{\chi_\gamma(r)}{N_K(r)^s}\right)\right) \\
&= N_K\left(\prod_{\mathfrak{q}|S_\gamma} \mathfrak{q}^{n_{\mathfrak{q}}}\right)^s \left(\sum_{d|\prod_{\mathfrak{q}|S_\gamma} \mathfrak{q}^{n_{\mathfrak{q}}}} N_K(d)^{(1-2s)} \prod_{r|d'} \left(1 - \frac{\chi_\gamma(r)}{N_K(r)^s}\right)\right) \\
&= N_K(S_\gamma)^s \sum_{d|S_\gamma} N_K(d)^{(1-2s)} \prod_{r|d'} \left(1 - \frac{\chi_\gamma(r)}{N_K(r)^s}\right).
\end{aligned}$$

This concludes the proof. □

Notice that the right had side of equation (6) is precisely the function (3.2). In this way we have proven, by local methods, the functional equation required for the approximate functional equation. We also have from this theorem the following

Corollary 2. *The product of orbital integrals satisfies*

$$\prod_{\mathfrak{q}} \mathcal{O}(\gamma, f_{\mathfrak{q}}) = \sum_{d|S_\gamma} N_K(d) \prod_{r|d} \left(1 - \frac{\chi_\gamma(r)}{N_K(r)}\right).$$

Proof. Evaluating at $s = 1$ the previous proposition and making the change of variable d for d' (both go over the divisors of S_γ), we obtain

$$\prod_{\mathfrak{q}} \mathcal{O}(\gamma, f_{\mathfrak{q}}) = \sum_{d|S_\gamma} N_K(d) \prod_{r|d} \left(1 - \frac{\chi_\gamma(r)}{N_K(r)}\right)$$

□

This recovers the formula (3.1) that we mentioned in Chapter One. Furthermore, for $\mathbb{Q}$, once we use the explicit expression of the quadratic character via the Kronecker symbol, we recover the original formula of Langlands. That is, when $K = \mathbb{Q}$ we have

$$\chi_\gamma(p) = \left(\frac{D_\gamma}{p}\right),$$

and this transforms the previous corollary into

$$\sum_{d|S_\gamma} d \prod_{p|d} \left(1 - \frac{\left(\frac{D_\gamma}{p}\right)}{p} \right).$$

3.3 The value of the orbital integral

We now compute the values of the orbital integrals we have been considering so far. This result already appears in [10] as Lemma 1 for the case of $K = \mathbb{Q}$. We keep the assumptions we have been using throughout the chapter.

Proposition 30. *The values of $\mathcal{O}(\gamma, f_\mathfrak{q})$ are given as follows:*

(a) if $K_\mathfrak{q}(\gamma)$ is split, it is given by q^n,

(b) if $K_\mathfrak{q}(\gamma)$ is nonsplit and unramified, it is given by

$$q^n \frac{q+1}{q-1} - \frac{2}{q-1},$$

(c) if $K_\mathfrak{q}(\gamma)$ is nonsplit and ramified, it is given by

$$\frac{q^{n+1}}{q-1} - \frac{1}{q-1},$$

for certain integer $n \geq 0$ that depends on γ and $\mathfrak{q}$

Proof. By Theorem (5) we have that, upon evaluating at $s = 1$,

$$\tilde{J}_{\mathcal{O}_{K_\mathfrak{q}}[\gamma]}(1) = \mathcal{O}(\gamma, f_\mathfrak{q}).$$

