Hölder continuity of tangent cones and non-branching in RCD(K,N) spaces

by

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Abstract

This thesis is concerned with the study of the structure theory of metric measure spaces (X,d,m) satisfying the synthetic lower Ricci curvature bound condition $\mathsf{RCD}(K,N)$. We prove that such a space is non-branching and that tangent cones from the same sequence of rescalings are Hölder continuous along the interior of every geodesic in X. More precisely, we show that the geometry of balls of small radius centred in the interior of any geodesic changes in at most a Hölder continuous way along the geodesic in pointed Gromov-Hausdorff distance. This improves a result in the Ricci limit setting by Colding-Naber where the existence of at least one geodesic with such properties between any two points is shown. As in the Ricci limit case, this implies that the regular set of an $\mathsf{RCD}(K,N)$ space has m-a.e. constant dimension, a result recently established by Brué-Semola, and is m-a.e convex. It also implies that the top dimension regular set is weakly convex and, therefore, connected. In proving the main theorems, we develop in the $\mathsf{RCD}(K,N)$ setting the expected second order interpolation formula for the distance function along the Regular Lagrangian flow of some vector field using its covariant derivative.

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Chapter 1

Introduction

This thesis is concerned with the study of the structure theory of metric measure spaces satisfying the synthetic lower Ricci curvature bound condition $\mathsf{RCD}(K,N)$. The Ricci curvature of a Riemannian manifold (M,g) is a way of measuring the local deviation of the metric from an Euclidean metric and plays an important role in the study of Non-Euclidean geometries. Lower bounds on Ricci curvature, when combined with an upper bound on dimension, gives strong constraints on the topology and geometry of the space. This can be seen through, among others, the Bonnet-Myer estimate on maximal diameter, the Bishop-Gromov inequality on volume monotonicity [56], Li-Yau heat kernel bounds [69], and the Cheeger-Gromoll splitting principle [24].

The consequences of volume monotonicity are far-reaching. In particular, Gromov observed in [56] that it is sufficient to give the precompactness under the Gromov-Hausdorff topology of the family of Riemannian manifolds $\mathscr{M}_{K,N,D}$ with Ricci curvature bounded from below by K, dimension bounded from above by N and diameter bounded from above by D. Although the Gromov-Hausdorff topology is a somewhat rough topology on the space of metric spaces in general, behaving like a kind of C^0-convergence for metric spaces, it is quite restrictive when it comes to the aforementioned class of Riemannian manifolds. Indeed, one expects that the so-called *Ricci limit spaces*, which are metric spaces arising as Gromov-Hausdorff limits of Riemannian manifolds with uniform Ricci lower bounds and dimension upper bounds, will maintain much of the geometric analytic properties of the class. A simple analogy of this phenomenon is the fact that a pointwise limit of convex functions is still convex. On the other hand, it is easy to see that one can obtain non-smooth behaviours in the limit space as well, for example, conical singularities.

Therefore, it becomes a natural question to ask exactly what can be expected for Ricci limit spaces. Not only is an answer to such a question interesting in its own right, since it helps one to understand what it means for a metric space to have lower Ricci curvature bounds, there are also several motivations from the point of view of Riemannian geometry:

- To rule out certain bad quantitative behaviours in a class of Riemannian manifolds with Ricci curvature bounded from below and dimension bounded from above, one may hypothesize, by the way of contradiction, a sequence of manifolds in the class with increasingly worse behaviour. If one is able to show that some extremely bad behaviour passes to the limit but also, on the other hand, has a priori information about the limit space which precludes such a situation, then one can show that such a sequence is impossible. In other words, one is able to put a bound on "how bad things can get" for the class of Riemannian manifolds under consideration. For examples of such an argument, see [35, 59, 64].

- In the other direction, one may show that certain quantitative behaviours are achieved by a simple Ricci limit space, and then use the stability of these behaviours under Gromov-Hausdorff convergence to show the same quantitative behaviours are almost achieved by Riemannian manifolds in the class of interest. See [39] for an instance of this type of argument.

- Finally, Ricci limit spaces occur naturally in the study of solutions of the Ricci flow where singularities appear, as well as in degenerations of manifolds with special holonomy such as Calabi-Yau, Hyper-Kähler, G2 and Spin(7) manifolds, which are all necessarily Ricci flat. Ricci limits are also closely related to subsolutions of the Ricci flow, which can be viewed as the parabolic version of the theory of spaces with lower Ricci curvature bounds.

A program to understand Ricci limit spaces was started in the mid 90s by Cheeger and Colding in [20–23, 36]. Among many remarkable results, they were able to establish the almost everywhere existence of unique, Euclidean tangent cones, a quantitative almost splitting theorem, and estimates on the Hausdorff dimension of singularities. They also showed that the naturally defined renormalized limit volume measure, which may not be the Hausdorff measure, carries important geometric and analytic information about the Ricci limit. This matches a previous claim made by Gromov in that lower Ricci curvature bounds should be understood in terms of both the metric and the measure. Further progress was made in [26, 33, 34] and all in all we now have a fairly good picture of the structure of Ricci limit spaces. This thesis will build on, in particular, the methods of [33], where by proving the continuity of tangent cones along the interior of limit geodesics in a Ricci limit space, it was shown that not only are tangent cones at almost every point unique and Euclidean, they must also all be of the same dimension.

However, in spite of the very successful early developments in the structure theory of Ricci limit spaces and as pointed out in [21, Appendix 2], it still remained unclear exactly what it means for a metric measure space to have Ricci curvature bounded below. By this one means an intrinsic theory without appealing to smooth approximations by Riemannian manifolds, much in the same way that Alexandrov spaces capture the notion of sectional curvature bounded from below. In essence, one would like a way to say if a metric measure space exhibits the correct behaviours to be considered to have Ricci curvature bounded below by K, using only the metric and measure information. An answer to this question came from optimal transportation theory in the mid 2000s.

In a series of works [32, 70, 74, 86], it was discovered that the K-convexity of Shannon entropy over the L^2-Wasserstein space associated with a Riemannian manifold (M, g) completely characterizes a lower Ricci curvature bound of K. Noting that this characterization relies only on the metric measure structure of the Riemannian manifold, a purely synthetic definition on metric measure spaces, called the $\mathsf{CD}(K,\infty)$ *condition*, was introduced independently by Sturm in [82] and Lott and Villani in [68] using this characterization. A further refinement called the $\mathsf{CD}(K,N)$ *curvature-dimension condition* was given in [68, 83] using convexity conditions on the N-Renyi entropy and captures the notion that a metric measure space has Ricci curvature bounded from below by K and dimension bounded from above by N. The existence of such a notion is perhaps not too surprising due to the presence of dimensional terms in many important geometric inequalities under the presence of a lower Ricci curvature bound (for example, the Bishop-Gromov inequality and the Bochner inequality).

Metric measure spaces satisfying the various curvature-dimension conditions where shown to capture several important classes including (possibly weighted) Riemannian manifolds [83], Finsler manifolds [73] and Alexandrov spaces [75] satisfying the correct curvature bounds. The conditions were also shown to be stable under measured Gromov-Hausdorff convergence and imply several powerful theorems as in the Riemannian case, including the Bishop-Gromov inequality, the Bonnet-Myer Theorem and a local Poincaré inequality [76]. Nevertheless, several desirable features were still missing for such spaces, chief among these are a globalization property, shown to be false for general $\mathsf{CD}(K,N)$ spaces in [77], and a splitting theorem, the latter being essential for carrying out the same type of structure theory analysis done by Cheeger-Colding for Ricci limit spaces.

Several refinements were then introduced with better properties:

- The $\mathsf{CD}^*(K,N)$ *reduced curvature-dimension condition* was introduced in [17] and shown to have a globalization property under a non-branching assumption. Much of the subsequent work in the theory was done for this class of spaces instead.

- *Essentially non-branching* $\mathsf{CD}(K,\infty)$ spaces were introduced in [78], naturally extending also to the $\mathsf{CD}(K,N)$ case. As shown in [53], essentially non-branching $\mathsf{CD}(K,N)$ spaces have the same existence and uniqueness of optimal maps enjoyed by Riemannian manifolds [71]. Furthermore, such spaces have good curvature localisation properties with respect to needlepoint decompositions corresponding to 1-Lipschitz functions, see [29]. This 1D-localisation method has found use in proving many metric measure analogues of Riemannian inequalities, see, for example, [15, 29, 30, 60]. Finally, it was shown in [27] that for essentially non-branching spaces with finite measure (most likely an unnecessary technical assumption), the $\mathsf{CD}(K,N)$ and $\mathsf{CD}^*(K,N)$ conditions are equivalent. In particular, this shows that on these types of spaces, the $\mathsf{CD}(K,N)$ condition has the desired globalization property. However, neither the non-branching property nor the essentially non-branching property (in conjunction with the various curvature-dimension conditions) is stable under Gromov-Hausdorff convergence and so these definitions were not completely satisfactory either.

Observing that many tools such as the Bochner inequality and the splitting theorem are impossible in the presence of a non-Riemannian Finsler structure, an *infinitesimally Hilbertian* condition, roughly meaning that the space has Riemannian structure almost everywhere, was added to the $\mathsf{CD}(K,\infty)$, $\mathsf{CD}(K,N)$ and $\mathsf{CD}^*(K,N)$ conditions in [3, 6, 41, 46] to give the *Riemannian curvature-dimension conditions* $\mathsf{RCD}(K,\infty)$, $\mathsf{RCD}(K,N)$ and $\mathsf{RCD}^*(K,N)$. This rules out general Finsler structure and was enough to give the splitting theorem in the $\mathsf{RCD}(K,N)$ case as in [45]. Moreover, it was able to clarify the relationship between the curvature-dimension condition, which uses a *Lagrangian* formulation in that it focuses on behaviours along geodesics, and the *Bakry-Émery condition*, which is *Eulerian* in nature in that it focuses on functions and their gradients. Indeed, it was shown in [7,9,41] that the Riemannian curvature-dimension condition and the Bakry-Émery condition are equivalent, where the presence of infinitesimally Hilbertian structure was crucially used to establish the necessary calculus for the Bakry-Émery condition. Since $\mathsf{RCD}(K,N)$ spaces are known to be essentially non-branching from [78], (compact) $\mathsf{RCD}(K,N)$ spaces also have the desired globalization property due to [27].

Much work has been done for the structure theory of $\mathsf{RCD}(K,N)$ spaces over the last few years. [18, 37, 38, 46, 55, 63, 72], among others, have recovered much of what was known for Ricci limit spaces. Along the way, many new methods were developed since one no longer had smooth approximations with which to obtain quantitative estimates. In particular, a calculus framework, summarized in [47], was established which strongly resembles calculus on Riemannian manifolds in a measure-theoretic sense and up to the second order. The existence of such a framework has greatly simplified some arguments, for example, the proof of the splitting theorem as in [16]. This thesis is, in particular, concerned with using this calculus framework on $\mathsf{RCD}(K,N)$ spaces and its geometric implications as established in [54] to generalize the results of [33]. Along the way, we will see the power of direct intrinsic arguments on $\mathsf{RCD}(K,N)$ spaces in action when we show that such spaces are non-branching, a result which was previously unknown for Ricci limit spaces.

1.1 Summary of results

In this thesis, we prove that $\mathsf{RCD}(K,N)$ spaces are non-branching and generalize to the $\mathsf{RCD}(K,N)$ setting an improved version of the main result from Colding-Naber [33]. We begin by stating two formulations of the latter, which is be the main technical result of this thesis.

Theorem 1.1.1. *(Hölder continuity of geometry of small balls with same radius) Let (X,d,m) be an $\mathsf{RCD}(K,N)$ space for some $K \in \mathbb{R}$ and $N \in (1,\infty)$. Let $p,q \in X$ and $d(p,q) = \ell$. Define $K' = (\frac{K}{-(N-1)} \vee 1)^{1/2}$ and $\ell' = \ell \wedge 1$. For any unit speed geodesic $\gamma : [0,\ell] \to X$ between p and q, there exist constants $C(N)$, $\alpha(N)$ and $r_0(N) > 0$ so that for any $\delta > 0$ with $0 < r < r_0\frac{\delta\ell'}{K'}$ and $\delta\ell < s < t < \ell - \delta\ell$,*

$$d_{pGH}\Big((B_r(\gamma(s)),\gamma(s)),(B_r(\gamma(t)),\gamma(t))\Big) < \frac{CK'}{\delta\ell'}r|s-t|^{\alpha}. \tag{1.1}$$

In order to pass the result to tangents, we use the following terminology: Let $x_1, x_2 \in X$, $(Y,d_Y,m_Y,y) \in \mathrm{Tan}(X,d,m,x_1)$ and $(Z,d_Z,m_Z,z) \in \mathrm{Tan}(X,d,m,x_2)$. We say Y and Z come from the *same sequence of rescalings* if there exists $s_j \downarrow 0$ so that

$$(X, s_j^{-1}d, m_{s_j}^{x_1}, x_1) \xrightarrow{pmGH} (Y, d_Y, m_Y, y) \text{ and } (X, s_j^{-1}d, m_{s_j}^{x_2}, x_2) \xrightarrow{pmGH} (Z, d_Z, m_Z, z). \tag{1.2}$$

The following estimate on tangents from the same sequence of rescaling follows from Theorem 1.1.1.

Theorem 1.1.2. *(Hölder continuity of tangent cones) In the notations of Theorem 1.1.1, for any unit speed geodesic $\gamma : [0, \ell] \to X$ between p and q, there exist constants $C(N)$, $\alpha(N) > 0$ so that if $(Y_s, d_{Y_s}, m_{Y_s}, y_s) \in \mathrm{Tan}(X, d, m, \gamma(s))$ and $(Y_t, d_{Y_t}, m_{Y_t}, y_t) \in \mathrm{Tan}(X, d, m, \gamma(t))$ come from the same squence of rescalings, then*

$$d_{pGH}\Big((B_r(y_s), y_s), (B_r(y_t), y_t)\Big) < \frac{CK'}{\delta\ell'} r|s - t|^{\alpha} \tag{1.3}$$

for all $r > 0$.

To prove these we first construct at least one geodesic between any two points satisfying the conclusion of Theorem 1.1.1, which is the main result of [33]. We then use this construction to prove that $\mathsf{RCD}(K, N)$ spaces, and so, in particular, Ricci limit spaces, are non-branching in Section 6.1.

Theorem 1.1.3. *Let (X, d, m) be an $\mathsf{RCD}(K, N)$ space for some $K \in \mathbb{R}$ and $N \in (1, \infty)$. (X, d, m) is non-branching.*

This has been a natural open problem for Ricci limits since the seminal work of Cheeger-Colding in [20–23]. Potential branching which can come from some simple examples were ruled out in [22, Section 5]. To compensate for the lack of non-branching, the weaker notion of essentially non-branching was introduced and shown to hold for $\mathsf{RCD}(K, \infty)$ spaces in [78]. This has found wide use in RCD theory and serves as a suitable replacement for the non-branching condition in most measure theoretic arguments. We point out finally that non-branching would follow for Ricci limits directly from the results of [33] and the argument we use if one were able to, for example, prove that all geodesics in any Ricci limit space are limit geodesics. Our intrinsic construction of a geodesic in Section 5.2 which satisfies the conclusion of Theorem 1.1.1 offers a little more freedom than the extrinsic construction of limit geodesics and this was enough to prove Theorem 1.1.3. Theorem 1.1.3 is then used to pass from the existence of a geodesic between any two points which satisfies the conclusion of Theorem 1.1.1 to the theorem for all geodesics. We note that non-branching immediately implies that all geodesics of a Ricci limit space must be limit geodesics, see Corollary 6.1.2.

In the case of Ricci limits, the Hölder continuity of tangent cones had several key applications.

Theorem 1.1.4. *([33, Theorems 1.18, 1.20 and 1.21]) Let (X, d) be the Ricci limit of $(M_i^n, g_i)_{i \in \mathbb{N}}$ and m be its canonical limit measure. The following holds:*

1. *There is a unique* $k \in \mathbb{N}$, $0 \le k \le n$ *so that* $m(X \backslash \mathcal{R}_k) = 0$, *where* $\mathcal{R}_k$ *is the k-dimensional regular set;*

2. $\mathcal{R}_k$ *from statement 1 is m-a.e. convex and weakly convex. In particular,* $\mathcal{R}_k$ *is connected;*

3. *The isometry group of X is a Lie group.*

Statements 1 and 3 have since been proved by other means in the case of $\mathsf{RCD}(K, N)$ spaces, see [18] for 1 and [80, 81] for 3. We prove statement 2 in Section 6.2 following [33]. Since the proofs of statements 1 and 2 are intricately related, we will prove statement 1 as well. Using the theory of 1D-localisation developed in [29,31], we also give a proof of a small improvement of part 2 of the previous theorem in Theorem 6.2.5.

1.2 Outline of Thesis and Proof

We begin this section by introducing the strategy of the proof in [33], which we will largely follow. We then discuss the issues that arise when extending this to the metric measure setting and give an outline of their solutions.

The existence of at least one geodesic satisfying the conclusion of Theorem 1.1.1 was shown in [33]. The proof there was extrinsic and obtained by proving the theorem for manifolds. As such, consider a Riemannian manifold with (M^n, g) with $\mathrm{Ric}_M \ge -(n-1)$ and a unit speed minimizing geodesic $\gamma : [0, 1] \to M$ from some p to q. Fix some $\delta > 0$ and $\delta < s_0 < s_1 < 1 - \delta$. The desired Gromov-Hausdorff approximations in the proof of Theorem 1.1.1 for γ are constructed on a large subset of the ball $B_r(\gamma(s_1))$ from the gradient flow Ψ of $-d_p$ (more on the definition of this later). This is not altogether surprising since in the interior of γ, d is a smooth function and the Laplacian of d has a two-sided bound. A simple argument applying the Bochner formula to d gives

$$\int_{\delta}^{1-\delta} |\mathrm{Hess}\, d_p|^2(\gamma(t))\, dt \le \frac{c(n)}{\delta}. \tag{1.4}$$

The fact that $\mathrm{Hess}\, d_p = \frac{1}{2}\mathscr{L}_{\nabla d_p} g$ shows that $D(\Psi_{s_1 - s_0}) : T_{\gamma(s_1)}M \to T_{\gamma(s_0)}M$ satisfies the estimates of Theorem 1.1.2. Of course, Theorem 1.1.2 is completely trivial in the case of Riemannian manifolds but the point is that this map comes from a construction on the manifolds itself. Smoothness then allows one to use $\Psi_{s_1 - s_0}$ to construct Gromov-Hausdorff aproximations from $B_r(\gamma(s_1))$ to $B_r(\gamma(s_0))$, where r needs to be sufficiently small depending on γ and δ instead of just n and δ. This property does not pass through Ricci limits and so the challenge, then, is to remove the dependence on γ.

One might consider using $\mathrm{Hess}\, d_p$ to control the geometry of balls of uniform (i.e. independent of γ) radius under Ψ. However, we do not have estimates on $\mathrm{Hess}\, d_p$ for such balls along γ; we do not even have smoothness of d_p. We mention that due to this lack of smoothness, Ψ is not globally defined. The integral curves of Ψ starting at any $x \in M$ should be thought of simply as a (choice of) unit speed geodesic from x to p. In this way Ψ is defined locally away from p for a definite amount of time.

Nevertheless, it turns out that one can still control the geometry under Ψ by utilizing the next formula, which follows from the standard first variation formula and second order interpolation formula using the Hessian.

$$\left|\frac{d}{dt}d(\sigma_1(t),\sigma_2(t))\right| \leq |\nabla h - \sigma_1'|(\sigma_1(t)) + |\nabla h - \sigma_2'|(\sigma_2(t)) + \inf \int_{\gamma_{\sigma_1(t),\sigma_2(t)}} |\mathrm{Hess}\, h| \tag{1.5}$$

for a.e. t, where σ_1, σ_2 are unit speed geodesics in M, $h : M \to \mathbb{R}$ is a smooth function, and inf is taken across all minimizing geodesics connecting $\sigma_1(t), \sigma_2(t)$. This means as long as we can closely approximate d_p by a smooth function h where we have reasonable control on $|\nabla h - \nabla d_p|$ along two geodesics and on $\mathrm{Hess}\, h$, then we can control the distance between those two geodesics.

We mention a similar strategy was used to prove the almost splitting theorem in [20]. In that setting, a single ball $B_r(\gamma(s))$ of small radius for a fixed $s \in (\delta, 1-\delta)$ is considered. It turns out that the correct approximation to take for d_p is the harmonic replacement b of d_p on $B_{2r}(\gamma(s))$. One is able to obtain the better than scale invariant estimate

$$⨍_{B_r(\gamma(s))} |\mathrm{Hess}\, b|^2 \, dV \leq c(n,\delta) r^{-2+\alpha(n)} \tag{1.6}$$

using the Bochner formula, which is enough to prove almost splitting. The almost splitting theorem has since been proved through other means for RCD spaces in [45], see also [72].

For our purposes, (1.6) and the resulting almost splitting theorem is not good enough because it only allows one to compare two balls of radius r that are distance r away from each other. As discussed in detail in [33, Section 2], this estimate blows up as $r \to 0$ if one iterates along γ in r-length intervals. The crucial idea in [33] was then to use the heat flow approximation h to (some cutoff of) d_p instead. For such an approximation, they were able to obtain the estimate

$$\int_{\delta}^{1-\delta} ⨍_{B_r(\gamma(t))} |\mathrm{Hess}\, h|^2 \, dV \, dt \leq c(n,\delta), \tag{1.7}$$

where h is the heat flow taken to some time on the scale of r^2 (see Theorem 4.0.12, statement 4). Moreover, $\int_{\delta}^{1-\delta} |\nabla d_p - \nabla h|$ can also be bounded to the correct order (see Theorem 4.0.13) for most geodesics. These estimates can then be used along with the segment inequality of Cheeger-Colding [20, Theorem 2.11] and (1.5) to control the total integral change in distances between elements of two sets of large measure in $B_r(\gamma(s_1))$ under the flow Ψ. This is ultimately good enough to construct a Gromov-Hausdorff approximation using $\Psi_{s_1 - s_0}$. We mention that since we are using segment inequality and integral bounds, the smaller the relative measure of the sets compared to the region where we have Hessian estimates, the worse the control we have on total distance change for those sets under Ψ.

A crucial detail in this is that in order to make use of estimate (1.7), it is important that most of $B_r(\gamma(t))$ stays close, on the scale of r, to γ under the gradient flow for an amount of time independent of r and γ. Since the control one has over distance is for sets of large relative measure and γ is trivial in measure, one cannot guarantee using the argument outlined in the previous paragraph that most of $B_r(\gamma(t))$ does not simply drift away from γ quickly. In [33], this was overcome by using (1.4). As mentioned previously, by smoothness, (1.4) implies balls of sufficiently small radius depending on γ stays close to γ under Ψ for some fixed amount of time depending only on n and δ. Induction with geometrically increasing radii along with the argument from the previous paragraph can then be used to guarantee large proportions of balls up to some radius independent of γ also stay close to γ under the flow Ψ.

We now outline the issues with extending this argument to the metric measure setting and the ideas we will use to resolve them:

- (1.5) is essential in utilizing the Hessian and gradient estimates of the approximating function to control the geometry under the flow Ψ. In the smooth setting, it stems from the first variation formula along σ_1 and σ_2 and the following interpolation formula along a unit speed geodesic α, which should be thought of as going between $\sigma_1(t)$ and $\sigma_2(t)$ for some t in this application.

$$\langle \alpha'(\tau_1), V\rangle - \langle \alpha'(\tau_0), V\rangle = \int_{\tau_0}^{\tau_1} \langle \nabla_{\alpha'(\tau)} V, \alpha'(\tau)\rangle \, d\tau, \tag{1.8}$$

  where V is a vector field along α and ∇V is its covariant derivative. We will apply (1.5) to control the integral distance change between all elements of two sets under the flow Ψ and so an integral version of the formula suffices. The first variation formula for "almost every" pair of σ_1 and σ_2 we are interested in follows easily from the first order differentiation formula for Wasserstein geodesics (see [45]).

  In the direction of (1.8), the same formula (with obvious changes) was proved along Wasserstein geodesics with bounded density in [54, Theorem 5.13]. While this does most of the work, a suitable interpretation is required to obtain the integral interpolation formula between two sets S_1 and S_2. To see the difficulty, one might try to decompose the set of all geodesics between S_1 and S_2 by grouping together all geodesics that start at the same $x \in S_1$. In this way, one obtains a family of Wasserstein geodesics parameterized by $x \in S_1$. However, these end at a δ measure and therefore the interpolation formula of [54, Theorem 5.13] does not apply. This is, of course, expected because $\langle \nabla d_x, V\rangle$ is not well-defined at x. The correct decomposition then is to break all the geodesics between S_1 and S_2 down the middle, parameterize the half that start in S_2 by the elements of S_1 they each go toward and vice versa. We point out that the same decomposition is used in the proof of the segment inequality. Some work then needs to be done to check the boundary terms that arise in interpolating between each of the halves match correctly. These are the contents of Chapter 3.

- The Hessian estimate (1.7) and several other estimates on the heat flow approximation of the distance function need to be shown in the $\mathsf{RCD}(K,N)$ setting. This simply comes down to verifying the proofs of [33] all translate to the metric measure setting with minor adjustments. These are the contents of Chapter 4.

- Lastly, the argument in [33] relies on (1.4) to obtain estimates for small balls centred along in the interior γ under Ψ in order to start an induction process. Such an inequality is not available in the RCD setting since the Hessian is a measure-theoretic object, although progress has been made in this direction, see [12, 19, 29, 31]. Even if it were well-defined, one does not have Jacobi fields or smoothness arguments to translate such an inequality to a statement about tangent cones or small balls along γ. This is, in many ways, the main obstruction to extending the arguments of [33]. We will not attempt to develop all this theory. The key observation is that in fact we can do without the start of induction in radius.

  Recall that the need for this start of induction argument stems from the failure of (1.7) to control distance between a set of small measure and another set under Ψ. As such, it is possible for most of $\Psi_t(B_r(\gamma(s_1)))$ to distance from $\gamma(s_1 - t)$ quickly, after which we can no longer apply (1.7) to control the geometry of $\Psi_t(B_r(\gamma(s_1)))$. To deal with this, consider for each $x \in B_r(\gamma(s_1))$ a piecewise geodesic which goes from p to x and then x to q. It was shown in [33] that for a significant amount of x (those with relatively low excess) and their corresponding piecewise geodesics, the estimate (1.7) still holds on the scale of r. The same is true for $\mathsf{RCD}(-(N-1), N)$ spaces, see Theorem 4.0.12. Using this, one can make an induction argument in time instead. Suppose for some small time t most of $\Psi_t(B_r(\gamma(s_1)))$ stays close to $\gamma(s_1 - t)$, after which it leaves. Due to the control we had on the geometry of $\Psi_t(B_r(\gamma(s_1)))$ in that time, we can guarantee that $\Psi_t(B_r(\gamma(s_1)))$ is still very close to one of (in fact, much of) these other piecewise geodesics with a good estimate (1.7). Therefore, we can use the estimate for that piecewise geodesic for a little longer. The start of induction is trivial since the integral curves of Ψ are 1-Lipschitz. In this way, we arrive at an $x \in B_r(\gamma(s_1))$ whose trajectory under Ψ well represents the behaviour of $B_r(\gamma(s_1))$ under Ψ, in the sense that most of $B_r(\gamma(s_1))$ stays close to x on the scale of r under Ψ for a definite amount of time. Multiple limiting and gluing arguments then allow for the selection of a geodesic from p to q, perhaps different from γ, which well represents the behaviours of small balls centred in its interior under Ψ. The original argument of [33] gives the required Gromov-Hausdorff approximations in the interior of such a geodesic. Notice that, analogous to [33], we have at this point only shown the existence of a geodesic between p and q which satisfies the main theorem. These are the contents of Chapter 5.

The ideas outlined above overcome the difficulties of generalizing the arguments of [33] to the RCD setting. To finish, we will first show that RCD spaces are non-branching before proving Theorem 1.1.1. In order to prove non-branching, first notice that any two geodesics having the property above cannot branch. To see this, let γ_1 and γ_2 be two branching geodesics starting at some $p \in X$ which can be constructed by the methods of Chapter 5. In the interior, most of an arbitrarily small ball centred around γ_1 (resp. γ_2) must stay close to γ_1 (resp. γ_2) for some definite amount of time under the flow of Ψ, where the closeness is Hölder dependent on time. Moreover, it is possible to control how the volumes of balls changes along each geodesic. Combining these observations with the essentially non-branching property of $\mathsf{RCD}(K, N)$ spaces show that there cannot be any splitting because there is simply not enough room to flow disjoint small balls around γ_1 and γ_2

into a small ball around a branching point. While we do not initially claim all geodesics can be constructed with the methods of Chapter 5, our construction does give a certain amount of freedom. For any $\delta > 0$, it allows us to construct a geodesic γ^δ with nice properties on $[\delta, 1-\delta]$ which agrees with the initial geodesic γ at δ. As it turns out, combining this with the previous observation is enough to show that in fact no pair of geodesics can branch. Theorem 1.1.1 follows easily from the results of Chapter 5 and non-branching. These are the contents of Section 6.1. In Section 6.2, we generalize to the RCD setting the applications of the main result for Ricci limits outlined in [33] using verbatim arguments.

Chapter 2

Preliminaries

2.1 Curvature-dimension condition preliminaries

A *metric measure space (m.m.s.)* is a triple (X, d, m) where (X, d) is a complete, separable metric space and m is a nonnegative, locally finite Borel measure. As a matter of convention, m-measurable in this thesis means measurable with respect to the completion of $(X, \mathscr{B}(X), m)$. We take the same convention for all other Borel measures as well.

Given a complete and separable metric space (X, d), we denote by $\mathcal{P}(X)$ the set of Borel probability measures and by $\mathcal{P}_2(X)$ the set of Borel probability measure with finite second moment, that is, the set of $\mu \in \mathcal{P}(X)$ where $\int_X d(x, x_0) d\mu(x) < \infty$ for some $x_0 \in X$. Given $\mu_1, \mu_2 \in \mathcal{P}_2(X)$, the L^2-*Wasserstein distance* W_2 between them is defined as

$$W_2^2(\mu_1, \mu_2) := \inf_{\gamma} \int_{X \times X} d^2(x, y) d\gamma(x, y),$$

where the infimum is taken over all $\gamma \in \mathcal{P}(X \times X)$ with $(\pi_1)_*(\gamma) = \mu_1$ and $(\pi_2)_*(\gamma) = \mu_2$. Such measures γ are called *admissible plans* for the pair (μ_1, μ_2). $(\mathcal{P}_2(X), W_2)$ is called the L^2-Wasserstein space of (X, d) and has been well-studied in the theory of optimal transportation. A W_2-geodesic between $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ is any path $(\mu_t)_{t \in [0,1]}$ in $\mathcal{P}_2(X)$ satisfying $W_2(\mu_s, \mu_t) = |s - t| W_2(\mu_0, \mu_1)$ for any $s, t \in [0, 1]$. If (X, d) is a geodesic space then $(\mathcal{P}_2(X), W_2)$ is as well. A c-concave solution φ to the corresponding dual problem of maximizing $\int \varphi d\mu_0 + \int \varphi^c d\mu_1$ is called a *Kantorovich potential*. We refer to [2] and [84] for definitions and details.

The various notions of the classical curvature-dimension condition were first proposed independently in [68] and [82, 83] and are defined as certain convexity conditions on the L^2-Wasserstein space of a metric measure space. We follow closely the formulations of [17].

Given a m.m.s. (X, d, m), for any $\mu \in \mathcal{P}(X)$, the *Shannon-Boltzmann entropy* is defined as

$$\mathsf{Ent}_m(\cdot) : \mathcal{P}(X) \to (-\infty, \infty], \ \mathsf{Ent}_m(\mu) := \int \log \rho \, d\mu \ \text{, if } \mu = \rho m \text{ and } (\rho \log \rho)_- \text{ is } m\text{-integrable}$$

and ∞ otherwise.

Definition 2.1.1. ($\mathsf{CD}(K,\infty)$ condition) Let $K \in \mathbb{R}$. A m.m.s. (X,d,m) is a $\mathsf{CD}(K,\infty)$ space iff for any two absolutely continuous measures $\mu_0,\mu_1 \in \mathcal{P}(X)$ with bounded support, there exists a W_2-geodesic $\{\mu_t\}_{t\in[0,1]}$ such that for any $t \in [0,1]$,

$$\mathsf{Ent}_m(\mu_t) \leq (1-t)\,\mathsf{Ent}_m(\mu_0) + t\,\mathsf{Ent}_m(\mu_1) - \frac{K}{2}t(1-t)W_2^2(\mu_0,\mu_1).$$

The *N-Renyi entropy* is defined as

$$S_N(\cdot|m) : \mathcal{P}(X) \to (-\infty,0],\ S_N(\mu|m) := -\int \rho^{1-\frac{1}{N}}\,dm,$$

if $\rho^{1-\frac{1}{N}} \in L^1(m)$, where $\mu = \rho m$, and 0 otherwise.

Let $K \in \mathbb{R}$ and $N \in [1,\infty)$, the distortion coefficents $\sigma_{K,N}^{(t)}$ and $\tau_{K,N}^{(t)}$ are defined as follows:

$$(t,\theta) \in [0,1]\times\mathbb{R}^+ \to \sigma_{K,N}^{(t)}(\theta) := \begin{cases} \infty & \text{if } K\theta^2 \geq N\pi^2 \\ \frac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } 0 < K\theta^2 < N\pi^2, \\ t & \text{if } K\theta^2 = 0, \\ \frac{\sinh(t\theta\sqrt{K/N})}{\sinh(\theta\sqrt{K/N})} & \text{if } 0 < K\theta^2 < 0, \end{cases}$$

and

$$\tau_{K,N}^{(t)}(\theta) := t^{\frac{1}{N}}\sigma_{K,N}^{(t)}(\theta)^{1-\frac{1}{N}}.$$

The standard finite dimensional *curvature-dimension* condition was introduced in [68, 83].

Definition 2.1.2. ($\mathsf{CD}(K,N)$ condition) Let $K \in \mathbb{R}$ and $N \in [1,\infty)$. We say that a m.m.s. (X,d,m) is a $\mathsf{CD}(K,N)$ space if for any two absolutely continuous measures $\mu_0 = \rho_0 m, \mu_1 = \rho_1 m \in \mathcal{P}(X)$ with bounded support there exists a W_2-geodesic $\{\mu_t\}_{t\in[0,1]}$ and an associated optimal coupling π between μ_0 and μ_1 such that for any $t \in [0,1]$ and $N' \geq N$,

$$S_N(\mu_t|m) \leq -\int \left(\tau_{K,N'}^{(1-t)}(d(x,y))\rho_0(x)^{-\frac{1}{N'}} + \tau_{K,N'}^{(t)}(d(x,y))\rho_1(y)^{-\frac{1}{N'}}\right)d\pi(x,y).$$

The *reduced curvature-dimension* condition $\mathsf{CD}^*(K,N)$ was introduced in [17] for its seemingly better tensorization and globalization properties. It is defined by replacing τ with σ in Definition 2.1.2. The $\mathsf{CD}(K,N)$ and $\mathsf{CD}^*(K,N)$ conditions generalize to the metric measure setting the notion of Ricci curvature bounded below by K and dimension bounded above by N. Examples include (possibly weighted) Riemannian manifolds [83], Finsler manifolds [73] and Alexandrov spaces [75].

Remark 2.1.3. $\mathsf{CD}(K,N)$ implies $\mathsf{CD}(K',N')$ and $\mathsf{CD}^*(K',N')$ for all $K' \leq K$ and $N' \geq N$ as well as $\mathsf{CD}(K',\infty)$. A host of results that we cite were shown in the $\mathsf{RCD}^*(K,N)$ and $\mathsf{RCD}(K,\infty)$ setting, see 2.4.1 for definitions, and therefore apply in the $\mathsf{RCD}(K,N)$ setting. Going in the other direction, it was shown in [27] that $\mathsf{RCD}^*(K,N)$ is equivalent to $\mathsf{RCD}(K,N)$ when $m(X) < \infty$. It is believed that this argument can be taken to the noncompact case. We mention that the proofs of this thesis carry forward without modification to the $\mathsf{RCD}^*(K,N)$ setting. However, since several papers we cite use the stronger RCD assumption (though it can be checked this is not needed for the particular results we cite from them), we will do so as well to ease the burden of exposition.

It is known that if (X,d,m) is $\mathsf{CD}(K,N)$ then $\mathrm{supp}(m)$ is a geodesic space which also satisfies the $\mathsf{CD}(K,N)$ condition. Due to this, we will always assume $X = \mathrm{supp}(m)$. One can check that for any $\lambda, c > 0$, if (X,d,m) is $\mathsf{CD}(K,N)$, then $(X,\lambda d, cm)$ is $\mathsf{CD}(\frac{K}{\lambda^2}, N)$. $\mathsf{CD}(K,N)$ spaces, like their smooth counterparts, satisfy the standard Bishop-Gromov volume comparison.

Theorem 2.1.4. *(Bishop-Gromov volume comparison [83, Theorem 2.3]) Let (X,d,m) be a $\mathsf{CD}(K,N)$ space for some $K \in \mathbb{R}$ and $N \in (1,\infty)$. Then for all $x_0 \in X$ and all $0 < r < R \leq \pi\sqrt{N-1/(K\vee 0)}$ it holds:*

$$\frac{m(B_r(x_0))}{m(B_R(x_0))} \geq V_{K,N}(r,R) := \begin{cases} \dfrac{\int_0^r \left(\sin(t\sqrt{K/(N-1)})\right)^{N-1} dt}{\int_0^R \left(\sin(t\sqrt{K/(N-1)})\right)^{N-1} dt} & \text{if } K > 0, \\ \left(\frac{r}{R}\right)^N & \text{if } K = 0, \\ \dfrac{\int_0^r \left(\sinh(t\sqrt{K/(N-1)})\right)^{N-1} dt}{\int_0^R \left(\sinh(t\sqrt{K/(N-1)})\right)^{N-1} dt} & \text{if } K < 0. \end{cases} \tag{2.1}$$

In contexts where K and N are clear, we will simply write $V(r,R)$ for $V_{K,N}(r,R)$.

