Geometric behaviour of solutions to equations of Ginzburg-Landau type on Riemannian manifolds

by

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In this thesis, we demonstrate the existence of complex-valued solutions to the Ginzburg-Landau equation

$$-\Delta u + \frac{1}{\varepsilon^2} u(|u|^2 - 1) = 0 \quad \text{on } M,$$

for $\varepsilon \ll 1$, where $M$ is a three dimensional compact manifold without boundary, that have interesting geometric properties. Specifically, we argue the existence of solutions whose vorticity concentrates about an arbitrary closed nondegenerate geodesic on $M$. In doing this, we extend the work of [27] and [41] who showed that there are solutions whose energy converges, after rescaling, to the arclength of a geodesic as above.

An important ingredient in the proof is a heat flow argument, which requires detailed information about limiting behaviour of solutions of the parabolic Ginzburg-Landau equation. Providing the necessary limiting behaviour is the other contribution of this thesis. In fact, more is achieved. Provided that $N \geq 3$, we give a structural description of the limiting behaviour of solutions to the parabolic Ginzburg-Landau equation on an $N$-dimensional compact manifold without boundary $(M, g)$. More specifically, we are able to show that the limit of the renormalized energy measure orthogonally decomposes into a diffuse part, absolutely continuous with respect to the volume measure on $M$ induced by $g$, and a concentrated vortex part, supported on a codimension 2 surface contained in $M$. Moreover, the diffuse part of the limiting energy has its time evolution governed by the heat equation while the concentrated part evolves in time according to a measure theoretic version of mean curvature flow. This extends the work of [11] who proved this for $N$-dimensional Euclidean space provided that $N \geq 2$. 
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Chapter 1

Introduction

In this thesis we consider several problems related to the Ginzburg-Landau energy functional

\[ \int_M \left\{ \frac{1}{2} |\nabla u|^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2 \right\} \, dvol_g, \]  

(1.0.1)

for functions \( u: M \to \mathbb{C} \), where \( (M, g) \) is a compact Riemannian manifold without boundary of dimension \( N \geq 3 \). This functional is related to the Ginzburg-Landau model of superconductivity. It has been extensively studied in the Euclidean setting, particular since the seminal work of Bethuel, Brezis, and Hélein [8], which gave an exhaustive description of the behaviour of minimizers, with suitable Dirichlet data, on a 2-d Euclidean domain, in the limit \( \varepsilon \to 0 \).

We will study both the associated elliptic PDE

\[ -\Delta u + \frac{1}{\varepsilon^2} u(|u|^2 - 1) = 0 \]  

(1.0.2)

and the parabolic Ginzburg-Landau heat flow

\[ \partial_t u - \Delta u + \frac{1}{\varepsilon^2} u(|u|^2 - 1) = 0 \]  

(1.0.3)

in the limit as \( \varepsilon \to 0 \). In 3 or more dimensions, the former is known to be related in a variety of ways to the geometric problem of codimension 2 minimal surfaces. For example, given a sequence of solutions \( \{u_\varepsilon\}_{\varepsilon \in (0,1]} \) of (1.0.2) satisfying appropriate energy bounds, in a simply connected setting (a subset of \( \mathbb{R}^n \) or a closed manifold) the energy is known to concentrate in the limit \( \varepsilon \to 0 \) around a codimension 2 stationary varifold – a measure theoretic minimal surface. Similar, if
more complicated, results are known on non-simply connected manifolds, where the limiting energy measure may have a diffuse part, but the concentrated part again forms a codimension 2 stationary varifold.

In this thesis we prove a sort of converse in a particularly simple situation, that of a closed 3-dimensional Riemannian manifold \((M, g)\). We show that, given any smooth nondegenerate codimension 2 minimal surface \(\gamma\) on \(M\) — here, a closed geodesic — one can find a sequence of solutions of (1.0.2) whose rescaled energy densities converge, as \(\varepsilon \to 0^+\), to the varifold naturally associated to \(\gamma\). Results of this sort were previously known if only \(\gamma\) is assumed to be length-minimizing. In the general case, the best prior results [27, 41] construct sequences of solutions whose total energy is consistent with convergence of the energy measure to the (multiplicity 1) varifold associated to \(\gamma\), but with no further information about the limiting energy density.

Our construction of solutions of (1.0.2) relies on a heat flow argument, which rests on the deep connection between the parabolic equation (1.0.3) in the limit \(\varepsilon \to 0\) and Brakke flows, a measure theoretic version of the mean curvature flow of submanifolds, in this case of codimension 2. This connection was established for solutions of (1.0.3) on \(\mathbb{R}^N, N \geq 3\), for sequences of initial data satisfying only natural energy bounds, in an important paper of Bethuel, Orlandi and Smets [11]. This followed earlier significant contributions of a number of authors, including [25, 33, 5, 35, 52]. In this thesis we extend the results of [11] to the setting of a closed Riemannian manifold of dimension \(N \geq 3\). This is a natural generalization of a landmark result, and it also provides exactly the information about the Ginzburg-Landau heat flow needed for our construction of solutions of the elliptic problem.

### 1.1 Organization of the Thesis

In the remainder of Chapter 1 we provide some preliminary discussion regarding concepts from Riemannian geometry and Geometric Measure Theory as they will be needed in each of Chapters 2 and 3. We defer the discussion of other necessary background concepts and notation for chapters 2 and 3 as they are fairly specialized to each chapter.

The first contribution of this thesis is a detailed analysis of the energy measure associated to a sequence of solutions of the parabolic equation (1.0.3) on a closed manifold \((M, g)\) of dimension \(N \geq 3\), in the limit \(\varepsilon \to 0\). Our main result in this direction is stated in Theorem 2.1.1, which does not impose any restrictions on the topology of \(M\) assumes that the initial data for (1.0.3) satisfies only natural energy bounds. As mentioned above, this generalizes results from the important paper
[11], with substantial new considerations arising not only from non-Euclidean character of the metric but also from global topological properties of $M$, in particular the fact that it need not be simply connected.

The proof of Theorem 2.1.1 is presented in Chapter 2. Our overall strategy closely follows that developed in [11]. We present detailed discussions of points that differ materially from [11]. Inevitably, there are also numerous arguments that are essentially similar to corresponding points in [11]. In an effort to focus on the significant new elements, we omit from Chapter 2 discussions that would essentially duplicate prior arguments. However, we attempt to sketch these proofs well enough to make it believable when we assert that no significant new subtleties arise in the Riemannian case. As a result in places our exposition in Chapter 2 functions as a sort of reader’s guide to parts of [11]. This seems to us necessary for a reasonably complete account of the proof of Theorem 2.1.1.

Among the details omitted in Chapter 2, there are many whose essential similarity to [11] is not entirely clear until one has gone through the proofs line-by-line. As a convenience to the reader, we collect full proofs of all such points in Appendix A. These are arguments that we consider to involve few novel ingredients, and they are presented for completeness, and as a reference for the interested reader.

The other main result of this thesis is contained in Chapter 3, where we specialize to the 3-dimensional case and demonstrate the existence of solutions to the Ginzburg-Landau equation (1.0.2) whose vorticity and energy concentrate as $\varepsilon \to 0^+$ about a nondegenerate unstable geodesic. A precise statement appears in Theorem 3.1.1. To do this we make use of, among many other papers such as [2] and [41] and the work of Chapter 2. This chapter is based on joint work with R.L. Jerrard and P. Sternberg, see [16].

Chapters 2 and 3 both start with Introductions that discuss at greater length the history and other aspects of the two problems under consideration.

1.2 Preliminaries

1.2.1 Riemannian Geometry

In this subsection we present some basic concepts from Riemannian geometry that we will make use of throughout the document. We refer the reader to [28] and [18] for further discussion on Riemannian Geometry.
Throughout this document we will reserve $M$ to be a compact manifold without boundary of dimension $N \geq 3$ paired with a smooth metric $g$. At $x \in M$ we let $T_x M$ denote the tangent space to $M$ at $x$. At each $x \in M$ we use either $\langle \cdot, \cdot \rangle_g$ or $(\cdot, \cdot)_g$ to denote the inner product on $T_x M$ given by $g$ and $| \cdot |_g$ to denote the norm on $T_x M$ given by $g$. For $x, y \in M$ we use $d_g(x, y)$ to denote the distance between $x$ and $y$ induced by the metric $g$. For $p \in M$ and $r > 0$ we use the notation $B_{r,g}(p)$ to denote the geodesic ball about $p$ of radius $r$ in the metric $g$ which is defined by

$$B_{r,g}(p) := \{ x \in M : d_g(x, p) < r \}.$$  

We will write $\text{vol}_g$ to denote the unique Radon measure on $M$ with the property that $\text{vol}_g(A)$ is the Riemannian volume of $A$ for all sufficiently regular $A$, and for non-negative $f \in L^1(M; \text{vol}_g)$, we write $f \text{vol}_g$ to denote the measure defined by

$$f \text{vol}_g(A) := \int_A f \text{dvol}_g.$$  

We define the injectivity radius of $M$ according to the metric $g$, denoted $\text{inj}_g(M)$, by

$$\text{inj}_g(M) := \sup \left\{ r > 0 \mid \exp_x : T_x M \to M \text{ is a diffeomorphism onto } B_{r,g}(x) \text{ for all } x \in M \right\}. \quad (1.2.1)$$

We define the diameter of $M$ according to the metric $g$, denoted $\text{diam}_g(M)$, by

$$\text{diam}_g(M) := \sup \{ d_g(x, y) : \forall x, y \in M \}. \quad (1.2.2)$$

In the above notation we may, for convenience, remove the subscript $g$. We note for $p \in M$ and $0 < s < \text{inj}_g(M)$ that the function

$$r(x) := \frac{1}{2} (d_g(x, p))^2$$  

satisfies

$$\nabla r(x) = -\exp_x^{-1}(p)$$  

on $B_s(p)$, see Theorem 6.6.1 of [28]. Also, if the sectional curvature, $K$, of $M$ satisfies

$$\lambda \leq K \leq \mu, \quad \text{with } \lambda \leq 0 \leq \mu$$
then, for $0 < \rho < \min \left\{ \frac{\pi}{2 \sqrt{\mu}}, \text{inj}(M) \right\}$ if $\mu > 0$ and $0 < \rho < \text{inj}(M)$ otherwise, we have

$$\sqrt{\mu} d(x, p) \cot \left( \sqrt{\mu} d(x, p) \right) |v|^2 \leq \text{Hess}(r)(v, v) \leq \sqrt{\lambda} d(x, p) \coth \left( \sqrt{\lambda} d(x, p) \right) |v|^2$$  \hspace{1cm} (1.2.5)

for $x \in B_\rho(p)$ and $v \in T_x M$, see Theorem 6.6.1 of [28]. This is referred to as the Hessian Comparison Theorem.

### 1.2.2 Brakke flows

In this subsection we introduce Brakke flows.

**Definition 1.2.1.** A Radon measure $\nu$ on $M$ is said to be $k$-rectifiable if there exists a $k$-rectifiable set $\Sigma$, and a density function $\Theta \in L^1_{\text{loc}}(\mathcal{H}^k \llcorner \Sigma)$ such that

$$\nu = \Theta(\cdot) \mathcal{H}^k \llcorner \Sigma.$$  \hspace{1cm} (1.2.6)

Next, we define the distributional first variation of a rectifiable Radon measure. To do this, we remark that if $\Sigma$ is $k$-rectifiable then at $\mathcal{H}^k$-almost every point $x \in \Sigma$ there is a unique tangent space $T_x \Sigma$ belonging to the Grassmannian $G_{N,k,x}$. Similar to [11] we associate $G_{N,k,x}$ to projection operators onto $k$-dimensional subspaces of $T_x M$.

**Definition 1.2.2.** Let $\nu$ be a $k$-rectifiable Radon measure. Then we define the *distributional first variation of $\nu$* to be the distribution, $\delta \nu$, defined by

$$\delta \nu(X) := \int_{\Sigma} \text{div}_{T_x \Sigma}(X) d\nu \text{ for all } X \in \chi(M)$$  \hspace{1cm} (1.2.7)

where $\chi(M)$ denotes the space of smooth vector fields over $M$ and, following section 2 of [45], we define

$$\text{div}_{T_x \Sigma}(X) := \sum_{k=1}^{N-2} \langle D_{e_k} X(x), e_i \rangle$$  \hspace{1cm} (1.2.8)

where $\{e_1, e_2, \ldots, e_{N-2}\}$ denote any orthonormal basis of $T_x \Sigma$ and $D_{e_i} X$ denote the associated covariant derivatives. When $|\delta \nu|$ is absolutely continuous with respect to $\nu$, we say that $\nu$ has a *first variation* and we may write

$$\delta \nu = H \nu$$
where $H$ is the Radon-Nikodym derivative of $\delta \nu$ with respect to $\nu$. In this case, (1.2.7) becomes

$$
\int_{\Sigma} \text{div}_{T_x \Sigma} (X) d\nu = \int_{\Sigma} \langle H, X \rangle d\nu. \tag{1.2.9}
$$

Finally, in the special case that $H = 0$ and we have

$$
\delta \nu = 0 \tag{1.2.10}
$$

we refer to $\nu$ as stationary.

Next, we let $\{\nu_t\}_{t \geq 0}$ be a family of Radon measures on $M$. For $\chi \in C^2(M; (0, \infty))$, we define

$$
\overline{D}_t \nu_t^0 (\chi) := \limsup_{t \to t_0} \frac{\nu_t^0(\chi) - \nu_{t_0}^0(\chi)}{t - t_0}.
$$

If $\nu_t^{\chi} \subset \{ \chi > 0 \}$ is a $k$-rectifiable measure which has a first variation verifying $\chi |H|^2 \in L^1(\nu^t)$, then we set

$$
\mathcal{B}(\nu^t, \chi) := -\int \chi |H|^2 d\nu^t + \int \langle \nabla \chi, P(H) \rangle d\nu^t,
$$

where $P$, as in section 2 of [45] and consistent with our identification of the Grassmannian with projections, denotes $\mathcal{H}^k$-almost everywhere the orthogonal projection onto the tangent space to $\nu^t$, otherwise, we set

$$
\mathcal{B}(\nu^t, \chi) = -\infty.
$$

We are now in a position to give the definition of a Brakke flow.

**Definition 1.2.3.** Let $\{\nu^t\}_{t \geq 0}$ be a family of Radon measures on $M$. We say that $\{\nu^t\}_{t \geq 0}$ is a $k$-dimensional Brakke flow if and only if

$$
\overline{D}_t \nu^t(\chi) \leq \mathcal{B}(\nu^t, \chi), \tag{1.2.11}
$$

for every $\chi \in C^\infty(M; (0, \infty))$ and for all $t \geq 0$. 
Chapter 2

Limiting Behaviour of solutions of the Parabolic Ginzburg-Landau Equation

2.1 Introduction

In this chapter we extend the work of Bethuel, Orlandi, and Smets on the parabolic Ginzburg-Landau equation from [11] to the setting of a compact Riemannian manifold \((M, g)\) of dimension \(N \geq 3\). More specifically, we are interested in providing a detailed description of the limiting behaviour as \(\varepsilon \to 0^+\) of solutions of the PDE initial value problem

\[
\begin{cases}
\partial_t u_\varepsilon = \Delta u_\varepsilon + \frac{1}{\varepsilon} u_\varepsilon (1 - |u_\varepsilon|^2) & \forall x \in M \text{ and } \forall t > 0 \\
u_\varepsilon(x, 0) = u_\varepsilon^0(x) & \forall x \in M
\end{cases}
\tag{PGL}_\varepsilon
\]

for a given \(u_\varepsilon^0\) which, throughout this chapter, we assume satisfies

\[
\mathcal{E}_\varepsilon(u_\varepsilon^0) \leq M_0 |\log(\varepsilon)| \quad \text{where } M_0 \text{ is a fixed positive constant}
\tag{H_0}
\]

and where

\[
\mathcal{E}_\varepsilon(u) := \int_M e_\varepsilon(u) \, d\text{vol}_g, \quad e_\varepsilon(u) := \frac{1}{2} |\nabla u|^2 + V_\varepsilon(u). \tag{2.1.1}
\]
with
\[ V_\varepsilon(x) := \frac{1}{4\varepsilon^2} (1 - |x|^2)^2. \]

The asymptotics of solutions to the equation \((PGL)_\varepsilon\) has been extensively studied in the setting of Euclidean space. For \(N \geq 3\) it was shown in \([25, 34]\), in a variety of settings, including \(\mathbb{R}^N\) and bounded open subset of \(\mathbb{R}^N\), that for well-prepared initial data, the energy of solutions to \((PGL)_\varepsilon\) concentrates around a codimension 2 mean curvature flow, as long as that flow remains smooth. It was then shown in \([5]\) that if the limiting energy measure satisfies a lower density bound, then this result may be extended past the formation of singularities, thereby giving a conditional proof of convergence of rescaled energy measures, globally in time, to a codimension 2 Brakke flow – a measure-theoretic weak solution of the mean curvature flow.

Following this work results were obtained in \([36]\), for \(N = 3\) and on a bounded domain, relating a local energy condition with the local absence of vortex behaviour and using this to demonstrate energy concentration on a rectifiable 1-varifold. The relationship between the local energy condition and the absence of vortex behaviour was shown to hold for \(\mathbb{R}^4\) in \([52]\) where the energy is weighted by a Gaussian function.

Finally, this line of research concluded with \([11]\) which, among other improvements, removed the lower density bound imposed in \([5]\), giving an unconditional proof that the concentrated part of a limiting energy measure evolves via a Brakke flow in \(\mathbb{R}^N\), globally in \(t\), for every \(N \geq 3\), and without requiring well-prepared initial data.

The description of the dynamics of the limiting energy measure over \(\mathbb{R}^N\) in \([11]\) raised the question of possible extensions to other settings. One such extension is found in \([37]\) who demonstrated the conclusions of \([11]\) for the parabolic Ginzburg-Landau equation with magnetic potential in \(\mathbb{R}^3\). Related work in the Riemannian setting includes \([46]\) and \([45]\) for the Allen-Cahn equation over a compact Riemannian manifold without boundary as well as \([48]\) which extends the Monotonicity formula to a suitably restricted class of compact Riemannian manifolds, possibly with boundary. Despite these efforts, an extension of the results of \([11]\) to the case of a compact Riemannian manifold without boundary has not been shown.

The main result of this chapter is, in the setting of a compact smooth Riemannian manifold
(M, g) without boundary, a careful study of the family of energy measures

$$\mu_t^\varepsilon(x) := \frac{e^\varepsilon(u_\varepsilon(x, t))}{|\log(\varepsilon)|} \, dvol_g(x)$$

for $t > 0$ as $\varepsilon \to 0^+$. Of particular note is that we impose no topological or curvature restrictions on $M$ beyond what is guaranteed by compactness. As a result, our analysis applies to compact manifolds with possibly non-trivial topology. The result of this analysis, stated in Theorem 2.1.1, is that the limiting energy decomposes into a diffuse energy and a concentrated vortex energy which do not interact. The evolution of the diffuse energy will be governed by the heat equation while the vortex energy evolves according to a Brakke flow, a measure theoretic formulation of mean curvature flow. More specifically, we have:

**Theorem 2.1.1.** Let $M$ be of dimension $N \geq 3$ and suppose that $\{u_\varepsilon\}_{\varepsilon \in (0,1)}$ are a family of solutions to (PGL)$_\varepsilon$ for corresponding $\varepsilon$ and with respective initial data $\{u_0^\varepsilon\}_{\varepsilon \in (0,1)}$. Let $\mu_t^\varepsilon$ be, for each $t > 0$, the measure on $M$ defined by

$$\mu_t^\varepsilon := \frac{e^\varepsilon(u_\varepsilon(\cdot, t))}{|\log(\varepsilon)|} \, dvol_g.$$ 

Then, after perhaps passing to a subsequence $\{u_{\varepsilon_n}\}_{n \in \mathbb{N}}$, there exists a family of limiting measures $\{\mu_t^*\}_{t > 0}$ and subsets $\{\Sigma_t^\mu_*\}_{t > 0}$ in $M$, as well as a function $\Phi_* : M \times (0, \infty) \to \mathbb{R}/2\pi\mathbb{Z}$ such that the following properties hold:

1. $\mu_{\varepsilon_n}^\varepsilon \rightharpoonup \mu_t^*$ in $M(M)$ for each $t > 0$.

2. $\Phi_*$ satisfies the heat equation on $M \times (0, \infty)$.

3. For each $t > 0$, the measure $\mu_t^*$ can be exactly decomposed as

$$\mu_t^* = \left| \nabla \Phi_* \right|^2_2 \mathcal{H}^N + \nu_t^* \tag{2.1.2}$$

where

$$\nu_t^* = \Theta_*(x, t)\mathcal{H}^{N-2} \mathcal{L} \Sigma_t^\mu_* \tag{2.1.3}$$

and where $\Theta_*(\cdot, t)$ is a bounded measurable function.

4. There exists a positive function $\eta$ defined on $(0, \infty)$ such that, for $\mathcal{L}^1$-almost every $t > 0$, the set $\Sigma_t^\mu_*$ is $(N-2)$-rectifiable and

$$\Theta_*(x, t) = \Theta_{N-2}(\mu_t^*, x) = \lim_{r \to 0^+} \frac{\mu_t^*(B_r(x))}{\omega_{N-2} r^{N-2}} \geq \eta(t), \tag{2.1.4}$$
for $\mathcal{H}^{N-2}$-almost every $x \in \Sigma^t_{\mu_\ast}$.

5. The family of measures $t \mapsto \Theta_{\ast}(x,t)\mathcal{H}^{N-2} \mathcal{L}_{\Sigma^t_{\mu_\ast}}$ forms a Brakke flow.

These conclusions were first demonstrated in [11] for the, non-compact, smooth manifold $\mathbb{R}^N$ paired with the standard metric.

In general we follow the strategy developed in [11]. However, a number of details need to be adapted in order for the strategy to extend to the more general setting.

- When defining the weighted energy, which is used to establish a monotonicity formula, we use an approximation to the heat kernel as a weight. The form of the alteration that we employ differs from the earlier works [46], [45] and is designed to facilitate a comparison of the weighted energy at distinct points in space-time, see Lemma 2.3.7.

- A consequence of modifying the weighted energy is that new error terms $\Phi$ and $\Psi$ arise, see (2.3.7) and (2.3.8) for definitions. The error term $\Phi$, as seen in Theorem 1 of [22], corresponds to the fact that we are not working over Euclidean space while $\Psi$, as seen in [48], reflects the fact that we have replaced the heat kernel on $M$ with an approximation. These error terms are handled by appealing to the Hessian Comparison Theorem which is discussed in (1.2.5).

- When following the Hodge de Rham decomposition strategy from subsection 3.6 of [11] we need to solve a Poisson problem over $M$. Since we do not impose any topological restrictions on $M$ some care is needed to ensure that a solution exists. Specifically, we needed to modify the argument from [11] to account for the harmonic part of the data as well as provide additional estimates for the resultant error terms.

- When decomposing the solution to $(\text{PGL})_\varepsilon$, as in Theorem 3 of [11], we now have to account for the fact that no topological restrictions were placed on $M$. This, in particular, has the effect of adding an additional term, $u_{h,\varepsilon}: M \times (0, \infty) \to S^1$, which corresponds to the harmonic part of the Hodge de Rham decomposition of $u_\varepsilon \times du_\varepsilon$ at time $t = 0$. The presence of this additional term also has consequences on how we are able to express the limiting energy density in Theorem 2.1.1.

The use of the Hessian Comparison Theorem gives rise to curvature-dependent constants in many of our estimates. In our arguments, it is often convenient to rescale the metric $g$ to a dilated metric $g/a$ with $a \in (0, 1)$. All estimates that we need continue to hold with the same, often better,
constants after such rescaling. Indeed, such a rescaling decreases bounds on the curvature and hence improves all curvature-dependent constants.

While Theorem 2.1.1 is interesting in its own right it is worth noting that this result is a key ingredient in demonstrating the existence of solutions to the elliptic Ginzburg-Landau equation over $(M,g)$, when $N = 3$, for which the energy and a quantity associated to vorticity concentrate about a non-length minimizing geodesic as $\varepsilon \to 0^+$. This is demonstrated in chapter 3 which improves on earlier work such as [27] and [41].

We conclude this introduction by describing some issues in the proof of Theorem 2.1.1.

First, as in [11], an important intermediate result is the following “clearing out” theorem. It involves a weighted energy, $\tilde{E}_\varepsilon$, whose definition is provided in (2.2.6).

**Theorem 2.1.2.** For any $\sigma \in (0,1)$ and $T > 0$ there exists positive numbers $\varepsilon_0$, $R(\sigma)$, and $\eta(\sigma)$ such that if $u_\varepsilon$ is a solution to (PGL)$_\varepsilon$ on $M \times (0,T)$ satisfying $(H_0)$ for $0 < \varepsilon < \varepsilon_0$, $R$ satisfies $\sqrt{2\varepsilon} < R < \min\{R(\sigma), \sqrt{T}\}$, and $x_T$ is a point such that

$$\tilde{E}_\varepsilon(u_\varepsilon,(x_T,T),R) \leq \eta(\sigma)|\log(\varepsilon)|,$$

then

$$|u_\varepsilon(x_T,T)| \geq 1 - \sigma.$$

The overall strategy of the proof follows that of Theorem 1 in [11], on which Theorem 2.1.2 is modelled. We start by presenting in Section 2.4, drawing on the work of [11]. In this overview we highlight elements of the proof in which substantial new considerations arise. All such points are treated in detail in Section 2.5. The overview of Section 2.4 also identifies many aspects of the proof that carry over to the Riemannian setting with only superficial changes. Detailed verification of these points are omitted from this chapter. However, for the sake of completeness, and as a convenience to the reader, these details are provided in Appendix A. In addition, we attempt in Section 2.4 to describe the underlying ideas in sufficient detail to explain why the arguments of [11] do not involve any substantial changes in the Riemannian context.

The next result is an adaptation of Theorem 3 from [11]. New issues arise from the possibly non-trivial topology of $M$. This is reflected in the presence of the $S^1$-valued map $u_{h,\varepsilon}$. We refer the reader to (2.2.7) for the definition of $ju$ where $u : M \to \mathbb{C}$ which is used in the statement of the next theorem.
**Theorem 2.1.3.** Suppose \( u_\varepsilon \) satisfies (PGL) \( \varepsilon \) and (H\( _0 \)). Then there exists an \( \mathbb{S}^1 \)-valued function \( u_{h,\varepsilon} \), depending only on the initial data of \( u_\varepsilon \) such that, for any compact set \( \mathcal{K} \subset M \times (0, \infty) \) and \( \varepsilon \) sufficiently small, there is a real-valued function \( \phi_\varepsilon \) and a complex-valued function \( w_\varepsilon \) defined on a neighbourhood of \( \mathcal{K} \), such that

1. \( u_\varepsilon = w_\varepsilon e^{i\phi_\varepsilon} u_{h,\varepsilon} \) on \( \mathcal{K} \),
2. \( \phi_\varepsilon \) verifies the heat equation on \( \mathcal{K} \),
3. \( |\nabla \phi_\varepsilon(x,t)| \leq C(\mathcal{K}) \sqrt{(M_0 + 1) |\log(\varepsilon)|} \) for all \( (x,t) \in \mathcal{K} \),
4. \( \|\nabla w_\varepsilon\|_{L^p(\mathcal{K})} \leq C(p, \mathcal{K}) \), for any \( 1 \leq p < \frac{N+1}{N} \),
5. \( u_{h,\varepsilon} \) does not depend on \( t \), \( ju_{h,\varepsilon} \) is a harmonic 1-form on \( M \), and

\[
|\nabla u_{h,\varepsilon}(x,t)| \leq K_M \sqrt{M_0 |\log(\varepsilon)|} \text{ for all } (x,t) \in \mathcal{K}.
\]

Here, \( C(\mathcal{K}) \) and \( C(p, \mathcal{K}) \) are constants depending only on \( \mathcal{K} \) and \( \mathcal{K}, p \) (and \( M_0 \)) respectively and \( K_M \) is a constant depending only on \( M \).

This is proved in Section 2.6. Finally, the proof of Theorem 2.1.1 is completed in Section 2.7.

### 2.2 Preliminaries

In this section we record some of the specialized notation and definitions used throughout this section.

We use the notation \( \Lambda_\alpha(x_0, T, R, \Delta T) \) for \( 0 < \alpha \leq 1 \), \( x_0 \in M \), \( T \geq 0 \), \( \Delta T > 0 \), and \( R > 0 \) to refer to

\[
\Lambda_\alpha(x_0, T, R, \Delta T) := B_{\alpha R}(x_0) \times [T + (1 - \alpha^2) \Delta T, T + \Delta T]. \tag{2.2.1}
\]

We also use the abbreviations \( \Lambda_\alpha \) for (2.2.1) and \( \Lambda := \Lambda_1(x_0, T, R, \Delta T) \) when the other parameters are understood.

For \( y \in M \) we define the approximate heat kernel about \( y \) evaluated at \( (x,t) \in M \times (0, \infty) \), denoted \( K_{ap}(x,t;y) \), by

\[
K_{ap}(x,t;y) := \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp \left[ -\frac{(d_{+g}(x,y))^2}{4t} \right]. \tag{2.2.2}
\]
where $d_{+,g} : M \times M \to [0,\infty)$ is a smooth function defined so that

$$d_{+,g}(x,y) := \text{inj}_g(M)f\left(\frac{d_g(x,y)}{\text{inj}_g(M)}\right)$$

(2.2.3)

where $f : [0,\infty) \to [0,\infty)$ is a smooth function chosen so that

1. $f(s) = s$ for $s \in [0,\frac{1}{2}]$,
2. $f(s) = 1$ for $s \geq 1$,
3. $f(s) \geq s$ for $0 \leq s \leq 1$,
4. $f$ is non-decreasing,
5. $\|f'\|_{L^\infty(\mathbb{R})} < \sqrt{2}$.

We note that $d_{+,g}$ satisfies

$$c_+ d_g(x,y) \leq d_{+,g}(x,y) \leq 2d_g(x,y)$$

(2.2.4)

where

$$c_+ := \frac{\text{inj}_g(M)}{\text{diam}_g(M)}.$$

We will use the notation $K_{ap,g}(x,t; x_*)$ when we wish to explicitly indicate the dependence of $K_{ap}$ on the metric $g$. Also, for a fixed point $p \in M$ we use the notation $r_+$ to denote

$$r_+(x) := \frac{1}{2}(d_+(x,p))^2.$$

(2.2.5)

Next we introduce notation for energy weighted by the approximate heat kernel on $M$. For $z_* = (x_*, t_*) \in M \times (0, \infty)$ and $0 < R \leq \sqrt{t_*}$ we use the notation

$$\tilde{E}_e(z_*, R) := R^2 \int_M e_e(u(x,t_* - R^2))K_{ap}(x,R^2;x_*)d\text{vol}_g(x).$$

(2.2.6)

We may also use variations of this notation which include $g$ in the subscript to emphasize particular dependence on the metric.

Finally, for a given $u : M \to \mathbb{C}$ we introduce the notation $ju$ for the 1-form

$$ju := u \times du$$

(2.2.7)
which in coordinates can be expressed as
\[ u \times du := \sum_{i=1}^{N} u \times \frac{\partial u}{\partial x_i} dx^i. \]

## 2.3 Toolbox

We record a few helpful results that will be needed for the proof of Theorem 2.1.1. These are generalizations of corresponding results found in [11].

**Lemma 2.3.1.** Let \( \chi \) be a Lipschitz function on \( M \). Then, for any \( T \geq 0 \), at \( t = T \),
\[
\frac{d}{dt} \int_{M \times \{t\}} e_{\varepsilon}(u_{\varepsilon}) \chi(x) = -\int_{M \times \{T\}} |\partial_t u_{\varepsilon}|^2 \chi(x) - \int_{M \times \{T\}} \partial_t u_{\varepsilon} \cdot \langle \nabla u_{\varepsilon}, \nabla \chi \rangle
\] (2.3.1)

and
\[
\frac{1}{2} \int_{M \times \{t\}} |\partial_t u_{\varepsilon}|^2 \chi^2 + \frac{d}{dt} \int_{M \times \{t\}} e_{\varepsilon}(u_{\varepsilon}) \chi^2 \leq 4 \|\nabla \chi\|_L^2 \int_{\text{supp}(\chi)} e_{\varepsilon}(u_{\varepsilon}).
\] (2.3.2)

In particular, for any \( 0 \leq T_1 \leq T_2 \),
\[
\int_{M \times \{T_2\}} e_{\varepsilon}(u_{\varepsilon}) \chi(x) - \int_{M \times \{T_1\}} e_{\varepsilon}(u_{\varepsilon}) \chi(x) = -\int_{M \times [T_1, T_2]} |\partial_t u_{\varepsilon}|^2 \chi(x) - \int_{M \times [T_1, T_2]} \partial_t u_{\varepsilon} \cdot \langle \nabla u_{\varepsilon}, \nabla \chi \rangle.
\] (2.3.3)

**Proof.** The proof of (2.3.1) follows from differentiation under the integral while (2.3.3) follows by integrating (2.3.1) in \( t \). To see (2.3.2) we replace \( \chi \) with \( \chi^2 \) in Lemma 2.3.1 and use standard estimates.

The next result, the basis for a monotonicity formula, will play a fundamental role in the proof of Theorem 2.1.2.

**Lemma 2.3.2.** Suppose \((M, g)\) is an \( N \)-dimensional compact Riemannian manifold without boundary and suppose that \( u_{\varepsilon} \) solves \((\text{PGL})_{\varepsilon} \) on \( M \). Let \( K_{ap} \) be the approximate heat kernel as in (2.2.2).

Then for \( 0 < R < \sqrt{T} \) and \( y \in M \):
\[
Z'(R) = 2R \int_M \left[ V_{\varepsilon}(u_{\varepsilon}(x, T - R^2)) + \Xi(u_{\varepsilon}, (y, T))(x, T - R^2) \right] \] (2.3.4)
\[ + 2R \int_M \Psi(u_{\varepsilon}, (y, T))(x, T - R^2) \]
\[ + 2R \int_M \Phi(u_{\varepsilon}, (y, T))(x, T - R^2) \]
where

$$Z(R) := R^2 \int_M \epsilon(x, y, T - R^2) K_{ap}(x, R^2; y) d\text{vol}_g(x)$$

(2.3.5)

and where, for $0 < t < T$, we have set

$$\Xi(u_\epsilon, (y, T))(x, t) := (T - t) \left| \partial_t u_\epsilon(x, t) + \frac{\langle \nabla u_\epsilon(x, t), \nabla K_{ap}(x, T - t; y) \rangle}{K_{ap}(x, T - t; y)} \right|^2,$$

(2.3.6)

$$\Phi(u_\epsilon, (y, T))(x, t) := (T - t) \left[ \text{Hess}(K_{ap}(x, T - t; y))(\nabla u_\epsilon(x, t), \nabla u_\epsilon(x, t)),$$

$$- \frac{|\langle \nabla u_\epsilon(x, t), \nabla K_{ap}(x, T - t; y) \rangle|^2}{K_{ap}(x, T - t; y)} + \frac{|\nabla u_\epsilon|^2 K_{ap}(x, T - t; y)}{2(T - t)} \right].$$

(2.3.7)

$$\Psi(u_\epsilon, (y, T))(x, t) := (T - t) \epsilon(x, y, T) e_\epsilon(x, t) (\partial_t K_{ap})(x, T - t, y) - (\Delta K_{ap})(x, T - t, y).$$

(2.3.8)

We also have, for any $z_T = (x_T, T) \in M \times (0, \infty)$ and $R_* = \sqrt{T}$, that

$$\tilde{E}_\epsilon(z_T, R_*) = \int_{M \times [0, T]} (V_\epsilon(u_\epsilon) + \Xi(u_\epsilon, z_T)) K_{ap}(x, T - t; x_T) d\text{vol}_g(x) dt$$

$$+ \int_{M \times [0, T]} \Phi(u_\epsilon, z_T) d\text{vol}_g(x) dt + \int_{M \times [0, T]} \Psi(u_\epsilon, z_T) d\text{vol}_g(x) dt.$$

(2.3.9)

Proof. Computations like (2.3.4) are quite standard, and very similar ones can be found for example in the proof of Theorem 2.1 of [48]. Then (2.3.9) follows by integrating (2.3.4) from $R = 0$ to $R = \sqrt{T}$ and changing variables. For completeness we have provided a proof in A.3.1.1. 

As remarked in the introduction, the terms $\Phi$ and $\Psi$ reflect the non-Euclidean character of the metric and the use of the approximate, rather than exact, heat kernel. They are estimated using arguments that ultimately rely on the Hessian Comparison Theorem. We illustrate this first for $\Psi$.

Lemma 2.3.3. Let $(M, g)$ be an $N$-dimensional compact Riemannian manifold and suppose $y \in M$. Let $K_{ap}$ be the approximate heat kernel from (2.2.2) and $\Psi$ be as in (2.3.8). Then there is $c_0 > 0$ such that for all $0 < t < T$ we have

$$\int_M \Psi(u_\epsilon, (y, T))(x, T - t) \geq -\frac{N|\lambda|t^{\frac{N}{2}}}{4} \int_M e_\epsilon(u) K_{ap} - c_0 t \int_M e_\epsilon(u)$$

(2.3.10)

where the constants remain bounded when dividing the metric by $0 < a \leq 1$ and we have used the abbreviations $K_{ap}$ for $K_{ap}(x, t; y)$ and $u$ for $u(x, T - t)$. Similarly, there is $c_1 > 0$ such that for all $0 < t < T$ we have

$$\int_M \Psi(u_\epsilon, (y, T))(x, T - t) \leq -\frac{N|\lambda|t^{\frac{N}{2}}}{6} \int_M e_\epsilon(u) K_{ap} + c_1 t \int_M e_\epsilon(u).$$

(2.3.11)
where the constants remain bounded when dividing the metric by $0 < a \leq 1$. It is worth noting that we also have

$$\int_M \Psi(u, (y,T))(x,T-t) \leq \frac{N|\lambda|}{6} \int_M (d_+(x,y))^2 e_\varepsilon(u)K_{ap} + C_M \int_M e_\varepsilon(u)K_{ap} + C_0E_0 t$$

(2.3.12)

where $C_M, C_0$ remain bounded when dividing the metric by $0 < a \leq 1$.

Proof. By computing $\partial_t K_{ap} - \Delta K_{ap}$ we obtain, using the notation from (2.2.5), that

$$\partial_t K_{ap} - \Delta K_{ap} = \frac{[\Delta r_+(x) - N]}{2t} K_{ap} + \frac{r_+(x) - \frac{1}{2} |\nabla r_+(x)|^2}{2t^2} K_{ap}.$$  

First observe that if $s := \min\left\{ \frac{\pi}{4\sqrt{\mu}}, \text{inj}(M) \right\}$ and $x \in B_s(y)$ then the rightmost term is zero and by using the notation (1.2.3) as well as (1.2.5) we obtain

$$\frac{[\Delta r_+(x) - N]}{2t} K_{ap} = \frac{[\Delta r(x) - N]}{2t} K_{ap} \geq -\frac{N\mu(d(x,y))^2}{4t} K_{ap}.  \quad (2.3.13)$$

Next, observe that for $x \in M \setminus B_s(y)$ we have

$$\partial_t K_{ap} - \Delta K_{ap} \geq -\frac{C_M \max\{t,1\}}{t^2} K_{ap}.  \quad (2.3.14)$$

Using (2.3.13) and (2.3.14) leads to

$$\int_M e_\varepsilon(u) |\partial_t K_{ap} - \Delta K_{ap}| \geq -\frac{N\mu}{4t} \int_{B_s(y)} (d(x,y))^2 e_\varepsilon(u)K_{ap} - \frac{C_M \max\{t,1\}}{t^2} \int_{M \setminus B_s(y)} e_\varepsilon(u)K_{ap}.$$  

Note that, since $d_+(\cdot, y)$ is a function of distance from $y$, we have

$$\frac{C_M \max\{t,1\}}{t^2} \int_{M \setminus B_s(y)} e_\varepsilon(u)K_{ap} \leq \frac{C_M \max\{t,1\} e^{-\frac{t}{2}}} {t^2(4\pi t)^\frac{N}{2}} \int_{M \setminus B_s(y)} e_\varepsilon(u)$$

$$\leq C'_M e^{-\frac{t}{2}} \int_{M \setminus B_s(y)} e_\varepsilon(u)$$

$$\leq C'_M \int_M e_\varepsilon(u).$$

Note that if we rescale the metric by dividing by $0 < a \leq 1$ then the constant $C'_M$ as well as the exponential term only become smaller. Observe that we either have $t^\frac{1}{4} \geq s$ or $0 < t^\frac{1}{4} < s$. If $t^\frac{1}{4} \geq s$ then

$$-\frac{N\mu}{4t} \int_{B_s(y)} (d(x,y))^2 e_\varepsilon(u)K_{ap} \geq -\frac{N\mu}{4t^\frac{1}{2}} \int_{B_s(y)} e_\varepsilon(u)K_{ap}.$$
If \( 0 < t^\frac{1}{2} < s \) then we have, using the notation \( A_{t^\frac{1}{2}, s}(y) := B_s(y) \setminus B_{t^\frac{1}{2}}(y) \) for \( y \in M \), that

\[
- \frac{N\mu}{4t} \int_{B_s(y)} (d(x, y))^2 e_x(u) K_{ap} = \frac{N\mu}{4t} \int_{B_{t^\frac{1}{2}}(y)} (d(x, y))^2 e_x(u) K_{ap} - \frac{N\mu}{4t} \int_{A_{t^\frac{1}{2}, s}(y)} (d(x, y))^2 e_x(u) K_{ap} \\
\geq - \frac{N\mu}{4t^\frac{1}{2}} \int_{B_{t^\frac{1}{2}}(y)} e_x(u) K_{ap} - \frac{N\mu (\text{inj}(M))^2}{16} \int_{A_{t^\frac{1}{2}, s}(y)} e_x(u) \cdot \frac{e^{-\frac{d(x, y)^2}{4t}}}{(4\pi t)^{\frac{d}{2}}} \\
\geq - \frac{N\mu}{4t^\frac{1}{2}} \int_{B_{t^\frac{1}{2}}(y)} e_x(u) K_{ap} - \frac{N\mu (\text{inj}(M))^2}{16} \sup_{\epsilon > 0} \left\{ \frac{e^{-\frac{1}{2\epsilon^2}}}{t(4\pi t)^{\frac{d}{2}}} \right\} \cdot \frac{1}{\epsilon^{\frac{1}{2}}} \int_{A_{t^\frac{1}{2}, s}(y)} e_x(u) \\
\geq - \frac{N\mu}{4t^\frac{1}{2}} \int_{M} e_x(u) K_{ap} - C''_M \int_{M} e_x(u) \\
= \frac{N\mu}{4t^\frac{1}{2}} \int_{M} e_x(u) K_{ap} - C''_M \int_{M} e_x(u).
\]

Notice that \( C''_M \) is invariant under rescaling in the metric and \( \mu \) only becomes smaller if we divide the metric by \( a \) for \( 0 < a < 1 \). Putting this altogether gives

\[
\int_{M} e_x(u) [\partial_t K_{ap} - \Delta K_{ap}] \geq - \frac{N\mu}{4t^\frac{1}{2}} \int_{M} e_x(u) K_{ap} - 2 \max\{C'_M, C''_M\} \int_{M} e_x(u).
\]

Setting

\[
c_0 := 2 \max\{C'_M, C''_M\}
\]

and multiplying by \( t \) gives the desired result. Observe that a similar proof holds for (2.3.11) and that (2.3.12) is demonstrated through the proof of the upper bound. \( \square \)

We next record estimates of a similar character for \( \Phi \).

**Lemma 2.3.4.** Suppose \((M, g)\) is an \(N\)-dimensional compact Riemannian manifold without boundary. Let \( K_{ap} \) be the approximate heat kernel from (2.2.2). Then there is \( c_2 > 0 \) such that for all \( 0 < t < T \) that

\[
\int_{M} \Phi(u, (y, T))(x, T - t) \geq - \frac{|\lambda||t^\frac{1}{2}|}{3} \int_{M} e_x(u) K_{ap} - c_2 t \int_{M} e_x(u) .
\]

where the constants remain bounded when dividing the metric by \( 0 < a \leq 1 \) and where we have used the abbreviations \( K_{ap} \) for \( K_{ap}(x, t; y) \) and \( u \) for \( u(x, T - t) \). Similarly, there is \( c_3 > 0 \) such that for
all $0 < t < T$ that
\[
\int_M \Phi(u, (y, T))(x, T - t) \leq \frac{\mu t^2}{2} \int_M e_\varepsilon(u)K_{ap} + c_3 t \int_M e_\varepsilon(u). \tag{2.3.16}
\]
where the constants remain bounded when dividing the metric by $0 < a \leq 1$. It is worth noting that we have
\[
\int_M \Phi(u, T)(x, T - t) \leq \mu \int_M \frac{(d_+(x, y))^2}{4} e_\varepsilon(u)K_{ap} + D_M \int_M e_\varepsilon(u)K_{ap}. \tag{2.3.17}
\]
where $D_M$ remains bounded when dividing the metric by $0 < a \leq 1$.

**Proof.** The proof is similar to that of Lemma 2.3.3. More discussion is provided in A.3.1.2. \qed

We now prove a monotonicity formula for solutions to (PGL)$_\varepsilon$. As noted before, this result will be instrumental to demonstrating many of the estimates needed in the proof of Theorem 2.1.2.

**Proposition 2.3.5.** Let $K_{ap}$ be the approximate heat kernel and suppose that $y \in M$ and $T > 0$. Then there exists positive constants $C_1 \geq 1$ and $C_2$ such that if $0 \leq R_1 \leq R_2 \leq \min\{\sqrt{T}, 1\}$ then
\[
C_1 E_0 R_1 + \exp[C_2 R_1]Z(R_1) \leq C_1 E_0 R_2 + \exp[C_2 R_2]Z(R_2) \tag{2.3.18}
\]
where
\[
E_0 := \int_M e_\varepsilon(u_0^0(x))d\text{vol}_g(x). \tag{2.3.19}
\]
That is, the function $r \mapsto C_1 E_0 r + \exp[C_2 r]Z(r)$ is non-decreasing on $[0, \min\{\sqrt{T}, 1\}]$.

**Proof.** Combining (2.3.10) and (2.3.15) for $u = u_\varepsilon$ with the expression for $Z'(R)$ from Lemma 2.3.2 gives an inequality of the form
\[
Z'(R) \geq -\tilde{C} Z(R) - \tilde{D} E_0
\]
where $\tilde{C}$ and $\tilde{D}$ are positive constants that remain bounded when dividing the metric by $0 < a \leq 1$. Setting $C_2 := \tilde{C}$ and $C_1 := \tilde{D} e^{\tilde{C}}$ as well as using that $R \leq 1$ leads to (2.3.18). More details are provided in A.3.1.3. \qed

**Remark 2.3.6.** As one might guess from the appeal to the Hessian Comparison Theorem, the constants $C_1$ and $C_2$ from the above Proposition can all be estimated in terms of upper and lower bounds on the sectional curvature. As a result, all such constants are preserved by dividing the
metric \( g \) by factors smaller than one. This is generally the case for all curvature-dependent constants appearing in this chapter.

The next result facilitates comparison of the weighted energy centred about two different points in space-time.

**Lemma 2.3.7.** Let \( 0 < t_0 < T \), and \( z_0 = (x_*, t_0) \in M \times (0, \infty) \). Then,

\[
\widetilde{E}_\varepsilon(z_0, \sqrt{t_0}) \leq \left( \frac{T}{t_0} \right)^{\frac{\gamma}{2} - 1} \exp \left[ C_f \left( \frac{d_+(x_T, x_*)}{t_0} \right)^2 \right] \widetilde{E}_{w, \varepsilon}(u_\varepsilon, (x_T, T), \sqrt{T})
\]

for all \( x_T \in M \) where \( C_f := \max \{ 1, \| f' \|_{L^\infty([0, \infty))} \} \) and \( f \) is as defined below (2.2.3). In particular,

\[
\widetilde{E}_\varepsilon(z_0, \sqrt{t_0}) \leq \left( \frac{T}{t_0} \right)^{\frac{\gamma}{2} - 1} \exp \left[ \frac{4C_f (d(x_T, x_*)^2)}{T - t_0} \right] \widetilde{E}_{w, \varepsilon}(u_\varepsilon, (x_T, T), \sqrt{T})
\]

for all \( x \in M \) where \( C_f \) is as above.

**Proof.** The proof proceeds in the same way as the proof of Lemma 2.3 of [11] except a careful estimate of the supremum of the function

\[
x \mapsto \exp \left( \frac{(d_+(x, x_T))^2}{4T} - \frac{(d_+(x, x_*)^2)}{4t_0} \right)
\]

is required. The corresponding estimate in [11] is done completely explicitly. Here it is carried out by considering several cases, depending on the relative size of \( d_g(x, x_T), d_g(x, x_*), \) and \( \text{inj}_g(M) \). Details can be found in A.3.2.1. \( \square \)

The next proposition is an important localization method that converts information about the energy density on a small ball to information about the weighted energy. This will be helpful when analyzing the structure of the energy density measure in the proof of Theorem 2.1.1.

**Proposition 2.3.8.** Suppose \( T > 0 \) and \( \sqrt{2\varepsilon} < R < 1 \). Then for any \( \lambda > 0 \) and \( x_T \in M \) the following inequality holds

\[
\int_M e_\varepsilon(u_\varepsilon(\cdot, T)) e^{-\frac{(d_+(\cdot, x_T))^2}{4\lambda^2}} \leq \int_{B_{\lambda R}(x_T)} e_\varepsilon(u_\varepsilon(\cdot, T))
\]

\[
+ M_0 e^{-\frac{\varepsilon^2}{4}} \left[ e^{C_2 \left( \frac{2R^2}{T + 2R^2} \right)^{\frac{N-2}{2}}} + C_1 (4\pi)^{\frac{N}{2}} (\sqrt{2R})^{N-2} \sqrt{T} \right] |\log(\varepsilon)|.
\]
Proof. The proof is essentially the same as that of Proposition 2.3 of [11]. The point is to estimate
\[
\int_{M \setminus B_{\lambda R}(x_T)} e_{\epsilon}(u_{\epsilon}) e^{- \frac{(d_{\epsilon}(x,x_T))^2}{4R^2}}.
\]
This is achieved by applying the monotonicity formula, see Proposition 2.3.5, in addition to the properties of \(d_+\), as in (2.2.4). See A.3.5.1 for details.

In the next proposition we exploit the monotonicity formula to obtain good estimates of the solution of a nonhomogeneous heat equation when the right-hand side is dominated by \(V_{\epsilon}(u_{\epsilon})\).

**Proposition 2.3.9.** If \(0 < T < 1\), \(x_T \in M\), and \(\omega: M \times (0, \infty) \to \wedge^2 M\) solves
\[
\begin{cases}
\partial_t \omega - \Delta \omega = h & \text{on } M \times (0, \infty) \\
\omega(x, 0) = 0 & x \in M
\end{cases}
\]
where \(h \in L^\infty(M \times [0, T]; \wedge^2 M)\) satisfies
\[
|h(x, t)| \leq V_{\epsilon}(u_{\epsilon}(x, t)) \quad \text{for } (x, t) \in M \times [0, T] \tag{2.3.20}
\]
then for any \(z = (x, t) \in M \times [0, T]\), the following estimate holds:
\[
|\omega(z)| \leq C_3(T + 1) \left(\frac{T}{t} \right)^{\frac{N}{2} - 1} e^{\frac{C_f (d_{\epsilon}(x_T, x))^2}{4T}} \left(\tilde{E}_{\epsilon}(u_{\epsilon}, (x_T, T), \sqrt{T}) + C_1 E_0 T\right) \tag{2.3.21}
\]
\[
\leq C_3(T + 1) \left(\frac{T}{t} \right)^{\frac{N}{2} - 1} e^{\frac{4C_f (d(x_T, x))^2}{4T}} \left(\tilde{E}_{\epsilon}(u_{\epsilon}, (x_T, T), \sqrt{T}) + C_1 E_0 T\right)
\]
where \(C_f\) is as in Lemma 2.3.7 and \(C_3\) depends on \(M\).

Proof. The proof proceeds in the same way as Proposition 2.2 of [11], the idea being to represent \(\omega(z)\) by Duhamel’s formula and then exploit the fact that, since \(|f| \leq V_{\epsilon}(u_{\epsilon})\), the right-hand side of Duhamel’s formula is controlled by the weighted energy. In our setting we must estimate the heat kernel for 2-forms, appearing in Duhamel’s formula, by the approximate heat kernel \(K_{ap}\), appearing in \(\tilde{E}_{\epsilon}\). This may be done using estimates on the heat kernel for differential forms which are provided in [38]. Details can be found in A.3.3.1.

The next proposition is a localization method that originated from [35] and was used in [11]. As in [11] this result is vital to our proof of Theorem 2.1.2 as it permits us to localize our estimate of the weighted energy to a small coordinate ball. It is based on a Pohozaev type inequality.
Proposition 2.3.10. Let \((M, g)\) be an \(N\)-dimensional compact Riemannian manifold without boundary and suppose \(0 < t < T\) is chosen so that \((T - t)\) is small enough that

\[ 1 - C_5(T - t) \geq \frac{1}{2} \]

where \(C_5 > 0\) depends linearly on the sectional curvature of \(M\). Then, there is a constant \(C_6 > 0\) invariant under dilations of the metric \(g\) by factors larger than one such that if \(z_T = (x_T, T) \in M \times (0, \infty)\) then

\[
\int_{M \times \{t\}} e_x(u_x) \frac{(d_+(x, x_T))^2}{4(T - t)} e^{- \frac{(d_+(x, x_T))^2}{4(T - t)}} \leq 2 \int_{A \times \{t\}} e_x(u_x) e^{- \frac{(d_+(x, x_T))^2}{4(T - t)}} + \frac{(4\pi)^\frac{N}{2} C_6 (T - t)^\frac{N}{2} + 1}{N} E_0 \tag{2.3.22}
\]

and consequently

\[
\int_{M \times \{t\}} e_x(u_x) e^{- \frac{(d_+(x, x_T))^2}{4(T - t)}} \leq 2 \int_{A \times \{t\}} e_x(u_x) e^{- \frac{(d_+(x, x_T))^2}{4(T - t)}} + \frac{(4\pi)^\frac{N}{2} C_6 (T - t)^\frac{N}{2} + 1}{N} E_0 \tag{2.3.23}
\]

where

\[ A := \left\{ x \in M : \frac{(d_+(x, x_T))^2}{8(T - t)} \leq C_6 \right\}. \]

Proof. The proof is mostly similar to the one found in Proposition 2.4 of [11], the only exception being we replace usage of the distance function \(d\) with \(d_+\) and use properties relating to the definition of \(d_+\). Due to the similarity we refer the reader to Lemma A.3.4 and A.3.5.2 for more details. First, for \(0 < T_1 \leq T_2 < T\) and \(x_T \in M\), we establish the inequality

\[
\int_{T_1}^{T_2} \int_{M} \frac{(d_+(x, x_T))^2}{4(T - t)} e_x(u_x) e^{- \frac{(d_+(x, x_T))^2}{4(T - t)}} \leq \left\| f' \right\|_{L^\infty(\mathbb{R})}^2 D_f E_\varepsilon(z_T, \sqrt{T - T_1}) - \left\| f' \right\|_{L^\infty(\mathbb{R})}^2 D_f E_\varepsilon(z_T, \sqrt{T - T_2})
\]

where

\[ D_f := (2 - \left\| f' \right\|_{L^\infty(\mathbb{R})}^2)^{-1}. \]

To do this, we take the dot product of \((\text{PGL}_\varepsilon)\) with \(2(T - t) \partial_t u_x e^{- \frac{(d_+(x, x_T))^2}{4(T - t)}}\), integrate by parts in
time, and apply elementary inequalities. The only difference from Lemma 2.6 of [11] involves using properties of $d_+$, such as that

$$|\nabla r_+(x)| \leq \|f\|_{L^\infty(\mathbb{R})}d_+(x, y)$$

where $r_+$ is as in (2.2.5) and $y \in M$.

Then, setting $T_1 = t$, letting $T_2 \searrow t$, and using (2.3.9) leads to

$$\int_{M \times \{t\}} e_\varepsilon(u_\varepsilon) \frac{(d_+(x, x_T))^2}{4(T - t)} e^{-\frac{(d_+(x, x_T))^2}{4(T - t)}} \leq ND_f(4\pi)^{\frac{N}{2}}(T - t)^{\frac{N}{2} - 1}\tilde{E}_{w,\varepsilon}(z_T, \sqrt{T - t})$$

(2.3.24)

$$+ (4\pi)^{\frac{N}{2}}D_f(T - t)^{\frac{N}{2}} \int_{M \times \{t\}} [V_\varepsilon(u_\varepsilon) + \Xi(u_\varepsilon, z_T)]K_{ap}(x, T - t; x_T)$$

$$+ (4\pi)^{\frac{N}{2}}D_f(T - t)^{\frac{N}{2}} \int_{M \times \{t\}} [\Phi(u_\varepsilon, z_T)(x, t) + \Psi(u_\varepsilon, z_T)(x, t)]dvol(x).$$

Combining (2.3.12) and (2.3.16) with (2.3.24) leads to

$$\left[1 - (4\pi)^{\frac{N}{2}}D_f[\mu - \frac{2N\lambda}{3}](T - t)\right] \int_{M \times \{t\}} e_\varepsilon(u_\varepsilon) \frac{(d_+(x, x_T))^2}{4(T - t)} e^{-\frac{(d_+(x, x_T))^2}{4(T - t)}}$$

$$\leq (4\pi)^{\frac{N}{2}}D_f[N + C_M + D_M](T - t)^{\frac{N}{2} - 1}\tilde{E}_{w,\varepsilon}(z_T, \sqrt{T - t})$$

$$+ (4\pi)^{\frac{N}{2}}D_f(T - t)^{\frac{N}{2}} \int_{M \times \{t\}} [V_\varepsilon(u_\varepsilon) + \Xi(u_\varepsilon, z_T)]K_{ap}(x, T - t; x_T)$$

$$+ C_0(4\pi)^{\frac{N}{2}}D_f(T - t)^{\frac{N}{2} + 1}E_0.$$
such that

$$\{ C_7 E_0 R_1 + Z(R_1) \} - \{ C_7 E_0(\delta_0 R_1) + Z(\delta_0 R_1) \} \leq \frac{4C_7e^{C_8}|\log(\delta_0)|}{|\log(\epsilon)|}\left[ R E_0 + Z(R) \right]$$

(2.3.25)

where $C_7 := C_1 e^{2C_2}$ and $C_8 := 2C_2$. We also have

$$\int_{T - R_1}^{T - (\delta_0 R_1)^2} \int_M \left[ (\Xi(x, z_T))(x, t) + V_e(u_e(x, t)) \right] K_{ap}(x, T - t; x_T) \text{dvol}_g(x) \text{d}t$$

(2.3.26)

$$\leq \frac{4C_7e^{C_8}|\log(\delta_0)|}{|\log(\epsilon)|}\left[ R E_0 + Z(R) \right].$$

Proof. The proof is essentially the same as in Proposition 2.6 of [11]. The idea in proving (2.3.25) is to average increments of $r \mapsto C_7 e^{C_8} E_0 r + e^{C_8 r} Z(r)$ over time intervals $[\delta_j^0 R, \delta_j^{k-1} R]$ for $j = 2, 3, \ldots, k_0$ where

$$k_0 \approx \left\lfloor \frac{|\log(\epsilon)|}{2|\log(\delta_0)|} \right\rfloor$$

and to find an interval, $[\delta_j^{k_0} R, \delta_j^{k_0-1} R]$, for which the increment is small. This is achieved by repeatedly making use of Proposition 2.3.5. The inequality (2.3.26) then follows from (2.3.9), additional estimates on the error terms $\Phi$ and $\Psi$ in terms of the weighted energy and the initial energy due to (2.3.10), (2.3.15), as well as our choice of constants $C_7$ and $C_8$. For more details we refer the reader to A.3.6.1.

2.4 Overview of the proof of Theorem 2.1.2

We start by presenting a detailed outline of the proof of Theorem 2.1.2. We closely follow the proof presented in [11], so much so that this section may be used as a reader’s guide to the arguments found there. As we proceed, we will distinguish between

1. arguments that can be adapted from the Euclidean to the Riemannian setting with only cosmetic changes; we will describe these but not present them in detail; and

2. places where more effort is needed in order to adjust earlier arguments to the present setting.

These points will be discussed at greater length in Section 2.5.

Note that, unless otherwise specified, all metric related quantities will be associated to $g_{R_1}$ and the metric will be suppressed from the notation.
**Reduction via rescaling:**

Throughout the proof of the theorem, \(0 < \delta_0 < \frac{1}{16}\) will denote a fixed parameter whose precise value will not be specified until a late stage of the proof. Applying Proposition 2.3.11 with this choice of \(\delta_0\) and \(R, T\) as in the statement of Theorem 2.1.2 we find a suitable time interval, \([\delta_0 R, R]\), in which the weighted energy of \(u_\varepsilon\) satisfies (2.3.25) and (2.3.26). Next we define \(v_\varepsilon : M \times (0, \infty) \rightarrow \mathbb{C}\), a rescaling of \(u_\varepsilon\) in \(\varepsilon\) and in time, by

\[
v_\varepsilon(x, t) := u_\varepsilon(x, T + R^2[t - 1]) \tag{2.4.1}
\]

where \(\varepsilon := \frac{\varepsilon}{R^2}\). We also introduce the rescaled metric

\[
g_{R_1} := \frac{g}{R^2}. \tag{2.4.2}
\]

It follows from standard parabolic estimates that there is \(K > 0\) such that for \(x \in M\) and \(t > 0\) that

\[
|v_\varepsilon(x, t)| \leq 3, \quad |\nabla v_\varepsilon(x, t)| \leq \frac{K}{\varepsilon}, \quad |\partial_t v_\varepsilon(x, t)| \leq \frac{K}{\varepsilon^2}. \tag{2.4.3}
\]

Rescaling (2.3.25) to be written in terms of \(v_\varepsilon\) as well as applying (H0) and the assumptions of Theorem 2.1.2 we obtain

\[
\{\tilde{E}_{\varepsilon, g_{R_1}}(v_\varepsilon, (x, 1), t) + C_\varepsilon E_0 R_1\} - \{\tilde{E}_{\varepsilon, g_{R_1}}(v_\varepsilon, (x, 1), \delta_0) + C_\varepsilon E_0(\delta_0 R_1)\} \leq 4C_\varepsilon \varepsilon^C \log(\delta_0) \|RM_0 + \eta\|. \tag{2.4.4}
\]

Finally, a change of variables applied to (2.3.26) in addition to an application (H0) and the assumptions of Theorem 2.1.2 leads to

\[
\int_{M \times [0, 1 - \delta_0]} [V_\varepsilon(v_\varepsilon) + \Xi(v_\varepsilon, (x, 1))] K_{ap, g_{R_1}}(x, 1 - t; x, t) \leq 4C_\varepsilon \varepsilon^C \log(\delta_0) \|RM_0 + \eta\|. \tag{2.4.5}
\]

From here Theorem 2.1.2 is reduced to demonstrating the following result.

**Proposition 2.4.1.** Let \(T > 0\) and \(x_T \in M\). Then there exists constants \(0 < \delta_0 < \frac{1}{16}\), \(0 < \epsilon_0 < \frac{1}{2} \min\{1, \text{inj}_g(M)\}\), \(0 < R_0 < 1\), and \(\eta_0 > 0\) such that for \(0 < \eta \leq \eta_0\), \(0 < \epsilon < \epsilon_0\), and
0 \leq R < \min \{ \sqrt{T}, R_0 \} \text{ the following inequality holds:}

\[ \tilde{E}_{\epsilon,gR_1}(v_\epsilon, (x_T, 1), \delta_0) \leq \frac{1}{4} \left( \tilde{E}_{\epsilon,gR_1}(v_\epsilon, (x_T, 1), 1) + C_7 E_0 R_1 \right) + \mathcal{R}(\eta, R) \] (2.4.6)

where \( \mathcal{R}(\eta, R) \) tends to zero as \( \eta, R \to 0^+ \) and \( R_1 \) is as in Proposition 2.3.11.

The proof of Theorem 2.1.2 using Proposition 2.4.1 proceeds with only minor differences to the argument in the Euclidean setting from [11]. We refer the reader to section 2.5 for additional details.

We then reduce proving Proposition 2.4.1 to demonstrating the existence of \( 0 < \delta_0 < \frac{1}{16} \) for which there is some \( \delta \in [\delta_0, 2\delta_0] \) for which

\[ \tilde{E}_{\epsilon,gR_1}(v_\epsilon, (x_T, 1), \delta) \leq e^{-C_2} \left( \tilde{E}_{\epsilon,gR_1}(v_\epsilon, (x_T, 1), 1) + C_7 E_0 R_1 \right) + \mathcal{R}(\eta, R) \] (2.4.7)

where \( \mathcal{R}(\eta, R) \to 0^+ \) as \( \eta, R \to 0^+ \). The argument reducing the proof of Proposition 2.4.1 to (2.4.7) is essentially the same as in [11] and so we refer the reader to subsection 2.5.1 for more details.

**Preliminary choice of good time slice:**

By Chebyshev’s inequality applied to (2.4.5) in time over the interval \([1 - 4\delta_0^2, 1 - \delta_0^2]\) we see that there are a large number of time slices \( t = 1 - \delta^2 \), where \( \delta \in [\delta_0, 2\delta_0] \), for which

\[ \int_M V_\epsilon(v_\epsilon) K_{ap,gR_1}(x, 1 - t; x_T) \leq C(\delta_0)[RM_0 + \eta] \] (2.4.8)

\[ \int_M V_\epsilon(x, (x_T, 1)) K_{ap,gR_1}(x, 1 - t; x_T) \leq C(\delta_0)[RM_0 + \eta] \] (2.4.9)

The inequalities (2.4.8) and (2.4.9) will be used to determine \( \mathcal{R}(\eta, R) \) from (2.4.7) as well as obtain the coefficient of \( \tilde{E}_{\epsilon,gR_1}(v_\epsilon, (x_T, 1), 1) + C_7 E_0 R_1 \) from (2.4.7). We use the notation \( \Theta_1 \) to denote

\[ \Theta_1 := \left\{ t \in [1 - 4\delta_0^2, 1 - \delta_0^2] : (2.4.8) \text{ and } (2.4.9) \text{ both hold at } t \right\} \] (2.4.10)

We note that the set \( \Theta_1 \) is defined with more explicit constants in (A.4.1). In particular, we provide a more explicit estimate on the number of slices in Lemma A.4.1. The strategy in proving (2.4.7) will be to decompose \( \tilde{E}_{\epsilon,gR_1}(v_\epsilon, (x_T, 1), \delta) \) into suitable components and estimate the resultant terms using PDE techniques by showing that the data can be controlled by (2.4.8) and (2.4.9).

**Localization and decomposition:**
We make use of Proposition 2.3.10, applied through \( u_\varepsilon \), in combination with (2.4.8) and (2.4.9) to obtain

\[
\tilde{E}_{e,g_{R_1}}(v_\varepsilon, (x_T, 1), \delta) \leq \frac{2}{(4\pi)^{\frac{N}{2}}\delta^{N-2}} \int_{B_{\sqrt{\varepsilon}}^0(x_T)} e_\varepsilon(v_\varepsilon) \text{dvol} \tag{2.4.11}
\]

\[ + K_M \delta_0^3 R^3 \left[ \tilde{E}_{e,g_{R_1}}(v_\varepsilon, (x_T, 1), 1) + C_7 E_0 R_1 \right] + C(\delta_0)[RM_0 + \eta]. \]

Since \( C(\delta_0)[RM_0 + \eta] \) can be included in \( R(\eta, R) \) and \( \delta_0^3 R^4 \) can be chosen suitably small it suffices to estimate the remaining term from (2.4.11). To do this we decompose \( e_\varepsilon(u_\varepsilon) \). We first observe that there is a constant \( K > 0 \) such that

\[
e_\varepsilon(v_\varepsilon) \leq K \left( |v_\varepsilon \times dv_\varepsilon|^2 + |v_\varepsilon|^2 ||\nabla|v_\varepsilon||^2 + V_\varepsilon(v_\varepsilon) \right). \tag{2.4.12}
\]

We further decompose \( v_\varepsilon \times dv_\varepsilon \) by using a Hodge de Rham decomposition. To do this we first define \( H(\omega) \) to be the harmonic part of a 2-form \( \omega \). Explicitly,

\[
H(\omega):= \sum_{i=1}^{\beta_2(M)} \langle \omega, \gamma_{i,g_{R_1}} \rangle_{L^2 \gamma_{i,g_{R_1}}}. \tag{2.4.13}
\]

where \( \{\gamma_{i,g_{R_1}}\}_{i=1}^{\beta_2(M)} \) is an \( L^2 \)-orthonormal basis for the space of harmonic 2-forms on \( M \) in the metric \( g_{R_1} \), obtained by rescaling an \( L^2 \)-orthonormal basis \( \{\gamma_{i,g}\}_{i=1}^{\beta_2(M)} \) for the space of harmonic 2-forms on \( M \) in the metric \( g \), and \( \beta_2(M) \) denotes the 2nd Betti number of \( M \). We may then use a Hodge de Rham decomposition to find \( \phi_t, \psi_t, \) and \( \xi_t \) satisfying

\[
v_\varepsilon \times dv_\varepsilon = d\phi_t + d^*\psi_t + \xi_t \quad \text{on } B_{3r/2}(x_T) \times \{t\} \tag{2.4.14}
\]

\[
d^*\xi_t = 0 \quad \text{on } B_{3r/2}(x_T) \times \{t\} \tag{2.4.15}
\]

\[
d\xi_t = d^*d\psi_t + H(d|v_\varepsilon \times dv_\varepsilon| \chi) \quad \text{on } B_{3r/2}(x_T) \times \{t\} \tag{2.4.16}
\]

\[
-\Delta \psi_t = d|v_\varepsilon \times dv_\varepsilon| \chi - H(d|v_\varepsilon \times dv_\varepsilon| \chi), \quad \text{on } M \times \{t\} \tag{2.4.17}
\]

where \( r > 0 \) is chosen sufficiently small toward the end of the argument and \( \chi \) is a smooth cutoff function supported in \( B_{4r}(x_T) \) which is identically 1 on \( B_{2r}(x_T) \), \( 0 \leq \chi \leq 1 \), and \( \|\nabla \chi\|_{L^\infty} \leq \frac{2}{r} \). To make the notation more compact we set

\[
H^+(\omega):=\omega - H(\omega) \tag{2.4.18}
\]
where \( \omega \) is a 2-form over \( M \times \{ t \} \). We note that \( H \) is a new consideration for the Hodge de Rham decomposition that does not appear in the corresponding identities from [11]. This term arises because we impose no topological restrictions on \( M \).