The difference between the different cases, as we have explained before, is which of the polynomials we must use. In (a), (b) and (c), we must use respectively, S_n, U_n and R_n. We know that all of these polynomials satisfy

$$(qX^2)^n P\left(\frac{1}{qX}\right) = P(X),$$

which upon substitution of $X = q^{-1}$ (i.e $s = 1$), becomes

$$q^{-n} P(1) = P(q^{-1}),$$

Let us begin with the ramified case, since the other ones depend on it. We have

$$\begin{aligned}\mathcal{O}(\gamma, f_{\mathfrak{q}}) &= q^n R_n(q^{-1}) \\ &= q^n(q^{-n} R_n(1)) \\ &= R_n(1) \\ &= 1 + q + q^2 + \ldots + q^n \\ &= \frac{q^{n+1} - 1}{q - 1}.\end{aligned}$$

For S_n we get

$$\begin{aligned}\mathcal{O}(\gamma, f_{\mathfrak{q}}) &= q^n S_n(q^{-1}) \\ &= q^n(q^{-n} S_n(1)) \\ &= S_n(1) \\ &= (1 - 1)R_{n-1}(1) + q^n \\ &= q^n.\end{aligned}$$

Finally, for the unramified case

$$\begin{aligned}\mathcal{O}(\gamma, f_{\mathfrak{q}}) &= q^n U_n(q^{-1}) \\ &= q^n(q^{-n} U_n(1)) \\ &= S_n(1) \\ &= 2R_{n-1}(1) + q^n \\ &= \frac{2(q^n - 1)}{q - 1} + q^n \\ &= q^n \frac{q + 1}{q - 1} - \frac{2}{q - 1}.\end{aligned}$$

This concludes the proof. □

Thus, we realize that via the zeta functions of orders we can recover the values of orbital integrals. We know that the evaluation of such integrals is not an easy task and as such it is a sign of caution with regards to our expectations from the polynomials: it might be the case that finding polynomials in such specific detail as we have done in this thesis will be complicated in other situations.

Chapter 4

Conclusions

We will conclude this thesis by discussing four possible directions for future research that are related to the work we have presented so far.

4.1 The method of Altuğ for other functions

There are several directions in which we can expect to generalize the process described in chapter one. One important direction is to change the test function f. We will describe the idea for $\mathbb{Q}$. As before, we consider a factorizable function

$$f = \prod_q f_q \times f_\infty,$$

but this time we will select a finite set S, which includes the archimidean prime, but also some finite primes. Say

$$S = \{r_1, ..., r_l\}.$$

We will define our functions f_q in the same way as we did in chapter one for $q \notin S$. This includes the definition at our special prime p, which we must pick outside of S. For those primes on S we let f_r be general, up to some analytical conditions we don't specify here.

The regular elliptic part of the trace formula is still the same and the problems we need to deal with are similar, yet we cannot just invoke the results we used previously since the set S includes finite primes and the conclusions we had before are not exactly the same.

For instance, the condition $\mathcal{O}(\gamma, f) \neq 0$ does not imply $\det(\gamma) = \pm p^k$. Rather, this time it implies

$$\det(\gamma) = \pm p^k m,$$

where m is a positive rational number whose prime factorization only includes primes from S. Nevertheless we can recover several results if we put extra attention to the construction.

The quadratic extension

$$K_\gamma = \frac{K[X]}{(P_\gamma(X))}$$

is still defined in the same way and we can also put

$$S_\gamma^2 = \frac{\mathrm{Disc}(\gamma)}{D_\gamma}.$$

As we have said in Chapter Three, the previous quotient is actually local, and we can break up S_γ into its primes from S and those outside of S, that is,

$$S_\gamma = S_{\gamma,S} S_\gamma^S,$$

where $S_{\gamma,S}$ is the part consisting on those inside S and S_γ^S consists of those outside.

