Cluster Structure for Mirković-Vilonen Cycles and Polytopes

by

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Abstract

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Introduction

Canonical Bases

For a simple simply-connected complex algebraic group G with unipotent $N \subset G$, there have been a number of bases introduced for $\mathbb{C}[N]$ that have several nice properties. For instance, as it is possible to view every irreducible representation $L(\lambda)$ of G as a subspace of $\mathbb{C}[N]$, we would like a basis $\beta \subset \mathbb{C}[N]$ such that $\beta \cap L(\lambda)$ is a basis for each $L(\lambda)$.

Such bases include:

1. Lusztig's dual canonical basis [Lus90]

2. Lusztig's dual semicanonical basis [Lus00]

3. The MV basis [BKK19]

These bases are in general different from each other, and there is a well-known type A_5 example showcasing this [GLS05][BKK19]. As these bases are very useful, one would like to compute their basis elements. However, this computation is in general geometric and difficult. For example, the dual canonical basis comes from perverse sheaves, the dual semicanonical basis is derived from constructible functions on irreducible components of varieties of modules, and the MV basis originates from cohomology classes. As such, we would like a combinatorial way to write down all basis elements. A partial answer to this problem can be achieved through cluster algebras.

Cluster Algebras

Cluster algebras were introduced by Fomin and Zelevinsky in [FZ02] and it was shown in [GLS07a], using the construction of [BFZ05], that $\mathbb{C}[N]$ has a natural cluster algebra structure. This structure is described axiomatically from an initial set of generators, called an initial seed, which is a pair $\Sigma = ((x_1, \ldots, x_r), B)$ with B being a matrix. We then obtain new generators through a process called mutation, where for each mutable x_i, there is a unique element x_i^* satisfying an equation of the form

$$x_i x_i^* = x_+ + x_-.$$

This equation is called an exchange relation and the terms x_+ and x_- are certain products of the x_j, $j \neq i$, determined by B. We then acquire a new seed

$$\mu_i(\Sigma) = ((x_1, \ldots, x_{i-1}, x_i^*, x_{i+1}, \ldots, x_r), \mu_i(B))$$

where $\mu_i(B)$ is another matrix derived from B.

We call each of the variables $x_1, \ldots, x_r$ and all variables that are produced through successive mutations cluster variables. As computation of these cluster variables is combinatorial in nature, a natural question is whether or not the cluster variables are elements of the aforementioned bases. If so, then a large number of basis elements can be efficiently computed.

For type ADE, it has been shown in [KKKO18] that the cluster monomials, products of cluster variables that all belong to a single seed, belong to the dual canonical basis and in [GLS06] for the dual semicanonical basis. The question is still open for the MV basis, but in [BGL20], the authors showed that for certain seeds satisfying a special condition, the cluster monomials in those seeds belong to the MV basis. Furthermore, in type A_n, $n \leq 3$, the MV basis agrees with the dual canonical and dual semicanonical basis [BKK19] so the cluster monomials are in the MV basis in those cases. This leads to the main conjecture this thesis attempts to investigate:

Conjecture. *The cluster monomials in $\mathbb{C}[N]$ belong to the MV basis.*

Cycles, Modules, and Polytopes

Our approach to investigating the above conjecture will be relating the MV basis to the dual semicanonical basis. The MV basis is created from and indexed by stable MV cycles, certain subvarieties of the affine Grassmannian $\mathcal{G}r$. More specifically, there are certain orbits in $\mathcal{G}r$, denoted $\mathcal{G}r^\lambda$, S_-^μ, and S_+^μ, where λ is a dominant weight and μ is a weight, and MV cycles are the irreducible components of $\overline{\mathcal{G}r^\lambda \cap S_-^\mu}$. The stable MV cycles are the irreducible components of $\overline{S_+^0 \cap S_-^\mu}$.

The dual semicanonical basis is related to irreducible components of varieties of modules over the preprojective algebra Λ and there is a bijection between MV cycles and certain Λ-modules called generic, by first going through MV polytopes [BK10]. For an MV cycle Z, its MV polytope $\mathrm{Pol}(Z)$ can be thought of as a moment polytope for Z [And03], and for a Λ-module M, its polytope $\mathrm{Pol}(M)$ is equal to the convex hull of dimension vectors of submodules of M [BKT14]. Hence we have the following diagram:

$$\begin{array}{ccc} \mathbb{C}[N] & \longleftarrow & \text{Generic } \Lambda\text{-modules} \\ \uparrow & & \updownarrow \\ \text{Stable MV Cycles} & \longleftrightarrow & \text{MV Polytopes} \end{array} \qquad (\star)$$

For a stable MV cycle Z, let $b_Z \in \mathbb{C}[N]$ be the associated MV basis element. Each reduced expression $\mathbf{i}$ of the longest word in the Weyl group of G gives us an initial seed $\Sigma = ((x_1, \ldots, x_r), B)$ of $\mathbb{C}[N]$ [BFZ05][GLS07a]. In this case, the cluster variables x_i are in the MV basis [BKK19], say $x_i = b_{Z_i}$. We are interested in the exchange relation $x_i x_i^* = x_+ + x_-$ where x_i^* is the mutation of x_i.

Fusion Product

There is a way to multiply MV cycles, called the fusion product and denoted by $*$, that is compatible with the multiplication in $\mathbb{C}[N]$ [BKK19]. Geometrically, this fusion product is viewed inside the Beilinson-Drinfeld Grassmannian $\mathcal{G}r_{0,\mathbb{A}}$, which is a family $\mathcal{G}r_{0,\mathbb{A}} \to \mathbb{A}^1$ such that the generic fibre is isomorphic to $\mathcal{G}r \times \mathcal{G}r$ and the 0-fibre is isomorphic to $\mathcal{G}r$. Then to compute the fusion product of two MV cycles Z_1 and Z_2, we take the family of $\mathcal{G}r_{0,\mathbb{A}}$ over $\mathbb{A}^1 \setminus \{0\}$ whose fibres are isomorphic to $Z_1 \times Z_2$. Taking the closure of this, the top dimensional irreducible components of the 0-fibre constitute the terms in the fusion product of $Z_1 * Z_2$. We call this family $Z_1 *_{\mathbb{A}} Z_2$ and denote its 0-fibre by $Z_1 *_0 Z_2$.

If we are able to find a stable MV cycle Z_i^* such that

$$Z_i * Z_i^* = Z_+ + Z_-$$

where Z_+ and Z_- corresponds to x_+ and x_-, respectively, then we can immediately conclude $x_i^* = b_{Z_i^*}$ Hence, as all cluster variables are created through these exchange relations, we deduce that the cluster variables are in the MV basis.

Exchange Relation for Polytopes and Cycles

In [GLS06], the authors gave a categorification of the cluster structure in $\mathbb{C}[N]$ by associating the seed Σ with a basic maximal rigid Λ-module $T = \bigoplus_{i=1}^r T_i$ along with the same matrix B, the x_i with T_i, and the exchange relations with non-split short exact sequences

$$0 \to T_i \to T_+ \to T_i^* \to 0$$

$$0 \to T_i^* \to T_- \to T_i \to 0.$$

Our first main result translates these short exact sequences in Λ-mod to an equation of MV polytopes akin to the original exchange relations.

Theorem A. *(Theorem 5.1.1)* $\mathrm{Pol}(T_i) + \mathrm{Pol}(T_i^*) = \mathrm{Pol}(T_+) \cup \mathrm{Pol}(T_-)$

This equation allows us to compute $\mathrm{Pol}(T_i^*)$ very easily by noting that

$$\mathrm{Pol}(T_i^*) = \{x : x + \mathrm{Pol}(T_i) \subset \mathrm{Pol}(T_+) \cup \mathrm{Pol}(T_-)\}.$$

Theorem A is also a necessary condition for a corresponding exchange relation of MV cycles, as we explain in Section 5.2. In [BKK19], the authors considered a Duistermaat-Heckmann measure DH on MV cycles such that for an MV cycle Z, $DH(Z)$ is supported on $\mathrm{Pol}(Z)$. Using this, it is possible to show that by applying DH to a fusion product $Z_1 * Z_2 = Z_+ + Z_-$, we obtain the equation in Theorem A. Hence the only possible candidate for Z_i^* is the stable MV cycle whose MV polytope is $\mathrm{Pol}(T_i^*)$. With this candidate for Z_i^*, we show in Chapter 6 the following result.

Proposition B. *(Proposition 6.2.6) In type A_n for $n \leq 4$, $b_{Z_i^*} = ax_i^*$ for some $a \in \mathbb{Z}_{\geq 1}$.*

Extending Relationship Between Cycles and Modules

The proof of Proposition B relies on an interesting relationship between a stable MV cycle Z and its corresponding generic module M. Each MV cycle has numerical invariants called valuations where for each $x \in \mathbb{C}[N]$, we obtain an integer $valx(Z)$. Some of these invariants were used in [Kam10] to create combinatorial data, called BZ data, that showed exactly when a polytope was an MV polytope. In [BK10], the authors showed that if x was a cluster variable in an initial seed derived from a reduced expression $\mathbf{i}$ and corresponds to the generic Λ-module X, then

$$valx(Z) = -\dim \mathrm{Hom}(X, M).$$

We extend this result to all cluster monomials x and modules M corresponding to a cluster monomial.

Theorem C. *(Theorem 6.2.5) Let M be a generic module corresponding to a cluster monomial in $\mathbb{C}[N]$ and MV cycle Z. Let X be a module corresponding to a cluster monomial x. Then*

$$valx(Z) = -\dim \mathrm{Hom}(X, M).$$

Calculating the Fusion Product

In Chapter 7, in joint work with Anne Dranowski and Joel Kamnitzer, we present a way to compute the fusion product in type A. Normally, the product is computed by looking at intersection multiplicities [BKK19], a very geometric point of view and difficult to calculate in general, but our approach computes this product via matrices and semi-standard Young tableaux. We begin with showing a generalization of the Mirković-Vybornov isomorphism [MV07b] [MV19].

Theorem D. *(Theorem 7.4.6) We have an isomorphism*

$$\overline{\mathbb{O}}^{\lambda',\lambda''}_{0,\mathbb{A}} \cap \mathbb{U}^{\mu',\mu''}_{0,\mathbb{A}} \cong \overline{\mathcal{G}r}^{\lambda',\lambda''}_{0,\mathbb{A}} \cap S^{\mu',\mu''}_{0,\mathbb{A}}.$$

Both sides of the isomorphism are families over $\mathbb{A}^1$ and the original Mirković-Vybornov isomorphism is recovered when we take the 0-fibre of these families. The family $\overline{\mathbb{O}}^{\lambda',\lambda''}_{0,\mathbb{A}}$ denotes an orbit of matrices where the fibre over $s \neq 0$ are matrices X conjugate to a two block matrix whose first block is the Jordan block of size λ' with eigenvalue 0 and whose second block is the Jordan block of size λ'' with eigenvalue s. The 0-fibre of this family consists of matrices conjugate to the Jordan block of size $\lambda' + \lambda''$ with eigenvalue 0. The Mirković-Vybornov slice $\mathbb{U}^{\mu',\mu''}_{0,\mathbb{A}}$ denotes a certain set of block upper triangular matrices.

The generic fibre of $\overline{\mathcal{G}r}^{\lambda',\lambda''}_{0,\mathbb{A}} \cap S^{\mu',\mu''}_{0,\mathbb{A}}$ is isomorphic to

$$(\overline{\mathcal{G}r}^{\lambda'} \cap S^{\mu'}_-) \times (\overline{\mathcal{G}r}^{\lambda''} \cap S^{\mu''}_-)$$

while the 0-fibre is isomorphic to

$$\overline{\mathcal{G}r}^{\lambda'+\lambda''} \cap S^{\mu'+\mu''}_-.$$

When computing the fusion product of MV cycles Z_1 and Z_2, we have $Z_1 \times Z_2 \subset (\overline{\mathcal{G}r}^{\lambda'} \cap S^{\mu'}_-) \times (\overline{\mathcal{G}r}^{\lambda''} \cap S^{\mu''}_-)$ for some $\lambda', \lambda'', \mu', \mu''$ and the terms in the fusion product are in $\overline{\mathcal{G}r}^{\lambda'+\lambda''} \cap S^{\mu_1+\mu_2}_-$ so by using Theorem D, we are able to view this family of MV cycles as a family of matrices instead.

Following [Dra20a], it is possible to find the ideal of the MV cycles appearing as a factor or as a term in the fusion product through these matrices. These ideals can then be turned into a semi-standard Young tableaux and each of these corresponds to an MV cycle [Dra20b]. Putting all these ideas together, the steps involved in computing a fusion product of MV cycles $Z_1 * Z_2$ are:

1. Create a generic matrix A in $\mathbb{U}^{\mu',\mu''}_{0,\mathbb{A}}$ and use the fact that we also require A to be in $\overline{\mathbb{O}}^{\lambda',\lambda''}_{0,\mathbb{A}}$ to create the ideal I for $Z_1 *_{\mathbb{A}} Z_2$

2. Find the ideal J for the 0-fibre $Z_1 *_0 Z_2$ by adding s to I and deconstruct it into ideals J_α for each irreducible component of $Z_1 *_0 Z_2$

3. Translate each ideal J_α into a semi-standard Young tableau and find the corresponding MV cycle Z_α

4. We then conclude $Z_1 * Z_2 = \sum_\alpha Z_\alpha$.

We end with Chapter 8 by looking at the details of diagram ($\star$) in type A_3 and compute all fusion products corresponding to an exchange relation through the method developed in Chapter 7. In particular, we show that the integer a in Proposition B is always 1, thus showing directly that the cluster variables are in the MV basis up to type A_3, culminating in our final result.

Proposition E. *(Proposition 8.4.1) In type A_3, every cluster variable in $\mathbb{C}[N]$ is in the MV basis.*

Chapter 1

MV Basis

Let G be a simple simply-connected complex algebraic group with Borel subgroup B, unipotent radical N, and maximal torus T. In this chapter we consider the first class of objects we are interested in, Mirković-Vilonen (MV) cycles, and see how they index a basis of $\mathbb{C}[N]$.

1.1 MV Cycles

Let $G^\vee$ be the Langlands dual group of G with maximal torus $T^\vee$. Our choice of positive and negative roots give dual opposite Borel subgroups $B^\vee_\pm$ with unipotent radicals $N^\vee_\pm$. Let P denote the character lattice of T and P_+ the set of dominant weights. Note that P is also the cocharacter lattice of $T^\vee$.

Let $R \subset P$ be the root lattice and $R_+ \subset R$ be the $\mathbb{Z}_{\geq 0}$-span of the positive roots. For $\lambda, \mu \in P$, we say $\lambda \geq \mu$ if and only if $\lambda - \mu \in R_+$. Let $\{\alpha_i\}_{i \in I}$ denote the set of simple roots and $\{\alpha_i^\vee\}_{i \in I}$ denote the set of simple coroots. Let ρ, respectively $\rho^\vee$, be the half-sum of the positive roots, respectively coroots.

Let $\mathcal{O} = \mathbb{C}[[t]]$ be the ring of formal power series and $\mathcal{K} = \mathbb{C}((t))$ be its quotient field. The **affine Grassmannian** $\mathcal{G}r$ of $G^\vee$ is defined to be the quotient $G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$.

Given a cocharacter $\mu \in P$, we obtain a map $\mathcal{K}^\times \to T^\vee(\mathcal{K})$. Let $t^\mu \in T^\vee(\mathcal{K}) \subset G^\vee(\mathcal{K})$ denote the image of $t \in \mathcal{K}^\times$ under this map. We will also use t^μ to denote its image in Gr. These points are actually the $T^\vee$ fixed points of $\mathcal{G}r(\mathbb{C})$, that is,

$$(\mathcal{G}r(\mathbb{C}))^{T^\vee} = \{t^\lambda : \lambda \in P_+\}.$$

Definition 1.1.1. For a dominant weight $\lambda \in P_+$, the **spherical Schubert cell** is $\mathcal{G}r^\lambda = G^\vee(\mathcal{O})t^\lambda \subset \mathcal{G}r$, the $G^\vee(\mathcal{O})$-orbit through t^λ.

We call its closure $\overline{\mathcal{G}r}^\lambda$ a **spherical Schubert variety** and from [Zhu16, Proposition 2.1.5],

$$\overline{\mathcal{G}r}^\lambda = \bigsqcup_{\gamma \leq \lambda} \mathcal{G}r^\gamma.$$

Definition 1.1.2. For a weight $\mu \in P$, the **semi-infinite orbit** is $S^\mu_\pm = N^\vee_\pm(\mathcal{K})t^\mu \subset \mathcal{G}r$, the $N^\vee_\pm(\mathcal{K})$-orbit through t^μ.

Theorem 1.1.1. *[MV07a, Theorem 3.2a] For $\lambda \in P_+$ and $\mu \in P$, $\overline{\mathcal{G}r^\lambda \cap S^\mu_-}$ is non-empty precisely when $t^\mu \in \overline{\mathcal{G}r}^\lambda$ and in this case, $\overline{\mathcal{G}r^\lambda \cap S^\mu_-}$ is pure of dimension $\langle \rho^\vee, \lambda - \mu \rangle$.*

When $\overline{\mathcal{G}r^\lambda \cap S_-^\mu}$ is non-empty, its irreducible components are called the **Mirković-Vilonen (MV) cycles of type λ and coweight μ**. We let $Z(\lambda)_\mu$ denote the set of all such cycles, and let

$$Z(\lambda) := \bigsqcup_{\mu \in P} Z(\lambda)_\mu$$

be the set of all MV cycles of type λ.

Proposition 1.1.2. *[And03, Proposition 3] Let $\lambda \in P_+$ and $\mu \in P$. The irreducible components of $\overline{\mathcal{G}r^\lambda \cap S_-^\mu}$ are the irreducible components of $\overline{S_+^\lambda \cap S_-^\mu}$ contained in $\overline{\mathcal{G}r}^\lambda$.*

We also have an action of P on $\mathcal{G}r$ given by $\lambda \cdot [g] = [g]t^\lambda$ which induces an isomorphism

$$\overline{S_+^0 \cap S_-^\mu} \xrightarrow{\sim} \overline{S_+^\lambda \cap S_-^{\lambda+\mu}}.$$

We will often be working with the irreducible components of $\overline{S_+^0 \cap S_-^\mu}$, which we shall call **stable MV cycles of coweight μ**. We denote the set of such cycles as $Z(\infty)_\mu$, and the set of all stable MV cycles as

$$Z(\infty) := \bigsqcup_{\mu \in Q_+} Z(\infty)_\mu.$$

Thus by Proposition 1.1.2, we obtain a bijection

$$\{Z \in Z(\infty) : \lambda \cdot Z \subset \overline{\mathcal{G}r}^\lambda\} \leftrightarrow Z(\lambda).$$

Example 1.1.3. This will be our running example thoughout the thesis. Let $G = \mathbf{GL}_3$. It is known that $\overline{\mathcal{G}r^{(1,0,0)} \cap S_-^{(0,1,0)}}$ and $\overline{\mathcal{G}r^{(1,1,0)} \cap S_-^{(1,0,1)}}$ are both irreducible and isomorphic to $\mathbb{P}^1$. We also know $\overline{\mathcal{G}r^{(2,1,0)} \cap S_-^{(1,1,1)}}$ to have exactly two irreducible components, each isomorphic to $\mathbb{P}^2$.

1.2 MV Basis of $\mathbb{C}[N]$

For $\lambda \in P_+$, let $L(\lambda)$ be the irreducible representation of G with highest weight λ. We will now explain a construction of a basis of $\mathbb{C}[N]$ indexed by $Z(\infty)$ by first constructing bases for each $L(\lambda)$ and gluing them together. The basis of $L(\lambda)$ comes from the following theorem, the Geometric Satake correspondence, first proven in [Gin95] and later refined in [MV07a].

Theorem 1.2.1. *[Gin95, Theorem 1.4.1][MV07a, Theorem 7.3] Let $P_{G^\vee(\mathcal{O})}(\mathcal{G}r)$ denote the category of $G^\vee(\mathcal{O})$-equivariant perverse sheaves on $\mathcal{G}r$ with $\mathbb{C}$-coefficients and Rep(G) denote the category of finite-dimensional $\mathbb{C}$-representations of G. Then Rep(G) is equivalent to $P_{G^\vee(\mathcal{O})}(\mathcal{G}r)$ as tensor categories.*

The proof of Theorem 1.2.1 used weight functors

$$F_\mu := H_c^{\langle 2\rho,\mu\rangle}(S_\mu^+, -) : P_{G_\mathcal{O}^\vee}(\mathcal{G}r) \to \mathrm{Vect}_\mathbb{C}$$

where $\mu \in P$. Let $F = \bigoplus_{\mu \in P} F_\mu$. By [MV07a, Theorem 3.6], the fiber functor F is isomorphic to the cohomology functor $H^*(\mathcal{G}r, -)$. Using Tannakian formalism, the functor F fits inside a commutative diagram

$$\begin{array}{ccc} P_{G^\vee(\mathcal{O})}(\mathcal{G}r) & \xrightarrow{\cong} & \mathrm{Rep}(G) \\ & {\scriptstyle F}\searrow \quad \swarrow & \\ & \mathrm{Vect}_{\mathbb{C}} & \end{array}$$

where $\mathrm{Rep}(G) \to \mathrm{Vect}_{\mathbb{C}}$ is the forgetful functor.

For $\lambda \in P_+$, let IC_λ be the intersection cohomology sheaf of $\overline{\mathcal{G}r}^\lambda$. These are the simple objects in $P_{G^\vee(\mathcal{O})}(\mathcal{G}r)$ so under F, they correspond to $L(\lambda)$. In fact, we have $F_\mu(\mathrm{IC}_\lambda) = L(\lambda)_\mu$, the μ weight space of $L(\lambda)$.

Proposition 1.2.2. *[MV07a, Proposition 3.10]*

$$L(\lambda)_\mu \cong H_{top}(\overline{\mathcal{G}r^\lambda \cap S_-^\mu}).$$

Since $\overline{\mathcal{G}r^\lambda \cap S_-^\mu}$ is pure-dimensional, its irreducible components form a basis for $L(\lambda)_\mu$. Combining all of the weight spaces together, we thus get a basis for $L(\lambda)$ indexed by $Z(\lambda)$, which we denote by $\{[Z] : Z \in Z(\lambda)\}$.

In [BK07, Corollary 5.43], the authors define an embedding

$$\Psi_\lambda : L(\lambda) \to \mathbb{C}[N]$$

such that

$$\mathbb{C}[N] = \bigcup_{\lambda \in P_+} \Psi_\lambda(L(\lambda)).$$

Proposition 1.2.3. *[BKK19, Proposition 6.1] For each $Z \in Z(\infty)$, there exists a unique element $b_Z \in \mathbb{C}[N]$ such that for any $\lambda \in P_+$, we have*

$$t^\lambda Z \subset \overline{\mathcal{G}r}^\lambda \Rightarrow b_Z = \Psi_\lambda([t^\lambda Z]).$$

We are thus able to glue the bases of the $L(\lambda)$'s to form a basis of $\mathbb{C}[N]$ indexed by $Z(\infty)$, which we call the **MV basis**.

Example 1.2.1. Let $G = \mathbf{GL}_3$. Let $Z_1 = \overline{\mathcal{G}r^{(1,0,0)} \cap S_-^{(0,1,0)}}$, $Z_2 = \overline{\mathcal{G}r^{(1,1,0)} \cap S_-^{(1,0,1)}}$, and Z_+, Z_- be the two MV cycles making up $\overline{\mathcal{G}r^{(2,1,0)} \cap S_-^{(1,1,1)}}$ (c.f. Example 1.1.3). Consider $\mathbb{C}[N] = \mathbb{C}[x_{12}, x_{13}, x_{23}]$ where the x_{ij} are coordinates on

$$N = \left\{ \begin{bmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{bmatrix} : x_{ij} \in \mathbb{C} \right\}.$$

We have $b_{Z_1} = x_{12}$, $b_{Z_2} = x_{23}$, and $\{b_{Z_+}, b_{Z_-}\} = \{x_{13}, x_{12}x_{23} - x_{13}\}$. We shall take $b_{Z_+} = x_{13}$ and $b_{Z_-} = x_{12}x_{23} - x_{13}$. Every element of the MV basis is of the form $b_{Z_1}^\alpha b_{Z_+}^\beta b_{Z_-}^\gamma$ or $b_{Z_2}^\alpha b_{Z_+}^\beta b_{Z_-}^\gamma$ for some $\alpha, \beta, \gamma \in \mathbb{N}$.

1.3 Product of MV Cycles

We start with an alternative description of the affine Grassmannian. Let $s \in \mathbb{C}$, $\mathcal{O}_s = \mathbb{C}[[t-s]]$, and $\mathcal{K}_s = \mathbb{C}((t-s))$. Note that $\mathcal{O} = \mathcal{O}_0$ and $\mathcal{K} = \mathcal{K}_0$. As before, we can consider the affine Grassmannian $\mathcal{G}r_s = G^\vee(\mathcal{K}_s)/G^\vee(\mathcal{O}_s)$. By [LS97, Propositions 3.8 and 3.10], we can also identify $\mathcal{G}r_s$ with the set of pairs $(\mathcal{F}, \nu)$ where $\mathcal{F}$ is a G-bundle on $\mathbb{A} := \mathbb{A}$ and ν is a trivialization of $\mathcal{F}$ over $\mathbb{A}\backslash\{s\}$.

We can consider a more general "two-point version" of $\mathcal{G}r$, where we fix one of the points to be 0. Let

$$\mathcal{G}r_{0,\mathbb{A}} := \{(\mathcal{F}, \nu, s) : \mathcal{F} \text{ is a } G\text{-bundle on } \mathbb{A}, s \in \mathbb{A}, \nu \text{ a trivialization of } \mathcal{F} \text{ over } \mathbb{A}\backslash\{0, s\}\}$$

be the **Beilinson-Drinfeld Grassmannian**. We have a natural projection map $\pi : \mathcal{G}r_{0,\mathbb{A}} \to \mathbb{A}$. Let $U = \mathbb{A}\backslash\{0\}$.

Lemma 1.3.1. *[MV07a, Diagram 5.9]* $\pi^{-1}(U) \cong U \times \mathcal{G}r \times \mathcal{G}r$ *and* $\pi^{-1}(0) \cong \mathcal{G}r$.

We shall identify the fibre $\mathcal{G}r_{0,s}$ over $s \in \mathbb{A}\backslash\{0\}$ with $\mathcal{G}r_0 \times \mathcal{G}r_s$. From [BGL20, Section 5.1], this generic fibre can also be viewed as the quotient

$$G(\mathbb{C}[t, t^{-1}, (t-s)^{-1}]/G(\mathbb{C}[t]).$$

The following geometric viewpoint of the product of MV cycles is from [AK06, Section 3]. Take $Z_1 \in Z(\infty)_{-\nu_1}, Z_2 \in Z(\infty)_{-\nu_2}$. Then for $i = 1, 2$, we have $Z_i \subset \overline{S^0_+ \cap S^{-\nu_i}_-}$. For $\mu_1, \mu_2 \in P$, we can consider a family $S^{\mu_1}_\pm *_{\mathbb{A}} S^{\mu_2}_\pm$ in $\mathcal{G}r_{0,\mathbb{A}}$ whose fibres over U are isomorphic to $S^{\mu_1}_\pm \times S^{\mu_2}_\pm$ and whose fibre over 0 is isomorphic to $S^{\mu_1+\mu_2}_\pm$. We can define a family $Z_1 *_{\mathbb{A}} Z_2$ as the irreducible component of $\overline{S^0_+ *_{\mathbb{A}} S^0_+ \cap S^{-\nu_1}_- *_{\mathbb{A}} S^{-\nu_2}_-}$ whose fibres over U are isomorphic to $Z_1 \times Z_2$. The fibre of this family over 0, denoted $Z_1 *_0 Z_2$, is a subset of $\overline{S^0_+ \cap S^{-\nu_1-\nu_2}_-}$ and its top dimensional components are MV cycles. If these components are Z_α for α in some indexing set, then we will denote the fusion product of Z_1 and Z_2 as $Z_1 * Z_2 = \sum_\alpha Z_\alpha$.

The product of MV cycles Z_1, Z_2 corresponds to the product of b_{Z_1}, b_{Z_2} in $\mathbb{C}[N]$ [BKK19, Lemma 7.10]. For a precise description of the coefficients in this product we have the following result:

Theorem 1.3.2. *[BKK19, Theorem 7.11] Let* $Z_1 \in Z(\infty)_{-\nu_1}, Z_2 \in Z(\infty)_{-\nu_2}$. *Then*

$$b_{Z_1} b_{Z_2} = \sum_Z i(Z, \pi^{-1}(0) \cdot \overline{Z_1 \times Z_2 \times U}) b_Z$$

where the sum is over all $Z \in Z(\infty)_{-\nu_1-\nu_2}$, *and* $i(Z, D \cdot V)$ *denotes the intersection multiplicity of* Z *in* $D \cdot V$ *(see Section 7.5.2).*

In general, the fusion product is difficult to compute. However, with joint work with Dranowski and Kamnitzer [BDK21], we present a method of computing the product in type A via a generalization of the Mirković-Vybornov isomorphism [MV07b][MV19] in Chapter 7.

Chapter 2

Semicanonical Basis

In this chapter, we define the second basis of $\mathbb{C}[N]$ that we are concerned with. This was originally constructed by Lusztig in [Lus00]. We now assume that G is a group of type ADE. Let $\mathfrak{n} \subset \mathfrak{g}$ be the associated Lie algebras of $N \subset G$. Let n be the rank of G and r be the number of positive roots of G. Let $\mathcal{U}(\mathfrak{n})$ be the universal enveloping algebra of $\mathfrak{n}$. The semicanonical basis is actually a basis for $\mathcal{U}(\mathfrak{n})$, but dualizing it gives a basis for $\mathcal{U}(\mathfrak{n})^* \cong \mathbb{C}[N]$.

2.1 Preprojective Algebras

Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver of type ADE, where Q_0 denotes the set of vertices of Q, Q_1 the set of edges/arrows, and $s, t : Q_1 \to Q_0$ the source and target maps. For an edge $\alpha \in Q_1$, we will often draw it as $s(\alpha) \xrightarrow{\alpha} t(\alpha)$. Let $\overline{Q} = (Q_0, H = Q_1 \sqcup \overline{Q_1}, s, t)$ be the doubled quiver, where $\overline{Q_1} = \{\alpha^* : \alpha \in Q_1\}$ with $s(\alpha^*) = t(\alpha)$ and $t(\alpha^*) = s(\alpha)$. Define $\epsilon : H \to \{\pm 1\}$ by

$$\epsilon(\alpha) = \begin{cases} 1 & \text{if } \alpha \in Q_1 \\ -1 & \text{if } \alpha \in \overline{Q_1} \end{cases}.$$

If α_1 is a path in Q starting at i and ending at j, and α_2 is a path in Q starting at j and ending at k, we denote $\alpha_2\alpha_1$ as their concatenation, so a path starting at i and ending at k.

Definition 2.1.1. The **preprojective algebra** Λ is defined as the path algebra with relations

$$\Lambda := \mathbb{C}\overline{Q}/\left(\sum_{\alpha \in Q_1} (\alpha^*\alpha - \alpha^*\alpha)\right) = \mathbb{C}\overline{Q}/\left(\sum_{\alpha \in H} \epsilon(\alpha)\alpha\alpha^*\right).$$

A Λ-module M is a pair $((M_i)_{i \in Q_0}, (M_\alpha)_{\alpha \in H})$ where M_i is a vector space and $M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is a linear morphism. We will let Λ-mod denote the category of finite dimensional modules over Λ. Hence each simple, projective indecomposable, and injective indecomposable of Λ corresponds to a vertex in Q_0. Let S_i denote the simple Λ-module at vertex $i \in Q_0$ and P_i denote the projective(-injective) indecomposable Λ-module at vertex $i \in Q_0$. We define $\underline{\dim}\, M = \sum_{i \in Q_0} (\dim M_i)\alpha_i$.

Example 2.1.2. Suppose Q is an A_2 quiver. Then modules for the preprojective algebra associated to Q consist of representations of the quiver $\bullet \underset{\alpha^*}{\overset{\alpha}{\rightleftarrows}} \bullet$ subject to the relations $\alpha\alpha^* = 0$ and $\alpha^*\alpha = 0$. This algebra has exactly 4 indecomposables: $S_1 = \ \mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} 0$, $S_2 = \ 0 \underset{0}{\overset{0}{\rightleftarrows}} \mathbb{C}$, $P_1 = \ \mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C}$, and $P_2 = \ \mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C}$. We then have $\underline{\dim}\, S_1 = \alpha_1$, $\underline{\dim}\, S_2 = \alpha_2$, and $\underline{\dim}\, P_1 = \underline{\dim}\, P_2 = \alpha_1 + \alpha_2$.

We will need a few notions and results concerning Λ later in the thesis.

Definition 2.1.3. For $d = (d_i)_{i \in Q_0}, e = (e_i)_{i \in Q_0} \in \mathbb{Z}^{|Q_0|}$, define a symmetric bilinear form $(-,-)$ given by

$$(d, e) = 2 \sum_{i \in Q_0} d_i e_i - \sum_{\alpha \in Q_1} (d_{s(\alpha)} e_{t(\alpha)} + e_{s(\alpha)} d_{t(\alpha)}).$$

Lemma 2.1.1. *[CB00, Lemma 1] For Λ-modules X and Y we have*

$$\dim \operatorname{Ext}^1(X, Y) = \dim \operatorname{Hom}(X, Y) + \dim \operatorname{Hom}(Y, X) - (\underline{\dim}(X), \underline{\dim}(Y)).$$

In particular, $\dim \operatorname{Ext}^1(X, Y) = \dim \operatorname{Ext}^1(Y, X)$.

Theorem 2.1.2. *[GLS07c, Theorem 3] For finite dimensional Λ-modules X and Y, there is a functorial pairing*

$$\operatorname{Ext}^1(X, Y) \times \operatorname{Ext}^1(Y, X) \to \mathbb{C}.$$

In other words, there exists an isomorphism $\phi_{X,Y} : \operatorname{Ext}^1(X, Y) \xrightarrow{\sim} D \operatorname{Ext}^1(Y, X)$ *such that for* $\lambda \in \operatorname{Hom}(Y, Y')$, $[\eta] \in \operatorname{Ext}^1(X, Y), \rho \in \operatorname{Hom}(X', X), [\epsilon] \in \operatorname{Ext}^1(Y', X')$, *we have*

$$\phi_{X',Y'}(\lambda \circ [\eta] \circ \rho)([\epsilon]) = \phi_{X,Y}([\eta])(\rho \circ [\epsilon] \circ \lambda).$$

For a short exact sequence $\eta : 0 \to Y \to E \to X \to 0$, $\lambda \in \operatorname{Hom}(Y, Y'), \rho \in \operatorname{Hom}(X', X)$, $[\eta] \circ \rho$ is the short exact sequence obtained by pulling back $E \to X$ and ρ, while $\lambda \circ [\eta]$ is the short exact sequence obtained by pushing out $Y \to E$ and λ. Note that we have $(\lambda \circ [\eta]) \circ \rho = \lambda \circ ([\eta] \circ \rho)$.

The following result is considered classical. We will refer the reader to [DR92] and [GS03] for proofs.

Proposition 2.1.3. *For the preprojective algebra Λ, the following hold:*

1. *Λ is of finite representation type if and only if the underlying graph of Q is Dynkin of type A_n for $n \leq 4$.*

2. *Λ is of tame representation type if and only if the underlying graph of Q is Dynkin of type A_5 or D_4.*

Remark 2.1.4. Λ has exactly 4 indecomposable modules in type A_2, 12 in type A_3, and 40 in type A_4.

