Convergence Rates of Random Discrete Model Curves Approaching SLE Curves in the Scaling Limit

by

Larissa Richards

A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy
Graduate Department of Mathematics
University of Toronto

Abstract

Convergence Rates of Random Discrete Model Curves Approaching SLE Curves in the Scaling Limit

Larissa Richards

Doctor of Philosophy

Graduate Department of Mathematics

University of Toronto

2021

Recently, A. Kempannien and S. Smirnov provided a framework for showing convergence of discrete model interfaces to the corresponding SLE curves. They show that given a uniform bound on specific crossing probabilities one can deduce that the interface has subsequential scaling limits that can be described almost surely by Löwner evolutions. This leads to the natural question to investigate the rate of convergence to the corresponding SLE curves. F. Johansson Viklund has developed a framework for obtaining a power-law convergence rate to an SLE curve from a power-law convergence rate for the driving function provided some additional geometric information along with an estimate on the growth of the derivative of the SLE map. This framework is applied to the case of the loop-erased random walk. In this thesis, we show that if your interface satisfies the uniform annulus condition proposed by Kempannien and Smirnov then one can deduce the geometric information required to apply Viklund's framework. As an application, we apply the framework to the critical percolation interface. The first step in this direction for critical percolation was done by I. Binder, L. Chayes and H.K. Lei where they proved that the convergence rate of the Cardy-Smirnov observable is polynomial in the size of the lattice. It relies on a careful analysis of the boundary behaviour of conformal maps and their discrete analytic approximations as well as a Percolation construction of the *Harris systems*. Further, we exploit the toolbox developed by D. Chelkak for discrete complex analysis on isoradial graphs to show polynomial rate of convergence for the discrete martingale observables for harmonic explorer and the FK Ising model to the corresponding continuum objects. Then, we apply the framework developed above to gain a polynomial convergence rate for the corresponding curves.

Contents

For Dad and Delmie.

Acknowledgements

Many people have contributed to this thesis. I wish to thank my advisor Ilia Binder for his advice, encouragement, and support throughout these years. I have learned so much from you and thoroughly enjoyed our many discussions. Also, thank you for sending me to all the summer schools, workshops, and conferences. I have met many brilliant people thanks to you.

I am also greatly indebted to Michael Kozdron for introducing me to this subject, opening up my mind to mathematics as a career for me, and constant encouragement and support over all the years. Thank you for sharing your passion and excitement towards this subject. Thank you to Dmitry Chelkak for his discussions with us on the Ising model and Fredrick Viklund for spending time listening to me talk on this work and insightful questions.

Thanks to Jemima Merisca, Sonja Injac, Ashley Armogan, Patrina Seepersaud and the main office. Things would not have worked without you. My graduate studies at University of Toronto were partially funded by NSERC. During my time as a graduate student, I had the privilege of attending the Masters class in statistical physics at the University of Geneva which provided me not only the opportunity to learn but many friendships as well. I most gratefully acknowledge these institutions.

I would also like to give a special thanks to all my family and friends all over the world. I owe a special thank you to Grace, Rachel, Jackie, and Sydney for your continuous, invaluable, and essential support along the way. Sheldon, thank you for encouraging me towards academia and always inspiring me. Mom, Miranda, Chris and John thank you for being the most amazing family, always believing in me and sharing this experience with me from the very beginning. Elif, thank you for sharing this experience with us this past year, opening up your home, and being an inspiration as a successful academic and mother. Özgür, you always believe in me, support me, and love me. Thank you for sharing all the laughs and experiences with Delmie this past year. Delmie, thank you for writing this thesis with me, being the best possible distraction, and bringing so much joy in our life. This past year has made me appreciate more all the friendship, love and support that I have in my life. Thank you to all my family and friends for always contributing to my happiness and encouraging me in everything I attempt. I could not have done this without you.

Chapter 1

Introduction and Background

Introduced by Oded Schramm [51], SLE is a one-parameter family of conformally invariant random curves in simply connected planar domains. It is conjectured that these curves are the scaling limits of the various interfaces in critical lattice models. These two-dimensional lattice models describe a variety of physical phenomenon including percolation, the Ising model, loop-erased random walk and the Potts model. Physicists had predicted that conformal invariance would play a key role in understanding the universal behaviour of these two-dimensional systems. Universality essentially means that the global properties of the physical system do not depend on the detailed local description of the model such as the underlying lattice. Recently, there has been a number of remarkable breakthroughs in the study of these models. In fact, Fields Medals were awarded to W. Werner in 2006 and to S. Smirnov in 2010 for their contributions in the study of SLE and critical lattice models.

For several two dimensional lattice models at criticality, it has been shown that the discrete interfaces converge in the scaling limit to SLE curves [38, 40, 52, 54, 13, 56, 9]. The proofs of these all begin in the same manner, that is, by describing the scaling limit of some observable related to the interface. The limit is constructed from the interface itself through conformal invariance. Generally, the difficulty in the proof arises in how to deduce the strong convergence of interfaces from some weaker notions resulting in a need to solve some specific technical estimates. The goal of this thesis is to study the rate of the above-mentioned convergence. In particular, we obtain a power-law convergence rate to an SLE curve from a power-law convergence rate for a martingale observable under suitable conditions on the discrete curves.

The *Loewner equation* is a partial differential equation that produces a Loewner chain which is a family of conformal maps from a reference domain onto a continuously decreasing sequence of simply connected domains. A real-valued function called the *driving term* controls the Loewner evolution. If the driving term satisfies a smoothness assumption, then the Loewner equation generates a growing continuous curve. Conversely, given a suitable curve, one can define the associated conformal maps which satisfy Loewner equation and recover the driving term. Thus, there is a correspondence

$$\{\text{Loewner curves}\} \leftrightarrow \{\text{their driving terms}\}$$

Schramm-Loewner evolutions (SLE) is the one-parameter family of random fractal curves in a reference domain (either unit disk $\mathbb{D}$ or the upper half plane $\mathbb{H}$) whose Loewner evolution is driven by a scaled standard one-dimensional Brownian motion.

In the existing proofs of convergence to SLE, the following two schemes have been suggested:

- Show that the driving processes convergence and then extend this to convergence of paths or,
- Show that the probability measure on the space of discrete curves is precompact and then show that the limiting curve can be described by Loewner evolution.

Given a discrete random curve that is expected to have a scaling limit described by a variant of SLE, Kempannien and Smirnov in [28] provide the framework for both approaches building upon the earlier work of Aizenman and Burchard [1]. They show that under similar conditions to [1] one can deduce that an interface has subsequential scaling limits that can be described almost surely by Loewner evolutions. An important condition for Kempannien and Smirnov's results and our framework is what we call the *KS Condition*, a uniform bound on specific crossing probabilities. The KS Condition (or one of the equivalent conditions) has been shown to be satisfied for the following models: FK-Ising model, random cluster representation of q-Potts model for $1 \leq q \leq 4$, spin Ising model, percolation, harmonic explorer and chordal loop-erased random walk (as well as radial loop-erased random walk). The KS Condition fails for the uniform spanning tree, see [28].

In [27], Viklund examines the first approach to convergence in order to develop a framework for obtaining a power-law convergence rate to an SLE curve from a power-law convergence rate for the driving function provided some additional geometric information, related to crossing events, along with an estimate on the growth of the derivative of the SLE map. For the additional geometric information, Viklund introduces a geometric gauge of the regularity of a Loewner curve in the capacity parameterization called the *tip structure modulus*. In some sense, the tip structure modulus is the maximal distance the curve travels into a fjord with opening smaller than ϵ when viewed from the point toward which the curve is growing. The framework developed is quite general and can be applied to several models. In

[27], it is shown in the case of loop-erased random walk. However, it can be difficult to show the needed technical estimate on the tip structure modulus. In this thesis, the framework is applied to other models. We build upon these earlier works to show that if the condition required for Kempannien and Smirnov's framework [28] is satisfied then one is able to obtain the needed additional geometric information for Viklund's framework. The end result is to obtain a power-law convergence rate to an SLE curve from a power-law convergence rate for the martingale observable provided the discrete curves satisfy the KS Condition, a bound on annuli crossing events.

Theorem 1.0.1 (Binder-Richards)**.** *Given a discrete Loewner curve that satisfies the KS Condition and a suitable martingale observable satisfying a power-law convergence rate, one can obtain a power-law convergence rate to an SLE_κ curve for $\kappa \in (0, 8)$.*

As an application, we apply the above framework for rate of convergence to SLE for various statistical physics models: Percolation, Harmonic Explorer and Ising model.

1.1. The Space of Curves

We will define curves in the same way as in [1] and [28]: *planar curves* are equivalence classes of continuous mappings from $[0, 1]$ to $\mathbb{C}$ modulo reparameterizations. While it is possible to work with the entire space $C([0, 1], \mathbb{C})$, it is nicer if we work with

$$C' = \left\{ f \in C([0,1], \mathbb{C}) : \begin{array}{l} f \text{ is identically a constant or} \\ f \text{ is not constant on any sub-intervals} \end{array} \right\}$$

instead. On C' we define an equivalence relation $\sim$ as follows: two functions f_1 and f_2 in C' are equivalent if there exists an increasing homeomorphism $\psi : [0, 1] \to [0, 1]$ such that $f_2 = f_1 \circ \psi$ in which case we say f_2 is a reparameterization of f_1.

Thus, $\mathcal{S} := \{[f] \ : \ f \in C'\}$ is the space of curves where $[f]$ is the equivalence class of the function f. The metric $d_{\mathcal{S}}([f], [g]) = \inf \{||f_0 - g_0||_\infty \ : \ f_0 \in [f],\ g_0 \in [g]\}$ gives $\mathcal{S}$ the structure of a metric space. For a proof that $(\mathcal{S}, d_{\mathcal{S}})$ forms a metric space see Lemma 2.1 in [1]. For any domain $D \subset \mathbb{C}$, let

$$\begin{aligned} \mathcal{S}_{\text{simple}}(D) &= \{[f] \ : \ f \in C',\ f((0, 1)) \subset D,\ f \text{ injective}\}, \\ \mathcal{S}_0(D) &= \overline{\mathcal{S}_{\text{simple}}(D)}. \end{aligned}$$

Let $\text{Prob}(\mathcal{S})$ be the space of probability measures on $\mathcal{S}$ equipped with the Borel $\sigma-$algebra $\mathcal{B}_{\mathcal{S}}$ and the weak-* topology induced by continuous functions. Suppose $(\mathbb{P}_n)$ is a sequence in $\text{Prob}(\mathcal{S})$ and for each n, $\mathbb{P}_n$ is supported on a closed subset of $\mathcal{S}_{\text{simple}}$. This can be assumed without loss of generality for discrete curves. If $\mathbb{P}_n \to \mathbb{P}$ weakly then $1 = \limsup \mathbb{P}_n(\mathcal{S}_0) \leq \limsup \mathbb{P}(\mathcal{S}_0)$ by properties of weak convergence. So, $\mathbb{P}$ is supported on S_0. See [28] for more information and comments on this probability structure.

Typically, the random curves that we are interested in connect two boundary points $a, b \in \partial D$ in a simply connected domain D. We will denote $\mathcal{S}_{\text{simple}}(D, a, b)$ for curves in $\mathcal{S}_{\text{simple}}(D)$ whose endpoints are a and b. The curves we are considering in this paper satisfy the Loewner equation and so must either be simple or non self-traversing, i.e., curves that are limits of sequences of simple curves. The curves are usually defined on a lattice L with small but finite lattice mesh. So, we can safely assume the random curve is (almost surely) simple as we could perturb the lattice and curve to remove any self-intersections, if necessary.

Although we work with arbitrary simply connected domains D, it will be convenient to work with reference domains $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$ and $\mathbb{H} = \{z \in \mathbb{C} \ : \ \text{Im} z > 0\}$. We uniformize by the disk $\mathbb{D}$ in order to work with a bounded domain. The conditions are conformally invariant so the choice of a particular uniformization domain is not important. To do this, we encode a simply connected domain D other than $\mathbb{C}$ and curve end points $a, b, \in \partial D$, if necessary accessible prime ends, by a conformal map $\phi : D \to \mathbb{D}$ with $\phi(a) = -1, \ \phi(b) = 1$.

Following the generality outlined in [28], our main object of study is $(\mathbb{P}_{(D;a,b)})$ where

- $(D; a, b)$ is a simply connected domain with two distinct accessible prime ends $a, b \in \partial\mathbb{D}$ and we can define a conformal map $\phi : (D; a, b) \to (\mathbb{D}; -1, 1)$ as above.

- and $\mathbb{P}$ is a probability measure supported on a closed subset of

$$\{\gamma \in \mathcal{S}_{\text{simple}}(D; a, b) \ : \ \text{ beginning and end points of } \phi(\gamma) \text{ are } -1 \text{ and } 1, \text{ respectively}\}. \quad (1.1.1)$$

We assume that 1.1.1 is nonempty. In which case, there are plenty such curves, see Corollary 2.17 in [48]. Since $(\mathcal{S}, d_{\mathcal{S}})$ forms a metric space, we can think about our family of probability measures as a sequence $((\mathbb{P}^n_{(D^{\delta_n}; a^{\delta_n}, b^{\delta_n})}))_{n \in \mathbb{N}}$ when discussing convergence. Since our goal is to study convergence rates of interfaces from statistical physic models to SLE, in general, we are considering a sequence of interfaces for the same lattice model with shrinking lattice mesh $\delta_n \to 0$. So, $\mathbb{P}_n$ is supported on curves defined on the δ_n-mesh lattice.

THEORY OF PRIME ENDS. Here we will recall the basic definitions in the theory of prime ends. More information and proofs can be found in [48, §2.4 and 2.5]. A *crosscut* of a bounded domain D is an open Jordan arc $\mathcal{C}$ such that $\mathcal{C} \subset D$ and $\overline{\mathcal{C}} = \mathcal{C} \cup \{p_1, p_2\}$ where $p_j \in \partial D,\ j = 1, 2$. A sequence of crosscuts $\{\mathcal{C}_n\}$ is called a *null-chain* if

1. $\overline{\mathcal{C}}_n \cap \overline{\mathcal{C}}_{n+1} = \emptyset$ for any n
2. $\mathcal{C}_n$ separates $\mathcal{C}_{n+1}$ from $\mathcal{C}_1$ for any n
3. $\operatorname{diam} \mathcal{C}_n \to 0$ as $n \to \infty$.

We say that two null chains $(\mathcal{C}_n)$ and $(\mathcal{C}'_n)$ are equivalent if for any m there exists n so that $\mathcal{C}'_m$ separates $\mathcal{C}_n$ from $\mathcal{C}_1$ and $\mathcal{C}_m$ separates $\mathcal{C}'_n$ from $\mathcal{C}'_1$. This equivalence relation forms equivalence classes called *prime ends* of D.

Notice that for a Jordan domain such as $\mathbb{D}$ each boundary point corresponds one-to-one to a prime end. So, we can almost forget about the theory of prime ends when we use the definition 1.1.1 for what is meant by a set of simple curves that connect two fixed boundary points. The prime end theorem states that there is a one-to-one correspondence between the prime ends of a simply connected domain and the prime ends of $\mathbb{D}$: Given a mapping Ψ from D onto $\mathbb{D}$, there is a bijection f from $\partial\mathbb{D}$ to the prime ends of D so that for any $\xi \in \partial\mathbb{D}$, any null-chain of $f(\xi)$ in D is mapped by Ψ to a null-chain of ξ in $\mathbb{D}$.

Example 1.1.1. Consider the slit domain $\mathbb{H}\backslash[0, i]$. Using the concept of prime end, we can distinguish between the right-hand side and the left-hand side of a boundary point $iy,\ 0 < y < 1$.

We say that a prime end a of D is *accessible* if there is a Jordan arc $P : [0, 1] \to \mathbb{C}$ such that $P(0, 1] \subset U, P(0) \in \partial U$ and P intersect all but finitely many crosscuts of a null-chain of a. This boundary point $P(0)$ is called the *principal point* of a, denoted by $\Pi(a)$, and for an accessible prime end it is unique. Let $\Psi : D \to \mathbb{D}$ be a conformal map and $\xi \in \partial\mathbb{D}$ be the boundary point corresponding to a. Then a is accessible if and only if the radial limit $\lim_{t\to 1} \Psi^{-1}(t\xi)$ exists. If the limit exists, then $\lim_{t\to 1} \Psi^{-1}(t\xi) = \Pi(a)$, see [48, Corollary 2.17].

LATTICE APPROXIMATION. A *lattice approximation* of $(D; a, b)$ is constructed where a and b are accessible prime ends of D, being careful when working in neighbourhoods of a and b. Let L be a lattice, i.e. an infinite graph consisting of periodically repeating parts such as $L = \mathbb{Z}^2$. Let δL be L scaled by the constant $\delta > 0$.

If $\Psi : D \to \mathbb{D}$ is a conformal map and $\zeta \in \partial\mathbb{D}$ is the boundary point corresponding to a, then a is *accessible* if and only if the radial limit $\lim\limits_{t\to 1} \Psi^{-1}(t\zeta)$ exists. If the limit exists, denote it by $\Pi(a)$. Similarly, we can do the same for b and denote the limit as $\Pi(b)$, see [49] for the theory on prime ends.

Let $w_0 \in D$. For small δ, there are vertices of δL in a neighbourhood of w_0. Let D^δ be the maximal connected sub-graph of δL containing these vertices so that $V(D^\delta) \cup \{\Pi(a), \Pi(b)\}$ and the edges lie inside D except for the edges that end at a or b in the following sense. Let P be a curve from a to b. Let t_1 be the smallest t such that $P(t)$ intersects an edge or vertex of D^δ. Else remove the edge that $P(t_1)$ is lying on and choose one of the endpoints to be a^δ. If removing the edge cuts the graph into two disconnected components then discard the one that is not connected to vertices near w_0. The same thing can be done for the largest $t = t_2$ so that $P(t)$ intersects an edge or a vertex of D^δ to obtain b^δ. Let P_1 be the curve $P(t),\ t \in [0, t_1]$ with the piece of removed edge. That is, a simple curve connecting a to a^δ. Similarly, let P_2 be the curve $P(t),\ t \in [t_2, 1]$ with the piece of removed edge. That is, a simple curve connecting b^δ to b. The random curve γ^δ is the random path on D^δ joined with P_1 and P_2 connecting a and b to D^δ.

1.2. Introduction to the SLE-Quest via Loop-Erased Random Walk

To gain some intuition about the family of random curves that we are studying, it is helpful to have a discrete model in mind. The goal of this section is to develop insight and intuition as to how SLE arises as a scaling limit of discrete curves through the loop-erased random walk, which is a probabilistic model process known to converge to an SLE as well as the rate of this convergence in the capacity parameterization. We will show that this model satisfies a domain Markov property which is essential to characterizing SLE.

For any $\mathbf{x} = (x_0, \cdots, x_m)$, define the *loop-erasure* $L(\mathbf{x})$ inductively as follows:

- $L_0 = x_0$ for all $j \geq 0$
- Erase all loops of $\mathbf{x}$ in chronological order.

  Inductively, $n_j = \max\{n \leq m : x_n = L_j\}$ and $L_j + 1 = X_{1+n_j}$ until $j = \sigma$ where $L_\sigma := x_m$.

Suppose that $(X_n, n \geq 0)$ is a recurrent Markov chain on a discrete state space $\mathcal{S}$ started from $X_0 = \mathbf{x}$. Suppose that $A \subset \mathcal{S}$ is non-empty, and let τ_A denote the hitting time of A by X. Set $X[0, \tau_A] = (X_0, \cdots, X_{\tau_A})$ and define the loop-erasure $L = L(X[0, \tau_A]) = L^A$ up to the hitting time τ_A. This can be defined exactly as above. Let σ be the number of steps of L^A and for $x, y \in \mathcal{S}$, let $p(x, y)$ be the transition probabilities for the Markov chain $(X_n)_{n \geq 0}$. For $y \in A$ with positive probability $L^A(\sigma) = X(\tau_A) = y$, let $\mathcal{L}(x, y; A)$ denote the law of L^A conditioned on the event $\{L^A(\sigma) = y\}$. That is, the law of the loop-erasure of the Markov chain X conditioned to hit A at y.

Lemma 1.2.1 (Markov property of LERW)**.** *Consider* $y_0, \cdots, y_j \in \mathcal{S}$ *so that with positive probability* *for* $\mathcal{L}(x, y_0; A)$,

$$\{L_\sigma = y_0, L_{\sigma-1} = y_1, \cdots, L_{\sigma-j} = y_j\}.$$

The condition law of $L[0, \sigma - j]$ *given this event is* $\mathcal{L}(x, y_j; A \cup \{y_1, \cdots, y_j\})$.

Proof. For each A and $x \in A$, let $G(x, A)$ be the expected number of visits by the Markov chain X before τ_A with $X_0 = x$. Then, it is easy to see that for all $n \geq 1, w = (w_0, \cdots, w_n)$ with $w_0 = x,\ w_n \in A$ and $w_1, \cdots, w_{n-1} \in S \backslash A$,

$$\mathbb{P}[L^A = w] = \sum_{x : L(x) = w} \mathbb{P}[X[0, \tau_A] = x]$$
$$= G(w_0, A)p(w_0, w_1)G(w_1, A \cup \{w_0\})p(w_1, w_2) \cdots G(w_{n-1}, A \cup \{w_0, w_1, \cdots, w_{n-2}\})p(w_{n-1}, w_n).$$

This shows that the probability that the loop-erasure of the Markov chain $(X_n)_{n\geq 0}$ equals the path w equals the product of the expected number of returns of $(X_n)_{n\geq 0}$ to each step in w times the transition probabilities from one step to the next. Thus, it is natural to define the function

$$F(w_0, \cdots, w_{n-1}; A) = \prod_{j=0}^{n-1} G(w_j, A \cup \{w_0, \cdots, w_{j-1}\}).$$

It is easy to check that for all A', y and y',

$$G(y, A')G(y', A' \cup \{y\}) = G(y', A')G(y, A' \cup \{y'\}).$$

Thus, F is a symmetric function of its arguments. So,

$$\mathbb{P}[L_0^A = w_0, \cdots, L_\sigma^A = w_n \mid L_\sigma = w_n, L_{\sigma-1} = w_{n-1}]$$
$$= \frac{p(w_{n-1}, w_n)G(w_{n-1}, A)}{\mathbb{P}[L_\sigma = w_n, L_{\sigma-1} = w_{n-1}]}$$
$$\times \prod_{j=0}^{n-2} p(w_j, w_{j+1})G(w_j, (A \cup \{w_{n-1}\}) \cup \{w_0, \cdots, w_{j-1}\}).$$

Thus, we have the Lemma when $j = 1$. Iterating this j times gives the result. □

We can see from this Lemma that it is actual natural to index the loop-erased random path backwards: defined $\gamma_j = L^A_{\sigma-j}$ so that γ starts on A and goes back to $\gamma_0 = x$. Thus, the time-reversal of loop-erased Markov chains satisfy a Markovian-type property.

It has been shown that for a bounded domain $D \subset \mathbb{C}$ as $\delta \to 0$ the LERW on $\delta\mathbb{Z}^2 \cap D$ converges (polynomially fast) to SLE. Heuristically: Suppose that X is a simple random walk on $\delta\mathbb{Z}^2 \cap D$ where D is a simply connected domain with $0 \in D$ and $D \neq \mathbb{C}$. Let $A = \delta\mathbb{Z}^2 \backslash D$. We are interested in the time-reversed loop-erasure of $X[0, \tau_A]$. Notice that the law of X_{τ_A} converges to harmonic measure on ∂D from 0. So, one can study the behaviour of γ^δ conditioned on $\{\gamma^\delta = y_0^\delta\}$ where $y_0^\delta \to y \in \partial D$ as $\delta \to 0$. Secondly, we can try to argue that since simple random walk converges to planar Brownian motion which is conformally invariant and the loop-erasing procedure is purely topological, that the law of γ^δ when $\delta \to 0$ should convergence to a conformal invariant curve that should be a loop-erasure of planar Brownian motion. However, the geometry of planar Brownian motion is very complicated and there is no algorithm to loop erase a Brownian motion path in chronological order. However, this heuristics strongly suggest that the scaling limit of LERW should be invariant under conformal transformations and the domain Markov property should still be valid. It turns out that these two properties characterize a family of continuous processes known as SLE.

Let us explain heuristically how these properties characterize the random family of curves SLE. So far in our story, we have seen that we are looking for a random continuous curve $(\gamma_t, t \geq 0)$ with no self crossings in the unit disk $\mathbb{D}$ with $\gamma_0 = 1$, $\lim_{t\to\infty} \gamma_t = 0$ which could be the scaling limit for the LERW on a grid approximation of $\mathbb{D}$ (conditioned to exit $\mathbb{D}$ near 1). For $t \geq 0$, let $f_t : \mathbb{D}\backslash\gamma[0,t] \to \mathbb{D}$ be the conformal map normalized by $f_t(0) = 0$ and $f_t(\gamma_t) = 1$ (assume for convenience that γ is simple). Then $t \mapsto |f_t'(0)|$ is an increasing continuous function which goes to ∞ as $t \to \infty$. Thus, one can reparameterize γ such that $|f_t^{'}(0)| = e^t$. By the domain Markov property, the condition law of $\gamma[t, \infty)$ given $\gamma[0,t]$ is the law of the scaling limit in the slit domain $\mathbb{D}\backslash\gamma[0,t]$ conditioned to exit at γ_t. Then, conformal invariance says that (modulo time-reparameterization) this is the same as the image under $z \mapsto f_t^{-1}(z)$ of an independent copy $\tilde{\gamma}$ of γ. Thus, for all fixed $t \geq 0$

$$(f_{t+s}, s \geq 0) = (\tilde{f}_s \circ f_t, s \geq 0) \text{ in law.}$$

By iterating this process, we see that f_t is obtained by iterating infinitely many independent conformal maps that are infinitesimally close to the identity.

In the 1920s, Loewner observed that if $\gamma[0, \infty)$ is a simple continuous curve starting at 1 in $\mathbb{D}$, then there is a continuous function ζ_t on the unit circle which naturally encodes γ. Define $\zeta_t = \left(f_t^{'}(0)/|f_t^{'}(0)|\right)^{-1}$. Then if $g_t : D_t := \mathbb{D}\backslash\gamma[0,t] \to \mathbb{D}$ is a conformal map with $g_t(0) = 0$ and $g_t^{'}(0) = e^t \in (0, \infty)$ then $\zeta_t = g_t(\gamma_t)$ and $g_t(z) = \zeta_t f_t(z)$. Also, for all $z \notin \gamma[0,t]$, we have

$$\partial_t g_t(z) = -g_t(z)\frac{g_t(z) + \zeta_t}{g_t(z) - \zeta_t}.$$

This process also reverses: given ζ_t we can reconstruct γ. For all $z \in \mathbb{D}$, define $g_t(z)$ as the solution to the PDE above with initial condition $g_0(z) = z$. If $g_t(z) = \zeta_t$ for some t, then define $\gamma_t = g_t^{-1}(\zeta_t)$ (we can do this because we know γ is a simple curve and g_t^{-1} extends continuously to the boundary). Notice that, in this case, $g_s(z)$ is not well-defined for $s \geq t$. Thus, in order to define the random curve γ, it is enough to define the random function $\zeta_t = \exp(iW_t)$ where $(W_t, t \geq 0)$ is real-valued.