For those $q \notin S$, we have $\det(\gamma)$ and $\mathrm{trace}(\gamma)$ are integers in $\mathbb{Z}_\mathfrak{q}$. As such, we can invoke all the theory of chapters Two and Three, and conclude by Theorem 5 that

$$\tilde{J}_{\mathcal{O}_{K_q}[\gamma]}(1) = \mathcal{O}(\gamma, f_q)$$

and so

$$\prod_{q \notin S} \mathcal{O}(\gamma, f_q) = \sum_{d | S_\gamma^S} d \prod_{l | d} \left(1 - \frac{(D_\gamma / l)}{l}\right),$$

where l runs over the primes dividing d. Furthermore, the function of a complex variable that satisfies a functional equation obtained with the same method of Chapter Three is

$$\mathcal{O}(s, \gamma) = N_K(S_\gamma^S)^s \sum_{d | S_\gamma^S} N_K(d)^{1-2s} \prod_{l | d'} \left(1 - \frac{(D_\gamma / l)}{l^s}\right).$$

This is ready to be subjected to the approximate functional equation once the step of distributing the volume term has been carried out.

The question now is how to handle

$$\prod_{q \in S} \mathcal{O}(\gamma, f_q).$$

For the case of Altuğ, as well as that studied in [7], this consists only of the archimidean places. What we must now address is what occurs in the finite places in S. The answer seems to lie in the Shalika germ expansion which guarantees that

$$\mathcal{O}(\gamma, f_q) = \mu_0(\gamma)\mathcal{O}_o(f_q) + \mu_1(\gamma)\mathcal{O}_1(f_q),$$

where $\mathcal{O}_o(f_q)$ and $\mathcal{O}_1(f_q)$ are the unipotent orbital integrals, that is, the orbital integrals of f_q over the unipotent conjugacy classes in $\mathrm{GL}(2, \mathbb{Q}_q)$. On the other hand, $\mu_0(\gamma)$ and $\mu_1(\gamma)$ are the germs associated to the trivial unipotent class and to the regular unipotent class, respectively. For $\mathrm{GL}(2, K)$, where K is a local field, the germs are known (see [13]). The hope is to be able to apply a similar procedure to the one carried out in step three of the five stages of the process described in Chapter One. That is, to be able to merge what comes from the approximate functional equation with what comes from the Shalika germ expansions and the expansions in the archimidean places into a Schwartz Function evaluated at integer points.

After this, the completion stage seems to go along the same lines as the one where the set S consists only of the archimidean places since what is being completed is the character of a global extension via local methods. Those extensions might be constructed locally in the same way as before. For example, for quadratic extensions of $\mathbb{Q}$ we would still use the Kronecker symbol.

Once this is done the Poisson summation has to be performed and the question of which will be the relevant lattice remains to be answered. It might be desirable to not complete to the whole set of integers but only to a lattice that includes integers divisible only by primes not in S.

For the evaluation of Kloosterman sums, since the process is local, one expects the same results would carry over. One detail that deserves attention is that in the last step we explicitly use that the Kloosterman sums deliver an expression over zeta functions, while in the case for general S it is not clear how would we get this, since only the Euler Factors from $q \notin S$ seem to appear. At the moment of writing, how this will resolve is not clear to the author, but nevertheless appears to be a question that can be readily written down as a question on the analytic behaviour of incomplete zeta functions, which (up to a point) is a studied theory.

Of course, the details that we have given can be modified for general number fields. Quite possibly there will be obstacles we are not describing at this time, but we hope we have succeeded in conveying to the reader the general strategy and why it seems achievable.

4.2 Polynomials associated to GL(n) for $n > 2$

The regular elliptic part for $\mathrm{GL}(n, K)$, where K is a global field, is still

$$\sum_{\gamma} \mathrm{vol}(\gamma)\mathcal{O}(\gamma, f).$$

To carry out a similar procedure as the one done for $n = 2$, we will need to find a generalization to the multiplicative formula of Langlands as well as its version as a complex valued function. Based on Chapter Three, an option that we expect will work is

$$\mathcal{O}(s, \gamma) = \prod_{q|S_\gamma} \tilde{J}_{\mathcal{O}_{K_\mathfrak{q}}[\gamma]}(s),$$

where S_γ is defined in the same way as before. The issue for $n > 2$ is that we do not have the polynomials available so we cannot construct a multiplicative formula. To be more precise, we need the previous function to be written as a Dirichlet product so that it leads to a sum to which to apply Poisson Summation.