From (2.4.11), (2.4.12), (2.4.14) it will suffice to estimate each of the following:

\[
\int_{B_{\delta \sqrt{\frac{8}{C_6}}}(x_T)} |v_\epsilon|^2 |\nabla|v_\epsilon||^2 + V_\epsilon(v_\epsilon). \tag{2.4.19}
\]

\[
\int_{B_{\delta \sqrt{\frac{8}{C_6}}}(x_T)} |d\varphi_t|^2 \tag{2.4.20}
\]

\[
\int_{B_{\delta \sqrt{\frac{8}{C_6}}}(x_T)} \{|d\psi_t|^2 + |d^*\psi_t|^2\} \tag{2.4.21}
\]

\[
\int_{B_{\delta \sqrt{\frac{8}{C_6}}}(x_T)} |\xi_t|^2. \tag{2.4.22}
\]

Since \( H (d[v_\epsilon \times d\psi_\epsilon] \chi) \) is present in both (2.4.16) and (2.4.17) then to achieve our goal we will also need to provide estimates related to \( H \). Below we outline the strategy for estimating (2.4.19)–(2.4.22). Where necessary, we will provide additional details in section 2.5.

**Modulus Estimate:**

As in [11] the goal is to demonstrate that (2.4.19) satisfies an estimate of the form

\[
\int_{B_{\delta_0, \Theta_1}(x_T)} \left\{ |v_\epsilon|^2 |\nabla|v_\epsilon||^2 + V_\epsilon(v_\epsilon) \right\} \leq C(\delta_0, r)[R M_0 + \eta] \frac{1}{2} \left[ E_{v_\epsilon, \Theta_1}(v_\epsilon, (x_T, 1), \delta) + C R E_0 + 1 \right]. \tag{2.4.23}
\]

The proof of (2.4.23) follows the same procedure as section 3.5 of [11]. As such, we will describe it briefly and refer the reader to subsection A.4.2. We first define \( \sigma_\epsilon := 1 - |v_\epsilon|^2 \) and observe that \( \sigma_\epsilon \) solves the PDE

\[
\partial_t \sigma_\epsilon - \Delta \sigma_\epsilon = 2|\nabla v_\epsilon|^2 - \frac{2}{\epsilon^2} \sigma_\epsilon (1 - \sigma_\epsilon). \tag{2.4.24}
\]

By moving \( \partial_t \sigma_\epsilon \) to the right-hand side we can treat this as a Poisson problem so that elliptic techniques can be applied to obtain an interior estimate for \( \nabla \sigma_\epsilon \). We then apply various algebraic manipulations to estimate the terms on the right-hand side of the Poisson problem by quantities involving \( \Xi \) and \( V_\epsilon \). From there we make use of the assumption that \( t \in \Theta_1 \).

**Estimate of \( \xi_t \):**

As in [11] we use that \( \xi_t \) solves (2.4.15) and (2.4.16) in addition to elliptic estimates, see Lemma 5.2
of [3] and note the correction found in [4], to obtain
\[
\|\xi_t\|_{L^2(B_{2r}(x_T))} \leq K \left[ \|d\psi_t\|_{L^2(B_{2r}(x_T))} + \|H(d[v \times dv])\|_{L^2(M)} \right].
\] (2.4.25)

The argument for (2.4.25) is the same as Lemma 3.4 of [11].

**Estimate of \(\varphi_t\):**
As in [11] the goal in estimating \(\varphi_t\) is to obtain an inequality of the form
\[
\int_{B_{3r}(x_T)} |\nabla \varphi_t|^2 e^{-\frac{(d(x,x_T))^2}{4r^2}} \leq \frac{K_M \delta \tilde{N}}{r} \left[ \tilde{E}_{c,R_1}(v, (x_1, 1)) + C_7 R_1 E_0 \right] (2.4.26)
\]
\[+ C(\delta_0, r) \left[ (RM_0 + \eta) + (RM_0 + \eta)^{\frac{3}{2}} \tilde{E}_{c,R_1}(v, (x_1, 1)) + C_7 R_1 E_0 \right]^{\frac{1}{2}} + R_2(t) \]
where
\[
R_2(t) := \int_{B_{3r}(x_T)} (|d^* \psi_t|^2 + |\xi_t|^2)
\]
\[+ \left( \int_{B_{3r}(x_T)} (|d^* \psi_t|^2 + |\xi_t|^2) \right)^{\frac{1}{2}} (\tilde{E}_{c,R_1}(v, (x_1, 1)) + C_7 R_1 E_0)^{\frac{1}{2}}. \]

Much of the proof extends with little change to the Riemannian setting with the exception of a computation done in coordinates. We present this new ingredient in section 2.5.2 and provide a brief outline of the general argument below.

To achieve (2.4.26), we introduce an elliptic PDE that \(\varphi_t\) solves over \(B_s(x_T)\) for \(s \in [r, \frac{3r}{2}]\) where \(s\) will be carefully chosen to ensure good properties. As shown in section 2.5.2 we have that for each \(s \in [r, \frac{3r}{2}]\), \(\varphi_t\) solves
\[
\begin{cases}
L_\delta \varphi = h & \text{on } B_s(x_T) \times \{t\} \\
\frac{\partial \varphi}{\partial r} = g & \text{on } \partial B_s(x_T) \times \{t\},
\end{cases}
\] (2.4.27)
where \(L_\delta\) is defined, using the abbreviation \(K_{ap} := K_{ap,g,R_1}(x; \delta^2; x_T)\), by
\[
L_\delta := \frac{-1}{K_{ap}} \text{div}[K_{ap} \nabla].
\]
and
\[ h := v_\epsilon \times \left( \frac{-\left( \nabla K_{ap}, \nabla v_\epsilon \right)}{K_{ap}} - \partial_t v_\epsilon \right) + \left\langle \frac{dK_{ap}}{K_{ap}}, d^* \psi_t + \xi_t \right\rangle \text{ on } B_s(x_T) \times \{t\} \]
\[ g := v_\epsilon \times \frac{\partial v_\epsilon}{\partial r} - (d^* \psi_t + \xi_t)_N \text{ on } \partial B_s(x_T) \times \{t\}, \]
where \( \omega_N \) denotes the normal part of \( \omega \) and we recall that \( t = 1 - \delta^2 \) is assumed to be an element of \( \Theta_1 \). Next, we make use of elliptic estimates for (2.4.27) to obtain
\[ \int_{B_s(x_T)} |\nabla \varphi|^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \leq C(\delta, r) \left[ \int_{B_s(x_T)} h^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} + \left( \int_{B_s(x_T)} h^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \right)^{\frac{1}{2}} \left( \int_{\partial B_s(x_T)} g^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \right)^{\frac{1}{2}} \right] + K_M \int_{\partial B_s(x_T)} g^2 e^{-\frac{(d(x,x_T))^2}{4s^2}}. \]

Finally, to obtain (2.4.26) we estimate the data, \( h \) and \( g \), in terms of \( \Xi, V_\epsilon(v_\epsilon), d^* \psi_t \), and \( \xi_t \) with one exception in which we obtain an estimate in terms of \( v_\epsilon \). In particular, in estimating \( g \) an averaging process is used for \( s \in [r, 3r^2] \) to estimate integrals over \( \partial B_s(x_T) \) in terms of integrals over \( B_{\frac{s}{r}}(x_T) \setminus B_r(x_T) \). We note that the described exception results in the first term on the right-hand side of (2.4.26). This is the only term where \( \delta \) will be needed to manufacture the leading coefficient of (2.4.7). We also note that the procedure for estimating \( g \) is the same as in [11] except an application of (1.2.5) is needed. We refer the reader to section 2.5 for more details.

**Estimate of \( \psi_t \):**

Estimating \( \psi_t \) is more involved and will be outlined through a number of steps. The goal of each step will be to successively decompose \( \psi_t \) into terms with more specific information that can be utilized. Unlike previous estimates, subterms of \( \psi_t \) are not always estimated by appealing to PDE techniques. Instead we may make use of detailed pointwise information as well as the the work of [26].

**Step 1: Decomposition of \( \psi_t \)**

Before proceeding with the decomposition we introduce some notation as well as some useful pointwise estimates. We introduce a real valued function defined on \( M \times (0, \infty) \) defined in terms of \( |v_\epsilon| \)
so that if $\tilde{v}_\epsilon = \tau v_\epsilon$ then

$$|1 - \tau^2(x, t)| \leq K|1 - |v_\epsilon(x, t)|^2|$$  

(2.4.29)

$$\tilde{v}_\epsilon = v_\epsilon \quad \text{if} \quad |v_\epsilon| \leq \frac{1}{4}$$  

(2.4.30)

$$|\tilde{v}_\epsilon| = 1 \quad \text{if} \quad |v_\epsilon| \geq \frac{1}{2}.$$  

(2.4.31)

Next we decompose $\psi_t$, which solves (2.4.17), as $\psi_t = \psi_{1,t} + \psi_{2,t}$ where $\psi_{1,t}$ and $\psi_{2,t}$ solve

$$-\Delta \psi_{1,t} = H^\perp (d[\tilde{v}_\epsilon \times d\tilde{v}_\epsilon] \chi) \quad \text{on} \quad M \times \{t\}$$  

(2.4.32)

$$-\Delta \psi_{2,t} = H^\perp (d[(1 - \tau^2)v_\epsilon \times dv_\epsilon] \chi) \quad \text{on} \quad M \times \{t\}.$$  

(2.4.33)

The smallness (2.4.29) of $1 - \tau^2$ will aid in estimates of $\psi_{2,t}$. A key point in estimates of $\psi_{1,t}$ is that

$$|d(\tilde{v}_\epsilon \times d\tilde{v}_\epsilon)| \leq K\frac{(1 - |v_\epsilon|^2)^2}{4\epsilon^2} = KV_\epsilon(v_\epsilon) \quad \text{on} \quad M \times (0, \infty).$$  

(2.4.34)

To prove this, note that

$$d(\tilde{v}_\epsilon \times d\tilde{v}_\epsilon) = \sum_{i \neq j} \frac{\partial \tilde{v}_\epsilon}{\partial x_i} \times \frac{\partial \tilde{v}_\epsilon}{\partial x_j} dx^i \wedge dx^j.$$  

(2.4.35)

By (2.4.3), the right-hand side is always bounded by $K/\epsilon^2$ and vanishes when $|\tilde{v}_\epsilon| = 1$, that is, when $|v_\epsilon| \geq 1/2$, so (2.4.34) follows from the definition of $V_\epsilon$. We also see from (2.4.35) that $d(\tilde{v}_\epsilon \times d\tilde{v}_\epsilon)$ has a Jacobian structure [26], which we will exploit.

**Step 2: Decomposition and estimate of $\psi_{2,t}$**

We further decompose $\psi_{2,t}$ as $\psi_{2,t} = \psi_{2,t}^1 + \psi_{2,t}^2$ where $\psi_{2,t}^1$, $\psi_{2,t}^2$ solve

$$-\Delta \psi_{2,t}^1 = d[(1 - \tau^2)(v_\epsilon \times dv_\epsilon) \chi] \quad \text{on} \quad M \times \{t\}$$  

(2.4.36)

$$-\Delta \psi_{2,t}^2 = H^\perp ((1 - \tau^2)[v_\epsilon \times dv_\epsilon] \wedge d\chi) \quad \text{on} \quad M \times \{t\}.$$  

(2.4.37)

where in (2.4.36) we have used that $H^\perp$ is the identity on exact forms. The argument to estimate $\psi_{2,t}^1$ and $\psi_{2,t}^2$ is similar in style to that of Lemma 3.8 of [11] though executed differently. In addition, the data of (2.4.37) requires additional estimates due to the harmonic projection term. We provide more details in section 2.5.

Appealing to, among other things, elliptic regularity, the pointwise estimates (2.4.29), (2.4.3),
and that $t \in \Theta_1$ we can estimate both terms in the decomposition to find that

$$\int_{M \times \{t\}} \{|d\psi_{2,t}|^2 + |d^*\psi_{2,t}|^2\} \leq C(\delta_0, r)[RM_0 + \eta].$$  (2.4.38)

**Step 3: Decomposition of $\psi_{1,t}$**

The decomposition and estimates presented here represent an additional step required to extend the argument of [11] to the manifold setting. This arises due to the presence of the harmonic part in (2.4.32). More details are provided in section 2.5. Since $\psi_{1,t}$ solves (2.4.32) then we may write

$$\psi_{1,t}(x) = \int_M \langle G(x, y), H^\perp(d[\tilde{v}_x \times d\tilde{v}_x] \chi) \rangle \quad (2.4.39)$$

$$= \int_M \langle G(x, y), d[\tilde{v}_x \times d\tilde{v}_x] \chi \rangle - \int_M \langle G(x, y), H(d[\tilde{v}_x \times d\tilde{v}_x] \chi) \rangle$$

$$=: \tilde{\psi}_{1,t}(x) - (G \ast H(d[\tilde{v}_x \times d\tilde{v}_x] \chi))(x)$$

where $G$ is the Dirichlet kernel for 2-forms on $M$. In addition, we can use (2.4.32) to establish

$$\int_M \{|d\psi_{1,t}|^2 + |d^*\psi_{1,t}|^2\} = \int_M \langle \psi_{1,t}, H^\perp(d[\tilde{v}_x \times d\tilde{v}_x] \chi) \rangle.$$  (2.4.40)

Combining (2.4.39), (2.4.40), and estimates on the harmonic projection gives

$$\int_M \{|d\tilde{\psi}_{1,t}|^2 + |d^*\tilde{\psi}_{1,t}|^2\} \leq \int_M \langle \tilde{\psi}_{1,t}, d[\tilde{v}_x \times d\tilde{v}_x] \chi \rangle + C(\delta_0, r)[RM_0 + \eta]^2$$  (2.4.41)

and so it suffices to estimate

$$\int_M \langle \tilde{\psi}_{1,t}, d[\tilde{v}_x \times d\tilde{v}_x] \chi \rangle.$$  (2.4.42)

The advantage of (2.4.42) is that both the “convolution” defining $\tilde{\psi}$ as well as the integral defining (2.4.42) can be taken over small geodesic ball. This facilitates the use of normal coordinates and allows for the use of a detailed coordinate description of the Green’s function.

**Step 4: Decomposition of $\tilde{\psi}_{1,t}$**

Here we follow the decomposition technique presented in [11] which adapts to the setting of a Riemannian manifold with only minor differences. A more complete discussion can be found in section 2.5.

Following [11] we choose an appropriate $\alpha > 0$ and carefully construct a Lipschitz function
$l: [0, \infty) \rightarrow [0, \infty)$ supported on $[e^{\alpha r}, 32r]$ such that $l \equiv 1$ on $[2e^{\alpha r}, 16r]$. From this we define $m: [0, \infty) \rightarrow [0, \infty)$ by

$$m(s) := \begin{cases} 1 - l(s) & \text{for } s \in [0, 16r] \\ 0 & \text{for } s \in (16r, \infty) \end{cases}$$

and set

$$G(x, y) = m(d(x,y))G(x, y) + (1 - m(d(x,y)))G(x, y) =: G^i(x, y) + G^e(x, y).$$

This decomposition allows us to define

$$\tilde{\psi}_1^i, t := \int_{B_{2r}(x_T)} \langle G^i(x, y), d[\tilde{\nu}_e \times d\tilde{\nu}_e] \chi \rangle$$

(2.4.44)

$$\tilde{\psi}_1^e, t := \int_{B_{32r}(x)} \langle G^e(x, y), d[\tilde{\nu}_e \times d\tilde{\nu}_e] \chi \rangle.$$  

(2.4.45)

Using (2.4.44) and (2.4.45) we reduce estimating (2.4.42) to estimating

$$\int_M \langle \tilde{\psi}_1^e, t, d[\tilde{\nu}_e \times d\tilde{\nu}_e] \chi \rangle \leq C(\delta_0, r) \left( E_{\epsilon, g_0} (v_e, (x_T, 1), 1) + C_7 R_1 E_0 + 1 \right) [RM_0 + \eta].$$

(2.4.48)

These computations use detailed information on the form of the Green’s function in normal coordinates.
Step 6: Auxiliary parabolic problem

Unfortunately, the obtainable estimates for $\tilde{\psi}_{1,t}$ are insufficient to make use of duality in (2.4.47).
As a result we, following [11], introduce $\psi^*_1$ the solution to the parabolic problem

$$\begin{cases}
\partial_t \psi^*_1 - \Delta \psi^*_1 = d[\tilde{v}_t \times d\tilde{v}_t] \chi & \text{on } M \times (0, \infty) \\
\psi^*_1(\cdot, 0) = 0 & \text{on } M \times \{0\},
\end{cases}$$ (2.4.49)

and use this to replace $d[\tilde{v}_t \times d\tilde{v}_t] \chi$ with $\partial_t \psi^*_1 - \Delta \psi^*_1$ in (2.4.47). Thus, it suffices to estimate each of

$$\int_M \langle \tilde{\psi}_{1,t}, \partial_t \psi^*_1 \rangle$$ (2.4.50)

$$\int_M \langle \tilde{\psi}_{1,t}, -\Delta \psi^*_1 \rangle.$$ (2.4.51)

The arguments to estimate (2.4.50) and (2.4.51) extend to the Riemannian setting with a bit of additional work. For $\tilde{\psi}_{1,t}$ the main obstacle is the need for local coordinate expressions of the Green’s function and direct computations of the distributional Laplacian of $G^t$. The techniques for estimating $\psi^*_1$ to the Riemannian setting but a bit of care is needed. We provide more details in section 2.5.

Using (2.4.49) it is possible to find a time slice for which we have $L^2$ estimates on $\partial_t \psi^*_1$ as well as be a member of $\Theta_1$. For such $t$ we can estimate (2.4.50) by Cauchy-Schwarz. We estimate (2.4.51) by making use of information about the distributional laplacian of $G^t$, the integral kernel defining $\tilde{\psi}_{1,t}$. These arguments closely follow [11], but some adjustments are needed to adapt them to the Riemannian setting.

2.5 Clearing Out Proof

In this section we present the details omitted from the outline presented in section 2.4. As in section 2.4, unless otherwise specified, all metric related quantities will be associated to $g_{Rt}$ and the metric will be suppressed from the notation.

2.5.1 Reduction via rescaling

Following [11] we first reduce the proof of Theorem 2.1.2 to that of Proposition 2.4.1, which is stated in section 2.4. We refer the reader to section 2.4 for the relevant definitions surrounding the rescaled
solution $v_\epsilon$ as well as to A.4.5.1 for a detailed account of this reduction.

**Proof of Theorem 2.1.2, assuming Proposition 2.4.1.** Using the conclusion of Proposition 2.4.1 as well as (2.4.4) leads to

$$
\tilde{E}_{e,gr_1}(v_\epsilon, (x_T, 1), 1) + C_7 E_0 R_1 \leq R_1(\eta, R)
$$

where $R_1(\eta, R) \to 0^+$ as $\eta, R \to 0^+$. It is then possible to choose $T_\epsilon$, also dependent on $\sigma$, such that $T_\epsilon = 1 + O_\sigma(\epsilon^2)$ and, by an extension of Lemma (III.3) of [9] to our setting,

$$
1 - |v_\epsilon(x_T, T_\epsilon)| \leq C_M \left[ \frac{1}{\epsilon^N} \int_{B_\epsilon(x_T)} (1 - |v_\epsilon(x, T_\epsilon)|^2)^2 \right]^{\frac{1}{2N}} \leq D_M R_1(\eta, R) \tag{2.5.1}
$$

where $C_M$ and $D_M$ are constants that depend on $M$. Next, using the time derivative estimate from (2.4.3) and our choice of $T_\epsilon$ we have

$$
|v_\epsilon(x_T, T_\epsilon) - v_\epsilon(x_T, 1)| \leq \frac{\sigma}{2}. \tag{2.5.2}
$$

Hence, after combining (2.5.1) and (2.5.2), as well as choosing $R_0$ and $\eta_0$ small enough that we can ensure $D_M R_1(\eta, R) \leq \frac{\sigma}{2}$, we will have the desired result. $\square$

As remarked in Section 2.4, to prove Proposition 2.4.1 it suffices to demonstrate that for some $0 < \delta_0 < \frac{1}{16}$ there is $\delta \in [\delta_0, 2\delta_0]$ such that

$$
\tilde{E}_{e,gr_1}(v_\epsilon, (x_T, 1), \delta) \leq \frac{e^{-C_2}}{8} \left( \tilde{E}_{e,gr_1}(v_\epsilon, (x_T, 1), 1) + C_7 E_0 R_1 \right) + R(\eta, R)
$$

where $R(\eta, R)$ tends to zero as $\eta, R \to 0^+$ and $R_1$ is as in Proposition 2.3.11. To see this, observe that by applying Proposition 2.3.5 through $u_\epsilon$, using (2.4.7), as well as that $C_1 \leq C_7$ and $\delta < 2\delta_0 < \frac{1}{8}$ we have

$$
\begin{align*}
\tilde{E}_{e,gr_1}(v_\epsilon, (x_T, 1), \delta_0) &\leq e^{C_2} \tilde{E}_{e,gr_1}(v_\epsilon, (x_T, 1), \delta) + \delta [C_1 E_0 R_1] \\
&\leq e^{C_2} \tilde{E}_{e,gr_1}(v_\epsilon, (x_T, 1), 1) + \delta [C_7 E_0 R_1] + e^{C_2} R(\eta, R) \\
&\leq \frac{1}{4} \left( \tilde{E}_{e,gr_1}(v_\epsilon, (x_T, 1), 1) + C_7 E_0 R_1 \right) + e^{C_2} R(\eta, R).
\end{align*}
$$
2.5.2 Estimate of $\varphi_t$

To obtain the PDE (2.4.27) we begin by taking the cross product of (PGL)$_{e}$ with $v_e$ we obtain

$$0 = v_e \times \partial_t v_e + d^*(v_e \times dv_e) \quad \text{on} \ M \times (0, \infty). \quad (2.5.3)$$

By applying $d^*$ to (2.4.14) we see that

$$d^*(v_e \times dv_e) = -\Delta \varphi_t \quad \text{on} \ B_{\frac{3r}{2}}(x_T) \times \{t\}. \quad (2.5.4)$$

Rewriting $v_e \times \partial_t v_e$ as

$$v_e \times \partial_t v_e = v_e \times \left( \frac{\langle \nabla K_{ap}, \nabla v_e \rangle}{K_{ap}} + \partial_t v_e \right) - \left( \frac{dK_{ap}}{K_{ap}}, d\varphi_t + d^* \psi_t + \xi_t \right)$$

and then applying (2.5.3) and (2.5.4) leads to

$$-\Delta \varphi_t - \left( \frac{\nabla K_{ap}}{K_{ap}}, \nabla \varphi_t \right) = v_e \times \left( -\frac{\langle \nabla K_{ap}, \nabla v_e \rangle}{K_{ap}} - \partial_t v_e \right) + \left( \frac{dK_{ap}}{K_{ap}}, d^* \psi_t + \xi_t \right). \quad (2.5.5)$$

Finally, noting that

$$-\Delta \varphi_t - \left( \frac{\nabla K_{ap}}{K_{ap}}, \nabla \varphi_t \right) = -\frac{1}{K_{ap}} \text{div}(K_{ap} \nabla \varphi_t)$$

we may rewrite (2.5.5) as well as introduce boundary conditions coming from (2.4.14) to obtain (2.4.27). We use elliptic PDE techniques to estimate the $L^2$ norm of $\nabla \varphi_t$ on $B_s(x_T)$, where $s \in [r, \frac{3r}{2}]$ will be chosen later, in terms of the data from (2.4.27). First, we multiply the PDE from (2.4.27) by $-2\delta^2 \langle \nabla v, \nabla K_{ap} \rangle$ and integrate by parts to obtain

$$-2\delta^2 \int_{B_s(x_T)} h \langle \nabla \varphi, \nabla K_{ap} \rangle = 2\delta^2 \int_{B_s(x_T)} \text{div}(K_{ap} \nabla \varphi) \frac{\langle \nabla \varphi, \nabla K_{ap} \rangle}{K_{ap}} \quad (2.5.6)$$

$$= -2\delta^2 \int_{B_s(x_T)} K_{ap} \left\langle \nabla \varphi, \nabla \left( \frac{\langle \nabla \varphi, \nabla K_{ap} \rangle}{K_{ap}} \right) \right\rangle + 2\delta^2 \int_{\partial B_s(x_T)} g \langle \nabla \varphi, \nabla K_{ap} \rangle.$$

Then one can verify, for example by a pointwise computation in normal coordinates as demonstrated in Lemma A.4.4, that

$$K_{ap} \left\langle \nabla \varphi, \nabla \left( \frac{\langle \nabla \varphi, \nabla K_{ap} \rangle}{K_{ap}} \right) \right\rangle = \text{Hess}(K_{ap}) \langle \nabla \varphi, \nabla \varphi \rangle - \frac{\langle \nabla \varphi, \nabla K_{ap} \rangle^2}{K_{ap}} + \frac{1}{2} \langle \nabla (|\nabla \varphi|^2), \nabla K_{ap} \rangle. \quad (2.5.7)$$
Using (2.5.7) in (2.5.6) followed by integrating by parts leads to

\[ 2\delta^2 \int_{B_s(x_T)} \left[ -\frac{\Delta K_{ap}}{2} |\nabla \varphi|^2 + \text{Hess}(K_{ap})(\nabla \varphi, \nabla \varphi) - \frac{\langle \nabla \varphi, \nabla K_{ap} \rangle^2}{K_{ap}} \right] - 2\delta^2 \int_{B_s(x_T)} h(\nabla \varphi, \nabla G) \]

\[ = -\delta^2 \int_{\partial B_s(x_T)} |\nabla \varphi|^2 \frac{\partial K_{ap}}{\partial r} + 2\delta^2 \int_{\partial B_s(x_T)} g(\nabla \varphi, \nabla K_{ap}). \]

(2.5.8)

We then compute the integrand of the braced term of (2.5.8) more explicitly in terms of the function (1.2.3) and apply (1.2.5) to obtain

\[ \int_{B_s(x_T)} \frac{(N - 2)}{2} |\nabla \varphi|^2 K_{ap} - \int_{B_s(x_T)} \frac{(d(x, x_T))^2}{4\delta^2} \left[ 1 + \frac{2N\lambda\delta^2}{3} - 2\mu\delta^2 \right] |\nabla \varphi|^2 K_{ap} - 2\delta^2 \int_{B_s(x_T)} h(\nabla \varphi, \nabla K_{ap}) \]

\[ \geq -\delta^2 \int_{\partial B_s(x_T)} |\nabla \varphi|^2 \frac{\partial K_{ap}}{\partial r} + 2\delta^2 \int_{\partial B_s(x_T)} g(\nabla \varphi, \nabla K_{ap}). \]

(2.5.9)

More details can be found in Lemma A.4.4. From there we appeal to standard estimates and choose \( \delta_0 \) such that \( 0 < 2\delta_0 \leq \frac{1}{2\sqrt{\frac{1}{2}\frac{N\lambda\delta^2}{3} - \frac{2\mu\delta^2}} |\nabla \varphi|^2 K_{ap} \]

\[ \leq \frac{1}{2} \int_{B_s(x_T)} h^2 K_{ap} \]

which, when combined with (2.5.9), leads to

\[ \int_{B_s(x_T)} \frac{(N - 2)}{2} |\nabla \varphi|^2 K_{ap} + \frac{1}{2} \int_{B_s(x_T)} h^2 K_{ap} \]

\[ \geq -\delta^2 \int_{\partial B_s(x_T)} |\nabla \varphi|^2 \frac{\partial K_{ap}}{\partial r} + 2\delta^2 \int_{\partial B_s(x_T)} g(\nabla \varphi, \nabla K_{ap}). \]

(2.5.10)

Then an explicit computation of \( \frac{\partial K_{ap}}{\partial r} \) and \( \langle \nabla \varphi, \nabla K_{ap} \rangle \), analogous to [11], give

\[ \int_{\partial B_s(x_T)} |\nabla \perp \varphi|^2 K_{ap} \leq \int_{B_s(x_T)} \frac{(N - 2)}{s} |\nabla \varphi|^2 K_{ap} + \frac{1}{s} \int_{B_s(x_T)} h^2 K_{ap} \]

\[ + \int_{\partial B_s(x_T)} g^2 K_{ap} \]

(2.5.11)

for each \( s \in [r, 3r/2] \) and where

\( \nabla \perp \varphi := \nabla \varphi - \frac{\partial \varphi}{\partial r} \frac{\partial}{\partial r}. \)
More details can be found in Corollary A.4.5. This estimate will permit us to obtain control over the $L^2$ norm of $\varphi$ on $\partial B_s(x_T)$ in terms of $h$ and $g$. Next if we multiply the PDE of (2.4.27) by $\varphi$, integrate by parts, and make use of (2.5.11), the Poincaré-Wirtinger, and Young’s inequality we obtain

$$
\int_{B_s(x_T)} |\nabla \varphi|^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \leq C(\delta, r) \left[ \int_{B_s(x_T)} h^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} + \left( \int_{B_s(x_T)} h^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \right)^{\frac{1}{2}} \left( \int_{\partial B_s(x_T)} g^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \right)^{\frac{1}{2}} \right]
+ K_M r \int_{\partial B_s(x_T)} g^2 e^{-\frac{(d(x,x_T))^2}{4s^2}}
$$

for each $s \in [r, 3r/2]$. We notice that when $h$ is the data from (2.4.27) then we have the pointwise estimate

$$
h^2 \leq C(\delta_0, r) \left[ \Xi(v_e, (x_T, 1) + |d^* \psi_t|^2 + |\xi_t|^2) \right].
$$

As a result of (2.5.13) and the assumption that $t \in \Theta_1$ we have

$$
\int_{B_s(x_T)} h^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \leq C(\delta_0, r) \left[ RM_0 + \eta + \int_{B_{3r}(x_T)} (|d^* \psi_t|^2 + |\xi_t|^2) \right].
$$

A similar pointwise estimate for the data $g$ from (2.4.27) leads to

$$
\int_{\partial B_s(x_T)} g^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \leq K_M \int_{\partial B_s(x_T)} \left[ |\nabla v_e|^2 + |d^* \psi_t|^2 + |\xi_t|^2 \right] e^{-\frac{(d(x,x_T))^2}{4s^2}}.
$$

Averaging to find a suitable $s \in [r, 3r/4]$ and manipulating Gaussian functions gives

$$
\int_{\partial B_s(x_T)} g^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \leq \frac{K_M}{r} \int_{B_s(x_T)} |\nabla v_e|^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} + \frac{K_M}{r} \int_{B_s(x_T)} \left[ |d^* \psi_t|^2 + |\xi_t|^2 \right]
= \frac{(4\pi)^N}{r} K_M \delta^N \int_{B_s(x_T)} |\nabla v_e|^2 K_{ap,gR_1} + \frac{K_M}{r} \int_{B_s(x_T)} \left[ |d^* \psi_t|^2 + |\xi_t|^2 \right].
$$

More details are provided in A.4.3.1. Combining (2.5.12) with (2.5.14) and (2.5.16) gives (2.4.26) if we choose $\delta_0$ small enough that $2\delta_0 \sqrt{SC_0} < r$.

### 2.5.3 Estimate of $\psi_t$

The strategy from [11] to estimate $\psi_t$ extends to our setting with a few modifications mostly caused by the use of coordinates and the possibility of non-trivial homology. In particular, some additional
work is due to the presence of the harmonic projection $H$.

We first observe that since $\psi_t$ solves (2.4.17) then we can represent $\psi_t$ as

$$\psi_t(x) = \int_M \langle G(x,y), H^\perp (d[v_\epsilon(y) \times dv_\epsilon(y)] \chi(y)) \rangle dvol(y)$$

(2.5.17)

where $G$ is the Dirichlet kernel for 2-forms on $M$ with the metric $g_{R^1}$, which can be constructed in coordinates using the results of [6] and [12]. In particular, for each $x, y \in M$ we have $G(\cdot, y), G(x, \cdot) \in W^{1,1}(M; \wedge^{(2,2)}M)$, $\Delta x G(\cdot, y) \in L^1(M; \wedge^{2}M)$, and when $x, y$ are elements of a normal coordinate neighbourhood then

$$|G(x,y)| \leq K(d(x,y))^{2-N}, \quad |DG(x,y)| \leq K(d(x,y))^{1-N}$$

(2.5.18)

$$|\Delta x G(x,y)| \leq K(d(x,y))^{2-N}.$$  

(2.5.19)

In particular, if we were to rescale the metric by a factor $a^2$ for $a > 0$ we would find

$$G_{a^2 g} = a^{6-N} G_g$$

(2.5.20)

where $G_g$ denotes the Green’s function constructed using the metric $g$. Next we introduce a smooth function $\rho$ such that

$$\rho(s) = 1 \text{ for } s \in [0, 1/4], \quad \rho(s) = \frac{1}{s} \text{ for } s \geq 1/2, \quad \|\rho'||_{L^\infty(\mathbb{R})} \leq 4$$

(2.5.21)

as well as the localized version of $v_\epsilon$

$$\tilde{v}_\epsilon(x,t) := \tau(x,t)v_\epsilon(x,t), \quad \tau(x,t) := \rho(|v_\epsilon(x,t)|).$$

(2.5.22)

Noting that $\tilde{v}_\epsilon \times d\tilde{v}_\epsilon = \tau^2 v_\epsilon \times dv_\epsilon$, we split (2.5.17) into

$$\psi_t(x) = \int_M \langle G(x,y), H^\perp (d[\tilde{v}_\epsilon \times d\tilde{v}_\epsilon] \chi) \rangle + \int_M \langle G(x,y), H^\perp (d[(1-\tau^2)v_\epsilon \times dv_\epsilon] \chi) \rangle$$

$$=: \psi_{1,t} + \psi_{2,t}.$$
From the definitions of $\psi_{1,t}$ and $\psi_{2,t}$ we see that

\begin{align}
-\Delta \psi_{1,t} &= H^\perp (d[\tilde{v}_\epsilon \times d\tilde{v}_\epsilon] \chi) \quad \text{on } M \times \{t\} \\
-\Delta \psi_{2,t} &= H^\perp ([1 - \tau^2]v_\epsilon \times dv_\epsilon] \chi) \quad \text{on } M \times \{t\}.
\end{align}

We observe that there is $K > 0$ such that

\begin{align}
|1 - \tau^2(x, t)| &\leq K |1 - |v_\epsilon(x, t)|^2 | \\
|d[\tilde{v}_\epsilon \times d\tilde{v}_\epsilon]| &\leq KV_\epsilon(v_\epsilon)
\end{align}

over $M \times \{t\}$.

### 2.5.3.1 Estimate of Harmonic Projection

Before we begin estimating $\psi_{1,t}$ and $\psi_{2,t}$ we first obtain estimates for $H(d[v_\epsilon \times dv_\epsilon] \chi)$. Note that

\begin{equation}
d[v_\epsilon \times dv_\epsilon] \chi = d[\tilde{v}_\epsilon \times d\tilde{v}_\epsilon] \chi + d([1 - \tau^2](v_\epsilon \times dv_\epsilon) \chi) + (1 - \tau^2)[v_\epsilon \times dv_\epsilon] \wedge d\chi
\end{equation}

and so the definition of $H$ implies that

\begin{equation}
H(d[v_\epsilon \times dv_\epsilon] \chi) = H(d[\tilde{v}_\epsilon \times d\tilde{v}_\epsilon] \chi) + H((1 - \tau^2)[v_\epsilon \times dv_\epsilon] \wedge d\chi).
\end{equation}

We now estimate each of the terms on the right-hand side of (2.5.29). By straightforward estimates using (2.4.13), the definition of $H$, we may find a constant $K$ such that

\begin{align}
\|H(\omega)\|_{L^2(\wedge^2 M)} &\leq KR_1^{\frac{\delta}{2}} \|\omega\|_{L^1(\wedge^2 M)} \\
\|H(\omega)\|_{L^\infty(\wedge^2 M)} &\leq KR_1^{N} \|\omega\|_{L^1(\wedge^2 M)}
\end{align}

for all 2-forms $\omega$, where the exponents on $R_1$ are due to the scaling properties of the basis $\{\gamma_{B_{\frac{R_1}{2}}(x_T)}\}_{i=1}^{2^N(M)}$ appearing in (2.4.13). Next observe that by (2.5.27) and manipulations with Gaussian functions using that $\chi$ is supported on $B_{4r}(x_T)$, we have

\begin{equation}
\|d(\tilde{v}_\epsilon \times d\tilde{v}_\epsilon) \chi\|_{L^1(\wedge^2 M)} \leq C(\delta_0, r) \int_M V_\epsilon(v_\epsilon) K_{ap, B_{\frac{R_1}{2}}}.
\end{equation}

Now we consider $(1 - \tau^2)[v_\epsilon \times dv_\epsilon] \wedge d\chi$. Using (2.4.3), Cauchy-Schwarz, (2.5.26), and manipulations
with Gaussian functions using that $\chi$ is supported on $B_{4r}(x_T)$ leads to

$$\|(1 - \tau^2)[v_e \times dv_e] \wedge d\chi\|_{L^1(\mathbb{R}^2 \times M)} \leq C(\delta_0, r) \left( \int_M V_e(v_e) K_{ap_gr_1} \right)^{\frac{1}{2}}. \quad (2.5.33)$$

We will make use of various combinations of (2.5.30) and (2.5.31) with (2.5.32) and (2.5.33).

### 2.5.3.2 Estimate of $\psi_{2,t}$

The aim of this subsection is to establish the following estimate:

$$\int_{M \times \{t\}} \left[ |d\psi_{2,t}|^2 + |d^*\psi_{2,t}|^2 \right] \leq C(\delta_0, r) [RM_0 + \eta]. \quad (2.5.34)$$

This will be achieved by making use of the Poisson problem (2.5.25), appealing to elliptic regularity, and applying our assumption that $t \in \Theta_1$. We note that, as in previous estimates, the goal is to show that the data from the PDE (2.5.25) can be estimated in terms of (2.4.8) and (2.4.9).

We notice that $\psi_{2,t}$ can be further decomposed as $\psi_{2,t} = \psi_{2,t}^1 + \psi_{2,t}^2$ where $\psi_{2,t}^1, \psi_{2,t}^2$ satisfy

$$-\Delta \psi_{2,t}^1 = d[(1 - \tau^2)(v_e \times dv_e)\chi] \text{ on } M \times \{t\} \quad (2.5.35)$$

$$-\Delta \psi_{2,t}^2 = H^\perp((1 - \tau^2)[v_e \times dv_e] \wedge d\chi) \text{ on } M \times \{t\} \quad (2.5.36)$$

where in (2.5.35) we have used that $H^\perp$ is the identity on exact forms. Since $\psi_{2,t}^1$ solves (2.5.35) elliptic regularity gives

$$\int_{M \times \{t\}} \left[ |d\psi_{2,t}^1|^2 + |d^*\psi_{2,t}^1|^2 \right] \leq K \|1 - \tau^2|[v_e \times dv_e]\chi\|^2_{L^2(\mathbb{R}^2 \times M)}. \quad (2.5.37)$$

Using (2.4.3), (2.5.26), and manipulations with Gaussian functions using that the support of $\chi$ is $B_{4r}(x_T)$ gives

$$\int_{M \times \{t\}} |1 - \tau^2|[v_e \times dv_e]\chi|^2 \leq C(\delta_0, r) \int_{M \times \{t\}} V_e(v_e) K_{ap_gr_1}. \quad (2.5.38)$$

Combining (2.5.37), (2.5.38), and using that $t \in \Theta_1$ leads to

$$\int_{M \times \{t\}} \left[ |d\psi_{2,t}^1|^2 + |d^*\psi_{2,t}^1|^2 \right] \leq C(\delta_0, r) [RM_0 + \eta]. \quad (2.5.39)$$
Next, since $\psi^2_{2,t}$ solves (2.5.36) then elliptic regularity gives
\[
\int_{M \times \{t\}} \{|d\psi^2_{2,t}|^2 + |d^*\psi^2_{2,t}|^2\} \leq K \|H^\perp((1 - \tau^2)[v_\epsilon \times dv_\epsilon] \wedge d\chi)\|_{L^2(M)}^2.
\] (2.5.40)

It then follows from (2.5.30), (2.5.33), and similar considerations as in (2.5.38) that we have
\[
\|H^\perp((1 - \tau^2)[v_\epsilon \times dv_\epsilon] \wedge d\chi)\|_{L^2(M)}^2 \leq C(\delta_0, r) \int_{M \times \{t\}} V_\epsilon(v_\epsilon) K_{ap,GB}.
\] (2.5.41)

From (2.5.40), (2.5.41), and the assumption that $t \in \Theta_1$ it follows that
\[
\int_{M \times \{t\}} \{|d\psi^2_{2,t}|^2 + |d^*\psi^2_{2,t}|^2\} \leq C(\delta_0, r)[RM_0 + \eta].
\] (2.5.42)

Finally, combining (2.5.39) and (2.5.42) gives (2.5.34).

2.5.3.3 Estimate of $\psi_{1,t}$

Next we estimate $\psi_{1,t}$. We proceed with the a slightly modified version of the strategy applied in [11]. The main difference is the need to estimate terms relating to the harmonic projection $H$. In addition, some computations need to be done in coordinates, for example the estimate of the low frequency term $\tilde{\psi}_{1,t}$.

Taking the inner product of (2.5.24) with $\psi_{1,t}$ and integrating by parts we obtain
\[
\int_{M \times \{t\}} \left[|d\psi_{1,t}|^2 + |d^*\psi_{1,t}|^2\right] = \int_{M \times \{t\}} \langle \psi_{1,t}, H^\perp(d[\tilde{v}_\epsilon \times d\tilde{v}_\epsilon]\chi) \rangle.
\] (2.5.43)

Thus, to obtain control of the $L^2$ norms of the differential and codifferential of $\psi_{1,t}$ it suffices to estimate
\[
\int_{M \times \{t\}} \langle \psi_{1,t}, H^\perp(d[\tilde{v}_\epsilon \times d\tilde{v}_\epsilon]\chi) \rangle.
\] (2.5.44)

As a result, we focus on obtaining an upper bound of (2.5.44). We proceed through a series of steps.

**Step 1: Localization**

Due to the presence of the harmonic projection we will need a few additional estimates not needed
in [11]. We begin by noting that \( \psi_{1,t} \) can be decomposed as

\[
\psi_{1,t} = \int_M \langle G, H \perp (d[\tilde{\nu}_r \times d\tilde{\nu}_r] \chi) \rangle = \int_M \langle G, d[\tilde{\nu}_r \times d\tilde{\nu}_r] \chi \rangle - \int_M \langle G, H (d[\tilde{\nu}_r \times d\tilde{\nu}_r] \chi) \rangle =: \tilde{\psi}_{1,t} - G \ast H (d[\tilde{\nu}_r \times d\tilde{\nu}_r] \chi).
\]

We then use (2.5.45) to rewrite (2.5.44) as

\[
\int_M \langle \psi_{1,t}, H \perp (d[\tilde{\nu}_r \times d\tilde{\nu}_r] \chi) \rangle = \int_M \langle \tilde{\psi}_{1,t}, d[\tilde{\nu}_r \times d\tilde{\nu}_r] \chi \rangle - \int_M \langle G \ast H (d[\tilde{\nu}_r \times d\tilde{\nu}_r] \chi), d[\tilde{\nu}_r \times d\tilde{\nu}_r] \chi \rangle - \int_M \langle \psi_{1,t}, H (d[\tilde{\nu}_r \times d\tilde{\nu}_r] \chi) \rangle.
\]

We now estimate the last two terms of (2.5.46). We will start with

\[
\int_{M \times \{t\}} \langle \psi_{1,t}, H (d[\tilde{\nu}_r \times d\tilde{\nu}_r] \chi) \rangle.
\]

Observe that by the integral representation of \( \psi_{1,t} \), standard integral kernel estimates, as well as \( W^{1,1} \) estimates on \( G \) we obtain

\[
\int_{M \times \{t\}} |\psi_{1,t}| \leq KR_1^{-2} \left\{ \int_M |d[\tilde{\nu}_r \times d\tilde{\nu}_r] \chi| + \|H (d[\tilde{\nu}_r \times d\tilde{\nu}_r] \chi)\|_{L^\infty(B_{4r}(x_T))} \text{vol}(M) \right\}.
\]

Combining (2.5.31), (2.5.32), (2.5.27), manipulations of Gaussians that use that \( \chi \) is supported on \( B_{4r}(x_T) \), as well as the assumption that \( t \in \Theta_1 \) with (2.5.47) we obtain

\[
\int_M |\psi_{1,t}| \leq C(\delta_0, r)R_1^{-2} [RM_0 + \eta].
\]

Finally, by (2.5.48), (2.5.31), and (2.5.32) we have

\[
\left| \int_M \langle \psi_{1,t}, H (d[\tilde{\nu}_r \times d\tilde{\nu}_r] \chi) \rangle \right| \leq C(\delta_0, r)R_1^{N-2} [RM_0 + \eta]^2.
\]
Gaussian functions which use that \( \chi \) is supported on \( B_{4r}(x_T) \) we have

\[
\|G \ast H(d[\tilde{v}_x \times d\tilde{v}_y])\chi\|_{L^\infty(M)} \leq C(\delta_0, r)R_1^{N-2} \int_{M \times \{t\}} V_\varepsilon(v)K_{ap,gr_1}.
\]

Thus, combining this with (2.5.27), similar Gaussian function manipulations to those in (2.5.48), and the assumption that \( t \in \Theta_1 \), we have

\[
\left| \int_M \langle G \ast H(d[\tilde{v}_x \times d\tilde{v}_y])\chi, d[\tilde{v}_x \times d\tilde{v}_y]\chi \rangle \right| \leq C(\delta_0, r)R_1^{N-2}[RM_0 + \eta]^2. \quad (2.5.50)
\]

**Step 2: Decomposition of \( \tilde{\psi}_{1,t} \)**

As a result of the previous step we focus on estimating

\[
\int_{M \times \{t\}} \langle \tilde{\psi}_{1,t}, d[\tilde{v}_x \times d\tilde{v}_y]\chi \rangle. \quad (2.5.51)
\]

To achieve this, we proceed as in [11] and decompose \( \tilde{\psi}_{1,t} \) by splitting \( G \) into its high and low frequency parts. For the low frequency terms, \( \tilde{\psi}_{1,t}^e \), we will be interested in establishing an \( L^\infty \) estimate by appealing to the work of [26]. For the high frequency terms, \( \tilde{\psi}_{1,t}^i \), we will be interested in \( L^2 \) estimates in addition to an operator norm bound on the distributional Laplacian of \( G^i \), the integral kernel of \( \tilde{\psi}_{1,t}^i \).

Given \( \alpha \in (2/3, 3/4) \) and assuming that \( 36r < \frac{\text{inj}(M)}{2} \) we consider the function \( l: [0, \infty) \rightarrow [0, \infty) \) defined by

\[
l(s):=\begin{cases} 
0 & \text{if } s \leq \epsilon^\alpha r \\
\left(\frac{s}{\epsilon^\alpha r}\right)^{N-1} \left(2^{N-1} - 1\right)^{-1} & \text{if } \epsilon^\alpha r \leq s \leq 2\epsilon^\alpha r \\
1 & \text{if } 2\epsilon^\alpha r \leq s \leq 16r \\
\left(2^{N-1} - \left(\frac{s}{16r}\right)^{N-1}\right) \left(2^{N-1} - 1\right)^{-1} & \text{if } 16r \leq s \leq 32r \\
0 & \text{if } s \geq 32r.
\end{cases} \quad (2.5.52)
\]

From this we define \( m: [0, \infty) \rightarrow [0, \infty) \) by

\[
m(s):=\begin{cases} 
1 - l(s) & \text{for } s \in [0, 16r] \\
0 & \text{for } s \in (16r, \infty)
\end{cases} \quad (2.5.53)
\]
and note that \( m \) satisfies
\[
\begin{cases}
  m(s) \equiv 1 & \text{for } s \in (0, \epsilon^\alpha r) \\
  m(s) \equiv 0 & \text{for } s \in (2\epsilon^\alpha r, \infty) \\
  |m'(s)| \leq \frac{K}{\epsilon^\alpha r}.
\end{cases}
\]

Then we set
\[
G(x, y) = m(d(x, y))G(x, y) + (1 - m(d(x, y)))G(x, y) =: G^i(x, y) + G^e(x, y).
\]

This decomposition allows us to define
\[
\tilde{\psi}^i_{1,t} := \int_{B_{2\epsilon^\alpha r}(x_T)} \langle G^i(x, y), d[\tilde{\nu}_x \times d\tilde{\nu}_y] \rangle (2.5.54)
\]
\[
\tilde{\psi}^e_{1,t} := \int_{B_{32\epsilon^\alpha r}(x)} \langle G^e(x, y), d[\tilde{\nu}_x \times d\tilde{\nu}_y] \rangle (2.5.55)
\]

In the above \( \tilde{\psi}^i_{1,t} \) represents the high frequencies of \( \tilde{\psi}_{1,t} \) while \( \tilde{\psi}^e_{1,t} \) represents the low frequencies of \( \tilde{\psi}_{1,t} \).

Next, we note that by (2.5.18), the definition of (2.5.53), and computations in normal coordinates we obtain
\[
\left\| G^i \right\|_{L^1_\infty(\wedge^2 M)} \left\| G^i \right\|_{L^\infty(\wedge^2 M)} \leq K_M \epsilon^\alpha r (2.5.56)
\]
\[
\left\| DG^i \right\|_{L^1_\infty(M)} \left\| DG^i \right\|_{L^\infty(M)} \leq K_M \epsilon^\alpha r, (2.5.57)
\]

where \( K_M \) is a constant depending only on \( M \). Similar computations in normal coordinates for the more delicate estimate of \( \tilde{\psi}^i_{1,t} \) are presented in detail below, see for example (2.5.63). We refer the reader to Lemma A.4.9 for more details.

In addition, through direct computations related to the distributional Laplacian of \( G^i \) we obtain
\[
\left\| \langle G^i(\cdot, y), -\Delta h \rangle \right\|_{L^\infty(\wedge^2 M)} \leq K_M \left\| h \right\|_{L^\infty(\wedge^2 M)} (2.5.58)
\]
for all \( h \in C^2(M; \wedge^2 M) \) and each \( y \in M \). We refer the reader to Lemma A.4.9 for more details.
Estimates (2.5.56) and (2.5.57) along with the integral kernel expression for \( \tilde{\psi}_{i,t} \) allow us to obtain
\[
\int_M |\tilde{\psi}_{i,t}|^2 \leq K C(\delta_0, r) e^{2\alpha} \left( \tilde{E}_{1,GR_1} (v_e, (x_T, 1), 1) + C_7 R_1 E_0 \right)
\] (2.5.59)
for \( t \in \Theta_1 \). We refer the reader to Lemma A.4.11 for more details.

Finally, we use the decomposition of \( \tilde{\psi}_{1,t} \) in (2.5.51) to conclude that it suffices to estimate
\[
\int_M \langle \tilde{\psi}_{1,t}, d[\tilde{v}_e \times d\tilde{v}_e] \chi \rangle \tag{2.5.60}
\]
\[
\int_M \langle \tilde{\psi}_{1,t}, d[\tilde{v}_e \times d\tilde{v}_e] \chi \rangle. \tag{2.5.61}
\]

**Step 3: Estimate of \( \tilde{\psi}_{1,t} \)**

We now focus on estimating the \( L^\infty \) norm of
\[
\tilde{\psi}_{1,t} = \int_M \langle G_e, d[\tilde{v}_e \times d\tilde{v}_e] \chi \rangle.
\]

Doing this will permit us to provide an upper bound on (2.5.60). We proceed in the same way as in [11] except we need to make use of normal coordinates in order to have an explicit expression for the integrand. The idea is to rewrite \( \tilde{\psi}_{1,t} \) into a distributional pairing of the coordinate components of \( d[\tilde{v}_e \times d\tilde{v}_e] \) and a Lipschitz function. Then, the work of [26] is able to provide an \( O(1) \) estimate for the Lipschitz dual norm of the coordinate components of \( d[\tilde{v}_e \times d\tilde{v}_e] \).

We will estimate \( \tilde{\psi}_{1,t} \) at a fixed \( x \in B_{4r}(x_T) \). We let \( y \) denote normal coordinates centered at \( x \). In these coordinates, of course \( x \) corresponds to zero, and if \( p \in M \) is the point corresponding to the coordinate \( y \), then \( d(x,p) = |y| \). In the coordinates, \( y \), we will write a 2-form as
\[
\omega = \sum_{j_1 \neq j_2} \omega_{j_1,j_2}(y) dy^{j_1} \wedge dy^{j_2} =: \sum_J \omega_J(y) dy^J. \tag{2.5.62}
\]
We recall that the Green’s function \( G \) is a tensor of type \((2,2)\) such that, if the first and second components \( z \) and \( p \) are written respectively in normal coordinates \( \bar{y} \) and \( y \) centred on \( x \), then \( G \) acts on 2-forms \( \omega \) via
\[
\langle G(z,p), \omega(p) \rangle = \sum_I \left( \sum_J G^I_J(\bar{y},y) \omega_J(y) \right) dy^I.
\]
In particular,

$$\langle G(x,p), \omega(p) \rangle = \sum_i \left( \sum_j G_I^j(0,y) \omega_J(y) \right) d\tilde{y}^I.$$  

In these coordinates, the Green’s function for 2-forms, evaluated with one argument fixed at $x$, has components

$$G^j_I(0,y) = |y|^{2-N} H^j_I(y), \quad I = (i_1, i_2), \quad J = (j_1, j_2)$$

where $H^j_I$ is a Lipschitz function in $y$ for each $I$ and $J$, see [12] and Proposition 4.12 from [6], $(g_{R_1})_{ij}$ and $g_{R_1}^{ij}$ denote, respectively, the metric tensor and its inverse with respect to these coordinates.

Due to our choice of $r$, $G^e$ has support in the domain of this coordinate system, and so the above discussion gives

$$\tilde{\psi}_{e,t}(x,t) \leq \sum_i \left[ \int_{B_{32r}(0)} l(|y|)|y|^{2-N} H^j_I(y) \frac{\partial \tilde{v}_e}{\partial x_{j_1}} \times \frac{\partial \tilde{v}_e}{\partial x_{j_2}} \chi \sqrt{|g_{R_1}(y)|} dy \right] d\bar{y}^I$$

(2.5.63)

where we have set $|g_{R_1}(y)| = \det((g_{R_1})_{ij})$. We now estimate each of the summands from (2.5.63).

Following the proof of Lemma 3.12 of [11] and applying Fubini’s Theorem we obtain, for all $k \in L^1(M)$, that

$$\int_{B_{32r}(0)} l(|y|)|y|^{2-N} k(y) dy = \int_{\epsilon \alpha r}^{16r} s^{-1} \mathcal{J}^k_s ds + \frac{1}{N-2} \left[ \mathcal{J}^{k}_{16r} - \mathcal{J}^{k}_{\epsilon \alpha r} \right]$$

(2.5.64)

$$\mathcal{J}^k_s := s^{2-N} \int_{B_{2s}(0)} k(y) h(|y|, s) dy$$

and

$$h(u, s) := \frac{(N-1)(N-2)}{2^{N-1}-1}, \quad \begin{cases} 1 & 0 \leq u \leq s, \\ \frac{2-u}{s} & if \ s \leq u \leq 2s, \\ 0 & if \ u \geq 2s. \end{cases}$$

We refer the reader to A.4.4.1 for more details. We then use (2.5.64) with $k = a_I$. As in [11], we exploit the Jacobian structure of $\frac{\partial \tilde{v}_e}{\partial x_{j_1}} \times \frac{\partial \tilde{v}_e}{\partial x_{j_2}}$ by applying Theorem 2.1 of [26] to $\mathcal{J}^a_{s}$ to obtain

$$\sup_{s \in [\epsilon \alpha r, 16r]} \{ \mathcal{J}^a_s \} \leq C(\delta_0, r) \left( \frac{E_{\epsilon R_1}(v_e, (x_T, 1), 1) + C_7 R_1 E_0}{|\log(\epsilon)|} + \epsilon^\beta \right)$$

(2.5.65)

for some $\beta > 0$. We refer the reader to Lemma A.4.10 for more details. Combining (2.5.63), (2.5.64),
and (2.5.65) leads to
\[\|\tilde{\psi}_{1}\|_{L^\infty(\lambda^2 M)} \leq C(\delta_0, r) \left( \tilde{E}_{\epsilon, gR_1}(v_\epsilon, (x_T, 1), 1) + C_7 R E_0 + 1 \right). \tag{2.5.66}\]

Observe that by (2.5.66), (2.5.27), manipulations of Gaussian functions that use that the support of \(\chi\) is \(B_4 r(x_T)\), in addition to using the assumption that \(t \in \Theta_1\) we have
\[|(2.5.60)| \leq C(\delta_0, r) \left( \tilde{E}_{\epsilon, gR_1}(v_\epsilon, (x_T, 1), 1) + C_7 R E_0 + 1 \right) [R M_0 + \eta]. \tag{2.5.68}\]

\[\|D\psi^*_1\|_{L^2(\lambda^2 M \times [0, 1])} \leq C(\delta_0, r) \left( \tilde{E}_{\epsilon, gR_1}(v_\epsilon, (x_T, 1), 1) + C_7 R_1 E_0 \right). \tag{2.5.69}\]

We refer the reader to Lemma A.4.12 for more details. Next, by using (2.3.22), Proposition 2.3.11, and Proposition 2.3.5 as well as its proof we obtain
\[
\int_{M \times [0, 1]} |D^2 \psi^*_1|^2 e^{-\frac{(d_{\epsilon}(x, x_T))^2}{4(1-t)}} \leq K_M \left( \tilde{E}_{\epsilon, gR_1}(v_\epsilon, (x_T, 1), 1) + C_7 R_1 E_0 \right). \tag{2.5.70}\]

We refer the reader to A.4.4.2 for additional details. Finally, we argue that we can find \(t \in |1 -
\[
\int_{M \times \{t\}} |\partial_t \psi_1^*|^2 \leq C(\delta_0, r) \epsilon^{-1} \left( \tilde{E}_{\epsilon, g_{R_1}}(v_\epsilon, (x_T, 1), 1) + C \gamma R_1 E_0 \right).
\]  

(2.5.71)

We introduce the notation \( \Theta_2 \) to refer to

\[
\Theta_2 := \{ t \in [1 - 4\delta_0^2, 1 - \delta_0^2] : \text{(2.5.71) holds at } t \}. \tag{2.5.72}
\]

To show (2.5.71) we proceed as in [11]. Taking the inner product of (2.5.67) with \( \partial_t \psi_1^* \), integrating over \( M \times [0, 1 - \delta_0^2] \), and integrating by parts leads to

\[
\int_{M \times [0, 1 - \delta_0^2]} |\partial_t \psi_1^*|^2 = -\frac{1}{2} \int_{M \times \{1 - \delta_0^2\}} \{ |d\psi_1^*|^2 + |d^* \psi_1^*|^2 \} + \int_{M \times [0, 1 - \delta_0^2]} (\partial_t \psi_1^*, d\tilde{v}_\epsilon \times d\tilde{v}_\epsilon) \chi. \tag{2.5.73}
\]

Next introducing normal coordinates, \( y \), centred at \( x_T \) and expressing \( \psi_1^* \) in these coordinates, similar to (2.5.62), as

\[
\psi_1^* = \sum_I \psi_{1,I}(y) dy^I
\]

we may write \( (d\tilde{v}_\epsilon \times d\tilde{v}_\epsilon) \chi, \partial_t \psi_1^* \) as

\[
\sum_I \sum_{j_1 < j_2} g_{R_1}^{IJ}(y) \partial_t \psi_{1,I}(y) \left[ \partial_{j_1}(\tilde{v}_\epsilon(y) \times \partial_{j_2} \tilde{v}_\epsilon(y)) - \partial_{j_2}(\tilde{v}_\epsilon(y) \times \partial_{j_1} \tilde{v}_\epsilon(y)) \right] \chi(y) \sqrt{|g_{R_1}(y)|}
\]

where \( I = (i_1, i_2) \), \( J = (j_1, j_2) \), \( |g_{R_1}(y)| \) is as in (2.5.63), and we have set

\[
g_{R_1}^{IJ}(y) := \det \begin{pmatrix} g_{R_1}^{i_1 j_1} & g_{R_1}^{i_1 j_2} \\ g_{R_1}^{i_2 j_1} & g_{R_1}^{i_2 j_2} \end{pmatrix}
\]

where \( g_{R_1}^{ij} \) denotes the \( i, j \) component of the inverse of metric tensor, \( g_{R_1} \), in these coordinates. By our choice of \( \chi \) and \( r > 0 \) we see that the support of \( \chi \) is contained in the domain of this coordinate system. Setting \( \chi_{g_{R_1}}^{IJ} := \chi g_{R_1}^{IJ} \sqrt{|g_{R_1}|} \), integrating by parts repeatedly as in [11], and using that \( \chi_{g_{R_1}}^{IJ} \) is supported on \( B_{4r}(0) \) we can write

\[
\int_{M \times [0, 1 - \delta_0^2]} (\partial_t \psi_1^*, d\tilde{v}_\epsilon \times d\tilde{v}_\epsilon) \chi = T_1 + T_2 + T_3 + T_4
\]
where
\[
T_1 := 2 \sum_I \sum_{j_1 < j_2} \int_{B_{4r}(0) \times [0, 1 - \delta_0^2]} \left[ \partial_{j_1} \psi_{1,I}^*(\partial_t \tilde{v}_e \times \partial_{j_2} \tilde{v}_e) - \partial_{j_2} \psi_{1,I}^*(\partial_t \tilde{v}_e \times \partial_{j_1} \tilde{v}_e) \right] \chi_{gR_1}^{IJ}.
\]
\[
T_2 := - \sum_I \sum_{j_1 < j_2} \int_{B_{4r}(0) \times [0, 1 - \delta_0^2]} (\tilde{v}_e \times \partial_t \tilde{v}_e) \left[ \partial_{j_1} \psi_{1,I}^* \partial_{j_2} \chi_{gR_1}^{IJ} - \partial_{j_2} \psi_{1,I}^* \partial_{j_1} \chi_{gR_1}^{IJ} \right] \]
\[
T_3 := - \sum_I \sum_{j_1 < j_2} \int_{B_{4r}(0) \times [0, 1 - \delta_0^2]} \partial_t \psi_{1,I}^* \left[ \partial_{j_1} \chi_{gR_1}^{IJ} (\tilde{v}_e \times \partial_{j_2} \tilde{v}_e) - \partial_{j_2} \chi_{gR_1}^{IJ} (\tilde{v}_e \times \partial_{j_1} \tilde{v}_e) \right] \]
\[
T_4 := - \sum_I \sum_{j_1 < j_2} \int_{B_{4r}(0) \times [1 - \delta_0^2]} \left[ \partial_{j_1} \psi_{1,I}^* (\tilde{v}_e \times \partial_{j_2} \tilde{v}_e) - \partial_{j_2} \psi_{1,I}^* (\tilde{v}_e \times \partial_{j_1} \tilde{v}_e) \right] \chi_{gR_1}^{IJ}.
\]

We estimate $T_1$, $T_2$, $T_3$, and $T_4$ as in [11] by using (2.5.68), (2.5.69), (2.4.3), (2.5.70), and Proposition 2.3.5. The only change required is in the estimate of $T_4$ in which an additional appeal to Gaffney’s inequality and $L^2$ estimates of $\psi_1^*$ obtained from the proof of (2.5.69) are applied. Proceeding in this way we obtain
\[
\int_{M \times [0, 1 - \delta_0^2]} |\partial_t \psi_1^*|^2 \leq C(\delta_0, r) \epsilon^{-1} \left( \tilde{E}_{e,gR_1}(v_e, (x_T, 1), 1) + C_7 R_1 E_0 \right).
\] (2.5.74)

An application of Chebyshev’s inequality then allows us to find $t \in [1 - 4\delta_0^2, 1 - \delta_0^2]$ for which (2.5.71) holds. We record this in Corollary A.4.13.

**Step 5: Estimate of $\tilde{\psi}_{1,t}$**

We assume that $t \in \Theta_1 \cap \Theta_2$ which is possible due to Lemma A.4.1 and Corollary A.4.13. Using (2.5.67) we can write
\[
\int_{M \times \{t\}} \left\langle \tilde{\psi}_{1,t}, d[\tilde{v}_e \times d\tilde{v}_e] \chi \right\rangle = \int_{M} \left\langle \tilde{\psi}_{1,t}, \partial_t \psi_1^* \right\rangle + \int_{M} \left\langle \tilde{\psi}_{1,t}, -\Delta \psi_1^* \right\rangle.
\]

The first term can be estimated using (2.5.59) and (2.5.71) to obtain
\[
\left| \int_{M \times \{t\}} \left\langle \tilde{\psi}_{1,t}, \partial_t \psi_1^* \right\rangle \right| \leq C(\delta_0, r) \epsilon^{\alpha - 1/2} \left( \tilde{E}_{e,gR_1}(v_e, (x_T, 1), 1) + C_7 R_1 E_0 \right).
\] (2.5.75)

The second term can be estimated using (2.5.58), the proof of Proposition 2.3.9, and (2.4.8) to obtain
\[
\left| \int_{M \times \{t\}} \left\langle \tilde{\psi}_{1,t}, -\Delta \psi_1^* \right\rangle \right| \leq C(\delta_0, r) \left( \tilde{E}_{e,gR_1}(v_e, (x_T, 1), 1) + C_7 R_1 E_0 \right)[RM_0 + \eta].
\] (2.5.76)
Combining estimates (2.5.34), (2.5.45), (2.5.50), (2.5.75), and (2.5.76) with (2.5.43) gives
\[
\int_{M \times \{t\}} \{|d\psi_{1,t}|^2 + |d^* \psi_{1,t}|^2\} \leq C(\delta_0, r) \epsilon^{\alpha - \frac{1}{2}} \left( \tilde{E}_{\epsilon, gr_1}(v_\epsilon, (x_T, 1), 1) + C_\gamma R_1 E_0 \right) + C(\delta_0, r) \left( \tilde{E}_{\epsilon, gr_1}(v_\epsilon, (x_T, 1), 1) + C_\gamma R_1 E_0 + 1 \right) (R \lambda + 1 + [R \lambda + 1]^2).
\]

Finally, combining (2.4.25) (2.4.26), (2.5.34), (2.4.41), the estimate of (2.5.60), and (2.5.77) and choosing the parameters sufficiently small completes the proof of Proposition 2.4.1.

### 2.6 Energy Decompositions

In this section we present the proof of Theorem 2.1.3. Compared to [11], there are new considerations related to the homology of $M$. More specifically, when applying the Hodge de Rham decomposition we must, since we impose no homological restrictions on $M$, consider the harmonic part. In particular, these considerations are responsible for the presence of $u_{h, \epsilon}$ in the conclusions of Theorem 2.1.3.

We start by stating a local energy decomposition for solutions of $(\text{PGL})_{\epsilon}$, valid in a region where the modulus is bounded away from zero.