By our argument above, W_t should:

- be almost surely continuous.
- have stationary increments (since g_t is iterations of identically distributed conformal maps).
- and the laws of W and $-W$ should be identical (since the laws of L and the laws of the complex conjugate of L are identical).

The theory of Markov processes tells us that the only possible candidate is $W_t = B_{\kappa t}$ where B is a standard Brownian motion and $\kappa \geq 0$ is a fixed constant. Thus, if the scaling limit of LERW exists and is conformally invariant then for some fixed constant κ which has been proven to be 2, we can define $\zeta_t = \exp(i\sqrt{2}B_t)$ for $t \geq 0$. Then solve the Loewner equation with initial condition $g_0(z) = z$ for each $z \in \mathbb{D}$ and construct γ by $\gamma_t = g_t^{-1}(\zeta_t)$. In the next section, we will make these arguments rigorous and which will allow us to define SLE.

1.2.1. Uniform Spanning Tree

Another famous model, the uniform spanning tree was studied in the same paper by Oded Schramm [51] which led to the introduction of SLE. The uniform spanning tree is intrinsically related to the loop-erased random walk and the connections between these two processes allow each to be used as an aid to study the other. A *spanning tree* T of a connected graph G is a subgraph G such that for every pair of vertices v, u in G there is a unique simple path (i.e. self-avoiding) in T with these vertices as endpoints. A *uniform spanning tree* (UST) in a finite, connected graph G is a sample from the uniform probability measure on spanning trees of G. It has been shown that the law of the self-avoiding path with end points a and b in UST is the same as the law of the LERW from a to $\{b\}$. There is an even stronger connection between these two processes via Wilson's algorithm which gives an algorithm to generate UST's using LERW, see [47].

Wilson's algorithm runs as follows. Pick an arbitrary ordering $v_0, v_1, \cdots v_m$ for vertices in G. Let $T_0 = \{v_0\}$. Inductively, for $n = 1, 2, \cdots m$, define T_n to be the union of T_{n-1} and a (conditionally independent) LERW path from v_n to T_{n-1} (If $v_n \in T_{n-1}$, then $T_n = T_{n-1}$). Regardless of the chosen order of the vertices, T_m is a UST on G, see [59] or [32] for an alternative proof.

Since UST can be built from the LERW via Wilson's algorithm, the conformal invariance of the UST scaling limit follows from that of the LERW scaling limit [38]. Then it can be shown to converge to SLE_8.

Theorem 1.2.2. *[38, Theorem 1.3] The UST Peano curve scaling limit in a simply connected domain D with Dobrushin boundary conditions is equal to the image of the chordal SLE_8 path.*

Wilson's algorithm gives us a algorithm for generating a spanning tree uniformly at random (without knowing the number of spanning trees). Given a finite connected graph, there are a lot of spanning trees! In general, it is not easy to calculate the number of spanning trees. In 1847, Gustav Kirchhoff gave a formula for the number of spanning trees in terms of the graph Laplacian matrix $\mathcal{L} = \mathcal{D} - \mathcal{A}$ where $\mathcal{D}$ is the diagonal matrix whose ith entry is the degree of vertex i and $\mathcal{A}$ is the adjacency matrix that is the (i, j) entry is 1 if there is an edge between i and j and is 0 else, see [30]. As a nice example of the usefulness of Wilson's algorithm, let us explore how one can use Wilson's algorithm to prove Kirchhoff's Matrix Tree Theorem. We will follow the strategy developed by Greg Lawler: Wilson's algorithm uses the Markov chain determined by the transition matrix $\mathcal{P} = \mathcal{D}^{-1}\mathcal{A}$ to explore the graph. Indeed, with probability 1, Wilson's algorithm produces a spanning tree uniformly at random. Thus, the probability of producing any particular spanning tree is the same for all spanning trees, the number of spanning trees must be equal to the reciprocal of the probability that Wilson's algorithm produces a particular one. Thus, if one can show that the probability that Wilson's algorithm produces a particular spanning tree is the reciprocal of Kirchhoff's expression in terms of $\mathcal{L}$, then one can recover Kirchhoff's result.

Theorem 1.2.3 (Kirchhoff's Matrix Tree Theorem)**.** *The number of spanning trees in a graph G, $\mathcal{T}$, is given by $\det(\mathcal{L}_G[i])$ for any i where $\mathcal{L}_G[i]$ is the Laplacian matrix with ith row and column removed.*

Here, we will give a sketch of the proof via Wilson's algorithm for full details see [31].

Sketch of Proof. Suppose that $T \in \mathcal{T}$ was produced by Wilson's algorithm with branches $\Delta_0 = \{v_0\}$, $\Delta_1 = [x_{1,1}, \cdots, x_{1,k_1}], \cdots, \Delta_L = [x_{L,1}, \cdots, x_{L,k_L}]$.

Each branch in Wilson's algorithm is generated by a loop-erased random walk. Thus,

$$\mathbb{P}\left(T \text{ is generated by Wilson's algorithm}\right) = \prod_{l=1}^{L} P^{\Delta^l}(x_{l,1}, \cdots, x_{l,k_l})$$

where $\Delta^l = \Delta_0 \cup \cdots \cup \Delta_{l-1}$ for $l = 1, \cdots, L$ and $P^{\Delta}(x_1, \cdots, x_{K+1})$ is the probability the loop erasure on Δ is exactly $[x_1, \cdots, x_{K+1}]$.

Notice that for a loop erasure to be exactly $[x_1, \cdots, x_{K+1}]$ we must have that: The simple random walk started at x_1, then made a number of loops back to x_1 without entering Δ, then took a step from

x_1 to x_2 and made a number of loops at x_2 without entering $\Delta \cup \{x_1\}$, $\cdots$, made a number of loops back to x_K without entering $\Delta \cup \{x_1, x_2, \cdots, x_{K-1}\}$ then took a step from x_K to $x_{K+1} \in \Delta$. So,

$$\begin{aligned} P^{\Delta}(x_1, \cdots, x_{K+1}) &= \sum_{m_1, \cdots, m_K = 0}^{\infty} r_{\Delta}(x_1)^{m_1} p(x_1, x_2) r_{\Delta \cup \{x_1\}}(x)^{m_2} p(x_2, x_3) \cdots r_{\Delta \cup \{x_1, \cdots, x_{K-1}\}}(x)^{m_K} p(x_K, x_{K+1}) \\ &= \prod_{j=1}^{K} \frac{1}{\deg(x_j)} \frac{1}{1 - r_{\Delta_j}(x_j)} \\ &= \prod_{j=1}^{K} \frac{1}{\deg(x_j)} G_{\Delta_j}(x_j, x_j) \end{aligned}$$

where $p(i,j)$ is the transition probability of a simple random walk, $\Delta_j = \Delta \cup \{x_1, \cdots, x_{j-1}\}$ for $j = 2, \cdots, K$, and $G_{\Delta}(x,y)$ is the discrete Green's function. Thus,

$$\mathbb{P}\left(T \text{ is generated by Wilson's algorithm}\right) = \prod_{l=1}^{L} \prod_{j=1}^{K_l - 1} \frac{1}{\deg(x_{l,j})} G_{\Delta_j^l}(x_{l,j}, x_{l,j})$$

where $\Delta_j^l = \Delta^l \cup \{x_{l,1}, \cdots, x_{l,j-1}\}$ for $j = 2, \cdots, k_l - 1$.

By Cramer's rule we get that $\mathbb{P}\left(T \text{ is generated by Wilson's algorithm}\right) = \det[\mathbb{G}^{\{v_0\}}]$ where $\mathbb{G}^{\Delta} = [G_{\Delta}(x,y)]_{x,y \in V \setminus \Delta}$. Thus,

$$\begin{aligned} \mathbb{P}\left(T \text{ is generated by Wilson's algorithm}\right) &= \frac{\det(\mathbb{G}^{\{v\}})}{\det(\mathcal{D}^{\{v\}})} \\ &= \frac{1}{\det(\mathcal{D}^{\{v\}}) \det(\mathbb{I}^{\{v\}} - \mathbb{P}^{\{v\}})} \\ &= \det[\mathcal{L}^{\{v\}}]^{-1} \end{aligned}$$

where $\mathcal{L}^{\{v\}}$ is the submatrix of $\mathcal{L}$ obtained by deleting the row and column corresponding to the vertex v. We can see that the righthand side is independent of reordering the remaining n vertices. Hence, we have that $|\mathcal{T}| = \det|\mathcal{L}^{\{v_0\}}|$ and the choice of v_0 was arbitrary. □

1.3. LOEWNER EVOLUTION AND SLE

Schramm's insight was that, under mild assumptions in addition to conformal invariance, the only possible scaling limits is a one of a one-parameter family of measures on curves, now known as *Schramm Loewner Evolution*. The aim of this section is to define these random curves and state a few fundamental properties.

1.3.1. Loewner Evolution

To begin, we sketch a derivation of the half-plane version of the Loewner equation, called the *chordal* Loewner equation, more details can be found in [4] and [35].

Let $\gamma_{\mathbb{D}}$ be some continuous nonself-crossing (possibly self touching) curve in $\overline{\mathbb{D}}$ parameterized by $s \in [0,1]$ such that $\gamma_{\mathbb{D}}(0) = -1$ and $\gamma_{\mathbb{D}}(1) = 1$. Fix conformal transformation $\Phi : \mathbb{D} \to \mathbb{H}$ such that $z \mapsto i\frac{z+1}{1-z}$. Then $\gamma_{\mathbb{H}} = \Phi(\gamma_{\mathbb{D}})$ is a simple curve in $\mathbb{H}$ with $\gamma_{\mathbb{H}}(0) = 0 \in \mathbb{R}$, $\gamma_{\mathbb{H}}((0,1)) \subset \mathbb{H}$, and $|\gamma_{\mathbb{H}}(t)| \to \infty$ as $t \to 1$. One can encode continuous simple curves $\gamma_{\mathbb{H}}$ from 0 to ∞ in the closed upper half plane $\overline{\mathbb{H}}$ via Loewner's evolution. As a convention, the driving term of a random curve in $(\mathbb{D}, -1, 1)$ means the driving term in $\mathbb{H}$ after the transformation Φ using the half-plane capacity parameterization.

To this end, we will start by defining the compact hulls of $\mathbb{H}$. Let K_s denote the *hull* of $\gamma_{\mathbb{H}}[0,s]$. That is, K_s is the complement of the connected component of $\overline{\mathbb{H}} \backslash \gamma_{\mathbb{H}}[0,s]$ containing ∞ and let $t(s) = \text{hcap}(K_s)$ be the *half plane capacity* of K_s. Observe that $t(s)$ is non-decreasing but it could remain constant for some nonzero time and the hulls K_s could remain the same even if $\gamma_{\mathbb{D}}$ is the limit of simple curves. What could happen is that: (a) for some $s \in (0,1)$, the tip of $\gamma_{\mathbb{H}}(s)$ is not visible from ∞ or (b) $\gamma_{\mathbb{H}}(s + s_0)$ travels along the boundary of K_s for a time $s_0 > 0$ which would not change the hull or (c) $\gamma_{\mathbb{D}}$ reaches $+1$ for the first time before $s = 1$ or (d) $t(s)$ remains bounded as $s \to 1$ which can happen if $\gamma_{\mathbb{H}}$ goes to ∞ very close to $\mathbb{R}$. If none of (a)-(d) happen and $t(s)$ is strictly increasing then, $\gamma_{\mathbb{D}}$ can be described by Loewner evolution.

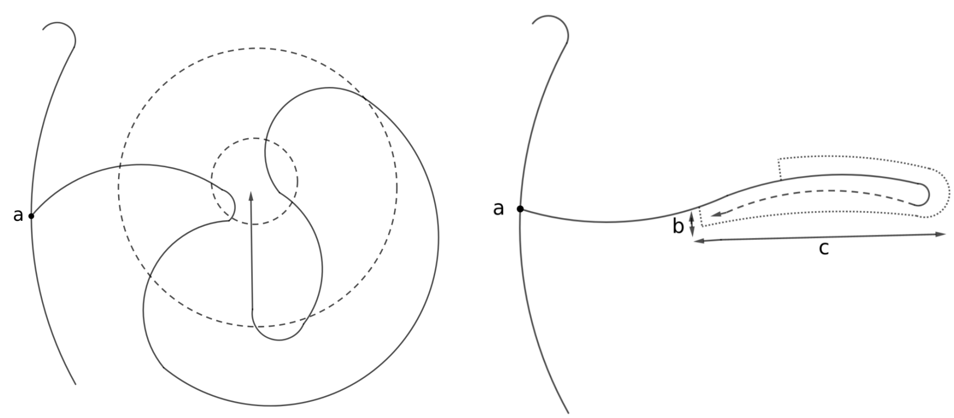


Figure 1.1: In order for a curve to be described by Loewner equation, the tip of the curve must remain visible at all times which excludes a certain 6-arm event: When the radius of the inner circle goes to zero, the portion of the curve that has gone inside the fjord is no longer visible from a distant reference point and so Loewner equation will not describe this portion of the curve. It also excludes long runs along the boundary of the domain or along the earlier part of the curve. If this is allowed to happen, then the driving process is discontinuous. So, it fails to be described by Loewner equation.

In this case, for each time $t \geq 0$, there is a unique conformal map g_t from $H_t := \mathbb{H} \backslash \gamma_{\mathbb{H}}[0,t]$ to $\mathbb{H}$

satisfying the hydrodynamic normalization: $g_t(\infty) = \infty$ and $\lim_{z\to\infty}[g_t(z) - z] = 0$. Then around infinity we have:

$$g_t(z) = z + \frac{a_1(t)}{z} + \frac{a_2(t)}{z^2} + \cdots$$

where $a_1(t) = \text{hcap}(K_t)$. Thus, $a_1(t)$ is monotone increasing and continuous. We can reparameterize $\gamma_{\mathbb{H}}$ in such a way that $a_1(t) = 2t$. This is called *parameterization by capacity.* Assuming the above normalization and parameterization, the family of mappings $(g_t)_{t\in[0,T]}$ satisfies the upper half plane version of Loewner differential equation:

$$\frac{\partial g_t}{\partial t}(z) = \frac{2}{g_t(z) - W_t} \quad t \in [0, T] \tag{1.3.1}$$

where $t \mapsto W_t$ is continuous and real-valued. W_t is called the *driving function.* It can be shown that g_t extends continuously to $\gamma_{\mathbb{H}}(t)$ and $W_t = g_t(\gamma_{\mathbb{H}}(t))$. We say that $\gamma_{\mathbb{H}}$ is determined by W.

On the other hand, we can associate a continuous function to any suitable increasing family of hulls via the Loewner equation. Suppose that W is a real valued continuous function. For $z \in \overline{\mathbb{H}}$, solve the Loewner differential equation 1.3.1 with $g_0(z) = z$ up to $\tau(z) = \inf\{t > 0 : g_t(z) - W(z) = 0\}$. Let $K_t := \{z \in \overline{\mathbb{H}} \ : \ \tau(z) \leq t\}$. Then $g_t : \mathbb{H}\backslash K_t \to \mathbb{H}$ is a conformal map and $g_0(z) = z$. The necessary and sufficient condition for a family of hulls K_t to be described by Loewner equation with a continuous driving function is given in the following proposition.

Proposition 1.3.1. *Let $T > 0$ and $(K_t)_{t\in[0,T]}$ be a family of hulls such that $K_s \subset K_t$ for any $s < t$ and let $H_t = \mathbb{H}\backslash K_t$.*

- *If $(K_t\backslash K_S) \cap \mathbb{H} \neq \emptyset$ for all $s < t$ then $t \mapsto hcap(K_t)$ is strictly increasing.*
- *If $t \mapsto H_t$ is continuous in Caratheodory kernel convergence, then $t \mapsto hcap(K_t)$ is continuous.*
- *Assume that $hcap(K_t) = 2t$ (this time reparameterization is possible given the first two assumptions). Then there is a continuous driving function W_t such that g_t satisfies Loewner equation 1.3.1 if and only if for each $\delta > 0$ there exists $\epsilon > 0$ so that for any $0 \leq s < t \leq T$, $|t - s| < \delta$, a connected set $C \subset H_s$ can be chosen such that $diam(C) < \epsilon$ and C separates $K_t\backslash K_s$ from infinity.*

The first two facts on capacity are straightforward to deduce, see [35]. A proof of the third claim can be found in [37]. We call the curves generated by such hulls $\mathbb{H}-$*Loewner curves.* Similarly, we can define $\mathbb{D}-$*Loewner curves* with $\mathbb{H}$ replaced by $\mathbb{D}$, $\gamma(0)$ is on $\partial\mathbb{D}$ and $\lim_{t\to\infty}\gamma(t) = 0$. These are exactly the curves which can be described using radial Loewner equation driven by a continuous driving function, see Theorem 1 of [49].

At some point we will need the following observation which is Lemma 2.1 of [38].

Lemma 1.3.2 (Diameter bounds on K_t)**.** *There is a constant $C > 0$ such that the following always holds. Let $W : [0,\infty) \to \mathbb{R}$ be continuous and let $(K_t,\ t \geq 0)$ be the corresponding hull for Loewner's chordal equation with driving function W. Set $k(t) := \sqrt{t} + \max\{|W(s) - W(0)\ :\ s \in [0,t]\}$, then*

$$\forall t \geq 0\ \ C^{-1}k(t) \leq \operatorname{diam}(K_t) \leq Ck(t).$$

Similarly, when $K_t \subset \overline{\mathbb{D}}$ is the radial hull for a continuous driving function $W : [0,\infty) \to \partial\mathbb{D}$, then

$$\forall t \geq 0\ \ C^{-1}\min\{k(t), 1\} \leq \operatorname{diam}(K_t) \leq Ck(t).$$

The inverse mapping $f_t := g_t^{-1}$ where g_t is the Loewner flow given above satisfies the reverse Loewner partial differential equation

$$\partial_t f_t = -f_t' \frac{2}{z - W_t} \quad f_0(z) = z. \tag{1.3.2}$$

A *Loewner pair* (f, W) consists of a function $f(t,z)$ and a continuous function $W(t),\ t \geq 0$ where f is a solution to the reverse Loewner equation 1.3.2 with W as the driving term. A sufficient condition for (f, W) to be generated by a curve γ is that the limit

$$\gamma(t) = \lim_{d \to 0^+} f(t, W(t) + id)$$

exists for all $t \geq 0$ and that $t \mapsto \gamma(t)$ is continuous. In the radial case, $\gamma(t) = \lim_{d\to 0^+} f(t,(1-d)W(t))$, see Theorem 4.1 of [50].

Note that it may be the case that K_t corresponds to a continuous driving function while not being generated by a curve. In [45], Marshall and Rohde proved the existence of a Hölder$-1/2$ driving function that corresponds to a logarithmic spiral which is not a curve. That is, there is some point at which the limit above defining γ does not exist. However, they prove that a Loewner chain is always generated by a simple curve if the driving function is Hölder$-1/2$ with a sufficiently small Hölder$-1/2$ norm.

If the driving term is Hölder continuous, then the existence of the curve and its regularity in the capacity parameterization is determined by the local behaviour at the tip, i.e., the growth of the derivative of the conformal map close to preimage of the tip. For a proof, see [27, Proposition 2.2].

Proposition 1.3.3. *Let (f, W) be a $\mathbb{D}-$Loewner pair and assume that $W(t) = e^{i\theta(t)}$ where $\theta(t)$ is a*

Hölder-α on $[0,T]$ for some $\alpha \leq 1/2$. Then the following holds. Suppose there are $c < \infty$, $d_0 > 0$, and $0 \leq \beta < 1$ such that

$$\sup_{t\in[0,T]} d|f_t'((1-d)W(t))| \leq cd^{1-\beta} \quad \text{for all } d \leq d_0.$$

Then (f, W) is generated by a curve that is Hölder-$\alpha(1-\beta)$ continuous on $[0,T]$. The analogous statement holds for $\mathbb{H}-$Loewner pairs.

1.3.2. Schramm Loewner Evolution

A random driving function produces a random Loewner chain. Schramm, in [51], when trying to find a scaling limit for LERW was led to Brownian motion with speed $\kappa > 0$ as the driving function for Loewner equation, see §1.2. A *Schramm Loewner evolution*, SLE_κ, $\kappa > 0$ is the random process $(K_t, t \geq 0)$ with random driving function $W(t) = \sqrt{\kappa}B_t$ where $B : [0,\infty) \to \mathbb{R}$ is a standard one-dimensional Brownian motion. In this section, we will mainly consider *chordal* SLE_κ, that is solutions to the chordal Loewner equation with driving function $\sqrt{\kappa}B_t$, where B_t is a standard Brownian motion.

Levy's theorem tells us that Brownian motion is Hölder continuous of order strictly less than $1/2$. As previously discussed, this is not enough to guarantee the existence of a curve via properties of Loewner equation. Combining with the properties of Brownian motion, Rohde and Schramm estimated the derivatives of the conformal maps to prove that SLE is generated by a curve with $\kappa \neq 8$, see [50]. For $\kappa = 8$, it follows from [38] since SLE_8 is a scaling limit of a random planar curve.

Theorem 1.3.4. *Let $(K_t)_{t>0}$ be an SLE_κ for some $\kappa \in [0,\infty)$. Write $(g_t)_{t>0}$ and $(\xi_t)_{t>0}$ for the associated Loewner flow and transform. The map $g^{-1}(t) : \mathbb{H} \to H_t$ extends continuous to $\overline{\mathbb{H}}$ for all $t > 0$, almost surely. Moreover, if we set $\gamma_t = g_t^{-1}(\xi_t)$, then $(\gamma_t)_{t>0}$ is continuous and generated $(K_t)_{t>0}$, almost surely.*

Due to conformal invariance, we can define in reference domains the upper half-plane $\mathbb{H}$ or the unit disk $\mathbb{D}$. The chordal version of SLE is defined in $\mathbb{H}$ and is a family of random curves connecting two boundary points 0 and ∞. With probability 1, this path is transient, i.e. tends to ∞ as $t \to \infty$. We can define *chordal* SLE between any two fixed boundary points a, b in any simply connected domain Ω via a Riemann map with the law defined by the pushforward by a conformal map $\varphi : \mathbb{H} \to \Omega$ with $\varphi(0) = a$ and $\varphi(\infty) = b$. This map is unique up to scaling which only affects the time parameterization of the curve. Similarly, *radial* SLE defines a conformally invariant family of random curves connecting a boundary point with an interior point.

Schramm's revolutionary observation that these processes were the unique possible scaling limits for a range of lattice based planar random systems at criticality, such as loop-erased random walk, percolation, Ising model, and self-avoiding walk. These limits had been conjectured but Schramm offered the candidate for the limit object. Any scaling limit is scale invariant.

Theorem 1.3.5 (Scale Invariance). *Let $\gamma(t)$ be the chordal SLE$_\kappa$ path in $\mathbb{H}$ and $c > 0$. Then for $t > 0$, $c\gamma[0,t]$ equals $\gamma[0,c^2t]$ in distribution.*

The proof follows using the scale invariance of Brownian motion. Note that this scale invariance is not present in the radial case, as is one reason why chordal SLE can be easier to work with than radial SLE (although, there is a version of radial SLE called whole plane SLE which satisfies some scale invariance). Moreover, the local determination of certain paths in the lattice models suggests a form of the *domain Markov property.* The Markov property of Brownian motion translates into the domain Markov property for SLE, see [35] for more details.

Theorem 1.3.6 (Domain Markov Property). *Let γ be the chordal SLE path. Conditionally on $\gamma[0,T]$, the curve $t \mapsto \gamma(T+t)$ has the same distribution as chordal SLE in H_T between $\gamma(T)$ and ∞.*

In fact it was widely conjectured that there would be limit objects, associated to some class of planar domains, with a stronger property of invariance under conformal maps. Similarly, the conformal invariance of Brownian motion translates into the conformal invariance for SLE.

Schramm's Principle. *Schramm-Loewner Evolutions are the only random curves satisfying conformal invariance and domain Markov property.*

Given a collection of probability measures $(\mathbb{P}_{(D,a,b)})$ where $\mathbb{P}$ is the law of a random curve $\gamma : [0,\infty) \to \mathbb{C}$ such that $\gamma([0,\infty)) \subset \overline{D}$ and $\gamma(0) = a,\ \gamma(\infty) = b$. The family $(\mathbb{P}_{(D,a,b)})$ satisfies *conformal invariance* if for all (D,a,b) and conformal maps ϕ

$$\mathbb{P}_{(D,a,b)} \circ \phi^{-1} = \mathbb{P}_{(\phi(D),\phi(a),\phi(b))}.$$

Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by $(\gamma(t))_{t\geq 0}$. The family $(\mathbb{P}_{(D,a,b)})$ satisfies the *domain Markov property* if: for all (D,a,b) for every $t \geq 0$ and for any measurable set B in the space of curves

$$\mathbb{P}_{(D,a,b)}(\gamma[t,\infty) \in B|\mathcal{F}_t) = \mathbb{P}_{(D\setminus\gamma[0,t],\gamma(t),b)}(\gamma \in B).$$

For more details and discussions see [51] and [57].