This in itself is a problem that is worthy of attention, but we need to be careful to phrase the correct question. We have seen in Chapter Three that we can recover the values of the orbital integrals by evaluating at $s = 1$ the corresponding functions $R_n(q^{-s}), U_n(q^{-s}), S_n(q^{-s})$. For higher rank, the orbital integrals are harder to compute and require more cases to be described entirely. This means that we should not expect a lot of explicitness in the polynomials associated to the functions $\tilde{J}_{\mathcal{O}_{K_\mathfrak{q}}[\gamma]}(s)$.

Rather, we might need to study abstractly the zeta functions of orders to develop extra properties, similar to those we are observing when we write

$$R_n(X) = 1 + qX^2 + q^2X^4 + ... + q^nX^{2n},$$
$$U_n(x) = (1 + X)R_{n-1}(X) + q^nX^{2n},$$
$$S_n(X) = (1 - X)R_{n-1}(X) + q^nX^{2n}.$$

In Chapter Three we saw that the appearance of the linear terms $1, 1+X, 1-X$ in these polynomials was what allowed us to give a uniform description of them when we had a global order. We can abstractly ask whether this is true in general, that is: *can we describe all polynomials associated to reduced $K-$algebras of a fixed dimension in a similar way as the quadratic case?* We might need to be flexible with what we mean by "similar" in this question, but I believe it is clear that the above description is far from unstructured.

Just to give an example of an explicit questions that go in the spirit of the above paragraph: *is it true that all the polynomials associated to reduced $K-$algebras of a fixed dimension can be written in terms of the one associated to a totally ramified extension?* Another possible question in this direction: *is it true that all extensions with the same e and f have the same polynomial?* The answer to both of these questions for $n = 2$ is yes.

4.3 Universal construction of sequence of orders.

In our analysis we have considered a sequence of orders for which a particular recurrence relation led to the explicit nature of the polynomials. Motivated by this, we ask the following: *What is the correct definition that should be satisfied by a sequence of orders to be able to carry out a similar analysis like ours successfully?*

The reason to care about the answer of the above is that of generality. After giving careful attention to what occurs in the different cases of GL(2), we can see that the results that lead to the recurrence relations, that is, the existence of representatives for ideals of an order and the existence of the travelling map, rest on very similar ideas yet have to be modified slightly according to case (i.e split vs non split).

It is plausible that there exists a general approach to this, by providing the answer to the previous question, with a set of axioms defining a structure on the reduced $K-$algebra that leads to the recurrence relations as a formal consequence, albeit probably not trivial, from the axioms.

To appreciate how this matters, let's briefly go to the cubic case and list the cases to consider:

Field extensions: When L/K is a cubic field extension of ramification e and inertia f, which satisfy $ef = 3$. This leads to two possible cases, the ramified and unramified case.

Product of a quadratic extension and K: $L = K_1 \times K$, where K_1 is a quadratic field extension and, again, it can be a ramified and unramified one.

Product of three copies of K: When $L = K \times K \times K$.

It would be highly preferable to have a general approach to handle all of these situations uniformly, since as n increases the number of cases also do, and all of them will appear as we vary the characteristic polynomials of regular elliptic classes.

Furthermore, the existence of such polynomials related to zeta functions of orders was also proven by Yun in [17] for function fields, and there might be reasons to be interested in the structure and behaviour of these that can be approached by similar methods as ours, and that might be handled uniformly if a general theory exists.

Related to this is also the following: *What is the correct notion of equivalence between sequences of orders?* This question is related to the issues of choice on our constructions. We used all the time a particular sequence of orders, that we called the *main sequence of orders*, and that was built out of a particular generator Δ of the ring of integers of L over K. We proved that this generator doesn't change the main sequence of orders, and furthermore the depth at which the order generated by γ lies also is independent of it.