2.2 Semicanonical Basis

For a dimension vector $\vec{d} = (d_i)_{i \in Q_0}$, let $V_{\vec{d}} = \bigoplus_{i \in I} V_i$ be the graded vector space where $\dim V_i = d_i$. We define $\Lambda_{\vec{d}}$ to be the variety of all possible Λ-module structures on $V_{\vec{d}}$, which can be viewed as

$$\Lambda_{\vec{d}} = \left\{ M \in \Lambda\text{-mod} : \underline{\dim}\, M = \sum_{i \in Q_0} d_i \alpha_i \right\},$$

all $\vec{d}$-dimensional Λ-modules. This is acted on by $G_{\vec{d}} = \prod_{i\in Q_0} \mathbf{GL}_{d_i}$ via conjugation. If M is a Λ-module, we will let $O(M)$ denote its orbit under the associated group action. Note that two modules are in the same orbit if and only if they are isomorphic to each other.

Let $M(\Lambda_{\vec{d}})$ denote the set of constructible functions on $\Lambda_{\vec{d}}$ and $M(\Lambda_{\vec{d}})^{G_{\vec{d}}}$ denote those constructible functions that are constant on the orbits of $G_{\vec{d}}$. Let $\widetilde{\mathcal{M}} = \bigoplus_{\vec{d}} M(\Lambda_{\vec{d}})^{G_{\vec{d}}}$. We can turn this into a $\mathbb{N}^{|Q_0|}$-graded associative $\mathbb{C}$-algebra with an operation $*$ as follows. Let $f_1 \in M(\Lambda_{\vec{d_1}})$, $f_2 \in M(\Lambda_{\vec{d_2}})$, and let $\vec{d} = \vec{d_1} + \vec{d_2}$. For $M \in \Lambda_{\vec{d}}$, let V_M be the variety of all Λ-submodules of M of dimension $\vec{d_2}$. Define a function $\phi_{M,f_1,f_2} : V_M \to \mathbb{C}$ by

$$\phi_{M,f_1,f_2}(U) = f_1(M/U)f_2(U).$$

As in [Lus00, Section 2.1], define

$$(f_1 * f_2)(M) = \sum_{m\in\mathbb{C}} m\chi(\phi^{-1}_{M,f_1,f_2}(m) \cap V_M)$$

where χ is the Euler characteristic.

Example 2.2.1. Let $O_1 \subset \Lambda_{\vec{d_1}}$ and $O_2 \subset \Lambda_{\vec{d_2}}$ be $G_{\vec{d_1}}$- and $G_{\vec{d_2}}$-orbits. For $M \in \Lambda_{\vec{d}}$, define

$$\mathcal{F}(O_1, O_2, M) = \{U \in V_M : U \in O_2, M/U \in O_1\}.$$

Then we have

$$(1_{O_1} * 1_{O_2})(M) = \chi(\mathcal{F}(O_1, O_2, M))$$

where 1_X is the characteristic function on X.

Define $\mathcal{M}$ to be the subalgebra of $\widetilde{\mathcal{M}}$ generated by the functions 1_{Λ_i} where Λ_i denotes the variety consisting of the single point S_i. There is an algebra isomorphism $U(\mathfrak{n}) \to \mathcal{M}$ given by sending e_i to 1_{Λ_i}. Let $\mathcal{M}_{\vec{d}} = \mathcal{M} \cap M(\Lambda_{\vec{d}})^{G_{\vec{d}}}$. This has a basis

$$\{f_Z : Z \in Irr(\Lambda_{\vec{d}})\}$$

where $Irr(\Lambda_{\vec{d}})$ denotes the irreducible components of $\Lambda_{\vec{d}}$ [Lus00, Theorem 2.7]. The function f_Z is the unique element of $\mathcal{M}_{\vec{d}}$ that is equivalent to 1 on a dense open subset U_Z of Z and 0 on a dense open subset of all other irreducible components of $\Lambda_{\vec{d}}$. Take U_Z to be the maximal possible subset in Z on which f_Z equals 1. For some $Z \in Irr(\Lambda_{\vec{d}})$ and $M \in Z$, we call the module M **generic** if $M \in U_Z$.

The basis of $U(\mathfrak{n})$ corresponding to

$$\bigcup_{\vec{d}} \{f_Z : Z \in Irr(\Lambda_{\vec{d}})\}$$

is called the **semicanonical basis**. We then get a dual basis of $U(\mathfrak{n})^* \cong \mathbb{C}[N]$ called the dual semicanonical basis. While this basis is indexed by irreducible components, we will often associate the basis elements to a corresponding generic module.

A large class of generic modules are rigid modules. A module M is called **rigid** if $\mathrm{Ext}^1_\Lambda(M, M) = 0$.

Proposition 2.2.1. *[GLS06, Corollary 3.15] For a Λ-module M with dimension vector $\vec{d}$, the following are equivalent:*

- *The closure $\overline{O(M)}$ of $O(M)$ is an irreducible component of $\Lambda_{\vec{d}}$*

- *The orbit $O(M)$ is open in $\Lambda_{\vec{d}}$*
- *M is rigid*

In particular, rigid modules are generic.

Example 2.2.2. Consider the same setup as in Example 2.1.2. The modules S_1, S_2, P_1, and P_2 are all rigid, so also generic. In fact, we have $\Lambda_{(1,1)} = O(P_1) \sqcup O(P_2) \sqcup \{S_1 \oplus S_2\}$. The irreducible components of $\Lambda_{(1,1)}$ are $\overline{O(P_1)} = O(P_1) \sqcup \{S_1 \oplus S_2\}$ and $\overline{O(P_2)} = O(P_2) \sqcup \{S_1 \oplus S_2\}$. If we let $b_M \in \mathbb{C}[N] = \mathbb{C}[x_{12}, x_{13}, x_{23}]$ denote the dual semicanonical basis element corresponding to a generic module M, then $b_{S_1} = x_{12}, b_{S_2} = x_{23}, b_{P_1} = x_{12}x_{23} - x_{13}$, and $b_{P_2} = x_{13}$. Each element of the dual semicanonical basis is of the form $b_{S_1}^{\alpha} b_{P_1}^{\beta} b_{P_2}^{\gamma}$ or $b_{S_2}^{\alpha} b_{P_1}^{\beta} b_{P_2}^{\gamma}$ for some $\alpha, \beta, \gamma \in \mathbb{N}$.

Chapter 3

Polytopes

In this chapter we look at some combinatorics associated to MV cycles and Λ-modules.

3.1 Polytopes from MV Cycles

Recall that the action of the torus $T^\vee$ acting on $\mathcal{G}r$ have fixed points $\{t^\mu : \mu \in P_+\}$. For an MV cycle Z, we can associate to Z the unique polytope

$$\mathrm{Pol}(Z) := \mathrm{Conv}\{\mu : t^\mu \in Z\}$$

called its **MV polytope**. This polytope can also be defined in two different ways using certain combinatorial data.

3.1.1 BZ Datum

Let $\omega_1, \ldots, \omega_n$ denote the fundamental weights of $G^\vee$ and let $W = N(T^\vee)/T^\vee$ be the Weyl group of $G^\vee$, generated by the simple reflections $s_1, \ldots, s_n$. Let $w_0 \in W$ be the longest word. Let $I = \{1, \ldots, n\}$. For each $i \in I$, let $\psi_i : \mathrm{SL}_2 \to G^\vee$ denote the ith root subgroup of $G^\vee$. Define $\overline{s_i} = \psi_i\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) \in G^\vee$ to be the lift of the simple reflection s_i. This allows us to define the lift of $w = s_{i_1} \cdots s_{i_k} \in W$ as $\overline{w} = \overline{s_{i_1}} \cdots \overline{s_{i_k}}$. Let $C = (a_{ij})$ denote the Cartan matrix of G. Define

$$\Gamma = \bigcup_{w \in W, i \in I} w \cdot \omega_i$$

to be the chamber weights.

Let V be a vector space over $\mathbb{C}$. Note that the space $V \otimes \mathcal{K}$ has a filtration

$$\cdots \subset V \otimes t\mathcal{O} \subset V \otimes \mathcal{O} \subset V \otimes t^{-1}\mathcal{O} \subset \cdots.$$

Define $val : V \otimes \mathcal{K} \to \mathbb{Z}$ by $val(u) = k$ if $u \in V \otimes t^k\mathcal{O}$ and $u \notin V \otimes t^{k+1}\mathcal{O}$. Choose a highest weight vector v_{ω_i} in the fundamental representation $V(\omega_i)$ for $G^\vee$. For $\gamma = w \cdot \omega_i \in \Gamma$, let $v_\gamma = \overline{w} \cdot \omega_i$. Note that as $G^\vee$ acts on $V(\omega_i)$, $G^\vee(\mathcal{K})$ acts on $V(\omega_i) \otimes \mathcal{K}$. For $\gamma \in \Gamma$, define constructible functions $D_\gamma : \mathcal{G}r \to \mathbb{Z}$

by $D_\gamma([g]) = val(g \cdot v_\gamma)$.

Given a collection of integers $M_\bullet = \{M_\gamma\}_{\gamma\in\Gamma}$, consider the sets

$$A(M_\bullet) = \{L \in \mathcal{G}r : D_\gamma(L) = M_\gamma \text{ for all } \gamma \in \Gamma\}$$

and

$$P(M_\bullet) = \{x \in \mathfrak{t}_\mathbb{R} : \langle x, \gamma\rangle \geq M_\gamma \text{ for all } \gamma \in \Gamma\}.$$

Definition 3.1.1. [Kam10, Section 3.3] The integers $M_\bullet$ is called a **BZ datum** of weight (λ, μ) if

1. (Edge inequalities) For each $w \in W$ and $i \in I$,

$$M_{ws_i\cdot\omega_i} + M_{w\cdot\omega_i} + \sum_{j\neq i} a_{ji} M_{w\cdot\omega_j} \leq 0$$

2. (Tropical Plücker relations) For $w \in W$, $i, j \in I$ with $ws_i > w$, $ws_j > w$, $i \neq j$, and $a_{ij} = a_{ji} = -1$,

$$M_{ws_i\cdot\omega_i} + M_{ws_j\cdot\omega_j} = \min(M_{w\cdot\omega_i} + M_{ws_is_j\cdot\omega_j}, M_{ws_js_i\cdot\omega_i} + M_{w\cdot\omega_j})$$

3. $M_{\omega_i} = \langle\lambda, \omega_i\rangle$ and $M_{w_0\cdot\omega_i} = \langle\mu, w_0 \cdot \omega_i\rangle$ for all i.

Remark 3.1.2. The tropical Plücker relations for groups that are not simply-laced are more complicated and can be found in [Kam10].

Theorem 3.1.1. *[Kam10, Theorem 3.1] Let $M_\bullet$ be a BZ datum of weight (λ, μ). Then $\overline{A(M_\bullet)}$ is an MV cycle of type λ and weight μ, and each MV cycle arises this way for a unique BZ datum $M_\bullet$. The associated MV polytope is equal to $P(M_\bullet)$.*

From [Kam10, Proposition 2.2], the vertices of $P(M_\bullet)$ are $\{\mu_w\}_{w\in W}$ which are given by

$$\mu_w = \sum_i M_{w\cdot\omega_i} w \cdot \alpha_i.$$

If $M_\bullet$ is a BZ datum of weight (λ, μ), then the lowest vertex of $P(M_\bullet)$ is $\mu_e = \lambda$ and the highest vertex is $\mu_{w_0} = \mu$.

If Z_1 and Z_2 are MV cycles in the same equivalence class, that is, $Z_1 = t^\lambda Z_2$ for some $\lambda \in P$, then their polytopes differ by translation by λ. When working with stable MV cycles, we will use the polytope whose lowest vertex is the origin, and we call such polytopes stable MV polytopes.

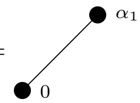
Example 3.1.3. Consider the MV cycles in Example 1.2.1. The chamber weights are

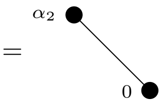
$$\Gamma = \{\omega_1 = (1,0,0), \omega_2 = (1,1,0), s_1\omega_1 = (0,1,0), s_2\omega_2 = (1,0,1), s_2s_1\omega_1 = (0,0,1), s_1s_2\omega_2 = (0,1,1)\}$$

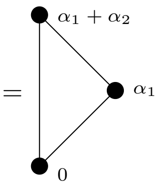
and the BZ data for the stable MV cycles are

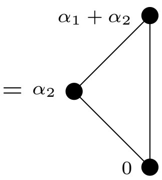
	M_{ω_1}	M_{ω_2}	$M_{s_1\omega_1}$	$M_{s_2\omega_2}$	$M_{s_2s_1\omega_1}$	$M_{s_1s_2\omega_2}$
Z_1	0	0	-1	0	0	-1
Z_2	0	0	0	-1	-1	0
Z_+	0	0	-1	0	-1	-1
Z_-	0	0	0	-1	-1	-1

This gives the stable MV polytopes

$\mathrm{Pol}(Z_1) = \mathrm{Conv}\{0, \alpha_1\} =$ 

$\mathrm{Pol}(Z_2) = \mathrm{Conv}\{0, \alpha_2\} =$ 

$\mathrm{Pol}(Z_+) = \mathrm{Conv}\{0, \alpha_1, \alpha_1+\alpha_2\} =$ 

$\mathrm{Pol}(Z_-) = \mathrm{Conv}\{0, \alpha_2, \alpha_1+\alpha_2\} =$ 

Moreover, every other stable MV polytope in type A_2 is a Minkowski sum of $\mathrm{Pol}(Z_1)$, $\mathrm{Pol}(Z_+)$, and $\mathrm{Pol}(Z_-)$, or of $\mathrm{Pol}(Z_2)$, $\mathrm{Pol}(Z_+)$, and $\mathrm{Pol}(Z_-)$.

3.1.2 Lusztig Datum

Let $\mathbf{i} = (i_1, \ldots, i_r)$ be a reduced expression of w_0. For each $1 \le k \le r$, define $w_k^{\mathbf{i}} = s_{i_1} \cdots s_{i_k}$ and $w_0^{\mathbf{i}} = e$. For $1 \le k \le r$, define the positive root $\beta_k^{\mathbf{i}} = w_{k-1}^{\mathbf{i}} \alpha_{i_k}$. Every MV polytope P has a unique path in its 1-skeleton that goes through its vertices $\mu_e, \mu_{w_1^{\mathbf{i}}}, \ldots, \mu_{w_r^{\mathbf{i}}} = \mu_{w_0}$. Let $n_1, \ldots, n_r$ denote the lengths of the edges in this path. We call the sequence $n_\bullet = (n_1, \ldots, n_r)$ the **i-Lusztig** datum for P. Note that the vector between vertices $\mu_{w_{k-1}^{\mathbf{i}}}$ and $\mu_{w_k^{\mathbf{i}}}$ is a multiple, including zero multiple, of $\beta_k^{\mathbf{i}}$. Hence if our MV polytope is stable, then the vertices in the distinguished path can be calculated using the **i**-Lusztig datum: we have $\mu_e = 0$ and for $1 \le k \le r$,

$$\mu_{w_k^{\mathbf{i}}} = \sum_{i=1}^{k} n_k \beta_k^{\mathbf{i}}.$$

We can obtain the BZ datum for a stable MV polytope through its Lusztig datum and vice-versa as well. We call a chamber weight $\gamma \in \Gamma$ an **i**-chamber weight if $\gamma = w_k^{\mathbf{i}} \cdot \omega_j$ for some k and j. Let $\Gamma^{\mathbf{i}}$ be the set of all **i**-chamber weights and let $\gamma_k^{\mathbf{i}} = w_k^{\mathbf{i}} \cdot \omega_{i_k}$. From [Kam10, Section 4.3], the relationship between the BZ datum $M_\bullet$ and **i**-Lusztig datum $n_\bullet$ for a stable MV polytope is that

$$M_{\gamma_k^{\mathbf{i}}} = \sum_{l \le k} \langle \beta_k^{\mathbf{i}}, \gamma_k^{\mathbf{i}} \rangle n_l$$

and

$$n_k = -M_{w_{k-1}^{\mathbf{i}} \cdot \omega_{i_k}} - M_{w_k^{\mathbf{i}} \cdot \omega_{i_k}} - \sum_{j \neq i_k} a_{j,i_k} M_{w_k^{\mathbf{i}} \cdot \omega_j}.$$

Note that since we are dealing with a stable MV polytope here, we always have $M_{\omega_i} = 0$ for all $i \in I$, which is needed in calculating the **i**-Lusztig datum.

Although to calculate the entire BZ datum we would need to use different reduced expressions for w_0, we can obtain the associated MV cycle from just one reduced expression $\mathbf{i}$. Take $M_{\omega_i} = 0$ for all $i \in I$ and let $M_{\gamma_k^{\mathbf{i}}}$ be the integers calculated from the **i**-Lusztig datum $n_\bullet$. Noting that $\Gamma^{\mathbf{i}} = \{\omega_i\}_{i=1}^n \cup \{\gamma_k^{\mathbf{i}}\}_{i=1}^r$, consider the set

$$A^{\mathbf{i}}(n_\bullet) = \{L \in \mathcal{G}r : D_\gamma(L) = M_\gamma \text{ for all } \gamma \in \Gamma^{\mathbf{i}}\}.$$

Theorem 3.1.2. *[Kam10, Theorem 4.2] For each* **i***-Lusztig datum* $n_\bullet$*,* $\overline{A^{\mathbf{i}}(n_\bullet)}$ *is an MV cycle and each*

MV cycle arises this way for a unique **i***-Lusztig datum.*

Example 3.1.4. Consider the setup in Example 3.1.3. For the reduced expression $\mathbf{i} = (1, 2, 1)$, we obtain

$$\beta_1^{\mathbf{i}} = \alpha_1,\ \beta_2^{\mathbf{i}} = \alpha_1 + \alpha_2,\ \beta_3^{\mathbf{i}} = \alpha_2.$$

Since each of these have length one, we see we have the following **i**-Lusztig data:

	$\mathrm{Pol}(Z_1)$	$\mathrm{Pol}(Z_2)$	$\mathrm{Pol}(Z_+)$	$\mathrm{Pol}(Z_-)$
$n_\bullet$	(1,0,0)	(0,0,1)	(1,0,1)	(0,1,0)

3.2 Polytopes from Λ-Modules

As in [BKT14, Section 1.3], we define the polytope of a Λ-module M to be

$$\mathrm{Pol}(M) := \mathrm{Conv}\left\{\underline{\dim}\, N : N \subset M \text{ is a submodule}\right\}.$$

This polytope was shown to be an MV polytope in [BK10].

Theorem 3.2.1. *[BK10, Theorem 5] There exists a family of Λ-modules $N(\gamma)$ indexed by weights $\gamma \in P$ characterized up to isomorphism by the following properties:*

- *If γ is antidominant, then $N(\gamma) = 0$.*
- *Let $i \in I$ and γ be a W-conjugate of $-\omega_i$ with $\gamma \neq -\omega_i$. Then $N(\gamma)$ satisfies*

$$\underline{\dim}\, N(\gamma) = \gamma + \omega_i \text{ and } \mathrm{soc}\, N(\gamma) \cong S_i.$$

For a Λ-module M and a weight $\gamma \in P$, define $D_\gamma(M) = -\dim \mathrm{Hom}_\Lambda(N(\gamma), M)$.

Theorem 3.2.2. *[BK10, Proposition 12, Remark 14(i), Theorem 19] If M is a generic Λ-module, then the collection of integers $(D_\gamma(M))_{\gamma\in\Gamma}$ is a BZ datum.*

Then another formulation of $\mathrm{Pol}(M)$ is

$$\mathrm{Pol}(M) = \mathrm{Conv}\left\{x \in \mathfrak{t}_{\mathbb{R}} : \langle x, \gamma\rangle \geq D_\gamma(M) \text{ for all } \gamma \in \Gamma\right\},$$

which we now know is a stable MV polytope by the previous theorem.

Example 3.2.1. Consider the situation in Example 2.1.2. The non-zero modules $N(\gamma)$ are $N(-s_1\omega_1) = S_1$, $N(-s_2\omega_2) = S_2$, $N(-s_2s_1\omega_1) = P_2 =: P_+$, and $N(-s_1s_2\omega_2) = P_1 =: P_-$. For each indecomposable Λ-module, we have the D_γ values to be:

	$D_{-\omega_1}$	$D_{-\omega_2}$	$D_{-s_1\omega_1}$	$D_{-s_2\omega_2}$	$D_{-s_2s_1\omega_1}$	$D_{-s_1s_2\omega_2}$
S_1	0	0	-1	0	0	-1
S_2	0	0	0	-1	-1	0
P_+	0	0	-1	0	-1	-1
P_-	0	0	0	-1	-1	-1

Comparing this with Example 3.1.3, we see that $\mathrm{Pol}(Z_1) = \mathrm{Pol}(S_1)$, $\mathrm{Pol}(Z_2) = \mathrm{Pol}(S_2)$, $\mathrm{Pol}(Z_+) = \mathrm{Pol}(P_+)$, and $\mathrm{Pol}(Z_-) = \mathrm{Pol}(P_-)$.

Given an MV polytope, we can also construct a generic module using its **i**-Lusztig datum [BKT14, Theorem 5.11]. For each $i \in I$, define an endofunctor Σ_i on Λ-mod as follows. For a Λ-module $M = ((M_i), (M_\alpha))$, we have the following diagram at vertex i:

$$\bigoplus_{\alpha \in H, t(\alpha)=i} M_{s(\alpha)} \xrightarrow{(\epsilon(\alpha)M_\alpha)} M_i \xrightarrow{(M_{\alpha^*})} \bigoplus_{\alpha \in H, t(\alpha)=i} M_{s(\alpha)}.$$

To simplify the notation, we shall rewrite this diagram as

$$\tilde{M}_i \xrightarrow{M_{in(i)}} M_i \xrightarrow{M_{out(i)}} \tilde{M}_i$$

Note that the preprojective relation implies we have $M_{in(i)} \circ M_{out(i)} = 0$, so $\operatorname{im} M_{out(i)} \subset \ker M_{in(i)}$. Let $\overline{M}_{out(i)} : M_i \to \ker M_{in(i)}$ be induced by $M_{out(i)}$. We then replace the above diagram with

$$\tilde{M}_i \xrightarrow{\overline{M}_{out(i)} M_{in(i)}} \ker M_{in(i)} \hookrightarrow \tilde{M}_i$$

to obtain the module $\Sigma_i M$.

Given a reduced expression $\mathbf{i} = (i_1, \ldots, i_r)$ of w_0, by [BKT14, Proposition 7.4], we obtain a sequence of indecomposable modules

$$B_1^{\mathbf{i}} = S_{i_1}, B_2^{\mathbf{i}} = \Sigma_{i_1} S_{i_2}, \ldots, B_r^{\mathbf{i}} = \Sigma_{i_1} \Sigma_{i_2} \cdots \Sigma_{i_{r-1}} S_{i_r}.$$

Note that we also have $\underline{\dim}\, B_k^{\mathbf{i}} = \beta_k^{\mathbf{i}}$ for each $1 \leq k \leq r$. For a generic Λ-module M, let $n_\bullet = (n_1, \ldots, n_r)$ be the **i**-Lusztig datum for $\mathrm{Pol}(M)$. We then obtain a filtration

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_r = M$$

where $F_i/F_{i-1} \cong (B_i^{\mathbf{i}})^{\oplus n_i}$. Moreover, up to isomorphism, there is a unique generic Λ-module with such a filtration [BKT14, Theorem 4.4].

Example 3.2.2. Consider the situation in Example 3.1.4. With $\mathbf{i} = (1, 2, 1)$, we have

$$B_1^{\mathbf{i}} = S_1, B_2^{\mathbf{i}} = P_-, B_3^{\mathbf{i}} = S_2.$$

From the table in Example 3.1.4, the only one that gives a non-trivial filtration is $(1, 0, 1)$, which corresponds to a module of dimension $\alpha_1 + \alpha_2$ with S_1 as a submodule. There are exactly two modules that satisfy this: $S_1 \oplus S_2$ and P_+. Of these, only P_+ is generic, so the **i**-Lusztig datum $(1, 0, 1)$ corresponds to P_+.

Chapter 4

Cluster Algebras

In this chapter we define cluster algebras and look at how they are related to the various objects we defined earlier.

4.1 Background

Fix integers $1 \leq m \leq n$. Let $\mathcal{F}$ be the field of rational functions over $\mathbb{Q}$ in n variables. A **seed** of $\mathcal{F}$ is a pair $\Sigma = (\mathbf{x}, Q)$ where $\mathbf{x} = (x_1, \ldots, x_n)$ is a free generating set of $\mathcal{F}$ and Q is a quiver with vertices Q_0 labelled by $\{1, \ldots, n\}$. The variables $x_1, \ldots, x_m$ are called **mutable** and the variables $x_{m+1}, \ldots, x_n$ are called **frozen**. We assume Q has no loops and no 2-cycles, and that there are no edges between the vertices $m+1, \ldots, n$. Note that instead of a quiver, we can use a $n \times m$ matrix $B = (b_{ij})$ where

$$b_{ij} = \#\{\text{arrows } i \to j\} - \#\{\text{arrows } j \to i\}.$$

This matrix is called the **exchange matrix**.

For each $1 \leq k \leq m$, we define a new seed $\mu_k(\Sigma) = (\mu_k(\mathbf{x}), \mu_k(Q))$ called the **mutation of Σ in direction k**. We have $\mu_k(\mathbf{x}) = (x_1, \ldots, x_k^*, \ldots, x_n)$ where

$$x_k^* = \frac{\prod_{i \to k} x_i + \prod_{k \to i} x_i}{x_k} = \frac{\prod_{b_{ik}>0} x_i + \prod_{b_{ik}<0} x_i}{x_k}.$$

These mutation equations are called **exchange relations**. The quiver $\mu_k(Q)$ is constructed from Q by

1. adding a new arrow $i \to j$ for each pair of arrows $i \to k$ and $k \to j$;
2. reversing the orientation of every arrow with target or source k;
3. and erasing every pair of opposite arrows.

In terms of matrices, we obtain a new matrix $\mu_k(B) = (b'_{ij})$ where

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \operatorname{sgn}(b_{ik}) \max\{0, b_{ik}b_{kj}\} & \text{otherwise} \end{cases}$$

Note that $\mu_k(\mu_k(\Sigma)) = \Sigma$.

If we have a seed $\Sigma = (\mathbf{x}, Q)$, then $\mathbf{x} = \{x_1, \dots, x_n\}$ is called a **cluster** and its elements are called **cluster variables**. The **cluster monomials** are the elements that are a product of cluster variables in a fixed cluster. The **cluster algebra** is defined as the subalgebra generated by all cluster variables.

Fix a seed $\Sigma = (\mathbf{x} = \{x_1, \dots, x_n\}, B = (b_{ij}))$ with $x_1, \dots, x_m$ the mutable variables, $x_{m+1}, \dots, x_n$ the frozen variables, and let $Q = (Q_0, Q_1, s, t)$ be the quiver corresponding to B. For $1 \leq j \leq m$, define polynomials

$$y_j = \prod_{i \in Q_0} x_i^{-b_{ij}}.$$

Proposition 4.1.1. *[FZ07, Proposition 7.8] If B has full rank, then every cluster monomial z can be written uniquely as*

$$z = R(y_1, \dots, y_m) \prod_{j=1}^{n} x_j^{g_j}$$

*where R is primitive, that is, R is a ratio of two polynomials, neither of which divisible by any y_j. In this case we define the g-**vector** of z to be $g^{\Sigma}(z) = (g_1, \dots, g_n)$.*

The following result shows that these g-vectors determine cluster monomials and also obey a mutation rule (note the sign differences due to a different convention).

Theorem 4.1.2. *[DWZ10, Nag13] Let $\Sigma = (\mathbf{x}, B)$ be a seed. Then different cluster monomials have different g-vectors. Furthermore, suppose z is a cluster monomial with $g^{\Sigma}(z) = (g_1, \dots, g_n)$. Assume Σ is mutable at vertex k and that $g^{\mu_k(\Sigma)}(z) = (g_1', \dots, g_n')$. Then*

$$g_j' = \begin{cases} -g_k & \text{if } j = k \\ g_j + \max(-B_{jk}, 0) g_k + B_{jk} \min(g_k, 0) & \text{if } j \neq k \end{cases}.$$

4.2 Cluster Structure for $\mathbb{C}[N]$

As before, let n be the rank of G and r be the number of positive roots. For each $L(\lambda)$, choose a highest weight vector v_λ. In [BZ97], the authors defined for each fundamental weight ω_i and for each $w \in W$, the generalized minor

$$\Delta_{\omega_i, w(\omega_i)} = \overline{w} \cdot v_{\omega_i}.$$

If $G = SL_{n+1}(\mathbb{C})$, then the generalized minors are ordinary minors of a matrix. In this case, we have $\Delta_{\omega_i, w(\omega_i)}$ to be the minor whose rows, respectively columns, are labelled by $\{1, .., i\}$, respectively $\{w(1), \dots, w(i)\}$.

The following is a construction of an initial seed for $\mathbb{C}[B_+ \cap B_- \overline{w}_0 B_-]$ [BFZ05]. Let Q be a quiver for the Dynkin diagram corresponding to G. Let $\mathbf{i} = (i_1, \dots, i_r)$ be a reduced expression for $w_0 \in W$ adapted to Q. This means that i_1 is a source in Q, and for each $1 \leq k \leq r-1$, i_{k+1} is a source in $s_{i_k} \cdots s_{i_1}(Q)$ where $s_i(Q)$ is the quiver obtained from Q by reversing each arrow starting at i. Let $k \in \{-n, \dots, -1\} \cup \{1, \dots, r\}$ and let

$$v_{>k} = \begin{cases} s_{i_r} s_{i_{r-1}} \cdots s_{i_{k+1}} & \text{if } k \geq 1 \\ w_0 & \text{if } k \leq -1 \end{cases}.$$

Set

$$\Delta(k, \mathbf{i}) = \Delta_{\omega_{|i_k|}, v_{>k}(\omega_{|i_k|})}$$

where $i_k = -k$ if $k \leq -1$. Define

$$k^+ = \begin{cases} r+1 & \text{if } |i_\ell| \neq |i_k| \text{ for all } \ell > k \\ \min\{\ell : \ell > k \text{ and } |i_\ell| = |i_k|\} & \text{otherwise} \end{cases}.$$

We say k is $\mathbf{i}$-exchangeable if $k, k^+ \in \{1, .., r\}$. Let $e(\mathbf{i})$ denote the set of all $\mathbf{i}$-exchangeable elements. Note that $e(\mathbf{i})$ contains exactly $r-n$ elements. Define a quiver $\tilde{A}_{\mathbf{i}}$ with vertices $\{-n, \ldots, -1\} \cup \{1, \ldots, r\}$. For two vertices k, ℓ in $\tilde{A}_{\mathbf{i}}$ such that $k < \ell$ and $\{k, \ell\} \cap e(\mathbf{i}) \neq \emptyset$, we have an arrow $k \to \ell$ if $k^+ = \ell$, and an arrow $\ell \to k$ if $\ell < k^+ < \ell^+$ and $a_{|i_k|,|i_\ell|} = -1$ where $a_{|i_k|,|i_\ell|}$ is the corresponding entry in the Cartan matrix of G.

Theorem 4.2.1. *[BFZ05, Theorem 2.10] We have*

$$\left\{\{\Delta(k, \mathbf{i}) : k \in \{-n, \ldots, -1\} \cup \{1, \ldots, r\}\}, \tilde{A}_{\mathbf{i}}\right\}$$

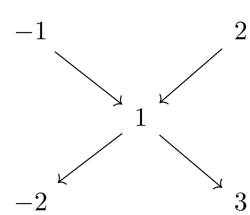
to be an initial seed for $\mathbb{C}[B_+ \cap B_- \overline{w}_0 B_-]$. *The mutable variables are those* $\Delta(k, \mathbf{i})$ *with* $k \in e(\mathbf{i})$.

In the case when G is simply-laced, the cluster algebra obtained is independent of the choice of the reduced expression $\mathbf{i}$ [BFZ05, Remark 2.14]. In [GLS07a, Section 4.4], it is shown that if we remove the variables in the initial seed corresponding to the n non-$\mathbf{i}$-exchangeable elements in $\{1, \ldots, r\}$, remove the corresponding vertices in $\tilde{A}_{\mathbf{i}}$, and restrict the generalized minors $\Delta(k, \mathbf{i})$ to N, then we obtain an initial seed for $\mathbb{C}[N]$, also independent of the choice of reduced expression $\mathbf{i}$. In particular, each cluster has r cluster variables, n of them being frozen.

Example 4.2.1. Let $G = SL_3$ and $\mathbf{i} = (1, 2, 1)$. Then we have the following pieces of data:

k	-2	-1	1	2	3
i_k	-2	-1	1	2	1
$v_{>k}$	w_0	w_0	$s_1 s_2$	s_1	e
$\Delta(k, \mathbf{i})$	$\begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix}$	x_{13}	x_{12}	$\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$	x_{11}

We have $e(\mathbf{i}) = \{1\}$ and the quiver $\tilde{A}_{\mathbf{i}}$ is



so the matrix $\tilde{B}_{\mathbf{i}}$ associated to $\tilde{A}_{\mathbf{i}}$ is $\tilde{B}_{\mathbf{i}} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ where the column is indexed by $\{1\}$ and the rows are indexed by $\{-2,-1,1,2,3\}$. The non-exchangeable elements in $\{1,2,3\}$ are $\{2,3\}$ so the initial seed for $\mathbb{C}[N]$ that we obtain is

$$\Sigma = \left(\left\{ \begin{vmatrix} x_{12} & x_{13} \\ 1 & x_{23} \end{vmatrix}, x_{13}, x_{12}, \right\}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) = \left(\{x_{12}, x_{13}, x_{12}x_{23} - x_{13}\}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right).$$

Note that this seed has only 1 exchangeable variable, x_{12}, and the other variables are frozen. Mutating in direction x_{12}, we get the exchange relation

$$x_{12}^* = \frac{x_{13} + (x_{12}x_{23} - x_{13})}{x_{12}} = x_{23}.$$

We then obtain the seed

$$\mu_{x_{12}}(\Sigma) = \left(\{x_{23}, x_{13}, x_{12}x_{23} - x_{13}\}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right)$$

and these are all the seeds in the cluster algebra. In fact, the seed $\mu_{x_{12}}(\Sigma)$ would be the initial seed if we instead took $\mathbf{i} = (2,1,2)$. Notice that each cluster monomial is of the form $x_{12}^{\alpha} x_{13}^{\beta} (x_{12}x_{23} - x_{13})^{\gamma}$ or $x_{23}^{\alpha} x_{13}^{\beta} (x_{12}x_{23} - x_{13})^{\gamma}$ for some $\alpha, \beta, \gamma \in \mathbb{N}$. Comparing this with Examples 1.2.1 and 2.2.2, we see that each cluster monomial is in both the MV basis and the dual semicanonical basis.

To determine the g-vectors with respect to Σ, we have the polynomial

$$y_1 = x_{13}^{-1}(x_{12}x_{23} - x_{13}).$$

The g-vectors of the cluster variables in Σ correspond to the standard basis vectors, so

$$g^{\Sigma}(x_{12}) = (1,0,0);\ g^{\Sigma}(x_{13}) = (0,1,0);\ g^{\Sigma}(x_{12}x_{23} - x_{13}) = (0,0,1).$$

We have

$$x_{23} = (y_1 + 1)x_{12}^{-1}x_{13}$$

so $g^{\Sigma}(x_{23}) = (-1,1,0)$, which can also be calculated from the mutation rule for g-vectors, knowing that $g^{\mu_{x_{12}}(\Sigma)}(x_{23}) = (1,0,0)$.

Remark 4.2.2. As we just saw, in type A_2, there are exactly 4 cluster variables. There are 12 in type A_3, 40 in type A_4, and infinitely many otherwise. This matches the number of indecomposables in Λ for the corresponding type.

4.3 Cluster Structure for Λ-mod

We now give a categorification of the cluster structure of $\mathbb{C}[N]$ via Λ-mod. This was constructed in [GLS06] to show that the cluster variables in $\mathbb{C}[N]$ are also in the dual semicanonical basis.