For $N < \infty$, $\mathsf{CD}(K,N)$ spaces are locally doubling, by Theorem 2.1.4, and are therefore proper. They satisfy a 1-1 Poincaré inequality by [76, Theorem 1.1].

2.2 First order calculus on metric measure spaces

We follow the framework for calculus on metric measure space developed by Ambrosio, Gigli and Savaré in [4–6, 46, 47]. Let (X,d,m) be a metric measure space. Let $\mathrm{lip}(X)$, $\mathrm{lip}_{loc}(X)$, $\mathrm{lip}_b(X)$ be its class of Lipschitz, locally Lipschitz, and bounded Lipschitz functions respectively. Given $f \in \mathrm{lip}_{loc}(X)$, the *local Lipschitz constant* (or *local slope*) $\mathrm{lip}(f) : X \to \mathbb{R}$ is defined by

$$\mathrm{lip}(f)(x) := \limsup_{y\to x} \frac{|f(y)-f(x)|}{d(y,x)}. \tag{2.2}$$

By convention, $\mathrm{lip}(f)(x) := 0$ at any isolated point x. Given $f \in L^2(m)$, a function $g \in L^2(m)$ is called a *relaxed gradient* if there exists a sequence $f_n \in \mathrm{lip}(X)$ and $\tilde{g} \in L^2(m)$ so that

1. $f_n \to f$ in $L^2(m)$ and $\mathrm{lip}(f_n)$ converges weakly to $\tilde{g}$ in $L^2(m)$

2. $g \geq \tilde{g}$ m-a.e..

A *minimal relaxed gradient* is a relaxed gradient that is minimal in L^2-norm in the family of relaxed gradients of f. If this family is non-empty, one can check that such a function exists and is unique m-a.e.. The minimal relaxed gradient is denoted by $|Df|$. The *domain of the Cheeger energy* $D(\mathsf{Ch}) \subseteq L^2(m)$ is the subset of L^2 functions with a minimal relaxed gradient. For $f \in L^2(m)$, the Cheeger energy is defined as

$$\mathsf{Ch}(f) = \begin{cases} \frac{1}{2}\int |Df|^2\, dm & \text{if } f \in D(\mathsf{Ch}), \\ \infty & \text{otherwise.} \end{cases}$$

Ch is a convex and lower semicontinuous functional on $L^2(m)$. The Cheeger energy first introduced in [25] was defined using a slightly different relaxation procedure. It is also possible to define a similar functional using the idea of *minimal weak upper gradients*, see [5, Section 5.1]. It is shown in [5, Section 6] that under mild assumptions on the metric measure space, satisfied, for example, by the various curvature-dimension conditions, all these notions are equivalent.

Remark 2.2.1. Let (X, d, m) be an $\mathsf{CD}(K, N)$ space with $N < \infty$. For any Lipschitz function f on X, $\mathrm{lip}(f) = |Df|$ m-a.e. This follows from [25], where it is shown that a metric measure space satisfying a Poincaré inequality and a doubling inequality has $\mathrm{lip}(f) = |Df|$ m-a.e..

$W^{1,2}(X) := D(\mathsf{Ch})$ is a Banach space endowed with the norm $\| f\|^2_{W^{1,2}(X)} := \| f\|^2_{L^2(m)} + \| |Df| \|^2_{L^2(m)}$. We define $W^{1,2}_{Loc}(X)$ as the space of all function $f \in L^2(X, m)$ so that $gf \in W^{1,2}(X)$ for every compactly supported, Lipschitz g. By the strong locality property of the minimal relaxed gradient (ie. $|Dg| = |Dh|$ m-a.e. in $\{g = h\}$ for any $g, h \in W^{1,2}(X)$), any $f \in W^{1,2}_{Loc}(X)$ has an associated differential $|Df| \in L^2_{loc}(m)$.

(X, d, m) is said to be *infinitesimally Hilbertian* if $W^{1,2}(X)$ is a Hilbert space. In this case, for $f, g \in W^{1,2}(X)$, one may define $\langle Df, Dg\rangle$ using polarization: $\langle Df, Dg\rangle := \frac{1}{2}(|D(f + g)|^2 - |Df|^2 - |Dg|^2) \in L^1(m)$.

From here one can define the *Laplacian*. $f \in W^{1,2}(X)$ is said to be in the *domain of the Laplacian* ($f \in D(\Delta)$) if there exists $\Delta f \in L^2(m)$ so that

$$\int g\Delta f\, dm + \int \langle Dg, Df\rangle\, dm = 0 \quad \text{for any } g \in W^{1,2}(X).$$

Given a subspace $V \in L^2(m)$, we denote $D_V(\Delta) := \{f \in D(\Delta) : \Delta(f) \in V\}$. More generally, one may define the *measure valued Laplacian*.

Definition 2.2.2. (Measure valued Laplacian [47, Definition 3.1.2]) The space $D(\mathbf{\Delta}) \subset W^{1,2}(X)$ is the space of $f \in W^{1,2}(X)$ such that there is a signed Radon measure μ satisfying

$$\int g d\mu = -\int \langle Dg, Df\rangle dm \qquad \forall g : X \to \mathbb{R} \text{ Lipschitz with bounded support.}$$

In this case the measure μ is unique and is denoted by $\mathbf{\Delta} f$.

2.3 Tangent, cotangent, and tensor modules

A technical framework for describing first order calculus on metric measure spaces and second order calculus on $\mathsf{RCD}(K, N)$ spaces was developed by Gigli in [47]. While aspects of second order calculus can be effectively developed without this framework (see for example [7, 79]), [47] crucially gives constructions which generalize the notion of tensor fields. In the next few sections, we will quickly introduce, sometimes informally, the necessary definitions given in [47] and refer to the original article for details and insights.

Let (X, d, m) be a metric measure space. The various collections of tensor fields of interest will be objects in the category of *$L^p(m)$-normed $L^\infty(m)$-modules*.

Definition 2.3.1. (L^p-normed L^∞-premodules [47, Definition 1.2.1 1.2.10]) Let $p \in [0, \infty]$. Let $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ be a Banach space endowed with a bilinear map $L^\infty(m) \times \mathcal{M} \ni (f, v) \mapsto f \cdot v \in \mathcal{M}$ and a function $|\cdot|\colon \mathcal{M} \to L^p_+(m)$. We say $(\mathcal{M}, \|\cdot\|_{\mathcal{M}}, \cdot, |\cdot|)$ is an L^p-normed L^∞-premodule iff the following holds

1. $(fg) \cdot v = f \cdot (g \cdot v)$ for all $f, g \in L^\infty(m)$ and $v \in \mathcal{M}$
2. $\mathbf{1} \cdot v = v$ for all $v \in \mathcal{M}$ where $\mathbf{1}$ is the constant function equal to 1
3. $\| |v| \|_{L^p(m)} = \| v \|_{\mathcal{M}}$ for all $v \in \mathcal{M}$
4. $|f \cdot v| = |f||v|$ m-a.e. for all $f \in L^\infty(m)$ and $v \in \mathcal{M}$.

We will often simply write fv for $f \cdot v$ and call $|\cdot|$ the pointwise norm. If an *$L^p(m)$-normed $L^\infty(m)$-premodule* satisfies additional locality and gluing properties ([47, Definition 1.2.1]), we say it is an *$L^p(m)$-normed $L^\infty(m)$-module*. One may localize such an object to some $A \in \mathscr{B}(X)$ by defining $\mathcal{M}|_A := \{v \in \mathcal{M} : |v| = 0 \; m\text{-a.e. on}\, A^c\}$, which is again canonically an $L^p(m)$-normed $L^\infty(m)$-module.

An L^2-normed L^∞-module which is a Hilbert space under $\|\cdot\|_{\mathcal{M}}$ is called a *Hilbert module*. In this case one can define a pointwise inner product by polarizing the pointwise norm $|\cdot|$. The prototypical example of a Hilbert module one has in mind is the collection of L^2 vector fields on a Riemannian manifold, where $|\cdot|$ is the Riemannian pointwise norm.

Given L^∞-modules $\mathcal{M}$ and $\mathcal{N}$, we say a map $T : \mathcal{M} \to \mathcal{N}$ is a *module morphism* if it is a bounded linear map between $\mathcal{M}$ and $\mathcal{N}$ as Banach spaces satisfying in addition $T(fv) = fT(v)$ for all $f \in L^\infty(m)$ and $v \in \mathcal{M}$. The *dual module* $\mathcal{M}^*$ is the space of all module morphisms between $\mathcal{M}$ and $L^1(m)$ and is an $L^{p_*}(m)$-normed $L^\infty(m)$-module, where $\frac{1}{p} + \frac{1}{p_*} = 1$. A Hilbert module $\mathcal{H}$ is canonically isomorphic to its dual.

Given two Hilbert modules $\mathcal{H}_1$ and $\mathcal{H}_2$, one can construct the Hilbert modules: the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ ([47, Definition 1.5.1]) and the exterior product $\mathcal{H}_1 \Lambda \mathcal{H}_2$ ([47, Definition 1.5.4]).

Definition 2.3.2. ([47, Definition 2.19]) Let $\mathcal{M}$ be an $L^2(m)$-normed and $V \subseteq \mathcal{M}$. Span(V) is defined as the collection of $v \in \mathcal{M}$ for which there is a Borel decomposition $(X_n)_{n \in \mathbb{N}}$ of X and, for each $n \in \mathbb{N}$, collections $v_{1,n}, ..., v_{k_n,n} \in V$ and $f_{1,n}, ..., f_{k_n,n} \in L^\infty(m)$ so that

$$\mathbf{1}_{X_n} v = \sum_{i=1}^{k_n} f_{i,n} v_{i,n} \text{ , for each } n \in \mathbb{N},$$

where $\mathbf{1}_{X_n}$ is the characteristic function of X_n. We say that V genereates $\mathcal{M}$ iff $\overline{\text{Span}\, V} = X$.

From this point we will assume (X, d, m) is a infinitesimally Hilbertian metric measure space. We now define the tangent and cotangent modules of (X, d, m).

Theorem 2.3.3. *([47, Proposition 2.2.5]) There exists a unique, up to isomorphism, Hilbert module $\mathcal{H}$ endowed with a linear map $d : W^{1,2}(X) \to \mathcal{H}$ satisfying:*

1. $|df| = |Df|$ *m-a.e. for all* $f \in W^{1,2}(X)$
2. $d(W^{1,2}(X))$ *generates* $\mathcal{H}$.

Such an $\mathcal{H}$ is called the cotangent module of (X, d, m) and denoted by $L^2(T^*X)$.
The dual of $L^2(T^*X)$ is called the tangent module of (X, d, m) and denoted by $L^2(TX)$. Elements of $L^2(TX)$ are called vector fields.
We denote by ∇f the dual of df (i.e. the unique element $\nabla f \in L^2(TX)$ so that $v(\nabla f) = \langle v, df \rangle$ for all $v \in L^2(T^*X)$).

Notice $\langle df, dg \rangle = df(\nabla g) = \langle \nabla f, \nabla g \rangle = \langle Df, Dg \rangle$ m-a.e. and we will use these interchangeably in the rest of the thesis depending on the convention of the theorems we are quoting. For a discussion of the philosophical differences of the objects involved, see [5, Section 2.2].

Definition 2.3.4. [47, Definition 2.3.11] $D(\text{div}) \subseteq L^2(TX)$ is the space of all vector fields $v \in L^2(TX)$ for which there exists $f \in L^2(m)$ so that for any $g \in W^{1,2}(X)$ the equality

$$\int fg \, dm = -\int dg(v) \, dm$$

holds. In this case f is called the *divergence of v* and denoted by $\text{div}(v)$. In particular, if $f \in D(\Delta)$, then $\nabla f \in D(\text{div})$ and $\text{div}(\nabla f) = \Delta f$.

Definition 2.3.5. $L^2((T^*)^{\otimes 2}(X))$ denotes the tensor product of $L^2(T^*X)$ with itself (see [47, Definition 1.5.1])). Similarly, $L^2(T^{\otimes 2}(X))$ denotes the tensor product of $L^2(TX)$ with itself. We will use $|\cdot|_{\text{HS}}$ and $\cdot : \cdot$ to denote the pointwise norm (Hilbert-Schmidt norm) and the pointwise inner product of Hilbert modules which arise from tensors.

$L^2((T^*)^{\otimes 2}(X))$ and $L^2(T^{\otimes 2}(X))$ are Hilbert module duals of each other. We mention that for any element $A \in L^2((T^*)^{\otimes 2}(X))$, we will often write $A(V, W) = A(V \otimes W)$. We will sometimes write this even when V and W are so that $V \otimes W$ is not in $L^2(T^{\otimes 2}(X))$. In all these cases, $V \otimes W$ when multiplied by the characteristic function of a compact set will be in $L^2(T^{\otimes 2}(X))$ and so $A(V, W)$ is well-defined as a measurable function by locality and satisfies

$$A(V, W) \leq |A|_{\mathrm{HS}}|V||W|. \tag{2.3}$$

We will usually have additional assumptions on $|V|$ and $|W|$ so that $A(V, W) \in L^1_{loc}(m)$.

2.4 RCD(K,N) and Bakry-Émery conditions

We now introduce the notion of RCD spaces, which are the main objects of interest for this thesis. These were proposed and carefully analyzed in a series of papers including [9, 41, 46] in the finite dimensional case and [3, 6] in the infinite dimensional case.

Definition 2.4.1. ([6, 41, 46]) Let (X, d, m) be a metric measure space, $K \in \mathbb{R}$, and $N \in [1, \infty)$. We say (X, d, m) satisfies the *Riemannian curvature-dimension condition* $\mathsf{RCD}(K, N)$ iff (X, d, m) satisfies the $\mathsf{CD}(K, N)$ condition and is infinitesimally Hilbertian. Similarly one defines the $\mathsf{RCD}^*(K, N)$ and $\mathsf{RCD}(K, \infty)$ conditions using $\mathsf{CD}^*(K, N)$ and $\mathsf{CD}(K, \infty)$ respectively.

The RCD condition is stable under measured Gromov-Hausdorff convergence and tensorization. Examples of RCD spaces include Ricci limits and Alexandrov spaces but non-Riemannian Finsler geometries are ruled out. We now state some equivalent formulations of the $\mathsf{RCD}(K, N)$ property. We will in general assume (X, d, m) is infinitesimally Hilbertian in this section.

As in [7], we define the *Carré du champ operator* for $f \in D_{W^{1,2}(X)}(\Delta)$ and $\varphi \in D_{L^\infty(m)\cap L^2(m)}(\Delta) \cap L^\infty(m)$ by

$$\Gamma_2(f; \varphi) := \int \frac{1}{2}|\nabla f|^2 \Delta\varphi dm - \int \langle \nabla f, \nabla \Delta f \rangle \varphi dm.$$

This enables us to state the non-smooth Bakry-Émery condition $\mathrm{BE}(K, N)$.

Definition 2.4.2. (Bakry-Émery condition [7, 41]) Let $K \in \mathbb{R}$ and $N \in [1, \infty]$. We say (X, d, m) satisfies the $\mathrm{BE}(K, N)$ condition iff

$$\Gamma_2(f; \varphi) \geq \frac{1}{N} \int (\Delta f)^2 \varphi dm + K \int |\nabla f|^2 \varphi dm.$$

$\mathrm{BE}(K, N)$ is closely related to $\mathsf{CD}(K, N)$. We say (X, d, m) satisfies the *Sobolev-to-Lipschitz property*, [44, Definition 3.15], if any function $f \in W^{1,2}(X)$ with $|\nabla f| \in L^\infty(m)$ has a Lipschitz representative $\tilde{f} = f$ m-a.e. with Lipschitz constant equal to $\operatorname{ess\,sup}(|\nabla f|)$.

Theorem 2.4.3. *([7, 9, 41]) Let (X, d, m) be a metric measure space satisfying an exponential growth condition (see [7, section 3]), $K \in \mathbb{R}$, and $N \in (1, \infty)$. (X, d, m) is* $\mathsf{RCD}(K, N)$ *iff (X, d, m) is infinitesimally Hilbertian, satisfies the Sobolev-to-Lipschitz property and the* $\mathrm{BE}(K, N)$ *condition.*

2.5 Heat flow and Bakry-Ledoux estimates

By applying the theory of the gradient flow of convex functionals on Hilbert spaces to Ch as in [5], one obtains for each $f \in L^2(m)$ a unique continuous curve $(H_t(f))_{t\in[0,\infty)}$ in $L^2(m)$ which is locally absolutely continuous in $(0,\infty)$ with $H_0(f) = f$ so that

$$\frac{d}{dt} H_t(f) = \Delta' H_t(f) \text{ for a.e. } t \in (0,\infty), \tag{2.4}$$

where $\Delta' g$ is defined as the minimizer in L^2 energy in $\partial^-(\mathsf{Ch})$ at g provided it is non-empty (see [5, Section 4.2]).

If (X, d, m) is infinitesimally Hilbertian, then for any $t > 0$, $H_t(f) \in D(\Delta)$ and one has the a priori estimates

$$\| H_t(f) \|_{L^2} \leq \| f \|_{L^2}, \quad \| |DH_t(f)| \|^2_{L^2} \leq \frac{\| f \|^2_{L^2}}{2t^2}, \quad \| \Delta H_t(f) \|_{L^2} \leq \frac{\| f \|_{L^2}}{t}.$$

$H_t(f)$ is linear and satisfies $\Delta(H_t(f)) = H_t(\Delta(f))$ for any $t > 0$. In particular, (2.4) is true for all $t \in (0,\infty)$ and

$$H_t(f) = f + \int_0^t \Delta(H_s(f))\, ds.$$

If (X, d, m) is $\mathsf{RCD}(K,\infty)$, H_t can be identified with $\mathscr{H}_t$, the gradient flow of Ent_m on $\mathcal{P}_2(X)$. Due to contraction properties coming from the RCD condition, $\mathscr{H}_t$ can be extended from $D(\mathsf{Ent}_m)$ to all of $\mathcal{P}_2(X)$.

Definition 2.5.1. For $t > 0$ and $x \in X$, $\mathscr{H}_t(\delta_x)$ is absolutely continuous with respect to m. Then $\mathscr{H}_t(\delta_x) = H_t(x,\cdot)m$, where $H_t(\cdot,\cdot)$ is the *heat kernel*.

$H_t(x, y)$ is symmetric and continuous in both variables. For each $f \in L^2(m)$, one has the representation formula, [6, Theorem 6.1],

$$H_t(f)(x) = \int f(y) H_t(x, y)\, dm. \tag{2.5}$$

The $\mathsf{RCD}(K,N)$ condition implies the Bakry-Ledoux estimate, which is a finite dimensional analogue of the Bakry-Émery contraction estimate ([6, Theorem 6.2]). Moreover, it was shown in [41] that one has equivalence in Theorem 2.4.3 with the BE(K,N) condition replaced by the (K,N) Bakry-Ledoux estimate.

Theorem 2.5.2. *(Dimensional Bakry-Ledoux L^2 gradient-Laplacian estimate [41, Theorem 4.3]) Let (X, d, m) be an $\mathsf{RCD}(K,N)$ space for some $K \in \mathbb{R}$ and $N \in [1,\infty)$. For any $f \in W^{1,2}(X)$ and $t > 0$,*

$$|\nabla(H_t(f))|^2 + \frac{4Kt^2}{N(e^{2Kt} - 1)} |\Delta H_t(f)|^2 \leq e^{-2Kt} H_t(|\nabla(f)|^2) \qquad m\text{-a.e.}.$$

Remark 2.5.3. If $|\nabla f|\in L^\infty$, one can take continuous representatives of $\Delta H_t(f)$ and $H_t(|\nabla(f)|^2)$ and identify $|\nabla(H_t(f))|$ canonically with the local Lipschitz constant of $H_t(f)$ to obtain a pointwise Bakry-Ledoux bound, see [41, Proposition 4.4]).

This implies the Sobolev-to-Lipschitz property by [6, Theorem 6.2]. H_t also has a L^∞-to-Lipschitz property by [6, Theorem 6.8], where it was shown for $t > 0$ and $f \in L^2(m)$ that

$$2I_{2K}(t)|\nabla H_t(f)|^2\leq H_t(f^2)\ \text{, } m\text{-a.e.,} \tag{2.6}$$

where $I_{2K}(t) := \int_0^t e^{2Ks}\,ds$.

2.6 Second order calculus and improved Bochner inequality

The class of *test functions* was introduced in [79] as

$$\mathrm{TestF}(X) := \left\{ f \in D(\Delta) \cap L^\infty \ :\ |Df|\in L^\infty \ \text{and}\ \Delta f \in W^{1,2}(X) \right\}.$$

It is known that TestF(X) is an algebra and, on $\mathsf{RCD}(K,\infty)$ spaces, it was shown by the results of [6] mentioned in the previous section that the heat flow approximations of an $L^\infty \cap L^2$ function are test functions and so TestF(X) is dense in $W^{1,2}(X)$.

In [57, 79], it was shown that under the BE(K,N) condition, $|\nabla f|^2\in D(\mathbf{\Delta})$ for any $f \in \mathrm{TestF}(X)$ and so one may define $\mathbf{\Gamma}_2(f) := \frac{1}{2}\mathbf{\Delta}|\nabla f|^2 - \langle \nabla f, \nabla\Delta f\rangle$. One can then define a Hessian for $f \in \mathrm{TestF}(X)$ and show the improved Bochner inequality (see 2.6.2). In Section 3 of [47], the same calculations were carried out in the framework proposed therein. We outline the main definitions and results from there. In this section, we assume (X, d, m) is an $\mathsf{RCD}(K,N)$ space.

Definition 2.6.1. ([47, Definition 3.3.1]) $W^{2,2}(X) \subseteq W^{1,2}(X)$ is the space of all functions $f \in W^{1,2}(X)$ for which there exists $A \in L^2((T^*)^{\otimes 2}(X))$ so that for any $g_1, g_2, h \in \mathrm{TestF}(X)$ the equality

$$\begin{aligned} 2\int hA(\nabla g_1, \nabla g_2)\,dm \\ = \int -\langle \nabla f, \nabla g_1\rangle \operatorname{div}(h\nabla g_2) - \langle \nabla f, \nabla g_2\rangle \operatorname{div}(h\nabla g_1) - h\langle \nabla f, \nabla\langle \nabla g_1, \nabla g_2\rangle\rangle\,dm \end{aligned}$$

holds. In this case A is called the *Hessian of f* and denoted by Hess f. $W^{2,2}(X)$ is a Hilbert space under the norm

$$\| f \|^2_{W^{2,2}(X)} := \| f \|^2_{L^2(m)} + \| df \|^2_{L^2(T^*X)} + \| \mathrm{Hess}\, f \|_{L^2((T^*)^{\otimes 2}(X))}.$$

It turns out that the test functions are contained in $W^{2,2}(X)$ and one has the improved Bochner inequality as in [57, 79].

Theorem 2.6.2. *(Improved Bochner inequality [47, Theorem 3.3.8]) Let (X, d, m) be an $\mathsf{RCD}(K,N)$ space for $K \in \mathbb{R}$ and $N \in [1,\infty)$ and $f \in \mathrm{TestF}(X)$. Then $f \in W^{2,2}(X)$ and*

$$\mathbf{\Gamma}_2(f) \geq [K|\nabla f|^2 + |\mathrm{Hess}(f)|^2_{\mathrm{HS}}]m.$$

$H^{2,2}(X)$ is then defined as the closure of TestF(X) in $W^{2,2}(X)$. An approximation argument gives the following.

Corollary 2.6.3. *([47, Corollary 3.3.9]) $D(\Delta) \subseteq W^{2,2}(X)$ and for $f \in D(\Delta)$,*

$$\int |\text{Hess } f|^2_{\text{HS}}\, dm \leq \int \left[(\Delta f)^2 - K|\nabla f|^2\right] dm.$$

Finally, we introduce the analogue of vector fields which have a first order (covariant) derivative.

Definition 2.6.4. ([47, Definition 3.4.1]) $W^{1,2}_C(TX) \subseteq L^2(TX)$ is the space of all $v \in L^2(TX)$ for which there exists $T \in L^2(T^{\otimes 2}(X))$ so that for any $g_1.g_2, h \in \text{TestF}(X)$ the equality

$$\int hT : (\nabla g_1 \otimes \nabla g_2) = \int -\langle v, \nabla g_2\rangle \operatorname{div}(h\nabla g_1) - h\,\text{Hess}(g_2)(v, \nabla g_2)\, dm$$

holds. In this case T is called the *covariant derivative* of v and denoted by ∇v. $W^{1,2}_C(TX)$ is a Hilbert space under the norm

$$\| v \|^2_{W^{1,2}_C(TX)} := \| v \|^2_{L^2(TX)} + \| \nabla v \|^2_{L^2(T^{\otimes 2}(X))}.$$

The class of *test vector fields* is defined as

$$\text{TestV}(X) := \left\{ \sum_{i=1}^{n} g_i \nabla f_i \ : \ n \in \mathbb{N},\ f_i, g_i \in \text{TestF}(X) \right\}.$$

By [47, Theorem 3.4.2], TestV(X) $\subseteq W^{1,2}_C(TX)$ and so $H^{1,2}_C(TX)$ is defined to be the closure of TestV(X) in $W^{1,2}_C(TX)$. For any $f \in W^{2,2}(X)$, Hess f and $\nabla(\nabla f)$ are dual under the duality of $L^2((T^*)^{\otimes 2}(X))$ and $L^2(T^{\otimes 2}(X))$.

2.7 Non-branching and essentially non-branching spaces

Given a geodesic metric space (X, d), we define the space of constant speed geodesics

$$\text{Geo}(X) := \{\gamma \in C([0,1], X) \ : \ d(\gamma(s), \gamma(t)) = |s - t| d(\gamma(0), \gamma(1))\ \forall s, t \in [0,1]\}.$$

For each $t \in [0, 1]$, $e_t : \text{Geo}(X) \to X$ defined by $e_t(\gamma) := \gamma(t)$ denotes the evaluation map at time t. On a complete and separable metric space (X, d), any W_2-geodesic has a *lifting* to a measure on the space of geodesics in the following sense.

Theorem 2.7.1. *[66, Theorem 3.2] Let $(\mu_t)_{t\in[0,1]}$ be a W_2-geodesic. Then there exists $\boldsymbol{\pi} \in \mathcal{P}(\text{Geo}(X))$ so that*

$$(e_t)_*(\boldsymbol{\pi}) = \mu_t \ \ \forall t \in [0,1],$$

$$|\dot{\mu}_t|^2 = \int |\dot{\gamma}_t|^2\, d\boldsymbol{\pi}(\gamma), \ \ \textit{for a.e. } t \in [0.1],$$

where $e_t(\gamma) := \gamma(t)$ is the evaluation map at time t.

These are called *optimal dynamical plans*. This motivates the following definition: for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, we denote by $\mathrm{OptGeo}(\mu_0, \mu_1)$ the space of all optimal dynamical plans from μ_0 to μ_1.

Definition 2.7.2. Given two geodesics $\gamma^1 \neq \gamma^2$ on a geodesic metric space (X, d). Assume γ^1, γ^2 are constant speed and parameterized on the unit interval. we say γ^1 and γ^2 *branch* if there exists $0 < t < 1$ such that $\gamma^1_s = \gamma^2_s$ for all $s \in [0, t]$. A subset $S \subseteq \mathrm{Geo}(X)$ is called a *set of non-branching geodesics* if there are no branching pairs in S. A geodesic metric space for which $\mathrm{Geo}(X)$ is itself a set of non-branching geodesics is called *non-branching*.

Many results were shown for various types of CD spaces under the additional non-branching assumption. These include the local-to-global property, tensorization property and local Poincaré inequality, see [17, 67, 82]. A weaker assumption was introduced in [78] for which these results generalize.

Definition 2.7.3. A metric measure space (X, d, m) is called *essentially non-branching* if for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ absolutely continuous with respect to m, any element of $\mathrm{OptGeo}(\mu_0, \mu_1)$ is concentrated on a set of non-branching geodesics.

$\mathsf{RCD}(K, \infty)$ spaces are shown to be essentially non-branching by the results of [6, 40] and [78]. We will frequently refer to the following theorem from [53] shown for finite dimensional $\mathsf{RCD}(K, N)$ spaces, see also [76, 78] for related results and [28] for the same result in the case of essentially non-branching $\mathsf{MCP}(K, N)$ spaces.

Theorem 2.7.4. *([53], [28, Theorem 1.1]) Let (X, d, m) be an $\mathsf{RCD}(K, N)$ space for some $K \in \mathbb{R}$ and $N \in [1, \infty)$. If $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ with $\mu_0 = \rho_0 m \ll m$, then there exists a unique $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$. $(e_t)_*(\nu) \ll m$ for any $t \in [0, 1)$ and such ν is given by a unique map $S : \mathrm{supp}(\mu_0) \to \mathrm{Geo}(X)$ in the sense that $\nu = S_*(\mu)$. Moreover, if μ_0, μ_1 have bounded support and $\|\rho_0\|_{L^\infty(m)} < \infty$, then*

$$\|\rho_t\|_{L^\infty(m)} \leq \frac{1}{(1-t)^N} e^{Dt\sqrt{(N-1)K^-}} \|\rho_0\|_{L^\infty(m)}, \quad \forall t \in [0, 1), \tag{2.7}$$

where $D := \mathrm{diam}(\mathrm{supp}(\mu_0) \cup \mathrm{supp}(\mu_1))$ and $K^- := \max\{-K, 0\}$

Remark 2.7.5. In particular, this implies for any $p \in X$, there is a unique geodesic between p and x for m-a.e. $x \in X$. Using the Kuratowski and Ryll-Nardzewski measurable selection theorem, one may select constant speed geodesics $\gamma_{x,p}$ from all $x \in X$ to p so that the map $X \times [0, 1] \ni (x, t) \mapsto \gamma_{x,p}(t)$ is Borel. This then guarantees such a choice is unique up to a set of measure 0. Similarly, one may select constant speed geodesics $\gamma_{x,y}$ for all $x, y \in X$ so that the map $X \times X \times [0, 1] \ni (x, y, t) \mapsto \gamma_{x,y}(t)$ is Borel. Using, for example, the arguments in [19, Section 4], the set of points $(x, y) \in X \times X$ connected by non-unique geodesics is analytic. $(m \times m)$-almost everywhere uniqueness of the Borel selection then follows using Fubini's theorem. In cases where we fix geodesics in this manner, we say we take a *Borel selection* of geodesics $\gamma_{x,p}$ from all $x \in X$ to p (or $\gamma_{x,y}$ from all $x \in X$ to all $y \in X$).

2.8 RCD(K,N) structure theory

We review of the structure theory of RCD spaces in this section. We will assume basic familiarity with pointed Gromov-Hausdorff (pGH) and pointed measure Gromov-Hausdorff (pmGH) convergence and refer to [11, 51, 84] for details.

A notion of considerable interest for $\mathsf{RCD}(K, N)$ spaces is that of measured tangents. Similar objects have been well-studied in the setting of Alexandrov spaces and Ricci limits (see, for example, [14] and [34] for an overview). Given a m.m.s. (X, d, m), $\bar{x} \in X$ and $r \in (0, 1)$, consider the *normalized rescaled pointed m.m.s. (p.m.m.s.)* $(X, r^{-1}d, m_r^{\bar{x}}, \bar{x})$ where

$$m_r^{\bar{x}} := \Big(\int_{B_r(\bar{x})} 1 - \frac{d(x, y)}{r} dm(y) \Big)^{-1} m.$$

In what follows let (X, d, m) be an $\mathsf{RCD}(K, N)$ space for some $K \in \mathbb{R}$ and $N \in (1, \infty)$. We define

Definition 2.8.1. (The collection of tangent spaces $\mathrm{Tan}(X, d, m, \bar{x})$) Let $\bar{x} \in X$. A p.m.m.s. $(Y, d_Y, m_Y, \bar{y})$ is called a *tangent (cone)* of (X, d, m) at $\bar{x}$ if there exists a sequence of radii $r_i \downarrow 0$ so that $(X, r_i^{-1}d, m_{r_i}^{\bar{x}}, \bar{x}) \to (Y, d_Y, m_Y, \bar{y})$ as $i \to \infty$ in the pmGH topology. The collection of all tangents of (X, d, m) at $\bar{x}$ is denoted $\mathrm{Tan}(X, d, m, \bar{x})$.

A standard compactness argument by Gromov shows that $\mathrm{Tan}(X, d, m, \bar{x})$ is non-empty for any $\bar{x} \in X$. The rescaling and stability properties of the $\mathsf{RCD}(K, N)$ condition under pmGH convergence (see [51, Theorem 7.2] and [6, Theorem 6.11]) show that every element of $\mathrm{Tan}(X, d, m, \bar{x})$ is an $\mathsf{RCD}(0, N)$ space.

Let $c_k = \int_{B_1(0)} 1 - |x| \, d\mathscr{L}(x)$ be the normalization constant of the k-dimensional Lebesgue measure and define the *k-dimensional regular set* $\mathcal{R}_k$ by

$$\mathcal{R}_k := \{x \in X \, : \, \mathrm{Tan}(X, d, m, x) = \{(\mathbb{R}^k, d_E, c_k \mathscr{H}^k, 0^k)\}\}.$$

Define $\mathcal{R}_{\mathrm{reg}} := \bigcup_{k=1}^{\lfloor N \rfloor} \mathcal{R}_k$ the *regular set* of X and $\mathcal{S} := X \backslash \mathcal{R}_{\mathrm{reg}}$ the *singular set*. In [50], it was shown that, for m-a.e. $x \in X$, there exists some $k \in \mathbb{N}$, $1 \le k \le N$ so that $(\mathbb{R}^k, d_E, c_k \mathscr{H}^k, 0^k) \in \mathrm{Tan}(X, d, m, x)$. This was improved in [72], where it was shown that each $\mathcal{R}_k$ is m-measurable and $m(\mathcal{S}) = 0$. A final improvement was made in [18] with the following theorem.

Theorem 2.8.2. *(Constancy of the dimension [18, Theorem 3.8]) Let (X, d, m) be an $\mathsf{RCD}(K, N)$ m.m.s. for some $K \in \mathbb{R}$ and $1 \le N < \infty$. Assume X is not a point. There exists a unique $n \in \mathbb{N}$, $1 \le n \le N$ so that $m(X \backslash \mathcal{R}_n) = 0$.*

The same theorem was proved in the case of Ricci limits in [33] from the main result there and as such also follows from Theorem 1.1.2, see Theorem 6.2.1. By [61, Theorem 1.2], see also [62, Theorem 1.9] in the case of Ricci limits, it is known that the unique n in Theorem 2.8.2 is also the largest integer n for which $\mathcal{R}_n$ is non-empty.

2.9 Additional RCD(K,N) theory

In this section we record several theorems for $\mathsf{RCD}(K,N)$ spaces which will be of use later.

Define the coefficients $\tilde{\sigma}_{K,N}(\cdot) : [0,\infty) \to \mathbb{R}$ by

$$\tilde{\sigma}_{K,N}(\theta) := \begin{cases} \theta\sqrt{\frac{K}{N}}\operatorname{cotan}\left(\theta\sqrt{\frac{K}{N}}\right), & \text{if } K > 0,\\ 1 & \text{if } K = 0,\\ \theta\sqrt{-\frac{K}{N}}\operatorname{cotanh}\left(\theta\sqrt{-\frac{K}{N}}\right), & \text{if } K < 0, \end{cases}$$

then one has the following sharp bound on the measure valued Laplacian of distance functions.

Theorem 2.9.1. *(Laplacian comparison for the distance function [46, Corollary 5.15]) Let* (X, d, m) *be a compact* $\mathsf{RCD}(K,N)$ *space for some* $K \in \mathbb{R}$ *and* $N \in (1,\infty)$. *For* $x_0 \in X$ *denote by* $d_{x_0} : X \to [0,\infty)$ *the function* $x \mapsto d(x, x_0)$. *Then*

$$\frac{d_{x_0}^2}{2} \in D(\mathbf{\Delta}) \quad \textit{with} \quad \mathbf{\Delta}\frac{d_{x_0}^2}{2} \le N\tilde{\sigma}_{K,N}(d_{x_0})m \quad \forall x_0 \in X$$

and

$$d_{x_0} \in D(\mathbf{\Delta}, X\backslash\{x_0\}) \quad \textit{with} \quad \mathbf{\Delta} d_{x_0}|_{X\backslash\{x_0\}} \le \frac{N\tilde{\sigma}_{K,N}(d_{x_0}) - 1}{d_{x_0}} m \quad \forall x_0 \in X.$$

$\mathbf{\Delta} d_{x_0}|_{X\backslash\{x_0\}}$ is defined similar to Definition 2.2.2, the difference being that the test functions g must be compactly supported in $X\backslash\{x_0\}$.

Remark 2.9.2. We mention, and will use the fact, that in the case where X is noncompact one can make essentially the same statement. Some small adjustments are needed since the Laplacians of d_{x_0} on $X\backslash\{x_0\}$ and $d_{x_0}^2$ on X are not naturally guaranteed to be signed Radon measures. To accomodate this, a weakening of the definition of the measure valued Laplacian was given in [31, Definition 2.11, 2.12]. This definition allows the Laplacian to be, more generally, a Radon functional (i.e. in $(C_C(X))'$). The difference is that a Radon functional has a representation as the difference of two possibly infinite positive Radon measures by the Riesz-Markov-Kakutani representation theorem, whereas in the case of a signed Radon measure, at least one of these must be finite. It was shown in [31, Corollary 4.17, 4.19] that the Laplacian comparison for the distance function holds as stated with this weaker definition. As such, we will, by a slight abuse of notation, treat the Laplacian of the distance function d_{x_0} on $X\backslash\{x_0\}$ as a signed Radon measure in the few instances where we use Theorem 2.9.1 in integration against compactly supported functions. Note that due to the comparison theorem, this Laplacian is locally a signed Radon measure, having at most an infinite negative part.

