TOPICS ON BASED LOOP GROUPS

BY

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To my parents,
  who have supported me along this journey.

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CONTENTS

INTRODUCTION

A well-studied object in symplectic geometry and mathematical physics is the loop group.

1.1 SUMMARY OF RESULTS

We prove two independent contributions concerning the based loop groups.

First, we compute positive-energy representations of ΩG in the case of $G = SU(2)$ as a limit of equivariant indices of Bott-Samelson manifolds, on which we may employ the Atiyah-Bott fixed point formula. Kumar proved that the character formula for the Bott-Samelson manifolds was the same as the character formula for the Schubert varieties. So we prove a character formula for the Bott-Samelson manifolds, and take a limit as the dimension of the Bott-Samelson manifolds goes to infinity.

As a corollary, we obtain an **effective** character formula for Demazure modules of the affine Lie algebra $\hat{\mathfrak{sl}}_2$ (i.e. a formula containing explicit weights). The character of Demazure modules was previously only known as a composition of Demazure operators. We also obtain an affine analogue of the Kostant multiplicity formula for semisimple Lie algebras.

Our main results are contained in chapters **??** and **??**.

Second, we generalize work done by R. Picken [**?**]. We give a generalization of the Duistermaat-Heckman formula for oscillatory integrals on the based loop group G by using the Wiener measure. Previously work has been done by R. Wendt for the generic coadjoint orbit LG/T, we carry out a similar computation for the degenerate coadjoint orbit $LG/G \cong \Omega G$ where $G = SU(n)$.

1.2 MAIN THEOREMS OF THIS THESIS

The first main theorem of this thesis is **??**, which is proven in Chapter **??**. The theorem computes the K-theoretic equivariant multiplicites of affine

Schubert varieties in the based loop group. Then there is a short section explaining its application to computing the characters of Demazure modules and an affine version of the Kostant multiplicity formula in Proposition **??**. As a corollary, we also obtain equivariant multiplicities for homology.

In Chapter **??**, the main theorem is a Duistermaat-Heckman type formula for the based loop group which relates to orbital integrals on ΩG with respect to the Wiener measure. Our main result is Theorem **??**.

EQUIVARIANT (CO)HOMOLOGY

The study of homology and cohomology of topological spaces has a very long history. It also became important to study the (co)homology of quotient spaces, particularly orbit spaces (such as...). Explicitly, given a compact Lie group G acting on a space X, there is a cohomology theory with the property that

$$H_G^*(X) \cong H^*(X/G)$$

as rings if the G-action is free.

The upshot of this theory is that even when the G-action is not free and the resulting quotient may not be a manifold, the cohomology theory $H_G^*(X)$ is still well-defined. Moreover, just as there is the de Rham model for singular cohomology, there are various models for equivariant cohomology.

2.1 THE HOMOTOPY QUOTIENT

The original construction of equivariant cohomology was due to Borel, and is hence known as the **Borel construction**. Given a group G, there is a classical construction called the classifying space which in a sense classifies all principle G-bundles on a space. The original construction of the classifying space (and the universal bundle) is due to Milnor [**?**], which we outline here.

First we begin with a topological construction.

Definition 2.1 (Milnor Join). Given n topological spaces $A_1, \cdots, A_n$, form the **join** $A_1 \circ \cdots \circ A_n$ is defined as the set of points in the form of: $(a_1, \cdots, a_n; t_1, \cdots, t_n)$ such that $t_i \geq 0, t_1 + \cdots + t_n = 1$ and $a_i \in A_i$ for each i such that $t_i \neq 0$. Each such point will be denoted as $t_1 a_1 \oplus \cdots \oplus t_n a_n$. (Note: If $t_i = 0$ for some i then the corresponding point a_i can be omitted).

For an arbitrary topological group G, let $E_n = G \circ \cdots \circ G$ be the join of $(n+1)$ copies of G with the strong topology. We define the right-translation $R : E_n \times G \to E_n$ by

$$R(t_0 g_0 \oplus \cdots \oplus t_n g_n, g) := t_0(g_0 g) \oplus \cdots \oplus t_n(g_n g).$$

Now let X_n denote the quotient space $E_n / \sim$ where $e \sim e'$ iff $e' = R(e, g)$ for some $g \in G$ (in other words the orbit space of E_n under right-translation. Let $p : E_n \to X_n$ denote the natural map from E_n to X_n.

Definition 2.2. The **classifying space** of the group G, denoted by BG is the base space X_n given above. The **universal bundle** EG is the limit $\bigcup_{n=1}^{\infty} E_n$. The map $p : EG \to BG$ is a principal G-bundle.

By the same construction, it can be shown that EG is a contractible space.

Example 2.3. If $G = S^1$, then $ES^1 = S^\infty$, the infinite-dimensional sphere in the Hilbert space R^∞ and $BS^1 = \mathbb{CP}^\infty$, the infinite projective space.

In general we will not need to know the spaces EG and BG explicitly, but they are necessarily in the definition of equivariant cohomology.

Definition 2.4. The **homotopy quotient** of a G-space X is given by

$$EG \times_G X$$

i.e. it is the space $EG \times X$ with equivalence relation $(p, x) \sim (p \cdot g, g^{-1} \cdot x)$.

Definition 2.5. The G-equivariant cohomology of a G-space X is defined by

$$H_G^*(X) = H^*(EG \times_G X). \tag{2.1}$$

Example 2.6.
- If X is a point, the $H_G^*(pt) = H^*(EG \times_G \{pt\}) = H^*(BG)$.

- If G is a group which acts on itself by left multiplication, then since there is only one orbit, $H_G^*(G) = H^*(pt)$.

2.2 THE CARTAN MODEL

This section follows M. Audin's book "Torus Actions on Symplectic Manifolds" [**?**].
Let G be a compact Lie group and $\mathfrak{g}$ be its Lie algebra. The Borel construction involves infinite-dimensional spaces EG and can be hard to work

with in practice. However, in the case that M and G are smooth manifolds, there is a model for $H_G^*(M)$ in terms of differential forms, just as there is the De Rham model for $H^*(M)$.

Define $\Omega_G(M) := (\Omega(M) \otimes S(\mathfrak{g}^*))^G$ as the G-invariant forms. Here $S(\mathfrak{g}^*)$ is the space of polynomials in $\mathfrak{g}$. If $G = T$ is abelian, then this construction simplifies to $\Omega_T(M) = \Omega^*(M)^T \otimes S(\mathfrak{t}^*) = \Omega^*(M)^T \otimes \mathbb{C}[x_1, \cdots, x_\ell]$. We may think of an element in $\Omega_G(M)$ as a G-equivariant map $\mathfrak{g} \to \Omega(M)$, such that the dependence on $X \in \mathfrak{g}$ is polynomial. When $G = T$ is abelian, we can also view $\Omega_T(M)$ as differential forms on M with coefficients in $S(\mathfrak{t}^*)$.

Example 2.7. On the sphere S^2, with coordinates (z, θ) and the circle acting on it by rotation, $f(X) = dz d\theta \otimes X^2$ is an equivariant form.

There is a $\mathbb{Z}$-grading on $\Omega_G(M)$: for an equivariant form $f = \sum \alpha_i \otimes f_i(X_1, \cdots, X_n)$ where f_i are polynomials in the X_ℓ of degree p and α_i are differential forms of degree i on M, the degree of f is given by $i + 2p$.

Now we define a differential $D : \Omega_G^*(M) \to \Omega_G^*(M)$ by

$$(Df)(X) = d(f(X)) - \iota_{X^\#} f(X)$$

where $X^\#$ is the fundamental vector field of $X \in \mathfrak{g}$ and $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is the interior product on differential forms.

Theorem 2.8. $D \circ D = 0$.

Proof. Since D is $\mathbb{C}$-linear it suffices to check that $D^2(f)(X) = 0$ for an equivariant form $f(X) = \beta \otimes P(X)$ with $\alpha \in \Omega(G), P(X) \in S[\mathfrak{g}^*]$. We have

$$\begin{aligned}
D^2(f)(X) &= D(d(f(X)) - \iota_{X^\#} f(X)) \\
&= d^2(f(X)) - \iota_{X^\#} d(f(X)) - d(\iota_{X^\#} f(X))) + \iota_{X^\#}(\iota_{X^\#} f(X)) \\
&= 0 - \mathcal{L}_{X^\#} f(X) + 0 \\
&= 0 \quad \text{(since by assumption, the form } \alpha \text{ is } G\text{-invariant)}
\end{aligned}$$

$\square$

Then it follows that $\Omega_G^*(M)$ is a chain complex, and we have the following result due to Cartan:

Theorem 2.9 (Cartan, [?]). *The cohomology of this complex is isomorphic to* $H_G^*(M)$.

There is a distinguished equivariant form that is of great interest in symplectic geometry. Let (M, ω) be a symplectic manifold and Φ a moment map. Define the extended symplectic form by $\tilde{\omega}(X) = \omega + \Phi(X)$. A routine computation verifies that $D\tilde{\omega} = 0$. Therefore $[\tilde{\omega}] \in H_G^2(M)$ is a well-defined equivariant class.

2.3 THE PUSHFORWARD MAP

Given a G-equivariant map $f : X \to Y$, there is naturally a pullback map $H_G^*(Y) \to H_G^*(X)$ given by $\alpha(X) \mapsto f^*(\alpha(X))$. However, using the Pontryagin-Thom construction, Atiyah-Bott have constructed a "wrong-way" map, or pushforward $f_* : H_G^*(X) \to H_G^*(Y)$. If $\pi : X \to Y$ is a G-equivariant fibration, then π_* can be considered a more general version of integration over fibres.

Definition 2.10 (GS, susy and equiv de rham). Let $\pi : X \to Y$ be a fibre bundle where the fibres have dimension k. The map $\pi_* : H_G^\ell(X) \to H_G^{\ell-k}(Y)$ is the map uniquely characterized by

$$\int_Y \pi^* \beta \wedge \mu = \int_X \beta \wedge \pi_* \mu$$

Let $\pi : M \to pt$ denote the constant map to a point. Define the integration over a G-manifold M by

$$\int_M \alpha := \pi_*(\alpha)$$

for all equivariant forms in $H_G^*(M)$.

The Thom and Euler classes

For this section we will denote compactly supported **equivariant** forms on X by $\Omega(X)_0$ (we suppress the notation for the G-action).

Definition 2.11. Let Z be a manifold and X be a codimension-k submanifold. Then the Thom class associated with X is the unique cohomology class $\tau(X) \in H_G^k(Z)$ such that

$$\int_Z \alpha \wedge \tau = \int_X \alpha$$

for all $\alpha \in H_G^*(X)$.

Using properties of fibre integration, we see that a form τ is a Thom form if $\pi_* \tau = 1$.

Guillemin and Sternberg prove the existence of the equivariant Thom class by using an equivariant version of the Mathai-Quillen construction.

The **Euler class** of a real orientable vector bundle V is denoted by $e(V)$. We will not give the definition here, but rather list some useful properties (see [**?**]):

- Let E be a complex vector bundle of (complex) rank m, let E' denote the same total space seen as a real vector bundle, then $e(E') = c_m(E)$, the m-th Chern class of E.

- If $E = E_1 \oplus E_2$ then $e(E) = e(E_1)e(E_2)$. Combined with the previous property, if $E = L_1 \oplus \cdots \oplus L_m$ is the direct sum of line bundles, then $e(E) = c_1(L_1) \cdots c_1(L_m)$.

- If E is a complex vector bundle with a T action, and $E = \bigoplus_j L_j$ where the L_j are complex line bundles with T action given by weights $\beta_j : T \to U(1)$, then the equivariant Euler class of E is

$$e^T(E) = \prod_j c_1^T(L_j)$$

  which is represented in the Cartan model by

$$e^T(E)(\xi) = \prod_j (d\theta_j - \beta_j)(\xi).$$

Now suppose X has even codimension in Z, then the following proposition illustrates the relationship between the Thom class and the Euler class.

Proposition 2.12 ([**?**])**.** Let $i : X \to Z$ be the inclusion map, and let $\tau(X)$ be the Thom class of X and $e(N)$ the equivariant Euler class of its normal bundle N in Z. Then $i^*\tau(X) = e(N)$.

Example 2.13. A very useful example of the Euler class arises when considering the normal bundle N_p of a point p inside a manifold M. In this case $N_p = T_pM$. Suppose that M is equipped with a T-action and p is a T-fixed point. Since T fixes p, there is an induced action on T_pM. This is called the **isotropy representation** of T on T_pM. Since T is abelian, the isotropy representation on T_pM decomposes into one-dimensional subspaces L_j, each with weight β_j. Then the Euler class $e(N_p) = e(T_pM)$ is given by $\prod_j \langle -\beta_j, \xi \rangle$ where ξ is an element of $\mathfrak{g}$. We see that in this case, the Euler class is a degree $dim_{\mathbb{C}} M$ homogeneous polynomial in $\mathfrak{g}$.

2.4 THE LOCALIZATION FORMULA

Using the definitions from the previous section, we can compute the fiber integration(or pushforward) of any equivariant form by using local coordinates. However, if the G-action has fixed points, then the data around the fixed point set is sufficient to yield the global integral $\int_M \alpha$. This celebrated theorem was discovered independently by Atiyah-Bott (1983) and Berline-Vergne (1983). Although the result is true for the action of a general compact Lie group G, we only state the version of the theorem for $G = T$ a torus.

Theorem 2.14 (ABBV localization). *Suppose that T acts on the compact manifold M with isolated fixed points. For a closed equivariant form $\alpha \in H_T^*(M)$, the following identity holds in $H_T^*(pt) \cong \mathbb{C}[\mathfrak{t}^*]$:*

$$\int_M \alpha(\xi) = \sum_{p \in M^T} \frac{i_p^* \alpha}{e_p(N_p)}(\xi) \tag{2.2}$$

where N_p denotes the normal bundle around the point p and $\xi \in \mathfrak{g}$.

   Remark: One notable fact about this theorem is that a priori, the RHS of the equation is in the fraction ring of $\mathbb{C}[\mathfrak{t}^*]$. However, part of the power of the localization theorem is that the denominators in the RHS actually cancel out and as a result, the quantity lives in $\mathbb{C}[\mathfrak{t}^*]$.

2.5 DUISTERMAAT-HECKMAN THEOREM

From previous sections we have seen that we can use the moment map $\Phi : M \to \mathfrak{g}^*$ to extend the symplectic form to an equivariant form by setting $\tilde{\omega}_m(\xi) := \omega_m + \langle \Phi(m), \xi \rangle$. This form is closed. Formally speaking we may also consider the "equivariant form" $e^{i\tilde{\omega}} = \sum_{k=0}^{\infty} \frac{(i\tilde{\omega})^k}{k!}$. We have that $e^{i\tilde{\omega}}(\xi) = e^{i\omega}e^{i\Phi(\xi)}$. The first factor is given by $e^{i\omega} = \sum_{k=0}^{\infty} \frac{(i\omega)^k}{k!}$. It is clear that ω^k vanishes for $k > dim M/2$. Therefore the sum is finite and $e^{i\tilde{\omega}}$ is a well-defined equivariant form. Applying the ABBV localization theorem for isolated fixed points, we obtain the following formula

$$\frac{1}{(2\pi)^{dim M/2}} \int_M e^{i\tilde{\omega}}(\xi) = \sum_{p \in M^T} \frac{e^{i\Phi_p}(\xi)}{e_p(N_p)} \tag{2.3}$$

This is the well-known **Duistermaat-Heckman formula**.

Example 2.15. The simplest example is $M = S^2$. We put cylindrical coordinates on it with symplectic form $dzd\theta$ and the counterclockwise rotation action of S^1. Then this action has two fixed points: one at the north pole N and one at the south pole S. The moment map for the Hamiltonian circle action is given by the height map $\phi(z, \theta) = z$. Then we see that $e^{i\tilde{\omega}} = e^{i\omega}e^{i\Phi} = \left(\sum_{k=0}^{\infty} \frac{(i\omega)^k}{k!} \right) e^{i\Phi} = (1 + \omega)e^{i\Phi}$. Now let ξ be a vector in $Lie(S^1) \cong \mathbb{R}$. As a map $\mathfrak{g} \to \Omega(M)$. We have $e^{i\tilde{\omega}}(\xi)_{(z,\theta)} = (1 + i\omega)e^{i\langle z, \xi \rangle} = (1 + i\omega)e^{iz\xi}$.

When we integrate this form over M, only the degree 2 form is considered, so

$$
\begin{aligned}
\int_M e^{i\tilde{\omega}}(\xi) &= \int_M e^{i\phi(\xi)}\omega \\
&= \int_0^{2\pi} \int_{-1}^1 e^{iz\xi}dzd\theta \\
&= \int_0^{2\pi} \left(\frac{e^{i\xi} - e^{-i\xi}}{i\xi} \right) d\theta \\
&= 2\pi \sin(\xi)
\end{aligned}
$$

On the other hand, let us apply the Duistermaat-Heckman formula. First we observe that at the north pole N, the isotropy representation is a (complex) line bundle given by weight 1. At the south pole S the isotropy representation is given by weight -1. Therefore we obtain:

$$
\begin{aligned}
\frac{1}{2\pi} \int_M e^{i\tilde{\omega}}(\xi) &= \frac{e^{i\xi}}{\xi} + \frac{e^{-i\xi}}{-\xi} \\
&= \sin(\xi)
\end{aligned}
$$

$\frac{1}{(2\pi)^{dimM/2}} \int_M e^{i\tilde{\omega}}(\xi)$ is an example of an **oscillatory integral**.