For a Λ-module M, define add(M) to be the full subcategory of Λ-mod consisting of all modules isomorphic to direct summands of finite direct sums of copies of M. We say a rigid module T is **maximal** if for every indecomposable T' such that $T \oplus T'$ is rigid, $T' \in \mathrm{add}(T)$.

Let $\Sigma(M)$ be the number of isomorphism classes of indecomposable direct summands of M.

Theorem 4.3.1. *[GS05, Corollary 1.2] For every rigid Λ-module T, $\Sigma(T) \le r$.*

A rigid module T is called **complete** if $\Sigma(T) = r$. Note that complete implies maximal.

An additive full subcategory $\mathcal{T}$ of Λ-mod is called **maximal 1-orthogonal** if for every module M, the following are equivalent:

- $M \in \mathcal{T}$
- $Ext^1(M,T) = 0$ for all $T \in \mathcal{T}$
- $Ext^1(T,M) = 0$ for all $T \in \mathcal{T}$.

We say a module T is **maximal 1-orthogonal** if $\mathrm{add}(T)$ is maximal 1-orthogonal. Note that a maximal 1-orthogonal T is a generator-cogenerator of Λ-mod, that is, T has all indecomposable projective and injective modules as direct summands. Since all projectives are also injectives in Λ, this is equivalent to saying T has $P_1, \ldots, P_n$ as direct summands.

Theorem 4.3.2. *[GLS06, Theorem 2.2] Let T be a Λ-module. The following are equivalent:*

- *T is maximal rigid*
- *T is complete rigid*
- *T is maximal 1-orthogonal*

Let $T = T_1 \oplus \cdots \oplus T_r$ be a basic complete rigid Λ-module where each T_i is indecomposable. Without loss of generality, $T_{r-n+1}, \ldots, T_r$ are projective. We can define a matrix $C_T = (c_{ij})$ where

$$c_{ij} = \dim \mathrm{Hom}_\Lambda(T_i, T_j).$$

Let $B(T) = (b_{ij})$ be the matrix obtained from $-C_T^{-t}$, the inverse transpose of $-C_T$, after removing the last n columns. For each $1 \le k \le r-n$, it was shown in [GLS06, GLS12] that there exists an indecomposable rigid Λ-module T_k^* with:

- $\dim \mathrm{Ext}^1_\Lambda(T_k, T_k^*) = \dim \mathrm{Ext}^1_\Lambda(T_k^*, T_k) = 1$,
- $T_k^* \oplus T/T_k$ is again a basic complete rigid Λ-module,
- and the non-split short exact sequences with end terms T_k, T_k^* are of the form

$$0 \to T_k \to \bigoplus_{b_{ik}>0} T_i^{b_{ik}} \to T_k^* \to 0$$

$$0 \to T_k^* \to \bigoplus_{b_{ik}<0} T_i^{b_{ik}} \to T_k \to 0.$$

Define the mutation of T in direction T_k to be $\mu_{T_k}(T) = T/T_k \oplus T_k^*$.

Theorem 4.3.3. *[GLS06, Theorem 2.6] For a basic complete rigid Λ-module T, $1 \le k \le r-n$, we have*

$$B(\mu_{T_k}(T)) = \mu_k(B(T)).$$

We see that Λ-mod is a categorification of the cluster structure on $\mathbb{C}[N]$, where the clusters correspond to a basic maximal rigid module T, the cluster variables correspond to the indecomposable summands of T, the frozen variables correspond to the indecomposable projectives, the cluster monomials correspond to elements in $\mathrm{add}(T)$, the exchange matrix is $B(T)$, and the exchange relations correspond to two non-split short exact sequences. We will call the basic maximal rigid modules **seeds**, the indecomposable summands of T **cluster modules**, and the non-split short exact sequences **exchange sequences**.

The initial seed for this cluster structure is constructed in [GLS07a] which we will present below. Let $Q = (Q_0 = \{1, \ldots, n\}, Q_1, s, t)$ be a quiver for the Dynkin diagram corresponding to G. We form another quiver $\mathbb{Z}Q = \{\mathbb{Z}Q_0, \mathbb{Z}Q_1, s, t\}$ where $\mathbb{Z}Q_0 = \mathbb{Z} \times Q_0$, $\mathbb{Z}Q_1 = \mathbb{Z} \times \overline{Q_1}$, and for $(i, \alpha), (i, \alpha^*) \in \mathbb{Z}Q_1$, we have

$$\begin{array}{cc} s(i,\alpha) = (i+1, s(\alpha)) & s(i,\alpha^*) = (i, t(\alpha)) \\ t(i,\alpha) = (i, t(\alpha)) & t(i,\alpha^*) = (i, s(\alpha)). \end{array}$$

Let $I \subset \mathbb{C}[\mathbb{Z}Q]$ be the ideal generated by the relations

$$\sum_{\alpha \in Q_1, s(\alpha)=q} (i,\alpha^*)(i,\alpha) - \sum_{\alpha \in Q_1, t(\alpha)=q} (i,\alpha)(i+1,\alpha^*)$$

for each $(i, q) \in \mathbb{Z}Q_0$, and let $\tilde{\Lambda} = \mathbb{C}[\mathbb{Z}Q]/I$, known as the universal, or Galois, cover of Λ. There is a natural covering functor $F : \tilde{\Lambda} \to \Lambda$ given by $(i, q) \mapsto q$ which extends to a push-down functor $F : \tilde{\Lambda}\text{-mod} \to \Lambda\text{-mod}$ where for $q \in Q_0$, $(FM)_q = \bigoplus_{i \in \mathbb{Z}} M_{(i,q)}$.

We have a translation automorphism τ on $\tilde{\Lambda}$ defined by $\tau(i, q) = (i + 1, q)$ for $(i, q) \in \mathbb{Z}Q_0$. We also have a Nakayama permutation ν on $\tilde{\Lambda}$ defined by

$$\nu(i,q) = \begin{cases} (i+q-1+\ell(\mu(q))-\ell(q), \mu(q)) & \text{in type } A_n \\ (i+n-2+\ell(\mu(q))-\ell(q), \mu(q)) & \text{in type } D_n \\ (p+q+2+\ell(\mu(q))-\ell(q), \mu(q)) & \text{in type } E_6 \text{ and } q \leq 5 \\ (p+5, \mu(q)) & \text{in type } E_6 \text{ and } q = 6 \\ (p+8, \mu(q)) & \text{in type } E_7 \\ (p+14, \mu(q)) & \text{in type } E_8 \end{cases}$$

where $\ell(q)$ is the number of arrows pointing towards 1 on the unique path from 1 to q in Q, and

$$\mu(q) = \begin{cases} n+1-q & \text{in type } A_n \\ 2n-1-q & \text{in type } D_n \text{ with } n \text{ odd and } q \geq n-1 \\ 6-q & \text{in type } E_6 \text{ and } q \leq 5 \\ q & \text{otherwise.} \end{cases}$$

Define a function $N : Q_0 \to \mathbb{N}$ by the property that $\tau^{N(q)}(0, q) = \nu(0, \mu(q))$. Let Γ_Q be the full subquiver of $\tilde{\Lambda}$ which has vertices satisfying the equation

$$(i,q) = \tau^{i-N(q)}\nu(0,\mu(q))$$

where $q \in Q_0$ and $0 \leq i \leq N(q)$. This is equivalent to taking the full subquiver between two copies of

Q^{op} in $\mathbb{Z}Q$ obtained by the embeddings $q \mapsto (0,q)$ and $q \mapsto \nu(0,q)$.

Remark 4.3.1. If we restrict the translation functor τ to Γ_Q, then we obtain the Auslander—Reiten quiver for $\mathbb{C}Q$.

We have a total ordering

$$x(-n) < x(-n+1) < \cdots < x(-1) < x(1) < \cdots < x(n-r)$$

of the vertices in Γ_Q defined by $x(i) < x(j)$ if there are no paths in Γ_Q from $x(i)$ to $x(j)$. Then $\mathbf{i} = (F(x(-n)), \ldots, F(x(-1)), F(x(1)), \ldots, F(x(n-r)))$ will be a reduced expression for $w_0 \in W$ adapted to Q.

For each vertex $(i,q) \in \Gamma_Q$, let $I_{(i,q)}$ be the injective indecomposable in $\mathbb{C}\Gamma_Q$. Viewing $\mathbb{C}\Gamma_Q$-mod $\subset \tilde{\Lambda}$-mod we then have

$$T = \bigoplus_{(i,q)\in\Gamma_Q} FI_{(i,q)}$$

to be a basic maximal rigid Λ-module with the projective indecomposables to be $\{FI_{(0,q)} : q \in Q_0\}$. The seed $(\{FI_{(i,q)}\}_{(i,q)\in\Gamma_Q}, B(T))$ will correspond to the seed in $\mathbb{C}[N]$ obtained from $\mathbf{i}$ where the correspondence is $\Delta(k,\mathbf{i}) \leftrightarrow FI_{x(k)}$ for each $k \in \{1,\ldots,r\}$ and $\Delta(k,\mathbf{i}) \leftrightarrow FI_{x(-n-k-1)}$ for each $k \in \{-n,\ldots,-1\}$.

There is also an analogue for the g-vectors of cluster monomials. Let $T = T_1 \oplus \cdots \oplus T_r$ be a basic maximal rigid Λ-module. For a Λ-module M, define the r**-vector** $r^T(M)$ of M to be the vector whose kth coordinate is

$$r^T(M)_k = \dim \mathrm{Hom}(T_k, M).$$

Then the g-vector of M with respect to T is defined as

$$g^T(M) = r^T(M)C_T^{-t}.$$

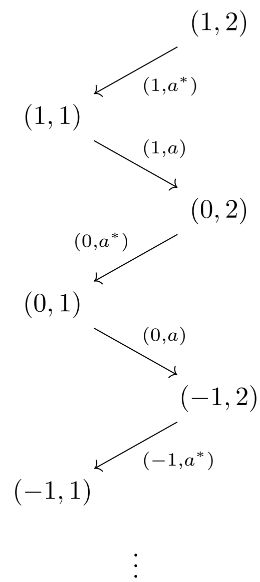
Proposition 4.3.4. *[GLS12, Lemma 6.4] Let T be a basic maximal rigid Λ-module with exchange matrix $B(T)$. Let T_i be an indecomposable summand of T that is not projective. Let M be a Λ-module with g-vectors $g^T(M) = (g_1,\ldots,g_r)$ and $g^{\mu_{T_i}(T)}(M) = (g'_1,\ldots,g'_r)$. Then*

$$g'_j = \begin{cases} -g_k & \text{if } j = k \\ g_j + \max((C_T^{-t})_{jk}, 0)g_k - (C_T^{-t})_{jk}\min(g_k, 0) & \text{if } j \neq k. \end{cases}$$

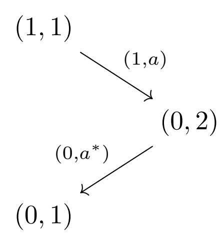
It is easy to see that the g-vectors of the indecomposable summands of T correspond to the standard basis vectors. Since the mutation for these g-vectors is the same as the mutation for g-vectors of cluster variables in $\mathbb{C}[N]$, we see that we also have a correspondence of g-vectors.

Example 4.3.2. Let $G = SL_3$. We will take our A_2 quiver Q to be the quiver $1 \xrightarrow{a} 2$. The Galois cover of Q is the quiver $\tilde{\Lambda} =$

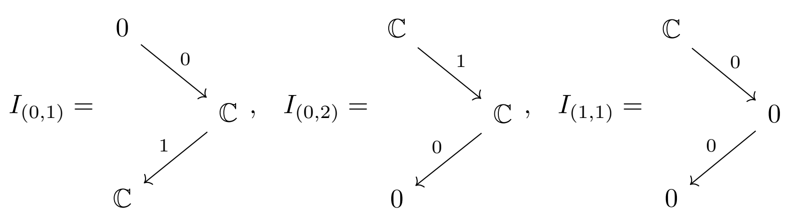
$$\vdots$$



with the relations that any composition of two arrows is 0. We then have Γ_Q to be the subquiver



The ordering of the vertices is $(0,1) < (0,2) < (1,1)$, so our reduced expression adapted to Q is $\mathbf{i} = (F(0,1), F(0,2), F(1,1)) = (1,2,1)$. The injective $\mathbb{C}\Gamma_Q$-modules are:



Thus the basic maximal rigid Λ-module we get is

$$
\begin{aligned}
T &= FI_{(1,1)} \oplus FI_{(0,1)} \oplus FI_{(0,2)} \\
&= \left(\mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} 0 \right) \oplus \left(\mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C} \right) \oplus \left(\mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C} \right) \\
&= S_1 \oplus P_+ \oplus P_-.
\end{aligned}
$$

The matrix C_T is

$$
\begin{bmatrix}
\dim \operatorname{Hom}_\Lambda(S_1, S_1) & \dim \operatorname{Hom}_\Lambda(S_1, P_+) & \dim \operatorname{Hom}_\Lambda(S_1, P_-) \\
\dim \operatorname{Hom}_\Lambda(P_+, S_1) & \dim \operatorname{Hom}_\Lambda(P_+, P_+) & \dim \operatorname{Hom}_\Lambda(P_+, P_-) \\
\dim \operatorname{Hom}_\Lambda(P_-, S_1) & \dim \operatorname{Hom}_\Lambda(P_-, P_+) & \dim \operatorname{Hom}_\Lambda(P_-, P_-)
\end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}.
$$

Then

$$C_T^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

Removing the last 2 columns of $-C_T^{-t}$, we have $B(T) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. We have S_1 to be the only direction we can mutate T in, and we see that the two non-split short exact sequences are

$$0 \to S_1 \to P_+ \to S_2 \to 0$$

and

$$0 \to S_2 \to P_- \to S_1 \to 0.$$

Hence $\mu_{S_1}(T) = S_2 \oplus P_+ \oplus P_-$ and $\mu_{S_1}(B(T)) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. Note that the seed $(\mu_{S_1}(T), \mu_{S_1}(B(T)))$ is what we would get if we instead used the quiver $1 \xleftarrow{a} 2$ and obtained the reduced expression $\mathbf{i} = (2,1,2)$.

The r-vectors are

$$r^T(S_1) = (1,0,1);\ r^T(S_2) = (0,1,0);\ r^T(P_+) = (1,1,1);\ r^T(P_-) = (0,1,1)$$

so the corresponding g-vectors after left multiplication by C_T^{-t} are

$$g^T(S_1) = (1,0,0);\ g^T(S_2) = (-1,1,0);\ g^T(P_+) = (0,1,0);\ g^T(P_-) = (0,0,1).$$

Comparing with Example 4.2.1, we see that the correspondence between the cluster variables of $\mathbb{C}[N]$ and the cluster modules in type A_2 also preserves the entire cluster structure.

Remark 4.3.3. We see that all indecomposable modules in type A_2 are also cluster modules. This is the case in types A_3 and A_4 as well. In other types, we have infinitely many cluster modules, but not all indecomposables will be cluster modules. We refer the reader to [GS05, Section 5] for an example of an indecomposable A_5 module that is not rigid, and hence cannot be a cluster module.

4.4 Lusztig Datum and g-vectors

Recall in Section 3.2 that for a reduced expression of w_0, $\mathbf{i} = (i_1, \ldots, i_r)$, and a generic Λ-module M, we obtained a filtration for M where the successive quotients in the filtration were of the form $(B_k^{\mathbf{i}})^{\oplus n_k}$. The numbers n_k form the $\mathbf{i}$-Lusztig datum for Pol(M), and we will denote $n^{\mathbf{i}}(M) = (n_1, \ldots, n_r)$. Consider the module $V^{\mathbf{i}} = V_1^{\mathbf{i}} \oplus \cdots \oplus V_r^{\mathbf{i}}$ where $V_k^{\mathbf{i}}$ is the unique module whose socle is S_{i_k} with dimension vector $\omega_{i_k} - s_{i_1} \cdots s_{i_k} \omega_{i_k}$. These summands are indecomposable and are also $N(\gamma)$'s. The module $V^{\mathbf{i}}$ is known to be a maximal rigid module [GLS11, Section 2.4]. The following result is due to Joel Kamnitzer.

Theorem 4.4.1. *For every generic Λ-module M, we have*

$$g^{V^{\mathbf{i}}}(M)_k = n^{\mathbf{i}}(M)_k - n^{\mathbf{i}}(M)_{k^+}$$

where we take $n^{\mathbf{i}}(M)_{r+1} = 0$.

Proof. The $\mathbf{i}$-Lusztig datum $n^{\mathbf{i}}(M)$ is linear in the BZ datum and part of the BZ datum is given by the r-vector $r^{V^{\mathbf{i}}}(M)$. Since the g-vector $g^{V^{\mathbf{i}}}(M)$ is linear in the r-vector, we thus have the Lusztig datum to also be linear in the g-vector. As this linear transformation is independent of the module M, we may assume $M = V_k^{\mathbf{i}}$ for some k. In this case, it is known that

$$n^{\mathbf{i}}(V_k^{\mathbf{i}})_j = \begin{cases} 1 & \text{if } j \leq k \text{ and } i_j = i_k \\ 0 & \text{otherwise} \end{cases}$$

Since $g^{\mathbf{i}}(V_k^{\mathbf{i}})$ is the kth standard basis vector, we then have

$$n^{\mathbf{i}}(M)_j = \sum_{k \geq j, i_k = i_j} g^{V^{\mathbf{i}}}(M)_k.$$

Inverting this equation, we conclude that $g^{V^{\mathbf{i}}}(M)_k = n^{\mathbf{i}}(M)_k - n^{\mathbf{i}}(M)_{k^+}$ as required. □

We will let $\varphi_{\mathbf{i}} : \mathbb{Z}^r \to \mathbb{N}^r$ denote the map sending g-vectors for $V^{\mathbf{i}}$ to $\mathbf{i}$-Lusztig datum.

Example 4.4.1. Let $\mathbf{i} = (1, 2, 1)$. Then $V^{\mathbf{i}} = S_1 \oplus P_- \oplus P_+$. Note that although $V^{\mathbf{i}}$ is the same module as T in Example 4.3.2, the order that we are labelling the summands is different. We have the following information of the indecomposable Λ-modules:

	S_1	S_2	P_+	P_-
$n^{\mathbf{i}}$	(1,0,0)	(0,0,1)	(1,0,1)	(0,1,0)
$g^{V^{\mathbf{i}}}$	(1,0,0)	(-1,0,1)	(0,0,1)	(0,1,0)

so we see that the linear transformation sending the $\mathbf{i}$-Lusztig datum to the g-vector corresponding to $V^{\mathbf{i}}$ is given by the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

There is also a relation between the Lusztig datum and mutations of g-vectors. We say the reduced expressions $\mathbf{i} = (i_1, \ldots, i_r)$ and $\mathbf{i}' = (i'_1, \ldots, i'_r)$ of w_0 differ by a 2-move at position k if

$$i_k = i'_{k+1}, i_{k+1} = i'_k, a_{i_k, i_{k+1}} = 0, \text{ and } i_j = i'_j \text{ for all } j \neq k, k+1,$$

and they differ by a 3-move at position k if

$$i_{k-1} = i_{k+1} = i'_k, i_k = i'_{k-1} = i'_{k+1}, a_{i_{k-1}, i_k} = -1, \text{ and } i_j = i'_j \text{ for all } j \neq k-1, k, k+1.$$

We say $\mathbf{i}$ and $\mathbf{i}'$ differ by a braid move at position k if they differ by a 2-move or a 3-move at position k.

Proposition 4.4.2. *[Kam10, Proposition 5.2] Let* $\mathbf{i}$ *and* $\mathbf{i}'$ *be two reduced expressions for* w_0*. Let* P *be an MV polytope with* $\mathbf{i}$*-Lusztig datum* $n^{\mathbf{i}} = (n_1, \ldots, n_r)$ *and* $\mathbf{i}'$*-Lusztig datum* $n^{\mathbf{i}'} = (n'_1, \ldots, n'_r)$*. Then*

1. *If* $\mathbf{i}$ *and* $\mathbf{i}'$ *differ by a 2-move at position* k*, then* $n'_k = n_{k+1}$*,* $n'_{k+1} = n_k$*, and* $n'_j = n_j$ *for all* $j \neq k, k+1$*.*

2. *If* $\mathbf{i}$ *and* $\mathbf{i}'$ *differ by a 3-move at position* k*, then for* $p = \min\{n_{k-1}, n_{k+1}\}$*,* $n'_{k-1} = n_k + n_{k+1} - p$*,* $n'_k = p$*,* $n'_{k+1} = n_{k-1} + n_k - p$*, and* $n'_j = n_j$ *for all* $j \neq k-1, k, k+1$*.*

We will let $\mu_{\mathbf{i}}^{\mathbf{i}'} : \mathbb{N}^r \to \mathbb{N}^r$ be the function that sends $\mathbf{i}$-Lusztig datum to $\mathbf{i}'$-Lusztig datum, assuming the reduced expressions $\mathbf{i}$ and $\mathbf{i}'$ differ by a single braid move.

Proposition 4.4.3. *[GKS16, Proposition 5.12] Let $\mathbf{i}$ and $\mathbf{i}'$ be two reduced expressions of w_0 that differ by a braid move at position k. Then for a Λ-module M, the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{Z}^r & \xrightarrow{\mu_k} & \mathbb{Z}^r \\ {\scriptstyle \varphi_{\mathbf{i}}}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathbf{i}'}} \\ \mathbb{N}^r & \xrightarrow{\mu_{\mathbf{i}}^{\mathbf{i}'}} & \mathbb{N}^r \end{array}$$

where the map μ_k is given by mutation of g-vectors in direction k along with a possible reordering.

The reason why a reordering is needed in the above proposition can be seen from the following example. In type A_2, if we let $\mathbf{i} = (1, 2, 1)$ and $\mathbf{i}' = (2, 1, 2)$, then $V^{\mathbf{i}} = S_1 \oplus P_- \oplus P_+$ and $V^{\mathbf{i}'} = S_2 \oplus P_+ \oplus P_-$. However, $\mu_1(V^{\mathbf{i}}) = S_2 \oplus P_- \oplus P_+$ so even though $V^{\mathbf{i}'} \cong \mu_1(V^{\mathbf{i}})$ as modules, the order of their summands is different which leads to different g-vectors. Hence the relation between $g^{V^{\mathbf{i}}}(M)$ and $g^{V^{\mathbf{i}'}}(M)$ is the mutation in direction k along with a possible reordering.

4.5 Summary

We have so far outlined the following diagram:

$$\begin{array}{ccc} \mathbb{C}[N] & \longleftarrow & \{\text{generic } \Lambda\text{-modules}\} \\ \uparrow & & \updownarrow \\ \{\text{stable MV cycles}\} & \longleftrightarrow & \{\text{stable MV polytopes}\} \end{array}$$

However, this diagram does not commute in general. It is known to commute in types A_n for $n \leq 3$ and conjectured for type A_4. In types A_5 and D_4, there exists a stable MV cycle Z and generic Λ-module M such that $\mathrm{Pol}(Z) = \mathrm{Pol}(M)$, but their corresponding elements in $\mathbb{C}[N]$ are different. The type A_5 situation is discussed in [BKK19, Theorem 1.7]. In particular, this shows that the dual semicanonical basis and the MV basis are different.

Nonetheless, the diagram does commute if we restrict to initial seeds. That is, if $\mathbf{i}$ is a reduced expression for w_0, then by [BKK19, Remark 2.10, Theorem 5.2], each generalized minor $\Delta(k, \mathbf{i})$ corresponds to an MV cycle with stable MV polytope the same as $\mathrm{Pol}(FI_{x(k)})$. Hence to show that the diagram commutes for all cluster variables, equivalently that the cluster variables are in the MV basis, then it suffices to find an analogue of the exchange relations for MV cycles and polytopes.

Chapter 5

Cluster Structure for MV Polytopes

In this section we show the analogue of the exchange sequences appearing in Section 4.3, and show that they give a necessary condition if the cluster variables in $\mathbb{C}[N]$ are also elements of the MV basis.

5.1 Exchange Relation coming from Λ-mod

Let $T = T_1 \oplus \cdots \oplus T_r$ be a basic maximal rigid Λ-module with indecomposable T_i and assume that $T_{r-n+1}, \ldots, T_r$ are projective. Recall that for $1 \leq i \leq r-n$, $\dim \operatorname{Ext}(T_i, T_i^*) = \dim \operatorname{Ext}(T_i^*, T_i) = 1$. Let the corresponding exchange sequences be

$$\epsilon_+ : 0 \to T_i \to T_+ \to T_i^* \to 0$$
$$\epsilon_- : 0 \to T_i^* \to T_- \to T_i \to 0.$$

This section is devoted to proving the following result, which can be thought of as an interpretation of exchange relations for MV polytopes.

Theorem 5.1.1. *We have*

$$\operatorname{Pol}(T_i) + \operatorname{Pol}(T_i^*) = \operatorname{Pol}(T_+) \cup \operatorname{Pol}(T_-).$$

The following inclusion is straightforward and true for all short exact sequences.

Proposition 5.1.2. *We have* $\operatorname{Pol}(T_i) + \operatorname{Pol}(T_i^*) \supset \operatorname{Pol}(T_+) \cup \operatorname{Pol}(T_-)$.

Proof. Let $M \subset T_+$ be a submodule. Viewing T_i as a submodule of T_+, consider the exact sequence

$$0 \to T_i \cap M \xrightarrow{\iota} M \to \operatorname{coker} \iota \to 0.$$

As $\operatorname{coker} \iota \cong M/T_i \cap M$ and $T_i^* \cong T_+/T_i$, we have a well-defined injective morphism $M/T_i \cap M \to T_+/T_i$ sending $[m] \mapsto [m]$. Then $\operatorname{coker} \iota$ is a submodule of T_i^*. Thus $\underline{\dim}\, M = \underline{\dim}\, T_i \cap M + \underline{\dim} \operatorname{coker} \iota \in \operatorname{Pol}(T_i) + \operatorname{Pol}(T_i^*)$ so $\operatorname{Pol}(T_+) \subset \operatorname{Pol}(T_i) + \operatorname{Pol}(T_i^*)$. The argument for $\operatorname{Pol}(T_-)$ is similar. $\square$

The other inclusion will require more work and uses specific properties of the preprojective algebra Λ.

Definition 5.1.1. Let $\epsilon : 0 \to T_1 \to T \to T_2 \to 0$ be a short exact sequence and $A \subset T_1$, $B \subset T_2$ be submodules. We say (A, B) is a **good pair** for ϵ if there exists a short exact sequence $0 \to A \to M \to B \to 0$ and morphism $M \to T$ making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & M & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_1 & \longrightarrow & T & \longrightarrow & T_2 & \longrightarrow & 0 \end{array}$$

Lemma 5.1.3. *Let $\epsilon : 0 \to T_1 \to T \to T_2 \to 0$ be a short exact sequence and $A \subset T_1, B \subset T_2$ be submodules. Suppose we have a short exact sequence $0 \to A \to M \to B \to 0$. Then there exists a morphism $M \to T$ causing (A, B) to be a good pair for ϵ if and only if the pushout S of $A \to M$ and $A \to T_1$ is equal to the pullback P of $B \to T_2$ and $T \to T_2$, as extensions of B by T_1.*

Proof. Suppose $S = P$ as extensions of B by T_1. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & M & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & T_1 & \longrightarrow & P & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_1 & \longrightarrow & T & \longrightarrow & T_2 & \longrightarrow & 0 \end{array}$$

The composition $M \to P \to T$ will make (A, B) a good pair for ϵ.

Now suppose we have a morphism $M \to T$ causing (A, B) to be a good pair for ϵ. We have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & M & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & T_1 & \longrightarrow & S & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ 0 & \longrightarrow & T_1 & \longrightarrow & P & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_1 & \longrightarrow & T & \longrightarrow & T_2 & \longrightarrow & 0 \end{array}$$

By the five lemma, it suffices to construct a morphism $S \to P$ that makes the diagram commute. Since the morphism $M \to T$ makes the square

$$\begin{array}{ccc} M & \longrightarrow & B \\ \downarrow & & \downarrow \\ T & \longrightarrow & T_2 \end{array}$$

commute and as P is the pullback of $B \to T_2$ and $T \to T_2$, we have a unique morphism $M \to P$ such that the square

$$\begin{array}{ccc} M & \longrightarrow & B \\ \downarrow & & \| \\ P & \longrightarrow & B \end{array}$$

commutes. As $A = \ker(M \to B)$ and $T_1 = \ker(P \to B)$, we have a commutative square

$$\begin{array}{ccc} A & \longrightarrow & M \\ \downarrow & & \downarrow \\ T_1 & \longrightarrow & P \end{array}$$

Since S is the pushout of $A \to M$ and $A \to T_1$, we have a unique morphism $S \to P$. □

Lemma 5.1.4. *Let $\epsilon : 0 \to T_1 \to T \to T_2 \to 0$ be a short exact sequence and let $A \subset T_1, B \subset T_2$ be submodules. Suppose (A, B) is a good pair for ϵ. Let A', B' be modules such that $A \subset A' \subset T_1$ and $B' \subset B$. Then (A', B) and (A, B') are good pairs for ϵ.*

Proof. As (A, B) is a good pair, there exists a module M such that the following diagram commutes and has exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & M & \xrightarrow{f} & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_1 & \longrightarrow & T & \xrightarrow{g} & T_2 & \longrightarrow & 0 \end{array}$$

Then the exact sequence $0 \to A \to f^{-1}(B') \to B' \to 0$ will allow (A, B') to be a good pair for ϵ.

By Lemma 5.1.3, as (A, B) is a good pair for ϵ and $g^{-1}(B)$ is the pullback of $B \to T_2$ and $T \to T_2$, we have the following commutative diagram with exact rows and exact columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & M & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & T_1 & \longrightarrow & g^{-1}(B) & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_1/A & \longrightarrow & g^{-1}(B)/M & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Consider the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & T_1 & \longrightarrow & g^{-1}(B)/M & \longrightarrow & 0 \\ & & \downarrow & & \| & & & & \\ 0 & \longrightarrow & A' & \xrightarrow{\iota} & T_1 & \longrightarrow & \operatorname{coker}\iota & \longrightarrow & 0 \end{array}$$

As $\operatorname{coker}\iota \cong T_1/A'$, we have a well-defined surjective morphism $\pi : g^{-1}(B)/M \cong T_1/A \to T_1/A' \cong \operatorname{coker}\iota$. Then $\operatorname{coker}\iota \cong (g^{-1}(B)/M)/\ker\pi$. As $\ker\pi$ is a submodule of $g^{-1}(B)/M$, we have $\ker\pi \cong M'/M$ for some module $M \subset M' \subset g^{-1}(B)$. Then $\operatorname{coker}\iota \cong g^{-1}(B)/M'$ and M' will allow (A', B) to be a good pair for ϵ. □

Proposition 5.1.5. *Let $A \subset T_i$ and $B \subset T_i^*$ be submodules. Then (A, B) is a good pair for ϵ_+ or (B, A) is a good pair for ϵ_-.*

Proof. Without loss of generality, suppose (A, B) is not a good pair for ϵ_+. Then by Lemma 5.1.4, (A, T_i^*) is not a good pair for ϵ_+. Then by Lemma 5.1.3, for every short exact sequence $0 \to A \to M \to T_i^* \to 0$, we have the pushout of $A \to M$ and $A \to T_i$ to not be equal to the pullback of $T_i^* \xrightarrow{=} T_i^*$ and $T \to T_i^*$, which is T, as extensions of T_i^* by T_i. As $\dim \operatorname{Ext}^1_\Lambda(T_i^*, T_i) = 1$, this means the pushout is the split extension for all such M. Let $\iota : A \to T_1$ be the injection. Then by Theorem 2.1.2, we have for all $[\mu] \in \operatorname{Ext}^1_\Lambda(T_1^*, A)$,

$$\phi_{T_i^*, T_i}(\iota \circ [\mu])([\epsilon_-]) = \phi_{T_i^*, A}([\mu])([\epsilon_-] \circ \iota_A).$$

As $\iota \circ [\mu]$ is the pushout of μ along ι, we get the split extension so $\iota \circ [\mu] = 0$ in $\operatorname{Ext}^1_\Lambda(T_1^*, A)$. Then

$$\phi_{T_i^*, A}([\mu])([\epsilon_-] \circ \iota) = 0$$

for all $[\mu]$ so $[\epsilon_-] \circ \iota = 0$. This means the pullback of $T_- \to T_i$ and $A \to T_i$ is the split extension in $\operatorname{Ext}^1_\Lambda(A, T_i^*)$, which is also the pushout of $0 \to A$ and $0 \to T_i^*$. Thus $(0, A)$ is a good pair for ϵ_-. Then by Lemma 5.1.4, (B, A) is a good pair for ϵ_-. □

Remark 5.1.2. The condition that $\dim \operatorname{Ext}^1_\Lambda(T_i, T_i^*) = 1$ can be weakened to $\dim \operatorname{Ext}^1_\Lambda(T_i^*, T_i) \neq 0$. The proof will then show that if (A, B) is not a good pair for any non-split extension of T_i^* by T_i, then (B, A) is a good pair for every non-split extension of T_i by T_i^*.

Combining Propositions 5.1.2 and 5.1.5 gives us the proof of Theorem 5.1.1.

Corollary 5.1.6. $\operatorname{Pol}(T_+) \cup \operatorname{Pol}(T_-)$ *is a convex polytope.*

Remark 5.1.3. It is possible for one of $\operatorname{Pol}(T_+)$ or $\operatorname{Pol}(T_-)$ to be equal to $\operatorname{Pol}(T_i) + \operatorname{Pol}(T_i^*)$. The smallest example is in type A_3, where $T_i = \mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C}$, $T_i^* = \mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C}$, $T_+ = S_1 \oplus S_3 \oplus P_2$, and $T_- = P_1 \oplus P_3$. In this case, $\operatorname{Pol}(T_+) = \operatorname{Pol}(T_i) + \operatorname{Pol}(T_i^*)$ and $\operatorname{Pol}(T_-) \subsetneq \operatorname{Pol}(T_+)$.

Using Theorem 5.1.1, we can construct the MV polytopes for the rigid modules obtained from mutation. For sets of vectors A and B in a vector space V, define

$$A - B := \{x \in V : \{x\} + B \subset A\}.$$

If we already know $P_i + P_i^* = Q$ as an equality of polytopes, then $P_i^* = Q - P_i$ [Sch13, Lemma 3.1.11]. Thus we can combinatorially construct the MV polytope of a module obtained from mutation as

$$\operatorname{Pol}(T_i^*) = (\operatorname{Pol}(T_+) \cup \operatorname{Pol}(T_-)) - \operatorname{Pol}(T_i).$$

5.2 Exchange Relation Coming from MV Cycles

In the previous section, we showed that Theorem 5.1.1 follows from the exchange sequences in Λ-mod. We will now explain why it also is a necessary condition for a corresponding exchange relation of MV cycles, in other words, the following result.

Proposition 5.2.1. *Suppose we have a fusion product of MV cycles*

$$Z_1 * Z_2 = Z_+ + Z_-.$$

Then

$$\mathrm{Pol}(Z_1) + \mathrm{Pol}(Z_2) = \mathrm{Pol}(Z_+) \cup \mathrm{Pol}(Z_-).$$

The following is based on Sections 9 and 10 of [BKK19]. It is well-known that there exists a projective embedding $\mathcal{G}r \to \mathbb{P}(V)$ where V is an infinite-dimensional vector space (see [BKK19, Section 4.3]). For $n \in \mathbb{Z}$, let $\mathcal{O}(n)$ denote the usual line bundle on $\mathbb{P}(V)$. Recall that for an MV cycle $Z \subset \mathcal{G}r$, Z is a $T^\vee$-invariant subvariety. Let $[\Gamma(Z, \mathcal{O}(n))]$ denote the class of the space of sections $\Gamma(Z, \mathcal{O}(n))$ in $R(T^\vee)$, the complexified representation ring of $T^\vee$. We view $R(T^\vee)$ inside the space of distributions of $(\mathfrak{t}_\mathbb{R}^\vee)^* = \mathfrak{t}_\mathbb{R}$ using the map

$$[U] \mapsto \sum_{\mu \in P^\vee} \dim U_\mu \delta_\mu.$$

Let $\tau_n : \mathfrak{t}_\mathbb{R} \to \mathfrak{t}_\mathbb{R}$ be the automorphism corresponding to scaling by $\frac{1}{n}$.