**Theorem 2.6.1.** Suppose that $0 < R < \text{inj}_g(M)$, $T > 0$, and $\Delta T > 0$ are given. Consider the cylinder
\[
\Lambda := B_R(x_0) \times [T, T + \Delta T].
\]
There exists a constant $0 < \sigma \leq \frac{1}{2}$ and $\beta > 0$ depending only on $N$, such that if
\[
|u_\epsilon| \geq 1 - \sigma \quad \text{on} \ \Lambda,
\]
then
\[
e_\epsilon(u_\epsilon)(x,t) \leq C(\Lambda) \int_\Lambda e_\epsilon(u_\epsilon),
\]
for any $(x, t) \in \Lambda_{\frac{1}{2}}$. Moreover,
\[
e_\epsilon(u_\epsilon) = \frac{|\nabla \Phi_\epsilon|^2}{2} + \kappa_\epsilon \quad \text{in} \ \Lambda_{\frac{1}{2}},
\]
where the functions $\Phi_\varepsilon$ and $\kappa_\varepsilon$ are defined on $\Lambda_{\frac{1}{2}}$ and verify

$$
\partial_t \Phi_\varepsilon - \Delta \Phi_\varepsilon = 0 \quad \text{in} \; \Lambda_{\frac{1}{2}},
$$

$$
\|\kappa_\varepsilon\|_{L^\infty(\Lambda_{\frac{1}{2}})} \leq C(\Lambda)e^\beta, \quad \|\nabla \Phi_\varepsilon\|^2_{L^\infty(\Lambda_{\frac{1}{2}})} \leq C(\Lambda)M_0|\log(\varepsilon)|.
$$

(2.6.4) \hspace{1cm} (2.6.5)

In addition, it follows from our choice of $\Phi_\varepsilon$ that if $u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$ on $\Lambda$ then

$$
\|\nabla \Phi_\varepsilon - \nabla \varphi_\varepsilon\|_{L^\infty(\Lambda_{\frac{1}{2}})} \leq C(\Lambda)e^\beta.
$$

(2.6.6)

This is an adaptation to the present setting of Theorem 2 in [11]. Since the analysis is entirely local, and because it does not involve any delicate properties of test functions adapted to the metric, the proof ends up being essentially identical in the Riemannian case. This being the case, we omit all details here. An interested reader can consult [11], or A.6.0.1, where it is verified in detail that the arguments of [11] remain valid on a manifold.

As was done in [11], we record a straightforward consequence obtained by combining the results of Theorem 2.1.2 with Proposition 2.3.8.

**Proposition 2.6.2.** Let $u_\varepsilon$ be a solution of $(\text{PGL})_\varepsilon$ verifying assumption $(H_0)$ and $\sigma > 0$ be given. Let $x_T \in M$, $T > 0$, and $0 < 2\varepsilon < R^2 < R(\sigma)$ where $R(\sigma)$ is as in Theorem 2.1.2. There exists a positive continuous function $\lambda$ defined on $(0, \infty)$ such that, if

$$
\check{\eta}(x_T, T, R):= \frac{1}{(4\pi)^{\frac{N}{2}} R^{N-2}|\log(\varepsilon)|} \int_{B_{\lambda(T)R}(x_T)} e_\varepsilon(u_\varepsilon(\cdot, T)) \leq \frac{\eta_1(\sigma)}{2} \frac{\eta_1(\sigma)}{2}
$$

then

$$
|u_\varepsilon(x, t)| \geq 1 - \sigma \quad \text{for} \; t \in [T + T_0, T + T_1] \; \text{and} \; x \in B_{\frac{1}{2}}(x_T).
$$

Here $T_0$ and $T_1$ are defined by

$$
T_0:= \left(\frac{2\check{\eta}}{\eta_1}\right)^{\frac{N-2}{2}} R^2, \quad T_1:= R^2.
$$

In particular, a more precise estimate shows that we can find $\lambda$ defined on $(0, \infty) \times (0, \infty)$ satisfying

$$
\lambda(T, R) \sim \left|\left[\frac{8}{\varepsilon^2} \log \left(\frac{(4\pi)^{\frac{N}{2}}}{M_0 e^{C_2}} \left[\frac{2}{T + 2R^2}\right]^{\frac{N-2}{2}}\right)\right]\right|
$$

as $(T, R) \to (0, 0)$. In particular, $\lambda(T, R)R$ is bounded as $R \to 0^+$ for any $T > 0$.
Following [11] we also record the following consequence of Proposition 2.6.2 combined with Theorems 2.1.2 and 2.6.1 for future use.

**Proposition 2.6.3.** For each \( \sigma > 0 \) there exists positive constants \( \eta_2(\sigma) \) and \( R(\sigma) \) as well as a positive function \( \lambda \) defined on \( (0, \infty) \) such that if, for \( x \in M, t > 0, \) and \( \sqrt{2\varepsilon} < r < R(\sigma) \) we have

\[
\int_{B_{\lambda(t)}(x)} e_\varepsilon(u_\varepsilon) \leq \eta_2 r^{N-2}|\log(\varepsilon)|,
\]

then

\[
e_\varepsilon(u_\varepsilon) = \frac{|\nabla \Phi_\varepsilon|^2}{2} + \kappa_\varepsilon
\]

in \( \Lambda_4(x,t,r) := B_\varepsilon(x) \times \left[ t + \frac{15}{16} r^2, t + r^2 \right] \), where \( \Phi_\varepsilon \) and \( \kappa_\varepsilon \) are as in Theorem 2.6.1. In particular,

\[
\mu_\varepsilon = \frac{e_\varepsilon(u_\varepsilon)}{|\log(\varepsilon)|} \leq C(t,r) \quad \text{on} \quad \Lambda_4(x,t,r).
\]

We use the remainder of this section to prove Theorem 2.1.3. We begin by introducing some notation. We let \( \Omega := M \times (t_1, t_2) \), where \( 0 < t_1 < t_2 < \infty \) and use \( \delta \) to denote the exterior derivative on \( M \times (0, \infty) \). In addition we let \( \delta^* \) denote its formal adjoint with respect to the natural product metric. If \( \eta \) is a \( k \)-form on \( M \times (0, \infty) \) and \( \Sigma \) is a smooth hypersurface, we will write \( \eta_T \) to denote the \( k \)-form on \( \Sigma \) defined by

\[
\eta_T := i^* \eta = \text{the tangential part of } \eta \text{ on } \Sigma,
\]

where \( i : \Sigma \to M \times (0, \infty) \) is the inclusion map. We also write

\[
\eta_N := \eta - \eta_T = \text{the normal part of } \eta \text{ on } \Sigma.
\]

We note that if \( u_\varepsilon \) solves \( (PGL)_\varepsilon \) and satisfies (H0) then standard parabolic estimates give, for sufficiently small \( \varepsilon \), that

\[
\int_{M \times \{t\}} e_\varepsilon(u_\varepsilon) \leq M_0|\log(\varepsilon)| \quad \forall t > 0, \tag{2.6.7}
\]

\[
|u_\varepsilon(x,t)| \leq 3 \quad \forall (x,t) \in \Omega. \tag{2.6.8}
\]
In particular, (2.6.7) allows us to conclude that

\[
\int_{M \times [0, t_2]} e_\varepsilon(u\varepsilon) \leq C(\Omega)M_0|\log(\varepsilon)|
\]

(2.6.9)

\[
\int_{\partial\Omega} e_\varepsilon(u\varepsilon) \leq 2M_0|\log(\varepsilon)|.
\]

(2.6.10)

The next result is the main decomposition tool used in the proof of Theorem 2.1.3.

**Proposition 2.6.4.** Assume that \( u_\varepsilon \) is a solution to \((PGL)_\varepsilon\) on \( M \times (0, \infty)\) that satisfies \((H_0)\).

Then there is a smooth 1-form \( \gamma \) dependent only on the initial data of \( u_\varepsilon \) such that, on \( \Omega \), there exists a smooth function \( \Phi \), a smooth 1-form \( \zeta \), and a smooth 2-form \( \Psi \) for which

\[
(u_\varepsilon \times \delta u_\varepsilon) = \delta\Phi + \delta^*\Psi + \gamma + \zeta, \quad \delta\Psi = 0 \text{ in } \Omega, \quad \Psi_T = 0 \text{ on } \partial\Omega,
\]

(2.6.11)

and

\[
\|\Phi\|_{W^{1,2}(\Omega)} + \|D\Psi\|_{L^2(\Omega)} + \|\gamma\|_{L^2(\Sigma)} \leq C(\Omega)\sqrt{(M_0 + 1)|\log(\varepsilon)|}.
\]

(2.6.12)

In addition, we have that \( \gamma \) is constant in time, independent of \( \varepsilon \), a harmonic 1-form on \( M \) for all \( t > 0 \), and there is a time independent \( S^1 \)-valued function \( u_{h,\varepsilon} \) such that \( ju_{h,\varepsilon} = \gamma \). Moreover, for any \( 1 \leq p < \frac{N+1}{N} \),

\[
\begin{cases}
\|D\Psi\|_{L^p(\Omega)} \leq C(p, \Omega)(M_0 + 1), \\
\|\zeta\|_{L^p(\Omega)} \leq C(p, \Omega)(M_0 + 1)\varepsilon^{\frac{1}{2}},
\end{cases}
\]

(2.6.13)

where \( C(p, \Omega) \) is a constant depending only on \( p \) and \( \Omega \).

**Proof.** As in [11] we split the proof into two steps. We begin by dealing with \( \Sigma := \partial\Omega \). Notice that \( \Sigma = (M \times \{t_1\}) \cup (M \times \{t_2\}) \).

**Step 1:** \( \text{HdR decompositions on } \Sigma \). Since \( \partial\Sigma = \emptyset \) then a standard Hodge-de Rham decomposition applied to the tangential part of \( u_\varepsilon \times \delta u_\varepsilon \) allows us to write

\[
(u_\varepsilon \times \delta u_\varepsilon)_T = u_\varepsilon \times d u_\varepsilon = d\Phi^i + d^*\Psi^i + \gamma^i \text{ on } M \times \{t_i\},
\]

(2.6.14)
for \( i = 1, 2 \), with
\[
d\Psi^i_\varepsilon = 0 \quad \text{on} \quad M \times \{ t_i \} \quad \text{for} \quad i = 1, 2, \tag{2.6.15}
\]
\[
d\gamma^i_\varepsilon = 0 = d^*\gamma^i_\varepsilon \quad \text{on} \quad M \times \{ t_i \} \quad \text{for} \quad i = 1, 2, \tag{2.6.16}
\]
\[
\int_{M \times \{ t_i \}} \Phi^i_\varepsilon \, d\operatorname{vol}_g = 0 \quad \text{for} \quad i = 1, 2. \tag{2.6.17}
\]
See for example Theorem 5 of section 5.2.5 of [21], which also shows that
\[
\| \Phi^i_\varepsilon \|_{W^{1,2}(M \times \{ t_i \})} + \| \Psi^i_\varepsilon \|_{W^{1,2}(M \times \{ t_i \})} + \| \gamma^i_\varepsilon \|_{L^2(M \times \{ t_i \})} \leq K M_0 |\log(\varepsilon)| \tag{2.6.18}
\]
for \( i = 1, 2 \). Next observe that by applying \( d \) to (2.6.14) at \( t = t_i \) for \( i = 1, 2 \) and using (2.6.15) we obtain
\[
- \Delta_M \Psi^i_\varepsilon = J_M u_\varepsilon \quad \text{on} \quad M \times \{ t_i \} \tag{2.6.19}
\]
for \( i = 1, 2 \) where \( J_M u_\varepsilon := \frac{1}{2} d[u_\varepsilon \times d u_\varepsilon] \) and \( \Delta_M \) is the Laplacian on \( M \). Thus, by Theorem 2.1 of [26], the Sobolev Embedding Theorem, duality, and elliptic regularity, see Lemma 2.9 of [9] and Propositions 5.17 and 6.5 of [24], we have that for all \( q > N \), \( p := \frac{q}{q-1} \), and \( \alpha := 1 - \frac{N}{q} \) that
\[
\| \Psi^i_\varepsilon \|_{W^{1,p}(M \times \{ t_i \})} \leq C(p, M) \| J_M u_\varepsilon \|_{W^{-1,p}(M \times \{ t_i \})} \tag{2.6.20}
\]
\[
\leq C(p, M) \| J_M u_\varepsilon \|_{[C^{0,\alpha}(M \times \{ t_i \})]^*} \leq C(p, M)(M_0 + 1)
\]
for \( i = 1, 2 \).

Next we provide an approximation to the harmonic parts from (2.6.14) that stores most of the energy. This is a new ingredient needed to extend the corresponding result of [11] to our setting. As a result, we go over the associated estimates in more detail.

We consider a collection of closed curves, \( \{ c_j \}_{j=1}^{\beta_1(M)} \) where \( \beta_1(M) \) is the first Betti number of \( M \), generating the first homology group \( H_1(M) \). It follows from item (ii) of Theorem 4 of section 5.3.2 and Theorem 6 of section 5.2.5 of [21] that, associated to these curves, we can find a basis \( \{ \psi^k \}_{k=1}^{\beta_1(M)} \) for \( H^1(M) \), the space of harmonic 1-forms on \( M \), satisfying
\[
\int_{c_j} \psi^k = 2\pi \delta_{jk}.
\]
Using this basis we can express \( \gamma_0^\varepsilon \), the harmonic part of \( u_\varepsilon \times d u_\varepsilon \) at \( t = 0 \), as

\[
\gamma_0^\varepsilon = \sum_{k=1}^{\beta_1(M)} a_k^0(\varepsilon) c^k
\]

(2.6.21)

where \( a_k^0(\varepsilon) \) may depend on \( \varepsilon \). From the representation (2.6.21) we may define

\[
\lfloor \gamma_0^\varepsilon \rfloor := \sum_{k=1}^{\beta_1(M)} \lfloor a_k^0(\varepsilon) \rfloor c^k
\]

(2.6.22)

which is a harmonic 1-form on \( M \). Notice that we may extend (2.6.22) to \( M \times (0, \infty) \), in particular to \( \Omega \), by being constant in time to obtain

\[
\gamma_\varepsilon(x,t) := \lfloor \gamma_0^\varepsilon \rfloor(x).
\]

(2.6.23)

We observe that this extension has no term corresponding to \( dt \). We also note that by construction we have

\[
\| \gamma_0^\varepsilon - \lfloor \gamma_0^\varepsilon \rfloor \|_{L^2(\wedge^1 M)} = \left\| \sum_{k=1}^{\beta_1(M)} (a_k^0(\varepsilon) - \lfloor a_k^0(\varepsilon) \rfloor) c^k \right\|_{L^2(\wedge^1 M)} \leq \sum_{k=1}^{\beta_1(M)} \| c^k \|_{L^2(\wedge^1 M)}
\]

(2.6.24)

where the rightmost quantity is not dependent on \( \varepsilon \). Next we establish that \( \gamma_1^\varepsilon \) and \( \gamma_2^\varepsilon \) are not too far from \( \lfloor \gamma_0^\varepsilon \rfloor \). We only demonstrate this for \( \gamma_1^\varepsilon \) as the proof is similar for \( \gamma_2^\varepsilon \). To do this we first extend \( \gamma_0^\varepsilon, \gamma_1^\varepsilon \) to \( M \times (0, \infty) \), in particular \( \Omega \), by being constant in time and, respectively, use \( \Gamma_0^\varepsilon, \Gamma_1^\varepsilon \) to denote this. Next we define the 2-form \( \eta \) by

\[
\eta := (\Gamma_1^\varepsilon - \Gamma_0^\varepsilon) \wedge dt.
\]

(2.6.25)

Observe that

\[
\delta^* \eta = -\left[ d^* (\Gamma_1^\varepsilon - \Gamma_0^\varepsilon) \right] \wedge dt = 0
\]

where we have identified \( \Gamma_1^\varepsilon - \Gamma_0^\varepsilon \) with an element of \( H^1(M) \) since this 1-form is independent of \( t \) and \( dt \). From this computation we can now see that, after integrating by parts, we obtain

\[
2 \int_{M \times [0,t_1]} \langle Ju_\varepsilon, \eta \rangle_{M \times [0,t_1]} = \int_{M \times [0,t_1]} \langle \delta(u_\varepsilon \times \delta u_\varepsilon), \eta \rangle_{M \times [0,t_1]}
\]

(2.6.26)

\[
= \int_{M \times \{t_1\}} (u_\varepsilon \times du_\varepsilon) \wedge * \eta_N + \int_{M \times \{0\}} (u_\varepsilon \times du_\varepsilon) \wedge * \eta_N
\]
where \( J u_\varepsilon := \frac{1}{2} \delta[u_\varepsilon \times \delta u_\varepsilon] \). By noting that \( \eta_N = (\gamma_\varepsilon^1 - \gamma_\varepsilon^0) \wedge dt \) at \( M \times \{ t_1 \} \) and \( \eta_N = -(\gamma_\varepsilon^1 - \gamma_\varepsilon^0) \wedge dt \) at \( M \times \{ 0 \} \) we can rewrite this last expression as

\[
\int_{M \times \{ t_1 \}} (u_\varepsilon \times du_\varepsilon) \wedge \star \eta_N + \int_{M \times \{ 0 \}} (u_\varepsilon \times du_\varepsilon) \wedge \star \eta_N \tag{2.6.27}
\]

\[
= (-1)^{N-1} \left[ \int_{M \times \{ t_1 \}} \langle u_\varepsilon \times du_\varepsilon, \gamma_\varepsilon^1 - \gamma_\varepsilon^0 \rangle_M - \int_{M \times \{ 0 \}} \langle u_\varepsilon \times du_\varepsilon, \gamma_\varepsilon^1 - \gamma_\varepsilon^0 \rangle_M \right]
\]

\[
= (-1)^{N-1} \left[ \int_{M} (\gamma_\varepsilon^1, \gamma_\varepsilon^1 - \gamma_\varepsilon^0)_M - \int_{M} (\gamma_\varepsilon^0, \gamma_\varepsilon^1 - \gamma_\varepsilon^0)_M \right]
\]

\[
= (-1)^{N-1} \int_{M} \gamma_\varepsilon^1 - \gamma_\varepsilon^0 \|^2_M
\]

where we have used (2.6.14) in the third line. Putting (2.6.26) and (2.6.27) together and using the Jerrard-Soner estimate, Theorem 2.1 of [26], together with equivalence of norms on \( H^1(M) \) gives

\[
\| \gamma_\varepsilon^1 - \gamma_\varepsilon^0 \|_{L^2(\Lambda^1 M)}^2 = 2 \int_{M \times [0,t_1]} \langle J u_\varepsilon, \eta \rangle_{M \times [0,t_1]} \leq 2 \| J u_\varepsilon \| (C^{0,\alpha}(\Lambda^2 M \times [0,t_1]))^* \| \eta \| (C^{0,\alpha}(\Lambda^2 M \times [0,t_1]))
\]

\[
\leq C(\alpha, \Omega) \| \eta \|_{L^2(\Lambda^2 M \times [0,t_1])} \left[ \int_{M \times [0,t_1]} \frac{e_\varepsilon(u_\varepsilon)}{\log(\varepsilon)} + 1 \right]
\]

\[
= C(\alpha, \Omega) \| \gamma_\varepsilon^1 - \gamma_\varepsilon^0 \|_{L^2(\Lambda^1 M)} \left[ \int_{M \times [0,t_1]} \frac{e_\varepsilon(u_\varepsilon)}{\log(\varepsilon)} + 1 \right]
\]

\[
\leq C(\alpha, \Omega)(M_0 + 1) \| \gamma_\varepsilon^1 - \gamma_\varepsilon^0 \|_{L^2(\Lambda^1 M)}.
\]

Thus, we obtain

\[
\| \gamma_\varepsilon^1 - \gamma_\varepsilon^0 \|_{L^2(\Lambda^1 M)} \leq C(\alpha, \Omega)(M_0 + 1). \tag{2.6.28}
\]

It then follows from (2.6.24) and (2.6.28) that for \( i = 1, 2 \)

\[
\| \gamma_i^1 - \gamma_i^0 \|_{L^2(\Lambda^1 M)} \leq C(\alpha, \Omega)(M_0 + 1). \tag{2.6.29}
\]

Next we notice that since the integral of \( \gamma_\varepsilon \) over every closed loop in \( M \) has a value in \( 2\pi \mathbb{Z} \) then there exists \( u_{h_\varepsilon} : M \to S^1 \) such that

\[
u_{h_\varepsilon} \times du_{h_\varepsilon} = \gamma_\varepsilon. \tag{2.6.30}
\]

We may extend \( u_{h_\varepsilon} \) to be constant in time to obtain \( u_{h_\varepsilon} : \Omega \to S^1 \) such that

\[
u_{h_\varepsilon} \times du_{h_\varepsilon} = \gamma_\varepsilon. \tag{2.6.31}
\]
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\( u_{h,\varepsilon} \) is independent of \( t \) and \( u_{h,\varepsilon} \times \delta u_{h,\varepsilon} \) is independent of \( dt \). We also consider the linear extension \( \Phi_{1,2} \) of \( \Phi_1 \) to \( \Phi_2 \) in \( \Omega \) defined by

\[
\Phi_{1,2}(x,t) = \left( \frac{t_2 - t}{t_2 - t_1} \right) \Phi_1(x) + \left( \frac{t - t_1}{t_2 - t_1} \right) \Phi_2(x).
\]

Note that by (2.6.18) this extension satisfies

\[
\| \Phi_{1,2} \|_{W^{1,2}(\Omega)} \leq K(\Omega) \sqrt{M_0 |\log(\varepsilon)|}.
\] (2.6.32)

Step 2: “Gauge transformation” of \( u_\varepsilon \). On \( \Omega \) we consider the map \( w_\varepsilon \) defined by

\[
w_\varepsilon := u_\varepsilon e^{-i\Phi_{1,2} \varepsilon \overline{w_{h,\varepsilon}}} \text{ in } \Omega.
\]

Notice that \( |w_\varepsilon| = |u_\varepsilon| \). Moreover, one can show

\[
w_\varepsilon \times \delta w_\varepsilon = u_\varepsilon \times \delta u_\varepsilon - \delta \Phi_{1,2} \varepsilon \gamma_\varepsilon + (1 - |u_\varepsilon|^2)(\delta \Phi_{1,2} \varepsilon \gamma_\varepsilon).
\] (2.6.33)

Since \( |u_\varepsilon| \leq 3 \) then

\[
|\nabla_x w_\varepsilon| \leq |\nabla_x u_\varepsilon| + 3|\nabla_x \Phi_{1,2} \varepsilon| + 3|\gamma_\varepsilon|
\] (2.6.34)

and hence

\[
\| \nabla w_\varepsilon \|_{L^2(\Omega)}^2 + \varepsilon^{-2} \| 1 - |w_\varepsilon|^2 \|_{L^2(\Omega)}^2 \leq K M_0 |\log(\varepsilon)|.
\] (2.6.35)

By Hölder’s inequality, (2.6.8), (2.6.9), (2.6.18), and (2.6.32) we have that for \( 1 \leq p < 2 \)

\[
\| (1 - |u_\varepsilon|^2) \delta \Phi_{1,2} \varepsilon \|_{L^p(\Omega)}^p \leq K(\Omega) M_0 \varepsilon^{2-p} |\log(\varepsilon)|
\] (2.6.36)

\[
\| (1 - |u_\varepsilon|^2) \gamma_\varepsilon \|_{L^p(\Omega)}^p \leq K(\Omega) M_0 \varepsilon^{2-p} |\log(\varepsilon)|
\] (2.6.37)

and similarly

\[
\| (1 - |u_\varepsilon|^2) \delta \Phi_i \varepsilon \|_{L^p(\Omega \times \{ t_i \})}^p \leq K(\Omega) M_0 \varepsilon^{2-p} |\log(\varepsilon)|
\] (2.6.38)

\[
\| (1 - |u_\varepsilon|^2) \gamma_i \varepsilon \|_{L^p(\Omega \times \{ t_i \})}^p \leq K(\Omega) M_0 \varepsilon^{2-p} |\log(\varepsilon)|
\] (2.6.39)

for \( i = 1, 2 \). Next, by Corollary 5.6 of [24], we have the following Hodge decomposition of \( w_\varepsilon \times \delta w_\varepsilon \)
on \( \Omega \):
\[
\begin{aligned}
&\begin{cases}
  w_\varepsilon \times \delta w_\varepsilon = \delta \Phi_\varepsilon + \delta^* \Psi_\varepsilon + \eta & \text{in } \Omega, \\
  \delta \Psi_\varepsilon = 0 & \text{in } \Omega, \\
  (\Phi_\varepsilon)_T = 0, (\Psi_\varepsilon)_T = 0 & \text{on } \Sigma \\
  \delta \eta = \delta^* \eta = 0 & \text{on } \Omega
\end{cases}
\end{aligned}
\tag{2.6.40}
\]
also satisfying
\[
\| \Phi_\varepsilon \|_{W^{1,2}(\Omega)} + \| \Psi_\varepsilon \|_{W^{1,2}(\Omega)} \leq K(\Omega) \sqrt{M_0 \log(\varepsilon)}.
\tag{2.6.41}
\]

Next we include another new estimate needed to extend the argument of \([11]\) to the setting of a Riemannian manifold. As this represents an addition to the argument from \([11]\) we provide a more detailed discussion. We will write
\[
H^1_T(\Omega) := \{ \text{1-forms } \eta \text{ on } \Omega : \delta \eta = \delta^* \eta = 0 \text{ in } \Omega, \eta_T = 0 \text{ on } \partial \Omega \}.
\]
It follows from the discussion in Lemma 10 of section 5.3 of \([7]\) that \(H^1_T(\Omega)\) is a real vector space of dimension \((\# \text{ of components of } \partial \Omega) = 1\). Since \(H^1_T(\Omega)\) clearly includes all 1-forms of the form \(\eta = \text{const} \, dt\), we deduce that
\[
\eta = a_\varepsilon dt
\tag{2.6.42}
\]
where \(a_\varepsilon \in \mathbb{R}\) that may depend on \(\varepsilon\). Next we observe that
\[
a_\varepsilon \times \text{vol}_g(M)(t_2 - t_1) = \int_{\Omega} \langle j_\Omega w_\varepsilon, dt \rangle_{\Omega}
\tag{2.6.43}
\]
where we have used the abbreviation \(j_\Omega w_\varepsilon := w_\varepsilon \times \delta w_\varepsilon\). By (2.6.33) and the fact that \(\gamma_\varepsilon\) is independent of \(dt\) we can rewrite (2.6.43) as
\[
\int_{\Omega} \langle j_\Omega w_\varepsilon, dt \rangle_{\Omega} = \int_{\Omega} \langle j_\Omega u_\varepsilon, dt \rangle_{\Omega} - \int_{\Omega} \langle \delta \Phi^{1,2}_\varepsilon, dt \rangle_{\Omega} + \int_{\Omega} \langle (1 - |u_\varepsilon|^2) \delta \Phi^{1,2}_\varepsilon, dt \rangle_{\Omega}
=: (A) + (B) + (C).
\]
Observe that since \(u_\varepsilon\) solves (PGL)\(_{\varepsilon}\) then we have
\[
(A) = \int_{t_1}^{t_2} \int_M u_\varepsilon \times \partial_t u_\varepsilon d\text{vol}_g(x) dt = \int_{t_1}^{t_2} \int_M u_\varepsilon \times \Delta_M u_\varepsilon d\text{vol}_g(x) dt
\tag{2.6.44}
\]
\[
= -\int_{t_1}^{t_2} \left[ \int_M \langle d^*(u_\varepsilon \times du_\varepsilon), 1 \rangle_M \right] dt = -\int_{t_1}^{t_2} \left[ \int_M \langle u_\varepsilon \times du_\varepsilon, d(1) \rangle_M \right] dt = 0
\]
where we integrated by parts over $M$. Next, observe that by (2.6.17)
\[(B) = \int_M \int_{t_1}^{t_2} \partial_t \Phi_{\varepsilon}^{1,2} \, dt \, d\text{vol}_g(x) = \int_M \left[ \Phi_{\varepsilon}^2 - \Phi_{\varepsilon}^1 \right] \, d\text{vol}_g(x) = 0. \tag{2.6.45}\]

Finally, observe that by Cauchy-Schwarz, (2.6.9), and (2.6.32)
\[|C| \leq \int_{t_1}^{t_2} \int_M \left| 1 - |u_{\varepsilon}|^2 \right| \left| \partial_t \Phi_{\varepsilon}^{1,2} \right| \leq C(\Omega) M_0 \varepsilon |\log(\varepsilon)|. \tag{2.6.46}\]

Combining (2.6.44), (2.6.45), and (2.6.46) with (2.6.43) we obtain
\[|a_{\varepsilon}| \leq C(\Omega) M_0 \varepsilon |\log(\varepsilon)|. \tag{2.6.47}\]

Next, note that $\Psi_{\varepsilon}$ satisfies
\[
\begin{align*}
-\Delta_\Omega \Psi_{\varepsilon} &= \omega_{\varepsilon} := J w_z \\
(\Psi_{\varepsilon})_T &= 0 \\
(\delta^* \Psi_{\varepsilon})_T &= A_{\varepsilon} := d^* \Psi_{\varepsilon}^i + (\gamma_{\varepsilon}^i - \gamma_{0}^i) \left( 1 - |u_{\varepsilon}|^2 \right) (d\Phi_{\varepsilon}^i + |\gamma_{0}^i|) 
\end{align*}
\tag{2.6.48}
\]
for $i = 1, 2$ where $\Delta_\Omega$ is the Laplacian on $\Omega$. By (2.6.20), (2.6.28), (2.6.29), (2.6.38), and (2.6.39) we have, for $i = 1, 2$, $1 \leq p < \frac{N+1}{N}$, and $q = \frac{p}{p-1}$ that
\[\|A_{\varepsilon}\|_{W^{1, q}((\partial \Omega))} \leq \|A_{\varepsilon}\|_{L^p(\partial \Omega)} \leq C(p, \Omega)(M_0 + 1). \tag{2.6.49}\]
Arguing as in (2.6.20) we also have
\[\|\omega_{\varepsilon}\|_{W^{1, q}((\Omega))} \leq C(p, \Omega) \|\omega_{\varepsilon}\|_{C^0, \alpha((\Omega))} \leq C(\alpha, \Omega)(M_0 + 1). \]

Thus, by elliptic regularity, obtained by a Stampacchia duality argument obtained by combining Proposition A.2 of [9] and Corollary 5.6 of [24], we have
\[\|\Psi_{\varepsilon}\|_{W^{1, p}((\Omega))} \leq C(p, \Omega)(M_0 + 1). \tag{2.6.50}\]

We refer the reader to [11] as well as Proposition 2.6.4 for more details regarding this estimate. We set
\[
\Psi := \Psi_{\varepsilon}, \quad \Phi := \Phi_{\varepsilon}^{1,2} + \overline{\Phi}_{\varepsilon}, \quad \gamma := \gamma_{\varepsilon}, \quad \zeta := - \left( 1 - |u_{\varepsilon}|^2 \right) \left( \delta \Phi_{\varepsilon}^{1,2} + \gamma_{\varepsilon} \right) + \eta.
\]
Then
\[
\begin{align*}
    u_\varepsilon \times \delta u_\varepsilon &= w_\varepsilon \times \delta w_\varepsilon + |u_\varepsilon|^2 (\delta \Phi_\varepsilon^{1,2} + \gamma_\varepsilon) \\
    &= \delta \Phi_\varepsilon + \delta^* \Psi_\varepsilon + \eta + \delta \Phi_\varepsilon^{1,2} + \gamma_\varepsilon - (1 - |u_\varepsilon|^2) (\delta \Phi_\varepsilon^{1,2} + \gamma_\varepsilon) \\
    &= \delta \Phi + \delta^* \Psi + \gamma + \zeta.
\end{align*}
\]

The conclusion follows from (2.6.40), (2.6.32), (2.6.36), (2.6.37), (2.6.41), (2.6.47), and (2.6.50).

Next we demonstrate, following [11], that the phase portion, \( \Phi \), of \( u_\varepsilon \) is close to satisfying the heat equation.

**Lemma 2.6.5.** Suppose \( u_\varepsilon \) satisfies \((\text{PGL})_\varepsilon\) on \( M \times (0, \infty) \) and \((H_0)\), and suppose, for \( 0 < t_1 < t_2 < \infty \), we set \( \Omega = M \times (t_1, t_2) \). For \( \varepsilon > 0 \) sufficiently small we let \( \Phi, \Psi, \gamma, \) and \( \zeta \) satisfy the conclusions of Proposition 2.6.4. Then the function \( \Phi \) verifies the equation

\[
\begin{align*}
    \partial_t \Phi - \Delta \Phi &= -d^* (\delta^* \Psi + \zeta - P_t(\delta^* \Psi + \zeta)) - P_t(\delta^* \Psi + \zeta) \quad \text{in } \Omega. \tag{2.6.51}
\end{align*}
\]

*Here, for a 1-form \( \omega \) on \( \Omega \), \( P_t(\omega) \) denotes its dt component.*

*Proof.* By Proposition 2.6.4 we have

\[
\begin{align*}
    u_\varepsilon \times \delta u_\varepsilon &= \delta \Phi + \delta^* \Psi + \gamma + \zeta, \tag{2.6.52}
\end{align*}
\]

where \( \Phi, \Psi, \gamma, \) and \( \zeta \) verify the conclusions of Proposition 2.6.4. Taking the cross product of \((\text{PGL})_\varepsilon\) with \( u_\varepsilon \) leads to

\[
\begin{align*}
    u_\varepsilon \times \partial_t u_\varepsilon &= -d^*(u_\varepsilon \times du_\varepsilon) \quad \text{in } \Omega. \tag{2.6.53}
\end{align*}
\]

On the other hand, we also have by (2.6.52)

\[
\begin{align*}
    \begin{cases}
        u_\varepsilon \times du_\varepsilon &= d\Phi + \gamma + (\delta^* \Psi + \zeta) - P_t(\delta^* \Psi + \zeta), \\
        u_\varepsilon \times \partial_t u_\varepsilon &= \Phi_t + P_t(\delta^* \Psi + \zeta). \tag{2.6.54}
    \end{cases}
\end{align*}
\]

Notice that \( d^* \gamma = 0 \) since \( \gamma \) is a harmonic 1-form on \( M \times \{t\} \) for all \( t \). As a result of this last observation along with (2.6.53) and (2.6.54) we obtain the conclusion.

With the above ingredients in hand, the proof of Theorem 2.1.3 exactly follows arguments in [11]. We recall some details for the convenience of the reader.
Proof of Theorem 2.1.3

Let \( u_\varepsilon \) be a solution of (PGL)\( \varepsilon \) verifying (H\( 0 \)) on \( M \times (0, \infty) \). Let \( K \) be a compact subset of \( M \times (0, \infty) \). Choose \( 0 < t_1 < t_2 < \infty \) so that \( K \subset M \times (t_1, t_2) \). Let \( \Omega := M \times (t_1, t_2) \) and suppose that \( \Phi, \Psi, \gamma, \) and \( \zeta \) be as in Proposition 2.6.4 and Lemma 2.6.5. We choose \( t_3 \) and \( t_4 \) such that \( t_1 < t_3 < t_4 < t_2 \) and so that \( K \subset M \times (t_3, t_4) =: \Lambda \). By perhaps perturbing \( t_3 \) and \( t_4 \) we may assume

\[
\int_{\partial \Lambda} |\Phi|^2 + \int_{\partial \Lambda} |\nabla_{x,t} \Phi|^2 \leq C(K)(M_0 + 1)|\log(\varepsilon)|.
\]  

(2.6.55)

This is possible because of (2.6.12). We split the proof into two steps.

Step 1: Defining \( \varphi_\varepsilon \). Let \( \varphi_\varepsilon \) verify the homogeneous heat equation

\[
\begin{align*}
\partial_t \varphi_\varepsilon - \Delta \varphi_\varepsilon &= 0 & \text{in } \Lambda \\
\varphi_\varepsilon &= \Phi & \text{on } \Omega_1.
\end{align*}
\]  

(2.6.56)

and define

\[
w_\varepsilon := u_\varepsilon e^{-i\varphi_\varepsilon} \bar{u}_{h,\varepsilon}.
\]

where \( u_{h,\varepsilon} \) is the \( \mathbb{S}^1 \)-valued function described in Proposition 2.6.4 satisfying

\[
j u_{h,\varepsilon} = \gamma.
\]

From the standard regularity theory for the heat equation, see Theorems 8 and 9 of [19], in addition to (2.6.12) we have

\[
\|\nabla \varphi_\varepsilon\|_{L^\infty(\Lambda)}^2 \leq C(\mathcal{K}) \|\Phi\|_{W^{1,2}(\Omega_0)}^2 \leq C(\mathcal{K})(M_0 + 1)|\log(\varepsilon)|.
\]  

(2.6.57)

Next, for later use we set \( \Phi_1 = \Phi - \varphi_\varepsilon \). Then \( \Phi_1 \) solves

\[
\begin{align*}
\partial_t \Phi_1 - \Delta \Phi_1 &= -d^*(\delta^* \Psi + \zeta - P_t(\delta^* \Psi + \zeta)) - P_t(\delta^* \Psi + \zeta) & \text{in } \Lambda, \\
\Phi_1 &= 0 & \text{on } \Omega_1.
\end{align*}
\]  

(2.6.58)

Since by (2.6.13) we have

\[
\|\delta^* \Psi + \zeta - P_t(\delta^* \Psi + \zeta)\|_{L^p(\Lambda)} + \|P_t(\delta^* \Psi + \zeta)\|_{L^p(\Lambda)} \leq C(p, \mathcal{K})(M_0 + 1)
\]
it follows from standard estimates for the non-homogeneous heat equation that

\[ \| \nabla \Phi_1 \|_{L^p(\Lambda)} \leq C(p,K)(M_0 + 1). \]  

(2.6.59)

**Step 2:** $W^{1,p}$ estimates for $w_{\varepsilon}$. First observe that

\[ |w_{\varepsilon}|^2 |\nabla w_{\varepsilon}|^2 = |w_{\varepsilon}|^2 |\nabla |w_{\varepsilon}| |^2 + |w_{\varepsilon} \times \nabla w_{\varepsilon}|^2, \]

and hence

\[ \int_{K \cap \{|u_{\varepsilon}| \geq \frac{1}{2}\}} |\nabla w_{\varepsilon}|^p \leq C(p) \left[ \int_K |w_{\varepsilon} \times \delta w_{\varepsilon}|^p + \int_K |\nabla |w_{\varepsilon}| |^p \right]. \]  

(2.6.60)

On the other hand, by standard estimates for $(PGL)_\varepsilon$, (2.6.57), (2.6.12), and equivalence of norms for $\gamma$ we have

\[ |\nabla w_{\varepsilon}| \leq |\nabla u_{\varepsilon}| + 3|\nabla \varphi_{\varepsilon}| + 3|\gamma| \leq C(K)M_0 \varepsilon^{-1}, \]

where we have used that since $|u_{h,\varepsilon}| = 1$ then $|\nabla u_{h,\varepsilon}| = |ju_{h,\varepsilon}| = |\gamma|$. As a result, we have

\[ \int_{K \cap \{|u_{\varepsilon}| \leq \frac{1}{2}\}} |\nabla w_{\varepsilon}|^p \leq C(p,K)M_0^{p+1} \int_K V_{\varepsilon}(u_{\varepsilon}) \leq C(p,K)M_0^{p+1}. \]  

(2.6.61)

By the definition of $w_{\varepsilon}$ and Proposition 2.6.4 we have

\[ w_{\varepsilon} \times \delta w_{\varepsilon} = \delta^* \Psi + \delta \Phi_1 + \zeta + (1 - |u_{\varepsilon}|^2)(\delta \varphi_{\varepsilon} + \gamma). \]  

(2.6.62)

By Hölder’s inequality we have

\[ \|(1 - |u_{\varepsilon}|^2) \delta \varphi_{\varepsilon}\|_{L^p(K)} + \|(1 - |u_{\varepsilon}|^2) \gamma\|_{L^p(K)} \leq C(p,K)M_0 \varepsilon^{\frac{2-p}{p}} \log(\varepsilon), \]

and hence, when combined with (2.6.62), Proposition 2.6.4, and (2.6.59), we have

\[ \int_K |w_{\varepsilon} \times \delta w_{\varepsilon}|^p \leq C(p,K)(M_0 + 1). \]  

(2.6.63)

The proof for the gradient of the modulus remains the same as in [11] except we use a cutoff function in time, $\chi_K$, and work over a set $K' := M \times [t_4, t_5] \subset \Omega$ containing $K$. Following this procedure we
have, for $1 \leq p < 2$, that

$$
\int_{B_K} |\nabla w_\varepsilon|^p \leq C(K)(M_0 + 1)\varepsilon^{1-\frac{p}{2}} |\log(\varepsilon)|.
$$

(2.6.64)

We refer the reader to [11] for additional details. Combining (2.6.64) with (2.6.60), (2.6.61), and (2.6.63) completes the proof.

\[\square\]

### 2.7 Analysis of Limiting Measures

In this section we complete the proof of Theorem 2.1.1. To do this we will, as in [11], combine the results of Theorems 2.1.2, 2.6.1, 2.1.3 as well as their consequences and apply a detailed analysis of the limiting energy measure. Much of the corresponding proof used in [11] carries over to the general setting with minor variations. However, new ingredients are needed in the globalization of $\Phi_*$ due to the presence of $u_{h,\varepsilon}$ from Theorem 2.1.3. We refer the reader to section A.7 for more detail.

We fix solutions $\{u_\varepsilon\}_{0 < \varepsilon < 1}$ of (PGL)$_\varepsilon$ satisfying assumption (H$_0$), and we define Radon measures over $M \times [0, \infty)$ and its time slices by

$$
\mu_\varepsilon(x,t) := e^{u_\varepsilon(x,t)} |\log(\varepsilon)| d\text{vol}_g(x) dt
$$

$$
\mu^t_\varepsilon(x) := e^{u_\varepsilon(x,t)} |\log(\varepsilon)| d\text{vol}_g(x).
$$

As a result of assumption (H$_0$) and standard estimates for (PGL)$_\varepsilon$, together with well-known arguments from [13, 23], there is a subsequence $\varepsilon_n \to 0^+$ and Radon measures $\mu_*$ and $\mu_*^t$, defined on $M \times [0, \infty)$ and on $M$ respectively, such that

$$
\mu_{\varepsilon_n} \rightharpoonup \mu_* \quad \text{as measures,}
$$

$$
\mu_{\varepsilon_n}^t \rightharpoonup \mu_*^t \quad \text{as measures for all } t > 0, \quad \text{where } \mu_* = \mu_*^t dt.
$$

(2.7.1)

We will write $\varepsilon$ instead of $\varepsilon_n$ when this is not misleading. We also identify the measure $\mu_*^t$ with a measure on $M \times \{t\}$, and we will sometimes identify $M$ with $M \times \{t\}$. We record a consequence of the monotonicity formula on the limit measures.
Lemma 2.7.1. For each $t > 0$ and $x \in M$, the function $r \mapsto \mathcal{G}_\mu((x,t),\cdot)$ defined on $(0,\infty)$ by

$$
\mathcal{G}((x,t),r):= \frac{e^{C_2r}}{(4\pi)^{\frac{N}{2}} r^{N-2}} \int_M e^{-\frac{(d_\mu(x,y))^2}{4r^2}} d\mu_{L^{-r^2}}(y) + C_1M_0r,
$$

where $C_1$ and $C_2$ are determined by Proposition 2.3.5, is non-decreasing for $0 < r < \min\{\sqrt{t},1\}$.

Next, we record an important consequence of the previous analysis.

Theorem 2.7.2. There exists an absolute constant $\eta_2 > 0$, and a positive continuous function $\lambda$ defined on $(0,\infty)$ such that if, for $x \in M$, $t > 0$, and $r > 0$ sufficiently small, and

$$
\mu_s^*\left(B_{\lambda(t)r}(x)\right) < \eta_2 r^{N-2}, \tag{2.7.2}
$$

then for every $s \in \left[t + \frac{15}{16}r^2, t + r^2\right]$, $\mu_s^*$ is absolutely continuous with respect to the volume measure on the ball $B_{\frac{r}{2}}(x)$. More precisely,

$$
\mu_s^* = \frac{1}{2} |\nabla \Phi_s|^2 d\text{vol}_g(x) \text{ on } B_{\frac{r}{2}}(x),
$$

where $\Phi_s$ satisfies the heat equation in $\Lambda_{\frac{r}{2}} = \Lambda_{\frac{r}{2}}(x,t,r,r^2)$.

Proof. The proof is the same as in [11] and is a straightforward consequence of Proposition 2.6.3. \qed

2.7.1 Densities and the concentration set

We begin by introducing some notation for measure densities.

Definition 2.7.3. Let $\nu$ be a Radon measure on $M$. For $m \in \mathbb{N}$, the $m$-dimensional lower and upper densities of $\nu$ at the point $x$, denoted $\Theta_{s,m}(\nu,x)$ and $\Theta_{m}^*(\nu,x)$ respectively, are defined by

$$
\Theta_{s,m}(\nu,x):= \liminf_{r \to 0^+} \frac{\nu(B_r(x))}{\omega_m r^m}, \quad \Theta_{m}^*(\nu,x):= \limsup_{r \to 0^+} \frac{\nu(B_r(x))}{\omega_m r^m}
$$

where $\omega_m$ denotes the volume of the $m$-dimensional Euclidean unit ball in the standard metric. When both quantities coincide, $\nu$ admits an $m$-dimensional density $\Theta_m(\nu,x)$ at the point $x$, defined as the common value.

Next, following [11], we record a lemma regarding upper bounds on measure densities.
Lemma 2.7.4. For all \( x \in M \) and for all \( t > 0 \),

\[
\Theta_{s,N-2}(\mu^t, x) \leq \Theta_{N-2}^*(\mu^t, x) \leq \frac{M_0 e^{\frac{1}{2}}}{\omega_{N-2}} \left[ e^{C_2 t \frac{2^N}{2^N}} + (4\pi)^{\frac{N}{2}} C_1 \sqrt{t} \right].
\]

Proof. The first inequality follows from the definition of lower and upper densities while the second inequality follows from the fact that \( d \) agrees with \( d_+ \) on \( B_r(x) \) for each \( x \in M \) and \( 0 < r < \min\left\{ 1, \frac{\text{inj}(M)}{2} \right\} \) combined with Lemma 2.7.1 and (H_0). We refer the reader to A.7.1.1 for a more elaborate proof.

Proceeding as in [11] we introduce a suitable notion (not the usual one in this context) of parabolic \( m \)-dimensional lower density of a Radon measure \( \nu \).

Definition 2.7.5. Let \( \nu \) be a Radon measure on \( M \times [0, \infty) \) such that \( \nu = \nu^t dt \). For \( t > 0 \) and \( m \in \mathbb{N} \), the parabolic \( m \)-dimensional density of \( \nu \) at the point \((x,t)\) is defined by

\[
\Theta^P_m(\nu, (x,t)) := \lim_{r \to 0^+} \frac{1}{(4\pi)^{\frac{m}{2}} r^m} \int_M e^{-\frac{(d_+^t(x,y))^2}{4r^2}} \frac{\nu^t - r}{\nu^t}(y)
\]

when it exists.

Observe that since \( r \mapsto \mathcal{E}_\mu((x,t),r) \) is non-decreasing then \( \Theta^P_m(\mu^t, (x,t)) \) is defined everywhere in \( M \times (0, \infty) \). Next, analogously to [11], we will relate the parabolic density to the lower \((N-2)\)-dimensional measure density.

Lemma 2.7.6. Suppose \( x \in M \) and \( t > 0 \). Then, there exists \( K_M > 0 \), depending on \( M \), such that

\[
\Theta_{N-2}^*(\mu^t, (x,t)) \geq K_M \Theta_{s,N-2}(\mu^t, x)). \tag{2.7.3}
\]

Proof. The proof is the same as found in subsection 6.2 of [11]. We refer the reader to 2.7.7 for additional details.

Let \((x,t) \in M \times (0, \infty)\) be given. Let \( 0 < r < \min\left\{ t, 1, \frac{\text{inj}(M)}{2} \right\} \) be fixed. Similar to the proof of Lemma 2.7.4, we conclude from Lemma 2.7.1 that

\[
\frac{\mu^t(B_r(x))}{r^{N-2}} \leq e^{\frac{1}{2} + C_2 r} \int_M e^{-\frac{(d_+^t(x,y))^2}{4(r^2 + r)}} \frac{1}{r^{N-2}} \frac{\nu^t - r}{\nu^t}(y) + (4\pi)^{\frac{N}{2}} e^{\frac{1}{4}} C_1 M_0 \sqrt{r^2 + r}.
\]

Observe that on \( B_{\frac{\text{inj}(M)}{2}}(x) \) that

\[
e^{-\frac{(d_+^t(x,y))^2}{4(r^2 + r)}} = e^{-\frac{(d_+^t(x,y))^2}{4r^2}} e^{-\frac{(d_+^t(x,y))^2}{4(r^2 + r)}} \leq e^{-\frac{(d_+^t(x,y))^2}{4r^2}} e^{-\frac{(d_+^t(x,y))^2}{4}}.
\]
On \( M \setminus B_{\text{inj}(M)}(x) \) we have
\[
\int_{M \setminus B_{\text{inj}(M)}(x)} e^{-\frac{(d_x(x,y))^2}{4(r^2+r)}} \, d\mu_*^{-r}(y) \leq e^{-\frac{[\text{inj}(M)]^2}{16(r^2+r)}} M_0
\]
Putting these together we obtain
\[
\mu_*^t(B_r(x)) \leq e^{\frac{1}{4} + \frac{[\text{inj}(M)]^2}{16}} + C_2 \int_M e^{-\frac{(d_x(x,y))^2}{r}} \, d\mu_*^t-r
\]
\[
+ \frac{e^{\frac{1}{4}} + C_2}{(r^2 + r)^{\frac{N-2}{2}}} M_0 + (4\pi)^{\frac{N}{2}} e^{\frac{1}{4}} C_1 M_0 \sqrt{r^2 + r}.
\]
Letting \( r \to 0^+ \) gives the conclusion.

Just as in [11] we define
\[
\Sigma_\mu := \{(x,t) \in M \times (0, \infty) : \Theta_{N-2}^p(\mu_*, (x,t)) > 0\},
\]
\[
\Sigma_t^\mu := \Sigma_\mu \cap (M \times \{t\}). \text{ for } t > 0,
\]
A consequence of Lemma 2.7.6 is
\[
\Theta_{*,N-2}(\mu_*, x) \equiv 0 \text{ on } M \setminus \Sigma_t^\mu.
\]
Next we record, for future use, that the function \((x,t) \mapsto \Theta_{N-2}^p(\mu_*, (x,t))\) is upper semi-continuous on \( M \times (0, \infty) \). We note that the proof of this is the same as in [11]. More detail regarding its extension can be found in A.7.1.1.

**Lemma 2.7.7.** The map \((x,t) \mapsto \Theta_{N-2}^p(\mu_*, (x,t))\) is upper semi-continuous on \( M \times (0, \infty) \).

### 2.7.2 First properties of \( \Sigma_\mu \)

We begin this subsection by demonstrating a lower bound estimate on the \((N-2)\)-dimensional lower density over the set \( \Sigma_\mu \). The proof follows [11] closely so we refer the reader to A.7.2.1 for more details.

**Lemma 2.7.8.** Suppose \( 0 < r < \sqrt{t} \) and \( x \in M \). Then, if \((x,t) \in \Sigma_\mu \) it follows that
\[
r^{2-N} \mu_*^{t-r^2}(B_{\lambda(t-r^2)}(x)) > \eta_2,
\]
where \( \eta_2 \) is the constant in Theorem 2.7.2.

Proof. We proceed by proving the contrapositive statement. Suppose there is \((x, t) \in M \times (0, \infty)\) and \(0 < r < \sqrt{t}\) for which

\[
r^{2-N} \mu^t-r^2(B_{\lambda(t-r^2)}r(x)) \leq \eta_2.
\]

By Theorem 2.7.2, for all \( \tau \in [t - \frac{r^2}{16}, t] \) we have

\[
\mu^\tau = \frac{|\nabla \Phi_*|^2}{2} \text{dvol}_g(x) \quad B_\tau(x)
\]

where \( \Phi_* \) is smooth. Straightforward computations then show that \( \Theta^\mu_P(\mu_*, (x, t)) = 0 \).

Next we prove a clearing out lemma related to the set \( \Sigma_\mu \).

Theorem 2.7.9. There exists a positive continuous function \( \eta_3 \) defined on \((0, \infty)\), such that for any \((x, t) \in M \times (0, \infty)\) and any \(0 < r < \sqrt{t}\), if

\[
\mathcal{F}_\mu((x, t), r):= \frac{1}{r^{N-2}} \int_M e^{-\frac{(d_+(x, y))^2}{r^2}} \text{d} \mu^t-r^2(y) \leq \eta_3(t - r^2)
\]

then \((x, t) \not\in \Sigma_\mu \).

Proof. The proof extends to our setting without change to the argument from the proof of Theorem 6 of [11]. We refer the reader to [11] or A.7.2.2 for additional details.

Following [11] we note that a consequence of Theorem 2.7.9, for which details can be found in A.7.2.3, is the following:

Corollary 2.7.10. For any \((x, t) \in \Sigma_\mu \),

\[
\Theta^\mu_{N-2}(\mu_*, (x, t)) \geq \eta_3(t).
\]

Next we provide a decomposition for \( \mu_*^t \) and demonstrate a few properties of \( \Sigma_*^t \) and \( \Sigma_\mu \). The proof of this proposition is the same as in [11] with the exception that we rescale the metric instead of the function in the argument for (2). More details can be found in A.7.2.4.

Proposition 2.7.11.

1. The set \( \Sigma_\mu \) is closed in \( M \times (0, \infty) \).
2. For any \( t > 0 \) we have
\[
\mathcal{H}^{N-2}(\Sigma^t_\mu) \leq K M_0 < \infty.
\]

3. For any \( t > 0 \), the measure \( \mu^t_\ast \) can be decomposed as
\[
\mu^t_\ast = g(x,t)\mathcal{H}^N + \Theta_\ast(x,t)\mathcal{H}^{N-2} \setminus \Sigma^t_\mu,
\]
where \( g \) is some smooth function defined on \([M \times (0,\infty)] \setminus \Sigma_\mu\) and \( \Theta_\ast \) verifies the bound
\[
\Theta_\ast(x,t) \leq K M M_0 [e^{C_M t^{2-\frac{2-N}{2}}} + D_M \sqrt{t}] \text{ for } C_M, D_M, K_M > 0 \text{ depend on } M.
\]

### 2.7.3 Regularity of \( \Sigma^t_\mu \)

Next we record that the \((N - 2)\)-dimensional parabolic density of \( \mu_\ast \) is controlled by \( \Theta_\ast,N-2(\mu^t_\ast, x) \) for most \( t \) and \( x \). This gives the reverse relationship illustrated in Lemma 2.7.6. The proof is very similar to the corresponding one from [11] the only exceptions are that we invoke the Besicovitch-Federer Covering Theorem, see Theorem 2.8.14 of [20], and we do not restrict our analysis to a finite region of time. As a result, we refer the reader to A.7.3.1 for more details.

**Proposition 2.7.12.** For \( \mathcal{L}^1 \)-almost every \( t > 0 \), the following inequality holds:
\[
\Theta_\ast,N-2(\mu^t_\ast, x) \geq K \Theta^P_{N-2}(\mu_\ast,(x,t)) \tag{2.7.7}
\]
for \( \mathcal{H}^{N-2} \)-almost every \( x \in M \).

Next we show that a lower density bound holds on \( \Sigma^t_\mu \) for most points.

**Corollary 2.7.13.** For \( \mathcal{L}^1 \)-almost every \( t \geq 0 \)
\[
\Theta_\ast,N-2(\mu^t_\ast, x) \geq K \eta_3(t) \tag{2.7.8}
\]
for \( \mathcal{H}^{N-2} \)-almost every \( x \in \Sigma^t_\mu \).

**Proof.** The corollary follows from Corollary 2.7.10 and Proposition 2.7.12. Details can be found in A.7.3.2.

Finally, we show that for \( \mathcal{L}^1 \)-almost every \( t > 0 \) and \( \mathcal{H}^{N-2} \)-almost every \( x \in \Sigma^t_\mu \) the upper and lower densities of \( \mu^t_\ast \) agree. As a result, for \( \mathcal{L}^1 \)-almost every \( t > 0 \) the set \( \Sigma^t_\mu \) is \((N-2)\)-rectifiable.
Proposition 2.7.14. For $\mathcal{L}^1$-almost every $t > 0$,

$$\Theta^*_{s,N-2}(\mu^t_s, x) = \Theta^*_{N-2}(\mu^t_s, x) \geq K\eta_3(t)$$

for $\mathcal{H}^{N-2}$-almost every $x \in \Sigma^t_{\mu}$. Consequently, for $\mathcal{L}^1$-almost every $t > 0$ the set $\Sigma^t_{\mu}$ is $(N-2)$-rectifiable.

Proof. The proof essentially follows ideas from [11]. One begins by defining the vector space, $F$, for a fixed $(x, t) \in \Omega_{\omega}$ by

$$F := \left\{ g \in L^\infty((0, \infty); \mathbb{R}) : I(g) := \lim_{r \to 0^+} I_r(g) \text{ exists and is finite} \right\}$$

where for $r > 0$,

$$I_r(y) := \frac{1}{r^{N-2}} \int_M g \left( \frac{d_+(x, y)}{r} \right) d\mu_s(y).$$

The same definition appears in [11], with the Euclidean distance in place of $d_+$.

To prove the proposition, it suffices to show that $\chi_{[0,1]} \in F$. The starting point is the fact that, if we write $e_s(\ell) = e^{-s\ell^2}$, then $e_{1/4} \in F$; this is established in the proof of Proposition 2.7.12. One can then proceed using the same technique as in [11], which involves a number of steps.

Then, it is shown that if $g \in F$ then for $s > 0$ the rescaling $g_s : \ell \mapsto g(\ell \sqrt{s})$ belongs to $F$ as well. Since $e_{1/4} \in F$ this shows that $e_s \in F$ for all $s$. Next, we proceed to inductively demonstrate that functions of the form $\ell \mapsto \ell^{2k} e^{-\ell^2}$ for $k \in \mathbb{N} \cup \{0\}$ are member of $F$. We then show that $g \in C^2_c((0, \infty))$ satisfying $g'(0) = 0$ are also members of $F$ by appealing to Hermite polynomials and an approximation argument. Finally, we use members of $C^2_c((0, \infty))$ to approximate $\chi_{[0,1]}$ and show that $\chi_{[0,1]} \in F$.

We refer the reader to the proof of Proposition 8 from [11] as well as Step 4 of A.7.3.1, Lemma A.7.1, Lemma A.7.2, Lemma A.7.3, and A.7.3.3 and simply note that the proof presented there only depends on functions over the real line.
2.7.4 Globalizing $\Phi_*$

In this subsection we demonstrate that the function $\Phi_*$ has a globally defined differential and partial derivative in $t$ even though its construction was merely local.

Lemma 2.7.15. The locally defined function $\Phi_*$ from Theorem 2.7.2 extends to a function $\Phi_* : M \times (0, \infty) \to \mathbb{R}/2\pi\mathbb{Z}$. In particular, $\Phi_*$ has a differential, $d\Phi_*$, that is globally defined and satisfies $d\Phi_* = d\phi_* + \gamma_*$ where $\phi_*, \gamma_*$ are globally defined so that $\phi_*$ solves the heat equation on $M \times (0, \infty)$ and $\gamma_*$ is a harmonic 1-form on $M \times (0, \infty)$ that is only a function of $x$ and has no term corresponding to $dt$. In addition, $\partial_t \Phi_*$ is globally defined and equal to $\partial_t \phi_*$. 

Proof. For $m \in \mathbb{N} \setminus \{1\}$ we set $K_m = M \times \left[\frac{1}{m}, m\right]$, so that $\bigcup_{m \geq 2} K_m = M \times (0, \infty)$. Applying Theorem 2.1.3 to $K = K_m$ we may write, for $\varepsilon$ sufficiently small,

$$u_\varepsilon = e^{i\phi^m_\varepsilon} w^m_\varepsilon u_{h,\varepsilon} \text{ on } K_m,$$  

(2.7.9)

where $\phi^m_\varepsilon$ solves the heat equation on $K_m$, $u_{h,\varepsilon} \times du_{h,\varepsilon} = \gamma_\varepsilon$ is a harmonic 1-form on $M$ not dependent, as a function, on $t$ and $m$ and has no component corresponding to $dt$. Theorem 2.1.3 yields the estimates

$$\|\nabla \phi^m_\varepsilon\|_{L^\infty(K_m)} + \|\nabla u_{h,\varepsilon}\|_{L^\infty(K_m)} \leq C(m) \sqrt{(M_0 + 1)|\log(\varepsilon)|}$$  

(2.7.10)

$$\|\nabla w^m_\varepsilon\|_{L^p(K_m)} \leq C(m, p) \text{ for any } 1 \leq p < \frac{N + 1}{N}. \quad (2.7.11)$$

For fixed $m$, we may pass to a further subsequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ such that

$$\frac{\phi^m_\varepsilon}{\sqrt{|\log(\varepsilon)|}} \to \phi_*^m \text{ in } C^2(K_{m-1})$$  

(2.7.12)

$$\frac{\gamma_\varepsilon}{\sqrt{|\log(\varepsilon)|}} \to \gamma_* \text{ in } C^2(K_{m-1}). \quad (2.7.13)$$

where $\phi_*^m$ also satisfies the heat equation on $K_{m-1}$, and $\gamma_*$ is a harmonic 1-form. We have used the fact that the space of harmonic forms is finite dimensional. Note also that $\gamma_*$ does not depend on $m$ or $t$ and has no component corresponding to $dt$.

Next, let $x_0 \in \Omega_\mu := (M \times (0, \infty)) \setminus \Sigma_\mu$. By (1) of Proposition 2.7.11 we have that $\Omega_\mu$ is open. Thus, we can find a set $\Lambda x_0 = B_R(x_0) \times [t_0, t_1]$ contained in $\Omega_\mu$. For $m_0$ large enough we will have,
for \( m \geq m_0 \), that \( \Lambda_{x_0} \subset \mathcal{K}_m \). For \( \varepsilon \) sufficiently small we have

\[
|u_\varepsilon| \geq 1 - \sigma \geq \frac{1}{2} \text{ on } \Lambda_{x_0}
\]  

(2.7.14)

where \( \sigma \) is the constant in Theorem 2.6.1. This lower bound on the norm allows us to write

\[
u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}
\]

(2.7.15)

for some real-valued function \( \varphi_\varepsilon : M \times (0, \infty) \to \mathbb{R}/2\pi\mathbb{Z} \). By (2.7.14) we may apply (2.6.6) of Theorem 2.6.1 to demonstrate that there exists a solution \( \Phi_\varepsilon \) of the heat equation on \( \Lambda_{x_0} \) such that

\[
\|\nabla \Phi_\varepsilon - \nabla \varphi_\varepsilon\|_{L^{\infty}(\Lambda_{x_0})} \leq C\varepsilon^\beta.
\]

(2.7.16)

On the other hand, since \( |w_\varepsilon^m| = |u_\varepsilon| \) we may write, for \( m \geq m_0 \)

\[
w_\varepsilon^m = \rho_\varepsilon e^{i\psi_\varepsilon^m} \text{ on } \Lambda_{x_0}
\]

(2.7.17)

where \( \psi_\varepsilon^m : M \times (0, \infty) \to \mathbb{R}/2\pi\mathbb{Z} \). Combining (2.7.9), (2.7.15), and (2.7.17) we obtain

\[
d\varphi_\varepsilon = d\phi_\varepsilon^m + \gamma_\varepsilon + d\psi_\varepsilon^m.
\]

(2.7.18)

By (2.7.16) for fixed \( m \) we have

\[
\left| \frac{d\phi_\varepsilon^m + \gamma_\varepsilon - d\Phi_\varepsilon}{\sqrt{|\log(\varepsilon)|}} \right| \leq \left| \frac{d\psi_\varepsilon^m}{\sqrt{|\log(\varepsilon)|}} \right| + C\varepsilon^\beta \text{ on } (\Lambda_{x_0})_{\frac{1}{2}}.
\]

By (2.7.11) we obtain

\[
\left\| \frac{d\phi_\varepsilon^m}{\sqrt{|\log(\varepsilon)|}} + \frac{\gamma_\varepsilon}{\sqrt{|\log(\varepsilon)|}} - \frac{d\Phi_\varepsilon}{\sqrt{|\log(\varepsilon)|}} \right\|_{L^p((\Lambda_{x_0})_{\frac{1}{2}})} \to 0 \text{ as } \varepsilon \to 0^+.
\]

Since \( \frac{d\phi_\varepsilon^m}{\sqrt{|\log(\varepsilon)|}} \to \phi_*^m \) and \( \frac{\gamma_\varepsilon}{\sqrt{|\log(\varepsilon)|}} \to \gamma_* \) from (2.7.12) and (2.7.10) then we deduce that \( \frac{d\Phi_\varepsilon}{\sqrt{|\log(\varepsilon)|}} \to d\Phi_* \) on \( (\Lambda_{x_0})_{\frac{1}{2}} \) and

\[
d\Phi_* = d\phi_*^m + \gamma_* \text{ on } (\Lambda_{x_0})_{\frac{1}{2}}.
\]

Observe that since \( \gamma_* \) and \( \Phi_* \) are independent of \( m \) then by changing \( \phi_*^m \) by a constant we may assume that all \( \phi_*^m \) coincide on \( (\Lambda_{x_0})_{\frac{1}{2}} \). By analyticity, for each \( n \geq m_0 \) the functions \( \{\phi_*^m\}_{m \geq n} \)
coincide on $K_m$. Letting $n$ go to infinity, we define their common value $\phi_*$ on $M \times (0, \infty)$. We then have

$$d\Phi_* = d\phi_* + \gamma_*$$

on $(\Lambda_{x_0})^\theta$. Since the right-hand-side is globally defined we can then extend $\Phi$, by integrating, to be the sum of a solution to the heat equation, $\phi_*$, and $\gamma_*$, a harmonic 1-form on $M \times (0, \infty)$ that is only a function of $x$ and has no term corresponding to $dt$. We also note that $\partial_t \Phi_*$ is globally defined and equal to $\partial_t \phi_*$.

2.7.5 Mean Curvature Flows

The goal of this subsection is to prove (5) from Theorem 2.1.1. In particular, we focus on studying the properties of the singular parts of $\{\mu_t\}_{t>0}$, denoted $\{\nu_t\}_{t>0}$, which for each $t>0$ satisfy

$$\nu_t = \Theta_*(x,t) \mathcal{H}^{N-2} \mathcal{L}^t \Sigma^t$$

where $\Theta_*$ and $\Sigma^t$ are as in (2.1.2). As in [11], and following the same proof, we will study limiting behaviour of

$$\omega^t := \frac{\partial_t u_\varepsilon}{|log(\varepsilon)|} dvol_g(x)$$

and

$$\sigma^t := -\frac{\partial_t u_\varepsilon \cdot \nabla u_\varepsilon}{|log(\varepsilon)|} dvol_g(x).$$

2.7.5.1 Convergence of $\sigma^t$

By the Cauchy-Schwarz inequality $\sigma_\varepsilon$ is uniformly bounded on $M \times [0, T]$ for every $T > 0$. By perhaps passing to a further subsequence, we may assume that $\sigma_\varepsilon \rightharpoonup \sigma_*$ as measures. The Radon-Nikodym derivative of $|\sigma_\varepsilon|$ with respect to $\mu_\varepsilon$ verifies

$$\frac{d|\sigma_\varepsilon|}{d\mu_\varepsilon} = \frac{\partial_t u_\varepsilon \cdot \nabla u_\varepsilon}{e_\varepsilon(u_\varepsilon)} \leq \sqrt{2} \frac{\partial_t u_\varepsilon}{e_\varepsilon(u_\varepsilon)} \frac{e_\varepsilon(u_\varepsilon)}{e_\varepsilon(u_\varepsilon)} = \frac{\sqrt{2} |\partial_t u_\varepsilon|}{e_\varepsilon(u_\varepsilon)}.$$
On the other hand,

\[
\frac{\|\partial_t u_\varepsilon\|}{\sqrt{\varepsilon u_\varepsilon}} \in L^2(M \times [0,T], d\mu_\varepsilon) = \int_{M \times [0,T]} \frac{\|\partial_t u_\varepsilon\|^2}{\varepsilon u_\varepsilon} d\mu_\varepsilon dt = \int_{M \times [0,T]} \frac{\varepsilon u_\varepsilon}{\|\log(\varepsilon)\|} d\mu_\varepsilon dt = \int_{M \times [0,T]} \frac{\|\partial_t u_\varepsilon\|^2}{\|\log(\varepsilon)\|} d\mu_\varepsilon dt \leq M_0
\]

where we used standard energy estimates for (PGL)$_\varepsilon$ and assumption (H$_0$) for the last inequality.