We will also need the Li-Yau Harnack inequality [69] and the Li-Yau gradient inequality [13,69]. These were proved for the RCD setting, in the finite measure case in [49], and in general in [58].

Theorem 2.9.3. *(Li-Yau Harnack inequality [49,58]) Let (X, d, m) be an $\mathsf{RCD}(K, N)$ space for some $K \in \mathbb{R}$ and $N \in [1, \infty)$. Let $f \in L^p(m)$ for some $p \in [1, \infty)$ be non-negative. If $K \geq 0$, then for every $x, y \in X$ and $0 < s < t$ it holds that*

$$(H_t f)(y) \geq (H_s f)(x) e^{-\frac{d^2(x,y)}{4(t-s)e^{\frac{2Ks}{3}}}} \Big(\frac{1 - e^{\frac{2K}{3}s}}{1 - e^{\frac{2K}{3}t}} \Big)^{\frac{N}{2}}.$$

If instead $K < 0$, then

$$(H_t f)(y) \geq (H_s f)(x) e^{-\frac{d^2(x,y)}{4(t-s)e^{\frac{2Kt}{3}}}} \Big(\frac{1 - e^{\frac{2K}{3}s}}{1 - e^{\frac{2K}{3}t}} \Big)^{\frac{N}{2}}.$$

Theorem 2.9.4. *(Li-Yau gradient inequality [49,58]) Let (X, d, m) be an $\mathsf{RCD}(K, N)$ space for some $K \in \mathbb{R}$ and $N \in [1, \infty)$. Let $f \in L^p(m)$ for some $p \in [1, \infty)$ be non-negative. Then for every $t > 0$ it holds that*

$$|\nabla H_t f|^2 \leq e^{-\frac{2Kt}{3}} (\Delta H_t f) H_t f + \frac{NK}{3} \frac{e^{-\frac{4Kt}{3}}}{1 - e^{-\frac{2Kt}{3}}} (H_t f)^2 \qquad m\text{-a.e.}.$$

$\mathsf{RCD}(K, N)$ spaces also satisfy the parabolic maximum principle, see [65, Section 3] and [42, Section 4.1] for full details.

Definition 2.9.5. ([65, Definition 3.1]) Let (X, d, m) be an $\mathsf{RCD}(K, N)$ space with $K \in \mathbb{R}$ and $N \in (1, \infty)$. Let I be an open interval in $\mathbb{R}$, Ω be an open subset of X, and $g \in L^2(\Omega)$. We say that a function $u : I \to W^{1,2}(\Omega)$ satisfies the parabolic equation

$$\frac{\partial}{\partial t} u - \Delta u \leq g, \quad \text{weakly in } I \times \Omega,$$

if for every $t \in I$, the Fréchet derivative of u, denoted by $\frac{\partial}{\partial t} u$, exists in $L^2(\Omega)$ and for any nonnegative function $\psi \in W^{1,2}(\Omega)$, it holds

$$\int_\Omega \frac{\partial}{\partial t} u(t, \cdot) \psi \, dm + \mathcal{E}(u(t, \cdot), \psi) \leq \int_\Omega g \psi \, dm.$$

Theorem 2.9.6. *(Parabolic maximum principle [65, Lemma 3.2]) Let (X, d, m) be an $\mathsf{RCD}(K, N)$ space with $K \in \mathbb{R}$ and $N \in (1, \infty)$. Fix $T \in (0, \infty]$ and open subset $\Omega \subseteq X$. Assume that a function $u : (0, T) \to W^{1,2}(\Omega)$, with $u_+(t, \cdot) = \max\{u(t, \cdot), 0\} \in W^{1,2}(\Omega)$ for any $t \in (0, T)$, satisfies the following equation with initial value condition:*

$$\begin{cases} \frac{\partial}{\partial t} u - \Delta u \leq 0, & \text{weakly in } (0, T) \times \Omega, \\ u_+(t, \cdot) \to 0, & \text{in } L^2(\Omega) \text{ as } t \to 0. \end{cases}$$

Then $u(t, x) \leq 0$ for any t in $(0, T)$ and m-a.e. x in Ω.

2.10 Mean value and integral excess inequalities

We refer to [33] in the smooth case, and [72] in the RCD case, for the proofs of the statements in this section. We start with the existence of good cut off functions.

Lemma 2.10.1. *(Existence of good cut off functions [72, Lemma 3.1]) Let* (X, d, m) *be an* $\mathsf{RCD}(K, N)$ *space for some* $K \in \mathbb{R}$ *and* $N \in [1, \infty)$*. Then for every* $x \subset X$*, for every* $R > 0$ *and* $0 < r < R$ *there exists a Lipschitz function* $\psi^r : X \to \mathbb{R}$ *satisfying:*

1. $0 \leq \psi^r \leq 1$ *on* X*,* $\psi^r \equiv 1$ *on* $B_r(x)$ *and* $supp(\psi) \subset B_{2r}(x)$*;*
2. $r^2|\Delta\psi^r| + r|\nabla\psi^r| \leq C(K, N, R)$ *m-a.e..*

For any subset C in a metric space, we denote by $T_r(C)$ the r-tubular neighbourhood of C and for $r_1 > r_0 > 0$, $A_{r_0,r_1}(C) := T_{r_1}(C)\backslash\overline{T_{r_0}(C)}$ the (r_0, r_1)-annular neighbourhood of C.

Lemma 2.10.2. *(Existence of good cut off functions on annular neighbourhoods [72, Lemma 3.2]) Let* (X, d, m) *be an* $\mathsf{RCD}(K, N)$ *space for some* $K \in \mathbb{R}$ *and* $N \in [1, \infty)$*. Then for every closed subset* $C \subset X$*, for every* $R > 0$ *and* $0 < 10r_0 < r_1 \leq R$ *there exists a Lipschitz function* $\psi : X \to \mathbb{R}$ *satisfying:*

1. $0 \leq \psi \leq 1$ *on* X*,* $\psi \equiv 1$ *on* $A_{3r_0,\frac{r_1}{3}}(C)$ *and* $supp(\psi) \subset A_{2r_0,\frac{r_1}{2}}(C)$*;*
2. $r_0^2|\Delta\psi| + r_0|\nabla\psi| \leq C(K, N, R)$ *m-a.e. on* $A_{2r_0,3r_0}(C)$*;*
3. $r_1^2|\Delta\psi| + r_1|\nabla\psi| \leq C(K, N, R)$ *m-a.e. on* $A_{\frac{r_1}{3},\frac{r_1}{2}}(C)$*.*

Remark 2.10.3. Note that the gradient bounds in 2.10.1 and 2.10.2 are naturally m-a.e.. However, since the proof involves only 2.5.2, by Remark 2.5.3, choosing continuous representatives and using the local Lipschitz constant of ψ for $|\nabla\psi|$, these statements can be made pointwise.

As demonstrated in [33], and later for the RCD setting in [72], several key estimates, including heat kernel bounds, the mean value, L^1-Harnack, and integral Abresch-Gromoll inequalities can be proved starting from the existence of good cut off functions and the Li-Yau Harnack inequality 2.9.3.

Lemma 2.10.4. *(Heat kernel bounds [72, lemma 3.3]) Let* (X, d, m) *be an* $\mathsf{RCD}(K, N)$ *space for some* $K \in \mathbb{R}$*,* $N \in (1, \infty)$ *and let* $H_t(x, y)$ *be the heat kernel for some* $x \in X$*. Then for every* $R > 0$*, for all* $0 < r < R$ *and* $t \leq R^2$*,*

1. *if* $y \in B_{10\sqrt{t}}(x)$*, then* $\frac{C^{-1}(K,N,R)}{m(B_{10\sqrt{t}}(x))} \leq H_t(x, y) \leq \frac{C(K,N,R)}{m(B_{10\sqrt{t}}(x))}$
2. $\int_{X\backslash B_r(x)} H_t(x, y)dm(y) \leq C(K, N, R)r^{-2}t.$

Lemma 2.10.5. *(Mean value and L^1-Harnack inequality [72, Lemma 3.4]) Let (X, d, m) be an $\mathsf{RCD}(K, N)$ space for some $K \in \mathbb{R}$, $N \in (1, \infty)$ and let $0 < r < R$. If $u : X \times [0, r^2] \to \mathbb{R}$, $u(x, t) = u_t(x)$, is a nonnegative Borel function with compact support at each time t and satisfying $(\partial_t - \Delta)u \geq -c_0$ in the weak sense, then,*

$$\fint_{B_r(x)} u_0 \leq C(K, N, R)[u_{r^2}(x) + c_0 r^2] \text{ for } m\text{-a.e. } x.$$

More generally the following L^1-Harnack inequality holds

$$\fint_{B_r(x)} u_0 \leq C(K, N, R)[\operatorname*{ess\,inf}_{y \in B_r(x)} u_{r^2}(y) + c_0 r^2] \ \ \forall x \in X.$$

Remark 2.10.6. Lemma 3.4 of [72], whose proof follows [33, Lemma 2.1], treats the continuous case of u. However, in what follows we will want to use this inequality for $u_t = |\nabla h_t(f)|$, which is not known to have a continuous representative. The proof of this statement for Borel u follows exactly as in the continuous case with the obvious measure-theoretic adjustments.

Applying 2.10.5 to a function which is constant in time gives the following classical mean value inequality.

Corollary 2.10.7. *(Classical mean value inequality [72, Corollary 3.5]) Let (X, d, m) be an $\mathsf{RCD}(K, N)$ space for some $K \in \mathbb{R}$, $N \in (1, \infty)$ and let $0 < r < R$. If $u : X \to \mathbb{R}$ is a nonnegative Borel function with compact support with $u \in D(\mathbf{\Delta})$ and satisfies $\mathbf{\Delta} u \leq c_0 m$ in the sense of measures, then for $0 < r \leq R$,*

$$\fint_{B_r(x)} u \leq C(K, N, R)[u(x) + c_0 r^2] \text{ for } m\text{-a.e. } x.$$

This, in combination with the existence of good cut off functions and Laplacian estimates on distance functions, allows one to prove an integral Abresch-Gromoll inequality. For points p and q in a metric space, we define the excess function $e_{p,q}(x) := d(p, x) + d(x, q) - d(p, q)$.

Theorem 2.10.8. *(Integral Abresch-Gromoll inequality [72, Theorem 3.6]) Let (X, d, m) be an $\mathsf{RCD}(K, N)$ space for some $K \in \mathbb{R}$, $N \in (1, \infty)$; let $p, q \in X$ with $d_{p,q} := d(p, q) \leq 1$ and fix $0 < \epsilon < 1$.*
If $x \in A_{\epsilon d_{p,q}, 2d_{p,q}}(\{p, q\})$ satisfies $e_{p,q}(x) \leq r^2 d_{p,q} \leq \bar{r}(K, N, \epsilon)^2 d_{p,q}$, then

$$\fint_{B_{rd_{p,q}}(x)} e_{p,q}(y) dm(y) \leq C(K, N, \epsilon) r^2 d_{p,q}.$$

Combined with Bishop-Gromov volume comparison, this immediately implies the classical Abresch-Gromoll inequality [1], see also [48].

Corollary 2.10.9. *(Classical Abresch-Gromoll inequaltiy [72, Corollary 3.7]) Let (X, d, m) be an $\mathsf{RCD}(K, N)$ space for some $K \in \mathbb{R}$, $N \in (1, \infty)$; let $p, q \in X$ with $d_{p,q} := d(p, q) \leq 1$ and fix*

$0 < \epsilon < 1$.

If $x \in A_{\epsilon d_{p,q}, 2d_{p,q}}(\{p,q\})$ *satisfies* $e_{p,q}(x) \leq r^2 d_{p,q} \leq \bar{r}(K,N,\epsilon)^2 d_{p,q}$, *then there exists* $\alpha(N) \in (0,1)$ *such that*

$$e_{p,q}(y) \leq C(K,N,\epsilon) r^{1+\alpha(N)} d_{p,q}\,,\ \forall y \in B_{r d_{p,q}}(x).$$

Chapter 3

Differentiation formulas for Regular Lagrangian flows

3.1 Regular Lagrangian flow

In what follows, we will always be on some $\mathsf{RCD}(K, N)$ space (X, d, m) for $K \in \mathbb{R}$ and $N \in [1, \infty)$. In [33], the crucial idea is to understand the geometric properties of the gradient flow with respect to heat flow approximations of the distance function. We will do the same with the Regular Lagrangian flow which was first introduced by Ambrosio in [8] on $\mathbb{R}^d$ and generalized to the metric measure setting by Ambrosio-Trevisan in [10]. The setup is quite general and we refer to [10] for full details. We will be interested in applying this theory specifically for vector fields in the L^2 tangent module of an RCD space.

Definition 3.1.1. (Time-dependent L^2 vector fields) Let $T > 0$. $V : [0, T] \to L^2(TX)$ is a time-dependent L^2 vector field iff the map $[0, T] \ni t \mapsto V_t \in L^2(TX)$ is Borel.
V is bounded iff

$$\| V \|_{L^\infty} := \| |V| \|_{L^\infty([0,T]\times X)} < \infty.$$

$V \in L^1([0, T], L^2(TX))$ iff

$$\int_0^T \| V_t \|_{L^2(TX)} dt < \infty.$$

Definition 3.1.2. (Regular Lagrangian flow) Given a time-dependent L^2 vector field (V_t). A Borel map $F : [0, T] \times X \to X$ is a *Regular Lagrangian flow (RLF)* to V_t iff the following holds:

1. $F_0(x) = x$ and $[0, T] \ni t \mapsto F_t(x)$ is continuous for every $x \in X$;

2. For every $f \in \mathrm{TestF}(X)$ and m-a.e. $x \in X$, $t \mapsto f(F_t(x))$ is in $W^{1,1}([0, T])$ and

$$\frac{d}{dt} f(F_t(x)) = df(V_t)(F_t(x)) \quad \text{for a.e. } t \in [0, T]; \tag{3.1}$$

3. There exists a constant $C := C(V)$ so that $(F_t)_* m \leq Cm$ for all t in $[0, T]$.

Remark 3.1.3. In the case where $V \in L^1([0, T], L^2(TX))$ and F is an RLF of V, using a standard Fubini's theorem argument and that TestF(X) is dense in $W^{1,2}(X)$, we have for every $f \in W^{1,2}_{loc}(X)$, $t \mapsto f(F_t(x))$ is in $W^{1,1}([0, T])$ and

$$\frac{d}{dt} f(F_t(x)) = df(V_t)(F_t(x)) \quad \text{for a.e. } t \in [0, T]. \tag{3.2}$$

[10] gives the existence and uniqueness of RLFs to (V_t) in a certain class of vector fields. We use the following weaker formulation of their result and note that only a bound on the symmetric part of ∇V_t is needed.

Theorem 3.1.4. *(Existence and uniqueness of Regular Lagrangian flow [10]) Let* $(V_t) \in L^1([0, T],$ $L^2(TX))$ *satisfy* $V_t \in D(div)$ *for a.e.* $t \in [0, T]$ *with*

$$div(V_t) \in L^1([0, T], L^2(m)) \quad (div(V_t))^- \in L^1([0, T], L^\infty(m)) \quad \nabla V_t \in L^1([0, T], L^2(T^{\otimes 2} X)).$$

There exists a unique, up to m-a.e. equality, RLF $(F_t)_{t \in [0,T]}$ *for* (V_t)*. The bound*

$$(F_t)_*(m) \leq exp\left(\int_0^t \| \operatorname{div}(V_s)^- \|_{L^\infty(m)} \, ds\right) m \tag{3.3}$$

holds for every $t \in [0, T]$.

Remark 3.1.5. It was pointed out to the author by Nicola Gigli that the estimate (3.3) can be localized for any $S \in \mathscr{B}(X)$,

$$(F_t)_*(m|_S) \leq exp\left(\int_0^t \| \operatorname{div}(V_s)^- \|_{L^\infty((F_s)_*(m|_S))} \, ds\right) m.$$

This follows from [10, (4-22)] choosing $\beta(z) := z^p$ for $p \to \infty$.

For (F_t) an RLF to some (V_t), we will be interested in expressing $\frac{d}{dt} d(F_t(x), F_t(y))$ in two ways: using V_t in a first order variation formula and ∇V_t in a second order formula, see (1.8), which we show in section 3.3.

Proposition 3.1.6. *(First order differentiation formula along RLFs) Let* $T > 0$ *and* $U, V \in L^1([0, T],$ $L^2(TX))$*. If* $(F_t), (G_t)$ *are the Regular Lagrangian flows of* $(U_t), (V_t)$ *respectively, then for m-a.e.* $x, y \in X$, $d(F_t(x), G_t(y)) \in W^{1,1}([0, T])$ *and*

$$\frac{d}{dt} d(F_t(x), G_t(y)) = \langle \nabla d_{G_t(y)}, U_t \rangle (F_t(x)) + \langle \nabla d_{F_t(x)}, V_t \rangle (G_t(y)) \quad \text{for a.e. } t \in [0, T].$$

Proof. It is known that $\mathsf{RCD}(K, \infty)$ spaces have the tensorization of Cheeger energy property from [6, Theorem 6.17] and the density of the product algebra property from [18, Proposition A.1], see also [52, Definition 3.8, 3.9] for definitions. Consider the vector field (W_t) defined by requiring, for all $f \in W^{1,2}(X \times X)$,

$$\langle W_t, \nabla f\rangle(x,y) = \langle U_t, \nabla f_y\rangle(x) + \langle V_t, \nabla f_x\rangle(y),$$

for $(m \times m)$-a.e. $(x, y) \in X \times X$. The tensorization of Cheeger energy is used implicitly in this definition and the vector field is naturally in $L^1([0,T], L^2_{loc}(T(X \times X)))$. We refer to [52] for a rigorous treatment of locally L^2 vector fields and the corresponding theory of RLFs. We mention a slightly more careful, alternative definition of (W_t) was also given in [52, Proposition 3.7, Theorem 3.13], where the expected decomposition of the module $L^0(T^*(X \times X))$ was shown for spaces with tensorization of Cheeger energy and density of the product algebra properties. By [18, Proposition A.2], (F_t, G_t) is an RLF of (W_t), from which the proposition follows by definition of (W_t). □

3.2 Continuity equation

We give a brief summary of the theory of continuity equations in this section. These are intimately related to Regular Lagrangian flows but provide a more convenient language for the discussion of local flows in cases where RLFs, which as defined are of a global nature, may not exist.

Definition 3.2.1. Curves of bounded compression [47, Definition 2.3.21] We say a curve $(\mu_t)_{t\in[0,T]} \subseteq \mathcal{P}_2(X)$ is a curve of bounded compression iff

1. It is W_2-continuous;
2. For some $C > 0$, $\mu_t \leq Cm$ for every $t \in [0,T]$.

Definition 3.2.2. (Solutions of continuity equation [43], [47, Definition 2.3.22]) Let $(\mu_t)_{t\in[0,T]} \subseteq \mathcal{P}_2(X)$ be a curve of bounded compression and $V \in L^1([0,T], L^2(TX))$. We say that (μ_t, V_t) solves the continuity equation

$$\frac{d}{dt}\mu_t + \mathrm{div}(V_t\mu_t) = 0$$

iff, for every $f \in W^{1,2}(X)$, the map $t \mapsto \int f\, d\mu_t$ is absolutely continuous and satisfies

$$\frac{d}{dt}\int f\, d\mu_t = \int df(V_t)\, d\mu_t \quad \text{for a.e. } t \in [0,T]. \tag{3.4}$$

Remark 3.2.3. By abuse of notation we will sometimes say (μ_t) solves the continuity equation $\frac{d}{dt}\mu_t + \mathrm{div}(V_t\mu_t) = 0$ for some vector field (V_t) which is only locally L^2, for example, $V_t = -\nabla d_p$ for some $p \in X$. In this case, (μ_t) is always compactly supported for every t and it is understood that we cut off the vector field V_t outside of this support.

As shown in [10], RLFs are very closely related to the solutions of continuity equations; they can be thought of as realizations of these solutions as maps on the space itself.

Theorem 3.2.4. *([10]) Let (V_t) satisfy the conditions of 3.1.4 and (F_t) be the corresponding unique Regular Lagrangian flow. If $\mu_0 \in \mathcal{P}_2(X)$ with bounded density, then $\mu_t := (F_t)_*(\mu_0)$ is a distributional solution of the continuity equation $\frac{d}{dt}\mu_t + \mathrm{div}(V_t\mu_t) = 0$.*

Remark 3.2.5. The existence and uniqueness of solutions to $\frac{d}{dt}\mu_t + \text{div}(V_t\mu_t) = 0$ starting at some μ_0 of bounded density is proved in [10] for (V_t) satisfying the conditions of Theorem 3.1.4. In fact, the existence and uniqueness of RLFs in 3.1.4 is shown in part by using the existence and uniqueness on the level of continuity equations combined with a superposition principle.

RLFs from vector fields with a two-sided divergence bound are m-a.e. invertible. To be precise,

Proposition 3.2.6. *Let $(V_t) \in L^1([0,T], L^2(TX))$ satisfy $V_t \in D(div)$ for a.e. $t \in [0,T]$ with*

$$div(V_t) \in L^1([0,T], L^2(m)) \quad div(V_t) \in L^1([0,T], L^\infty(m)) \quad \nabla V_t \in L^1([0,T], L^2(T^{\otimes 2}X)).$$

Let (F_t) be the unique RLF of $(V_t)_{t\in[0,T]}$ and (G_t) be the unique RLF of $(-V_{T-t})_{t\in[0,T]}$. For m-a.e. $x \in X$ and any $0 \le t \le T$,

$$G_t(F_T(x)) = F_{T-t}(x).$$

Proof. We first show $G_T(F_T(x)) = x$ for m-a.e. $x \in X$. Define the time-dependent L^2 vector field $(W_t)_{t\in[0,2T]}$ by

$$W_t := \begin{cases} V_t & \text{if } 0 \le t \le T \\ -V_{2T-t} & \text{if } T < t \le 2T. \end{cases}$$

For any μ with compact support and bounded density, $(\mu_t)_{t\in[0,2T]}$ defined by

$$\mu_t := \begin{cases} (F_t)_*(\mu) & \text{if } 0 \le t \le T \\ (G_{t-T})_*((F_T)_*(\mu)) & \text{if } T < t \le 2T \end{cases}$$

solves the continuity equation $\frac{d}{dt}\mu_t + \text{div}(W_t\mu_t) = 0$ by Theorem 3.2.4. This in particular means $(\mu_t)_{t\in[0,T]}$ solves the continuity equation $\frac{d}{dt}\mu_t + \text{div}(V_t\mu_t) = 0$ on $[0,T]$. It is then easy to check by definition that

$$\mu'_t := \begin{cases} \mu_t & \text{if } 0 \le t \le T \\ \mu_{2T-t} & \text{if } T < t \le 2T \end{cases}$$

solves the continuity equation $\frac{d}{dt}\mu'_t + \text{div}(W_t\mu'_t) = 0$ as well. By uniqueness, see Remark 3.2.5, $(G_T)_*((F_T)_*(\mu)) = \mu$. Since this is true for any μ with compact support and bounded density, we conclude $G_T(F_T(x)) = x$ for m-a.e. $x \in X$.

By the same argument for each t in the countable set $\mathbb{Q} \cap [0,T]$, we have for m-a.e. $x \in X$ and any $t \in \mathbb{Q} \cap [0,T]$,

$$G_t(F_T(x)) = F_{T-t}(x).$$

The proposition follows by continuity of $G_t(x)$ and $F_t(x)$ in t for all $x \in X$. □

We recall the following result from [45] which in particular implies W_2-geodesics with uniformly bounded densities are solutions of continuity equations.

Theorem 3.2.7. *([54, Theorem 1.1]) Let μ_t be a W_2-geodesic with compact support and $\mu_t \leq Cm$ for every $t \in [0,1]$ and some $C > 0$. If $f \in W^{1,2}(X)$ then the map $[0,1] \ni t \mapsto \int f d\mu_t$ is $C^1([0,1])$ and*

$$\frac{d}{dt}\int f d\mu_t = \int \langle \nabla f, \nabla \phi_t \rangle \, d\mu_t,$$

where ϕ_t is any function such that for some $s \neq t$, $s \in [0,1]$, the function $-(s-t)\phi_t$ is a Kantorovich potential from μ_t to μ_s.

The corollary below then follows by making the same type of arguments as in [10, Section 7].

Corollary 3.2.8. *Let $p \in X$ and $f \in W^{1,2}(X)$ fixing a representative. For m-a.e. $x \in X$, the map $t \mapsto f(\gamma_{x,p}(t))$ is in $W^{1,1}_{loc}([0, d_{x,p}))$ and*

$$\frac{d}{dt} f(\gamma_{x,p}(t)) = -df(\nabla d_p)(\gamma_{x,p}(t)) \;\; \textit{for a.e. } t \in [0, d_{x,p}),$$

where $\gamma_{x,p}$ is a unit speed geodesic from x to p.

Proof. By Remark 2.7.5, we take a Borel selection of $\gamma_{x,p}$ which is unique for m-a.e. x. For each $x \in X$, let $\tilde{\gamma}_{x,p} : [0,1] \to X$ be the constant speed reparameterization of γ.

First consider a Lipschitz representative of $f \in \mathrm{TestF}(X)$. Clearly $f(\tilde{\gamma}_{x,p}(t))$ is continuous on $t \in [0,1]$ for each x. We show

1. for m-a.e. x, $\frac{-d(\tilde{\gamma}_{x,p}(t),p)}{1-t}\langle \nabla f, \nabla d_p \rangle(\tilde{\gamma}_{x,p}(t)) \in L^1_{loc}([0,1))$;
2. for m-a.e. x, $f(\tilde{\gamma}_{x,p}(b)) - f(\tilde{\gamma}_{x,p}(a)) = \int_a^b \frac{-d(\tilde{\gamma}_{x,p}(t),p)}{1-t}\langle \nabla f, \nabla d_p \rangle(\tilde{\gamma}_{x,p}(t))\, dt$ for any $0 \leq a < b < 1$.

For any μ with compact support and bounded density with respect to m, define $\mu_t := (\tilde{\gamma}_{\cdot,p}(t))_*(\mu)$ for $t \in [0,1]$. $(\mu_t)_{t\in[0,1]}$ is a W_2-geodesic. By 2.7.4, for any $\delta > 0$, $(\mu_t)_{t\in[0,1-\delta]}$ is of uniformly bounded density. By Theorem 3.2.7, the map $[0,1-\delta] \ni t \mapsto \int f d\mu_t$ is in $C^1([0,1-\delta])$ and

$$\frac{d}{dt}\int f d\mu_t = \int \frac{-d(x,p)}{1-t}\langle \nabla f, \nabla d_p \rangle(x) d\mu_t(x). \tag{3.5}$$

Fix a representative of $\langle \nabla f, \nabla d_p \rangle \in L^2(m)$.

Proof of 1: The map $t \mapsto \int \frac{-d(x,p)}{1-t}\langle \nabla f, \nabla d_p \rangle(x)\, d\mu_t(x)$ is in $L^1_{loc}([0,1))$ since $\langle \nabla f, \nabla d_p \rangle(x) \in L^1_{loc}(m)$ and μ_t has uniformly bounded support and locally uniformly bounded density on $[0,1)$. By Fubini's theorem, this implies for μ-a.e. x, $\frac{-d(\tilde{\gamma}_{x,p}(t),p)}{1-t}\langle \nabla f, \nabla d_p \rangle(\tilde{\gamma}_{x,p}(t)) \in L^1_{loc}([0,1))$. Since this is true starting at any measure μ with compact support and bounded density with respect to m, statement 1 follows.

Proof of 2: By continuity of $f(\tilde{\gamma}_{x,p}(t))$ in t, it is enough to show that, for m-a.e. x,

$$f(\tilde{\gamma}_{x,p}(\frac{k}{n})) - f(\tilde{\gamma}_{x,p}(\frac{k-1}{n})) = \int_{\frac{k-1}{n}}^{\frac{k}{n}} \frac{-d(\tilde{\gamma}_{x,p}(t),p)}{1-t}\langle \nabla f, \nabla d_p \rangle(\tilde{\gamma}_{x,p}(t))\, dt$$

for any $n \in \mathbb{N}$ and $1 \leq k \leq n-1$. Assume this is not the case, then there exists some n, k and a bound set S with $0 < m(S) < \infty$ so that for each $x \in S$, without loss of generality,

$f(\tilde{\gamma}_{x,p}(\frac{k}{n})) - f(\tilde{\gamma}_{x,p}(\frac{k-1}{n})) > \int_{\frac{k-1}{n}}^{\frac{k}{n}} \frac{-d(\tilde{\gamma}_{x,p}(t),p)}{1-t} \langle \nabla f, \nabla d_p \rangle (\tilde{\gamma}_{x,p}(t))\, dt$. Applying (3.5) to a part of the Wasserstein geodesic from the normalization of $m|_S$ to δ_p gives a contradiction.

The general case of $f \in W^{1,2}(X)$ then follows by an approximation argument. Choose a sequence $f_i \in \mathrm{TestF}(X)$ converging to f in $W^{1,2}(X)$. Using a diagonalization argument with Borel-Cantelli lemma and Fubini's theorem, there exists some subsequence f_i so that for m-a.e. $x \in X$, $f_i(\tilde{\gamma}_{x,p}(t)) \to f(\tilde{\gamma}_{x,p}(t))$ in $L^1_{loc}([0,1))$ as functions of t.

For any μ with compact support and bounded density, and μ_t defined as before, we also have

$$\int \frac{-d(x,p)}{1-t} \langle \nabla f_i, \nabla d_p \rangle (x)\, d\mu_t(x) \to \int \frac{-d(x,p)}{1-t} \langle \nabla f, \nabla d_p \rangle (x)\, d\mu_t(x) \quad \text{in} \quad L^1_{loc}([0,1)).$$

Another diagonalization argument with Borel-Cantelli lemma and Fubini's theorem gives a further subsequence f_i so that for m-a.e. $x \in X$,

$$\frac{-d(\tilde{\gamma}_{x,p}(t),p)}{1-t} \langle \nabla f_i, \nabla d_p \rangle (\tilde{\gamma}_{x,p}(t)) \to \frac{-d(\tilde{\gamma}_{x,p}(t),p)}{1-t} \langle \nabla f, \nabla d_p \rangle (\tilde{\gamma}_{x,p}(t)) \quad \text{in} \quad L^1_{loc}([0,1)).$$

Combining these with statements 1 and 2, we have for any $f \in W^{1,2}(X)$, for m-a.e. $x \in X$, the map $t \mapsto f(\tilde{\gamma}_{x,p}(t))$ is in $W^{1,1}_{loc}([0,1))$ and

$$\frac{d}{dt} f(\tilde{\gamma}_{x,p}(t)) = \frac{-d(\tilde{\gamma}_{x,p}(t),p)}{1-t} df(\nabla d_p)(\tilde{\gamma}_{x,p}(t)) \quad \text{for a.e.} \quad t \in [0,1).$$

The corollary then follows by a reparameterization of $\tilde{\gamma}_{x,p}$. □

We will be particularly interested in the following type of object: Let $p \in X$ and $\mu \in \mathcal{P}_2(X)$ be of bounded density with respect to m. Take a Borel selection (2.7.5) of unit speed geodesics $\gamma_{x,p}$ from all $x \in X$ to p and define $T := d(\mathrm{supp}(\mu), p)$. For $0 \le t \le T$, define $\mu_t := (\gamma_{\cdot,p}(t))_*(\mu)$. (μ_t) defined this way are more naturally considered L^1-Wasserstein geodesics and are well-studied in the theory of needle decomposition of RCD spaces, see [12, 19, 29]. We record some properties of these objects which will be needed later.

Theorem 3.2.9. *Let $0 < \delta < T$ and $\mu \le Am$ for some $A > 0$. Let $(\mu_t)_{t\in[0,T-\delta]}$ be as defined in the previous paragraph and $D := \sup_{x\in \mathrm{supp}(\mu)} d(x,p) \le \bar{D}$. Then*

1. *$(\mu_t)_{t\in[0,T-\delta]}$ is a W_2-geodesic;*
2. *There exists $C(K,N,\bar{D},\delta)$ so that $\mu_t \le A(1+Ct)^N m$ for all $t \in [0, T-\delta]$. In particular, the densities of $(\mu_t)_{t\in[0,T-\delta]}$ are uniformly bounded with respect to m;*
3. *$(\mu_t)_{t\in[0,T-\delta]}$ solves the continuity equation*

$$\frac{d}{dt}\mu_t + \mathrm{div}(-\nabla d_p \mu_t) = 0.$$

Proof. Statement 1 was proved in [19, Lemma 4.4], where d-monotonicity of $\left\{(x, \gamma_{x,p}(T-\delta)) : x \in \mathrm{supp}\,\mu\right\}$ is used to show its d^2-monotonicity. Statement 2 was proved in [12, Section 9], see also [31, Section 3.2] for a discussion. Statements 1 and 2 give that (μ_t) is of bounded compression. Statement 3 then follows from Corollary 3.2.8. □

In order to have terminology which includes the globally defined RLFs as well as the type of locally defined flows such as the example above, we will use the following definition.

Definition 3.2.10. Let $S \in \mathscr{B}(X)$ with $m(S) > 0$ and $(V_t)_{t\in[0,T]} \in L^1([0,T], L^2(TX))$. A Borel map $F : [0,T] \times S \to X$ is a *local flow of V_t from S* if the following holds:

1. $F_0(x) = x$ and $[0,T] \ni t \mapsto F_t(x)$ is continuous for every $x \in S$;
2. For every $f \in \mathrm{TestF}(X)$ and m-a.e. $x \in S$, $t \mapsto f(F_t(x))$ is in $W^{1,1}([0,T])$ and

$$\frac{d}{dt} f(F_t(x)) = df(V_t)(F_t(x)) \text{ for a.e. } t \in [0,T]; \tag{3.6}$$

3. There exists a constant $C := C(V,S)$ so that $(F_t)_*(m|_S) \leq Cm$ for all t in $[0,T]$.

As before, by abuse of notation we will often say (F_t) is the local flow of (V_t) from S for some vector field V which is only locally L^2. In this case, it is understood that $F_t(S)$ is essentially bounded for each t and we cut off V_t outside of this region.

Remark 3.2.11. We will be primarily interested in the following examples:

1. For any $p \in X$ and bounded $S \in \mathscr{B}(X)$, $F_t(x) := \gamma_{x,p}(t)$ defined on $(t,x) \in [0, T-\delta] \times S$, where $T := \operatorname{ess\,inf}_{x\in S} d(x,p)$, $\delta > 0$, and $\gamma_{x,p}$ is a unit speed geodesic from x to p is Borel selected (see 2.7.5), is a local flow of $-\nabla d_p$ from S by Corollary 3.2.8 and Theorem 3.2.9.
2. The restriction of any RLF onto some $S \in \mathscr{B}(X)$ is a local flow of the corresponding (V_t) from S by definition.

The following differentiation formula follows by the same argument for RLFs in Proposition 3.1.6.

Proposition 3.2.12. *(First order differentiation formula for distance along local flows) Let $T > 0$. If $(F_t)_{t\in[0,T]}$, $(G_t)_{t\in[0,T]}$ are local flows of (U_t), (V_t) from S_1 and S_2 respectively, then for $(m\times m)$-a.e. $(x,y) \in S_1 \times S_2$, $d(F_t(x), G_t(y)) \in W^{1,1}([0,T])$ and*

$$\frac{d}{dt} d(F_t(x), G_t(y)) = \langle \nabla d_{G_t(y)}, U_t\rangle(F_t(x)) + \langle \nabla d_{F_t(x)}, V_t\rangle(G_t(y)) \quad \textit{for a.e. } t \in [0,T].$$

We mention that if (W_t) is defined as in proposition 3.1.6 from (U_t) and (V_t), then it is straightforward to check using the arguments of [18] that (F_t, G_t) is a local flow of (W_t) from $S_1 \times S_2$. Again, (W_t) here naturally belongs in $L^1([0,T], L^2_{loc}(T(X\times X)))$ so Definition 3.2.10 needs to be altered to allow for this. We refer to [52] for relevant definitions.

The next proposition gives control on the metric speeds of the curves $t \mapsto F_t(x)$ of a local flow F. As pointed out in [54, (A.22)], it follows from a similar argument as in [47, Theorem 2.3.18] after a small adjustment since we do not a priori assume the absolute continuity of the curves $F_\cdot(x)$.

Proposition 3.2.13. *Let $V \in L^1([0,T], L^2(TX))$ and let (F_t) be a local flow of (V_t) from S. For m-a.e. $x \in S$ the curve $t \mapsto F_t(x)$ is absolutely continuous and its metric speed $ms_t(F_\cdot(x))$ at time t satisfies*

$$ms_t(F_\cdot(x)) = |V_t|(F_t(x)) \quad \text{for a.e. } t \in [0,T].$$

3.3 Second order interpolation formula

The proof of a second order interpolation formula for the distance function (see (1.8) and (1.5)) along flows requires the results of [54]. The hard work is done there and their result immediately implies an analogous second order interpolation formula (Theorem 3.3.1) for the Wasserstein distance. It is our goal to pass this formula from Wasserstein distance to distance on the space itself.

For the rest of the section we will always be in the setting of some $\mathsf{RCD}(K,N)$ space (X,d,m) with $K \in \mathbb{R}$ and $N \in [1,\infty)$. We fix a Borel selection (2.7.5) of constant speed geodesics $\tilde{\gamma}_{x,y}$ from all $x \in X$ to all $y \in X$ parameterized on the unit interval. We denote by $\gamma_{x,y}$ the unit speed reparameterization of $\tilde{\gamma}_{x,y}$ to $[0, d(x,y)]$. We start with the following formulation of the main result from [54].