Definition 5.2.1. For an MV cycle Z, define its Duistermaat-Heckmann measure $DH(Z)$ as the weak-limit

$$DH(Z) = \lim_{n \to \infty} \frac{1}{n^{\dim Z}} (\tau_n)_* [\Gamma(Z, \mathcal{O}(n))]$$

in the space of distributions of $\mathfrak{t}_\mathbb{R}$.

For each $p \in Z^{T^\vee}$, let $\Phi_{T^\vee}(p)$ be the weight of the action of $T^\vee$ on the fibre of $\mathcal{O}(1)$ at p. Define $pol(Z) = \mathrm{Conv}\{\Phi_{T^\vee}(p) : p \in Z^{T^\vee}\} \subset \mathfrak{t}_\mathbb{R}$.

Proposition 5.2.2. *[BP90] The measure $DH(Z)$ is well-defined and has support $pol(Z)$.*

In [BKK19, Section 4.3], they defined a map $\iota : P \to \mathfrak{t}_\mathbb{R}$, and after extending this to a linear bijection $\iota : \mathfrak{t}_\mathbb{R}^* \to \mathfrak{t}_\mathbb{R}$, they showed that for every stable MV cycle Z, we have an equality $\iota(\mathrm{Pol}(Z)) = pol(Z)$. Hence we will think of the support of $DH(Z)$ as $\mathrm{Pol}(Z)$.

Suppose we had an equation of MV cycles $Z_1 * Z_2 = Z_+ + Z_-$. Geometrically, this means that $Z_1 \circ Z_2 = Z_+ \cup Z_-$. Note that Definition 5.2.1 can be extended to the case where Z is a subscheme, as in [BKK19, Section 9.1], so we can make sense of $DH(Z_1 \circ Z_2)$ and $DH(Z_+ \cup Z_-)$. By [BKK19, Theorem 9.8], we have

$$DH(Z_+ \cup Z_-) = DH(Z_+) + DH(Z_-)$$

which is supported on $\mathrm{Pol}(Z_1) \cup \mathrm{Pol}(Z_2)$. From [BKK19, Theorem 10.2], as DH is an algebra map, we conclude

$$DH(Z_1 \circ Z_2) = DH(Z_1) * DH(Z_2)$$

where the product on the right is the convolution of measures, and this is supported on $\mathrm{Pol}(Z_1) + \mathrm{Pol}(Z_2)$ [BKK19, Section 8.3]. Combining these, we thus have

$$\mathrm{Pol}(Z_1) + \mathrm{Pol}(Z_2) = \mathrm{Pol}(Z_+) \cup \mathrm{Pol}(Z_-).$$

Chapter 6

Investigating the Cluster Structure for MV Cycles

In this chapter we look at the exchange relations from the point of view of MV cycles. We will do this by relating the fusion product with short exact sequences in Λ-mod.

6.1 Valuations

Recall the definition of val in Section 3.1.1. We will use val_s to denote the same function val, but constructed using $\mathcal{O}_s$ and $\mathcal{K}_s$ instead. We will look at how the valuation of terms in a fusion product compares with the valuations of the factors of the product, but we will first need to recall the ind-structure of $\mathcal{G}r$ and $\mathcal{G}r_{0,\mathbb{A}}$. Since we are considering stable MV cycles $Z \subset \overline{S_+^0 \cap S_-^\mu}$, we will be focused on the dense subset $Z \cap S_+^0 = Z \cap N^\vee(\mathcal{K})/N^\vee(\mathcal{O})$ of Z. Hence we will be only looking at the parts of the affine Grassmannian and Beilinson-Drinfeld Grassmannian defined using $N^\vee$ instead of $G^\vee$.

Take a faithful representation $\iota : G^\vee \hookrightarrow GL(V)$ for a suitable m-dimensional vector space V. The ind-structure on $G^\vee(\mathcal{K})$ is defined using functors $(G^\vee)^{(n)} : \mathbb{C}\text{-alg} \to \text{Set}$, $n \in \mathbb{N}$, where

$$(G^\vee)^{(n)}(R) = \{A \in G(R((t))) : \text{the matrix coefficients of } A, A^{-1} \text{ have poles of order} \leq n\}.$$

The ind-structure on $N^\vee(\mathcal{K})$ is defined similarly using functors $(N^\vee)^{(n)}$ whose definition is analogous to $(G^\vee)^{(n)}$.

Now restrict the faithful representation $\iota : N^\vee \hookrightarrow GL(V) \subset \text{End}(V)$. We will also use $N^\vee$ to denote the image of $N^\vee$ in $\text{End}(V)$. As $N^\vee$ is unipotent, by [Bor91, Corollary 4.8], we can choose a basis for V such that $N^\vee$ is mapped into a closed subgroup of U, the group of upper triangular matrices with 1 along the diagonal, which is closed in $\text{End}(V)$. Then by transitivity, $N^\vee$ is closed in $\text{End}(V)$.

From now on, assume the closed embedding $\iota : N^\vee \hookrightarrow M_m$, where $M_m := M_m(\mathbb{C})$ is the set of $m \times m$ matrices, factors through U. Let $x \in \mathbb{C}[N^\vee]$. As ι is a closed embedding, we have $\iota^* : \mathbb{C}[M_m] \to \mathbb{C}[N^\vee]$ to be a surjection. Note that for $\widetilde{x} \in (\iota^*)^{-1}(x)$ and $\eta \in N^\vee(\mathcal{K}_s)$, we have

$$x(\eta) = \iota^*(\widetilde{x})(\eta) = \widetilde{x}(\iota(\eta))$$

so $val_s x(\eta) = val_s \widetilde{x}(\iota(\eta))$.

For $s \in \mathbb{C}$, define

$$V_k^s = \{g \in N^\vee(\mathcal{K}_s) : val_s x(g) \geq k\}.$$

Lemma 6.1.1. *V_k^s is closed in $N^\vee(\mathcal{K}_s)$.*

Proof. Let $x \in \mathbb{C}[N^\vee]$. Viewing x as a polynomial in matrix entries, it is clear that there exists functions $c_i \in (N^\vee)^{(n)}$ such that for every $A \in (N^\vee)^{(n)}(R)$, we have

$$x(A) = \sum_{i \geq -n \deg x} c_i(A)(t-s)^i.$$

Then $val_s x(A) \geq k$ if and only if $c_i(A) = 0$ for $-n \deg x \leq i < k$. These vanishing equations show that $V_k^s \cap (N^\vee)^{(n)}$ is closed in $(N^\vee)^{(n)}$ so V_k^s is closed in $N^\vee(\mathcal{K}_s)$. □

In the proof above, if k is chosen so that $k < -n \deg x$, then this would give $(N^\vee)^{(n)} \subset V_k^s$ so each stratum is contained in some V_k^s. Fix a stratum $(N^\vee)^{(n)}$ and let $A = (a_{ij}) \in (N^\vee)^{(n)}(R)$, $B = (b_{ij}) \in N^\vee(\mathcal{O}_s) = N^{(0)}(R)$. Consider

$$AB = (\sum_t a_{it} b_{tj}).$$

Since a_{it} has poles of order at most n and b_{tj} has no poles, then $a_{it}b_{tj}$ has poles of order at most n, so $\sum_t a_{it}b_{tj}$ has poles of order at most n. Thus $(N^\vee)^{(n)}$ is stable under right multiplication by $N^\vee(\mathcal{O}_s)$. This implies that if $Z \cap S_+^0 \subset N^\vee(\mathcal{K}_s)/N^\vee(\mathcal{O}_s)$ is dense in an MV cycle Z and $\pi_s : N^\vee(\mathcal{K}_s) \to N^\vee(\mathcal{K}_s)/N^\vee(\mathcal{O}_s)$ is the quotient map, then $\pi_s^{-1}(Z \cap S_+^0)$ is contained in some stratum $(N^\vee)^{(n)}$ so $\pi_s^{-1}(Z \cap S_+^0) \subset V_k^s$ for some k. Choose k maximal with respect to this property and define $val_s x(Z) = k$.

From [BGL20, Section 5.1], instead of viewing the Beilinson-Drinfeld Grassmannian as G-bundles with a trivialization, we can instead use

$$\mathcal{G}r_{0,\mathbb{A}} = \{(s,[g]) : s \in \mathbb{C}, [g] \in G^\vee(\mathbb{C}[t,t^{-1},(t-s)^{-1}])/G^\vee(\mathbb{C}[t])\}.$$

As we are focused on the part of the Grassmannian defined using $N^\vee$ instead of $G^\vee$, we will instead have $\mathcal{G}r_{0,\mathbb{A}}$ denote

$$\{(s,[g]) : s \in \mathbb{C}, [g] \in N^\vee(\mathbb{C}[t,t^{-1},(t-s)^{-1}])/N^\vee(\mathbb{C}[t])\}.$$

Let

$$\widetilde{\mathcal{G}r_{0,\mathbb{A}}} = \{(s,g) : s \in \mathbb{C}, g \in N^\vee(\mathbb{C}[t,t^{-1},(t-s)^{-1}])\}$$

and let $q : \widetilde{\mathcal{G}r_{0,\mathbb{A}}} \to \mathcal{G}r_{0,\mathbb{A}}$ be the quotient map. Define for $s \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{Z}$,

$$V_k^\circ = \{(s,g) \in \widetilde{\mathcal{G}r_{0,\mathbb{A}}} : val_0 x(g) + val_s x(g) \geq k\}$$

and let V_k be the closure of V_k° in $\widetilde{\mathcal{G}r_{0,\mathbb{A}}}$. We are interested in the 0-fibre of V_k, but first a technical lemma, whose proof was partly communicated by Marko Riedel.

Lemma 6.1.2. *Let $p(t) = \sum_{i=0}^d a_i t^i$ be a polynomial. Let $k \geq 0$ be an integer. In the coordinate ring $\mathbb{C}[a_0, \dots, a_d, s, s^{-1}]$ of a polynomial and a non-zero scalar, consider the ideal*

$$I = \langle a_0, \dots, a_{k-1} \rangle \cap \langle a_0, \dots, a_{k-2}, p(s) \rangle \cap \cdots \cap \langle p(s), \dots, p^{(k-1)} \rangle$$

where $p^{(j)}(s)$ denotes the jth derivative of p evaluated at s. Let $J = (I \cap \mathbb{C}[a_0, \dots, a_d, s]) + \langle s \rangle$. Then $a_i \in \sqrt{J}$ for all $i = 0, \dots, k-1$.

Proof. We see that $q_0 := -a_0 p(s) \in I$ so $a_0^2 \in J$. Hence $a_0 \in \sqrt{J}$. Define

$$q_1 := s^{-1}(a_0 p^{(1)}(s) - a_1 p(s)) = 2a_0 a_2 - a_1^2 + s(...) \in I.$$

Then $2a_0a_2 - a_1^2 \in J \subset \sqrt{J}$ so $a_1 \in \sqrt{J}$. For $n > 1$, define

$$q_n := s^{-n} \left(\sum_{j=0}^{n} \frac{(-1)^{j+1}}{j!} a_{n-j} p^{(j)}(s) + \sum_{n/2 \le j \le n-1} (-1)^{n-j+1} \binom{j}{n-j} s^{2j-n} q_j \right).$$

It is easy to see that $q_n \in I$ by induction on n. Define

$$p_n := \sum_{j=n}^{d} \left(\sum_{k=0}^{n} c_{j,k} a_{n-k} a_{j+k} \right) s^{j-n}$$

where

$$c_{j,k} = \begin{cases} -1 & \text{if } j = n, k = 0 \\ 2 & \text{if } j = n, k = 1 \\ 0 & \text{if } j < n \text{ or } k < 0 \\ c_{j-1,k} - c_{j,k-1} & \text{otherwise.} \end{cases}$$

We will show by induction on n that $q_n = p_n$ and $a_n \in \sqrt{J}$. Note first that for $m \geq 1$, if we let $A_k^m = c_{m+n-k,k}$, then up to sign, the numbers $A_0^m, \dots, A_m^m$ constitute the mth row of the Lucas triangle [Fei67]. In particular, we have

$$A_k^m = (-1)^{k+1} \frac{m+k}{m} \binom{m}{k}.$$

Then if we define $A_0^0 = -1$, we have

$$\begin{array}{rcccccccc} p_n & = & A_0^0 a_n^2 & + & A_1^1 a_{n-1} a_{n+1} & + & \cdots & + & A_n^n a_0 a_{2n} \\ & + & (A_0^1 a_n a_{n+1} & + & A_1^2 a_{n-1} a_{n+2} & + & \cdots & + & A_n^{n+1} a_0 a_{2n+1}) s \\ & & & & \vdots & & & & \\ & + & (A_0^k a_n a_{n+k} & + & A_1^{k+1} a_{n-1} a_{n+k+1} & + & \cdots & + & A_n^{k+n} a_0 a_{2n+k}) s^k \\ & & & & \vdots & & & & \end{array}$$

It is easy to check that $q_0 = p_0$, $q_1 = p_1$, and we have shown $a_0, a_1 \in \sqrt{J}$, so suppose $q_j = p_j$ and $a_j \in \sqrt{J}$ for each $j = 0, \dots, n-1$. Fix $j \le \frac{n}{2}$ and $k < n$. We will show that the coefficient of $a_j a_{n-j+k} s^k$ in $s^n q_n$ is 0. This coefficient is comprised of three possible values: one acquired from $a_j p^{(n-j)}(s)$, another from $a_{n-j+k} p^{(j-k)}(s)$ if $j \ge k$. and the third from $q_{n/2+\ell}$ for each $0 \le \ell \le \lfloor k/2 \rfloor$ if n is even, otherwise $q_{(n+1)/2+\ell}$ for each $0 \le \ell \le \lfloor (k-1)/2 \rfloor$ if n is odd. Through induction, these coefficients are respectively

$$(-1)^{n-j+1} \binom{n-j+k}{k}, (-1)^{j-k+1} \binom{j}{k}, \begin{cases} (-1)^{\frac{n}{2}+\ell+1} \binom{\frac{n}{2}+\ell}{\frac{n}{2}-\ell} A^{\frac{n}{2}-j+k-\ell}_{\frac{n}{2}-j+\ell} & \text{if } n \text{ is even} \\ (-1)^{\frac{n+1}{2}+\ell} \binom{\frac{n+1}{2}+\ell}{2\ell+1} A^{\frac{n+1}{2}-j+k-\ell}_{\frac{n+1}{2}-j+\ell} & \text{if } n \text{ is odd} \end{cases}.$$

Suppose that n is even. Then

$$\begin{aligned}
&\sum_{\ell=0}^{\lfloor k/2\rfloor}(-1)^{\frac{n}{2}+\ell+1}\binom{\frac{n}{2}+\ell}{\frac{n}{2}-\ell}A^{\frac{n}{2}-j+k-\ell}_{\frac{n}{2}-j+\ell}\\
=&\sum_{\ell=0}^{\lfloor k/2\rfloor}(-1)^{n-j}\binom{\frac{n}{2}+\ell}{2\ell}\frac{n-2j+k}{\frac{n}{2}-j+k-\ell}\binom{\frac{n}{2}-j+k-\ell}{\frac{n}{2}-j+\ell}\\
=&\sum_{\ell=0}^{\lfloor k/2\rfloor}(-1)^{j}\binom{\frac{n}{2}+\ell}{2\ell}\left(\binom{\frac{n}{2}-j+k-\ell}{\frac{n}{2}-j+\ell}+\binom{\frac{n}{2}-j+k-1-\ell}{\frac{n}{2}-j-1+\ell}\right)\\
=&\sum_{\ell=0}^{\lfloor k/2\rfloor}(-1)^{j}\binom{\frac{n}{2}+\ell}{2\ell}\left(\binom{\frac{n}{2}-j+k-\ell}{k-2\ell}+\binom{\frac{n}{2}-j+k-1-\ell}{k-2\ell}\right).
\end{aligned}$$

Multiplying each coefficient by $(-1)^j$, it suffices to show that

$$\binom{n-j+k}{k}-\sum_{\ell=0}^{\lfloor k/2\rfloor}\binom{\frac{n}{2}+\ell}{2\ell}\left(\binom{\frac{n}{2}-j+k-\ell}{k-2\ell}+\binom{\frac{n}{2}-j+k-1-\ell}{k-2\ell}\right)=\begin{cases}0 & \text{if } j<k\\ (-1)^{k+1}\binom{j}{k} & \text{if } j\geq k\end{cases}.$$

The Egorychev method [Ego84] gives us the following relationship between binomial coefficients and contour integrals:

$$\binom{n}{k}=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^n}{z^{k+1}}dz.$$

Using this, we obtain

$$\begin{aligned}
&\sum_{\ell=0}^{\lfloor k/2\rfloor}\binom{\frac{n}{2}+\ell}{2\ell}\binom{\frac{n}{2}-j+k-\ell}{k-2\ell}\\
=&\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{\frac{n}{2}-j+k}}{z^{k+1}}\sum_{\ell=0}^{\lfloor k/2\rfloor}\binom{\frac{n}{2}+\ell}{2\ell}\frac{z^{2\ell}}{(1+z)^{\ell}}dz\\
=&\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{\frac{n}{2}-j+k}}{z^{k+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{\frac{n}{2}}}{w}\sum_{\ell\geq 0}\frac{z^{2\ell}(1+w)^{\ell}}{(1+z)^{\ell}w^{2\ell}}dwdz\\
=&\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{\frac{n}{2}-j+k}}{z^{k+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{\frac{n}{2}}}{w}\frac{1}{1-\frac{z^2(1+w)}{w^2(1+z)}}dwdz\\
=&\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{\frac{n}{2}-j+k+1}}{z^{k+1}}\frac{1}{2\pi i}\int_{|w|=\gamma}(1+w)^{\frac{n}{2}}\frac{w}{(w-z)(w(1+z)+z)}dwdz.
\end{aligned}$$

We need to choose a contour containing the simples poles $w=z$ and $w=\frac{-z}{1+z}$. Then we require

$$\left|\frac{z}{1+z}\right|<\gamma.$$

As $|z|<\epsilon$, this means

$$\left|\frac{z}{1+z}\right|<\frac{\epsilon}{1-\epsilon}$$

so it suffices to have

$$\frac{\epsilon}{1-\epsilon}<\gamma.$$

For convergence of the geometric series, we also require

$$\left|\frac{z^2(1+w)}{w^2(1+z)}\right| < 1,$$

which holds if

$$\frac{\epsilon^2}{1-\epsilon} < \frac{\gamma^2}{1+\gamma}.$$

To have both $\frac{\epsilon}{1-\epsilon} < \gamma$ and $\epsilon(\frac{\epsilon}{1-\epsilon}) < \frac{\gamma^2}{1+\gamma}$, it suffices to have

$$\epsilon\gamma < \frac{\gamma^2}{1+\gamma}.$$

Hence choose ϵ such that $\epsilon < \frac{\gamma}{1+\gamma}$. Then $\epsilon < \gamma$ so we also have the pole at $w = z$ in this contour. For the residue at the pole $w = \frac{-z}{1+z}$, we have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{\frac{n}{2}-j+k}}{z^{k+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} (1+w)^{\frac{n}{2}} \frac{w}{(w-z)(w+z/(1+z))} dwdz \\
=&\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{\frac{n}{2}-j+k}}{z^{k+1}} (1+z)^{-\frac{n}{2}} \frac{-z/(1+z)}{-z-z/(1+z)} dz \\
=&\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{k-j}}{z^{k+1}} \frac{1}{z+2} dz.
\end{aligned}$$

Similarly, we also obtain the residue for

$$\sum_{\ell=0}^{\lfloor k/2 \rfloor} \binom{\frac{n}{2}+\ell}{2\ell} \binom{\frac{n}{2}-j+k-1-\ell}{k-2\ell}$$

at $w = \frac{-z}{1+z}$ to be

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{k-j-1}}{z^{k+1}} \frac{1}{z+2} dz.$$

Hence the residue at $w = \frac{-z}{1+z}$ for the entire sum is

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{k-j-1}(1+z+1)}{z^{k+1}} \frac{1}{z+2} dz \\
&= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{k-j-1}}{z^{k+1}} dz = \binom{k-j-1}{k}.
\end{aligned}$$

For the residue at the pole $w = z$, we have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{\frac{n}{2}-j+k+1}}{z^{k+1}} \frac{1}{2\pi i}\int_{|w|=\gamma} (1+w)^{\frac{n}{2}} \frac{w}{(w-z)(w(1+z)+z)} dwdz \\
=&\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{\frac{n}{2}-j+k+1}}{z^{k+1}} (1+z)^{\frac{n}{2}} \frac{z}{z(1+z)+z} dz \\
=&\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{(1+z)^{n-j+k+1}}{z^{k+1}} \frac{1}{z+2} dz.
\end{aligned}$$

Similarly, for the other sum, the residue is

$$\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{n-j+k}}{z^{k+1}}\frac{1}{z+2}dz.$$

Adding these two residues, we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{k-j-1}(1+z+1)}{z^{k+1}}\frac{1}{z+2}dz\\ &=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{(1+z)^{k-j-1}}{z^{k+1}}dz=\binom{n-j+k}{k}.\end{aligned}$$

In summary,

$$\begin{aligned}&\binom{n-j+k}{k}-\sum_{\ell=0}^{\lfloor k/2\rfloor}\binom{\frac{n}{2}+\ell}{2\ell}\left(\binom{\frac{n}{2}-j+k-\ell}{k-2\ell}+\binom{\frac{n}{2}-j+k-1-\ell}{k-2\ell}\right)\\ =&\binom{n-j+k}{k}-\binom{k-j-1}{k}-\binom{n-j+k}{k}\\ =&-\binom{k-j-1}{k}\\ =&\frac{-(k-j-1)(k-j-2)\cdots(-j)}{k!}.\end{aligned}$$

If $j<k$, one of the terms in the numerator is 0 so the entire term is 0. If $j\geq k$, then all terms in the numerator are negative and we obtain $(-1)^{k+1}\binom{j}{k}$ as required. The n being odd case is similar where we require

$$\binom{n-j+k}{k}-\sum_{\ell=0}^{\lfloor (k-1)/2\rfloor}\binom{\frac{n+1}{2}+\ell}{2\ell+1}\frac{n-2j+k}{\frac{n-1}{2}+k-\ell-j}\binom{\frac{n-1}{2}+k-\ell-j}{k-2\ell-1}=\begin{cases}0 & \text{if } j<k\\ (-1)^k\binom{j}{k} & \text{if } j\geq k\end{cases}.$$

Hence all coefficients of the s^j terms in $s^n q_n$ with $j<n$ are 0. In the alternating sum

$$\sum_{j=0}^{n}\frac{(-1)^{j+1}}{j!}a_{n-j}p^{(j)}(s),$$

let $C_{j,k}$ be the coefficient of $a_{n-k}a_{j+k}s^j$ when $j\geq n$. Then

$$C_{j,k}=(-1)^{k+1}\binom{j+k}{k}.$$

We see that

$$\begin{aligned}C_{j-1,k}-C_{j,k-1}&=(-1)^{k+1}\binom{j-1+k}{k}-(-1)^k\binom{j+k-1}{k-1}\\ &=(-1)^{k+1}\left(\binom{j+k-1}{k}+\binom{j+k-1}{k-1}\right)\\ &=(-1)^{k+1}\binom{j+k}{k}=C_{j,k},\end{aligned}$$

which is the same recursion formula as p_n. Since each of the q_j for $j \leq n-1$ also satisfy the same formula, so does q_n. The term a_n^2 only comes from $-a_n p(s)$, so it has coefficient -1. The coefficient of the term $a_{n-1}a_{n+1}$ comes from $a_{n-1}p^{(1)}(s)$, which is $n+1$, and q_{n-1}, which is

$$(-1)^{n-(n-1)}\binom{n-1}{n-(n-1)} = -(n-1).$$

Hence the coefficient of $a_{n-1}a_{n+1}$ is 2. As these are the initial conditions of p_n, we have shown that $q_n = p_n$.

It is clear that $p_n = -a_n^2 + p + sP$ where p is a polynomial where every term has at least one of $a_0, \dots, a_{n-1}$ as a factor and P is some polynomial. Then as $p_n = q_n \in I$, we have $-a_n^2 + p \in J \subset \sqrt{J}$. As $a_0, \dots, a_{n-1} \in \sqrt{J}$, we thus have $a_n \in \sqrt{J}$. □

Lemma 6.1.3. *We have*

$$V_k|_{s=0} = \{(0,g) \in \widetilde{\mathcal{G}r_{0,\mathbb{A}}} : val_0 x(g) \geq k\}.$$

Proof. We can define an ind-structure on $\widetilde{\mathcal{G}r_{0,\mathbb{A}}}$ using functors $\widetilde{(N^\vee)^{(n)}} : \mathbb{C}\text{-alg} \to \text{Set}$ where

$$\begin{aligned}\widetilde{(N^\vee)^{(n)}}(R) =& \{(s,A) : s \in R, A \in N^\vee(R[t,t^{-1},(t-s)^{-1}]), \text{ and} \\ & A, A^{-1} \text{ have the form} \sum_{-n\leq i,j\leq n} M_{ij}t^i(t-s)^j \text{ where } M_{ij} \in M_m(R)\}.\end{aligned}$$

Let $f_s = t(t-s)$. To see how $\widetilde{(N^\vee)^{(n)}}$ is affine, for a matrix $A = \sum_{-n\leq i,j\leq n} A_{ij}t^i(t-s)^j$ with $A_{ij} \in M_m$, we can consider the matrix

$$f_s^n A = \sum_{i,j\geq 0} A_{ij}t^i(t-s)^j = \sum_{i\geq 0} A_i t^i$$

for some $A_i \in M_m$.

Choose a lift of x in $\mathbb{C}[M_m]$, denoted by $\widetilde{x}$. Since $\iota : N^\vee \to M_m$ is a closed embedding, then we can instead consider $val_{0,s}\widetilde{x}(A) := val_0\widetilde{x}(A) + val_s\widetilde{x}(A)$ for $A \in \iota(N^\vee)(\mathbb{C}[t,t^{-1},(t-s)^{-1}])$ so consider the set

$$\widetilde{V_k^\circ} = \left\{(s,A) : s \in \mathbb{C}\setminus\{0\}, A \in \iota(N^\vee)(\mathbb{C}[t,t^{-1},(t-s)^{-1}]), val_{0,s}\widetilde{x}(A) \geq k\right\}.$$

Let $(s,A) \in \widetilde{(N^\vee)^{(n)}}(R) \cap \widetilde{V_k^\circ}$ and suppose is homogeneous of degree d. Then

$$\widetilde{x}(f_s^n A) = f_s^{dn}\widetilde{x}(A)$$

so

$$val_{0,s}\widetilde{x}(f_s^n A) = val_{0,s} f_s^{dn}\widetilde{x}(A) = 2dn + val_{0,s}\widetilde{x}(A).$$

Suppose $\widetilde{x}$ is not homogeneous, but of degree d. Let $M = (m_{ij}) \in M_m$ be an arbitrary matrix and suppose that

$$\widetilde{x}(m_{ij}) = \sum_{i=0}^{d} P_i(m_{ij})$$

where $P_i(m_{ij})$ is a polynomial of degree i. Let $x' \in \mathbb{C}[M_m]$ be such that

$$x'(m_{ij}) = \sum_{i=0}^{d} P_i(m_{ij}) m_{11}^{d-i}.$$

Note that x' is homogeneous of degree d. As $\iota : N^\vee \hookrightarrow M_r$ factors through U, we have the diagonal entries of $A = (a_{ij})$ to all be equivalent to 1. In particular, $a_{11} \equiv 1$ so $\widetilde{x}(A) = x'(A)$. Hence $val_{0,s}\widetilde{x}(A) \geq k$ if and only if $val_{0,s}x'(A) \geq k$. As x' is homogeneous of degree d, this is equivalent to wanting $val_{0,s}x'(f_s^n A) \geq 2dn + k$.

Let $p(t) = x'(f_s^n A)$. Then $val_{0,s}p \geq 2dn + k$ corresponds to the ideal

$$\langle p(0), \ldots, p^{(2dn+k-1)}(0)\rangle \cap \langle p(0), \ldots, p^{(2dn+k-2)}(0), p(s)\rangle \cap \cdots \cap \langle p(s), \ldots, p^{(2dn+k-1)}(s)\rangle.$$

Hence by Lemma 6.1.2, the 0-fibre corresponds to the ideal

$$\langle p(0), \cdots, p^{(2dn+k-1)}(0)\rangle$$

so we see that we obtain the set $\{(0, p) : val_0 p \geq 2dn + k\}$.

□

Proposition 6.1.4. *Let Z_1 and Z_2 be MV cycles. We will view $Z_2 \subset \mathcal{G}r_s$, for some $s \neq 0$, through the isomorphism $\mathcal{G}r \cong \mathcal{G}r_s$. Suppose*

$$Z_1 * Z_2 = \sum_\alpha Z_\alpha.$$

Then for each α and $x \in \mathbb{C}[N^\vee]$, we have

$$val_0 x(Z_\alpha) \geq val_0 x(Z_1) + val_s x(Z_2).$$

Proof. We can assume our MV cycles are stable, so take $Z_1 \cap S_+^0 \subset N^\vee(\mathcal{K})/N^\vee(\mathcal{O})$ and $Z_2 \cap N_+^\vee(\mathcal{K}_s)t^0 \subset N^\vee(\mathcal{K}_s)/N^\vee(\mathcal{O}_s)$. For $p \in \mathbb{C}$, let $\pi_p : N^\vee(\mathcal{K}_p) \to N^\vee(\mathcal{K}_p)/N^\vee(\mathcal{O}_p)$ be the quotient map. Suppose $\pi_0^{-1}(Z_1 \cap S_+^0) \subset V_k^0$ and $\pi_s^{-1}(Z_2 \cap N_+^\vee(\mathcal{K}_s)t^0) \subset V_l^s$ for some k and l. Then

$$q^{-1}((Z_1 *_{\mathbb{A}^1} Z_2)|_U) \subset U \times V_k \times V_l$$

by using the isomorphism $N^\vee(\mathbb{C}[t, t^{-1}(t-s)^{-1}])/\mathbb{C}[t] \cong N^\vee(\mathbb{C}[t, t^{-1}])/\mathbb{C}[t] \times N^\vee(\mathbb{C}[t, (t-s)^{-1}])/\mathbb{C}[t]$. Since V_{k+l} is closed and

$$V_{k+l}|_{s\neq 0} \cong \bigcup_{a+b=k+l} V_a^0 \times V_b^s,$$

we have

$$q^{-1}(Z_1 *_{\mathbb{A}^1} Z_2) \subset \overline{\mathbb{C}^* \times V_k^0 \times V_l^s} \subset V_{k+l}$$

so $q^{-1}(Z_1 \circ Z_2) \subset V_{k+l}|_0 = V_{k+l}^0$. In particular, this means that for any MV cycle $Z \subset Z_1 \circ Z_2$, we have $val_0 x(Z) \geq val_0 x(Z_1) + val_s x(Z_2)$. □

6.2 Terms in the Fusion

We will determine the MV cycles in a fusion product by finding their associated generic Λ-modules. To do this, we will translate the valuation inequality of Proposition 6.1 into a short exact sequence.

Let M and N be Λ-modules with $\underline{\dim}\, M = \underline{\dim}\, N$. Consider three partial orders on Λ-mod, the first of which was created by [AdF85].

Definition 6.2.1. We say $M \leq_{ext} N$ if there exists modules M_i, U_i, V_i and short exact sequences

$$0 \to U_i \to M_i \to V_i \to 0$$

such that $M = M_1$, $M_{i+1} = U_i \oplus V_i, 1 \leq i \leq s$, and $N = M_{s+1}$ for some s.

Definition 6.2.2. We say M **degenerates** to N, denoted $M \leq_{deg} N$, if $N \in \overline{O(M)}$, the closure of the orbit of M.

Definition 6.2.3. We say $M \leq N$ if $[X, M] \coloneqq \dim \operatorname{Hom}(X, M) \leq [X, N]$ for all modules X.

We have $M \leq N$ if and only if $[M, X] \leq [N, X]$ for all modules X by [AR85]. In general, we have $M \leq_{ext} N \implies M \leq_{deg} N \implies M \leq N$ (see [Bon96] for details), but if every indecomposable is rigid, then the partial orders are equivalent to each other.

Theorem 6.2.1. *[Zwa99, Theorem 2] Let A be an algebra. If every indecomposable of A is rigid, then $\leq$, $\leq_{deg}$, and $\leq_{ext}$ coincide for all A-modules.*

Let $\{x_1, \ldots, x_r\}$ be the initial cluster in $\mathbb{C}[N]$ corresponding to a reduced expression $\mathbf{i}$ of w_0 with $x_{r-n+1}, \ldots, x_r$ being the frozen variables. Let Z_i be the corresponding MV cycle [BKK19, Remark 2.10, Theorem 5.2], so $x_i = b_{Z_i}$. Let T_i, be the corresponding rigid indecomposable module and $T \coloneqq \bigoplus_{i=1}^r T_i$, a basic maximal rigid module. Let x_i be a mutable variable with mutation x_i^*. Similarly, let T_i^* be the mutation of T_i. Let Z_i^* be the MV cycle whose MV polytope is $\operatorname{Pol}(T_i^*)$. The question is if $b_{Z_i^*} = x_i^*$.

From the exchange relation in $\mathbb{C}[N]$, we have

$$x_i x_i^* = x_+ + x_-$$

where x_+ and x_- are monomials in the $x_j, j \neq i$. We also have nonsplit short exact sequences

$$0 \to T_i \to T_+ \to T_i^* \to 0$$

$$0 \to T_i^* \to T_- \to T_i \to 0$$

where T_+ and T_- are in $\operatorname{add}(T/T_i)$. In particular, T_+ and T_- are both rigid. We also have $T_\pm$ corresponding to $x_\pm$.

It suffices to show that $b_{Z_i} b_{Z_i^*} = x_+ + x_-$. Suppose $b_{Z_i} b_{Z_i^*} = \sum_\alpha b_{Z_\alpha}$ where Z_α are MV cycles. Let M_α be the generic Λ-module such that $\operatorname{Pol}(M_\alpha) = \operatorname{Pol}(Z_\alpha)$.

Lemma 6.2.2. *If $M_\alpha \leq_{ext} T_i \oplus T_i^*$, then $M_\alpha \in \{T_+, T_-\}$.*

Proof. By definition, there exist modules M_j, U_j, V_j and short exact sequences $0 \to U_j \to M_j \to V_j \to 0$ such that $M_\alpha = M_1, M_{j+1} = U_j \oplus V_j, 1 \leq j \leq s$, and $T_i \oplus T_i^* = M_{s+1}$ for some s. In particular, we have a short exact sequence of the form

$$0 \to T_i \to M_s \to T_i^* \to 0$$

or

$$0 \to T_i^* \to M_s \to T_i \to 0$$

As the only middle term in extensions of T_i by T_i^* or of T_i^* by T_i are T_+ and T_-, we have $M_s \in \{T_+, T_-, T_i \oplus T_i^*\}$.

Suppose $M_s \in \{T_+, T_-\}$, so in particular, M_s is rigid. Then as $U_{s-1} \oplus V_{s-1} = M_s$ is rigid, the short exact sequence

$$0 \to U_{s-1} \to M_{s-1} \to V_{s-1} \to 0$$

is split. Hence $M_{s-1} = U_{s-1} \oplus V_{s-1} = M_s$. Repeating this argument, we get that $M_j = M_s$ for all $j = 1, \ldots, s$. In particular, $M_1 = M_\alpha \in \{T_+, T_-\}$.