We will need the following derivative estimate for both chordal and radial SLE. See Appendix A in

[27] for a detailed analysis and proof of these bounds. Suppose $\kappa > 0$, let

$$\begin{aligned} \lambda_C &= 1 + \frac{2}{\kappa} + \frac{3\kappa}{32}, \\ q(\beta) &= \min\left\{\lambda_C \beta, \beta + \frac{2(1+\beta)}{\kappa} + \frac{\beta^2\kappa}{8(1+\beta)} - 2\right\}, \\ \text{and } \beta_+ &= \max\left\{0, \frac{4(\kappa\sqrt{8+\kappa} - (4-\kappa))}{(4+\kappa)^2}\right\}. \end{aligned}$$

Proposition 1.3.7. *Let $T < \infty$ be fixed and let (f_t) be the reverse chordal SLE$_\kappa$ Loewner chain, $\kappa \in (0,8)$. Let $\beta \in (\beta_+, 1)$ and $q < q(\beta)$. There exists a constant $0 < c < \infty$ depending only on T, κ, q such that for every $y_* < 1$*

$$\mathbb{P}\left\{\forall y \leq y_*, \sup_{t\in[0,T]} y \ |f'(t, W(t) + iy)| \leq c y^{1-\beta}\right\} \geq 1 - c y_*^q.$$

Let (f_s, W_s) be a radial Loewner pair generated by the curve $\gamma(s)$ with W continuous. Recall that $f_s : \mathbb{D} \to D_s$ satisfies the reverse radial Loewner equation. Let $g_s = f_s^{-1}$ and $z_s = g_s(-1)\overline{W}_s$. Fix $\epsilon > 0$ and $T < \infty$ and define $\sigma = \inf\{s \geq 0 : \ |1 - z_s| \leq \epsilon\} \wedge T$. This is a measure of the "disconnection time" σ' when K_s first disconnects -1 from 0 in $\mathbb{D}$, i.e. the first time z_s hits 1. Clearly, $\sigma < \sigma'$.

Proposition 1.3.8. *Let $\kappa \in (0,8)$. Let $\epsilon > 0$ be fixed and let (f_s), $0 \leq s \leq \sigma$, be the radial SLE$_\kappa$ Loewner chain stopped at σ. For every $\beta \in (\beta_+, 1)$ and $q < q(\beta)$, there exists a constant $c = c(\beta, \kappa, q, \epsilon, T) < \infty$ such that for $d_* < 1$*

$$\mathbb{P}\left\{\forall d \leq d_*, \sup_{t\in[0,\sigma]} d \ |f'(t, (1-d)W(t))| \leq c d^{1-\beta}\right\} \geq 1 - c d_*^q.$$

The properties of SLE$_\kappa$ depend on the value of κ. One of the most striking examples is the phase transitions of the SLE path.

Theorem 1.3.9 (Phases of SLE). *Let γ be the chordal SLE$_\kappa$ path and let K_t be the associated hulls. The following statements hold almost surely:*

- *γ is a simple curve for $0 < \kappa \leq 4$ and $\gamma[0,\infty) \subset \mathbb{H} \cup \{0\}$.*
- *γ is generated by a curve for $4 < \kappa < 8$ and $\gamma[0,t] \subset K_t$ has double points and touches the real line. Moreover, for every $z \in \overline{\mathbb{H}}$ there exists a random $\tau = \tau(z) < \infty$ such that $z \in K_\tau$.*
- *and γ is a space filling curve for $\kappa \geq 8$.*

Actually, some properties of SLE only hold for certain values of κ. This allows one to predict the

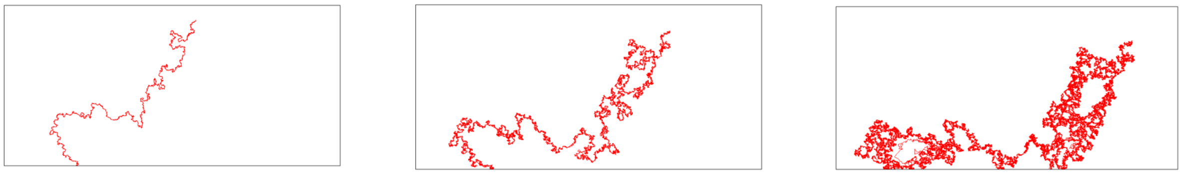


Figure 1.2: Chordal SLE_κ with $\kappa = 2, 4, 6$ corresponding to the same Brownian motion. Pictures by T. Kennedy.

κ associated to the discrete lattice model based on the properties of the model that corresponds to the specific κ. For example, a random path γ in $\mathbb{H}$ satisfies the *restriction* property if for any hull A such that $\mathrm{dist}(0, A) > 0$, the probability law of γ conditioned not to intersect A is the same as the law of the path defined in the smaller domain $\mathbb{H} \backslash A$.

Remark 1.3.10. The self-avoiding walk measure has the same property at the discrete level which is one reason that it is predicted to converge to $\mathrm{SLE}_{8/3}$ in the scaling limit. However, this presumes the existence of a scaling limit and its conformal invariance.

Another example, a random curve can satisfy is the *locality* property if the law of γ stopped when hitting A is the same as the law of the curve in the smaller domain $\mathbb{H} \backslash A$ also stopped when hitting A. In other words, as long as γ does not touch the boundary of the domain, it does not know if it is growing in $\mathbb{H}$ or $\mathbb{H} \backslash A$. Notice that the exploration process of percolation (see §4.1) satisfies the same property at the discrete level. This is one way to predict that its scaling limit is SLE_6.

Theorem 1.3.11 (Restriction and locality)**.** *The SLE_κ path has the restriction property if and only if $\kappa = 8/3$ and locality property if and only if $\kappa = 6$.*

An interesting corollary of the locality property is the following.

Corollary 1.3.12. *Let D be a simply connected domain in the plane with $a, b, c \in \partial D$. Then until their first hitting time of the boundary arc bc, an SLE_6 in D from a to b and an SLE_6 in D from a to c have (up to time-change) the same distribution.*

That is, SLE_6 not only does not know which domain it is growing in, it does not know where it is going either! More details can be found in [35], [37], and [34].

SLE is a random fractal object that has been rigorously studied and there are many interesting properties that we could discuss. However, that is outside the scope of this thesis. We will simply end this section with the following result, see [3] and [33] for two different proofs.

Theorem 1.3.13 (Hausdorff dimension)**.** *Let γ be the chordal SLE$_\kappa$ path $\kappa \geq 0$. Then almost surely*

$$\dim_H(\gamma[0,1]) = \min\left\{1 + \frac{\kappa}{8}, 2\right\}$$

1.3.3. Carathéodory convergence

In this section, we present a known implication of driving process convergence, namely Carathéodory convergence.

Definition 1.3.14. Suppose D_n is a sequence of domains in $\mathcal{D}$ the set of simply connected domains other than $\mathbb{C}$ containing the origin. Let f_n be the conformal transformation of $\mathbb{D}$ onto D_n with $f_n(0) = 0,\ f_n'(0) > 0$. We say that D_n *converges in the Carathéodory sense* to $D \in \mathcal{D}$ if f_n converges to f uniformly on compact sets: for each compact $K \subset D$

$$f_n \to f \quad \text{uniformly on } K$$

If D_n is a sequence of simply connected domains containing z_n we say that D_n *converges to D in the Caratheéodory sense with respect to z_n and z* if $D_n - z_n \to D - z$ in the Carathéodory sense.

The Carathéodory topology can also be defined on the set of doubly-connected domains with a market point using the same geometric description. We can also talks about the compact hulls converging in the sense of Carathéodory. Suppose A_n and A are compact $\mathbb{H}$-hulls and g_{A_n} and g_A defined as in §1.3.1. Then $A_n \to A$ in Carathéodory sense if $g_{A_n}^{-1} \to g_A^{-1}$ uniformly away from $\mathbb{R}$. That is, for every $\epsilon > 0,\ g_{A_n}^{-1} \to g_A^{-1}$ uniformly on $\mathbb{H}\backslash\{\text{dist}(z,\mathbb{R}) < \epsilon\}$

Proposition 1.3.15. *Suppose A_n is a compact $\mathbb{H}-$hull with $\text{rad}(A_n)$ uniformly bounded. Then there exists a compact $\mathbb{H}-$hull A (possibly empty) and a subsequence A_{n_j} such that $A_{n_j} \to A$ in the Carathéodory sense.*

The following proposition relates the driving function convergence to Carathéodory convergence. The result for chordal Loewner chains is proved in [35, §4.7] and the proof for radial Loewner chains is entirely similar. Suppose that W_t^n and W_t are continuous functions from $[0,\infty)$ into $\mathbb{R}$. Let g_t^n and g_t be the corresponding chordal Loewner chains and K_t be the closure of $\mathbb{H}\backslash H_t$ where $H_t := \mathbb{H}\backslash\gamma[0,t]$. We say that the chain g_t^n converges to the chain g_t *in the Carathéodory sense* if for every $\epsilon > 0$ and every $T < \infty$, $g_t^n \to g_t$ uniformly on $[0,T] \times \{z \in \mathbb{H} : \text{dist}(z, K_T) \leq \epsilon\}$.

Proposition 1.3.16. *If W_t^n converges to W_t uniformly on compact intervals $[0,T]$ then $g^n \to g$ in the Carathéodory sense.*

We also know that if we have a continuous driving function and corresponding Loewner chain, then there is a sequence of Loewner chains generated by simple curves which converge to it in the Carathéodory sense.

Proposition 1.3.17. *If $W : [0, \infty) \to \mathbb{R}$ is a continuous function and g_t is the corresponding Loewner chain, then there exists a sequence of Loewner chains g_t^n generated by simple curves γ^n such that $g^n \to g$ in the Carathéodory sense.*

From this proposition, we cannot conclude that the Loewner chain itself is generated by a path. However, if the sequence γ^n is precompact then we know this sequence of curves has subsequential limits and can conclude that γ^n converges to γ uniformly on compact sets. If that is the case, then the Loewner chain is generated by a curve γ. The following proposition collects these results, see [35, §4.7].

Proposition 1.3.18. *Suppose g^n are Loewner chains generated by curves γ^n.*

- *If γ^n converges uniformly to γ on compact intervals, then $g^n \to g$ in the Carathéodory sense where g is the Loewner chain generated by γ.*
- *If $g^n \to g$ in the Carathéodory sense and for every t_0, $\{\gamma_t^n : 0 \le t \le t_0\}$ is an equicontinuous family, then there exists a γ such that γ^n converges uniformly to γ on compact intervals and g is generated by γ.*

1.4. Approaches to Proving Convergence to SLE and Main Results in the Literature

The two main schemes used for proving convergence are as follows:

Scheme 1. (a) Establish precompactness for the sequence of probability measures describing the discrete curves which provides the existence of subsequential limits.

(b) Choose a converging sub-sequence and check that we can describe the limiting curve by Loewner evolution.

(c) Show that the convergence is equivalent to the uniqueness of the subsequential limit.

An example of this approach can be found in [56]. The main part of this scheme is proving uniqueness which involves finding an observable with a well-behaved scaling limit. The observable needs to be a martingale with respect to the information generated by the growing curve and so must be closely related to the interface. In general, one finds an observable by solving a discrete boundary value problem defined on the same or related graph as the interface typically resulting in a preharmonic function. Kempannien and Smirnov's main result shows that for $\kappa < 8$ a certain, uniform bound on the probability of an annulus crossing event implies the existence of subsequential limits (part(a) of scheme 1) and that they can be described by Loewner equation with random driving forces (part (b) of scheme 1). This is referred to as the *precompactness part* of the scheme.

Recall the general set up: Let $D^\delta_{\mathbb{C}}$ be the polygonal domain (union of tiles) corresponding to D^δ and $\phi^\delta : (D^\delta_{\mathbb{C}}; a^\delta, b^\delta) \to (\mathbb{D}; -1, 1)$ be some conformal map. This map is not normalized in any specific way yet. We equip the space of continuous oriented curves by the following metric:

$$d(\gamma_1, \gamma_2) = \inf_{\phi_1, \phi_2} ||\gamma_1 \circ \phi_1 - \gamma_2 \circ \phi_2||_\infty,$$

where the infimum is taken over all orientation-preserving reparameterizations of γ_1 and γ_2.

Theorem 1.4.1 (See [28]). *Let D be a bounded simply connected domain with two distinct accessible prime ends a and b. If the family of probability measures $\{\gamma^\delta\}$ satisfies the KS Condition, given below, then both $\{\gamma^\delta\}$ and $\{\gamma^\delta_{\mathbb{D}}\}$, the family of probability measures in $\mathcal{S}_{\text{simple}}(\mathbb{D})$ connecting -1 and 1 defined by the push-forward by ϕ, are tight in the topology associated with the curve distance. Moreover, if $\gamma^\delta_{\mathbb{D}}$ is converging weakly to some random curve $\gamma_{\mathbb{D}}$ then the following statements hold:*

1. *Almost surely, the curve $\gamma_{\mathbb{D}}$ can be fully described by the Loewner evolution and the corresponding maps g_t satisfying the Loewner equation with driving process W_t which is α-Hölder continuous for any $\alpha < 1/2$.*

2. *The driving process W^δ_t corresponding to $\gamma^\delta_{\mathbb{D}}$ convergences in law to W_t with respect to uniform norm on finite intervals; moreover, $\sup_{\delta>0} \mathbb{E}[\exp(\epsilon|W^\delta_t|/\sqrt{t})] < \infty$ for some $\epsilon > 0$ and all t.*

Remark 1.4.2. This theorem combines several of the main results from [28]. Note that if the prime ends a, b are accessible, as assumed, then the convergence of γ^δ outside of their neighbourhood implies the convergence of the whole curves.

CROSSING BOUNDS. The following definitions will be useful throughout this paper.

Definition 1.4.3. A curve γ^δ *crosses* an annulus $A(z_0, r, R) = B(z_0, R) \setminus \overline{B(z_0, r)}$ if it intersects both its inner and outer boundaries $\partial B(z_0, r)$ and $\partial B(z_0, R)$. An *unforced crossing* is a crossing that can be avoided by deforming the curve inside D^δ. That is, the crossing occurs along a sub-arc of γ^δ contained in a connected component of $A(z_0, r, R) \cap D^\delta$ that does not disconnect a^δ and b^δ.

KS Condition. The curves γ^δ satisfy a *geometric bound on unforced crossings*, if there exists $C > 1$ such that for any $\delta > 0$, for any stopping time $0 \le \tau \le 1$ and for any annulus $A(z_0, r, R)$ where $0 < Cr < R$

$$\mathbb{P}^\delta\left(\gamma^\delta[\tau, 1] \text{ makes an unforced crossing of } A(z_0, r, R) \mid \gamma^\delta[0, \tau]\right) < 1/2.$$

Remark 1.4.4. As the interfaces γ^δ we are interested in satisfy the domain Markov property, it is sufficient to check the time zero condition for all domains D^δ simultaneously.

It is shown in [28] that this condition is equivalent to a conformal bound on an unforced crossing and hence it is conformally invariant, see Proposition 2.6 in [28]. Thus, if the conditions hold for the curves γ^δ then they hold for $\gamma^\delta_{\mathbb{D}}$ too.

Equivalent Conditions.

Condition G3. The curves γ^δ satisfy a *geometric power-law bound on any unforced crossings* if there exists $K > 0$ and $\Delta > 0$ such that for any $\delta > 0$, for any stopping time $0 \le \tau \le 1$ for any annulus $A(z_0, r, R)$ where $0 < r \le R$

$$\mathbb{P}^\delta\left(\gamma^\delta[\tau, 1] \text{ makes an unforced crossing of } A(z_0, r, R) \mid \gamma^\delta[0, \tau]\right) < K\left(\frac{r}{R}\right)^\Delta.$$

Instead of an annuli, one can consider all conformal rectangles Q, i.e., conformal images of rectangles $\{z : \operatorname{Re} z \in (0, l), \operatorname{Im} z \in (0, 1)\}$. For fixed Q, define the marked sides as the images of the segments $[0, i]$ and $[l, l + i]$ and the other sides as unmarked. Let the uniquely defined quantity $l(Q)$ be the extremal length of Q. Say that γ^δ crosses Q if γ^δ intersects both of its marked sides.

Condition C2. The curves γ^δ satisfy a *conformal bound on any unforced crossings* if there exists $L > 0$ such that for any $\delta > 0$, for any stopping time $0 \le \tau \le 1$ for any conformal rectangle $Q \subset D^\delta$ that does not disconnect a^δ and b^δ. If $l(Q) > L$ and unmarked sides of Q lie on ∂D^δ then

$$\mathbb{P}^\delta\left(\gamma^\delta[\tau, 1] \text{ makes a crossing of } Q \mid \gamma^\delta[0, \tau]\right) < K\left(\frac{r}{R}\right)^\Delta.$$

Remark 1.4.5. Since all the conditions are equivalent, the constant $1/2$ above and in the KS Condition can be replaced by any other from $(0, 1)$.

Condition C3. The curves γ^δ satisfy a *conformal power-law bound on any unforced crossings* if there exists $K > 0$ and $\epsilon > 0$ such that for any $\delta > 0$, for any stopping time $0 \le \tau \le 1$ for any conformal rectangle Q that does not disconnect a^δ and b^δ. If the unmarked sides of Q lie on ∂D^δ then

$$\mathbb{P}^\delta\left(\gamma^\delta[\tau,1] \text{ makes a crossing of } Q \mid \gamma^\delta[0,\tau]\right) < K\exp\left(-\epsilon l(Q)\right).$$

For a discussion on these conditions, see [28]. The KS Condition (or one of the equivalent conditions) has been shown to be satisfied for the following models: FK-Ising model, spin Ising model, percolation, harmonic explorer, chordal loop-erased random walk (as well as radial loop-erased random walk), and random cluster representation of q-Potts model for $1 \leq q \leq 4$. The KS Condition fails for the uniform spanning tree, see [28].

Scheme 2. (a) Describe the discrete curve by Loewner evolution. (A discrete curve can always be described by Loewner evolution).

(b) Use the observable to show that the driving force is approximately $\sqrt{\kappa}B_t$ for the appropriate κ.

(c) Show convergence of the driving process in Loewner characterization.

(d) Improve to convergence of curves.

An example of this scheme being implemented can be found in [38]. The main part of this scheme involves showing the convergence of driving processes in Loewner characterization. Convergence of driving processes is not sufficient to obtain path-wise convergence. One needs what is known as *a priori estimates of interface regularity.* Kempannien and Smirnov in [28] show that these *a priori bounds* can be derived from a certain, uniform bound on the probability of an annulus crossing event, the KS condition. The following corollary formulates the relationship between convergence of random curves and the convergence of their driving processes. In particular, if the driving processes of Loewner chains satisfying this uniform bound on the probability of a certain annulus crossing event converge, then the limiting Loewner chain is also generated by a curve (part (d) of scheme 2). In this corollary, it is assumed that $\mathbb{H}$ is endowed with a bounded metric. Otherwise, map $\mathbb{H}$ onto a bounded domain.

Corollary 1.4.6 (Corollary 1.7 [28])**.** *Suppose that (W^δ) is a sequence of driving processes of random Loewner chains that are generated by simple random curves (γ^δ) in $\mathbb{H}$, satisfying the KS Condition. Suppose also that (γ^δ) are parameterized by capacity. Then*

- *(W^δ) is tight in the metrizable space of continuous functions on $[0,\infty)$ with the topology of uniform convergence on the compact subsets of $[0,\infty)$.*
- *(γ^δ) is tight in the space of curves $\mathcal{S}$.*

- (γ^δ) *is tight in the metrizable space of continuous functions on* $[0,\infty)$ *with the topology of uniform convergence on the compact subsets of* $[0,\infty)$.

Moreover, if the sequence converges in any of the topologies above it also converges in the two other topologies and the limits agree in the sense that the limiting random curve is driven by the limiting driving process.

In [27], Viklund examines the second approach to develop a framework for obtaining a power law convergence rate to an SLE curve from a power law convergence rate for the driving function provided some additional geometric information, related to crossing events, along with an estimate on the growth of the derivative of the SLE map. For the additional geometric information, Viklund introduces a geometric gauge of the regularity of a Loewner curve in the capacity parameterization called the *tip structure modulus.*

Definition 1.4.7. For $s,t\in[0,T]$ with $s\le t$, we let $\gamma_{s,t}$ denote the curve determined by $\gamma(r),\ r\in[s,t]$. Let $S_{t,\delta}$ to be the collection of crosscuts $\mathcal{C}$ of H_t of diameter at most δ that separate $\gamma(t)$ from ∞ in H_t. For a crosscut $\mathcal{C}\in S_{t,\delta}$,

$$s_{\mathcal{C}} := \inf\{s>0 : \gamma[t-s,t]\cap\overline{\mathcal{C}}\neq\emptyset\},\quad \gamma_{\mathcal{C}} := (\gamma(r), r\in[t-s_C,t]).$$

Define $s_{\mathcal{C}}$ to be t if γ never intersects $\overline{\mathcal{C}}$. For $\delta>0$, the *tip structure modulus* of $(\gamma(t), t\in[0,T])$ in H, denoted by $\eta_{\mathrm{tip}}(\delta)$, is the maximum of δ and

$$\sup_{t\in[0,T]}\ \sup_{\mathcal{C}\in S_{t,\delta}} \mathrm{diam}\gamma_{\mathcal{C}}.$$

(In the radial setting, it is defined similarly.) In some sense, $\eta_{\mathrm{tip}}(\delta)$ is the maximal distance the curve travels into a fjord with opening smaller than δ when viewed from the point toward which the curve is growing.

The following lemma collects the results in [27] and tailors them for the situation where a discrete model Loewner curve approaches an SLE curve in the scaling limit, see Lemma 3.4 in [27].

Lemma 1.4.8. *Consider* (f_j, W_j) $\mathbb{H}-$*Loewner pair generated by the curves* γ_j *where* f_j *satisfies the reverse Loewner flow with continuous driving term* W_j 1.3.2 *for* $j=1,2$. *Fix* $T<\infty$ *and* $\rho>1$. *Assume that there exists* $\beta<1,\ r\in(0,1),\ p\in(0,1/\rho)$ *and* $\epsilon>0$ *such that the following holds with* $d_*=\epsilon^p$.

1. *There is a polynomial rate of convergence of driving processes:*

$$\sup_{t\in[0,T]} |W_1(t) - W_2(t)| \leq \epsilon.$$

2. *An estimate on the growth of derivative of the SLE map: There exists a constant $c' < \infty$ such that the derivative estimate*

$$\sup_{t\in[0,T]} y \;\; |f_2'(t, W_2(t) + id)| \leq c' d^{1-\beta} \qquad \forall\ d \leq d_*.$$

3. *Additional quantitative geometric information related to crossing probabilities: There exists a constant $c < \infty$ such that the tip structure modulus for $(\gamma_1(t),\ t \in [0,T])$ in $\mathbb{H}$ satisfies*

$$\eta_{tip}(d_*) \leq c d_*^r.$$

Then there is a constant $c'' = c''(T, \beta, r, p, c, c') < \infty$ such that

$$\sup_{t\in[0,T]} |\gamma_1(t) - \gamma_2(t)| \leq c'' \max\{\epsilon^{p(1-\beta)r}, \epsilon^{(1-\rho p)r}\}.$$

The analogous statement holds for $\mathbb{D}-$Loewner pairs.

Remark 1.4.9. In [27], an estimate is given explicitly in terms of d_* and β on the probability of the event that the estimate in 2 holds uniformly in $t \in [0,T]$ when $f(t,z)$ is the chordal (and radial) SLE_κ Loewner chain. This is the derivative bound for chordal (and radial) SLE, see Proposition 1.3.7 and 1.3.8.

Let us explore a bit the main ideas in the chordal setting that gives the previous lemma. Suppose that γ_n parameterized by capacity is the conformal image of a discrete-model curve on a $1/n$-lattice approximation of a smooth domain D and the driving term of γ_n, denoted W_n, is coupled with a scaled Brownian motion W driving the chordal SLE curve γ so that the driving terms are at distance at most $\epsilon = n^{-q}$ for some $q < 1$. That is, $\sup_{t\in[0,T]} |W(t) - W_n(t)| \leq \epsilon$, where ϵ is small and fixed for now. Let $f_t : \mathbb{H} \to H_t$ and $f_t^n : \mathbb{H} \to H_t^n$ are the solutions to the (reverse) chordal Loewner equation with driving terms W and W_n, respectively. For each t, H_t and H_t^n are unbounded components of $\mathbb{H}\backslash\gamma[0,t]$ and $\mathbb{H}\backslash\gamma_n[0,t]$. The theorem says that with some regularity on the tip of the curve (which hold uniformly in $t \in [0,T]$ with high probability in terms of ϵ), we can obtain a power-law bound in terms of ϵ on $\sup_{t\in[0,T]} |\gamma(t) - \gamma_n(t)|$. Let us give a little intuition into this theorem by a sketch of the ideas involved in the proof. Let $y > 0$ which will be chosen later depending on ϵ. Let $t \in [0, T]$. Then we can write

$$\begin{aligned} |\gamma(t) - \gamma_n(t)| &\le |\gamma(t) - f(t, W(t) + iy)| \\ &+ |f(t, W(t) + iy) - f(t, W_n(t) + iy)| \\ &+ |f(t, W_n(t) + iy) - f_n(t, W_n(t) + iy)| \\ &+ |f_n(t, W_n(t) + iy) - \gamma_n(t)| \\ &=: A_1 + A_2 + A_3 + A_4. \end{aligned}$$

To prove the theorem, one needs to carefully estimate the A_j's in terms of ϵ (with the constants only depend on the parameters and not on ϵ, y, etc.). By the derivative bound assumption 2 (which is shown to be satisfied for SLE, see Proposition 1.3.7), there are $\beta < 1$ and $c < \infty$ such that

$$|f'(t, W(t) + id)| \le cd^{-\beta} \quad \text{all } d \le y$$

Then by integrating, we get $A_1 \le cy^{1-\beta}$. Further, the distortion theorem, gives the same bound for A_2 if $y \ge \epsilon$. The third term, A_3, is the distance between two solutions to the Loewner equation having driving terms at supremum distance at most ϵ and evaluated at the same point. Using the reverse-time Loewner flow, one can get that if $\operatorname{Im} z = y$, then

$$|f_t(z) - f_t^n(z)| \le c\epsilon y^{-1}$$

with c only depending on T. Thus, we have $A_3 \le c\epsilon y^{-1}$ and by Cauchy's integral formula we get

$$|y||f'(t, z)| - y|f_n'(t, z)| \le c\epsilon y^{-1}.$$

If $|\Delta_n(t, y) := \text{dist}[f_n(t, W_n(t) + iy), \partial H^n(t)]$ then using Koebe's estimate and derivative estimate one gets

$$\Delta_n(t, y) \le cy|f_n'(t, W_n(t) + iy)| \le cy^{1-\beta} + c\epsilon y^{-1}.$$

Notice that this requires no explicit assumptions of $|f_n'|$, see [27, Proposition 2.4].