2.6 EQUIVARIANT HOMOLOGY

One may also consider equivariant homology, which is a dual theory to equivariant cohomology. An exposition of the theory can be found in Brion [?]. All we will require is the following localization theorem, which tolds for both equivariant homology and equivariant cohomology. Our results in chapter **??** will involve equivariant homology.

Theorem 2.16 (Brion [?] Lemma 1). *Let $\iota : X^T \to X$ denote the inclusion of the fixed point set into the variety X, then the maps*

$$\iota_* : H_*^T(X^T) \to H_*^T(X), \qquad \iota^* : H_T^*(X) \to H_T^*(X^T)$$

become isomorphisms after inverting finitely many prime ideals.

EQUIVARIANT K-THEORY

The material in this section follows Chriss-Ginzburg [?] and Segal.

Definition 3.1. Given an additive category $\mathcal{C}$, the **Grothendieck group** of $\mathcal{C}$ is given by the free abelian group generated by the isomorphism classes $[A]$ of objects $[A]$ in $\mathcal{C}$, subject to the relations $[A] = [A'] + [A'']$ whenever $A \cong A' \oplus A''$.

Example 3.2. Let *Vect* denote the category of finite dimensional vector spaces over $\mathbb{C}$, then the Grothendieck group of *Vect* is isomorphic to $\mathbb{Z}$, corresponding to the dimensions of the vector spaces.

3.1 EQUIVARIANT SHEAVES

Let $f : X \to Y$ be a morphism of complex algebraic varieties and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules on Y. Then form the **pullback sheaf** $f^*(\mathcal{F})$ by setting $f^*(\mathcal{F})(U) := \mathcal{O}_X(U) \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}$.

For an algebraic G-action on a variety X, we have two maps $G \times X \to X$ given by the action map a and the projection p onto X.

Definition 3.3. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on an algebraic variety X is G-equivariant if the following conditions hold:

1. There is a given isomorphism of sheaves on $G \times X$

$$I : \alpha^* \mathcal{F} \xrightarrow{p^*} \mathcal{F}$$

   (i.e. the isomorphism is part of the data for an G-equivariant sheaf)

2. The pullbacks of the isomorphism I satisfy

$$p_{23}^* I \circ (id_G \times a) I = (m \times id_x)^* I$$

   where $p_{23} : G \times G \times X \to G \times X$ is the projection along the first factor G.

Example 3.4. The structure sheaf $\mathcal{O}_X$ is G-equivariant, with the isomorphism given by $a^* \mathcal{O}_X \equiv \mathcal{O}_{G \times X} \equiv p^* \mathcal{O}_X$.

If X is smooth and $\mathcal{F}$ is a locally-free sheaf, then we can state the definition in terms of G-equivariant vector bundles:

Definition 3.5. A G-equivariant structure on a vector bundle $V \to X$ is the same as giving a G-action over the total space V such that

1. The projection $\pi : V \to X$ commutes with the G-action, in particular any $g \in G$ takes the fibre V_x to $V_{g \cdot x}$.

2. The map given by $g : V_x \to V_{g \cdot x}$ is a linear map of vector spaces.

We now consider the category $Coh_G(X)$ of G-equivariant coherent sheaves on X. This is an abelian category. We will now use the following definition G-equivariant K-theory.

Definition 3.6. The G-equivariant K-theory of X, denoted by $K_G(X)$, is given by the Grothendieck group of $Coh_G(X)$.

Of course when X is smooth we can consider G-equivariant vector bundles instead of G-equivariant sheaves, and if $G = \{1\}$ then we recover ordinary K-theory $K(X)$.

When $X = pt$, $K_G(X) = R(G)$ the representation ring of G.

We will henceforth use square brackets $[\mathcal{F}]$ to denote the K-theory class corresponding to the G-equivariant sheaf $\mathcal{F}$.

3.2 PULLBACK AND PUSHFORWARD MAPS

For a functor $F : \mathcal{C} \to \mathcal{D}$, denote the right-derived functor of f by $R_i F$.

Let $f : X \to Y$ be a proper G-equivariant map. We define a pushforward map $f_* : K_G^*(X) \to K_G^*(Y)$ by setting $f_*[\mathcal{F}] = \sum_i (-1)^i [R^i f_* \mathcal{F}]$ (here f_* on the RHS is the direct image functor on sheaves). This map descends to K-theory by using the long exact sequence of right derived functors. (See Chriss-Ginzburg, p. 248)

We can also define a natural pullback map. Let $f : X \to Y$ be a (not necessarily proper) G-equivariant map. To give the definition in full, we consider separate cases:

1. f is an open embedding. Then $f^* : K_G(Y) \to K_G(X)$ is defined by $\mathcal{F} \mapsto f^* \mathcal{F}$.

2. f is a closed embedding and X, Y are smooth and quasi-projective. It turns out that for any G-equivariant sheaf $\mathcal{F}$ on Y, there exists a finite locally free G-equivariant resolution $F^{\cdot}$ given by

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to \mathcal{F} \to 0$$

.

Then $f^* : K_G(Y) \to K_G(X)$ is defined by

$$\mathcal{F} \mapsto \sum_i (-1)^i H^i(f_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} F^\cdot) = \sum_i (-1)^i Tor_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F})$$

.

3.3 LOCALIZATION

Let $R(T)$ denote the representation ring of T. Define S to be the multiplicative set generated by $\{1 - e^\lambda\}$ for all $\lambda \in M$. For a variety X denote its T-fixed point set by K_T and denote $\iota : X^T \to X$ by its inclusion into X. Then we have the following theorem:

Theorem 3.7 (Localization)**.** *[Theorem 2.1 in [?]] The homomorphism*

$$S^{-1}\iota_* : S^{-1}K_T(X^T) \to S^{-1}K_T(X)$$

is an isomorphism.

Although the theorem above is true for arbitrary fixed point sets, henceforth we will assume that the fixed point set is isolated and that each fixed point is **nondegenerate**. This means that for any point $p \in X^T$, the isotropy weights of T on $T_p X$ are all nonzero.

Equivariant Multiplicities

We follow the work of W. Rossman [?] and D. Anderson [?].

The localization theorem states that we have an isomorphism $K_T(X) \to K_T(X^T)$ after localizing at certain prime ideals. This means that for any class $[V] \in K_T(X_T)$, we can write it as a sum

$$[V] = \sum_{p \in K^T} \varepsilon_p^K [\mathcal{O}_p]$$

The coefficients ε_p^K live in $R(T)$ and are called **equivariant multiplicities**. Similarly we denote the coefficients appearing in the localization formala in equivariant homology by ε_p^H. It is an analogue of the Todd class, which appears in the Grothendieck-Riemann-Roch theorem.

Example 3.8. An important example is when X is smooth, in that case, we have $\varepsilon_p^K = \prod_\alpha \frac{1}{1 - e^\alpha}$ where α ranges over the isotropy weights of $T_p X$. For equivariant homology the result is $\varepsilon_p^K = \prod_\alpha \frac{1}{\alpha}$.

When there is a resolution of singularities $\pi : Y \to X$, then we have that $\pi_* \mathcal{O}_Y = \mathcal{O}_X$ and $\pi_*[Y] = [X]$.

Proposition 3.9 (Anderson, [**?**])**.** If there is a resolution of singularities $\pi : Y \to X$ then we have

$$\varepsilon_p^K = \sum_{q \in Y, \pi(q) = p \in X^T} \frac{1}{\prod\limits_{i=1}^{n} \left(1 - e^{\alpha_n^q}\right)} \quad \text{and} \quad \varepsilon_p^H = \sum_{q \in Y, \pi(q) = p \in X^T} \frac{1}{\prod\limits_{i=1}^{n} \alpha_n^p}$$

where $\alpha_1^p, \cdots, \alpha_n^p$ are the isotropy weights on $T_p Y$.

Therefore we can compute the equivariant multiplicities of points in X by using Atiyah-Bott-Lefschetz on Y. This is the technique we will employ to derive our results in chapter **??**.

INFINITE-DIMENSIONAL PRELIMINARIES

In this section we introduce background on the free and based loop groups.

4.1 LOOP GROUPS

Let G be a compact Lie group. Consider the set $Map(X, G)$ of all smooth maps from a compact smooth manifold X to G with the topology of uniform convergence. There is a natural group structure on $Map(X, G)$ given by pointwise multiplication. In other words for two maps $f, g : X \to G$, $(f \cdot g)(x) := f(x) \cdot g(x)$. Inverses are given by $f^{-1}(x) := f(x)^{-1}$.

We are primarily interested in the case where $X = S^1$, as it is the simplest well-studied example of mapping groups. Denote $Map(S^1, G)$ by LG. This is called the **free loop group** and it is an infinite-dimensional Lie group. The Lie algebra of LG is given by $Map(S^1, \mathfrak{g})$, the set of all smooth maps from S^1 to the Lie algebra of G, which we denote by $L\mathfrak{g}$.

We also study the **based loop group**, which is a subgroup of LG consisting of all smooth loops which are based at the identity $id \in G$, i.e. $f(1) = id$. This group can also be considered as a quotient LG/G of the free loop group.

We will study multiple variants of loop groups in this thesis, among them are

- The continuous loop space $L_{cts}G$ of all **continuous** maps from S^1 to G. This allows us to use the well-known Wiener measure.

- The polynomial loop space $L_{poly}G$ of all smooth loops from S^1 to G with finite Fourier expansions. This allows us to use a filtration of finite-dimensional varieties. This space also has connections to the affine Grassmannian Gr_G.

4.2 HAMILTONIAN GROUP ACTIONS ON ΩG

In this section we follow [?] chapter 8. It turns out that we can think of ΩG has an infinite-dimensional symplectic manifold. Define a 2-form on ΩG by first setting the following bilinear form on $T_e \Omega G$:

$$\omega(X(t), Y(t)) = \frac{1}{2\pi} \int_0^{2\pi} \langle X(t), Y'(t) \rangle \, dt \qquad (4.1)$$

where $X(t), Y(t) \in T_e \Omega G \cong L\mathfrak{g}/\mathfrak{g} = \Omega\mathfrak{g}$, the space of loops in $\mathfrak{g}$ based at 0. For elements in the tangent space around an arbitrary point $T_p \Omega G$ the symplectic form is computed by pulling back the tangent vectors to the identity (via left or right multiplication), i.e. to elements of $\Omega\mathfrak{g}$ (see [?] chapter 8 for more details).

There are two natural Lie groups acting on ΩG: First, the maximal torus $T \subset G$ acts by pointwise conjugation $k \cdot \theta(t) = k\theta(t)k^{-1}$ for $k \in T$; second the circle group S^1 acts by rotating the loops. However, since the loops must be based, we instead have a "based rotation" given by $\theta \cdot \gamma(t) = \gamma(t + \theta)\gamma(\theta)^{-1}$ for $\theta \in S^1$. These two actions commute with each other. (See [?], [?] for more details.)

This $(T \times S^1)$-action is in fact Hamiltonian with respect to ω, and the corresponding moment map is given by $\mu = (E, \rho) : \Omega_G \to Lie(T \times S^1)^*$ with

$$E(\gamma) = \frac{1}{4\pi} \int_0^{2\pi} ||\gamma(t)^{-1}\gamma'(t)||^2 dt,$$

$$\rho(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} pr_{Lie(T)}(\gamma(t)^{-1}\gamma'(t)) dt.$$

E is referred to as the **energy function** and ρ is referred to as the **momentum function**. The moment map and symplectic form restrict to the polynomial loops as well.

It is well-known that ΩG is a Kahler manifold. It has the following complex structure

$$I\left(\sum_{k=1}^{\infty} \xi_k e^{ikt} + \overline{\xi_k} e^{-ikt}\right) = \sum_{k=1}^{\infty} i\xi_k e^{ikt} - i\overline{\xi_k} e^{-ikt} \qquad (4.2)$$

4.3 BOTT AND MITCHELL'S FILTRATIONS

Although ΩG is infinite-dimensional, it is fortunately **tamely** infinite-dimensional, meaning that it can be filtered by a sequence of finite-dimensional spaces $pt = X_0 \subset X_1 \subset \cdots$ where $X_i \subset \Omega G$ and $\bigcup_{i=0}^{\infty} X_i = \Omega G$. Bott used a filtration of Ω to compute its homology groups for the case of $G = SU(2)$:

Theorem 4.1 (Bott). *The integer homology $H_*(\Omega SU(2), \mathbb{Z})$ equipped with the Pontryagin product is isomorphic as algebras to the polynomial algebra $\mathbb{Z}[x_0, x_1, \cdots,]$. Each x_i corresponds to the unique generator in $H_{2i}(\Omega SU(2)) \cong \mathbb{Z}$.*

There are subspaces which represent the classes x_i inside $\Omega SU(2)$, and these subspaces can be taken as a filtration. Mitchell gave a systematic way to find the filtrations in ΩG for all Lie types. In fact, the spaces in Mitchell's filtration correspond to certain Schubert varieties in the based loop group. We use Mitchell's filtration for our results in Chapter **??**.

4.4 AFFINE LIE ALGEBRAS

This section follows Kac [**?**].

Let $\mathcal{L} = \mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials in t. Define the $\mathbb{C}$-bilinear function $\phi : \mathcal{L}^2 \to \mathbb{C}$ given by $\phi(P, Q) = res \frac{dP}{dt} Q$ where res denotes the residue of a Laurent polynomial.
Let $\mathring{\mathfrak{g}}$ be a simple Lie algebra. Consider the **loop algebra**

$$\mathcal{L}(\mathring{\mathfrak{g}}) = \mathcal{L} \otimes_{\mathbb{C}} \mathring{\mathfrak{g}}.$$

with bracket defined by

$$[P \otimes x, Q \otimes y]_0 = PQ \otimes [x, y].$$

Now we extend the loop algebra to obtain a construction of affine Lie algebras. First we extend the nondegenerate bilinear form $(\cdot | \cdot)$ on $\mathring{\mathfrak{g}}$: define the $\mathcal{L}$-valued bilinear form $(\cdot | \cdot)_t$ on $\mathcal{L}(\mathring{\mathfrak{g}})$ by

$$(P \otimes x | Q \otimes y)_t = PQ(x | y).$$

First we define a 2-cocycle ψ on the Lie algebra $\mathcal{L}(\mathring{\mathfrak{g}})$ by

$$\phi(P \otimes x, Q \otimes y) = (x | y)\phi(P, Q).$$

Denote by $\hat{\mathcal{L}}(\mathring{\mathfrak{g}})$ the Lie algebra gotten by: extending $\mathcal{L}(\mathring{\mathfrak{g}})$ by a 1-dimensional center associated to the cocycle ϕ, and adjoining the derivation operator $d = t\frac{d}{dt}$. We construct this Lie algebra by setting $\tilde{\mathcal{L}}(\mathring{\mathfrak{g}}) = \mathcal{L}(\mathring{\mathfrak{g}}) \oplus \mathbb{C}K \oplus \mathbb{C}d$ as a vector space, with bracket given by

$$[t^m \otimes x \oplus \lambda_1 K \oplus \mu_1 d, t^n \otimes y \oplus \lambda_2 K \oplus \mu_2 d]$$
$$= (t^{m+n} \otimes [x,y] + \mu_1 n t^n \otimes y - \mu_2 m t^m \otimes x) \oplus m\delta_{m,-n}(x|y)K. \quad (4.3)$$

Note that K is the central element in $\hat{\mathcal{L}}(\mathring{\mathfrak{g}})$ and d kills K.
For the case where $\mathfrak{g} = \mathfrak{sl}_n$, we denote $\hat{\mathcal{L}}(\mathring{\mathfrak{g}})$ by $\hat{\mathfrak{sl}}_\mathfrak{n}$.

The Affine Weyl Group

In this section we follow the notes of Magyar [?]. First we recall that the Weyl group associated to type A_{N-1} is isomorphic to the symmetric group S_N. Given the finite Weyl group W associated to type A_{N-1}, the affine Weyl group $\overline{W}$ is the semidirect product $W \ltimes Q$, where Q denotes the coroot lattice of $SU(N)$. Another way of expressing this is that each element in $\overline{W}$ is given by the product of a reflection r_α and a translation t^β, where α is a root and $\beta \in Q$.

The generators include the standard generators for the finite Weyl group: $s_i = r_{\alpha_i}$ for $i = 1, \cdots, N-1$. There is an extra generator in the affine Weyl group given by $s_0 = r_\theta t^{-\theta^\vee}$ where θ is the highest root. For these generators, the multiplication is given by

$$\left(r_{\alpha_{i_1}} t^{\beta_{i_1}}\right)\left(r_{\alpha_{i_2}} t^{\beta_{i_2}}\right) = r_{\alpha_{i_1}} r_{\alpha_{i_2}} t^{\beta_{i_1} + r_{\alpha_{i_1}}(\beta_{i_2})}.$$

Example 4.2. The group $SU(2)$ has a coroot lattice isomorphic to $\mathbb{Z}$ and its Weyl group is the group $\mathbb{Z}_2$ with generator r, so the Weyl group for affine $\mathfrak{sl}_2$ is the semidirect product $\mathbb{Z}_2 \ltimes \mathbb{Z}$.