Theorem 3.3.1. *([54, Theorem 5.13]) Let $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ be compactly supported and satisfy $\mu_0, \mu_1 \leq Cm$ for some $C > 0$. Let (μ_t) be the unique W_2-geodesic connecting μ_0 to μ_1. For every $t \in [0,1]$, let ϕ_t be any function so that for some $s \neq t$, $s \in [0,1]$, the function $-(s-t)\phi_t$ is a Kantorovich potential from μ_t to μ_s. For any $V \in H^{1,2}_C(TX)$, the map $[0,1] \ni t \mapsto \int \langle V, \nabla\phi_t\rangle d\mu_t$ is in $C^1([0,1])$ and*

$$\frac{d}{dt}\Big(\int \langle V, \nabla\phi_t\rangle d\mu_t\Big) = \int (\nabla V : (\nabla\phi_t \otimes \nabla\phi_t))\, d\mu_t \quad \text{for all } t \in [0,1]. \tag{3.7}$$

The next lemma follows from the previous theorem.

Lemma 3.3.2. *Let $p \in X$ and $\nu \leq Cm$ be a nonnegative, compactly supported measure. For any $V \in H^{1,2}_C(TX)$,*

$$\begin{aligned}\int \langle V, \nabla d_p\rangle(x)\, d\nu(x) - \int \langle V, \nabla d_p\rangle(\tilde{\gamma}_{p,x}(\tfrac{1}{2}))\, d\nu(x) = \\ \int_{\frac{1}{2}}^{1} \Big(\int d(p,x)(\nabla V : (\nabla d_p \otimes \nabla d_p))(\tilde{\gamma}_{p,x}(t))\, d\nu(x)\Big)\, dt,\end{aligned} \tag{3.8}$$

where $\tilde{\gamma}_{x,y}$ is as defined in the beginning of this section.

Note that although $\nabla d_p \otimes \nabla d_p$ is not in $L^2(T^{\otimes 2}(X))$, it is locally (i.e. it is after multiplication by the characteristic function of any compact set). Therefore, by the locality properties of the objects involved, $\nabla V : (\nabla d_p \otimes \nabla d_p)$ is well-defined and is in $L^2(m)$.

Proof. Fix representatives for $\langle V, \nabla d_p\rangle$ and $\nabla V : (\nabla d_p \otimes \nabla d_p) \in L^2(m)$.

We claim for m-a.e. $x \in X$,

$$\langle V, \nabla d_p \rangle(x) - \langle V, \nabla d_p \rangle(\tilde{\gamma}_{p,x}(\frac{1}{2})) = \int_{\frac{1}{2}}^{1} d(p,x)(\nabla V : (\nabla d_p \otimes \nabla d_p))(\tilde{\gamma}_{p,x}(t))\, dt. \tag{3.9}$$

The right side integral is finite for m-a.e. x by using a Fubini's theorem argument along with Theorem 2.7.4 (2.7).

Suppose (3.9) does not hold for m-a.e. $x \in X$, then without loss of generality we may assume there exists a bounded set S with $0 < m(S) < \infty$ so that

$$\langle V, \nabla d_p \rangle(x) - \langle V, \nabla d_p \rangle(\tilde{\gamma}_{p,x}(\frac{1}{2})) > \int_{\frac{1}{2}}^{1} d(p,x)(\nabla V : (\nabla d_p \otimes \nabla d_p))(\tilde{\gamma}_{p,x}(t))\, dt \tag{3.10}$$

for each $x \in S$. Let $\mu := \frac{1}{m(S)} m|_S$. Multiplying both sides of (3.10) by $d(p,x)$ and integrating with respect to μ, we immediately contradict Theorem 3.3.1. Therefore, (3.9) holds and so the lemma follows for any ν with compact support and bounded density. $\square$

To proceed we state the segment inequality for $L^1(m)$ functions, first introduced by Cheeger-Colding in [20, Theorem 2.11]. This has been established for the metric measure setting in [85] but we will give a self-contained proof since the decomposition procedure for the family of geodesics used in the proof will be used again in the proof of Proposition 3.3.4.

Theorem 3.3.3. *(Segment inequality for L^1 functions on RCD spaces) Let (X, d, m) be an* $\mathsf{RCD}(K,N)$ *space with $K \in \mathbb{R}$ and $N \in [1, \infty)$. Let $\bar{R} > 0$. Let $f \in L^1_{loc}(X)$ be nonnegative and $\mu \le A(m \times m)$ be a nonnegative measure on $X \times X$ supported on $B_R(p) \times B_R(p)$ for some $0 < R \le \bar{R}$ and $p \in X$. Then*

$$\begin{aligned} &\int_0^1 \Big(\int f(\tilde{\gamma}_{x,y}(t)) d(x,y)\, d\mu(x,y) \Big) dt \\ &\qquad \le AC(K,N,\bar{R}) R [m(\pi_1(\mathrm{supp}(\mu))) + m(\pi_2(\mathrm{supp}(\mu)))] \int_{B_{2R}(p)} f(z)\, dm(z), \end{aligned} \tag{3.11}$$

where π_1, π_2 are projections onto the first and the second coordinate respectively and $\tilde{\gamma}_{x,y}$ is as defined in the beginning of this section.

Proof. By Radon-Nikodym theorem, $\mu = g(m \times m)$ for some compactly supported $g \in L^\infty(X \times X, m \times m)$. We denote $g^1_x(\cdot) := g(x, \cdot)$ and $g^2_y(\cdot) := g(\cdot, y)$. By Fubini's Theorem, $\| g^1_x \|_{L^\infty} \le A$ for m-a.e. x and similarly, $\| g^2_y \|_{L^\infty} \le A$ for m-a.e. y.

For each $t \in [\frac{1}{2}, 1]$ and m-a.e. $x \in \pi_1(\mathrm{supp}(\mu))$, by Theorem 2.7.4 (2.7),

$$(\tilde{\gamma}_{x,\cdot}(t))_*(g^1_x m) \le AC(K,N,\bar{R}) m|_{B_{2R}(p)}. \tag{3.12}$$

Similarly for $t \in [0, \frac{1}{2}]$ and m-a.e. $y \in \pi_2(\mathrm{supp}(\mu))$,

$$(\tilde{\gamma}_{\cdot,y}(t))_*(g^2_y m) \le AC(K,N,\bar{R}) m|_{B_{2R}(p)}. \tag{3.13}$$

We conclude

$$\begin{aligned}
&\int_0^1 \Big(\int f(\tilde{\gamma}_{x,y}(t)) d(x,y)\, d\mu(x,y)\Big)\, dt \\
&\leq 2R \int_0^1 \Big(\int f(\tilde{\gamma}_{x,y}(t))\, d\mu(x,y)\Big)\, dt \\
&= 2R\Big(\int_{\frac{1}{2}}^1 \Big(\int f(\tilde{\gamma}_{x,y}(t))\, d\mu(x,y)\Big)\, dt + \int_0^{\frac{1}{2}} \Big(\int f(\tilde{\gamma}_{x,y}(t))\, d\mu(x,y)\Big)\, dt\Big) \\
&\leq AC(K,N,\bar{R})R[m(\pi_1(\mathrm{supp}(\mu))) + m(\pi_2(\mathrm{supp}(\mu)))] \int_{B_{2R}(p)} f(z)\, dm(z),
\end{aligned}$$

where (3.12), (3.13) and Fubini's theorem was used in the last line.

□

We now prove the main interpolation formula of this section. The main idea is to use Lemma 3.3.2 along with the decomposition procedure from the proof of the segment inequality. Notice that the right side of (3.14) makes sense due to the segment inequality.

Proposition 3.3.4. *(Second order interpolation formula) Let $\mu \leq C(m \times m)$ be a nonnegative and compactly supported measure on $X \times X$. Let $V \in H^{1,2}_C(TX)$. Then*

$$\int (\langle V, \nabla d_x\rangle(y) + \langle V, \nabla d_y\rangle(x))\, d\mu(x,y) = \int_0^1 \int d(x,y)(\nabla V : \nabla d_x \otimes \nabla d_x)(\tilde{\gamma}_{x,y}(t))\, d\mu(x,y)\, dt, \quad (3.14)$$

where $\tilde{\gamma}_{x,y}$ is as defined in the beginning of this section.

Proof. Let $\mu = g(m \times m)$ for compactly supported $g \in L^\infty(X \times X, m \times m)$ with $g^1_x(\cdot) := g(x,\cdot)$ and $g^2_y(\cdot) := g(\cdot, y)$. $\| g^1_x \|_{L^\infty} \leq C$ for m-a.e. x and $\| g^2_y \|_{L^\infty} \leq C$ for m-a.e. y. We have

$$\begin{aligned}
&\int \langle V, \nabla d_x\rangle(y)\, d\mu(x,y) - \int \langle V, \nabla d_x\rangle(\tilde{\gamma}_{x,y}(\frac{1}{2}))\, d\mu(x,y) \\
&= \int \int \Big(\langle V, \nabla d_x\rangle(y) - \langle V, \nabla d_x\rangle(\tilde{\gamma}_{x,y}(\frac{1}{2}))\Big)\, d(g^1_x m)(y)\, dm(x) \quad \text{. by Fubini's theorem} \\
&= \int \int_{\frac{1}{2}}^1 \Big(\int d(x,y)(\nabla V : (\nabla d_x \otimes \nabla d_x))(\tilde{\gamma}_{x,y}(t))\, d(g^1_x m)(y)\Big)\, dt\, dm(x) \quad \text{, by 3.3.2} \\
&= \int_{\frac{1}{2}}^1 \Big(\int \Big(d(x,y)(\nabla V : (\nabla d_x \otimes \nabla d_x))(\tilde{\gamma}_{x,y}(t))\Big) d\mu(x,y)\Big)\, dt \quad \text{, by Fubini's theorem.}
\end{aligned} \quad (3.15)$$

Making the analogous argument on $t \in [0, \frac{1}{2}]$, we obtain

$$\begin{aligned}
&\int \langle V, \nabla d_y\rangle(x)\, d\mu(x,y) - \int \langle V, \nabla d_y\rangle(\tilde{\gamma}_{x,y}(\frac{1}{2}))\, d\mu(x,y) \\
&= \int_0^{\frac{1}{2}} \Big(\int \Big(d(x,y)(\nabla V : (\nabla d_y \otimes \nabla d_y))(\tilde{\gamma}_{x,y}(t))\Big) d\mu(x,y)\Big)\, dt.
\end{aligned} \quad (3.16)$$

The claim then follows if

$$\int \langle V, \nabla d_x\rangle(\tilde{\gamma}_{x,y}(\frac{1}{2}))\, d\mu(x,y) + \int \langle V, \nabla d_y\rangle(\tilde{\gamma}_{x,y}(\frac{1}{2}))\, d\mu(x,y) = 0 \tag{3.17}$$

and

$$\begin{aligned}&\int_0^{\frac{1}{2}} \Big(\int \Big(d(x,y)(\nabla V : (\nabla d_y \otimes \nabla d_y))(\tilde{\gamma}_{x,y}(t))\Big) d\mu(x,y) \Big)\, dt \\ = &\int_0^{\frac{1}{2}} \Big(\int \Big(d(x,y)(\nabla V : (\nabla d_x \otimes \nabla d_x))(\tilde{\gamma}_{x,y}(t))\Big) d\mu(x,y) \Big)\, dt,\end{aligned} \tag{3.18}$$

which we show in Lemma 3.3.6 and Remark 3.3.7. □

The following two lemmas show that, in a measure-theoretic sense, ∇d_p and ∇d_q in the interior of a geodesic between p and q "point in opposite directions".

Lemma 3.3.5. *Let $p, q \in X$. Let $\gamma_{p,q} : [0,1] \to X$ be a constant speed geodesic from p to q and let $z = \gamma(t_0)$ for some $t_0 \in (0,1)$. Let f be a locally Lipschitz function, then* $\mathrm{lip}(f+d_p)(z) = \mathrm{lip}(f-d_q)(z)$.

Proof. By classical Abresch-Gromoll inequality 2.10.9, for x in a sufficiently small neighbourhood of z, $e_{p,q}(x) \leq Cd(x,z)^{1+\alpha}$. Therefore, for any such x,

$$\begin{aligned}&\frac{|(f(x)+d_p(x)) - (f(z)+d_p(z))|}{d(x,z)} - \frac{|(f(x)-d_q(x)) - (f(z)-d_q(z))|}{d(x,z)} \\ \leq &\frac{Cd(x,z)^{1+\alpha}}{d(x,z)} = Cd(x,z)^{\alpha}.\end{aligned}$$

This shows that $\mathrm{lip}(f+d_p)(z) = \mathrm{lip}(f-d_q)(z)$ since $Cd(x,z)^{\alpha} \to 0$ as $d(x,z) \to 0$. □

Lemma 3.3.6. *In the notations of 3.3.4, for any $t \in (0,1)$,*

$$\int \langle V, \nabla d_x\rangle(\tilde{\gamma}_{x,y}(t))\, d\mu(x,y) + \int \langle V, \nabla d_y\rangle(\tilde{\gamma}_{x,y}(t))\, d\mu(x,y) = 0.$$

Proof. Fix $t \in (0,1)$. Let $\mu = g(m \times m)$ for compactly supported $g \in L^\infty(X \times X, m \times m)$ with $g_x^1(\cdot) := g(x,\cdot)$ and $g_y^2(\cdot) := g(\cdot,y)$. $\| g_x^1 \|_{L^\infty} \leq C$ for m-a.e. x and $\| g_y^2 \|_{L^\infty} \leq C$ for m-a.e. y.

We first prove the claim for $V = \nabla f$ where f is locally Lipschitz. The general claim then follows by approximation. We observe the following:

1. By Remark 2.2.1, for each $x \in X$,

$$\begin{aligned}\langle \nabla f, \nabla d_x\rangle &= \frac{|\nabla(f+d_x)|^2 - |\nabla f|^2 - |\nabla d_x|^2}{2} \\ &= \frac{(\mathrm{lip}(f+d_x))^2 - (\mathrm{lip}(f))^2 - (\mathrm{lip}(d_x))^2}{2} \quad m-\text{a.e.}.\end{aligned}$$

2. Similarly, for each $y \in X$,

$$\langle \nabla f, -\nabla d_y \rangle = \frac{|\nabla(f-d_y)|^2 - |\nabla f|^2 - |-\nabla d_y|^2}{2} = \frac{(\mathrm{lip}(f-d_y))^2 - (\mathrm{lip}(f))^2 - (\mathrm{lip}(-d_y))^2}{2} \quad m-\text{a.e.}.$$

3. $(\tilde{\gamma}_{x,\cdot}(t))_*(g_x^1 m)$ is of bounded density w.r.t. m for m-a.e. x by Theorem 2.7.4 (2.7).

Lemma 3.3.5 and Fubini's Theorem then gives the claim for ∇f,

$$\begin{aligned}
&\int \langle \nabla f, \nabla d_x \rangle(\tilde{\gamma}_{x,y}(t))\, d\mu(x,y) \\
&= \int \Big(\int \langle \nabla f, \nabla d_x \rangle(\tilde{\gamma}_{x,y}(t))\, d(g_x^1 m)(y) \Big) dm(x) \\
&= \int \Big(\int \frac{(\mathrm{lip}(f+d_x))^2 - (\mathrm{lip}(f))^2 - (\mathrm{lip}(d_x))^2}{2} (\tilde{\gamma}_{x,y}(t))\, d(g_x^1 m)(y) \Big) dm(x) \text{ , by observations 1 and 3} \\
&= \int \Big(\int \frac{(\mathrm{lip}(f-d_y))^2 - (\mathrm{lip}(f))^2 - (\mathrm{lip}(-d_y))^2}{2} (\tilde{\gamma}_{x,y}(t))\, d(g_x^1 m)(y) \Big) dm(x) \text{ , by 3.3.5 and observ. 3} \\
&= \int \langle \nabla f, -\nabla d_y \rangle(\tilde{\gamma}_{x,y}(t))\, d\mu(x,y) \text{ , by observations 2 and 3.}
\end{aligned}$$

□

Remark 3.3.7. 3.3.6 implies (3.18) as well. This is because Lemma 3.3.2 is true for any interval $[a,b] \subseteq (0,1]$ and so one can use the same argument as in Proposition 3.3.4 to say that, for $0 < s < \frac{1}{2}$,

$$\begin{aligned}
&\int_s^{\frac{1}{2}} \Big(\int \Big(d(x,y)(\nabla V : (\nabla d_y \otimes \nabla d_y))(\tilde{\gamma}_{x,y}(t)) \Big) d\mu(x,y) \Big) dt \\
&= \int \langle V, \nabla d_y \rangle(\tilde{\gamma}_{x,y}(\tfrac{1}{2}))\, d\mu(x,y) - \int \langle V, \nabla d_y \rangle(\tilde{\gamma}_{x,y}(s))\, d\mu(x,y) \\
&= -\int \langle V, \nabla d_x \rangle(\tilde{\gamma}_{x,y}(\tfrac{1}{2}))\, d\mu(x,y) + \int \langle V, \nabla d_y \rangle(\tilde{\gamma}_{x,y}(s))\, d\mu(x,y) \\
&= \int_s^{\frac{1}{2}} \Big(\int \Big(d(x,y)(\nabla V : (\nabla d_x \otimes \nabla d_x))(\tilde{\gamma}_{x,y}(t)) \Big) d\mu(x,y) \Big) dt.
\end{aligned}$$

Taking a limit $s \to 0$ gives (3.18).

The second order interpolation formula 3.3.4 and first order differentiation formula for distance along local flows 3.2.12 immediately give an integral version of (1.5). They also give the following related estimate which will be used heavily in Chapter 5. Let $U, V \in L^1([0,T], L^2(TX))$ be bounded (see Definition 3.1.1) and S_1, S_2 be bounded sets of positive measure. Let $(F_t)_{t\in[0,T]}$, $(G_t)_{t\in[0,T]}$ be local flows of U, V from S_1, S_2 respectively. Let $r > 0$ and for each $t \in [0,T]$, define $dt_r^{F,G}(t) : S_1 \times S_2 \to [0,r]$ the *distance distortion on scale r at t* by

$$dt_r^{F,G}(t)(x,y) := \min\Big\{ r, \max_{0\le\tau\le t} |d(x,y) - d(F_\tau(x), G_\tau(y))| \Big\}. \tag{3.19}$$

Define $\Gamma_r^{F,G}(t) := \left\{(x,y) \in S_1 \times S_2 : dt_r^{F,G}(t)(x,y) < r\right\}$. The terminology and definition of the distance distortion function comes from [64], where it was used in a similar way as in this thesis to analyze the geometry of gradient flows.

Proposition 3.3.8. *Let $W \in H_C^{1,2}(TX)$. The map $t \mapsto \int_{S_1\times S_2} dt_r^{F,G}(t)(x,y)\, d(m\times m)(x,y)$ is Lipschitz on $[0,T]$ and satisfies*

$$\begin{aligned}
&\frac{d}{dt}\int\limits_{S_1\times S_2} dt_r^{F,G}(t)(x,y)\, d(m\times m)(x,y)\\
&\leq \int_{\Gamma_r^{F,G}(t)} (|U_t - W|(F_t(x)) + |V_t - W|(G_t(y)))\, d(m\times m)(x,y)\\
&\qquad + \int_0^1 \int_{\Gamma_r^{F,G}(t)} d(F_t(x), G_t(y))|\nabla W|_{\mathrm{HS}}(\tilde{\gamma}_{F_t(x),G_t(y)}(s))\, d(m\times m)(x,y)\, ds
\end{aligned}$$

for a.e. $t \in [0,T]$, where $\tilde{\gamma}_{\cdot,\cdot}$ is as defined in the beginning of this section.

Proof. First fix representatives for all involved measure-theoretic objects. For any $(x,y) \in S_1 \times S_2$, $dt_r^{F,G}(t)(x,y)$ is continuous, monotone non-decreasing and bounded between 0 and r as a function of $t \in [0,T]$. Therefore, $dt_r^{F,G}(t)(x,y)$ is differentiable for a.e. $t \in [0,T]$ and $\frac{d^+}{dt} dt_r^{F,G}(t)(x,y) = 0$ for all t where $(x,y) \notin \Gamma_r^{F,G}(t)$.

Furthermore, by boundedness of U, V and Proposition 3.2.13, $F_\cdot(x)$, $G_\cdot(y)$ are uniformly Lipschitz curves for m-a.e. $x \in S_1$ and m-a.e. $y \in S_2$ respectively. In particular, the functions $[0,T] \ni t \mapsto d(F_t(x), G_t(y))$ are uniformly Lipschitz for $(m\times m)$-a.e. $(x,y) \in S_1 \times S_2$. The same is true for the functions $t \mapsto dt_r^{F,G}(t)(x,y)$. Therefore, $t \mapsto \int_{S_1\times S_2} dt_r^{F,G}(t)(x,y)\, d(m\times m)(x,y)$ is Lipschitz on $[0,T]$ as well.

By Proposition 3.2.12, for $(m\times m)$-a.e.$(x,y) \in S_1 \times S_2$, $d(F_t(x), G_t(y)) \in W^{1,1}([0,T])$ and

$$\frac{d}{dt} d(F_t(x), G_t(y)) = \langle \nabla d_{G_t(y)}, U_t\rangle(F_t(x)) + \langle \nabla d_{F_t(x)}, V_t\rangle(G_t(y)). \tag{3.20}$$

At any point of differentiability for both $d(F_t(x), G_t(y))$ and $dt_r^{F,G}(t)(x,y)$, it is clear from definition that $\frac{d}{dt} dt_r^{F,G}(t)\,(x,y) \leq (\frac{d}{dt} d(F_t(x), G_t(y)))_+$

For any $t \in [0,T]$, let $\tilde{\Gamma}_r^{F,G}(t) := \left\{(x,y) : \langle \nabla d_{G_t(y)}, U_t\rangle(F_t(x)) + \langle \nabla d_{F_t(x)}, V_t\rangle(G_t(y)) \geq 0\right\} \cap \Gamma_r^{F,G}(t)$. Then $\mu_t := (F_t, G_t)_*((m\times m)|_{\tilde{\Gamma}_r^{F,G}(t)})$ is compactly supported by Proposition 3.2.13 and has bounded density with respect to $(m\times m)$ by definition 3.2.10 of a local flow. By the second order interpolation formula 3.3.4,

$$\int (\langle W, \nabla d_y\rangle(x) + \langle W, \nabla d_x\rangle(y))\, d\mu_t(x,y) = \int_0^1 \int d(x,y)(\nabla W : \nabla d_x \otimes \nabla d_x)(\tilde{\gamma}_{x,y}(s))\, d\mu_t(x,y)\, ds. \tag{3.21}$$

Therefore,

$$
\begin{aligned}
&\int (\langle U_t, \nabla d_y\rangle(x) + \langle V_t, \nabla d_x\rangle(y))\, d\mu_t(x,y) \\
&= \int (\langle U_t - W, \nabla d_y\rangle(x) + \langle V_t - W, \nabla d_x\rangle(y))\, d\mu_t(x,y) \\
&\qquad + \int_0^1 \int d(x,y)(\nabla W : \nabla d_x \otimes \nabla d_x)(\tilde{\gamma}_{x,y}(s))\, d\mu_t(x,y)\, ds \quad \text{, by (3.21)} \\
&\leq \int (|U_t - W|(x) + |V_t - W|(y))\, d\mu_t(x,y) \\
&\qquad + \int_0^1 \int d(x,y) |\nabla W|_{\mathrm{HS}}(\tilde{\gamma}_{x,y}(s))\, d\mu_t(x,y)\, ds \quad \text{, since } |\nabla d_x|, |\nabla d_y| = 1 \ m\text{-a.e.} \\
&\leq \int_{\Gamma_r^{F,G}(t)} (|U_t - W|(F_t(x)) + |V_t - W|(G_t(y)))\, d(m\times m)(x,y) \\
&\qquad + \int_0^1 \int_{\Gamma_r^{F,G}(t)} d(F_t(x), G_t(y)) |\nabla W|_{\mathrm{HS}}(\tilde{\gamma}_{F_t(x),G_t(y)}(s))\, d(m\times m)(x,y)\, ds,
\end{aligned}
\tag{3.22}
$$

by definition of $\mu_t = (F_t, G_t)_*((m\times m)|_{\tilde{\Gamma}_r^{F,G}(t)})$, $\tilde{\Gamma}_r^{F,G}(t) \subseteq \Gamma_r^{F,G}(t)$ and the fact that all integrands are positive.

To conclude, we have for a.e. $t \in [0,T]$, for $(m\times m)$-a.e. $(x,y) \in S_1 \times S_2$, $d(F_t(x), G_t(y))$ and $dt_r^{F,G}(t)(x,y)$ are both differentiable in t. For any such t,

$$
\begin{aligned}
&\frac{d}{dt} \int_{S_1\times S_2} dt_r^{F,G}(t)(x,y)\, d(m\times m)(x,y) \\
&= \int_{S_1\times S_2} \frac{d}{dt} dt_r^{F,G}(t)(x,y)\, d(m\times m)(x,y) \quad \text{, by DCT since } dt_r \text{ is uniformly Lipschitz for a.e. } (x,y) \\
&= \int_{\Gamma_r^{F,G}(t)} \frac{d}{dt} dt_r^{F,G}(t)(x,y)\, d(m\times m)(x,y) \\
&\leq \int_{\Gamma_r^{F,G}(t)} (\frac{d}{dt} d(F_t(x), G_t(y)))_+\, d(m\times m)(x,y) \\
&= \int_{\tilde{\Gamma}_r^{F,G}(t)} \langle \nabla d_{G_t(y)}, U_t\rangle(F_t(x)) + \langle \nabla d_{F_t(x)}, V_t\rangle(G_t(y))\, d(m\times m)(x,y) \quad \text{, by (3.20) and definition of } \tilde{\Gamma} \\
&\leq \int_{\Gamma_r^{F,G}(t)} (|U_t - W|(F_t(x)) + |V_t - W|(G_t(y)))\, d(m\times m)(x,y) \\
&\qquad + \int_0^1 \int_{\Gamma_r^{F,G}(t)} d(F_t(x), G_t(y)) |\nabla W|_{\mathrm{HS}}(\tilde{\gamma}_{F_t(x),G_t(y)}(s))\, d(m\times m)(x,y)\, ds \quad \text{, by (3.22).}
\end{aligned}
$$

□

Chapter 4

Estimates on the heat flow approximations of distance and excess functions

Here we collect estimates on the heat flow approximations of distance and excess functions, all of which were established in [33]. All their arguments translate directly to the RCD setting due to the availability of the improved Bochner inequality, the Li-Yau Harnack and gradient inequalities, and the various estimates of Section 2.10. We record their proofs for the sake of completeness, making minor regularity and measure-theoretic adjustments as needed.

In this chapter we fix (X, d, m) an $\mathsf{RCD}(-(N-1), N)$ space for $N \in (1, \infty)$, $0 < \delta < \frac{1}{2}$, and two points $p, q \in X$ with $d(p, q) \le 1$. Any time we use c it is always a constant depending only on N and δ unless specified otherwise. We fix the following notations:

1. $d_{p,q} = d(p, q) \le 1$ and for any $\epsilon > 0$, $d_\epsilon := \epsilon d_{p,q}$.
2. $d^-(x) := d(p, x)$.
3. $d^+(x) := d(p, q) - d(x, q)$.
4. $e(x) := d(p, x) + d(x, q) - d(p, q) = d^-(x) - d^+(x)$.

We will consider these functions multiplied by some appropriate cut off functions. Let $\psi^\pm : X \to \mathbb{R}$ be the good cut off functions as in 2.10.2 satisfying

$$\psi^- = \begin{cases} 1 & \text{on } A_{\frac{\delta}{8}d_{p,q}, 8d_{p,q}}(p) \\ 0 & \text{on } X \setminus A_{\frac{\delta}{16}d_{p,q}, 16d_{p,q}}(p) \end{cases}, \quad \psi^+ = \begin{cases} 1 & \text{on } A_{\frac{\delta}{8}d_{p,q}, 8d_{p,q}}(q) \\ 0 & \text{on } X \setminus A_{\frac{\delta}{16}d_{p,q}, 16d_{p,q}}(q) \end{cases}.$$

Let $\psi := \psi^+\psi^-$, $e_0 := \psi e$ and $h_0^\pm := \psi d^\pm$. We denote

5. $h_t^\pm := H_t(h_0^\pm)$ and $e_t := H_t(e_0)$.

6. $X_{r,s} := A_{rd_{p,q},sd_{p,q}}(p) \cap A_{rd_{p,q},sd_{p,q}}(q)$.

By definition $e_0 = e$, $h_0^\pm = h^\pm$ on $X_{\frac{\delta}{8},8}$ and $e_t = h_t^- - h_t^+$ by uniqueness of heat flow.

We will always take the continuous representative whenever possible. This in particular applies to $e_t, h_t^\pm, \Delta e_t$, and $\Delta h_t^\pm$ for $t > 0$. We remark that since $h_t^\pm$ and e_t are Lipschitz, one can also take the local Lipschitz constant as the representatives of $|\nabla h_t^\pm|$ and $|\nabla e_t|$ by 2.2.1. These have a sufficiently nice continuity property, see Lemma 4.0.1, which makes most of our m-a.e. statements about $|\nabla h_t^\pm|$ and $|\nabla e_t|$ pointwise and ease certain measure-theoretic difficulties in the arguments for this chapter.

Lemma 4.0.1. *Let (X, d, m) be an $\mathsf{RCD}(K, N)$ space for $K \in \mathbb{R}$ and $N \in [1, \infty)$. Let $f : X \to \mathbb{R}$ be a Lipschitz function. Fix $U \subseteq X$ open and $x \in U$. Then*

$$\operatorname{lip}(f)(x) \le \operatorname*{ess\,sup}_U \operatorname{lip}(f).$$

Proof. For any $\epsilon > 0$, there exists $y \in U$ so that $\frac{|f(y)-f(x)|}{d(y,x)} \ge \operatorname{lip}(f)(x) - \frac{\epsilon}{2}$. By continuity of f, there exists $r > 0$ so that $B_r(y) \subseteq U$ and for any $z \in B_r(y)$, $\frac{|f(y)-f(x)|}{d(y,x)} \ge \operatorname{lip}(f)(x) - \epsilon$. Let $\gamma_{z,x} : [0, 1] \to X$ be a constant speed geodesic from z to x. The local Lipschitz constant is an upper gradient of f, see [5, Remark 2.7], and therefore,

$$\fint_{B_r(y)} |f(z) - f(x)|\, dm(z) \le \fint_{B_r(y)} \int_0^1 d(z, x) \operatorname{lip}(f)(\gamma_{z,x}(s))\, ds\, dm(z).$$

Since we know $\fint_{B_r(y)} |f(z) - f(x)| \ge \operatorname{lip}(f)(x) - \epsilon$ and for each $s < 1$, $(\gamma_{\cdot,x}(s))_*(m)$ is absolutely continuous w.r.t. m by Theorem 2.7.4, we conclude $\operatorname{ess\,sup}_U \operatorname{lip}(f) \ge \operatorname{lip}(f)(x) - \epsilon$. □

We proceed with our estimates for e_t and $h_t^\pm$.

Lemma 4.0.2. *There exists a constant $c(N, \delta)$ such that for all $t > 0$,*

$$\Delta h_t^-, -\Delta h_t^+, \Delta e_t \le \frac{c(N, \delta)}{d_{p,q}}. \tag{4.1}$$

Proof. We show the claim for e_t; the proof is analogous for others. By Laplacian comparison theorem for the distance function 2.9.1, see also Remark 2.9.2, and the definition ψ, $e_0 \in D(\boldsymbol{\Delta})$ with

$$\boldsymbol{\Delta} e_0 = \Delta\psi e m + \langle \nabla\psi, \nabla e \rangle m + \psi \boldsymbol{\Delta} e \le \frac{c(N, \delta)}{d_{p,q}} m. \tag{4.2}$$

We know $e_t(x) = \int H_t(x,y)e_0(y)dm(y)$. For $t > 0$, $e_t \in D(\Delta)$ and

$$\begin{aligned} \Delta e_t &= \int \Delta_x H_t(x,y)e_0(y)\,dm(y) \\ &= \int \Delta_y H_t(x,y)e_0(y)\,dm(y) \quad \text{, by symmetry of } H_t \\ &= \int H_t(x,y)d\mathbf{\Delta} e_0(y) \quad \text{, since } e_0 \text{ is compactly supported} \\ &\leq \frac{c(N,\delta)}{d_{p,q}}. \end{aligned} \tag{4.3}$$

□

Lemma 4.0.3. *There exists a constant $c(N,\delta)$ such that for all $0 < \epsilon \leq \bar{\epsilon}(N,\delta)$ and $x \in X_{\frac{\delta}{4},5}$ the following holds:*

1. $|e_{d_\epsilon^2}(y)| \leq c(\epsilon^2 d_{p,q} + e(x))$ *for every* $y \in B_{10d_\epsilon}(x)$;
2. $|\nabla e_{d_\epsilon^2}|(y) \leq c\Big(\epsilon + \frac{\epsilon^{-1}e(x)}{d_{p,q}}\Big)$ *for m-a.e.* $y \in B_{10d_\epsilon}(x)$;
3. $|\Delta e_{d_\epsilon^2}(y)| \leq c\Big(\frac{1}{d_{p,q}} + \frac{\epsilon^{-2}e(x)}{d_{p,q}^2}\Big)$ *for every* $y \in B_{10d_\epsilon}(x)$;
4. $⨍_{B_{d_\epsilon}(y)} |\mathrm{Hess}\, e_{d_\epsilon^2}|^2_{\mathrm{HS}} \leq c\Big(\frac{1}{d_{p,q}} + \frac{\epsilon^{-2}e(x)}{d_{p,q}^2}\Big)^2$ *for every* $y \in B_{10d_\epsilon}(x)$.

Proof. $e_t(x) = e_0(x) + \int_0^t \Delta e_s(x)ds$ pointwise by definition of the heat flow and the continuity e_s. By Lemma 4.0.2,

$$e_t(x) \leq e_0(x) + \frac{c}{d_{p,q}}t = e(x) + \frac{c}{d_{p,q}}t. \tag{4.4}$$

Setting $s = d_\epsilon^2$, $t = 2d_\epsilon^2$ and $y \in B_{10d_\epsilon}(x)$ in the statement of the Li-Yau Harnack inequality, 2.9.3, we conclude

$$\begin{aligned} e_{d_\epsilon^2}(y) &\leq c(N)e_{2d_\epsilon^2}(x) \text{ , by 2.9.3 and } d_{p,q} \leq 1 \\ &\leq c(e(x) + \epsilon^2 d_{p,q}) \text{ , by (4.4).} \end{aligned} \tag{4.5}$$

This proves statement 1 of the lemma.

To prove 3 of the lemma, first notice that we need only establish a lower bound on $\Delta e_{d_\epsilon^2}(y)$ since 4.0.2 already gives us the desired upper bound. This is an application of the Li-Yau gradient inequality 2.9.4 and statement 1. The bound holds pointwise even though 2.9.4 holds only a.e. due to the existence of a continuous representative of Δe_t.

Statement 2 of the lemma follows from another application of Li-Yau gradient inequality 2.9.4 along with the bounds from statements 1 and 3.

For the last statement take good cut off function ϕ supported on $B_{2d_\epsilon}(y)$ with $\phi \equiv 1$ on $B_{d_\epsilon}(y)$.

We have

$$\begin{aligned}\int_X (\Delta e_{d_\epsilon^2})^2 \phi dm &= -\int_X \langle \nabla e_{d_\epsilon^2}, \nabla(\Delta e_{d_\epsilon^2}\phi)\rangle dm \\ &= -\int_X \langle \nabla e_{d_\epsilon^2}, \nabla \Delta e_{d_\epsilon^2}\rangle \phi dm - \int_X \langle \nabla e_{d_\epsilon^2}, \nabla \phi\rangle \Delta e_{d_\epsilon^2} dm.\end{aligned} \tag{4.6}$$

Integrating ϕ with $|\mathrm{Hess}\, e_{d_\epsilon^2}|^2_{\mathrm{HS}}$ and applying the improved Bochner inequality 2.6.2, we get

$$\begin{aligned}&\int_{B_{d_\epsilon}} |\mathrm{Hess}\, e_{d_\epsilon^2}|^2_{\mathrm{HS}}\, dm \le \int_X |\mathrm{Hess}\, e_{d_\epsilon^2}|^2_{\mathrm{HS}} \phi\, dm \\ &\le \int_X \frac{1}{2}\phi d\mathbf{\Delta}|\nabla e_{d_\epsilon^2}|^2 - \int_X \langle \nabla e_{d_\epsilon^2}, \nabla \Delta e_{d_\epsilon^2}\rangle \phi\, dm + \int_X (N-1)|\nabla e_{d_\epsilon^2}|^2 \phi\, dm \quad \text{, by 2.6.2} \\ &\le \int_X \frac{1}{2}\Delta\phi |\nabla e_{d_\epsilon^2}|^2 dm + \int_X (\Delta e_{d_\epsilon^2})^2 \phi dm + \int_X |\nabla e_{d_\epsilon^2}||\nabla \phi|\Delta e_{d_\epsilon^2} dm + \int_X (N-1)|\nabla e_{d_\epsilon^2}|^2 \phi dm \text{ , by (4.6).}\end{aligned} \tag{4.7}$$

Applying to this computation properties 1 - 3 of the lemma, property 2 of good cut off functions 2.10.1 and Bishop-Gromov volume comparison 2.1.4, we obtain statement 4 of the lemma. □

Next we prove estimates on the heat flow approximation of the distance functions.