If $M_s = T_i \oplus T_i^*$, then $M_{s-1} \in \{T_+, T_-, T_i \oplus T_i^*\}$ and we repeat the argument. For the case where $M_\alpha = M_1 = T_i \oplus T_i^*$, note that M_α is generic as it comes from an MV cycle. Since $T_+ \leq_{ext} T_i \oplus T_i^*$ and $T_- \leq_{ext} T_i \oplus T_i^*$, we have both rigid modules T_+ and T_- degenerate to $T_i \oplus T_i^*$. This implies that $T_i \oplus T_i^*$ is an element of both irreducible components $\overline{O(T_+)}$ and $\overline{O(T_-)}$. If $T_i \oplus T_i^*$ is also generic, then $T_i \oplus T_i^*$ is in a unique irreducible component but $\overline{O(T_+)} \neq \overline{O(T_-)}$ else $O(T_+) \cap O(T_-) = \emptyset$ is open and dense. Hence $M_\alpha \neq T_i \oplus T_i^*$. □

We now begin the process of showing $[X, M_\alpha] \leq [X, T_i \oplus T_i^*]$ for all cluster modules X. Then in the finite representation type case, as all indecomposables are cluster modules so also rigid, the previous lemma will imply that the only terms that can show up in $Z_i * Z_i^*$ correspond to T_+ or T_-.

Lemma 6.2.3. *Let $x, x' \in \mathbb{C}[N]$ be cluster monomials and Z an MV cycle. Then*

$$val(x + x')(Z) = \min\{valx(Z), valx'(Z)\}.$$

Proof. In [BZ01, Theorem 5.8], the authors used a Lusztig parametrization $x_{\mathbf{i}} : (\mathbb{C}^\times)^r \to N$ to place a positive atlas on N so that if x is a cluster variable in an initial seed coming from a reduced expression $\mathbf{i}$ and $b \in \mathcal{K}^r$, then $x(x_{\mathbf{i}}(b))$ is a positive polynomial, that is, a polynomial with positive coefficients. Since every cluster monomial is a rational function of positive polynomials in the cluster variables of an initial seed, we have

$$val(x + x')(x_{\mathbf{i}}(b)) = \min\{valx(x_{\mathbf{i}}(b)), valx'(x_{\mathbf{i}}(b))\}.$$

Instead of using the parametrization $x_{\mathbf{i}}$, we use the parametrization $y_{\mathbf{i}}$ defined in [Kam10, Section 4.4]. Using [BZ01, Theorem 5.7], if $x_{\mathbf{i}}(b) = y_{\mathbf{i}}(b')$, then b' is a rational function of positive polynomials in terms of b, so $x(y_{\mathbf{i}}(b'))$ is a rational function of positive polynomials in terms of b'. If $b' = (b'_1, \cdots, b'_r) \in \mathcal{K}^r$ is such that for each i, the leading coefficient of b'_i is positive, then we are able to conclude that

$$val(x + x')(y_{\mathbf{i}}(b')) = \min\{valx(y_{\mathbf{i}}(b')), valx'(y_{\mathbf{i}}(b'))\}$$

as there will be no cancellations involving the leading coefficients. Using [Kam10, Theorem 4.2, Theorem 4.5], if $val\, b'_i = n_i$, then the image of $y_{\mathbf{i}}(b')$ in $\mathcal{G}r$ is a generic point of the MV cycle Z' with $\mathbf{i}$-Lusztig datum $n_\bullet$ and each MV cycle has a generic point of this form.

Let $n_\bullet$ be the $\mathbf{i}$-Lusztig datum of Z and choose b' such that $val\, b' = n_i$ and the leading coefficient of each b'_i is in $\mathbb{R}_{>0}$. Then we have

$$\begin{aligned} val(x+x')(Z) &= val(x+x')(y_{\mathbf{i}}(b')) \\ &= \min\{valx(y_{\mathbf{i}}(b')), valx'(y_{\mathbf{i}}(b'))\} \\ &= \min\{valx(Z), valx'(Z)\} \end{aligned}$$

as required. □

Lemma 6.2.4. *Let $T = \bigoplus T_i$ and $R = \bigoplus R_k$ be basic maximal rigid modules and for each i, let $x_i \in \mathbb{C}[N]$ be the cluster variable corresponding to T_i. Fix a mutable module T_i with exchange sequences*

$$0 \to T_i \to T_+ \to T_i^* \to 0 \text{ and } 0 \to T_i^* \to T_- \to T_i \to 0$$

and corresponding exchange relation $x_i x_i^* = x_+ + x_-$. Let Z be an MV cycle with corresponding generic module M such that $M \in add(R)$. If $[T_i, M] = -valx_i(Z)$ for all i, then $[T_i^*, M] = -valx_i^*(Z)$.

Proof. From [GLS07b, Section 7.3] and [GLS11, Proposition 12.4], we have $[T_+, R] \neq [T_-, R]$ and for all k,

$$[T_i \oplus T_i^*, R_k] = \max\{[T_+, R_k], [T_-, R_k]\}$$

so we obtain from [GLS11, Proposition 12.4(iii)] that for each k,

$$[T_i \oplus T_i^*, R_k] = \begin{cases} [T_+, R_k] & \text{if } [T_+, R] > [T_-, R] \\ [T_-, R_k] & \text{if } [T_+, R] < [T_-, R]. \end{cases}$$

In particular, since $M \in \text{add}(R)$,

$$[T_i \oplus T_i^*, M] = \max\{[T_+, M], [T_-, M]\}.$$

Then by Lemma 6.2.3,

$$\begin{aligned} [T_i^*, M] &= \max\{[T_-, M], [T_+, M]\} - [T_i, M] \\ &= \max\{-valx_+(Z), -valx_-(Z)\} + valx_i(Z) \\ &= -\min\{valx_+(Z), valx_-(Z)\} + valx_i(Z) \\ &= -val\left(\frac{x_+ + x_-}{x_i}\right)(Z) \\ &= -valx_i^*(Z) \end{aligned}$$

□

Theorem 6.2.5. *Let M be a generic module corresponding to an MV cycle Z and assume that M underlies a cluster monomial in $\mathbb{C}[N]$. Let X be a module corresponding to a cluster monomial x. Then*

$$-valx(Z) = [X, M].$$

Proof. Let $(x_1, \ldots, x_r)$ be an initial seed coming from a reduced expression $\mathbf{i}$, so each x_i is a generalized minor. From the proof of [Kam10, Theorem 4.5] as well as the results in [BK10], for each i, there exists

$\gamma \in \Gamma$ such that

$$valx_i(Z) = D_\gamma(Z) = -[N(\gamma), M] = -[T_i, M].$$

Since M underlies a cluster monomial, $M \in add(R)$ for some basic maximal rigid module R. Then by Lemma 6.2.4, inducting on mutation, we have $[X', M] = -valx'(Z)$ for all cluster modules X' with corresponding cluster variable x'. Let $(x_1, \cdots, x_r)$ be the cluster x is supported on and $T = T_1 \oplus \cdots \oplus T_r$ be the basic maximal rigid module with $X \in add(T)$ such that x_i corresponds to T_i. Then

$$X = \bigoplus_{i=1}^{r} T_i^{\oplus n_i} \text{ and } x = \prod_{i=1}^{r} x_i^{n_i}$$

so

$$valx(Z) = val\left(\prod_i x_i^{n_i}\right)(Z) = \sum_i n_i valx_i(Z) = -\sum_i n_i [T_i, M] = -[X, M].$$

□

Proposition 6.2.6. *In type A_n for $n \leq 4$, we have $b_{Z_i^*} = ax_i^*$ for some integer $a \geq 1$.*

Proof. In the finite representation type case, it is known that a module is generic if and only if it is an element of $\mathrm{add}(R)$ for some basic maximal rigid module R [CBS02, Theorem 1.1, Theorem 1.2] so using Theorem 6.2.5 and Proposition 6.1.4, we obtain

$$[X, M_\alpha] \leq [X, T_i] + [X, T_i^*] = [X, T_i \oplus T_i^*]$$

for all cluster modules X. Since all indecomposables are cluster modules in the finite representation type case, we have $M_\alpha \leq T_i \oplus T_i^*$. As every indecomposable is rigid, Theorem 6.2.1 implies $M_\alpha \leq_{ext} T_i \oplus T_i^*$. As $M_\alpha \in \{T_+, T_-\}$ by Lemma 6.2.2, we have

$$b_{Z_i} b_{Z_i^*} = ax_+ + bx_-$$

for some $a, b \in \mathbb{Z}_{\geq 0}$. Without loss of generality, suppose $a \leq b$. Then

$$b_{Z_i}(b_{Z_i^*} - ax_i^*) = (b - a)x_-.$$

If $a < b$, then $b_{Z_i} = x_i$ is a factor of x_-, a contradiction [GLS13, Theorem 1.3]. Thus $a = b$ so $b_{Z_i^*} = ax_i^*$ for some $a \geq 1$. □

Chapter 7

Fusion Product in Type A

We show a method of computing fusion products in type A using a generalization of the Mirković-Vybornov isomorphism [MV07b][MV19]. The following is also in [BDK21].

7.1 Notation

7.1.1 Rings and discs

Recall that for $s \in \mathbb{C}$, $\mathcal{O}_s := \mathbb{C}[\![t-s]\!]$ and $\mathcal{K}_s := \mathbb{C}(\!(t-s)\!)$, as well as $\mathcal{O} := \mathcal{O}_0$, $\mathcal{K} := \mathcal{K}_0$, $\mathcal{O}_\infty := \mathbb{C}[\![t^{-1}]\!]$, and $\mathcal{K}_\infty := \mathbb{C}(\!(t^{-1})\!)$. For any point $s \in \mathbb{P} = \mathbb{P}^1$, $\mathcal{O}_s$ is the completion of the local ring $\mathcal{O}_{\mathbb{P},s}$ and thus the formal spectrum of $\mathcal{O}_s$ is the formal neighbourhood D_s of s, also known as the formal disc centered at s. Similarly, the formal spectrum of the field $\mathcal{K}_s$ is the deleted formal neighbourhood, or punctured disc, denoted $D_s^\times$.

7.1.2 Groups

Let H be an algebraic group over $\mathbb{C}$. We will be interested in $H(R)$ where R is a $\mathbb{C}$-algebra. Note that evaluation at $t = s$ provides a group homomorphism $H(\mathcal{O}_s) \to H$. We denote the kernel of this map by $H_1(\mathcal{O}_s)$, often called the first congruence subgroup. We will be particularly interested in this construction in the case $s = \infty$, which gives us the group $H_1(\mathcal{O}_\infty)$.

Throughout this chapter, fix $m \in \mathbb{N}$. We let $G = \mathbf{GL}_m$ and we let $T \subset G$ be the maximal torus of diagonal matrices. We identify $\mathbb{Z}^m$ with the coweight lattice of G and so a coweight $\nu = (\nu_1, \dots, \nu_m)$ is **dominant** if $\nu_1 \geq \cdots \geq \nu_m$ and **effective** if $\nu_j \geq 0$ for all j. If ν is both effective and dominant, then it is a partition of size $|\nu| = \nu_1 + \cdots + \nu_m \in \mathbb{N}$. Recall the positive root cone $R_+ \subset \mathbb{Z}^m$. Explicitly, we have

$$R_+ = \{(\nu_1, \dots, \nu_m) : \nu_1 + \cdots + \nu_j \geq 0 \text{ for } j = 1, \dots, m-1 \text{ and } \nu_1 + \cdots + \nu_m = 0\}.$$

Given $s \in \mathbb{C}$ we define $(t-s)^\nu$ to be the diagonal matrix

$$\begin{bmatrix} (t-s)^{\nu_1} & & & \\ & (t-s)^{\nu_2} & & \\ & & \ddots & \\ & & & (t-s)^{\nu_m} \end{bmatrix}$$

which we can view in $G(K)$ for any ring K containing $(t-s)^{-1}$ and $\mathbb{C}[t]$. For example, $(t-s)^{\nu} \in G(K)$ for $K = \mathcal{K}_s$ or $\mathbb{C}(t)$.

We will also be interested in the affine space M_m of $m \times m$ matrices. Note that for any $\mathbb{C}$-algebra R, $G(R)$ consists of those matrices $M \in M_m(R)$ whose determinant is invertible in R. Thus, for example, $(t-s)^{\mu} \in M_m(\mathbb{C}[t])$ for all effective $\mu \in \mathbb{Z}^m$ but $(t-s)^{\mu} \in G(\mathbb{C}[t])$ if and only if $\mu = 0$.

7.1.3 Lattices

We will use the lattice model for the affine Grassmannian, so it is useful to recall the following definition. Let $R \subset K$ be two $\mathbb{C}$-algebras. Consider K^m as a K-module. By restriction, K^m can be viewed as an R-module. An **R-lattice** in K^m is an R-submodule $L \subset K^m$ which is a free R-module of rank m and satisfies $L \otimes_R K = K^m$. Equivalently, $L = \mathrm{Span}_R(v_1, \ldots, v_m)$ where $v_1, \ldots, v_m$ are free generators of K^m.

We call $R^m \subset K^m$ the standard lattice. The group $\mathbf{GL}_m(K)$ acts transitively on the set of R-lattices in K^m, thus giving a bijection between this set and $\mathbf{GL}_m(K)/\mathbf{GL}_m(R)$, since $\mathbf{GL}_m(R)$ is the stabilizer of the standard lattice.

We will be particularly interested in $\mathbb{C}[t]$-lattices in $\mathbb{C}(t)^m$. Given such a lattice L and a point $a \in \mathbb{C}$, the **specialization** of L at a is the lattice in $\mathcal{K}_a^m$ defined as $L(a) := L \otimes_{\mathbb{C}[t]} \mathcal{O}_a$. If $L(a) = \mathcal{O}_a^m$ then L is said to be **trivial at** a. For example, the lattice $(t-s)^{-1}\mathbb{C}[t] \subset \mathbb{C}(t)$ is trivial at any $a \neq s$, since $t-s$ is invertible in $\mathcal{O}_a$.

7.2 Affine Grassmannians

We now redefine and define various versions of the affine Grassmannian. Each definition is made group-theoretically and then restated as a moduli space of vector bundles and as a moduli space of lattices. We also sketch how to pass between descriptions.

In these definitions, V_{triv} denotes the trivial rank m vector bundle.

Recall we have the affine Grassmannian $\mathcal{G}r_s = G(\mathcal{K}_s)/G(\mathcal{O}_s)$ for $s \in \mathbb{C}$ with $\mathcal{G}r = \mathcal{G}r_0$. It can also be viewed as the moduli space of vector bundles with trivializations,

$$\mathcal{G}r_s = \left\{ (V, \varphi) : V \text{ is a rank } m \text{ vector bundle on } D_s,\ \varphi : V \xrightarrow{\sim} V_{\mathrm{triv}} \text{ on } D_s^{\times} \right\} .$$

As a moduli space of lattices,

$$\mathcal{G}r_s = \{ L \subset \mathcal{K}_s^m :\ L \text{ is a } \mathcal{O}_s\text{-lattice} \} .$$

We obtain a lattice from a pair (V, φ) by setting $L = \Gamma(D_s, V)$ which is embedded into $\mathcal{K}_s^m = \Gamma(D_s^{\times}, V_{\mathrm{triv}})$ using φ. On the other hand, to get a lattice from the group-theoretic description $G(\mathcal{K}_s)/G(\mathcal{O}_s)$, we set $L = g\mathcal{O}_s^m$ for $g \in G(\mathcal{K}_s)$.

Definition 7.2.1. The **thick affine Grassmannian $\boldsymbol{Gr}$** $= G(\mathcal{K}_\infty)/G(\mathbb{C}[t])$.

Again, we have the following modular and lattice descriptions:

$$\boldsymbol{Gr} = \left\{(V,\varphi) : V \text{ is a rank } m \text{ vector bundle on } \mathbb{P},\ \varphi : V \xrightarrow{\sim} V_{\mathrm{triv}} \text{ on } D_\infty \right\},$$
$$\boldsymbol{Gr} = \{L : L \subset \mathcal{K}_\infty^m \text{ is a } \mathbb{C}[t]\text{-lattice}\}\,.$$

Recall the two-point Beilinson–Drinfeld Grassmannian

$$\pi : \mathcal{G}r_{0,\mathbb{A}} \to \mathbb{A}$$

described in modular terms by

$$\mathcal{G}r_{0,\mathbb{A}} = \left\{(V,\varphi,s) : V \text{ is a rank } m \text{ vector bundle on } \mathbb{P}, \varphi : V \xrightarrow{\sim} V_{\mathrm{triv}} \text{ on } \mathbb{P}\setminus\{0,s\}\right\}.$$

The fibre $\mathcal{G}r_{0,s}$ over s is given by

$$G(\mathbb{C}[t,t^{-1},(t-s)^{-1}]/G(\mathbb{C}[t])\,.$$

We also have the lattice descriptions:

$$\mathcal{G}r_{0,\mathbb{A}} = \{(L,s) : L \subset \mathbb{C}(t)^m \text{ is a } \mathbb{C}[t]\text{-lattice trivial at any } a \neq 0, s\}\,,$$
$$\mathcal{G}r_{0,s} = \left\{L : L \subset \mathbb{C}[t,t^{-1},(t-s)^{-1}]^m \text{ is a } \mathbb{C}[t]\text{-lattice}\right\}.$$

Definition 7.2.2. The **positive part** of $\mathcal{G}r$, respectively $\boldsymbol{Gr}$, is defined by

$$\mathcal{G}r^+ = (M_m(\mathcal{O}) \cap G(\mathcal{K}))\,/G(\mathcal{O})\,,\ \text{respectively } \boldsymbol{Gr}^+ = (M_m(\mathbb{C}[t]) \cap G(\mathcal{K}_\infty))\,/G(\mathbb{C}[t])\,.$$

In modular terms, $\mathcal{G}r^+$ (resp. $\boldsymbol{Gr}^+$) is the set of those (V,φ) where $\varphi : V \to V_{\mathrm{triv}}$ extends to an inclusion of coherent sheaves over D_0 (resp. over $\mathbb{P}$). In lattice terms, $\mathcal{G}r^+$ (resp. $\boldsymbol{Gr}^+$) contains those lattices L which are contained in $\mathcal{O}^m$ (resp. $\mathbb{C}[t]^m$).

7.2.1 Relations between different affine Grassmannians

These different versions of the affine Grassmannian are related as follows.

Proposition 7.2.1. *There is a map $\mathcal{G}r_{0,\mathbb{A}} \to \boldsymbol{Gr}$ defined in the following equivalent ways:*

1. *in modular terms as*
$$(V,\varphi,s) \mapsto (V,\varphi|_{D_\infty})\,;$$
2. *in the fibre over $s \in \mathbb{A}$ as the inclusion*
$$G(\mathbb{C}[t,t^{-1},(t-s)^{-1}])/G(\mathbb{C}[t]) \to G(\mathcal{K}_\infty)/G(\mathbb{C}[t])\,;$$
3. *in terms of lattices as*
$$(L,s) \mapsto L\,,$$
using the inclusion $\mathbb{C}[t,t^{-1},(t-s)^{-1}]^m \to \mathcal{K}_\infty^m$ on ambient spaces.

The following result is the factorization property of the BD Grassmannian (see [Zhu16, Prop. 3.13]).

Proposition 7.2.2. *The fibres $\mathcal{G}r_{0,s}$ of $\mathcal{G}r_{0,\mathbb{A}} \to \mathbb{A}$ can be described as*

$$\mathcal{G}r_{0,s} \cong \begin{cases} \mathcal{G}r \times \mathcal{G}r_s & s \neq 0 \\ \mathcal{G}r & s = 0\,. \end{cases}$$

In the modular realization, this isomorphism is given by restricting the vector bundle and trivialization.

Suppose that $s \neq 0$. In the lattice realization, this is given by forming the specializations

$$L \mapsto (L(0), L(s))\,.$$

Note that if $L = g\mathbb{C}[t]^m$ for $g \in G([t, t^{-1}, (t-s)^{-1}])$, then

$$(L(0), L(s)) = (g\mathcal{O}^m, g\mathcal{O}_s^m)\,.$$

In the $s = 0$ case, the isomorphism is described in the same way, except that we just need to form $L(0)$.

We will let θ_s denote the isomorphism

$$\theta_s : \mathcal{G}r_{0,s} \to \mathcal{G}r \times \mathcal{G}r.$$

7.2.2 The fusion construction

We now consider a generalization of the fusion construction introduced in Section 1.3.

Let $\mathcal{G}r_{0,\mathbb{A}^\times}$ denote the preimage of $\mathbb{A}^\times = \mathbb{A} \setminus \{0\}$ under the map $\mathcal{G}r_{0,\mathbb{A}} \to \mathbb{A}$. The isomorphisms θ_s glue together to a projection map

$$\theta : \mathcal{G}r_{0,\mathbb{A}^\times} \to \mathcal{G}r \times \mathcal{G}r\,.$$

Let $X_1, X_2 \subset \mathcal{G}r$ be two subschemes, and consider the subscheme

$$X_1 *_{\mathbb{A}^\times} X_2 := \theta^{-1}(X_1 \times X_2) \subset \mathcal{G}r_{0,\mathbb{A}^\times}$$

Given a point $s \in \mathbb{A}^\times$, we write $X_1 *_s X_2$ for the fibre of $X_1 *_{\mathbb{A}^\times} X_2$ over s. The map θ_s identifies $X_1 *_s X_2 \subset \mathcal{G}r_{0,s}$ with $X_1 \times X_2 \subset \mathcal{G}r \times \mathcal{G}r$.

We define $X_1 *_{\mathbb{A}} X_2$ to be the scheme-theoretic closure of $X_1 *_{\mathbb{A}^\times} X_2$ in $\mathcal{G}r_{0,\mathbb{A}}$. By construction, this is a flat family over $\mathbb{A}$. The fibre $X_1 *_0 X_2$ of $X_1 *_{\mathbb{A}} X_2$ over 0 is a subscheme of $\mathcal{G}r_{0,0}$, but regarded as a subscheme of $\mathcal{G}r$ using the isomorphism $\mathcal{G}r_{0,0} \cong \mathcal{G}r$. We call this subscheme the (geometric) **fusion** of X_1 and X_2.

7.2.3 Some subschemes of affine Grassmannians

Going forward, we fix a pair of arbitrary effective dominant coweights λ', λ'' of sizes N', N'' and a pair of effective coweights μ', μ'' also of sizes N', N'' such that $\mu = \mu' + \mu''$ is dominant. Let $\lambda = \lambda' + \lambda''$ and $N = N' + N''$.

Using $\lambda, \lambda', \lambda''$ and μ, μ', μ'' we define the subschemes of $\mathcal{G}r$, $\boldsymbol{Gr}$, and $\mathcal{G}r_{0,\mathbb{A}}$ which will be considered, give some properties, and say how they are related.

Recall the spherical Schubert cell $\mathcal{G}r^\lambda = G(\mathcal{O})t^\lambda$ and semi-infinite orbits $S^\mu_\pm = N_\pm(\mathcal{K})t^\mu$.

Definition 7.2.3. The **family of two spherical Schubert varieties** $\overline{\mathcal{G}r}^{\lambda',\lambda''}_{0,\mathbb{A}} = \overline{\mathcal{G}r}^{\lambda'} *_{\mathbb{A}} \overline{\mathcal{G}r}^{\lambda''}$.

By a theorem of Zhu, $\overline{\mathcal{G}r}^{\lambda'} *_0 \overline{\mathcal{G}r}^{\lambda''}$ is reduced and equal to $\overline{\mathcal{G}r}^{\lambda}$ ([Zhu16, Proposition 3.1.14]). If $s \neq 0$, then the fibre $\overline{\mathcal{G}r}^{\lambda',\lambda''}_{0,s}$ contains the open locus $\mathcal{G}r^{\lambda',\lambda''}_{0,s} = \mathcal{G}r^{\lambda'} *_s \mathcal{G}r^{\lambda''}$ a $G(\mathbb{C}[t])$-orbit whose lattice description is given in Lemma 7.4.2 below. Because we consider effective dominant λ', λ'', the spherical Schubert varieties and their fusions lie in the positive part of the thick affine Grassmannian.

Lemma 7.2.3. *The map $\mathcal{G}r_{0,\mathbb{A}} \to \boldsymbol{Gr}$ restricts to a map $\overline{\mathcal{G}r}^{\lambda',\lambda''}_{0,\mathbb{A}} \to \boldsymbol{Gr}^+$.*

Proof. Let $s \neq 0$. Since λ', λ'' are effective, $t^{\lambda'}, (t-s)^{\lambda''} \in \boldsymbol{Gr}^+$. Since $\boldsymbol{Gr}^+$ is closed and invariant under the action of $G(\mathbb{C}[t])$, $\overline{\mathcal{G}r}^{\lambda',\lambda''}_{0,s} \subset \boldsymbol{Gr}^+$.

For the $s = 0$ fibre, a similar reasoning applies (or we can simply conclude by taking closure). □

Definition 7.2.4. The **Kazhdan–Lusztig slice** $\mathcal{W}_\mu = G_1(\mathcal{O}_\infty)t^\mu \subset \boldsymbol{Gr}$.

In modular terms, $\mathcal{W}_\mu$ corresponds to the locus of those (V, φ) such that V is isomorphic to the trivial vector bundle on $\mathbb{P}$ and such that φ preserves the Harder–Narasimhan filtration of V at ∞. The lattice description of $\mathcal{W}_\mu \cap \boldsymbol{Gr}_+$ is given in Lemma 7.4.3 below.

Definition 7.2.5. The **family of two semi-infinite orbits** $S^{\mu',\mu''}_{0,\mathbb{A}} \subset \mathcal{G}r_{0,\mathbb{A}}$ which we define to have fibres $S^{\mu',\mu''}_{0,s} := \theta_s^{-1}(S^{\mu'}_- \times S^{\mu''}_-)$ for $s \neq 0$ and fibre $\theta_0^{-1}(S^\mu)$ over $s = 0$.

We can also describe the fibres as orbits

$$S^{\mu',\mu''}_{0,s} = N_-(\mathbb{C}[t, t^{-1}, (t-s)^{-1}])t^{\mu'}(t-s)^{\mu''} \subset G(\mathbb{C}[t, t^{-1}, (t-s)^{-1}])/G(\mathbb{C}[t])\,.$$

See [BGL20, Section 5.2] for further details where this fibre is also described as the attracting locus for a $\mathbb{C}^\times$ action.

In [KWWY14, Prop 2.6], it was observed that $S^\mu \subset \mathcal{W}_\mu$ when μ is dominant. More generally, we have the following result.

Lemma 7.2.4. *Under the map $\mathcal{G}r_{0,\mathbb{A}} \to \boldsymbol{Gr}$, the image of $S^{\mu',\mu''}_{0,\mathbb{A}}$ lands in $\mathcal{W}_\mu$.*

Proof. Let $L \in \boldsymbol{Gr}$ be a lattice in the image of $S^{\mu',\mu''}_{0,\mathbb{A}}$. So we can write $L = gt^{\mu'}(t-s)^{\mu''}\mathbb{C}[t]^m$ for some $g \in N_-(\mathbb{C}[t, t^{-1}, (t-s)^{-1}])$.

Let $h = (t-s)^{\mu''}t^{-\mu''}$. Note that $h \in T_1(\mathcal{O}_\infty)$. Moreover, $L = h(h^{-1}gh)t^\mu\mathbb{C}[t]^m$. Since $h^{-1}gh \in N_-(\mathcal{K}_\infty)$ we can factor $h^{-1}gh$ as n_1n_2 for some $n_1 \in (N_-)_1(\mathcal{O}_\infty)$ and $n_2 \in N_-(\mathbb{C}[t])$.

As μ is dominant, we see that $t^{-\mu}n_2t^\mu \in N_-(\mathbb{C}[t])$, and so $L = hn_1t^\mu\mathbb{C}[t]^m$. Since $hn_1 \in G_1(\mathcal{O}_\infty)$, the result follows. □

7.3 Matrices

We now consider some subvarieties of the space of $N \times N$ matrices, again using the coweights λ, λ', λ'', μ, μ', μ'' fixed in the previous section.

7.3.1 Adjoint orbits and their deformations

Recall that $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a partition of N. Given a point $s \in \mathbb{A}$ we write $J_{s,\lambda}$ for the Jordan form matrix with eigenvalue s and Jordan blocks of sizes $\lambda_1, \ldots, \lambda_m$.

Definition 7.3.1. The nilpotent (adjoint) orbit $\mathbb{O}^\lambda \subset M_N(\mathbb{C})$ of matrices conjugate to $J_{0,\lambda}$. These matrices and the linear operators they represent will be said to have Jordan type λ.

Definition 7.3.2. For $s \in \mathbb{A}^\times$, the (adjoint) orbit $\mathbb{O}_{0,s}^{\lambda',\lambda''}$ of matrices conjugate to $J_{0,\lambda'} \oplus J_{s,\lambda''}$. These matrices and the linear operators they represent will be said to have Jordan type $((0, \lambda'), (s, \lambda''))$.

We recall that these orbits have closures which are given by rank conditions. More precisely, we have

$$\overline{\mathbb{O}}^\lambda = \{A \in M_N(\mathbb{C}) : \operatorname{rank} A^c \leq N - \#\text{ boxes in first } c \text{ columns of } \lambda, \text{ for } c \in \mathbb{N}\}$$

and

$$\begin{aligned} \overline{\mathbb{O}}_{0,s}^{\lambda',\lambda''} = \{A \in M_N(\mathbb{C}) : \operatorname{rank} A^c \leq N - \#\text{ boxes in first } c \text{ columns of } \lambda', \text{ for } c \in \mathbb{N}, \text{ and} \\ \operatorname{rank}(A - s)^c \leq N - \#\text{ boxes in first } c \text{ columns of } \lambda'', \text{ for } c \in \mathbb{N}\}\,. \end{aligned} \tag{7.3.1}$$

Proposition 7.3.1. *There exists a flat family* $\overline{\mathbb{O}}_{0,\mathbb{A}}^{\lambda',\lambda''} \to \mathbb{A}$ *whose fibre over* $s \in \mathbb{A}$ *is reduced and given by* $\overline{\mathbb{O}}_{0,s}^{\lambda',\lambda''}$ *if* $s \neq 0$ *and* $\overline{\mathbb{O}}^\lambda$ *if* $s = 0$.

In order to prove this proposition, we will need to recall some results from Eisenbud–Saltman [ES89].

Let r be a decreasing, non-negative function of $\mathbb{N}$ with $r(0) = N$, which we call a **rank function**. Let k be maximal such that $r(k) \neq 0$. Let W be a subspace of $\mathbb{A}^k$. Eisenbud and Saltman define and study the flat family $X_{r,W} \to W$ whose fibres are reduced and defined by rank conditions. We will now describe these fibres.

Let $z \in \mathbb{A}^k$ be a point. Some of the coordinates of z may be equal, so we introduce the following data to keep track of these equalities. Let ℓ denote the number of distinct coordinates of z. There exist a sequence of integers $\underline{k} = (k_0, \ldots, k_\ell)$ such that $0 = k_0 < k_1 < \cdots < k_\ell = k$, a point $\underline{s} \in \mathbb{A}^\ell$, such that $s_a \neq s_b$ for any $a \neq b$, and a permutation p of $\{1, \ldots, k\}$ such that

$$s_a = z_{p(k_{a-1}+1)} = z_{p(k_{a-1}+2)} = \cdots = z_{p(k_a)}$$

for each $a = 1, \ldots, \ell$. Given the choice of s we require that p is of minimal length. Given z, the data $\underline{k}, \underline{s}, p$ is unique up to a permutation of $\{1, \ldots, \ell\}$.

Next for $a = 1, \ldots, \ell$ and $c = 1, \ldots, k_{a+1} - k_a$, we define

$$r_{z,a}(c) := N + \sum_{d=1}^{c} r(p(k_a + d)) - r(p(k_a + d) - 1)\,.$$

By [ES89, Corollary 2.2], the fibre of the flat family $X_{r,W}$ over z is given by

$$X_{r,z} = \{A \in M_N(\mathbb{C}) : \operatorname{rank}(A - s_a)^c \leq r_{z,a}(c) \text{ for all } c, a \text{ as above}\}. \tag{7.3.2}$$

We will now apply these ideas to prove our Proposition 7.3.1.

Proof. For $c \in \mathbb{N}$, let

$$r(c) = N - \#\text{ boxes in first } c \text{ columns of } \lambda\,.$$

It is easy to see that this is a rank function, with $k = \lambda_1$, the number of columns of λ.

Since $\lambda = \lambda' + \lambda''$, there exists a permutation p of $\{1, \dots, k\}$ (the columns of λ) such that $p(1), \dots, p(k_1)$ are the columns of λ' and $p(k_1+1), \dots, p(k)$ are the columns of λ''. Here $k_1 = \lambda'_1$ the number of columns of λ'.

For $s \in \mathbb{A}$, we define $z(s) \in \mathbb{A}^k$ by

$$\begin{cases} z(s)_{p(c)} = 0 & c = 1, \dots, k_1 \\ z(s)_{p(c)} = s & c = k_1 + 1, \dots, k\,. \end{cases}$$

For $s \neq 0$, we see that the equalities in $z(s)$ give rise to the data of $\ell = 2$, $\underline{k} = (0, \lambda'_1)$, $\underline{s} = (0, s)$ and the permutation p. We also see that

$$\begin{aligned} r_{z(s),1}(c) &= N - \#\text{ boxes in first } c \text{ columns of } \lambda' \\ r_{z(s),2}(c) &= N - \#\text{ boxes in first } c \text{ columns of } \lambda''\,. \end{aligned} \tag{7.3.3}$$

On the other hand, for $z(0)$, we get that $\ell = 1$ and that

$$r_{z(0),1}(c) = r(c) = N - \#\text{ boxes in first } c \text{ columns of } \lambda\,. \tag{7.3.4}$$

Let $W = \{z(s) : s \in \mathbb{C}\}$. Combining Equations (7.3.1) to (7.3.4), we see that the Eisenbud–Saltman family $X_{r,W} \to W$ gives our family $\overline{\mathbb{O}}^{\lambda',\lambda''}_{0,\mathbb{A}}$. ☐

7.3.2 The Mirković–Vybornov slice

Recall that $\mu = (\mu_1, \dots, \mu_m)$ is a partition of N. The **Mirković–Vybornov slice** $\mathbb{T}_\mu$ is the affine space of $N \times N$ matrices of the form $A = J_{0,\mu} + X$ where X is a $\mu \times \mu$ block matrix with possibly nonzero entries $A^1_{ij}, \dots, A^{\min(\mu_i,\mu_j)}_{ij}$ in the first $\min(\mu_i, \mu_j)$ columns of the last row of each $\mu_i \times \mu_j$ block.

By example, if $\mu = (3, 2)$, then $A \in \mathbb{T}_\mu$ looks like

$$\left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ A^1_{11} & A^2_{11} & A^3_{11} & A^1_{12} & A^2_{12} \\ \hline 0 & 0 & 0 & 0 & 1 \\ A^1_{21} & A^2_{21} & 0 & A^1_{22} & A^2_{22} \end{array}\right]$$

for some $A^k_{ij} \in \mathbb{C}$.

To $A \in \mathbb{T}_\mu$ we will associate the $m \times m$ matrix of polynomials $g(A)$ in $M_m(\mathbb{C}[t])$ whose (i,j)th entry is defined as follows.

$$g(A)_{ij} = \begin{cases} t^{\mu_i} - \sum_{k=1}^{\mu_i} A^k_{ji} t^{k-1} & i = j\,, \\ -\sum_{k=1}^{\mu_i} A^k_{ji} t^{k-1} & i \neq j\,. \end{cases} \tag{7.3.5}$$

Continuing with the $\mu = (3,2)$ example,

$$g(A) = \begin{bmatrix} t^3 - A_{11}^3 t^2 - A_{11}^2 t - A_{11}^1 & -A_{21}^2 t - A_{21}^1 \\ -A_{12}^2 t - A_{12}^1 & t^2 - A_{22}^2 t - A_{22}^1 \end{bmatrix}.$$

In other words, the $\mu_i \times \mu_j$ block of A is used to produce a polynomial which is inserted in the (j,i) entry of the $m \times m$ matrix $g(A)$.