Root Systems

In analogy with finite-dimensional Lie algebras, affine Lie algebras have their own root systems. Let $\Pi = \{\alpha_1, \cdots, \alpha_n\}, \Pi^\vee = \{\alpha_1^\vee, \cdots^\vee, \alpha_n^\vee\}$ be two subsets of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively, satisfying the following conditions:

1. Π and $\Pi^\vee$ are linearly independent in $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively

2. $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$

3. $n - \ell$ and $dim\ \mathfrak{h} - n$

The elements α_i and $\alpha_i^\vee$ are called **simple roots** (denoted by $\triangle^{re}$) and simple coroots (denoted by $\triangle^{im}$) respectively. A root α is called **real** if there is a $w \in \overline{W}$ such that $w(\alpha)$ is a simple root. A root which is not real is called **imaginary**.

For the affine Lie algebras $\hat{\mathfrak{sl}}_n$ we have an explicit description of both types of roots ([?], Proposition 6.13). Let $\mathring{\triangle}$ be the root system of $\mathfrak{sl}_n$ (the finite-dimensional Lie algebra), then we have:

$$\triangle^{re} = \{\alpha + n\delta | \alpha \in \mathring{\triangle}, n \in \mathbb{Z}\}$$

$$\triangle^{im} = \{n\delta, n \in \mathbb{Z} \setminus \{0\}\}$$

where $\delta = \sum_{n=0}^{\ell} \alpha_i$.

In contrast to finite-dimensional Lie algebras, affine Lie algebras have infinitely many roots.

4.5 THE WEYL-KAC CHARACTER FORMULA

Theorem 4.3 (Pressley-Segal p. 280, Theorem 14.3.1). *Let λ be an (anti)-dominant weight then the character of V_λ is given by*

$$ch\, V_\lambda = \prod_{\alpha>0}(1 - e^{i\alpha})^{-1} \sum_{w \in \overline{W}} (-1)^{\ell(w)} e^{i(w \cdot \lambda + s(w))} \tag{4.4}$$

where $s(w)$ denotes the sum of all positive roots α of LG for which $w^{-1}\alpha$ is negative. The dot action of the affine Weyl group is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$. where ρ is half the sum of positive roots of G.

WIENER MEASURES AND HEAT KERNELS

The heat kernel is a solution to the famous heat equation $\frac{\partial}{\partial t} = \triangle_x$. For $\mathbb{R}^n$ the heat kernel has been worked out to be $e^{\frac{1}{4}t\|x\|^2}$. The construction is similar for a Riemannian manifold M.

Throughout this section we follow the paper of H. Urakawa [?].

Definition 5.1. Let (M, g) be a Riemannian manifold, and $\triangle$ the Laplace-Beltrami operator on M. Then the *fundamental solution* to the heat equation is a solution $v_t(m, m')$ to the equation

$$\triangle_x u(t, x) = \frac{\partial}{\partial t} u(t, x) \tag{5.1}$$

such that

1. $v_t(m, m') \in C^\infty((0, \infty) \times M \times M)$ is a positive function

2. $\frac{\partial v}{\partial t} = \triangle_m v = \triangle'_m v$

3. $\lim\limits_{t \to 0^+} \int_M v_t(m, m') f(m') dVol(m') = f(m)$ for any function $f \in C^\infty(M)$
   and $dVol$ is the Riemannian volume on M.

Also define the trace of $v_t(m, m')$ as $v(t) := \int_M v_t(m, m) dVol(m)$.

For a compact Lie group G, we get a Riemannian metric from the Killing form on its Lie algebra $\mathfrak{g}$. For a dominant integral weight $\lambda \in D$, denote χ_λ as the character of the unique up to isomorphism irreducible representation V_λ of G with highest-weight λ. Denote $d(\lambda)$ as the degree of V_λ. We also denote the maximal torus of G as T and their Lie algebras by $\mathfrak{g}, \mathfrak{t}$ respectively.

Theorem 5.2 (Urakawa, Theorem 1)**.** *Let G be a compact simply-connected Lie group. Let $V(t, g) := \sum\limits_{\lambda \in D} d(\lambda) e^{-\langle \lambda, \lambda + 2\rho \rangle t} \chi_\lambda(g)$, $t > 0, g \in G$, where ρ is half*

the sum of the positive roots. Then $V(t, g)$ is the fundamental solution to the heat equation on G. In other words then $v_t(x, y) := V(t, xy^{-1})$ is a solution to the heat equation on G.

It will be convenient for us to write the character $\chi_\lambda(x)$ in another form. Let

$$j(x) = \sum_{w \in W} (-1)^w e^{2\pi i (w(\rho), x)}$$

. Then

$$\chi_\lambda(x) = \frac{\sum_{w \in W} (-1)^w e^{2\pi i (w(\lambda + \rho), x)}}{j(x)}$$

.

We also have an explicit formula for the degree. Define the function

$$d(\lambda) = \frac{\prod_{\alpha > 0}(\lambda + \rho, \alpha)}{\prod_{\alpha > 0}(\rho, \alpha)},$$ then the degree $dim V_\lambda$ is equal to $d(\lambda + \rho)$. For dominant weights $\lambda, d(\lambda) = dim V_\lambda$, but the function $d(\cdot)$ also extends to other weights in t^*.

By using the denominator $j(x)$ we can write the heat kernel as follows:

Theorem 5.3 (Fegan, [?]).

$$V(t, x) = \frac{e^{-2\pi i |\rho| 2t} \left(\sum_{\gamma \in P} d(\gamma) e^{2\pi i |\gamma|^2 t + \langle \gamma, x \rangle} \right)}{j(x)} \tag{5.2}$$

where $\lambda \in t^*$ is the dual element identified with $H \in t$ via the Killing form and $h = \exp(H)$.

Let Q denote the coroot lattice of G and define $\pi(\lambda) = \prod_{\alpha \in \Phi^+} (\lambda, \alpha)$. Then we have the following **inversion formula**, which is a consequence of the Poisson summation formula applied to the dual lattices P and Q.

Theorem 5.4 (Fegan, [?]).

$$V(-1/t, x) = \frac{e^{\frac{2\pi i ||\rho||^2}{t}}}{j(x)} \left(\frac{t}{i} \right)^{dim G/2} \frac{i^{-n}}{vol P} \sum_{\lambda \in Q} d\left(\lambda - \frac{1}{2} x \right) e^{2\pi i ||\lambda - \frac{1}{2} x||^2 t} \tag{5.3}$$

where $Q = P^*$ denotes the coweight lattice and volP denotes the volume of a fundamental cell of the lattice P.

The case of $G = SU(2)$

We now compute the formula in Theorem (**??**) for the group $G = SU(2)$. Recall that irreducible representations of $SU(2)$ are indexed by integers, and we denote the representation with the highest weight $\lambda \in \mathbb{Z}$ by V_λ. Its degree is $d_{\lambda+\rho} = \lambda + 1$. Moreover since every dominant weight is a multiple of the highest root α, we have $(\lambda + \alpha, \lambda) = \frac{\lambda^2}{2} + \lambda$ where $(\cdot, \cdot)$ is the Killing form on $\mathfrak{g}$ and $\alpha = 2$.

Finally since every element of $SU(2)$ is conjugate to an element in the maximal torus T, it suffices to compute the character χ_λ on T. There the formula is $\chi_\lambda(\theta) = \dfrac{\sin((\lambda+1)\theta)}{\sin(\theta)} = e^{-\lambda\theta} + e^{-(\lambda-2)\theta} + \cdots + e^{(\lambda-2)\theta} + e^{\lambda\theta}$.

5.1 THE WIENER MEASURE

In this section we follow I. Frenkel [**?**].

Let $C_G(T)$ denote the space of continuous paths on the compact group G, where paths are parametrized by the interval $[0, T]$. Let $C_G(Z; T)$ denote the subspace of all continuous paths with the fixed endpoint $f(T) = Z$.

A cylinder set is a subset of $C_G(T)$ of the form

$$\{f \in C_G(T) : f(T_1) \in A_1, \cdots, f(T_n) \in A_n; 0 \le T_1 < \cdots < T_n \le T\} \quad (5.4)$$

where A_i are Borel sets of G.

Definition 5.5. Let dz be the Haar measure on G, $\triangle z_k = z_k z_{k-1}^{-1}, z_0 = id, \triangle T_k = T_k - T_{k-1}, T_0 = 0$, then we have the following measures on the continuous loop spaces

1. The **Wiener measure** of variance $t > 0$ is defined on the cylinder sets of $C_G(T)$ by

$$w_G^t(f(T_1) \in A_1, \cdots, f(T_n) \in A_n)$$
$$:= \int_{A_1} \cdots \int_{A_n} v_t(\triangle z_1, \triangle T_1) \cdot \ldots \cdot v_t(\triangle z_n, \triangle T_n) \, dz_1 \cdots dz_n \quad (5.5)$$

2. The **conditional Wiener measure** of variance $t > 0$ s defined on the cylinder sets of $C_G(Z; T)$ by

$$w_{G,Z}^t(f(T_1) \in A_1, \cdots, f(T_{n-1}) \in A_{n-1})$$
$$:= \int_{A_1} \cdots \int_{A_{n-1}} v_t(\triangle z_1, \triangle T_1) \cdot \ldots \cdot v_t(\triangle z_n, \triangle T_n) \, dz_1 \cdots dz_{n-1}$$

$$(5.6)$$

in this case $z_n = Z$ and $T_n = T$.

Unlike the Lebesgue measure on Euclidean space, the Wiener measure is not translation-invariant. However, there is a useful transformation formula under the (right) translation of a group element $g \in G$.

Theorem 5.6 (Frenkel, [?], p. 333). *Let $K, Y \in C(\mathfrak{g}), g \in G$ be elements such that $K = gYg^{-1} - g'g^{-1}$ and $g(0) = id$, then one has*

$$e^{-\frac{||Y||^2}{2t}} \int_{C_G(Z;T)} e^{\frac{1}{t}(z^{-1}z',Y)} dw^t_{G,Z}(z) = e^{-\frac{||K||^2}{2t}} \int_{C_G(Z;T)} e^{\frac{1}{t}(z^{-1}z',K)} dw^t_{G,Z}(z)$$

$$= v_t(Za_0^{-1}, T)$$

This formula will be used in chapter **??** to derive an identity involving a Duistermaat-Heckman type formula on the based loop group.

AN ATIYAH-BOTT TYPE FORMULA FOR THE BASED LOOP GROUP

It is known that $\Omega_{poly}G$ is homotopy equivalent to the affine Grassmannian of $G_{\mathbb{C}}$ defined by $Gr = G_{\mathbb{C}}((t))/G_{\mathbb{C}}[[t]]$, where $G_{\mathbb{C}}((t))$ and $G_{\mathbb{C}}[[t]]$ denote Laurent series with values in $G_{\mathbb{C}}$ and power series with values in $G_{\mathbb{C}}$ respectively.

6.1 BOTT-SAMELSON MANIFOLDS

First we state the important Bruhat decomposition. Let $\mathcal{B}$ denote the Iwahori subgroup $\{f \in G_{\mathbb{C}}[[t]] \mid f(0) \in B\}$ where B is a th e standard Borel subgroup of G. For $G = SU(N)$, B is given by the group of upper triangular matrices. For general G we can embed G into $SU(N)$ for some N and then take upper triangular matrices intersected with the image of G. Let $\ell = rank\, G + 1$, then we have the following definition.

Definition 6.1 (Kumar, [**?**] Definition 6.1.18, p. 186)**.** For any subset $Y \subset \{1, \cdots, \ell\}$ define the **standard parabolic subgroup** $\mathcal{P}_Y$ as $\mathcal{P}_Y = \mathcal{B}W_Y\mathcal{B}$ where W_Y is the subgroup of the affine Weyl group generated by $\{s_i\}_{i \in Y}$.

We consider the case where $Y = \{1, \cdots, \ell\}$ and we denote the corresponding subgroup $\mathcal{P}_Y$ simply by $\mathcal{P}$. Of course, in this case, the group W_Y is simply the finite Weyl group W.

Theorem 6.2 (Bruhat Decomposition)**.** $Gr = \sqcup_{w \in \overline{W}}\mathcal{B}w\mathcal{P}/\mathcal{P}$ *as sets, and each component* $\mathcal{B}w\mathcal{P}/\mathcal{P}$ *is isomorphic to an affine space of dimension* $\ell(w)$.

For any element x in a poset S, one may consider its lower order ideal defined by $I_x = \{y \in S \mid y \leq x\}$. The affine Weyl group is a poset with order given by the Bruhat order. Therefore for any Weyl group element $w \in \overline{W}$, we can take its lower order ideal $I_w := \{v \in \overline{W} \mid v \leq w\}$. Then the disjoint union $\sqcup_{v \in I_w}\mathcal{B}v\mathcal{P}/\mathcal{P}$ is known as the Schubert variety associated to w. It is denoted by X_w.
In general X_w may have singularities. In fact in the case of the finite-dimensional flag manifold $G_{\mathbb{C}}/B$, X_w is smooth if and only if w, seen as an

element of S_n, avoids (1324) and (2143) (see [?] for example). Therefore in order to use Atiyah and Bott's formula, we must use the desingularization given by Bott-Samelson manifolds.

In this section we will use Magyar's topological description of Bott-Samelson manifolds (as opposed to the description using lattices in Gr). This approach allows us to use well-known coordinates on the Bott-Samelson manifolds to compute fixed point data.

A reduced word is one without redundant pairs (i.e. expressions like xx^{-1}). For each reduced word w, denote by M_w the Bott-Samelson manifold associated to w. It is a desingularization of X_w.

Magyar's Topological Description

For this section it suffices to give a description in type A. Let $\alpha = e_i - e_j$ be a root and k be an integer. Define $K_{(e_i - e_j, k)}$ as the subgroup of LG generated by the maximal torus $T \subset G$ and matrices with a copy of $SU(2)$ in the (ij)-block, in the form

$$\begin{pmatrix} a & -\bar{b}t^k \\ bt^{-k} & \bar{a} \end{pmatrix}.$$

In type A, the root $e_i - e_{i+1}$ corresponds to the simple root α_i for $i = 1, \cdots, n-1$.

By abuse of notation, let $\alpha_0 = (-\sum_i^{n-1} \alpha_i, 1)$ and $\alpha_i = (\alpha_i, 0)$ for $i \geq 1$ denote the affine simple root and the finite simple roots respectively. We now denote by $K_{(i)}$ the group K_{α_i}. These groups K_{α_i} are subgroups of the free loop group LK.

Definition 6.3. Let $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ be a reduced word. The Bott-Samelson manifold associated to w is given by the group

$$M_w := \left(K_{(i_1)} \times K_{(i_2)} \times \cdots \times K_{(i_n)} \right) / T^n$$

where $K_{(i_1)} \times K_{(i_2)} \times \cdots \times K_{(i_n)}$ has the following action of T^n:

$$(x_1, \cdots, x_n) \cdot (t_1, \cdots, t_n) = (x_1 t_1, t_1^{-1} x_2 t_2, \cdots, t_{n-1}^{-1} x_n t_n)$$

We also give a topological description for the affine Schubert varieties living in ΩG:

Definition 6.4. Let $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ be a reduced word. Consider the group $K_w := K_{(i_1)} \cdot K_{(i_2)} \cdot \ldots \cdot K_{(i_n)}$ which lives in LG. Then the Schubert variety

is defined to be the image of K_w under the basepoint map $b : LG \to \Omega G$ given by $f(t) \mapsto f(t) \cdot f(1)^{-1}$.

Coordinates on M_w

For a Bott-Samelson manifold M_w there is a birational multiplication map $\eta : M_w \to X_w$ to the corresponding Schubert variety. This map is given by

$$\eta(x_1(t), \cdots, x_n(t)) = x_1(t) \cdot \ldots \cdot x_n(t) \, (x_1(1) \cdot \ldots \cdot x_n(1))^{-1}$$

The Schubert variety embeds into the based loop group and it inherits the $(T \times S^1)$-action, where T acts by conjugation and S^1 acts by rotation. We want an action on M_w that is equivariant with respect to η. The action that does the job is

$$t \cdot (k_1, \cdots, k_n) = (tk_1, k_2, \cdots, k_{n-1}, k_n), \tag{6.1}$$

$$\theta \cdot (k_1, \cdots, k_n) = (\theta \cdot k_1, \theta \cdot k_2, \cdots, \theta \cdot k_{n-1}, \theta \cdot k_n). \tag{6.2}$$

We observe that M_w is defined by the equivalence relation

$$(k_1, \cdots, k_n) = (k_1 t_1, t_1^{-1} k_2 t_2, \cdots, t_{n-1}^{-1} k_n t_n) \tag{6.3}$$

Hence the fixed points are $(k_1, \cdots, k_n)$ such that $k_j = \dot{s}_{i_j}$ or 1. We define $\dot{w}$ as follows: If $w = s_i$ for some $i \in \{1, \cdots, n-1\}$, then let $\dot{w}$ denote a lift of an element of the Weyl group $N_G(T)/T$ to $N_G(T)$. An explicit formula for a canonical lift of s_α is given by

$$\exp(F_\alpha) \exp(-E_\alpha) \exp(F_\alpha),$$

where $E_\alpha, F_\alpha, H_\alpha$ is the $\mathfrak{sl}_2$-triple associated to the root α. Therefore we have a total of 2^n isolated fixed points.