Lemma 4.0.4. *There exists $c(N,\delta)$ such that for every $\epsilon \le \bar{\epsilon}(N,\delta)$ and $x \in X_{\frac{\delta}{4},5}$,*

$$|h^{\pm}_{d_\epsilon^2} - d^{\pm}|(x) \le c(\epsilon^2 d_{p,q} + e(x)).$$

Proof. From the Laplacian bounds in 4.0.2, for $x \in X_{\frac{\delta}{4},5}$,

$$h^-_{d_\epsilon^2}(x) - d^-(x) = \int_0^{d_\epsilon^2} \Delta h^-_t(x)\, dt \le c\epsilon^2 d_{p,q}, \tag{4.8}$$

and

$$d^+(x) - h^+_{d_\epsilon^2}(x) = \int_0^{d_\epsilon^2} -\Delta h^+_t(x)\, dt \le c\epsilon^2 d_{p,q}. \tag{4.9}$$

To obtain bounds in the other direction, we note that

$$h^-_{d_\epsilon^2} - d^-(x) = h^+_{d_\epsilon^2} - d^+(x) + e_{d_\epsilon^2}(x) - e(x).$$

We conclude using this with the bound $|e_{d_\epsilon^2}(x)| \le c(\epsilon^2 d_{p,q} + e(x))$ from statement 1 of Lemma 4.0.3 and bounds (4.8) or (4.9). □

We will end up wanting to establish appropriate gradient and Hessian bounds along curves that are close to being a geodesic between p and q. This requires the following definition.

Definition 4.0.5. A unit speed, piecewise geodesic curve σ between p and q is called an ϵ-geodesic between p and q if $||\sigma| - d_{p,q}| \leq \epsilon^2 d_{p,q}$, where $|\sigma|$ is the length of σ.

Remark 4.0.6. Notice that x lies on an ϵ-geodesic iff $e(x) \leq \epsilon^2 d_{p,q}$.

The previous lemma 4.0.4 can now be restated in terms of ϵ-geodesics.

Corollary 4.0.7. *There exists $c(N, \delta)$ such that for every ϵ-geodesic between p and q with $\epsilon \leq \bar{\epsilon}(N, \delta)$, and $\frac{\delta}{3} \leq t \leq 1 - \frac{\delta}{3}$,*

$$|h^{\pm}_{d_\epsilon^2} - d^{\pm}|(\sigma(t)) \leq c(\epsilon^2 d_{p,q}).$$

Proof. This follows from the 4.0.4 since

1. $e(\sigma(t)) \leq \epsilon^2 d_{p,q}$
2. Since σ is unit speed and an ϵ-geodesic, as long as $\epsilon < \sqrt{\frac{\delta}{12}}$, we will have $\sigma(t) \in X_{\frac{\delta}{4},5}$.

□

We establish an upper bound on the norm of the gradient of $h^{\pm}_t$ for $x \in X_{\frac{\delta}{5},6}$.

Lemma 4.0.8. *There exists $c(N, \delta)$ such that for $\epsilon \leq \bar{\epsilon}(N, \delta)$ and m-a.e. $x \in X_{\frac{\delta}{5},6}$,*

$$|\nabla h^{\pm}_{d_\epsilon^2}| \leq 1 + c d_\epsilon^2.$$

Proof. By Bakry-Ledoux estimate 2.5.2, for any $t > 0$,

$$|\nabla h^{\pm}_t| \leq e^{2(N-1)t} H_t(|\nabla h^{\pm}_0|) \quad m\text{-a.e.}. \tag{4.10}$$

By definition of $h^{\pm}_0 = \psi d^{\pm}$, we have the following a.e. bounds on $|\nabla h^{\pm}_0|$.

1. $|\nabla h^{\pm}_0| = 1$ in $X_{\frac{\delta}{8},8}$
2. $|\nabla h^{\pm}_0| = 0$ in $X \backslash X_{\frac{\delta}{16},16}$
3. In $X_{\frac{\delta}{16},16} \backslash X_{\frac{\delta}{8},8}$,

$$|\nabla h^{\pm}_0| = |\nabla\psi||d^{\pm}| + |\psi||\nabla d^{\pm}| \leq \frac{c(N)}{\delta d_{p,q}} |d^{\pm}| + 1 \leq c(N, \delta). \tag{4.11}$$

Finally, for any $x \in X_{\frac{\delta}{2},4}$,

$$\begin{aligned}
H_t(|\nabla h^{\pm}_0|)(x) &= \int_X H_t(x,y) |\nabla h^{\pm}_0|(y)\, dm(y) \\
&= \int_{X_{\frac{\delta}{16},16} \backslash X_{\frac{\delta}{8},8}} H_t(x,y)|\nabla h^{\pm}_0|(y)\, dm(y) + \int_{X_{\frac{\delta}{4},8}} H_t(x,y)|\nabla h^{\pm}_0|(y)\, dm(y) \\
&\leq c(N,\delta) \int_{X_{\frac{\delta}{16},16} \backslash X_{\frac{\delta}{8},8}} \nabla H_t(x,y)\, dm(y) + 1 \quad \text{, by (4.11)} \\
&\leq c(N,\delta)(\frac{\delta}{8})^{-2} t + 1,
\end{aligned} \tag{4.12}$$

where the last lines uses statement 2 of the heat kernel bounds 2.10.4.

The lemma follows by combining (4.10) and (4.12), with $t = d_\epsilon^2$ for small ϵ. □

We will now establish some integral bounds on $|\nabla h_t^\pm|$. Roughly, we want to apply the L^1-Harnack inequality 2.10.5 to $|\nabla h^\pm|$. To this effect, we give some regularity of the heat flow in the time parameter.

Lemma 4.0.9. *Let (X, d, m) be an $\mathsf{RCD}(K, N)$ space for some $K \in \mathbb{R}$, $N \in [1, \infty)$. Let $f \in L^2(m)$.*

1. *$H_t(f) \in C^1((0, \infty), L^2(m))$ and*

$$\frac{d}{dt} H_t(f) = \Delta H_t(f) \quad \forall t > 0; \tag{4.13}$$

2. *$H_t(f) \in C^0((0, \infty), W^{1,2}(X))$;*

3. *$H_t(f) \in C^1((0, \infty), W^{1,2}(X))$ and in particular $|\nabla H_t(f)|^2 \in C^1((0, \infty), L^1(m))$ with*

$$\frac{d}{dt} |\nabla H_t(f)|^2 = 2\langle \nabla H_t(f), \nabla \Delta H_t(f)\rangle \quad \forall t > 0. \tag{4.14}$$

If f or $|\nabla f|$ is in $L^\infty(X, m)$, then $|\nabla H_t(f)|^2 \in C^1((0, \infty), L^2(m))$ and the same formula holds.

Proof. Since $H_t(f) = H_{t'}(f) + \int_{t'}^t \Delta H_s(f) ds$ for $t > t' > 0$ and $\Delta H_s(f) \in C^0((0, \infty), L^2(m))$, statement 1 follows by fundamental theorem of calculus.

For $t > 0$,

$$\begin{aligned} \lim_{s\to 0} \int_X |\nabla H_{t+s}(f) - \nabla H_t(f)|^2 &= \lim_{s\to 0} \int_X (H_{t+s}(f)\Delta H_{t+s}(f)) + 2(H_{t+s}(f)\Delta H_t(f)) - (H_t(f)\Delta H_t(f)) \\ &= 0 \text{ , since all terms involved are continuous from } s \to L^2(m). \end{aligned}$$

We already know $H_t(f) \in C^0((0, \infty), L^2(X))$, so statement 2 follows.

Applying statement 2 to $\Delta H_\epsilon(f)$ for arbitrarily small positive ϵ, we see that $\Delta H_t(f) \in C^0((0, \infty), W^{1,2}(X))$. The first part of statement 3 then follows by applying fundamental theorem of calculus to $H_t(f) = H_{t'}(f) + \int_{t'}^t \Delta H_s(f) ds$ viewed as a $W^{1,2}(X)$-valued Bochner integral. The second part follows by a direct computation. Notice that if f or $|\nabla f|$ is in $L^\infty(X, m)$, then $|\nabla h_t(f)| \in L^\infty$ by L^∞-to-Lipschitz regularization (2.6) or by Bakry-Ledoux estimate 2.5.2. □

Lemma 4.0.10. *Let $\phi \in D(\Delta)$ be nonnegative, compactly supported, time independent with $|\phi|, |\nabla\phi|, |\Delta\phi| \leq K_1$. If h is the heat flow of some $h_0 \in L^2(m) \cap L^\infty(m)$ and $|\nabla h| \leq K_2$ on $\{\phi > 0\}$, then $(\frac{\partial}{\partial t} - \Delta)[\phi^2 |\nabla h|^2] \leq c(N, K_1, K_2)$ weakly in $(0, \infty) \times X$ as in Definition 2.9.5.*

Proof. Let $t > 0$, $h_t \in \mathrm{TestF}(X)$ by the L^∞-to-Lipschitz regularization property of H_t (2.6). Let $f \in \mathrm{TestF}(X)$. By [47, Proposition 3.3.22], $\langle \nabla h_t, \nabla h_t\rangle \in W^{1,2}(X)$ and

$$\langle \nabla\langle \nabla h_t, \nabla h_t\rangle, \nabla f\rangle = 2\,\mathrm{Hess}(h_t)(\nabla h_t, \nabla f) \quad m\text{-a.e..}$$

Therefore, $|\nabla\langle\nabla h_t,\nabla h_t\rangle\nabla f|\leq 2|\mathrm{Hess}(h_t)|_{\mathrm{HS}}|\nabla h_t||\nabla f|$ m-a.e..

We then have, for any $\epsilon>0$ and m-a.e.,

$$\begin{aligned}4\phi|\langle\nabla|\nabla h_t|^2,\nabla\phi\rangle| &\leq 8\phi|\mathrm{Hess}(h_t)|_{\mathrm{HS}}|\nabla h_t||\nabla\phi| \quad \text{, by above and the density of TestF}(X)\text{ in } W^{1,2}(X).\\ &\leq 4\epsilon\phi^2|\mathrm{Hess}(h_t)|^2_{\mathrm{HS}}|\nabla h_t|^2+\frac{4}{\epsilon}|\nabla\phi|^2.\end{aligned}$$

Choosing $\epsilon>0$ small so that $4\epsilon K_2^2<2$,

$$\begin{aligned}\boldsymbol{\Delta}(\phi^2|\nabla h_t|^2) &= \phi^2\boldsymbol{\Delta}|\nabla h_t|^2+(2\langle\nabla|\nabla h_t|^2,\nabla\phi^2\rangle+|\nabla h_t|^2\Delta\phi^2)m\\ &\geq(2\phi^2|\mathrm{Hess}(h_t)|^2_{\mathrm{HS}}+2\phi^2\langle\nabla\Delta h_t,\nabla h_t\rangle-2(N-1)\phi^2|\nabla h_t|^2+4\phi\langle\nabla|\nabla h_t|^2,\nabla\phi\rangle+|\nabla h_t|^2\Delta\phi^2)m\\ &\geq(2\phi^2\langle\nabla\Delta h_t,\nabla h_t\rangle-c)m,\end{aligned}$$

where the improved Bochner inequality 2.6.2 is used for line 2 and the previous estimate with ϵ was used for line 3.

Finally, by Lemma 4.0.9, $\frac{d}{dt}\phi^2|\nabla h_t|^2=2\phi^2\langle\nabla\Delta h_t,\nabla h\rangle$. This lets us conclude. □

Theorem 4.0.11. *There exists a constant $c(N,\delta)$ such that for all $\epsilon\leq\bar{\epsilon}(N,\delta)$,*

1. *if $x\in X_{\frac{\delta}{2},3}$ with $e(x)\leq\epsilon^2 d_{p,q}$ then $⨍_{B_{10d_\epsilon}(x)}||\nabla h^{\pm}_{d_\epsilon^2}|-1|\leq c\epsilon$;*
2. *if σ is an ϵ-geodesic connecting p and q, then $\int_{\frac{\delta}{2}d_{p,q}}^{(1-\frac{\delta}{2})d_{p,q}}⨍_{B_{10d_\epsilon}(\sigma(s))}||\nabla h^{\pm}_{d_\epsilon^2}|-1|\leq c\epsilon^2 d_{p,q}$.*

Proof. We prove the theorem in the case of h_t^-. The h_t^+ case is similar. We will take the local Lipschitz constant representatives for $|\nabla h_t^{\pm}|$. All statements made will be for ϵ sufficiently small depending on N and δ so we will forgo repeating this.

By Lemma 4.0.8, we choose $c'(N,\delta)$ so that $|\nabla h_t^-|\leq 1+c't^2$ for all $x\in X_{\frac{\delta}{5},6}$ and $t\leq\epsilon'(N,\delta)^2 d_{p,q}^2$. This means there exists $c''(N,\delta)$ so that

$$w_t:=1+c''t-|\nabla h_t^-|^2\geq 0 \text{ on } X_{\frac{\delta}{5},6}.$$

Let $\phi=\phi^+\phi^-$, where $\phi^{\pm}$ are annular good cutoff functions (2.10.2) around p and q respectively so that $\phi=1$ on $X_{\frac{\delta}{4},5}$ and $\phi=0$ on $X\backslash X_{\frac{\delta}{5},6}$. By Lemma 4.0.10,

$$(\partial_t-\Delta)|\phi^2 w_t|\geq -c \ \text{ weakly in } (0,d_{\epsilon'}^2)\times X.$$

Applying the L^1-Harnack inequality 2.10.5, we have, for $x\in X_{\frac{\delta}{3},4}$,

$$⨍_{B_{10d_\epsilon}(x)} w_{d_\epsilon^2}\leq c[\operatorname*{ess\,inf}_{B_{10d_\epsilon}(x)} w_{2d_\epsilon^2}+d_\epsilon^2]. \tag{4.15}$$

We will show that the right side is sufficiently small. Let $e(x)\leq\epsilon^2 d_{p,q}$ and let $\gamma(t)$ be a unit speed geodesic from x to p. By Corollary 4.0.7,

$$\begin{aligned} &|h^-_{2d_\epsilon^2}(x) - h^-_{2d_\epsilon^2}(\gamma(10d_\epsilon)) - 10d_\epsilon| \\ &\leq |h^-_{2d_\epsilon^2}(x) - d^-(x)| + |h^-_{2d_\epsilon^2}(\gamma(10d_\epsilon)) - d^-(\gamma(10d_\epsilon))| + |d^-(x) - d^-(\gamma(10d_\epsilon)) - 10d_\epsilon| \\ &\leq c\epsilon^2 d_{p,q}. \end{aligned} \tag{4.16}$$

The local Lipschitz constant $|\nabla h^-_{2d_\epsilon^2}|$ is an upper gradient, [5, Remark 2.7]. Therefore,

$$|h^-_{2d_\epsilon^2}(\sigma) - h^-_{2d_\epsilon^2}(\gamma(10d_\epsilon))| \leq \int_0^{10d_\epsilon} |\nabla h^-_{2d_\epsilon^2}|(\gamma(s))\, ds. \tag{4.17}$$

We have

$$\begin{aligned} \int_0^{10d_\epsilon} w_{2d_\epsilon^2}(\gamma(s))\, ds &= \int_0^{10d_\epsilon} (1 + cd_\epsilon^2 - |\nabla h^-_{2d_\epsilon^2}|^2(\gamma(s)))\, ds \\ &\leq 10d_\epsilon + cd_\epsilon^3 - \frac{1}{10d_\epsilon}(\int_0^{10d_\epsilon} |\nabla h^-_{2d_\epsilon^2}|(\gamma(s))\, ds)^2 \quad \text{, by Cauchy-Schwarz} \\ &\leq 10d_\epsilon + cd_\epsilon^3 - \frac{1}{10d_\epsilon}(h^-_{2d_\epsilon^2}(x) - h^-_{2d_\epsilon^2}(\gamma(10d_\epsilon)))^2 \quad \text{, by (4.17)} \\ &\leq 10d_\epsilon + cd_\epsilon^3 - \frac{1}{10d_\epsilon}(10d_\epsilon - c\epsilon^2 d_{p,q})^2 \quad \text{, by (4.16)} \\ &\leq c\epsilon \quad \text{, if } \epsilon < 1 \end{aligned} \tag{4.18}$$

In particular, there exists $s \in [0, 10d_\epsilon]$ so that $w_{2d_\epsilon^2}(\gamma(s)) \leq c\epsilon$. Applying Lemma 4.0.1 to $|\nabla h_t^\pm|$ and $\gamma(s) \in \overline{B_{10d_\epsilon}(x)}$, we conclude $\operatorname*{ess\,inf}_{B_{10d_\epsilon}(x)} w_{2d_\epsilon^2} \leq c\epsilon$ and so statement 1 is proved by (4.15).

By Corollary 4.0.7, arguing as in (4.16) ,

$$|h^-_{2d_\epsilon^2}\Big(\sigma((1-\frac{\delta}{2})d_{p,q})\Big) - h^-_{2d_\epsilon^2}\Big(\sigma(\frac{\delta}{2}d_{p,q})\Big) - (1-\delta)d_{p,q}| \leq c\epsilon^2 d_{p,q}. \tag{4.19}$$

Arguing as in (4.17) and (4.18),

$$\int_{\frac{\delta}{2}d_{p,q}}^{(1-\frac{\delta}{2})d_{p,q}} w_{2d_\epsilon^2}(\sigma(s))\, ds \leq c\epsilon^2 d_{p,q}. \tag{4.20}$$

Finally,

$$\begin{aligned}\int_{\frac{\delta}{2}d_{p,q}}^{(1-\frac{\delta}{2})d_{p,q}}\Big(\fint_{B_{10d_\epsilon}(\sigma(s))}\big||\nabla h^-_{d_\epsilon^2}|^2-1\big|\Big)\,ds &\le \int_{\frac{\delta}{2}d_{p,q}}^{(1-\frac{\delta}{2})d_{p,q}}\Big(\fint_{B_{10d_\epsilon}(\sigma(s))} w_{d_\epsilon^2}+c\epsilon^2 d_{p,q}\Big)\,ds\\ &\le c\int_{\frac{\delta}{2}d_{p,q}}^{(1-\frac{\delta}{2})d_{p,q}}\Big(\operatorname*{ess\,inf}_{B_{10d_\epsilon^2}(\sigma(s))} w_{2d_\epsilon^2}+c\epsilon^2 d_{p,q}\Big)\,ds \quad \text{, by (4.15)}\\ &\le c\int_{\frac{\delta}{2}d_{p,q}}^{(1-\frac{\delta}{2})d_{p,q}}\Big(w_{2d_\epsilon^2}(\sigma(s))+c\epsilon^2 d_{p,q}\Big)\,ds \quad \text{, by Lemma 4.0.1}\\ &\le c\epsilon^2 d_{p,q}.\end{aligned} \tag{4.21}$$

□

We now prove the main Hessian estimate for $h_t^\pm$.

Theorem 4.0.12. *There exists a constant $c(N,\delta)$ such that for any $0<\epsilon\le\bar{\epsilon}(N,\delta)$, any $x\in X_{\frac{\delta}{2},3}$ with $e(x)\le\epsilon^2 d_{p,q}$, or any ϵ-geodesic σ connecting p and q, there exists $r\in[\frac{1}{2},2]$ with*

1. $|h^\pm_{rd_\epsilon^2}-d^\pm|\le c\epsilon^2 d_{p,q}$;
2. $\fint_{B_{d_\epsilon}(x)}\big||\nabla h^\pm_{rd_\epsilon^2}|^2-1\big|\le c\epsilon$;
3. $\int_{\frac{\delta}{2}d_{p,q}}^{(1-\frac{\delta}{2})d_{p,q}}\Big(\fint_{B_{d_\epsilon}(\sigma(s))}\big||\nabla h^\pm_{rd_\epsilon^2}|^2-1\big|\Big)\,ds\le c\epsilon^2 d_{p,q}$;
4. $\int_{\frac{\delta}{2}d_{p,q}}^{(1-\frac{\delta}{2})d_{p,q}}\Big(\fint_{B_{d_\epsilon}(\sigma(s))}|\mathrm{Hess}\,h^\pm_{rd_\epsilon^2}|^2\Big)\,ds\le\frac{c}{d_{p,q}^2}$.

Proof. Statement 1 follows from Lemma 4.0.4 and statements 2 and 3 follow from Theorem 4.0.11 with Bishop-Gromov. Note that any $r\in[\frac{1}{2},2]$ works in the first 3 statements.

Using 2.10.1, we fix, for each $s\in(\frac{\delta}{2}d_{p,q},(1-\frac{\delta}{2})d_{p,q})$, good cut off function ϕ with $\phi\equiv 1$ on $B_{d_\epsilon}(\sigma(s))$, vanishing outside of $B_{3d_\epsilon}(\sigma(s))$, and $d_\epsilon|\nabla\phi|$, $d_\epsilon^2|\Delta\phi|\le c(N)$. Similarly, fix $\alpha(t)$ a smooth function in time so that $0\le\alpha(t)\le 1$, $\alpha(t)\equiv 1$ for $t\in[\frac{1}{2}d_\epsilon^2,2d_\epsilon^2]$, vanishing for t outside of $[\frac{1}{4}d_\epsilon^2,4d_\epsilon^2]$, and satisfying $|\alpha'|\le 10d_\epsilon^{-2}$.

Applying the improved Bochner inequality 2.6.2 to $h_t^\pm$, we obtain, for each s,t

$$\begin{aligned}\int\alpha(t)\phi|\mathrm{Hess}\,h_t^\pm|^2\,dm &\le \int\alpha(t)\phi d(\mathbf{\Delta}|\nabla h_t^\pm|^2)+2\int\alpha(t)\phi\Big((N-1)|\nabla h_t^\pm|^2-\langle\nabla h_t^\pm,\nabla\Delta h_t^\pm\rangle\Big)\,dm\\ &=\int\alpha(t)(|\nabla h_t^\pm|^2-1)\Delta(\phi)\,dm+2(N-1)\int\alpha(t)\phi|\nabla h_t^\pm|^2\,dm-\int\alpha(t)\phi\partial_t(|\nabla h_t^\pm|^2)\,dm.\end{aligned} \tag{4.22}$$

In the last line, we used the definition of the Laplacians along with the fact that $\int\Delta\phi dm=0$ for the first term and Lemma 4.0.9 for the third term. Integrating in time using integration by parts and

$\int_0^\infty \alpha'(t)dt = 0$ on the third term of the previous line,

$$\begin{aligned}&\int_0^\infty \int \alpha(t)\phi|\text{Hess}\, h_t^\pm|^2\, dm\, dt \\ &\leq \int_0^\infty \Big(\int \alpha(t)(|\nabla h_t^\pm|^2{-}1)\Delta(\phi)\, dm + 2(N-1)\int \alpha(t)\phi|\nabla h_t^\pm|^2\, dm + \int \alpha'(t)\phi(|\nabla h_t^\pm|^2{-}1)\, dm\Big)\, dt.\end{aligned} \tag{4.23}$$

Using what we know about ϕ and α and using Bishop-Gromov in line 2 of the following, we obtain

$$\begin{aligned}&\int_{\frac{1}{2}d_\epsilon^2}^{2d_\epsilon^2} \fint_{B_{d_\epsilon}(\sigma(s))} |\text{Hess}\, h_t^\pm|^2\, dm\, dt \\ &\leq \int_{\frac{1}{4}d_\epsilon^2}^{4d_\epsilon^2} \Big(\fint_{B_{3d_\epsilon}(\sigma(s))} \Big((|\nabla h_t^\pm|^2{-}1)\Delta(\phi) + 2(N-1)|\nabla h_t^\pm|^2{+}\alpha'(t)(|\nabla h_t^\pm|^2{-}1)\Big)\, dm\Big)\, dt \\ &\leq \int_{\frac{1}{4}d_\epsilon^2}^{4d_\epsilon^2} \Big(\fint_{B_{3d_\epsilon}(\sigma(s))} 2(N-1) + cd_\epsilon^{-2}||\nabla h_t^\pm|^2{-}1|\, dm\Big)\, dt.\end{aligned} \tag{4.24}$$

Integrating across σ for $s \in [\frac{\delta}{2}d_{p,q}, (1-\frac{\delta}{2})d_{p,q}]$,

$$\begin{aligned}&\int_{\frac{1}{2}d_\epsilon^2}^{2d_\epsilon^2} \Big(\int_{\frac{\delta}{2}d_{p,q}}^{(1-\frac{\delta}{2})d_{p,q}} \fint_{B_{d_\epsilon}(\sigma(s))} |\text{Hess}\, h_t^\pm|^2\, dm\, ds\Big)\, dt \\ &\leq cd_\epsilon^{-2} \int_{\frac{1}{4}d_\epsilon^2}^{4d_\epsilon^2} \Big(\int_{\frac{\delta}{2}d_{p,q}}^{(1-\frac{\delta}{2})d_{p,q}} \fint_{B_{3d_\epsilon}(\sigma(s))} cd_\epsilon^2 + ||\nabla h_t^\pm|^2{-}1|\, dm\, ds\Big)\, dt \\ &\leq c\epsilon^2 d_{p,q} \quad \text{, by statement 2 of Theorem 4.0.11.}\end{aligned} \tag{4.25}$$

Therefore, statement 4 holds for some $r \in [\frac{1}{2}, 2]$ and $t = rd_\epsilon^2$. □

Lemma 4.0.13. *Let $\epsilon \leq \bar{\epsilon}(N, \delta)$. Let $\gamma_{x,p}$ be any unit speed geodesic from $x \in X$ to p. Then for m-a.e. $x \in X_{\frac{\delta}{2},3}$ and any $0 \leq t_1 < t_2 \leq d_{x,p} - \frac{\delta}{2}$, the following estimates hold:*

1. $\int_0^{d_{x,p}-\frac{\delta}{2}} ||\nabla h_{d_\epsilon^2}^-|^2{-}1|(\gamma_{x,p}(s))\, ds \leq \frac{c(N,\delta)}{d_{p,q}}(e(x) + d_\epsilon^2)$;
2. $\int_0^{d_{x,p}-\frac{\delta}{2}} |\langle \nabla h_{d_\epsilon^2}^-, \nabla d^-\rangle - 1|(\gamma_{x,p}(s))\, ds \leq \frac{c(N,\delta)}{d_{p,q}}(e(x) + d_\epsilon^2)$;
3. $\int_{t_1}^{t_2} |\nabla h_{d_\epsilon^2}^- - \nabla d^-|(\gamma_{x,p}(s))\, ds \leq \frac{c(N,\delta)\sqrt{t_2 - t_1}}{\sqrt{d_{p,q}}}(\sqrt{e(x)} + d_\epsilon)$.

Proof. The bounds on $(|\nabla h_{d_\epsilon^2}^-|^2{-}1)_+$ and $(\langle \nabla h_{d_\epsilon^2}^-, \nabla d^-\rangle{-}1)_+$ for statements 1 and 2 come from Lemma 4.0.8, Fubini's theorem and statement 2 of Theorem 3.2.9. The bound on the negative part comes from an estimate like (4.19) combined with Corollary 3.2.8 for statement 2, and then an additional application Cauchy-Schwarz for statement 1. We note that if one traces the proof of (4.19) back to Lemma 4.0.4, it is clear that one can obtain bounds where the excess is not related to the heat flow time as they have been for the past several claims.

For statement 3,

$$|\nabla h^-_{d^2_\epsilon} - \nabla d^-|^2 = |\nabla h^-_{d^2_\epsilon}|^2 + 1 - 2\langle \nabla h^-_{d^2_\epsilon}, \nabla d^- \rangle \leq ||\nabla h^-_{d^2_\epsilon}|^2 - 1| + 2|\langle \nabla h^-_{d^2_\epsilon}, \nabla d^- \rangle - 1| \ m\text{-a.e.}.$$

Therefore, statement 1, 2, Cauchy-Schwarz and an argument by Fubini's theorem using statement 2 of Theorem 3.2.9 gives statement 3. □

Chapter 5

Gromov-Hausdorff approximation

This chapter will be divided into three sections. The main lemma proved in the first section gives a way of overcoming the lack of start of induction in the arguments of [33] generalized to the RCD setting. In the second section we use the main lemma to construct geodesics with nice properties in its interior. Finally, we prove Theorem 1.1.1 in the third section. To be precise, we prove a slightly weaker version of the main theorem analogous to the main result of [33], which will be used to prove non-branching in Chapter 6 and, subsequently, the main theorem.

Fix (X, d, m), an $\mathsf{RCD}(-(N-1), N)$ metric measure space for $N \in (1, \infty)$ and $p, q \in X$ with d(p,q)=1. Fix $0 < \delta < 0.1$. For any $x_1, x_2 \in X$, we fix a constant speed geodesic from x_1 to x_2 parameterized on $[0, 1]$ and denote it $\tilde{\gamma}_{x_1,x_2}$. By Remark 2.7.5, we may assume the map $X \times X \times [0, 1] \ni (x_1, x_2, t) \mapsto \tilde{\gamma}_{x_1,x_2}(t)$ is Borel. The unit speed reparameterizations of $\tilde{\gamma}_{x_1,x_2}$ to the interval $[0, d(x_1, x_2)]$ will be denoted γ_{x_1,x_2}. γ will denote $\gamma_{p,q}$. For each $x \in X$, define $\Psi : X \times [0, \infty) \to X$ by

$$(x, s) \mapsto \Psi_s(x) = \begin{cases} \gamma_{x,p}(s) & \text{if } d(x, p) \geq s, \\ p & \text{if } d(x, p) < s. \end{cases} \tag{5.1}$$

Similarly, define $\Phi : X \times [0, \infty) \to X$ by

$$(x, s) \mapsto \Phi_s(x) = \begin{cases} \gamma_{x,q}(s) & \text{if } d(x, q) \geq s, \\ q & \text{if } d(x, q) < s. \end{cases} \tag{5.2}$$

By integral Abresch-Gromoll inequality 2.10.8, for any sufficiently small $r \leq \bar{r}(N, \delta)$ and any $\delta \leq t_0 \leq 1 - \delta$,

$$\fint_{B_r(\gamma(t_0))} e \leq c_0(N, \delta) r^2.$$

Therefore, there exists a subset $S \subseteq B_r(\gamma(t_0))$ so that

1. $\frac{m(S)}{m(B_r(\gamma(t_0)))} \geq 1 - \frac{V(1,10)}{3}$ (see 2.1.4 for the definiton of $V := V_{-(N-1),N}$)
2. $\forall z \in S, e(z) \leq c_1(N, \delta)^2 r^2$.

We fix such a c_1 for the rest of this chapter and assume in addition $c_1 > 100$.

In all sections the letter c will be used to represent different constants which only depend on N and δ. Any constant which will be used repeatedly will be given a subscript. We will continue using the notations of Chapter 4.

5.1 Proof of main lemma

Lemma 5.1.1. *(Main lemma) There exists $\epsilon_1(N,\delta) > 0$ and $\bar{r}_1(N,\delta) > 0$ so that for all $r \le \bar{r}_1$ and $\delta \le t_0 \le 1-\delta$, there exists $z \in B_r(\gamma(t_0))$ so that*

1. *$V(1,100) \le \frac{m(B_r(\Psi_s(z)))}{m(B_r(z))} \le \frac{1}{V(1,100)}$ for any $s \le \epsilon_1$.*
2. *There exists $A \subseteq B_r(z)$ with $m(A) \ge (1 - V(1,10))m(B_r(z))$ and $\Psi_s(A) \subseteq B_{2r}(\Psi_s(z))$ for any $s \le \epsilon_1$.*
3. *$e(z) \le c_1^2 r^2$.*

Proof. Fix $\delta \le t_0 \le 1-\delta$ and a scale $r \le \bar{r}_1(N,\delta)$. $\bar{r}_1$ need only be chosen smaller than the radius bounds required for the application of various theorems in the proof; most notably Theorem 2.10.8 and the estimates of Chapter 4. It will be clear that all the required radius bounds only depend on N and δ so we will not address this each time for the sake of brevity. In addition, we assume $\bar{r}_1 \le \frac{\delta}{10}$.

By Bishop-Gromov volume comparison 2.1.4, integral Abresch-Gromoll inequality 2.10.8, and the fact that Ψ is defined using unit speed geodesics, it is clear there exist ϵ depending on N, δ and r, and $z \in B_r(\gamma(t_0))$ satisfying

1. $V(1,100) \le \frac{m(B_r(\Psi_s(z)))}{m(B_r(z))} \le \frac{1}{V(1,100)}$ for any $s \le \epsilon$.
2. There exists $A \subseteq B_r(z)$ with $m(A) \ge (1 - V(1,10))m(B_r(z))$ and $\Psi_s(A) \subseteq B_{2r}(\Psi_s(z))$ for any $s \le \epsilon$.
3. $e(z) \le c_1^2 r^2$.

We will remove the dependence of ϵ on r.

To this effect, we will show that if 1, 2 and 3 hold for some $z \in B_r(\gamma(t_0))$ and all $s \le \epsilon$ less than or equal to some $\epsilon_1(N,\delta)$ to be fixed later, then in fact we can choose $z' \in B_r(\gamma(t_0))$ satisfying 3 which significantly improves the estimates in 1 and 2 for $s \le \epsilon$. To be precise, we find $z' \in B_r(\gamma(t_0))$ satisfying

1'. $2V(1,100) \le \frac{m(B_r(\Psi_s(z')))}{m(B_r(z'))} \le \frac{1}{2V(1,100)}$ for any $s \le \epsilon$.

2'. There exists $A' \subseteq B_r(z')$ with $m(A') \ge (1 - V(1,10))m(B_r(z'))$ and $\Psi_s(A') \subseteq B_{\frac{3r}{2}}(\Psi_s(z'))$ for any $s \le \epsilon$.

3'. $e(z') \le c_1^2 r^2$.

We a priori assume $\epsilon_1 \le \frac{\delta}{10}$ and impose more bounds on ϵ_1 as the proof continues. Let z satisfy 1, 2 and 3 for some $\epsilon \le \epsilon_1$.

Let w be the midpoint of z and $\gamma(t_0)$. We know $B_{\frac{r}{2}}(w) \subseteq B_r(\gamma(t_0)) \cap B_r(z)$. By Bishop-Gromov and since $r \le \bar{r}$ which was assumed to be less than 0.01,

$$\frac{m(B_r(\gamma(t_0)) \cap B_r(z))}{m(B_r(z))} \ge \frac{m(B_{\frac{r}{2}}(w))}{m(B_r(z))} \ge \frac{m(B_{\frac{r}{2}}(w))}{m(B_{\frac{3r}{2}}(w))} \ge V(\frac{r}{2}, \frac{3r}{2}) > V(\frac{1}{2}, \frac{3}{2}).$$

Using $m(A) \ge (1 - V(1, 10))m(B_r(z)) > (1 - \frac{V(\frac{1}{2},\frac{3}{2})}{3})m(B_r(z))$ and the previous estimate,

$$\frac{m(A \cap B_r(\gamma(t_0)))}{m(B_r(z))} > \frac{2}{3}V(\frac{1}{2}, \frac{3}{2}).$$

Therefore,

$$\begin{aligned}\frac{m(A \cap B_r(\gamma(t_0)))}{m(B_r(\gamma(t_0)))} &= \frac{m(A \cap B_r(\gamma(t_0)))}{m(B_r(z))}\frac{m(B_r(z))}{m(B_r(\gamma(t_0)))} \\ &> \frac{2}{3}V(\frac{1}{2}, \frac{3}{2})V(r, 2r) > \frac{2}{3}V(\frac{1}{2}, \frac{3}{2})V(\frac{3}{2}, 3) > \frac{2}{3}V(1, 10).\end{aligned} \tag{5.3}$$

Define the set

$$D_1 := A \cap B_r(\gamma(t_0)) \cap \left\{e(x) \le c_1^2 r^2\right\}, \tag{5.4}$$

where c_1 is as fixed earlier. We will choose a z' satisfying properties 1' - 3' from D_1. From (5.3) and the definition of c_1,

$$\frac{m(D_1)}{m(B_r(\gamma(t_0)))} > \frac{1}{3}V(1, 10).$$

Therefore, by Bishop-Gromov,

$$\frac{m(D_1)}{m(B_r(z))} \ge c(N). \tag{5.5}$$

Since $e(z) \le c_1^2 r^2$ by property 3 of z, the curve traversing $\gamma_{z,p}$ in reverse and then $\gamma_{z,q}$ is a $c_1 r$-geodesic from p to q. Fix $h^- \equiv h^-_{\rho(c_1 r)^2}$ satisfying statement 4 of Theorem 4.0.12 for the balls of radius $c_1 r$ along this curve, where $\rho \in [\frac{1}{2}, 2]$.

Since z has low excess, by integral Abresch-Gromoll, there exists $B_{2r}(z)' \subseteq B_{2r}(z)$ so that

$$e(x) \le c(N, \delta)r^2 \ \forall x \in B_{2r}(z)' \ \text{ and } \ \frac{m(B_{2r}(z)')}{m(B_{2r}(z))} \ge 1 - \frac{1}{2}V(1, 10)^2. \tag{5.6}$$

For all $s \in [0, \epsilon]$ and $(x, y) \in X \times X$, define

$$dt_1(s)(x, y) := \min\left\{r, \max_{0 \le \tau \le s}|d(x, y) - d(\Psi_\tau(x), \Psi_\tau(y))|\right\} \tag{5.7}$$

and

$$U_1^s := \{(x, y) \in D_1 \times B_{2r}(z)' | dt_1(s)(x, y) < r\}. \tag{5.8}$$

Consider $\int_{D_1 \times B_{2r}(z)'} dt_1(s)(x, y)\, d(m \times m)(x, y)$ for $0 \le s \le \epsilon$. Since $r \le \bar{r}_1 \le \frac{\delta}{10}$, $\epsilon \le \epsilon_1 \le \frac{\delta}{10}$, and $t_0 \geq \delta$, $(\Psi_s)_{s\in[0,\epsilon]}$ is a local flow of $-\nabla d_p$ from both D_1 and $B_{2r}(z)'$. Therefore, $s \mapsto \int_{D_1\times B_{2r}(z)'} dt_1(s)(x,y)\, d(m\times m)(x,y)$ is Lipschitz and

$$\begin{aligned}
&\frac{d}{ds}\int_{D_1\times B_{2r}(z)'} dt_1(s)(x,y)\, d(m\times m)(x,y)\\
&\leq \int_{U_1^s}\Big(|\nabla h^- - \nabla d_p|(\Psi_s(x)) + |\nabla h^- - \nabla d_p|(\Psi_s(y))\Big)\, d(m\times m)(x,y)\\
&\qquad + \int_0^1\int_{U_1^s} d(\Psi_s(x),\Psi_s(y))|\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde\gamma_{\Psi_s(x),\Psi_s(y)}(\tau))\, d(m\times m)(x,y)\, d\tau,
\end{aligned}\tag{5.9}$$

for a.e. $s \in [0,\epsilon]$ by Proposition 3.3.8.