Choose $y(\epsilon) = \epsilon^p$ for some $p \in (0, 1)$. Then

$$A_1 + A_2 + A_3 \le c\epsilon^{p(1-\beta)} + c\epsilon^{1-p}.$$

So, all that is left to do is to estimate A_4. Notice that $A_4 \geq \Delta_n(t, \epsilon^p)$ but we really want an upper bound in terms of $\Delta_n(t, \epsilon^p)$. To do this, there is some additional information of the boundary behaviour of f_n required which is the tip structure modulus (1.4.7). Using this definition, Viklund shows that

$$|f_n(t, W_n(t) + iy) - \gamma_n(t)| \leq c_1 \eta_{\text{tip}}(c\Delta_n(t, y)),$$

where η_{tip} is the tip structure modulus for γ_n, see [27, Proposition 3.2]. Thus, if we have a power-law bound on the tip structure evaluated at $c_n \Delta_n(t, \epsilon^p)$ for some $r \in (0, 1)$, see condition (3), then by (1.4)

$$A_4 \leq c\epsilon^{p(1-\beta)r} + c\epsilon(1-p)r.$$

It is worth noticing that the estimate on the tip structure modulus is only required on the scale of $\Delta_n(t, \epsilon^p)$ and later we will discuss that the failure of the existence of such a bound implies certain crossing events for the curve which we will exploit. In order to implement these ideas in our setting, we will need to show that the assumptions are satisfied uniformly in $t \in [0, T]$ with high probability in terms of ϵ.

We build upon these earlier works to show that if the condition required for Kempannien and Smirnov's [28] framework is satisfied then one is able to obtain the needed additional geometric information for Viklund's framework. The end result is to obtain a power-law convergence rate to an SLE curve from a power-law convergence rate for the driving terms provided the discrete curves satisfy the KS Condition, a bound on annuli crossing events.

1.5. Main Theorem

The proof of the main theorem involves two key steps. The first step is to derive a rate of convergence for the driving terms. Then we extend the result based on the work of Viklund in [27] to a rate of convergence for the curves. All the a priori estimates required to extend from a convergence of driving terms to a convergence of paths follows from the discrete curve satisfying the Kempannien-Smirnov condition. The method developed here is a general framework for obtaining the rate of convergence of scaling limit of interfaces for various models in statistical physics known to converge to SLE curves, provided the martingale (or almost martingale) observable converges polynomially to its continuous counterpart. All proofs of convergence of scaling limits of interfaces to SLE begin by describing some observable closely related to the interface so that the observable is a martingale (or almost martingale) with respect to the information generated by the growing curve. Typically, observables are solutions to some discrete boundary value problem. For instance, it might be a discrete harmonic function with prescribed boundary values defined on the same or related graph to the observable. In order to obtain our polynomial rate of convergence, we require the observable to be able to be well estimated so that one can obtain a polynomial rate of convergence to its continuous counterpart, to be able to verify the KS condition, and to satisfy the domain Markov property.

The martingale property together with the convergence to a conformally covariant object is sufficient to imply the convergence of the interface to an SLE.

Schramm's Principle. *Schramm-Loewner Evolutions are the only random curves satisfying conformal invariance and domain Markov property.*

We can define conformal invariance for a model with many different definitions. The usual way it is defined in the literature is as conformal invariance of interfaces. That is, conformal invariance of the law of the random curves. An alternative definition is to have conformal invariance refer to the fact that relevant observables of the model are conformally covariant in the scaling limit.

Definition 1.5.1. A family of functions $f_{D,a_1,\cdots,a_n} : D \to \mathbb{C}$ indexed by simply connected domains with marked points $a_1, \cdots, a_n \in \overline{D}$ is *conformally covariant* if there exists $\alpha, \alpha', \beta_1, \beta_1', \cdots, \beta_n, \beta_n' > 0$ such that for any domain D and any conformal map $\varphi : D \to \mathbb{C}$ for every $z \in D$.

$$f_{\varphi(D),\varphi(a_1),\cdots,\varphi(a_n)}(\varphi(z)) = \varphi'(z)^{\alpha}\overline{\varphi'(z)^{\alpha'}}\varphi'(a_1)^{\beta_1}\overline{\varphi'(a_1)^{\beta_1'}}\cdots\varphi'(a_n)^{\beta_n}\overline{\varphi'(a_n)^{\beta_n'}} \cdot f_{D,a_1,\cdots,a_n}(z).$$

If $\alpha = \beta_1 = \beta_1' = \cdots = \beta_n = \beta_n' = 0$, then the family is *conformally invariant.*

Let $D = (D; v, \cdots; a, b, \cdots)$ be a simply connected bounded domain with several marked interior points $v, \cdots \in \text{Int}D$ and boundary points (distinct prime ends) $a, b, \cdots \in \partial D$. For each D let some C^3 function

$$h(\cdot, D) = h(\cdot, v, \cdots; a, b, \cdots; D) : D \to \mathbb{R}$$

be defined.

Let $D^n = (D^n; v^n, \cdots; a, b, \cdots; D) : D \to \mathbb{R}$ be $1/n$-lattice approximation of D on $n^{-1}L$ lattice with several marked vertices $v^n, \cdots \in \text{Int}D^n$ and $a^n, b^n, \cdots \in \partial D^n$. Let

$$H^n(\cdot, D^n) = H^n(\cdot, v^n, \cdots; a^n, b^n, \cdots; D^n) : D^n \to \mathbb{R}$$

be some discrete in D^n function.

Definition 1.5.2. We say H^n are *polynomially close* to h inside D^n if there exists $C < \infty$ and $s \in (0,1)$ such that

$$|H^n(u^n, v^n, \cdots; a^n, b^n, \cdots; D^n) - h(u^n, v^n, \cdots; a^n, b^n, \cdots; D^n)| \le Cn^s.$$

Fix conformal transformation $\Phi : \mathbb{D} \to \mathbb{H}$ such that $z \mapsto i\frac{z+1}{1-z}$. Let $d_*(\cdot,\cdot)$ be the metric on $\overline{\mathbb{H}} \cup \{\infty\}$ defined by $d_*(z,w) = |\Phi^{-1}(z) - \Phi^{-1}(w)|$. If $z \in \overline{\mathbb{H}}$ then $d_*(z_n, z) \to 0$ is equivalent to $|z_n - z| \to 0$ and $d_*(z_n, \infty) \to 0$ is equivalent to $|z_n| \to \infty$. This is a metric corresponding to mapping $(\mathbb{H}, 0, \infty)$ to $(\mathbb{D}, -1, 1)$ which is convenient because it is compact.

Let D be a simply connected bounded domain with distinct accessible prime ends $a, b, \in \partial D$, let $D_n \subset D$ denote the n^{-1}-lattice approximation of D. Fix the maps $\phi : (D, a, b) \to (\mathbb{H}, 0, \infty)$ and $\phi^n : (D^n, a^n, b^n) \to (\mathbb{H}, 0, \infty)$ so that $\phi^n(z) \to \phi(z)$ as $n \to \infty$ uniformly on compact subsets of D with $\phi^n(a^n) \to \phi(a)$ and $\phi^n(b^n) \to \phi(b)$ and satisfying the hydrodynamic normalization : $\phi^n \circ \phi^{-1}(z) - z \to 0$ as $z \to \infty$ in $\mathbb{H}$. Then let $\tilde{\gamma}^n = \phi^n(\gamma^n)$ where γ^n is a random curve on $n^{-1}L$ lattice between a^n and b^n. Thus, $\tilde{\gamma}^n$ is a random curve in $(\mathbb{H}, 0, \infty)$ parameterized by capacity. Let $D^n_t = D^n \backslash \gamma^n[0,t]$, and $g^n_t : H^n_t = \phi^n(D^n_t) \to \mathbb{H}$ be the corresponding Loewner equation with driving terms W^n_t. Thus, $(f^n_t = (g^n_t)^{-1}, W^n_t)$ is a $\mathbb{H}-$Loewner pair generated by $\tilde{\gamma}^n$.

Theorem 1.5.3 (Main Theorem)**.** *Suppose that the family of probability measures $\{\gamma^n\}$ satisfies the KS Condition and the domain Markov property. Assume that for any map ϕ there exists a stopping time $T > 0$, $s \in (0,1)$ and $n_0(D)$ such that for every $n \ge n_0$ and all $v \in V(D)$ the following holds.*

1. *There is a discrete almost martingale observable $H^n_D = H^n_{(D^n, a^n, b^n)}$ associated with the curve γ^n. That is, $H^n_t = H^n_{(D^n_t, \gamma^n(t), b^n)}$ is almost a martingale (for any fixed v) with respect to the (discrete) interface γ^n growing from a^n: $\left| H^n_{D_t}(v) - \mathbb{E}\left[H^n_{D^n_{t'}}(v) \,|D^n_t \right] \right| \le n^{-s}$, for $0 \le t \le t' \le T$*
2. *There is a continuous $\mathcal{C}^3$ function h on the upper half plane $\mathbb{H}$ such that*
   - *(a) $\partial_x h - \partial_x^2 h$ is not locally constant in any neighbourhood*
   - *(b) H^n is polynomially close to its continuous conformally invariant counterpart hfor any v with distance from the boundary bigger than $n^{-\beta}$:*

$$\left| H^n_{D_t}(v) - h(\phi^n_t(v) - W^n_t) \right| \le n^{-s} \text{ for any } t \le T$$

Then for some $0 < \kappa < 8$ there is a coupling of $\tilde{\gamma}^n$ with Brownian motion $\sqrt{\kappa} B(t)$, $t \ge 0$, with the property that for $n \ge N$,

$$\mathbb{P}\left\{ \sup_{t\in[0,T]} d_*\left(\tilde{\gamma}^n(t), \tilde{\gamma}(t)\right) > n^{-u} \right\} < n^{-u}$$

for some $u \in (0,1)$ where $\tilde{\gamma}$ denotes the chordal SLE$_\kappa$ path for $\kappa \in (0,8)$ in $\mathbb{H}$ driven by $\sqrt{\kappa}B(t)$ and both curves are parameterized by capacity. Here N depends on s, n_0, and T. u depends only on s.

Moreover, if D is an α-Hölder domain, then under the same coupling, the SLE curve in the image is polynomially close to the original discrete curve:

$$\mathbb{P}\left\{\sup_{t\in[0,T]} d_*\left(\gamma^n(t), \phi^{-1}(\tilde{\gamma}(t))\right) > n^{-v}\right\} < n^{-v}, \text{ for } n \geq N$$

where v depends only on α and u.

In the future, when we work with $\mathbb{H}-$Loewner pairs, it will be easier for us to work with the bounded version of $\mathbb{H}$, that is, $\mathbb{D}$ with two marked boundary points -1 and 1. Let $\phi_n : (D_n, a^n, b^n) \to (\mathbb{D}, -1, 1)$ be the conformal map. Note that the sequence of domains D^n converge *in the Caratheodory sense*, i.e. the mappings ϕ_n^{-1} converge uniformly in the compact subsets of $\mathbb{D}$ to ϕ^{-1}. Moreover, as we show later (see Lemma 3.4.1), the convergence is polynomially fast if the domain D is Hölder. In applications (Chapter 4), we have sequence of stopping times tending to ∞ and n_0 will tend to ∞.

Question 1.5.4. *In [28], Kempannien and Smirnov make the following heuristic observation: in general, a non-degenerate martingale observable should suffice to verify the KS condition as indicated by the work done in [28]. What precisely should the condition be on a non-degenerate martingale observable that would make this observation hold?*

Question 1.5.5. *We have weak convergence of the curves to SLE$_\kappa$ with respect to the supremum norm on curves modulo reparameterizations. Our discrete curves were parameterized by capacity. Instead, we could consider parameterizing by length. That is, the discrete curve takes one lattice step in one unit of time. One edge traversed in time cN^{-d} where d is the growth exponent, $d = 5/4$ for loop-erased random walk and $d = 7/4$ for percolation. The natural time parameterization of the limiting measure SLE$_\kappa$ is defined by the non-trivial $2 \wedge (1 + \frac{\kappa}{8})$ dimensional Minkowski content. This is called the natural time parameterization. SLE with its natural parameterizatoin is uniquely characterized by conformal invariance and domain Markov property under the constraint that the parameterization is rescaled in a covariant way by conformal maps. SLE with its natural parameterization is believed to be the scaling limit of cures in statistical physics models parameterized by length in the Prokhorov topology, see [39] [43] and [36]. This has been proved for the case of loop-erased random walk to SLE$_2$ and percolation to SLE$_6$, see [41] [42] and [25], respectively. Is it possible to find a rate of convergence in the natural time parameterization?*

Question 1.5.6. *Given a uniform measure on equivalence classes of $n-$vertex triangulations of the* *sphere, it is possible to view a sample from this measure with the graph metric as a random geometry on the sphere. Such models are called quantum gravity in physics literature. In addition, one can impose statistical physics models on such random triangulations. There are two models for random surfaces: random planar maps (RPM) and Liouville Quantum Gravity (LQG). Their conjectured relationship has been used by physicists to predict and calculate the dimension of random fractals and exponents of statistical physics models. Another motivation comes from deep conjectures which state that LQG should describe the large-scale behaviours of random planar maps. It has been shown that there is an equivalence*

$$\left\{\sqrt{8/3}\ LQG\right\} \leftrightarrow \{Brownian\ surfaces\}\ [46].$$

This equivalence allows one to define SLE on Brownian surfaces in a canonical way. Thus, it is possible to prove that certain statistical physics models on uniformly random planar maps converge to SLE. Recently, there have been several convergence results for RPM decorated with a statistical physics model to SLE on a random surface [22, 23, Gromov-Hausdorff Topology] [18, 24, 21, 29, 20, 44, Peanosphere Sense] [19, Joint convergence in both senses].

Currently, the rate of convergence is unknown for any model. This could be a direction to understanding rate of convergence of quantum gravity. I want to understand and work with the imaginary geometry and combine classical techniques to find a rate of convergence and to see if this leads to new results for models on random surfaces. A hope is that techniques from classical approaches can be used to give a better understanding of certain aspects.

Percolation would serve as a good model to begin with as there have been many recent developments in understanding convergence.

- *Gwynne and Miller use the $\sqrt{8/3}$ LQG metric to prove the convergence of percolation on RPM toward SLE_6 on a Brownian surface. [23]*
- *Bernardi, Holden and Sun proved that a number of observables associated with critical site percolation on the triangulation converge jointly in law to the associated observables of SLE_6 on an independent $\sqrt{8/3}$ LQG surface. [6]*
- *In his renowned work on the critical site percolation on the hexagonal lattice, Smirnov's proof of Cardy's formula gives a discrete approximation of the conformal embedding called Cardy embedding. Holden and Sun show that the uniform triangulation under Cardy embedding converges to the*

Brownian disk under the conformal embedding. In addition, they prove a quenched scaling limit result for critical percolation on uniform triangulations [26].

1.5.1. Outline of Thesis

In this section, I will briefly explain the methods used to obtain the main theorem and the outline of the thesis. In this thesis, we present a general framework is outlined for identifying the rate of convergence of scaling limit of interfaces for various models in statistical physics, provided that the martingale observable converges polynomially to its continuous counterpart and the discrete-model curve satisfies the KS Condition. To find a rate of convergence, we re-examine each step of second scheme listed above keeping track of the rate at each step.

The thesis begins with showing that under our assumptions on the almost martingale observable, one is able to extract a power-law convergence rate of the corresponding driving terms. The approach for the derivation of a rate of convergence of driving terms follows almost directly from the scheme outlined for the loop-erased random walk in [5]. The main contributions to the convergence rate are: the rate of convergence of the martingale observable, the rate acquired in transferring this to information about the driving function for a mesoscopic piece of the curve, and the rate obtained after applying Skorokhod embedding to couple with Brownian motion.

The approach for this step is outlined for LERW from [5] and we follow the same approach. Let $\varphi := \Phi(\phi) : D \to \mathbb{H}$ be the conformal map of D onto $\mathbb{H}$ which maps a to 0 and b to ∞. Let the random curve $\tilde{\gamma} = \Phi(\phi(\gamma))$ be parameterized by capacity and $W(t)$ be the Loewner driving term for $\tilde{\gamma}$. For $j \geq 0$, define $D_j = D \setminus \gamma[0, j]$ and let $\varphi_j : D_j \to \mathbb{H}$ be conformal map satisfying $\varphi_j(z) - \varphi(z) \to 0$ and $z \to b$ within D_j. Note that $W(t_j) = \varphi_j(\gamma(j)) \in \mathbb{R}$. Also let $t_j := \mathrm{cap}_\infty(\tilde{\gamma}[0, j])$ be the half plane capacity of $\tilde{\gamma}$.

Consider the lattice scale $\frac{1}{n}$ and the assumptions with rate of convergence of the martingale observable: $\epsilon = n^{-s}$. Then the main contributions for the rate follow from:

- The rate acquired in transferring the rate of convergence for the martingale observable into information about the driving function for a mesoscopic piece (at scale $\epsilon^{2/3}$) of the curve. The convergence rate at this step $\epsilon^{1/3}$.

  Proposition 1.5.7 (Key Estimate)**.** *Suppose the family of probability measures γ^n satisfies the KS condition. For $j \geq 0$, let $\tilde{\gamma}$, D_j, φ_j, $W(t_j)$ and t_j be defined as above. Set $p_j = \varphi_j^{-1}(i + W(t_j))$ and $s \in (0, 1)$. Set $R = \mathrm{rad}_{p_j}(D)$. Under the assumptions above on the martingale observable,* *there exists $c > 0$ and $R_0 > 1$ such that if $\text{rad}_{p_j}(D) > R_0$, the following holds. Fix $k \in \mathbb{N}$ and let*

$$m = \min\{j \geq k \ : \ t_j - t_k \geq R^{-\frac{2s}{3}} \text{ or } |W(t_j) - W(t_k)| \geq R^{-\frac{s}{3}}\}.$$

Then

$$|\mathbb{E}[W_{t_m} - W_{t_j} \mid D_j]| \leq cR^{-s} \tag{1.5.1}$$

$$|\mathbb{E}\left[(W_{t_m} - W_{t_j})^2 - \kappa\mathbb{E}[t_m - t_j] \mid D_j\right]| \leq cR^{-s}. \tag{1.5.2}$$

The analogous statement holds for $\mathbb{D}-$Loewner pairs.

- And the rate obtained after applying Skorokhod embedding to couple with Brownian motion. The resulting error is approximately $\epsilon^{1/3 \cdot 1/2}$.

  Theorem 1.5.8. *Suppose the family of probability measures γ^n satisfies the KS condition and the assumptions above on martingale observables hold. For every $T > 0$, there exists $n_0 = n_0(T, s) < \infty$ such that the following holds. For each $n \geq n_0$, there is a coupling of γ^n with Brownian motion $B(t)$, $t \geq 0$ with the property that*

$$\mathbb{P}\left\{\sup_{t\in[0,T]} |W^n(t) - W(t)| > n^{-s}\right\} < n^{-s}$$

  where $W(t) = B(\kappa t)$ for some $\kappa \in (0, 8)$.

  The analogous statement holds for $\mathbb{D}-$Loewner pair.

In Chapter 3, we begin by establishing the required bound on the tip structure modulus in order to implement Viklund's framework [27]. Then using the framework, we obtain a power-law rate for the corresponding curves. Finally, we argue that if our domain D is a Hölder domain then there is a powerlaw convergence rate for the corresponding conformal maps. The following lemma collects the results in [27] and tailors them for the situation where a discrete model Loewner curve approaches an SLE curve in the scaling limit, see Lemma 3.4 in [27].

Definition 1.5.9. In some sense, *the tip structure modulus*, $\eta_{\text{tip}}(\delta)$, is the maximal distance the curve travels into a fjord with opening smaller than δ when viewed from the point toward which the curve is growing.

Lemma 1.5.10 (Viklund, 2015)**.** *Assume that γ_1, γ_2 are two Loewner curves with driving functions W_1, W_2. Fix $\rho > 1$. Assume that there exists $\beta < 1$, $r \in (0, 1)$, $p \in (0, 1/\rho)$ and $\epsilon = n^{-s}$ for some $s > 0$*

such that the following holds with $d_* = \epsilon^p$*.*

1. *There is a polynomial rate of convergence of driving processes:* $\sup_{t\in[0,T]} |W_1(t) - W_2(t)| \le \epsilon$.

2. *An estimate on the growth of derivative of the SLE map.*

3. *Additional quantitative geometric information related to crossing probabilities: There exists a constant* $c < \infty$ *such that the tip structure modulus for* $(\gamma_1(t),\ t \in [0,T])$ *in* $\mathbb{H}$ *satisfies* $\eta_{tip}(d_*) \le c d_*^r$

Then there is a constant $c'' < \infty$ *such that*

$$\sup_{t\in[0,T]} |\gamma_1(t) - \gamma_2(t)| \le c'' \max\{\epsilon^{p(1-\beta)r}, \epsilon^{(1-\rho p)r}\}.$$

Remark. Condition 2 is established by Viklund for any $\kappa < 8$.

The following proposition shows that a curve satisfying the KS Condition satisfies the required bound on the tip structure modulus.

Proposition 1.5.11 (Binder-Richards)**.** *Suppose the random family of curves* γ_n *(transformed to* $\mathbb{D}$*) satisfies the KS condition. Let* $\eta_{\text{tip}}^{(n)}(\delta)$ *be the tip structure modulus for* γ_n*. Then for some* $p > 0,\ r \in (0,1)$*, and* $\alpha = \alpha(r) > 0$*. There exists* $C, c < \infty$ *independent of* n *and* $n_2 < \infty$ *such that if* $n \ge n_2$ *then*

$$\mathbb{P}_n\left(\eta_{\text{tip}}^{(n)}(n^{-p}) > cn^{-pr}\right) \le Cn^{-\alpha(r)}.$$

The following lemma relates the structure modulus on a sufficiently large mesoscopic scale for the curve in $(\mathbb{D}; -1, 1)$ to the image curve in $(\mathbb{H}; 0, \infty)$.

Lemma 1.5.12. *Suppose* D *is a simply connected domain. Let* D_n *be the* $n^{-1}L$ *grid domain approximation of* D *and let* γ_n *be the Loewner curve transformed onto* $(\mathbb{D}; -1, 1)$*. There is a constant* $c, c' < \infty$ *such that the following holds. Set* $0 < r < 1/2$ *and* $d_n = n^{-r}$ *and let* $\eta_{\text{tip}}^{(n)}(\delta; \mathbb{D})$ *be the tip structure modulus for* γ_n*. Then for all* n *sufficiently large (independently of* γ_n*) the tip structure modulus* $\eta_{\text{tip}}^{(n)}(\delta; \mathbb{H})$ *for* $\Phi(\gamma_n)$ *in* $\mathbb{H}$ *satisfies*

$$\eta_{\text{tip}}^{(n)}(c' d_n; \mathbb{D}) \le c\eta_{\text{tip}}^{(n)}(d_n; \mathbb{H}).$$

To summarize:

- By our assumptions, item 1 in Lemma 1.5.10 is satisfied.

- Item 2 is established in [27] for any SLE_κ, $\kappa < 8$.

- Proposition 1.5.11 establishes the required estimate for the tip structure modulus, item 3.

Thus, we obtain the desired power-law convergence rate for the curves.

Question 1.5.13. *We are able to show that the conformal maps to the discrete approximations for Hölder simply connected domains converge to the corresponding maps polynomially fast, up to the boundary. Is it possible to extend to a wider class of domains by showing that there is polynomial convergence rate in a part of the boundary that is hit by* SLE_κ*? For* $\kappa < 4$*, the boundary should not play a role. Suppose that* $\kappa < 4$ *with no boundary. Is it possible to show polynomial rate of convergence?*

In Chapter 4, we show how we can apply the main theorem in the case of specific models: percolation, harmonic explorer and FK-Ising model.

Chapter 2

Convergence of Driving Term

For the first step, the derivation of a rate of convergence of driving terms follows almost directly from the scheme outlined in [5]. The main contributions to the convergence rate are: the rate of convergence of the martingale observable, the rate acquired in transferring this to information about the driving function for a mesoscopic piece of the curve, and the rate obtained after applying Skorokhod embedding to couple with Brownian motion. Thus, there are essentially four different scales coming into play here:

1. The microscopic scale which is essentially the lattice size $\frac{1}{n}$.
2. The martingale observable scale, $\delta = \delta(n)$, corresponding to the rate of convergence of the martingale observable.
3. The mesoscopic scale, $\delta^{\frac{2}{3}}$, on which the discrete driving function is close to a martingale. The convergence rate at this step is essentially determined by the maximal step size of the discrete driving function, $\delta^{\frac{1}{3}}$.
4. The macroscopic scale which is of constant order. One can iterate the estimate for the mesoscopic pieces of the curve through repeated applications of the domain Markov property in order to "build" a macroscopic piece of it. The resulting error after Skorokhod embedding is roughly, $\delta^{(\frac{1}{3})(\frac{1}{2})}$.

Recall the set-up described before Theorem 1.5.3. Fix $s \in (0, 1)$. Then we have the following result on the convergence of the driving terms.

Theorem 2.0.1. *For every $T > 0$, there exists $n_0 = n_0(T, s) < \infty$ such that the following holds. For each $n \geq n_0$, there is a coupling of γ^n with Brownian motion $B(t)$, $t \geq 0$ with the property that*

$$\mathbb{P}\left\{\sup_{t\in[0,T]} |W^n(t) - W(t)| > n^{-s}\right\} < n^{-s}$$

where $W(t) = B(\kappa t)$ for some $\kappa \in (0, 8)$.

The analogous statement holds for $\mathbb{D}-$Loewner pair.

Remark 2.0.2. Throughout the proofs in the remainder of this section, instead of rescaling the grid, we will consider larger and larger domains. The main theorems in the paper use n^{-1} to denote the lattice spacing and the results are presented as $n \to \infty$. Instead, we will rephrase this as the condition that there is an R_0 such that for domains D with $\text{rad}_{p_0}(D) > R_0$ where $\text{rad}_{p_0}(D) := \inf\{|w - p_0| \ : \ w \notin D\}$ is the inner radius of D about p_0. With this view, we take the scaling limit by using the domain nD with $n \to \infty$ and using a unit lattice. Let $\varphi_n : nD \to \mathbb{H}$ be the normalized conformal map. We can see immediately that $\varphi(z) = \varphi_n(z/n)$. So, the image of the curve in nD on unit lattice under φ_n is trivially the same as the image of the curve on a lattice spacing $1/n$ in D. Also, $\text{rad}_{p_0}(nD) = n\text{rad}_{p_0}(D) \to \infty$. So the condition $\text{rad}_{p_0}(D) > R_0$ is applied to nD and is really the condition that n is large or in other words that the lattice spacing is small.

2.1. KEY ESTIMATE

As in [38] and [5], the idea is to use the polynomial convergence of the discrete observable to its continuous counterpart in order to transfer the fact that we have a discrete martingale observable to information about the Loewner driving function for a mesoscopic piece of the path.