Let G be a compact Lie group; for any simple root $\alpha \in \mathfrak{g}^*$ there is an $SU(2)$-subgroup associated to α. Let Ψ_α denote the embedding of that $SU(2)$-subgroup into G. Then we have the following description of the affine coordinates on the Bott-Samelson manifold (see [?]):

Proposition 6.5. There are affine charts around each fixed point $k = (k_1, \cdots, k_n) \in M_w$ given by

$$\Phi_{(k_1, \cdots, k_n)} : (z_1, \cdots, z_n) \mapsto (\Psi_{\alpha_{i_1}}(m_{z_1}) \cdot k_1, \cdots, \Psi_{\alpha_{i_n}}(m_{z_n}) \cdot k_n) \tag{6.4}$$

where

$$m_{z_i} := \frac{1}{\sqrt{1 + |z_i|^2}} \begin{pmatrix} 1 & -\bar{z}_i \\ z_i & 1 \end{pmatrix}$$

if $k_i = e$ and

$$m_{z_i} := \frac{i}{\sqrt{1 + |z_i|^2}} \begin{pmatrix} z_i & 1 \\ 1 & -\bar{z}_i \end{pmatrix}$$

if $k_i = \dot{s}_i$.

Even though these coordinates are not holomorphic, we can still treat M_w as a real manifold of dimension $2n$ and use the real coordinates x_i, y_i for each z_i.

Each Bott Samelson manifold M_w has $2^{\ell(w)}$ fixed points, indexed by what are called **galleries**. We now give the following definition due to Härterich [?]:

Definition 6.6. Let $w = s_{i_1} \cdots s_{i_n}$ be a reduced word. A **gallery** associated to w is an element in the set $\{0,1\}^n$. We denote a gallery γ by an n-tuple $(\gamma_1, \cdots, \gamma_n)$. We define the realization of a gallery γ by the word $s_{i_1}^{\gamma_1} \cdots s_{i_n}^{\gamma_n}$; it is denoted by $u(\gamma)$.

Example 6.7. Let $w = s_1 s_2 s_1$, then there are $2^3 = 8$ possible galleries associated to w. If we take the gallery $\gamma = (0,1,1)$, then its realization is $u(\gamma) = s_1^0 s_2^1 s_1^1 = s_2 s_1$. Note that two galleries may have the same realization; e.g. $u((0,0,0))$ and $u((1,0,1))$ are both equal to the identity word e.

Proposition 6.8. Let w be a reduced word. The fixed points of M_w under the $(T \times S^1)$-action are in bijection with the set of galleries associated to w. In particular M_w has $2^{\ell(w)}$ fixed points.

Euler class of tangent spaces at fixed points

In the general situation where a torus T acts on a manifold M, suppose that the T-action has isolated fixed points. Then for every $p \in M^T$, there is an induced T-action on the tangent space $T_p M$. Furthermore if M is even dimensional, then $T_p M$ splits into a direct sum of copies of $\bigoplus_{i=1}^{n} \mathbb{C}_i$, and T acts on each $\mathbb{C}_i$ by a weight β_i. This is what is known as the **isotropy representation** of T.

With the coordinates on M_w from the previous section, we can then compute the weights of the isotropy representation of $T \times S^1$ on the fixed points of M_w.

At the fixed point $k = (k_1, \cdots, k_n)$, we notice that the affine chart sends $(z_1, \cdots, z_n)$ to $(\Psi_{\alpha_{i_1}}(m_{z_1}) \cdot k_1, \cdots, \Psi_{\alpha_{i_n}}(m_{z_n}) \cdot k_n)$. By the definition of Bott-Samelson manifolds, we have

$$(\Psi_{\alpha_{i_1}}(m_{z_1})k_1, \Psi_{\alpha_{i_1}}(m_{z_2})k_2, \cdots, \Psi_{\alpha_{i_n}}(m_{z_n}) \cdot k_n)$$
$$= (\Psi_{\alpha_{i_1}}(m_{z_1}), k_1\Psi_{\alpha_{i_1}}(m_{z_2})k_2, \cdots, \Psi_{\alpha_{i_n}}(m_{z_n}) \cdot k_n)$$
$$= (\Psi_{\alpha_{i_1}}(m_{z_1}), k_1\Psi_{\alpha_{i_1}}(m_{z_2})k_1^{-1}, \cdots, (k_1k_2\cdots k_{n-1})\Psi_{\alpha_{i_n}}(m_{z_n})(k_1k_2\cdots k_{n-1})^{-1}k_n)$$

(here we multiplied $k_2^{-1}k_1^{-1}$ to the end of the second component and multiplied k_1k_2 at the beginning of the third component, and so on...)

$$= (\Psi_{\alpha_{i_1}}(m_{z_1}), k_1\Psi_{\alpha_{i_1}}(m_{z_2})k_1^{-1}, \cdots, (k_1k_2\cdots k_{n-1})\Psi_{\alpha_{i_n}}(m_{z_n})(k_1k_2\cdots k_{n-1})^{-1})$$

(in this final step we multiplied k_n^{-1} to the end of the last component).

If one of the k_i is id, then it does not change the $\Psi_{\alpha_{i_n}}(m_{z_n})$ factor. However if $k_i = \dot{s}_i$ then the conjugation $k_i\Psi_{\alpha_{i_n}}(m_{z_n})k_i^{-1}$ will have the effect of the simple reflection s_{α_i}. Therefore $(T \times S^1)$ acts by

$$(t, \theta) \cdot (z_1, \cdots, z_n) = (t^{-k_1(\alpha_{i_1})} \cdots z_1, t^{-k_1k_2(\alpha_{i_2})}, \cdots, t^{-k_1k_2\cdots k_n(\alpha_{i_n})}) \quad (6.5)$$

At the fixed point $k = (k_1, \cdots, k_n)$, we differentiate the affine chart $\Phi_{(k_1, \cdots, k_n)}$ at o. First we compute $\frac{\partial}{\partial z_i}m_{z_i}$ for each z_i. If $k_i = e$ then

$$\frac{\partial}{\partial z_i}m_{z_i} = \frac{\partial}{\partial z_i}\frac{1}{\sqrt{1+|z_i|^2}}\begin{pmatrix} 1 & -\bar{z}_i \\ z_i & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{-\bar{z}_i}{2(1+|z_i|^2)^{\frac{3}{2}}} & \frac{\bar{z}_i^2}{2(1+|z_i|^2)^{\frac{3}{2}}} \\ -\frac{\frac{|z_i|^2}{2\sqrt{1+|z_i|^2}}+1}{(1+|z_i|^2)} & \frac{-\bar{z}_i}{2(1+|z_i|^2)^{\frac{3}{2}}} \end{pmatrix}$$

Evaluating at $z_i = 0$ we get $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Likewise if we take $\frac{\partial}{\partial \bar{z}_i}m_{z_i}|_{z_i=0}m_{z_i} =$ then we get the matrix $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$

Each Ψ_α is an embedding into the $SU(2)$-subgroup of G corresponding to the root α. The matrices computed above get sent via $T\Psi$ to F_α and $-E_\alpha$ respectively.

Now suppose $k_i = \dot{s}_i$ then since s_i is the simple reflection corresponding

to the simple root α_i. This means that the tangent vectors at $(\cdots, \dot{s}_i, \cdots)$ are $(T\Psi_{\alpha_i})_{k_i} \frac{\partial}{\partial z_i} m_{z_i}$ and $(T\Psi_{\alpha_i})_{k_i} \frac{\partial}{\partial \overline{z_i}} m_{z_i}$. By a similar calculation as above, we get that $\frac{\partial}{\partial z_i} m_{z_i}|_{z_i=0} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\frac{\partial}{\partial \overline{z_i}}|_{z_i=0} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$. Then we see that $T\Psi(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \cdot \dot{s}_i$ equals exactly E_{α_i} and $T\Psi(\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}) \cdot \dot{s}_i$ equals exactly $-F_{\alpha_i}$.

Now taking into account the $(k_1 k_2 \cdots k_{n-1})(\ldots)(k_1 k_2 \cdots k_{n-1})^{-1}$ conjugation action on the n-th component, we get the following corollary.

Corollary 6.9. *The isotropy weights of the tangent space of M_w at the fixed point $k = (k_1, \cdots, k_n)$ are given by $\left\{ s_{i_1}^{e_1} \cdot (-\alpha_{i_j}), \ldots, \left(\prod_{k=0}^{n} s_{i_k}^{e_k} \right) \cdot (-\alpha_{i_n}) \right\}$, where $e_j = 1$ if $k_j = \dot{s}_{i_j}$ and $e_j = 0$ if $k_j = \dot{1}$.*
The equivariant Euler class of the tangent space of M_w at the fixed point $k = (k_1, \cdots, k_n)$ is given by $\left(\prod_{j=1}^{n} (\prod_{k=1}^{j} s_{i_k}^{e_k}) \cdot (-\alpha_{i_j}) \right)$.

6.2 GEOMETRIC QUANTIZATION

Prequantum line bundles

A prequantum line bundle over a symplectic manifold (M, ω) is a Hermitian line bundle with connection $(L, \langle , \rangle, \nabla)$, such that the curvature Ω of the connection is cohomologous to ω. Let Φ denote the moment map of a Hamiltonian G-action on M; then for all $X \in \mathfrak{g}$ we define the quantum operator $H_X := \nabla_{X^\#} + i\langle \Phi, X \rangle$, where $X^\#$ is the fundamental vector field associated to X.
The map $X \mapsto H_X$ is a Lie algebra homomorphism, and each H_X acts on the space of sections of L. Therefore we can consider the sections of L as a representation of G (through the exponential map). Note that if $p \in M$ is a fixed point, then $\nabla_{X^\#} = 0$ for all $X \in \mathfrak{g}$. So the character of $g = \exp(X)$ is $\exp(i\langle \Phi, X \rangle)$.
If in addition M is Kahler, then we can use the Kahler polarization on M to get a representation of G on the space of holomorphic sections.

Line bundles on ΩG

Let $\mathcal{L}(\lambda)$ denote the prequantum line bundle on ΩG associated to the dominant weight λ. This bundle can be restricted to Schubert varieties X_w and then pulled back to the Bott-Samelson manifold M_w to get a prequantum line bundle $\eta^* \mathcal{L}(\lambda)|_{X_w}$ on M_w.

Proposition 6.10 (Kumar)**.** The line bundles on ΩG are parametrized by $k\overline{\omega_0}, k \in \mathbb{Z}$ where $\overline{\omega_0}$ is the 0-th fundamental weight of $\hat{\mathfrak{g}}$. Therefore $Pic(\Omega G) \cong \mathbb{Z}$.

6.3 THE ATIYAH-BOTT FIXED POINT THEOREM

In this section we state a special case of the Atiyah-Bott Lefschetz formula for equivariant indices of elliptic operators. Let T be a torus and let M be an almost-complex T-manifold equipped with a T-equivariant vector bundle V. Then the operator which applies to our case is the Dirac-Dolbeault operator $\overline{\partial}_V + \overline{\partial}_V^*$. The index of this operator is given by the alternating sum $\sum_{i=1}(-1)^i tr(H^i(M, V))$ of the traces of the natural T-action on the sheaf cohomology groups $H^i(M, V)$.

Furthermore suppose that $p \in M^T$ is a fixed point; then there is a natural action of T on $T_p M$ (see Section 3.3). This gives rise to a dual T-action on the cotangent space $T_p^* M$. Likewise, since V is T-equivariant, we get a T-action on the fiber V_p as well.

Now we state a very important formula by Atiyah and Bott [**?**]:

Theorem 6.11. *Let T be a torus. Assume that M is a manifold with a Hamiltonian T-action and that the fixed point set M^T is finite. Then for any T-equivariant vector bundle V on M, we get that for all $t \in T$:*

$$\sum_i (-1)^i tr(t|_{H^i(M,V)}) = \sum_{p \in M^T} \frac{tr(t|_{V_p})}{\det(Id - t|_{T_p^* M})}. \tag{6.6}$$

We derive an Atiyah-Bott type formula for the based loop group $\Omega SU(2)$ in the following form:

Theorem 6.12. *For $V = \mathcal{L}(\lambda)$ for a dominant weight λ, we have the following equality:*

$$\sum_i (-1)^i tr((t,\theta)|_{H^i(\Omega SU(2),V)}) = \sum_{w \in \overline{W}} tr((t,\theta)|_{V_w}) \cdot R_w(\Omega SU(2)) \tag{6.7}$$

for all $(t,\theta) \in T \times S^1$.

Remark: We will explicitly compute the functions $R_w(\Omega SU(2))$ in Theorem **??**.

Remark: From this point on we will abuse notation slightly and suppress the torus elements in the computations. I.e., $tr((t,\theta)|_{V_w})$ will simply be denoted as $tr(V_w)$. Equations containing such terms will be understood

to hold for all elements of the torus.

Here $R_w(\Omega SU(2)) : T \times S^1 \to \mathbb{C}$ are rational functions in $T \times S^1$ which live in Kostant and Kumar's nil-Hecke ring (see Section **??**). Since the fixed points of the $(T \times S^1)$-action are indexed by the affine Weyl group elements, this formula is in the exact same form as the Atiyah-Bott formula, except that it contains an infinite sum on the RHS. These functions will be limits of certain K-theoretic equivariant multiplicities we will compute in Lemma **??**.

Proof.

$$\sum_i (-1)^i tr((t,\theta)|_{H^i(\Omega SU(2),\mathcal{L}(\lambda))})$$

$$= tr((t,\theta)|_{H^0(\Omega SU(2),\mathcal{L}(\lambda))})$$

(by Proposition **??**)

$$= tr((t,\theta)|_{\varprojlim_{w \in \overline{W}} H^0(X_w,\mathcal{L}(\lambda)|_{X_w})})$$

(by equation **??**)

$$= \varprojlim_{w \in \overline{W}} \sum_{\text{cosets } vW, v \leq w} e^{i(v \cdot \lambda)} \sum_{v' \in \overline{W}, vW = v'W} \overline{b_{w,v'}}$$

(by Proposition **??**)

$$= \lim_{\ell(w) \to \infty} \sum_{\text{cosets } vW, v \leq w} e^{i(v \cdot \lambda)} \sum_{v' \in \overline{W}, vW = v'W} \overline{b_{w,v'}}$$

(since the limit on words $w \in \overline{W}$ are ordered by length)

$$= \lim_{\ell(w) \to \infty} \sum_{v \in \overline{W}/W, v \leq w} tr\mathcal{L}(\lambda)|_{\mathcal{L}_v} \, \varepsilon_v^K(X_w)$$

(since $e^{i(v \cdot \lambda)} = tr\mathcal{L}(\lambda)|_v$)

$$= tr\mathcal{L}(\lambda)|_{\mathcal{L}_v} \left(\lim_{\ell(w) \to \infty} \sum_{v \in \overline{W}/W, v \leq w} \varepsilon_v^K(X_w) \right)$$

(by Lemma **??**)

$$= tr\mathcal{L}(\lambda)|_{\mathcal{L}_v} \left(\sum_{v \in \overline{W}/W} \lim_{\ell(w) \to \infty} \varepsilon_v^K(X_w) \right)$$

(by taking $\ell(w) \to \infty$ the sum will range over all words $w \in \overline{W}$)

$\square$

We run into a small technical difficulty: the based loop group has a filtration by Schubert varieties, which are in general very singular. Therefore we cannot apply this theorem directly. However, we can apply this result to the Bott-Samelson manifolds, which are desingularizations of Schubert varieties. We are allowed to do so due to the following result:

Proposition 6.13 (Grossberg-Karshon, Kumar, [?], [?]).

$$H^p(M_w, \eta^* \mathcal{L}(\lambda)|_{X_w}) \cong H^p(X_w, \mathcal{L}(\lambda)|_{X_w}).$$

Furthermore, Kumar proved that

Theorem 6.14 ([?] Theorem 8.1.25 p. 271).

$$\varprojlim_{w \in \overline{W}} H^0(M_w, \eta^* \mathcal{L}(\lambda)|_{X_w})^* \cong L_\lambda$$

where L_λ is the irreducible representation of the Kac-Moody group with highest weight λ. Here elements of $\overline{W}$ are ordered by the Bruhat order.

Moreover, Kumar gave results which lead to a vanishing theorem in the higher cohomology groups:

Theorem 6.15 ([?] Corollary 8.3.12 p. 291, Exercise 8.3.E (2) p. 292). *For λ a dominant weight,*

$$H^p(\mathcal{G}/\mathcal{P}, \mathcal{L}(\lambda)) = 0, \text{for all } p \neq 0.$$

where $\mathcal{P}$ is the standard parabolic subgroup defined in the beginning of the chapter.