In $\mathbb{T}_\mu$, we will be interested in a certain family of block upper-triangular matrices.

Definition 7.3.3. The **upper-triangular** Mirković–Vybornov slice $\mathbb{U}_{0,\mathbb{A}}^{\mu',\mu''} \to \mathbb{A}$ is defined by

$$\mathbb{U}_{0,\mathbb{A}}^{\mu',\mu''} := \{(A,s) \in \mathbb{T}_\mu \times \mathbb{A} : g(A)_{ii} = t^{\mu_i'}(t-s)^{\mu_i''}, g(A)_{ij} = 0 \text{ for } j < i\}.$$

So a matrix in $\mathbb{U}_{0,\mathbb{A}}^{\mu',\mu''}$ is weakly block upper-triangular and its diagonal blocks are given by the companion matrices for the polynomials $t^{\mu_i'}(t-s)^{\mu_i''}$ where $i = 1, \ldots, m$.

For example, elements of $\mathbb{U}_{0,s}^{(1,1,0),(2,1,1)}$ look like

$$\left[\begin{array}{ccc|cc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -s^2 & 2s & A_{12}^1 & A_{12}^2 & A_{13}^1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & s & A_{23}^1 \\ \hline 0 & 0 & 0 & 0 & 0 & s \end{array}\right].$$

Note that the fibre $\mathbb{U}_{0,0}^{\mu',\mu''}$ is the same as the intersection $\mathbb{T}_\mu \cap \mathfrak{n}$, where $\mathfrak{n}$ denotes the set of strictly upper-triangular matrices.

7.4 Mirković–Vybornov Generalization

Throughout this section, we continue our notation of the previous sections. So $\lambda, \lambda', \lambda''$ denote dominant effective coweights, with $\lambda' + \lambda'' = \lambda$. Also μ, μ', μ'' are effective coweights with $\mu = \mu' + \mu''$ and we assume that μ is dominant.

7.4.1 Linear Operators from lattices

Lemma 7.4.1. *Let V be a $\mathbb{C}[t]$-module which is finite-dimensional as a complex vector space. For any $a \in \mathbb{A}$, the map*

$$V \to V \otimes_{\mathbb{C}[t]} \mathcal{O}_a$$

restricts to an isomorphism between the generalized a-eigenspace of t and $V \otimes_{\mathbb{C}[t]} \mathcal{O}_a$

Proof. For any $b \in \mathbb{A}$, let E_b denote the generalized b-eigenspace of t. Then $V = \oplus_{b \in \mathbb{C}} E_b$. Since $t - b$ is invertible in $\mathcal{O}_a$ and $t - b$ acts nilpotently on E_b, we see that $E_b \otimes_{\mathbb{C}[t]} \mathcal{O}_a = 0$.

So it suffices to show that $E_a \to E_a \otimes_{\mathbb{C}[t]} \mathcal{O}_a$ is an isomorphism. By the classification of modules over $\mathbb{C}[t]$, it suffices to check this when $E_a = \mathbb{C}[t]/(t-a)^k$, where it is clearly true. □

Lemma 7.4.2. *Let $L \in \boldsymbol{Gr}^+$. Let $s \in \mathbb{A}^\times$. The following are equivalent:*

(i) L is in the image of the map $\mathcal{G}r_{0,s}^{\lambda',\lambda''} \to \boldsymbol{Gr}^+$.

(ii) The linear operator t on $\mathbb{C}[t]^m/L$ has Jordan type $((0,\lambda'),(s,\lambda''))$.

(iii) $L \in G(\mathbb{C}[t])t^{\lambda'}(t-s)^{\lambda''}$.

Proof. First we recall that for $L \in \mathcal{G}r^+$, $L \in \mathcal{G}r^\lambda = G(\mathcal{O})t^\lambda$ if and only if $t|_{\mathcal{O}^m/L}$ has Jordan type λ.

Now assume that $(L,s) \in \mathcal{G}r_{0,s}^{\lambda',\lambda''}$. By definition, $L(0) \in \mathcal{G}r^{\lambda'}$ and $L(s) \in \mathcal{G}r^{\lambda''}$. This means that t acting on $\mathcal{O}^m/L(0)$ has Jordan type λ' and t acting on $\mathcal{O}_s^m/L(s)$ has Jordan type λ''. For $a = 0, s$, we see that

$$\mathbb{C}[t]^m/L \otimes_{\mathbb{C}[t]} \mathcal{O}_a \cong \mathcal{O}_a^m/L(a).$$

Lemma 7.4.1 shows that the map $\mathbb{C}[t]^m/L \to \mathcal{O}^m/L(0)$ induces an isomorphism between the 0-generalized eigenspace of t and $\mathcal{O}^m/L(0)$. The same thing holds for the s-generalized eigenspace of t and $\mathcal{O}_s^m/L(s)$. This shows that item (i) implies item (ii) and the logic can be reversed to see that item (ii) implies item (i).

On the other hand, if $L = gt^{\lambda'}(t-s)^{\lambda''}\mathbb{C}[t]^m$ for some $g \in G(\mathbb{C}[t])$, then $L(0) = gt^{\lambda'}\mathcal{O}^m$ since $(t-s)^{\lambda''} \in G(\mathcal{O})$. In this way, we see that item (iii) implies item (ii) and the logic can be reversed to get equivalence. □

7.4.2 Matrices from lattices

Lemma 7.4.3. *Let $L \in \boldsymbol{Gr}^+$. The following are equivalent:*

(i) $L \in \mathcal{W}_\mu$.

(ii) $L = \mathrm{Span}_{\mathbb{C}[t]}(v_1,\dots,v_m)$ for some v_i of the form $v_i = t^{\mu_i}e_i + \sum_{j=1}^m p_{ij}e_j$ where $p_{ij} \in \mathbb{C}[t]$ has degree less than $\min(\mu_i,\mu_j)$.

(iii) For all i,

$$t^{\mu_i}e_i \in \mathrm{Span}_{\mathbb{C}}(\{t^k e_j : 0 \le k < \min(\mu_i,\mu_j), 1 \le j \le m\}) + L.$$

Moreover, for such L, $\beta_\mu := \{[t^k e_i] : 0 \le k < \mu_i, 1 \le i \le m\}$ forms a basis for $\mathbb{C}[t]^m/L$.

Proof. Let $L \in \mathcal{W}_\mu$. Then $L = \mathrm{Span}_{\mathbb{C}[t]}(v_1,\dots,v_m)$ for some v_i with $v_i = t^{\mu_i}e_i + \sum_{j=1} q_{ij}t^{\mu_i}e_j$ and $q_{ij} \in t^{-1}\mathcal{O}_\infty$. Since $L \in \boldsymbol{Gr}^+$, we see that $v_i \in \mathbb{C}[t]^m$ which means that $p_{ij} := q_{ij}t^{\mu_i}$ lies in $\mathbb{C}[t]$. By construction, the polynomial p_{ij} has degree less than μ_i.

Fix i and suppose that for some j, $\mu_j < \mu_i$. In this case, we can alter our basis to $v_i' = v_i - qv_j$ for some polynomial $q \in \mathbb{C}[t]$. This gives us new polynomials $p_{ij}' = p_{ij} - q(t^{\mu_j} + p_{jj})$. In this way, we can ensure that p_{ij} has degree less than $\min(\mu_i,\mu_j)$. Thus item (i) implies item (ii).

Suppose that $L = \mathrm{Span}_{\mathbb{C}[t]}(v_1,\dots,v_m)$ as in item (ii). Then

$$t^{\mu_i}e_i - v_i \in \mathrm{Span}_{\mathbb{C}}(\{t^k e_j : k < \min(\mu_i,\mu_j), 1 \le j \le m\})\,.$$

Hence item (ii) implies item (iii).

Finally, given item (iii), then we can see $v_i := t^{\mu_i}e_i - \sum_{j=1}^m p_{ij}e_j \in L$ for some $p_{ij} \in \mathbb{C}[t]$ of degree less than $\min(\mu_i, \mu_j)$. It is easy to see that $L = \operatorname{Span}_{\mathcal{O}}(v_1, \dots, v_m)$ and so $L \in \mathcal{W}_\mu$. To show that β_μ forms a basis for $\mathbb{C}[t]^m/L$, it suffices to show that for each i,

$$t^{\mu_i}e_i \in \operatorname{Span}_{\mathbb{C}}(t^k e_i : 0 \le k < \mu_i, 1 \le i \le m) + L\,.$$

This follows immediately from item (iii). □

Given $A \in \mathbb{T}_\mu$, recall the definition of $g(A)$ given in Equation (7.3.5). Note that $g(A)t^{-\mu} \in G_1(\mathcal{O}_\infty)$. Since $g(A) \in M_m(\mathbb{C}[t]) \cap G_1(\mathcal{O}_\infty)t^\mu$, we will regard $g(A)$ as giving an element of $\boldsymbol{Gr}^+ \cap \mathcal{W}_\mu$. Alternatively, we can see that $g(A)\mathbb{C}[t]^m$ satisfies the condition of item (ii) from Lemma 7.4.3.

The following result is called the Mirković–Vybornov isomorphism [MV07b]. In its present form, it can be found in [CK18, Theorem 3.2], except that we have tweaked both maps with a matrix transpose $[\ \ ]^{\mathrm{tr}}$.

Theorem 7.4.4. *The map $\mathbb{T}_\mu \to \boldsymbol{Gr}^+ \cap \mathcal{W}_\mu$ given by $A \mapsto g(A)\mathbb{C}[t]^m$ is an isomorphism with inverse given by*

$$L \mapsto [t|_{\mathbb{C}[t]^m/L}]^{\mathrm{tr}}_{\beta_\mu}\,.$$

7.4.3 Upper-triangularity and the Mirkovic-Vybornov isomorphism

For the next result, we will consider the "intersection" of $\overline{\mathcal{G}r}^{\lambda',\lambda''}_{0,\mathbb{A}}$ with $\mathcal{W}_\mu$. As $\overline{\mathcal{G}r}^{\lambda',\lambda''}_{0,\mathbb{A}}$ is not a subscheme of $\boldsymbol{Gr}$, by this intersection, we really mean the preimage of $\mathcal{W}_\mu$ under the composition

$$\overline{\mathcal{G}r}^{\lambda',\lambda''}_{0,\mathbb{A}} \hookrightarrow \mathcal{G}r_{0,\mathbb{A}} \to \boldsymbol{Gr}\,.$$

This is not a very serious abuse of notation, since the map $\mathcal{G}r_{0,\mathbb{A}} \to \boldsymbol{Gr}$ is almost injective. In a similar way, we will write $\overline{\mathbb{O}}^{\lambda',\lambda''}_{0,\mathbb{A}} \cap \mathbb{T}_\mu$ using the "almost injective" map $\overline{\mathbb{O}}^{\lambda',\lambda''}_{0,\mathbb{A}} \to M_N(\mathbb{C})$.

The following refinement of the Mirković–Vybornov isomorphism is a special case of [MV07b, Theorem 5.3].

Theorem 7.4.5. *There is an isomorphism*

$$\overline{\mathbb{O}}^{\lambda',\lambda''}_{0,\mathbb{A}} \cap \mathbb{T}_\mu \cong \overline{\mathcal{G}r}^{\lambda',\lambda''}_{0,\mathbb{A}} \cap \mathcal{W}_\mu$$

given by $(A, s) \mapsto (g(A)\mathbb{C}[t]^m, s)$.

Proof. Since we already have the isomorphism from Theorem 7.4.4, it suffices to show that for any $A \in \mathbb{T}_\mu$,

$$(A, s) \in \overline{\mathbb{O}}^{\lambda',\lambda''}_{0,\mathbb{A}} \cap \mathbb{T}_\mu \text{ if and only if } (g(A)\mathbb{C}[t]^m, s) \in \overline{\mathcal{G}r}^{\lambda',\lambda''}_{0,\mathbb{A}} \cap \mathcal{W}_\mu\,.$$

This follows immediately from Lemma 7.4.2. □

Theorem 7.4.6. *The isomorphism from Theorem 7.4.5 restricts to an isomorphism*

$$\overline{\mathbb{O}}^{\lambda',\lambda''}_{0,\mathbb{A}} \cap \mathbb{U}^{\mu',\mu''}_{0,\mathbb{A}} \cong \overline{\mathcal{G}r}^{\lambda',\lambda''}_{0,\mathbb{A}} \cap S^{\mu',\mu''}_{0,\mathbb{A}}\,.$$

Proof. We could prove this by observing the both sides are the attracting locus of an appropriate $\mathbb{C}^\times$ action. However, we will give the following more algebraic proof.

Let $A \in \mathbb{T}_\mu$ and $s \in \mathbb{C}$. We must show that $(A, s) \in \mathbb{U}_{0,\mathbb{A}}^{\mu',\mu''}$ if and only if $(g(A)\mathbb{C}[t]^m, s) \in S_{0,\mathbb{A}}^{\mu',\mu''}$.

On the one hand, if $(A, s) \in \mathbb{U}_{0,\mathbb{A}}^{\mu',\mu''}$, then $g(A)$ is lower-triangular with diagonal $t^{\mu'}(t-s)^{\mu''}$, and so $g(A) \in N_-[t, t^{-1}, (t-s)^{-1}]t^{\mu'}(t-s)^{\mu''}$.

On the other hand, if $(g(A)\mathbb{C}[t]^m, s) \in S_{0,\mathbb{A}}^{\mu',\mu''}$, then we can write

$$gt^\mu r = nt^{\mu'}(t-s)^{\mu''}$$

for some $r \in G(\mathbb{C}[t]), n \in N_-(\mathbb{C}[t, t^{-1}, (t-s)^{-1}])$ and $g = g(A)t^{-\mu}$. Let $h = (t-s)^{\mu''}t^{-\mu''}$ which lies in $T_1(\mathcal{O}_\infty)$. Note that $h^{-1}nh \in N_-(\mathcal{K}_\infty)$, so we can factor it as $h^{-1}nh = n_1 n_2$, where $n_1 \in N_{-,1}(\mathcal{O}_\infty), n_2 \in N_-(\mathbb{C}[t])$. So then after doing a bit of algebra, we reach

$$t^\mu r(t^{-\mu}n_2^{-1}t^\mu)t^{-\mu} = g^{-1}hn_1.$$

Since $g, h, n_1 \in G_1(\mathcal{O}_\infty)$, the right hand side $g^{-1}hn_1$ lies in $G_1(\mathcal{O}_\infty)$. Since μ is dominant, $t^{-\mu}n_2t^\mu \in N_-(\mathbb{C}[t])$, and so the left hand side lies in $t^\mu G(\mathbb{C}[t])t^{-\mu}$.

Moreover, since μ is dominant, we know that

$$t^\mu G(\mathbb{C}[t])t^{-\mu} \cap G_1(\mathcal{O}_\infty) = N_{-,1}(\mathcal{O}_\infty)\,.$$

Thus, we deduce that $g^{-1}hn_1 \in N_{-,1}(\mathcal{O}_\infty)$ and hence $g(A) \in t^{\mu'}(t-s)^{\mu''}N_{-,1}(\mathcal{O}_\infty)$. Since $A \in \mathbb{T}_\mu$, this implies that $(A, s) \in \mathbb{U}_{0,\mathbb{A}}^{\mu',\mu''}$ as desired. $\square$

Restricting to the zero fibre and applying θ_0 together with Proposition 7.3.1, we recover the following result from [Dra20a].

Corollary 7.4.7. *([Dra20a, Corollary 5.2.2]) The map $A \mapsto g(A)\mathcal{O}^m$ gives an isomorphism*

$$\overline{\mathbb{O}}^\lambda \cap \mathbb{T}_\mu \cap \mathfrak{n} \cong \overline{\mathcal{G}r}^\lambda \cap S_-^\mu\,.$$

7.5 Application to the Fusion Product

7.5.1 MV cycles and tableaux

Continuing now with the notation of the previous sections, we add the assumption that $\mu \le \lambda$ and forget the assumption that μ is dominant. Let $r = m(m-1)/2$ be the number of positive roots in $\mathbf{GL}_m$. We will now explain how to index MV cycles for $\mathbf{GL}_m$ using Young tableaux.

Denote by $[a, b]$ the interval $\{a, a+1, \ldots, b-1, b\} \subset \mathbb{N}$. We begin with the following definitions.

Definition 7.5.1. Let $L \subset \mathcal{O}^m$ be a point in $\mathcal{G}r_+$. We define the relative dimension of L by

$$\operatorname{rdim} L := \dim_{\mathbb{C}} \mathcal{O}^m/L\,.$$

If $\gamma \subset [1, m]$, we define two lattices L^γ, L_γ in $\mathcal{O}^\gamma := \operatorname{Span}_{\mathcal{O}}(e_i : i \in \gamma)$ by

$$L^\gamma := L \cap \mathcal{O}^\gamma \qquad L_\gamma := L/L^{\gamma^c}$$

where $\gamma^c := [1,m] \setminus \gamma$.

We will also need the analogous definitions when $\mathcal{O}$ is replaced by $\mathbb{C}[t]$ and $L \in \boldsymbol{Gr}_+$. We record the following easy observations.

Lemma 7.5.1. *Let $L \in \mathcal{G}r_+$ or $\boldsymbol{Gr}_+$.*

1. *For any $\gamma \subset [m]$, we have*
$$\operatorname{rdim} L_\gamma + \operatorname{rdim} L^{\gamma^c} = \operatorname{rdim} L\,.$$

2. *For any $\gamma_1 \subset \gamma_2 \subset [m]$, we have*
$$(L_{\gamma_2})_{\gamma_1} = L_{\gamma_1}\,.$$

For any $L \in \mathcal{G}r_+$ and $\gamma \subset [m]$, we define $D_\gamma(L) := \operatorname{rdim} L^\gamma = \operatorname{rdim} L - \operatorname{rdim} L_{\gamma^c}$. By [Kam10, Proposition 9.3], this coincides with the definition of D_γ used to compute the hyperplanes of the associated MV polytopes.

Lemma 7.5.2. *Let $L \in \mathcal{G}r_+$. The following are equivalent:*

1. *$L \in S_+^\mu$*

2. *$\operatorname{rdim} L^{[1,i]} = \mu_1 + \cdots + \mu_i$ for all $i = 1, \ldots, m$.*

3. *$\operatorname{rdim} L_{[i+1,m]} = \mu_{i+1} + \cdots + \mu_m$ for $i = 1, \ldots, m$.*

Similarly, the following are equivalent:

1'. *$L \in S_-^\mu$*

2'. *$\operatorname{rdim} L^{[i+1,m]} = \mu_{i+1} + \cdots + \mu_m$ for all $i = 1, \ldots, m$.*

3'. *$\operatorname{rdim} L_{[1,i]} = \mu_1 + \cdots + \mu_i$ for $i = 1, \ldots, m$.*

Proof. We prove the first statement as the second is similar. Let $L = nt^\mu \mathbb{C}[t]^m$ for some $n \in N_+(\mathcal{K})$ and let $v_1, \ldots, v_m$ denote the columns of the matrix nt^μ. Then $L = \operatorname{Span}_{\mathcal{O}}(v_1, \ldots, v_m)$. Fix $i \leq m$. Then $L^{[1,i]} = \operatorname{Span}_{\mathcal{O}}(w_1, \ldots, w_i)$ where w_j denotes the first i entries of v_j (by upper-triangularity, the rest of the entries are 0, in any case).

Assume that $L \subset \mathcal{O}^m$. Then $L^{[1,i]} \subset \mathcal{O}^i$ and $\operatorname{rdim} L^{[1,i]}$ is the valuation of the determinant of the matrix whose columns are $w_1, \ldots, w_i$. Since this matrix is upper-triangular with diagonal entries $t^{\mu_1}, \ldots, t^{\mu_i}$, this determinant is $t^{\mu_1 + \cdots + \mu_i}$ and hence $\operatorname{rdim} L^{[1,i]} = \mu_1 + \cdots + \mu_i$.

Thus item 1 implies item 2. The converse follows from the fact that every L lies in some S_+^ν and by the above reasoning, ν is determined by the values of $\dim L^{[1,i]}$.

The equivalence of item 2 and item 3 follows from Lemma 7.5.1. □

We now introduce some notation related to Young tableaux. Denote by $YT(\lambda)$ the set of Young tableaux of shape λ and by $YT(\lambda)_\mu$ the subset of $YT(\lambda)$ of tableaux having weight μ. Let $YT = \bigcup_\lambda YT(\lambda)$ and denote by $YT_+ \subset YT$ the subset of those tableaux having dominant weight.

Given $\tau \in YT(\lambda)_\mu$ and $i \in \{1, \ldots m\}$, denote by $\lambda(i)$ (resp. $\mu(i)$) the shape (resp. the weight) of $\tau(i)$, the tableau got from τ by discarding all boxes of weight exceeding i. Note that $\mu(i)$ only depends on μ, while $\lambda(i)$ depends on the tableau τ. We will regard $\lambda(i)$ (resp. $\mu(i)$) as an effective dominant coweight (resp. effective coweight) for $\mathbf{GL}_i$.

The Lusztig datum $n_\bullet(\tau)$ of the tableau τ is a list of r non-negative integers defined from its **Gelfand–Tsetlin pattern** (see [BZ88, Sect. 4]) $\mathrm{gt}(\tau) = (\lambda(i)_j)_{1\le j\le i\le m}$ by the formula

$$n_\bullet(\tau)_{(a,b)} = \lambda(b)_a - \lambda(b-1)_a = \text{ \# of boxes on row } a \text{ of } \tau \text{ of weight } b$$
$$(a,b) = (1,2),\dots,(1,m),(2,3),\dots,(2,m),\dots,(m-1,m)\,.$$

Note that $\lambda - \mu = \sum_{1\le a<b\le m} n_\bullet(\tau)_{(a,b)}\beta_{a,b}$ where $\beta_{a,b}$ denotes the positive root of G with a 1 in the a slot and a -1 in the b slot. The pattern $\mathrm{gt}(\tau)$ is recorded as a lower-triangular matrix (the array of shapes of subtableaux $\tau(i)$) and the datum $n_\bullet(\tau)$ is recorded as a sequence, unless noted otherwise. Below is an example with $\lambda = (4,2)$ and $\mu = (3,2,1)$.

$$\tau = \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2&3 \\ \cline{1-2}\end{array} \qquad \mathrm{gt}(\tau) = \begin{matrix} 3 & & \\ 4 & 1 & \\ 4 & 2 & 0 \end{matrix} \qquad n_\bullet(\tau) = (1,0,1)$$

We can associate to τ the locus

$$\mathring{Z}(\tau) = \{L \in S_-^\mu : L_{[1,i]} \in \mathcal{G}r^{\lambda(i)} \text{ for } i = 1,\dots m\}\,.$$

The following result is closely related to Theorem 5.4.3 from [Dra20a]. It is also closely related to the description of MV cycles in terms of Kostant data obtained by Anderson–Kogan [AK04] (see [Kam10, Section 9]). We remark that a different map from Young tableaux to MV cycles was obtained by Gaussent, Littelmann and Nguyen [GLN13, Theorem 2]; we are not certain of the relation with our construction.

Proposition 7.5.3. *$\mathring{Z}(\tau)$ has a unique irreducible component of dimension $\rho(\lambda-\mu)$. Let $Z(\tau)$ denote the closure of this component. Then, $Z(\tau)$ is the MV cycle whose Lusztig datum (with respect to the standard reduced word) is $n_\bullet(\tau)$.*

Remark 7.5.2. We believe that $\mathring{Z}(\tau)$ is irreducible, so that in fact $Z(\tau) = \overline{\mathring{Z}(\tau)}$.

Proof. The proof follows the same strategy as in [Dra20a]. First, we consider

$$\mathring{Z}(\tau)_1 := \{L \in S_-^\mu \cap \mathcal{G}r_+ : L_{[1,i]} \in S_+^{\lambda(i)} \text{ for } i = 1,\dots m\}\,.$$

Using Lemma 7.5.1(2) and Lemma 7.5.2, we see that

$$\mathring{Z}(\tau)_1 = \{L \in \mathcal{G}r_+ : \mathrm{rdim}\, L_{[a,b]} = \lambda(b)_a + \cdots + \lambda(b)_b \text{ for all } 1 \le a \le b \le m\}$$

where note that $\lambda(b)_a + \cdots + \lambda(b)_b$ is the number of boxes on rows $a,\dots,b$ of weight $1,\dots,b$.

From the proof of [Kam10, Proposition 9.6], we see that $\mathring{Z}(\tau)_1$ is equal to $t^\mu A(n_\bullet(\tau))$, where $A(n_\bullet)$ is defined in [Kam10, Sect. 4.3]. In particular, it is irreducible and of dimension $\rho(\lambda-\mu)$. Moreover, its closure is the MV cycle whose Lusztig datum is $n_\bullet(\tau)$. Now, by results of [Kam08] (see especially the proof of Theorem 1.4), we note that $\mathring{Z}(\tau)_1 \cap \mathcal{G}r^\lambda$ is dense in $\mathring{Z}(\tau)_1$.

Fix $i \in \{1,\dots,m\}$. If $L \in S_-^\mu$, then $L_{[1,i]} \in S_-^{\mu(i)}$ so we get a map

$$f_i : \mathring{Z}(\tau)_1 \to S_-^{\mu(i)} \cap S_+^{\lambda(i)}\,.$$

From the definition of $\mathring{Z}(\tau)_1$, we see that $f_i(\mathring{Z}(\tau)_1) = \mathring{Z}(\tau(i))_1$. From above, $f_i(\mathring{Z}(\tau(i))_1) \cap \mathcal{G}r^{\lambda(i)}$ is a dense constructible subset of $\mathring{Z}(\tau(i))_1$. Since $\mathring{Z}(\tau)_1$ is irreducible, this implies that $f_i^{-1}(S_-^{\mu(i)} \cap S_+^{\lambda(i)} \cap \mathcal{G}r^{\lambda(i)})$ is a dense constructible subset of $\mathring{Z}(\tau)_1$.

Working with all i at once, we conclude that

$$\mathring{Z}(\tau)_2 := \bigcap_{i=1}^{m} f_i^{-1}(S_-^{\mu(i)} \cap S_+^{\lambda(i)} \cap \mathcal{G}r^{\lambda(i)})$$

is a dense constructible subset of $\mathring{Z}(\tau)_1$. Thus $\overline{\mathring{Z}(\tau)_2} = \overline{\mathring{Z}(\tau)_1}$.

On the other hand, by definition

$$\mathring{Z}(\tau)_2 \subset \mathring{Z}(\tau).$$

Since $\dim \mathring{Z}(\tau)_2 = \rho(\lambda - \mu)$, we see that $\mathring{Z}(\tau)$ must have at least one component of the maximal dimension. To see that it cannot have any other components of maximal dimension, we note that

$$\bigsqcup_{\tau \in YT(\lambda)_\mu} \mathring{Z}(\tau) \subset \mathcal{G}r^\lambda \cap S_-^\mu$$

and the number of irreducible components of the right hand side equals $|YT(\lambda)_\mu|$, thus on the left hand side, each $\mathring{Z}(\tau)$ can only have one irreducible component of dimension $\rho(\lambda - \mu)$. □

7.5.2 Fusion of MV cycles via fusion of generalized orbital varieties

Given $A \in M_N(\mathbb{C})$ we denote by $A\big|_{\mathbb{C}^p}$ the restriction of A to the subspace spanned by the first p standard basis vectors of $\mathbb{C}^N$. If $A\big|_{\mathbb{C}^p}(\mathbb{C}^p) \subset \mathbb{C}^p$ then we identify it with the $p \times p$ upper-left submatrix of A.

Let $\tau \in YT(\lambda)_\mu$ with μ dominant. We define

$$\mathring{X}(\tau) = \left\{A \in \mathbb{T}_\mu \cap \mathfrak{n} : A\big|_{\mathbb{C}^{|\mu(i)|}} \in \mathbb{O}^{\lambda(i)} \text{ for each } i = 1, \dots, m\right\}.$$

Lemma 7.5.4. *Under the Mirković–Vybornov isomorphism $\mathring{X}(\tau)$ is mapped isomorphically onto $\mathring{Z}(\tau)$.*

Proof. Fix $A \in \mathring{X}(\tau)$ and $i \in \{1, \dots, m\}$. Let $\mathcal{O}^i \subset \mathcal{O}^m$ denote the submodule generated by the first i standard basis vectors. Let $l = g(A\big|_{\mathbb{C}^{|\mu(i)|}})\mathcal{O}^i$ and $L = g(A)\mathcal{O}^m$. By definition of the map $A \mapsto g(A)$, since A is upper-triangular we have that $l = L_{[1,i]}$. Moreover, since elements of $\mathbb{T}_\mu \cap \mathfrak{n}$ are upper triangular $A\big|_{\mathbb{C}^{|\mu(i)|}} \in \mathbb{T}_{\mu(i)} \cap \mathfrak{n}_i$ where $\mathfrak{n}_i$ denotes the subalgebra of upper-triangular matrices in $M_{|\mu(i)|}(\mathbb{C})$. By definition of $\mathring{X}(\tau)$ this principal submatrix has Jordan type $\lambda(i)$. It follows that $l = L_{[1,i]} \in \mathcal{G}r^{\lambda(i)} \cap S^{\mu(i)}$ for each $i = 1, \dots, m$ so that $L \in \mathring{Z}(\tau)$.

Conversely, let $L \in \mathring{Z}(\tau)$. By Corollary 7.4.7, $L = g(A)\mathcal{O}^m$ for some $A \in \mathbb{T}_\mu \cap \mathfrak{n}$. Running the above argument in reverse, we see that $A \in \mathring{X}(\tau)$. □

By [Dra20a, Prop. 4.5.4] (or Proposition 7.5.3 above), $\mathring{X}(\tau)$ has a unique irreducible component of dimension $\rho(\lambda - \mu)$. We write $X(\tau)$ for the closure of this component. It is an irreducible component of $\overline{\mathbb{O}}^\lambda \cap \mathbb{T}_\mu \cap \mathfrak{n}$ and will be called a **generalized orbital variety of type** λ. The collection of all possible generalized orbital varieties of type λ will be denoted $\mathcal{X}(\lambda)$.

Theorem 7.5.5. *([Dra20a, Theorem 4.8.2]) $\{X(\tau) : \tau \in YT(\lambda)_\mu\}$ is a complete set of irreducible components of $\overline{\mathbb{O}}^\lambda \cap \mathbb{T}_\mu \cap \mathfrak{n}$.*

Our next goal is to describe the fusion of two MV cycles using a "fusion" of generalized orbital varieties, which we will now define. Let $\lambda, \lambda', \lambda''$ and μ, μ', μ'' be as in previous sections, once again assuming that μ is dominant.

Given a point $s \in \mathbb{A}^\times$ and a pair of tableaux $\tau' \in YT(\lambda')_{\mu'}$ and $\tau'' \in YT(\lambda'')_{\mu''}$, we define

$$\mathring{X}(\tau', \tau'')_{0,s} = \left\{ A \in \mathbb{U}_{0,s}^{\mu',\mu''} : A\big|_{\mathbb{C}^{|\mu(i)|}} \text{ has Jordan type } ((\lambda'(i), 0), (\lambda''(i), s)) \text{ for } i = 1, \dots, m \right\}$$

and

$$\mathring{X}(\tau', \tau'')_{0,\mathbb{A}^\times} = \left\{ (A, s) \in M_N \times \mathbb{A}^\times : A \in \mathring{X}(\tau', \tau'')_{0,s} \right\} .$$

Note that elements $A \in \mathring{X}(\tau', \tau'')$ do not in general correspond to pairs $(A', A'') \in \mathring{X}(\tau') \times X(\tau'')$ because if μ' (resp. μ'') is not dominant then $X(\tau')$ (resp. $X(\tau'')$) is not well-defined.

Proposition 7.5.6. *The image of $\mathring{X}(\tau', \tau'')_{0,\mathbb{A}^\times}$ under the isomorphism of Theorem 7.4.6 is $\mathring{Z}(\tau') *_{\mathbb{A}^\times} \mathring{Z}(\tau'')$.*

Proof. Fix $A \in \mathring{X}(\tau', \tau'')_{0,s}$ and $i \in \{1, \dots, m\}$. Let $\mathbb{C}[t]^i \subset \mathbb{C}[t]^m = \mathbb{C}^m \otimes_{\mathbb{C}} \mathbb{C}[t]$ denote the $\mathbb{C}[t]$-submodule generated by the first i standard basis vectors.

The lattices $l = g(A\big|_{\mathbb{C}^{|\mu(i)|}})\mathbb{C}[t]^i$ and $L = g(A)\mathbb{C}[t]^m$ are related by the equation $l = L_{[1,i]}$ where recall $L_{[1,i]} = (L + \mathbb{C}[t]^{[i+1,m]})/\mathbb{C}[t]^{[i+1,m]} \subset \mathbb{C}[t]^m/\mathbb{C}[t]^{[i+1,m]} = \mathbb{C}[t]^i$.

By definition of $\mathbb{U}_{0,\mathbb{A}}^{\mu',\mu''}$ we have $A\big|_{\mathbb{C}^{|\mu(i)|}} \in \mathbb{U}_{0,s}^{\mu'(i),\mu''(i)}$ and by definition of $\mathring{X}(\tau', \tau'')_{0,s}$, the Jordan type of $A\big|_{\mathbb{C}^{|\mu(i)|}}$ is $((\lambda'(i), 0), (\lambda''(i), s))$. So, $l \in \mathcal{G}r_{0,s}^{\lambda'(i),\lambda''(i)}$ by Theorem 7.4.6. On the other hand, since $L \in S_{0,s}^{\mu',\mu''}$, $l \in S_{0,s}^{\mu'(i),\mu''(i)}$.

Therefore, as i varies, we see that the pair $(L(0), L(s)) \in S^{\mu'} \times S^{\mu''}$ is such that

$$(L_{[1,i]}(0), L_{[1,i]}(s)) = (L(0)_{[1,i]}, L(s)_{[1,i]}) \in \mathcal{G}r^{\lambda'(i)} \times \mathcal{G}r^{\lambda''(i)}$$

and so $(L(0), L(s)) \in \mathring{Z}(\tau') \times \mathring{Z}(\tau'')$.

Conversely, given $L \in \mathring{Z}(\tau') *_s \mathring{Z}(\tau'')$, we can write $L = g(A)\mathbb{C}[t]^m$ by Theorem 7.4.5. Then reversing the above logic, we conclude that $A \in \mathring{X}(\tau', \tau'')_{0,s}$. ☐

Now, in analogy with the fusion of Section 7.2.2, we define $X(\tau', \tau'')_{0,\mathbb{A}}$ to be the Zariski closure of the unique top-dimensional component of $\mathring{X}(\tau', \tau'')_{0,\mathbb{A}^\times}$ in $M_N \times \mathbb{A}$. (As noted in remark 7.5.2, we expect that all these varieties $\mathring{X}(\tau) \cong \mathring{Z}(\tau)$ are irreducible, which would mean that taking this "top-dimensional component" irrelevant.) Then we define the fusion of generalized orbital varieties to be the scheme-theoretic intersection

$$X(\tau', \tau'')_{0,0} = X(\tau', \tau'')_{0,\mathbb{A}} \cap M_N \times \{0\} \text{ in } M_N \times \mathbb{A}$$

By Proposition 7.3.1, it is contained in $\overline{\mathbb{O}}^\lambda \cap \mathbb{T}_\mu \cap \mathfrak{n}$.