Let us recall our setup: For D be a simply connected bounded domain with distinct accessible prime ends $a, b, \in \partial D$, let $D_n \subset D$ denote the n^{-1}-lattice approximation of D. Fix the maps $\phi : (D, a, b) \to (\mathbb{D}, -1, 1)$ and $\phi^n : (D^n, a^n, b^n) \to (\mathbb{D}, -1, 1)$ so that $\phi^n(z) \to \phi(z)$ as $n \to \infty$ uniformly on compact subsets of D with $\phi^n(a^n) \to \phi(a)$ and $\phi^n(b^n) \to \phi(b)$. Let $\varphi^n := \Phi(\phi^n) : D^n \to \mathbb{H}$ be the conformal map satisfying the hydrodynamic normalization: $\varphi^n \circ \varphi^{-1}(z) - z \to 0$ as $z \to \infty$ in $\mathbb{H}$. Then let $\tilde{\gamma}^n = \varphi^n(\gamma^n)$ where γ^n is a random curve on $n^{-1}L$ lattice between a^n and b^n. Thus, $\tilde{\gamma}^n$ is a random curve in $(\mathbb{H}, 0, \infty)$ parameterized by capacity. Let $D^n_t = D^n \backslash \gamma^n[0, t]$, and $g^n_t : H^n_t = \Phi(\phi^n(D^n_t)) \to \mathbb{H}$ be the corresponding Loewner equation with driving terms W^n_t.

For the rest of this section, let D denote the n^{-1} lattice approximation in order to simplify notation. For the conformal map $\phi^n : (D, a^n, b^n) \to (\mathbb{D}, -1, 1)$ and the family of curves $\{\gamma^n\}$ on D, let $\varphi := \Phi(\phi^n) : D \to \mathbb{H}$ be the associated conformal map on $\mathbb{H}$ and $p_0 = \varphi^{-1}(i)$. Assume the family of probability measure $\{\gamma^n\}$ satisfies the KS condition.

In this setting, $\rho = \text{rad}_{p_0}(D)$ is the appropriate indicator of the size of D from the perspective of the map φ. For instance, if ρ is small, then the image under φ of D is not fine near i and we cannot expect $\tilde{\gamma} = \varphi(\gamma)$ to look like an SLE_κ for some $\kappa \in (0, 8)$. (In $\mathbb{D}$, we measure inner radius from the preimage of

0).

Let the random curve $\tilde{\gamma} = \Phi(\phi^n(\gamma))$ be parameterized by capacity and $W(t)$ be the Loewner driving term for $\tilde{\gamma}$. For $j \geq 0$, define $D_j = D \backslash \gamma[0, j]$ and $\varphi_j : D_j \to \mathbb{H}$ be conformal map satisfying $\varphi_j(z) - \varphi(z) \to 0$ and $z \to b$ within D_j. Note that $W(t_j) = \varphi_j(\gamma(j)) \in \mathbb{R}$. Also let $t_j := \mathrm{cap}_\infty(\tilde{\gamma}[0, j])$ be the half plane capacity of $\tilde{\gamma}$.

Proposition 2.1.1. *For $j \geq 0$, let $\tilde{\gamma},\ D_j,\ \varphi_j,\ W(t_j)$ and t_j be defined as above. Set $p_j = \varphi_j^{-1}(i + W(t_j))$ and $s \in (0, 1)$. Set $R = \mathrm{rad}_{p_j}(D)$. Under the assumptions of Theorem 1.5.3, there exists $c > 0$ and $R_0 > 1$ such that if $\mathrm{rad}_{p_j}(D) > R_0$, the following holds. Fix $k \in \mathbb{N}$ and let*

$$m = \min\{j \geq k \ : \ t_j - t_k \geq R^{-\frac{2s}{3}} \ \textit{or} \ |W(t_j) - W(t_k)| \geq R^{-\frac{s}{3}}\}.$$

Then

$$|\mathbb{E}[W_{t_m} - W_{t_j} \mid D_j]| \leq cR^{-s}, \textit{and} \tag{2.1.1}$$

$$|\mathbb{E}\left[(W_{t_m} - W_{t_j})^2 - \kappa \mathbb{E}[t_m - t_j] \mid D_j\right]| \leq cR^{-s}. \tag{2.1.2}$$

Proof. Notice that $\mathrm{rad}_{p_j}(D_m) \leq \frac{1}{2}\mathrm{rad}_{p_j}(D_j) - 1$ provided R is large enough. Indeed, let z be on $|z| = \frac{1}{2}\mathrm{rad}_{p_j}(D_j)$. Since $\mathrm{Im}\varphi_j(p_j) = 1$, the Koebe distortion theorem implies a positive constant lower bound for $\mathrm{Im}\varphi_j(z)$. Let $g_t : \mathbb{H}_t := \varphi(D_j) \to \mathbb{H}$ be the corresponding Loewner evolution driven by $W(t)$ then $\varphi_k = g_{t_k} \circ \varphi$. By chordal Loewner equation, $\frac{d}{dt}\mathrm{Im}g_t(z) \geq \frac{-2}{\mathrm{Im}g_t(z)}$ which implies $\frac{d}{dt}\left(\mathrm{Im}g_t(z)\right)^2 \geq -4$. Thus, $\tau(z) \geq t_j + \left(\mathrm{Im}\varphi_j(z)\right)^2/4$ where $\tau(z)$ is the time beyond which the solution to the Loewner ODE does not exist. Since $t_{m-1} - t_j \leq R^{-\frac{2s}{3}}$, it follows that $z \notin \gamma[0, m-1]$ for R large enough. Hence, $\mathrm{rad}_{p_j}(D_{m-1}) \geq \frac{1}{2}\mathrm{rad}_{p_j}(D_j)$ which implies that $\mathrm{rad}_{p_j}(D_m) \geq \frac{1}{2}\mathrm{rad}_{p_j}(D_j) - 1$.

Fix $w_0 \in V(D)$ satisfying $|w_0 - p_j| < \mathrm{rad}_{p_j}(D)/6$. Let $R > 100\max\{1, R_0^{2s'}\}$ for large enough R_0. Notice that we can use Beurling estimate to see that the corresponding vertex is at least $R_0^{s'}$-away from the boundary.

Since H_j is almost a martingale, we get $\mathbb{E}[H_m(w_0) \mid D_j] = H_j(w_0) + O(R^{-s})$. So,

$$\mathbb{E}[h_{\mathbb{H}}(\varphi_m(w_0) - W(t_m))|D_j] = h_{\mathbb{H}}(\varphi_j(w_0) - W(t_j)) + O(R^{-s}).$$

Claim. For all $t \in [t_j, t_m]$, $|W_t - W_{t_j}| = O(R^{-\frac{s}{3}})$ and $t_m - t_j = O(R^{-\frac{2s}{3}})$. Indeed, by the definition of m, we get the relations when $t \in [t_j, t_{m-1}]$ and the second when t_{m-1} replaces t_m. Assuming that R is large enough, if $k \in \{j, \cdots, m-1\}$ then, by Beurling projection theorem, the harmonic measure from p_j of $\gamma[k,k+1]$ in D is $O(R^{-\frac{s}{3}})$. By conformal invariance of harmonic measure, the harmonic measure from $\varphi_k(p_j)$ of $\varphi_k \circ \gamma[k,k+1]$ in $\mathbb{H}$ is $O(R^{-\frac{s}{3}})$. Note that $\varphi_k(p_j) = g_{t_k} \circ \varphi(p_j)$ and by above there is a constant positive lower bound for $\mathrm{Im}\varphi_{m-1}(p_j)$. By Loewner equation, $\mathrm{Im}g_t(z)$ is monotone decreasing in t. Hence, $\mathrm{Im}g_t \circ \varphi(p_j)$ has constant positive lower bound for $t \le t_{m-1}$. By Loewner equation again, $|\partial_t(g_t \circ \varphi(p_j))| = O(1)$ for $t \le t_{m-1}$. Integrating this gives $|\varphi_k(p_j) - \varphi_j(p_j)| \le O(R^{-\frac{2s}{3}})$ for $k = j, \cdots, m-1$. As $W_k \in \varphi_k \circ \gamma[k,k+1]$, the distance from $\varphi_k(p_j)$ to $\varphi_k \circ \gamma[k,k+1]$ is $O(1)$. So the harmonic measure estimate gives $\mathrm{diam}(\varphi_k \circ \gamma[k,k+1]) = O(R^{-\frac{s}{3}})$. Since $\varphi_k(\gamma[k,k+1])$ is the set of points hitting the real line under Loewner equation in time interval $[t_k, t_{k+1}]$, the claim follows by 1.3.2.

Let $z_t := g_t \circ \varphi(w_0)$. Since $\varphi_k(w_0) = z_{t_k}$, by flowing from $\varphi_j(w_0)$ according to Loewner equation between times t_j and t_m, we get that $\varphi_m(w_0) = z_{t_m}$. From the Loewner equation we get that

$$z_{t_m} - z_{t_j} = \varphi_m(w_0) - \varphi_j(w_0) = \frac{2(t_m - t_j)}{\varphi_j(w_0) - W_{t_j}} + O(R^{-s}).$$

Define $F(z,W) = h_{\mathbb{H}}(z-W)$. By the assumptions of the main theorem 1.5.3, $h_{\mathbb{H}}(z)$ is a $\mathcal{C}^3$ function which is conformally invariant in $\mathbb{H}$. Our goal is to estimate $F(z_{t_m}, W(t_m)) = h_{\mathbb{H}}(\varphi_m(w_0) - W(t_m))$ up to $O(R^{-s})$ terms. Then Taylor expanding about (z_{t_j}, W_{t_j}) in first order with respect to z (since $|z - z_{t_j}| = O(R^{-2s/3})$) and second order with respect to W (since $|W(t) - W(t_j)| = O(R^{-s/3})$), we get

$$\begin{aligned} h_{\mathbb{H}}(\varphi_m(w_0) - W(t_m)) - h_{\mathbb{H}}(\varphi_j(w_0) - W(t_j)) &= \partial_z F_{(z_{t_j}, W(t_j))}(z_{t_m} - z_{t_j}) + \partial_W F_{(z_{t_j}, W(t_j))}(W_{t_m} - W_{t_j}) \\ &+ \frac{1}{2}\partial_W^2 F_{(z_{t_j}, W(t_j))}(W_{t_m} - W_{t_j})^2 + O(R^{-s}). \end{aligned}$$

We know that the conditional expectation given D_j of the left hand side is $O(R^{-s})$ and applying the bound for $z_{t_m} - z_{t_j}$, we get an equation in terms of $\mathbb{E}[t_m - t_j|D_j]$, $\mathbb{E}[W_{t_m} - W_{t_j}|D_j]$ and $\mathbb{E}[(W_{t_m} - W_{t_j})^2|D_j]$.

$$\begin{aligned} O(R^{-s}) = {} & 2\frac{\partial_z F(z_{t_j}, W(t_j))}{\varphi_j(w_0) - W(t_j)}\mathbb{E}[(t_m - t_j)|D_j] + \partial_W F(z_{t_j}, W(t_j))\mathbb{E}[(W(t_m) - W(t_j))|D_j] \\ &+ \frac{1}{2}\partial_W^2 F(z_{t_j}, W(t_j))\mathbb{E}[(W(t_m) - W(t_j))^2|D_j]. \end{aligned}$$

By Koebe's distortion theorem, we can find $v_1, v_2 \in V(D)$ satisfying $|v_j - p_j| < \mathrm{rad}_{p_j}(D)/6$. If h is non constant, then F is non constant and $\partial_W F - \partial_W^2 F \ne c$ for some constant c (given by the assumptions of the main theorem), then $\partial_W F$ and $\partial_W^2 F$ are not multiples of each other. Thus, we can find two values

v_1 and v_2 such that

$$\begin{bmatrix} \partial_W^2 F(v_1) & \partial_W^2 F(v_2) \\ \partial_W F(v_1) & \partial_W F(v_2) \end{bmatrix}$$

is invertible. If $\partial_z F$ is nonzero then

$$\begin{bmatrix} \partial_W^2 F(v_1) \\ \partial_W F(v_1) \\ \partial_z F(v_1) \end{bmatrix} \text{ and } \begin{bmatrix} \partial_W^2 F(v_2) \\ \partial_W F(v_2) \\ \partial_z F(v_2) \end{bmatrix}$$

are still linearly independent and gives two linearly independent equations in the variables $\mathbb{E}[\kappa(t_m - t_j) - (W(t_m) - W(t_j))^2 | D_j]$ and $\mathbb{E}[W(t_m) - W(t_j) | D_j]$ and thus proves 2.1.1 and 2.1.2. □

2.2. Proof of Theorem 2.0.1

The goal of this section is to show that driving term W of the previous section is close to a standard Brownian motion with speed κ. The standard tool for proving convergence to Brownian motion is to use the Skorokhod embedding theorem.

Lemma 2.2.1 (Skorokhod Embedding Theorem)**.** *Suppose $(M_k)_{k\le K}$ is and $(\mathcal{F}_k)_{k\le K}$ with $||M_{k+1} - M_k||_\infty \le \delta$ and $M_0 = 0$ a.s. There are stopping times $0 = \tau_0 \le \tau_1 \le \cdots \le \tau_K$ for standard Brownian motion $B(t)$, $t \ge 0$, such that $(M_0, M_1, \cdots, M_k)$ and $(B(\tau_0), B(\tau_1), \cdots, B(\tau_K))$ have the same law. Moreover, we have for $k = 0, 1, \cdots K - 1$*

$$\mathbb{E}[\tau_{k+1} - \tau_k | B[0, \tau_k]] = \mathbb{E}[(B(\tau_{k+1}) - B(\tau_k))^2 | B[0, \tau_k]] \tag{2.2.1}$$

$$\mathbb{E}[(\tau_{k+1} - \tau_k)^p | B[0, \tau_k]] \le C_p \mathbb{E}[(B(\tau_{k+1}) - B(\tau_k))^{2p} | B[0, \tau_k]] \tag{2.2.2}$$

for constants $C_p < \infty$ and also

$$\tau_{k+1} \le \inf\{t \ge \tau_k \ : \ |B(t) - B(\tau_k) \ge \delta\}.$$

The proof of Skorokhod embedding theorem can be found in many textbooks including [16]. We will now prove Theorem 2.0.1 using the Skorokhod embedding theorem above and Proposition 2.1.1. The proof of Theorem 2.0.1 follows almost identically to the proof in [5] and is only included for the completeness of the exposition. The outline of the proof is as follows. First, use the domain Markov property to iterate the key estimate to construct sequences of random variables that almost form martingales and adjust the sequence to make it a martingale so that it can be coupled with Brownian motion via Skorokhod embedding. Then show that the stopping times τ_k obtained by Skorokhod embedding theorem are likely to be close to capacities κt_{m_k} for all $k \leq K$ for some appropriate K. Indeed, one can show that each of these has high probability of being close to the quadratic variation of the martingale. Now that they are running on similar clocks, all that is left is to show that they are close at all times with high probability. The key tools for this is the following estimate on the modulus of continuity of Brownian motion.

Lemma 2.2.2. *Let $B(t),\ t \geq 0$, be the standard Brownian motion. For each $\epsilon > 0$ there exists a constant $C = C(\epsilon) > 0$ such that the inequality*

$$\mathbb{P}\left(\sup_{t\in[0,T-h]} \sup_{s\in(0,h]} |B(t+s) - B(t)| \leq v\sqrt{h} \right) \geq 1 - \frac{CT}{h} e^{-\frac{y^2}{2+\epsilon}}$$

holds for every positive v, T and $0 < h < T$.

The proof also requires the following maximal inequality for martingales.

Lemma 2.2.3. *Let $\xi_k,\ k = 1, \cdots, K$, be a martingale difference sequence with respect to the filtration $\mathcal{F}_k$. If $\sigma, u, v > 0$ then it follows that*

$$\mathbb{P}\left(\max_{1\leq j\leq K} |\sum_{k=1}^{j} \xi_k| \geq \sigma \right) \leq \sum_{k=1}^{K} \mathbb{P}\left(|\xi_k| > u\right) + 2\mathbb{P}\left(\sum_{k=1}^{K} \mathbb{E}[\xi_k^2|\mathcal{F}_{k-1}] > v \right)$$
$$+ 2\exp\{\sigma u^{-1}(1 - \log(\sigma u v^{-1}))\}.$$

Proof of Theorem 2.0.1. Depending on the model being considered, it may happen that $p_0 = \varphi^{-1}(i)$ is "swallowed" before time $\bar{t}$. Assume that $\bar{t}$ is small enough so that

$$\bar{t} \leq \frac{1}{100} \text{ and } \mathbb{P}\left[B[0,\bar{t}] \subset \left[-\frac{1}{10}, \frac{1}{10} \right] \right] > 1 - R^{-s} \tag{2.2.3}$$

where B is standard Brownian motion.

Let R_0 be as in Proposition 2.1.1. Let $k \in \mathbb{N}$ be the first integer where $\text{rad}_{p_0}(D_k) \leq R_0$ and define $\bar{t}_0 := \min\{\bar{t}, t_k\}$ where t_j is as in the proposition. Then following the proof of Theorem 1.1 in [5] (see below), Proposition 2.1.1 implies we may couple W with a Brownian motion B in such a way that

$$\mathbb{P}\left(\sup_{t\in[0,\bar{t}_0]} |W(t) - B(\kappa t)| > c_1 R^{-s/6} \bar{t}_0 \right) < c_2 R^{-s/6}$$

if $\text{rad}_{p_0}(D) \geq R_1$ and R_1 large enough. By the assumptions regarding $\bar{t}$, with high probability for all $t \in [0, \bar{t}_0]$, $W(t) \in [-\frac{1}{5}, \frac{1}{5}]$. If R_1 is chosen large enough, then $\mathbb{P}(\bar{t}_0 \neq \bar{t}) < R^{-s}$ and we have the theorem when 2.2.3 is satisfied.

Now, consider a more general case. Let $\epsilon \in (0,1)$ and $\bar{t}_0 := \sup\{t \in [0,T] : |W(t)| \leq \epsilon^{-1}\}$ and $I := \{k \in \mathbb{N} : \ t_k \leq \bar{t}_0\}$. In order to apply Proposition 2.1.1 at every $k \in I$, we need to verify that $\text{rad}_{p_k}(D) \geq R_0$ for such k. Since $\varphi_k(p_k) = i + W(t_k)$ and $g_{t_k} = \varphi_k \circ \varphi^{-1}$, we have $g_{t_k} \circ \varphi(p_k) = i + W(t_k)$. As $g_t \circ \varphi(p_k)$ flows according to Loewner evolution (1.3.1) starting from $\varphi(p_k)$ at $t = 0$ to $i + W(t_k)$ at $t = t_k$, for every $t \in [0, t_k]$, $\text{Im} g_t \circ \varphi(p_k) \geq 1$ which shows that $|\partial_t g_t \circ \varphi(p_k)| = O(1)$. Thus, $|\varphi(p_k)| \leq 1 + |W(t_k)| + O(T) \leq 1 + \epsilon^{-1} + O(T)$. Take $K = \{z \in \mathbb{C} : \ \text{Im}(z) \geq 1, \ |z| \leq O(T + \epsilon^{-1})\}$ compact and $\varphi(p_k) \in K$ for each $k \in I$. Thus, the Koebe distortion theorem implies $\text{rad}_{p_0}(D) \leq O(1)\text{rad}_{p_k}$. So, we can assume that $\text{rad}_{p_k}(D) \geq R_0$ for all $k \in I$ provided we take $\text{rad}_{p_0} \geq R'$ for some constant $R' = R'(\epsilon, T)$. Consequently, we can apply the previous argument with base point moved from p_0 to a vertex near p_k. So, we have

$$\mathbb{P}\left(\sup_{t \in [0, \bar{t}_0]} |W(t) - B(\kappa t)| > c_1 R^{-s/6} \bar{t}_0\right) < c_2 R^{-s/6}$$

if $\text{rad}_{p_0}(D) \geq R'$. Finally, since standard Brownian motion is unlikely to hit $\{-\epsilon^{-1}, \epsilon^{-1}\}$ before time T if ϵ is small, we can take a limit as $\epsilon \to 0$ to get

$$\mathbb{P}\left(\sup_{t \in [0, T]} |W(t) - B(\kappa t)| > c_1 R^{-s/6} T\right) < c_2 R^{-s/6}$$

if $\text{rad}_{p_0}(D) \geq R'$.

For completeness of exposition, we include the following proof.

Proof of Theorem 1.1 in [5]. Choose without loss of generality $T \geq 1$ and assume $R > R_1 := 100TR_0$ where R_0 is the constant from Proposition 2.1.1. Hence if $\text{rad}_{p_0}(D) \geq R_1$, Proposition 2.1.1 is not only valid for the initial domain D, but also for domain D slit by subarcs of γ up to capacity $50T$. From here on, we will not distinguish between most constants instead denoting them by c, which may depend on T. Define $m_0 = 0$ and $m_1 = m$ where m is defined as above. Inductively for $k = 1, 2, \cdots$, define

$$m_{k+1} = \{j > m_k \ : \ |t_j - t_{m_k}| \geq R^{-\frac{2s}{3}} \text{ or } |W_j - W_{m_k}| \geq R^{-\frac{s}{3}}\}.$$

Define $K = \lceil 25TR^{\frac{2s}{3}} \rceil$ and note that $t_{m_K} \leq 50T$. Set $\eta = R^{-\frac{s}{3}}$. Then by the assumption and the domain Markov property of γ, we can find a universal c such that

$$|\mathbb{E}\left[W_{m_{k+1}} - W_{m_k}|\mathcal{F}_k\right]| \leq c\eta^3 \tag{2.2.4}$$

$$|\mathbb{E}\left[(W_{m_{k+1}} - W_{m_k})^2 - \kappa(t_{m_{k+1}} - t_{m_k})|\mathcal{F}_k\right]| \leq c\eta^3 \tag{2.2.5}$$

for $k = 0, \cdots, K-1$ where $\mathcal{F}_k$ is the filtration generated by $\gamma_n[0, m_k]$. For $j = 1, \cdots, K$ define the martingale difference sequence

$$\xi_j = W_{m_j} - W_{m_{j-1}} - \mathbb{E}\left[W_{m_j} - W_{m_{j-1}}|\mathcal{F}_{j-1}\right]$$

and define the martingale M with respect to $\mathcal{F}_k$ by $M_0 = 0$ and

$$M_k = \sum_{j=1}^{k} \xi_j \text{ for } k = 1, \cdots, K.$$

Notice that by Lemma 1.3.2 we get $t_m \leq R^{-2s/3} + O(R^{-1})$ and $|W_m| \leq R^{-s/3} + O(R^{-1/2})$ and so $||M_k - M_{k-1}||_\infty \leq 4\eta$ for R sufficiently large. Now that we have a martingale, we can use Skorokhod embedding to find stopping times $\{\tau_k\}$ for standard Brownian motion B and a coupling of B with the martingale M such that $M_k = B(\tau_k),\ k = 0, \cdots, K$. Now, we need to show that these times $\{\tau_k\}$ are close to the time we run γ at κt_{m_k} for all $k \leq K$. To do this, we show separately that each of these times are close to the natural time of the martingale Y_k. Consider the quadratic variation of M

$$Y_k = \sum_{j=1}^{k} \xi_j^2 \ k = 1, \cdots, K.$$

First, we will show that Y_k is close to κt_{m_k} for every $k \leq K$.

Claim.

$$\mathbb{P}\left(\max_{1 \leq k \leq K} |Y_k - \kappa t_{m_k}| \geq 3\eta|\log \eta|\right) = O(\eta) \tag{2.2.6}$$

for all R large.

The claim follows almost directly from [5] and is only included for completeness of the exposition.

Indeed, set $\sigma_k = \kappa t_{m_k} - \kappa t_{m_{k-1}}$. For $\phi = 3\eta|\log\eta|$, we have

$$\begin{aligned}
\mathbb{P}\left(\max_{1\le k\le K}|\sum_{j=1}^{k}(\xi_j^2-\sigma_j)|\ge\phi\right) &\le \mathbb{P}\left(\max_{1\le k\le K}|\sum_{j=1}^{k}(\xi_j^2-\mathbb{E}[\xi_j^2\mid\mathcal{F}_{j-1}])|\ge\phi/3\right)\\
&+\mathbb{P}\left(\max_{1\le k\le K}|\sum_{j=1}^{k}(\xi_j^2-\mathbb{E}[\sigma_j\mid\mathcal{F}_{j-1}])|\ge\phi/3\right)\\
&+\mathbb{P}\left(\max_{1\le k\le K}|\sum_{j=1}^{k}(\sigma_j-\mathbb{E}[\sigma_j\mid\mathcal{F}_{j-1}])|\ge\phi/3\right)\\
&=: p_1+p_2+p_3.
\end{aligned}$$

Applying Lemma 2.2.3 with $\sigma = \eta|\log\eta|,\ u=\eta$ and $v = e^{-2}\sigma u$,

$$\begin{aligned}
p_1 &\le \sum_{j=1}^{k}\mathbb{P}\left((\xi_j^2-\mathbb{E}[\xi_j^2\mid\mathcal{F}_{j-1}])|\ge\eta\right)\\
&+2\mathbb{P}\left(\sum_{j=1}^{k}\mathbb{E}[\left(\xi_j^2-\mathbb{E}[\xi_j^2\mid\mathcal{F}_{j-1}]\right)^2\mid\mathcal{F}_{j-1}])|\ge e^{-2}\eta|\log\eta|\right)+2\eta.
\end{aligned}$$

Since $\max_j|\xi_j|\le 4\eta$, we get that the first sum is zero for R sufficiently large. Combining this and the definition of K we obtain:

$$\sum_{j=1}^{K}\mathbb{E}[\left(\xi_j^2-\mathbb{E}[\xi_j^2\mid\mathcal{F}_{j-1}]\right)^2\mid\mathcal{F}_{j-1}]\le 16\lceil 50T\rceil\eta^2$$

and so the second sum is also zero for R sufficiently large. In order to bound p_2, we notice that by 2.2.4 and 2.2.5

$$\left|\mathbb{E}[\xi_j^2\mid\mathcal{F}_{j-1}]-\mathbb{E}[\sigma_j\mid\mathcal{F}_{j-1}]\right| = \left|\mathbb{E}[(W_{m_j}-W_{m_{j-1}})^2\mid\mathcal{F}_{j-1}]-\kappa\mathbb{E}[t_{m_j}-t_{m_{j-1}}\mid\mathcal{F}_{j-1}]+O(\eta^4)\right|\le c\eta^3.$$

Apply the triangle inequality and then sum over j to see that $p_2 = 0$ if R is large enough. Lastly, p_3 can be estimated similarly to p_1 and using the inequality $\max_k\sigma_k\le 7\eta^2$. This shows the claim.