This implies that $H^0(\Omega SU(2), \mathcal{L}(\lambda))$ is the only cohomology group we need to consider in the LHS of Theorem **??**.

We will also need the following theorem

Theorem 6.16 ([?] Corollary 8.2.2. (3) (d) p. 274). *For any $v \leq w$, the restriction map $H^p(X_w, \mathcal{L}(\lambda)|_{X_w}) \to H^p(X_v, \mathcal{L}(\lambda)|_{X_v})$ is surjective.*

This means that the lim^1 term vanishes and we have that

$$H^p(\Omega SU(2), \mathcal{L}(\lambda)) \cong \varprojlim_{w \in W} H^p(X_w, \mathcal{L}(\lambda)|_{X_w}) \tag{6.8}$$

Therefore it suffices to apply the Atiyah-Bott fixed point theorem to the Bott-Samelson manifolds to obtain the character of the virtual representation $\sum_i (-1)^i H^i(\mathcal{L}(\lambda)|_{X_w})$. Then we will take a limit over words w in the affine Weyl group to obtain $\sum_i (-1)^i H^i(\Omega SU(2), \mathcal{L}(\lambda))$ in the following sections.

Application to the Bott-Samelson Manifolds

To summarize our set-up so far, we now have:

- A smooth presymplectic manifold $(M_w, \eta^*(\omega|_{X_w}))$

- A Hamiltonian $(T \times S^1)$-action on M_w with moment map $\mu \circ \eta : M_w \to Lie(T \times S^1)^*$, and 2^n fixed points parametrized by **galleries** $\{\gamma \in \Gamma_w\}$

- A prequantum line bundle $\eta^* \mathcal{L}(\lambda)|_{X_w}$ on M_w

Applying the Atiyah-Bott formula yields

$$\sum_i (-1)^i tr(H^i(M_w, \eta^* \mathcal{L}(\lambda)|_{X_w}))$$

$$= \sum_{p \in M_w^{T \times S^1}} \frac{tr(\eta^* \mathcal{L}(\lambda)|_{X_w}|_p)}{\det(1 - T_p^* M_w)}$$

$$= \sum_{\gamma \in \Gamma_w} \frac{tr(\eta^* \mathcal{L}(\lambda)|_{X_w}|_\gamma)}{\det(1 - T_\gamma^* M_w)} \quad \text{(the fixed points are galleries associated to } w)$$

$$= \sum_{\gamma \in \Gamma_w} \frac{e^{-i\mu(\gamma)}}{\det(1 - T_\gamma^* M_w)} \quad \text{(by Section \textbf{??})}$$

$$= \sum_{v \leq w} e^{i(v \cdot \lambda)} \sum_{\gamma : u(\gamma) = v} \frac{1}{\det(1 - T_\gamma^* M_w)}$$

$$= \sum_{v \leq w} e^{i(v \cdot \lambda)} \sum_{\gamma : u(\gamma) = v} \frac{1}{\left(1 - e^{s_{i_1}^{\gamma_1} \alpha_{i_1}}\right) \cdots \left(1 - e^{s_{i_1}^{\gamma_1} \cdots s_{i_n}^{\gamma_n} \alpha_{i_n}}\right)}.$$

Here the last equality is due to the fact that $\det(1 - T_\gamma^* M_w) = \prod_\alpha (1 - e^{-\alpha})$, and α ranges over all isotropy weights of $T_\gamma^* M_w$. These were computed in the section on Bott-Samelson manifolds.

Note that by Proposition **??**, the last line is also equal to $\sum_i (-1)^i tr(H^i(X_w, \mathcal{L}(\lambda)|_{X_w}))$.

6.4 THE LOCALIZATION THEOREM IN EQUIVARIANT K-THEORY

The following two subsections are due to Kumar. (See [**?**] Chapters 11 and 12.)

Kostant-Kumar nil-Hecke ring

Let $\mathcal{G}$ be a Kac-Moody group and $\mathcal{T}$ be a maximal torus. Then denote by $A[\mathcal{T}]$ the group algebra of the character group of T. Denote by $Q[\mathcal{T}]$ the fraction field of $A[T]$ and let $\tilde{Q}[\mathcal{T}]$ be the completion of $A[\mathcal{T}]$: the set of formal infinite sums $\sum\limits_{e^\lambda \in X(\mathcal{T})} n_\lambda e^\lambda$ ($n_\lambda \in \mathbb{Z}$) with coefficients in $Q[\mathcal{T}]$.

Then form the $Q[\mathcal{T}]$-vector space $Q[\mathcal{T}]_{\overline{W}}$ with basis $\{\delta_w\}_{w \in \overline{W}}$. The space $Q[\mathcal{T}]_{\overline{W}}$ is an associative ring with multiplication given by

$$\left(\sum_v q_v \delta_v \right) \cdot \left(\sum_w q_w \delta_w \right) = \sum_w q_v(vq_w)\delta_{vw} \tag{6.9}$$

and identity δ_e (see [?], p. 450).

A special element

Let s_i be an element in $\overline{W}$ which is a simple reflection. Then there is a special element $T_{s_i} = \dfrac{1}{1 - e^{\alpha_i}}\delta_{s_i} + \dfrac{1}{1 - e^{-\alpha_i}}\delta_e \in Q[\mathcal{T}]_{\overline{W}}$. For a general word $w \in \overline{W}$, we define $T_w = T_{s_{i_1}} \cdots T_{s_{i_n}}$. Then we can calculate the coefficients of T_w with respect to the $Q[\mathcal{T}]$-basis. Write $T_w = \sum\limits_{v \leq w} b_{w,v}\delta_v$, then we have

$$b_{w,v} = \sum \left(\left(1 - e^{-s_{i_1}^{\varepsilon_1}\alpha_{i_1}}\right) \cdots \left(1 - e^{-s_{i_1}^{\varepsilon_1}\cdots s_{i_n}^{\varepsilon_n}\alpha_{i_n}}\right) \right)^{-1} \tag{6.10}$$

where the sum is over all n-tuples $(\varepsilon_1, \cdots, \varepsilon_n) \in \{0,1\}^n$ satisfying $s_{i_1}^{\varepsilon_1} \cdots s_{i_n}^{\varepsilon_n} = v$.

Example 6.17. Let $\mathcal{G}$ be the Kac-Moody group associated to the affine Lie algebra $\hat{\mathfrak{sl}}_2$. We would like to compute $b_{s_1s_0,s_1}$. The only subword of s_1s_0 equalling s_1 is $s_1^1s_0^0$. So $b_{s_1s_0,s_1} = \dfrac{1}{1 - e^{-s_1\alpha_1}} \cdot \dfrac{1}{1 - e^{-s_1\alpha_0}} = \dfrac{1}{1 - e^{\alpha_1}} \cdot \dfrac{1}{1 - e^{-(\alpha_0 + 2\alpha_1)}}$.

We also define an involution on $Q[\mathcal{T}]_{\overline{W}}$ by $\overline{e^\lambda} := e^{-\lambda}$.

Character of the tangent cone

Let $Y = \mathcal{G}/\mathcal{B}$ denote the affine flag variety. Its fixed points under the T action by left multiplication are parametrized by $v \in \overline{W}$. Kumar proved the following statement about the T-characters of tangent cones in the Schubert varieties X_w around these points.

Theorem 6.18 (Kumar, [?] Theorem 12.1.3, p. 451)**.**

$$ch(gr(\mathcal{O}_{v,X_w})) = \overline{b_{w,v}}. \tag{6.11}$$

However we are working with the case of the affine Grassmannian $Gr = \mathcal{G}/\mathcal{P}$ where $\mathcal{P}$ is the maximal parabolic containing all the finite simple roots. Here the T-fixed points are parametrized by minimal length cosets in $\overline{W}/W$. So we make a slight modification to the formula above.

Recall the application of the Atiyah-Bott fixed point formula to the Bott-Samelson manifold M_w given in section (**??**). The last line contained expressions which may be written in the form of $b_{w,v}$ for certain words w,v. Indeed we may rewrite it as

$$\sum_{v \leq w} e^{i(v \cdot \lambda)} \sum_{\gamma; u(\gamma)=v} \frac{1}{\left(1 - e^{s_{i_1}^{\gamma_1} \alpha_{i_1}}\right) \cdots \left(1 - e^{s_{i_1}^{\gamma_1} \cdots s_{i_n}^{\gamma_n} \alpha_{i_n}}\right)}$$

$$= \sum_{v \leq w} e^{i(v \cdot \lambda)} \cdot \overline{b_{w,v}}$$

$$= \sum_{\text{cosets } vW, v \leq w} e^{i(v \cdot \lambda)} \sum_{v' \in \overline{W}, vW = v'W} \overline{b_{w,v'}}.$$

The last equality follows because the fixed points in the Schubert variety are indexed by cosets vW and galleries whose realizations belong to the same cosets are mapped to the same value under μ.

By virtue of Proposition **??**, we now have a fixed point formula for the Schubert varieties as well:

Proposition 6.19.

$$\sum_i (-1)^i tr(H^i(X_w, \mathcal{L}(\lambda)|_{X_w})) = \sum_{\text{cosets } vW, v \leq w} e^{i(v \cdot \lambda)} \sum_{v' \in \overline{W}, vW = v'W} \overline{b_{w,v'}}.$$

$$\tag{6.12}$$

In the language of Section 4, $\varepsilon_v^K(X_w) = \sum_{v' \in \overline{W}, vW = v'W} \overline{b_{w,v'}}$. In the next section we will derive an effective formula for $\varepsilon_v^K(X_w)$ for the Schubert varieties of $\Omega SU(2)$. Notice that if we take a limit on both sides as $\ell(w) \to \infty$, we get the formula we desired in equation **??**.

6.5 CALCULATIONS FOR G = SU(2)

In this section we present our results regarding the equivariant multiplicities of Schubert varieties.

We can consider the space of polynomial based loops in $SU(2)$ as the affine

Grassmannian $\mathcal{G}/\mathcal{P}$ where $\mathcal{P}$ is the maximal parabolic subgroup corresponding to the set $I = \{1\}$. For the reduced Weyl group $\overline{W}/W_{\{1\}}$ (see Section **??** for definitions), it is totally ordered w.r.t Bruhat order, with an ascending chain $s_0 \leq s_1 s_0 \leq s_0 s_1 s_0 \leq ...$ (see Chapter **??**). Then we denote the length n element of this chain by w_n. In this way, the $(T \times S^1)$-fixed points of $\Omega SU(2)$ are indexed by the integers.

Moreover $\Omega SU(2)$ has a filtration by Schubert varieties, and in this case they correspond exactly to the elements of the ascending chain described above, i.e. $X_{s_0} \subset X_{s_1 s_0} \subset X_{s_0 s_1 s_0} \subset \cdots \subset \Omega SU(2)$. For more details, see [**?**].

Remark: Note that the chain we use above consists of every **second** space in Mitchell's filtration (see [**?**], [**?**]).

For a reduced word $w = s_{i_1} \cdots s_{i_n}$ define $f(w) = \prod_{j=1}^{n} e^{s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})}$. Denote $\varepsilon_{w_n}, X_{w_n}$ by ε_n, X_n respectively. Also define the rational fucnction $R_m(\Omega G) := \lim_{n \to \infty} \varepsilon_m(X_n)$.

Definition 6.20. We define the following restricted partitions:
$P_{n,k}(m) :=$ the number of integer partitions of m into at most n parts, all of which are at most k.
$q_{n,k}(m) :=$ the number of integer partitions of m into exactly n parts, all of which are at most k.

Example 6.21. $P_{3,2}(4) = 2$ since we have $2 + 2$, $2 + 1 + 1$, $q_{3,2}(4) = 1$ since we only have the partition $2 + 1 + 1$.

We also denote the number of all integer partitions of m by $P(m)$.

We take a moment here to give a reminder on the roots of the affine Lie algebra $\hat{\mathfrak{sl}}_2$. In contrast to semisimple Lie algebras, affine Lie algebras have both real and imaginary roots (roots which have nonpositive length). Let $\delta = \alpha_0 + \alpha_1$ denote the smallest positive imaginary root of $\hat{\mathfrak{sl}}_2$. The real roots of $\hat{\mathfrak{sl}}_2$ are given by the set $\triangle_{re} = \{\pm\alpha_1 + n\delta \mid n \in \mathbb{Z}\}$ and the positive real roots form the set $\triangle_{re} = \{\alpha_1\} \cup \{\pm\alpha_1 + n\delta \mid n \in \mathbb{N}\}$. For more details on roots of affine Lie algebras, see [**?**].

Theorem 6.22. *The rational functions in the localization formula for the affine Grassmannian ΩG are given by*

$$R_m(\Omega G) = \frac{(-1)^m f(w_m)(1 - e^{w_m \cdot \alpha_1}) \sum\limits_{n=0}^{\infty} P(n)e^{n\delta}}{\prod\limits_{\alpha \in \triangle_{re}^+} (1 - e^\alpha)} \qquad (6.13)$$

*where $\triangle_{re}^+$ denotes the **positive real** roots of the affine Lie algebra $\hat{\mathfrak{sl}}_2$.*

Theorem 6.23. *(Second version) The rational functions in the localization formula for the affine Grassmannian ΩG are given by*

$$R_m(\Omega G) = \frac{(-1)^m f(w_m)(1 - e^{w_m \cdot \alpha_1})}{\prod\limits_{\alpha \in \triangle^+} (1 - e^\alpha)} \qquad (6.14)$$

*where $\triangle^+$ denotes the **positive** roots of the affine Lie algebra $\hat{\mathfrak{sl}}_2$.*

The sum in the numerator in Theorem **??** is simply the infinite product $\left(\prod\limits_{n=0}^{\infty}(1 - e^{n\delta})\right)^{-1}$, and each factor encodes an imaginary root, complementing the infinite product over all positive real roots in Theorem **??**.

Let us do a few sample computations.

$$\varepsilon_1^K(X_3) = \frac{(-1)S_1(1 - e^{w_1 \cdot \alpha_1})(1 + e^\delta + e^{2\delta})}{(1 - e^{\alpha_1})(1 - e^{\alpha_0})(1 - e^{\alpha_0 + \delta})(1 - e^{\alpha_0 + 2\delta})}$$

$$\varepsilon_2^K(X_4) = \frac{S_2(1 - e^{w_2 \cdot \alpha_1})(1 + e^\delta + e^{2\delta} + e^{3\delta})}{(1 - e^{\alpha_0})(1 - e^{\alpha_1})(1 - e^{\alpha_1 + \delta})(1 - e^{\alpha_1 + 2\delta})(1 - e^{\alpha_1 + 3\delta})}$$

$$\varepsilon_0^K(X_4) = \frac{(1 - e^{\alpha_1})(1 + e^\delta + e^{2\delta})(1 + e^{2\delta})}{(1 - e^{\alpha_0})(1 - e^{\alpha_0 + \delta})(1 - e^{\alpha_1})(1 - e^{\alpha_1 + \delta})}$$

$$\varepsilon_0^K(X_3) = \frac{(1 + e^\delta + e^{2\delta})}{(1 - e^{\alpha_0})(1 - e^{\alpha_0 + \delta})(1 - e^{\alpha_1 + \delta})}$$

Notice that $\varepsilon_0^K(X_4)$ is an instance when the (restricted) partition function can be observed.

To prove the theorem we first prove a finite-dimensional version. Let $\overline{m}$ denote $m(mod 2)$ for any integer m.

Lemma 6.24. *The K-theoretic equivariant multiplicies in the localization formula for the Schubert varieties are given by*

$$\varepsilon_m^K(X_n) = \frac{(-1)^m f(w_m)(1 - e^{w_m \cdot \alpha_1}) \left(\sum_{j=0}^{\left(n - \left\lfloor \frac{n-m}{2} \right\rfloor\right)\left\lfloor \frac{n-m}{2} \right\rfloor} P_{\left\lfloor \frac{n-m}{2} \right\rfloor, n - \left\lfloor \frac{n-m}{2} \right\rfloor}(j) e^{j\delta} \right)}{\left(\prod_{i=0}^{m + \left\lfloor \frac{n-m}{2} \right\rfloor} 1 - e^{\alpha_{\overline{m+1}} + j\delta} \right) \left(\prod_{j=0}^{\left\lceil \frac{n-m}{2} \right\rceil - 1} 1 - e^{\alpha_{\overline{m}} + j\delta} \right)}.$$

Remark: It is possible to write the sum in the numerator with j ranging from 0 to ∞. However, since we are dealing with restricted partitions there is an effective limit ℓ above which $P_{a,b}(N) = 0$ for all $N > \ell$. This limit is given exactly by $\ell = ab$, so we have written that limit for the sum in this lemma.