For any $s \in [0,\epsilon]$ and $(x,y) \in U_1^s$,

1. $d(x,y) < 3r$ since $D_1 \subseteq B_r(z)$ and $B_{2r}(z)' \subseteq B_{2r}(z)$;
2. $d(\Psi_s(x),\Psi_s(z)) < 2r$ since $D_1 \subseteq A$ by definition (5.4);
3. $dt_1(s)(x,y) < r$ by definition of U_1^s (5.8).

Therefore, $\Psi_s(y) \in B_{6r}(\Psi_s(z))$ by triangle inequality and so $(\Psi_s,\Psi_s)(U_1^s) \subseteq B_{\frac{c_1}{2}r}(\Psi_s(z))\times B_{\frac{c_1}{2}r}(\Psi_s(z))$ since we assumed $c_1 > 100$. Therefore,

$$\begin{aligned}
&\int_0^1\int_{U_1^s} d(\Psi_s(x),\Psi_s(y))|\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde\gamma_{\Psi_s(x),\Psi_s(y)}(\tau))\, d(m\times m)(x,y)\, d\tau\\
&\leq c(N,\delta)\int_0^1\int_{(\Psi_s,\Psi_s)(U_1^s)} d(x,y)|\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde\gamma_{x,y}(\tau))\, d(m\times m)(x,y)\, d\tau \quad \text{, by Theorem 3.2.9, 2}\\
&\leq c(N,\delta) r m(B_{\frac{c_1}{2}r}(\Psi_s(z)))\int_{B_{c_1 r}(\Psi_s(z))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm \quad \text{, by segment inequality 3.3.3}\\
&\leq c(N,\delta) r m(B_r(z))^2 \fint_{B_{c_1 r}(\Psi_s(z))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm \quad \text{, by Bishop-Gromov and property 1 of } z.
\end{aligned}\tag{5.10}$$

Integrating in $s \in [0,\epsilon]$,

$$\begin{aligned}
&\int_0^\epsilon\Big(\int_0^1\int_{U_1^s} d(\Psi_s(x),\Psi_s(y))|\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde\gamma_{\Psi_s(x),\Psi_s(y)}(\tau))\, d(m\times m)(x,y)\, d\tau\Big)\, ds\\
&\leq c r m(B_r(z))^2\int_0^\epsilon \fint_{B_{c_1 r}(\Psi_s(z))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm\, ds\\
&\leq c(N,\delta) r m(B_r(z))^2\sqrt{\epsilon},
\end{aligned}\tag{5.11}$$

where the last line follows from the definition of h^-, statement 4 of Theorem 4.0.12, and Cauchy-Schwarz.

By statement 3 of 4.0.13, the excess bound on the elements of D_1 (5.4) (and therefore also on the elements $\Psi_s(D_1)$), and Bishop-Gromov,

$$\int_0^{\epsilon}\int_{U_1^s}|\nabla h^- - \nabla d_p|(\Psi_s(x))\, d(m\times m)(x,y)\, ds \le c(N,\delta) r m(B_r(z))^2\sqrt{\epsilon}. \tag{5.12}$$

Similarly by the excess bounds on the elements of $B_{2r}(z)'$ (5.6),

$$\int_0^{\epsilon}\int_{U_1^s}|\nabla h^- - \nabla d_p|(\Psi_s(y))\, d(m\times m)(x,y)\, ds \le c(N,\delta) r m(B_r(z))^2\sqrt{\epsilon}. \tag{5.13}$$

Combining (5.11) - (5.13) with the bound (5.9) on $\frac{d}{ds}\int_{D_1\times B_{2r}(z)'} dt_1(s)(x,y)$, we obtain

$$\begin{aligned}\int_{D_1\times B_{2r}(z)'} dt_1(\epsilon)(x,y)\, d(m\times m)(x,y) &= \int_0^{\epsilon}[\frac{d}{ds}\int_{D_1\times B_{2r}(z)'} dt_1(s)(x,y)\, d(m\times m)(x,y)]\, ds \\ &\le c(N,\delta) r m(B_r(z))^2\sqrt{\epsilon}.\end{aligned} \tag{5.14}$$

Since D_1 takes a non-trivial portion of the measure of $B_r(z)$ by (5.5),

$$⨍_{D_1}\int_{B_{2r}(z)'} dt_1(\epsilon)(x,y)\, dm(y) \le c(N,\delta) r m(B_r(z))\sqrt{\epsilon}.$$

In particular, there exists $z' \in D_1$ so that

$$\int_{B_{2r}(z)'} dt_1(\epsilon)(z',y)\, dm(y) \le c r m(B_r(z))\sqrt{\epsilon}.$$

By definition of D_1, property 3' of z' is satisfied.

We next check property 2' is satisfied for the chosen z' as well if ϵ is sufficiently small. Define $B_r(z')' := B_r(z')\cap B_{2r}(z)'$. By the previous estimate and Bishop-Gromov,

$$\int_{B_r(z')'} dt_1(\epsilon)(z',y)\, dm(y) \le c r m(B_r(z))\sqrt{\epsilon} \le c(N,\delta) r m(B_r(z'))\sqrt{\epsilon}. \tag{5.15}$$

Using this, we bound ϵ_1 sufficiently small depending on N and δ so that for $\epsilon \le \epsilon_1$,

$$\int_{B_r(z')'} dt_1(\epsilon)(z',y)\, dm(y) \le \frac{1}{4} r m(B_r(z'))V(1,10). \tag{5.16}$$

For example, $\epsilon_1 \leq (\frac{V(1,10)}{4c})^2$ suffices, where c is the last one from (5.15). Moreover, $B_{2r}(z)'$ takes significant mass in $B_{2r}(z)$ from (5.6) and so by Bishop-Gromov,

$$\frac{m(B_r(z')')}{m(B_r(z'))} \geq 1 - \Big(\frac{m(B_{2r}(z)')}{m(B_{2r}(z))}\frac{m(B_{2r}(z))}{m(B_r(z'))}\Big) \geq 1 - \Big(\frac{1}{2}\frac{(V(1,10)^2)}{V(1,3)}\Big) \geq 1 - \frac{1}{2}V(1,10). \tag{5.17}$$

Combining (5.16) and (5.17), we conclude there exists $A' \subseteq B_r(z')'$ so that

$$\frac{m(A')}{m(B_r(z'))} \geq 1 - V(1,10) \;\text{ and }\; dt_1(\epsilon)(z',y) \leq \frac{1}{2}r \;\forall y \in A'. \tag{5.18}$$

The latter implies $\Psi_s(A') \subseteq B_{\frac{3r}{2}}(\Psi_s(z'))$ for any $s \leq \epsilon$ and so property 2' of z' is satisfied.

This also gives one direction of the bound in property 1' for z'. For each $s \leq \epsilon$,

$$\begin{aligned}\frac{m(B_r(\Psi_s(z')))}{m(B_r(z'))} &\geq V(1,\frac{3}{2})\frac{m(B_{\frac{3r}{2}}(\Psi_s(z')))}{m(B_r(z'))} \quad \text{, by Bishop-Gromov}\\ &\geq V(1,\frac{3}{2})\frac{m(\Psi_s(A'))}{m(B_r(z'))}\\ &\geq V(1,\frac{3}{2})(1+c(N,\delta)s)^{-N}\frac{m(A')}{m(B_r(z'))} \quad \text{, by Theorem 3.2.9, 2}\\ &\geq V(1,\frac{3}{2})(1+c(N,\delta)s)^{-N}(1-V(1,10)).\end{aligned}$$

We bound ϵ_1 sufficiently small depending on N and δ so that for $s \leq \epsilon \leq \epsilon_1$, the last line is greater than $2V(1,100)$.

The other direction of the bound in property 1' of z' will be proved similarly by sending a sufficiently large portion of $B_r(\Psi_s(z'))$ close to z' (in fact z) using a flow which does not decrease measure significantly and then using Bishop-Gromov. To do this, we first use the RLF associated to $-\nabla h^-$ to send a portion of $B_r(z')$ close to $\Psi_s(z')$. We then use the inverse flow (i.e. the RLF associated to ∇h^-) on the image of that portion to make sure a large enough portion of $B_r(\Psi_s(z'))$ indeed ends up close to z' under the inverse flow.

$|\nabla h_0^-| \in L^\infty(m)$ by (4.11) and so $|\nabla h^-|, \Delta h^- \in L^\infty(m)$ by Bakry-Ledoux estimate 2.5.2 and $h^- \in W^{2,2}(X)$ by Corollary 2.6.3 of the improved Bochner inequality. Therefore, the time-independent vector fields $-\nabla h^-$ and ∇h^- are bounded and satisfy the conditions of the existence and uniqueness of RLFs Theorem 3.1.4. Let $(\tilde{\Psi}_t)_{t\in[0,1]}$ and $(\tilde{\Psi}_{-t})_{t\in[0,1]}$ be the associated RLFs of $-\nabla h^-$ and ∇h^- for $t \in [0,1]$ respectively. The choice of notation is due to Proposition 3.2.6, which says $\tilde{\Psi}_{-t}$ and $\tilde{\Psi}_t$ are m-a.e. inverses of each other.

Since $e(z') \leq c_1^2 r^2$ and $m(A') \geq (1-V(1,10))m(B_r(z'))$, integral Abresch-Gromoll gives $A'' \subseteq A'$ so that

$$e(x) \leq c(N,\delta)r^2 \;\forall x \in A'' \;\; \textit{and} \;\; \frac{m(A'')}{m(B_r(z'))} \geq 1 - 2V(1,10). \tag{5.19}$$

For all $s \in [0,\epsilon]$ and $(x,y) \in X \times X$, define

$$dt_2(s)(x,y) := \min\Big\{r, \max_{0\leq\tau\leq s}|d(x,y) - d(\Psi_\tau(x),\tilde{\Psi}_\tau(y))|\Big\} \tag{5.20}$$

and

$$U_2^s := \{(x,y) \in A'' \times B_r(z') | dt_2(s)(x,y) < r\}. \tag{5.21}$$

Consider $\int_{A''\times B_r(z')} dt_2(s)(x,y)\, d(m\times m)(x,y)$ for $0 \le s \le \epsilon$. By Proposition 3.3.8, for a.e. $s \in [0,\epsilon]$,

$$\begin{aligned} &\frac{d}{ds} \int_{A''\times B_r(z')} dt_2(s)(x,y)\, d(m\times m)(x,y) \\ &\le \int_{U_2^s} |\nabla h^- - \nabla d_p|(\Psi_s(x))\, d(m\times m)(x,y) \\ &\qquad + \int_0^1 \int_{U_2^s} d(\Psi_s(x), \tilde{\Psi}_s(y)) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\Psi_s(x),\tilde{\Psi}_s(y)}(\tau))\, d(m\times m)(x,y)\, d\tau. \end{aligned} \tag{5.22}$$

For any $s \in [0,\epsilon]$ and $(x,y) \in U_2^s$,

1. $d(x,y) < 2r$ since $A'' \subseteq B_r(z')$;
2. $d(\Psi_s(x), \Psi_s(z)) \le d(\Psi_s(x), \Psi_s(z')) + d(\Psi_s(z'), \Psi_s(z)) < \frac{7}{2}r$ by definition of A' (5.18) and $z' \in D_1 \subseteq A$ (5.4);
3. $dt_2(s)(x,y) < r$ by definition of U_2^s (5.22).

Therefore, $\tilde{\Psi}_s(y) \in B_{\frac{13r}{2}}(\Psi_s(z))$ by triangle inequality and so $(\Psi_s, \tilde{\Psi}_s)(U_2^s) \subseteq B_{\frac{c_1}{2}r}(\Psi_s(z)) \times B_{\frac{c_1}{2}r}(\Psi_s(z))$ since $c_1 > 100$. Therefore,

$$\begin{aligned} &\int_0^1 \int_{U_2^s} d(\Psi_s(x), \tilde{\Psi}_s(y)) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\Psi_s(x),\tilde{\Psi}_s(y)}(\tau))\, d(m\times m)(x,y)\, d\tau \\ &\le c(N,\delta) \int_0^1 \int_{(\Psi_s,\tilde{\Psi}_s)(U_2^s)} d(x,y) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{x,y}(\tau))\, d(m\times m)(x,y)\, d\tau \text{ , by 3.2.9 2, 4.0.2 and 3.1.4 (3.3)} \\ &\le c(N,\delta) r m(B_{\frac{c_1}{2}r}(\Psi_s(z))) \int_{B_{c_1 r}(\Psi_s(z))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm \text{ , by segment inequality 3.3.3} \\ &\le c(N,\delta) r m(B_r(z))^2 \fint_{B_{c_1 r}(\Psi_s(z))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm \text{ , by Bishop-Gromov and property 1 of } z. \end{aligned} \tag{5.23}$$

Integrating in $s \in [0, \epsilon]$,

$$\begin{aligned}&\int_0^{\epsilon} \Big(\int_0^1 \int_{U_2^s} d(\Psi_s(x), \tilde{\Psi}_s(y)) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\Psi_s(x), \tilde{\Psi}_s(y)}(\tau))\, d(m \times m)(x, y)\, d\tau \Big)\, ds \\ &\leq crm(B_r(z))^2 \int_0^{\epsilon} \fint_{B_{c_1 r}(\Psi_s(z))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm\, ds \\ &\leq c(N, \delta) rm(B_r(z))^2 \sqrt{\epsilon},\end{aligned} \tag{5.24}$$

where the last line follows from the definition of h^-, statement 4 of Theorem 4.0.12, and Cauchy-Schwarz.

By statement 3 of 4.0.13, the excess bound on the elements of A'' (5.19), and Bishop-Gromov,

$$\int_0^{\epsilon} \int_{U_2^s} |\nabla h^- - \nabla d_p|(\Psi_s(x))\, d(m \times m)(x, y)\, ds \leq c(N, \delta) rm(B_r(z))^2 \sqrt{\epsilon}. \tag{5.25}$$

Combining (5.24), (5.25) with the bound (5.22) we obtain,

$$\begin{aligned}\int_{A'' \times B_r(z')} dt_2(\epsilon)(x, y)\, d(m \times m)(x, y) &= \int_0^{\epsilon} [\frac{d}{ds} \int_{A'' \times B_r(z')} dt_2(s)(x, y)\, d(m \times m)(x, y)]\, ds \\ &\leq c(N, \delta) rm(B_r(z))^2 \sqrt{\epsilon}.\end{aligned} \tag{5.26}$$

A'' is comparable in measure to $B_r(z')$ by (5.19) and hence also to $B_r(z)$ by Bishop-Gromov. Therefore, there exists $z_1 \in A''$ so that

$$\int_{B_r(z')} dt_2(\epsilon)(z_1, y)\, dm(y) \leq c(N, \delta) rm(B_r(z)) \sqrt{\epsilon}.$$

By Bishop-Gromov, $\frac{m(B_r(z))}{m(B_r(z'))} \leq c(N)$ and so

$$\fint_{B_r(z')} dt_2(\epsilon)(z_1, y)\, dm(y) \leq c(N, \delta) r \sqrt{\epsilon}.$$

Using this, we bound ϵ_1 sufficiently small depending on N and δ so that there exists $D_2 \subseteq B_r(z')$ with

$$\frac{m(D_2)}{m(B_r(z'))} \geq 1 - V(1, 10) \text{ and } dt_2(\epsilon)(z_1, y) \leq \frac{r}{2}\ \forall y \in D_2. \tag{5.27}$$

For each $y \in D_2$ and $s \in [0, \epsilon]$,

1. $d(z_1, y) < 2r$;
2. $d(\Psi_s(z_1), \Psi_s(z)) \leq d(\Psi_s(z_1), \Psi_s(z')) + d(\Psi_s(z'), \Psi_s(z)) < \frac{7}{2} r$;

3. $dt_2(\epsilon)(z_1, y) \le \frac{r}{2}$,

and so

$$\tilde{\Psi}_s(D_2) \subseteq B_{4r}(\Psi_s(z')) \subseteq B_{6r}(\Psi_s(z)). \tag{5.28}$$

Moreover, $\tilde{\Psi}_s(D_2)$ is non-trivial in measure compared to $B_r(z)$.

$$\begin{aligned} \frac{m(\tilde{\Psi}_s(D_2))}{m(B_r(z))} &\ge e^{-c(N,\delta)\frac{\delta}{10}} \frac{m(D_2)}{m(B_r(z))} \quad \text{, by 4.0.2, 3.1.4 (3.3), and } \epsilon_1 \le \frac{\delta}{10} \\ &\ge c(N,\delta) \quad \text{, by definition of } D_2 \text{ and Bishop-Gromov.} \end{aligned} \tag{5.29}$$

We will now flow $\tilde{\Psi}_s(D_2)$ back by $\tilde{\Psi}_{-t}$ and use that to control the flow of $B_r(\Psi_s(z'))$ under $\tilde{\Psi}_{-t}$. Fix $s \in [0, \epsilon]$. By Proposition 3.2.6, we may assume, up to choosing a full measure subset, that D_2 satisfies

$$\tilde{\Psi}_{-t}(\tilde{\Psi}_s(x)) = \tilde{\Psi}_{s-t}(x) \ \forall t \in [0, s] \text{ and } \forall x \in D_2. \tag{5.30}$$

For all $t \in [0, s]$ and $(x, y) \in X \times X$, define

$$dt_3(t)(x, y) := \min\left\{r, \max_{0 \le \tau \le t} |d(x, y) - d(\tilde{\Psi}_{-\tau}(x), \tilde{\Psi}_{-\tau}(y))|\right\} \tag{5.31}$$

and

$$U_3^t := \{(x, y) \in \tilde{\Psi}_s(D_2) \times B_r(\Psi_s(z')) | dt_3(t)(x, y) < r\}. \tag{5.32}$$

We note that U_3^t implicitly depends on s. Consider $\int_{\tilde{\Psi}_s(D_2)\times B_r(\Psi_s(z'))} dt_3(t)(x, y)\, d(m \times m)(x, y)$ for $0 \le t \le s$. By proposition 3.3.8, for a.e. $t \in [0, s]$,

$$\begin{aligned} &\frac{d}{dt} \int\limits_{\tilde{\Psi}_s(D_2)\times B_r(\Psi_s(z'))} dt_3(t)(x, y)\, d(m \times m)(x, y) \\ &\le \int_0^1 \int\limits_{U_3^t} d(\tilde{\Psi}_{-t}(x), \tilde{\Psi}_{-t}(y)) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\tilde{\Psi}_{-t}(x), \tilde{\Psi}_{-t}(y)}(\tau))\, d(m \times m)(x, y)\, d\tau, \end{aligned} \tag{5.33}$$

For any $t \in [0, s]$, $\omega \in [0, t]$ and $(x, y) \in U_3^t$,

1. $d(x, y) < 5r$ since $\tilde{\Psi}_s(D_2) \subseteq B_{4r}(\Psi_s(z'))$ by (5.28);

2. $d(\tilde{\Psi}_{-\omega}(x), \Psi_{s-\omega}(z)) = d(\tilde{\Psi}_{s-\omega}(x'), \Psi_{s-\omega}(z)) < 6r$ for some $x' \in D_2$ by (5.28) and (5.30);

3. $dt_3(t)(x, y) < r$ by definition of U_3^t (5.32).

Hence,

$$\tilde{\Psi}_{-\omega}(y) \in B_{12r}(\Psi_{s-\omega}(z)) \tag{5.34}$$

by triangle inequality. Therefore, $(\tilde{\Psi}_{-\omega}, \tilde{\Psi}_{-\omega})(U_3^t) \subseteq B_{\frac{c_1}{2}r}(\Psi_{s-\omega}(z)) \times B_{\frac{c_1}{2}r}(\Psi_{s-\omega}(z))$ for all $\omega \in [0, t]$

since $c_1 > 100$. For any $(x,y) \in U_3^t$,

$$\begin{aligned}\Delta h^-(\tilde{\Psi}_{-\omega}(x)) &= \Delta h^+(\tilde{\Psi}_{-\omega}(x)) + \Delta\hat{e}(\tilde{\Psi}_{-\omega}(x)) \\ &\geq -c(N,\delta) \quad \text{, by Lemma 4.0.2 and Lemma 4.0.3 3, using } e(z) \leq c_1^2 r^2,\end{aligned} \tag{5.35}$$

where h^+, $\hat{e}$ are heat flow approximations of h_0^+ and e_0 respectively up to the same time as h^-. We have the same bound for $\Delta h^-(\tilde{\Psi}_{-\omega}(y))$. Therefore,

$$\begin{aligned}&\int_0^1 \int_{U_3^t} d(\tilde{\Psi}_{-t}(x), \tilde{\Psi}_{-t}(y)) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\tilde{\Psi}_{-t}(x),\tilde{\Psi}_{-t}(y)}(\tau))\, d(m\times m)(x,y)\, d\tau \\ &\leq c(N,\delta) \int_0^1 \int_{(\tilde{\Psi}_{-t},\tilde{\Psi}_{-t})(U_3^t)} d(x,y) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{x,y}(\tau))\, d(m\times m)(x,y)\, d\tau, \text{ by (5.35) and Remark 3.1.5} \\ &\leq c(N,\delta) r m(B_{\frac{c_1}{2}r}(\Psi_{s-t}(z))) \int_{B_{c_1 r}(\Psi_{s-t}(z))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm \quad \text{, by segment inequality 3.3.3} \\ &\leq c(N,\delta) r m(B_r(z))^2 \fint_{B_{c_1 r}(\Psi_{s-t}(z))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm \quad \text{, by Bishop-Gromov and property 1 of } z.\end{aligned} \tag{5.36}$$

Integrating in $t \in [0,s]$,

$$\begin{aligned}&\int_0^s \Big(\int_0^1 \int_{U_3^t} d(\tilde{\Psi}_{-t}(x), \tilde{\Psi}_{-t}(y)) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\tilde{\Psi}_{-t}(x),\tilde{\Psi}_{-t}(y)}(\tau))\, d(m\times m)(x,y)\, d\tau \Big)\, dt \\ &\leq c r m(B_r(z))^2 \int_0^s \fint_{B_{c_1 r}(\Psi_{s-t}(z))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm\, ds \\ &\leq c(N,\delta) r m(B_r(z))^2 \sqrt{s},\end{aligned} \tag{5.37}$$

where the last line follows from the definition of h^-, statement 4 of Theorem 4.0.12, and Cauchy-Schwarz. Therefore,

$$\begin{aligned}\int_{\tilde{\Psi}_s(D_2)\times B_r(\Psi_s(z'))} dt_3(s)(x,y)\, d(m\times m)(x,y) &= \int_0^s \Big[\frac{d}{dt} \int_{\tilde{\Psi}_s(D_2)\times B_r(\Psi_s(z'))} dt_3(t)(x,y)\, d(m\times m)(x,y)\Big]\, dt \\ &\leq c(N,\delta) r m(B_r(z))^2 \sqrt{s}.\end{aligned} \tag{5.38}$$

We previously computed that $\tilde{\Psi}_s(D_2)$ is non-trivial in measure compared to $B_r(z)$ in (5.29) and so there exists $z_2 \in \tilde{\Psi}_s(D_2)$ with

$$\int_{B_r(\Psi_s(z'))} dt_3(s)(z_2, y)\, dm(y) \le c(N,\delta) r m(B_r(z)) \sqrt{s}.$$

By Bishop-Gromov, $\frac{m(B_r(\Psi_s(z')))}{m(B_r(\Psi_s(z)))} \ge c(N)$ and so by property 1 of z,

$$\fint_{B_r(\Psi_s(z'))} dt_3(s)(z_2, y)\, dm(y) \le c(N,\delta) r \sqrt{s}.$$

Using this, we bound ϵ_1 sufficiently small depending on N and δ so there exists $D_3 \subseteq B_r(\Psi_s(z'))$ with

$$\frac{m(D_3)}{m(B_r(\Psi_s(z')))} \ge 1 - V(1,10) \;\text{ and }\; dt_3(s)(z_2, y) \le \frac{1}{2} r \;\forall y \in D_3. \tag{5.39}$$

For each $y \in D_3$,

1. $d(z_2, y) < 5r$ by (5.28);
2. $d(\tilde{\Psi}_{-s}(z_2), z') = d(z_2', z') < r$, where $z_2' \in D_2 \subseteq B_r(z')$ is so that $\tilde{\Psi}_s(z_2') = z_2$;
3. $dt_3(s)(z_2, y) \le \frac{1}{2} r$,

and so $\tilde{\Psi}_{-s}(D_3) \subseteq B_{7r}(z')$. Notice also for the next calculation that $\tilde{\Psi}_{-t}(D_3) \subseteq B_{12r}(z)$ for any $t \in [0, s]$ by the calculations of (5.34). Therefore, one has the same lower bound for the Laplacian of h^- on $\tilde{\Psi}_{-t}(D_3)$ as in (5.35).

We estimate

$$\begin{aligned}
\frac{m(B_r(\Psi_s(z')))}{m(B_r(z'))} &\le \frac{1}{V(1,7)} \frac{m(B_r(\Psi_s(z')))}{m(B_{7r}(z'))} \quad \text{, by Bishop-Gromov} \\
&\le \frac{1}{V(1,7)} \frac{1}{1 - V(1,10)} \frac{m(D_3)}{m(B_{7r}(z'))}, \text{ by property (5.39) of } D_3 \\
&\le \frac{1}{V(1,7)} \frac{1}{1 - V(1,10)} e^{c(N,\delta)s} \frac{m(\tilde{\Psi}_{-s}(D_3))}{m(B_{7r}(z'))} \quad \text{, by Remark 3.1.5} \\
&\le \frac{1}{V(1,7)} \frac{1}{1 - V(1,10)} e^{cs}.
\end{aligned}$$

We bound ϵ_1 sufficiently small depending on N and δ so that for $s \le \epsilon \le \epsilon_1$, the last line is less than $\frac{1}{2V(1,100)}$.

All this imply that if 1, 2 and 3 hold for $z \in B_r(\gamma(t_0))$ and $\epsilon \le \epsilon_1(N,\delta)$, then there exists $z' \in B_r(\gamma(t_0))$ satisfying 1', 2' and 3' for the same ϵ. By Bishop-Gromov volume comparison and the fact that Ψ is defined using unit speed geodesics, there is some $T(N,\delta,r) > 0$ so that 1 and 2 hold for z', A', and $0 \le s \le \epsilon + T$. Combining this with the existence of some z and ϵ depending on N, δ, and r satisfying 1, 2 and 3 mentioned at the beginning of the proof, we conclude there exists $z \in B_r(\gamma(t_0))$ so that 1, 2 and 3 hold for $\epsilon = \epsilon_1$. □

5.2 Construction of limit geodesics

In this section we construct a geodesic between p and q which has properties 1 and 2 of Lemma 5.1.1 on the geodesic itself. Roughly, this means we construct $\bar{\gamma}$ so that small balls centered on $\bar{\gamma}$ between δ and $1-\delta$ stay close to the geodesic itself for a short amount time under the flows Ψ and Φ.

We start by showing that for any fixed scale r we can find points z arbitrarily close to $\gamma(t_0)$ which have the properties 1 - 3 as in Lemma 5.1.1. In order to do this we will prove the following lemma which will form our induction step.

Lemma 5.2.1. *There exists $\epsilon_2(N,\delta) > 0$ and $\bar{r}_2(N,\delta) > 0$ so that for any $r \le \bar{r}_2$, $\delta \le t_0 \le 1-\delta$, if there exists $z \in B_r(\gamma(t_0))$ and $\epsilon \le \epsilon_2$ so that*

1. $V(1,100) \le \frac{m(B_r(\Psi_s(z)))}{m(B_r(z))} \le \frac{1}{V(1,100)}$ *for any $s \le \epsilon$;*
2. *There exists $A_r \subseteq B_r(z)$ with $m(A_r) \ge (1-V(1,10))m(B_r(z))$ and $\Psi_s(A_r) \subseteq B_{2r}(\Psi_s(z))$ for any $s \le \epsilon$;*
3. $e(z) \le c_1^2 r^2$,

then for the same z and any $r' \in [4r, 16r]$,

i. $V(1,100) \le \frac{m(B_{r'}(\Psi_s(z)))}{m(B_{r'}(z))} \le \frac{1}{V(1,100)}$ *for any $s \le \epsilon$;*

ii. *There exists $A_{r'} \subseteq B_{r'}(z)$ with $m(A_{r'}) \ge (1-V(1,10))m(B_{r'}(z))$ and $\Psi_s(A_{r'}) \subseteq B_{2r'}(\Psi_s(z))$ for any $s \le \epsilon$;*

iii. $e(z) \le c_1^2 r^2 < c_1^2 (r')^2$.

Proof. Fix $\delta \le t_0 \le 1-\delta$. We assume $\bar{r}_2 \le \frac{\delta}{1000}$ and $\epsilon_2 \le \frac{\delta}{1000}$ to begin with but will impose more bounds on both depending on N and δ as the proof continues. We will not keep track of $\bar{r}_2$ for the sake of brevity. Fix a scale $r \le \bar{r}_2$ and $r' \in [4r, 16r]$. Fix z, A_r, and $\epsilon \le \epsilon_2$ so that 1, 2 and 3 hold.

Since $e(z) \le c_1^2 r^2$, by integral Abresch-Gromoll, there exists $B_{r'}(z)' \subseteq B_{r'}(z)$ so that

$$e(x) \le c(N,\delta) r^2 \ \forall x \in B_{r'}(z)' \ \text{ and } \ \frac{m(B_{r'}(z)')}{m(B_{r'}(z))} \ge 1 - \frac{1}{2}V(1,10). \tag{5.40}$$

Similarly, there exists $A' \subseteq A_r$ so that

$$e(x) \le c(N,\delta) r^2 \ \forall x \in A' \ \text{ and } \ \frac{m(A')}{m(B_r(z))} \ge 1 - 2V(1,10). \tag{5.41}$$

As in the previous lemma, the curve traversing $\gamma_{z,p}$ in reverse and then $\gamma_{z,q}$ is a $c_1 r$-geodesic from p to q. Fix $h^- \equiv h^-_{\rho(c_1 r)^2}$ satisfying statement 4 of Theorem 4.0.12 for the balls of radius $c_1 r$ along this curve, where $\rho \in [\frac{1}{2}, 2]$.

For all $s \in [0, \epsilon]$ and $(x, y) \in X \times X$, define

$$dt_1(s)(x, y) := \min\left\{r, \max_{0\le\tau\le s}|d(x, y) - d(\Psi_\tau(x), \Psi_\tau(y))|\right\} \tag{5.42}$$

and

$$U_1^s := \{(x, y) \in A' \times B_{r'}(z)' | dt_1(s)(x, y) < r\}. \tag{5.43}$$

Consider $\int_{A'\times B_{r'}(z)'} dt_1(s)(x, y)\, d(m \times m)(x, y)$ for $0 \le s \le \epsilon$. For any $s \in [0, \epsilon]$ and $(x, y) \in U_1^s$,

1. $d(x, y) < r' + r$;
2. $d(\Psi_s(x), \Psi_s(z)) < 2r$ by definition of $A' \subseteq A_r$;
3. $dt_1(s)(x, y) < r$.

Therefore, $\Psi_s(y) \in B_{r'+4r}(\Psi_s(z)) \subseteq B_{\frac{c_1}{2}r}(\Psi_s(z))$ since $c_1 > 100$.

Using exactly the same type of computation as the first part of the proof of the main lemma, by interpolating between the two local flows of Ψ_s from A' and $B_{r'}(z)'$ with ∇h^-, we obtain

$$\int_{A'\times B_{r'}(z)'} dt_1(\epsilon)(x, y)\, d(m \times m)(x, y) \le c(N, \delta) r m(B_r(z))^2 \sqrt{\epsilon}. \tag{5.44}$$

Since A' takes a significant portion of the measure of $B_r(z)$ by (5.41),

$$⨍_{A'} \int_{B_{r'}(z)'} dt_1(\epsilon)(x, y)\, dm(y) \le c(N, \delta) r m(B_r(z)) \sqrt{\epsilon},$$

and so there exists $z' \in A'$ so that

$$\int_{B_{r'}(z)'} dt_1(\epsilon)(z', y)\, dm(y) \le crm(B_r(z)) \sqrt{\epsilon}.$$

Therefore, by the fact that $\frac{m(B_{r'}(z)')}{m(B_{r'}(z))} \ge 1 - \frac{1}{2}V(1, 10)$ and Bishop-Gromov,

$$⨍_{B_{r'}(z)'} dt_1(\epsilon)(z', y)\, dm(y) \le c(N, \delta) r \sqrt{\epsilon}.$$

Using this, we bound ϵ_2 sufficiently small depending on N and δ so that there exists $A_{r'} \subseteq B_{r'}(z)'$ with

$$\frac{m(A_{r'})}{m(B_{r'}(z))} \ge 1 - V(1, 10) \quad\text{and}\quad dt_1(\epsilon)(z', y) \le \frac{r}{2}\ \forall y \in A_{r'}. \tag{5.45}$$

For each $y \in A_{r'}$ and $s \in [0, \epsilon]$,

1. $d(z', y) < r' + r$;
2. $d(\Psi_s(z'), \Psi_s(z)) < 2r$;

3. $dt_1(\epsilon)(z', y) \leq \frac{r}{2}$,

and so

$$\Psi_s(A_{r'}) \subseteq B_{r'+\frac{7r}{2}}(\Psi_s(z)) \subseteq B_{2r'}(\Psi_s(z)). \tag{5.46}$$

This proves property ii.

This also gives one direction of the bound in property i. For each $s \leq \epsilon$,

$$\begin{aligned}
\frac{m(B_{r'}(\Psi_s(z)))}{m(B_{r'}(z))} &\geq V(1,2)\frac{m(B_{2r'}(\Psi_s(z)))}{m(B_{r'}(z'))} \quad \text{, by Bishop-Gromov} \\
&\geq V(1,2)\frac{m(\Psi_s(A_{r'}))}{m(B_{r'}(z'))} \\
&\geq V(1,2)(1+c(N,\delta)s)^{-N}\frac{m(A_{r'})}{m(B_{r'}(z'))} \quad \text{, by Theorem 3.2.9, 2} \\
&\geq V(1,2)(1+c(N,\delta)s)^{-N}(1-V(1,10)).
\end{aligned}$$

We bound ϵ_2 sufficiently small depending on N and δ so that for $s \leq \epsilon \leq \epsilon_2$, the last line is greater than $V(1, 100)$.

To obtain the other direction of the bound in property i, we employ the same strategy as the proof of the main lemma as well. Let $(\tilde{\Psi}_t)_{t\in[0,1]}$ and $(\tilde{\Psi}_{-t})_{t\in[0,1]}$ be the RLFs of the time-independent vector fields $-\nabla h^-$ and ∇h^- respectively as before.

For all $s \in [0, \epsilon]$ and $(x, y) \in X \times X$, define

$$dt_2(s)(x,y) := \min\left\{r, \max_{0\leq\tau\leq s}|d(x,y) - d(\Psi_\tau(x), \tilde{\Psi}_\tau(y))|\right\} \tag{5.47}$$

and

$$U_2^s := \{(x,y) \in A' \times B_r(z) | dt_2(s)(x,y) < r\}. \tag{5.48}$$

Consider $\int_{A'\times B_r(z)} dt_2(s)(x,y)\, d(m\times m)(x,y)$ for $0 \leq s \leq \epsilon$. For any $s \in [0,\epsilon]$ and $(x,y) \in U_2^s$,

1. $d(x, y) < 2r$;

2. $d(\Psi_s(x), \Psi_s(z)) < 2r$ by definition of $A' \subseteq A_r$;

3. $dt_2(s)(x, y) < r$.

Therefore, $\tilde{\Psi}_s(y) \in B_{5r}(\Psi_s(z)) \subseteq B_{\frac{c_1}{2}r}(\Psi_s(z))$.

Using exactly the same type of computation as the second part of the proof of the main lemma,

$$\int_{A'\times B_r(z)} dt_2(\epsilon)(x,y)\, d(m\times m)(x,y) \leq c(N,\delta) r m(B_r(z))^2 \sqrt{\epsilon}. \tag{5.49}$$

A' takes a significant portion of the measure of $B_r(z)$ by (5.41). The same considerations as before gives the existence of $D_1 \subseteq B_r(z)$ so that

$$\frac{m(D_1)}{m(B_r(z))} \geq 1 - V(1,10) \quad \text{and} \quad \tilde{\Psi}_s(D_1) \subseteq B_{5r}(\Psi_s(z)), \tag{5.50}$$

for any $s \in [0, \epsilon]$ after we bound ϵ_2 sufficiently small depending only on N and δ. Moreover, $\tilde{\Psi}_s(D_1)$ is non-trivial in measure compared to $B_r(z)$.

$$\begin{aligned}\frac{m(\tilde{\Psi}_s(D_1))}{m(B_r(z))} &\geq e^{-c(N,\delta)\frac{\delta}{1000}} \frac{m(D_1)}{m(B_r(z))} \quad \text{, by 4.0.2, 3.1.4 (3.3), and } \epsilon_2 \leq \frac{\delta}{1000} \\ &\geq c(N, \delta) \quad \text{, by definition of } D_1.\end{aligned} \tag{5.51}$$

Fix $s \in [0, \epsilon]$. By Proposition 3.2.6, we may assume, up to choosing a full measure subset, that D_1 satisfies

$$\tilde{\Psi}_{-t}(\tilde{\Psi}_s(x)) = \tilde{\Psi}_{s-t}(x)\ \forall t \in [0, s] \text{ and } \forall x \in D_1. \tag{5.52}$$

For all $t \in [0, s]$ and $(x, y) \in X \times X$, define

$$dt_3(t)(x, y) := \min\left\{r, \max_{0\leq\tau\leq t} |d(x, y) - d(\tilde{\Psi}_{-\tau}(x), \tilde{\Psi}_{-\tau}(y))|\right\} \tag{5.53}$$

and

$$U_3^t := \{(x, y) \in \tilde{\Psi}_s(D_1) \times B_{r'}(\Psi_s(z)) | dt_3(t)(x, y) < r\}. \tag{5.54}$$

Consider $\int_{\tilde{\Psi}_s(D_1)\times B_{r'}(\Psi_s(z))} dt_3(t)(x, y)\, d(m \times m)(x, y)$ for $0 \leq t \leq s$. For any $t \in [0, s]$, $\omega \in [0, t]$, and $(x, y) \in U_3^t$,

1. $d(x, y) < r' + 5r$ by (5.50);
2. $d(\tilde{\Psi}_{-\omega}(x), \Psi_{s-\omega}(z)) = d(\tilde{\Psi}_{s-\omega}(x'), \Psi_{s-\omega}(z)) < 5r$ for some $x' \in D_1$ by (5.50) and (5.52);
3. $dt_3(t)(x, y) < r$.