We conclude that $\frac{d|\sigma|}{d\mu_\varepsilon}$ is uniformly bounded in $L^2(M \times [0,T], d\mu_\varepsilon)$. Arguing as in Theorem 2.2 of [14], but adapting to the case of a compact Riemannian manifold without boundary, it follows that $\sigma_*$ is absolutely continuous with respect to $\mu_*$. Therefore, we may write

\[
\sigma_* = h \mu_* dt
\]

where $h \in L^2(M \times [0,T], \mu_* dt)$. We use (2.1.2) from Theorem 2.1.1 to decompose $\mu_*$ into its absolutely continuous part with respect to $d\nu_\varepsilon$ and its singular part $\nu_\varepsilon$ satisfying (2.7.20). Arguing as in Proposition 3.1 of [10] combined with Theorem 2.6.1 and Lemma 2.7.15 we see that the part of $\sigma_*$ absolutely continuous with respect to $d\nu_\varepsilon$ has density $-\partial_t \Phi_* \cdot \nabla \Phi_*$. We now have

**Lemma 2.7.16.** The measure $\sigma_*$ decomposes as $\sigma_* = \sigma_*^t dt$, where for $\mathcal{L}^1$-almost every $t \geq 0$,

\[
\sigma_*^t = -\partial_t \Phi_* \cdot \nabla \Phi_* d\nu_\varepsilon(x) + h \nu_*^t.
\]

Next we observe that for every $t \geq 0$, by appealing to the ideas found in Lemmas 7.5 and 7.6 of [45], we have for all smooth vector fields, $X$, that

\[
\frac{1}{|\log(\varepsilon)|} \int_{M \times \{t\}} [e_\varepsilon(u_\varepsilon)I - \nabla u_\varepsilon \otimes du_\varepsilon] : DX d\nu_\varepsilon(x) = \int_M \langle X, \frac{\partial u_\varepsilon \cdot \nabla u_\varepsilon}{|\log(\varepsilon)|} \rangle d\nu_\varepsilon(x) = -\int_M \langle X, \sigma_*^t \rangle
\]

where $I$ is the identity operator, $\nabla u_\varepsilon \otimes du_\varepsilon = \nabla u^1_\varepsilon \otimes du^1_\varepsilon + \nabla u^2_\varepsilon \otimes du^2_\varepsilon$, $DX$ is the $(1,1)$-tensor field defined at a point $p \in M$ by

\[
DX_p: v \in T_p M \rightarrow D_v X,
\]
and we use the notation \( A : B \) to denote the inner product of (1,1)-tensor fields on \( T_x M \) defined by

\[
A : B := \sum_{i=1}^{N} \sum_{j=1}^{N} \langle A(e_i), e_j \rangle \langle B(e_i), e_j \rangle
\]  

where \( \{e_1, e_2, \ldots, e_N\} \) is any orthonormal basis for \( T_x M \). Following [11] we use (2.7.23) as motivation to analyze the weak limit of \( \alpha^t \epsilon \approx (I - \nabla u \epsilon \otimes du \epsilon) (du \epsilon) \).

Since \( |\alpha^t \epsilon| \leq K N \mu^t \epsilon \) then we may assume that, by perhaps passing to a subsequence, that

\[
\alpha^t \epsilon \rightharpoonup \alpha^t \star \equiv A \cdot \mu^t \epsilon
\]

where \( A \) is a symmetric (1,1)-tensor field and where a (1,1)-tensor field is symmetric if for each \( x \in M \) and each \( u, v \in T_x M \) we have

\[
\langle A_x(u), v \rangle = \langle u, A_x(v) \rangle.
\]

We also recall that a symmetric (1,1)-tensor is referred to as positive-semidefinite if for each \( x \in M \) and each \( u \in T_x M \) we have

\[
\langle A_x(u), u \rangle \geq 0.
\]

Finally, notice that if \( A, B \) are symmetric (1,1)-tensor fields then we write

\[
A \leq B
\]

if \( B_x - A_x \) is positive semi-definite for each \( x \in M \). We now notice that since \( \nabla u \epsilon \otimes du \epsilon \) is a positive semidefinite (1,1)-tensor field then

\[
A \leq I. 
\]  

(2.7.26)

On the other hand, computing in normal coordinates about a point \( x \in M \), we have, at \( x \), that

\[
\text{tr}_g[\{e_\epsilon(u_\epsilon)I - \nabla u_\epsilon \otimes du_\epsilon\}] = (N - 2)e_\epsilon(u_\epsilon) + 2V_\epsilon(u_\epsilon).
\]
Therefore, since the trace is a linear operation, passing to the limit we obtain

\[ \text{tr}_g(A) = (N - 2) + 2 \frac{dV_*}{d\mu_*} \tag{2.7.27} \]

where \( \frac{dV_*}{d\mu_*} \) is the non-negative limiting measure, obtained after passing to a subsequence, of \( \frac{V_*}{\varepsilon^m} \).

Taking the limit \( \varepsilon \to 0^+ \) in (2.7.23), decomposing \( \mu_* \) using (2.1.2) of Theorem 2.1.1, and using the pointwise estimates provided by Theorem 2.6.1 we obtain for \( L^1 \)-almost every \( t \geq 0 \)

\[ \int_M A : DX \, d\nu_*^t + \int_M \left[ \frac{\nabla \Phi_*}{2} I - \nabla \Phi_* \otimes d\Phi_* \right] : DX \, d\nu_g(x) = - \int_M \langle X, h \rangle \, d\nu_*^t - \int_M \langle X, \partial_t \Phi_* \nabla \Phi_* \rangle \, d\nu_g(x). \tag{2.7.28} \]

Since \( \Phi_* \) solves the heat equation then we also have, by multiplying the heat equation by \( \langle X, \nabla \Phi_* \rangle \) and arguing in coordinates similar to Lemmas 7.5 and 7.6 of [45], that

\[ \int_M \left[ \frac{\nabla \Phi_*}{2} I - \nabla \Phi_* \otimes d\Phi_* \right] : DX \, d\nu_g(x) = - \int_M \langle X, \partial_t \Phi_* \cdot \nabla \Phi_* \rangle \, d\nu_g(x). \tag{2.7.29} \]

Combining (2.7.28) and (2.7.29) now gives the following

**Lemma 2.7.17.** For \( L^1 \)-almost every \( t \geq 0 \) and for every smooth vector field \( X \) we have

\[ \int_M A : DX \, d\nu_*^t = - \int_M \langle X, h \rangle \, d\nu_*^t. \tag{2.7.30} \]

We see that the conclusion of Lemma 2.7.17 is close to (1.2.9). Thus, if we can show that \( A \) is the orthogonal projection operator from \( T_x M \) onto \( T_x \Sigma_*^t \) then we will have shown that \( \nu_*^t \) has first variation with mean curvature \( h \). Following [11] we proceed in this direction by first demonstrating that \( A \) is perpendicular to normal vectors to \( T_x \Sigma_*^t \).

**Lemma 2.7.18.** For \( L^1 \)-almost every \( t \geq 0 \) and \( \mathcal{H}^{N-2} \)-almost every \( x \in \Sigma_*^t \) we have

\[ A_x \left[ \int_{T_x \Sigma_*^t} \nabla \chi(y) \, d\mathcal{H}^{N-2}(y) \right] = 0 \tag{2.7.31} \]

where \( \chi \) is a compactly support smooth function supported in \( T_x M \) where we use the exponential map to identify a neighbourhood of zero in \( T_x M \) with subsets of \( M \).

**Proof.** As in the corresponding proof from [11] we choose \( t \geq 0 \) for which (2.7.30) holds and \( x \in \Sigma_*^t \) such that \( T_x \Sigma_*^t \) exists and such that \( x \) is a Lebesgue point for \( \Theta_* \), with respect to \( \mathcal{H}^{N-2} \), and of
A with respect to $\nu_t^\ast$. We now consider a smooth function $\chi$ with support contained in a normal coordinate neighbourhood centred at $x$. We then consider, written in normal coordinates centred at $x$, the vector field defined by

$$X_{r,l}(y) := \chi(-y) \frac{\partial}{\partial x^l}$$

for $l \in \{1, 2, \ldots, N\}$. Inserting $X_{r,l}$ into (2.7.30), taking the limit $r \to 0^+$, and appealing to the difference of homogeneity of the right-hand side as in Theorem 3.8 of [5], we conclude that

$$A_x \left[ \int_{T_x \Sigma_t^\mu} \nabla \chi(y) d\mathcal{H}^{N-2}(y) \right] = 0.$$ 

We have, due to the arguments of section 6 of [5], the following consequence:

**Corollary 2.7.19.** For $t$ and $x$ as in Lemma 2.7.18,

$$(T_x \Sigma_t^\mu)^\perp \subset \ker(A_x).$$

We now show that $A_x = P$ where $P$ is the orthogonal projection of $T_x M$ onto $T_x \Sigma_t^\mu$.

**Corollary 2.7.20.** For $t$ and $x$ as in Lemma 2.7.18, $A_x = P$ is the orthogonal projection onto the tangent space $T_x \Sigma_t^\mu$.

**Proof.** By (2.7.26) we have $A_x \leq I_x$ for each $x \in M$, and therefore all the eigenvalues of $A_x$ are less than or equal to 1. By (2.7.27), $\text{tr}_g(A_x) \geq N - 2$ so that the sum of the eigenvalues of $A_x$ is at least $N - 2$. By Corollary 2.7.19 and our choice of $x$ and $t$ we know that $A_x$ has at least two zero eigenvalues. Combining the above information allow us to conclude that $A_x$ has precisely two zero eigenvalues and $(N - 2)$ eigenvalues equal to 1. In particular, since the kernel is $(T_x \Sigma_t^\mu)^\perp$ then $A_x$ is the orthogonal projection onto $T_x \Sigma_t^\mu$. \qed

Combining Lemma 2.7.17 and Corollary 2.7.20 we obtain:

**Proposition 2.7.21.** For $L^1$-almost every $t \geq 0$, $\nu_t^\ast$ has a first variation and

$$\delta \nu_t^\ast = h \nu_t^\ast.$$ 

That is, $h$ is the mean curvature of $\nu_t^\ast$.

Next, following [11], we demonstrate the semi-continuity of $\omega_t^\ast$ defined in (2.7.21). First, we introduce the bundle $B$ whose fiber over $x \in M$ is the space of linear maps $T_x M \to \mathbb{R}^2$, which we
identify with \((T_xM)^2\). On \(B\) we define the measure

\[
\tilde{\omega}_t^\varepsilon := \delta_{p_\varepsilon(x)} \frac{|\partial_t u_\varepsilon \cdot p_\varepsilon|}{|\log(\varepsilon)|} \, d\nu_g(x)
\]

where \(p_\varepsilon := \nabla u_\varepsilon / |\nabla u_\varepsilon|\). By perhaps passing to a further subsequence, we may assume that \(\tilde{\omega}_t^\varepsilon dt \to \omega_*\) as measures. We deduce from the decomposition provided by Theorem 2.6.1 and the Portmanteau Theorem that:

**Lemma 2.7.22.** The measure \(\tilde{\omega}_*\) decomposes as \(\tilde{\omega}_* = \tilde{\omega}_*^t \, dt\), and for \(L^1\)-almost every \(t \geq 0\)

\[
\tilde{\omega}_*^t = \Pi_{*,x}^t(p) |\partial_t \Phi_*|^2 d\nu_g(x) + \mathcal{M}_*^t,
\]

where \(\Pi_{*,x}^t\) is a probability measure on \((T_xM)^2\) with support on the unit ball and \(\mathcal{M}_*^t = \tilde{\omega}_*^t \Sigma_\mu^t\).

We borrow the following proposition, after adapting it to the case of a manifold, from section 6 of [5]

**Proposition 2.7.23.** For \(L^1\)-almost every \(t \geq 0\) and every smooth function \(\chi\) we have

\[
\int_B \chi(x) M^t_* (x, p) \geq \int_M |h|^2 \, d\nu_*.
\]

We are now ready to prove (5) of Theorem 2.1.1.

**Proof.** We begin by using Lemma 2.3.1, integrating over \([T_0, T_1]\), and dividing by \(|\log(\varepsilon)|\). Next we let \(\varepsilon \to 0^+\). Then by combining Lemma 2.7.16, Proposition 2.7.21, Lemma 2.7.22, and Theorem 2.6.1 we obtain

\[
\nu_*^{T_1} - \nu_*^{T_1} + \int_{M \times [T_0, T_1]} \chi \frac{|\nabla \Phi_*|^2}{2} \, d\nu_g(x) - \int_{M \times [T_0, T_1]} \chi \frac{|\nabla \Phi_*|^2}{2} \, d\nu_g(x) \quad (2.7.32)
\]

\[
\leq - \int_{M \times [T_0, T_1]} \chi |h|^2 d\nu_* + \int_{M \times [T_0, T_1]} (\nabla \chi \cdot P(h)) d\nu_*
\]

\[
- \int_{M \times [T_0, T_1]} \chi |\partial_t \Phi_*|^2 \, d\nu_g(x) dt + \int_{M \times [T_0, T_1]} (\nabla \chi \cdot \partial_t \Phi_* \nabla \Phi_*).
\]

Since \(\Phi_*\) solves the heat equation, we have the identity

\[
\int_{M \times \{T_1\}} \chi \frac{|\nabla \Phi_*|^2}{2} \, d\nu_g(x) - \int_{M \times \{T_0\}} \chi \frac{|\nabla \Phi_*|^2}{2} \, d\nu_g(x) \quad (2.7.33)
\]

\[
= \int_{M \times [T_0, T_1]} \chi |\partial_t \Phi_*|^2 \, d\nu_g(x) dt + \int_{M \times [T_0, T_1]} (\partial_t \Phi_* \cdot \nabla \Phi_* \nabla \chi) \, d\nu_g(x) dt.
\]
Combining (2.7.32) and (2.7.33) gives

\[ \nu^T - \nu^T_1 \leq -\int_{M \times [T_0, T_1]} |h|^2 d\nu_s + \int_{M \times [T_0, T_1]} \langle \nabla \chi, P(h) \rangle d\nu_s. \]

Applying Theorem 4.4 of [5], whose proof extends to the case of a compact Riemannian manifold, then this completes the proof of the theorem. \( \square \)
Chapter 3

Solutions of the Ginzburg-Landau equations with vorticity concentrating near a nondegenerate geodesic

3.1 Introduction

In this chapter we construct certain geometrically meaningful solutions of the Ginzburg-Landau equations

\[- \Delta u_\varepsilon + \frac{1}{\varepsilon^2} (|u_\varepsilon|^2 - 1) u_\varepsilon = 0\]  \hspace{1cm} (3.1.1)

for $u_\varepsilon : M \to \mathbb{C}$, where $(M,g)$ is a closed $n$-dimensional Riemannian manifold, with $n = 3$ in our main results. Such solutions are critical points of the Ginzburg-Landau functional

$$E_\varepsilon(u_\varepsilon):= \frac{1}{\pi |\log \varepsilon|} \int_M e_\varepsilon(u_\varepsilon) \, \text{vol}_g, \quad e_\varepsilon(u_\varepsilon):= \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{1}{4 \varepsilon^2} (|u_\varepsilon|^2 - 1)^2.$$

We note that the energy functional here and in the rest of this chapter is normalized and hence will differ in convention from the energy used in chapter 2. If $M$ is simply connected, then given a
sequence of solutions \((u_\varepsilon)\) of (3.1.1) satisfying the energy bound

\[ E_\varepsilon(u_\varepsilon) \leq C, \tag{3.1.2} \]

the rescaled energy density \(\frac{1}{|\log \varepsilon|^1} e_\varepsilon(u_\varepsilon)\) is known to concentrate as \(\varepsilon \to 0\), after possibly passing to a subsequence, around an \((n-2)\)-dimensional stationary varifold — a weak, measure-theoretic minimal surface. This is proved in an appendix in [51], following earlier results in simply-connected Euclidean domains, such as those in [9, 32, 36]. Similar but more complicated results hold when \(M\) is not simply connected; in this case, the limiting energy measure may have a diffuse part, but any concentrated part must again be an \((n-2)\)-dimensional stationary varifold.

In this chapter we address a sort of converse question:

*When can a given codimension 2 minimal surface be realized as the energy concentration set of a sequence of solutions of (3.1.1)?*

A first answer is provided by Gamma-convergence results, see [26, 1], that relate the Ginzburg-Landau functional and, roughly speaking, the \((n-2)\)-dimensional area (with multiplicity) of a limiting vorticity concentration set, where the vorticity associated to a wave function \(u\), denoted \(J u\), is the 2-form defined by

\[ J u := du^1 \wedge du^2, \quad \text{where } u = u^1 + iu^2 \text{ and } u^1, u^2 \text{ are real-valued.} \tag{3.1.3} \]

(We will also sometimes refer to \(J u\) as the Jacobian of \(u\).) These results imply as a general principle that one should be able to find solutions \(u_\varepsilon\) of (3.1.1) whose energy and vorticity concentrate around a *locally area-minimizing* minimal surface of codimension 2. In the Euclidean setting, specific instances of this general principle, for particular compatible choices of boundary conditions on the minimal surface and the solutions \(u_\varepsilon\) of (3.1.1), have been established in [1, 49, 43]. However, arguments based on Gamma-convergence are of limited use for capturing the behaviour of non-minimizing critical points.

The corresponding question is also very well-understood for minimal *hypersurfaces* and the Allen-Cahn equation, i.e. the scalar counterpart of (3.1.1), see for example [29, 47, 30, 17] among many others. Many of these results are based on gluing techniques and elliptic PDE arguments, which can be used to construct a great variety of solutions and establish detailed descriptions of them. These techniques seem to be hard to implement for the Ginzburg-Landau equation in 3 or more dimensions.
A particularly basic case in which our question remains open concerns the Ginzburg-Landau equation (3.1.1) on a smooth bounded domain $\Omega \subset \mathbb{R}^3$ containing an unstable geodesic with respect to natural boundary conditions, i.e. a line segment in $\Omega$ meeting $\partial \Omega$ orthogonally at both ends, admitting perturbations that decrease the arclength quadratically, and satisfying a natural nondegeneracy condition.

In this situation one would like to prove the existence of a sequence $(u_\varepsilon)$ of solutions of the Ginzburg-Landau equations, also with natural (Neumann) boundary conditions, whose energy and vorticity concentrate around the given line segment. Such solutions would satisfy

$$\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = L =: \text{the length of the geodesic.}$$

Partial progress toward this goal was achieved in [27], which develops a general framework for using Gamma-convergence to study convergence, not of critical points, but of critical values, then uses this framework to prove the existence of solutions of (3.1.1), in the situation described above, that satisfy (3.1.4), but without control over the limiting concentration set. An example in the same paper (Remark 4.5) shows that the general framework is too weak to characterize asymptotic behaviour of critical points — in this context, to determine where the energy and vorticity concentrate. For this, more detailed information about the sequence of solutions is needed.

The results of [27] were extended to the Riemannian setting in the Ph.D. thesis of Jeffrey Mesaric in [41] which, starting with a nondegenerate unstable closed geodesic on a closed, oriented 3 dimensional Riemannian manifold $(M,g)$, uses machinery from [27] to construct solutions to (3.1.1) satisfying (3.1.4). Again, this result does not establish whether the energy of the solutions concentrates along the geodesic.

In the main result of this chapter, we fill in this gap in the Riemannian case. Our main result is the following theorem.

**Theorem 3.1.1.** Let $(M,g)$ be a closed oriented 3-dimensional Riemannian manifold, and let $\gamma$ be a closed, embedded, nondegenerate geodesic of length $L$. Assume in addition that $\gamma = \partial S$ in the sense of Stokes’ Theorem for some 2-dimensional submanifold $S$ of $M$.

Then there exists $\varepsilon_1 > 0$ such that for every $0 < \varepsilon < \varepsilon_1$, there is a solution $u_\varepsilon$ of the Ginzburg-Landau equation (3.1.1) such that

$$\frac{1}{\pi} \int_M \varphi \wedge J u_\varepsilon \to \int_\gamma \varphi \quad \text{for every smooth 1-form } \varphi \text{ on } M$$
and
\[\frac{1}{\pi \log \epsilon} \int_M \phi e^\epsilon(u_\epsilon) \to \int_\gamma \phi dH^1\] for every \(\phi \in C^\infty(M)\) as \(\epsilon \to 0\), where \(H^k\) denotes \(k\)-dimensional Hausdorff measure.

In fact we will prove a slightly stronger result; see Theorem 3.5.1 for the full statement.

We briefly sketch the main ideas, not in the order in which they appear in the body of the chapter. Terminology such as “nondegenerate” and “stationary varifold” are defined in Section 3.2 below.

• In Section 4 we show that for any \(\delta > 0\), there exists \(\epsilon_0 > 0\) such that for \(0 < \epsilon < \epsilon_0\) and any \(\tau > 0\), one can find a solution \(u_\epsilon\) of the Ginzburg-Landau heat flow whose vorticity is initially concentrated near the geodesic \(\Gamma := \gamma([0, L])\), and such that
\[L - \delta \leq E_\epsilon(u_\epsilon(\cdot, t)) \leq L + \delta\] for all \(t \in [0, \tau]\).

See Proposition 3.4.1. This relies heavily on tools developed in the earlier papers [27, 41].

The main point of the proof of Theorem 3.1.1 is to strengthen this by showing that for such solutions, if \(\epsilon\) and \(\delta\) are small enough, the vorticity \(\frac{1}{\pi} Ju_\epsilon(\cdot, t)\) does not stray very far from \(\Gamma\) for any \(t \in [0, \tau]\).

• We carry this out in Section 3.5, using an argument by contradiction and passing to limits to obtain a stationary 1-dimensional varifold that is close, but not equal, to the varifold associated to \(\Gamma\). This argument requires, among other ingredients, an extension to the Riemannian setting of an important theorem of Bethuel, Orlandi, and Smets [11] which is stated in Theorem 2.1.1 and is proved in Chapter 2. The stationary varifold satisfies additional good properties, notably including lower density bounds.

• To obtain a contradiction, we prove that this stationary varifold cannot exist. This is the content of Proposition 3.3.1, which is a measure theoretic strengthening of the classical fact that a nondegenerate closed geodesic is isolated; it is the only closed geodesic in a tubular neighborhood of itself. The proof relies, among other ingredients, on results from [2] about the structure of stationary 1-dimensional varifolds on Riemannian manifolds.

We believe that something like Theorem 3.1.1 should be valid in much greater generality, including on higher-dimensional manifolds and on smooth, bounded subsets of \(\mathbb{R}^n, n \geq 3\), with natural
boundary conditions both for the geodesic $\Gamma$ (or codimension 2 minimal surface, for $n \geq 4$) and the Ginzburg-Landau equation. Our proof does not adapt in a straightforward way to either of these settings.

- Our strategy requires a sufficiently good version of Theorem 2.1.1. On a bounded set $\Omega \subset \mathbb{R}^n$, even for $n = 3$, such a result is not known. If $\Omega$ is convex, a result of this type for the scalar parabolic Allen-Cahn equation was proved several years ago in [42]. A similar strategy could probably be pursued for the Ginzburg-Landau heat flow, but convexity is not a natural assumption for any analog of Theorem 3.1.1.

- Our reliance on results from [2] about stationary 1-dimensional varifolds would seriously complicate any effort to adapt our argument to dimensions $n \geq 4$, where one would confront stationary varifolds of dimension $n - 2 \geq 2$.

3.2 Background and notation

3.2.1 Geometric notions regarding a non-degenerate geodesic

Throughout this chapter we use $M$ or $(M, g)$ to denote a closed oriented three dimensional Riemannian manifold where “closed” means compact and without boundary. We let $TM$ be the bundle over $M$ whose fiber $T_p M$ at $p \in M$ is the tangent space to $M$ at $p$. We use the notation $(\cdot, \cdot)_g$ to denote the inner product on $TM$ given by $g$. We also use $|\cdot|_g$ to denote the corresponding norm, where we will omit mention of $g$ when no confusion will arise. We write $\text{vol}_g$ to denote the Riemannian volume form associated to the metric $g$.

We will write $r_0 > 0$ to be a number, fixed throughout this chapter, such that

$$r_0 < \frac{1}{2} \text{ (injectivity radius of } M\text{).} \quad (3.2.1)$$

Throughout this chapter, a central role will be played by a geodesic $\gamma$ that we take to be parametrized by arclength. That is, we will assume the existence of an injective map $\gamma : \mathbb{R}/L\mathbb{Z} \to M$ whose range consists of a simple closed curve $\Gamma :=\{\gamma(t) : t \in \mathbb{R}/L\mathbb{Z}\}$ of length $L$ such that

$$|\gamma'| = 1, \quad \nabla_{\gamma'} \gamma = 0 \quad \text{everywhere in } \mathbb{R}/L\mathbb{Z}. \quad (3.2.2)$$
We will insist that this curve $\Gamma$ bounds an orientable smooth surface $S_\Gamma \subset M$, i.e.

$$\Gamma = \partial S_\Gamma. \quad (3.2.3)$$

We introduce here the notation

$$d_\Gamma(x):= \text{dist}(x, \Gamma):= \inf \left\{ \int_0^1 |\lambda'(t)| dt : \lambda \in \text{Lip}([0,1]; M), \lambda(0) = x, \lambda(1) \in \Gamma \right\} \quad (3.2.4)$$

as well as

$$K_r := \{ x \in M : d_\Gamma(x) < r \}$$

for a neighborhood of $\Gamma$.

For $t \in \mathbb{R}/L\mathbb{Z}$, we then let

$$N_\gamma(t) \Gamma := \{ u \in T_\gamma(t)M : (u, \gamma'(t))_g = 0 \}.$$

A normal vector field along $\gamma$ is a map $\xi : \mathbb{R}/L\mathbb{Z} \to TM$ such that $\xi(t) \in N_\gamma(t) \Gamma$ for every $t$. We also introduce the coordinates $\psi : B_r(0) \times (0, L) \to K_r$ defined by

$$\psi(y,t) := \exp_{\gamma(t)} \left( \sum_{i=1}^2 y^i \Xi_i(t) \right) \quad (3.2.5)$$

where $\Xi_1, \Xi_2$ are fixed normal vector fields which are orthogonal for each $t \in (0, L)$. We note for $r < r_0$, this map is smoothly invertible. For future use, we will use the notation $\psi^{-1}(x) = (y(x), \tau(x)) \in B_r(0) \times (0, L)$, so that for $x \in K_r$,

$$\psi(y,t) = x \iff y(x) = y \quad \text{and} \quad \tau(x) = t. \quad (3.2.6)$$

We observe that the mapping $\tau$ simply assigns to an $x \in K_r$ the parameter value $t$ corresponding to the closest point on $\Gamma$ to $x$.

Given two normal vector fields along $\gamma$, denoted by $\xi, \tilde{\xi}$, we can define the $L^2$ inner product in the natural way:

$$\langle \xi, \tilde{\xi} \rangle_{L^2} := \int_{\mathbb{R}/L\mathbb{Z}} (\xi(t), \tilde{\xi}(t))_g \, dt.$$

We will write $L^2(\Gamma)$ to denote the space of square integrable normal vector fields, a Hilbert space with the above inner product.
For $\xi \in L^2(NT)$, we will use the notation

$$\gamma(t) := \exp_{\gamma(t)} \xi(t), \quad (3.2.7)$$

where $\exp$ denotes the exponential map.

We next recall the Jacobi operator $L_J$ which acts on smooth normal vector fields $\xi$ along $\gamma$, and is defined by

$$L_J \xi := -\xi'' + R(\xi, \gamma')\gamma', \quad (3.2.8)$$

where $R$ denotes the curvature tensor. We say that a geodesic is nondegenerate if 0 is not an eigenvalue of $L_J$.

With this notion in hand, we add another crucial hypothesis on the geodesic by assuming henceforth that

$$\gamma: \mathbb{R}/LZ \to M$$

is a simple, closed, nondegenerate geodesic with $|\gamma'| \equiv 1$. \quad (3.2.9)

One says that $\gamma$ has finite index if the total number (algebraic multiplicity) of negative eigenvalues of $L_J$ is finite. Since $M$ is closed, this is always true, as a consequence of standard Sturm-Liouville theory. Our standing assumption (3.2.9) that $\gamma$ is nondegenerate then implies there exists some $\ell \geq 0$ and a nondecreasing sequence of eigenvalues

$$\lambda_1 \leq \ldots \leq \lambda_\ell < 0 < \lambda_{\ell+1} \leq \ldots \quad (3.2.10)$$

of $L_J$, together with an associated orthonormal basis of $L^2(NT)$ consisting of (smooth) eigensections $\{\xi_j\}_{j=1}^\infty$. We will always assume that $\ell > 0$, since otherwise the results presented here admit much simpler proofs. We define

$$H_- := \text{span}\{\xi_1, \ldots, \xi_\ell\}, \quad H_+ := H_-^\perp. \quad (3.2.11)$$

We will say that $\xi$ is Lipschitz, and we will write $\xi \in Lip$, if $\gamma_\xi$ is Lipschitz continuous. It is clear that

$$H_- (r_0) := \{\xi \in H_- : \|\xi\|_{L^\infty} \leq r_0\} \subset Lip$$

for $r_0$ and $H_-$ from (3.2.1) and (3.2.11) respectively.

The standard fact that the Jacobi operator, cf. (3.2.8), is the second variation of arclength, together with the definition (3.2.11) of $H_-$, implies that there exist $c_0, r_0 > 0$ such that for all...
∥ξ∥_{L^\infty} \leq r_0$$ one has
\[
\int_{\mathbb{R} / L^2} |\gamma'_\xi(t)| \, dt \leq L - c_0 \|\xi\|^2_{L^2} \quad \text{if } \xi \in H_-(r_0)
\]
\[
\int_{\mathbb{R} / L^2} |\gamma'_\xi(t)| \, dt \geq L + c_0 \|\xi\|^2_{L^2} \quad \text{if } \xi \in H_+ \cap \text{Lip}.
\] (3.2.12)

### 3.2.2 Forms and currents

We denote, for $$k \in \mathbb{N} \cup \{0\}$$, the space of smooth $$k$$-forms on $$M$$ by
\[
\mathcal{D}^k(M) := \{ \phi \in C^\infty(M; \wedge^k M) \}
\]
where $$\wedge^k M$$ is an abbreviated notation for $$\wedge^k T^* M$$. We denote the dual space of $$\mathcal{D}^k(M)$$, for $$k \in \mathbb{N} \cup \{0\}$$, by
\[
\mathcal{D}_k(M) := \{ k\text{-currents on } M \}.
\]
We refer to the elements of $$\mathcal{D}_k(M)$$ as $$k$$-currents. For a $$k$$-current $$T$$, we define

the mass of $$T = M(T) := \sup \{ T(\phi) : \|\phi\|_\infty \leq 1 \} \in [0, +\infty]$$.

We will be most interested in 1-currents. A basic class of examples consists of 1-currents we shall write as $$T_\lambda$$ whose action on $$\phi \in \mathcal{D}^1(M)$$ takes the form
\[
T_\lambda(\phi) := \int_\lambda \phi, \quad \text{where } \lambda : (a, b) \to M \text{ is a Lipschitz curve.}
\] (3.2.13)

We will say a 1-current is integer multiplicity rectifiable if it can be written as a finite or countable sum of currents of the form (3.2.13). We will write
\[
\mathcal{R}_1(M) := \{ T \in \mathcal{D}_1(M) : M(T) < \infty, \ T \text{ is integer multiplicity rectifiable } \}.
\]

For a 1-current $$J$$, we write $$\|J\|$$ to denote the associated total variation measure, defined through its action on continuous, nonnegative functions $$f : M \to \mathbb{R}$$ via
\[
\int f \, d\|J\| := \sup \{ J(\phi) : \phi \in \mathcal{D}^1(M), \ |\phi|_g \leq f \}.
\] (3.2.14)
For a \( k \)-current \( S \), the boundary of \( S \) is the \((k-1)\)-current \( \partial S \) defined by
\[
\partial S(\phi) := S(d\phi), \quad \text{for all } \phi \in \mathcal{D}^{k-1}(M).
\]

We define
\[
\mathcal{F}_1'(M) := \{ T \in \mathcal{D}_1(M) : T = \partial S \text{ for some } S \in \mathcal{D}_2(M), M(S) < \infty \}.
\]
and for \( T \in \mathcal{F}_1'(M) \), we will write
\[
\|T\|_\mathcal{F} := \inf\{ M(S) : T = \partial S \}.
\]

We also define
\[
\mathcal{R}_1'(M) := \mathcal{R}_1(M) \cap \mathcal{F}_1'(M).
\]

We note that the 1-current \( T_{\gamma} \) associated with the geodesic \( \gamma \) via (3.2.13), in particular, bounds a finite mass 2-current; that is,
\[
T_{\gamma} \in \mathcal{R}_1'(M), \quad (3.2.15)
\]
in light of the assumption (3.2.3).

Lastly, we will at times wish to identify the Jacobian (i.e. vorticity) of a map \( u \in H^1(M; \mathbb{C}) \) with an element of \( \mathcal{D}_1(M) \), which we denote \( \star J(u) \), and which is defined through its action on 1-forms \( \phi \) by
\[
\star J(u)(\phi) = \int \phi \wedge J(u), \quad (3.2.16)
\]
where \( J(u) = du^{(1)} \wedge du^{(2)} \) for \( u = u^{(1)} + iu^{(2)} \) where \( u^{(1)}, u^{(2)} \) are real-valued.

### 3.2.3 Gamma-limit of the Ginzburg-Landau functional

Below we state the version we will need of standard Gamma-convergence results for the Ginzburg-Landau functional.

We first fix the notation \( V = \mathcal{F}_1'(M) \), with the flat norm \( \|v\|_V := \|v\|_\mathcal{F} \). We also define the functional
\[
E_V(T) := \begin{cases} 
M(T) & \text{if } T \in \mathcal{R}_1'(M) \\
+\infty & \text{if not.}
\end{cases} \quad (3.2.17)
\]
Thus \( E_V \) is an extension to \( V \) of the “arclength functional” in the sense that if \( \lambda: (a,b) \to M \) is an injective Lipschitz continuous curve and \( T_\lambda \) is the corresponding current, then \( E_V(T_\lambda) = \)
The following result is deduced in [41], Theorem 5.1 from corresponding Euclidean results, cf. [1, 26].

**Theorem 3.2.1.** Let $(M, g)$ be a closed 3-dimensional Riemannian manifold.

1. Let $(u_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ be a sequence in $H^1(M; \mathbb{C})$. If there exists $C > 0$ such that $E_{\varepsilon}(u_\varepsilon) \leq C$ for all $\varepsilon \in (0, \varepsilon_0)$, then $(\frac{1}{\varepsilon} \ast J u_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ is precompact in $V$, and any limit as $\varepsilon \to 0$ belongs to $\mathcal{R}'_1(M)$.

2. Let $(u_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ be a sequence in $H^1(M; \mathbb{C})$. If $T \in V$ and
\[
\|\frac{1}{\varepsilon} \ast J u_\varepsilon - T\|_F \to 0 \quad \text{as} \quad \varepsilon \to 0,
\]
then $\lim \inf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) \geq E_V(T)$.

3. For any $T \in V$, there exists a sequence $(u_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ in $H^1(M; \mathbb{C})$ such that $\|\frac{1}{\varepsilon} \ast J u_\varepsilon - T\|_F \to 0$ and $\lim \sup_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) \leq E_V(T)$.

The geodesic $\gamma$ is a saddle point of the arclength with respect to smooth perturbations, as reflected in (3.2.12). For use in combination with Theorem 3.2.1, one needs to identify a sense in which the corresponding current $T_\gamma$ is a saddle point of $E_V$. We defer a discussion of this and related issues to Section 3.4.

### 3.2.4 Varifolds

We briefly recall the definition of a rectifiable 1-varifold and introduce some notation that will be used later. After doing this we will introduce the definition of a general 1-varifold. We note that the general definition will only be used in the proof of Proposition 3.3.1. For general varifolds we will follow [2] with some terminology from [50].

For any 1-dimensional rectifiable set $\Sigma$, basic theory (see for example [50], Lemma 11.1) shows that there exists a countable family of $C^1$ curves $(\Lambda_j)_{j \in \mathbb{N}}$ in $M$ such that

\[\Sigma \subset N_0 \cup (\bigcup_{j \in \mathbb{N}} \Lambda_j), \quad \text{and} \quad \mathcal{H}^1(N_0) = 0,\]

and every point in $\Sigma \setminus N_0$ is contained in exactly one $\Lambda_j$. We then define, for $x \in \Sigma \setminus N_0$

\[\text{ap}T_x \Sigma = T_x \Lambda_j \text{ for the unique } j \text{ such that } x \in \Lambda_j.\]

We will write $\tau_\Sigma(x)$ to denote a unit vector in $\text{ap}T_x \Sigma$.

First, we recall that if $\mathcal{S}$ is a countably 1-rectifiable, $\mathcal{H}^1$ measurable subset of $M$ and $\Theta : \mathcal{S} \to (0, \infty)$ is a locally $\mathcal{H}^1$-integrable function on $\mathcal{S}$ then we can use the pair $(\mathcal{S}, \Theta)$ to form the measure

\[\Sigma \subset N_0 \cup \bigcup_{j \in \mathbb{N}} \Lambda_j, \quad \text{and} \quad \mathcal{H}^1(N_0) = 0,\]

and every point in $\Sigma \setminus N_0$ is contained in exactly one $\Lambda_j$. We then define, for $x \in \Sigma \setminus N_0$

\[\text{ap}T_x \Sigma = T_x \Lambda_j \text{ for the unique } j \text{ such that } x \in \Lambda_j.\]

We will write $\tau_\Sigma(x)$ to denote a unit vector in $\text{ap}T_x \Sigma$.
\(\mathcal{H}^1 \sqcup \Theta\), where we have extended \(\Theta\) to be zero outside of \(S\). We refer to such a measure as a \textit{rectifiable 1-varifold}. We also refer to the function \(\Theta\) as the \textit{multiplicity function} of this rectifiable 1-varifold and, at times, we will write \(\Theta_S\) to emphasize the association. We will also sometimes use the alternate notation \(\Theta \mathcal{H}^1 \sqcup S\) for \(\mathcal{H}^1 \sqcup \Theta_S\). If \(\Theta\) happens to be integer-valued \(\mathcal{H}^1\)-almost everywhere then we will say this rectifiable varifold is of \textit{integer multiplicity}. Finally, if there is a \(\lambda > 0\) such that \(\Theta \geq \lambda\) at \(\mathcal{H}^1\)-almost every point then we say that the rectifiable varifold has \textit{density bounded below}. A particular example of an integer multiplicity rectifiable 1-varifold that we will be interested in will be integration over a countable collection of geodesics.

Next, for a smooth Riemannian manifold, \(M\), we let \(PM\) be the bundle whose fiber \(P_a M\) at \(a \in M\) consists of the lines through the origin in \(T_a M\). If \(x \in M\) and \(\xi\) is a unit vector in \(T_x M\), we will sometimes abuse notation slightly and write \((x, \xi)\) to denote the element of \(PM\)

\[
(x, \xi) \sim \{s\xi : s \in \mathbb{R}\} \subset T_x M. \tag{3.2.18}
\]

Thus \((x, \xi)\) and \((x, -\xi)\) correspond to the same element of \(PM\). Suppose that \(\eta\) is a smooth function on \(M\). When representing points in \(PM\) as described above, a mapping such as \((x, \xi) \in PM \mapsto |\nabla \xi \eta(x)|^2\) is well-defined as a function \(PM \to \mathbb{R}\), since it is independent of the choice of sign for the unit vector \(\xi\).

We let \(\pi: PM \to M\) be the bundle projection.

We refer to a measure \(V \in \mathcal{M}(PM)\) as a \textit{1-varifold}.

Observe that to a rectifiable 1-varifold \(V = \mathcal{H}^1 \sqcup \Theta_S\), we may associate a 1-varifold \(\mathcal{V}\) defined by

\[
\mathcal{V}(A) := V(\{a \in M : apT_a S \in A\}) = \int_{\{a \in S : apT_a S \in A\}} \Theta_S(a) d\mathcal{H}^1. \tag{3.2.19}
\]

Roughly speaking, the difference between a rectifiable 1-varifold and the associated general 1-varifold is that the latter explicitly records information about the approximate tangent spaces to the set \(S\) on which the former lives.

### 3.2.5 Definitions: first variation, stationarity, Brakke flow

We remark that in light of (3.2.2), of course it follows from an integration by parts that one can associate a multiplicity-one stationary varifold with the geodesic \(\gamma\). Properties of stationary varifolds will be recalled later as needed.

For \((\nu^t)_{t>0}\) a Brakke flow in \(M\) satisfying (1.2.6) at each \(t > 0\), it is an immediate consequence
of (1.2.11) that
\[ t \mapsto \nu^t(M) \text{ is nonincreasing.} \] (3.2.20)

Another simple fact we will need is the following.

**Lemma 3.2.2.** If there exist numbers \(0 \leq a < b\) such that
\[ t \mapsto \nu^t(M) \text{ is constant for } a < t < b \] (3.2.21)
then
\[ \exists \text{ a stationary varifold } \nu_* \text{ in } M \text{ such that } \nu^t = \nu_* \text{ for all } a < t < b. \] (3.2.22)

**Proof.** Clearly, if (3.2.21) holds, then by taking \(\chi = 1\) in (1.2.11), we find that \(H = 0\) a.e. in \(\Sigma^t\) for every \(t \in (a,b)\). It follows that \(\nu^t\) is stationary for such \(t\). It is also easy to see that \(t \mapsto \nu^t\) is constant for \(t \in (a,b)\). Indeed, given any nonnegative \(\chi_1 \in C^2(M)\), choose \(\chi_2 \in C^2(M)\) such that \(\chi_1 + \chi_2\) is constant on \(M\). Then it follows from (3.2.21) that
\[ \int_M (\chi_1 + \chi_2) d\nu^t = \int_M \chi_1 d\nu^t + \int_M \chi_2 d\nu^t = c\nu^t(M) \text{ is constant for } t \in (a,b). \]

On the other hand, since \(H = 0\), it follows from (1.2.11) that
\[ \int_M \chi_j d\nu^t \text{ is nonincreasing for } j = 1,2, \text{ for } t \in (a,b). \]

These together imply that \(t \mapsto \int_M \chi_j d\nu^t\) is constant for \(j = 1,2\). Since this holds for all nonnegative \(\chi_1 \in C^2(M)\), it easily follows that \(\nu^t\) does not depend on \(t \in (a,b)\), proving (3.2.22). \(\square\)

We will make heavy use of results from a paper of Allard and Almgren [2] on stationary 1-dimensional varifolds with positive density in a Riemannian manifold. Among other results, they prove that a stationary 1-d varifold with density bounded away from 0 is supported on a finite or countable union of geodesic segments terminating at singular points. From these singular points multiple segments emanate, with a balance condition on the weighted sum, at each singular point, of the tangent vectors generating the geodesics that meet there. Other results from [2] will be cited as the need arises.
3.3 A non-existence result for stationary 1-varifolds near a non-degenerate geodesic

The proof of our main result hinges crucially on showing there is no stationary varifold sitting over a 1-current that is nearby $T_\gamma$, the 1-current associated with the non-degenerate geodesic $\gamma$. While the non-degeneracy assumption (3.2.9) easily precludes the existence of another nearby smooth geodesic, it is the need to rule out proximity in the weaker sense of (3.3.2), (3.3.3) below and within the larger class of varifolds that makes the result below much more challenging to establish.

Proposition 3.3.1. Let $T_\gamma$ be the 1-current in $M$ corresponding to integration over the non-degenerate geodesic $\gamma$, and let $\eta > 0$ be given. Then there exists $r_0 > 0$ depending on $M, \gamma,$ and $\eta$, such that for $0 < r < r_0$ there is no stationary 1-dimensional rectifiable varifold $V_*$ and 1-current $J_1 \in R_1 \cap F_1(M)$ satisfying the conditions

$$V_* = \Theta_*(x)\mathcal{H}^1 \sqcap \Sigma_*, \quad (3.3.1)$$

for $\Sigma_*$ 1-rectifiable and $\Theta_* \geq \eta > 0 \mathcal{H}^1$ a.e. in $\Sigma_*$,

$$\|J_1 - T_\gamma\|_{\mathcal{F}} = r, \quad (3.3.2)$$

and

$$V_* \geq \|J_1\|, \quad V_*(M) \leq L. \quad (3.3.3)$$

The starting point of the proof is provided by the following lemma, established by Mesarić [41].

Lemma 3.3.2. For $T_\gamma$ as above, let $J_1 \in R_1 \cap F_1(M)$ be a current satisfying (3.3.2), and such that

$$\partial J_1 = 0 \quad \text{and} \quad M(J_1) \leq L.$$

Then provided $r$ is taken sufficiently small, there is a 1-current $J_1^* \in R_1(M)$ such that the support of $J_1^*$, denoted by $\Gamma^*$, consists of a single Lipschitz curve with no boundary satisfying

$$\Gamma^* \subset K_2 \varphi \cap \text{spt}(J_1), \quad (3.3.4)$$

and

$$\Gamma^* \cap \tau^{-1}(t) \neq \emptyset \text{ for all } t \in \mathbb{R}/L\mathbb{Z}, \quad (3.3.5)$$
(cf. (3.2.6) for the definition of $\tau$).

In addition,

$$M(J_1 - J_1^*) = M(J_1) - M(J_1^*), \quad (3.3.6)$$

there exists a constant $C_1 > 0$ such that

$$M(J_1^*) \geq L - C_1 \sqrt{r}, \quad (3.3.7)$$

and

$$J_1^* - T_\gamma = \partial S^* \text{ for some 2-current } S^* \text{ with}$$

$$\text{spt}(S^*) \subset K_{\sqrt{r}} \text{ and } M(S^*) < \infty. \quad (3.3.8)$$

This is demonstrated in Lemma 4.4 and comments following Lemma 4.6 of [41]. The proof is an adaptation to the Riemannian setting of arguments from [27], Lemma 5.5. The idea is that (3.3.2) and the definition of the flat norm imply that a large set of transverse slices to $\Gamma$ must intersect $J_1$ at exactly one point, and this point must be close to $\Gamma$. Behavior of $J_1$ on other slices is constrained by the assumption that $\partial J_1 = 0$ and the mass bound. The proof also uses Federer’s decomposition of integral 1-currents, [20], 4.2.25.

**Proof of Proposition 3.3.1**

**Step 1:** First we show that the rectifiable varifold $V_\ast$ does not have any mass outside of $K_{\sqrt{r}}$. The idea is that if this fails, then the monotonicity formula and (3.3.7) would contradict the assumption $M(V_\ast) \leq L$. This argument relies crucially on the uniform lower density bound for $V_\ast$.

In view of the lower density bound $\Theta_\ast \geq \eta$, $V_\ast$ almost everywhere, it follows from a Riemannian version of the Monotonicity Formula (established with different notation in [2] in item (5) of Theorem 1 on pg. 87) that there exists $r_{con}(M) > 0$ such that for every $0 < s < r_{con}$,

$$\eta \leq \frac{1}{2s} \int_{B_s(p)} \text{Hess}_g \left( \frac{\rho^2(x)}{2} \right) (v,v) dV_\ast \quad (3.3.9)$$

for $V_\ast$-almost every $p \in M$, where $v$ is a unit vector in $\text{ap} \mathcal{T}_{\ast} \Sigma_\ast$. (Clearly the value of $\text{Hess}_g \left( \frac{\rho^2(x)}{2} \right) (v,v)$ does not depend on which unit tangent is chosen.)
It follows from (1.2.5) that
\[ 2s\eta \leq (1 + \mu s^2)V_\ast (B_s(p)) \]
for all \( 0 < s < \frac{1}{2} \min \left\{ r_0, r_{\text{con}}, \frac{\pi}{2\sqrt{\mu}} \right\} \) and \( V_\ast \)-almost every \( p \in M \). We conclude that for all \( 0 < s < \frac{1}{2} \min \left\{ r_0, r_{\text{con}}, \frac{\pi}{2\sqrt{\mu}}, 1 \right\} \) we have
\[ V_\ast (B_s(p)) \geq \frac{2s\eta}{1 + \mu}, \quad \text{for } V_\ast \text{ a.e. } p \in M. \] (3.3.10)

We now use this to prove that if \( r \) is chosen sufficiently small, then
\[ V_\ast (M \setminus K_{\sqrt{r}}) = 0. \] (3.3.11)

To verify (3.3.11), suppose to the contrary; then \( V_\ast (M \setminus K_{\sqrt{r}}) > 0 \) and so there is a point \( p \) of \( \text{spt}(V_\ast) \) in \( M \setminus K_{\sqrt{r}} \) for which (3.3.10) holds. By (3.3.10) we have that
\[ V_\ast \left( B_{\frac{1}{2}\sqrt{r}}(p) \right) \geq \frac{\eta}{1 + \mu} \sqrt{r}, \quad \text{if } r < \min \left\{ r_{\text{con}}, r_0, \frac{\pi}{2\sqrt{\mu}}, 1 \right\}. \]

By shrinking \( r_0 \) if necessary, we may assume that \( B_{\frac{1}{2}\sqrt{r}}(p) \cap K_{\sqrt{r}} = \emptyset \). Hence, appealing to (3.3.7), we find that
\[ L \geq V_\ast (M) \geq V_\ast (K_{\sqrt{r}}) + V_\ast \left( B_{\frac{1}{2}\sqrt{r}}(p) \right) > L - C_1 \sqrt{r} + \frac{\eta}{1 + \mu} \sqrt{r}. \]

Choosing \( r \) smaller if necessary, depending on \( C_1, \eta, \mu \), this yields a contradiction. We conclude that (3.3.11) holds. Since \( \|J_1\| \leq V_\ast \), we remark that \( J_1 \) is also supported in \( K_{\sqrt{r}} \).

**Step 2:** Next we demonstrate that
\[ |\nabla \tau(x)| = 1 + O(\sqrt{r}), \quad [\text{Hess}_g(\tau)(x)](v,v) = O(\sqrt{r}) \] (3.3.12)
where \( v \in T_x M \) is a unit vector, \( x \in K_{\sqrt{r}} \), and \( \tau \) is the mapping defined in (3.2.6).

We prove only the statement about the Hessian, as the gradient estimate follows by similar arguments. In coordinates introduced by \( \psi: B_{\sqrt{r}}(0) \times (0, L) \to K_{\sqrt{r}} \) as defined in (3.2.5), we can
write $\tau$ as

$$\tilde{\tau}(y, t) = t$$

where $\tilde{\tau} = \tau \circ \psi$. These are what are called Fermi coordinates, and a basic fact, proved for example in Section V of [39], is that the vectors $\Xi_1, \Xi_2$ in (3.2.5) can be chosen so that all Christoffel symbols vanish along the central geodesic, that is, when $y = 0$:

$$\Gamma^k_{ij}(0, t) = 0 \quad \text{for } i, j, k = 1, \ldots, 3$$

where $x^k = y^k$ for $k = 1, 2$, and $x^3 = t$. In general, the expression for the Hessian in coordinates is

$$\text{Hess}_g(\tau)(\psi(0, t)) = \sum_{i=1}^{3} \sum_{j=1}^{3} \left( \frac{\partial^2 \tilde{\tau}}{\partial x^i \partial x^j}(0, t) - \sum_{k=1}^{3} \Gamma^k_{ij}(0, t) \frac{\partial \tilde{\tau}}{\partial x^k}(0, t) \right) dx^i \otimes dx^j,$$

see for example [28], Definition 4.3.5. By combining these, we readily deduce that

$$\text{Hess}_g(\tau)(\psi(0, t)) = 0$$

and thus that $\text{Hess}_g(\tau)(\psi(y, t))_{ij} = O_t(|y|)$ for $1 \leq i, j \leq 3$. The Hessian estimate in (3.3.12) follows directly.

**Step 3:** Let $\mathcal{V}_* \subset PM$ be the 1-varifold associated as in (3.2.19) to the rectifiable 1-varifold $V_*$. We next demonstrate that for each $\delta > 0$ there is $r_1 > 0$ such that if $0 < r < r_1$ in (3.3.2) and hence in (3.3.11), then

$$\mathcal{V}_* \left( \{(x, \xi) \in PM : |\nabla_\xi \tau(x)|^2 \leq (1 - \delta)^2 \} \right) < \delta,$$

where we recall our convention that a generic element of $PM$ — that is, a line in $T_xM$ for some $x \in M$ — is represented by a pair $(x, \xi)$, where $\xi$ is a unit vector in $T_xM$ spanning the given line, see (3.2.18). This will establish that most tangent vectors to the support of $V_*$ are, according to the measure $\mathcal{V}_*$, nearly parallel to $\nabla \tau$.

We suppose toward a contradiction that there is a $\delta > 0$, a sequence $(r_k)_{k \in \mathbb{N}}$ tending to 0 from the right, and a sequence of stationary rectifiable varifolds $(V_k)_{k \in \mathbb{N}}$ on $M$ satisfying the hypotheses of Proposition 3.3.1 with $r$ replaced by $r_k$ in (3.3.2), and such that the associated 1-varifolds $\mathcal{V}_k$
satisfy
\[ V_k \left( \{ (x, \xi) \in PM : |\nabla_\xi \tau(x)|^2 \leq (1 - \delta)^2 \} \right) \geq \delta, \quad (3.3.14) \]
for all \( k \in \mathbb{N} \). In particular we have
\[ V_k(M) \leq L, \quad \text{spt}(V_k) \subset K \sqrt[4]{r_\tau}, \quad \text{and} \ \Theta_{V_k}(x) \geq \eta \text{ for } x \in \text{spt}(V_k). \quad (3.3.15) \]

Since \((V_k)_{k \in \mathbb{N}}\) is a sequence of stationary rectifiable varifolds we may combine (3.3.15) with the compactness result of Theorem 42.7, pg. 247 of [50] to conclude that there is a subsequence \((V_{k_j})_{j \in \mathbb{N}}\) and a rectifiable varifold \(V\) with associated multiplicity \(\Theta_V\) such that

1. \(V_{k_j} \rightharpoonup V\) weakly as measures
2. \(\Theta_V(x) \geq \eta\) on \(\text{spt}(V)\)
3. \(\delta V(W) \leq \liminf_{j \to \infty} \delta V_{k_j}(W)\) for all \(W \subset \subset M\)

It follows from (3) and the fact that each \(V_{k_j}\) is stationary that \(V\) is also stationary. Then from (1) and the fact that \(\text{spt}(V_{k_j}) \subset K \sqrt[4]{r_\tau_{k_j}}\) we conclude that \(\text{spt}(V) \subset \Gamma\). Next, we observe that, due to (3.3.5), each \(V_{k_j}\) has support that contains a closed curve that meets every level set of \(\tau\). Hence, \(\text{spt}(V) = \Gamma\) as a result of (1). Observe that since \(V\) is a stationary varifold with density bounded below and \(\text{spt}(V) = \Gamma\), then by the Theorem on page 89 of [2] we have that \(V\) is simply a constant multiplicity multiple of the stationary varifold \(V_\Gamma\) associated with the geodesic \(\gamma\). Applying the weak convergence (1) to (3.3.14), however, we see that
\[ V \left( \{ (x, \xi) \in PM : |\nabla_\xi \tau(x)|^2 \leq (1 - \delta)^2 \} \right) \geq \delta, \]
an impossibility given that all tangent vectors \(\xi\) along \(\Gamma\) coincide with \(\pm \nabla \tau\).

**Step 4:** Next we introduce three sets corresponding to slices normal to the central geodesic that are in some sense bad. We will argue that two of them correspond to sets of \(t\)-values of measure zero while the third is of small measure.

We introduce the first such set, \(B_1\), through the function \(h : (0, L) \to \mathbb{R}\) given by
\[ h(t) = \int_{\pi^{-1}(\{0 \leq \tau \leq t\} \cap K \sqrt[4]{r_\tau})} |\nabla_\xi \tau(x)|^2 \, dV_\gamma(x, \xi), \]
with \(B_1\) defined by
\[ B_1 := \{ a \in (0, L) : h \text{ is not differentiable at } a \}. \quad (3.3.16) \]
Since \( h \) is non-decreasing, it is differentiable \( L^1 \)-almost everywhere and consequently \( L^1(B_1) = 0 \).

Now we recall that the singular set \( S_{V_*} \), as defined in [2], is the set of points of \( M \) near which \( \Theta_{V_*} \), restricted to \( \text{spt}(V_*) \), is not constant. Then we introduce the set \( B_2 \) as the set of slices meeting the singular set:

\[
B_2 := \{ t \in (0, L) : \{ \tau = t \} \cap S_{V_*} \neq \emptyset \} = \{ \tau(x) : x \in S_{V_*} \cap K_{\sqrt{r}} \}.
\]

We claim that \( L^1(B_2) = 0 \) as well.

To this end, we note that in the remark following the theorem on page 89 of [2], it is stated that

\[
V_* (S_{V_*}) = 0,
\]

and so by (4) and (3.3.17) we have that

\[
0 = \int_{S_{V_*}} \Theta(x) d\mathcal{H}^1(x) \geq \eta \mathcal{H}^1(S_{V_*}).
\]

We conclude that

\[
\mathcal{H}^1(S_{V_*}) = 0.
\]

Since \( B_2 \) is the image of a subset of \( S_{V_*} \) by the Lipschitz map \( \tau : K_{\sqrt{r}} \to (0, L) \), it follows that \( L^1(B_2) = 0 \) as claimed.

The final ‘bad’ set of slices is \( B_3 \) defined by

\[
B_3 := \{ t \in (0, L) : \exists (x, \xi) \in \text{spt}(V_*) \text{ s.t. } |\nabla_{\xi} \tau(x)|^2 < (1 - \delta)^2, \tau(x) = t \}.
\]

Replacing the role of the equality (3.3.17) by the inequality (3.3.13) in the argument above, the same line of reasoning goes to show that there is a constant \( C_2 > 0 \) such that

\[
L^1(B_3) < \frac{C_2 \delta}{\eta},
\]

where \( r \) is chosen sufficiently small.

**Step 5:** We now use the results obtained in Steps 1 and 2 to show that, for \( r \) and \( \delta \) chosen
sufficiently small, and \( a, b \in (0, L) \setminus B_1 \) we have

\[ h'(b) = h'(a) + O(\sqrt{r}) \]

where \( h \) is as defined in Step 4.

For \( s \) small and positive, we define the function

\[
H_{a,b,s}(t) = \begin{cases} 
1 & \text{if } a + s < t < b - s \\
0 & \text{if } 0 < t < a \text{ or } b < t < L \\
\frac{t-a}{s} & \text{if } a \leq t \leq a + s \\
\frac{b-t}{s} & \text{if } b - s \leq t \leq b,
\end{cases}
\]

and we let \( X \) be a smooth vector field on \( M \) such that \( X(x) = H_{a,b,s}(\tau(x))\nabla \tau(x) \) for \( x \in K_{r_0/2} \).

The fact that \( V_* \) is stationary implies that \( \delta V_*(X) = 0 \), cf. (1.2.7), and this means that

\[
\int_{\pi^{-1}(K_{\frac{4}{\sqrt{r}}})} H'_{a,b,s}(\tau) |\nabla \xi \tau|^2 dV_*(x,\xi) + 
\int_{\pi^{-1}(K_{\frac{4}{\sqrt{r}}})} H_{a,b,s}(\tau) \text{Hess}_g(\tau)(\xi,\xi) dV_*(x,\xi) = 0, \tag{3.3.19}
\]

We have used Step 1 to obtain that \( V_* \) is concentrated on \( K_{\frac{4}{\sqrt{r}}} \). Next, we use Step 2, \( V_*(M) \leq L \), and the fact that \( \| H_{a,b,s} \|_{L^\infty(\mathbb{R})} = 1 \) to conclude that

\[
\int_{\pi^{-1}(K_{\frac{4}{\sqrt{r}}})} H_{a,b,s}(\tau) \text{Hess}(\tau)(\xi,\xi) dV_*(x,\xi) = O(\sqrt{r}). \tag{3.3.20}
\]

Then we observe that, by the definition of \( H_{a,b,s} \), we have

\[
\int_{\pi^{-1}(K_{\frac{4}{\sqrt{r}}})} H'_{a,b,s} |\nabla \xi \tau|^2 dV_*(x,\xi) = \frac{1}{s} \int_{\pi^{-1}(\{a \leq \tau \leq a+s\} \cap K_{\frac{4}{\sqrt{r}}})} |\nabla \xi \tau|^2 dV_*(x,\xi) - \frac{1}{s} \int_{\pi^{-1}(\{b-s \leq \tau \leq b\} \cap K_{\frac{4}{\sqrt{r}}})} |\nabla \xi \tau|^2 dV_*(x,\xi).
\]

Combing this with (3.3.19) and (3.3.20) yields

\[
\frac{h(b) - h(b - s)}{s} = \frac{h(a + s) - h(a)}{s} + O(\sqrt{r}).
\]
Letting $s$ tend to zero, we find that
\begin{equation}
    h'(b) = h'(a) + O(\sqrt{r}).
\end{equation}
(3.3.21)

**Step 6:** We next introduce a set of ‘good’ slices via
\[
    \mathcal{G} := \{ a \in (0, L) : \mathcal{H}^0 (\{ \tau = a \} \cap \text{spt}(V_*)) = 1 \},
\]
and in this step we will demonstrate that
\begin{equation}
    (0, L) \setminus (B_1 \cup B_2 \cup B_3) \subset \mathcal{G}.
\end{equation}
(3.3.22)

From this and Step 4 it will follow that
\begin{equation}
    \mathcal{L}^1 (\mathcal{G}) \geq L - \frac{C_2 \delta}{\eta}
\end{equation}
(3.3.23)
provided that $r$ and $\delta$ are chosen sufficiently small.

Suppose by way of contradiction that there exists a value $a$ such that
\begin{equation}
    a \in (0, L) \setminus (B_1 \cup B_2 \cup B_3 \cup \mathcal{G}).
\end{equation}
(3.3.24)

Then in light of (3.3.5) we have that
\[
    \mathcal{H}^0 (\{ \tau = a \} \cap \text{spt}(V_*)) \geq 2
\]
for some $a \in (0, L) \setminus (B_1 \cup B_2 \cup B_3)$. We first argue that for such an $a$ and every $c, \delta \in (0, 1)$ it must hold that
\begin{equation}
    h'(a) \geq (1 - c)(1 - \delta)^2(1 + \eta)
\end{equation}
(3.3.25)
provided $r$ is chosen sufficiently small, depending on $c$. Fix $0 < c, \delta < 1$, and choose $1 < \alpha < \frac{1}{2}$. Note that since $a \not\in B_2$ then by item (3) of the theorem on page 89 of [2] we have that $\{ \tau = a \} \cap \text{spt}(V_*)$ consists only of interior points of the constituent geodesics (or “intervals” as they are referred to in [2]) making up $\mathcal{V}_*$. Moreover, the endpoints of these geodesics cannot accumulate at $a$. We conclude
by compactness of $\text{spt}(V_\alpha)$ that $\{\tau = a\} \cap \text{spt}(V_\alpha)$ can intersect only finitely many of these constituents of $\text{spt}(V_\alpha)$. Also since $a \notin B_3$, it follows that this slice must intersect $\text{spt}(V_\alpha)$ transversally, so that $\{\tau = a\} \cap \text{spt}(V_\alpha)$ consists of finitely many points, say $x_1, \ldots, x_K$, where $K \geq 2$ by the choice of $a$.

It follows from the gradient estimate $|\nabla \tau| = 1 + O(4\sqrt{r})$ in $K_{4\sqrt{r}}$, established in (3.3.12), that for $0 < \alpha < \beta < L$, the geodesic distance between the sets $\{\tau = \alpha\} \cap K_{4\sqrt{r}}$ and $\{\tau = \beta\} \cap K_{4\sqrt{r}}$ is at least $(\beta - \alpha)/(1 + O(\sqrt{r}))$, if $\beta - \alpha < L/2$. Hence if $r$ is small enough, then

$$B_{(1-c)s}(x_i) \cap K_{4\sqrt{r}} \subset \{x \in K_{4\sqrt{r}}: a - s < \tau(x) < a + s\}.$$ 

(Here and below, we tacitly assume that $0 < a - s < a + s < L$ and $s < L/2$.)

Next we again use that $a \notin B_3$ to choose $s_0 > 0$ small enough so that

$$|\nabla_\xi \tau(y)|^2 \geq (1 - \alpha \delta)^2$$ 

if $y \in \bigcup_{i=1}^K B_{s_0}(x_i)$ and $(y, \xi) \in \text{spt}(V_\alpha)$. Combining these facts, for each $0 < s \leq s_0$ we estimate

$$\frac{h(a + s) - h(a - s)}{2s} \geq (1 - \alpha \delta)^2 \sum_{i=1}^K V_\alpha(B_{(1-c)s}(x_i))$$

We now apply item (5) of the theorem on page 87 of [2] and let $s \to 0^+$, using the differentiability of $h$ at $a$ guaranteed by the assumption $a \notin B_1$, to find

$$h'(a) \geq (1 - c)(1 - \alpha \delta)^2 \sum_{i=1}^K \Theta_\alpha(x_i) \geq (1 - c)(1 - \alpha \delta)^2(1 + \eta)$$

Here we have used Lemma 3.3.2 to assert that $\Gamma^*$ intersects each level set of $\tau$ with $\Theta_\alpha(x) \geq 1$ for $x \in \Gamma^*$ by (3.3.3), and that $\Theta_\alpha \geq \eta$ in general by (3.5.12). Since $\alpha > 1$ was arbitrary we may let $\alpha \to 1^+$ to obtain (3.3.25).

In light of (3.3.21), it then follows from (3.3.25) that for any $b \notin B_1$ we obtain

$$h'(b) \geq (1 - c)(1 - \delta)^2(1 + \eta) + O(\sqrt{r})$$

Thus, choosing $c, r,$ and $\delta$ sufficiently small and recalling that $\mathcal{L}^1(B_1) = 0$, we deduce that

$$h'(b) \geq 1 + \frac{\eta}{2} \quad \text{for a.e. } b \in (0, L).$$ 

(3.3.27)
Thus, if there were a value \( a \in (0, L) \) satisfying (3.3.24), then

\[
[1 + C_3 \sqrt{r}] L \geq [1 + C_3 \sqrt{r}] V_\ast (\pi^{-1} \{ 0 \leq \tau \leq L \})
\]

\[
\geq h(L) - h(0) = \int_0^L h'(s)ds \geq \left(1 + \frac{\eta}{2}\right) L.
\]

Here we use (3.3.12) in the second inequality and the constant \( C_3 \) depends only on \( M \) and \( \Gamma \). If we choose \( r \) sufficiently small, the contradiction is reached, establishing (3.3.22) and (3.3.23).

**Step 7:** In this step we show that

\[ S_{V_\ast} = \emptyset. \]

It will immediately follow that the Lipschitz curve \( \Gamma^* \) guaranteed by Lemma 3.3.2 represents the entire rectifiable varifold and is in fact a closed geodesic.

Crucial use in this step will be made of the following general property of stationary 1-varifolds (cf. [2], pg. 88.):

*Every point \( p \in M \) is contained in an open set \( U_p \) such that if \( V \) is any stationary varifold on \( M \) with support in \( U_p \), and if the support of \( \delta V \) consists of exactly two points, then \( V \) is the varifold corresponding to a constant multiple of the geodesic joining these two points.*

This result is proved in [2] for possibly noncompact manifolds. Since \( M \) is compact, we may invoke the Lebesgue Number Lemma to conclude that there exists \( \lambda > 0 \) such that for any \( p \in M \), the geodesic ball \( B_\lambda(p) \) has the stated property.

For any \( A \subset (0, L) \), we will write

\[
K_{\sqrt{r}}(A) := \{ x \in K_{\sqrt{r}} : \tau(x) \in A \} = \{ \psi(y, t) : |y| \leq \sqrt{r}, \ t \in A \}.
\]

By extending \( \psi \) to be periodic with respect to the \( t \) variable in the natural way, we can define \( K_{\sqrt{r}}(A) \) for any \( A \subset \mathbb{R} \).

By shrinking \( r \) and \( \delta \), we may arrange that if \( I \) is any interval of length at most \( 2C_2 \delta / \eta \), where \( C_2 \) is the constant appearing in (3.3.23), then \( K_{\sqrt{r}}(I) \) is contained in a ball of radius \( \lambda \).

We will prove the claim by showing that

for any \( t \in [0, L) \), \( K_{\sqrt{r}}(\{ t \}) \cap S_{V_\ast} = \emptyset. \)
Indeed, for any $t$ we can appeal to (3.3.23) to find some $s_1, s_2 \in G$ such that

$$s_1 < t < s_2, \quad s_2 - s_1 < \frac{2C_2\delta}{\eta}.$$  

We now apply the result stated at the outset of this step to the varifold

$$\tilde{V} = V_\ast \subseteq K_{\sqrt{r}}([s_1, s_1]),$$

in an open ball $B_\lambda(p)$ that contains $K_{\sqrt{r}}([s_1, s_1])$. The definition of $G$ implies that $V_\ast$ intersects \{\tau = s_j\} in exactly one point, say $x_j$, and that $\delta \tilde{V}$ is supported in $\{x_1, x_2\}$. Hence, this restriction of $V_\ast$ consists of a multiple of the geodesic joining these two points. This immediately implies the claim.

Since $S_{V_\ast} = \emptyset$, $V_\ast$ must simply be the rectifiable varifold associated with a single, closed, smooth geodesic.