Before we prove **??** we will need the following identity on restricted partitions:

Proposition 6.25. $P_{a,b}(j) + P_{a+1,b-1}(j - a - 1) = P_{a+1,b}(j).$

Proof. Notice that by definition,

$$P_{a+1,b}(j) = \sum_{n=0}^{a+1} q_{n,b}(j) = \sum_{n=0}^{a} q_{n,b}(j) + q_{a+1,b}(j) = P_{a,b}(j) + q_{a+1,b}(j);$$

therefore it suffices to prove that $P_{a+1,b-1}(j - a - 1) = q_{a+1,b}(j)$. First we establish some bounds on j: If $j > (a+1)b$, then both sides equal 0. Also if $j < a + 1$, then both sides equal 0. So we assume that $a + 1 \leq j \leq (a+1)b$. Then $P_{a+1,b-1}(j - a - 1)$ counts the number of Young diagrams with at most $a + 1$ rows and at most $b - 1$ columns, with $j - a - 1 = j - (a+1)$ total blocks. We call this set of Young diagrams $Y_{a+1,b-1}(j - (a+1))$. On the other hand $q_{a+1,b}(j)$ counts the number of Young diagrams with exactly $a + 1$ rows and at most $b - 1$ columns, with j total blocks, call this set of Young diagrams $y_{a+1,b-1}(j)$. Then we have a map

$$Y_{a+1,b-1}(j - (a+1)) \to y_{a+1,b-1}(j)$$
$$(\lambda_1, \cdots, \lambda_{a+1}) \mapsto (\lambda_1 + 1, \cdots, \lambda_{a+1} + 1)$$

which is in fact a bijection. Therefore, $P_{a+1,b-1}(j - a - 1) = q_{a+1,b}(j)$. □

Proposition 6.26. $P_{a+1,b-1}(j) + P_{a,b}(j - b) = P_{a+1,b}(j)$

Proof. By the correspondence with Young diagrams, we see that $P_{a,b}(j) = P_{b,a}(j)$ for all a, b, j just by conjugation. Then the identity follows from Proposition **??** by setting $a = b - 1, b = a + 1$. □

Finally we need the following elementary identities:

Proposition 6.27. If k, m are integers with $k > m$ and $k - m$ odd, then

- $\left(k - \left\lfloor \frac{k-m}{2} \right\rfloor\right) \left\lfloor \frac{k-m}{2} \right\rfloor + \left(m + \left\lfloor \frac{k-m}{2} \right\rfloor + 1\right) = \left(k - \left\lfloor \frac{k-m}{2} \right\rfloor\right) \left(\left\lfloor \frac{k-m}{2} \right\rfloor + 1\right)$

- $\left(k - \left\lfloor \frac{k-(m-1)}{2} \right\rfloor\right) \left\lfloor \frac{k-(m-1)}{2} \right\rfloor + \left\lceil \frac{k-m}{2} \right\rceil = \left(k - \left\lfloor \frac{k-m}{2} \right\rfloor\right) \left(\left\lfloor \frac{k-m}{2} \right\rfloor + 1\right)$

- $m + \left\lfloor \frac{k-m}{2} \right\rfloor + 1 = k - \left\lfloor \frac{k-m}{2} \right\rfloor$

Proof. An elementary calculation. □

Proof of lemma ??:

The proof of the lemma is by induction on n.

For $n = 1$, $X_{s_0} = M_{s_0} = \begin{pmatrix} a & -\bar{b}t \\ bt^{-1} & \bar{a} \end{pmatrix}$ where $a, b \in \mathbb{C}$. This space is diffeomorphic to $\mathbb{CP}^1$ and the only fixed points of X_1 are the north and south poles whose moment map images are id and s_0 respectively (i.e. the correspond to the Weyl group elements id and s_0). Using formula (**??**) we get that $R_0(X_{s_0}) = R_{id}(X_{s_0}) = \frac{1}{1-e^{\alpha_0}}$ and $R_1(X_{s_0}) = R_{s_0}(X_{s_0}) = \frac{1}{1-e^{-\alpha_0}} = \frac{-e^{\alpha_0}}{1-e^{\alpha_0}}$. So the lemma is true for $n = 1$.

Now assume that the lemma is true for $n = k$, now we wish to compute the rational functions $R_m(X_{k+1})$ for all $m \leq k + 1$. Notice that $w_{k+1} = s_\alpha \cdot w_k$ for some simple root α, and we recall that $R_m(X_{k+1})$ is a sum of the rational functions $\sum_{mW=m'W} \overline{b_{k+1,m'}}$. Notice that if $w = s_i \cdot w'$ then we have the identity $b_{w,v} = (1 - e^{\alpha_i})^{-1} \left(b_{w',v} - e^{-\alpha_i} s_i \cdot b_{w',s_i \cdot v}\right)$, since the subwords of w that equal v are either: **(1)** id concatenated with the subwords of w' that form v, or **(2)** s_i concatenated with the subwords of w' that form $s_i \cdot v$. And because s_i turns α_i into a negative root, we "polarize" the second term with a factor of $-e^{\alpha_i}$. Therefore putting together the last two formulas we get $R_m(X_{k+1}) = (1 - e^{\alpha_i})^{-1} \left(R_m(X_k) - e^{\alpha_i} s_i \cdot R_{s_i \cdot w_m}(X_k)\right)$.

Notice that when $m = 0$ and $k + 1$ is even (i.e. the fixed point is the identity coset and $s_i = s_1$), then s_1 and e belong to the same coset, so in this case $R_0(X_{k+1}) = (1 - e^{\alpha_1})^{-1}(1 - e^{\alpha_1} s_1) R_0(X_k)$, which amounts to applying the Demazure operator D_{s_1} to $R_0(X_k)$.

There are two cases, either, $s_i \cdot w_m = w_{m-1}$ or $s_i \cdot w_m = w_{m+1}$.
Case 1: $s_i \cdot w_m = w_{m-1}$ which implies that $k - m$ is odd.
In this case we have

1. $\left\lfloor \frac{k-m}{2} \right\rfloor + 1 = \left\lceil \frac{k-m}{2} \right\rceil = \left\lfloor \frac{k-(m-1)}{2} \right\rfloor = \left\lceil \frac{k-(m-1)}{2} \right\rceil$

2. $s_i = s_{\alpha_{\overline{m+1}}}$

3. $s_i \cdot (\alpha_k + j\delta) = s_i \cdot \alpha_k + j\delta$

4. $s_i \cdot f(w_{m-1}) = -e^{-\alpha_i} \cdot f(w_m)$ by definition

Now by the induction hypothesis we can compute $R_m(X_{k+1})$ readily as

$$
\begin{aligned}
R_m(X_{k+1}) &= (1 - e^{\alpha_i})^{-1} \left(R_m(X_k) - e^{-\alpha_i} s_i \cdot R_{s_i \cdot w_m}(X_k) \right) \\
&= (1 - e^{\alpha_i})^{-1} \left(R_m(X_k) - e^{-\alpha_i} s_i \cdot R_{m-1}(X_k) \right)
\end{aligned}
$$

Before carrying out the induction step, let us simplify notation by letting

$$
A = \sum_{j=0}^{\left(k - \left\lfloor \frac{k-m}{2} \right\rfloor \right) \left\lfloor \frac{k-m}{2} \right\rfloor} P_{\left\lfloor \frac{k-m}{2} \right\rfloor, k - \left\lfloor \frac{k-m}{2} \right\rfloor}(j) e^{j\delta}
$$

and

$$
B = \sum_{j=0}^{\left(k - \left\lfloor \frac{k-(m-1)}{2} \right\rfloor \right) \left\lfloor \frac{k-(m-1)}{2} \right\rfloor} P_{\left\lfloor \frac{k-(m-1)}{2} \right\rfloor, k - \left\lfloor \frac{k-(m-1)}{2} \right\rfloor}(j) e^{j\delta}.
$$

We now give a detailed computation of $R_m(X_k) - e^{-\alpha_i} s_i \cdot R_{s_i \cdot w_m}(X_k)$:

$$
\begin{aligned}
&R_m(X_k) - e^{-\alpha_i} s_i \cdot R_{m-1}(X_k) \\
&= \frac{(-1)^m f(w_m)(1 - e^{w_m \cdot \alpha_1}) A}{\left(\prod_{j=0}^{m + \left\lfloor \frac{k-m}{2} \right\rfloor} 1 - e^{\alpha_{\overline{m+1}} + j\delta} \right) \left(\prod_{j=0}^{\left\lceil \frac{k-m}{2} \right\rceil - 1} 1 - e^{\alpha_{\overline{m}} + j\delta} \right)} \\
&\quad - e^{\alpha_i} s_i \cdot \frac{(-1)^{m-1} f(w_{m-1})(1 - e^{w_{m-1} \cdot \alpha_1}) B}{\left(\prod_{j=0}^{m + \left\lfloor \frac{k-m}{2} \right\rfloor} 1 - e^{\alpha_{\overline{(m-1)+1}} + j\delta} \right) \left(\prod_{j=0}^{\left\lceil \frac{k-m}{2} \right\rceil - 1} 1 - e^{\alpha_{\overline{m-1}} + j\delta} \right)} \\
&= \frac{(-1)^m f(w_m)(1 - e^{w_m \cdot \alpha_1}) A}{\left(\prod_{j=0}^{m + \left\lfloor \frac{k-m}{2} \right\rfloor} 1 - e^{\alpha_{\overline{m+1}} + j\delta} \right) \left(\prod_{j=0}^{\left\lceil \frac{k-m}{2} \right\rceil - 1} 1 - e^{\alpha_{\overline{m}} + j\delta} \right)} \\
&\quad - e^{\alpha_i} \frac{s_i \cdot (-1)^{m-1} f(w_{m-1})(1 - e^{w_{m-1} \cdot \alpha_1}) B}{\left(\prod_{j=0}^{m + \left\lfloor \frac{k-m}{2} \right\rfloor} 1 - e^{s_i \cdot \alpha_{\overline{m}} + j\delta} \right) \left(\prod_{j=0}^{\left\lceil \frac{k-m}{2} \right\rceil - 1} 1 - e^{s_i \cdot \alpha_{\overline{m+1}} + j\delta} \right)}
\end{aligned}
$$

(by definition, s_i acting on $f(w_{m-1})$ turns it into $f(w_m)$. The reflection s_i also acts on the weights e^{λ} and transforms them by reflecting across the hyperplane associated to the root α_i)

$$= \frac{(-1)^m f(w_m)(1 - e^{w_m \cdot \alpha_1})A}{\left(\prod_{j=0}^{m+\lfloor \frac{k-m}{2} \rfloor} 1 - e^{\alpha_{\overline{m+1}}+j\delta}\right)\left(\prod_{j=0}^{\lceil \frac{k-m}{2} \rceil - 1} 1 - e^{\alpha_{\overline{m}}+j\delta}\right)}$$

$$+ \frac{(-1)^m f(w_m)(1 - e^{w_m \cdot \alpha_1})B}{\left(\prod_{j=1}^{m+\lfloor \frac{k-m}{2} \rfloor+1} 1 - e^{\alpha_{\overline{m+1}}+j\delta}\right)\left(\prod_{j=-1}^{\lceil \frac{k-m}{2} \rceil - 2} 1 - e^{\alpha_{\overline{m}}+j\delta}\right)}$$

(renumbering the indices to start at 1 and -1)

(Now notice that $\alpha_{\overline{m}} - \delta = -\alpha_{\overline{m+1}}$, so)

$$= \frac{(-1)^m f(w_m)(1 - e^{w_m \cdot \alpha_1})A}{\left(\prod_{j=0}^{m+\lfloor \frac{k-m}{2} \rfloor} 1 - e^{\alpha_{\overline{m+1}}+j\delta}\right)\left(\prod_{j=0}^{\lceil \frac{k-m}{2} \rceil - 1} 1 - e^{\alpha_{\overline{m}}+j\delta}\right)}$$

$$+ \frac{(-1)^m f(w_m)(1 - e^{w_m \cdot \alpha_1})B}{\left(\prod_{j=1}^{m+\lfloor \frac{k-m}{2} \rfloor+1} 1 - e^{\alpha_{\overline{m+1}}+j\delta}\right)\left(\prod_{j=0}^{\lceil \frac{k-m}{2} \rceil - 2} 1 - e^{\alpha_{\overline{m}}+j\delta}\right)(1 - e^{-\alpha_{\overline{m+1}}})}$$

$$= \frac{(-1)^m f(w_m)(1 - e^{w_m \cdot \alpha_1})A}{\left(\prod_{j=0}^{m+\lfloor \frac{k-m}{2} \rfloor} 1 - e^{\alpha_{\overline{m+1}}+j\delta}\right)\left(\prod_{j=0}^{\lceil \frac{k-m}{2} \rceil - 1} 1 - e^{\alpha_{\overline{m}}+j\delta}\right)}$$

$$+ \frac{(-1)^m f(w_m)(1 - e^{w_m \cdot \alpha_1})B(-e^{\alpha_{\overline{m}}})}{\left(\prod_{j=0}^{m+\lfloor \frac{k-m}{2} \rfloor+1} 1 - e^{\alpha_{\overline{m+1}}+j\delta}\right)\left(\prod_{j=0}^{\lceil \frac{k-m}{2} \rceil - 2} 1 - e^{\alpha_{\overline{m}}+j\delta}\right)}$$

(in the previous step we polarized the negative weight $1 - e^{-\alpha_{\overline{m+1}}}$ into $(1 - e^{\alpha_{\overline{m+1}}})(-e^{\alpha_{\overline{m+1}}})^{-1}$ and absorbed $1 - e^{\alpha_{\overline{m+1}}}$ into the bottom right sum)

$$= \frac{(-1)^m f(w_m)(1 - e^{w_m \cdot \alpha_1})C}{\left(\prod_{j=0}^{m+\lfloor \frac{k-m}{2} \rfloor+1} 1 - e^{\alpha_{\overline{m+1}}+j\delta}\right)\left(\prod_{j=0}^{\lceil \frac{k-m}{2} \rceil - 1} 1 - e^{\alpha_{\overline{m}}+j\delta}\right)}.$$

where $C = \left((1 - e^{\alpha_{\overline{m+1}} + (m + \lfloor \frac{k-m}{2} \rfloor + 1)\delta}) A + (-e^{\alpha_{\overline{m+1}}})(1 - e^{\alpha_{\overline{m}} + (\lceil \frac{k-m}{2} \rceil - 1)\delta}) B \right)$

We calculate the numerator of the last expression above. We set $a = \lfloor \frac{k-m}{2} \rfloor$ and $b = k - \lfloor \frac{k-m}{2} \rfloor$ for more clarity in the following formulae:

$$\left((1 - e^{\alpha_{\overline{m+1}} + (m + \lfloor \frac{k-m}{2} \rfloor + 1)\delta}) \sum_{j=0}^{(k - \lfloor \frac{k-m}{2} \rfloor) \lfloor \frac{k-m}{2} \rfloor} P_{\lfloor \frac{k-m}{2} \rfloor, k - \lfloor \frac{k-m}{2} \rfloor}(j) e^{j\delta} \right)$$

$$- \left(e^{\alpha_{\overline{m+1}}} (1 - e^{\alpha_{\overline{m}} + (\lceil \frac{k-m}{2} \rceil - 1)\delta}) \sum_{j=0}^{\lfloor \frac{k-(m-1)}{2} \rfloor} P_{\lfloor \frac{k-(m-1)}{2} \rfloor, k - \lfloor \frac{k-(m-1)}{2} \rfloor}(j) e^{j\delta} \right)$$

$$= \left((1 - e^{\alpha_{\overline{m+1}} + (m+a+1)\delta}) \sum_{j=0}^{ab} P_{a,b}(j) e^{j\delta} \right) - \left(e^{\alpha_{\overline{m+1}}} (1 - e^{\alpha_{\overline{m}} + a\delta}) \sum_{j=0}^{a+1} P_{a+1,b-1}(j) e^{j\delta} \right)$$

$$= \sum_{j=0}^{ab} P_{a,b}(j) e^{j\delta} + e^{(a+1)\delta} \sum_{j=0}^{(b-1)(a+1)} P_{a+1,b-1}(j) e^{j\delta} -$$

$$e^{\alpha_{\overline{m+1}}} \left\{ \sum_{j=0}^{(a+1)(b-1)} P_{a+1,b-1}(j) e^{j\delta} + e^{(m+a+1)\delta} \sum_{j=0}^{ab} P_{a,b}(j) e^{j\delta} \right\}$$

(now we re-index the sums and partitions: In the second sum we reindex $j \leftarrow j + b + 1$ and in the fourth sum we reindex $j \leftarrow j + (m + a + 1)$. Then we can change the lower limits in the (reindexed) sums to 0 because the partitions of integers below $a + 1$ and $(m + a + 1)$ will both be 0)

$$= \sum_{j=0}^{ab} P_{a,b}(j) e^{j\delta} + \sum_{j=0}^{b(a+1)} P_{a+1,b-1}(j - a - 1) e^{j\delta}$$

$$- e^{\alpha_{\overline{m+1}}} \left\{ \sum_{j=0}^{(a+1)(b-1)} P_{a+1,b-1}(j) e^{j\delta} + \sum_{j=0}^{ab+m+a+1} P_{a,b}(j - (m + a + 1)) e^{j\delta} \right\}$$

(using Proposition **??** we can rewrite the partitions and upper limits of the sums as)

$$= \sum_{j=0}^{b(a+1)} (P_{a,b}(j) + P_{a+1,b-1}(j - a - 1)) e^{j\delta}$$

$$- e^{\alpha_{\overline{m+1}}} \cdot \left\{ \sum_{j=0}^{b(a+1)} (P_{a+1,b-1}(j) + P_{a,b}(j - b)) e^{j\delta} \right\}.$$

We see that by using Proposition **??** and Corollary **??**, the partitions in both sums become $P_{a+1,b}(j)$. Finally after factoring out the $\left(1 - e^{\alpha_{\overline{m+1}}}\right) = \left(1 - e^{\alpha_i}\right)$ term we arrive at the correct formula for X_{k+1}, thereby proving the induction step.