Hence,

$$\tilde{\Psi}_{-\omega}(y) \in B_{r'+11r}(\Psi_{s-\omega}(z)) \tag{5.55}$$

by triangle inequality. Therefore, $(\tilde{\Psi}_{-\omega}, \tilde{\Psi}_{-\omega})(U_3^t) \subseteq B_{\frac{c_1}{2}r}(\Psi_{s-\omega}(z)) \times B_{\frac{c_1}{2}r}(\Psi_{s-\omega}(z))$ for any $\omega \in [0, t]$ since $c_1 > 100$.

Using exactly the same type of computation as the third part of the proof of the main lemma,

$$\int_{\tilde{\Psi}_s(D_1)\times B_{r'}(\Psi_s(z))} dt_3(\epsilon)(x, y)\, d(m \times m)(x, y) \leq c(N, \delta) r m(B_r(z))^2 \sqrt{\epsilon}. \tag{5.56}$$

By (5.51), $\tilde{\Psi}_s(D_1)$ is non-trivial in measure compared to $B_r(z)$. By Bishop-Gromov and property 1 of z, $B_{r'}(\Psi_s(z))$ is also non-trivial in measure compared to $B_r(z)$. The same considerations as before gives the existence of $D_2 \subseteq B_{r'}(\Psi_s(z))$ so that

$$\frac{m(D_2)}{m(B_{r'}(\Psi_s(z)))} \geq 1 - V(1, 10) \text{ and } \tilde{\Psi}_{-s}(D_2) \subseteq B_{r'+7r}(z) \subseteq B_{3r'}(z), \tag{5.57}$$

for any $s \in [0, \epsilon]$ after we bound ϵ_2 sufficiently small depending only on N and δ.

We estimate

$$\begin{aligned}
\frac{m(B_{r'}(\Psi_s(z)))}{m(B_{r'}(z))} &\leq \frac{1}{V(1,3)}\frac{m(B_{r'}(\Psi_s(z)))}{m(B_{3r'}(z))} \quad \text{, by Bishop-Gromov}\\
&\leq \frac{1}{V(1,3)}\frac{1}{1-V(1,10)}\frac{m(D_2)}{m(B_{3r'}(z))}, \text{ by property (5.57) of } D_2\\
&\leq \frac{1}{V(1,3)}\frac{1}{1-V(1,10)}e^{c(N,\delta)s}\frac{m(\tilde{\Psi}_{-s}(D_2))}{m(B_{3r'}(z))} \quad \text{, by Remark 3.1.5}\\
&\leq \frac{1}{V(1,7)}\frac{1}{1-V(1,10)}e^{cs},
\end{aligned}$$

where for the third inequality we used the fact that $\tilde{\Psi}_{-t}(D_2) \subseteq B_{27r}(\Psi_{s-t}(z))$ for $0 \leq t \leq s$ by the calculations of (5.55); On these sets Δh^- is bounded below by $-c(N,\delta)$ by the same argument as (5.35). Using this, we bound ϵ_2 sufficiently small depending on N and δ so that for $s \leq \epsilon \leq \epsilon_2$, the last line is less than $V(1,100)$. This shows the other half of the bound in property i. Property iii is obvious and so we conclude. □

Combined with the main lemma, this gives

Lemma 5.2.2. *There exists $\epsilon_3(N,\delta) > 0$ and $\bar{r}_3(N,\delta) > 0$ so that for any $r \leq \bar{r}_3$ and $\delta \leq t_0 \leq 1-\delta$, there exists $z \in B_r(\gamma(t_0))$ so that for any $r \leq r' \leq \bar{r}_3$:*

1. $V(1,100) \leq \frac{m(B_{r'}(\Psi_s(z)))}{m(B_{r'}(z))} \leq \frac{1}{V(1,100)}$ *for any $s \leq \epsilon_3$;*
2. *There exists $A \subseteq B_{r'}(z)$ with $m(A) \geq (1-V(1,10))m(B_{r'}(z))$ and $\Psi_s(A) \subseteq B_{2r'}(\Psi_s(z))$ for any $s \leq \epsilon_3$;*
3. $e(z) \leq c_1^2 r^2$.

Proof. Choose $\epsilon_3 = \min\{\epsilon_1, \epsilon_2\}$ and $\bar{r}_3 := \min\{\bar{r}_1, \bar{r}_2\}$. Apply 5.1.1 to find some $z \in B_{\frac{r}{4}}(\gamma(t_0))$ which satisfies properties 1 - 3 on the scale of $\frac{r}{4}$ and then use 5.2.1 repeatedly to conclude. □

The following corollary immediately follows from Lemma 5.2.2 by a limiting argument.

Corollary 5.2.3. *There exists $\epsilon_3(N,\delta) > 0$ and $\bar{r}_3(N,\delta) > 0$ so that if γ is the unique geodesic between p and q, then for all $r \leq \bar{r}_3$ and $\delta \leq t_0 \leq 1-\delta$,*

1. $V(1,100) \leq \frac{m(B_r(\gamma(t_0-s)))}{m(B_r(\gamma(t_0)))} \leq \frac{1}{V(1,100)}$ *for any $0 \leq s \leq \epsilon_3$.*
2. *There exists $A \subseteq B_r(\gamma(t_0))$ with $m(A) \geq (1-V(1,10))m(B_r(\gamma(t_0)))$ and $\Psi_s(A) \subseteq B_{2r}(\gamma(t_0-s))$ for any $0 \leq s \leq \epsilon_3$.*

We can use the corollary directly in the proof of Theorem 5.3.1 to prove the main result for all unique geodesics. Taking Theorem 2.7.4 and Remark 2.7.5 into account, this is already enough for several applications. Nevertheless, we will next prove the existence of a geodesic between any $p, q \in X$ with Hölder continuity on the geometry of small radius balls in its interior, which is the full result of [33]. The desired geodesic will be constructed using multiple limiting and gluing arguments.

Lemma 5.2.4. *There exists $\epsilon_4(N,\delta) > 0$ and $\bar{r}_4(N,\delta) > 0$ so that for any unit speed geodesic γ from p to q, there exists a unit speed geodesic γ^δ from p to q with $\gamma^\delta \equiv \gamma$ on $[1-\delta, 1]$ so that for all $r \leq \bar{r}_4$ and $\delta \leq t_0 \leq t_1 \leq 1-\delta$, if $t_1 - t_0 \leq \epsilon_4$ then,*

1. *$V(1,100)^4 \leq \frac{m(B_r(\gamma^\delta(t_1)))}{m(B_r(\gamma^\delta(t_0)))} \leq \frac{1}{V(1,100)^4}$;*
2. *There exists $A \subseteq B_r(\gamma^\delta(t_1))$ so that $m(A) \geq (1-V(1,10))m(B_r(\gamma^\delta(t_1)))$ and $\Psi_s(A) \subseteq B_{2r}(\gamma^\delta(t_1 - s))$ for all $s \in [0, t_1 - t_0]$.*

Proof. Let $\epsilon_3, \bar{r}_3$ be from Lemma 5.2.2. We begin by assuming $\bar{r}_4 \leq \bar{r}_3$ and $\epsilon_4 \leq \frac{\epsilon_3}{2}$ but will impose more bounds on both as the proof continues. We will not keep track of $\bar{r}_4$ for the sake of brevity.

Partition $[\delta, 1-\delta]$ by $\{\sigma_0 = \delta, \sigma_1, \ldots, \sigma_k = 1-\delta\}$ for some $k \in \mathbb{N}$ so that all subintervals have the same width equal to some $\omega \in [\frac{\epsilon_3}{2}, \epsilon_3]$. This is always possible since the width of the original interval is $1 - 2\delta > 0.8$ and ϵ_3 is much smaller than 0.4. We construct γ^δ inductively as follows

- Define $\gamma^\delta \equiv \gamma$ for $t \in [1-\delta, 1]$.
- Let $1 \leq i \leq k$. Assume γ^δ has been constructed on the interval $[\sigma_i, 1]$. Fix a sequence of $r_j \to 0$. For each j, choose $z_j \in B_{r_j}(\gamma^\delta(\sigma_i))$ as in the statement of Lemma 5.2.2. Use Arzelà-Ascoli Theorem to take a limit, after passing to a subsequence, of the unit speed geodesics from z_j to p. This limit $\gamma^{\delta,i}$ is a unit speed geodesic from $\gamma^\delta(\sigma_i)$ to p. For $t \in [\sigma_{i-1}, \sigma_i]$, define $\gamma^\delta(t) = \gamma^{\delta,i}(\sigma_i - t)$.
- For $t \in [0,\delta]$, define γ^δ to be any geodesic from p to $\gamma^\delta(\sigma_0)$.

For any $1 \leq i \leq k$, $\sigma_i - \sigma_{i-1} \leq \epsilon_3$ so it follows from the construction that γ^δ satisfies, for any $r \leq \bar{r}_4$,

i. $V(1,100) \leq \frac{m(B_r(\gamma^\delta(\sigma_i - s)))}{m(B_r(\gamma^\delta(\sigma_i)))} \leq \frac{1}{V(1,100)}$ for any $0 \leq s \leq \omega$;

ii. There exists $A \subseteq B_r(\gamma^\delta(\sigma_i))$ with $m(A) \geq (1-V(1,10))m(B_r(\gamma^\delta(\sigma_i)))$ and $\Psi_s(A) \subseteq B_{2r}(\gamma^\delta(\sigma_i - s))$ for any $0 \leq s \leq \omega$.

Fix $\delta \leq t_0 \leq t_1 \leq 1-\delta$ with $t_1 - t_0 \leq \epsilon_4$ and a scale $r \leq \bar{r}_4$. Since $\epsilon_4 \leq \frac{\epsilon_3}{2}$ is no greater than the widths of the subintervals of the partition $\omega \geq \frac{\epsilon_3}{2}$, t_0 and t_1 must be contained in $[\sigma_{i-2}, \sigma_i]$ for some $2 \leq i \leq k$. Statement 1 of the lemma then follows trivially from property i of γ^δ.

t_0 and t_1 must then be either contained in a single subinterval or two neighbouring subintervals of the partition. We will assume the second case; the first case follows from a similar and simpler argument. Let $t_1 \in (\sigma_{i-1}, \sigma_i]$ and $t_0 \in (\sigma_{i-2}, \sigma_{i-1}]$ for some i. By property ii of γ^δ and Abresch-Gromoll, there exists $A_1 \subseteq B_{\frac{r}{16}}(\gamma^\delta(\sigma_i))$ so that

1. $\frac{m(A_1)}{m(B_{\frac{r}{16}}(\gamma^\delta(\sigma_i)))} \geq 1 - 2V(1,10)$;
2. $\Psi_s(A_1) \subseteq B_{\frac{r}{8}}(\gamma^\delta(\sigma_i - s))\ \forall s \in [0,\omega]$;
3. $e(x) \leq c(N,\delta)r^2\ \forall x \in A_1$.

Simiarly, there exists $A_2 \subseteq B_{\frac{r}{16}}(\gamma^\delta(\sigma_{i-1}))$ so that

1. $\frac{m(A_2)}{m(B_{\frac{r}{16}}(\gamma^\delta(\sigma_{i-1})))} \geq 1 - 2V(1, 10)$;

2. $\Psi_s(A_2) \subseteq B_{\frac{r}{8}}(\gamma^\delta(\sigma_{i-1} - s))\ \forall s \in [0, \omega]$;

3. $e(x) \leq c(N, \delta)r^2\ \forall x \in A_2$.

The plan is as follows: first we show that a significant portion of A_1 can be flowed by Ψ a non-trivial amount of time past $\gamma^\delta(\sigma_{i-1})$ while staying close γ^δ, then we use the flow of A_1 under Ψ to control the flow of $B_r(\gamma^\delta(t_1))$ under Ψ.

Fix $h^- \equiv h^-_{\rho(8r)^2}$ satisfying statement 4 of Theorem 4.0.12 for the balls of radius $8r$ along γ^δ, where $\rho \in [\frac{1}{2}, 2]$. For all $s \in [0, \omega]$ and $(x, y) \in X \times X$, define

$$dt(s)(x, y) := \min\left\{r, \max_{0\leq\tau\leq s}|d(x, y) - d(\Psi_\tau(x), \Psi_\tau(y))|\right\} \tag{5.58}$$

and

$$U_1^s := \{(x, y) \in A_2 \times A_1 | dt(s)(x, \Psi_\omega(y)) < r\}. \tag{5.59}$$

Consider $\int_{A_2\times A_1} dt(s)(x, \Psi_\omega(y))\, d(m \times m)(x, y)$ for $0 \leq s \leq \omega$.

For any $s \in [0, \omega]$ and $(x, y) \in U_1^s$,

1. $d(x, \Psi_\omega(y)) < \frac{3r}{16}$;

2. $d(\Psi_s(x), \gamma^\delta(\sigma_{i-1} - s)) < \frac{r}{8}$;

3. $dt(s)(x, \Psi_\omega(y)) < r$.

Therefore, $\Psi_s(\Psi_\omega(y)) \in B_{r+\frac{5r}{16}}(\gamma^\delta(\sigma_{i-1} - s))$ and so $(\Psi_s, \Psi_s \circ \Psi_\omega)(U_1^s) \subseteq B_{4r}(\gamma^\delta(\sigma_{i-1} - s)) \times B_{4r}(\gamma^\delta(\sigma_{i-1} - s))$. By Remark 2.7.5, we have $\Psi_s \circ \Psi_\omega = \Psi_{s+\omega}$ m-a.e.. Since we may always choose subsets of full measure where the equality is satisfied, we will replace the former with the latter freely.

We have,

$$\begin{aligned}
&\int_0^1 \int_{U_1^s} d(\Psi_s(x), \Psi_{s+\omega}(y)))|\text{Hess}\, h^-|_{\text{HS}}(\tilde{\gamma}_{\Psi_s(x),\Psi_{s+\omega}(y)}(\tau))\, d(m \times m)(x, y)\, d\tau \\
&\leq c(N, \delta) \int_0^1 \int_{(\Psi_s,\Psi_{s+\omega})(U_1^s)} d(x, y)|\text{Hess}\, h^-|_{\text{HS}}(\tilde{\gamma}_{x,y}(\tau))\, d(m \times m)(x, y)\, d\tau \quad \text{, by Theorem 3.2.9, 2} \\
&\leq c(N, \delta) r m(B_{4r}(\gamma^\delta(\sigma_{i-1} - s))) \int_{B_{8r}(\gamma^\delta(\sigma_{i-1}-s))} |\text{Hess}\, h^-|_{\text{HS}}\, dm \quad \text{, by segment inequality 3.3.3} \\
&\leq c(N, \delta) r m(B_r(\gamma^\delta(\sigma_i)))^2 \fint_{B_{8r}(\gamma^\delta(\sigma_{i-1}-s))} |\text{Hess}\, h^-|_{\text{HS}}\, dm \quad \text{, by Bishop-Gromov and property i of } \gamma^\delta.
\end{aligned} \tag{5.60}$$

Integrating in $s \in [0, \omega']$ for some $\omega' \in (0, \omega]$ to be fixed later, we have

$$\begin{aligned}
&\int_0^{\omega'} \Big(\int_0^1 \int_{U_1^s} d(\Psi_s(x), \Psi_{s+\omega}(y)) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\Psi_s(x),\Psi_{s+\omega}(y)}(\tau))\, d(m \times m)(x,y)\, d\tau \Big)\, ds \\
&\leq crm(B_r(\gamma^\delta(\sigma_i)))^2 \int_0^{\omega'} \fint_{B_{8r}(\gamma^\delta(\sigma_{i-1}-s))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm\, ds \\
&\leq c(N,\delta) rm(B_r(\gamma^\delta(\sigma_i)))^2 \sqrt{\omega'},
\end{aligned} \tag{5.61}$$

where the last line follows from the definition of h^-, statement 4 of Theorem 4.0.12, and Cauchy-Schwarz.

By statement 3 of 4.0.13, the excess bound on the elements of A_2 and property i of γ^δ,

$$\int_0^{\omega'} \int_{U_1^s} |\nabla h^- - \nabla d_p|(\Psi_s(x))\, d(m \times m)(x,y)\, ds \leq c(N,\delta) rm(B_r(\gamma^\delta(\sigma_i)))^2 \sqrt{\omega'}. \tag{5.62}$$

Similarly by the excess bounds on the elements of A_1,

$$\int_0^{\omega'} \int_{U_1^s} |\nabla h^- - \nabla d_p|(\Psi_{s+\omega}(y))\, d(m \times m)(x,y)\, ds \leq c(N,\delta) m(B_r(\gamma^\delta(\sigma_i)))^2 r \sqrt{\omega'}. \tag{5.63}$$

By Proposition 3.3.8,

$$\int_{A_2 \times A_1} dt(\omega')(x, \Psi_\tau(y)) \leq c(N,\delta) m(B_r(\gamma^\delta(\sigma_i)))^2 r \sqrt{\omega'}. \tag{5.64}$$

Arguing as in the proof of Lemma 5.1.1 and using property i of γ^δ, we can then fix ω' sufficiently small depending only on N and δ so that there exists $z \in A_2$ and $A_1' \subseteq A_1$ with

$$\frac{m(A_1')}{m(B_{\frac{r}{16}}(\gamma^\delta(\sigma_i)))} \geq 1 - 3V(1,10) \text{ and } dt(\omega')(z, \Psi_\omega(y)) \leq \frac{r}{16}\ \forall y \in A_1'. \tag{5.65}$$

The latter implies

$$\Psi_{s+\omega}(A_1') \subseteq B_{\frac{3r}{8}}(\gamma^\delta(\sigma_{i-1} - s))\ \forall s \in [0, \omega']. \tag{5.66}$$

Notice by definiton of A_1, $\Psi_s(A_1') \subseteq B_{\frac{r}{8}}(\gamma^\delta(\sigma_i - s))$ for any $s \in [0, \omega]$.

We now compare the flow of $B_r(\gamma^\delta(t_1))$ to that of $\Psi_{\sigma_i - t_1}(A_1')$ under Ψ_s for $s \in [0, \omega']$. By integral Abresch-Gromoll there exists $B_r(\gamma^\delta(t_1))' \subseteq B_r(\gamma^\delta(t_1))$ so that

$$e(x) \leq c(N,\delta) r^2\ \forall x \in B_r(\gamma^\delta(t_1))' \text{ and } \frac{m(B_r(\gamma^\delta(t_1))')}{m(B_r(\gamma^\delta(t_1)))} \geq 1 - \frac{1}{2} V(1,10). \tag{5.67}$$

For all $s \in [0, \omega']$, define

$$U_2^s := \{(x, y) \in A_1' \times B_r(\gamma^\delta(t_1))' | dt(s)(\Psi_{\sigma_i - t_1}(x), y) < r\}. \tag{5.68}$$

Consider $\int_{A_1' \times B_r(\gamma^\delta(t_1))'} dt(s)(\Psi_{\sigma_i - t_1}(x), y)\, d(m \times m)(x, y)$ for $0 \le s \le \omega'$. For any $s \in [0, \omega']$ and $(x, y) \in U_2^s$,

1. $d(\Psi_{\sigma_i - t_1}(x), y) < r + \frac{r}{8}$ by definition of A_1;
2. $d(\Psi_s(\Psi_{\sigma_i - t_1}(x)), \gamma^\delta(t_1 - s)) < \frac{3r}{8}$ by (5.66) and the line below it;
3. $dt(s)(\Psi_{\sigma_i - t_1}(x), y) < r$.

Therefore, $\Psi_s(y) \in B_{\frac{5r}{2}}(\gamma^\delta(t_1 - s))$ and so $(\Psi_s \circ \Psi_{\sigma_i - t_1}, \Psi_s)(U_2^s) \subseteq B_{4r}(\gamma^\delta(t_1 - s)) \times B_{4r}(\gamma^\delta(t_1 - s))$.

By the same type of computations as the first part of this proof, for some $\omega'' \in (0, \omega']$ to be fixed later,

$$\int_{A_1' \times B_r(\gamma(t_1))'} dt(\omega'')(\Psi_{\sigma_i - t_1}(x), y) \le c(N, \delta) m(B_r(\gamma^\delta(\sigma_i)))^2 r \sqrt{\omega''}. \tag{5.69}$$

Arguing as in the proof of Lemma 5.1.1 and using property i of γ^δ, we can then fix ω'' sufficiently small depending only on N and δ so that there exists $z' \in A_1'$ and $A \subseteq B_r(\gamma(t_1))'$ with

$$\frac{m(A)}{m(B_r(\gamma^\delta(t_1)))} \ge 1 - V(1, 10) \text{ and } dt(\omega'')(\Psi_{\sigma_i - t_1}(z'), y) \le \frac{r}{2} \; \forall y \in A. \tag{5.70}$$

The latter implies

$$\Psi_s(A) \subseteq B_{2r}(\gamma^\delta(t_1 - s)) \; \forall s \in [0, \omega'']. \tag{5.71}$$

We bound $\epsilon_4 \le \omega''$ and so statement 1 of the lemma is proved. □

We may also apply Lemma 5.2.4 in the other direction of γ towards q. However, there is no guarantee the two geodesics we end up with in the two applications of the lemma are the same geodesics. Therefore, we will show that a geodesic which has the properties of 5.2.4 necessarily has the same properties going in the other direction. The reason for this is Lemma 4.0.3, which roughly implies the local flows of h^+ and h^- are close to each other near a geodesic on the scale of r.

Lemma 5.2.5. *There exists $\epsilon_5(N, \delta) > 0$ and $\bar{r}_5(N, \delta) > 0$ so that if there exists a unit speed geodesic γ from p to q, $\epsilon \le \epsilon_5$ and $\bar{r} \le \bar{r}_5$ which satisfy, for all $r \le \bar{r}$ and $\delta \le t_0 \le t_1 \le 1 - \delta$ with $t_1 - t_0 \le \epsilon$,*

1. *$V(1, 100)^4 \le \frac{m(B_r(\gamma(t_1)))}{m(B_r(\gamma(t_0)))} \le \frac{1}{V(1,100)^4}$;*
2. *There exists $A_1 \subseteq B_r(\gamma(t_1))$ so that $m(A_1) \ge (1 - V(1, 10)) m(B_r(\gamma(t_1)))$ and $\Psi_s(A_1) \subseteq B_{2r}(\gamma(t_1 - s))$ for all $s \in [0, t_1 - t_0]$,*

then for the same geodesic γ, ϵ and $\bar{r}$, for all $r \le \bar{r}$ and $\delta \le t_0 \le t_1 \le 1 - \delta$ with $t_1 - t_0 \le \epsilon$, there exists $A_2 \subseteq B_r(\gamma(t_0))$ so that

$$\frac{m(A_2)}{m(B_r(\gamma(t_0)))} \ge 1 - V(1, 10) \;\; \textit{and} \;\; \Phi_s(A_2) \subseteq B_{2r}(\gamma(t_0 + s)) \; \forall s \in [0, t_1 - t_0]. \tag{5.72}$$

Proof. As a reminder, Φ is defined by (5.2) and is the local flow of $-\nabla d_q$, at least from the sets and for the time interval we are concerned with.

We assume $\bar{r}_5 \le \frac{\delta}{10}$ and $\epsilon_5 \le \frac{\delta}{10}$ to begin with but will impose more bounds on both as the proof continues. We will not keep track of $\bar{r}_5$ for the sake of brevity. Fix γ, $\epsilon \le \epsilon_5$ and $\bar{r} \le \bar{r}_5$ which satisfy conditions 1 and 2. Fix $r \le \bar{r}$ and $\delta \le t_0 \le t_1 \le 1-\delta$ with $t_1 - t_0 \le \epsilon$.

Fix $h^- \equiv h^-_{\rho(8r)^2}$ satisfying statement 4 of Theorem 4.0.12 for the balls of radius $8r$ along γ, where $\rho \in [\frac{1}{2}, 2]$. Let $(\tilde{\Psi}_t)_{t\in[0,1]}$ and $(\tilde{\Psi}_{-t})_{t\in[0,1]}$ be the RLFs of the time-independent vector fields $-\nabla h^-$ and ∇h^- respectively as before.

The plan is as follows: first we use property 2 to make sure a significant portion of $B_{\frac{r}{16}}(\gamma(t_1))$ stays close to γ from t_1 to t_0 under the flow of $\tilde{\Psi}_s$, then we reverse flow the image of this portion under $\tilde{\Psi}_{-s}$ and use it to make sure a significant portion of $B_r(\gamma(t_0))$ stays close to γ from t_0 to t_1 under Φ_s.

By condition 2 and integral Abresch-Gromoll, there exists $A_1 \subseteq B_{\frac{r}{16}}(\gamma(t_1))$ so that

1. $\frac{m(A_1)}{m(B_{\frac{r}{16}}(\gamma(t_1)))} \ge 1 - 2V(1, 10)$;
2. $\Psi_s(A_1) \subseteq B_{\frac{r}{8}}(\gamma(t_1 - s))\ \forall s \in [0, t_1 - t_0]$;
3. $e(x) \le c(N, \delta)r^2\ \forall x \in A_1$.

For all $s \in [0, t_1 - t_0]$ and $(x, y) \in X \times X$, define

$$dt_1(s)(x,y) := \min\left\{r, \max_{0\le\tau\le s} |d(x,y) - d(\Psi_\tau(x), \tilde{\Psi}_\tau(y))|\right\} \tag{5.73}$$

and

$$U_1^s := \{(x,y) \in A_1 \times B_{\frac{r}{16}}(\gamma(t_1)) | dt_1(s)(x,y) < r\}. \tag{5.74}$$

Consider $\int_{A_1 \times B_{\frac{r}{16}}(\gamma(t_1))} dt_1(s)(x,y)\, d(m \times m)(x,y)$ for $0 \le s \le t_1 - t_0$. For any $s \in [0, t_1 - t_0]$ and $(x, y) \in U_1^s$,

1. $d(x, y) < \frac{r}{8}$;
2. $d(\Psi_s(x), \gamma(t_1 - s)) < \frac{r}{8}$ by definition of A_1;
3. $dt_1(s)(x, y) < r$.

Therefore, $\tilde{\Psi}_s(y) \in B_{\frac{5r}{4}}\gamma(t_1 - s)$ by triangle inequality and so $(\Psi_s, \tilde{\Psi}_s)(U_1^s) \subseteq B_{4r}(\Psi_s(z)) \times B_{4r}(\Psi_s(z))$.

Using exactly the same type of computation as the second part of the proof of the main lemma,

$$\int_{A_1 \times B_{\frac{r}{16}}(\gamma(t_1))} dt_1(t_1 - t_0)(x,y)\, d(m \times m)(x,y) \le c(N, \delta) r m(B_r(\gamma(t_1)))^2 \sqrt{t_1 - t_0}. \tag{5.75}$$

A_1 takes a significant portion of the measure of $B_r(\gamma(t_1))$ by definition. The same considerations as in the proof of Lemma 5.1.1 gives the existence of $z \in A_1$ and $D_1 \subseteq B_r(\gamma(t_1))$ so that

$$\frac{m(D_1)}{m(B_{\frac{r}{16}}(\gamma(t_1)))} \geq 1 - V(1,10) \text{ and } dt_1(t_1 - t_0)(z,y) \leq \frac{r}{16} \ \forall y \in D_1, \tag{5.76}$$

after we bound ϵ_5 sufficiently small depending only on N and δ. The latter implies

$$\tilde{\Psi}_s(D_1) \subseteq B_{\frac{5r}{16}}(\gamma(t_1 - s)) \ \forall s \in [0, t_1 - t_0]. \tag{5.77}$$

By Proposition 3.2.6, we may assume in addition that D_1 satisfies

$$\tilde{\Psi}_{-s}(\tilde{\Psi}_{t_1-t_0}(x)) = \tilde{\Psi}_{t_1-t_0-s}(x) \ \forall s \in [0, t_1 - t_0] \text{ and } \forall x \in D_1. \tag{5.78}$$

Furthermore, $\tilde{\Psi}_s(D_1)$ is non-trivial in measure compared to $B_r(\gamma(t_1))$ for all $s \in [0, t_1 - t_0]$.

$$\begin{aligned} \frac{m(\tilde{\Psi}_s(D_1))}{m(B_r(\gamma(t_1)))} &\geq e^{-c(N,\delta)\frac{\delta}{10}} \frac{m(D_1)}{m(B_r(z))} \quad \text{, by 4.0.2, 3.1.4 (3.3), and } \epsilon_5 \leq \frac{\delta}{10} \\ &\geq c(N,\delta) \quad \text{, by definition of } D_1 \text{ and Bishop-Gromov.} \end{aligned} \tag{5.79}$$

By integral Abresch-Gromoll, there exists $B_r(\gamma(t_0))' \subseteq B_r(\gamma(t_0))$ so that

$$e(x) \leq c(N,\delta) r^2 \ \forall x \in B_r(\gamma(t_0))' \text{ and } \frac{m(B_r(\gamma(t_0))')}{m(B_r(\gamma(t_0)))} \geq 1 - \frac{1}{2} V(1,10). \tag{5.80}$$

For all $s \in [0, t_1 - t_0]$ and $(x,y) \in X \times X$, define

$$dt_2(s)(x,y) := \min\left\{ r, \max_{0 \leq \tau \leq s} |d(x,y) - d(\tilde{\Psi}_{-\tau}(x), \Phi_\tau(y))| \right\} \tag{5.81}$$

and

$$U_2^s := \{(x,y) \in \tilde{\Psi}_{t_1-t_0}(D_1) \times B_r(\gamma(t_0))' | dt_2(s)(x,y) < r\}. \tag{5.82}$$

Consider $\int_{\tilde{\Psi}_{t_1-t_0}(D_1) \times B_r(\gamma(t_0))'} dt_2(s)(x,y)\, d(m \times m)(x,y)$ for $0 \leq s \leq t_1 - t_0$. By proposition 3.3.8, for a.e. $s \in [0, t_1 - t_0]$,

$$\begin{aligned} &\frac{d}{ds} \int_{\tilde{\Psi}_{t_1-t_0}(D_1) \times B_r(\gamma(t_0))'} dt_2(s)(x,y)\, d(m \times m)(x,y) \\ &\leq \int_{U_2^s} |\nabla h^- + \nabla d_q|(\Phi_s(y))\, d(m \times m)(x,y) \\ &\qquad + \int_0^1 \int_{U_2^s} d(\tilde{\Psi}_s(x), \Phi_s(y)) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\tilde{\Psi}_s(x), \Phi_s(y)}(\tau))\, d(m \times m)(x,y)\, d\tau. \end{aligned} \tag{5.83}$$

For any $s \in [0, t_1 - t_0]$, $s' \in [0, s]$, and $(x,y) \in U_2^s$,

1. $d(x,y) < r + \frac{5r}{16}$ by (5.77);
2. $d(\tilde{\Psi}_{-s'}(x), \gamma(t_0+s')) = d(\tilde{\Psi}_{t_1-t_0-s'}(x'), \gamma(t_0+s')) < \frac{5r}{16}$ for some $x' \in D_1$ by (5.77) and (5.78);
3. $dt_2(s)(x,y) < r$ by definition of U_2^s (5.82).

Hence,

$$\Phi_{s'}(y) \in B_{2r+\frac{5r}{8}}(\gamma(t_0+s')) \tag{5.84}$$

by triangle inequality. Therefore, $(\tilde{\Psi}_{-s'}, \Phi_{s'})(U_2^s) \in B_{4r}(\gamma(t_0+s')) \times B_{4r}(\gamma(t_0+s'))$ for all $s' \in [0,s]$. For any $(x,y) \in U_2^s$,

$$\begin{aligned} \Delta h^-(\tilde{\Psi}_{-s'}(x)) &= \Delta h^+(\tilde{\Psi}_{-s'}(x)) + \Delta\hat{e}(\tilde{\Psi}_{-s'}(x)) \\ &\geq -c(N,\delta) \quad \text{, by Lemma 4.0.2 and Lemma 4.0.3 3,} \end{aligned} \tag{5.85}$$

where h^+, $\hat{e}$ are heat flow approximations of h_0^+ and e_0 respectively up to the same time as h^-. Therefore,

$$\begin{aligned} &\int_0^1 \int_{U_2^s} d(\tilde{\Psi}_{-s}(x), \Phi_s(y)) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\tilde{\Psi}_{-s}(x), \Phi_s(y)}(\tau))\, d(m\times m)(x,y)\, d\tau \\ &\leq c(N,\delta) \int_0^1 \int_{(\tilde{\Psi}_{-s}, \Phi_s)(U_2^s)} d(x,y) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{x,y}(\tau))\, d(m\times m)(x,y)\, d\tau, \text{ by 3.2.9 2, (5.85) and 3.1.5} \\ &\leq c(N,\delta) r m(B_{4r}(\gamma(t_0+s))) \int_{B_{8r}(\gamma(t_0+s))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm \quad \text{, by segment inequality 3.3.3} \\ &\leq c(N,\delta) r m(B_r(\gamma(t_1)))^2 \fint_{B_{8r}(\gamma(t_0+s))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm \quad \text{, by Bishop-Gromov and property 1 of } \gamma. \end{aligned} \tag{5.86}$$

Integrating in $s \in [0, t_1 - t_0]$,

$$\begin{aligned} &\int_0^{t_1-t_0} \Big(\int_0^1 \int_{U_2^s} d(\tilde{\Psi}_{-s}(x), \Phi_s(y)) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\tilde{\Psi}_{-s}(x), \Phi_s(y)}(\tau))\, d(m\times m)(x,y)\, d\tau \Big)\, dt \\ &\leq crm(B_r(\gamma(t_1)))^2 \int_0^{t_1-t_0} \fint_{B_{8r}(\gamma(t_0+s))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm\, ds \\ &\leq c(N,\delta) r m(B_r(\gamma(t_1)))^2 \sqrt{t_1-t_0}, \end{aligned} \tag{5.87}$$

where the last line follows from the definition of h^-, statement 4 of Theorem 4.0.12, and Cauchy-Schwarz.

We have

$$\int_0^{t_1-t_0}\int_{U_2^s}|\nabla h^+ + \nabla d_q|(\Phi_s(y))\,d(m\times m)(x,y)\,ds \le c(N,\delta)rm(B_r(\gamma(t_1)))^2\sqrt{t_1-t_0}, \tag{5.88}$$

where the first inequality is from statement 3 of Lemma 4.0.13 and the excess bound on the elements of $B_r(\gamma(t_0))'$ (5.80), and the second inequality is from property 1 of γ and Bishop-Gromov. Similarly, using statement 3 of Lemma 4.0.3 with (5.84),

$$\begin{aligned}\int_0^{t_1-t_0}\int_{U_2^s}|\nabla h^- - \nabla h^+|(\Phi_s(y))\,d(m\times m)(x,y)\,ds &\le c(N,\delta)rm(B_r(\gamma(t_1)))^2(t_1-t_0)\\ &\le crm(B_r(\gamma(t_1)))^2\sqrt{t_1-t_0}.\end{aligned} \tag{5.89}$$

Combining (5.86) - (5.88) with the bound (5.83) on $\frac{d}{ds}\int_{\tilde{\Psi}_{t_1-t_0}(D_1)\times B_r(\gamma(t_0))'} dt_2(s)(x,y)$, we obtain

$$\begin{aligned}&\int_{\tilde{\Psi}_{t_1-t_0}(D_1)\times B_r(\gamma(t_0))'} dt_2(t_1-t_0)(x,y)\,d(m\times m)(x,y)\\ &=\int_0^{t_1-t_0}[\frac{d}{ds}\int_{\tilde{\Psi}_{t_1-t_0}(D_1)\times B_r(\gamma(t_0))'} dt_2(s)(x,y)\,d(m\times m)(x,y)]\,ds\\ &\le c(N,\delta)rm(B_r(\gamma(t_1)))^2\sqrt{t_1-t_0}.\end{aligned} \tag{5.90}$$

Both $\tilde{\Psi}_{t_1-t_0}(D_1)$ and $B_r(\gamma(t_0))'$ are non-trivial in measure compared to $B_r(\gamma(t_1))$ by (5.79) and property 1 respectively. The same considerations as in the proof of Lemma 5.1.1 gives the existence of $z'\in\tilde{\Psi}_{t_1-t_0}(D_1)$ and $A_2\subseteq B_r(\gamma(t_0))'$ so that

$$\frac{m(A_2)}{m(B_r(\gamma(t_0)))}\ge 1-V(1,10)\ \text{ and }\ dt_2(t_1-t_0)(z',y)\le\frac{3r}{8}\ \forall y\in A_2, \tag{5.91}$$

after we bound ϵ_5 sufficiently small depending only on N and δ. The latter implies

$$\Phi_s(A_2)\subseteq B_{2r}(\gamma(t_0+s))\ \forall s\in[0,t_1-t_0]. \tag{5.92}$$

This finishes the proof of the lemma. □

Lemmas 5.2.4 and 5.2.5 give

Corollary 5.2.6. *For any $\delta\in(0,0.1)$, there exists $\epsilon_6(N,\delta)>0$ and $\bar{r}_6(N,\delta)>0$ so that for any unit speed geodesic γ from p to q, there exists a unit speed geodesic γ^δ from p to q with with $\gamma^\delta\equiv\gamma$ on $[1-\delta,1]$ so that for all $r\le\bar{r}_6$ and $\delta\le t_0\le t_1\le 1-\delta$, if $t_1-t_0\le\epsilon_6$, then*

1. $V(1,100)^4\le\frac{m(B_r(\gamma^\delta(t_1)))}{m(B_r(\gamma^\delta(t_0)))}\le\frac{1}{V(1,100)^4}$;

2. *There exists* $A_1 \subseteq B_r(\gamma^\delta(t_1))$ *so that* $m(A_1) \geq (1 - V(1,10))m(B_r(\gamma^\delta(t_1)))$ *and* $\Psi_s(A_1) \subseteq B_{2r}(\gamma^\delta(t_1 - s))$ *for all* $s \in [0, t_1 - t_0]$*;*

3. *There exists* $A_2 \subseteq B_r(\gamma^\delta(t_0))$ *so that* $m(A_2) \geq (1 - V(1,10))m(B_r(\gamma^\delta(t_0)))$ *and* $\Phi_s(A_2) \subseteq B_{2r}(\gamma^\delta(t_0 + s))$ *for all* $s \in [0, t_1 - t_0]$.