By perhaps shrinking $r$ one more time and applying the Morse-Palais Lemma, cf. [44], pg. 307, we may conclude that the central geodesic $\Gamma$, being a nondegenerate critical point of length, is isolated and so necessarily $\text{spt}(V_\ast) = \Gamma$. However, this contradicts (3.5.13) since $r > 0$, and the proof of Proposition 3.3.1 complete.

\[ \square \]

### 3.4 Finding good trajectories

The critical points of Ginzburg-Landau that we seek will be obtained as limits of certain carefully chosen trajectories of the Ginzburg-Landau heat flow. In this section we identify these trajectories.

**Proposition 3.4.1.** Given $\delta > 0$, there exists $\varepsilon_0(\delta) > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and every $\tau > 0$, there is a solution $u_\varepsilon = u_\varepsilon(x, t; \delta, \tau)$ of the Ginzburg-Landau heat flow

\begin{align*}
\partial_t u_\varepsilon &= \Delta u_\varepsilon - \frac{1}{\varepsilon^2}(|u_\varepsilon|^2 - 1)u_\varepsilon \quad \text{in } M \times (0, \infty), \\
u_\varepsilon(x, 0) &= u_\varepsilon^0(x) \quad \text{for } x \in M,
\end{align*}

such that $|u_\varepsilon(x, t)| \leq 1$ for every $(x, t) \in M \times [0, \infty)$, and

\begin{align*}
\left\| \frac{1}{\pi} \ast J u_\varepsilon^0 - T_\gamma \right\|_F &< \delta, \\
E_\varepsilon(u_\varepsilon^0) &\leq L + \delta, \\
E_\varepsilon(u_\varepsilon(t)) &\geq L - \delta \quad \text{for all } t \in [0, \tau].
\end{align*}

(3.4.2)
The proposition follows from small modifications of the asymptotic minmax theory developed in [27, 41]. Indeed, in the end the proof amounts to this:

*short proof of Proposition 3.4.1.* This follows from arguments in [27, 41].

In the remainder of this section we expand on this, aiming to provide enough detail to convey the main ideas, to explain where we depart from [27, 41], and to make it possible, in principle, to check the terse proof given above.

We remark that the main difference between [27, 41] and our present treatment is that those earlier works use a pseudo-gradient flow for the energy $E_\epsilon$, whereas we employ a small modification of the Ginzburg-Landau heat flow (3.4.1) for similar purposes. The use of (3.4.1) is necessary for our approach, due to our reliance in Section 3.5 below on Theorem 2.1.1.

### 3.4.1 Saddle point property of $E_V$

Our assumptions about $\gamma$ imply, roughly speaking, that the “arc-length functional” has a local min-max geometry near $\gamma$, as reflected in (3.2.12), with respect to smooth perturbations. In particular, there is an $\ell$-parameter family of arc-length-decreasing perturbations of $\gamma$, and arc-length increases for sufficiently transverse smooth perturbations. Here and below, $\ell$ is the index of $\gamma$, see (3.2.10).

The result below states that the “generalized arc-length functional” $E_V$ defined in (3.2.17) has a saddle point, in a suitable weak sense, at the current $T_\gamma \in V$ corresponding to $\gamma$. The relevant notion of saddle point was first introduced in [27].

**Lemma 3.4.2** (cf. Theorem 4.1, [41]). *For the geodesic $\gamma$ satisfying (3.2.9) and (3.2.15), the associated current $T_\gamma$ is a saddle point of $E_V$ in the sense that there exist $R, \delta_0 > 0$ and continuous functions

$$P_{WV} : V \to \mathbb{R}^\ell, \quad Q_{VW} : W \to V$$

for

$$W = B^\ell_R = \{ w \in \mathbb{R}^\ell : |w| < R \}$$

(3.4.3)*
such that $P_{WV}(T_\gamma) = 0$, and the following conditions are satisfied:

\begin{align}
E_V(T_\gamma) &< E_V(v) \text{ for } \{v \in V : 0 < \|v - T_\gamma\| \leq \delta_0, P_{WV}(v) = 0\}, \quad (3.4.4) \\
Q_{VW}(0) &= T_\gamma, \quad (3.4.5) \\
P_{WV} \circ Q_{VW}(w) &= w \quad \text{for all } w \in W, \quad (3.4.6) \\
\text{and for every } r > 0, \quad \sup_{\{w \in W : |w| \geq r\}} E_V(Q_{VW}(w)) < E_V(T_\gamma). \quad (3.4.7)
\end{align}

We sketch the proof from [41], although we note that this will not play any role in what follows, except that the notation $\gamma_w$ for the curve defined in (3.4.9) and $T_{\gamma_w}$ for the associated 1-current, cf. (3.2.13), will be used below.

To start, for $w \in W$ we define

$$\xi(w) = w_1 \xi_1 + \ldots + w_\ell \xi_\ell$$

(3.4.8)

where $\xi_j$ denote eigenfunctions of the Jacobi operator, see in particular (3.2.10). We then define the curve $\gamma_w$ via

$$\gamma_w(t) := \exp_{\gamma(t)} \xi(w)(t),$$

(3.4.9)

cf. (3.2.7). We will always assume that $R$ is small enough that

$$\|\xi(w)\|_{L^\infty} \leq r_0, \quad \text{in other words, } \gamma_w \in H_-(r_0), \quad \text{for } w \in W.$$

With this in hand, we define $Q_{VW} : B_\ell^R \to R^1_1$ via

$$Q_{VW}(w) := T_{\gamma_w} \quad \text{for } w \in W.$$

Then (3.4.5) is immediate, and (3.4.7) follows directly from (3.2.12).

The construction of $P_{WV}$ is carried out in [41], Lemma 4.3, by designing an $\mathbb{R}^\ell$-valued 1-form $\Phi$ such that

$$T_{\gamma_\xi}(\Phi) = ((\xi, \xi_1)_{L^2}, \ldots, (\xi, \xi_\ell)_{L^2})$$

(3.4.10)

for $\xi \in L^2(NT) \cap \text{Lip}$, as long as $\|\xi\|_{L^\infty} \leq r_0$, a condition that can be guaranteed by a suitable choice of $R$. Here we recall that $\gamma_\xi$ is the curve given by (3.2.7) and $T_{\gamma_\xi}$ is its associated 1-current. We then simply define $P_{WV}(T) = T(\Phi)$. With this choice, (3.4.6) follows directly from (3.4.8), (3.4.9),
and (3.4.10).

The hard part of the proof of Lemma 3.4.2 is the verification of (3.4.4). This is carried out in Proposition 4.1 of [41], to which we refer for the details. We remark that the main ideas in this proof, including the construction of \( \Phi \), are similar to elements in the proof of Theorem 5 in [53].

Note that we may shrink at will the parameter \( R \) in the definition (3.4.3) of \( W \), and the conclusions of the lemma remain valid.

### 3.4.2 An \( \ell \)-parameter family of solutions of (3.4.1)

To prove Proposition 3.4.1, we will define an \( \ell \)-parameter family of solutions of (3.4.1) for every sufficiently small \( \varepsilon > 0 \). In the final step, given \( \varepsilon \) and \( \tau \) (where ultimately we will take \( \varepsilon < \varepsilon_0(\delta) \)), we will choose from this family one solution \( u_\varepsilon \) such that \( E_\varepsilon(u_\varepsilon(t)) \geq L - o(1) \) for all times \( t \in [0, \tau] \).

The initial data for this family of solutions is provided by the following result.

**Lemma 3.4.3.** There exist \( R, \varepsilon_1 > 0 \) such that for every \( \varepsilon \in (0, \varepsilon_1) \) and \( w \in B^R_\ell \), there exists a function \( U_{\varepsilon,0} \in H^2(M) \) satisfying the conditions:

1. \( w \mapsto U_{\varepsilon,0} \) is Lipschitz continuous from \( B^R_\ell \) into \( H^2(M) \)
2. \( \|U_{\varepsilon,0}\|_{L^\infty} \leq 1 \) and \( \|U_{\varepsilon,0}\|_{H^2} \leq C_\varepsilon \) for all \( \varepsilon \in (0, \varepsilon_1) \), \( w \in B^R_\ell \)
3. \( E_\varepsilon(U_{\varepsilon,0}) \leq L - c_0|w|^2 + o(1) \) as \( \varepsilon \to 0 \)
4. \( \|\frac{1}{\pi} \star JU_{\varepsilon,0} - T_{\gamma_w}\|_V \to 0 \)

uniformly for \( w \in B^R_\ell \).

For fixed \( w \), in view of (3.2.12) and the construction of \( \gamma_w \), conclusions (3) and (4) hold if \( U_{\varepsilon,0} \) is a recovery sequence for the current \( T_{\gamma_w} \) the Gamma-limit in Theorem 3.2.1. Such constructions are rather standard. It is easy to arrange that \( \|U_{\varepsilon,0}\|_{L^\infty} \leq 1 \). The only points requiring attention are that the construction has to be carried out so that it depends continuously on \( w \), in the \( H^1 \) norm, and with some control over the \( H^2 \) norm. The former point is carried out in [41], and the latter can be achieved by a small modification of the construction of [41]. We defer a more detailed discussion to Appendix 3.6.

The \( H^2 \) estimate facilitates the proof of Lemma 3.4.4 below, whose need arises because we require the Ginzburg-Landau heat flow rather than the pseudo-gradient flows employed in [27, 41].

Having constructed appropriate initial data for the Ginzburg-Landau flow, we are now ready to define the flow that we will use in our arguments below.
Lemma 3.4.4. For $\varepsilon \in (0, \varepsilon_1)$ and $w \in W$, let $U_{w}^{\varepsilon,1}(x, t)$ solve the Ginzburg-Landau heat flow with initial data $U_{w}^{\varepsilon,0}$. Then

$$(t, w) \in [0, \infty) \times W \mapsto U_{w}^{\varepsilon,1}(\cdot, t) \in H^3(M; \mathbb{C})$$

is continuous.

See Appendix 3.6 for the proof, which involves rather standard parabolic estimates.

Finally, we define $U_{\varepsilon} : [0, \infty) \times W \to H^1(M; \mathbb{C})$ by

$$U_{\varepsilon}(t, w):= U_{w}^{\varepsilon,1}(\cdot, \chi(w)t). \quad (3.4.11)$$

for a smooth, compactly supported $\chi : B^t_{R} \to [0, 1]$ such that $\chi = 1$ in $B^t_{R/2}$. We point out that

$$\text{if } |w| = R, \quad \text{then } U_{\varepsilon}(t, w) = U_{w}^{\varepsilon,0} \text{ for all } t \geq 0. \quad (3.4.12)$$

We can guarantee that $E_{U}(U_{\varepsilon}(0, w)) < c_{\varepsilon} - \delta$ whenever $|w| > R/2$ and $\varepsilon$ is small enough, for suitable $\delta$.

3.4.3 Choosing a good trajectory

We finally make use of the asymptotic saddle point geometry of $E_{\varepsilon}$, inherited from $E_{V}$ via the Gamma-convergence Theorem 3.2.1, to complete the proof of Proposition 3.4.1.

We will use the notation

$$P_{WU}(u) = P_{WV}(\frac{1}{\pi} \star Ju) \quad \text{for } u \in H^1(M; \mathbb{C}).$$

Lemma 3.4.5. There exist $\delta_0 > 0$ and $R_0 > 0$ such that for every $R \in (0, R_0)$, there is some $\delta = \delta(R) > 0$ such that if we define

$$a_{\varepsilon}:= \max\{E_{\varepsilon}(U_{w}^{\varepsilon,0}) : |w| = R\}$$

$$c_{\varepsilon}:= \min\{E_{\varepsilon}(u) : P_{WU}(u) = 0, \|\frac{1}{\pi} \star Ju - T\|_V \leq \delta_0\}$$

$$d_{\varepsilon}:= \max\{E_{\varepsilon}(U_{w}^{\varepsilon,0}) : |w| \leq R\},$$

then

$$a_{\varepsilon} \to L - 2\delta, \quad \liminf_{\varepsilon \to 0} c_{\varepsilon} \geq L, \quad d_{\varepsilon} \to L \quad (3.4.13)$$
as \( \varepsilon \to 0 \), where \( L \) is the length of the geodesic \( \gamma \).

The assertions about \( a_\varepsilon \) and \( d_\varepsilon \) follow directly from (3.2.12) and Lemma 3.4.3, and Step 3 of the proof of Theorem 4.4 of [27] shows exactly that \( c_\varepsilon \to L \). The proof uses only ingredients that we have collected in Theorem 3.2.1 and Lemma 3.4.2 below.

Below we will not refer explicitly to the assertion about \( \lim_{\varepsilon \to 0} a_\varepsilon \), but it plays a role in the proof of Lemma 3.4.6, and together with the lower bound for \( \lim \inf c_\varepsilon \), it reflects the asymptotic minmax geometry of \( E_\varepsilon \).

Proposition 3.4.1 will essentially follow from the next fact.

**Lemma 3.4.6.** For each \( r > 0 \) there exists \( \varepsilon_0 > 0 \) and \( R > 0 \) such that for every \( 0 < \varepsilon < \varepsilon_0 \) and every \( \tau > 0 \), there exists \( w = w(\varepsilon, \tau) \) such that

\[
P_{WU}(U_\varepsilon(\tau, w)) = 0, \quad \frac{1}{\tau} \ast JU_\varepsilon(\tau, w) - J_\gamma \leq r.
\]

(3.4.14)

As a result, \( w = w(\varepsilon, \tau) \) satisfies

\[
d_\varepsilon \geq E_\varepsilon(U_\varepsilon(t, w)) \geq E_\varepsilon(U_\varepsilon(\tau, w)) \geq c_\varepsilon \text{ for all } t \in [0, \tau] \text{ and } \varepsilon \in (0, \varepsilon_0).
\]

(3.4.15)

Finally, \( w(\varepsilon, \tau) \to 0 \) as \( \varepsilon \to 0 \).

**Proof.** One may prove (3.4.14) by simply repeating the arguments from Steps 5-8 of the proof of Theorem 4.4 in [27], for \( r > 0 \) such that \( 0 < r < \delta_0 \), where \( \delta_0 \) is the constant from item (3.4.4) of Lemma 3.4.2. Some comments are in order:

First, the argument in [27] is stated for a pseudo-gradient flow (see Lemma 4.8, [27]) with certain properties that our flow \((t, w) \to U_\varepsilon(t, w)\) does not possess. These are in fact not needed for the proof of (3.4.14), and some of them may appear in [27] only because there Lemma 4.8 is quoted directly from a standard text, which provides more than is actually needed. These are the only properties of the flow that are required for the proof of (3.4.14):

- \( t \mapsto E_\varepsilon(U_\varepsilon(t, w)) \) is nonincreasing.
- \( t \mapsto U_\varepsilon(t, w) \) is constant for \( w \in \partial W \), see (3.4.12).
- continuity properties of the flow, as summarized in Lemma 3.4.4.

All of these are available here.
Without going into detail, we remark that the basic strategy of the proof is to apply degree theory arguments to the maps $w \mapsto P_{WU}(U_{\varepsilon}(t, w)): W \to W$ as $t$ varies from 0 to $\tau$ (where $\tau = 1$ in [27], a harmless normalization).

Next, (3.4.15) follows directly from (3.4.14) and Lemma 3.4.5. Finally, we deduce from (3.4.15) and Lemma 3.4.5 that
\[ E_{\varepsilon}(U_{\varepsilon}(0, w)) = E_{\varepsilon}(U_{w}^{\varepsilon, 0}) \to L \quad \text{as} \ \varepsilon \to 0. \]

Then conclusion (3) of Lemma 3.4.3 implies that $w = w(\varepsilon, \tau) \to 0$ as $\varepsilon \to 0$. \hfill \Box

We are now in a position to present:

Proof of Proposition 3.4.1. Let $0 < \delta < \delta_0$ be given where $\delta_0$ is as in item (3.4.4) of Lemma 3.4.2. We take $\varepsilon_0(\delta)$ and $R(\delta)$ as defined in Lemma 3.4.6 and set $u_{\varepsilon}(x, t; \delta, \tau) = U_{\varepsilon}(t, w(\varepsilon, \tau))$. By shrinking $\varepsilon_0$ if necessary, we may ensure that $|w(\varepsilon, \tau)| < R(\delta)$, as shown in Lemma 3.4.6. Thus, $u_{\varepsilon}(x, t; \delta, \tau)$ solves (3.4.1) since
\[ U_{\varepsilon}(t, w(\varepsilon, \tau)) = U_{w}^{\varepsilon, 0}(t, x) \]
for such $\varepsilon > 0$.

Since $\|U_{w}^{\varepsilon, 0}\|_{L^\infty} \leq 1$, it is clear that $|u_{\varepsilon}(x, t)| \leq 1$ everywhere, and all other conclusions of the Proposition follow directly from Lemmas 3.4.5 and 3.4.6. \hfill \Box

3.5 Proof of the main result

The main result of this chapter, stated more informally in the introduction as Theorem 3.1.1, can now be phrased precisely as

Theorem 3.5.1. Let $(M, g)$ be a closed oriented 3-dimensional Riemannian manifold, and let $\gamma$ be a closed, embedded, nondegenerate geodesic of length $L$. Assume in addition that $\gamma = \partial S$ (in the sense of Stokes’ Theorem) for some 2-dimensional submanifold $S$ of $M$.

Then for every $r > 0$, there exists $\varepsilon_1(r) > 0$ such that if $0 < \varepsilon < \varepsilon_1(r)$, then there is a solution $u_{\varepsilon}$ of the Ginzburg-Landau equations
\[ -\Delta u_{\varepsilon} + \frac{1}{\varepsilon^2}(|u_{\varepsilon}|^2 - 1)u_{\varepsilon} = 0 \quad \text{on} \ M \quad (3.5.1) \]
such that
\[ \| \frac{1}{\pi} \star J u_\varepsilon - T_\gamma \|_F \leq r \]
and
\[ |E_\varepsilon(u_\varepsilon) - L| < r. \]

As a result, there exists a sequence \((u_\varepsilon)_{\varepsilon > 0} \subset H^1(M; \mathbb{C})\) of solutions of the Ginzburg-Landau equations such that
\[ \| \frac{1}{\pi} \star J u_\varepsilon - T_\gamma \|_F \to 0, \quad E_\varepsilon(u_\varepsilon) \to L \quad \text{as} \ \varepsilon \to 0. \quad (3.5.2) \]

We remark that standard Gamma-convergence results (see Theorem 3.2.1) imply that the sequence of solutions in (3.5.2) satisfies
\[ \frac{e_\varepsilon(u_\varepsilon)}{\pi|\log \varepsilon|} \to \| T_\gamma \| = \mathcal{H}^1 \Gamma \quad \text{weakly as measures}, \quad (3.5.3) \]
which is the last conclusion of Theorem 3.1.1. Indeed, since \(E_\varepsilon(u_\varepsilon)\) is uniformly bounded, there exists some measure \(\mu\) such that
\[ \frac{e_\varepsilon(u_\varepsilon)}{\pi|\log \varepsilon|} \to \mu \quad \text{weakly as measures}, \]
after perhaps passing to a subsequence, and \(\mu(M) = \lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = L\). Standard Gamma-convergence results and (3.5.2) imply
\[ \mu \geq \| T_\gamma \|, \]
and since \(\| T_\gamma \|(M) = L = \mu(M)\), it follows that \(\mu = \| T_\gamma \|\), proving (3.5.3).

**Proof.** The proof relies on an improvement on the properties of the flow defined in the previous section. The assertion is that the trajectory solving the Ginzburg-Landau flow identified in Proposition 3.4.1 remains close to \(T_\gamma\) in the flat norm. More precisely, we will show:

**Claim:**
For every \(r > 0\), there exist positive constants \(\delta_1(r)\) and \(\varepsilon_1(r) < \varepsilon_0(\delta_1)\), depending on \(r\), such that for every \(\varepsilon \in (0, \varepsilon_1)\) and \(\delta \in (0, \delta_1)\), and for every \(\tau > 0\), if \(u_\varepsilon = u_\varepsilon(x, t; \delta, \tau)\) is the solution of (3.4.1) found in Proposition 3.4.1, then
\[ \| \frac{1}{\pi} \star J u_\varepsilon(t) - T_\gamma \|_F < r \quad \text{for all} \quad t \in [0, \tau]. \quad (3.5.4) \]
To establish this, we assume toward a contradiction that there exists \( r > 0 \) and sequences \( \delta_k, \varepsilon_k \to 0 \) and \( 0 < t_k \leq \tau_k \) such that \( u_k(x, t) := u_{\varepsilon_k}(x, t; \delta_k, \tau_k) \) satisfies
\[
\| \frac{1}{\pi} \star J u_k(\cdot, t_k) - T_\gamma \|_F = r. \tag{3.5.5}
\]
Clearly, we may assume that \( r < r_0 \) for some \( r_0 \) to be chosen below.

**Step 1:** Under the assumption (3.5.5), we will argue that necessarily \( t_k \to \infty \) as \( k \to \infty \). (In fact, below we only need to know that \( t_k \) is bounded away from 0.)

Assume toward a contradiction that \( \lim \inf_k t_k < K \), for some \( K > 0 \). By passing to subsequences, relabelling, and invoking standard compactness, continuity, and Gamma-convergence results (i.e. Theorem 3.2.1), we may assume that the following hold.

First, \( t_k \leq K \) for all \( k \).

Second, there exists a 1-current \( J_1 \in \mathcal{R}_1 \cap \mathcal{F}_1^\prime(M) \) such that
\[
\| \frac{1}{\pi} \star J u_k(\cdot, t_k) - J_1 \|_F \to 0.
\]
Hence, by (3.5.5),
\[
\| J_1 - T_\gamma \|_F = r. \tag{3.5.6}
\]
Third, there exists a Radon measure \( \mu_1 \) such that
\[
\frac{e_{\varepsilon_k}(u_k(\cdot, t_k))}{\pi |\log \varepsilon|} \to \mu_1, \quad \text{weakly as measures.}
\]
and in addition
\[
\mu_1 \geq \| J_1 \|, \tag{3.5.7}
\]
cf. (3.2.14).

Finally, since \( \delta_k \to 0 \) and \( M \) is compact, it follows from Proposition 3.4.1 that
\[
\mu_1(M) = L = \| T_\gamma \|_F(M). \tag{3.5.8}
\]
We next claim that
\[
\frac{e_{\varepsilon_k}(u^0_k(\cdot))}{\pi |\log \varepsilon|} \to \mu_0 = \| T_\gamma \|, \quad \text{weakly as measures.}
\]
Indeed, we may assume that $\frac{\varepsilon_k(u_k^0(\cdot))}{\pi|\log \varepsilon_k|}$ converges weakly as measures to a limit $\mu_0$ as $k \to \infty$. Then recalling that $\| \frac{1}{\pi} \star J u_k^0 - T_\gamma \|_\infty \leq \delta_k \to 0$, standard Gamma-convergence results as in (3.5.7) imply that $\mu_0 \geq \| T_\gamma \|$. On the other hand, as in (3.5.8),

$$\mu_0(M) = L = \text{length}(\gamma) = \| T_\gamma \|(M),$$

so it follows that in fact $\mu_0 = \| T_\gamma \|$ as claimed. It then follows from (3.5.6) and (3.5.7) that $\mu_0 \neq \mu_1 = \| T_\gamma \|$. We will obtain a contradiction, completing Step 1, by showing that under our assumptions $\mu_0$ and $\mu_1$ must be equal. Indeed, after taking the inner product of (3.4.1) with $\partial_t u_k$, standard computations show that

$$\partial_t \varepsilon_k(u_k) = -|\partial_t u_k|^2 + \text{div}(\partial_t u_k \cdot \nabla u_k).$$

Multiplying by a function $\phi \in C^1(M)$ and integrating by parts,

$$-\int_M \phi \varepsilon_k(u_k) \text{ vol} \bigg|^{t_k}_0 = \int_0^{t_k} \int_M \phi |\partial_t u_k|^2 - (\nabla \phi, \partial_t u_k \cdot \nabla u_k)_g \text{ vol} dt$$

(3.5.9)

Clearly

$$\frac{1}{\pi|\log \varepsilon_k|} \int_0^{t_k} \int_M \phi \varepsilon_k(u_k) \text{ vol} \bigg|^{t_k}_0 \to \int_M \phi (\mu_0 - \mu_1).$$

On the other hand, it is not hard to see that after dividing by $\pi|\log \varepsilon_k|$, the right-hand side of (3.5.9) tends to 0 as $k \to \infty$. First, taking $\phi = 1$ in (3.5.9), we see that

$$\frac{1}{\pi|\log \varepsilon_k|} \int_0^{t_k} \int_M |\partial_t u_k|^2 \text{ vol} dt = E_{\varepsilon_k}(u_k^0) - E_{\varepsilon_k}(u_k(\cdot, t_k)) \leq 2\delta_k \to 0$$

as $k \to \infty$. It immediately follows that

$$\left| \int_0^{t_k} \int_M \frac{\phi |\partial_t u_k|^2}{\pi|\log \varepsilon|} \text{ vol} dt \right| \to 0 \quad \text{as } k \to \infty$$

Similarly, from Cauchy-Schwarz, the fact that

$$\int_M \frac{|\nabla u_k(\cdot, t)|^2}{\pi|\log \varepsilon_k|} \text{ vol} dt \leq E_{\varepsilon_k}(u_k(\cdot, t)) \leq L + \delta_k \quad \text{for all } t \geq 0$$
and the assumption that $t_k \leq K$, we easily see that

$$\frac{1}{\pi |\log \varepsilon_k|} \left| \int_0^{t_k} \int_M (\nabla \phi, \partial_t u_k \cdot \nabla u_k)_g \text{vol} \, dt \right| \to 0 \quad \text{as } k \to \infty.$$  

Combining these, we conclude that $\int_M \phi (\mu_1 - \mu_0) = 0$ for all $\phi \in C^1(M)$, and hence that $\mu_0 = \mu_1$. This contradiction yields the conclusion that if (3.5.5) holds, then $t_k \to \infty$, completing Step 1.

**Step 2**: We will now reach a contradiction to (3.5.5), and so obtain Claim 3.5.4.

To this end, we apply Theorem 2.1.1 to the sequence of functions

$$\tilde{u}_k(x, t) = u_k(x, t + t_k - 1),$$

which, in particular, satisfies (3.4.1) on $M \times [0, \infty)$ with $\varepsilon = \varepsilon_k \to 0$. This yields a function $\Phi_* : M \times (0, 1] \to \mathbb{R}$ that solves the heat equation, as well as measures $\mu^t_* = \mu^t_*$ such that

$$\frac{\varepsilon_k (\tilde{u}_k(\cdot, t))}{\pi |\log \varepsilon_k|} \to \mu_*^t = \frac{1}{2} |\nabla \Phi_* (\cdot, t)|^2 \text{vol} + \nu_*^t$$

weakly as measures and $(\nu_*^t)_{0 < t \leq 1}$ is a 1-dimensional Brakke flow satisfying (2.1.3) and (4) (with $n = 2$).

As with (3.5.8), and using Claim 1, it follows that

$$\mu_*^t (M) = L \quad \text{for all } t \in (0, 1]. \quad (3.5.10)$$

We recall the standard estimate

$$\frac{d}{dt} \int_M |\nabla \Phi_* (\cdot, t)|^2 \text{vol} = -\int_M |\partial_t \Phi_*|^2 \, dx \leq 0 \quad (3.5.11)$$

(the counterpart for the linear heat equation of (3.5.9)). Since $t \mapsto \nu_*^t (M)$ is also nonincreasing, as noted in (3.2.20), we conclude from (3.5.10) that both

$$\nu_*^t (M) \quad \text{and} \quad \int_M |\nabla \Phi_* (\cdot, t)|^2 \text{vol}$$

are both independent of $t \in (0, 1]$. It then follows from (3.5.11) that $\partial_t \Phi_* = 0$ and hence that $\Phi_* = \Phi_* (x)$ is independent of $t$ and harmonic. Similarly, it follows from (3.2.21), (3.2.22) that there exists a stationary 1-dimensional
varifold $V_*$ such that

$$\nu_\varepsilon^t = V_* \quad \text{for all } t \in (0, 1],$$

and then by continuity at $t = 1$ as well. Also, (2.1.3) and (4) imply that there exists a 1-rectifiable set $\Sigma_* \subset M$ and a function $\Theta_*$ such that

$$V_* = \Theta_*(x)\mathcal{H}^1|_{\Sigma_*}, \quad \Theta_* \geq \eta > 0 \quad \mathcal{H}^1 \text{ a.e. in } \Sigma_*.$$  \hfill (3.5.12)

Moreover, as in the proof of Claim 1, there exists a 1-current $J_1 \in \mathcal{R}_1 \cap \mathcal{F}_1(M)$ such that

$$\|\frac{1}{\pi} \star J\varepsilon(\cdot, 1) - J_1\|_\mathcal{F} = \|\frac{1}{\pi} \star J\varepsilon(\cdot, t_k) - J_1\|_\mathcal{F} \to 0$$

and thus

$$\|J_1 - T_\gamma\|_\mathcal{F} = r.$$ \hfill (3.5.13)

We claim that in addition

$$V_* \geq \|J_1\|, \quad V_*(M) \leq L.$$ \hfill (3.5.14)

The second assertion follows from (3.5.10), and the first assertion is a consequence of standard Gamma-convergence results, which imply that

$$\mu_*^1 = \frac{1}{2} |\nabla \Phi_*(x)|^2 \text{ vol} + V_* \geq \|J_1\|.$$  

Since $\frac{1}{2} |\nabla \Phi_*(x)|^2 \text{ vol}$ is absolutely continuous with respect to vol and $\|J_1\|$ is concentrated on a 1-rectifiable set, this implies that $V_* \geq \|J_1\|$, as claimed.

However, through an appeal to Proposition 3.3.1, we see that no such varifold can exist. Claim 3.5.4 is established.

**Step 3:** Fix $r > 0$, and let $\delta_1(r)$ and $\varepsilon_1(r)$ be as provided in Claim 3.5.4. We may assume that $\delta_1(r) \leq r$. For $\varepsilon \in (0, \varepsilon_1(r))$, let

$$u_k = u_\varepsilon(\cdot, \cdot; \frac{1}{2}\delta_1(r), 2^k).$$

It follows from Proposition 3.4.1 that

$$\delta_1(r) \geq E_{k_\varepsilon}(u_k(\cdot, 0)) - E_{k_\varepsilon}(u_k(\cdot, 2^k)) = \frac{1}{\pi |\log \varepsilon|} \int_0^{2^k} \int_M |\partial_t u_k|^2 \text{ vol} dt,$$
so there exists $\sigma_k \in (0, 2^k)$ such that $w_k := u_k(\cdot, \sigma_k)$ satisfies

$$\int_M |\Delta w_k - \frac{1}{\varepsilon^2}(|w_k|^2 - 1)w_k|^2 \, \text{vol} = \int_M |\partial_t u_k|^2 \, \text{vol} \bigg|_{t=\sigma_k} \leq \delta_1 \pi |\log \varepsilon|^{-k}.$$

Also, it follows from Claim 3.5.4 that

$$\|\frac{1}{\pi} \star J w_k - T_\gamma\|_F = \|\frac{1}{\pi} \star J u_k(\cdot, \sigma_k) - T_\gamma\|_F < r.$$

Since $|u_k| \leq 1$ everywhere, we have that $\|\Delta w_k\|_{L^2(M)} \leq C_\varepsilon$, and hence by elliptic regularity, $\|w_k\|_{H^2} \leq C_\varepsilon$. One may thus extract a subsequence and a function $u_\varepsilon \in H^2(M; \mathbb{C})$ such that $w_k \to u_\varepsilon$ weakly in $H^2$, and it easily follows from the above that

$$-\Delta u_\varepsilon + \frac{1}{\varepsilon^2}(|u_\varepsilon|^2 - 1)u_\varepsilon = 0 \quad \text{and} \quad \|\frac{1}{\pi} \star J u_\varepsilon - T_\gamma\|_F \leq r.$$

Finally, we may insist that $\delta < r$, and then it follows from Proposition 3.4.1 that $|E_\varepsilon(u_\varepsilon) - L| < r$. \hfill \Box

3.6 Appendix

3.6.1 On the proof of Lemma 3.4.3

As remarked above, this lemma is essentially proved in [41]. We describe the proof given there and the extremely small modifications that we need.

The idea of the proof is first to construct $U^\varepsilon_0$, with its vorticity concentrating around the central geodesic $\Gamma$, then for $w \in W$, to define

$$U^\varepsilon_w := U^\varepsilon_0 \circ O_w^{-1} \quad (3.6.1)$$

where $O_w : M \to M$ is a suitable family of diffeomorphisms indexed by $w \in W$ such that $(w, x) \to O_w(x)$ is smooth, described below.

**Construction of $U^\varepsilon_0$** Recall that in (3.2.6) we defined a map $y : K_{r_0} \to \mathbb{R}^2$, smooth and nonvanishing away from $\Gamma$. Let $y^0 : M \setminus K_{r_0}/2 \to S^1$ be any smooth function such that $y^0(x) = y(x)/|y(x)|$ in $K_{r_0} \setminus K_{r_0}/2$. The existence of such a function is a consequence of the topological assumption (3.2.3).

Then we set $\tilde{v}^\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ by $\tilde{v}^\varepsilon(p) = f\left(\frac{|p|}{\varepsilon}\right) \frac{p}{|p|}$ where $f : [0, \infty) \to [0, 1]$ is a smooth nondecreas-
ing function such that $f(s) = s$ for $s \in [0, 1/2]$ and $f(s) = 1$ for $s \geq 1$. Finally we define

$$U_0^{\varepsilon, 0}(x) := \begin{cases} \tilde{v}^\varepsilon(y(x)) & \text{for } x \in K_{\varepsilon 0}, \\ y^0(x) & \text{for } x \in M \setminus K_{\varepsilon 0}. \end{cases}$$

The only way in which this construction differs from that in [41] is that there, $f$ is chosen to be $f(s) = \min(s, 1)$, which is Lipschitz continuous but not smooth. With this change, $U_0^{\varepsilon, 0}$ is smooth.

Construction of $O_w$. We take $O_w$ in (3.6.1) to be exactly the same map as in [41], see pg. 62.

The construction easily implies that $(w, x) \mapsto U_0^{\varepsilon, 0}(x)$ is smooth and hence that $\|U_0^{\varepsilon, 0}\|_{H^2} \leq C\varepsilon$ for all $w \in W$. All other conclusions are proved in [41], and some are obvious anyway, such as that $\|U_0^{\varepsilon, 0}\|_{L^\infty} \leq 1$. In particular, (3), which follows from a Gamma-limsup type estimate together with (3.2.12), is verified in Lemma 5.5 from [41]. Finally, (4) follows from Lemma 5.4 of [41].

### 3.6.2 Proof of Lemma 3.4.4

**Proof.** The maximum principle and standard energy estimates imply that for every $t > 0$,

$$\|U_0^{\varepsilon, 1}(\cdot, t)\|_{L^\infty(M)} \leq 1,$$

$$E_\varepsilon(U_0^{\varepsilon, 1}(\cdot, t)) + \frac{1}{\pi|\log \varepsilon|} \int_0^t \int_M |\partial_t U_0^{\varepsilon, 1}(y)|^2 d\text{vol} \: dx \leq E_\varepsilon(U_0^{\varepsilon, 1}(\cdot, 0)) \leq L + 1$$

for all $|w| \leq R$, provided $\varepsilon$ and $R$ are small enough. We next claim that for every $t > 0$, there exists $C = C_{\varepsilon, \tau}$ such that

$$\|U_0^{\varepsilon, 1}(\cdot, t)\|_{H^2} \leq C_{\varepsilon, \tau} \quad \text{for all } t \in [0, \tau] \text{ and } |w| \leq R. \quad (3.6.2)$$

To specify the norm, we fix an open cover \( \{U_j\}_{j \in J} \) of \( M \), with local coordinates \( \varphi_j : U_j \to V_j \subset \mathbb{R}^3 \) on each patch, and a finite partition of unity \( \{\eta_j\} \) subordinate to \( \{U_j\} \). We then define

$$\|u\|_{H^2}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \sum_{j \in J} \sum_{k=1}^3 \|\sqrt{\eta_j} \nabla \partial_k(u \circ \varphi_j^{-1})\|_{L^2(V_j)}^2,$$

where \( \partial_k \) denotes differentiation with respect to local coordinates on \( V_j \). To prove (3.6.2), we write (3.4.1) in local coordinates on each patch, apply \( \partial_k \) to derive an equation for \( V_k := \partial_k U_0^{\varepsilon, 1} \) of the form

$$\partial_t V_k - \Delta_g V_k = \text{terms involving } U_0^{\varepsilon, 1}, \nabla U_0^{\varepsilon, 1}.$$
Multiplying by $\partial_t V_k$, using the fact from Lemma 3.4.3 point (2) that $\|\nabla V_k(\cdot,0)\| \leq C_\varepsilon$, integrating by parts, and carrying out rather standard estimates leads to (3.6.2).

It follows from the above estimates and the equation that $\|\partial_t U_{w_1}^{\varepsilon,1}(\cdot,t)\|_{L^2} \leq C_{\varepsilon,\tau}$ for $0 < s \leq \tau$. Thus, for $0 \leq t_1 < t_2 \leq \tau$ and any $w \in W$, we have

$$
\|U_{w_1}^{\varepsilon,1}(\cdot,t_2) - U_{w_1}^{\varepsilon,1}(\cdot,t_1)\|_{L^2}^2 \leq (t_2 - t_1) \int_{M \times [t_1,t_2]} |\partial_t U_{w_1}^{\varepsilon,1}(x,t)|^2 \text{ vol } dt \leq C_{\varepsilon,\tau}(t_2 - t_1).
$$

Then the interpolation estimate $\|u\|_{H^1} \leq C\|u\|_{L^2}^{1/2}\|u\|_{H^2}^{1/2}$ and (3.6.2) imply that

$$
\|U_{w_1}^{\varepsilon,1}(\cdot,t_2) - U_{w_1}^{\varepsilon,1}(\cdot,t_1)\|_{H^1} \leq C_{\varepsilon,\tau}(t_2 - t_1)
$$

for $w \in W$, $0 \leq t_1 < t_2 \leq \tau$.

Now consider $w_1, w_2 \in W$. Writing $f_\varepsilon(u) = \frac{1}{\varepsilon^2}(1 - |u|^2)u$ and using the identity

$$
f_\varepsilon(b) - f_\varepsilon(a) = \int_0^1 \frac{d}{d\sigma}f_\varepsilon(\sigma b + (1 - \sigma)a) d\sigma = \int_0^1 f_\varepsilon'(\sigma b + (1 - \sigma)a) d\sigma (b - a),
$$

we find that $V := U_{w_2}^{\varepsilon,1} - U_{w_1}^{\varepsilon,1}$ satisfies the equation

$$
\partial_t V - \Delta V = gV,
$$

where $\|g(\cdot,t)\|_{L^\infty} \leq C$ for every $t$. In addition, it follows from Lemma 3.4.3 that $\|V(\cdot,0)\|_{H^1} \leq C|w_2 - w_1|$. Thus carrying out further standard parabolic estimates (multiplying by $V$ or $\partial_t V$, integrating by parts ...) leads to

$$
\|U_{w_2}^{\varepsilon,1}(\cdot,t) - U_{w_1}^{\varepsilon,1}(\cdot,t)\|_{H^1} \leq C_{\varepsilon,\tau}|w_2 - w_1| \quad \text{for } 0 \leq t \leq \tau.
$$

We conclude that the map $(t,w) \in [0,\tau] \times W \mapsto U_{w_1}^{\varepsilon,1}(\cdot,t) \in H^1(M;\mathbb{C})$ is continuous, since it is separately uniformly continuous in $t$ and $w$. A more detailed reference for such parabolic estimates on manifolds can be found, e.g. in appendix A of [40].
Appendix A

Extended Proofs

A.1 Energy Identity

We note that \( u_\epsilon \) solving \((PGL)_\epsilon \) with \((H_0)\) satisfies

\[
\mathcal{E}_\epsilon(u_\epsilon(\cdot, T_2)) + \int_{T_1}^{T_2} \int_M |\nabla u_\epsilon(x, t)|^2 \, \text{dvol}_g(x) \, \text{d}t = \mathcal{E}_\epsilon(u_\epsilon(\cdot, T_1)).
\]  

(A.1.1)

for all \( 0 \leq T_1 \leq T_2 \).

A.2 Pointwise Estimates

**Proposition A.2.1.** Let \( u_\epsilon \) be a solution of \((PGL)_\epsilon \) verifying \( \mathcal{E}_\epsilon(u_\epsilon^0) < \infty \). Then, there exists a constant \( K > 0 \) depending only on \( M \) such that, for \( t \geq \epsilon^2 \) and \( x \in M \)

\[
|u_\epsilon(x,t)| \leq 3, \quad |\nabla u_\epsilon(x,t)| \leq \frac{K}{\epsilon}, \quad |\partial_t u_\epsilon(x,t)| \leq \frac{K}{\epsilon^2}.
\]  

(A.2.1)

**Proof.** This is proved, as in Proposition 1.1 of [11], by making use of the comparison principle for parabolic PDEs as well as regularity results for parabolic PDEs.

\[ \square \]

**Remark A.2.2.** We may also conclude that

\[
|u_\epsilon(x,t)|^2 \leq 1 + Ke^{-\frac{K}{\epsilon^2}}
\]

for \( t \geq \epsilon^2 \) and \( x \in M \). As in Remark 1.1 of [11] this is proved by comparing \( |u_\epsilon|^2 - 1 \) to a function
Let \( u_\varepsilon \) be a solution of \((PGL)_\varepsilon\) verifying \( E_\varepsilon(u_\varepsilon^0) < \infty \). Let \( x_0 \in M \), \( 0 < R < \text{inj}_g(M) \), \( T \geq 0 \), and \( \Delta T > 0 \) be given. Assume that

\[
|u_\varepsilon| \geq \frac{1}{2}, \quad \text{on } \Lambda_1(x_0, T, R, \Delta T);
\]

then, for \( 0 < \alpha < 1 \), we have

\[
1 - |u_\varepsilon| \leq C(\alpha, \Lambda)\varepsilon^2 \left( \|\nabla \phi_\varepsilon\|^2_{L^\infty(\Lambda)} + |\log(\varepsilon)| \right) \quad \text{on } \Lambda_\alpha, \tag{A.2.2}
\]

where \( \phi_\varepsilon \) is defined on \( \Lambda \), up to a multiple of \( 2\pi \), by \( u_\varepsilon = |u_\varepsilon|e^{i\phi_\varepsilon} \). We also have

\[
-(1 - |u_\varepsilon|) \leq e^{\frac{\alpha}{2}} \quad \text{on } \Lambda_\alpha \tag{A.2.3}
\]

and hence

\[
|1 - |u_\varepsilon|| \leq \max \left\{ C(\alpha, \Lambda)\varepsilon^2 \left( \|\nabla \phi_\varepsilon\|^2_{L^\infty(\Lambda)} + |\log(\varepsilon)| \right), e^{\frac{\alpha}{2}} \right\} \quad \text{on } \Lambda_\alpha. \tag{A.2.4}
\]

**Proof.** By perhaps considering a smaller parabolic cylinder (A.2.2) can be proved, as in Lemma 1.1 of [11], by obtaining a PDE for \( 1 - |u_\varepsilon| \) and finding a suitable comparison function. Through a similar reduction (A.2.3) follows from an application of Remark A.2.2 and (A.2.4) follows from (A.2.2) and (A.2.3).

---

**A.3 Toolbox Proofs**

**A.3.1 The Monotonicity Formula**

**Proof of Lemma 2.3.2**

For brevity we will use \( u \) in the following calculation as a placeholder for \( u_\varepsilon(x, T - R^2) \) and \( K_{ap} \) as a placeholder for \( K_{ap}(x, R^2; y) \). Observe that by differentiating under the integral we obtain

\[
\frac{d}{dR} \left\{ \int_M e_\varepsilon(u)K_{ap} \right\} = -2R \int_M \left[ \nabla u_i \nabla u_t + V'_\varepsilon(u) \cdot u_t \right] K_{ap} + 2R \int_M e_\varepsilon(u)(K_{ap})_t.
\]
Observe that
\[
\langle \nabla u, \nabla u_t \rangle_{Kap} = \langle \nabla u, \nabla (u_t K_{ap}) \rangle - \langle \nabla u, \nabla K_{ap} \rangle \cdot u_t
\]
so integrating by parts and using that \( u_t = \Delta u - V'_\varepsilon(u) \) leads to
\[
\int_M \langle \nabla u, \nabla u_t \rangle_{Kap} = - \int_M \Delta u \cdot u_t K_{ap} - \int_M \langle \nabla u, \nabla K_{ap} \rangle \cdot u_t
\]
Thus,
\[
\frac{d}{dR} \left\{ \int_M \varepsilon(u) K_{ap} \right\} = 2R \int_M \left[ |u_t|^2 K_{ap} + \langle \nabla u, \nabla K_{ap} \rangle \cdot u_t \right] + 2R \int_M \varepsilon(u)(K_{ap})_t.
\]
Next we observe that
\[
\frac{1}{2} \int_M |\nabla u|^2 \Delta K_{ap} = \int_M \langle \nabla u, \nabla K_{ap} \rangle \cdot u_t + \int_M \langle \nabla u, \nabla K_{ap} \rangle \cdot V'_\varepsilon(u) + \int_M \text{Hess}(K_{ap})(\nabla u, \nabla u) \quad (A.3.1)
\]
where if \( u = u^1 + iu^2 \) we have set
\[
\text{Hess}(K_{ap})(\nabla u, \nabla u) := \text{Hess}(K_{ap})(\nabla u^1, \nabla u^1) + \text{Hess}(K_{ap})(\nabla u^2, \nabla u^2).
\]
Since (A.3.1) is the sum of the same identity for \( u^1 \) and \( u^2 \), the real and imaginary part of \( u \), we prove the identity for \( u^1 \). Integrating by parts we obtain
\[
\frac{1}{2} \int_M |\nabla u^1|^2 \Delta K_{ap} = - \frac{1}{2} \int_M \langle \nabla |\nabla u^1|^2, \nabla K_{ap} \rangle.
\]
Next for each \( p \in M \) we consider normal coordinates centred at \( p \) and compute using Einstein summation convention
\[
\langle \nabla |\nabla u^1|^2 \rangle^i = g^{ij} \nabla_{e_j}(|\nabla u^1|^2) = 2g^{ij} \langle \nabla e_j, \nabla u^1 \rangle = 2g^{ij} \langle \nabla (\nabla_{e_j} u^1), \nabla u^1 \rangle = 2g^{ij} g_{kl} (\nabla (\nabla_{e_j} u^1))^k (\nabla u^1)^l = 2g^{ij} g_{kl} (g^{km} \nabla_{e_m} \nabla_{e_j} u^1)(g^{lp} \nabla_{e_p} u^1) = 2g^{ij} g^{lp} (\nabla_{e_l} \nabla_{e_j} u^1)(\nabla_{e_p} u^1).
\]
Thus, at $p$ we have

$$
g_{is} \left( \nabla |\nabla u^1|^2 \right)^i (\nabla K_{ap})^s = 2g_{is}g^{i,j}g^{lp}(\nabla_{ei} \nabla_{ej} u^1)(\nabla_{es} u^1)(g^{sr} \nabla_{er} K_{ap})
g_{is}g^{i,j}g^{lp}(\nabla_{ei} (\nabla u^1)^i (\nabla_{es} u^1)(g^{sr} \nabla_{er} K_{ap})
g^{lp} \nabla_{ei} (2g_{is} (\nabla u^1)^i (\nabla K_{ap})^s) [\nabla_{es} u]
g_{is} (\nabla u^1)^i (g^{lp} \nabla_{ei} (g^{sr} \nabla_{er} K_{ap}) \nabla_{es} u
= g^{lp} \nabla_{ei} (2(\nabla u^1, \nabla K_{ap})) [\nabla_{es} u^1] - 2g_{is} (\nabla u^1)^i (g^{lp} \nabla_{ei} (g^{sr} \nabla_{er} K_{ap}) \nabla_{es} u^1
= (\nabla [2(\nabla u^1, \nabla K_{ap})])^p \nabla_{es} u^1 - 2g_{is} (\nabla u^1)^i (g^{lp} \nabla_{ei} (g^{sr} \nabla_{er} K_{ap}) \nabla_{es} u^1
= g_{pr} (\nabla [2(\nabla u^1, \nabla K_{ap})])^p (\nabla u^1)^r - 2g_{is} (\nabla u^1)^i (g^{lp} \nabla_{ei} (g^{sr} \nabla_{er} K_{ap}) \nabla_{es} u^1
= (\nabla [2(\nabla u^1, \nabla K_{ap})], \nabla u^1) - 2g_{is} (\nabla u^1)^i (g^{lp} \nabla_{ei} (g^{sr} \nabla_{er} K_{ap}) \nabla_{es} u^1
= (\nabla [2(\nabla u^1, \nabla K_{ap})], \nabla u^1) - 2g_{is} (\nabla u^1)^i (\nabla (\nabla u^1)^i (\nabla K_{ap})^s
= (\nabla [2(\nabla u^1, \nabla K_{ap})], \nabla u^1) - 2g_{is} (\nabla u^1)^i (\nabla u^1 \nabla K_{ap})^s
= (\nabla [2(\nabla u^1, \nabla K_{ap})], \nabla u^1) - 2(\nabla u^1 \nabla K_{ap}, \nabla u^1)
= (\nabla [2(\nabla u^1, \nabla K_{ap})], \nabla u^1) - 2 \text{Hess}(K_{ap}) (\nabla u^1, \nabla u^1).
$$

Next observe that integrating by parts and the fact that $u$ solves $(\text{PGL})_\varepsilon$ gives

$$
\int_M \langle \nabla [2(\nabla u^1, \nabla K_{ap})], \nabla u^1 \rangle = -2 \int_M \langle \nabla u^1, \nabla K_{ap} \rangle \Delta u^1 = -2 \int_M \langle \nabla u^1, \nabla K_{ap} \rangle [u_1 + (V_\varepsilon(u))^1].
$$

Combining the previous observations gives the desired identity. Also, we note that

$$
\int_M V_\varepsilon(u) \Delta K_{ap} = - \int_M \langle \nabla u, \nabla K_{ap} \rangle \cdot V_\varepsilon'(u). \quad (A.3.2)
$$

Putting (A.3.1) and (A.3.2) together gives

$$
\int_M e_\varepsilon(u) \Delta K_{ap} = \int_M \langle \nabla u, \nabla K_{ap} \rangle \cdot u_t + \int_M \text{Hess}(K_{ap}) (\nabla u, \nabla u). \quad (A.3.3)
$$
Thus, multiplying by $2R$, using (A.3.3), and factoring gives

$$\frac{d}{dR} \left\{ \int_M c_\varepsilon(u)K_{ap} \right\} = 2R \int_M \left[ |u_t|^2 K_{ap} + 2(\nabla u, \nabla K_{ap}) \cdot u_t \right] + 2R \int_M \text{Hess}(K_{ap})(\nabla u, \nabla u)$$

$$+ 2R \int_M c_\varepsilon(u)((K_{ap})_t - \Delta K_{ap})$$

$$= \frac{1}{2R^3} \int_M \left[ 4R^4 |u_t|^2 K_{ap} + 8R^4 \langle \nabla u, \nabla K_{ap} \rangle \cdot u_t \right] + 2R \int_M \text{Hess}(K_{ap})(\nabla u, \nabla u)$$

$$+ 2R \int_M c_\varepsilon(u)((K_{ap})_t - \Delta K_{ap})$$

$$= \frac{1}{2R^3} \int_M \left[ |\nabla u|^2 K_{ap} + \frac{1}{K_{ap}} \langle \nabla u, 2R^2 \nabla K_{ap} \rangle \right]^2 K_{ap}$$

$$- 2R \int_M \frac{|\langle \nabla u, \nabla K_{ap} \rangle|^2}{K_{ap}} + 2R \int_M \text{Hess}(K_{ap})(\nabla u, \nabla u)$$

$$+ 2R \int_M c_\varepsilon(u)((K_{ap})_t - \Delta K_{ap}).$$

Finally, this permits us to conclude that

$$\frac{d}{dR} \left\{ R^2 \int_M c_\varepsilon(u)K_{ap} \right\} = \frac{1}{2R} \int_M 2R^2 u_t + \frac{\langle \nabla u, 2R^2 \nabla K_{ap} \rangle}{K_{ap}} \right|^2 K_{ap}$$

$$+ 2R^3 \int_M \left[ \text{Hess}(K_{ap})(\nabla u, \nabla u) - \langle \nabla u, \nabla K_{ap} \rangle \right]$$

$$\left[ \frac{|\langle \nabla u, \nabla K_{ap} \rangle|^2}{K_{ap}} + \frac{c_\varepsilon(u)}{R^2} K_{ap} \right]$$

$$+ 2R \int_M c_\varepsilon(u)((K_{ap})_t - \Delta K_{ap}).$$

\[\square\]

A.3.1.2 Proof of Lemma 2.3.4

We first notice, using the notation (2.2.5), that

$$\text{Hess}(K_{ap})(\nabla u, \nabla u) = \langle \nabla_{\nabla u} \nabla K_{ap}, \nabla u \rangle = -\frac{\text{Hess}(r_+)(\nabla u, \nabla u)}{2t} K_{ap} + \frac{|\langle \nabla r_+(x), \nabla u \rangle|^2}{4t^2} K_{ap},$$

$$\frac{|\langle \nabla u, \nabla K_{ap} \rangle|^2}{K_{ap}} = \frac{|\langle \nabla r_+(x), \nabla u \rangle|^2}{4t^2} K_{ap}.$$
Next observe that by (1.2.5) we have, for $i = 1, 2$ and $x \in B_s(y)$ where $s := \min\left\{ \frac{\pi}{\sqrt{n}}, \frac{\text{inj}(M)}{2} \right\}$, that

$$\text{Hess}(r_+)(\nabla u^i, \nabla u^i) \leq \sqrt{-\lambda d(x, y) \coth(\sqrt{-\lambda d(x, y)})} |\nabla u^i|^2 \leq \left(1 - \frac{\lambda(d(x, y))^2}{3}\right) |\nabla u^i|^2$$

where $u = u^1 + iu^2$. Thus,

$$|\nabla u|^2 - \text{Hess}(r_+)(\nabla u, \nabla u) \geq \frac{\lambda(d(x, y))^2}{3} |\nabla u|^2. \quad \text{So for } x \in B_s(y) \text{ we have that}$$

$$\text{Hess}(K_{ap})(\nabla u, \nabla u) - \frac{\langle \nabla u, K_{ap}\rangle^2}{K_{ap}} + \frac{|\nabla u|^2}{2t} K_{ap} \geq \frac{\lambda(d(x, y))^2 |\nabla u|^2}{6t} K_{ap}. \quad \text{Also for } x \in M \setminus B_s(y) \text{ we have}$$

$$\text{Hess}(K_{ap})(\nabla u, \nabla u) - \frac{\langle \nabla u, K_{ap}\rangle^2}{K_{ap}} + \frac{|\nabla u|^2}{2t} K_{ap} \geq -\frac{D_M |\nabla u|^2}{t} K_{ap}$$

where $D_M$ remains bounded when dividing the metric by $0 < a \leq 1$. Integrating this we find that

$$\int_M \left[ \text{Hess}(K_{ap})(\nabla u, \nabla u) - \frac{\langle \nabla u, K_{ap}\rangle^2}{K_{ap}} + \frac{|\nabla u|^2}{2t} K_{ap} \right] \geq \int_{B_{\text{inj}(M)/2}(y)} \frac{\lambda(d(x, y))^2 |\nabla u|^2}{6t} K_{ap} - \frac{D_M |\nabla u|^2}{t} e^{-\frac{s^2}{4t}} \int_{M \setminus B_s(y)} |\nabla u|^2$$

$$\geq \int_{B_{\text{inj}(M)/2}(y)} \frac{\lambda(d(x, y))^2 |\nabla u|^2}{6t} K_{ap} - D_M' \int_M |\nabla u|^2$$

where $D_M'$ only becomes smaller when we divide the metric by $a$ where $0 < a \leq 1$. We now consider two possibilities. Either $t^+ \geq \frac{\text{inj}(M)}{2}$ or $0 < t^+ < \frac{\text{inj}(M)}{2}$. If $t^+ \geq \frac{\text{inj}(M)}{2}$ then since $\lambda < 0$ we have

$$\int_{B_{\text{inj}(M)/2}(y)} \frac{\lambda(d(x, y))^2 |\nabla u|^2}{6t} K_{ap} \geq \frac{\lambda}{6t^+} \int_M |\nabla u|^2 K_{ap}$$
where we have used that \( d(x, y) \leq \frac{\text{inj}(M)}{2} \) in \( B_{\text{inj}(M)}(y) \). If \( 0 < t^4 \leq \frac{\text{inj}(M)}{2} \) then we have, using \( \lambda < 0 \) and the notation \( A_{t^4, \frac{\text{inj}(M)}{2}}(y) := B_{\frac{\text{inj}(M)}{2}} \setminus B_{\frac{1}{4}}(y) \), that

\[
\frac{\lambda}{6t} \int_{B_{\frac{\text{inj}(M)}{2}}(y)} (d(x, y))^2 |\nabla u|^2 K_{ap} \\
= \frac{\lambda}{6t} \int_{B_{\frac{1}{4}}(y)} (d(x, y))^2 |\nabla u|^2 K_{ap} + \frac{\lambda}{6t} \int_{A_{t^4, \frac{\text{inj}(M)}{2}}(y)} (d(x, y))^2 |\nabla u|^2 K_{ap} \\
\geq \frac{\lambda}{6t} \int_{B_{\frac{1}{4}}(y)} |\nabla u|^2 K_{ap} + \frac{\lambda (\text{inj}(M))^2}{24t} \int_{A_{t^4, \frac{\text{inj}(M)}{2}}(y)} |\nabla u|^2 \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{1}{8t}} \\
\geq \frac{\lambda}{6t} \int_{B_{\frac{1}{4}}(y)} |\nabla u|^2 K_{ap} + \frac{\lambda (\text{inj}(M))^2}{24} \sup_{t>0} \left\{ \frac{1}{t(4\pi t)^{\frac{1}{2}}} e^{-\frac{1}{8t}} \right\} \int_{A_{t^4, \frac{\text{inj}(M)}{2}}(y)} |\nabla u|^2 \\
\geq \frac{\lambda}{6t} \int_M |\nabla u|^2 K_{ap} + D''_M e^{-\frac{1}{8t}} \int_M |\nabla u|^2 \\
\geq \frac{\lambda}{6t} \int_M |\nabla u|^2 K_{ap} + D''_M e^{-\frac{1}{8t}} \int_M |\nabla u|^2.
\]

Notice that the constants remain bounded when dividing the metric by \( 0 < a \leq 1 \). Putting together the previous work we can now say that

\[
\int_M \left[ \text{Hess}(K_{ap}) \langle \nabla u, \nabla u \rangle - \frac{\langle \nabla u, K_{ap} \rangle^2}{K_{ap}} + \frac{|\nabla u|^2}{2t} K_{ap} \right] \geq \frac{\lambda}{6t} \int_M |\nabla u|^2 K_{ap} \\
- 2 \max \{-D''_M, D'_M\} \int_M |\nabla u|^2.
\]

Setting

\[
e_{2} := \max \{-D''_M, D'_M\}
\]

and using that

\[
\frac{1}{2} |\nabla u|^2 \leq e_{2}(u)
\]

we obtain the desired conclusion for \( 0 < t < T \). A similar proof holds for (2.3.16) and (2.3.17) is demonstrated in the proof of the upper bound.

\[\square\]

### A.3.1.3 Proof of Proposition 2.3.5

First we show that there is \( c_5 > 0 \) such that

\[
\frac{d}{dR} \left\{ R^2 \int_M e_{2}(u_{\varepsilon}) K_{ap} \right\} \geq \left[ \frac{2\lambda}{3} - \frac{N\mu}{2} \right] \left\{ R^2 \int_M e_{2}(u_{\varepsilon}) K_{ap} \right\} - c_5 \int_M e_{2}(u) \quad (A.3.4)
\]
for $0 < R < \sqrt{T}$ where we have used the abbreviations $u_\varepsilon$ and $K_{ap}$ for $u_\varepsilon(x, T-R^2)$ and $K_{ap}(x; R^2; y)$ respectively.

For $0 < t < T$, $u = u(x, T-t)$, and $K_{ap} = K_{ap}(x; t; y)$ we have, by Lemmas 2.3.3 and 2.3.4, that

$$\int_M \left[ \text{Hess}(K_{ap})(\nabla u, \nabla u) - \frac{(\nabla u, K_{ap})^2}{K_{ap}} + \frac{|\nabla u|^2}{2t} K_{ap} \right] \geq \frac{\lambda}{3t^2} \int_M e_\varepsilon(u) K_{ap} - c_2 \int_M e_\varepsilon(u)$$

(A.3.5)

$$\int_M e_\varepsilon(u)[\partial_t K_{ap} - \Delta K_{ap}] \geq -\frac{N\mu}{4t^2} \int_M e_\varepsilon(u) K_{ap} - c_0 \int_M e_\varepsilon(u).$$

(A.3.6)

Combining (A.3.5) and (A.3.6) with Lemma 2.3.2 we conclude that, at $t = R^2 < T$ and with $u = u_\varepsilon$, we have

$$\frac{d}{dR} \left\{ R^2 \int_M e_\varepsilon(u) K_{ap} \right\} \geq 2R^3 \left[ \frac{\lambda}{3R^2} \int_M e_\varepsilon(u) K_{ap} - c_2 \int_M e_\varepsilon(u) \right]$$

(A.3.7)

$$+ 2R^3 \left[ -\frac{N\mu}{4R^2} \int_M e_\varepsilon(u) K_{ap} - c_0 \int_M e_\varepsilon(u) \right]$$

$$\geq \left[ \frac{2\lambda}{3} - \frac{N\mu}{2} \right] \left\{ R^2 \int_M e_\varepsilon(u) K_{ap} \right\} - 2(c_0 + c_2) \int_M e_\varepsilon(u)$$

which is the desired conclusion provided we set $c_5 := 2(c_0 + c_2)$.

Now we prove (2.3.18). We have, by (A.3.7), that

$$Z'(R) \geq \left[ \frac{2\lambda}{3} - \frac{N\mu}{2} \right] Z(R) - c_5 E_0$$

(A.3.8)

where $Z(R)$ and $E_0$ are as defined in (2.3.5) and (2.3.19) respectively. Choosing $C_2 := -\frac{2\lambda}{3} + \frac{N\mu}{2}$ and multiplying (A.3.8) by $e^{C_2 R}$ and rearranging, for $0 < R < \min\{\sqrt{T}, 1\}$, gives

$$\left( e^{C_2 R} Z(R) \right)' \geq -c_5 e^{C_2 R} E_0 \geq -c_5 e^{C_2} E_0.$$

(A.3.9)

Finally, integrating (A.3.9) between $0 < R_1 < R_2 < \min\{\sqrt{T}, 1\}$ we obtain

$$e^{C_2 R_2} Z(R_2) - e^{C_2 R_1} Z(R_1) \geq -c_5 (R_2 - R_1) e^{C_2} E_0$$

and hence

$$e^{C_2 R_1} Z(R_1) \leq c_5 e^{C_2} E_0 (R_2 - R_1) + e^{C_2 R_2} Z(R_2).$$

(A.3.10)
Observe that by choosing $C_1 > 0$ such that
\[ C_1 := \max\{c_5 e^{C_2}, 1\} \]
then we obtain (2.3.18) from (A.3.10). Observe that we can extend the inequalities to the boundary cases by appealing to continuity.

**Lemma A.3.1.** Let $(M, g)$ be an $N$-dimensional compact Riemannian manifold without boundary and $u \equiv u_0$ be a solution to $(PGL)_\varepsilon$ verifying $E_\varepsilon(u_0) < \infty$. For any $z_* = (x_*, t_*) \in M \times (0, \infty)$, the following equality holds for $R_* = \sqrt{t_*}:
\[
\tilde{E}_\varepsilon(z_*, R_*) = \int_{M \times [0, t_*]} (V_\varepsilon(u) + \Xi(u, z_*)) K_{ap}(x, t_* - t; x*) d\text{vol}_g(x) dt
+ \int_{M \times [0, t_*]} \Psi(u, z_*) d\text{vol}_g(x) dt + \int_{M \times [0, t_*]} \Phi(u, z_*) d\text{vol}_g(x) dt
\]
where
\[
\Xi(u, z_*)(x, t) := |t - t_*| \left| \partial_t u + \frac{\langle \nabla u, \nabla K_{ap}(x, t_* - t; x_*) \rangle}{K_{ap}(x, t_* - t; x_*)} \right|^2
\]
\[
\Phi(u, z_*)(x, t) := |t - t_*| \left[ \text{Hess}(K_{ap}(x, t_* - t; x_*))(\nabla u, \nabla u) - \frac{|\langle \nabla u, \nabla K_{ap}(x, t_* - t; x_*) \rangle|^2}{K_{ap}(x, t_* - t; x_*)} \right.
+ \left. \frac{|\nabla u|^2 K_{ap}(x, t_* - t; x_*)}{2(t_* - t)} \right]
\]
\[
\Psi(u, z_*)(x, t) := |t - t_*| e_\varepsilon(u) \left[ (\partial_t K_{ap})(x, t_* - t; x_*) - (\Delta K_{ap})(x, t_* - t; x_*) \right].
\]

In particular, if $t_* < 1$, then
\[
\int_{M \times [0, t_*]} \left[ \Xi(u, z_*) + V_\varepsilon(u) \right] K_{ap} + \int_{M \times [0, t_*]} \Phi(u, z_*) + \left( \frac{-\lambda}{9} + 1 \right) C_1 E_0 t_* + \frac{-\lambda e^{C_2}}{6} \tilde{E}_\varepsilon(u, t_*^\frac{1}{2})
+ \int_{M \times [0, t_*]} \Psi(u, z_*) + \left( \frac{N \mu}{4} + 1 \right) C_1 E_0 t_* + \frac{N \mu e^{C_2}}{4} \tilde{E}_\varepsilon(u, t_*^\frac{1}{2})
\]
\[
= \left( \frac{1 + N \mu e^{C_2}}{4} + \frac{-\lambda e^{C_2}}{6} \right) \tilde{E}_\varepsilon(u, t_*^\frac{1}{2}) + \left( \frac{2 + N \mu}{4} + \frac{-\lambda}{6} \right) C_1 E_0 t_*.
\]

**Proof.** Following the proof of Lemma 2.2 from [11] we integrate the monotonicity formula found in
Lemma 2.3.2 in $R$ between 0 and $R_*$ gives

$$
\tilde{E}_g(z_*, R_*) = \int_0^{R_*} \int_M 2R(\Xi(u, z_*))(x, t_*) - R^2)K_{ap}(x, R^2; x_*) + \int_0^{R_*} \int_M 2RV_e(u)K_{ap}(x, R^2; x_*)
$$

$$
+ \int_0^{R_*} \int_M 2R(\Phi(u, z_*))(x, t_*) - R^2) + \int_0^{R_*} \int_M 2R(\Psi(u, z_*))(x, t_*) - R^2)
$$

$$
= \int_0^{R_*} \int_M \Xi(u, z_*)K_{ap}(x, t_* - t; x_*) + \int_0^{R_*} \int_M V_e(u)K_{ap}(x, t_* - t; x_*)
$$

$$
+ \int_0^{R_*} \int_M (\Phi(u, z_*))(x, t) + \int_0^{R_*} \int_M (\Psi(u, z_*))(x, t)
$$

where we have used the change of variables $t = t_* - R^2$. This gives the desired equality. Next, observe that for $0 < t < t_* < 1$ we have by applying Lemma 2.3.5 that

$$
0 \leq \int_{M \times [0, t_*]} \Psi(u, z_*) + \frac{N\mu}{4} \int_0^{t_*} \tilde{E}_g(u, |t - t_*|^\frac{3}{2}) dt + c_0 t_* E_0
$$

$$
0 \leq \int_{M \times [0, t_*]} \Phi(u, z_*) + \frac{-\lambda}{3} \int_0^{t_*} \tilde{E}_g(u, |t - t_*|^\frac{3}{2}) dt + c_2 t_* E_0.
$$

We also note that by Proposition 2.3.5 we have, since $t_* < 1$

$$
\int_0^{t_*} \tilde{E}_g(u, |t - t_*|^\frac{3}{2}) dt \leq \int_0^{t_*} \left[ C_1 E_0 t_*^\frac{3}{2} + e^{C_2} \tilde{E}_g(u, t_*^\frac{3}{2}) \right] dt \leq C_1 E_0 t_*^\frac{3}{2} + e^{C_2} \tilde{E}_g(u, t_*^\frac{3}{2}).
$$

Thus, since $C_1 \geq \max\{c_0, c_2\}$ and $t_* < 1$, we have that

$$
0 \leq \int_{M \times [0, t_*]} \Psi(u, z_*) + \left( \frac{N\mu}{4} + 1 \right) C_1 E_0 t_* + \frac{N\mu C_2}{4} \tilde{E}_g(u, t_*^\frac{3}{2})
$$

$$
0 \leq \int_{M \times [0, t_*]} \Phi(u, z_*) + \left( \frac{-\lambda}{3} + 1 \right) C_1 E_0 t_* + \frac{-\lambda C_2}{3} \tilde{E}_g(u, t_*^\frac{3}{2}).
$$

Thus,

$$
\int_{M \times [0, t_*]} \Xi(u, z_*) + V_e(u)K_{ap} + \left[ \int_{M \times [0, t_*]} \Phi(u, z_*) + \left( \frac{-\lambda}{3} + 1 \right) C_1 E_0 t_* + \frac{-\lambda C_2}{3} \tilde{E}_g(u, t_*^\frac{3}{2}) \right]
$$

$$
+ \left[ \int_{M \times [0, t_*]} \Psi(u, z_*) + \left( \frac{N\mu}{4} + 1 \right) C_1 E_0 t_* + \frac{N\mu C_2}{4} \tilde{E}_g(u, t_*^\frac{3}{2}) \right]
$$

$$
= \left( 1 + \frac{N\mu C_2}{4} + \frac{-\lambda C_2}{3} \right) \tilde{E}_g(u, t_*^\frac{3}{2}) + \left( 2 + \frac{N\mu}{4} + \frac{-\lambda}{3} \right) C_1 E_0 t_0.
$$
A.3.2 Weighted Energy Comparison

A.3.2.1 Proof of Lemma 2.3.7

We follow the proof of Lemma 2.3 from [11] but a careful estimate is required to obtain the exponential term. By definition we have

\[\tilde{E}_\varepsilon (z_*, \sqrt{t_*}) = \frac{1}{(4\pi)^{N/2} T^{N/2 - 1}} \int_M e_\varepsilon (u_\varepsilon (x, 0)) e^{-\frac{(d_+ (x, x_*))^2}{4t_*}} \]

\[= \left( \frac{T}{t_*} \right)^{N/2 - 1} \frac{1}{(4\pi)^{N/2} T^{N/2 - 1}} \int_M e_\varepsilon (u_\varepsilon (x, 0)) e^{-\frac{(d_+ (x, x_0))^2}{4t_*}} Q(x)\]

where \(Q\) is defined as

\[Q(x) := e^{\frac{(d_+ (x, x_0))^2}{4t_*}} - e^{\frac{(d_+ (x, x_*))^2}{4t_*}}.\]

We will now demonstrate that

\[\|Q\|_{L^\infty (M)} \leq e^{\frac{C f (d_+ (x, x_0))^2}{4t_*}}.\]

To do this it suffices to estimate the \(L^\infty\) norm of \(x \mapsto (d_+ (x, x_0))^2 - (d_+ (x, x_*))^2\). We achieve this by considering a few cases.