The other case is similar. □

Theorems **??** and **??** follow from taking the limit as $k \to \infty$ of the expression in Lemma **??**. The restricted partitions become partitions outright. We get the rational functions $R_w(\Omega SU(2)) : T \times S^1 \to \mathbb{C}$ for each fixed point of $\Omega SU(2)$.

Remark: Strictly speaking, we should have taken the complex conjugate of the quantities in Theorems **??** and **??**. This is because the character of the positive energy representation was equal to the inverse limit of the **dual** representations given by the pullback bundles on the Bott-Samelson manifolds (see Theorem **??**). However, we note that this is mostly a matter of convention (using lowest weights vs. highest weights). The standard convention is with highest weights, as in Kumar [**?**]. However, Pressley and Segal use the lowest weight convention in their book *Loop Groups* (see [**?**]).

6.6 DEMAZURE MODULES

A consequence of Lemma **??** is that it allows us to write an effective character formula for Demazure modules associated to the Schubert varieties. Previously, these characters were only given in terms of iterated Demazure operators. In this section we let $\mathfrak{g}$ be a Kac-Moody algebra and $\mathfrak{b}$ a Borel subalgebra.

Definition 6.28. Let λ be a dominant weight, and let $V(\lambda)$ be the irreducible representation with highest-weight λ. For any $w \in \overline{W}$, form $E_w(\lambda) = \mathfrak{b} \cdot v_{w(\lambda)}$; it is a submodule of $V(\lambda)$ called the Demazure submodule of $V(\lambda)$ associated to w.

The Demazure submodule of $V(\lambda)$ associated to w is isomorphic to the restriction of the prequantum line bundle L_λ to the Schubert variety X_w (see Kumar [**?**]). Therefore Lemma **??** allows us to effectively compute the character of $E_w(\lambda)$ for $\lambda = k\overline{\omega_0}$ and for any $w \in \overline{W}$.

Remark: Previously, characters of Demazure modules were computed using the Demazure character formula. This formula allows one to write the character *ch E_w* as iterated applications of certain **Demazure operators**

to the weight e^λ, i.e. $ch\ E_w = D_{s_{i_1}} \cdots D_{s_{i_n}} e^\lambda$ where $w = s_{i_1} \cdots s_{i_n}$ is a decomposition of the reduced word w. These Demazure operators act on the group algebra $A[T]$ (see Section **??**). For a simple reflection s_i, the Demazure operator is defined by

$$D_{s_i}(e^\lambda) = \frac{e^\lambda - e^{s_i \lambda - \alpha_i}}{1 - e^{-\alpha_i}}.$$

6.7 A KOSTANT MULTIPLICITY FUNCTION

Each factor $(1 - e^\alpha)^{-1}$ can be written as $\sum_{n=0}^{\infty} e^{n\alpha}$, so we can compute the multiplicities of the character in the last section as a sum of partition functions, defined below:

Definition 6.29. The partition function of a weight $\mu \in Lie(\mathcal{T})^*$ (see Section 6 for this notation) is defined as $N(\mu) = $# of solutions of the equation $\sum_{\alpha \in \triangle^+} n_\alpha \alpha = \mu$ where each $n_\alpha \in \mathbb{N} \cup \{0\}$.

Also define $N_\beta(\mu) = $# of solutions of the equation $\sum_{\alpha \in \triangle^+ \setminus \{\beta\}} n_\alpha \alpha = \mu$ where each $n_\alpha \in \mathbb{N} \cup \{0\}$.

For a reduced word $w = s_{i_1} \cdots s_{i_n}$ we also define $g(w) = \sum_{j=1}^{n} s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$.

Then by reading off the equivariant multiplicities derived in the last section, we obtain the following Kostant multiplicity formula for $\Omega SU(2)$:

Proposition 6.30. Let $\lambda = k\overline{w}_0, k \in \mathbb{Z}_{>0}$, then the multiplicity of the weight α in the irreducible representation L_λ is given by

$$m(\alpha, L_\lambda) = \sum_{m=0}^{\infty} (-1)^m \left\{ N(\alpha - (w_m \cdot \lambda + g(w_m))) - N(\alpha - (w_m \cdot (\lambda + \alpha_1) + g(w_m))) \right\}$$

$$= \sum_{m=0}^{\infty} (-1)^m N_{w_m \cdot \alpha_1}(\alpha - (w_m \cdot \lambda + g(w_m)))$$

An immediate corollary is the following identity:

Corollary 6.31.

$$N(\alpha - (w_m \cdot \lambda + g(w_m))) - N(\alpha - (w_m \cdot (\lambda + \alpha_1) + g(w_m)))$$
$$= N_{w_m \cdot \alpha_1}(\alpha - (w_m \cdot \lambda + g(w_m))). \quad (6.15)$$

Remark: The sum given above will always be finite. This is because only finitely many terms will be partitions of positive weights.

6.8 FROM K-THEORY TO HOMOLOGY

Once we have an equivariant multiplicity in K-theory, there is a straightforward way to compute the equivariant multiplicity in homology. The assumption we need is that the fixed point p is attractive, i.e. all the weights in the Zariski tangent space are in the same half-plane.

Theorem 6.32 (Chriss, Ginzburg,). *If $p \in X$ is an attractive fixed point then*

$$\varepsilon_p^H(X) = \lim_{t \to 0} t^{dim_{\mathbb{C}}(X)}(t \cdot \varepsilon_p^K(X))$$

where $t \cdot \varepsilon_p^K(X)$ means dilating the weights in $\varepsilon_p^K(X)$ by $t \in \mathbb{R}$.

Example 6.33. For $X = \mathbb{P}^1$, take the south pole S, then we have $\varepsilon_S^K(X) = \frac{1}{1-e^{2t}}$ so $\varepsilon_p^H(X) = \lim_{t \to 0} \frac{t}{1-e^{2t}} = \frac{1}{2t}$.

We use our previous results in *K*-theory to derive the following formula for the equivariant multiplicities in homology:

Theorem 6.34.

$$\varepsilon_{w_m}^H(X_{w_n}) = \frac{(-1)^m \binom{n}{\lceil \frac{n+m}{2} \rceil}}{\left(\prod_{i=0}^{m+\lfloor \frac{n-m}{2} \rfloor} \alpha_{\overline{m+1}} + j\delta \right) \left(\prod_{j=0}^{\lceil \frac{n-m}{2} \rceil - 1} \alpha_{\overline{m}} + j\delta \right)}$$

Unfortunately this does not converge to a limit as we take $n \to \infty$. However, in the next chapter we will explore a different approach — the theory of heat kernels on Riemannian manifolds — in order to provide a solution.

7

THE OSCILLATORY INTEGRAL

In [**?**], Picken gives a thorough exposition of applying the Duistermaat-Heckman integral to the flag manifolds G/B, and was able to derive formulas based on his calculations. We will apply the Duistermaat-Heckman integral **??** to the based loop group ΩG. However, one obvious issue props up: since the based loop group is infinite-dimensional, it is not obvious how to define the Liouville measure "$\frac{\omega^\infty}{\infty!}$" in the following integral

$$\int_{\Omega G} e^{i\phi} \frac{\omega^\infty}{\infty!}.$$

However, there is a well known measure on the (continuous) based loop group — the Wiener measure. We employ the Wiener measure (see Chapter **??** for details) as a substitute for the Liouville measure.

A similar idea as in the works of Frenkel [**?**] and Wendt [**?**] on orbital integrals on loop groups. The two authors worked with the generic coadjoint orbit LG/T, we will work with the "degenerate" orbit $LG/G \cong \Omega G$. We obtain a Duistermaat-Heckman type formula which computes a quantity for the oscillatory integral by using the Wiener measure and zeta regularization.

7.1 FIXED POINTS OF THE $(T \times S^1)$-ACTION

The fixed points of the $(T \times S^1)$-action are precisely the homomorphisms from $S^1 \to T$. Therefore the fixed points are in bijective correspondence of the coroot lattice Q of G. Moreover these fixed points are isolated.

Example 7.1. For $G = SU(2)$ these fixed points are isolated, and they come in the form of $P_k := \begin{pmatrix} e^{itk} & 0 \\ 0 & e^{-itk} \end{pmatrix}$.

Tangent space of ΩG

We have that $T_e \Omega G = \{f : S^1 \to \mathfrak{g} | f(1) = 0\}$. A loop in $T_e \Omega G$ can be written as a Fourier series $\zeta(t) := -\sum_{k=1}^{\infty} (\xi_k + \overline{\xi_k}) + \sum_{k=1}^{\infty} \left(\xi_k e^{ikt} + \overline{\xi_k} e^{-ikt}\right)$ where $\xi_k \in \mathfrak{g}_{\mathbb{C}}$ for each k. (The extra first term is to ensure that the loop sums to 0 at 1).

The other fibres of the tangent bundle $T\Omega G$ can be obtained from $T_e \Omega G$ by right-translation, just as in the case of compact Lie groups.

It is well-known that ΩG is a Kahler manifold. It has the following complex structure

$$I \left(\sum_{k=1}^{\infty} \xi_k e^{ikt} + \overline{\xi_k} e^{-ikt} \right) = \sum_{k=1}^{\infty} i \xi_k e^{ikt} - i \overline{\xi_k} e^{-ikt} \tag{7.1}$$

The maximum torus $T \subset G$ of course acts on loops in the tangent space by the adjoint action. The circle action is trickier because we are in the situation of based loops, and so we must use the based rotation action

$$\theta \cdot \gamma(t) := \gamma(t + \theta)\gamma(\theta)^{-1}$$

. For $u \in Q$, the tangent vectors at the fixed point $P_u \in \Omega G$ are given by left (or right) translating the tangent vectors at the identity element. We choose to identify the tangent space $T_{P_u} \Omega G$ by the tangent space isomorphism induced by the left translation $dL_{P_u} : T_e \Omega G \to T_{P_u} \Omega G$. I.e. tangent vectors living in $T_{P_u} \Omega G$ are given by $P_u(t)X(t)$ where $X(t) \in T_e \Omega G \cong \Omega \mathfrak{g}$.

Proposition 7.2. The induced action on tangent vectors of $T_{P_u} \Omega G$ is given by:

$$\theta \cdot (P_u(t)X(t)) = P_u(t)P_u(\theta)\left(X(t + \theta) - X(\theta)\right)P_u(\theta)^{-1}$$

Proof. Let $X(t) \in T_e \Omega G$ be a tangent vector. Let $\gamma_\varepsilon : (-\varepsilon, \varepsilon) \times S^1 \to G$ be a path in ΩG such that $\gamma_0 : S^1 \to G$ is the identity loop and $\frac{d}{d\varepsilon}\gamma_\varepsilon|_{\varepsilon=0} : S^1 \to \mathfrak{g}$ is equal to $X(t)$. For each fixed point $P_u(t)$, the tangent space $T_{P_u} \Omega G$ is gotten by right translating the tangent space $T_e \Omega G$. Therefore

to calculate the induced isotropy action on $T_{P_u}\Omega G$, we need to compute $\frac{d}{d\varepsilon}\theta \cdot P_u(t)\gamma_\varepsilon(t)|_{\varepsilon=0}$.

$$\frac{d}{d\varepsilon}\theta \cdot P_u(t)\gamma_\varepsilon(t)|_{\varepsilon=0} = \frac{d}{d\varepsilon}\left(P_u(t+\theta)\gamma_\varepsilon(t+\theta)\gamma_\varepsilon(\theta)^{-1}P_u(\theta)^{-1}\right)|_{\varepsilon=0}$$

$$= P_u(t+\theta)\left(\frac{d}{d\varepsilon}\gamma_\varepsilon(t+\theta)\right)|_{\varepsilon=0}\gamma_0(\theta)^{-1}P_u(\theta)^{-1}$$

$$+ P_u(t+\theta)\gamma_0(t+\theta)\left(\frac{d}{d\varepsilon}\gamma_\varepsilon(\theta)^{-1}\right)|_{\varepsilon=0}P_u(\theta)^{-1}$$

$$= P_u(t+\theta)\left(\frac{d}{d\varepsilon}\gamma_\varepsilon(t+\theta)\right)|_{\varepsilon=0}P_u(\theta)^{-1}$$

$$- P_u(t+\theta)\gamma_0(t+\theta)\gamma_0(\theta)^{-1}X(\theta)\gamma_0(\theta)^{-1}P_u(\theta)^{-1}$$

$$= P_u(t+\theta)\left(X(t+\theta) - X(\theta)\right)P_u(\theta)^{-1}$$

$$= P_u(t)P_u(\theta)\left(X(t+\theta) - X(\theta)\right)P_u(\theta)^{-1}$$

$$\square$$

Suppose that $X(t) \in T_e\Omega G$ is an eigenvector for the θ action, then on $T_e\Omega G$ the θ-action is given by $\theta \cdot X(t+\theta) - X(\theta)$. Therefore $\theta \cdot X(t) = e^{i\alpha\theta}X(t)$ for some weight $\alpha : S^1 \to \mathbb{R}$. Then since $P_u(t)$ is a homomorphism from $T \to S^1$, conjugation of $X(t)$ by $P_u(\theta)$ yields $e^{i\beta\theta}X(t)$ for some weight $\beta : S^1 \to \mathbb{R}$. So left-translation takes eigenvectors of θ in $T_e\Omega G$ to eigenvectors of θ in $T_{P_u}\Omega G$. (Right-translation however, does not take eigenvectors to eigenvectors).

In the tangent space, θ acts on the basis vectors in the tangent space by it's rotation number. I.e. it acts by θ^k on k-th degree homogeneous trigonometric polynomials. On the other hand the torus action T acts on these basis vectors by the root corresponding to the matrix coefficient.

An Eigenbasis for the $T \times S^1$ action

Define the following loops in $\Omega\mathfrak{su}(2) \otimes \mathbb{C} = \Omega\mathfrak{sl}(2)$:

$$X_k(t) = \begin{pmatrix} 0 & ie^{ikt} - i \\ ie^{-ikt} - i & 0 \end{pmatrix}$$

$$Y_k(t) = \begin{pmatrix} 0 & e^{ikt} - 1 \\ 1 - e^{-ikt} & 0 \end{pmatrix}$$

$$H_k(t) = \begin{pmatrix} ie^{ikt} - i & 0 \\ 0 & i - ie^{-ikt} \end{pmatrix}$$

for $k \in \mathbb{Z} \setminus \{0\}$.

Notice that $IX_k(t) = Y_k(t)$ and $IH_k(t) = H_{-k}(t)$. So $\{E_k(t), k \in \mathbb{Z} \setminus \{0\}\} \cup \{H_k(t), k \in \mathbb{N}\}$ is a **complex** basis for $\Omega\mathfrak{g} \otimes \mathbb{C}$. For the tangent spaces $T_p\Omega SU(2)$, these basis vectors are right translated by the loop p where p is a group homomorphism $S^1 \to T$.

The maximal torus T acts by conjugation, and S^1 acts by the action described in equation (**??**). Let α be the simple root of $SU(2)$ and δ be the imaginary root of $\hat{\mathfrak{sl}}_2$ The weights of the $(T \times S^1)$ are given by $\pm\alpha + k\delta$ for the basis $E_k(t), F_k(t)$ respectively. For the basis $H_k(t)$ the weights are $k\delta$.

For general compact simply-connected G, for a choice of positive roots, let $\Phi_\alpha : \mathfrak{sl}_2 \to \mathfrak{g} \otimes \mathbb{C}$ be the embedding associated to the $\mathfrak{sl}_2$-triple X, Y_α, H_α (here α is a positive root of G). The let $E_k^\alpha(t) = \Phi_\alpha(E_k(t)), F_k^\alpha(t) = \Phi_\alpha(F_k(t)), H_k^\alpha(t) = \Phi_\alpha(H_k(t))$. These basis vectors of $T_{id}\Omega G$ will have weights $\alpha + k\delta, -\alpha + k\delta, k\delta$ respectively.

Root Multiplicities

For $SU(n)$ where $n > 2$, there is an additional nuance: the imaginary root spaces will not be multiplicity free. This is because the Cartan subalgebra $\mathfrak{t}$ of the Lie algebra of G has dimension greater than one. In fact, the $T \times S^1$-action acts with weight $(k, 0)$ for all elements in the form $t^k H$ for all $H \in \mathfrak{t}_\mathbb{C}$. Therefore the multiplicity of the weight $(k, 0)$ is equal to the rank of G.