For example, once more when we consider GL(3), and higher rank of course, we can copy the construction of the main sequence of orders and create the order generated by γ in the same way. The main sequence of orders will be those of the form

$$O_K[p^n\Delta] = \left\{ x + yp^n\Delta + zp^{2n}\Delta^2 \mid x, y, z \in O_K \right\},$$

but now all the arithmetic that we carried out before is not so immediate.

To put an example of how we get immediately burdened, let us discuss the proof of proposition 8 on page 46. Recall that a representative of an ideal I in the split case is an element $(x, y) \in I$ that has minimal valuation on each entry among all elements of I. What we did was to minimize entrywise and, in case of not producing a representative in these choices, to add them. The ultrametric triangle inequality implied this sum is a representative.

Let us consider this same process for the cubic case in which $L = \mathbb{Q}_2 \times \mathbb{Q}_2 \times \mathbb{Q}_2$. It might happen that the ideal I has elements

$$(1, 1, 2), (1, 2, 1), (2, 1, 1).$$

Each of these minimizes one of the different entries, yet when we add them we get $(4, 4, 4)$ which does not minimize the valuation entrywise. The issue is that the possible repetition of valuations along the entries avoids us from using the equality in the ultametric traingle inequality. We see that we cannot guarantee the existence of representatives and hence the form of the traveling map and recurrence relation, if there is any version of them, is unclear.

Furthermore, if γ is written in this coordinates we will get an expression

$$\gamma = a + b\Delta + c\Delta^2,$$

and *a priori* there seems to be no reason why there should exist an n such that $p^n \mid b$ and $p^{2n} \mid c$ and, simultaneously, $\{1, \gamma, \gamma^2\}$ generates $O_K[p^n\Delta]$. Whether this is true for this or more general cases at this point is not clear to the author, but in case it is not, it hints at the possible existence of other sequences of orders that can be studied and that are different than what we are calling the main one.

Due to this fact, it would be desirable to understand how to define an equivalence between sequences of orders with a finite number of representatives, or maybe better still, find that there is just a finite number of possible sequences where our order $O_K[\gamma]$ can land and, associated to this, to be able to tell explicitly where it lands in the sequence from information associated to γ (we would want S_γ to contain all the information for this task, but maybe we need other additional consructions).

Hidden in the arguments of Chapter Three is the fact that we have enough Galois conjugates, because quadratic extensions are Galois, to obtain the equation

$$(\gamma - \overline{\gamma}) = b(\Delta - \overline{\Delta}).$$

If we do not have enough Galois conjugates, then we will have fewer equations available, and we will not be able to solve for the valuations of the different coordinates of γ in this way. Yet, it was this result what told us that where γ lands in the main sequence of orders is really independent of our generator and only depends on γ. This leads to the question: *How do we relate sequences of orders in one extension to those of another bigger one?*

The immediate case that comes to mind is that of relating embedding an extension that is not Galois into its Galois closure, do the analysis there, and then bring down the results to the original result via some sort of Galois Descent (reminiscent of how computing orbital integrals requires first to compute in a larger building and then descend). Maybe when the extension is Galois, the main sequence of orders is indeed unique, and it is when descending to non Galois extensions that the dependence on the generator becomes apparent, probably because some of the orders in the sequence are not defined in the lower extension.

The previous justifications given to make the above questions stem from the immediate perspectives of our interests in generalizing the work of Altuğ in Beyond Endoscopy, but the study of these possibly existent structures and its properties is interesting by itself.