**Case 1:** \(d(x_T, x_*) \geq \frac{\text{inj} (M)}{2}\)

In this case, it follows from (4) that we have \(f \left( \frac{d(x_T, x_*)}{\text{inj} (M)} \right) \geq f \left( \frac{1}{2} \right) = \frac{1}{2} \) and so

\[\frac{(d_+ (x, x_T))^2}{4T} - \frac{(d_+ (x, x_*))^2}{4t_*} \leq \frac{\text{inj} (M))^2}{4T} \leq \frac{(d_+ (x_T, x_*))^2}{T - t_*}.\]

Thus, we may assume that \(d(x_T, x_*) < \frac{\text{inj} (M)}{2}\). This implies that \(B_{\text{inj} (M)} (x_T) \cap B_{\text{inj} (M)} (x_*) \neq \emptyset\).

**Case 2:** \(d(x, x_T) \geq \text{inj} (M)\)

In this case we have that

\[\frac{(d_+ (x, x_T))^2}{4T} - \frac{(d_+ (x, x_*))^2}{4t_*}\]

is maximized at \(x_0 \in \partial B_{\text{inj} (M)} (x_T)\) lying on the geodesic ray from \(x_T\) through \(x_*\). In particular, at
this point we have

\[
\frac{(d_+(x_0, x))^2}{4T} - \frac{(d_+(x_0, x_*)^2}{4t_*} = \frac{(d(x_0, x))^2}{4T} - \frac{(d_+(x_0, x_*))^2}{4t_*} = \frac{(d(x_T, x_* + d(x_0, x_*))^2}{4T} - \frac{(d(x_0, x_*))^2}{4t_*}.
\]

Notice that for all \(z, w \in M\) we have that, due to the definition of \(d_+\) and (3), if \(d(z, w) \leq \text{inj}(M)\) then

\[
d_+(z, w) = \text{inj}(M) \frac{(d(z, w)}{	ext{inj}(M)} \geq d(z, w).
\]

Thus,

\[
\frac{(d(x_T, x_*) + d(x_0, x_*))^2}{4T} - \frac{(d_+(x_0, x_*))}{4t_*} \leq \frac{(d(x_T, x_*) + d(x_0, x_*))^2}{4T} - \frac{(d(x_0, x_*))^2}{4t_*}.
\]

We note that for a given \(a \geq 0\) the function

\[
s \mapsto \frac{(s + a)^2}{4T} - \frac{s^2}{4t_*}
\]

has a maximum value of \(\frac{a^2}{4(T - t_*)}\). Thus, for \(x \in M \setminus B_{\text{inj}(M)}(x_T)\) we have

\[
\frac{(d_+(x, x_*))^2}{4T} - \frac{(d_+(x, x_*))^2}{4t_*} \leq \frac{(d(x_T, x_*))^2}{4T} - \frac{(d(x_0, x_*))^2}{4t_*}.
\]

**Case 3: \(d(x, x_T) \leq d(x_T, x_*)\)**

In this case we have, since we are assuming that \(d(x_T, x_*) < \frac{\text{inj}(M)}{2}\),

\[
\frac{(d_+(x, x_T))^2}{4T} - \frac{(d_+(x, x_*))^2}{4t_*} \leq \frac{(d(x, x_T))^2}{4T} \leq \frac{(d(x_*, x_*))^2}{4(T - t_*)} = \frac{(d_+(x_T, x_*))^2}{4(T - t_*)}.
\]

Thus, we may also assume that \(d(x_T, x_*) < d(x, x_T)\).

**Case 4: \(x \in B_{\text{inj}(M)}(x_T)\) and \(d(x_T, x_*) < d(x, x_T)\)**

First observe that since \(d_+(x, x_T)\) and \(d_+(x, x_*\) are functions of distance from \(x_T\) and \(x_*\) respectively
then we note that for $x \in \partial B_{r \cdot \text{inj}(M)}(x_T)$ that the function
\[ \frac{(d_+(x, x_T))^2}{4T} - \frac{(d_+(x, x_*))^2}{4t_*} \]
is maximized when $x$ is the point of distance $r \cdot \text{inj}(M)$ from $x_T$ on the geodesic ray from $x_T$ through $x_*$. Since this can argument works for each $\frac{d(x_T, x_*)}{\text{inj}(M)} \leq r < 1$ then we may assume that $x$ lies on the geodesic ray emanating from $x_T$ through $x_*$. We note that distance is additive along a geodesic ray.

Notice that the function we are optimizing can be rewritten as
\[ \frac{(d_+(x, x_T))^2}{4T} - \frac{(d_+(x, x_*))^2}{4t_*} = \frac{(d_+(x, x_T))^2}{4T} - \frac{(d_+(x, x_*))^2}{4t_*} + \frac{(d_+(x, x_*))^2}{4} \left[ \frac{1}{T} - \frac{1}{t_*} \right]. \]

Observe that
\[
\frac{(d_+(x, x_T))^2}{4T} - \frac{(d_+(x, x_*))^2}{4t_*} = \frac{\left( \text{inj}(M) f \left( \frac{d(x, x_T)}{\text{inj}(M)} \right) \right)^2}{4T} - \frac{\left( \text{inj}(M) f \left( \frac{d(x, x_*)}{\text{inj}(M)} \right) \right)^2}{4T} \]
\[
= \frac{\left( \text{inj}(M) f \left( \frac{d(x, x_T) + t(x, x_T)}{\text{inj}(M)} \right) \right)^2}{4T} - \frac{\left( \text{inj}(M) f \left( \frac{d(x, x_*)}{\text{inj}(M)} \right) \right)^2}{4T} \]
\[
= \frac{d(x, x_T) \int_0^1 \left( \text{inj}(M) f \left( \frac{d(x, x_T) + t(x, x_T)}{\text{inj}(M)} \right) \right) f' \left( \frac{d(x, x_T) + t(x, x_T)}{\text{inj}(M)} \right) dt}{2T} \]
\[
\leq \frac{\| f' \|_{L^\infty([0, \infty))} d(x, x_T) (d(x, x_T) + d(x, x_*))}{T} \]
\[
= \frac{\| f' \|_{L^\infty([0, \infty))} \| f \|_{L^\infty([0, \infty))} d(x, x_T) d(x, x_*) + d(x, x_T)}{T} \]
\[
= \frac{\| f' \|_{L^\infty([0, \infty))} \left( \text{inj}(M) \frac{d(x, x_T)}{\text{inj}(M)} \right)^2 + \| f' \|_{L^\infty([0, \infty))} \left[ \text{inj}(M) \frac{d(x, x_T)}{\text{inj}(M)} \right] \left[ \text{inj}(M) \frac{d(x, x_*)}{\text{inj}(M)} \right]}{T} \]
\[
+ \frac{(d_+(x, x_*))^2}{4} \left[ \frac{1}{T} - \frac{1}{t_*} \right] \]
\[
\leq \frac{\| f' \|_{L^\infty([0, \infty))} \| f \|_{L^\infty([0, \infty))} d(x, x_T) d(x, x_*) + d(x, x_T)}{T} \]
where we have used (2.2.4). Thus, we have
\[
\frac{(d_+(x, x_T))^2}{4T} - \frac{(d_+(x, x_*))^2}{4t_*} \leq \frac{\| f' \|_{L^\infty([0, \infty))} \| f \|_{L^\infty([0, \infty))} d(x, x_T) d(x, x_*) + d(x, x_T)}{T} \]
where we have used (3) and that by assumption we have $d(x_T, x_*) < \frac{\text{inj}(M)}{2}$. We next observe that,
for $b \geq 0$, the function
\[
s \mapsto \frac{\|f\|_{L^\infty([0,\infty))} s b}{T} - \frac{s^2}{4} \left[ \frac{1}{t_*} - \frac{1}{T} \right]
\]
attains a maximum value of
\[
\frac{\|f\|_{L^\infty([0,\infty))}}{T(T-t_*)} t_* b^2
\]
where I have used that $t_* < T$. This implies
\[
\frac{\|f\|_{L^\infty([0,\infty))}}{T} \left( d_+(x_*, x_T) \right)^2 + \frac{\|f\|_{L^\infty([0,\infty))}}{T} d_+(x_*, x_t) d_+(x_*, x) + \frac{(d_+(x, x_*))^2}{4} \left[ \frac{1}{T} - \frac{1}{t_*} \right]
\leq \frac{\|f\|_{L^\infty([0,\infty))}}{T} \left( d_+(x_*, x_T) \right)^2 + \frac{\|f\|_{L^\infty([0,\infty))}}{T(T-t_*)} (d_+(x_T, x_*))^2
\leq \max \left\{ \frac{\|f\|_{L^\infty([0,\infty))}}{T-t_*}, \frac{\|f\|_{L^\infty([0,\infty))}}{T} \right\} \left( d_+(x_*, x_T) \right)^2.
\]
Putting this altogether gives
\[
\bar{E}_\varepsilon(z_*, \sqrt{t_*}) \leq \left( \frac{T}{t_*} \right)^{\frac{N-1}{2}} e^{C_f(d_+(x_T, x_*))^2} \bar{E}_\varepsilon(u_\varepsilon(x_T, 0), \sqrt{T})
\]
where $C_f := \max \left\{ 1, \|f\|_{L^\infty([0,\infty))}, \|f\|_{L^\infty([0,\infty))}^2 \right\} = \max \left\{ 1, \|f\|_{L^\infty([0,\infty))}^2 \right\}$ as we wanted. The second inequality follows from (2.2.4).

\textbf{A.3.3 Auxiliary Problem}

\textbf{A.3.3.1 Proof of Lemma 2.3.9}

We follow the proof of Lemma 2.4 from [11] but add additional estimates due to the error terms from the monotonicity formula. Let $z_* = (x_*, t_*) \in M \times [0, T]$. By Duhamel’s formula we have
\[
\omega(z_*) = \int_{M \times [0, t_*)} (K(x, t_* - t; x_*), f(x, t))
\]
and so that by (2.3.20) we have
\[
|\omega(z_*)| \leq \int_{M \times [0, t_*]} V_\varepsilon(u_\varepsilon(x, t))|K(x, t_* - t; x_*)|
\leq \int_0^{t_*} \int_{B_{\text{inj}(M)}(x_*)} V_\varepsilon(u_\varepsilon(x, t))|K(x, t_* - t; x_*)|
+ \int_0^{t_*} \int_{M \setminus B_{\text{inj}(M)}(x_*)} V_\varepsilon(u_\varepsilon(x, t))|K(x, t_* - t; x_*)|. \tag{A.3.11}
\]
We now estimate each of the integrals in (A.3.11). First notice that, by Theorem 1.1 of [38], that on \( K = \{ (x, y) \in M : d(x, y) \leq \frac{\text{inj}(M)}{2} \} \) there exists a constant, \( C_M \), dependent on \( M \), such that, for \((x, y) \in K\)

\[
|K(x, t_s - t; y)| \leq C_M(T + 1) \frac{1}{(4\pi(t_s - t))^\frac{1}{2}} e^{-\frac{(d(x, y))^2}{4(t_s - t)}}
\]

and so, since \( B_{\text{inj}(M)}(x_s) \times \{ x_s \} \subset K \), we have

\[
\int_0^{t_s} \int_{B_{\text{inj}(M)}(x_s)} V_\varepsilon(u_\varepsilon(x, t))|K(x, t_s - t; x_s)|
\]

\[
\leq C_M(T + 1) \int_0^{t_s} \int_{B_{\text{inj}(M)}(x_s)} V_\varepsilon(u_\varepsilon(x, t)) \left\{ \frac{1}{(4\pi(t_s - t))^\frac{1}{2}} e^{-\frac{(d(x, y))^2}{4(t_s - t)}} \right\}
\]

\[
= C_M(T + 1) \int_0^{t_s} \int_{B_{\text{inj}(M)}(x_s)} V_\varepsilon(u_\varepsilon(x, t))K_{ap}(x, t_s - t; x_s)
\]

\[
\leq C_M(T + 1) \int_{[0, t_s]} V_\varepsilon(u_\varepsilon(x, t))K_{ap}(x, t_s - t; x_s)
\]

\[
\leq C_M(T + 1) \left\{ e^{C_2 \bar{E}_\varepsilon(u_\varepsilon, \sqrt{t_s})} + C_1 E_0 t_s \right\}
\]

\[
\leq C_M e^{C_2}(T + 1) \left\{ \bar{E}_\varepsilon(u_\varepsilon, \sqrt{t_s}) + C_1 E_0 t_s \right\}
\]

where the second last inequality follows from Proposition 2.3.5. Next, by Theorem 3.5 of [38] that there is a constant \( D_M \)

\[
\int_0^{t_s} \int_{M \setminus B_{\text{inj}(M)}(x_s)} V_\varepsilon(u_\varepsilon(x, t))|K(x, t_s - t; x_s)|
\]

\[
\leq D_M \int_0^{t_s} \int_{M \setminus B_{\text{inj}(M)}(x_s)} V_\varepsilon(u_\varepsilon(x, t)) \left\{ \frac{1}{(t_s - t)^{N+1}} e^{-\frac{(d(x, y))^2}{4(t_s - t)^2}} \right\}
\]

\[
\leq D_M \sup_{0 < t < t_s} \left\{ e^{-\frac{(\text{inj}(M))^2}{4(t_s - t)^2}} \right\} \int_0^{t_s} \int_{M \setminus B_{\text{inj}(M)}(x_s)} V_\varepsilon(u_\varepsilon(x, t))
\]

\[
\leq D_M' E_0 t_s.
\]

Combining the previous two estimates and setting

\[
C_3 := 2 \max \{ C_M e^{C_2}, D_M' \}
\]

gives the desired conclusion. \( \square \)

**Proposition A.3.2.** Suppose \( 0 < T < 1 \) and \( x_T \in M \). For any \( z = (x, t) \in M \times [0, T] \), the following
estimate holds:

\[
|\omega(z)| \leq C_3(T + 1) \left( \frac{T}{t_*} \right)^{\frac{N}{2} - 1} e^{\frac{C_f(d(x_T,x_*))^2}{T-t_*}} \left( \tilde{E}_\varepsilon(u_x,(x_T,0),\sqrt{T}) + E_0 T \right) \quad (A.3.12)
\]

where \( C_f \) is as in Lemma 2.3.7.

**Proof.** As in Proposition 2.2 from [11] the proof follows by combining Lemmas 2.3.7 and 2.3.9. \( \square \)

### A.3.4 Bounds for the weighted energy

**Lemma A.3.3.** Let \( 0 < T < 1 \), and \((x,t) \in M \times [0,T)\). Then

\[
\tilde{E}_\varepsilon((x,t+R^2),R) \leq e^{C_2} \left( \frac{T}{t+R^2} \right)^{\frac{N}{2} - 1} e^{\frac{C_f(d(x_T,x_*))^2}{T-t-R^2}} \left\{ \tilde{E}_\varepsilon((x_T,T),\sqrt{T}) + C_1 E_0 \sqrt{T} \right\} \quad (A.3.13)
\]

for any \( x_T \in M \), and for \( 0 < R < \sqrt{T-t} \) where \( C_2 \) and \( C_f \) are as in Proposition 2.3.5 and Lemma 2.3.7 respectively.

**Proof.** The proof, as in Lemma 2.5 of [11], follows by combining the monotonicity formula from Proposition 2.3.5 and Lemma 2.3.7. \( \square \)

### A.3.5 Localizing the energy

#### A.3.5.1 Proof of Proposition 2.3.8

We follow the proof of Proposition 2.3 from [11]. Let \( R \) and \( \lambda \) be fixed as in the statement. Observe that it suffices to estimate

\[
\int_{M \setminus B_{\lambda R}(x_T)} e_x(u_x(\cdot,T)) e^{-\frac{(d_x(x_T,x)^2)}{4R^2}}.
\]

Notice that

\[
e^{-\frac{(d_x(x_T,x)^2)}{4R^2}} \leq e^{-\frac{c_2^2 (d(x_T,x)^2)(d_x(x_T,x)^2)}{8R^2}} \leq e^{-\frac{\lambda^2}{8}} e^{-\frac{(d_x(x_T,x)^2)}{8R^2}}.
\]
Thus, by Proposition 2.3.5 and \((H_0)\) we have
\[
\frac{1}{(4\pi)^{\frac{N}{2}}(\sqrt{2}R)^{N-2}} \int_{M \setminus B_N(x_T)} e_x(u_x(x,T)) e^{-\frac{(d_+(x,x_T))^2}{4t}} \\
\leq e^{-\frac{\lambda^2}{8}} \frac{e^{C_2}}{(4\pi)^{\frac{N}{2}}(T + 2R^2)^{\frac{N-2}{2}}} \int_M e_x(u_x(\cdot,0)) e^{-\frac{(d_+(x,x_T))^2}{4T}} + C_1 E_0 [\sqrt{T + 2R^2} - \sqrt{2R^2}] \\
\leq e^{-\frac{\lambda^2}{8}} M_0 \left[ e^{C_2} \frac{e^{C_2}}{(4\pi)^{\frac{N}{2}}(T + 2R^2)^{\frac{N-2}{2}} + C_1 \sqrt{T}} \right] \log(\varepsilon).
\]

Rearranging gives the desired inequality.

\[\square\]

**Lemma A.3.4.** Let \((M,g)\) be an \(N\)-dimensional compact Riemannian manifold without boundary and suppose that \(0 < T_1 \leq T_2 < T\), \(x_T \in M\), and \(z_T = (x_T,T)\). Then,
\[
\int_{T_1}^{T_2} \int_M (d_+(x,x_T))^2 \frac{e_x(u_x(x,T)) e^{-\frac{(d_+(x,x_T))^2}{4(t - t)}}}{4(T - t)} \\
\leq [4\pi(T - T_1)]^{\frac{N}{2}} D_f \tilde{E}_x(z_T,\sqrt{T - T_1}) - [4\pi(T - T_2)]^{\frac{N}{2}} D_f \tilde{E}_x(z_T,\sqrt{T - T_2})
\]
where \(D_f := (2 - \|f'\|^2_{L^\infty(\Omega)})^{-1}\).

**Proof.** We follow the proof of Lemma 2.6 from [11]. We will use the notation (2.2.5) with \(p = x_T\). Taking the dot product of equation (PGL)_x and \(2(T-t)\partial_t u_x e^{-\frac{(d_+(x,x_T))^2}{4(t - t)}}\), as well as integrating over \(M \times [T_1, T_2]\) one obtains, after integrating by parts, that
\[
\int_{T_1}^{T_2} \int_M 2(T-t) |\partial_t u_x|^2 e^{-\frac{(d_+(x,x_T))^2}{4(t - t)}} dv(x)dt \\
= \int_{T_1}^{T_2} \int_M 2(T-t) (\partial_t u_x \cdot \Delta u_x) e^{-\frac{(d_+(x,x_T))^2}{4(t - t)}} dv(x)dt \\
- \int_{T_1}^{T_2} \int_M 2(T-t) \frac{\partial}{\partial t} [V_x(u_x)] e^{-\frac{(d_+(x,x_T))^2}{4(t - t)}} dv(x)dt \\
= - \int_{T_1}^{T_2} \int_M (T-t) \frac{\partial}{\partial t} \left[ |\nabla u_x|^2 + 2V_x(u_x) \right] e^{-\frac{(d_+(x,x_T))^2}{4(t - t)}} dv(x)dt \\
+ \int_{T_1}^{T_2} \int_M \left[ (\nabla u_x, \nabla r_+(x)) \cdot \partial_t u_x \right] e^{-\frac{(d_+(x,x_T))^2}{4(t - t)}} dv(x)dt.
\]
Integrating the first term of the final equality by parts in time leads to

\[- \int_{T_1}^{T_2} \int_M (T-t) \frac{\partial}{\partial t} \left| \nabla u_x \right|^2 + 2V_x(u_x) e^{-\frac{(d_x(x,x_T))^2}{4(T-t-\tau)}} \ dv(x) \ dt\]

\[= -(T-T_2) \int_{M \times \{T_2\}} \left| \nabla u_x \right|^2 + 2V_x(u_x) e^{-\frac{(d_x(x,x_T))^2}{4(T-t-\tau)}} \ dv(x)\]

\[+ (T-T_1) \int_{M \times \{T_1\}} \left| \nabla u_x \right|^2 + 2V_x(u_x) e^{-\frac{(d_x(x,x_T))^2}{4(T-t-\tau)}} \ dv(x)\]

\[ - \int_{T_1}^{T_2} \int_M \left[ \nabla u_x \right]^2 + 2V_x(u_x) e^{-\frac{(d_x(x,x_T))^2}{4(T-t-\tau)}} \ dv(x) dt\]

\[- \int_{T_1}^{T_2} \int_M \frac{(d_x(x,x_T))^2}{4(T-t)} \left[ \nabla u_x \right]^2 + 2V_x(u_x) e^{-\frac{(d_x(x,x_T))^2}{4(T-t-\tau)}} \ dv(x) dt\]

Inserting the previous computation and adding

\[\int_{T_1}^{T_2} \int_M \frac{1}{2(T-t)} \left| \langle \nabla u_x, \nabla r_+(x) \rangle \right|^2 e^{-\frac{(d_x(x,x_T))^2}{4(T-t-\tau)}} \ dv(x) \ dt\]

to both sides leads to

\[\int_{T_1}^{T_2} \int_M \frac{1}{2(T-t)} \left| \langle \nabla u_x, \nabla r_+(x) \rangle \right|^2 - 2(T-t) \frac{\partial_t u_x}{2(t-t-\tau)} e^{-\frac{(d_x(x,x_T))^2}{4(T-t-\tau)}} \ dv(x) \ dt\]

\[+ \int_{T_1}^{T_2} \int_M \left( \frac{1}{2(T-t)} + \frac{(d_x(x,x_T))^2}{4(T-t)} \right) \left| \nabla u_x \right|^2 + 2V_x(u_x) e^{-\frac{(d_x(x,x_T))^2}{4(T-t-\tau)}} \ dv(x) dt\]

\[= -(T-T_2) \int_{M \times \{T_2\}} \left| \nabla u_x \right|^2 + 2V_x(u_x) e^{-\frac{(d_x(x,x_T))^2}{4(T-t-\tau)}} \ dv(x)\]

\[+ (T-T_1) \int_{M \times \{T_1\}} \left| \nabla u_x \right|^2 + 2V_x(u_x) e^{-\frac{(d_x(x,x_T))^2}{4(T-t-\tau)}} \ dv(x)\]

\[+ \int_{T_1}^{T_2} \int_M \frac{\langle \nabla u_x, \nabla r_+(x) \rangle}{2(T-t)} \cdot \left[ \left| \langle \nabla u_x, \nabla r_+(x) \rangle \right| - 2(T-t) \frac{\partial_t u_x}{e^{-\frac{(d_x(x,x_T))^2}{4(T-t-\tau)}}} \ dv(x) dt.\]

Next, using the Cauchy-Schwarz inequality for complex numbers as well as the inequality \(ab \leq \frac{a^2}{4} + b^2\) where we choose

\[a = \frac{\left| \langle \nabla u_x, \nabla r_+(x) \rangle \right|}{\sqrt{2(T-t)}} e^{-\frac{(d_x(x,x_T))^2}{8(T-t-\tau)}}\]

\[b = \frac{\left| \langle \nabla u_x, \nabla r_+(x) \rangle \right| - 2(T-t) \frac{\partial_t u_x}{e^{-\frac{(d_x(x,x_T))^2}{4(T-t-\tau)}}}}{\sqrt{2(T-t)}}\]
to obtain, after applying the Cauchy-Schwarz inequality for the metric inner product, that
\[
\int_{T_1}^{T_2} \frac{1}{M} 2(1-T) \left( \nabla u_{e}, \nabla r_+(x) \right) - 2(T-t) \partial_t u_{e}^2 e^{-\frac{(d_{e}(x,x_{T}))^2}{2(T-t)}} \text{dvol}_g(x)dt
\]
\[
+ \int_{T_1}^{T_2} \int_{M} \left[ 1 + \frac{(d_{e}(x,x_{T}))^2}{(4(T-t)-t)^2} \right] \left( \nabla u_{e} \right)^2 e^{-\frac{(d_{e}(x,x_{T}))^2}{(4(T-t)-t)^2}} \text{dvol}(x)dt
\]
\[
\leq - (T-T_2) \int_{M \times \{T_2\}} \left( \nabla u_{e} \right)^2 e^{-\frac{(d_{e}(x,x_{T}))^2}{(4(T-t)-t)^2}} \text{dvol}(x)
\]
\[
+ (T-T_1) \int_{M \times \{T_1\}} \left( \nabla u_{e} \right)^2 e^{-\frac{(d_{e}(x,x_{T}))^2}{(4(T-t)-t)^2}} \text{dvol}(x)
\]
\[
+ \int_{T_1}^{T_2} \int_{M} \left[ \frac{1}{2} + \frac{\| f' \|^2_{L^\infty(\mathbb{R})}}{8(T-t)} \right] \left( \nabla u_{e} \right)^2 e^{-\frac{(d_{e}(x,x_{T}))^2}{(4(T-t)-t)^2}} \text{dvol}(x)dt
\]

Using the definition of $e_{e}(u_{e})$ we obtain, after bounding the right-hand side, that
\[
\int_{T_1}^{T_2} \int_{M} \left( 2 + \frac{(d_{e}(x,x_{T}))^2}{2(T-t)} \right) e_{e}(u_{e}) e^{-\frac{(d_{e}(x,x_{T}))^2}{(4(T-t)-t)^2}} \text{dvol}(x)dt
\]
\[
\leq - (T-T_2) \int_{M \times \{T_2\}} \left( \nabla u_{e} \right)^2 e^{-\frac{(d_{e}(x,x_{T}))^2}{(4(T-t)-t)^2}} \text{dvol}(x)
\]
\[
+ (T-T_1) \int_{M \times \{T_1\}} \left( \nabla u_{e} \right)^2 e^{-\frac{(d_{e}(x,x_{T}))^2}{(4(T-t)-t)^2}} \text{dvol}(x)
\]
\[
+ \int_{T_1}^{T_2} \int_{M} \left[ \frac{(d_{e}(x,x_{T}))^2}{4(T-t)} \right] e_{e}(u_{e}) e^{-\frac{(d_{e}(x,x_{T}))^2}{(4(T-t)-t)^2}} \text{dvol}(x)dt.
\]

Rearranging now gives
\[
\int_{T_1}^{T_2} \int_{M} \left( 2 + \frac{(d_{e}(x,x_{T}))^2}{2(T-t)} \right) e_{e}(u_{e}) e^{-\frac{(d_{e}(x,x_{T}))^2}{(4(T-t)-t)^2}} \text{dvol}(x)dt
\]
\[
\leq - (T-T_2) \int_{M \times \{T_2\}} \left( \nabla u_{e} \right)^2 e^{-\frac{(d_{e}(x,x_{T}))^2}{(4(T-t)-t)^2}} \text{dvol}(x)
\]
\[
+ (T-T_1) \int_{M \times \{T_1\}} \left( \nabla u_{e} \right)^2 e^{-\frac{(d_{e}(x,x_{T}))^2}{(4(T-t)-t)^2}} \text{dvol}(x).
\]

We can use this inequality to now conclude that
\[
\int_{T_1}^{T_2} \int_{M} \frac{(d_{e}(x,x_{T}))^2}{4(T-t)} e_{e}(u_{e}) e^{-\frac{(d_{e}(x,x_{T}))^2}{(4(T-t)-t)^2}} \text{dvol}(x)dt
\]
\[
\leq - D_{f}(T-T_2) \int_{M \times \{T_2\}} \left( \nabla u_{e} \right)^2 e^{-\frac{(d_{e}(x,x_{T}))^2}{(4(T-t)-t)^2}} \text{dvol}(x)
\]
\[
+ D_{f}(T-T_1) \int_{M \times \{T_1\}} \left( \nabla u_{e} \right)^2 e^{-\frac{(d_{e}(x,x_{T}))^2}{(4(T-t)-t)^2}} \text{dvol}(x)
\]
where \( D_f := 2 - \|f''\|_{L^\infty(\mathbb{R})}^2 \). Note that this is the inequality we were interested in.

### A.3.5.2 Proof of Proposition 2.3.10

We follow the proof of Proposition 2.4 of [11] except we require additional estimates for error terms coming from Proposition 2.3.2. Let \( 0 < t < T \) be given. We apply Lemma A.3.4 with \( T_1 = t \), \( T_2 = t + \Delta t \), for \( 0 < \Delta t < T - t \), divide by \( \Delta t \), and let \( \Delta t \) tend to zero from the right to obtain

\[
\int_{M \times \{t\}} e(x) (d_+(x, x_T))^2 \frac{4(T-t)}{e^{-\frac{(d_+(x, x_T))^2}{4(T-t)}}} \text{dvol}(x) \\
\leq g'(T-t),
\]

where

\[
g(s):=(4\pi s)^{\frac{N}{2}} D_f \tilde{E}_e(z_T, \sqrt{s}).
\]

Since

\[
g'(T-t) = \frac{ND_f(4\pi)^{\frac{N}{2}}}{2} (T-t)^{\frac{N-2}{2}} \tilde{E}_e(z_T, \sqrt{T-t}) + \frac{(4\pi)^{\frac{N}{2}} D_f(T-t)^{\frac{N-1}{2}}}{2} \frac{d\tilde{E}_e}{dR}(\sqrt{T-t}),
\]

we obtain, using 2.3.15

\[
\int_{M \times \{t\}} e(x) (d_+(x, x_T))^2 \frac{4(T-t)}{e^{-\frac{(d_+(x, x_T))^2}{4(T-t)}}} \leq \frac{N(4\pi)^{\frac{N}{2}} D_f(T-t)^{\frac{N-1}{2}}}{2} \tilde{E}_e(z_T, \sqrt{T-t}) + \int_{M \times \{t\}} [V_e(u_e) + \Xi(u_e, z_T)] K_{ap}(x, T-t; x_T) \\
+ \frac{(4\pi)^{\frac{N}{2}} D_f(T-t)^{\frac{N}{2}}}{2} \int_{M \times \{t\}} [\Phi(u_e, z_T)(x,t) + \Psi(u_e, z_T)(x,t)] \text{dvol}(x).
\]

Next we use inequalities (2.3.12) and (2.3.17) to obtain

\[
\int_{M \times \{t\}} [\Phi(u_e, z_T)(x,t) + \Psi(u_e, z_T)(x,t)] \\
\leq \left[ \frac{-2N\lambda}{3} + \mu \right] (T-t) \int_{M} (d_+(x, y))^2 \frac{e(x)}{4(T-t)} e(x)(x, T-t; x_T) \\
+ (C_M + D_M) \int_{M} e(x)(x, T-t; x_T) \\
\leq \left[ \frac{-2N\lambda}{3} + \mu \right] (T-t) \int_{M} (d_+(x, y))^2 \frac{e(x)}{4(T-t)} e(x)(x, T-t; x_T) \\
+ \frac{(C_M + D_M)}{T-t} \tilde{E}_e(z_T, \sqrt{T-t}) + C_0 [(T-t) E_0].
\]
Assuming that $t$ is chosen so that $1 - C_5(T - t) \geq \frac{1}{2}$, where $C_5 := (4\pi)^{\frac{N}{2}} D_f \left[ \mu - \frac{2N \lambda}{3} \right]$, then we can combine the previous work to obtain

$$
\int_{M \times \{t\}} e_c(u_e) \frac{(d_+ \langle x, x_T \rangle)^2}{4(T - t)} e^{- \frac{(d_+(x,x_T))^2}{4(T - t)}}
$$

$$
\leq 2(4\pi)^{\frac{N}{2}} D_f [N + C_M + D_M] (T - t)^{-\frac{N}{2} - 1} e_c(z_T, \sqrt{T - t}) + 2(4\pi)^{\frac{N}{2}} D_f C_0 [(T - t)^{\frac{N}{2} + 1} E_0]
$$

$$
+ 2(4\pi)^{\frac{N}{2}} D_f (T - t)^{\frac{N}{2}} \int_{M \times \{t\}} [V_c(u_e) + \Xi(u_e, z_T)] K_{ap}(x, T - t; x_T).
$$

This establishes the first inequality. Now we consider the region

$$
A := \left\{ x \in M : \frac{(d_+ \langle x, x_T \rangle)^2}{8(T - t)} \leq C_6 \right\}
$$

where $C_6 := 2D_f [N + C_M + D_M]$. From (2.3.22) we obtain

$$
\int_{M \times \{t\}} e_c(u_e) \frac{(d_+ \langle x, x_T \rangle)^2}{4(T - t)} e^{- \frac{(d_+(x,x_T))^2}{4(T - t)}} \leq 2D_f [N + C_M + D_M] \int_{A \times \{t\}} e_c(u_e) e^{- \frac{(d_+(x,x_T))^2}{4(T - t)}} \quad (A.3.15)
$$

$$
+ \int_{(M \setminus A) \times \{t\}} e_c(u_e) \frac{(d_+ \langle x, x_T \rangle)^2}{8(T - t)} e^{- \frac{(d_+(x,x_T))^2}{4(T - t)}}
$$

$$
+ 2(4\pi)^{\frac{N}{2}} D_f (T - t)^{\frac{N}{2}} \int_{M \times \{t\}} [V_c(u_e) + \Xi(u_e, z_T)] K_{ap}(x, T - t; x_T)
$$

$$
+ 2(4\pi)^{\frac{N}{2}} D_f C_0 [(T - t)^{\frac{N}{2} + 1} E_0].
$$

Thus, rearranging and bounding below gives

$$
\int_{(M \setminus A) \times \{t\}} 2D_f [N + C_M + D_M] e_c(u_e) e^{- \frac{(d_+(x,x_T))^2}{4(T - t)}} \quad (A.3.16)
$$

$$
\leq 2D_f [N + C_M + D_M] \int_{A \times \{t\}} e_c(u_e) e^{- \frac{(d_+(x,x_T))^2}{4(T - t)}}
$$

$$
+ 2(4\pi)^{\frac{N}{2}} D_f (T - t)^{\frac{N}{2}} \int_{M \times \{t\}} [V_c(u_e) + \Xi(u_e, z_T)] K_{ap}(x, T - t; x_T)
$$

$$
+ 2(4\pi)^{\frac{N}{2}} D_f C_0 [(T - t)^{\frac{N}{2} + 1} E_0].
$$
and finally

\[
\int_{M \times \{t\}} e_\varepsilon(u_\varepsilon)\ e^{-\frac{(d_x(x,x_T))^2}{4(t-t')}} \leq 2 \int_{A \times \{t\}} e_\varepsilon(u_\varepsilon)\ e^{-\frac{(d_x(x,x_T))^2}{4(t-t')}}
\]

\[
+ \frac{(4\pi)^{\frac{N}{2}}}{N + C_M + D_M} \int_{M \times \{t\}} [V_\varepsilon(u_\varepsilon) + \Xi(u_\varepsilon, z_T)]K_{ap}(x, T-t; x_T)
\]

\[
+ \frac{(4\pi)^{\frac{N}{2}} D_f C_0}{N + C_M + D_M} \left[(T-t)^{\frac{N}{2} + 1} + C_7 E_0\right]
\]

\[
\leq 2 \int_{A \times \{t\}} e_\varepsilon(u_\varepsilon)\ e^{-\frac{(d_x(x,x_T))^2}{4(t-t')}} + \frac{(4\pi)^{\frac{N}{2}}}{N} \int_{M \times \{t\}} [V_\varepsilon(u_\varepsilon) + \Xi(u_\varepsilon, z_T)]K_{ap}(x, T-t; x_T).
\]

\[
+ \frac{(4\pi)^{\frac{N}{2}} C_0}{N} \left[(T-t)^{\frac{N}{2} + 1} + C_7 E_0\right]
\]

A.3.6 Choice of an appropriate scaling

A.3.6.1 Proof of Proposition 2.3.11

We follow the proof of Proposition 2.5 of [11] with slight modifications due to the error terms from Proposition 2.3.5. We let \(0 < R \leq \min\{\sqrt{T}, 1\}\) and we introduce the function \(\phi: [0, \infty) \to \mathbb{R}\) defined by

\[
\phi(r) := e^{C_8 r} Z(r) + C_7 e^{C_8 E_0} r
\]

where \(C_7 := C_1 e^{2C_2}\) and \(C_8 := 2C_2\). From (A.3.8) and the choices made for \(C_1\) and \(C_2\) we obtain for \(r^2 \leq \min\{T, 1\}\)

\[
Z'(r) \geq -C_2 Z(r) - c_5 E_0 \geq -C_8 Z(r) - C_1 E_0.
\]

and hence

\[
(e^{C_8 r} Z(r))' \geq -C_1 e^{2C_2 E_0} = -C_7 E_0 \geq -C_1 e^{C_8 E_0}.
\]

Thus, \(\phi\) is non-decreasing on \([0, \min\{\sqrt{T}, 1\}]\). We make the choice \(\varepsilon_1 := R^4 \delta^8\). Now, given \(0 < \varepsilon \leq \varepsilon_1\) we set

\[
k_0 := \left\lfloor \frac{\log(\varepsilon) - 2\log(R)}{2\log(\delta)} \right\rfloor.
\]
Observe that, by our choices, we have

\[
\log(\epsilon) + \frac{\log(R)}{4 \log(\delta)} - \log(\delta) - 2 = \log(\epsilon) = \frac{\log(\epsilon)}{2} - \log(\delta) - 2 = \log(\epsilon) + \frac{\log(R)}{4 \log(\delta)} - \log(\delta) - 2 \leq 0.
\] (A.3.19)

Observe that by monotonicity of \( \phi \)

\[
\sum_{j=2}^{k_0} \left[ \phi(\delta_0^{j-1}R) - \phi(\delta_0^j R) \right] = \phi(\delta_0^k R) - \phi(\delta_0^{k_0} R) \leq \phi(R). \] (A.3.20)

From (A.3.20) and (A.3.19) we can find \( k_1 \in \{2, 3, \ldots, k_0\} \) for which

\[
\phi(\delta_0^{k_1-1} R) - \phi(\delta_0^k R) \leq \frac{\phi(R) - \phi(\delta_0^{k_0} R)}{k_0 - 1} \leq 4 \frac{\log(\delta_0)}{\log(\epsilon)} \phi(R). \] (A.3.21)

Setting \( R_1 = \delta_0^{k_1-1} R \) we can rewrite (A.3.21) as

\[
C_7 e^{C_8} E_0(1 - \delta_0) R_1 + e^{C_8 R_1} Z(R_1) - e^{C_8 \delta_0 R_1} Z(\delta_0 R_1) \leq 4 \frac{\log(\delta_0)}{\log(\epsilon)} \left[ e^{C_8 R Z(R)} + R C_7 e^{C_8 E_0} \right]. \] (A.3.22)

Multiplying both sides of (A.3.22) by \( e^{-C_2 \delta_0 R_1} \) and estimating the right-hand side leads to

\[
C_7 e^{C_8 (1 - \delta_0 R_1)} E_0(1 - \delta_0) R_1 + e^{C_8 (1 - \delta_0) R_1} Z(R_1) - Z(\delta_0 R_1) \leq 4 C_7 e^{C_8 \frac{\log(\delta_0)}{\log(\epsilon)}} \left[ Z(R) + R E_0 \right]. \] (A.3.23)

Using that \( 0 < \delta_0, R_1 < 1 \) in (A.3.23) gives

\[
C_7 E_0(1 - \delta_0) R_1 + Z(R_1) - Z(\delta_0 R_1) \leq 4 C_7 e^{C_8 \frac{\log(\delta_0)}{\log(\epsilon)}} \left[ Z(R) + R E_0 \right] \] (A.3.24)
which can be rearranged to give (2.3.25). Finally, noting that by Corollary 2.3.2 we have

\[
Z(R_1) - Z(\delta R_1) = \int_{\delta R_1}^{R_1} \frac{dZ}{ds}(s)ds = \int_{\delta R_1}^{R_1} 2s \int_M (\Xi(u_z, z_T))(x, T - s^2)Kap(x, s^2; x_T)dvol_g(x)ds \\
+ \int_{\delta R_1}^{R_1} 2s \int_M V\varepsilon(u_z(x, T - s^2))Kap(x, s^2; x_T)dvol_g(x)ds \\
+ \int_{\delta R_1}^{R_1} 2s \int_M (\Phi(u_z, z_T))(x, T - s^2)dvol_g(x)ds \\
+ \int_{\delta R_1}^{R_1} 2s \int_M (\Psi(u_z, z_T))(x, T - s^2)dvol_g(x)ds \\
= \int_{T - R_1^2}^{T - (\delta R_1)^2} \int_M (\Xi(u_z, z_T))(x, t) + V\varepsilon(u_z(x, t)))Kap(x, T - t; x_T)dvol_g(x)dt \\
+ \int_{T - R_1^2}^{T - (\delta R_1)^2} (\Phi(u_z, z_T))(x, t)dvol_g(x)dt \\
+ \int_{T - R_1^2}^{T - (\delta R_1)^2} (\Psi(u_z, z_T))(x, t)dvol_g(x)dt.
\]

We note that by Lemma 2.3.5 we have, for \(T - R_1^2 \leq t \leq T - (\delta R_1)^2\), that

\[
\int_M (\Psi(u_z, z_T))(x, t)dvol_g(x) \geq -\frac{N\mu\sqrt{T-t}}{4}Z(\sqrt{T-t}) - c_0(T-t)E_0 \geq -\frac{N\mu\sqrt{T-t}}{4}Z(\sqrt{T-t}) - c_0E_0 \\
\geq -\frac{N\mu\varepsilon C_2(1-\delta)R_1}{4}Z(R_1) - \left(c_0 + \frac{C_1N\mu(1-\delta)R_1}{4}\right)E_0
\]

and so

\[
\int_{T - R_1^2}^{T - (\delta R_1)^2} \int_M [(\Phi(u_z, z_T))(x, t) + (\Psi(u_z, z_T))(x, t)]dvol_g(x)dt \\
\geq - (1 - \delta^2)R_1^2 \left[\frac{-\lambda}{3} + \frac{N\mu}{4}\right] e^{C_2(1-\delta)R_1} Z(R_1) \\
- (1 - \delta^2)R_1^2 E_0 \left(c_0 + c_2 + \frac{C_1N\mu(1-\delta)R_1}{4} - \frac{C_1\lambda(1-\delta)R_1}{3}\right)E_0 \\
\geq - (1 - \delta)R_1 \left[\frac{-2\lambda}{3} + \frac{N\mu}{2}\right] e^{C_2(1-\delta)R_1} Z(R_1) \\
- (1 - \delta)R_1 E_0 \left(2[c_0 + c_2] + C_1 \left[\frac{N\mu}{2} - \frac{2\lambda}{3}\right] (1 - \delta)R_1\right) \\
\geq - C_2(1 - \delta)R_1 e^{C_2(1-\delta)R_1} Z(R_1) - (1 - \delta)R_1 E_0(C_1 + C_1C_2(1 - \delta)R_1) \\
\geq - C_2(1 - \delta)R_1 e^{C_2(1-\delta)R_1} Z(R_1) - (1 - \delta)R_1 E_0C_1(C_2 + 1)
\]
where \( C_1 \) and \( C_2 \) are the constants as defined in Proposition 2.3.5. Next observe that, by our previous work, we have

\[
e^{C_8(1-\delta)R_1} Z(R_1) - Z(\delta R_1) = (e^{C_8(1-\delta)R_1} - 1) Z(R_1) + (Z(R_1) - Z(\delta R_1))
\]

\[
\geq (e^{C_8(1-\delta)R_1} - 1) Z(R_1)
\]

\[
+ \int_{T-(\delta R_1)^2}^{T} \int_M ((\Xi(u_\varepsilon, z\varepsilon))(x, t) + V_\varepsilon(u_\varepsilon(x, t))) K_{ap}(x, T-t; x_T) dvol_\varepsilon(x) dt
\]

\[
- C_2(1-\delta)R_1 e^{C_2(1-\delta)R_1} Z(R_1) - (1-\delta)R_1 E_0 C_1 (C_2 + 1).
\]

Combining the above with (A.3.24), our choice of \( C_8 \), and that \( C_7 \geq C_1 (C_2 + 1) \) gives

\[
\left[ e^{2C_2(1-\delta)R_1} - C_2(1-\delta)R_1 e^{C_2(1-\delta)R_1} - 1 \right] Z(R_1)
\]

\[
+ \int_{T-(\delta R_1)^2}^{T} \int_M ((\Xi(u_\varepsilon, z\varepsilon))(x, t) + V_\varepsilon(u_\varepsilon(x, t))) K_{ap}(x, T-t; x_T) dvol_\varepsilon(x) dt
\]

\[
\leq \frac{4C_7 e^{C_8}}{|\log(\varepsilon)|} [RE_0 + Z(R)].
\]

Finally, note that if we let \( z = C_2(1-\delta)R_1 \) then

\[
e^{2C_2(1-\delta)R_1} - C_2(1-\delta)R_1 e^{C_2(1-\delta)R_1} - 1 = e^{2z} - ze^z - 1.
\]

Since, for \( z \geq 0 \), we have

\[
(e^{2z} - ze^z - 1)' = 2e^{2z} - e^z - ze^z = e^z(e^z - 1) + e^z(e^z - z) > 0
\]

and since at \( z = 0 \) we have \( e^{2z} - ze^z - 1 = 0 \) then for all \( z \geq 0 \) we obtain

\[
e^{2z} - ze^z - 1 \geq 0.
\]

Using this inequality with \( z = C_2(1-\delta)R_1 \) gives

\[
\int_{T-(\delta R_1)^2}^{T} \int_M ((\Xi(u_\varepsilon, z\varepsilon))(x, t) + V_\varepsilon(u_\varepsilon(x, t))) K_{ap}(x, T-t; x_T) dvol_\varepsilon(x) dt
\]

\[
\leq \frac{4C_7 e^{2C_8}}{|\log(\varepsilon)|} [RE_0 + Z(R)]
\]

which proves (2.3.26).
A.4 Proof of Theorem 2.1.2

In this section many metric dependent quantities will be in terms of the rescaled metric $g_{R_1}$. As a result, we may omit mention of $g_{R_1}$ when no confusion will arise.

A.4.1 Localizing the energy on appropriate time slices

Next we introduce the notation

$$A(t) := \int_M V_t(v_{\epsilon}) K_{ap,g_{R_1}}(x, 1 - t; x_{T})dvol_{g_{R_1}}(x)$$

$$B(t) := \int_M \Xi(v_{\epsilon}, (0, 1), (x, t)) K_{ap,g_{R_1}}(x, 1 - t; x_{T})dvol_{g_{R_1}}(x)$$

as well as consider the set $\Theta_1$ defined by

$$\Theta_1 := \left\{ t \in [1 - 4\delta_0^2, 1 - \delta_0^2] : A(t) + B(t) \leq \frac{32C_7eC_8|\log(\delta_0)|[RM_0 + \eta]}{3\delta_0^2} \right\} \quad (A.4.1)$$

Lemma A.4.1. We have the following:

$$L^1(\Theta_1) \geq \frac{3}{4}L^1([1 - 4\delta_0^2, 1 - \delta_0^2]).$$

Proof. This follows from (2.4.5) combined with Chebyshev’s inequality. \qed

A.4.2 Improved energy decay estimate for the modulus

Lemma A.4.2. At $t = 1 - \delta^2$ we have

$$\int_{B_r(x_T) \times \{t\}} |\nabla \sigma_{e}|^2 \leq \frac{K_M}{r\delta_0^2} e^{\frac{\tau^2}{7M}} \left( \int_{M \times \{t\}} V_t(v_{\epsilon}) e^{\frac{(d(x,x_T))^2}{4\delta^2}} \right)^{\frac{1}{2}} \left( \int_{M \times \{t\}} \left[ |\nabla v_{\epsilon}|^2 + |\partial_t v_{\epsilon} + \frac{\langle \nabla v_{\epsilon}, \nabla K_{ap,g_{R_1}} \rangle}{K_{ap,g_{R_1}}} \right]^2 e^{-\frac{(d(x,x_T))^2}{4\delta^2}} \right)^{\frac{1}{2}} \quad (A.4.2)$$

where $0 < r < \frac{\text{inj}(M)}{2}$.

Proof. The proof follows by multiplying (2.4.24) by $\sigma_{e}$, integrating by parts, and using (2.4.3). \qed
Proposition A.4.3. For any $t \in \Theta_1$ and $0 < r < \frac{\text{inj}_s(M)}{2}$ we have

$$\int_{B_r(x_T) \times \{t\}} \left\{ \frac{1}{2} \nabla(|v_r|^2)|^2 + \frac{(1 - |v_r|^2)^2}{4\epsilon^2} \right\} \leq C(\delta_0, r) \left[ (\eta + M_0 R)^{\frac{1}{2}} \left( \bar{E}_r(v_r, (x_T, 1), 1) + CR_1 E_0 + 1 \right) \right],$$

where we assume that $\eta < \frac{1}{2}$ and $R < \frac{1}{2M_0}$.

Proof. This follows from Lemma A.4.4, the definition of $\Theta_1$, and the monotonicity formula. 

\section{A.4.3 Estimate for $\varphi_t$ Proofs}

Lemma A.4.4. Let $\varphi$, $h$, and $g$ be as in (2.4.27) and suppose that $0 < r < s < \frac{3r}{2} < \frac{\text{inj}_s(M)}{2}$. Then

$$\int_{B_s(x_T)} \frac{(N - 2)}{2} |\nabla \varphi|^2 K_{ap} - \int_{B_r(x_T)} \frac{(d(x, x_T))^2}{4\delta^2} \left[ 1 + \frac{2N\lambda\delta^2}{3} - 2\mu\delta^2 \right] |\nabla \varphi|^2 K_{ap} - 2\delta^2 \int_{B_s(x_T)} h(\nabla \varphi, \nabla K_{ap}) \geq -\delta^2 \int_{\partial B_s(x_T)} |\nabla \varphi|^2 \frac{\partial K_{ap}}{\partial r} + 2\delta^2 \int_{\partial B_s(x_T)} g(\nabla \varphi, \nabla K_{ap}).$$

Proof. We start by multiplying both sides of PDE of (2.4.27) by $-2\delta^2 \langle \nabla \varphi, \nabla K_{ap} \rangle$ and integrating by parts over $B_s(x_T)$ to obtain

$$-2\delta^2 \int_{B_s(x_T)} h(\nabla \varphi, \nabla K_{ap}) = 2\delta^2 \int_{B_s(x_T)} \text{div}(K_{ap} \nabla \varphi) \frac{\langle \nabla \varphi, \nabla K_{ap} \rangle}{K_{ap}} \quad (A.4.3)$$

$$= -2\delta^2 \int_{B_s(x_T)} K_{ap} \langle \nabla \varphi, \nabla \left( \frac{\langle \nabla \varphi, \nabla K_{ap} \rangle}{K_{ap}} \right) \rangle$$

$$+ 2\delta^2 \int_{\partial B_s(x_T)} g(\nabla \varphi, \nabla K_{ap}).$$

Next note that, if we consider normal coordinates about a fixed $x \in M$ we obtain, at $x$, that

$$g^{ij}_{R_i} \nabla_{e_j} \left( \frac{\nabla \varphi, \nabla K_{ap}}{K_{ap}} \right) = g^{ij}_{R_i} \nabla_{e_j} \left[ (g^{kp}_{R_i} \nabla \varphi)^{k} \nabla K_{ap} \right] - \frac{\langle \nabla \varphi, \nabla K_{ap} \rangle}{K_{ap}^2} g^{ij}_{R_i} \nabla_{e_j} K_{ap}$$

$$= g^{ij}_{R_i} \nabla_{e_j} \left[ (g^{kp}_{R_i} \nabla \varphi)^{k} \nabla K_{ap} \right] - \frac{\langle \nabla \varphi, \nabla K_{ap} \rangle}{K_{ap}^2} g^{ij}_{R_i} \nabla_{e_j} K_{ap}$$

$$= g^{ij}_{R_i} \nabla_{e_j} \left[ g^{mp}_{R_i} \nabla \varphi \nabla K_{ap} \right] - \frac{\langle \nabla \varphi, \nabla K_{ap} \rangle}{K_{ap}^2} g^{ij}_{R_i} \nabla_{e_j} K_{ap}$$

$$= \frac{1}{K_{ap}} g^{mp}_{R_i} g^{ij}_{R_i} \nabla \varphi \nabla K_{ap} + \frac{1}{K_{ap}} g^{mp}_{R_i} g^{ij}_{R_i} \nabla \varphi \nabla K_{ap}$$

$$= (1)^i + (2)^i + (3)^i.$$
First, observe that
\[
\langle K_{ap} \nabla \varphi, (3)^i \rangle = -\frac{\langle \nabla \varphi, \nabla K_{ap} \rangle^2}{K_{ap}}.
\] (A.4.4)

Next note that, since we are using normal coordinates centred at \( x \), then
\[
(g_{R_1})_{ki}(K_{ap} \nabla \varphi)^k(2)^i = (g_{R_1})_{ki}(\nabla \varphi)^k(g_{R_1}^{mp} \nabla_{ep} \varphi[g_{R_1}^{ij} \nabla_{e_j} (\nabla_{e_m} K_{ap})])
\] (A.4.5)
\[
= (\nabla \varphi)^i((\nabla \varphi)^m \nabla_{e_j} (\nabla_{e_m} K_{ap}))
\]
\[
= (\nabla \varphi)^m \nabla \varphi \nabla_{e_m} K_{ap}
\]
\[
= (g_{R_1})_{mr}(\nabla \varphi)^m \nabla \varphi (\nabla K_{ap})^r
\]
\[
= (\nabla, \nabla \varphi \nabla K_{ap})
\]
\[
= \text{Hess}(K_{ap})(\nabla \varphi, \nabla \varphi).
\]

Finally observe, since we are working in normal coordinates centred at \( x \), that
\[
(g_{R_1})_{ki}(K_{ap} \nabla \varphi)^k(1)^i = (g_{R_1})_{ki}g_{R_1}^{kr} \nabla_{e_r} \varphi(g_{R_1}^{mp} \nabla_{e_p} \varphi(g_{R_1}^{ij} \nabla_{e_j} (\nabla_{e_m} K_{ap})))
\] (A.4.6)
\[
= \nabla_{e_r} \varphi(g_{R_1}^{mp} \nabla_{e_p} \varphi(g_{R_1}^{ij} \nabla_{e_j} (\nabla_{e_m} K_{ap})))
\]
\[
= \frac{1}{2} g_{R_1}^{mp} \nabla_{e_p} (|\nabla \varphi|^2) \nabla_{e_m} K_{ap}
\]
\[
= \frac{1}{2} \langle \nabla (|\nabla \varphi|^2), \nabla K_{ap} \rangle.
\]

Putting together (A.4.3), (A.4.4), (A.4.5), and (A.4.6) we obtain
\[
-2\delta^2 \int_{B_s(x_T)} h(\nabla \varphi, \nabla K_{ap}) = -2\delta^2 \int_{B_s(x_T)} \text{Hess}(K_{ap})(\nabla \varphi, \nabla \varphi) + 2\delta^2 \int_{B_s(x_T)} \frac{\langle \nabla \varphi, \nabla K_{ap} \rangle^2}{K_{ap}}
\] (A.4.7)
\[
- \delta^2 \int_{B_s(x_T)} \langle \nabla (|\nabla \varphi|^2), \nabla K_{ap} \rangle + 2\delta^2 \int_{\partial B_s(x_T)} g(\nabla \varphi, \nabla K_{ap}).
\]

Next we integrate by parts to obtain
\[
\int_{B_s(x_T)} \langle \nabla (|\nabla \varphi|^2), \nabla K_{ap} \rangle = -\int_{B_s(x_T)} |\nabla \varphi|^2 \Delta K_{ap} + \int_{\partial B_s(x_T)} |\nabla \varphi|^2 \frac{\partial K_{ap}}{\partial r}.
\] (A.4.8)
Combining (A.4.8) with (A.4.7) gives
\[
2\delta^2 \int_{B_s(x_T)} \left[ \frac{-\Delta K_{ap}}{2} |\nabla \varphi|^2 + \text{Hess}(K_{ap})(\nabla \varphi, \nabla \varphi) - \frac{(\nabla \varphi, \nabla K_{ap})^2}{K_{ap}} \right] - 2\delta^2 \int_{B_s(x_T)} h(\nabla \varphi, \nabla K_{ap}) \tag{A.4.9}
\]
\[
= -\delta^2 \int_{\partial B_s(x_T)} |\nabla \varphi|^2 \frac{\partial K_{ap}}{\partial r} + 2\delta^2 \int_{\partial B_s(x_T)} g(\nabla \varphi, \nabla K_{ap}).
\]

Next, by similar computations to those used in A.3.1.2 regarding the gradient and Hessian of $K_{ap}$, we have
\[
\frac{-\Delta K_{ap}}{2} |\nabla \varphi|^2 + \text{Hess}(K_{ap})(\nabla \varphi, \nabla \varphi) - \frac{(\nabla \varphi, \nabla K_{ap})^2}{K_{ap}} \tag{A.4.10}
\]
\[
= \Delta r(x)\frac{|\nabla \varphi|^2}{4\delta^2} K_{ap} - \frac{(d(x, x_T))^2}{8\delta^4} |\nabla \varphi|^2 K_{ap} - \frac{\text{Hess}(r)(\nabla \varphi, \nabla \varphi)}{2\delta^2} K_{ap}.
\]

Observe that combining (A.4.10) with (1.2.5) gives
\[
\frac{\Delta r(x)}{4\delta^2} |\nabla \varphi|^2 K_{ap} - \frac{(d(x, x_T))^2}{8\delta^4} |\nabla \varphi|^2 K_{ap} - \frac{\text{Hess}(r)(\nabla \varphi, \nabla \varphi)}{2\delta^2} K_{ap} \tag{A.4.11}
\]
\[
\leq \frac{N(1 - \frac{\lambda(d(x, x_T))^2}{3})}{4\delta^2} |\nabla \varphi|^2 K_{ap} - \frac{(d(x, x_T))^2}{8\delta^4} |\nabla \varphi|^2 K_{ap} - \frac{\left(1 - \frac{\mu(d(x, x_T))^2}{2}\right)}{2\delta^2} |\nabla \varphi|^2 K_{ap}
\]
\[
= \frac{(N - 2)}{4\delta^2} |\nabla \varphi|^2 K_{ap} - \frac{(d(x, x_T))^2}{8\delta^4} \left[ 1 + \frac{2N\lambda\delta^2}{3} - 2\mu\delta^2 \right] |\nabla \varphi|^2 K_{ap}.
\]

Combining (A.4.11) and (A.4.9) gives the desired inequality.

**Corollary A.4.5.** If $\varphi$, $h$, and $g$ satisfy 2.4.27 then, if $0 < \delta < 2\delta_0 < \frac{1}{2\sqrt{\frac{1}{s} - \frac{s}{4}}} + \frac{\delta_0}{2}$ and $0 < r < s < \frac{3r}{4} < \frac{\text{inj}(M)}{2}$, we have
\[
\int_{\partial B_s(x_T)} |\nabla_{\perp} \varphi|^2 K_{ap} \leq \int_{B_s(x_T)} \frac{(N - 2)}{r} |\nabla \varphi|^2 K_{ap} + \frac{1}{r} \int_{B_s(x_T)} h^2 K_{ap}
\]
\[
+ \int_{\partial B_s(x_T)} g^2 K_{ap}.
\]
Proof. We first note that

\[-2\delta^2 \int_{B_r(x_T)} h(\nabla \varphi, \nabla K_{ap}) - \int_{B_r(x_T)} \frac{(d(x, x_T))^2}{4\delta^2} \left[ 1 + \frac{2N\lambda\delta^2}{3} - 2\mu\delta^2 \right] |\nabla \varphi|^2 K_{ap} \]

\[= \int_{B_r(x_T)} hK_{ap}(\nabla \varphi, \exp_{x_T}^{-1}(x_T)) - \int_{B_r(x_T)} \frac{(d(x, x_T))^2}{4\delta^2} \left[ 1 + \frac{2N\lambda\delta^2}{3} - 2\mu\delta^2 \right] |\nabla \varphi|^2 K_{ap} \]

\[\leq \int_{B_r(x_T)} hK_{ap}(\nabla \varphi|d(x, x_T) - \int_{B_r(x_T)} \frac{(d(x, x_T))^2}{4\delta^2} \left[ 1 + \frac{2N\lambda\delta^2}{3} - 2\mu\delta^2 \right] |\nabla \varphi|^2 K_{ap} \]

\[\leq \left( \int_{B_r(x_T)} h^2 K_{ap} \right)^{\frac{1}{2}} \left( \int_{B_r(x_T)} (d(x, x_T))^2 |\nabla \varphi|^2 K_{ap} \right)^{\frac{1}{2}} - \int_{B_r(x_T)} \frac{(d(x, x_T))^2}{4\delta^2} \left[ 1 + \frac{2N\lambda\delta^2}{3} - 2\mu\delta^2 \right] |\nabla \varphi|^2 K_{ap} \]

If we ensure that \(0 < \delta \leq \frac{1}{2\sqrt{\frac{1}{2} - \frac{\delta^2}{4\delta^2}}} \) then by Young’s inequality we have

\[\left( \int_{B_r(x_T)} h^2 K_{ap} \right)^{\frac{1}{2}} \left( \int_{B_r(x_T)} (d(x, x_T))^2 |\nabla \varphi|^2 K_{ap} \right)^{\frac{1}{2}} - \int_{B_r(x_T)} \frac{(d(x, x_T))^2}{4\delta^2} \left[ 1 + \frac{2N\lambda\delta^2}{3} - 2\mu\delta^2 \right] |\nabla \varphi|^2 K_{ap} \]

\[\leq \frac{1}{2} \int_{B_r(x_T)} h^2 K_{ap} + \left[ \frac{1}{2} - \frac{1}{4\delta^2} \left( 1 + \frac{2N\lambda\delta^2}{3} - 2\mu\delta^2 \right) \right] \int_{B_r(x_T)} (d(x, x_T))^2 |\nabla \varphi|^2 K_{ap} \]

\[\leq \frac{1}{2} \int_{B_r(x_T)} h^2 K_{ap}. \]

Thus, by Lemma A.4.4 we have

\[\delta^2 \int_{\partial B_r(x_T)} |\nabla \varphi|^2 \frac{\partial K_{ap}}{\partial r} - 2\delta^2 \int_{\partial B_r(x_T)} g(\nabla \varphi, \nabla K_{ap}) \]

\[\leq \int_{B_r(x_T)} \frac{(N - 2)}{2} |\nabla \varphi|^2 K_{ap} + \frac{1}{2} \int_{B_r(x_T)} h^2 K_{ap}. \]

Observe that since \(\nabla K_{ap} \) is locally radial about \(x_T \) then

\[g(\nabla \varphi, \nabla K_{ap}) = g^2 \frac{\partial K_{ap}}{\partial r} \]

and so by splitting \(\nabla \varphi \) into its radial and perpendicular parts we obtain

\[\delta^2 \int_{\partial B_r(x_T)} |\nabla \varphi|^2 \frac{\partial K_{ap}}{\partial r} - 2\delta^2 \int_{\partial B_r(x_T)} g(\nabla \varphi, \nabla K_{ap}) \]

\[= \delta^2 \int_{\partial B_r(x_T)} |\nabla \varphi|^2 \frac{\partial K_{ap}}{\partial r} - \delta^2 \int_{\partial B_r(x_T)} g^2 \frac{\partial K_{ap}}{\partial r}. \]
Hence, rearranging gives
\[
\delta^2 \int_{\partial B_s(x_T)} |\nabla \varphi|^2 \frac{\partial K_{ap}}{\partial r} \leq \int_{B_s(x_T)} \frac{(N - 2)}{2} |\nabla \varphi|^2 K_{ap} + \frac{1}{2} \int_{B_s(x_T)} h^2 K_{ap} + \int_{\partial B_s(x_T)} g^2 \frac{\partial K_{ap}}{\partial r}.
\]

Finally, observing that on \( \partial B_s(x_T) \)
\[
\frac{\partial K_{ap}}{\partial r} = \frac{s}{2\delta^2} K_{ap}
\]
gives
\[
\int_{\partial B_s(x_T)} |\nabla \varphi|^2 K_{ap} \leq \int_{B_s(x_T)} \frac{(N - 2)}{s} |\nabla \varphi|^2 K_{ap} + \frac{1}{s} \int_{B_s(x_T)} h^2 K_{ap} + \int_{\partial B_s(x_T)} g^2 K_{ap}.
\]

This gives the desired inequality.

\(\square\)

**Lemma A.4.6.** There exists a constant \( C(\delta, r) \) depending continuously, and exclusively on, \( \delta \) and \( r \), such that if \( \varphi, h, \) and \( g \) verify (2.4.27), \( 0 < r < s < 3r^2 \), and \( 0 < \delta < \frac{1}{2\sqrt{2 - \frac{3}{4} + \frac{1}{2}}} \) then
\[
\int_{B_s(x_T)} |\nabla \varphi|^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \leq C(\delta, r) \left[ \int_{B_s(x_T)} h^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} + \left( \int_{B_s(x_T)} h^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \right)^{\frac{1}{2}} \left( \int_{\partial B_s(x_T)} g^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \right)^{\frac{1}{2}} \right]
+ K_M r \int_{\partial B_s(x_T)} g^2 e^{-\frac{(d(x,x_T))^2}{4s^2}},
\]
where \( K_M \) depends only possibly on \( M \) but not on \( \delta \) or \( r \).

**Proof.** The proof follows by multiplying equation (2.4.27) by \( \varphi \), integrating by parts, and using Lemma A.4.4 to obtain control of the weighted \( L^2 \) norm of \( \nabla \varphi \).