We quickly recall the definition of intersection multiplicity that we will need following [Ful16, Example 2.6.5]. We consider a fibre square of $\mathbb{C}$-schemes

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ D & \longrightarrow & Y \end{array}$$

with D an effective Cartier divisor and V an irreducible variety of dimension k. We assume that V is not contained in the support of D. Let Z be an irreducible component of W; it is a subvariety of V of codimension 1. The **multiplicity** of Z in the intersection $D \cap V$ is defined to be the length of the module $\mathcal{O}_{V,Z}/(f)$ over the local ring $\mathcal{O}_{V,Z}$ of V along Z, where f is a local equation of $D|_V$ on an affine open subset of V which meets Z. Following [Ful16, chap. 7], this multiplicity is denoted by $i(Z, D \cdot V)$.

For our purposes, the zero fibre in $\mathcal{G}r_{0,\mathbb{A}}$ is an effective divisor D. For Z', Z'' subvarieties of $\mathcal{G}r$, $V = Z' *_{\mathbb{A}} Z'' \subset \mathcal{G}r_{0,\mathbb{A}}$ is a variety not contained in the support of D. Then we choose an irreducible component Z of $W := D \cap V = Z' *_0 Z''$. Thus, we may consider the intersection multiplicity $i(Z, Z' *_0 Z'') := i(Z, D \cdot V)$.

In a similar way, we may consider the intersection multiplicities of generalized orbital varieties in $X(\tau', \tau'')_{0,0} = M_N \times \{0\} \cap X(\tau', \tau'')_{0,\mathbb{A}}$.

Corollary 7.5.7. $i(X(\tau), X(\tau', \tau'')_{0,0}) = i(Z(\tau), Z(\tau') *_0 Z(\tau''))$.

Proof. By the isomorphism of Lemma 7.5.4, $X(\tau)$ is a dense constructible subset of $Z(\tau)$. On the other hand, Proposition 7.5.6 gives us an identification of $\mathring{X}(\tau', \tau'')_{0,\mathbb{A}^\times}$ and $\mathring{Z}(\tau') *_{\mathbb{A}^\times} Z(\tau'')$; the schemes $X(\tau', \tau'')_{0,\mathbb{A}}$ and $Z(\tau') *_{\mathbb{A}} Z(\tau'')$ are constructed from these by taking the top-dimensional irreducible component and then closure. Thus, we conclude that $X(\tau', \tau'')_{0,\mathbb{A}}$ is a dense constructible subset of $Z(\tau') *_{\mathbb{A}} Z(\tau'')$. From the definition of intersection multiplicity, it is clear that the multiplicity can be computed locally and so the result follows. ☐

7.5.3 Multiplying MV basis elements in $\mathbb{C}[N]$

As $\mathbf{GL}_m^\vee = \mathbf{GL}_m$ (so also $N^\vee = N$) we will ignore the distinction between weights and coweights.

Recall that structure constants of multiplication in $\mathbb{C}[N]$ with respect to the MV basis elements $b_{Z'}, b_{Z''} \in Z(\infty)$ are given by intersection multiplicities

$$b_{Z'} \cdot b_{Z''} = \sum_{Z \in Z(\infty)} i\left(Z, Z' *_0 Z''\right) b_Z. \tag{7.5.1}$$

Our next goal is to show that these structure constants can be computed using generalized orbital varieties. Consider the following commutative diagram.

$$\begin{array}{ccccc}
YT_+ & \xrightarrow{\tau \mapsto X(\tau)} & \bigcup \mathcal{X}(\lambda) & & \\
\big\downarrow & & \big\downarrow \scriptstyle{\text{Lemma 7.5.4}} & & \\
YT & \xrightarrow{\tau \mapsto Z(\tau)} & \bigcup Z(\lambda) & \longrightarrow & \bigoplus V(\lambda) \\
\sigma \uparrow \big\downarrow n_\bullet & & \big\downarrow t^{-\lambda} & & \big\downarrow \Psi_\lambda \\
\mathbb{N}^r & \longrightarrow & Z(\infty) & \longrightarrow & \mathbb{C}[N]
\end{array}$$

The sequence $\mu^0 = (\mu_1^0, \ldots, \mu_{m-1}^0)$ keeping track of the number of boxes of weight i in row i of a given tableau is called its **padding**. Two tableaux have equal Lusztig data if and only if they are related by padding, which is to say that we can change the padding of one (removing or adding boxes) to recreate the other. Moreover, a tableau is completely determined by its Lusztig datum and padding.

For example

$$\bar{\tau} = \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline \end{array} \quad \text{and} \quad \tau = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

have equal Lusztig data, since we can increase the padding of $\bar{\tau}$ (by adding to it the padding of τ) to get τ, or forget the padding of τ to get $\bar{\tau}$.

We call a tableau **stable** if its padding cannot be decreased. The following lemma shows that stable tableaux are in bijection with Lusztig data.

Lemma 7.5.8. *Let $n_\bullet = (n_{(a,b)})$ be a Lusztig datum and let $\mu^0 = (\mu_i^0)$ be a padding. If $\lambda = (\sum_b n_{(a,b)} + \mu_a^0)_a$ then the smallest μ^0 such that λ is dominant effective (and μ is effective) is:*

$$\mu_m^0, \mu_{m-1}^0 = 0$$
$$\mu_i^0 = \max\{0, \mu_{i+1}^0 + \sum_a n_{(a,i+1)} - \sum_a n_{(a,i)}\} \quad i = 1, \dots, m-2.$$

This choice of μ^0 defines a section $\sigma : \mathbb{N}^r \to YT$.

Proof sketch. To produce a tableau with Lusztig datum $n_\bullet$ of smallest possible shape and weight:

- we can always take the number of m's in row m, and the number of $m-1$'s in row $m-1$ to be zero;
- we can take the number of $m-2$'s in row $m-2$ to be zero, unless $n_\bullet$ tells us that there are more boxes in row $m-1$ than there are in row $m-2$, and then we take μ_{m-2}^0 to offset the difference;

and iterate the last step up to μ_1^0. □

Corollary 7.5.9. *Given two stable MV cycles Z', Z'' with Lusztig data $n'_\bullet, n''_\bullet$ resp. there exists a pair of tableaux $(\tau', \tau'') \in YT(\lambda')_{\mu'} \times YT(\lambda'')_{\mu''}$ for some λ', λ'' dominant effective and some μ', μ'' effective, such that $\mu = \mu' + \mu'' \leq \lambda = \lambda' + \lambda''$ are partitions and*

$$b_{Z'} \cdot b_{Z''} = \sum_{\tau \in YT(\lambda)} i(X(\tau), X(\tau', \tau'')_{0,0}) b_{t^{-\lambda} Z(\tau)}.$$

Proof. Choose $\tau' = \sigma(n'_\bullet)$, $\tau'' = \sigma(n''_\bullet)$ and suppose $\tau' \in YT(\lambda')_{\mu'}$, $\tau'' \in YT(\lambda'')_{\mu''}$. Note that $Z' = t^{-\lambda'} Z(\tau'), Z'' = t^{-\lambda''} Z(\tau'')$ so the intersection multiplicity of a stable MV cycle Z in $Z' *_0 Z''$ can be computed as the intersection multiplicity of an MV cycle $Z(\tau)$ in $Z(\tau') *_0 Z(\tau'')$ for some $\tau \in YT(\lambda)_\mu$, because these two situations isomorphic by translation by t^λ.

In case μ is not dominant, increase the padding of the larger of the two tableaux by $(\max(0, \mu_2 - \mu_1), \max(0, \mu_3 - \mu_2), \dots, \max(0, \mu_m - \mu_{m-1}))$.

Now, by Corollary 7.5.7, the intersection multiplicity of $Z(\tau)$ in $Z(\tau') *_0 Z(\tau'')$ can in turn be computed as the intersection multiplicity of the generalized orbital variety $X(\tau)$ in $X(\tau', \tau'')_{0,0}$. (We have ensured μ is dominant, in order so that $X(\tau)$ will be defined.) □

7.6 Examples

In this section, we will compute some examples of multiplication of MV basis elements using the method provided by Corollary 7.5.9.

We use the section defined by Lemma 7.5.8 to abbreviate MV basis elements to tableaux and view $\mathbb{C}[N]$ as an algebra in these, rewriting Equation (7.5.1) as an equation in tableaux. The coefficients are found using generalized orbital varieties. Suppose we have two tableaux τ' and τ'', with respective weights λ', μ' and λ'', μ'', and we wish to form their (geometric) fusion product. Once we have applied any necessary padding (as in the proof of Corollary 7.5.9) so that $\mu = \mu' + \mu''$ is dominant, we take a generic matrix $A \in X(\tau', \tau'')_{0,s}$.

The requirement that $A_i := A\big|_{\mathbb{C}^{|\mu(i)|}} \in \mathbb{O}_{0,s}^{\lambda'(i),\lambda''(i)}$ for each i imposes rank conditions on A in the form of vanishing and non-vanishing minors. We form the ideal $I \subset \mathbb{C}[A_{ij}^k, s^{\pm}] = \mathbb{C}[\mathbb{U}_{0,\mathbb{A}^\times}^{\mu',\mu''}]$ generated by these relations. The ideal of the fusion $X(\tau', \tau'')_{0,0}$ is

$$J = (I \cap \mathbb{C}[A_{ij}^k, s]) + (s)\,.$$

Taking the primary decomposition (to account for possible multiplicities) of J gives us ideals $J_1, \ldots, J_n$, each corresponding to a generalized orbital variety occurring in the fusion. We use Theorem 7.5.5 to find a tableau for each J_i and then forget all unnecessary padding to get the corresponding stable MV cycle. Each J_i also tells us the multiplicity of each MV cycle in the fusion product, where we have multiplicity 1 if and only if $J_i = \sqrt{J_i}$.

Example 7.6.1. Consider the MV cycles in Example 1.2.1 and the exchange relation in Example 4.2.1. The stable tableau for each MV cycle is given below:

$$Z_1 = Z\left(\boxed{2}\right) \quad Z_2 = Z\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\right) \quad Z_+ = Z\left(\boxed{3}\right) \quad Z_- = Z\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)$$

We will verify the exchange relation $b_{Z_1}b_{Z_2} = b_{Z_+} + b_{Z_-}$ through the fusion of Z_1 and Z_2. We have the weights to be $\lambda' = (1,0,0)$, $\lambda'' = (1,1,0)$, $\mu' = (0,1,0)$, and $\mu'' = (1,0,1)$. The matrix under consideration is therefore

$$A = \left[\begin{array}{c:c:c} s & A_{12}^1 & A_{13}^1 \\ \hdashline & 0 & A_{23}^1 \\ \hdashline & & s \end{array}\right].$$

There are no relations from the submatrices A_1 and A_2. From the submatrix $A_3 = A$, we require

$$A_{12}^1 A_{23}^1 + sA_{13}^1 = 0.$$

Then the ideal J is

$$(A_{12}^1 A_{23}^1, s) = (A_{12}^1, s) \cap (A_{23}^1, s).$$

The corresponding tableaux are

Ideal of $X(\tau)$	τ
(A_{12}^1, s)	$\begin{array}{\|c\|c\|}\hline 1 & 3\\\hline 2 \\\cline{1-1}\end{array}$
(A_{23}^1, s)	$\begin{array}{\|c\|c\|}\hline 1 & 2\\\hline 3 \\\cline{1-1}\end{array}$

After removing the padding, we see that we get the tableaux for Z_+ and Z_-, and each has multiplicity

1 as the corresponding ideal is radical.

Example 7.6.2. $\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array} \cdot \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 3 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 3 & 3 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array} + 2 \cdot \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1} \end{array}$: The requirement that

$$A = \left[\begin{array}{cc:cc:cc} 0 & 1 & & & & \\ -s^2 & 2s & A^1_{12} & A^2_{12} & A^1_{13} & A^2_{13} \\ \hdashline & & 0 & 1 & & \\ & & & 0 & A^1_{23} & A^2_{23} \\ \hdashline & & & & 0 & 1 \\ & & & & -s^2 & 2s \end{array}\right] \text{ is contained in } \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array} *_{\mathbb{A}^\times} \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 3 \\ \hline \end{array}$$

imposes the following nontrivial relations on A.

$$\begin{aligned} &2A^2_{12}A^2_{23}s + 3A^2_{13}s^2 + A^2_{12}A^1_{23} + A^1_{12}A^2_{23} + 2A^1_{13}s, A^2_{13}s^3 - A^2_{12}A^1_{23}s - A^1_{12}A^2_{23}s - 2A^1_{12}A^1_{23}, \\ &A^2_{12}A^2_{13}A^1_{23}s^2 + A^1_{12}A^2_{13}A^2_{23}s^2 - (A^2_{12}A^1_{23})^2 + 2A^1_{12}A^2_{12}A^1_{23}A^2_{23} - (A^1_{12}A^2_{23})^2 + 6A^1_{12}A^2_{13}A^1_{23}s + 4A^1_{12}A^1_{13}A^1_{23}, \\ &A^1_{12}A^2_{13}(A^2_{23})^2s^2 - (A^2_{12}A^1_{23})^2A^2_{23} + 2A^1_{12}A^2_{12}A^1_{23}(A^2_{23})^2 - (A^1_{12})^2(A^2_{23})^3 - 2A^2_{12}A^2_{13}(A^1_{23})^2s \\ &\quad + 4A^1_{12}A^2_{13}A^1_{23}A^2_{23}s - A^1_{13}A^2_{13}A^1_{23}s^2 - 3A^1_{12}A^2_{13}(A^1_{23})^2 + 4A^1_{12}A^1_{13}A^1_{23}A^2_{23}, \\ &(A^2_{12})^3(A^1_{23})^2A^2_{23} - 2A^1_{12}A^1_{23}(A^2_{12}A^2_{23})^2 + A^2_{12}(A^1_{12})^2(A^2_{23})^3 + 2A^2_{13}(A^2_{12}A^1_{23})^2s + 2A^2_{13}(A^1_{12}A^2_{23})^2s \\ &\quad + 3A^1_{12}(A^2_{13})^2A^1_{23}s^2 + A^1_{13}(A^2_{12}A^1_{23})^2 + 4A^1_{12}A^2_{12}A^2_{13}(A^1_{23})^2 - 6A^1_{12}A^2_{12}A^1_{13}A^1_{23}A^2_{23} \\ &\quad + 4(A^1_{12})^2A^2_{13}A^1_{23}A^2_{23} - 4A^1_{12}A^1_{13}A^2_{13}A^1_{23}s - 4A^1_{12}(A^1_{13})^2A^1_{23} + A^1_{13}(A^1_{12}A^2_{23})^2 \end{aligned}$$

At $s = 0$ the ideal generated by the above relations decomposes as a union of the following generalized orbital varieties, where the last component occurs with multiplicity 2.

Ideal of $X(\tau)$	τ
(A^1_{12}, A^2_{12}, s)	$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 3 \\ \hline 2 & 2 \\ \cline{1-2} \end{array}$
(A^1_{23}, A^2_{23}, s)	$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 3 & 3 \\ \cline{1-2} \end{array}$
$\sqrt{((A^1_{23})^2, A^2_{12}A^1_{23} + A^1_{12}A^2_{23}, A^1_{12}A^1_{23}, (A^1_{12})^2, s)}$	$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}$

Chapter 8

A_3 Case Study

In this chapter we will study the cluster structure in the A_3 case.

8.1 Clusters in $\mathbb{C}[N]$

Our initial seed will be derived from the reduced expression $\mathbf{i} = (1, 2, 3, 1, 2, 1)$. We obtain the following pieces of data:

k	-3	-2	-1	1	2	3	4	5	6
i_k	-3	-2	-1	1	2	3	1	2	1
k^+	3	2	1	4	5	7	6	7	7
$v_{>k}$	w_0	w_0	w_0	$s_1s_2s_1s_3s_2$	$s_1s_2s_1s_3$	$s_1s_2s_1$	s_1s_2	s_1	e

with $e(\mathbf{i}) = \{1, 2, 4\}$.

The generalized minors are

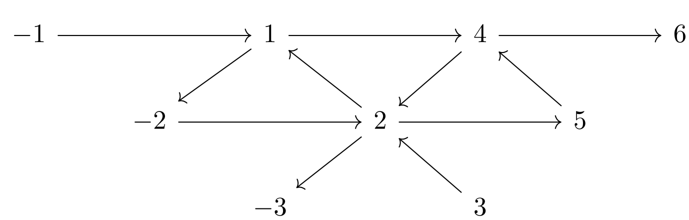
$$\Delta(-3,\mathbf{i}) = \begin{vmatrix} x_{12} & x_{13} & x_{14} \\ x_{22} & x_{23} & x_{24} \\ x_{32} & x_{33} & x_{34} \end{vmatrix} \qquad \Delta(1,\mathbf{i}) = x_{13} \qquad \Delta(4,\mathbf{i}) = x_{12}$$

$$\Delta(-2,\mathbf{i}) = \begin{vmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{vmatrix} \qquad \Delta(2,\mathbf{i}) = \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} \qquad \Delta(5,\mathbf{i}) = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$$

$$\Delta(-1,\mathbf{i}) = x_{14} \qquad \Delta(3,\mathbf{i}) = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} \qquad \Delta(6,\mathbf{i}) = x_{11}$$

with $\Delta(-3,\mathbf{i}), \Delta(-2,\mathbf{i})$, and $\Delta(-1,\mathbf{i})$ as the frozen variables. The corresponding exchange quiver is

$\tilde{A}_{\mathbf{i}} =$



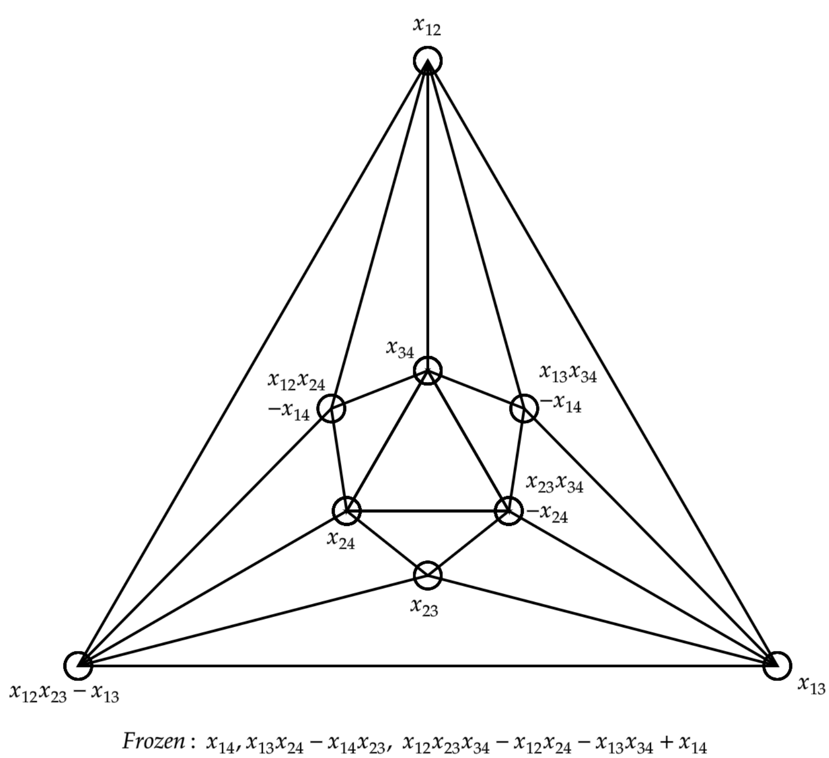
so the exchange matrix is

$$\tilde{B}_{\mathbf{i}} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

where the columns are indexed by $\{1, 2, 4\}$ and the rows are indexed by $\{-3, -2, -1, 1, 2, 4, 3, 5, 6\}$. As the non-exchangeable elements are $\{3, 5, 6\}$, when we restrict to $\mathbb{C}[N]$, the initial seed is

$$\Sigma = \left(\left\{ \begin{vmatrix} x_{12} & x_{13} & x_{14} \\ 1 & x_{23} & x_{24} \\ 0 & 1 & x_{34} \end{vmatrix}, \begin{vmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{vmatrix}, x_{14}, x_{13}, \begin{vmatrix} x_{12} & x_{13} \\ 1 & x_{23} \end{vmatrix}, x_{12} \right\}, \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \right)$$

After successively mutating this seed, there are a total of 14 seeds and 12 total cluster variables. The diagram below illustrates the cluster structure.



Each cluster has exactly 3 mutable variables, which constitute the vertices of a triangle and the initial seed Σ corresponds to the outer triangle. For a seed corresponding to the triangle with vertices a, b, c, mutation in say direction a will correspond to the unique other triangle that has b and c as its vertices. The mutation of a is given by the third vertex of the triangle. For example, if we mutate Σ in direction x_{12}, then from the diagram we have $\mu(x_{12}) = x_{23}$. Note that every cluster variable except $x_{13}x_{34} - x_{14}$ corresponds to a generalized minor.

Remark 8.1.1. Originally noticed by Anderson, a similar diagram to the above can be built by looking at MV polytopes in type A_3. We create a graph where each vertex represents a prime MV polytope, that is, an MV polytope that can not be written as the sum of two MV polytopes, and an edge joins MV polytopes P and Q if $P + Q$ is also an MV polytope. The resulting graph will be the same as the diagram above except it will have an additional edge, which joins the polytopes corresponding to the polynomials $x_{12}x_{24} - x_{14}$ and $x_{13}x_{34} - x_{14}$. The reason for this is because the associated Λ-modules are $M_1 = \mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C}$ and $M_2 = \mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C}$, and we have

$$\mathrm{Pol}(M_1) + \mathrm{Pol}(M_2) = \mathrm{Pol}(S_1) + \mathrm{Pol}(S_3) + \mathrm{Pol}(P_2).$$

The polynomials in $\mathbb{C}[N]$ corresponding to the Λ-modules S_1, S_3, and P_2 are, respectively, x_{12}, x_{34}, and $x_{13}x_{24} - x_{14}x_{23}$, which are all in one cluster. Hence even though M_1 and M_2 are not both in a cluster, the sum of their polytopes is still an MV polytope as it can be written as a sum of polytopes of other modules that are all in a cluster.

8.2 Clusters in Λ-mod

We will use the quiver $Q = 1 \longrightarrow 2 \longrightarrow 3$. The universal quiver $\tilde{\Lambda}$ is then

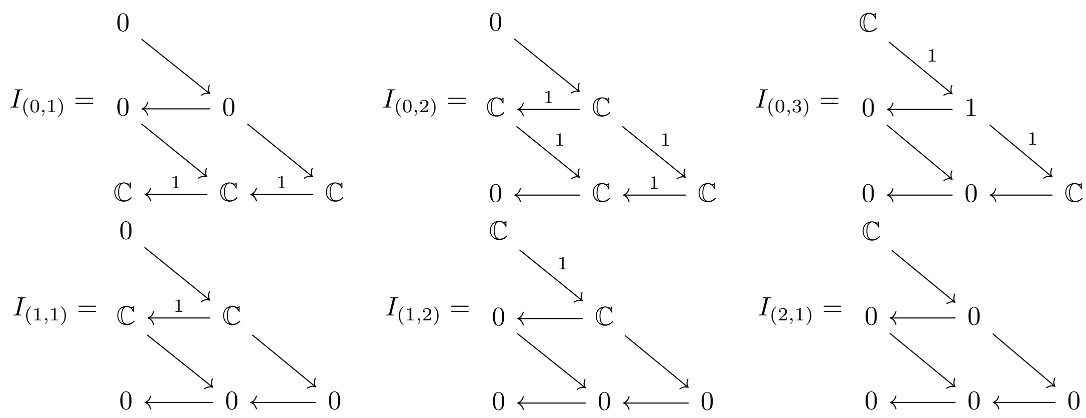
$$\begin{array}{ccccc}
 & & \vdots & & \\
(1,1) & \longleftarrow & (1,2) & \longleftarrow & (1,3) \\
 & \searrow & & \searrow & \\
(0,1) & \longleftarrow & (0,2) & \longleftarrow & (0,3) \\
 & \searrow & & \searrow & \\
(-1,1) & \longleftarrow & (-1,2) & \longleftarrow & (-1,3) \\
 & & \vdots & &
\end{array}$$

The subquiver Γ_Q is then

$$\begin{array}{ccccc}
(2,1) & & & & \\
 & \searrow & & & \\
(1,1) & \longleftarrow & (1,2) & & \\
 & \searrow & & \searrow & \\
(0,1) & \longleftarrow & (0,2) & \longleftarrow & (0,3)
\end{array}$$

The ordering of the vertices of Γ_Q is $(0,1) < (0,2) < (0,3) < (1,1) < (1,2) < (2,1)$ so applying the pushdown functor to this ordering, we get the reduced expression $\mathbf{i} = (1,2,3,1,2,1)$ that is adapted to

Q. The injective indecomposables of $\mathbb{C}\Gamma_Q$ are



Hence the basic maximal rigid Λ-module is

$$
\begin{aligned}
T \;=\; & \left(\mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C} \right) \oplus \left(\mathbb{C} \underset{[1\ \ 0]}{\overset{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}{\rightleftarrows}} \mathbb{C}^2 \underset{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}{\overset{[1\ \ 0]}{\rightleftarrows}} \mathbb{C} \right) \oplus \left(\mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C} \right) \\
& \oplus \left(\mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} 0 \right) \oplus \left(\mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} 0 \right) \oplus \left(\mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} 0 \underset{0}{\overset{0}{\rightleftarrows}} 0 \right)
\end{aligned}
$$

Labelling the indecomposable summands of $T = T_1 \oplus \cdots \oplus T_6$, we have the matrix

$$
C_T = (\dim \operatorname{Hom}_\Lambda(T_i, T_j)) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}
$$

Removing the three columns of $-C_T^{-t}$ corresponding to the projective indecomposables, so the first three, we obtain the matrix

$$
B(T) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}
$$

After reordering the cluster variables, as $\Delta(-3, \mathbf{i})$ corresponds to $FI_{(0,1)}$ and $\Delta(-1, \mathbf{i})$ corresponds to $FI_{(0,3)}$, we obtain the same matrix, and hence same seed, as the one computed for $\mathbb{C}[N]$.

The 12 cluster modules along with their corresponding element in $\mathbb{C}[N]$ are presented below:

Module in Λ-mod	Corresponding element in $\mathbb{C}[N]$
$S_1 = \ \mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} 0 \underset{0}{\overset{0}{\rightleftarrows}} 0$	x_{12}
$S_2 = \ 0 \underset{0}{\overset{0}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} 0$	x_{23}
$S_3 = \ 0 \underset{0}{\overset{0}{\rightleftarrows}} 0 \underset{0}{\overset{0}{\rightleftarrows}} \mathbb{C}$	x_{34}
$1 \leftarrow 2 := \ \mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} 0$	x_{13}
$1 \rightarrow 2 := \ \mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} 0$	$x_{12}x_{23} - x_{13}$
$2 \leftarrow 3 := \ 0 \underset{0}{\overset{0}{\rightleftarrows}} \mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C}$	x_{24}
$2 \rightarrow 3 := \ 0 \underset{0}{\overset{0}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C}$	$x_{23}x_{34} - x_{24}$
$1 \rightarrow 2 \leftarrow 3 := \ \mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C}$	$x_{12}x_{24} - x_{14}$
$1 \leftarrow 2 \rightarrow 3 := \ \mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C}$	$x_{13}x_{34} - x_{14}$
$P_1 = \ \mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C}$	$x_{12}x_{23}x_{34} - x_{12}x_{24} - x_{13}x_{34} + x_{14}$
$P_2 = \ \mathbb{C} \underset{[1\ \ 0]}{\overset{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}{\rightleftarrows}} \mathbb{C}^2 \underset{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}{\overset{[1\ \ 0]}{\rightleftarrows}} \mathbb{C}$	$x_{13}x_{24} - x_{14}x_{23}$
$P_3 = \ \mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C}$	x_{14}

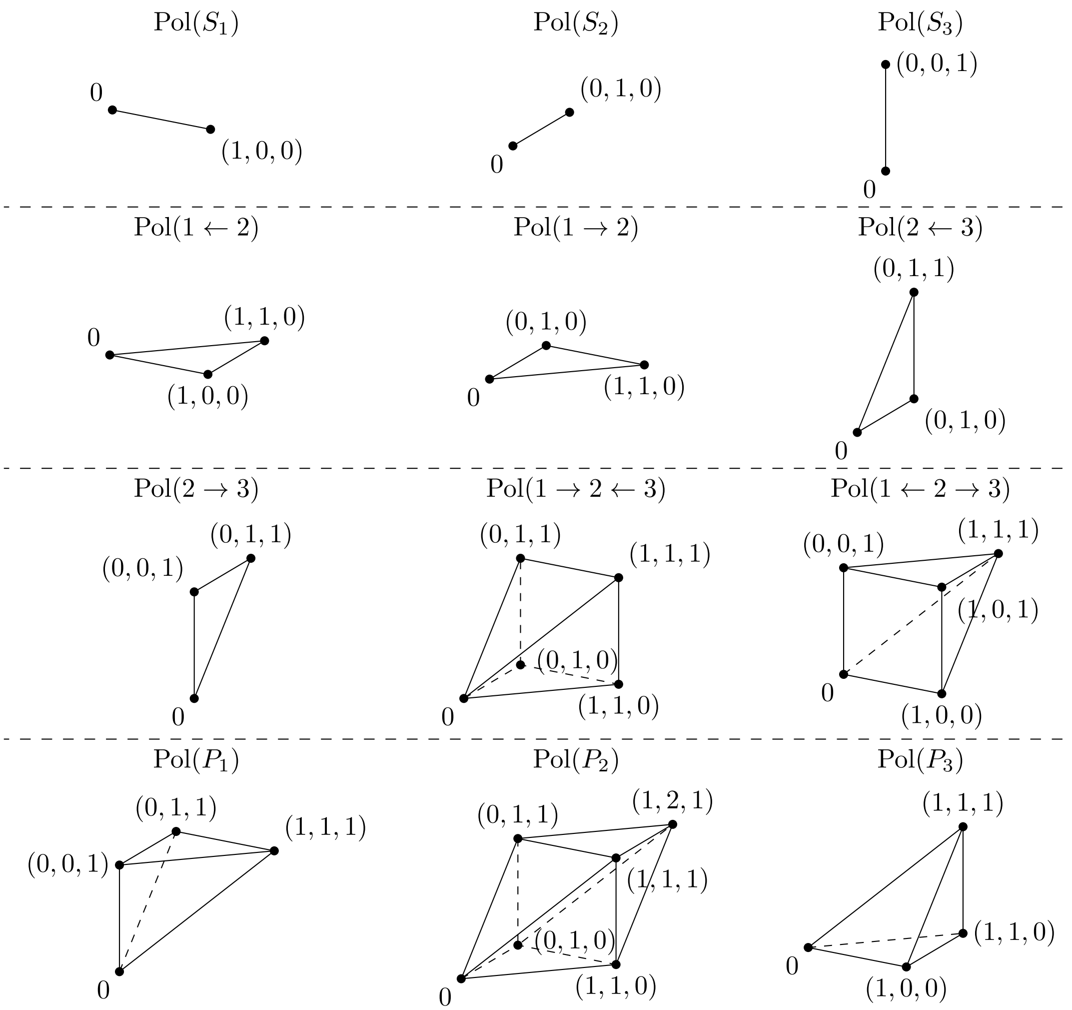
We mentioned that the element in $\mathbb{C}[N]$ corresponding to the module $1 \leftarrow 2 \rightarrow 3$ was the only one that is not a generalized minor. In addition to this, all the cluster modules are of the form $N(\gamma)$ for some $\gamma \in \Gamma$ except $1 \leftarrow 2 \rightarrow 3$.