Now, we need that Y_k is close to τ_k for every $k\le K$.

Claim.

$$\mathbb{P}\left(\max_{1\le k\le K}|Y_k-\tau_k|\ge 3\eta|\log\eta|\right)=O(\eta) \tag{2.2.7}$$

for large enough R.

Again, the claim follows directly from [5] and is only included for completeness of the exposition. Set $\zeta_k = \tau_k - \tau_{k-1}$ and let $\mathcal{G}_k$ denote the sigma algebra generated by $B[0, \tau_k]$. Again, let $\phi = 3\eta|\log \eta|$. Then

$$\begin{aligned}
\mathbb{P}\left(\max_{1\le k\le K} |\sum_{j=1}^{k}(\xi_j^2 - \zeta_j)| \ge \phi\right) &\le \mathbb{P}\left(\max_{1\le k\le K} |\sum_{j=1}^{k}(\xi_j^2 - \mathbb{E}[\xi_j^2 \mid \mathcal{G}_{j-1}])| \ge \phi/3\right) \\
&+ \mathbb{P}\left(\max_{1\le k\le K} |\sum_{j=1}^{k}(\xi_j^2 - \mathbb{E}[\zeta_j \mid \mathcal{G}_{j-1}])| \ge \phi/3\right) \\
&+ \mathbb{P}\left(\max_{1\le k\le K} |\sum_{j=1}^{k}(\zeta_j - \mathbb{E}[\zeta_j \mid \mathcal{G}_{j-1}])| \ge \phi/3\right) \\
&=: p_4 + p_5 + p_6.
\end{aligned}$$

The estimate of p_4 is identical to the estimate done for p_1 above. By the first estimate in Skorokhod embedding theorem and noting that $\xi_j^2 = (B(\tau_j) - B(\tau_{j-1}))^2$, one gets that $p_5 = 0$ for n sufficiently large. So, we just need to estimate p_6. Applying Lemma 4.3, we obtain

$$\begin{aligned}
p_6 \le& \sum_{j=1}^{k} \mathbb{P}\left((\zeta_j - \mathbb{E}[\zeta_j \mid \mathcal{G}_{j-1}])| > \eta\right) \\
&+ 2\mathbb{P}\left(\sum_{j=1}^{k} \mathbb{E}[(\zeta_j - \mathbb{G}[\zeta_j \mid \mathcal{G}_{j-1}])^2 \mid \mathcal{G}_{j-1}])| > e^{-2}\eta|\log \eta|\right) + 2\eta.
\end{aligned}$$

By Chebyshev's inequality, estimates 2.2.1 and 2.2.2, and the definition of K, we obtain

$$\sum_{j=1}^{K} \mathbb{P}\left((\zeta_j - \mathbb{E}[\zeta_j \mid \mathcal{G}_{j-1}])| > \eta\right) \le \sum_{j=1}^{K} \eta^{-3}\mathbb{E}[|\zeta_j - \mathbb{E}[\zeta_j \mid \mathcal{G}_{j-1}]|^3] \le C\eta.$$

Since $\mathbb{E}[(\zeta_j - \mathbb{E}[\zeta_j \ : \ \mathcal{G}_{j-1}])^2 \mid \mathcal{G}_{j-1}] = O(\eta^4)$, the second probability equals 0 for large enough n. Hence, $p_6 = O(\eta)$ and we have our claim. Combining equations 2.2.6 and 2.2.7 we get

$$\mathbb{P}\left(\max_{1\le k\le K} |\kappa t_{m_k} - \tau_k| > 6\eta|\log \eta|\right) = O(\eta). \tag{2.2.8}$$

Now that we have them running on similar clocks, we just need to show that they are close at all times with high probability. Observe that the last estimate in Skorokhod embedding implies that for $k \le K$

$$\sup\{|B(t) - B(\tau_{k-1})| \ : \ t \in [\tau_{k-1}, \tau_k]\} \le 4\eta. \tag{2.2.9}$$

Similarly, by the definition of m_k and (3.2), we get for large enough n

$$\sup\{|W_{m_k} - t| \;:\; t \in [t_{k-1}, t_k]\} \leq 2\eta.$$

By summing over k and using the definitions of ξ_j and K, we get from 2.2.4

$$\sup\{|W_{m_k} - M_k| \;:\; k \leq K\} \leq cT\eta.$$

By the definition of t_{m_k} we have $Y_{k+1} - Y_k + t_{m_{k+1}} - t_{m_k} \geq \eta^2$. Summing over k gives $Y_k + t_{m_K} \geq K\eta^2 \geq 50T$. Hence, the event that $t_{m_K} < \kappa T$ is contained in the event that $|Y_K - \kappa t_{m_K}| \geq 2\kappa T$. Hence, 2.2.6 implies that

$$\mathbb{P}[t_{m_K} < \kappa T] = O(\eta). \tag{2.2.10}$$

Set $h = \eta|\log\eta|$ and consider the event

$$\mathcal{E} = \{t_{m_K} \geq \kappa T\} \cap \left\{ \sup_{t\in[0,\kappa T-h]} \sup_{s\in(0,h]} |B(t+s) - B(t)| \leq \sqrt{6h|\log h|} \right\} \cap \left\{ \max_{k\leq K} |\tau_k - \kappa t_{m_k}| \leq 6h \right\}.$$

Then applying Lemma 2.2.3 with $\epsilon = 1$ and $v = \sqrt{6|\log h|}$ along with 2.2.10 and 2.2.8, we get $\mathbb{P}(\mathcal{E}^c) = O(\eta|\log\eta|)$. Observe that on $\mathcal{E}$

$$\begin{aligned}\sup\left\{|W(t) - B(\kappa t)| \;:\; t \in [0,T]\right\} \leq \max_{1\leq k\leq K} \Big(&\sup\left\{|W(t) - W_{m_k}| \;:\; t \in [t_{m_{k-1}}, t_{m_k}]\right\} \\ &+ |W_{m_k} - B(\tau_k)| + \sup\left\{|B(\tau_k) - B(\kappa t)| \;:\; t \in [t_{m_{k-1}}, t_{m_k}]\right\}\Big)\end{aligned}$$

and the first two terms are $O(T\eta)$ uniformly in k. It remains to show that the last term is $O(\eta)$. On $\mathcal{E}$, by 2.2.9 we have

$$\begin{aligned}&\sup\{|B(\tau_k) - B(\kappa t)| \;:\; t \in [t_{m_{k-1}}, t_{m_k}]\} \\ &= \sup\{|B(\tau_k) - B(s)| \;:\; s \in [\kappa t_{m_{k-1}}, \kappa t_{m_k}]\} \\ &\leq \sup\{|B(\tau_k) - B(s)| \;:\; s \in [\tau_{k-1} - 6h, \tau_k + 6h]\} \\ &\leq 4\eta + \sup\{|B(\tau_{k-1}) - B(s)| \;:\; s \in [\tau_{k-1} - 6h, \tau_{k-1}]\} + \sup\{|B(\tau_k) - B(s)| \;:\; s \in [\tau_k, \tau_k + 6h]\} \\ &\leq 4\eta + c(\eta\varphi(1/\eta))^{1/2}\end{aligned}$$

where $\varphi(x) = o(x^\epsilon)$ for any $\epsilon > 0$. Thus, we can couple $W(t)$ and B so that

$$\mathbb{P}\left(\sup_{t\in[0,T]}\{|Wt - B(\kappa t)|\} > c_1 T \eta^{1/2}\varphi_1(1/\eta)\right) < c_2\eta|\log\eta|$$

where we recall that $\eta = R^{-\alpha/3}$ and φ_1 is also a subpower function. Hence, there are constants c_1, c_2 such that for all n sufficiently large,

$$\mathbb{P}\left(\sup_{t\in[0,T]}\{|Wt - B(\kappa t)|\} > c_1 R^{-\alpha/6} T\right) < c_2 R^{-\alpha/6}.$$

□

□

Chapter 3

From Convergence of Driving Terms to Convergence of Paths

This section proves a convergence rate result for the interfaces given a convergence rate for the driving processes. In [27], Viklund develops a framework for obtaining a powerlaw convergence rate to an SLE curve from a powerlaw convergence rate for the driving function provided some additional estimates hold. For this, Viklund introduces a geometric gauge of the regularity of a Loewner curve in the capacity parameterization called the *tip structure modulus*.

Definition 3.0.1. For $s, t \in [0, T]$ with $s \leq t$, we let $\gamma_{s,t}$ denote the curve determined by $\gamma(r),\ r \in [s, t]$. Let $S_{t,\delta}$ to be the collection of crosscuts $\mathcal{C}$ of H_t of diameter at most δ that separate $\gamma(t)$ from ∞ in H_t. For a crosscut $\mathcal{C} \in S_{t,\delta}$,

$$s_{\mathcal{C}} := \inf\{s > 0 : \gamma[t - s, t] \cap \overline{\mathcal{C}} \neq \emptyset\}, \quad \gamma_{\mathcal{C}} := (\gamma(r), r \in [t - s_C, t]).$$

Define $s_{\mathcal{C}}$ to be t if γ never intersects $\overline{\mathcal{C}}$. For $\delta > 0$, the *tip structure modulus* of $(\gamma(t), t \in [0, T])$ in H, denoted by $\eta_{\text{tip}}(\delta)$, is the maximum of δ and

$$\sup_{t\in[0,T]} \sup_{\mathcal{C}\in S_{t,\delta}} \text{diam}\gamma_{\mathcal{C}}.$$

In the radial setting, it is defined similarly.

3.1. Main Estimate for the Tip Structure Modulus

In order to apply the framework outlined in the previous section, we need to establish the estimate for the tip structure modulus. We will show that this follows provided γ_n satisfies the KS condition.

Proposition 3.1.1. *Suppose the random family of curves $\{\gamma_n\}$ satisfies the KS condition. Let D_n be a $1/n$-lattice approximation and assume that $1 \leq \mathrm{inrad}(D_n) \leq 2$ and that $\mathrm{diam}(D_n) \leq R < \infty$ where R is given. Let $\gamma_{\mathbb{D}}^n$ be the curve transformed to $(\mathbb{D}; -1, 1)$. Let $\eta_{\mathrm{tip}}^{(n)}(\delta)$ be the tip structure modulus for γ_n stopped when first reaching distance $\rho > 0$ from 1. There exists a universal constant $c_0 > 0$ such that for some $\epsilon > 0$ and $\alpha > 0$. If $\delta = O(\eta^{1+\tilde{\epsilon}})$ where $\tilde{\epsilon} \in \left(0, \frac{4(1+\epsilon)}{\Delta}\right)$. If n is sufficiently large and $\delta > c_0/n$ then*

$$\mathbb{P}_n(\eta_{\mathrm{tip}}^{(n)}(\delta) > \eta) \leq 2\eta^{\alpha}.$$

Corollary 3.1.2. *Suppose the random family of curves $\{\gamma_n\}$ satisfies the KS condition. Let $\eta_{\mathrm{tip}}^{(n)}(\delta)$ be the tip structure modulus for $\gamma_{\mathbb{D}}^n$ stopped when first reaching distance $\rho > 0$ from 1. Then for some $p > 0$, $r \in (0, 1)$, and $\alpha = \alpha(r) > 0$. There exists $C, c < \infty$ independent of n and $n_2 < \infty$ such that if $n \geq n_2$ then*

$$\mathbb{P}_n(\eta_{\mathrm{tip}}^{(n)}(n^{-p}) > cn^{-pr}) \leq Cn^{-\alpha(r)}.$$

For a simple curve γ in $\mathbb{H}$, let $(g_t)_{t\in\mathbb{R}_+}$ and $(W(t))_{t\in\mathbb{R}_+}$ be its Loewner chain and driving function. Then define the *hyperbolic geodesic from ∞ to the tip* $\gamma(t)$ as $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \overline{\mathbb{H}}$ by

$$F(t, y) = g_t^{-1}(W(t) + iy)$$

and the corresponding geodesic in $\mathbb{D}$ for the curve $\Phi^{-1}\gamma$ by

$$F_{\mathbb{D}}(t, y) = \Phi^{-1} \circ F(t, y).$$

Fix a small constant $\rho > 0$ and let τ be the hitting time of $\mathcal{B}(1, \rho)$, i.e. τ is the smallest t such that $|\gamma_{\mathbb{D}}(t) - 1| \leq \rho$. Define $E(\delta, \eta)$ subset of $\mathcal{S}_{\mathrm{simple}}(\mathbb{D})$ as the event that there exists $(s, t) \in [0, \tau]^2$ with $s < t$ such that

- diam $(\gamma[s, t]) \geq \eta$ and
- there exists a crosscut $\mathcal{C}$, $\mathrm{diam}(\mathcal{C}) \leq \delta$, that separates $\gamma(s, t]$ from $\mathcal{B}(1, \rho)$ in $\mathbb{D}\backslash\gamma(0, s]$.

Denote the set of such pairs (s, t) by $\mathcal{T}(\delta, \eta)$.

Recall that for $0 < \delta \leq \eta$, we say γ has a *nested* (δ, η) *bottleneck in* D if there exists $t \in [0,T]$ and $\mathcal{C} \in S_{t,\delta}$ with $\mathrm{diam}\gamma_{\mathcal{C}} \geq \eta$. Observe that η is a bound for the tip structure modulus for γ in D if and only if γ has no nested (δ, η) bottleneck in D. So, in other words, $E(\delta, \eta)$ is the event that there exists a nested (δ, η)-bottleneck somewhere in $\mathbb{D}$.

The following proposition relates an annulus-crossing type event $E(\delta, \eta) \subset \mathcal{S}_{\text{simple}}(\mathbb{D})$ to the speed of convergence of radial limit of a conformal map towards the tip of $\tilde{\gamma}$ in $\mathbb{H}$.

Proposition 3.1.3. *There exists a constant $C > 0$ and increasing function $\mu : [0,1] \to \mathbb{R}_{\geq 0}$ such that $\lim_{\delta \to 0} \mu(\delta) = 0$ and the following holds. Let $\delta < \min(2, \rho)$, $\eta \geq 2\delta$. Assume $\gamma_{\mathbb{D}}(0,t) \subset \mathbb{D} \backslash \mathcal{B}(1, 2\rho)$. Let $\eta_{\text{tip}}(\delta)$ be the tip structure modulus for $\gamma_{\mathbb{D}}$. If $\gamma_{\mathbb{D}}$ is not in $E(\delta, \eta)$ then*

$$\sup_{y \in (0, \mu(\delta)]} |\tilde{\gamma}(t) - F(t,y)| \leq C\rho^{-2}\eta_{\text{tip}}(\delta). \tag{3.1.1}$$

For the proof, we will use Wolff Lemma (see [48]):

Lemma 3.1.4. *Let ϕ be conformal map from open set $U \subset \mathbb{C}$ into $\mathcal{B}(0,R)$. Let $z_0 \in \mathbb{C}$ and let $C(r) = U \cap \{z : |z - z_0| = r\}$ for any $r > 0$. Then*

$$\inf_{\rho < r < \sqrt{\rho}} \{\mathrm{diam}(\phi(C(r)))\} \leq \frac{2\pi R}{\sqrt{\log 1/\rho}}.$$

Proof of Proposition 3.1.3. Let $\mu(\delta) = \exp\left(-\frac{2\pi^2}{\delta^2}\right)$. Fix $t \in \mathbb{R}_{\geq 0}$ and let $C_y = \{\Phi^{-1} \circ g_t^{-1}(W_t + ye^{i\theta}) : \theta \in (0, \pi)\}$ and $z_y = \Phi^{-1} \circ g_t^{-1}(W_t + iy)$. By Lemma 3.1.4, for each $\delta > 0$, there exists $y_\delta \in [\mu(\delta), \sqrt{\mu(\delta)}]$ such that C_{y_δ} has diameter less than δ. Thus by the definition of tip structure, we have $\mathrm{dist}(\gamma_{\mathbb{D}}(t), C_{y_\delta}) \leq \eta_{\text{tip}}(\delta)$. Then by the assumption that $\gamma_{\mathbb{D}}$ is not in $E(\delta, \eta)$, the path with least diameter from z_{y_δ} to $\gamma(t)$ has diameter at most $\delta + \eta_{\text{tip}}(\delta) < 2\eta_{\text{tip}}(\delta)$.

By the Gehring-Hayman theorem (see [48, Theorem 4.20]), the diameter of $J := \Phi^{-1} \circ g_t^{-1}(\{W_t + iy : y \in (0, \mu(\delta)]\}) < C\eta_{\text{tip}}(\delta)$ where C is an absolute constant. Observe that $\mathrm{dist}(\mathbb{D} \backslash \mathcal{B}(1, 2\rho), \mathcal{B}(1, \rho)) > \rho$ and so $J \subset \mathbb{D} \backslash \mathcal{B}(1, \rho)$. Also, by definition of the map Φ, $|\Phi'(z)| = \frac{2}{|1-z|^2}$ and so $\frac{1}{2} < |\Phi'(z)| < \frac{1}{\rho^2}$. Hence, $\mathrm{diam}\Phi(J)$ is at most $C\rho^{-2}\eta_{\text{tip}}(\delta)$. Hence, we get the result (3.1.1).

□

3.2. RESULTS FROM AIZENMAN AND BURCHARD

In order to prove Proposition 3.1.1 we need some results from Aizenman and Burchard found in their paper [1]. Aizenman and Burchard [1] were studying the regularity of a curve which depends on how much the curve "wiggles back and forth." They concluded that the following assumption on a collection of probability measures on the space of curves is able to guarantee a certain degree of regularity of a random curve while remaining intrinsically rough.

Hypothesis H1. The family of probability measures $(\mathbb{P}_h)_{h>0}$ satisfies a *power-law bound on multiple crossings* if there exists $C > 0,\ K_n > 0$ and sequence $\Delta_n > 0$ with $\Delta_n \to \infty$ as $n \to \infty$ such that for any $0 < h < r \leq C^{-1}R$ and for any annulus $A = A(z_0, r, R)$

$$\mathbb{P}\left(\gamma \text{ makes } n \text{ crossings of } A\right) \leq K_n \left(\frac{r}{R}\right)^{\Delta_n}.$$

Recall the following definition which will be needed in the proceeding result.

Definition 3.2.1. A random variable U is said to be *stochastically bounded* if for each $\epsilon > 0$ there is $N > 0$ such that

$$\mathbb{P}_h\left(|U| > N\right) \leq \epsilon \text{ for all } h > 0$$

The following is a reformulation of the result of [1]. In particular, Lemma 3.1 and Theorem 2.5 with the equation (2.22) within the proof.

Denote

$$M(\gamma, l) = \min\left\{ n \in \mathbb{N} \;\middle|\; \begin{array}{c} \exists \text{ partition } 0 = t_0 < t_1 < \cdots < t_n = 1 \\ \text{s.t. } \operatorname{diam}\left(\gamma[t_{k-1}, t_k]\right) \leq l \text{ for } 1 \leq k \leq n \end{array} \right\}.$$

Theorem 3.2.2. *(Aizenman-Burchard). Suppose that for a collection of probability measures $(\mathbb{P}_h)_{h>0}$ there exists $\beta > 1,\ n \in \mathbb{N},\ \tilde{\Delta} > 0$ and $D > 0$ such that $\dfrac{\beta - 1}{\beta}\tilde{\Delta} > 2$ and*

$$\mathbb{P}_h\left(\gamma \textit{ crosses } A(z_0, \rho^\beta, \rho) \textit{ at least } n \textit{ times }\right) \leq D\rho^{\tilde{\Delta}}$$

for any z_0 and any $\rho > 0$. Then there exists a random variable $r_0 > 0$ which remains stochastically bounded as $h \to 0$ with

$$\mathbb{P}_h\left(r_0 < u\right) \leq C u^{(\beta-1)\tilde{\Delta} - 2\beta}$$

and

$$M(\gamma, l) \leq \tilde{C}(\gamma) l^{-2\beta}$$

where the random variable $\tilde{C}(\gamma) = c\left(\frac{l}{r_0}\right)^{2\beta}$ stays stochastically bounded as $h \to 0$. Furthermore, $(\mathbb{P}_h)_{h>0}$ is tight and γ can be reparameterized such that γ is Hölder continuous with any exponent less than $\frac{1}{2}$

with stochastically bounded Hölder norm.

Remark 3.2.3. If Hypothesis H1 holds then for any $\beta > 1$, one has

$$\mathbb{P}_h\left(\gamma \text{ crosses } A(z_0, \rho^\beta, \rho) \text{ at least n times }\right) \leq D\rho^{(\beta-1)\Delta_n}$$

and as $(\beta - 1)\Delta_n > 2\beta$ for large enough n, the previous theorem applies. In this case, Hypothesis H1 can only be applied for $0 < h < \rho^\beta$. Hence, $M(\gamma, l) \leq \tilde{C}(\gamma)l^{-2\beta}$ for $l \geq h^{1/\beta}$. In the other cases we use that:

$$M(\gamma, l) \leq c\left(\frac{h^{1/\beta}}{l}\right)^2 M(\gamma, h^{1/\beta}) \text{ for } 0 < l \leq h$$
$$\text{and } M(\gamma, l) \leq M(\gamma, h) \leq C' l^{-(4\beta-2)} \text{ for } h < l < h^{1/\beta}.$$

In all cases, we still obtain a power bound.

Kempannien and Smirnov show in [28, Proposition 3.6] that the assumption hypothesis H1 can be verified given the KS condition holds.

Proposition 3.2.4 ([28])**.** *If the random family of curves $\{\gamma_n\}$ satisfies the KS condition, then it (transformed to $\mathbb{D}$) satisfies Hypothesis H1.*

3.3. Proof of Main Estimate for Tip Structure Modulus

Proof of Proposition 3.1.1. Suppose for now that $0 < \delta < \eta/20$. Observe that η is a bound for the tip structure modulus for γ in D if and only if γ has no nested (δ, η) bottleneck in D. Thus, it is enough to look at the probability that there exists a nested (δ, η) bottleneck somewhere in D.

Define $E(\delta, \eta)$ as the event that there exists $(s, t) \in [0, \tau]^2$ with $s < t$ such that

- diam $(\gamma[s, t]) \geq \eta$ and
- there exists a crosscut $\mathcal{C}$, diam$(\mathcal{C}) \leq \delta$, that separates $\gamma(s, t]$ from $\mathcal{B}(1, \rho)$ in $\mathbb{D}\backslash\gamma(0, s]$.

Denote the set of such pairs (s, t) by $\mathcal{T}(\delta, \eta)$. In other words, $E(\delta, \eta)$ is the event that there exists a nested (δ, η)-bottleneck somewhere in $\mathbb{D}$.

STEP 1. Divide the curve γ into N arcs of diameter less than or equal to $\eta/4$.

Let σ_k be defined by $\sigma_k = 0$ for $k \leq 0$ and then recursively

$$\sigma_k = \sup\{t \in [\sigma_{k-1}, 1] \ : \ \mathrm{diam}(\gamma[\sigma_{k-1}, t]) < \eta/4\}.$$

Let $J_k = \gamma[\sigma_{k-1}, \sigma_k]$ and $J_0 = \partial\mathbb{D}$. Let $M(\gamma, l)$ be the minimum number of segments of γ with diameters less than or equal to l that are needed to cover γ. There is the following observation: If the curve is divided into pieces that have diameter at most $\eta/4 - \epsilon$ for $\epsilon > 0$, then none of these pieces can contain more than one of the $\gamma(\sigma_k)$ by how the curve was divided. Thus, $N \leq \inf_{\epsilon>0} M(\gamma, \eta/4 - \epsilon) \leq M(\gamma, \eta/8)$.

Step 2. For $E(\delta, \eta)$ to occur there has to be a fjord of depth η with mouth formed by some pair (J_j, J_k) $j < k$ and a piece of the curve enters the fjord resulting in an unforced crossing.

Define stopping times

$$\tau_{j,k} = \inf\{t \in [\sigma_{k-1}, \sigma_k] \ : \ \mathrm{dist}(\gamma(t), J_j) \leq 2r\} \text{ for } 0 \leq j < k.$$

where the distance is the infimum of numbers l such that $\gamma(t)$ can be connected to J_j by a path of diam $< l$ in $D_t = \mathbb{D}\backslash\gamma[0, t]$. If empty, define inf $= 1$.

Suppose $E(\delta, \eta)$ occurs. Take a crosscut C and pair of times $0 \leq s \leq t$ as in the definition.

Let $j < k$ be such that the end points of C are on J_j and J_k. Notice that $\mathrm{dist}(J_j, J_k) \leq \delta$ and so the stopping time $\tau_{j,k}$ is finite.

Set $z_1 = \gamma(\tau_{j,k})$ and let z_2 be any point on J_j such that $|z_1 - z_2| = 2\delta$ and

$$C' = [z_1, z_2] := \{\lambda z_1 + (1 - \lambda) z_2 \ : \ \lambda \in [0, 1]\}.$$

Let $V \subset D_s$ be connected component of $D_s \backslash C$ which is disconnected from $+1$ by C in D_s and let

$$V' = \{z \in V \ : \ z \text{ disconnected from 1 by } C' \text{ in } D_{\tau_{j,k}}\}$$
$$D' = D_s \backslash V'.$$

Claim There is an unforced crossing of $A_{j,k} := A(z_1, 2\delta, \eta/2)$ as observed at time $\tau_{j,k}$.

Indeed, consider the subpath of $J_j \cup J_k$ which connects end point of C to endpoint of C'. That is,

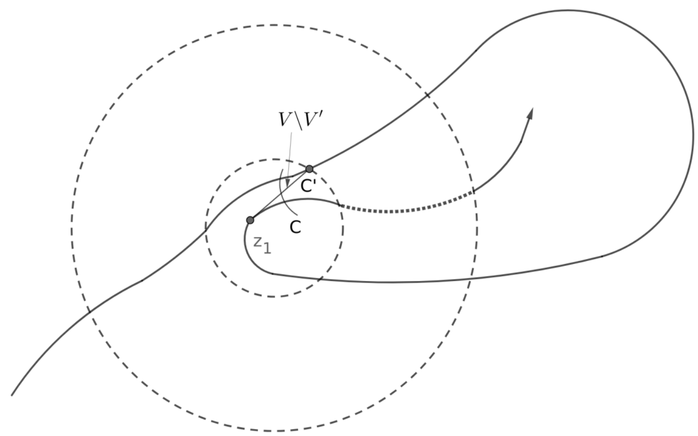


Figure 3.1: The boundary and the curve are cut into pieces of diameter $< \eta/4$. The dotted portion of the curve is an example of event $E_{j,k}$ and has diameter $> \eta/8$. The number of J_k's and the number of dotted pieces are both stochastically bounded as $h \to 0$.