\(\square\)

**A.4.3.1 Estimates for \( \varphi_t \)**

Recall that for every \( 0 < s < \frac{3r^2}{2} \), \( \varphi_t \) verifies the boundary value problem
\[
\begin{cases}
L_\delta \varphi_t = h & \text{on } B_s(x_T) \times \{t\} \\
\frac{\partial \varphi_t}{\partial r} = g & \text{on } \partial B_s(x_T) \times \{t\}
\end{cases}
\]
where $h$ and $g$ are defined by

$$ h := v_x \times \left( -\frac{\nabla K_{ap}}{K_{ap}} \cdot \nabla v_x - \partial_t v_x \right) + \left( \frac{dK_{ap}}{K_{ap}} d^* \psi_t + \xi_t \right) \quad \text{on } B_{\frac{3r}{2}}(x_T) \times \{t\} \quad (A.4.13) $$

and

$$ g := v_x \times \frac{\partial v_x}{\partial r} - (d^* \psi_t + \xi_t)_N \quad \partial B_r(x_T) \times \{t\}. \quad (A.4.14) $$

Using the notation $A_{r, \frac{3r}{2}}(x_T):= B_{\frac{3r}{2}}(x_T) \setminus B_r(x_T)$ and Chebyshev’s inequality there exists $s \in [r, \frac{3r}{2}]$ such that

$$ \int_{\partial B_r(x_T) \times \{t\}} |\nabla v_x|^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \, \leq \, \frac{12}{r} \int_{A_{r, \frac{3r}{2}}(x_T) \times \{t\}} |\nabla v_x|^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \quad (A.4.15) $$

$$ \int_{\partial B_r(x_T) \times \{t\}} |d^* \psi_t|^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \, \leq \, \frac{12}{r} \int_{A_{r, \frac{3r}{2}}(x_T) \times \{t\}} |d^* \psi_t|^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \quad (A.4.16) $$

$$ \int_{\partial B_r(x_T) \times \{t\}} |\xi_t|^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \, \leq \, \frac{12}{r} \int_{A_{r, \frac{3r}{2}}(x_T) \times \{t\}} |\xi_t|^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \quad (A.4.17) $$

which means that

$$ \int_{\partial B_r(x_T)} g_x e^{-\frac{(d(x,x_T))^2}{4s^2}} \, \leq \, \frac{108}{r} \int_{A_{r, \frac{3r}{2}}(x_T) \times \{t\}} \left( |\nabla v_x|^2 + |d^* \psi_t|^2 + |\xi_t|^2 \right) e^{-\frac{(d(x,x_T))^2}{4s^2}} \quad (A.4.18) $$

where I have used (2.4.3) to bound $v_x$. Our main estimate for $\varphi_t$ is the following proposition:

**Proposition A.4.7.** Let $(M,g)$ be an $N$-dimensional compact Riemannian manifold without boundary. Then, if $0 < 2r < \frac{\text{inj}(M)}{2}$, $x_T \in M$, and $\delta_0 < \delta < 2\delta_0 < \frac{1}{2\sqrt{1 - \frac{4\pi}{N}} + \frac{1}{4}}$ we have

$$ \int_{B_r(x_T)} |\nabla \varphi_t|^2 e^{-\frac{(d(x,x_T))^2}{4s^2}} \, \leq \, \frac{K_M \delta^N}{r} \bar{E}_c(v_x,(x_T,1),\delta) \quad (A.4.19) $$

$$ + \frac{K_M R^3 \delta^N}{r} \left[ \bar{E}_c(v_x,(x_T,1),\delta) + C(\delta R_1) E_0 \right] $$

$$ + C(\delta_0, r) \left[ R(t) + \sqrt{R(t)} \sqrt{\delta^N \bar{E}_c(v_x,(x_T,1),\delta)} + \sqrt{R(t)} \sqrt{\delta^N R^3 \left[ \bar{E}_c(v_x,(x_T,1),\delta) + C(\delta R_1) E_0 \right]} \right] $$

where $C(\delta_0, r)$ is a constant depending only on $\delta_0$ and $r$. We also have that $R(t)$ is defined as

$$ R(t) := \int_{M \times \{t\}} \left[ \Xi(v_x,(x_T,1)) + V_c(v_x) + \left( |d^* \psi_t|^2 + |\xi_t|^2 \right) \chi_{B_{\frac{3r}{2}}(x_T)} \right] e^{-\frac{(d_x(x,x_T))^2}{4s^2}}. $$
Proof. By Lemma A.4.12, the fact that
\[ \int_{B_s(x)} h^2 e^{-\frac{(d(x, x_T))^2}{4s^2}} \leq 2 \max \left\{ g_s \left( \frac{(\text{inj}_x(M))^2}{4} \right), \frac{1}{\delta^2} \right\} R(t) \] (A.4.20)
as well as, by (A.4.18) for appropriately chosen \( s \),
\[ \int_{\partial B_s(x)} g^2 e^{-\frac{(d(x, x_T))^2}{4s^2}} \leq \frac{108}{r} \int_{A_{r, \frac{3}{2}r}(x_T) \times \{ t \}} |\nabla v_t|^2 e^{-\frac{(d(x, x_T))^2}{4s^2}} + \frac{108}{r} R(t) \] (A.4.21)
then we have
\[
\int_{B_s(x)} |\nabla \varphi_t|^2 e^{-\frac{(d(x, x_T))^2}{4s^2}} \leq C(\delta, r) \left[ \int_{B_s(x)} h^2 e^{-\frac{(d(x, x_T))^2}{4s^2}} + \left( \int_{B_s(x)} h^2 e^{-\frac{(d(x, x_T))^2}{4s^2}} \right) \frac{1}{2} \left( \int_{\partial B_s(x)} g^2 e^{-\frac{(d(x, x_T))^2}{4s^2}} \right) \right]
\]
\[ + Kr \int_{\partial B_s(x)} g^2 e^{-\frac{(d(x, x_T))^2}{4s^2}} \leq C(\delta, r) \left[ C(\delta) R(t) + \sqrt{C(\delta)} (R(t))^{\frac{1}{2}} \left( \frac{K}{r} \int_{A_{r, \frac{3}{2}r}(x_T) \times \{ t \}} |\nabla v_t|^2 e^{-\frac{(d(x, x_T))^2}{4s^2}} + \frac{K}{r} R(t) \right)^{\frac{1}{2}} \right]
\]
\[ + K \int_{A_{r, \frac{3}{2}r}(x_T) \times \{ t \}} |\nabla v_t|^2 e^{-\frac{(d(x, x_T))^2}{4s^2}} + KR(t). \]

Next notice that by applying Lemma 2.3.10 through \( u_\epsilon \) we have
\[
\int_{A_{r, \frac{3}{2}r}(x_T) \times \{ t \}} |\nabla v_t|^2 e^{-\frac{(d(x, x_T))^2}{4s^2}} \leq \frac{4\delta^2}{r} \int_{A_{r, \frac{3}{2}r}(x_T) \times \{ t \}} \frac{(d(x, x_T))^2}{4\delta^2} |\nabla v_t|^2 e^{-\frac{(d(x, x_T))^2}{4s^2}}
\]
\[ \leq 4(4\pi)^{\frac{N}{2}} c_0 \frac{\delta^{N}}{r} E_0 (v_\epsilon, (x_T, 1), \delta) + \frac{8(4\pi)^{\frac{N}{2}} D_f c_0 R^3 \delta^{N+3}}{r} [\delta R_1 E_0]
\]
\[ + \frac{8(4\pi)^{\frac{N}{2}} D_f \delta^{N+2}}{r} R(t). \]

Note that \( r \) is a fixed quantity such that \( 0 < 2r < \frac{\text{inj}_x(M)}{2} \). Combining the previous estimates gives the desired inequality. \( \square \)

Corollary A.4.8. For \( t \in \Theta_1 \),
\[
\int_{B_{\sqrt{\text{inj}_x(M)/s}}(x_T)} |\nabla \varphi_t|^2 e^{-\frac{(d(x, x_T))^2}{4s^2}} \leq \frac{K_M \delta^N}{r} \left( E_0 (v_\epsilon, (x_T, 1), 1) + C^2 R_1 E_0 \right)
\]
\[ + C(\delta_0, r) \left[ (RM_0 + \eta) + \sqrt{(RM_0 + \eta)} \sqrt{\delta^N \left\{ E_0 (v_\epsilon, (x_T, 1), 1) + C^2 R_1 E_0 \right\} + R_2(t) \right] \]
provided that $\sqrt{8C_6}(2\delta_0) \leq r$ and where we have set

$$R_2(t) := C(\delta_0) \left[ \int_{B_{\sqrt{8C_6}(x_T)}} \left( |d^*\psi_t|^2 + |\xi_t|^2 \right) + \left( \int_{B_{\sqrt{8C_6}(x_T)}} \left( |d^*\psi_t|^2 + |\xi_t|^2 \right) \right)^\frac{1}{2} \left( \tilde{E}_\varepsilon(v_\varepsilon, (x_T, 1), 1) + C_7 R_1 E_0 \right)^\frac{1}{2} \right].$$

**Proof.** By Proposition A.4.7 we have

$$\int_{B_r(x_T)} |\nabla \varphi_t|^2 e^{-\frac{(d(x,x_T))^2}{4\delta^2}} \leq \frac{K_M \delta^N}{r} \tilde{E}_\varepsilon(v_\varepsilon, (x_T, 1), \delta)$$

(A.4.22)

$$+ \frac{K_M R^3 \delta^N}{r} \left[ \tilde{E}_\varepsilon(v_\varepsilon, (x_T, 1), \delta) + C_7 (\delta R_1) E_0 \right]$$

$$+ C(\delta_0, r) \left[ R(t) + \sqrt{R(t)} \sqrt{\delta^N \tilde{E}_\varepsilon(v_\varepsilon, (x_T, 1), \delta)} \right]$$

(A.4.23)

Hence, if we choose $\delta_0$ small enough that

$$\sqrt{8C_6}(2\delta_0) \leq r$$

then, for $\delta = \sqrt{1-t}$ and $t \in \Theta_1$ we have, after applying the monotonicity formula through $u_\varepsilon$

$$\int_{B_{\sqrt{8C_6}(x_T)}} |\nabla \varphi_t|^2 e^{-\frac{(d(x,x_T))^2}{4\delta^2}} \leq \frac{K_M \delta^N}{r} \tilde{E}_\varepsilon(v_\varepsilon, (x_T, 1), 1) + C_7 R_1 E_0$$

$$+ \frac{K_M R^3 \delta^N}{r} \left[ \tilde{E}_\varepsilon(v_\varepsilon, (x_T, 1), 1) + C_7 (\delta R_1) E_0 \right]$$

$$+ C(\delta_0, r) \left[ (R M_0 + \eta) + \sqrt{(R M_0 + \eta)} \sqrt{\delta^N \left( \tilde{E}_\varepsilon(v_\varepsilon, (x_T, 1), 1) + C_7 R_1 E_0 \right)} \right]$$

$$+ R_2(t).$$

\[ \square \]

### A.4.4 Estimate of $\psi_t$ Proofs

**Lemma A.4.9.** We have the following estimates:

$$\left\| \left\| G^i \right\|_{L^1(M)} \right\|_{L^\infty(M)} \right\|_{L^\infty(M)} \leq K e^\alpha r$$

(A.4.24)

$$\left\| \left\| \nabla G^i \right\|_{L^1(M)} \right\|_{L^\infty(M)} \right\|_{L^\infty(M)} \leq K e^\alpha r,$$

(A.4.25)
\(K\) is a constant depending only on \(M\). Also, for all \(f \in C^2(M; \wedge^2 M)\) and each \(y \in M\) we have

\[
\left| \langle G^i(\cdot, y), -\Delta f \rangle \right| \leq K \|f\|_{L^\infty(M)}.
\] (A.4.26)

**Proof.** Estimates (A.4.24) and (A.4.25) follow from the inequalities

\[
|G(x, y)| \leq K(d_{gR_1}(x, y))^{2-N}
\]

\[
|DG(x, y)| \leq K(d_{gR_1}(x, y))^{1-N}
\]

along with the definition of \(m\). Next one can compute that if \(f \in C^2(M; \wedge^2 M)\) and \(y\) is fixed then, by the definition of \(m\), we have

\[
\langle G^i, -\Delta f \rangle = \int_M \langle G^i(x, y), -\Delta_x f(x) \rangle = f(y) - H(f)
\]

\[- \int_{M \setminus B_{\alpha R_1}(y)} \langle G(x, y), -\Delta_x f(x, y) \rangle + \int_{B_{\alpha R_1}(y) \setminus B_{\alpha R_1}(y)} \langle G^i(x, y), -\Delta_x f(x) \rangle
\]

Next observe that by integrating by parts we have

\[
= - \int_{M \setminus B_{\alpha R_1}(y)} (-\Delta_x G(x, y), f(x)) \right) + \int_{B_{\alpha R_1}(y) \setminus B_{\alpha R_1}(y)} \langle -\Delta_x G^i(x, y), f(x) \rangle
\]

\[- \int_{\partial B_{\alpha R_1}(y)} (d^* G(x, y))_T \wedge \ast (f(x))_N + \int_{\partial B_{\alpha R_1}(y)} (f(x))_T \wedge \ast (dG(x, y))_N
\]

\[- \int_{\partial B_{\alpha R_1}(y)} (d^* G^i(x, y))_T \wedge \ast (f(x))_N + \int_{\partial B_{\alpha R_1}(y)} (d^i G^i(x, y))_T \wedge \ast (f(x))_N
\]

\[+ \int_{\partial B_{\alpha R_1}(y)} (f(x))_T \wedge \ast (dG^i(x, y))_N - \int_{\partial B_{\alpha R_1}(y)} (f(x))_T \wedge \ast (dG^i(x, y))_N.
\]

Notice that

\[
\left| \int_{M \setminus B_{\alpha R_1}(y)} \langle -\Delta_x G(x, y), f(x, y) \rangle \right| \leq K \|f\|_{L^\infty(M)}
\] (A.4.27)

\[
\left| \int_{\partial B_{\alpha R_1}(y)} (d^* G(x, y))_T \wedge \ast (f(x))_N \right| \leq K \|f\|_{L^\infty(M)}
\] (A.4.28)

\[
\left| \int_{\partial B_{\alpha R_1}(y)} (f(x))_T \wedge \ast (dG(x, y))_N \right| \leq K \|f\|_{L^\infty(M)}
\] (A.4.29)

due to (2.5.18) and (2.5.19). Using the coordinate expression, in normal coordinates suited to \(g_{R_1}\),
for the codifferential from equation 2.1.31 of subsection 2.1 of [28], the coordinate formula for the differential, and (2.5.18) it follows that the integrals over the boundary are bounded by the $L^\infty$ norm of $f$ multiplied by a constant. Similarly, it follows from the coordinate formula for the codifferential from equation 2.1.31 of section 2.1 of [28] that $-\Delta G^i$ can be written in terms of $m(d_{g_{R_1}})[-\Delta G]$ as well as terms involving partial derivatives of $m(d_{g_{R_1}})$ and $G$ such the sum of the orders of partial derivatives never exceeds 2 and there are no partial derivatives of order 2 on $G$. Observe that by (2.5.18), (2.5.19), the definition of $m$, and the region of integration, all such terms can be bounded by a constant times the $L^\infty$-norm of $f$.

\[ \text{A.4.4.1 Proof of (2.5.64)} \]

\[
\int_{B_{32r}(0)} l(|y|)|y|^{2-N}k(y)dy = \int_{B_{32r}(0)\setminus B_{o,r}(0)} l(|y|)|y|^{2-N}k(y)dy \\
= \int_{B_{32r}(0)\setminus B_{o,r}(0)} l(|y|)|y|^{-N}\{ |y|f(x,y) \}dy \\
= \int_{B_{32r}(0)\setminus B_{o,r}(0)} \frac{N-1}{2^{N-1}-1} \left( \int_{\max(|y|,16r)}^{\min(|y|,16r)} s^{-N} ds \right) \{ |y|f(x,y) \}dy \\
= \frac{N-1}{2^{N-1}-1} \int_{B_{32r}(0)\setminus B_{o,r}(0)} \left( \int_{\max(|y|,16r)}^{\min(|y|,16r)} s^{-N} |y|f(x,y)ds \right)dy \\
= \frac{N-1}{2^{N-1}-1} \left( \int_{0}^{\infty} \chi_{[\max\left\{ \frac{|y|}{2},e^\alpha r \right\},\min\{ |y|,16r \}]}(s)s^{-N}|y|f(x,y)ds \right)dy \\
= \frac{N-1}{2^{N-1}-1} \left( \int_{0}^{\infty} \chi_{[\max\left\{ \frac{|y|}{2},e^\alpha r \right\},\min\{ |y|,16r \}]}(s)s^{-N}|y|f(x,y)dy \right)ds \\
= \frac{N-1}{2^{N-1}-1} \int_{0}^{16r} \left( \int_{B_{32r}(0)\setminus B_{o,r}(0)} s^{-N} |y|f(x,y)dy \right)ds \\
= \frac{N-1}{2^{N-1}-1} \int_{0}^{16r} \left( \int_{B_{2r}(0)\setminus B_{s}(0)} s^{-N} |y|f(x,y)dy \right)ds \\
\]

where the last equality follows from the fact that for $s$ to be in the correct range of the indicator we must require that $\frac{|y|}{2} \leq s \leq |y|$. Next, we let

\[
h(u,s) := \frac{(N-1)(N-2)}{2^{N-1}-1} \begin{cases} 
1 & 0 \leq u \leq s, \\
2 - \frac{u}{s} & \text{if } s \leq u \leq 2s, \\
0 & \text{if } u \geq 2s
\end{cases}
\]

and observe that

\[
\frac{\partial h}{\partial s}(|y|,s) = \frac{(N-1)(N-2)}{2^{N-1}-1} \frac{|y|}{s^2} \chi_{[s,2s]}(|y|)
\]
so that the final equality from the previous work becomes:

\[
\int_{e^{0r}}^{e^{r}} \frac{s^{2-N}}{N-2} \left( \int_{B_{2r}(0)} f(y) \frac{\partial h}{\partial s}(|y|, s) dy \right) ds.
\]

We notice that \( h(2s, s) = 0 \) and hence \( h \equiv 0 \) on \( \partial B_{2r}(0) \). As a result we may integrate by parts to obtain

\[
\int_{e^{0r}}^{e^{r}} \frac{s^{2-N}}{N-2} \left( \int_{B_{2r}(0)} f(y) \frac{\partial h}{\partial s}(|y|, s) dy \right) ds = \int_{e^{0r}}^{e^{r}} s^{-1} J^{f}_{s}(x) ds + \frac{1}{N-2} [ J^{f}_{16r}(x) - J^{f}_{e^{0r}}(x) ]
\]

where we have introduced the notation

\[
J^{f}_{s}(x) := s^{2-N} \int_{B_{2r}(0)} f(y) h(|y|, s) dy.
\]

**Lemma A.4.10.** Let \( \beta > 0 \) be given as in Theorem 2.1 of [26]. Then, for any \( x \in B_{4r}(0) \),

\[
\sup_{s \in [e^{0r}, 16r]} \{ J^{a_{I}}(x) \} \leq C(\delta_0, r) \left( \frac{E_{\epsilon}(v_{\epsilon}, (xT, 1), 1) + C_{T}R_{1}E_{0}}{|\log(\epsilon)|} + \epsilon^{3} \right)
\]

provided that \( 2\delta_0 \leq \frac{1}{r} \).

**Proof.** First, notice that, by changing variables, we have

\[
J^{a_{I}}(x) = \sum_{I} \frac{1}{s^{N-2}} \int_{B_{2r}(0)} \frac{\partial \tilde{v}_{\epsilon}}{\partial x_{j_{1}}} \times \frac{\partial \tilde{v}_{\epsilon}}{\partial x_{j_{2}}} H^{I}_{f}(x, s) \sqrt{|g_{R_{1}(x,s)}|} h(|s|, s) \chi(s) ds.
\]

where \( \tilde{v}_{\epsilon}(s) := \tilde{v}_{\epsilon}(s) \) and \( \chi(s) = \chi(s) \). We also set \( \epsilon_{s} := \epsilon/s \) for \( s \in [e^{0r}, 16r] \) and observe that

\[
|\log(\epsilon_{s})| = |\log(\epsilon) - \log(s)| \geq (1 - \alpha)|\log(\epsilon)| - |\log(r)| = \frac{1 - \alpha}{2}|\log(\epsilon)|
\]

provided \( \epsilon \) is sufficiently small relative to \( r \), which will at most be dependent on \( M \). Next, observe that for each \( I \) and \( J \)

\[
\left\| h(|s|, 1) H^{I}_{f}(x, s(\cdot)) \sqrt{|g_{R_{1}(x,s(\cdot))}|} \chi(s) \right\|_{C^{0,1}(B_{2r}(0))} \leq K.
\]
Using the estimate of Jerrard and Soner with \( \varphi(z) = h(|z|, 1)H_\beta^1(x, sz)\chi_z(z)\sqrt{|g_{R_0}(x, sz)|} \) gives

\[
|J_\varepsilon(x)| \leq K \left( \frac{1}{s^{N-2}} \int_{B_{2s}(0)} e_{\varepsilon}(\tilde{\varepsilon}_x) \right) \leq K \left( \frac{1}{s^{N-2}} \int_{B_2(0)} e_{\varepsilon}(\tilde{\varepsilon}_x) \right).
\]

If \( e^{\delta r} \leq s \leq \frac{\delta}{2} \) then

\[
\frac{1}{s^{N-2}} \int_{B_{2s}(0)} e_{\varepsilon}(\tilde{\varepsilon}_x) dy = \frac{1}{s^{N-2}} \int_{B_{2s}(0) \cap \{|v| \leq \frac{1}{2}\}} e_{\varepsilon}(\tilde{\varepsilon}_x) dy + \frac{1}{s^{N-2}} \int_{B_{2s}(0) \cap \{|v| > \frac{1}{2}\}} e_{\varepsilon}(\tilde{\varepsilon}_x) dy \\
\leq \frac{1}{s^{N-2}} \int_{B_{2s}(0) \cap \{|v| \leq \frac{1}{2}\}} e_{\varepsilon}(\tilde{\varepsilon}_x) dy + \frac{1}{s^{N-2}} \int_{B_{2s}(0) \cap \{|v| > \frac{1}{2}\}} e_{\varepsilon}(v_x) dy \\
\leq \frac{8}{s^{N-2}} \int_{B_{2s}(0) \cap \{|v| \leq \frac{1}{2}\}} e_{\varepsilon}(v_x) dy + \frac{1}{s^{N-2}} \int_{B_{2s}(0) \cap \{|v| > \frac{1}{2}\}} e_{\varepsilon}(v_x) dy \\
\leq \frac{8}{s^{N-2}} \int_{B_{2s}(0)} e_{\varepsilon}(v_x) dy \\
\leq \frac{8e}{s^{N-2}} \int_{B_{2s}(0)} e_{\varepsilon}(v_x)e^{-\frac{|v|^2}{s^2}} dy \\
\leq KR_1^N \tilde{E}_{w,e}(v_x, (x, 1 - \delta^2 + s^2), s) = KR_1^N \tilde{E}_{w,e}(u_e, (x, T + R_1^2(s^2 - \delta^2)), R_1 s) = KR_1^N \tilde{E}_{w,e}(u_e, (x, T - R_1^2 + R_1^2(1 + s^2 - \delta^2)), R_1 s) \\
\leq Ke^{C_2 R_1^N} \tilde{E}_{w,e}(u_e, (x, (T - R_1^2 + R_1^2(1 + s^2 - \delta^2)), R_1(1 + s^2 - \delta^2)^\frac{1}{2}) + C_7 R_1 E_0) \\
= Ke^{C_2 R_1^N} \tilde{E}_{w,e}(u_e, (x, 1 + s^2 - \delta^2), (1 + s^2 - \delta^2)^\frac{1}{2}) + C_7 R_1 E_0) \\
\leq Ke^{C_2 R_1^N} \left( 1 + s^2 - \delta^2 \right) \left( 1 + s^2 - \delta^2 \right) + C_7 R_1 E_0) \\
\leq Ke^{C_2 R_1^N} \left( \frac{A}{3} \right)^{\frac{1}{s^{N-2}}} \left( 1 + s^2 - \delta^2 \right) R_1^N \left( \tilde{E}_{e}(v_x, (x, T, 1), 1) + C_7 R_1 E_0) \\
where we have rewritten the integral over a manifold, used that \( \delta \leq 2\delta_0 \leq \frac{1}{2} \), applied Lemma 2.3.7, and used that on \( |v_x| \leq \frac{1}{2} \) we have

\[
e_{\varepsilon}(\tilde{\varepsilon}_x) = \frac{1}{2} |p'(|v_x|)|^2 |\nabla(|v_x|)|^2 |v_x|^2 + \frac{1}{2} |p(|v_x|)|^2 |\nabla v_x|^2 + \frac{1}{4e^2} (1 - |v_x|^2)^2 \\
\leq 8|\nabla(|v_x|)|^2 |v_x|^2 + 2|\nabla v_x|^2 + \frac{1}{4e^2} \\
\leq 2|\nabla(|v_x|)|^2 + 2|\nabla v_x|^2 + V_e(v_x) \cdot \frac{1}{|1 - |v_x|^2|} \\
\leq 4|\nabla v_x|^2 + \frac{16}{9} V_e(v_x) \\
\leq 8e_{\varepsilon}(v_x).
\]
When $\frac{t}{2} \leq s \leq 16r$ we have
\[
\frac{1}{s^{N-2}} \int_{B_{2s}(0)} e_c(\tilde{v}_c) \, dy \leq \frac{2^{N-2}}{\delta^{N-2}} \int_{B_{2s}(0)} e_c(\tilde{v}_c) \, dy
\]
\[
\leq \frac{2^{N+1}}{\delta^{N-2}} \int_{B_{2s}(0)} e_c(v_c) \, dy
\]
\[
\leq \frac{2^{N+1}}{\delta^{N-2}} \int_{B_{2s}(0)} e_c(v_c) e^{-\frac{(\delta g_{R_1}(y,x))^2}{4s^2}} e^{\frac{(\delta g_{R_1}(y,x))^2}{4s^2}} \, dy
\]
\[
\leq 2^{N+1} e^{\frac{\delta g_{R_1}(x,y)^2 + (2s)^2}{4s^2}} R_1^N \tilde{E}_{c}(v_c, (x_T, 1), \delta)
\]
\[
\leq 2^{N+1} e^{C_2 \cdot \frac{32s^2}{46r}} R_1^N \tilde{E}_{c}(v_c, (x_T, 1), 1) + C_7 R_1 E_0
\]
respectively we see that

\[
\|\psi_{1,t}\|_{L^2(M)} \leq K\epsilon^\alpha \left( \int_{B_{4r}(x_T)} |\nabla \tilde{\psi}_{1,t}|^2 \right)^{1/2} \leq K'' \delta^{N-2} e^{\alpha} \left( \frac{1}{(4\pi)^{N/2}} \delta^{N-2} \int_{B_{4r}(x_T)} \epsilon_x(v_{\epsilon}) e^{-\frac{(d(x,y))^2}{4\delta}} e^{-\frac{(d(x,Y))^2}{\delta}} \right)^{1/2}
\]

\[
\leq K'' \delta^{N-2} e^{\alpha} \frac{4\delta^2}{\delta} \left( \tilde{E}_\epsilon(v_{\epsilon}, (x_T, 1), \delta) \right)^{1/2} 
\]

\[
\leq K'' \delta^{N-2} e^{\alpha} \frac{4\delta^2}{\delta} \left( \tilde{E}_\epsilon(v_{\epsilon}, (x_T, 1), 1) + C_7 R_1 E_0 \right)^{1/2}.
\]

Squaring both sides gives the desired conclusion.

\[ \square \]

**Lemma A.4.12.** For any \( t \in [1 - 4\delta_0^2, 1 - \delta_0^2] \),

\[
\|\psi_1(t, 1 - \delta^2)\|_{L^\infty(M)} \leq C(\delta_0, r) \left( \tilde{E}_\epsilon(v_{\epsilon}, (x_T, 1), 1) + C_7 R_1 E_0 \right) \tag{A.4.34}
\]

and

\[
\|D\psi_1^*\|_{L^2(M \times [0, 1 - \delta_0^2])}^2 \leq C(\delta_0, r) \left( \tilde{E}_\epsilon(v_{\epsilon}, (x_T, 1), 1) + C_7 R_1 E_0 \right) \tag{A.4.35}
\]

provided that \( 2\delta_0 \leq \frac{1}{2} \).

**Proof.** To show (A.4.34) we use the proof of Proposition 2.3.9, the monotonicity formula, and Lemma 2.3.7 that

\[
|\psi_1^*(x, 1 - \delta^2)| \leq K \left( \frac{1}{1 - \delta^2} \right)^{\frac{N}{2} - 1} e^{4C_f(d(x,y)^2)} \left( \tilde{E}_\epsilon(v_{\epsilon}, (x_T, 1), 1) + C_7 R_1 E_0 \right)
\]

\[
\leq K \left( \frac{1}{1 - \delta^2} \right)^{\frac{N}{2} - 1} e^{64C_f r^2} \left( \tilde{E}_\epsilon(v_{\epsilon}, (x_T, 1), 1) + C_7 R_1 E_0 \right)
\]

for \( x \in B_{4r}(x_T) \). Since \( \chi \) is supported in \( B_{4r}(x_T) \) then we deduce by Duhamel’s representation formula for \( \omega \) and Theorem 3.5 of [38] that

\[
\sup_{x \in M \setminus B_{4r}(x_T)} \{ \omega(x, 1 - \delta^2) \} \leq KR_1 E_0.
\]

Therefore, we obtain

\[
\sup_{x \in M} \{ \omega(x, 1 - \delta^2) \} \leq C(\delta_0, r) \left( \tilde{E}_\epsilon(v_{\epsilon}, (x_T, 1), 1) + C_7 R_1 E_0 \right)
\]

for \( 2\delta_0 \leq \frac{1}{2} \). Next we prove (A.4.35). Suppose \( s \in [0, 1 - \delta_0^2] \). Taking the inner product of (2.5.67)
by \( \psi^*_1 \), integrating over \( M \times [0, s] \), and integrating by parts gives

\[
\frac{1}{2} \int_{M \times \{s\}} |\psi^*_1|^2 + \int_{M \times [0, s]} \{|d\psi^*_1|^2 + |d^*\psi^*_1|^2\} = \int_{M \times [0, s]} \langle d[\tilde{\nu}_e \times d\tilde{\nu}_e] \chi, \psi^*_1 \rangle. \tag{A.4.36}
\]

Therefore, by (2.4.3) and the monotonicity formula we have

\[
\int_{M \times [0, s]} \langle d[\tilde{\nu}_e \times d\tilde{\nu}_e] \chi, \psi^*_1 \rangle \leq C(\delta_0, r) \int_{M \times [0, s]} |\tilde{\nu}_e \times d\tilde{\nu}_e| \left( |d\chi||\psi^*_1| + \chi|d^*\psi^*_1| \right) \tag{A.4.37}
\]

\[
\leq C(\delta_0, r) \int_{M \times [0, s]} |d\tilde{\nu}_e| \left( |d\chi||\psi^*_1| + \chi|d^*\psi^*_1| \right) \leq C(\delta_0, r) \|\nabla \tilde{\nu}_e\|_{L^2(B_{tr}(x_T \times 0, s))} \left( \|\psi^*_1\|_{L^2(M \times [0, s])} + \|d^*\psi^*_1\|_{L^2(M \times [0, s])} \right) \leq C(\delta_0, r) \|\nabla \tilde{\nu}_e\|_{L^2(B_{tr}(x_T \times 0, 1 - \delta_0^2))} \left( \|\psi^*_1\|_{L^2(M \times [0, 1 - \delta_0^2])} + \|d^*\psi^*_1\|_{L^2(M \times [0, s])} \right) \leq C(\delta_0, r) \|e_r(\tilde{\nu}_e)\|_{L^1(B_{tr}(x_T \times 0, 1 - \delta_0^2))} \left( \|\psi^*_1\|_{L^2(M \times [0, 1 - \delta_0^2])} + \|d^*\psi^*_1\|_{L^2(M \times [0, s])} \right) \leq C(\delta_0, r) \left( \bar{E}_r(v_e, (x_T, 1), 1) + CR_1 E_0 \right) \frac{1}{2} \left( \|\psi^*_1\|_{L^2(M \times [0, 1 - \delta_0^2])} + \|d^*\psi^*_1\|_{L^2(M \times [0, s])} \right) \leq \frac{1}{4} \int_{M \times [0, 1 - \delta_0^2]} |\psi^*_1|^2 + \frac{1}{2} \int_{M \times [0, s]} |d^*\psi^*_1|^2 + C(\delta_0, r) \left( \bar{E}_r(v_e, (x_T, 1), 1) + CR_1 E_0 \right).
\]

Rearranging terms and omitting the remaining integrals of \( d\psi^*_1 \) and \( d^*\psi^*_1 \) we obtain

\[
\frac{1}{2} \int_{M \times \{s\}} |\psi^*_1|^2 \leq C(\delta_0, r) \left( \bar{E}_r(v_e, (x_T, 1), 1) + CR_1 E_0 \right) + \frac{1}{4} \int_{M \times [0, 1 - \delta_0^2]} |\psi^*_1|^2. \tag{A.4.38}
\]

Integrating both sides for \( s \in [0, 1 - \delta_0^2] \) and rearranging gives

\[
\int_{M \times [0, 1 - \delta_0^2]} |\psi^*_1|^2 \leq C(\delta_0, r) \left( \bar{E}_r(v_e, (x_T, 1), 1) + CR_1 E_0 \right). \tag{A.4.39}
\]

Next, proceeding similarly to the previous estimate, except integrating over \( M \times [0, 1 - \delta_0^2] \), we can obtain, after using (A.4.39), that

\[
\frac{1}{2} \int_{M \times \{1 - \delta_0^2\}} |\psi^*_1|^2 + \int_{M \times [0, 1 - \delta_0^2]} \left\{|d\psi^*_1|^2 + |d^*\psi^*_1|^2\right\} \leq C(\delta_0, r) \left( \bar{E}_r(v_e, (x_T, 1), 1) + CR_1 E_0 \right) \frac{1}{2} \left( \|\psi^*_1\|_{L^2(M \times [0, 1 - \delta_0^2])} + \|d^*\psi^*_1\|_{L^2(M \times [0, 1 - \delta_0^2])} \right) \leq \frac{1}{2} \int_{M \times [0, 1 - \delta_0^2]} |d^*\psi^*_1|^2 + C(\delta_0, r) \left( \bar{E}_r(v_e, (x_T, 1), 1) + CR_1 E_0 \right). \tag{A.4.40}
\]
Rearranging and ignoring the integral of $\psi_1^*$ we obtain

$$
\int_{M \times [0,1 - \delta_0^2]} \left\{ |d\psi_1^*|^2 + |d^*\psi_1^*|^2 \right\} \leq C(\delta_0, r) \left( \overline{E}_c(v_1, (x_T, 1), 1) + C_7 R_1 E_0 \right). \tag{A.4.41}
$$

Next, by Gaffney’s inequality we have, for each $t \in [0, 1 - \delta_0^2]$, that

$$
\|D\psi_1^*\|_{L^2(M \times \{t\})}^2 \leq K \left( \|\psi_1^*\|_{L^2(M \times \{t\})}^2 + \|d\psi_1^*\|_{L^2(M \times \{t\})}^2 + \|d^*\psi_1^*\|_{L^2(M \times \{t\})}^2 \right). \tag{A.4.42}
$$

Integrating (A.4.42) over $[0, 1 - \delta_0^2]$ and applying (A.4.39) and (A.4.41) we obtain (A.4.35). \hfill \square

### A.4.4.2 Proof of (2.5.70)

By the definition of $\Xi$, Proposition 2.3.10, and Lemma A.3.1 we have

$$
\begin{align*}
\int_{M \times [0,1 - \delta_0^2]} |\partial_t v_1|^2 e^{-\frac{(d_+(x,x_T))^2}{4(1-t)}} \\
\leq 2 \int_{M \times [0,1 - \delta_0^2]} \frac{1}{1-t} \left[ (1-t)^{\frac{N}{2}} \|\psi_1^*\|_{L^2(M \times \{t\})}^2 + (1-t)^{\frac{N}{2}} \|d\psi_1^*\|_{L^2(M \times \{t\})}^2 \right] \\
= 2(4\pi)^{\frac{N}{2}} \int_{M \times [0,1 - \delta_0^2]} \left( 1-t \right)^{\frac{N}{2}} \Xi(v_1, (x_T, 1))K_{ap,gr_1}(x, 1-t; x_T) \\
+ 2 \int_0^{1-\delta_0^2} \frac{1}{1-t} \int_M \frac{(d_+(x,x_T))^2}{4(1-t)} |\nabla v_1|^2 e^{-\frac{(d_+(x,x_T))^2}{4(1-t)}} \\
\leq 2(4\pi)^{\frac{N}{2}} \int_{M \times [0,1 - \delta_0^2]} \left( 1-t \right)^{\frac{N}{2}} \Xi(v_1, (x_T, 1))K_{ap,gr_1}(x, 1-t; x_T) \\
+ (4\pi)^{\frac{N}{2}} C_6 \int_0^{1-\delta_0^2} (1-t)^{\frac{N-2}{2}} \int_M e_\varepsilon(v_1(x,t))K_{ap,gr_1}(x, 1-t; x_T) e_{vol}^{gr_1}(x) d\varepsilon \\
+ \frac{4D_1 C_0(4\pi)^{\frac{N}{2}} \delta_0^{N+4} R_1}{N+4} E_0 \\
+ 2(4\pi)^{\frac{N}{2}} D_1 \int_0^{1-\delta_0^2} (1-t)^{\frac{N-2}{2}} \int_M [\Xi(v_1(x,t)) + \Xi(v_1, (x_T, x_T))K_{ap,gr_1}(x, 1-t; x_T)] e_{vol}^{gr_1}(x) d\varepsilon \\
\leq (4\pi)^{\frac{N}{2}} C_6 \int_0^{1-\delta_0^2} (1-t)^{\frac{N-4}{2}} \overline{E}_c \left( v_1, (x_T, 1), \sqrt{1-t} \right) d\varepsilon \\
+ \frac{4D_1 C_0(4\pi)^{\frac{N}{2}} \delta_0^{N+4} R_1}{N+4} E_0 \\
+ 2(4\pi)^{\frac{N}{2}} (D_1 + 1) \int_0^{1-\delta_0^2} (1-t)^{\frac{N-2}{2}} \int_M [\Xi(v_1(x,t)) + \Xi(v_1, (x_T, 1))K_{ap,gr_1}(x, 1-t; x_T)] e_{vol}^{gr_1}(x) d\varepsilon.
\end{align*}
$$
Next we apply Lemma A.3.1 and the monotonicity formula to obtain

\[
(4\pi)^\frac{2}{3} C_0 \int_0^{1-\delta_0^2} (1-t)^{\frac{N}{2}+\frac{d}{2}} \tilde{E}_e(v_\epsilon, (x,T,1), 1-t) \, dt
\]

\[
+ 2 (4\pi)^{\frac{2}{3}} (D_f + 1) \int_0^{1-\delta_0^2} (1-t)^{\frac{N}{2}+\frac{d}{2}} \int_M [V_e(v_\epsilon(x,t)) + \Xi(v_\epsilon, (x,T,1))(x,t)] K_{ap,g_1}(x,1-t;x,T) \, dvol(x) \, dt
\]

\[
\leq (4\pi)^{\frac{2}{3}} e^{C_2} C_0 [\tilde{E}_e(v_\epsilon, (x,T,1), 1) + C_7 R_1 E_0]
\]

\[
+ 2 (4\pi)^{\frac{2}{3}} (D_f + 1) \int_M [V_e(v_\epsilon(x,t)) + \Xi(v_\epsilon, (x,T,1))(x,t)] K_{ap,g_1}(x,1-t;x,T) \, dvol(x) \, dt
\]

\[
\leq (4\pi)^{\frac{2}{3}} e^{C_2} C_0 [\tilde{E}_e(v_\epsilon, (x,T,1), 1) + C_7 R_1 E_0]
\]

\[
+ 2 (4\pi)^{\frac{2}{3}} (D_f + 1) \tilde{E}_e(v_\epsilon, (x,T,1), \sqrt{1-\delta_0^2})
\]

\[
- 2 (4\pi)^{\frac{2}{3}} (D_f + 1) \int_0^{1-\delta_0^2} \int_M [(\Phi(v_\epsilon, (x,T,1)))(x,t) + (\Psi(v_\epsilon, (x,T,1)))(x,t)] \, dvol(x) \, dt
\]

\[
\leq 2 (4\pi)^{\frac{2}{3}} e^{C_2} (C_0 + D_f + 1) [\tilde{E}_e(v_\epsilon, (x,T,1), 1) + C_7 R_1 E_0]
\]

\[
- 8 (4\pi)^{\frac{2}{3}} \int_0^{1-\delta_0^2} \int_M [(\Phi(v_\epsilon, (x,T,1)))(x,t) + (\Psi(v_\epsilon, (x,T,1)))(x,t)] \, dvol(x) \, dt.
\]

Next observe that by the proof of Corollary 2.3.3 and Lemma 2.3.4 as well as the monotonicity formula we have

\[
- 8 (4\pi)^{\frac{2}{3}} \int_0^{1-\delta_0^2} \int_M [(\Phi(v_\epsilon, (x,T,1)))(x,t) + (\Psi(v_\epsilon, (x,T,1)))(x,t)] \, dvol(x) \, dt
\]

\[
\leq C'_M [\tilde{E}_e(v_\epsilon, (x,T,1), 1) + C_7 R_1 E_0]
\]

where \( C'_M \) depends only on \( M \). Putting the above estimates together gives the desired estimate. \( \square \)

**A.4.4.3 Proof of (2.5.74)**

First, we take the inner product of (2.5.67) with \( \partial \psi_1^* \), integrate over \( M \times [0,1-\delta_0^2] \), and integrate by parts to obtain

\[
\int_{M \times [0,1-\delta_0^2]} |\partial \psi_1^*|^2 = -\frac{1}{2} \int_0^{1-\delta_0^2} \frac{d}{dt} \int_M \{ |d \psi_1^*|^2 + |d^* \psi_1^*|^2 \} + \int_{M \times [0,1-\delta_0^2]} \left( \partial_t \psi_1^*, d[\tilde{v}_\epsilon \times d\tilde{v}_\epsilon] \chi \right)
\]

\[
= -\frac{1}{2} \int_{M \times [1-\delta_0^2]} \{ |d \psi_1^*|^2 + |d^* \psi_1^*|^2 \} + \int_{M \times [0,1-\delta_0^2]} \left( \partial_t \psi_1^*, d[\tilde{v}_\epsilon \times d\tilde{v}_\epsilon] \chi \right).
\]
We now estimate the last term from (A.4.43). Since \( \chi \) is supported in \( B_{4r,GR_1}(x_T) \) then, assuming that \( 4r < \frac{\text{inj}(M)}{2} \), we can use normal coordinates about \( x_T \) to write

\[
\langle \partial \psi^*_1, d[\tilde{v}_r \times d\tilde{v}_r] \rangle = 2 \sum_I \sum_{j_1 < j_2} g_{R_1}^{IJ} \partial_t \psi^*_1 \left[ \partial_{j_1} \tilde{v}_r \times \partial_{j_2} \tilde{v}_r \right] \chi \sqrt{|g_{R_1}|}
\]

on \( B_{4r,GR_1}(x_T) \) where \( I = (i_1,i_2) \) and \( J = (j_1,j_2) \). We observe that this can be rewritten as

\[
\sum_I \sum_{j_1 < j_2} g_{R_1}^{IJ} \partial_t \psi^*_1 \left[ \partial_{j_1} (\tilde{v}_r \times \partial_{j_2} \tilde{v}_r) - \partial_{j_2} (\tilde{v}_r \times \partial_{j_1} \tilde{v}_r) \right] \chi \sqrt{|g_{R_1}|}.
\]

Letting \( \chi_{GR_1} := g_{R_1}^{IJ} \chi \sqrt{|g_{R_1}|} \) integrating over \( M \times [0,1-\delta^2] \), using that \( \chi_{GR_1} \) has support on \( B_{4r}(0) \), and integrating by parts gives

\[
\int_{M \times [0,1-\delta^2]} \langle \partial \psi^*_1, d[\tilde{v}_r \times d\tilde{v}_r] \rangle = \sum_I \sum_{j_1 < j_2} \int_{B_{4r}(0) \times [0,1-\delta^2]} \partial_t \psi^*_1 \left[ \partial_{j_1} (\tilde{v}_r \times \partial_{j_2} \tilde{v}_r) - \partial_{j_2} (\tilde{v}_r \times \partial_{j_1} \tilde{v}_r) \right] \chi_{GR_1}
\]

\[
= -\sum_I \sum_{j_1 < j_2} \int_{B_{4r}(0) \times [0,1-\delta^2]} \left[ \partial_{j_1} \psi^*_1 \left( \tilde{v}_r \times \partial_{j_2} \tilde{v}_r \right) - \partial_{j_2} \psi^*_1 \left( \tilde{v}_r \times \partial_{j_1} \tilde{v}_r \right) \right] \chi_{GR_1}
\]

Next, integrating by parts in time and using that \( \chi_{GR_1} \) does not depend on \( t \) gives

\[
-\int_{B_{4r}(0) \times [0,1-\delta^2]} \left[ \partial_{j_1} \psi^*_1 \left( \tilde{v}_r \times \partial_{j_2} \tilde{v}_r \right) - \partial_{j_2} \psi^*_1 \left( \tilde{v}_r \times \partial_{j_1} \tilde{v}_r \right) \right] \chi_{GR_1}
\]

and so combining this with the previous computation gives

\[
\int_{M \times [0,1-\delta^2]} \langle \partial \psi^*_1, d[\tilde{v}_r \times d\tilde{v}_r] \rangle = -\sum_I \sum_{j_1 < j_2} \int_{M \times [0,1-\delta^2]} \left[ \partial_{j_1} \psi^*_1 \left( \tilde{v}_r \times \partial_{j_2} \tilde{v}_r \right) - \partial_{j_2} \psi^*_1 \left( \tilde{v}_r \times \partial_{j_1} \tilde{v}_r \right) \right] \chi_{GR_1}
\]

\[
+ \sum_I \sum_{j_1 < j_2} \int_{B_{4r}(0) \times [0,1-\delta^2]} \left[ \partial_{j_1} \psi^*_1 \partial_t \left( \tilde{v}_r \times \partial_{j_2} \tilde{v}_r \right) - \partial_{j_2} \psi^*_1 \partial_t \left( \tilde{v}_r \times \partial_{j_1} \tilde{v}_r \right) \right] \chi_{GR_1}
\]

\[
- \sum_I \sum_{j_1 < j_2} \int_{B_{4r}(0) \times [0,1-\delta^2]} \partial_t \psi^*_1 \left[ \partial_{j_1} \chi_{GR_1} \left( \tilde{v}_r \times \partial_{j_2} \tilde{v}_r \right) - \partial_{j_2} \chi_{GR_1} \left( \tilde{v}_r \times \partial_{j_1} \tilde{v}_r \right) \right].
\]
Next we observe that

\[
\partial_t (\tilde{v}_e \times \partial_j \tilde{v}_e) = \partial_t \tilde{v}_e \times \partial_j \tilde{v}_e + \tilde{v}_e \times \partial^2_{j\alpha} \tilde{v}_e \\
= 2 \partial_t \tilde{v}_e \times \partial_j \tilde{v}_e + \partial_j (\tilde{v}_e \times \partial_t \tilde{v}_e)
\]

and similarly for \( \partial_t (\tilde{v}_e \times \partial_j \tilde{v}_e) \). Combined with our previous computations we obtain

\[
\int_{M \times [0,1 - \delta^2_0]} (\partial_t \psi_1^*, d[\tilde{v}_e \times d\tilde{v}_e] \chi)
\]

\[
= - \sum_{j_1 < j_2} \int_{M \times \{1 - \delta^2_0\}} \left[ \partial_{j_1} \psi_1^* f (\tilde{v}_e \times \partial_{j_2} \tilde{v}_e) - \partial_{j_2} \psi_1^* f (\tilde{v}_e \times \partial_{j_1} \tilde{v}_e) \right] \chi_{GR_1}
\]

\[
+ 2 \sum_{j_1 < j_2} \int_{B^{4*}(0) \times [0,1 - \delta^2_0]} \left[ \partial_{j_1} \psi_1^* f (\partial_t \tilde{v}_e \times \partial_{j_2} \tilde{v}_e) - \partial_{j_2} \psi_1^* f (\partial_t \tilde{v}_e \times \partial_{j_1} \tilde{v}_e) \right] \chi_{GR_1}
\]

\[
+ \sum_{j_1 < j_2} \int_{B^{4*}(0) \times [0,1 - \delta^2_0]} \left[ \partial_{j_1} \psi_1^* f \partial_{j_2} (\tilde{v}_e \times \partial_t \tilde{v}_e) - \partial_{j_2} \psi_1^* f \partial_{j_1} (\tilde{v}_e \times \partial_t \tilde{v}_e) \right] \chi_{GR_1}
\]

\[
- \sum_{j_1 < j_2} \int_{B^{4*}(0) \times [0,1 - \delta^2_0]} \partial_{j_1} \psi_1^* f \left[ \partial_{j_2} \chi_{GR_1} (\tilde{v}_e \times \partial_{j_2} \tilde{v}_e) - \partial_{j_2} \chi_{GR_1} (\tilde{v}_e \times \partial_{j_1} \tilde{v}_e) \right].
\]

Finally, we note that integrating by parts twice gives

\[
\int_{B^{4*}(0) \times [0,1 - \delta^2_0]} \left[ \partial_{j_1} \psi_1^* f \partial_{j_2} (\tilde{v}_e \times \partial_t \tilde{v}_e) - \partial_{j_2} \psi_1^* f \partial_{j_1} (\tilde{v}_e \times \partial_t \tilde{v}_e) \right] \chi_{GR_1}
\]

\[
= - \int_{B^{4*}(0) \times [0,1 - \delta^2_0]} (\tilde{v}_e \times \partial_t \tilde{v}_e) \left[ \partial_{j_1} \psi_1^* f \partial_{j_2} \chi_{GR_1} - \partial_{j_2} \psi_1^* f \partial_{j_1} \chi_{GR_1} \right].
\]

Putting this together with the previous computation gives

\[
\int_{M \times [0,1 - \delta^2_0]} (\partial_t \psi_1^*, d[\tilde{v}_e \times d\tilde{v}_e] \chi) = T_1 + T_2 + T_3 + T_4
\]

where

\[
T_1 = 2 \sum_{j_1 < j_2} \int_{B^{4*}(0) \times [0,1 - \delta^2_0]} \left[ \partial_{j_1} \psi_1^* f (\partial_t \tilde{v}_e \times \partial_{j_2} \tilde{v}_e) - \partial_{j_2} \psi_1^* f (\partial_t \tilde{v}_e \times \partial_{j_1} \tilde{v}_e) \right] \chi_{GR_1}
\]

\[
T_2 = - \sum_{j_1 < j_2} \int_{B^{4*}(0) \times [0,1 - \delta^2_0]} (\tilde{v}_e \times \partial_t \tilde{v}_e) \left[ \partial_{j_1} \psi_1^* f \partial_{j_2} \chi_{GR_1} - \partial_{j_2} \psi_1^* f \partial_{j_1} \chi_{GR_1} \right]
\]

\[
T_3 = - \sum_{j_1 < j_2} \int_{B^{4*}(0) \times [0,1 - \delta^2_0]} \partial_{j_1} \psi_1^* f \left[ \partial_{j_2} \chi_{GR_1} (\tilde{v}_e \times \partial_{j_2} \tilde{v}_e) - \partial_{j_2} \chi_{GR_1} (\tilde{v}_e \times \partial_{j_1} \tilde{v}_e) \right]
\]

\[
T_4 = - \sum_{j_1 < j_2} \int_{B^{4*}(0) \times \{1 - \delta^2_0\}} \left[ \partial_{j_1} \psi_1^* f (\tilde{v}_e \times \partial_{j_2} \tilde{v}_e) - \partial_{j_2} \psi_1^* f (\tilde{v}_e \times \partial_{j_1} \tilde{v}_e) \right] \chi_{GR_1}.
\]
First we estimate $T_1$. By (2.4.3), removing the coordinate description, Cauchy-Schwarz, (A.4.35), and the definition of $\tilde{v}_t$, we have

$$|T_1| \leq \frac{K M}{\epsilon} \left( \int_{B^{4_\epsilon}(0) \times [0,1 - \delta_0^2]} |\partial_t \tilde{v}_t|^2 \sqrt{|g_{\epsilon_1}|} \right)^{\frac{1}{2}} \left( \int_{B^{4_\epsilon}(0) \times [0,1 - \delta_0^2]} |D\psi_1^*|^2 \sqrt{|g_{\epsilon_1}|} \right)^{\frac{1}{4}}$$

$$\leq \frac{K M}{\epsilon} \left( \int_{B^{4_\epsilon}(0) \times [0,1 - \delta_0^2]} |\partial_t v_{\epsilon_1}| \right)^{\frac{1}{2}} \left( D\psi_1^* \right)_{L^2(M \times [0,1 - \delta_0^2])} \left( \int_{B^{4_\epsilon}(0) \times [0,1 - \delta_0^2]} |\partial_t v_{\epsilon_1}| e^{-\frac{4^{d(x,x_T)}^2}{M^2(1-t)}} \right)^{\frac{1}{2}} \left( E_{\epsilon}(v_{\epsilon_1}, (x_T,1), 1) + C R_1 E_0 \right)^{\frac{1}{2}}$$

$$\leq \frac{C(\delta_0, r)}{\epsilon} \left( \int_{B^{4_\epsilon}(0) \times [0,1 - \delta_0^2]} |\partial_t v_{\epsilon_1}| K_{a_p,r} (x, 1 - t; x_T) \right)^{\frac{1}{2}} \left( E_{\epsilon}(v_{\epsilon_1}, (x_T,1), 1) + C R_1 E_0 \right)^{\frac{1}{2}}.$$

Applying (2.5.70) gives

$$|T_1| \leq \frac{C(\delta_0, r)}{\epsilon} \left( E_{\epsilon}(v_{\epsilon_1}, (x_T,1), 1) + C R_1 E_0 \right).$$

A similar estimating technique gives

$$|T_2| \leq C(\delta_0, r) \left( E_{\epsilon}(v_{\epsilon_1}, (x_T,1), 1) + C R_1 E_0 \right).$$

For $T_3$ we use (2.4.3), Cauchy-Schwarz, remove the coordinate description, and use the monotonicity formula to obtain

$$|T_3| \leq K M \left( \int_{B^{4_\epsilon}(0) \times [0,1 - \delta_0^2]} |\nabla \tilde{v}_t|^2 \right)^{\frac{1}{2}} \left( \int_{B^{4_\epsilon}(0) \times [0,1 - \delta_0^2]} |\partial_t \psi_1^*|^2 \right)^{\frac{1}{2}}$$

$$\leq C(\delta_0, r) \left( \int_{0}^{1-\delta_0^2} E_{\epsilon}(v_{\epsilon_1}, (x_T,1), 1 - t) \right)^{\frac{1}{2}} \left( \partial_t \psi_1 \right)_{L^2(M \times [0,1 - \delta_0^2])} \left( \int_{B^{4_\epsilon}(0) \times [0,1 - \delta_0^2]} |\partial_t \psi_1^* | \right)^{\frac{1}{2}} \left( E_{\epsilon}(v_{\epsilon_1}, (x_T,1), 1) + C R_1 E_0 \right)^{\frac{1}{2}}$$

$$\leq C(\delta_0, r) \left( E_{\epsilon}(v_{\epsilon_1}, (x_T,1), 1) + C R_1 E_0 \right)^{\frac{1}{2}} \left( \partial_t \psi_1^* \right)_{L^2(M \times [0,1 - \delta_0^2])}.$$

Applying Young’s inequality then gives

$$|T_3| \leq C(\delta_0, r) \left( E_{\epsilon}(v_{\epsilon_1}, (x_T,1), 1) + C R_1 E_0 \right) + \frac{1}{2} \int_{M \times [0,1 - \delta_0^2]} |\partial_t \psi_1^*|^2.$$
Similar to the estimate for $T_3$ we use (2.4.3), Cauchy-Schwarz, remove the coordinate description, and as well as apply the monotonicity formula to obtain

$$|T_4| \leq C(\delta_0, r) \left( \bar{E}_\epsilon(v_\epsilon, (xT, 1, 1)) + CR_1 E_0 \right)^{\frac{1}{2}} \| D\psi_1^* \|_{L^2(M \times [0, 1 - \delta_0^2])}.$$

Appealing to Gaffney’s inequality and Young’s inequality we obtain

$$|T_4| \leq C(\delta_0, r) \left( \bar{E}_\epsilon(v_\epsilon, (xT, 1, 1)) + CR_1 E_0 \right)^{\frac{1}{2}} \left[ \| \psi_1^* \|_{L^2(M \times \{1-\delta_0^2\})} + \| d\psi_1^* \|_{L^2(M \times \{1-\delta_0^2\})} + \| d^* \psi_1^* \|_{L^2(M \times \{1-\delta_0^2\})} \right]
\leq C(\delta_0, r) \left( \bar{E}_\epsilon(v_\epsilon, (xT, 1, 1)) + CR_1 E_0 \right)^{\frac{1}{2}} \| \psi_1^* \|_{L^2(M \times \{1-\delta_0^2\})} + C(\delta_0, r) \left( \bar{E}_\epsilon(v_\epsilon, (xT, 1, 1)) + CR_1 E_0 \right)
+ \frac{1}{4} \int_{M \times \{1-\delta_0^2\}} \{ |d\psi_1^*|^2 + |d^* \psi_1^*|^2 \}.$$

Finally, by (A.4.38) and (A.4.39) combined with the previous computation we have

$$|T_4| \leq C(\delta_0, r) \left( \bar{E}_\epsilon(v_\epsilon, (xT, 1, 1)) + CR_1 E_0 \right) + \frac{1}{4} \int_{M \times \{1-\delta_0^2\}} \{ |d\psi_1^*|^2 + |d^* \psi_1^*|^2 \}.$$

Combining the estimates of $T_1$, $T_2$, $T_3$, and $T_4$ with (A.4.43) then shows that

$$\int_{M \times [0, 1 - \delta_0^2]} |\partial_t \psi_1^*|^2 \leq -\frac{1}{4} \int_{M \times \{1-\delta_0^2\}} \{ |d\psi_1^*|^2 + |d^* \psi_1^*|^2 \} + \frac{C(\delta_0, r)}{\epsilon} \left( \bar{E}_\epsilon(v_\epsilon, (xT, 1, 1)) + CR_1 E_0 \right)
\leq \frac{C(\delta_0, r)}{\epsilon} \left( \bar{E}_\epsilon(v_\epsilon, (xT, 1, 1)) + CR_1 E_0 \right).$$  \hfill (A.4.44)

**Corollary A.4.13.** There exists a set $\Theta_2 \subset [1 - 4\delta_0^2, 1 - \delta_0^2]$ such that

$$\mathcal{L}^1(\Theta_2) \geq \frac{3}{4} \mathcal{L}^1([1 - 4\delta_0^2, 1 - \delta_0^2])$$  \hfill (A.4.45)

where for each $t \in \Theta_2$ we have

$$\int_{M \times \{t\}} |\partial_t \psi_1^*|^2 \leq C(\delta_0, r)\epsilon^{-1} \left( \bar{E}_\epsilon(v_\epsilon, (xT, 1, 1)) + C_\gamma R_1 E_0 \right).$$  \hfill (A.4.46)
A.4.5 Clearing Out Theorem

A.4.5.1 Proof of Theorem 2.1.2

Assume $0 < \eta \leq \eta_0$ and set $\lambda(\sigma) := \sqrt{\frac{\sigma}{2K}}$, where $\sigma$ is the constant appearing in the theorem statement and where $K$ is the constant appearing in (2.4.3). We set $r_\epsilon := \min\{1, \lambda(\sigma)\epsilon\}$ and $T_\epsilon := \max\{0, 1 - \lambda^2(\sigma)\epsilon^2\} = 1 - r_\epsilon^2$. We claim that

$$\frac{1}{\epsilon^N} \int_{B_{\epsilon,\epsilon R_1}(x_T)} (1 - |v_\epsilon(x, T_\epsilon)|^2)^2 \leq \mathcal{R}_1(\eta, R),$$  \hspace{1cm} (A.4.47)

where $\mathcal{R}_1(\eta) \to 0$ as $\eta \to 0$. By (2.4.4) and Proposition 2.4.1 we have

$$\tilde{E}_\epsilon(v_\epsilon, (x_T, 1)) + CE_0 R_1 \leq \tilde{E}_\epsilon(v_\epsilon, (x_T, 1), \delta) + CE_0 \delta R_1 + C(\delta_0, r)[RM_0 + \eta]$$

$$\leq \frac{1}{4} (\tilde{E}_\epsilon(v_\epsilon, (x_T, 1), 1) + CE_0 R_1) + R(\eta, R) + CE_0 \delta R_1 + C(\delta_0, r)[RM_0 + \eta]$$

$$\leq \frac{1}{2} (\tilde{E}_\epsilon(v_\epsilon, (x_T, 1), 1) + CE_0 R_1) + R(\eta, R) + C(\delta_0, r)[RM_0 + \eta]$$

and so rearranging gives

$$\tilde{E}_\epsilon(v_\epsilon, (x_T, 1), 1) + CE_0 R_1 \leq 2R(\eta, R) + 2C(\delta_0, r)[RM_0 + \eta].$$  \hspace{1cm} (A.4.48)

Assume first that $\lambda(\sigma)\epsilon \leq 1$, so that $T_\epsilon = 1 - (\lambda(\sigma))^2\epsilon^2$. We deduce from the monotonicity formula that

$$\tilde{E}_\epsilon(v_\epsilon, (x_T, 1), \lambda(\sigma)\epsilon) + CE_0 \lambda(\sigma)\epsilon R_1 \leq K_M (\tilde{E}_\epsilon(v_\epsilon, (x_T, 1), 1) + CE_0 R_1)$$  \hspace{1cm} (A.4.49)

so that, combining (A.4.48) and (A.4.49) we obtain

$$\tilde{E}_\epsilon(v_\epsilon, (x_T, 1), r_\epsilon) + CE_0 \lambda(\sigma)\epsilon R_1 \leq 2K_M R(\eta, R) + 2K_M C(\delta_0, r)[RM_0 + \eta].$$

If $\lambda(\sigma)\epsilon \geq 1$, then $r_\epsilon = 1$ and $T_\epsilon = 0$ so that

$$\tilde{E}_\epsilon(v_\epsilon, (x_T, 1), 1) \leq \eta|\log(\epsilon)| = \eta \log \left(\frac{1}{\epsilon}\right) \leq \eta|\log(\lambda(\sigma))|.$$
Hence, when $\lambda(\sigma) \epsilon \leq 1$

\[
\frac{1}{\epsilon^N} \int_{B_{e,s_{R_1}}(x_T)} (1 - |v_e(x, T_e)|^2)^2 = \frac{1}{\epsilon^{N-2}} \int_{B_{e,s_{R_1}}(x_T)} \frac{(1 - |v_e(x, T_e)|^2)^2}{\epsilon^2} \\
= \frac{(\lambda(\sigma))^{N-2}}{r^N} \int_{B_{e,s_{R_1}}(x_T)} V_e(v_e(x, T_e)) \\
\leq \frac{(\lambda(\sigma))^{N-2}}{r^N} e^{\frac{\lambda(\sigma)}{4}} \int_{B_{e,s_{R_1}}(x_T)} V_e(v_e(x, T_e)) e^{-\frac{(d_e(r, x_T))^2}{2}} \\
\leq (4\pi)^\frac{N}{2} (\lambda(\sigma))^{N-2} e^{\frac{\lambda(\sigma)}{4}} \tilde{E}_e(v_e, (x_T, 1), r_e).
\]

and when $\lambda(\sigma) \epsilon \geq 1$, since $T_e = 0$ and $r_e = 1$, we have

\[
\frac{1}{\epsilon^N} \int_{B_{e,s_{R_1}}(x_T)} (1 - |v_e(x, T_e)|^2)^2 = \frac{1}{\epsilon^{N-2}} \int_{B_{e,s_{R_1}}(x_T)} \frac{(1 - |v_e(x, 0)|^2)^2}{\epsilon^2} \\
= \frac{(\lambda(\sigma))^{N-2}}{r^N} \int_{B_{e,s_{R_1}}(x_T)} V_e(v_e(x, 0)) \\
\leq \frac{(\lambda(\sigma))^{N-2}}{r^N} \int_{B_{e,s_{R_1}}(x_T)} V_e(v_e(x, 0)) e^{-\frac{(d_e(r, x_T))^2}{2}} \\
\leq (4\pi)^\frac{N}{2} (\lambda(\sigma))^{N-2} e^{\frac{\lambda(\sigma)}{4}} \tilde{E}_e(v_e, (x_T, 1), r_e).
\]

We conclude that in either instance we have

\[
\frac{1}{\epsilon^N} \int_{B_{e,s_{R_1}}(x_T)} (1 - |v_e(x, T_e)|^2)^2 \leq C(\sigma) \tilde{E}_e(v_e, (x_T, 1), r_e)
\]

for some constant dependent only on $\sigma$ and $M$. Arguing as in Lemma (III.3) from [9], except adapted to a Riemannian manifold, we obtain

\[
1 - |v_e(x_T, T_e)| \leq C_M \left( \frac{1}{\epsilon^N} \int_{B_{e}(x_T)} (1 - |v_e(x, T_e)|^2)^2 \right)^{\frac{1}{N+2}} \leq C'(\sigma)(R_1(\eta, R))^{\frac{1}{N+2}}. \quad (A.4.52)
\]

On the other hand by (2.4.3) we have

\[
|v_e(x_T, T_e) - v_e(x_T, 1)| \leq \frac{K}{\epsilon^2} (1 - T_e) = K(\lambda(\sigma))^2 = \frac{\sigma}{2}. \quad (A.4.53)
\]
Combining (A.4.52), (A.4.53), and the definition of \( v \) we obtain

\[
1 - |u_\varepsilon(x_T, T)| = 1 - |u_\varepsilon(x_T, 1)| \leq \frac{\sigma}{2} + C'(\sigma)(R_1(\eta, R))^{\frac{1}{\sqrt{2}}}
\]

so that the conclusion follows if \( \eta(\sigma) \) and \( R(\sigma) \) are chosen sufficiently small, since \( R_1(\eta, R) \to 0 \) as \( \eta, R \to 0 \).

A.5 Consequences of Theorem 2.1.2

Proposition A.5.1. Let \( u_\varepsilon \) be a solution of \((PGL)_\varepsilon\) verifying assumption \((H_0)\) and \( \sigma > 0 \) be given. Let \( x_T \in M, T > 0 \), and \( 0 < 2\varepsilon < R^2 < R(\sigma) \) where \( R(\sigma) \) is as in Theorem 2.1.2. There exists a positive continuous function \( \lambda \) defined on \((0, \infty)\) such that, if

\[
\tilde{\eta}(x_T, T, R) := \frac{1}{(4\pi)^{\frac{N}{2}} R^{N-2}} \log(\varepsilon) \int_{B(x_T, R)} e_\varepsilon(u_\varepsilon(\cdot, T)) \leq \frac{\eta_1(\sigma)}{2}
\]

then

\[
|u_\varepsilon(x, t)| \geq 1 - \sigma \quad \text{for } t \in [T + T_0, T + T_1] \text{ and } x \in B_R(x_T).
\]

Here \( T_0 \) and \( T_1 \) are defined by

\[
T_0 := \left( \frac{2\eta}{\eta_1} \right)^{\frac{1}{2}} R^2, \quad T_1 := R^2.
\]

In particular, a more precise estimate shows that we can find \( \lambda \) defined on \((0, \infty) \times (0, \infty)\) satisfying

\[
\lambda(T, R) \sim \sqrt{-\frac{8}{c^2} \log \left( \frac{(4\pi)^{\frac{N}{2}}}{M_0 e^{C_2}} \left[ \frac{2}{T + 2R^2} \right]^{\frac{N-2}{2}} \right)}
\]

as \((T, R) \to (0, 0)\). In particular, \( \lambda(T, R)R \) is bounded as \( R \to 0^+ \) for any \( T > 0 \).

Proof. Let \( x_0 \in B_{\frac{R}{2}}(x_T) \). Observe that by Proposition 2.3.8 we have that for any \( \lambda > 0 \) and \( \sqrt{T_0} < r < \sqrt{T_1} = R \) we have

\[
\widetilde{E}_\varepsilon((x_0, T + r^2), r) \leq \frac{1}{(4\pi)^{\frac{N}{2}} r^{N-2}} \int_{B_{r}(x_0) \times (T)} e_\varepsilon(u_\varepsilon) + M_0 e^{c_2 \frac{C_2 \frac{1}{2} \varepsilon^2}{(4\pi)^{\frac{N}{2}}} \left[ \frac{2}{T + 2r^2} \right]^{\frac{N-2}{2}}} + C_1(4\pi)^{\frac{N}{2}} (\sqrt{2})^{N-2} \sqrt{T} \log(\varepsilon)).
\]

(A.5.1)
We choose $\lambda_0(T)$ sufficiently large that

$$M_0 e^{-\frac{c_2(\lambda_0(T))^2}{2}} \left[ e^{C_2 \left( \frac{2}{T} \right)^{\frac{N-2}{2}}} + C_1 (4\pi)^{\frac{N}{2}} (\sqrt{2})^{N-2} \sqrt{T} \right] \leq \frac{\eta_1}{2}. \quad (A.5.2)$$

Next we set

$$\lambda(T) := \max\{2, 2\lambda_0(T)\}.$$

From this we obtain

$$\frac{1}{\left( \frac{4\pi}{2} \right)^{\frac{N-2}{2}} r^{N-2}} \int_{B_\lambda(T) \times \{x_0\} \times \{T\}} e^\epsilon(u_\epsilon) \leq \left( \frac{R}{r} \right)^{N-2} \left\{ \frac{1}{\left( \frac{4\pi}{2} \right)^{\frac{N-2}{2}} R^{N-2}} \int_{B_{2\lambda}(T) \times \{x_0\} \times \{T\}} e^\epsilon(u_\epsilon) \right\}$$

$$\leq \left( \frac{R}{r} \right)^{N-2} \tilde{\eta}(x_0, T, R) \log(\varepsilon) \leq \left( \frac{R}{\sqrt{T_0}} \right)^{N-2} \tilde{\eta} |\log(\varepsilon)|.$$

Then, if we choose $T_0 := K \tilde{\eta}^{\frac{N-2}{2}} R^2$ where $K := \left( \frac{\eta_1}{2} \right)^{\frac{N-2}{2}}$ then we obtain

$$\frac{1}{r^{N-2}} \int_{B_{2\lambda}(T) \times \{T\}} e^\epsilon(u_\epsilon) \leq \frac{\eta_1}{2} |\log(\varepsilon)|. \quad (A.5.3)$$

Combining (A.5.1) and (A.5.3) and using Theorem 2.1.2 gives the desired conclusion.