There are a total of 15 pairs of exchange sequences which we present only the three middle terms of below:

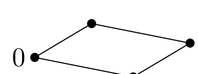
$S_1 \hookrightarrow (1 \leftarrow 2) \twoheadrightarrow S_2$	$S_2 \hookrightarrow (1 \rightarrow 2) \twoheadrightarrow S_1$
$S_2 \hookrightarrow (2 \leftarrow 3) \twoheadrightarrow S_3$	$S_3 \hookrightarrow (2 \rightarrow 3) \twoheadrightarrow S_2$
$(1 \leftarrow 2) \hookrightarrow P_3 \oplus (1 \rightarrow 2) \twoheadrightarrow (1 \rightarrow 2 \leftarrow 3)$	$(1 \rightarrow 2 \leftarrow 3) \hookrightarrow S_1 \oplus P_2 \twoheadrightarrow (1 \leftarrow 2)$
$(1 \rightarrow 2) \hookrightarrow S_1 \oplus P_2 \twoheadrightarrow (1 \leftarrow 2 \rightarrow 3)$	$(1 \leftarrow 2 \rightarrow 3) \hookrightarrow P_1 \oplus (1 \leftarrow 2) \twoheadrightarrow (1 \rightarrow 2)$
$(1 \leftarrow 2) \hookrightarrow S_2 \oplus P_3 \twoheadrightarrow (2 \leftarrow 3)$	$(2 \leftarrow 3) \hookrightarrow P_2 \twoheadrightarrow (1 \leftarrow 2)$
$(1 \rightarrow 2) \hookrightarrow P_2 \twoheadrightarrow (2 \rightarrow 3)$	$(2 \rightarrow 3) \hookrightarrow S_2 \oplus P_1 \twoheadrightarrow (1 \rightarrow 2)$
$(1 \rightarrow 2) \hookrightarrow (1 \rightarrow 2 \leftarrow 3) \twoheadrightarrow S_3$	$S_3 \hookrightarrow P_1 \twoheadrightarrow (1 \rightarrow 2)$
$(1 \leftarrow 2) \hookrightarrow P_3 \twoheadrightarrow S_3$	$S_3 \hookrightarrow (1 \leftarrow 2 \rightarrow 3) \twoheadrightarrow (1 \leftarrow 2)$
$(2 \leftarrow 3) \hookrightarrow (1 \rightarrow 2 \leftarrow 3) \twoheadrightarrow S_1$	$S_1 \hookrightarrow P_3 \twoheadrightarrow (2 \leftarrow 3)$
$(2 \rightarrow 3) \hookrightarrow P_1 \twoheadrightarrow S_1$	$S_1 \hookrightarrow (1 \leftarrow 2 \rightarrow 3) \twoheadrightarrow (2 \rightarrow 3)$
$(1 \rightarrow 2 \leftarrow 3) \hookrightarrow P_2 \twoheadrightarrow S_2$	$S_2 \hookrightarrow (1 \rightarrow 2) \oplus (2 \leftarrow 3) \twoheadrightarrow (1 \rightarrow 2 \leftarrow 3)$
$(1 \leftarrow 2 \rightarrow 3) \hookrightarrow (1 \leftarrow 2) \oplus (2 \rightarrow 3) \twoheadrightarrow S_2$	$S_2 \hookrightarrow P_2 \twoheadrightarrow (1 \leftarrow 2 \rightarrow 3)$
$(2 \leftarrow 3) \hookrightarrow S_3 \oplus P_2 \twoheadrightarrow (1 \leftarrow 2 \rightarrow 3)$	$(1 \leftarrow 2 \rightarrow 3) \hookrightarrow P_3 \oplus (2 \rightarrow 3) \twoheadrightarrow (2 \leftarrow 3)$
$(2 \rightarrow 3) \hookrightarrow P_1 \oplus (2 \leftarrow 3) \twoheadrightarrow (1 \rightarrow 2 \leftarrow 3)$	$(1 \rightarrow 2 \leftarrow 3) \hookrightarrow S_3 \oplus P_2 \twoheadrightarrow (2 \rightarrow 3)$
$(1 \rightarrow 2 \leftarrow 3) \hookrightarrow S_1 \oplus P_2 \oplus S_3 \twoheadrightarrow (1 \leftarrow 2 \rightarrow 3)$	$(1 \leftarrow 2 \rightarrow 3) \hookrightarrow P_1 \oplus P_3 \twoheadrightarrow (1 \rightarrow 2 \leftarrow 3)$

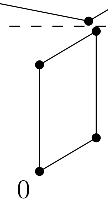
8.3 The MV Polytopes

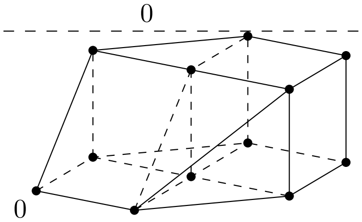
We list each cluster module and its corresponding MV polytope where we label a point $a\alpha_1 + b\alpha_2 + c\alpha_3$ as (a, b, c):



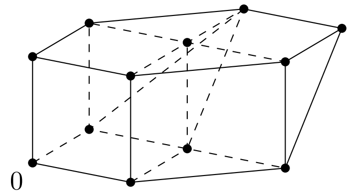
We have the following visualization of the exchange relation for MV Polytopes:

$\mathrm{Pol}(S_1) + \mathrm{Pol}(S_2) =$  $= \mathrm{Pol}(1 \leftarrow 2) \cup \mathrm{Pol}(1 \rightarrow 2)$

$\mathrm{Pol}(S_2) + \mathrm{Pol}(S_3) =$  $= \mathrm{Pol}(2 \leftarrow 3) \cup \mathrm{Pol}(2 \rightarrow 3)$

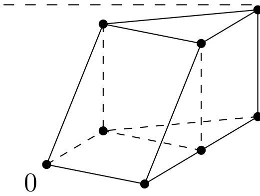
$\mathrm{Pol}(1 \leftarrow 2) +$
$\mathrm{Pol}(1 \rightarrow 2 \leftarrow 3)$ $=$  $= \mathrm{Pol}(P_3 \oplus (1 \rightarrow 2)) \cup \mathrm{Pol}(S_1 \oplus P_2)$

$\text{Pol}(1 \to 2) + \text{Pol}(1 \leftarrow 2 \to 3) =$



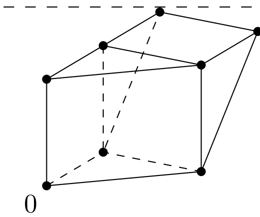
$= \text{Pol}(P_1 \oplus (1 \leftarrow 2)) \cup \text{Pol}(S_1 \oplus P_2)$

$\text{Pol}(1 \leftarrow 2) + \text{Pol}(2 \leftarrow 3) =$



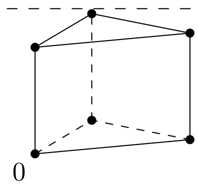
$= \text{Pol}(P_2) \cup \text{Pol}(S_2 \oplus P_3)$

$\text{Pol}(1 \to 2) + \text{Pol}(2 \to 3) =$



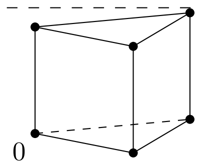
$= \text{Pol}(P_2) \cup \text{Pol}(S_2 \oplus P_1)$

$\text{Pol}(1 \to 2) + \text{Pol}(S_3) =$



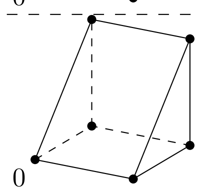
$= \text{Pol}(P_1) \cup \text{Pol}(1 \to 2 \leftarrow 3)$

$\text{Pol}(1 \leftarrow 2) + \text{Pol}(S_3) =$



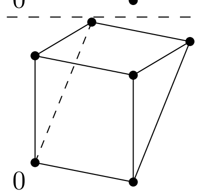
$= \text{Pol}(P_3) \cup \text{Pol}(1 \leftarrow 2 \to 3)$

$\text{Pol}(2 \leftarrow 3) + \text{Pol}(S_1) =$



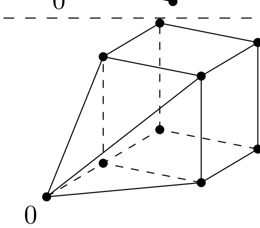
$= \text{Pol}(P_3) \cup \text{Pol}(1 \to 2 \leftarrow 3)$

$\text{Pol}(2 \to 3) + \text{Pol}(S_1) =$



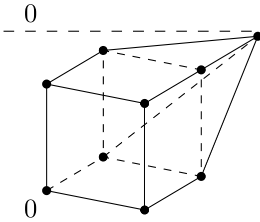
$= \text{Pol}(P_1) \cup \text{Pol}(1 \leftarrow 2 \to 3)$

$\text{Pol}(1 \to 2 \leftarrow 3) + \text{Pol}(S_2) =$



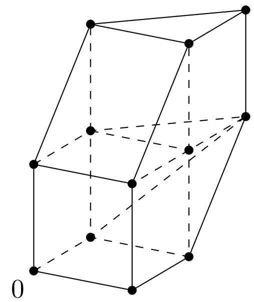
$= \text{Pol}(P_2) \cup \text{Pol}((2 \leftarrow 3) \oplus (1 \to 2))$

$\text{Pol}(1 \leftarrow 2 \to 3) + \text{Pol}(S_2) =$



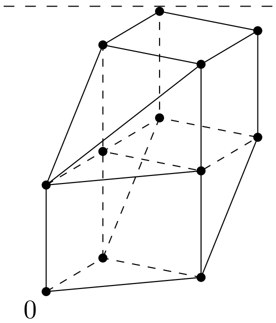
$= \text{Pol}(P_2) \cup \text{Pol}((2 \to 3) \oplus (1 \leftarrow 2))$

$\mathrm{Pol}(2 \leftarrow 3) + \mathrm{Pol}(1 \leftarrow 2 \rightarrow 3) =$



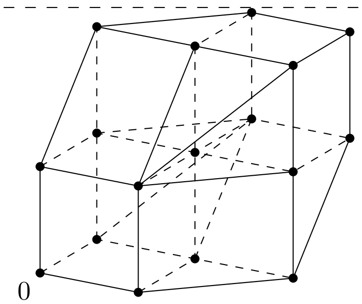
$= \mathrm{Pol}(S_3 \oplus P_2) \cup \mathrm{Pol}(P_3 \oplus (2 \rightarrow 3))$

$\mathrm{Pol}(2 \rightarrow 3) + \mathrm{Pol}(1 \rightarrow 2 \leftarrow 3) =$



$\mathrm{Pol}(S_3 \oplus P_2) \cup \mathrm{Pol}(P_1 \oplus (2 \leftarrow 3))$

$\mathrm{Pol}(1 \rightarrow 2 \leftarrow 3) + \mathrm{Pol}(1 \leftarrow 2 \rightarrow 3) =$



$= \mathrm{Pol}(S_1 \oplus P_2 \oplus S_3) \cup \mathrm{Pol}(P_1 \oplus P_3)$

8.4 The MV Cycles

All except one of the MV cycles corresponding to cluster variables happen to be toric varieties whose MV polytopes are the same as their toric polytopes. The exception is the MV cycle corresponding to the module P_2.

Below we list each cluster module along with its stable tableau, the coordinate ring of its generalized orbital variety, and a variety that the MV cycle is isomorphic to:

Λ-Module	τ	Coordinate ring of $X(\tau)$	MV Cycle
S_1	2	$\mathbb{C}[A^1_{12}]$	$\mathbb{P}^1$
S_2	1 3	$\frac{\mathbb{C}[A^1_{12},A^1_{13},A^1_{23}]}{(A^1_{12},A^1_{13})}$	$\mathbb{P}^1$
S_3	1 2 4	$\frac{\mathbb{C}[A^1_{12},A^2_{12},A^1_{13},A^1_{14},A^1_{23},A^1_{24},A^1_{34}]}{(A^1_{12},A^2_{12},A^1_{13},A^1_{14},A^1_{23},A^1_{24})}$	$\mathbb{P}^1$
$1 \leftarrow 2$	3	$\frac{\mathbb{C}[A^1_{12},A^1_{13},A^1_{23}]}{(A^1_{12})}$	$\mathbb{P}^2$
$1 \rightarrow 2$	2 3	$\frac{\mathbb{C}[A^1_{12},A^1_{13},A^1_{23}]}{(A^1_{23})}$	$\mathbb{P}^2$
$2 \leftarrow 3$	1 4	$\frac{\mathbb{C}[A^1_{12},A^1_{13},A^1_{14},A^1_{23},A^1_{24},A^1_{34}]}{(A^1_{12},A^1_{13},A^1_{23},A^1_{34})}$	$\mathbb{P}^2$
$2 \rightarrow 3$	1 3 4	$\frac{\mathbb{C}[A^1_{12},A^1_{13},A^1_{14},A^1_{23},A^1_{24},A^1_{34}]}{(A^1_{12},A^1_{13},A^1_{14},A^1_{34})}$	$\mathbb{P}^2$
$1 \rightarrow 2 \leftarrow 3$	2 4	$\frac{\mathbb{C}[A^1_{12},A^2_{12},A^1_{13},A^1_{14},A^1_{23},A^1_{24},A^1_{34}]}{(A^1_{12},A^1_{13},A^1_{23},A^1_{24})}$	$S(2,4)$
$1 \leftarrow 2 \rightarrow 3$	1 3 2 4	$\frac{\mathbb{C}[A^1_{12},A^1_{13},A^1_{14},A^1_{23},A^1_{24},A^1_{34}]}{(A^1_{12},A^1_{34},A^1_{13}A^1_{24}-A^1_{23}A^1_{14})}$	$S(2,4)$
P_1	2 3 4	$\frac{\mathbb{C}[A^1_{12},A^1_{13},A^1_{14},A^1_{23},A^1_{24},A^1_{34}]}{(A^1_{23},A^1_{24},A^1_{34})}$	$\mathbb{P}^3$
P_2	3 4	$\frac{\mathbb{C}[A^1_{12},A^1_{13},A^1_{14},A^1_{23},A^1_{24},A^1_{34}]}{(A^1_{12},A^1_{34})}$	$Gr(2,4)$
P_3	4	$\frac{\mathbb{C}[A^1_{12},A^1_{13},A^1_{14},A^1_{23},A^1_{24},A^1_{34}]}{(A^1_{12},A^1_{13},A^1_{23})}$	$\mathbb{P}^3$

TABLE 8.1. The variety $S(2,4)$ denotes the non-smooth Schubert divisor in $Gr(2,4)$ the Grassmannian of planes in $\mathbb{C}^4$

In what follows we check that the exchange relations in $\mathbb{C}[N]$, written in terms of stable tableaux, are corroborated by the fusion of the corresponding MV cycles.

Example 8.4.1. $\begin{array}{|c|}\hline 2\\\hline\end{array} \cdot \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} = \begin{array}{|c|}\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$: The requirement that

$$A = \left[\begin{array}{c:c:c} s & A^1_{12} & A^1_{13} \\ \hdashline & 0 & A^1_{23} \\ \hdashline & & s \end{array}\right] \text{ is contained in } \begin{array}{|c|}\hline 2\\\hline\end{array} *_{\mathbb{A}^\times} \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}$$

imposes the relation $A^1_{12}A^1_{23} + sA^1_{13}$ on A and at $s = 0$ the ideal generated by this relation decomposes as a union of the following generalized orbital varieties.

Ideal of $X(\tau)$	τ
(A^1_{12}, s)	$\begin{array}{|c|c|}\hline 1 & 3\\\hline 2\\\cline{1-1}\end{array}$
(A^1_{23}, s)	$\begin{array}{|c|c|}\hline 1 & 2\\\hline 3\\\cline{1-1}\end{array}$

Example 8.4.2. $\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \cdot \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array} = \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}$: The requirement that

$$A = \left[\begin{array}{cc:c:c:c} 0 & 1 & & & \\ & s & A^1_{12} & A^1_{13} & A^1_{14} \\ \hdashline & & s & A^1_{23} & A^1_{24} \\ \hdashline & & & 0 & A^1_{34} \\ \hdashline & & & & s \end{array}\right] \text{ is contained in } \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} *_{\mathbb{A}^\times} \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}$$

results in the following relations on submatrices:

Submatrix	Relations
A_2	A^1_{12}
A_3	A^1_{13}
A_4	$A^1_{14}, A^1_{23}A^1_{34} + sA^1_{24}$

At $s = 0$ the ideal generated by the above relations decomposes as a union of the following generalized orbital varieties.

Ideal of $X(\tau)$	τ
$(A^1_{12}, A^1_{13}, A^1_{23}, A^1_{14}, s)$	$\begin{array}{|c|c|}\hline 1 & 1\\\hline 2 & 4\\\hline 3\\\cline{1-1}\end{array}$
$(A^1_{12}, A^1_{13}, A^1_{14}, A^1_{34}, s)$	$\begin{array}{|c|c|}\hline 1 & 1\\\hline 2 & 3\\\hline 4\\\cline{1-1}\end{array}$

Example 8.4.3. $\begin{array}{|c|}\hline 2\\\hline\end{array} \cdot \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array} = \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array} + \begin{array}{|c|c|}\hline 1 & 3\\\hline 2 & \\\cline{1-1} 4 & \\\cline{1-1}\end{array}$: The requirement that

$$A = \left[\begin{array}{c:c:c:c} s & A^1_{12} & A^1_{13} & A^1_{14} \\ \hdashline & 0 & A^1_{23} & A^1_{24} \\ \hdashline & & s & A^1_{34} \\ \hdashline & & & s \end{array}\right] \text{ is contained in } \begin{array}{|c|}\hline 2\\\hline\end{array} *_{\mathbb{A}^\times} \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}$$

results in the following relations on submatrices:

Submatrix	Relations
A_3	$A^1_{12}A^1_{23} + sA^1_{13}$
A_4	$A^1_{34}, A^1_{12}A^1_{24} + sA^1_{14}, A^1_{23}A^1_{14} - A^1_{13}A^1_{24}$

At $s = 0$ the ideal generated by the above relations decomposes as a union of the following generalized orbital varieties.

Ideal of $X(\tau)$	τ
$(A^1_{23}, A^1_{24}, A^1_{34}, s)$	$\begin{array}{|c|c|}\hline 1 & 2\\\hline 3 & \\\cline{1-1} 4 & \\\cline{1-1}\end{array}$
$(A^1_{12}, A^1_{34}, A^1_{23}A^1_{14} - A^1_{13}A^1_{24}, s)$	$\begin{array}{|c|c|}\hline 1 & 3\\\hline 2 & \\\cline{1-1} 4 & \\\cline{1-1}\end{array}$

Example 8.4.4. $\begin{array}{|c|}\hline 2\\\hline\end{array} \cdot \begin{array}{|c|c|}\hline 1 & 1\\\hline 2 & 4\\\hline 3 & \\\cline{1-1}\end{array} = \begin{array}{|c|}\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}$: The requirement that

$$A = \left[\begin{array}{cc:cc:c:c} 0 & 1 & & & & \\ -s^2 & 2s & A^1_{12} & A^2_{12} & A^1_{13} & A^1_{14} \\ \hdashline & & 0 & 1 & & \\ & & & s & A^1_{23} & A^1_{24} \\ \hdashline & & & & s & A^1_{34} \\ \hdashline & & & & & s \end{array}\right] \text{ is contained in } \begin{array}{|c|}\hline 2\\\hline\end{array} *_{\mathbb{A}^\times} \begin{array}{|c|c|}\hline 1 & 1\\\hline 2 & 4\\\hline 3 & \\\cline{1-1}\end{array}$$

results in the following relations on submatrices:

Submatrix	Relations
A_2	$A^1_{12} + sA^2_{12}$
A_3	A^1_{13}, A^1_{23}
A_4	$A^2_{12}A^1_{24} + sA^1_{14}, A^1_{12}A^1_{24} - s^2A^1_{14}$

At $s = 0$ the ideal generated by the above relations decomposes as a union of the following generalized orbital varieties.

Ideal of $X(\tau)$	τ
$(A^1_{12}, A^2_{12}, A^1_{13}, A^1_{23}, s)$	$\begin{ytableau} 1 & 1 & 4 \\ 2 & 2 \\ 3 \end{ytableau}$
$(A^1_{12}, A^1_{13}, A^1_{23}, A^1_{24}, s)$	$\begin{ytableau} 1 & 1 & 2 \\ 2 & 4 \\ 3 \end{ytableau}$

Example 8.4.5. $\begin{ytableau} 1 & 1 \\ 2 \\ 4 \end{ytableau} \cdot \begin{ytableau} 2 \\ 3 \end{ytableau} = \begin{ytableau} 1 \\ 4 \end{ytableau} + \begin{ytableau} 1 \\ 3 \\ 4 \end{ytableau}$: The requirement that

$$A = \left[\begin{array}{cc|cc|c|c} 0 & 1 & & & & \\ & 0 & A^1_{12} & A^2_{12} & A^1_{13} & A^1_{14} \\ \hline & & 0 & 1 & & \\ & & & s & A^1_{23} & A^1_{24} \\ \hline & & & & s & A^1_{34} \\ \hline & & & & & 0 \end{array}\right] \text{ is contained in } \begin{ytableau} 1 & 1 \\ 2 \\ 4 \end{ytableau} *_{\mathbb{A}^\times} \begin{ytableau} 2 \\ 3 \end{ytableau}$$

results in the following relations on submatrices:

Submatrix	Relations
A_2	A^1_{12}
A_3	A^1_{23}
A_4	$A^1_{24}, A^1_{13}A^1_{34} - sA^1_{14}$

At $s = 0$ the ideal generated by the above relations decomposes as a union of the following generalized orbital varieties.

Ideal of $X(\tau)$	τ
$(A^1_{12}, A^1_{13}, A^1_{23}, A^1_{24}, s)$	$\begin{ytableau} 1 & 1 & 2 \\ 2 & 4 \\ 3 \end{ytableau}$
$(A^1_{12}, A^1_{23}, A^1_{24}, A^1_{34}, s)$	$\begin{ytableau} 1 & 1 & 2 \\ 2 & 3 \\ 4 \end{ytableau}$

Example 8.4.6. $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array} \cdot \begin{array}{|c|}\hline 3\\\hline\end{array} = \begin{array}{|c|}\hline 4\\\hline\end{array} + \begin{array}{|c|c|}\hline 1 & 3\\\hline 2 \\\cline{1-1} 4 \\\cline{1-1}\end{array}$: The requirement that

$$A = \left[\begin{array}{c|c|c|c} 0 & A^1_{12} & A^1_{13} & A^1_{14} \\ \hline & 0 & A^1_{23} & A^1_{24} \\ \hline & & s & A^1_{34} \\ \hline & & & 0 \end{array}\right] \text{ is contained in } \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array} *_{\mathbb{A}^\times} \begin{array}{|c|}\hline 3\\\hline\end{array}$$

results in the following relations on submatrices:

Submatrix	Relations
A_2	A^1_{12}
A_4	$A^1_{23}A^1_{34} - sA^1_{24}, A^1_{13}A^1_{34} - sA^1_{14}, A^1_{13}A^1_{24} - A^1_{23}A^1_{14}$

At $s = 0$ the ideal generated by the above relations decomposes as a union of the following generalized orbital varieties.

Ideal of $X(\tau)$	τ
$(A^1_{12}, A^1_{13}, A^1_{23}, s)$	$\begin{array}{|c|c|}\hline 1 & 4\\\hline 2 \\\cline{1-1} 3 \\\cline{1-1}\end{array}$
$(A^1_{12}, A^1_{34}, A^1_{13}A^1_{24} - A^1_{23}A^1_{14}, s)$	$\begin{array}{|c|c|}\hline 1 & 3\\\hline 2 \\\cline{1-1} 4 \\\cline{1-1}\end{array}$

Example 8.4.7. $\begin{array}{|c|c|}\hline 1 & 1\\\hline 2 & 4\\\hline\end{array} \cdot \begin{array}{|c|}\hline 3\\\hline\end{array} = \begin{array}{|c|c|}\hline 1 & 4\\\hline 3 \\\cline{1-1}\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}$: The requirement that

$$A = \left[\begin{array}{cc|c|c|c} 0 & 1 & & & \\ & 0 & A^1_{12} & A^1_{13} & A^1_{14} \\ \hline & & 0 & A^1_{23} & A^1_{24} \\ \hline & & & s & A^1_{34} \\ \hline & & & & 0 \end{array}\right] \text{ is contained in } \begin{array}{|c|c|}\hline 1 & 1\\\hline 2 & 4\\\hline\end{array} *_{\mathbb{A}^\times} \begin{array}{|c|}\hline 3\\\hline\end{array}$$

results in the following relations on submatrices:

Submatrix	Relations
A_2	A^1_{12}
A_4	$A^1_{13}A^1_{34} - sA^1_{14}$

At $s = 0$ the ideal generated by the above relations decomposes as a union of the following generalized orbital varieties.

Ideal of $X(\tau)$	τ
(A^1_{12}, A^1_{13}, s)	$\begin{array}{\|c\|c\|c\|}\hline 1&1&4\\\hline 2&3\\\cline{1-2}\end{array}$
(A^1_{12}, A^1_{34}, s)	$\begin{array}{\|c\|c\|c\|}\hline 1&1&3\\\hline 2&4\\\cline{1-2}\end{array}$

Example 8.4.8. $\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline 4\\\cline{1-1}\end{array} \cdot \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} = \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline 4\\\cline{1-1}\end{array}$: The requirement that

$$A = \left[\begin{array}{cc|cc|cc|c} 0 & 1 & & & & & \\ & 0 & A^1_{12} & A^2_{12} & A^1_{13} & A^2_{13} & A^1_{14} \\ \hline & & 0 & 1 & & & \\ & & & s & A^1_{23} & A^2_{23} & A^1_{24} \\ \hline & & & & 0 & 1 & \\ & & & & & s & A^1_{34} \\ \hline & & & & & & 0 \end{array}\right] \text{ is contained in } \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline 4\\\cline{1-1}\end{array} *_{\mathbb{A}^\times} \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$$

results in the following relations on submatrices:

Submatrix	Relations
A_2	A^1_{12}
A_3	$A^2_{12}A^1_{23} - sA^1_{13}, A^1_{23} + sA^2_{23}, A^2_{12}A^2_{23} + A^1_{13}$
A_4	$A^1_{34}, A^2_{12}A^1_{24} - sA^1_{14}, A^1_{14}A^1_{23} - A^1_{13}A^1_{24}$

At $s = 0$ the ideal generated by the above relations decomposes as a union of the following generalized orbital varieties.

Ideal of $X(\tau)$	τ
$(A^1_{12}, A^2_{12}, A^1_{13}, A^1_{23}, A^1_{34}, s)$	$\begin{array}{\|c\|c\|c\|}\hline 1&1&3\\\hline 2&2&4\\\hline 3\\\cline{1-1}\end{array}$
$(A^1_{12}, A^1_{23}, A^1_{24}, A^1_{34}, A^2_{12}A^2_{23} + A^1_{13}, s)$	$\begin{array}{\|c\|c\|c\|}\hline 1&1&2\\\hline 2&3&3\\\hline 4\\\cline{1-1}\end{array}$

Example 8.4.9. $\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2\\\cline{1-2} 4\\\cline{1-1}\end{array} \cdot \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} = \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array} = \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array} \cdot \begin{array}{|c|}\hline 3\\\hline\end{array}$: The requirement that

$$A = \left[\begin{array}{ccc|cc|cc|c} 0 & 1 & & & & & & \\ & 0 & 1 & & & & & \\ & & s & A_{12}^1 & A_{12}^2 & A_{13}^1 & A_{13}^2 & A_{14}^1 \\ \hline & & & 0 & 1 & & & \\ & & & & 0 & A_{23}^1 & A_{23}^2 & A_{24}^1 \\ \hline & & & & & 0 & 1 & \\ & & & & & & s & A_{34}^1 \\ \hline & & & & & & & 0 \end{array}\right] \text{ is contained in } \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2\\\cline{1-2} 4\\\cline{1-1}\end{array} *_{\mathbb{A}^\times} \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}$$

results in the following relations on submatrices:

Submatrix	Relations
A_2	A_{12}^1, A_{12}^2
A_3	$A_{13}^1 + sA_{13}^2$
A_4	$A_{34}^1, A_{14}^1 A_{23}^1 - A_{13}^1 A_{24}^1$

At $s = 0$ the ideal generated by the above relations decomposes as a union of the following generalized orbital varieties.

Ideal of $X(\tau)$	τ
$(A_{12}^1, A_{12}^2, A_{13}^1, A_{23}^1, A_{34}^1, s)$	$\begin{array}{\|c\|c\|c\|c\|}\hline 1&1&1&3\\\hline 2&2&4\\\cline{1-3} 3\\\cline{1-1}\end{array}$
$(A_{12}^1, A_{12}^2, A_{13}^1, A_{14}^1, A_{34}^1, s)$	$\begin{array}{\|c\|c\|c\|c\|}\hline 1&1&1&3\\\hline 2&2&3\\\cline{1-3} 4\\\cline{1-1}\end{array}$

Example 8.4.10. $\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \cdot \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} = \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}$: The requirement that

$$A = \left[\begin{array}{c|c|c|c} 0 & A_{12}^1 & A_{13}^1 & A_{14}^1 \\ \hline & s & A_{23}^1 & A_{24}^1 \\ \hline & & 0 & A_{34}^1 \\ \hline & & & s \end{array}\right] \text{ is contained in } \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} *_{\mathbb{A}^\times} \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}$$

results in the following relations on submatrices:

Submatrix	Relations
A_3	$A^1_{12}A^1_{23} - sA^1_{13}$
A_4	$A^1_{23}A^1_{34} + sA^1_{24}, A^1_{12}A^1_{24} + A^1_{13}A^1_{34}$

At $s = 0$ the ideal generated by the above relations decomposes as a union of the following generalized orbital varieties.

Ideal of $X(\tau)$	τ
(A^1_{12}, A^1_{34}, s)	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$
$(A^1_{23}, A^1_{12}A^1_{24} + A^1_{13}A^1_{34}, s)$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$

Example 8.4.11. $\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \cdot \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} + \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$: The requirement that

$$A = \left[\begin{array}{cc|cc|cc|c} 0 & 1 & & & & & \\ & 0 & A^1_{12} & A^2_{12} & A^1_{13} & A^2_{13} & A^1_{14} \\ \hline & & 0 & 1 & & & \\ & & & s & A^1_{23} & A^2_{23} & A^1_{24} \\ \hline & & & & 0 & 1 & \\ & & & & & s & A^1_{34} \\ \hline & & & & & & 0 \end{array}\right] \text{ is contained in } \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} *_{\mathbb{A}^\times} \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array}$$

results in the following relations on submatrices:

Submatrix	Relations
A_2	A^1_{12}
A_3	$A^1_{23} + sA^2_{23}$
A_4	$A^1_{34}, A^1_{13}A^1_{24} - A^1_{23}A^1_{14}$

At $s = 0$ the ideal generated by the above relations decomposes as a union of the following generalized orbital varieties.

Ideal of $X(\tau)$	τ
$(A^1_{12}, A^1_{13}, A^1_{23}, A^1_{34}, s)$	$\begin{array}{\|c\|c\|c\|c\|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$
$(A^1_{12}, A^1_{23}, A^1_{24}, A^1_{34}, s)$	$\begin{array}{\|c\|c\|c\|c\|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$

Example 8.4.12. $\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} \cdot \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} = \begin{array}{|c|c|}\hline 2 & 4\\\hline 3 \\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 3\\\hline 4 \\\cline{1-1}\end{array}$: The requirement that

$$A = \left[\begin{array}{c:c:c:c} s & A^1_{12} & A^1_{13} & A^1_{14} \\ \hdashline & 0 & A^1_{23} & A^1_{24} \\ \hdashline & & s & A^1_{34} \\ \hdashline & & & 0 \end{array}\right] \text{ is contained in } \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} *_{\mathbb{A}^\times} \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}$$

results in the following relations on submatrices:

Submatrix	Relations
A_4	$A^1_{23}A^1_{34} - sA^1_{24}$

At $s = 0$ the ideal generated by the above relations decomposes as a union of the following generalized orbital varieties.

Ideal of $X(\tau)$	τ
(A^1_{23}, s)	$\begin{array}{|c|c|c|}\hline 1 & 2 & 4\\\hline 3 \\\cline{1-1}\end{array}$
(A^1_{34}, s)	$\begin{array}{|c|c|c|}\hline 1 & 2 & 3\\\hline 4 \\\cline{1-1}\end{array}$

Example 8.4.13. $\begin{array}{|c|c|c|}\hline 1 & 1 & 3\\\hline 2 & 2 \\\cline{1-2} 3 & 4 \\\cline{1-2}\end{array} \cdot \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} = \begin{array}{|c|c|}\hline 1 & 4\\\hline 3 \\\cline{1-1} 4 \\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 3\\\hline 2 & 4\\\hline 4 \\\cline{1-1}\end{array}$: The requirement that

$$A = \left[\begin{array}{ccc:cc:cc:cc} 0 & 1 & & & & & & & \\ & 0 & 1 & & & & & & \\ & & s & A^1_{12} & A^2_{12} & A^1_{13} & A^2_{13} & A^1_{14} & A^2_{14} \\ \hdashline & & & 0 & 1 & & & & \\ & & & & 0 & A^1_{23} & A^2_{23} & A^1_{24} & A^2_{24} \\ \hdashline & & & & & 0 & 1 & & \\ & & & & & & 0 & A^1_{34} & A^2_{34} \\ \hdashline & & & & & & & 0 & 1 \\ & & & & & & & & s \end{array}\right] \text{ is contained in } \begin{array}{|c|c|c|}\hline 1 & 1 & 3\\\hline 2 & 2 \\\cline{1-2} 3 & 4 \\\cline{1-2}\end{array} *_{\mathbb{A}^\times} \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}$$

results in the following relations on submatrices:

Submatrix	Relations
A_2	A^1_{12}, A^2_{12}
A_3	A^1_{13}, A^1_{23}
A_4	$A^1_{34}, A^2_{13}A^2_{34} + A^1_{14} + sA^2_{14}, A^1_{14}A^2_{23} - A^2_{13}A^1_{24}$

At $s = 0$ the ideal generated by the above relations decomposes as a union of the following generalized orbital varieties.

Ideal of $X(\tau)$	τ
$(A_{12}^1, A_{12}^2, A_{13}^1, A_{13}^2, A_{23}^1, A_{14}^1, A_{34}^1, s)$	$\begin{array}{\|c\|c\|c\|c\|}\hline 1&1&1&4\\\hline 2&2&3\\\cline{1-3} 3&4\\\cline{1-2}\end{array}$
$(A_{12}^1, A_{12}^2, A_{13}^1, A_{23}^1, A_{34}^1, A_{23}^2A_{34}^2 + A_{24}^1, A_{13}^2A_{34}^2 + A_{14}^1, A_{13}^2A_{24}^1 - A_{23}^2A_{14}^1, s)$	$\begin{array}{\|c\|c\|c\|c\|}\hline 1&1&1&3\\\hline 2&2&4\\\cline{1-3} 3&4\\\cline{1-2}\end{array}$

Example 8.4.14. $\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline 3&4\\\hline\end{array} \cdot \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} = \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 4\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 4\\\cline{1-1}\end{array}$: The requirement that

$$A = \left[\begin{array}{cc|cc|cc|cc} 0 & 1 & & & & & & \\ & 0 & A_{12}^1 & A_{12}^2 & A_{13}^1 & A_{13}^2 & A_{14}^1 & A_{14}^2 \\ \hline & & 0 & 1 & & & & \\ & & & s & A_{23}^1 & A_{23}^2 & A_{24}^1 & A_{24}^2 \\ \hline & & & & 0 & 1 & & \\ & & & & & 0 & A_{34}^1 & A_{34}^2 \\ \hline & & & & & & 0 & 1 \\ & & & & & & & s \end{array}\right] \text{ is contained in } \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline 3&4\\\hline\end{array} *_{\mathbb{A}^\times} \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}$$

results in the following relations on submatrices:

Submatrix	Relations
A_2	A_{12}^1
A_3	$A_{13}^1, A_{23}^1, A_{12}^2A_{23}^2 + sA_{13}^2$
A_4	$A_{34}^1, A_{23}^2A_{34}^2 + A_{24}^1 + sA_{24}^2, A_{12}^2A_{24}^2 + A_{13}^2A_{34}^2 + A_{14}^1, A_{12}^2A_{24}^1 - sA_{14}^1,$ $A_{14}^1A_{23}^2 - A_{13}^2A_{24}^1, A_{13}^2A_{24}^1A_{34}^2 + A_{14}^1A_{24}^1 + sA_{14}^1A_{24}^2$

At $s = 0$ the ideal generated by the above relations decomposes as a union of the following generalized orbital varieties.

Ideal of $X(\tau)$	τ
$(A_{12}^1, A_{12}^2, A_{13}^1, A_{23}^1, A_{34}^1, A_{23}^2A_{34}^2 + A_{24}^1,$ $A_{13}^2A_{34}^2 + A_{14}^1, A_{13}^2A_{24}^1 - A_{23}^2A_{14}^1, s)$	$\begin{array}{\|c\|c\|c\|}\hline 1&1&3\\\hline 2&2&4\\\hline 3&4\\\cline{1-2}\end{array}$
$(A_{12}^1, A_{13}^1, A_{23}^1, A_{23}^2, A_{24}^1,$ $A_{34}^1, A_{12}^2A_{24}^2 + A_{13}^2A_{34}^2 + A_{14}^1, s)$	$\begin{array}{\|c\|c\|c\|}\hline 1&1&2\\\hline 2&3&4\\\hline 3&4\\\cline{1-2}\end{array}$

Example 8.4.15. $\begin{ytableau} 1 & 1 & 1 & 3 \\ 2 & 2 \\ 3 & 4 \end{ytableau} \cdot \begin{ytableau} 2 \\ 4 \end{ytableau} = \begin{ytableau} 2 & 4 \\ 3 \\ 4 \end{ytableau} + \begin{ytableau} 1 & 2 & 3 \\ 2 & 4 \\ 4 \end{ytableau}$: The requirement that

$$A = \left[\begin{array}{ccc|ccc|cc|cc} 0 & 1 & & & & & & & & \\ & 0 & 1 & & & & & & & \\ & & 0 & A_{12}^1 & A_{12}^2 & A_{12}^3 & A_{13}^1 & A_{13}^2 & A_{14}^1 & A_{14}^2 \\ \hline & & & 0 & 1 & & & & & \\ & & & & 0 & 1 & & & & \\ & & & & & s & A_{23}^1 & A_{23}^2 & A_{24}^1 & A_{24}^2 \\ \hline & & & & & & 0 & 1 & & \\ & & & & & & & 0 & A_{34}^1 & A_{34}^2 \\ \hline & & & & & & & & 0 & 1 \\ & & & & & & & & & s \end{array}\right] \text{ is contained in } \begin{ytableau} 1 & 1 & 1 & 3 \\ 2 & 2 \\ 3 & 4 \end{ytableau} *_{\mathbb{A}^\times} \begin{ytableau} 2 \\ 4 \end{ytableau}$$

results in the following relations on submatrices:

Submatrix	Relations
A_2	A_{12}^1, A_{12}^2
A_3	A_{13}^1, A_{23}^1
A_4	$A_{34}^1, A_{23}^2A_{34}^2 + A_{24}^1 + sA_{24}^2, A_{14}^1A_{23}^2 - A_{13}^2A_{24}^1, A_{13}^2A_{24}^1A_{34}^2 + A_{14}^1A_{24}^1 + sA_{14}^1A_{24}^2$

At $s = 0$ the ideal generated by the above relations decomposes as a union of the following generalized orbital varieties.

Ideal of $X(\tau)$	τ
$(A_{12}^1, A_{12}^2, A_{13}^1, A_{23}^1, A_{23}^2, A_{24}^1, A_{34}^1, s)$	$\begin{ytableau} 1 & 1 & 1 & 2 & 4 \\ 2 & 2 & 3 \\ 3 & 4 \end{ytableau}$
$(A_{12}^1, A_{12}^2, A_{13}^1, A_{23}^1, A_{34}^1, A_{23}^2A_{34}^2 + A_{24}^1, A_{13}^2A_{34}^2 + A_{14}^1, A_{13}^2A_{24}^1 - A_{23}^2A_{14}^1, s)$	$\begin{ytableau} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 \end{ytableau}$

In conclusion, we obtain the following result:

Proposition 8.4.1. *In type A_3, every cluster variable in $\mathbb{C}[N]$ is in the MV basis.*

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