$\gamma[0, t_C]$ where $t_c \in [0,1]$ is the unique time such that $\{\gamma(t_C)\} = \overline{C} \cap J_k$. So,

$$\partial D' = \partial \mathbb{D} \cup \gamma[0, t_C] \cup C \subset (\partial \mathbb{D} \cup \gamma[0, \tau_{j,k}]) \cup (J_k \cup C)\,.$$

Hence, $J_k \cup C$ separates V from $+1$ in $D_{\tau_{j,k}}$. Since $V\backslash V'$ is the set of points disconnected by C from $+1$ in D_s but not by C' in $D_{\tau_{j,k}}$, we see that

$$V\backslash V' \subset \text{ union of bounded components of } \mathbb{C}\backslash\left(J_j \cup J_k \cup C \cup C'\right).$$

Thus, $V\backslash V' \subset \mathcal{B}\left(z_1, \eta/4 + 3\delta\right)$. Since $\gamma[s,t]$ is a connected subset of V, there are $[s', t'] \subset (\tau_{j,k}, t]$ such that $\gamma[s', t'] \subset \overline{V'}$ and it crosses $A_{j,k} := A(z_1, 2\delta, \eta/2)$. Thus, $\gamma[s,t]$ contains an unforced crossing of $A_{j,k}$ as observed at time $\tau_{j,k}$.

As $\gamma[s,t] \subset V$ and $\gamma[s,t]$ is connected, we can find $[s', t'] \subset (\tau_{j,k}, t]$ such that $\gamma[s', t'] \subset \overline{V'}$ and it crosses $A_{j,k} = A(z_1, 2\delta, \eta/2)$. Thus, $\gamma[s,t]$ contains an unforced crossing of $A_{j,k}$ as observed at time $\tau_{j,k}$.

STEP 3. Estimate $\mathbb{P}\left(E(\delta, \eta)\right)$.

Define

$$E_{j,k} = \left\{ \gamma \in \mathcal{S}_{\text{simple}}\left(\mathbb{D}, -1, +1\right) \;\middle|\; \begin{array}{l} \gamma[\tau_{j,k}, 1] \text{ contains a crossing of} \\ A_{j,k} \text{ contained in } \tilde{A}_{j,k} \end{array} \right\}$$

where $\tilde{A}_{j,k} := \left\{z \in A_{j,k} \cap D_{\tau_{j,k}} : \text{connected component of } z \text{ is disconnected from 1 by } C' \text{ in } D_{\tau_{j,k}}\right\}$. We have seen that

$$E(\delta, \eta) \subset \bigcup_{j=0}^{\infty} \bigcup_{k=j+1}^{\infty} E_{j.k}$$

and by *the KS Condition*,

$$\mathbb{P}\left(E_{j,k}\right) \leq K \left(\frac{\delta}{\eta}\right)^{\Delta}.$$

Let $\epsilon > 0$. Let $m \in \mathbb{N}$ then using the tortuosity bounds for $\mathbb{P}(N > m)$ and the facts that $\{N \leq m\} \cap E_{j,k} = \emptyset$ when $k > m$ and $\mathbb{P}(\{N \leq m\} \cap E_{j,k}) \leq \mathbb{P}(E_{j,k})$, we get

$$\begin{aligned}\mathbb{P}(E(\delta, \eta)) &\leq \mathbb{P}(N > m) + \mathbb{P}\left(\bigcup_{0 \leq j < k} \{N \leq m\} \cap E_{j,k}\right) \\ &\leq \text{ constant } \left[m^{-1/2(1+\epsilon)[\epsilon\Delta_k - 2(1+\epsilon)]} + m^2 \left(\frac{\delta}{\eta}\right)^{\Delta}\right].\end{aligned}$$

Choose $m = \eta^{-2(1+\epsilon)}$. Then

$$\mathbb{P}(E(\delta, \eta)) \leq \text{constant} \left[\eta^{\epsilon\Delta_k - 2(1+\epsilon)} + \eta^{-4(1+\epsilon)} \left(\frac{\delta}{\eta}\right)^{\Delta}\right].$$

If we choose $\delta = c\eta^{1+\tilde{\epsilon}}$ where $\tilde{\epsilon} \in \left(0, \frac{4(1+\epsilon)}{\Delta}\right)$ then

$$\mathbb{P}\left(E(c\eta^{1+\tilde{\epsilon}}, \eta)\right) \leq c\eta^{\alpha} \text{ for some } \alpha > 0.$$

□

3.4. Convergence of Discrete Domains

In this section, we prove that the conformal maps to the discrete approximations for Hölder simply-connected domains converge to the corresponding maps polynomially fast, up to the boundary.

Lemma 3.4.1. *Let D_n be a $1/n$-discrete lattice approximation of domain D, w_0 be a point in D and ψ, ψ_n be the conformal maps of the unit disc $\mathbb{D}$ to D and D_n respectively with $\psi(0) = \psi_n(w_0) = 0$, $\psi'(0) > 0$, $\psi_n'(0) > 0$.*

Suppose that ψ is α-Hölder. Then there are constants C, $\beta > 0$ and N depending only on the lattice in question, α, the α-Hölder norm of ψ, and on the conformal radius $\psi'(0)$, such that for $n > N$ and *any $z \in \mathbb{D}$ we have*

$$|\psi(z) - \psi_n(z)| < Cn^{-\beta}.$$

Remark 3.4.2. The same estimate is true for the maps ψ and ψ_n normalized to map $\{-1, 1\}$ to A, $B \in \partial D$ and their discrete approximations A_n, $B_n \in \partial D_n$ correspondingly, provided that they are additionally normalized to converge polynomially fast at 0.

Remark 3.4.3. One can actually show that the polynomial convergence of the approximating maps ψ_n to ψ up to the boundary implies that ψ is α-Hölder for some α.

Proof. Let

$$\Omega_n := \psi^{-1}(D_n) \subset \mathbb{D}.$$

Let also ρ_n be the confomal map of $\mathbb{D}$ onto Ω_n with $\rho_n(0) = 0$, $\rho_n'(0) > 0$. Thus $\psi_n = \psi \circ \rho_n$. Since the map ψ it self is Hölder, it is enough to check that

$$|\rho_n(z) - z| \leq n^{-\gamma} \text{ for some } \gamma > 0. \tag{3.4.1}$$

Observe that any point of ∂D_n is at distance at most $C_1 \frac{1}{n}$ from the boundary of ∂D, where C_1 depends only on the lattice. Thus, by the classical Beurling estimate (see for example Theorem III in [58]), for any $\zeta \in \partial\Omega_n$, we have $|\zeta| > 1 - \frac{C_2}{\sqrt{n}}$.

To see that, we will apply a restatement of another classical result, a lemma of Marchenko (see, for example, [58], Section 3 and Theorem IV):

Lemma 3.4.4 (Marchenko Lemma)**.** *Let Γ be a closed Jordan curve which lies in the ring $1-\varepsilon \leq |\zeta| \leq 1$. Let $\lambda = \lambda(\varepsilon)$ have the property that any two points $\zeta_1, \zeta_2 \in \Gamma$ with $|\zeta_1 - \zeta_2| \leq \varepsilon$ can be connected by an arc of Γ of diameter at most λ.*

Let ρ be the conformal map of $\mathbb{D}$ onto the interior of Γ with $\rho(0) = 0$ and $\rho'(0) > 0$. Then

$$|\rho(z) - z| \leq C_3 \varepsilon \log \frac{1}{\varepsilon} + C_4 \lambda,$$

where C_3 and C_4 are some absolute constants.

Let us show that for the curve $\partial\Omega_n$

$$\lambda\left(\frac{C_2}{\sqrt{n}}\right) \leq n^{-\eta} \tag{3.4.2}$$

for some $\eta > 0$. Indeed, let $\Gamma_0 := \Gamma_{[\zeta_1, \zeta_2]}$ be an arc of Γ with endpoints ζ_1, ζ_2 with $|\zeta_1 - \zeta_2| \leq \frac{C_2}{\sqrt{n}}$. We assume that Γ_0 is the shorter of two such arcs. Then, since $\psi(\zeta_1)$ and $\psi(\zeta_2)$ are $\frac{C_1}{n}$-close to ∂D, $\psi(\Gamma_0)$

is separated from $\psi(0)$ by some crosscut of the length at most

$$2\frac{C_1}{n} + C_5 n^{-\alpha/2}$$

where C_5 depends only on the α-Hölder norm of ψ. Applying Beurling Lemma again we see that $\operatorname{diam}\Gamma_0 \le C_6 n^{-\alpha/4}$, which implies (3.4.2).

Combining Marchenko Lemma and (3.4.1) gives us our result.

□

3.5. Proof of Main Theorem

The proof now follows almost directly from the proof of Theorem 4.3 in [27].

Proof of Theorem 1.5.3. For $\kappa \in (0, 8)$, let γ be the chordal SLE$_\kappa$ path in $\mathbb{H}$ corresponding to the Brownian motion in Theorem 2.0.1. Hence, there is a coupling of chordal SLE$_\kappa$ path and the image of the interface path $\tilde{\gamma}_n = \varphi_n(\gamma_n)$. The goal is to estimate the distance between these curves in this coupling. Take $s \in (0, 1)$ and $n > n_0$ where n_0 is as in Theorem 2.0.1. Fix $\rho > 1$ and for $p \in (0, 1/\rho)$, let

$$\epsilon_n = n^{-s} \quad d_n = (\epsilon_n)^p.$$

For each $n \ge n_0$, define three events each of which we have seen occur with large probability in our coupling. On the intersection of these events, we can apply the estimate from Lemma 1.4.8.

1. Let $\mathcal{A}_n = \mathcal{A}_n(s)$ be the event that the estimate

   $$\sup_{t\in[0,T]} |W_n(t) - W(t)| \le \epsilon_n$$

   holds. By Theorem 2.0.1 we know that there exists $n_0 < \infty$ such that if $n \ge n_0$ then

   $$\mathbb{P}(\mathcal{A}_n) \ge 1 - \epsilon_n.$$

2. For $\beta \in (\beta_+, 1)$ where β_+ is as in Proposition 1.3.7, let $\mathcal{B}_n = \mathcal{B}_n(s, \beta, T, c_B)$ be the event that the chordal SLE_κ reverse Loewner chain (f_t) driven by $W(t)$ satisfies the estimate

   $$\sup_{t\in[0,\widetilde{T}]} d\ |f'(t, W(t) + id)| \le c_B d^{1-\beta} \qquad \forall\ d \le d_n.$$

where $\widetilde{T} \leq T$ is the (stopping) time defined in Proposition 1.3.7 (and Proposition 1.3.8, for radial case). Then by Proposition 1.3.7 (1.3.8 respectively) there exists $c'_B < \infty$, independent of n, and $n_1 < \infty$ such that if $n \geq n_1$ then

$$\mathbb{P}(\mathcal{B}_n) \geq 1 - c'_B d_n^q$$

where $q < q_\kappa(\beta) = \min\left\{\beta\left(1 + \frac{2}{\kappa} + \frac{3\kappa}{32}\right), \beta + \frac{2(1+\beta)}{\kappa} + \frac{\beta^2\kappa}{8(1+\beta)} - 2\right\}$.

3. Let $\gamma_{\mathbb{D}}$ be γ_n transformed to $\Sigma_{\mathbb{D}}$. For $r \in \left(0, \frac{\Delta}{\Delta+4(1+\epsilon)}\right)$, let $\mathcal{C}_n = \mathcal{C}_n(s, r, p, \epsilon, c_{\mathcal{C}})$ be the event that the tip structure modulus for $\gamma_{\mathbb{D}}$, $t \in [0, T]$, in $\mathbb{D}$, $\eta_{\text{tip}}^{(n)}$, satisfies

$$\eta_{\text{tip}}^{(n)}(d_n) \leq c_{\mathcal{C}} d_n^r.$$

We know from Proposition 3.1.1 that there exists $C, c < \infty$, independent of n, and $n_2 < \infty$ such that if $n \geq n_2$ then

$$\mathbb{P}(\mathcal{C}_n) \geq 1 - c d_n^{\alpha(r)}.$$

Notice that by Proposition 3.1.3 we get $|\tilde{\gamma_n}(t) - g_t^{-1}(W_n(t) - i d_n)| \leq C\eta_{\text{tip}}^{(n)}(d_n)$.

Now we will look at intersection of these events. Thus, there exist $c_B, c_C < \infty$ and $c < \infty$, all independent of n (but dependent on $s, r, p, T, \beta, \epsilon$), such that for all n sufficiently large

$$\mathbb{P}(\mathcal{A}_n \cap \mathcal{B}_n \cap \mathcal{C}_n) \geq 1 - c(\epsilon_n + d_n^q + d_n^{\alpha(r)})$$

and on the event $\mathcal{A}_n \cap \mathcal{B}_n \cap \mathcal{C}_n$ we can apply Lemma 1.4.8 with constants $c = c_C$ and $c' = c_B$, independent of n, to see that there exists $c'' < \infty$ independent of n such that for all n sufficiently large

$$\sup_{t \in [0, \widetilde{T}]} |\tilde{\gamma}_n(t) - \tilde{\gamma}(t)| \leq c''(d_n^{r(1-\beta)} + \epsilon_n^{(1-\rho p)r}).$$

Since $d_n = \epsilon_n^p$, one can see that $d_n^{r(1-\beta)}$ dominates when $p \in (0, 1/(1+\rho-\beta)]$ and $\epsilon_n^{(1-\rho p)r}$ whenever $p \in [1/(1+\rho-\beta), 1]$. Suppose $p \in (0, 1/(1+\rho-\beta)]$. Set

$$\mu(\beta, r) = \min\{r(1-\beta), q(\beta), \alpha(r), (1-r)\Delta - 4r(1+\epsilon)\}$$

The optimal rate would be given by optimizing μ over β, r and then choosing p very close to $1/(1+\rho-\beta)$

as choosing p in $[1/(1+\rho-\beta),1]$ will not provide any better. Set

$$\mu(\beta_*,r_*) = \max\{\mu(\beta,r) : \beta_+ < \beta < 1, 0 < r < \Delta/(\Delta+4(1+\epsilon))\}$$

Consequently,

$$\mathbb{P}\left(\sup_{t\in[0,\widetilde{T}]} |\tilde{\gamma}_n(t) - \gamma(t)| > \epsilon_n^m\right) < \epsilon_n^m$$

where $m < m_* = \frac{\mu(r_*,\beta_*)}{2-\beta_*}$

Now assume that D is a Hölder domain. Let $\psi : \mathbb{D} \mapsto D$ be the corresponding Riemann map, and $\psi_n : \mathbb{D} \mapsto D_n$ be the map to the discrete lattice approximation. Observe that by Lemma 3.4.1, $\gamma_n = \psi_n(\tilde{\gamma}_n(t))$ is $n^{-\beta}$ close to $\psi(\tilde{\gamma}_n(t))$ for some $\beta > 0$. But since ψ itself is Hölder, $\psi(\tilde{\gamma}_n(t))$ is $n^{-\beta_1}$ close to $\psi(\gamma)$, which is the SLE curve in D. This concludes the proof of the last assertion. □

Chapter 4

Applications

In this section, we will apply the framework to the modified bond percolation model described in [11, §2.2] which includes the triangular site percolation problem described in [55].

4.1. Percolation

Let $\Omega \subset \mathbb{C}$ be a simply connected domain (whose boundary is a simple closed curve). Let a and c be two distinct points on $\partial\Omega$ or prime ends, if necessary, which separate it into a curve c_1 going from a to c and c_2 going from c to a such that $\partial\Omega\backslash\{a, c\} = c_1 \cup c_2$ and impose boundary conditions so that c_1 is coloured blue and the complementary portion, c_2 is coloured yellow. Consider the percolation model described as follows, for more details see [11, §2.2].

Review of the model.

Consider the hexagon tiling of the 2D triangular site lattice. A hexagon can be coloured blue, yellow, or in specific cases split. The model at hand depends on particular local arrangements of hexagons.

Definition 4.1.1. A *flower* is the union of a particular hexagon with its six neighbors. The central hexagon in each flower is called an *iris* and the outer hexagons are called *petals*. All hexagons which are not flowers are called *fillers*.
Consider a simply connected domain $\Omega \subset \mathbb{C}$ tiled by hexagons. A *floral arrangement*, denoted by $\Omega_{\mathcal{F}}$, is a designation of certain hexagons as irises. The irises satisfy three criteria: (i) no iris is a boundary hexagon, (ii) there are at least two non–iris hexagons between each pair of irises, and (iii) in infinite volume, the irises have a periodic structure with 60° symmetries.

Now, the general description of the model is as follows.

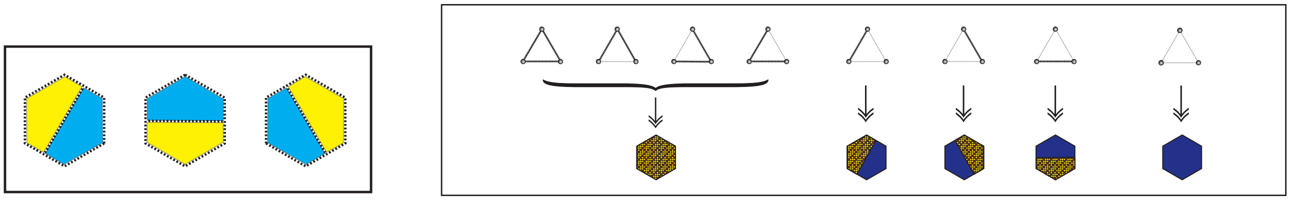


Figure 4.1: The three allowed mixed states of the hexagons corresponding to single-bond occupancy events. *Figures courtesy of Binder, Chayes and Lei.*

Definition 4.1.2. Let Ω be a domain with floral arrangement $\Omega_{\mathcal{F}}$.

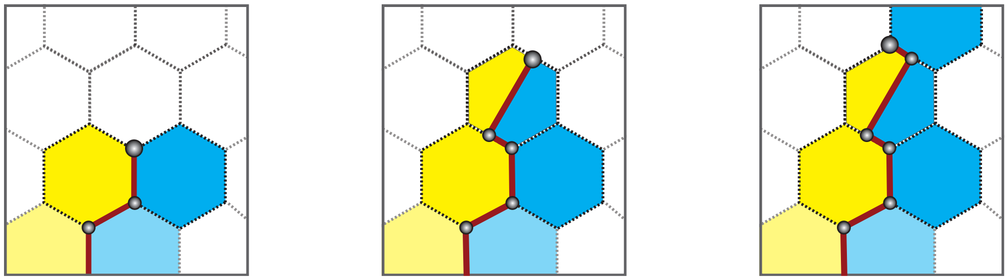
- Any background filler sites, as well as the petal sites, are yellow or blue with probability $\frac{1}{2}$.
- In "most" configurations of the petals, the iris can be in one of five states: yellow, blue, or three mixed states: horizontal split, 120° split, and 60° split with probability a, a, or s so that $2a+3s = 1$ and $a^2 \geq 2s^2$.
- The exceptional configurations, called *triggers* are configurations where there are three yellow petals and three blue petals with exactly one pair of yellow (and hence one pair of blue) petals contiguous. Under these circumstances, the iris is restricted to a pure form, i.e.,blue or yellow with probability $\frac{1}{2}$.

Remark 4.1.3. The only source of (local) correlation in this model is triggering. All petal arrangements are independent, all flowers are configured independently, and these in turn are independent of the background filler sites.

Notice that if we take $s = 0$ in this one-parameter family of models we are reduced to the site percolation on the triangular lattice. This model is shown to exhibit all the typical properties of the percolation model at criticality, see [11, Theorem 3.10]. As well as, the verification of Cardy's formula for this model, see [11, Theorem 2.4].

Exploration Process

Given Ω as above. Consider a hexagon $\epsilon-$tiling of $\mathbb{C}$ and assume that the location of all irises/flowers/fillers are predetermined. Let Ω_ϵ be the union of all fillers and flowers whose closure lies in the interior of Ω where ϵ is small enough so that both a and c are in the same lattice connected component. The boundary $\partial\Omega_\epsilon$ is the usual internal lattice boundary where if it cuts through a flower, the entire flower is included as part of the boundary.



time t "time $t+\frac{1}{2}$" time $t+1$

Figure 4.2: The multistep procedure by which the Exploration Process goes through a mixed hexagon. *Figures courtesy of Binder, Chayes and Lei.*

Admissible domains $(\Omega_\epsilon, \partial\Omega_\epsilon, a_\epsilon, c_\epsilon)$ satisfy the following properties:

- Ω_ϵ contains no partial flowers.
- $\partial\Omega_\epsilon$ can be decomposed into two lattice connected sets, c_1 and c_2, which consists of hexagons and/or halves of boundary irises, one coloured blue and one coloured yellow such that a_ϵ and c_ϵ lie at the points where c_1 and c_2 join and such that the blue and yellow paths are valid paths following the connectivity and statistical rules of the model; in particular, the coloring of these paths do not lead to flower configurations that have probability zero.
- a_ϵ and c_ϵ lie at the vertices of hexagons, such that of the three hexagons sharing the vertex, one of them is blue, one of them is yellow, and the third is in the interior of the domain.

Remark 4.1.4. One can see that $(\Omega_\epsilon, \partial\Omega_\epsilon, a_\epsilon, c_\epsilon)$ converges to $(\Omega, \partial\Omega, a, c)$ in the Caratheodory sense and one can find a_ϵ, c_ϵ which converge to a, c as $\epsilon \to 0$.

Let us now define the Exploration Process $\mathbb{X}_t^\epsilon$. Let $\mathbb{X}_0^\epsilon = a_\epsilon$. Colour the new interior hexagons in order to determine the next step of the Exploration Process according to the following rules:

- If the hexagon is a filler, colour blue or yellow with probability $\frac{1}{2}$.
- If the hexagon is a petal or iris, colour blue, yellow or mixed with the conditional distribution given by the hexagons of the flower already determined.
- If another petal needs to be uncovered, colour it according to the conditional distribution given by the iris and the other hexagons of the flower which have already been determined.

The Exploration Process $\mathbb{X}_t^\epsilon$ is determined from $\mathbb{X}_{t-1}^\epsilon$ as follows:

- If $\mathbb{X}^{\epsilon}_{t-1}$ is not adjacent to an iris, then $\mathbb{X}^{\epsilon}_{t}$ is equal to the next hexagon vertex entered when blue is always to the right of the segment $[\mathbb{X}^{\epsilon}_{t-1}, \mathbb{X}^{\epsilon}_{t}]$.

- If $\mathbb{X}^{\epsilon}_{t-1}$ is adjacent to an iris, then the colour of the iris is determined by the above rules. The exploration path then continues by keeping blue to the right until a petal is hit. The colour of this petal is determined according to the proper conditional distribution and $\mathbb{X}^{\epsilon}_{t}$ is one of the two possible vertices common to the iris and the new petal which keeps the blue region to the right of the final portion of the segments joining $\mathbb{X}^{\epsilon}_{t-1}$ and $\mathbb{X}^{\epsilon}_{t}$.

Thus, at each step, we arrive at a vertex of a hexagon. For this Exploration Process we still maintain the following properties.

Proposition 4.1.5 (Proposition 4.3, [7])**.** *Let $\gamma_\epsilon([0,t])$ be the line segments formed by the process up until time t, and $\Gamma_\epsilon([0,t])$ be the hexagons revealed by the Exploration Process. Let $\partial\Omega^t_\epsilon = \partial\Omega_\epsilon \cup \Gamma_\epsilon([0,t])$ and let $\Omega^t_\epsilon = \Omega_\epsilon \backslash \Gamma_\epsilon([0,t])$. Then, the quadruple $(\Omega^t_\epsilon, \partial\Omega^t_\epsilon, \mathbb{X}^t_\epsilon, c_\epsilon)$ is admissible. Furthermore, the Exploration Process in Ω^t_ϵ from $\mathbb{X}^t_\epsilon$ to c_ϵ has the same law as the original Exploration Process from a_ϵ to c_ϵ in Ω_ϵ conditioned on $\Gamma_\epsilon([0,t])$.*

PERCOLATION SATISFIES THE KS CONDITION.

It is well known that the Exploration Process produces in any critical percolation configuration Ω_ϵ, the unique interface connecting a_ϵ to c_ϵ denoted by γ_ϵ, i.e. the unique curve which separates the blue connected cluster of the boundary from the yellow connected cluster of the boundary. Let $\mathbb{P}_{\Omega_\epsilon}$ be the law of this interface. Let μ_ϵ be the probability measure on random curves induced by the Exploration Process on Ω_ϵ, and let us endow the space of curves with the sup-norm metric $\text{dist}(\gamma_1, \gamma_2) = \inf_{\varphi_1,\varphi_2} \sup_t |\gamma_1(\varphi_1(t)) - \gamma_2(\varphi_2(t))|$ over all possible parameterizations φ_1, φ_2.

The proof of the fact that the collection $(\mathbb{P}_\Omega \ : \ \Omega$ admissible$)$ satisfies the KS Condition follows directly from [28, Proposition 4.13] since this generalized percolation model still satisfies Russo-Seymour-Welsh (RSW) type correlation inequalities.

Remark 4.1.6. As long as $a^2 \geq 2s^2$, then a restricted form of Harris-FKG property holds for all paths and path type events, see [11, Lemma 6.2]. Since we have this essential ingredient in the RSW type arguments, we are indeed free to use RSW sort of correlation inequalities.

Proposition 4.1.7 (Proposition 4.13 in [28])**.** *The collection of the laws of the interface of the modified bond percolation model described above on the hexagonal lattice.*

$$\Sigma_{\text{Percolation}} = \{(\Omega_\epsilon, \phi(\Omega_\epsilon), \mathbb{P}_{\Omega_\epsilon}) : \Omega_\epsilon \textit{ an admissible domain}\} \tag{4.1.1}$$

satisfies the KS Condition.

Proof. First, notice that for percolation, we do not have to consider stopping times. Indeed, by Proposition 4.1.5 if $\gamma : [0, N] \to \Omega_\epsilon \cup \{a, c\}$ is the interface parameterized so that $\gamma(k),\ k = 0, 1, \cdots, N$ are vertices along the path, then $\Omega_\epsilon \backslash \gamma(0, k]$ is admissible for any $k = 0, 1, \cdots, N$ and there is no information gained during $(k, k+1)$. Also, the law of percolation satisfies the domain Markov property so the law conditioned to the vertices explored up to time n is the percolation measure in the domain where $\gamma(k),\ k = 0, 1, \cdots, n$ is erased. Thus, the family (4.1.1) is closed under stopping.