A.6 Proof of Theorem 2.6.1 and Consequences

A.6.0.1 Proof of Theorem 2.6.1

By (2.6.1) there are real-valued functions $\rho_\epsilon$ and $\varphi_\epsilon$ defined on $\Lambda$ such that

$$u_\epsilon = \rho_\epsilon e^{i\varphi_\epsilon} \quad \text{in} \ \Lambda, \quad (A.6.1)$$

where $\rho_\epsilon := |u_\epsilon|$. By multiplying $u_\epsilon$ by $e^{\frac{1}{\sqrt{N\pi}} \int_\Lambda \varphi_\epsilon}$ we may assume that

$$\frac{1}{|\Lambda|} \int_\Lambda \varphi_\epsilon = 0. \quad (A.6.2)$$

Using the representation (A.6.1) in (PGL)$_\epsilon$ we obtain the following two PDEs

$$\rho^2_\epsilon \partial_\epsilon \varphi_\epsilon - \text{div}(\rho^2_\epsilon \nabla \varphi_\epsilon) = 0 \quad (A.6.3)$$

$$\partial_\epsilon \rho_\epsilon - \Delta \rho_\epsilon + \rho |\nabla \varphi_\epsilon|^2 = \rho \left( \frac{1 - \rho^2_\epsilon}{\varepsilon^2} \right) \quad (A.6.4)$$
in $\Lambda$. We shall omit the $\varepsilon$ from the notation when no confusion will arise. We consider the truncated function $\tilde{\varphi}$ defined on $M \times [T, T + \Delta T]$ by

$$\tilde{\varphi}(x,t) := \varphi(x,t) \chi(x),$$

where $\chi$ is a smooth cutoff function such that $\chi \equiv 1$ on $B_{\frac{4}{3}}R_5(x_0)$ and $\chi \equiv 0$ on $M \setminus B_{\frac{5}{3}}R_6(x_0)$.

Then $\tilde{\varphi}$ satisfies the equation

$$\rho^2 \partial_t \tilde{\varphi} - \text{div}(\rho^2 \nabla \tilde{\varphi}) = -\text{div}(\rho^2 \varphi \nabla \chi) - \rho^2 \langle \nabla \chi, \nabla \varphi \rangle \quad \text{in } \Lambda. \quad (A.6.5)$$

Notice that, by construction, we have

$$\text{supp}(\tilde{\varphi}) \subset B_{\frac{4}{3}}R_5(x_0) \times [T, T + \Delta T].$$

In particular, $\tilde{\varphi} = 0$ on $\partial B_R(x_0) \times [T, T + \Delta T]$. Since

$$\int_T^{T+\Delta T} \int_{B_R(x_0)} e_\varepsilon(u_\varepsilon) \leq \int_{\Lambda} e_\varepsilon(u_\varepsilon)$$

then by a mean value argument there exists $t_0 \in [T, T + \Delta T]$ such that

$$\int_{B_R(x_0) \times \{t_0\}} e_\varepsilon(u_\varepsilon) \leq \frac{4}{\Delta T} \int_{\Lambda} e_\varepsilon(u_\varepsilon). \quad (A.6.6)$$

We set

$$\Lambda_0 := B_R(x_0) \times [t_0, T + \Delta T] \supset \Lambda_\frac{3}{4} = B_{\frac{4}{3}}R_5(x_0) \times \left[ T + \frac{\Delta T}{4}, T + \Delta T \right]. \quad (A.6.7)$$

We rewrite (A.6.5) in the following way to emphasize proximity to the heat equation

$$\partial_t \tilde{\varphi} - \Delta \tilde{\varphi} = \text{div}(\rho^2 - 1)\nabla \tilde{\varphi} + (1 - \rho^2) \partial_t \tilde{\varphi} - \text{div}(\rho^2 \varphi \nabla \chi) - \rho^2 \langle \nabla \chi, \nabla \varphi \rangle \quad (A.6.8)$$
We introduce the function \( \varphi_0 \) defined on \( \Lambda_0 \) as the solution to
\[
\begin{cases}
\partial_t \varphi_0 - \Delta \varphi_0 = -\text{div}(\rho^2 \varphi \nabla \chi) - \rho^2 \langle \nabla \chi, \nabla \varphi \rangle & \text{in } \Lambda_0, \\
\varphi_0(x, t_0) = \tilde{\varphi}(x, t_0) & \text{on } B_R(x_0) \times \{t_0\}, \\
\varphi_0(x, t) = 0 & \forall x \in \partial B_R(x_0), \forall t \geq t_0.
\end{cases}
\] (A.6.9)

We note that since \( \chi \equiv 1 \) on \( B_{4R}(x_0) \) then
\[
\partial_t \varphi_0 - \Delta \varphi_0 = 0 \quad \text{in } B_{4R}(x_0) \times [t_0, T + \Delta T].
\] (A.6.10)

We also note that \( \varphi_0 \) will eventually be our choice for \( \Phi_\varepsilon \). Next we set \( \varphi_1 := \tilde{\varphi} - \varphi_0 \). We now obtain our desired estimates through a series of steps. We begin by estimating \( \varphi_0 \).

**Step 1: Estimates for \( \varphi_0 \).** We will demonstrate that
\[
\| \nabla \varphi_0 \|^2_{L^2 L^{2^*}(\Lambda_0)} \leq C_1(\Lambda) \int_{\Lambda} e_\varepsilon(u_\varepsilon) \tag{A.6.11}
\]
\[
\| \nabla \varphi_0 \|^2_{L^\infty(A^{1/2}_0)} \leq C_2(\Lambda) \int_{\Lambda} e_\varepsilon(u_\varepsilon), \tag{A.6.12}
\]

where \( 2^* := \frac{2N}{N-2} \) is the Sobolev exponent in dimension \( N \), and, for \( 1 < p, q < \infty \), we use the notation
\[
L^p L^q(\Lambda_0) := \left\{ f \text{ measurable on } \Lambda_0 \text{ such that } \| f(\cdot, t) \|_{L^p(B_R(x_0))} \|_{L^q([t_0, T + \Delta T])} < \infty \right\}.
\]

**Proof.** We decompose \( \varphi_0 \) as \( \varphi_0 = \varphi_0^0 + \varphi_0^1 \), where \( \varphi_0^0 \) solves
\[
\begin{cases}
\partial_t \varphi_0^0 - \Delta \varphi_0^0 = 0 & \text{in } \Lambda \\
\varphi_0^0(x, t_0) = \tilde{\varphi}(x, t_0) & \text{on } B_R(x_0) \times \{t_0\}, \\
\varphi_0^0(x, t) = 0 & \forall x \in \partial B_R(x_0), \forall t \geq t_0.
\end{cases}
\] (A.6.13)

By Theorem 5 of section 7.1 of [19] we have
\[
\| \varphi_0^0 \|_{L^2([t_0, T + \Delta T]; W^{2,2}(B_R(x_0)))} \leq C(\Lambda) \| \nabla \tilde{\varphi} \|_{L^2(B_R(x_0) \times \{t_0\})} \leq C(\Lambda) \| e_\varepsilon(u_\varepsilon) \|_{L^{1/2}(\Lambda)},
\]
and therefore by the Sobolev Embedding Theorem
\[
\|\nabla \varphi^0\|_{L^2L^{2^*}(\Lambda_0)} \leq C(\Lambda) \|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)}^{\frac{1}{2}}. \quad (A.6.14)
\]

Next we estimate $\varphi^1_0$. As in [11] we let $T$ denote the linear mapping which, to any function $f$ defined on $\Lambda_0$, associates the unique solution $v = T(f)$ of the problem
\[
\begin{cases}
\partial_t v - \Delta v = f & \text{in } \Lambda, \\
v = 0 & \text{on } B_R(x_0) \times \{t_0\}, \\
v = 0 & \forall x \in \partial B_R(x_0), \forall t \geq t_0.
\end{cases}
\]

Recall that, see [31], the operators $f \mapsto T(f)$, $f \mapsto \nabla T(f)$, $f \mapsto \nabla^2(T(f))$, and $g \mapsto \nabla(T(\text{div}(g)))$ are linear continuous on $L^pL^q(\Lambda_0)$. Using this notation we can write
\[
\varphi^1_0 = T(f) + T(\text{div}(g))
\]
where
\[
f := -\rho^2(\nabla \chi, \nabla \varphi), \quad g := -\rho^2 \varphi \nabla \chi.
\]

Notice that we have the estimates
\[
\|f\|_{L^2(\Lambda_0)} \leq C(\Lambda) \|\nabla \varphi\|_{L^2(\Lambda_0)} \leq C(\Lambda) \|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)}^{\frac{1}{2}},
\]
and in view of (A.6.2), the Sobolev-Poincaré inequality in dimension $N + 1$, and (2.3.2), and (A.6.6)
\[
\|g\|_{L^{2^*}(\Lambda_0)} \leq C(\Lambda) \|\varphi\|_{L^{2^*}(\Lambda_0)} \leq C(\Lambda) \|\nabla_{x,t} \varphi\|_{L^2(\Lambda_0)} \leq C(\Lambda) \|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)}^{\frac{1}{2}},
\]
where $2^* := \frac{2(N+1)}{N-1}$ is the Sobolev conjugate in dimension $N + 1$. Therefore, by the linear theory for $T$ above,
\[
\|\nabla \varphi^1_0\|_{L^2L^{2^*}(\Lambda_0)} \leq \|\nabla(T(f))\|_{L^2L^{2^*}(\Lambda_0)} + \|\nabla(T(\text{div}(g)))\|_{L^2L^{2^*}(\Lambda_0)} \quad (A.6.15)
\]
\[\leq C(\Lambda)\left[\|f\|_{L^2(\Lambda_0)} + \|g\|_{L^2L^{2^*}(\Lambda_0)}\right] \leq C(\Lambda)\left[\|f\|_{L^2(\Lambda_0)} + \|g\|_{L^2L^{2^*}(\Lambda_0)}\right] \leq C(\Lambda) \|e_\varepsilon(u_\varepsilon)\|_{L^1(\Lambda)}^{\frac{1}{2}}.
\]
Combining (A.6.14) and (A.6.15) gives (A.6.11). Finally, (A.6.12) follows from (A.6.10), (A.6.11), and standard estimates for the homogeneous heat equation, see Theorem 4.45 from [6] as well as Theorems 8 and 9 from subsection 2.3 of [19]. 

\textit{Step 2: The equation for }\varphi_1\textit{ The function }\varphi_1\textit{ verifies the evolution problem}

\[
\begin{cases}
\partial_t \varphi_1 - \Delta \varphi_1 = \text{div}((\rho^2 - 1)\nabla \tilde{\varphi}) + (1 - \rho^2)\partial_t \tilde{\varphi} & \text{in } \Lambda, \\
\varphi_1(x,t_0) = 0 & \text{on } B_R(x_0) \times \{t_0\}, \\
\varphi_1(x,t) = 0 & \forall x \in \partial B_R(x_0), \forall t \geq t_0.
\end{cases}
\] (A.6.16)

For convenience we rewrite (A.6.16) as

\[
\partial_t \varphi_1 - \Delta \varphi_1 = \text{div}((\rho^2 - 1)\nabla \varphi_1) + f_0 + \text{div}(g_0)
\] (A.6.17)

where

\[f_0 := (1 - \rho^2)\partial_t \tilde{\varphi}\quad\text{and}\quad g_0 := (\rho^2 - 1)\nabla \varphi_0.\]

Observe that for \(1 \leq p < 2\) and any \(t \in [t_0, T + \Delta T]\), we have

\[
\int_{B_R(x_0) \times \{t\}} |f_0|^p = \int_{B_R(x_0) \times \{t\}} |1 - \rho^2|^p |\partial_t \tilde{\varphi}|^p \leq \left( \int_{B_R(x_0) \times \{t\}} |\partial_t \tilde{\varphi}|^2 \right)^{\frac{p}{2}} \left( \int_{B_R(x_0) \times \{t\}} |1 - \rho^2|^2 \right)^{\frac{2p}{2 - p}}.
\]

Using the hypothesis (H0) and identity (A.1.1) we find that

\[
\int_{t_0}^{T + \Delta T} \left( \int_{B_R(x_0) \times \{t\}} |f_0|^p \right)^{\frac{2}{p}} \leq C(\Lambda) \int_{t_0}^{T + \Delta T} \left\{ \left( \int_{B_R(x_0) \times \{t\}} |\partial_t \tilde{\varphi}|^2 \right) (M_0 |\varepsilon|^2 |\log(\varepsilon)|)^{\frac{2}{2 - p}} \right\}^{\frac{p}{2}}
\]

\[
= C(\Lambda) \left( \int_{\Lambda_0} |\partial_t \tilde{\varphi}|^2 \right)^{\frac{2}{p}} (M_0 |\varepsilon|^2 |\log(\varepsilon)|)^{\frac{2-p}{2-p}}
\]

\[
\leq C(\Lambda) [M_0 |\varepsilon|^2 |\log(\varepsilon)|]^{\frac{2}{p}} (M_0 |\log(\varepsilon)|)^{\frac{2}{2-p} - 1}
\]

\[
= C(\Lambda) [M_0 |\log(\varepsilon)|]^{\frac{2}{p}} (M_0 |\log(\varepsilon)|)^{\frac{2}{2-p}}.
\]

Hence,

\[
\|f_0\|_{L^p L^p(\Lambda_0)}^p \leq C(\Lambda) M_0 |\varepsilon|^{2-p} |\log(\varepsilon)|.
\] (A.6.18)
Similarly, for $2 \leq q < 2^*$ we have
\[ \|g_0\|_{L^2 L^q(\Lambda_0)}^q \leq C(\Lambda) \varepsilon^{\frac{2}{q^*}(2^*-q)} (M_0|\log(\varepsilon)|)^{\frac{2^*}{q^*}}. \quad (A.6.19) \]

We now estimate $\varphi_1$.

**Step 3: The fixed point argument.** Using the linear operator $T$ defined earlier we can rewrite \((A.6.16)\) as
\[ \varphi_1 = T(\div((\rho^2 - 1)\nabla \varphi_1)) + T(f_0 + \div(g_0)). \]

Rewriting this as
\[ (I - A)(\varphi_1) = b \]
where $A$ is the linear operator $v \mapsto T(\div((\rho^2 - 1)\nabla v))$ and $b := T(f_0 + \div(g_0))$. Next we set $I = [t_0, T + \Delta T]$, we fix $p$ and $q$ such that
\[ 1 < p < 2, \quad q = p^* = \frac{Np}{N-p}, \quad \text{and} \quad 2 < q < 2^* < 2^*, \]
and we consider the Banach space
\[ X_q = \{ v \in W^{1,2}(I, W^{-1,q}(B_R(x_0))) \cap L^2(I, W^{1,q}(B_R(x_0))) \text{ such that } v(t_0) = 0 \}. \]

It follows from the linear theory for $T$ mentioned earlier that $A : X_q \to X_q$ is linear and continuous and that
\[ \|A\|_{L(X_q)} \leq C(q) \|1 - \rho\|_{L^\infty(\Lambda_0)}. \]

In particular, we fix $\sigma > 0$ such that
\[ C(q) \|1 - \rho\|_{L^\infty(\Lambda_0)} \leq C(q)\sigma < \frac{1}{2}. \]

Thus, $I - A$ is invertible on $X_q$, and hence
\[ \|\varphi_1\|_{X_q} \leq C \|b\|_{X_q}. \quad (A.6.20) \]

Finally, by estimates $(A.6.18)$, $(A.6.19)$, the Sobolev Embedding Theorem, and [31] we have
\[ \|b\|_{X_q} \leq C(\Lambda) \left( \varepsilon^{\frac{2^*}{q^*}} (M_0|\log(\varepsilon)|)^{\frac{1}{q^*}} + \varepsilon^{\frac{2^*}{q^*}(2^*-q)} (M_0|\log(\varepsilon)|)^{\frac{2^*}{q^*}} \right). \]
Putting the previous information together gives
\[
\|\nabla \varphi_1\|_{L^2(L^q(\Lambda_0))} \leq C(\Lambda) \left( \varepsilon^{\frac{q}{q+2}} \{M_0|\log(\varepsilon)|\}^{\frac{1}{q}} + \varepsilon^{\frac{q}{q+2}}(2^q - q) \{M_0|\log(\varepsilon)|\}^{\frac{q}{q+2}} \right). \tag{A.6.21}
\]

**Step 4:** Improved integrability of \(\nabla \tilde{\varphi}\). Combining (A.6.11) and (A.6.21) we obtain the following estimate
\[
\|\nabla \tilde{\varphi}\|_{L^2(L^q(\Lambda_0))} \leq C(\Lambda)(M_0 + 1)|\log(\varepsilon)|. \tag{A.6.22}
\]

**Step 5:** Estimates for the modulus and potential terms. Following the same ideas presented in Step 5 of the proof of Theorem 2 in [11] shows that
\[
\int_{\Lambda_0} e_\varepsilon(|u_\varepsilon|) \leq C(\Lambda)(M_0 + 1)\left( \varepsilon^{\frac{q}{2}(q-2)} + \varepsilon |\log(\varepsilon)|^2 \right). \tag{A.6.23}
\]

**Step 6:** Proof of the \(L^\infty\) bound (2.6.2) for the energy. As in [11] we use Lemma 4.4 of [15].

**Proposition A.6.1.** Let \(0 < \varepsilon < 1\), \(x_0 \in M\), and let \(u_\varepsilon\) be a solution of \((PGL)_\varepsilon\) on the cylinder \(\Lambda_{R,x_0} := B_R(x_0) \times [0,R^2]\) for some \(R > 0\). Then there exists a constant \(\gamma_0 > 0\), depending only on \(M\) such that if \(R > \sqrt{\varepsilon}\) is sufficiently small and
\[
\frac{1}{R^N} \int_{\Lambda_{R}} e_\varepsilon(u_\varepsilon) \leq \gamma_0 \tag{A.6.24}
\]
thен
\[
e_\varepsilon(u_\varepsilon)(x,t) \leq \frac{K}{R^{N+2}} \int_{\Lambda_{R}} e_\varepsilon(u_\varepsilon) \tag{A.6.25}
\]
for any \((x,t) \in B_{\frac{R}{2}}(x_0) \times [\frac{3R}{4}, R]\).

The rest of the proof is similar to step 6 of the proof of Theorem 2 in [11].

**Step 7:** Improved estimates for \(\nabla \rho\) and \(V_\varepsilon(u_\varepsilon)\). Set \(\theta := 1 - \rho\). Applying Lemma A.2.3 to the cylinder \(\Lambda_{\frac{R}{8}}\), we obtain
\[
|\theta| \leq C(\Lambda)\varepsilon^2|\log(\varepsilon)| \quad \text{on} \quad \Lambda_{\frac{R}{16}}. \tag{A.6.26}
\]

Using (A.6.4) and Lemma A.2.3 we obtain
\[
|\partial_\theta - \Delta \theta| \leq C(\Lambda)|\log(\varepsilon)| \quad \text{on} \quad \Lambda_{\frac{R}{16}}. \tag{A.6.27}
\]

From (A.6.27) we deduce by standard linear theory, see Theorem 4.45 of [6] and also [31], that for
every $1 < q_1 < \infty$ and $1 < q_2 < \infty$,

$$\|\theta\|_{W^{1,q_1}(I,L^{q_2}(B))} \leq C(\Lambda)|\log(\varepsilon)|, \quad \|\theta\|_{L^{q_1}(I,W^{2,q_2}(B))} \leq C(\Lambda)|\log(\varepsilon)|,$$

where $I := [T + \frac{\Delta T}{2}, T + \Delta T]$ and $B := B_{2}\varepsilon(x_0)$. By interpolation, see [31] we obtain

$$\|\theta\|_{W^{1,4}(I,W^{4,2}(B))} \leq C(\Lambda)|\log(\varepsilon)|.$$

Choosing $q_1$ and $q_2$ sufficiently large we obtain that for every $0 < \gamma < 1$

$$\|\theta\|_{C^{0,\frac{1}{4}}(I,C^{1,\gamma}(B))} \leq C(\gamma, \Lambda)|\log(\varepsilon)|.$$

Notice that (A.6.26) shows that

$$\|\theta\|_{L^\infty(I,L^\infty(B))} \leq C(\Lambda)\varepsilon^2|\log(\varepsilon)|,$$

and hence by interpolation we have

$$\|\theta\|_{C^{0,\frac{1}{4}}(I,C^{1,\beta}(B))} \leq C(\Lambda)\varepsilon^\alpha \quad (A.6.28)$$

for some $\alpha > 0$. In particular, we have

$$|\nabla \rho|_{L^\infty(\Lambda_{\frac{1}{2}})} = |\nabla \theta|_{L^\infty(\Lambda_{\frac{1}{2}})} \leq C(\Lambda)\varepsilon^\alpha. \quad (A.6.29)$$

Finally, as a result of (A.6.26) and Proposition A.2.1 we obtain

$$V_\varepsilon(u_\varepsilon) \leq K\frac{\theta^2}{\varepsilon^2} \leq C(\Lambda)\varepsilon^2|\log(\varepsilon)|^2 \quad \text{on } \Lambda_{\frac{1}{2}}$$

so that

$$|\nabla \rho| + V_\varepsilon(u_\varepsilon) \leq C(\Lambda)\varepsilon^\alpha \quad \text{on } \Lambda_{\frac{1}{2}}. \quad (A.6.30)$$

**Step 8: Improved $L^\infty$ estimates for $\nabla \varphi_1$.** Recall that $\varphi$ satisfies the equation

$$\rho^2 \partial_t \varphi - \text{div}(\rho^2 \nabla \varphi) = 0.$$

From the $\alpha$-Hölder regularity bound for $\rho$ we infer $\alpha$-Hölder regularity bounds for $\partial_t \varphi$ of the order
Notice that on $\Lambda_{\frac{1}{2}}$ we have that $\varphi = \tilde{\varphi}$ and so $f_0 = (1 - \rho^2)\partial_1 \tilde{\varphi} = (1 - \rho^2)\partial_1 \varphi$. Thus, by (A.6.29) we have

$$\|f_0\|_{C^{0,\alpha}(\Lambda_{\frac{1}{2}})} \leq C(\Lambda)\varepsilon^\alpha |\log(\varepsilon)|^2.$$ 

Also, by (A.6.28) combined with (A.6.13) and the Hölder regularity of $\varphi$, we have that

$$\|g_0\|_{C^{0,\frac{1}{5}}(I,C^{1,\beta}(B))} \leq C(\Lambda)\varepsilon^\alpha |\log(\varepsilon)|.$$

By equation (A.6.16), the previous estimates on $f_0$ and $g_0$, as well as Schauder theory we obtain

$$\|\nabla \varphi_1\|_{C^{0,\alpha}(\Lambda_{\frac{1}{2}})} \leq C(\Lambda)\varepsilon^\beta$$

for some $\beta > 0$.

**Step 9: Estimate (2.6.5) completed.** On $\Lambda_{\frac{1}{2}}$ we write

$$2 \varepsilon(u_\varepsilon) = |\nabla u_\varepsilon|^2 + 2V_\varepsilon(u_\varepsilon)$$

$$= |\nabla \rho|^2 + \rho^2|\nabla \varphi|^2 + 2V_\varepsilon(u_\varepsilon)$$

$$= |\nabla \rho|^2 + (\rho^2 - 1)|\nabla \varphi|^2 + |\nabla \Phi_\varepsilon|^2 + 2(\nabla \Phi_\varepsilon, \nabla \varphi_1) + |\nabla \varphi_1|^2 + 2V_\varepsilon(u_\varepsilon)$$

$$= |\nabla \Phi_\varepsilon|^2 + \kappa_\varepsilon$$

and the conclusion follows from the previous steps. 

### A.7 Analysis of the measures $\mu_\ast^{t}$ Proofs

#### A.7.1 Densities and the concentration set Proofs

**A.7.1.1 Proof of Lemma 2.7.7**

Fix $(x, t) \in M \times (0, \infty)$ and consider a sequence $\{(x_n, t_n)\}_{n \in \mathbb{N}}$ in $M \times (0, \infty)$ converging to $(x, t)$. Fix $0 < r < \sqrt{t}$ and, for $n$ sufficiently large, we set $r_n = \sqrt{t^2 + t_n - t}$ so that $t - r^2 = t_n - r_n^2$. By
monotonicity of $\mathcal{G}$ and since $\Theta^\mathcal{P}_{N-2}$ exists everywhere

$$\Theta^\mathcal{P}_{N-2}(\mu_\star, (x_n, t_n)) \leq \frac{e^{C_2r_n}}{(4\pi)^\frac{N}{2} r_n^{N-2}} \int_M e^{-\frac{(d_\star(x_n, y))^2}{4r_n^2}} \mu_\star^{t_n-r_n^2}(y) + C_1M_0 r_n$$

Letting $n \to \infty$ we obtain

$$\limsup_{n \to \infty} \Theta^\mathcal{P}_{N-2}(\mu_\star, (x_n, t_n)) \leq \frac{e^{C_2r}}{(4\pi)^\frac{N}{2} r^{N-2}} \int_M e^{-\frac{(d_\star(x, y))^2}{4r^2}} \mu_\star^{t-r^2}(y) + C_1M_0 r.$$ 

Letting $r \to 0^+$ we obtain

$$\limsup_{n \to \infty} \Theta^\mathcal{P}_{N-2}(\mu_\star, (x_n, t_n)) \leq \Theta^\mathcal{P}_{N-2}(\mu_\star, (x, t))$$

which demonstrates upper semi-continuity of $(x, t) \mapsto \Theta^\mathcal{P}_{N-2}(\mu_\star, (x, t))$.  

\[\square\]

**A.7.2 First properties of $\Sigma_\mu$**

**A.7.2.1 Proof of Lemma 2.7.8**

We proceed by proving the contrapositive statement. Suppose there is $(x, t) \in M \times (0, \infty)$ and $0 < r < \sqrt{t}$ for which

$$r^{2-N} \mu_\star^{t-r^2}(B_\lambda(t-r^2)r(x)) \leq \eta_2.$$

By Theorem 2.7.2, for all $\tau \in [t - \frac{r^2}{16}, t]$ we have

$$\mu_\tau^\star = \frac{\lvert \nabla \Phi_\star \rvert^2}{2} dvol_\rho(x) \quad B_\xi(x)$$
where $\Phi_*$ is smooth. Observe that on $B_\frac{r}{4}(x)$ we have, if $\text{inj}_g(M) < \frac{r}{4}$ and we use the notation $A_{\text{inj}_g(M)} \left( \frac{x}{r} \right) = B_{\text{frac}(r)}(x) \setminus B_{\text{inj}_g(M)}(x)$, that

\[
\begin{align*}
&\quad s^{2-N} \int_{B_\frac{r}{4}(x)} e^{-\frac{(d_+(x,y))^2}{4s^2}} \, d\mu_*^{t-s^2}(y) \leq s^{2-N} \frac{2}{\eta^2} \left\| \nabla \Phi_* \right\|_{L^\infty}(B_\frac{r}{8}(x)) \int_{B_\frac{r}{8}(x)} e^{-\frac{(d_+(x,y))^2}{4s^2}} \, d\text{vol}_g(y) \\
&\quad \leq s^{2-N} \frac{\left\| \nabla \Phi_* \right\|_{L^\infty}(B_\frac{r}{8}(x))}{2} \left[ \int_{B_{\text{inj}_g(M)}(x)} e^{-\frac{(d_+(x,y))^2}{4s^2}} \, d\text{vol}_g(y) + \int_{A_{\text{inj}_g(M)} \left( \frac{x}{r} \right)} e^{-\frac{(d_+(x,y))^2}{4s^2}} \, d\text{vol}_g(y) \right] \\
&\quad \leq s^{2-N} \frac{\left\| \nabla \Phi_* \right\|_{L^\infty}(B_\frac{r}{8}(x))}{2} \left[ \int_{\text{inj}_g(M)} s^{-N} e^{-\frac{|y|^2}{4s^2}} \sqrt{|g|} \, d\text{vol}_g(y) + s^{-N} e^{-\frac{(\text{inj}_g(M))^2}{16s^2}} \, \text{vol}_g \left( A_{\text{inj}_g(M)} \left( \frac{x}{r} \right) \right) \right] \\
&\quad \leq s^{2-N} \frac{\left\| \nabla \Phi_* \right\|_{L^\infty}(B_\frac{r}{8}(x))}{2} \left[ (4\pi)^\frac{N}{2} \left\| \sqrt{|g|} \right\|_{L^\infty}(B_{\text{inj}_g(M)}(x)) + \sup_{s>0} \left\{ s^{-N} e^{-\frac{(\text{inj}_g(M))^2}{16s^2}} \right\} \, \text{vol}_g \left( A_{\text{inj}_g(M)} \left( \frac{x}{r} \right) \right) \right]
\end{align*}
\]

where we have used normal coordinates centred at $x$ for the third inequality. A similar estimate can be obtained when $\frac{r}{4} \leq \text{inj}_g(M)$. Next observe that on $M \setminus B_\frac{r}{4}(x)$ we have, by (2.2.4), that

\[
\begin{align*}
&\quad s^{2-N} \int_{M \setminus B_\frac{r}{4}(x)} e^{-\frac{(d_+(x,y))^2}{4s^2}} \, d\mu_*^{t-s^2}(y) \leq s^{2-N} e^{-\frac{r^2}{20s^2}} \, M_0. \\
&\quad \text{(A.7.2)}
\end{align*}
\]

As a result of (A.7.1) and (A.7.2) we have, as $s \to 0^+$, that

\[
\begin{align*}
&\quad s^{2-N} \int_{M} e^{-\frac{(d_+(x,y))^2}{4s^2}} \, d\mu_*^{t-s^2}(y) \to 0
\end{align*}
\]

which shows that $(x,t) \notin \Sigma_\mu$. \hfill \Box

A.7.2.2 Proof of Theorem 2.7.9

Let $(x,t) \in M \times (0, \infty)$ and $0 < r < \sqrt{t}$. Then we have

\[
\begin{align*}
&\quad r^{2-N} \mu_*(B_{\lambda(t-r^2)}(x)) \leq e^{\frac{|\lambda(t-r^2)|^2}{2}} \mathcal{F}_\mu((x,t), r). \\
&\quad \text{We consider the function } \eta_3 \text{ defined by}
\end{align*}
\]

\[
\begin{align*}
&\quad \eta_3(s) := e^{-\frac{\lambda(s)^2}{2}} \eta_2. \\
&\quad \text{Observe that if for some } 0 < r < \sqrt{t} \text{ we have}
\end{align*}
\]

\[
\mathcal{F}_\mu((x,t), r) \leq \eta_3(t - r^2)
\]
then by (A.7.3) we have
\[ r^{2-N} \mu_*^{t-r^2}(B_{\lambda(t-r^2)})(x) \leq \eta_2. \]
The conclusion then follows from Lemma 2.7.8. \(\square\)

A.7.2.3 Proof of Corollary 2.7.10

Suppose \((x,t) \in M \times (0, \infty)\). By Theorem 2.7.9 we have that for all \(0 < r < \sqrt{t}\) that
\[ \frac{1}{r^{N-2}} \int_M e^{-\frac{(d_{x}(x,y))^2}{4r^2}} \text{d} \mu_*^{t-r^2}(y) > \eta_3(t-r^2). \]
Letting \(r \to 0^+\) as using that \(\eta_3\) is a continuous function gives the desired conclusion. \(\square\)

A.7.2.4 Proof of Proposition 2.7.11

Proof of (1): By Corollary 2.7.10 we have
\[ \Sigma_{\mu} = \{(x,t) \in M \times (0, \infty) : \Theta^P_{N-2}(\mu_*, (x,t)) \geq \eta_3(t)\}. \]
Since \(\eta_3\) is continuous and \(\Theta^P_{N-2}(\mu_*, \cdot)\) is upper semi-continuous by Lemma 2.7.7 then \(\Theta^P_{N-2}(\mu_*, \cdot) - \eta_3(\cdot)\) is upper semi-continuous on \(M \times (0, \infty)\) and so \(\Sigma_{\mu}\) is closed due to the above characterization of \(\Sigma_{\mu}\). \(\square\)

Proof of (2): We prove this estimate first for \(t = 1\) and then argue by scaling.

Step 1: The case \(t = 1\). Let \(0 < \delta < \frac{1}{4} \min \{1, \text{inj}_g(M)\}\). We consider a covering of \(M\) by geodesic balls of radius \(\delta\) such that
\[ M \subset \bigcup_{j \in I} B_{\delta}(x_j), \quad \text{and} \quad B_{\frac{\delta}{2}}(x_i) \cap B_{\frac{\delta}{2}}(x_j) = \emptyset \quad \text{for} \ i \neq j. \]
Set
\[ I_\delta := \{i \in I : B_{\delta}(x_i) \cap \Sigma^1_{\mu} \neq \emptyset\}. \]
If \(i \in I_\delta\), then there is \(y_i \in \Sigma^1_{\mu} \cap B_{\delta}(x_i)\). By Lemma 2.7.8 we have that
\[ \mu_*^{1-\delta^2}(B_{\lambda(1-\delta^2)})(y_i) > \eta_2 \delta^{N-2}, \]
and hence
\[ \mu^1_\delta(B_{\lambda(1-\delta^2)+1}\delta(x_i)) > \eta_2 \delta^{N-2}. \]  
(A.7.5)

Since the balls \( B_\delta(x_i) \) are disjoint then the balls
\[ \{ B_{\lambda(1-\delta^2)+1}\delta(x_i) \}_{i \in I_\delta} \]
cover \( M \) at most \( K \) times where \( K \) is a constant depending on \( N \) for \( \delta < \frac{1}{4} \min \{ 1, \inj_g(M) \} \). Thus, we have
\[ \sum_{i \in I_\delta} \mu^1_\delta(B_{\lambda(1-\delta^2)+1}\delta(x_i)) \leq KM_0. \]  
(A.7.6)

From (A.7.5) and (A.7.6) we obtain
\[ (#I_\delta)\delta^{N-2} \leq \frac{KM_0}{\eta_2}. \]  
(A.7.7)

Using the definition of Hausdorff measure as well as (A.7.7) leads to
\[ \mathcal{H}^{N-2}(\Sigma^1_{\mu}) \leq \limsup_{\delta \to 0^+} (#I_\delta) \left(2 \left[ \| \lambda \|_{L^\infty([0,1])} + 1 \right] \right)^{N-2} \delta^{N-2} \]
\[ \leq \left(2 \left[ \| \lambda \|_{L^\infty([0,1])} + 1 \right] \right)^{N-2} \frac{KM_0}{\eta_2} \]
which is the desired conclusion.

**Step 2 Invariance by scaling.** For \( t_0 > 0 \) fixed, we consider the function
\[ v_\epsilon(x, t) := u_\epsilon(x, t_0 t) \]
where \( \epsilon := \frac{\epsilon}{\sqrt{t_0}}. \) Observe that for all \( x \in M \) we have
\[ v_\epsilon(x, 1) = u_\epsilon(x, t_0), \quad v_\epsilon(x, 0) = u_\epsilon(x, 0) \]
and that \( v_\epsilon \) verifies (PGL)_\epsilon when \( M \) has the metric \( g_{t_0} := \frac{g}{t_0} \). In particular, we have
\[ E_{\epsilon, g_{t_0}}(v_\epsilon(x, 0)) = t_0^{2-N} E_{\epsilon, g}(u_\epsilon(x, 0)). \]  
(A.7.8)
Notice that as $\varepsilon_n \to 0^+$ then $\varepsilon_n = \frac{\varepsilon_n}{\sqrt{t}_0} \to 0^+$. Finally, we observe that

$$ \sum_{\mu, g, t_0}^t (v_{\varepsilon}) = \sum_{\mu, g}^{t_0} (u_{\varepsilon}), $$

which means that

$$ \mathcal{H}_{g, t_0}^{N-2} (\Sigma_{\mu, g, t_0}^1 (v_{\varepsilon})) = \mathcal{H}_{g, t_0}^{N-2} (\Sigma_{\mu, g}^{t_0} (u_{\varepsilon})) = t_0^{\frac{2-N}{2}} \mathcal{H}_g^{N-2} (\Sigma_{\mu, g}^{t_0} (u_{\varepsilon})). $$

Observe that step 1 did not depend on what the metric was used. Hence, applying it to $v_{\varepsilon}$ as well as using (A.7.8) and $(H_0)$ gives

$$ \mathcal{H}_{g, t_0}^{N-2} (\Sigma_{\mu, g, t_0}^1 (v_{\varepsilon})) \leq K \sup_{n \in \mathbb{N}} \{ E_{\varepsilon_n} (v_{\varepsilon_n}) \} \leq K t_0^{\frac{2-N}{2}} M_0. $$

Therefore

$$ \mathcal{H}_g^{N-2} (\Sigma_{\mu, g}^{t_0} (u)) \leq K M_0. $$

Since $t_0$ was arbitrary the conclusion follows. \hfill \Box

Proof of (3): By (1) we know that $\Sigma_{\mu}^t$ is closed, and hence Borel measurable. Therefore, since $\mu_*^t$ is a Radon measure we have

$$ \mu_*^t = \mu_*^t \ll M \setminus \Sigma_{\mu}^t + \mu_*^t \ll \Sigma_{\mu}^t. \quad \text{(A.7.9)} $$

We will show that there is a smooth function $g$ defined on $(M \times (0, \infty)) \setminus \Sigma_{\mu}$ such that

$$ \mu_*^t \ll M \setminus \Sigma_{\mu}^t = g(\cdot, t) \mathcal{H}^N. $$

Observe that, by definition, if $x \in M \setminus \Sigma_{\mu}^t$ and $t > 0$ then

$$ \lim_{r \to 0^+} \mathcal{F}_\mu ((x, t), r) = 0. $$

Hence, if $r_0$ is chosen small enough then we can guarantee that

$$ \mathcal{F}_\mu ((x, t), r_0) \leq \eta_3 (t - r_0^2). $$
Therefore, by (A.7.3) and the choice (A.7.4) we may conclude that
\[ \mu_{r^2}(B_{\lambda(t-r_0^2)}(x)) \leq \eta_2 r_0^{N-2} \]
and we may infer from Theorem 2.7.2 that for all \( s \in [t - \frac{r_0^2}{16}, t] \),
\[ \mu_s \equiv g(\cdot, s)H^N \quad B_{\frac{t}{2}}(x), \]
for some smooth function \( g \). Since, by (2), \( H^{N-2}(\Sigma^t_\mu) < \infty \) then \( H^N(\Sigma^t_\mu) = 0 \) and hence
\[ \mu_t \sim M \setminus \Sigma^t_\mu = g(\cdot, t)H^N, \quad (A.7.10) \]
which establishes the claim. Next, by Lemma 2.7.4 we have that \( \mu_t \sim \Sigma^t_\mu \) is absolutely continuous with respect to the measure \( H^{N-2} \) and by (2) that \( H^{N-2} \Sigma^t_\mu \) is finite. Thus, by the Radon-Nikodym theorem we obtain
\[ \mu_t \sim \Sigma^t_\mu = \Theta_*(x, t)H^{N-2} \setminus \Sigma^t_\mu \quad (A.7.11) \]
where \( \Theta_* \) is the Radon-Nikodym derivative. By Lemma 2.7.4 \( \Theta_* \) verifies the bound
\[ \Theta_*(x, t) \leq K_M M_0 \left[ e^{C_M t^{\frac{2-N}{2}}} + D_M \sqrt{T} \right]. \quad (A.7.12) \]
Putting together (A.7.9), (A.7.10), (2.7.4), and (A.7.12) gives the desired conclusion. \( \square \)

A.7.3 Regularity of \( \Sigma^t_\mu \)

A.7.3.1 Proof of Proposition 2.7.12

We begin by noting that by (A.1.1) and (H0) we have
\[ \frac{1}{|\log(\varepsilon)|} \int_{M \times (0, T]} |\partial_t u_\varepsilon|^2 \leq M_0 \quad \text{for every } T > 0. \]
By perhaps passing to a further subsequence, we may assume that there exists a non-negative Radon measure, \( \omega_* \), defined on \( M \times (0, \infty) \) such that
\[ \frac{1}{|\log(\varepsilon)|} |\partial_t u_\varepsilon|^2 \rightarrow \omega_* \quad \text{as measures}, \]
so that for each $T > 0$

$$\omega_*(M \times (0, T]) \leq M_0. \quad \text{(A.7.13)}$$

For $l \in \mathbb{N}$ and $q \in (0, \infty)$ to be fixed later, we set

$$A_l(\omega_*) := \{(x, t) \in M \times (0, \infty) : \limsup_{r \to 0^+} \frac{1}{r^q} \int_{B_{r^2}(x) \times [t-r^2, t]} \omega_* \geq 1 \}. \quad \text{(A.7.14)}$$

To complete the proof we continue in a series of steps.

**Step 1:** We demonstrate that for each $l \in \mathbb{N},$

$$\mathcal{H}_P^q(A_l(\omega_*)) < \infty$$

where $\mathcal{H}_P^q$ denotes the $q$-dimensional Hausdorff measure with respect to the parabolic distance. We recall that the parabolic distance on $M \times [0, \infty)$ is defined so that if $(x, t), (x', t') \in M \times [0, \infty)$ then

$$d^P_g((x, t), (x', t')) := \max\{d_g(x, x'), |t - t'|^{1/2}\}.$$
where for fixed $i$, the sets $\Gamma_j := \Gamma_P^i(x_j, t_j, r(x_j, t_j))$ are disjoint. Consequently, it follows from (A.7.14) that for each $i = 1, 2, \ldots, m(l, N, M)$,

$$\sum_{j \in I_l^i} (r(x_j, t_j))^q \leq \sum_{j \in I_l^i} \int_{\Gamma_j} \omega_* \leq \int_{M \times (0, T_f]} \omega_* \leq C(M_0).$$

Therefore,

$$m(l, n, M) \sum_{i=1}^{m(l, n, M)} \sum_{j \in I_l^i} (r_P(\Gamma_j))^q \leq m(l, N, M) l^q C(M_0).$$

Since the right-hand side is independent of $\delta$ we may let $\delta \to 0^+$ to obtain

$$\mathcal{H}_{l}^q(A_l(\omega_*)) \leq m(l, N, M) l^q C(M_0)$$

which gives the desired conclusion.

We now fix $q = N - \frac{3}{2}$. Next we demonstrate some properties of the Hausdorff measure of $\bigcup_{l \in \mathbb{N}} A_l(\omega_*)$.

Step 2: We will demonstrate that

$$\mathcal{H}^{N-1} \left( \bigcup_{l \in \mathbb{N}} A_l(\omega_*) \right) = 0,$$

(A.7.15)

hence, for almost every $t > 0$

$$\mathcal{H}^{N-2} \left( \bigcup_{l \in \mathbb{N}} A_l^t(\omega_*) \right) = 0,$$

(A.7.16)

where $A_l^t(\omega_*) := A_l(\omega_*) \cap (M \times \{t\})$.

By the previous step we have $\mathcal{H}_{l}^{N-\frac{3}{2}}(A_l(\omega_*)) < \infty$. It follows that

$$\mathcal{H}_{l}^{N-1}(A_l(\omega_*)) = 0.$$

However, since

$$B^P_r(x, t) \subset B_{\sqrt{2}r}(x, t)$$

for $0 \leq r \leq 1$ and since Hausdorff measure only depends on small balls then it follows that

$$\mathcal{H}^{N-1} \left( \bigcup_{l \in \mathbb{N}} A_l(\omega_*) \right) = 0.$$
This completes the proof.

Before we begin the next step we introduce the set

$$
\Omega_\omega := (M \times (0, \infty)) \setminus \bigcup_{t \in \mathbb{N}} A_t(\omega_t).
$$

**Step 3:** Let $$\chi \in C^\infty(\mathbb{R})$$ such that $$\chi$$ is a bounded Lipschitz function with compact support. Then, for $$(x, t) \in \Omega_\omega$$,

$$
\lim_{r \to 0^+} \left( \frac{1}{r^{N-2}} \int_M \chi \left( \frac{d_+(x, y)}{r} \right) d\mu^t(y) - \frac{1}{r^{N-2}} \int_M \chi \left( \frac{d_+(x, y)}{r} \right) d\mu^{t-r^2}(y) \right) = 0.
$$

For $$0 < r < \min\left\{ \sqrt{\ell}, 1, \frac{\text{inj}_l(M)}{2} \right\}$$ we have by Lemma 2.3.1 that

$$
\int_{M \times \{ t \}} \frac{e_+(u_x)}{|\log(\varepsilon)|^2} \chi \left( \frac{d_+(x, y)}{r} \right) d\nu^t(y) - \int_{M \times \{ t-r^2 \}} \frac{e_+(u_x)}{|\log(\varepsilon)|} \chi \left( \frac{d_+(x, y)}{r} \right) d\nu^t(y) = - \int_{M \times \{ t-r^2, t \}} \frac{|\partial_t u_x|^2}{|\log(\varepsilon)|} \chi \left( \frac{d_+(x, y)}{r} \right) d\nu^t(y) dt
$$

$$
+ \frac{1}{r |\log(\varepsilon)|} \int_{M \times \{ t-r^2, t \}} \partial_t u_x \cdot \langle \nabla u_x, \nabla_y(d_+(x, y)) \rangle \left( \frac{d_+(x, y)}{r} \right) d\nu^t(y) dt.
$$

Observe that by the definition of $$d_+$$ and (1.2.4) we have

$$
\nabla_y(d_+(x, y)) = - f' \left( \frac{d(y, \infty)}{\text{inj}_t(M)} \right) \nabla_y \exp_y^{-1}(x) \frac{d(x, y)}{d(y, \infty)},
$$

where $$f'$$ is supported on $$[0, \text{inj}_t(M)]$$. Choose $$l \in \mathbb{N}$$ such that $$\text{supp}(\chi) \subset [-l, l]$$ and set $$\Lambda := B_{l_0}^+(x) \times [t-r^2, t]$$ where $$y \in B_{l_0}^+(x)$$ if and only if $$d_+(x, y) \leq r$$. Observe that by Cauchy-Schwarz we have

$$
\frac{1}{r |\log(\varepsilon)|} \left| \int_{\Lambda} \partial_t u_x \cdot \langle \nabla u_x, \nabla_y(d_+(x, y)) \rangle \chi \left( \frac{d_+(x, y)}{r} \right) \right|
$$

$$
\leq \left( \int_{\Lambda} \frac{|\partial_t u_x|^2}{|\log(\varepsilon)|^2} \right)^{\frac{1}{2}} \left( \int_{\Lambda} \frac{\nabla u_x^2}{r^2 |\log(\varepsilon)|} \right)^{\frac{1}{2}} \| \chi \|_{L^\infty(\mathbb{R})} \| f' \|_{L^\infty(0, \infty)}. \tag{A.7.17}
$$

Letting $$\varepsilon \to 0^+$$ gives the following inequality for measures

$$
\left| \frac{1}{r^{N-2}} \int_M \chi \left( \frac{d(x, y)}{r} \right) \left( d\mu^t - d\mu^{t-r^2} \right)(y) \right| \leq \left[ \frac{1}{r^{N-2}} \int_\Lambda \omega_t + \left( \frac{1}{r^{N-2}} \int_\Lambda \omega_t \right)^{\frac{1}{2}} \left( \frac{1}{r^N} \int_\Lambda d\mu^t \right)^{\frac{1}{2}} \right] \| \chi \|_{W^{1, \infty}(\mathbb{R})}.
$$
Observe that
\[
\frac{1}{r^{N-2}} \int_{\Lambda} \omega_\ast = r^{\frac{1}{2}} \left( \frac{1}{r^{N-2}} \int_{\Lambda} \omega_\ast \right)
\]
and that by (A.1.1), (2.7.1), and (H₀) we have
\[
\frac{1}{r^{N}} \int_{\Lambda} \omega_\ast \leq e^{\frac{r^2}{4} \int_{M} e^{-\frac{d_+(x,y)^2}{4r^2}} d\mu_\ast - \frac{r^2}{4} \int_{M} e^{-\frac{d_+(x,y)^2}{4r^2}} d\mu_\ast - \frac{r^2}{2} \int_{M} e^{-\frac{d_+(x,y)^2}{4r^2}} d\mu_\ast - r^2} \leq e^{\frac{r^2}{2} \int_{M} \frac{d_+(x,y)^2}{r^2} d\mu_\ast + (4\pi)^{\frac{N}{2}} C_1 M_0 \sqrt{t}}.
\]

Therefore, the right-hand side of (A.7.17) can be bounded by
\[
\mathcal{R}(r) := C(t, l, M_0) \|\chi\|_{W^{1, \infty}(\mathbb{R})} r^{\frac{1}{2}} \left[ \frac{1}{r^{N-2}} \int_{\Lambda} \omega_\ast + \left( \frac{1}{r^{N-2}} \int_{\Lambda} \omega_\ast \right)^{\frac{1}{2}} \right].
\]
Since by assumption \((x, t) \in \Omega_\omega\) then letting \(r \to 0^+\) we obtain
\[
\lim_{r \to 0^+} \mathcal{R}(r) \leq 2C(t, l, M_0) \|\chi\|_{W^{1, \infty}(\mathbb{R})} \lim_{r \to 0^+} r^{\frac{1}{2}} = 0
\]
which completes the proof.

Step 4: Next we show that the previous step extends to exponentially decaying functions. That is, for \((x, t) \in \Omega_\omega\) we have
\[
\lim_{r \to 0^+} \left[ \frac{1}{r^{N-2}} \int_{M} e^{-\frac{d_+(x,y)^2}{4r^2}} d\mu_\ast (y) - \frac{1}{r^{N-2}} \int_{M} e^{-\frac{d_+(x,y)^2}{4r^2}} d\mu_\ast - r^2 \right] = 0. \quad (A.7.18)
\]

To see this, we consider \(\zeta: \mathbb{R} \to \mathbb{R}\) is a smooth cut-off function such that \(0 \leq \zeta \leq 1, \zeta \equiv 1\) on \(B_1(0)\), and \(\zeta \equiv 0\) on \(\mathbb{R} \setminus B_2(0)\). From this we can construct the function \(\zeta_t\) defined by \(\zeta_t(y) := \zeta \left( \frac{y}{t} \right)\) as well as
\[
\chi_t(y) := e^{\frac{y^2}{4t}} \zeta_t(y)
\]
defined over \(\mathbb{R}\). Applying the previous step to this function allows us to conclude that
\[
\lim_{r \to 0^+} \left[ \frac{1}{r^{N-2}} \int_{M} \chi_t \left( \frac{d_+(x,y)}{r} \right) d\mu_\ast (y) - \frac{1}{r^{N-2}} \int_{M} \chi_t \left( \frac{d_+(x,y)}{r} \right) d\mu_\ast - r^2 \right] = 0. \quad (A.7.19)
\]
Observe that for $2r < \min\{\sqrt{t}, 1\}$ and $s \in [t - r^2, t]$ we have, by the properties of $\zeta$, that

\[
e^{-\frac{(d_+(x, y))^2}{4r^2}} - \chi_l \left( \frac{d_+(x, y)}{r} \right) \leq e^{-\frac{(d_+(x, y))^2}{4r^2}} \left[ 1 - \zeta_l \left( \frac{d_+(x, y)}{r} \right) \right] \leq e^{-\frac{(d_+(x, y))^2}{8r^2}} \cdot e^{-\frac{r^2}{8}}
\]

and so by (2.7.1) and (H0) we have

\[
\frac{1}{r^{N-2}} \int_M e^{-\frac{(d_+(x, y))^2}{4r^2}} - \chi_l \left( \frac{d_+(x, y)}{r} \right) \, d\mu^*_y(y) \leq e^{-\frac{r^2}{8}} \frac{1}{r^{N-2}} \int_M e^{-\frac{(d_+(x, y))^2}{8r^2}} \, d\mu^*_y(y)
\]

(A.7.20)

\[
\leq 2^{\frac{N-2}{2}} (4\pi)^{\frac{N}{2}} e^{-\frac{r^2}{8}} \varepsilon'^2 \mu_\varepsilon \left( (x, s + 2r^2), r \sqrt{2} \right)
\]

\[
\leq 2^{\frac{N-2}{2}} (4\pi)^{\frac{N}{2}} e^{-\frac{r^2}{8}} \varepsilon'^2 \mu_\varepsilon \left( (x, s + 2r^2), \sqrt{s + 2r^2} \right)
\]

\[
= 2^{\frac{N-2}{2}} e^{-\frac{r^2}{8}} \left[ \frac{e^{C_2(s + 2r^2)}}{(s + 2r^2)^{\frac{N-2}{2}}} \int_M e^{-\frac{(d_+(x, y))^2}{4(s + 2r^2)}} \, d\mu^*_y(y) + (4\pi)^{\frac{N}{2}} C_1 M_0 \sqrt{s + 2r^2} \right]
\]

\[
\leq 2^{\frac{N-2}{2}} M_0 e^{-\frac{r^2}{8}} \left[ \frac{e^{C_2(s + 2r^2)}}{(s + 2r^2)^{\frac{N-2}{2}}} + (4\pi)^{\frac{N}{2}} C_1 \sqrt{s + 2r^2} \right].
\]

Taking the limit supremum as $r \to 0^+$, and noting that $s \to t^-$ as $r \to 0^+$, in (A.7.20) allows us to conclude that

\[
\limsup_{r \to 0^+} \left| \frac{1}{r^{N-2}} \int_M e^{-\frac{(d_+(x, y))^2}{4r^2}} \left( d\mu^* - d\mu^{*-r^2} \right)(y) \right| \leq 2^{\frac{N-2}{2}} M_0 e^{-\frac{r^2}{8}} \left[ \frac{e^{C_2 t}}{t^{\frac{N-2}{2}}} + (4\pi)^{\frac{N}{2}} C_1 \sqrt{t} \right].
\]

Since $l$ was arbitrary the conclusion follows by letting $l \to \infty$.

**Step 5:** We are now ready to prove proposition A.7.3.1. For $(x, t) \in \Omega_\omega$ we set

\[
\tilde{\Theta}_{N-2}(\mu^*_x, x) := \lim_{r \to 0^+} \frac{1}{(4\pi)^{\frac{N}{2}} r^{N-2}} \int_M e^{-\frac{(d_+(x, y))^2}{4r^2}} \, d\mu^*_y(y).
\]

Note that by the previous step we have that $\tilde{\Theta}_{N-2}(\mu^*_x, x)$ exists on $\Omega_\omega$ and

\[
\tilde{\Theta}_{N-2}(\mu^*_x, x) = \Theta^P_{N-2}(\mu^*_x, (x, t)).
\]

(A.7.21)

If $(x, t) \notin \Sigma_\mu$ then, by definition, $\Theta^P_{N-2}(\mu^*_x, (x, t)) = 0$ and so (2.7.7) holds. Thus we may assume,
using (A.7.15), that \((x, t) \in \Sigma_\mu \cap \Omega_\mu\). Next observe that by Lemma 2.7.1 and \((H_0)\) we have

\[
\frac{1}{(lr)^{N-2}} \int_{B_{lr}(x)} d\mu^*_t(y) \geq \frac{1}{(lr)^{N-2}} \int_{B_{lr}(x)} e^{-\frac{(d_\mu(x,y))^2}{4r^4}} d\mu^*_t(y)
\]

\[
= \frac{1}{l^{N-2}} \cdot \frac{1}{r^{N-2}} \int_M e^{-\frac{(d_\mu(x,y))^2}{4r^4}} d\mu^*_t(y)
\]

\[
- \frac{1}{l^{N-2}} \cdot \frac{1}{r^{N-2}} \int_{M \setminus B_{lr}(x)} e^{-\frac{(d_\mu(x,y))^2}{4r^4}} d\mu^*_t(y)
\]

\[
\geq \frac{1}{l^{N-2}} \cdot \frac{1}{r^{N-2}} \int_M e^{-\frac{(d_\mu(x,y))^2}{4r^4}} d\mu^*_t(y)
\]

\[
- \frac{e^{-\pi}}{l^{N-2}} \cdot \frac{1}{r^{N-2}} \int_{M \setminus B_{lr}(x)} e^{-\frac{(d_\mu(x,y))^2}{4r^4}} d\mu^*_t(y)
\]

\[
\geq \frac{1}{l^{N-2}} \cdot \frac{1}{r^{N-2}} \int_M e^{-\frac{(d_\mu(x,y))^2}{4r^4}} d\mu^*_t(y)
\]

\[
- \frac{2^{N-2}}{l^{N-2}} \frac{(4\pi) \frac{\eta}{2} e^{-\pi}}{r^{N-2}} \mu^*_t((x, t + 2r^2), \sqrt{2})
\]

\[
\geq \frac{1}{l^{N-2}} \cdot \frac{1}{r^{N-2}} \int_M e^{-\frac{(d_\mu(x,y))^2}{4r^4}} d\mu^*_t(y)
\]

\[
- \frac{2^{N-2}}{l^{N-2}} e^{-\frac{\pi}{2}} \mu^*_t((x, t + 2r^2), \sqrt{t + 2r^2})
\]

\[
\geq \frac{1}{l^{N-2}} \cdot \frac{1}{r^{N-2}} \int_M e^{-\frac{(d_\mu(x,y))^2}{4r^4}} d\mu^*_t(y)
\]

\[
- \frac{2^{N-2}}{l^{N-2}} e^{-\frac{\pi}{2}} M_0 \left[ \frac{e^{C_2 \sqrt{t + 2l^2}}}{l^{N-2}} + (4\pi) \frac{\eta}{2} C_1 \sqrt{t + 2r^2} \right].
\]

Letting \(r \rightarrow 0^+\) and using (A.7.21) leads to

\[
\Theta_{\mu^*_t, x} \geq \frac{1}{l^{N-2}} \left[ \Theta_{\mu^*_t, x} - KM_0 e^{-\frac{\pi}{2}} \left( \frac{e^{C_2 \sqrt{t}}}{l^{N-2}} + D\sqrt{t} \right) \right].
\]

(A.7.22)

Since \((x, t) \in \Sigma_\mu\) then \(\Theta_{\mu^*_t, x} \geq \eta_3(t)\) then, if we choose \(l\) sufficiently large, we have that

\[
KM_0 e^{-\frac{\pi}{2}} \left( \frac{e^{C_2 \sqrt{t}}}{l^{N-2}} + D\sqrt{t} \right) \leq \frac{1}{2} \eta_3(t) \leq \frac{1}{2} \Theta_{\mu^*_t, x}.
\]

(A.7.23)

Combining (A.7.22) and (A.7.23) we may conclude that

\[
\Theta_{\mu^*_t, x} \geq \frac{1}{2l^{N-2}} \Theta_{\mu^*_t, x} \geq \frac{1}{2l^{N-2}} \Theta_{\mu^*_t, x} \geq \frac{1}{2l^{N-2}} \Theta_{\mu^*_t, x}
\]

which demonstrates (2.7.7).
A.7.3.2 Proof of Corollary 2.7.13

By proposition A.7.3.1 we have that for $L^1$-almost every $t > 0$

$$\Theta_{\ast,N-2}(\mu_{\ast},x) \geq K\Theta^P_{N-2}(\mu_{\ast},(x,t))$$

for $H^N$-almost every $x \in M$. In particular, if $(x,t) \in \Sigma_{\mu}$ then

$$\Theta^P_{N-2}(\mu_{\ast},(x,t)) \geq \eta_3(t).$$

Combining the previous two inequalities gives the desired conclusion. 

Lemma A.7.1.

1. If $g \in F$ then for every $s > 0$ the function $g_s(l) := g(l\sqrt{s})$ is a member of $F$.

2. If $A(s) := I(g_{s})$ then $A(s) = A(1)s^{\frac{2-N}{2}}$.

3. The function $e_s(l) := e^{-l^2s} \in F$ for all $s > 0$ and

$$I(e_s) = s^{\frac{2-N}{2}}I(e_1). \quad (A.7.24)$$

Proof. We prove (1) and (2) together. Let $s > 0$ and suppose $g \in F$. Observe that

$$I(g) = \lim_{r \to 0^+} \frac{1}{r^{N-2}} \int_M g \left( \frac{d_+(x,y)}{r} \right) d\mu^t_r(y)$$

$$= \lim_{r \to 0^+} \frac{1}{r^{N-2}} \left( \frac{r}{\sqrt{s}} \right)^{N-2} \int_M g \left( \frac{d_+(x,y)}{\sqrt{s}} \right) d\mu^t_{s_1}(y)$$

$$= s^{\frac{N-2}{2}} \lim_{r \to 0^+} \frac{1}{r^{N-2}} \int_M g_s \left( \frac{d_+(x,y)}{r} \right) d\mu^t_{s}(y)$$

$$= s^{\frac{N-2}{2}} I(g_s)$$

and so $g_s \in F$. Also, we have

$$A(s) = I(g_{s}) = s^{\frac{2-N}{2}}I(g) = s^{\frac{2-N}{2}}A(1)$$

which demonstrates (2). Observe that $e_s \in F$ is shown in step 4 of the proof of Proposition A.7.3.1 and that (2) allows us to conclude $e_s \in F$ for all $s > 0$. Finally, observe that (A.7.24) follows from (2). 

\[\square\]
Lemma A.7.2. For every $k \in \mathbb{N} \cup \{0\}$, the function $l \mapsto l^{2k}e^{-l^2}$ belongs to $F$.

Proof. The case $k = 0$ follows from step (3) of Lemma A.7.1 with $s = 1$. Next we show that if, for $k \in \mathbb{N} \cup \{0\}$ and $s > 0$, $\frac{d^k}{ds^k}(e_s(l)) \in F$ and we have shown

$$\frac{d^k}{ds^k}(A(s)) = \lim_{r \to 0^+} \frac{(-1)^k}{r^{N-2}} \int_M \left( \frac{d_+(x,y)}{r} \right)^{2k} e^{-\frac{(d_+(x,y))^2}{r^2}} d\mu_t(y)$$

then $\frac{d^{k+1}}{ds^{k+1}}(e_s(l)) \in F$ and we also have

$$\frac{d^{k+1}}{ds^{k+1}}(A(s)) = \lim_{r \to 0^+} \frac{(-1)^{k+1}}{r^{N-2}} \int_M \left( \frac{d_+(x,y)}{r} \right)^{2(k+1)} e^{-\frac{(d_+(x,y))^2}{r^2}} d\mu_t(y) \quad (A.7.25)$$

for each $s > 0$ and $k \in \mathbb{N} \cup \{0\}$. Observe that

$$\frac{d^k}{ds^k}(e_s(l)) = (-1)^k l^{2k} e^{-l^2s}$$

and that, for $\Delta s \in \mathbb{R}$ such that $s + \Delta s > 0$, we have

$$\left| (-1)^k l^{2k} e^{-l^2(s+\Delta s)} - (-1)^k l^{2k} e^{-l^2s} + (-1)^k l^{2k} e^{-l^2s} l^2 \Delta s \right|$$

$$= l^{2k} e^{-l^2s} \left| 1 - e^{-l^2\Delta s} - l^2 \Delta s \right|$$

$$\leq \left\| e^{-l^2(\cdot)} \right\|_{L^\infty(B_{r}(\Delta s)(s))} \frac{l^{2k+4} e^{-l^2s}}{2} (\Delta s)^2$$

$$\leq C(s,k) e^{-\frac{l^2s}{2}} (\Delta s)^2.$$
$0 < r < \sqrt{t}$, that

$$
\frac{1}{r^{N-2}} \left| \int_M \left[ \frac{h\left(s + \Delta s, \frac{d_+(x,y)}{r}\right) - h\left(s, \frac{d_+(x,y)}{r}\right)}{\Delta s} - \frac{\partial h}{\partial s} \left( s, \frac{d_+(x,y)}{r} \right) \right] d\mu^*_r(y) \right| 
\leq \frac{C(s, k) |\Delta s|}{r^{N-2}} \int_M e^{-\frac{(d_+(x,y))^2}{2r^2}} d\mu^*_r(y)
\leq (4\pi)^{\frac{N}{2}} C(s, k) |\Delta s| \left\| \phi \right\| \left( x, t + \frac{r^2}{2} \right), \frac{r}{\sqrt{2}}
\leq (4\pi)^{\frac{N}{2}} C(s, k)|\Delta s| \left\| \phi \right\| \left( x, t + \frac{r^2}{2} \right), \sqrt{t + \frac{r^2}{2}}
\leq \frac{C(s, k) M_0 |\Delta s|}{t^{N-2}} \left[ e^{C_2 \sqrt{\frac{r^2 + 2}{2}}} + (4\pi)^{\frac{N}{2}} C_1 \sqrt{t + \frac{r^2}{2}} \right]
\leq \frac{C(s, k) M_0 |\Delta s|}{t^{N-2}} \left[ e^{C_2 \sqrt{\frac{r^2}{2}}} + (4\pi)^{\frac{N}{2}} C_1 \sqrt{\frac{3t}{2}} \right].
$$

Since the right-hand side does not depend on $r$ then letting $\Delta s \to 0$ gives (A.7.25).

$$
\frac{d^{k+1}}{ds^{k+1}} (A(s)) = \lim_{r \to 0^+} \left( \frac{1}{r^{N-2}} \int_M \left( \frac{d_+(x,y)}{r} \right)^{2(k+1)} e^{-\frac{(d_+(x,y))^2}{r^2}} d\mu^*_r(y) \right).
$$

Setting $s = 1$ shows that $t^{2(k+1)} e^{-t^2} \in F$.

**Lemma A.7.3.** The set

$$
W := \left\{ g \in C^2_c((0, \infty)) : g'(0) = 0 \right\}
$$

is included in $F$.

**Proof.** We refer the reader to the proof of Lemma 6.7 from [11] and simply note that the proof presented there only depends on functions over the real line.

**A.7.3.3 Proof of Proposition 2.7.14**

We refer the read to the proof of Proposition 8 from [11] and simply note that the proof presented there only depends on functions over the real line.

**A.8 Integral Estimate**

**Lemma A.8.1.** Let $(M, g)$ be an $N$-dimensional compact Riemannian manifold. Suppose that $h \in L^2(M)$. Then, if for each $x, y \in M K(x, \cdot), K(\cdot, y) \in L^1(M)$, we have

$$
\|K * h\|_{L^2(M)} \leq \|K\|_{L^1(M)} \left\| \frac{1}{L^2(M)} \right\| \|K\|_{L^1(M)} \left\| \frac{1}{L^2(M)} \right\| \| h\|_{L^2(M)}
$$
where \((K \star h)(x)\) is defined by

\[
(K \star h)(x) := \int_M K(x, y)h(y)\text{dvol}_g(y).
\]

**Proof.** Observe that for each \(x \in M\) we have, by Cauchy-Schwarz, that

\[
|(K \star h)(x)| \leq \int_M |K(x, y)||h(y)|\text{dvol}_g(y) = \int_M |K(x, y)|^{\frac{1}{2}} \{|K(x, y)|^{\frac{1}{2}}|h(y)|\}\text{dvol}_g(y)
\]

\[
\leq \|K(x, \cdot)\|_{L^1_y(M)}^{\frac{1}{2}} \left( \int_M |K(x, y)||h(y)|^2\text{dvol}_g(y) \right)^{\frac{1}{2}}
\]

\[
\leq \left\| \|K\|_{L^1_y(M)} \right\|_{L^\infty_x(M)}^{\frac{1}{2}} \left( \int_M |K(x, y)||h(y)|^2\text{dvol}_g(y) \right)^{\frac{1}{2}}.
\]

Hence,

\[
\int_M |(K \star h)(x)|^2\text{dvol}_g(x) \leq \left\| \|K\|_{L^1_y(M)} \right\|_{L^\infty_x(M)} \int_M \left( \int_M |K(x, y)||h(y)|^2\text{dvol}_g(y) \right)\text{dvol}_g(x)
\]

\[
= \left\| \|K\|_{L^1_y(M)} \right\|_{L^\infty_x(M)} \int_M |h(y)|^2 \left( \int_M |K(x, y)|\text{dvol}_g(x) \right)\text{dvol}_g(y)
\]

\[
= \left\| \|K\|_{L^1_y(M)} \right\|_{L^\infty_x(M)} \int_M |h(y)|^2 \|K(\cdot, y)\|_{L^1_x(M)} \text{dvol}_g(y)
\]

\[
\leq \left\| \|K\|_{L^1_y(M)} \right\|_{L^\infty_x(M)} \left\| \|K\|_{L^1_y(M)} \right\|_{L^\infty_y(M)} \|h\|^2_{L^2(M)}.
\]

Taking the square root of both sides gives the desired inequality. \(\square\)
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