4.4 Dictionary between orders and buildings

This last direction is a broad one and asks whether new properties of the affine buildings associated to general linear groups can be deduced from the theory of zeta functions for orders. An example in this direction has been mentioned already but let us elaborate now on it. In proposition 29 on page 77 we have proven that if $\mathfrak{q}$ is a prime such that does not S_γ then

$$\mathcal{O}(\gamma, f_{\mathfrak{q}}) = 1.$$

If for a moment we suppose we know nothing about the approach via zeta functions of orders, we can try to prove this statement via building theory. We know that the values of orbital integrals are given as follows:

(a) if $K_{\mathfrak{q}}(\gamma)$ is split, it is given by q^n,

(b) if $K_{\mathfrak{q}}(\gamma)$ is nonsplit and unramified, it is given by

$$q^n \frac{q+1}{q-1} - \frac{2}{q-1},$$

(c) if $K_{\mathfrak{q}}(\gamma)$ is nonsplit and ramified, it is given by

$$\frac{q^{n+1}}{q-1} - \frac{1}{q-1},$$

for a certain integer $n \geq 0$. We know that this integer n is precisely the valuation of S_γ at the prime $\mathfrak{q}$. This can be proven just by studying the building and the lattices, and is implicitly done in the original paper of Langlands. Putting $n = 0$ in all cases we get 1 every time. Like this we have seen that only with building theory constructions we can recover this fact over GL(2) because we have the values of the orbital integrals.

Theorem 5 is true for any regular elliptic element of $\mathrm{GL}(n)$ for any n and so is the definition of S_γ. So, for any n we must still have the above implication, but now we do not have the values of the integrals available, yet the result must be true. We ask for a proof of this result entirely within building theory methods.

This particular problem might be also straightforward from a building theory perspective, the author doesn't know. The point, nevertheless, is not this specific problem but rather the spirit of which it is directing us. We know from analytic number theory (and other areas, of course) that the values of L-functions at certain points translate into general properties of objects that interest us. In here, we are studying zeta functions and this heuristic will probably be true as well and possibly a place to find implications will be in the building.

In the same spirit, we mention a question that was posed to the author when discussing the results of previous chapters:

Question: *What distinguishes the approach via buildings to the one via orders?*

We will conclude this chapter with a discussion related to this question. Our aim will be to explain what is the contribution that orders make to the computation of the polynomials that is not possible for the building approach to make. Let K be a local field and L a quadratic reduced K-algebra, just as in Chapter Two.

We have mentioned elsewhere that buildings have not been used at all in the proof of the Multiplicative Formula of Langlands as well as in the computations of the values of orbital integrals. This is not exactly precise: rather, what we should say is that the contribution of the building is encapsulated in the proof of Theorem 5. Once that theorem is proven, and used as a sort of a black box, we can use different tools to compute the polynomials.

That said, the shadow of the building and the lattices does linger on the whole method since we repeatedly use the structure of free $\mathcal{O}_K-$module of rank two. If we were to *only* use the $\mathcal{O}_K$ module structure we would be able to prove proposition 4, stating that the only $\mathcal{O}_n-$modules

$$\mathcal{O}_n \subseteq M \subseteq \mathcal{O}_0 \tag{4.1}$$

are precisely $\mathcal{O}_0, ..., \mathcal{O}_n$. Basically by using the same procedure, we would prove that the only $\mathcal{O}_K$-modules of rank 2 satisfying (4.1) are precisely $\mathcal{O}_0, ..., \mathcal{O}_n$. The reason why both approaches are the same is because this is done with the correspondence theorem on the same set $\mathcal{O}_n/p\mathcal{O}_n$ and the computation on both cases reduces to counting the ideals of $\mathcal{O}_K/p\mathcal{O}_K$.

The real issue appears when we want to prove the existence of representatives. That is, when we want to show that each ideal of I can be written as $x\mathcal{O}_i$ for some unique $0 \leq i \leq n$ and for some $x \in I$. As we have seen, this is used to organize ideals into principal and nonprincipal ones, which leads to the recurrence relations. Furthermore, it allows concentrating the whole of our study into the principal ideals, which we do by splitting them into high ideals and low ideals and finding their explicit contributions according to type.