Now we can see that the tangent space at the identity decomposes into $T_e\Omega G = \bigoplus_{k=1}^{\infty} \left(\bigoplus_{\alpha \in \triangle^+} \left(X_\alpha^k(t) \oplus F_\alpha^k(t) \right) \right) \oplus \left(\bigoplus_{j=1}^{rk\,G} H^k(t) \right)$. The complexified tangent space decomposes into $T_e\Omega G \otimes \mathbb{C} = \bigoplus_{k=1}^{\infty} \left(\bigoplus_{\alpha \in \triangle^+} X_\alpha^k(t) \right) \oplus \left(\bigoplus_{j=1}^{rk\,G} H^k(t) \right)$.

The Generalized Equivariant Euler class

For a compact manifold M with isolated fixed points, the equivariant Euler class $e(p)$ at a fixed point p is the product of the weights of the isotropy representation on the tangent space $T_p M$. Inspired by this, we try to construct a renormalized Euler class around each fixed point P_k of the

infinite-dimensional manifold ΩG.

Using our filtration $\{T_N \cdot P_u\}_{N \in \mathbb{N}}$, we compute the product of the isotropy representation of $T \times S^1$ on $T_N \cdot P_u$.

Since we have an explicit formula for computing the equivariant Euler class of the normal bundle of isolated fixed points in the finite-dimensional case, we will attempt to extend such a formula into our loop group situation. Let α denote the only simple root of $SU(2)$, then we can compute the normalized equivariant Euler by taking a limit of the Euler classes of the finite dimensional subbundles. To do so we need to use zeta regularization, in particular we will need the following identities:

Lemma 7.3. *[?]*

1. $\displaystyle\prod_{j=1}^{\infty} j^N = (2\pi)^{N/2}$

2. $\displaystyle\prod_{j=1}^{\infty} j^2 \cdot A = A^{-1/2}(2\pi)$

Now we take the (infinitely many) weights of the tangent space and calculate a renormalized product.

Isotropy weights on $T_{P_u}\Omega G$

First we work out the regularized Euler class of $T_e\Omega G$ (this corresponds to the point P_u with $u = 0$):

$$\left(\prod_{\alpha \in \triangle^+} \prod_{k=1}^{\infty} (\langle \alpha, X \rangle + kY)(\langle \alpha, X \rangle - kY) \right) \left(\prod_{k=1}^{\infty} (kY) \right)^{rk\ G}$$

$$= \prod_{\alpha \in \triangle^+} \prod_{k=1}^{\infty} ((\langle \alpha, X \rangle)^2 - k^2 Y^2) \left(\prod_{k=1}^{\infty} (kY) \right)^{rk\ G}$$

$$= \left((2\pi)^{1/2} Y^{-1/2} \right)^{rk\ G} \prod_{\alpha \in \triangle^+} \prod_{k=1}^{\infty} k^2 Y^2 \prod_{j=1}^{\infty} \left(\frac{(\frac{\langle \alpha, X \rangle}{Y})^2}{j^2} - 1 \right)$$

$$= \left((2\pi)^{1/2} Y^{-1/2} \right)^{rk\ G} \prod_{\alpha \in \triangle^+} (2\pi) Y^{-1} \left(\frac{-\sin(\frac{\pi \langle \alpha, X \rangle}{Y})}{\pi \langle \alpha, X \rangle / Y} \right)$$

$$= \frac{(2\pi)^{rk\ G/2 + |\triangle^+|} \displaystyle\prod_{\alpha \in \triangle^+} \sin\left(-\frac{\pi \langle \alpha, X \rangle}{Y} \right)}{\pi(X) Y^{rk\ G/2}}$$

Now at the fixed point $T_{P_u}\Omega G$, by Proposition **??**, we have for each tangent vector $P_u(t)X^k(t) \in T_{P_u}\Omega G$, $\theta \cdot P_u(t)X^k(t) = P_u(t)P_u(\theta)(X^k(t+\theta) - X^k(\theta))P_u(\theta)^{-1} = P_u(t)P_u(\theta)(e^{ik\theta}X^k(t))P_u(\theta)^{-1}$.

Notice that since $P_u(\theta)X^k(t))P_u(\theta)^{-1} = e^{i\langle \alpha, u\rangle}$, we have that $\theta \cdot P_u(t)X^k(t) = e^{i(k+\langle \alpha, u\rangle)\theta}P_u(t)X^k(t)$.

The tangent vector $P_u(t)H_i^k(t)$ however, will still have the same eigenvalue k. This allows us to compute the generalized Euler class of $T_{P_u}\Omega G$. Notice that for each positive simple root α, the eigenvector $E_\alpha^k(t) \in T_e\Omega G$ gets sent under left translation to $P_u(t)E_\alpha^k(t)$, which has weight $k + \langle u, \alpha\rangle$ under the θ-action.

$$\prod_{\alpha \in \triangle^+} \prod_{k \in \mathbb{Z}\backslash\{0\}}^{\infty} (\langle \alpha, X\rangle + (k + \langle \alpha, u\rangle)Y)(\langle \alpha, X\rangle - (k + \langle \alpha, u\rangle)Y) \left(\prod_{k=1}^{\infty}(kY)\right)^{rk\,G}$$

$$= \prod_{\alpha \in \triangle^+} \prod_{k=1}^{\infty} (\langle \alpha, X\rangle + kY)(\langle \alpha, X\rangle - kY)\frac{\langle \alpha, X\rangle}{\langle \alpha, X\rangle + \langle u, \alpha\rangle Y} \left((2\pi)^{1/2}Y^{-1/2}\right)^{rk\,G}$$

$$= \left((2\pi)^{1/2}Y^{-1/2}\right)^{rk\,G} \prod_{\alpha \in \triangle^+} \prod_{k=1}^{\infty} (\langle \alpha, X\rangle)^2 - k^2Y^2)\frac{\langle \alpha, X\rangle}{\langle \alpha, X\rangle + \langle u, \alpha\rangle Y}$$

$$= \left((2\pi)^{1/2}Y^{-1/2}\right)^{rk\,G} \prod_{\alpha \in \triangle^+} \prod_{k=1}^{\infty} k^2Y^2 \prod_{j=1}^{\infty} \left(\frac{(\frac{\langle \alpha, X\rangle}{Y})^2}{j^2} - 1\right)\frac{\langle \alpha, X\rangle}{\langle \alpha, X\rangle + \langle u, \alpha\rangle Y}$$

$$= \left((2\pi)^{1/2}Y^{-1/2}\right)^{rk\,G} \prod_{\alpha \in \triangle^+} \frac{-\sin(\frac{\pi\langle \alpha, X\rangle}{Y})}{\langle \alpha, X\rangle} \prod_{\alpha \in \triangle^+} \frac{\langle \alpha, X\rangle}{\langle \alpha, X\rangle + \langle u, \alpha\rangle Y}$$

$$= \frac{(2\pi)^{rkG/2+|\triangle^+|} \prod_{\alpha \in \triangle^+} \sin\left(-\frac{\pi\langle \alpha, X\rangle}{Y}\right)}{\pi(X + uY)Y^{rk\,G/2}}$$

Then we arrive at the following

Theorem 7.4. *The normalized equivariant Euler class at the fixed point P_u is given by*

$$e_{T \times S^1}(P_u) = \frac{(2\pi)^{rkG/2+|\triangle^+|} \prod_{\alpha \in \triangle^+} \sin\left(\frac{\pi\langle \alpha, X\rangle}{Y}\right)}{\pi(X + uY)Y^{rk\,G/2}} \tag{7.2}$$

Now we recall the root space decomposition of semisimple Lie algebras:

Theorem 7.5 ([?]). *Let $\mathfrak{g}$ be a semisimple Lie algebra. Let $\mathfrak{h}$ denote its maximal toral subalgebra. For each root $\alpha \in \triangle$ of $\mathfrak{g}$, denote its root space by $\mathfrak{g}_\alpha = \{X \in$*

$\mathfrak{g}|[H,X] = \alpha(H)X, \forall H \in \mathfrak{h}\}$. *Upon making a choice of positive roots, then* $\mathfrak{g}$ *can be written as the following direct sum*

$$\mathfrak{g} = \left(\bigoplus_{\alpha \in \triangle^+} \mathfrak{g}_\alpha \right) \oplus \mathfrak{h} \left(\bigoplus_{\alpha \in \triangle^+} \mathfrak{g}_{-\alpha} \right) \tag{7.3}$$

For a semisimple compact Lie group G, the maximal toral subalgebra $\mathfrak{h}$ is simply the Lie algebra of the maximal torus $T \subset G$. Therefore this suggests the following corollary:

Corollary 7.6. $dim G = rk\ G + 2|\triangle^+|$ *where* $|\triangle^+|$ *denotes the number of positive roots of G.*

Therefore we can rewrite the equivariant Euler class as follows:

$$\begin{aligned}
e_{T \times S^1}(P_u) &= \frac{(2\pi)^{dim\ G/2}}{Y^{rk\ G/2}\pi(X + uY)} \prod_{\alpha \in \triangle^+} \sin(\frac{\pi\langle \alpha, X \rangle}{Y}) \\
&= \frac{(2\pi)^{dim\ G/2}}{Y^{rk\ G/2}Y^{|\triangle^+|}\pi(X/Y + u)} \prod_{\alpha \in \triangle^+} \sin(\frac{\pi\langle \alpha, X \rangle}{Y}) \\
&= \frac{(2\pi)^{dim\ G/2}}{Y^{dim\ G/2}\pi(X/Y + u)} \prod_{\alpha \in \triangle^+} \sin(\frac{\pi\langle \alpha, X \rangle}{Y})
\end{aligned}$$

7.2 APPLYING DUISTERMAAT-HECKMAN

In [?], Atiyah and Pressley construct a moment map for the based loop group ΩG with respect to the Hamiltonian $(T \times S^1)$-action. The components are given by

$$E(f) = \frac{1}{4\pi} \int_0^{2\pi} ||f(t)^{-1}f'(t)||^2 dt \tag{7.4}$$

$$p(f) = pr_{Lie(T)} \left(\frac{1}{2\pi} \int_0^{2\pi} f(t)^{-1}f'(t)dt \right) \tag{7.5}$$

By a routine computation we get that for a cocharacter $u : S^1 \to T, t \mapsto t_1^{a_1} \cdots t_n^{a_n}$, $E(P_u) = -\frac{1}{2}||u||^2$, and $p(P_u) = (u, \cdot)$. Then a DH formula would look like

$$V_{X,Y}(\Omega G) = \sum_{\alpha \in \triangle^+} \frac{e^{-i(E,p)(P_u)}}{e_{T \times S^1}(P_u)}(X, Y)$$

$$= \frac{Y^{dim \; G/2}}{(2\pi)^{dim \; G/2} \prod_{\alpha \in \triangle^+} \sin\left(\dfrac{\pi \alpha X}{Y}\right)} \sum_{u \in Q} \pi(X/Y + u)e^{-i(||u||^2+(u,X))}$$

We will show that this formula is actually almost identical to the formula for the heat kernel on the compact Lie group G.

7.3 ORBITAL INTEGRALS ON LOOP GROUPS

From this point on, we will shorten notation and call $v_t(m, \tilde{m}) := Z(t, m, \tilde{m})$. Note that heuristically we can express the Wiener measure as $"dW_t(z) = e^{-\frac{1}{2t}\langle z^{-1}z', z^{-1}z'\rangle}dz"$ where dz is a sort of "infinite-dimensional Riemannian measure". For more details on the Wiener measure and pinned Wiener measure, see [?] and [?].

Now we use Frenkel's transformation formula Theorem ?? to write the following orbital integral on ΩG.

Theorem 7.7 (Frenkel Proposition 5.2.12). $e^{\frac{-||X||^2}{2t}} \displaystyle\int_{\Omega G} e^{\frac{1}{t}\langle z^{-1}z', X\rangle}dW_t(z) = v_t(\exp(2\pi X), 2\pi)$

Applying this formula to our oscillatory integral, we get

$$\int_{\Omega G} e^{-\langle z^{-1}z', X\rangle}e^{-Y\langle z^{-1}z', z^{-1}z'\rangle}dz \quad (\text{substituting } dW_t \text{ for } e^{-\frac{1}{2t}\langle z^{-1}z', z^{-1}z'\rangle}dz)$$

$$= \int_{\Omega G} e^{-\frac{1/(2Y)}{(2Y)^{-1}}\langle z^{-1}z', X\rangle}dW_t(z)$$

$$= \int_{\Omega G} e^{2Y\langle z^{-1}z', -\frac{X}{2Y}\rangle}dW_t(z)$$

$$= e^{||X||^2 Y}v_{(2Y)^{-1}}(\exp(2\pi(-X/2Y)), 2\pi)$$

where in the first step we let $Y = (2t)^{-1}$.

Corollary 7.8.

$$\int_{\Omega G} e^{-\langle z^{-1}z', X\rangle} e^{-Y\langle z^{-1}z', z^{-1}z'\rangle} dz = e^{||X||^2 Y}\left(v_{\frac{1}{2Y}}(\exp(2\pi(-X/Y)H), 2\pi)\right)$$

We can apply the results of section 1 to our corollary above. First we recall an important identity: We can write the denominator has a product as well:

Proposition 7.9 (Weyl Denominator Formula).

$$j(x) = \prod_{\alpha \in \triangle^+} \left(e^{i\pi\langle\alpha,x\rangle} - e^{-i\pi\langle\alpha,x\rangle}\right) = \prod_{\alpha \in \triangle^+} 2\sin(\pi\langle\alpha, x\rangle)$$

Now applying Fegan's formula to Corollary **??** with $t = \frac{1}{2Y}, x = -\frac{X}{2Y}$, we get:

$$v_{\frac{1}{2Y}}\left(-\frac{X}{Y}\right) = \frac{e^{-\frac{2\pi i||\rho||^2}{2Y}}}{j(-X/2Y)}\left(\frac{-2Y}{i}\right)^{dimG/2} \frac{i^{-n}}{volP} \sum_{\lambda \in Q} d\left(\lambda + \frac{X}{4Y}\right) e^{-2\pi i||\lambda + \frac{X}{4Y}||^2 2Y}$$

$$= \frac{e^{-\frac{2\pi i||\rho||^2}{2Y}}}{j(-X/2Y)}\left(\frac{-2Y}{i}\right)^{dimG/2} \frac{i^{-n}}{volP} \sum_{\lambda \in Q} d\left(\lambda + \frac{X}{4Y}\right) e^{-4\pi i\left(||\lambda||^2 + (\lambda, X/2Y) + ||\frac{X}{4Y}||^2\right)Y}$$

$$= \left\{\frac{e^{-\frac{2\pi i||\rho||^2}{2Y} + ||\frac{X}{4Y}||^2 Y}}{j(-X/2Y)}\left(\frac{-2Y}{i}\right)^{dimG/2} \frac{i^{-n}}{volP}\right\} \sum_{\lambda \in Q} d\left(\lambda + \frac{X}{4Y}\right) e^{-4\pi i\left(||\lambda||^2 Y + (\lambda, X/2)\right)}$$

$$= \left\{\frac{e^{-\frac{2\pi i||\rho||^2}{2Y} + ||\frac{X}{4Y}||^2 Y}}{\prod_{\alpha \in \triangle^+} \sin(-X/2Y)}\left(\frac{-2Y}{i}\right)^{dimG/2} \frac{i^{-n}}{volP}\right\} \sum_{\lambda \in Q} d\left(\lambda + \frac{X}{4Y}\right) e^{-4\pi i\left(||\lambda||^2 Y + (\lambda, X/2)\right)}$$

Now we may write a DH oscillatory integral in terms of fixed points as follows:

$$\int_{\Omega G} e^{-\langle z^{-1}z', X\rangle} e^{-Y\langle z^{-1}z', z^{-1}z'\rangle} dz = \sum_{u \in Q} R_u(X, Y) e^{-4\pi i\left(||u||^2 Y + (u, X/2)\right)}$$

where $R_u(X, Y) = \left\{\dfrac{e^{\frac{||X||^2 - 2\pi i||\rho||^2}{2Y}}}{\prod_{\alpha \in \triangle^+} \sin(-X/2Y)}\left(\dfrac{-2Y}{i}\right)^{dimG/2} \dfrac{i^{-n}}{volP} d\left(u + \dfrac{X}{4Y}\right)\right\}.$

The upshot of writing the denominator as a product is that we can relate it to a renormalized version of the equivariant Euler classes around the fixed points of ΩG. Indeed, we see that the quantities $R_u(X, Y)$ are almost

identical to the Euler classes $e_{T \times S^1}(P_u)$ we've computed up to a constant and the extra factor $e^{\frac{||X||^2 - 2\pi i ||\rho||^2}{2Y}}$. This suggests the following version of the Duistermaat-Heckman formula for the based loop group:

Theorem 7.10. *The following formula holds for any* $X \in Lie(T), Y > 0$

$$\int_{\Omega G} e^{-\langle z^{-1} z', X \rangle} e^{-Y \langle z^{-1} z', z^{-1} z' \rangle} dz = e^{\frac{||X||^2 - 2\pi i ||\rho||^2}{2Y}} \sum_{u \in Q} \frac{e^{-4\pi i \left(||u||^2 Y + (u, X/2) \right)}}{e_{T \times S^1}(P_u)(X, Y)}$$

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