We would like to now construct a geodesic that has this behaviour for all δ by taking a limit of γ^δ as $\delta \to 0$. To have the properties pass over to the limit, we need to make sure each γ^δ also satisfies the above properties for any $\delta' > \delta$. For all $\delta \in (0, 0.1)$, we fix the constants $\epsilon_6(N, \delta)$ and $\bar{r}_6(N, \delta)$ which come from taking the minimum of their respective counterparts in lemmas 5.2.4 and 5.2.5.

Lemma 5.2.7. *Let* $\delta \in (0, 0.1)$ *and assume some unit speed geodesic* γ^δ *from* p *to* q *satisfies properties 1 - 3 for* δ, $\epsilon_6(N,\delta)$ *and* $\bar{r}_6(N,\delta)$*. Then for any* $\delta' \in (\delta, 0.1)$*,* γ^δ *also satisfies properties 1 - 3 for* δ'*,* $\epsilon_6(N,\delta')$ *and* $\bar{r}_6(N,\delta')$

Proof. Fix $\delta' \in (\delta, 0.1)$. Use γ^δ to construct some $\gamma^{\delta'}$ with the construction of Lemma 5.2.4. $\gamma^{\delta'}$ satisfies properties 1 - 3 for δ', $\epsilon_6(N,\delta')$ and $\bar{r}_6(N,\delta')$ by lemmas 5.2.4 and 5.2.5. Furthermore, $\gamma^\delta(t) = \gamma^{\delta'}(t)$ for all $t \in [1-\delta', 1]$ by construction. We will show $\gamma^\delta(t) = \gamma^{\delta'}(t)$ for all $t \in [\delta', 1]$ which will allow us to conclude.

Assume this is not the case. Define $s_0 := \min\{s \in [\delta', 1-\delta'] : \gamma^\delta(t) = \gamma^{\delta'}(t) \, \forall \, t \in [s, 1]\}$. By assumption, $\delta' < s_0 \leq 1 - \delta'$. Therefore, there exists $t_0 \in [\delta', s_0)$ so that $t_1 - t_0 \leq \min\{\epsilon_6(N,\delta), \epsilon_6(N,\delta')\}$ and $\gamma^\delta(t_0) \neq \gamma^{\delta'}(t_0)$. Choose any $r < \min\{\bar{r}_6(N,\delta), \bar{r}_6(N,\delta'), \frac{d(\gamma^\delta(t_0), \gamma^{\delta'}(t_0))}{4}\}$ and consider $B_r(\gamma^\delta(t_1))$. On one hand, most of $B_r(\gamma^\delta(t_1))$ needs to end up in $B_{2r}(\gamma^\delta(t_0))$ under $\Psi_{t_1 - t_0}$ by property 2 for γ^δ. On the other hand, most of $B_r(\gamma^\delta(t_1))$ needs to end up in $B_{2r}(\gamma^{\delta'}(t_0))$ under $\Psi_{t_1 - t_0}$ by property 2 for $\gamma^{\delta'}$. These two balls are disjoint and so we have a contradiction. □

This immediately gives the existence of a geodesic with the desired properties for any $\delta \in (0, 0.1)$.

Theorem 5.2.8. *There exists a unit speed geodesic* $\bar{\gamma}$ *from* p *to* q *so that for any* $\delta \in (0, 0.1)$*, there exists* $\epsilon_6(N,\delta) > 0$ *and* $\bar{r}_6(N,\delta) > 0$ *so that for all* $r \leq \bar{r}_6$ *and* $\delta \leq t_0 \leq t_1 \leq 1 - \delta$*, if* $t_1 - t_0 \leq \epsilon_6$*, then*

1. $V(1,100)^4 \leq \frac{m(B_r(\bar{\gamma}(t_1)))}{m(B_r(\bar{\gamma}(t_0)))} \leq \frac{1}{V(1,100)^4}$*;*

2. *There exists* $A_1 \subseteq B_r(\bar{\gamma}(t_1))$ *so that* $m(A_1) \geq (1 - V(1,10))m(B_r(\bar{\gamma}(t_1)))$ *and* $\Psi_s(A_1) \subseteq B_{2r}(\bar{\gamma}(t_1 - s))$ *for all* $s \in [0, t_1 - t_0]$*;*

3. *There exists* $A_2 \subseteq B_r(\bar{\gamma}(t_0))$ *so that* $m(A_2) \geq (1 - V(1,10))m(B_r(\bar{\gamma}(t_0)))$ *and* $\Phi_s(A_2) \subseteq B_{2r}(\bar{\gamma}(t_0 + s))$ *for all* $s \in [0, t_1 - t_0]$.

Proof. Fix any unit speed geodesic γ from p to q and then use the construction of Lemma 5.2.4 to obtain a γ^δ for each $\delta \in (0, 0.1)$. By Arzelà-Ascoli theorem, we can take a limit $\bar{\gamma}$ of γ^δ as $\delta \to 0$ after passing to a subsequence. By Lemma 5.2.7 and Bishop-Gromov, $\bar{\gamma}$ will have the desired properties. □

5.3 Proof of main theorem

We now prove the Hölder continuity in pointed Gromov-Hausdorff distance of small balls along the interior of any geodesic between p and q constructed in Theorem 5.2.8 using essentially the same argument as in [33].

Theorem 5.3.1. *There exists a unit speed geodesic γ between p and q so that for any $\delta \in (0, 0.1)$, there exists $\epsilon(N, \delta) > 0$, $\bar{r}(N, \delta) > 0$ and $C(N, \delta)$ so that for any $r \leq \bar{r}$ and $t_0, t_1 \in [\delta, 1-\delta]$, if $|t_1 - t_0| \leq \epsilon$ then*

$$d_{pGH}\Big((B_r(\gamma(t_0)), \gamma(t_0)), (B_r(\gamma(t_1)), \gamma(t_1))\Big) \leq Cr|t_1 - t_0|^{\frac{1}{2N(1+2N)}}\,^{1}.$$

Proof. Let $\bar{\gamma}$, ϵ_6 and $\bar{r}_6$ be as in Theorem 5.2.8 and let $\gamma = \bar{\gamma}$. Fix $\delta \in (0, 0.1)$. We begin by assuming $\epsilon \leq \epsilon_6$ and $\bar{r} \leq \bar{r}_6$ but will impose more bounds as the proof continues. Fix $r \leq \bar{r}$ and $\delta \leq t_0 \leq t_1 \leq 1 - \delta$ with $\omega := t_1 - t_0 \leq \epsilon$.

We begin by showing that a large portion of $B_r(\gamma(t_1))$ is mapped by Ψ_ω close (on the scale of r) to $\gamma(t_0)$, where the closeness and the relative size of the portion are both Hölder dependent on ω. This also shows that the measure of $B_r(\gamma)$ is Hölder along $\gamma|_{[\delta, 1-\delta]}$ as a consequence.

Define $\eta := \omega^{\frac{N}{2(1+2N)}}$ and $\mu := \eta^{\frac{1}{N}} = \omega^{\frac{1}{2(1+2N)}}$. By property 2 of γ from Theorem 5.2.8 and integral Abresch-Gromoll, there exists $B_{\mu r}(\gamma(t_1))' \subseteq B_{\mu r}(\gamma(t_1))$ so that

1. $\frac{m(B_{\mu r}(\gamma(t_1))')}{m(B_{\mu r}(\gamma(t_1)))} \leq 1 - 2V(1, 10)$;
2. $\Psi_s(B_{\mu r}(\gamma(t_1))') \subseteq B_{2\mu r}(\gamma(t_1 - s))\ \forall s \in [0, \omega]$;
3. $e(x) \leq c(N, \delta)\mu^2 r^2 \leq cr^2\ \forall x \in B_{\mu r}(\gamma(t_1))'$.

By integral Abresch-Gromoll, there exists $B_r(\gamma(t_1))' \subseteq B_r(\gamma(t_1))$ so that

$$e(x) \leq c(N, \delta)\frac{1}{\eta}r^2\ \forall x \in B_r(\gamma(t_1))' \ \text{ and } \ \frac{m(B_r(\gamma(t_1))')}{m(B_r(\gamma(t_1)))} \geq 1 - \eta. \tag{5.93}$$

Fix $h^- \equiv h^-_{\rho(4r)^2}$ satisfying statement 4 of Theorem 4.0.12 for the balls of radius $4r$ along γ, where $\rho \in [\frac{1}{2}, 2]$. For all $s \in [0, \omega]$ and $(x, y) \in X \times X$, define

$$dt(s)(x, y) := \min\left\{4r, \max_{0 \leq \tau \leq s} |d(x, y) - d(\Psi_\tau(x), \Psi_\tau(y))|\right\}. \tag{5.94}$$

Note for any $(x, y) \in B_{\mu r}(\gamma(t_1))' \times B_r(\gamma(t_1))'$, $\Psi_s(x), \Psi_s(y) \in B_{2r}(\gamma(t_1 - s))$ and so $dt(s)(x, y) < 4r$.

[1] We note that the Hölder exponent is slightly different to that of [33] due to a minor error in the first line of equation (3.38).

By Proposition 3.3.8, for a.e. $s \in [0,\omega]$,

$$\begin{aligned}
&\frac{d}{ds}\int_{B_{\mu r}(\gamma(t_1))'\times B_r(\gamma(t_1))'} dt(s)(x,y)\,d(m\times m)(x,y)\\
&\leq \int_{B_{\mu r}(\gamma(t_1))'\times B_r(\gamma(t_1))'} \Big(|\nabla h^- - \nabla d_p|(\Psi_s(x)) + |\nabla h^- - \nabla d_p|(\Psi_s(y))\Big)\,d(m\times m)(x,y)\\
&\qquad + \int_0^1 \int_{B_{\mu r}(\gamma(t_1))'\times B_r(\gamma(t_1))'} d(\Psi_s(x),\Psi_s(y))|\mathrm{Hess}\,h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\Psi_s(x),\Psi_s(y)}(\tau))\,d(m\times m)(x,y)\,d\tau.
\end{aligned} \tag{5.95}$$

We estimate

$$\begin{aligned}
&\int_0^1 \int_{B_{\mu r}(\gamma(t_1))'\times B_r(\gamma(t_1))'} d(\Psi_s(x),\Psi_s(y))|\mathrm{Hess}\,h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\Psi_s(x),\Psi_s(y)}(\tau))\,d(m\times m)(x,y)\,d\tau\\
&\leq c(N,\delta)\int_0^1 \int_{(\Psi_s,\Psi_s)(B_{\mu r}(\gamma(t_1))'\times B_r(\gamma(t_1))')} d(x,y)|\mathrm{Hess}\,h^-|_{\mathrm{HS}}(\tilde{\gamma}_{x,y}(\tau))\,d(m\times m)(x,y)\,d\tau \quad \text{, by Theorem 3.2.9, 2}\\
&\leq c(N,\delta) r m(B_{2r}(\gamma(t_1-s))) \int_{B_{4r}(\gamma(t_1-s))} |\mathrm{Hess}\,h^-|_{\mathrm{HS}}\,dm \quad \text{, by segment inequality 3.3.3}\\
&\leq c(N,\delta) r m(B_r(\gamma(t_1)))^2 \fint_{B_{4r}(\gamma(t_1-s))} |\mathrm{Hess}\,h^-|_{\mathrm{HS}}\,dm \quad \text{, by Bishop-Gromov and property 1 of } \gamma.
\end{aligned} \tag{5.96}$$

Integrating in $s \in [0,\omega]$,

$$\begin{aligned}
&\int_0^\omega \Big(\int_0^1 \int_{B_{\mu r}(\gamma(t_1))'\times B_r(\gamma(t_1))'} d(\Psi_s(x),\Psi_s(y))|\mathrm{Hess}\,h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\Psi_s(x),\Psi_s(y)}(\tau))\,d(m\times m)(x,y)\,d\tau\Big)\,ds\\
&\leq c r m(B_r(\gamma(t_1)))^2 \int_0^\omega \fint_{B_{4r}(\gamma(t_1-s))} |\mathrm{Hess}\,h^-|_{\mathrm{HS}}\,dm\,ds\\
&\leq c(N,\delta) m(B_r(\gamma(t_1)))^2 \sqrt{\omega} r,
\end{aligned} \tag{5.97}$$

where the last line follows from the definition of h^-, statement 4 of Theorem 4.0.12. and Cauchy-Schwarz.

By statement 3 of 4.0.13 and the excess bound on the elements of $B_{\mu r}(\gamma(t_1))'$,

$$\int_0^\omega \int_{B_{\mu r}(\gamma(t_1))'\times B_r(\gamma(t_1))'} |\nabla h^- - \nabla d_p|(\Psi_s(x))\,d(m\times m)(x,y)\,ds \leq c(N,\delta) m(B_r(\gamma(t_1)))^2 \sqrt{\omega} r. \tag{5.98}$$

Similarly by the excess bound on the elemnts of $B_r(\gamma(t_1))'$ (5.93),

$$\int_0^{\omega} \int_{B_{\mu r}(\gamma(t_1))' \times B_r(\gamma(t_1))'} |\nabla h^- - \nabla d_p|(\Psi_s(y))\, d(m \times m)(x,y)\, ds \tag{5.99}$$
$$\leq c(N,\delta) \frac{1}{\sqrt{\eta}} m(B_r(\gamma(t_1))) m(B_{\mu r}(\gamma(t_1))) \sqrt{\omega} r.$$

Combining (5.97) - (5.99) with (5.95) we immediately obtain

$$\int_{B_{\mu r}(\gamma(t_1))' \times B_r(\gamma(t_1))'} dt(\omega)(x,y)\, d(m \times m)(x,y)$$
$$= \int_0^{\omega} [\frac{d}{ds} \int_{B_{\mu r}(\gamma(t_1))' \times B_r(\gamma(t_1))'} dt(s)(x,y)\, d(m \times m)(x,y)]\, ds \tag{5.100}$$
$$\leq c(N,\delta) m(B_r(\gamma(t_1))) \Big(m(B_r(\gamma(t_1))) + \frac{1}{\sqrt{\eta}} m(B_{\mu r}(\gamma(t_1))) \Big) \sqrt{\omega} r.$$

By Bishop-Gromov, $\frac{m(B_{\mu r}(\gamma(t_1)))}{m(B_r(\gamma(t_1)))} \geq C(N)\mu^N = c\eta$. Therefore, there exists $z \in B_{\mu r}(\gamma(t_1))'$ so that

$$\int_{B_r(\gamma(t_1))'} dt(\omega)(z,y)\, dm(y) \leq c(N,\delta) m(B_r(\gamma(t_1))) \Big(\frac{1}{\eta} + \frac{1}{\sqrt{\eta}} \Big) \sqrt{\omega} r \leq c(N,\delta) m(B_r(\gamma(t_1))) \frac{1}{\eta} \sqrt{\omega} r.$$

This means there exists $D \subseteq B_r(\gamma(t_1))'$ so that

$$\frac{m(D)}{m(B_r(\gamma(t_1)))} \geq 1 - 2\eta \text{ and } dt(\omega)(z,y) \leq c(N,\delta) \frac{1}{\eta^2} \sqrt{\omega} r \ \forall y \in D. \tag{5.101}$$

Since $\omega = (\eta^2 \mu)^2$, the latter combined with $z \in B_{\mu r}(\gamma(t_1))'$ implies

$$\Psi_\omega(D) \subseteq B_{(1+c(N,\delta)\mu) r}(\gamma(t_0)). \tag{5.102}$$

We have the following estimate on the volume of $B_r(\gamma(t_1))$ compared to volume of $B_r(\gamma(t_0))$ after possibly constraining ϵ further depending on N and δ.

$$\frac{m(B_r(\gamma(t_1)))}{m(B_r(\gamma(t_0)))} \leq (1 + c\eta) \frac{m(D)}{m(B_r(\gamma(t_0)))} \text{ , by (5.101)}$$
$$\leq (1 + c\eta)(1 + c(N,\delta)\omega)^N \frac{m(\Psi_\omega(D))}{m(B_r(\gamma(t_0)))} \text{ , by Theorem 3.2.9, 2} \tag{5.103}$$
$$\leq (1 + c\eta)(1 + c\omega)^N (1 + c(N,\delta)\mu)^N \frac{m(\Psi_\omega(D))}{m(B_{(1+c\mu)r}(\gamma(t_0)))} \text{ , by Bishop-Gromov}$$
$$\leq 1 + c(N,\delta)\mu = 1 + c\omega^{\frac{1}{2(1+2N)}}.$$

Making the same calculation with Φ in the other direction as well, we obtain the following Hölder estimate on volume

$$\Big|\frac{m(B_r(\gamma(t_1)))}{m(B_r(\gamma(t_0)))} - 1\Big| \le c(N,\delta)|t_1 - t_0|^{\frac{1}{2(1+2N)}}. \tag{5.104}$$

We now show the required Bishop-Gromov approximation can be constructed by using Ψ_ω on a $c\mu r$-dense subset of $B_r(\gamma(t_1))$.

Fix representatives for $|\mathrm{Hess}\, h^-|_{\mathrm{HS}}$ and $|\nabla h^- - \nabla d_p|$. Using the same calculation as before,

$$\begin{aligned}&\int_0^\omega \Big(\int_0^1 \int_{D\times D} d(\Psi_s(x),\Psi_s(y))|\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde\gamma_{\Psi_s(x),\Psi_s(y)}(\tau))\, d(m\times m)(x,y)\, d\tau\Big)\, ds \\ &\le c(N,\delta) m(B_r(\gamma(t_1)))^2 \sqrt{\omega} r,\end{aligned} \tag{5.105}$$

and so by Fubini's theorem, there exists $A \subseteq D$ so that

1. $\frac{m(A)}{m(B_r(\gamma(t_1)))} \ge 1 - 3\eta$;
2. $\int_0^\omega \Big(\int_0^1 \int_D d(\Psi_s(x),\Psi_s(y))|\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde\gamma_{\Psi_s(x),\Psi_s(y)}(\tau))\, dm(y)\, d\tau\Big)\, ds \;\le\; c\frac{1}{\eta} m(B_r(\gamma(t_1)))\sqrt{\omega} r$ for all $x \in A$.

For each $x \in A$, there exists $A_x \subseteq D$ so that

1. $\frac{m(A_x)}{m(B_r(\gamma(t_1)))} \ge 1 - 3\eta$;
2. $\int_0^\omega \Big(\int_0^1 d(\Psi_s(x),\Psi_s(y))|\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde\gamma_{\Psi_s(x),\Psi_s(y)}(\tau))\, d\tau\Big)\, ds \;\le c\frac{1}{\eta^2}\sqrt{\omega} r$ for all $y \in A_x$.

Since A and each A_x are contained in $B_r(\gamma(t_1))'$, their elements have an excess bound of $c\frac{1}{\eta}r^2$ by (5.93). By statement 3 of 4.0.13, for m-a.e. $x \in A$,

$$\int_0^\omega |\nabla h^- - \nabla d_p|(\Psi_s(x))ds \le c(N,\delta)\frac{1}{\sqrt{\eta}}\sqrt{\omega} r. \tag{5.106}$$

Similarly, for each $x \in A$ and m-a.e. $y \in A_x$,

$$\int_0^\omega |\nabla h^- - \nabla d_p|(\Psi_s(y))ds \le c(N,\delta)\frac{1}{\sqrt{\eta}}\sqrt{\omega} r. \tag{5.107}$$

By a Fubini's theorem argument, it is clear that the inequality in Proposition 3.3.8 holds pointwise for $(m\times m)$-a.e. (x,y). We first replace A with a full measure subset so that in addition the inequality in Proposition 3.3.8 holds for all $x \in A$ and m-a.e. $y \in A_x$. We then replace each A_x with a full measure subset so that the same inequality holds for all $x \in A$ and all $y \in A_x$. Therefore, for all $x \in A$, $y \in A_x$,

$$dt(\omega)(x,y) \le c(N,\delta)\Big(\frac{1}{\eta^2} + \frac{1}{\sqrt{\eta}}\Big)\sqrt{\omega} r \le c(N,\delta)\frac{1}{\eta^2}\sqrt{\omega} r \le c\mu r.$$

For any $x, y \in A$, $A_x \cap A_y$ is $c(N)\eta^{\frac{1}{N}} r$-dense in $B_r(\gamma(t_1))$ and so there exists some $z \in A_x \cap A_y$ where $d(x,z) < c\eta^{\frac{1}{N}} r = c\mu r$. Therefore,

$$\begin{aligned}
&|d(\Psi_\omega(x), \Psi_\omega(y)) - d(x,y))| \\
&\leq |d(\Psi_\omega(x), \Psi_\omega(y)) - d(\Psi_\omega(z), \Psi_\omega(y)| + |d(\Psi_\omega(z), \Psi_\omega(y)) - d(z,y)| + |d(z,y) - d(x,y)| \\
&\leq c(N,\delta)\mu r.
\end{aligned} \tag{5.108}$$

Moreover, we have the following estimate on the volume of $\Psi_\omega(A) \subseteq B_{(1+c\mu)r}(\gamma(t_0))$ after possibly constraining ϵ further depending on N and δ.

$$\begin{aligned}
&\frac{m(\Psi_\omega(A))}{m(B_{1+c\mu r}(\gamma(t_0)))} \geq \frac{1}{(1+c(N,\delta)\omega)^N} \frac{m(A)}{m(B_{(1+c\mu)r}(\gamma(t_0)))} \quad \text{, by Theorem 3.2.9, 2} \\
&\geq \frac{1}{(1+c\omega)^N} \frac{1}{(1+c(N,\delta)\mu)^N} \frac{m(A)}{m(B_r(\gamma(t_0)))} \quad \text{, by Bishop-Gromov} \\
&\geq \frac{1}{(1+c\omega)^N} \frac{1}{(1+c\mu)^N} \frac{1}{1+c(N,\delta)\mu} \frac{m(A)}{m(B_r(\gamma(t_1)))} \quad \text{, by (5.103)} \\
&\geq \frac{1}{(1+c\omega)^N} \frac{1}{(1+c\mu)^N} \frac{1}{1+c\mu} (1-3\eta) \geq 1 - c(N,\delta)\mu.
\end{aligned} \tag{5.109}$$

To summarize, $A \subseteq B_r(\gamma(t_1))$ is so that

1. $\Psi_\omega(A) \subseteq B_{(1+c(N,\delta)\mu)r}(\gamma(t_0))$;
2. $\forall x, y \in A$, $|d(\Psi_\omega(x), \Psi_\omega(y)) - d(x,y)| \leq c(N,\delta)\mu r$;
3. A is $c(N)\mu r$-dense in $B_r(\gamma(t_1))$;
4. $\Psi_\omega(A)$ is $c(N,\delta)\mu^{\frac{1}{N}} r$-dense in $B_{(1+c(N,\delta)\mu)r}(\gamma(t_0))$.

Moreover, there exists $c(N)$ so that $m(B_{c(N)\mu r}(\gamma(t_1))) \geq \frac{2\eta}{1-V(1,10)} m(B_r(\gamma(t_1)))$ by Bishop-Gromov. By property 2 of γ, there exists $B_{c\mu r}(\gamma(t_1))' \subseteq B_{c\mu r}(\gamma(t_1))$ so that

$$\frac{m(B_{c\mu r}(\gamma(t_1))')}{m(B_{c\mu r}(\gamma(t_1)))} \geq 1 - V(1,10) \quad \text{and} \quad \Psi_\omega(B_{c\mu r}(\gamma(t_1))') \subseteq B_{2c\mu r}(\gamma(t_0)).$$

Therefore, $B_{c(N)\mu r}(\gamma(t_1))' \cap A$ is non-empty by measure considerations. In other words, there is an element in A which is $c(N)\mu r$ close to $\gamma(t_1)$ and is mapped $2c(N)\mu r$ close to $\gamma(t_0)$ under Ψ_ω.

These facts about A allow for the construction of a $c(N,\delta)\mu^{\frac{1}{N}} r$ pointed Gromov-Hausdorff approximation which finishes the proof. □

Before we prove Theorem 1.1.1, we will first prove that $\mathsf{RCD}(K,N)$ spaces are non-branching in the next chapter using the construction we have developed so far. A corollary then follows which immediately gives Theorem 1.1.1.

Chapter 6

Applications

6.1 Non-branching

In this section, we prove that $\mathsf{RCD}(K, N)$ spaces are non-branching. The use of the essentially non-branching property of $\mathsf{RCD}(K, N)$ spaces in the proof was pointed out to the author by Vitali Kapovitch.

Proof of Theorem 1.1.3. Assume otherwise. By zooming in and cutting off geodesics if necessary, we may assume (X, d, m) is an $\mathsf{RCD}(-(N-1), N)$ space for some $N \in (1, \infty)$ and we have two unit speed geodesics $\gamma_1, \gamma_2 : [0, 1] \to X$ with

1. $\gamma_1(0) = \gamma_2(0) = p$ for $p \in X$;
2. $\gamma_1(1) = q_1$ and $\gamma_2(1) = q_2$ for $q_1, q_2 \in X$;
3. $\max\{t \in [0, 1] : \gamma_1(s) = \gamma_2(s)\ \forall s \in [0, t]\} = 0.5$.

Let $p' = \gamma_1(0.5)$ and Ψ_s be as in (5.1) towards p.

Since $(X, 2d, m)$ is again an $\mathsf{RCD}(-(N-1), N)$ space and $2d(p, p') = 1$, we may apply Theorem 5.2.8 to obtain a $2d$-unit speed geodesic $\tilde{\gamma} : [0, 1] \to X$ between p and p'. Reparameterize $\tilde{\gamma}$ to $\gamma : [0, 0.5] \to X$ so that γ is a d-unit speed geodesic.

Fix any $\delta \in (0, 0.1)$, use Corollary 5.2.6 to construct a unit speed geodesic $\gamma_1^\delta : [0, 1] \to X$ from p to q_1 with $\gamma_1^\delta(t) = \gamma(t)$ for all $t \in [0, \delta]$. Therefore, the proof of Theorem 5.3.1 passes for γ_1^δ for the same δ and in particular we have the estimates (5.101) - (5.104) for γ_1^δ. As a reminder, for $\delta \leq s_1 < s_2 \leq 1 - \delta$ and sufficiently small r,

- (5.101) and (5.102) imply a portion of $B_r(\gamma_1^\delta(s_2))$ is sent to a ball of radius slightly larger than r around $\gamma_1^\delta(s_1)$ by $\Psi_{s_2 - s_1}$, where the relative size of the portion and the increase in radius on the scale of r can be both made Hölder dependent on $s_2 - s_1$ and go uniformly to 1 and 0 respectively as $s_2 - s_1 \to 0$.
- (5.104) implies the ratio between the measures of $B_r(\gamma_1^\delta(s_1))$ and $B_r(\gamma_1^\delta(s_2))$ is Hölder dependent on $s_2 - s_1$ and in particular goes uniformly to 1 as $s_2 - s_1 \to 0$

We show that $\gamma_1^\delta(t) = \gamma(t)$ for all $t \in [0, 0.5]$. Suppose not, let $t_0 := \max\{t \in [0, 0.5] : \gamma_1^\delta(s) = \gamma(s)\ \forall s \in [0, t]\}$ and so $t_0 \in [\delta, 0.5)$.

We claim there exists $t_1 \in (t_0, 0.5)$ and $\bar{r} > 0$ so that for any $r \le \bar{r}$, there exists $A_1 \subseteq B_r(\gamma_1^\delta(t_1))$ and $A_2 \subseteq B_r(\gamma(t_1))$ so that

1. $\gamma_1^\delta(t_1) \neq \gamma(t_1)$.
2. $\Psi_{t_1 - t_0}(A_1) \subseteq B_r(\gamma_1^\delta(t_0))$ and $\Psi_{t_1 - t_0}(A_2) \subseteq B_r(\gamma(t_0))$;
3. $\frac{m(\Psi_{t_1 - t_0}(A_1))}{m(B_r(\gamma_1^\delta(t_0)))} > \frac{1}{2}$ and $\frac{m(\Psi_{t_1 - t_0}(A_2))}{m(B_r(\gamma(t_0)))} > \frac{1}{2}$.

We can choose t_1 arbitrarily close to t_0 so that statement 1 holds by definition of t_0. Statements 2 and 3 then follow from (5.101) - (5.104) for γ_1^δ, the same for $\tilde{\gamma}$, Bishop-Gromov and statement 2 of Theorem 3.2.9 to control the volume distortion of Ψ, as soon as t_1 is chosen close enough to t_0. Choosing $r \le \min\{\bar{r}, d(\gamma_1^\delta(t_1), \gamma(t_1))/4\}$, it is straightforward to check using the triangle inequality that any two points from $B_r(\gamma_1^\delta(t_1))$ and $B_r(\gamma(t_0))$ respectively cannot lie on the same geodesic towards p. As such, any point which lies in $\Psi_{t_1 - t_0}(A_1) \cap \Psi_{t_1 - t_0}(A_2)$ must be so that a geodesic from p to that point can be extended to two branching geodesics. It is known by [19, Proposition 4.5] that the subset of points $x \in X$ where a geodesic from p to x can be extended to two branching geodesics has measure 0 and so $m(\Psi_{t_1 - t_0}(A_1) \cap \Psi_{t_1 - t_0}(A_2)) = 0$. We now have a contradiction with properties 2 and 3 of A_1 and A_2.

Therefore, $\gamma_1^\delta(t) = \gamma(t)$ for all $t \in [0, 0.5]$. Since this is true for all $\delta \in (0, 0.1)$, taking $\delta \to 0$ and using Arzelà-Ascoli Theorem, after possibly passing to a subsequence, we obtain a geodesic $\bar{\gamma}_1$ satisfying Theorem 5.2.8 with $\bar{\gamma}_1 \equiv \gamma$ on $[0, 0.5]$ and $\bar{\gamma}_1(1) = q_1$. The same construction for γ_2 gives $\bar{\gamma}_2$ satisfying Theorem 5.2.8 with $\bar{\gamma}_2 \equiv \gamma$ on $[0, 0.5]$ and $\bar{\gamma}_2(1) = q_2$. Applying the previous argument again for $\bar{\gamma}_1$ and $\bar{\gamma}_2$ shows that they cannot split, which is a contradiction. □

As a corollary, we have the following improvement of Theorem 5.3.1.

Corollary 6.1.1. *Let (X, d, m) be an $\mathsf{RCD}(-(N-1), N)$ space for some $N \in (1, \infty)$ and $p, q \in X$ with $d(p, q) = 1$. For any $\delta \in (0, 0.1)$, there exists $\epsilon(N, \delta) > 0$, $\bar{r}(N, \delta) > 0$ and $C(N, \delta)$ so that for any unit speed geodesic γ between p and q, $r \le \bar{r}$, and $t_0, t_1 \in [\delta, 1-\delta]$, if $|t_1 - t_0| \le \epsilon$ then*

$$d_{pGH}\Big((B_r(\gamma(t_0)), \gamma(t_0)), (B_r(\gamma(t_1)), \gamma(t_1))\Big) \le Cr|t_1 - t_0|^{\frac{1}{2N(1+2N)}}.$$

Proof. Fix any $s \in [0, 0.5]$. Since $(X, 2d, m)$ is again an $\mathsf{RCD}(-(N-1), N)$ space and $2d(\gamma(s), \gamma(s + 0.5)) = 1$, we may use Theorem 5.3.1 to construct some 2d-unit speed geodesic γ^s between $\gamma(s)$ and $\gamma(s + 0.5)$. Since X is non-branching, γ^s and γ must coincide between $\gamma(s)$ and $\gamma(s + 0.5)$. Since this is true for all $s \in [0, 0.5]$ and all γ^s have the Hölder properties of Theorem 5.3.1 on $(X, 2d, m)$. The same is true for γ on (X, d, m) with slightly worse constants. □

Theorems 1.1.1 now follows immediately by rescaling.

Theorem 1.1.3 also implies that all geodesics on a Ricci limit space are limit geodesics.

Corollary 6.1.2. *Let (X, d, m) be a Ricci limit space. Any geodesic of X is a limit geodesic.*

Proof. Fix any geodesic γ of X. For any two points in the interior of γ, there must be a limit geodesic connecting them. Since X is non-branching, this limit geodesic must coincide with the portion of γ between the two points. Since this is true for any two points in the interior of γ, a simple diagonalization argument shows that γ itself must also be a limit geodesic. □

6.2 Dimension and weak convexity of the regular set

In this subsection we will extend to the $\mathsf{RCD}(K, N)$ setting the results of [33] on regular set. All proofs translate directly from [33]. We mention again that Theorem 6.2.1 has already been established using a new argument involving the Green's function in [18].

Theorem 6.2.1. *(Constancy of the dimension) Let (X, d, m) be an $\mathsf{RCD}(K, N)$ m.m.s. for some $K \in \mathbb{R}$ and $N \in (1, \infty)$. Assume X is not a point. There exists a unique $n \in \mathbb{N}$, $1 \leq n \leq N$ so that $m(X \backslash \mathcal{R}_n) = 0$.*

Proof. Let $A^1, A^2 \subseteq X \times X$ be the sets of $(x, y) \in X \times X$ so that geodesics from x to y are extendible past x and y respectively. For each $x \in X$, let A_x be the set of $y \in X$ so that geodesics from x to y are extendible past y. Using the arguments of [19, Section 4], A^1, A^2 are $(m \times m)$-measurable and A_x is m-measurable for all $x \in X$. $m(X \backslash A_x) = 0$ for any $x \in X$ by a standard argument using Bishop-Gromov. Let $A := A^1 \cap A^2$, Fubini's theorem then gives $(m \times m)((X \times X) \backslash A) = 0$.

Let $\gamma_{x,y} : [0, 1] \to X$ be a Borel selection (2.7.5) of constant speed geodesics from any $x \in X$ to any $y \in X$. Since $m(\mathcal{S}) = 0$, it follows from applying the segment inequality to the characteristic function of $\mathcal{S}$ that for $(m \times m)$-a.e. $(x, y) \in X \times X$, $\gamma_{x,y} \cap \mathcal{R}_{\text{reg}}$ has full measure, and therefore is also dense, in $[0, 1]$. By Theorem 1.1.1, for any geodesic γ and k, $\gamma \cap \mathcal{R}_k$ is closed relative to the interior of γ. Combining these with the fact that almost every $\gamma_{x,y}$ is extendible, we obtain for $(m \times m)$-a.e. $(x, y) \in X \times X$, there exists $k \in \mathbb{N}$ with $1 \leq k \leq N$ so that $\gamma_{x,y} \subseteq \mathcal{R}_k$. This leads to a contradiction if there are two regular sets of different dimension with positive measure. □

Definition 6.2.2. (m-a.e. convexity) Let (X, d, m) be a m.m.s.. Let S be an m-measurable set in X. S is *m-a.e. convex* iff for $(m \times m)$ almost every pair $(x, y) \in S \times S$, there exists a minimizing geodesic $\gamma \subseteq S$ connecting x and y.

Definition 6.2.3. (weak convexity) Let (X, d) be a metric space. $S \subseteq X$ is *weakly convex* iff for all $(x, y) \in S \times S$ and $\epsilon > 0$, and there exists an ϵ-geodesic (see Definition 4.0.5) $\gamma \subseteq S$ connecting x and y.

Theorem 6.2.4. *(m-a.e. and weak convexity of the regular set) Let $\mathcal{R}_n$ be as in Theorem 6.2.1, then*

1. *$\mathcal{R}_n$ is m-a.e. convex;*

2. *$\mathcal{R}_n$ is weakly convex.*

In particular, $\mathcal{R}_n$ is connected.

Proof. Statement 1 is contained in the proof of Theorem 6.2.1. The proof of statement 2 follows verbatim from [33, Theorem 1.20]. □

Part 1 of Theorem 6.2.4 can be improved slightly to

Theorem 6.2.5. *Let p be a point in an $\mathsf{RCD}(K,N)$ space (X,d,m). For m-a.e. $q \in X$, there exists a geodesic $\gamma_{p,q} : [0,1] \to X$ from p to q so that $\gamma(t) \in \mathcal{R}_n$ for any $t \in (0,1]$.*

Proof. The argument will make use of the theory of 1D-localisation. We refer to [31, Section 3] for details. Fixing $p \in X$, we use [31, Theorem 3.6] with the 1-Lipschitz function $u := d_p$ to give a disintegration of m strongly consistent with R_u^{nb}. We use the notations $Q, \mathfrak{q}, X_\alpha$, and m_α as in the statement of Theorem 3.6. By Fubini's Theorem and the fact that m-a.e. point in X is regular, we have that for $\mathfrak{q}$-a.e. $\alpha \in Q$, for m_α-a.e. $x \in X_\alpha$, x is regular. By Theorem 1.1.2, this implies that for $\mathfrak{q}$-a.e. $\alpha \in Q$, all points in the interior of X_α is regular. Since for $\mathfrak{q}$-a.e. $\alpha \in Q$, $m_\alpha \ll \mathscr{H}^1 \llcorner_{X_\alpha}$, the union of the interior of such X_α has full measure in X. This completes the proof. □

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