In order to see clearly how it is that the building will not help us in this direction, we need first to agree on *what do we mean when we say we "used the building"?* Probably, a good answer to this would be to understand whether we can prove the result, in this case the existence of representatives, via the methods associated to constructing the building.

Recall that the building consists of vertices representing homothety classes of lattices of rank 2 inside $K \times K$. Hence, proving representatives exist would go as follows: we need to show that for any contributing ideal $I \subseteq \mathcal{O}_n$ there exists one of $\mathcal{O}_0, ..., \mathcal{O}_n$ to which I is homothetic. In that case, there exists $k \in K^*$ such that

$$I = k\mathcal{O}_i. \tag{4.2}$$

Since $1 \in \mathcal{O}_i$, we deduce $k \in I$, and $k \in \mathcal{O}_K$. Hence k is a representative in the sense of proposition 9. To achieve this would be a proof that only used the building and it is the first direction the author tried when proving this theorem.

Let us then assume such result has been proved. The theory developed in Chapter Two guarantees that the i of equation (4.2) above is unique. In particular, for the principal ideals we have $i = n$ and we would be saying that all principal ideals have a generator that is in K. Under this assumption, what happens to the type of the high ideals? The type of an ideal is the type of its generators, which in turn is the vector of valuations entrywise. Taking type of the generator k we get, for the nonsplit case,

$$\varepsilon(k) = e\mathrm{val}_p(k),$$

while for the split case we get

$$\varepsilon(k) = (\mathrm{val}_p(k), \mathrm{val}_p(k)).$$

What we learned in Chapter Two is that high ideals contribute in *all* possible ways, while the types that we are finding in the ramified split case and in the split case, respectively, from the above computations is that those ideals generated by elements of K only produce types which are multiples e, or have entries that are equal, as if they were low ideals.

Let us give a picture for $\mathcal{O}_3$ in the split case. We have seen that the types are as follows:

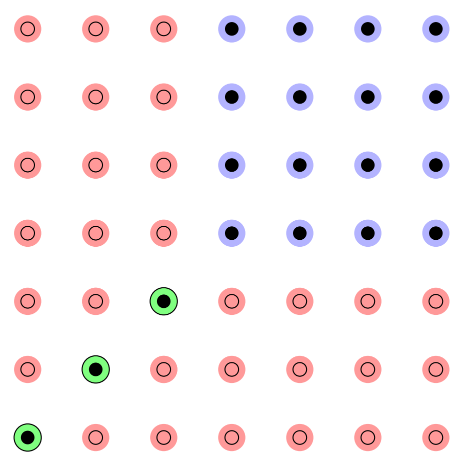

Figure 4.1: There are many high types.

Among all occurring types only those in the diagonal might be represented by an element of K.

There are several high types not in the main diagonal. (Please refer to table 2.1 on page 60, and the related discussion, for the explanation of the above drawing.)

With this, we see that there is no hope of proving such a result just from the building alone. To describe all the possible ideals of $\mathcal{O}_n$ from the perpsective of $\mathcal{O}_K$ modules, we need more than $\mathcal{O}_0, ..., \mathcal{O}_n$. This would destroy our hopes of studying this recursively. We conclude that the zeta functions of orders is really a theory of orders and not one of $\mathcal{O}_K$-modules, despite them being related.

The reason why orders succeed in this stage is because they have a ring structure. This is particularly evident in the split case when we work in $K \times K$. This is not a discrete valuation ring and there is no valuation that we can use to describe properly its ideals, rather we have to use the valuation of K entrywise and only then we can construct the representatives.

The author believes this result, the existence of representatives, is the most characteristic of what distinguishes both approaches, and it is this result that opens the door to the study of the principal part of the zeta function via high and low ideals, and their corresponding actions. I do not see how to prove analogous results or what shape would they take, if we wanted to stay within the building alone.

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