Since crossing an annuli is a translation invariant event for percolation, for any Ω_ϵ, we can apply a translation and consider the annuli around the origin. Let B_n be the set of points on the triangular lattice that are graph distance less than or equal to n from 0. Consider the annulus $B_{9^N n} \backslash B_n$ for any $n, N \in \mathbb{N}$. We can consider concentric balls B_{3n} inside the annulus $B_{9^N n} \backslash B_n$. Then for an open crossing of the annulus $B_{9^N n} \backslash B_n$, there needs to be an open path inside each annulus $A_n = B_{3n} \backslash B_n, A_{3n} = B_{9n} \backslash B_{3n}, \cdots$ etc. The probability that A_n contains an open path separating 0 from ∞ and A_{3n} contains a closed path separating 0 from ∞ are independent. Hence, by Russo-Seymour-Welsh (RSW) theory, we know that there exists a $q > 0$ for any n

$$\mu_\epsilon \left(\text{open path inside } A_n \cap \text{ closed path in } A_{3n} \text{ both separating 0 from } \infty\right) \geq q^2$$

Since a closed path in one of the concentric annuli prohibits an open crossing of $B_{9^N n} \backslash B_n$, we conclude that

$$\mathbb{P}_\epsilon \left(\gamma \text{ makes an unforced crossing of } B_{9^N n} \backslash B_n\right) \leq (1 - q^2)^N \leq \frac{1}{2}$$

for large enough N. □

THE OBSERVABLE IS "ALMOST" ANALYTIC.

Consider two addition marked points (or prime ends) b,d so that a,b,c,d are in cyclic order. Let Ω_n be the *admissible domain* described above at lattice scale n^{-1} to the domain Ω. More details of the construction can be found in [7, §3 and §4] and [8, §4.2]. Furthermore, the boundary arcs can be appropriately coloured and the lattice points a_n, b_n, c_n, d_n can be selected. The main objects of study for percolation is the crossing probability of the conformal rectangle Ω_n from (a_n, b_n) to (c_n, d_n), denoted by $\mathcal{C}_n$ and $\mathcal{C}_\infty$ its limit in the domain Ω, i.e., Cardy's formula in the limiting domain. Geometrically, $\mathcal{C}_n$ produces in any percolation configuration on Ω_n, the unique interface connecting a_n to c_n, i.e. the curve separating the blue lattice connected cluster of the boundary from the yellow. Let us temporarily forget the marked point a_n and consider the conformal triangle $(\Omega_n; b_n, c_n, d_n)$.

We will briefly recall the observable function introduced in [55] which we will denote by S_b, S_c, S_d. For a lattice point $z \in \Omega_n$, $S_d(z)$ is the probability of a yellow crossing from (c_n, d_n) to (d_n, b_n) separating z from (b_n, c_n). Notice that S_d has boundary value 0 on (b_n, c_n) and 1 at the point d_n. S_b and S_c are defined similarly. We define the complexified function $S_n := S_b + \tau S_c + \tau^2 S_d$ with $\tau = e^{2\pi i/3}$, called the *Carleson-Cardy-Smirnov* (CCS) function.

The CCS functions S_n are not *discrete analytic* but are "almost" discrete analytic in the following sense, see [9, §4]:

Definition 4.1.8 ($(\sigma, \rho)-$Holomorphic)**.** Let $\Lambda \subseteq \mathbb{C}$ be a simply connected domain and Λ_ϵ be the (interior) discretized domain given as $\Lambda_\epsilon := \bigcup_{h_\epsilon \subseteq \Lambda} h_\epsilon$ and let $(Q_\epsilon : \Lambda_\epsilon \to \mathbb{C})_{\epsilon \searrow 0}$ be a sequence of functions defined on the vertices of Λ_ϵ. We say that the sequence (Q_ϵ) is (σ, ρ)*–holomorphic* if there exist constants $0 < \sigma, \rho \leq 1$ such that for all ϵ sufficiently small:

1. Q_ε is Hölder continuous up to $\partial\Lambda_\epsilon$: There exists some small $\psi > 0$ and constants $c, C \in (0, \infty)$ (independent of domain and ϵ) such that
   (a) if $z_\epsilon, w_\epsilon \in \Lambda_\epsilon \setminus N_\psi(\partial\Lambda_\epsilon)$ such that $|z_\epsilon - w_\epsilon| < \psi$, then $|Q_\epsilon(z_\epsilon) - Q_\epsilon(w_\epsilon)| \leq c \left(\frac{|z_\epsilon - w_\epsilon|}{\psi}\right)^\sigma$ and
   (b) if $z_\epsilon \in N_\psi(\partial\Lambda_\epsilon)$, then there exists some $w_\epsilon^\star \in \partial\Lambda_\epsilon$ such that $|Q_\epsilon(z_\epsilon) - Q_\epsilon(w_\epsilon^\star)| \leq C \left(\frac{|z_\epsilon - w_\epsilon^\star|}{\psi}\right)^\sigma$.

2. For any simply closed lattice contour Γ_ϵ,

$$\left| \oint_{\Gamma_\epsilon} Q \; dz \right| = \left| \sum_{h_\epsilon \subseteq \Lambda'_\epsilon} \oint_{\partial h_\epsilon} Q \; dz \right| \leq c \cdot |\Gamma_\epsilon| \cdot \epsilon^\rho, \tag{4.1.2}$$

   with $c \in (0, \infty)$ (independent of domain and ϵ) and $\Lambda'_\epsilon, |\Gamma_\epsilon|$ denoting the region enclosed by Γ_ϵ and the Euclidean length of Γ_ϵ, respectively.

Proposition 4.1.9 (Proposition 4.3, [9])**.** *Let Λ denote a conformal triangle with marked points (or prime ends) b, c, d and let Λ_ϵ denote an interior approximation (see [8, Definition 3.1]) of Λ with $b_\epsilon, c_\epsilon, d_\epsilon$ the associated boundary points. Let $S_\epsilon(z)$ denote the CCS function defined on Λ_ϵ. Then for all ϵ sufficiently small, the functions $(S_\epsilon : \Lambda_\epsilon \to \mathbb{C})$ are (σ, ρ)–holomorphic for some $\sigma, \rho > 0$.*

POLYNOMIAL CONVERGENCE OF THE OBSERVABLE FUNCTION TO ITS CONTINUOUS COUNTERPART.

Observe that $\mathcal{C}_n$ can be realized from $S_d(a_n)$ as $\mathcal{C}_n = \frac{-2}{\sqrt{3}} \cdot \text{Im}[S_n(a_n)]$. Since it is already known that S_n converges to $H : D \to T$, a conformal map to equilateral triangle T which sends (b, c, d) to $(1, \tau, \tau^2)$, we can see that $\mathcal{C}_\infty = \frac{-2}{\sqrt{3}} \operatorname{Im}[H(a)]$ (see, [55], [2], and [8]). Thus, when establishing a rate of convergence of $\mathcal{C}_n$ to $\mathcal{C}_\infty$, it is sufficient to show that there exists $\psi > 0$ such that

$$|S_n(a_n) - H(a)| \le C_\psi \cdot n^{-\psi}$$

for some $C_\psi < \infty$ independent of the domain. Indeed, a polynomial rate of convergence is shown in [9, Main Theorem]. This is a slight reformulation of the theorem in which we have that the constant ψ is independent of the domain Ω. Indeed, a direct reconstruction of the proof in [9] gives this result.

Theorem 4.1.10. *Let Ω be a domain with two marked boundary points (or prime ends) a and c. Let (Ω_n, a_n, c_n) be its admissible discretization. Consider the site percolation model or the models introduced in [11] on the domain Ω_n. In the case of the latter we also impose the assumption that the boundary Minkowski dimension is less than 2 (in the former, this is not necessary). Let γ be the interface between a and c. Consider the stopping time $T := \inf\{t \ge 0 \;:\; \gamma$ enters a Δ-neighbourhood of $c\}$ for some $\Delta > 0$. Then there exists $n_0 < \infty$ depending only on the domain $(\Omega; a, b, c, d)$ and T such that the following estimate holds: There exists some $\psi > 0$ (which does not depend on the domain Ω) such that $\mathscr{C}_n$ converges to its limit with the estimate*

$$|\mathscr{C}_n - \mathscr{C}_\infty| \le C_\psi \cdot n^{-\psi},$$

for some $C_\psi < \infty$ provided $n \ge n_0(\Omega)$ is sufficiently large.

POLYNOMIAL CONVERGENCE OF CRITICAL PERCOLATION ON THE TRIANGULAR LATTICE.

Thus, by Proposition 4.1.7, Proposition 4.1.9, and Theorem 4.1.10, we can now apply Theorem 1.5.3 to obtain:

Theorem 4.1.11. *Let γ_n be the percolation Exploration Process defined above on the admissible triangular lattice domain Ω_n. Let $\tilde{\gamma}_n$ be its image in $(\mathbb{H}; 0, \infty)$ parameterized by capacity. There exists stopping time $T < \infty$ and n_1 such that $\sup_n \sup_{t \in [0,T]} n_1(\Omega_t) < \infty$. Then if $n \ge n_1$, there is a coupling of γ_n with Brownian motion $B(t)$, $t \ge 0$ with the property that if $\tilde{\gamma}$ denotes the chordal SLE_6 path in $\mathbb{H}$,*

$$\mathbb{P}\left\{ \sup_{t \in [0,T]} |\tilde{\gamma}_n(t) - \tilde{\gamma}(t)\,| \; > n^{-u} \right\} < n^{-u}$$

for some $u \in (0,1)$ and where both curves are parameterized by capacity.

Moreover, if Ω is an α-Hölder domain, then under the same coupling, the SLE curve in the image is *polynomially close to the original discrete curve:*

$$\mathbb{P}\left\{\sup_{t\in[0,T]} d_*\left(\gamma^n(t), \phi^{-1}(\tilde{\gamma}(t))\right) > n^{-v}\right\} < n^{-v}$$

where v depends only on α and u.

Remark 4.1.12. The authors believe that modifications of the arguments in [9] could lead to a full convergence statement.

Remark 4.1.13. Notice that under this modified percolation model, we still maintain the reversibility of the exploration path. Let ω be a simple polygonal path from a^δ to c^δ. Suppose that the corresponding *path designate* is the sequence

$$[H_{0,1}, (\mathcal{F}_1, h_1^e, h_1^x), H_{1,2}, (\mathcal{F}_2, h_2^e, h_2^x), H_{2,3}, \cdots, (\mathcal{F}_K, h_K^e, h_K^x), H_{K,K+1}]$$

where $\mathcal{F}_1, \cdots \mathcal{F}_K$ are flowers in Ω^δ with h_j^e and h_j^x are the entrance and exit petals in the j^{th} flower and for $1 \le j \le K-1$, $H_{j,j+1}$ is a path in the complement of flowers which connects h_j^x to h_{j+1}^e. That is, we are not viewing the microscopic description where we have to specifying how the path got between entry and exit petals. With a small loss of generality we are also assuming that the path only visits the flower once else we would have to specify the first entrance and exit petals, the second entrance and exit petals, etc.

Let γ^δ be a chordal exploration process from a^δ to c^δ in Ω^δ and $\hat{\gamma}^\delta$ be a chordal exploration process from c^δ to a^δ in Ω^δ. Recall that all petal arrangements are independent, all flowers are configured independently and these in turn are independent of the background filler sites. Thus the exploration process generated by the colouring algorithm given previously, excluding colouring of flowers, is independent and flowers are independent of background filler sites. Thus, by the colouring algorithm we have:

$$\mathbb{P}(\gamma^\delta = \omega) = \left(\frac{1}{2}\right)^{l(H_{0,1})} p_1 \left(\frac{1}{2}\right)^{l(H_{1,2})} \cdots p_K \left(\frac{1}{2}\right)^{l(H_{K,K+1})}$$

where $l(H_{j,j+1})$ is the number of coloured hexagons in $H_{j,j+1}$ produced by the colouring algorithm on the event $\gamma^\delta = \omega$ and p_j is the appropriate conditional probabilities on each flower of a petal or iris given by the colouring algorithm. Notice that on the event $\gamma^\delta = \hat{\gamma}^\delta = \omega$ for any hexagon in Ω^δ either it is coloured by both the colouring algorithm for γ^δ and the colouring algorithm for $\hat{\gamma}^\delta$ or by neither. Therefore, we have the following lemma:

Lemma 4.1.14. *Suppose Ω^δ is a simply connected domain in the δ-hexagonal lattice with a predetermined* *flower arrangement. For any simple polygonal path ω from a^δ to c^δ we have*

$$\mathbb{P}(\gamma^\delta = \omega) = \mathbb{P}(\hat{\gamma}^\delta = \omega)$$

This lemma directly implies the following lemma.

Lemma 4.1.15. *For any simply connected domain Ω^δ with predetermined flower arrangement, the percolation exploration path from a^δ to c^δ in Ω^δ has the same distribution as the time-reversal of the percolation exploration path from c^δ to a^δ in Ω^δ.*

Question 4.1.16. *Is it possible to use reversibility to extend the polynomial convergence for the whole curve percolation exploration process?*

4.2. Approximate harmonicity

The polynomial convergence of the functions H_{Ω^δ} to the solution of the continuous Dirichlet problem uniformly up to a thin $\delta^{1-\eta}$-strip away from the boundary on s-embeddings, as well as isoradial graphs, is a new result presented in [12, §4.1]. This is a needed improvement from convergence results in the bulk of Ω^δ, i.e. $O(1)$ away from the boundary $\partial\Omega^\delta$. We prove a similar statement to [12, Theorem 4.1] with a more general object. However, it becomes weaker in terms of the lattice. In the proof of the theorem, we will apply the following standard result with $\Omega' = \Omega^\delta_{\mathrm{int}(\eta)}$ and $\Omega = \Omega^\delta$, considered as subsets of $\mathbb{C}$.

Lemma 4.2.1. *[12, Lemma A.2] Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain, $\alpha \in (0, 2]$, and a function $h \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy the Dirichlet boundary conditions $h = 0$ at $\partial\Omega$. If $|\Delta h(u)| \leq d_u^{-2+\alpha}$ for all $u \in \Omega$, then*

$$|h(w)| \leq \mathrm{cst}(\alpha) \cdot d_w^{\frac{1}{5}\alpha} \cdot (\mathrm{diam}(\Omega))^{\frac{4}{5}\alpha} \quad \textit{for all} \quad w \in \Omega,$$

where $\mathrm{cst}(\alpha)) > 0$ *depend only on* α.

Theorem 4.2.2. *Let $\Omega^\delta \subset \mathbb{C}$ be a bounded simply connected discrete domain drawn on an isoradial graph. Assume that H_{Ω^δ} is an "almost" harmonic function in (the bulk of) Ω^δ, that is,*

$$|\Delta^\delta H_{\Omega^\delta}|(v) = O\left((\delta/d_u)^{2\beta} d_u^{-1}\right) \quad \textit{with} \quad \beta > \tfrac{1}{2}, \tag{4.2.1}$$

and that $|H_{\Omega^\delta}| \leq 1$ in Ω^δ.

Let $\eta \in (0, 1)$ and $\Omega^\delta_{\mathrm{int}(\eta)} \subset \mathbb{C}$ be (one of the connected components of) the $\delta^{1-\eta}$-interior of Ω^δ. Denote by $h_{\mathrm{int}(\eta)}$ the harmonic continuation of the function H_{Ω^δ} from the boundary to the bulk of the *domain* $\Omega^\delta_{\mathrm{int}(\eta)}$ *(i.e.,* $h_{\mathrm{int}(\eta)}$ *is the solution of the* continuous *Dirichlet problem in* $\Omega^\delta_{\mathrm{int}(\eta)}$ *with the boundary values given by* H_{Ω^δ}*).*

Then, there exists an exponent $\alpha(\eta) > 0$ *such that, provided that* δ *is small enough (depending only on* η*), the following estimate holds:*

$$|H_{\Omega^\delta} - h_{\mathrm{int}(\eta)}| \;=\; O(\delta^{\alpha(\eta)}) \quad \textit{uniformly in } \Omega^\delta_{\mathrm{int}(\eta)}, \tag{4.2.2}$$

where the implicit constant in the O-estimate depends only on the diameter of Ω^δ*.*

Proof. Let $\phi_0 \in C_0^\infty(\mathbb{C})$ be a fixed positive symmetric function with $\phi_0(u) = 0$ for $|u| \geq \frac{1}{2}$ and $\int_{\mathbb{C}} \phi_0(w)dA(w) = 1$. Let $0 < \epsilon \ll \eta$ be a small parameter which will be chosen later. Define

$$d_u := \mathrm{dist}(u, \partial\Omega^\delta) \text{ and } \rho_u := \delta^\epsilon \,\mathrm{crad}(u, \Omega^\delta) \asymp \delta^\epsilon d_u \gg \delta \text{ for } u \in \Omega^\delta_{\mathrm{int}(\eta)}$$

where crad is the conformal radius of u in Ω^δ. Recall that the mapping $u \mapsto \mathrm{crad}(u, \Omega^\delta)$ is smooth with gradient uniformly bounded and second derivative bounded by $O(d_u^{-1})$ with absolute constants which are independent of u and Ω^δ, see [12, Lemma A.1].

Define the running mollifier $\phi(w, u)$ as

$$\phi(w, u) := \rho_u^{-2}\phi_0(\rho_u^{-1}(w - u)), \quad u \in \Omega^\delta_{\mathrm{int}(\eta)},$$

and mollify the function H_{Ω^δ} as follows

$$\widetilde{H}_{\Omega^\delta}(u) := \int_{B(u,\rho_u)} \phi(w, u) H_{\Omega^\delta}(w) dA(w), \tag{4.2.3}$$

where H_{Ω^δ} is continued to $\Omega^\delta_{\mathrm{int}(\eta)}$ in a piecewise constant way. By the a priori regularity of the functions H_{Ω^δ}, we get that

$$|\widetilde{H}_{\Omega^\delta}(u) - H_{\Omega^\delta}(u)| = O(\rho_u \cdot d_u^{-1}) = O(\delta^\epsilon) \quad \text{uniformly in } \Omega^\delta_{\mathrm{int}(\eta)}. \tag{4.2.4}$$

In order to apply Lemma 4.2.1, we need a uniform estimate for $\Delta\widetilde{H}_{\Omega^\delta}$: That is,

Claim.

$$|\Delta\widetilde{H}_{\Omega^\delta}(u)| = O(\delta^p d_u^{-2-q}) + O(\delta^{1-s} d_u^{-3}) \quad \text{for all } u \in \Omega^\delta_{\mathrm{int}(\eta)} \tag{4.2.5}$$

where the exponents $p, q, s \geq 0$ satisfy $p \geq q(1-\eta)$ and $s < \eta$.

Indeed, since $\phi(\cdot, u)$ vanishes near the boundary of $B(u, \rho_u)$

$$\Delta \widetilde{H}_{\Omega^\delta}(u) = \int_{B(u,\rho_u)} (\Delta_u \phi(w,u)) H_{\Omega^\delta}(w) dA(w) \quad u \in \Omega^\delta_{\mathrm{int}(\eta)}$$

Thus, we can write

$$\Delta \widetilde{H}_{\Omega^\delta}(u) = \underbrace{\int H_{\Omega^\delta}(w) \Delta_w \phi(w,u) dA(w)}_{2} - \underbrace{\int H_{\Omega^\delta}(u) [\Delta_u - \Delta_w](\phi(w,u)) dA(u)}_{1}$$

and we will estimate 1 and 2 separately.

Estimate 1. We start with noting that H_{Ω^δ} is a linear function in the ball $B_{\rho_u}(u)$ up to error $O(\rho_u \cdot (\rho_u/d_u)^\beta d_u^{-1}) = O(\delta^{\epsilon(1+\beta)})$ where $\beta > 0$ comes from the a priori Hölder exponent of $\|\nabla H_{\Omega^\delta}\|$. We use that $\|\nabla H_{\Omega^\delta}\| \le \frac{|H_{\Omega^\delta}|}{d}$ and ∇H_{Ω^δ} is constant up to $\left(\frac{\rho}{d}\right)^\beta \cdot \|\nabla H_{\Omega^\delta}\|$. Notice that for a symmetric mollifier ϕ_0,

$$\begin{aligned} \textstyle\int_{B(u,\rho_u)} (\Delta_w \phi(w,u)) L(w) dA(w) &= \textstyle\int_{B(u,\rho_u)} \phi(w,u)(\Delta L)(w) dA(w) = 0, \\ \textstyle\int_{B(u,\rho_u)} (\Delta_u \phi(w,u)) L(w) dA(w) &= \Delta L(u) = 0. \end{aligned}$$

By a straightforward computation we get that

$$|\Delta_w \phi(w,u) - \Delta_u \phi(w,u)| = O(\delta^\varepsilon \cdot \rho_u^{-4}),$$

where we use that $u \mapsto \rho_u$ is smooth with gradient of order $O(\delta^\epsilon)$, second derivatives of order $O(\delta^\epsilon d_u^{-1}) = O(\delta^{2\epsilon} \rho_u^{-1})$, and the additional factor δ^ϵ comes from the differentiation of ρ_u. Thus, this is gives a total error of $O(\delta^\epsilon \delta^{\epsilon(1+\beta)} \rho_u^{-2})$. So, we have

$$\Delta \widetilde{H}_{\Omega^\delta}(u) = \int_{B(u,\rho_u)} (\Delta_w \phi(w,u)) H_{\Omega^\delta}(w) dA(w) + O(\delta^\epsilon \delta^{\epsilon(1+\beta)} \rho_u^{-2}).$$

Estimate 2. Cover $B(u, \rho_u)$ by squares of size $\delta^{1-\gamma} \ll \delta^{1-\eta} \le \rho_u$ where γ is chosen so that $\gamma > 2\epsilon$ and $3(\gamma + \epsilon) < \eta$. (This is possible if ϵ is chosen to be less than $\frac{1}{9}\eta$.) Notice that on each square H_{Ω^δ}

can be approximated by a constant up to error $O(\delta^{1-\gamma}d_u^{-1})$. Further, $\Delta_w\phi(w,u)$ is a constant on each square up to error $O(\delta^3|D^3\phi(w,u)|)$. Thus, we can replace $\Delta_w\phi(w,u)$ with $\Delta^\delta\phi(w,u)$. Since it contains $O(\delta^{-2}\rho_u^2)$ terms of order $O(\delta^3|D^3\phi(w,u)|)$, we get a total error of $O(\delta^{1-\gamma}d_u^{-1}\rho_u^{-2}) = O(d^{1-\gamma-2\epsilon}d_u^{-3})$. More precisely, we obtain

$$\int_{B(u,\rho_u)}(\Delta_w\phi(w,u))H_{\Omega^\delta}(w)dA(w) = \sum_{v:\,\Lambda(v)\in B(u,\rho_u)}\left[\Delta^\delta\phi(\cdot,u)\right](v)H_{\Omega^\delta}(v)\delta^2 + O(\delta^{1-\gamma-2\epsilon}d_u^{-3}).$$

By the "discrete integration by parts" formula,

$$\int_{B(u,\rho_u)}(\Delta_w\phi(w,u))H_{\Omega^\delta}(w)dA(w) = \sum_{v:\,\Lambda(v)\in B(u,\rho_u)}\phi(v,u)\Delta^\delta H_{\Omega^\delta}(v)\delta^2 + O(\delta^{1-\gamma-2\epsilon}d_u^{-3})$$

Thus, we get that

$$\Delta\widetilde{H}_{\Omega^\delta}(u) = O(\delta^{-2}\rho_u^2\cdot\delta\cdot\rho_u^{-2}\cdot\delta^{2\beta}d_u^{-1-2\beta}) = O(\delta^{2\beta-1}d_u^{-1-2\beta})$$

where we use our assumption that $|\Delta^\delta H_{\Omega^\delta}|(v) = O\left((\delta/d_u)^{2\beta}d_u^{-1}\right)$. The proof of the claim follows by choosing $\epsilon = \min\{\frac{1}{9},\frac{\beta}{2(1+\beta)}\}\cdot\eta$ where β is the Hölder exponent of $\|\nabla H_{\Omega^\delta}\|$.

Thus, since we have $d_u \geq \delta^{1-\eta}$ for all $w\in\Omega^\delta_{\text{int}(\eta)}$, one can find $\alpha = \alpha(\eta,p,q,s) > 0$ such that

$$|\Delta\widetilde{H}_{\Omega^\delta}(u)| = O(\delta^\alpha d_u^{-2+\alpha})\quad\text{uniformly for } u\in\Omega^\delta_{\text{int}(\eta)}.$$

Let $\widetilde{h}$ be the harmonic continuation of H_{Ω^δ} from $\partial\Omega^\delta_{\text{int}(\eta)}$ to $\Omega^\delta_{\text{int}(\eta)}$, that is, the continuous solution to the Dirichlet problem in $\Omega^\delta_{\text{int}(\eta)}$ with boundary value $\widetilde{H}_{\Omega^\delta}$. By applying Lemma 4.2.1 to $\widetilde{H}_{\Omega^\delta}-\widetilde{h}$ in the $\delta^{1-\eta}$ interior $\Omega^\delta_{\text{int}(\eta)}$, one can derive from (4.2) that

$$|\widetilde{H}_{\Omega^\delta}-\widetilde{h}| = O(\delta^\alpha)\quad\text{uniformly in } \Omega^\delta_{\text{int}(\eta)}$$

where the additional coefficient depends only on α and $\operatorname{diam}\Omega^\delta$. By the maximum principle, the estimate (4.2.4) also implies that $|\widetilde{h}-h| = O(\delta^\varepsilon)$ in $\Omega^\delta_{\text{int}(\eta)}$. Combining this with (4.2.4) and (4.2) we obtain the desired estimate (4.2.2) with the exponent $\min\{\alpha,\varepsilon\} > 0$.

□

At this point, we have a polynomial $O(\delta^{\beta/8})$ convergence of H to a harmonic function h_m in the bulk of Ω^δ.

4.3. HARMONIC EXPLORER

4.3.1. SETUP AND MAIN RESULTS

Throughout this paper we will work with the *isoradial lattice* or equivalently *rhombic lattices*.

Definition 4.3.1. A planar graph Γ embedded in $\mathbb{C}$ is called **δ-isoradial** if each face is inscribed into a circle of a common radius δ. If all the circle centers are inside the corresponding faces, then the dual graph Γ^* can be embedded in $\mathbb{C}$ isoradially with the same δ by taking the circle centers as vertices of Γ^*.

The corresponding bipartite graph $\Lambda = (V(\Lambda) = \Gamma \cup \Gamma^*, E(\Lambda) = \text{radii of the circles})$, called **rhombic lattice**, have rhombi faces with sides of length δ. The set of rhombi centers is denoted by $\diamondsuit$ with the mild assumption that the rhombi angles are uniformly bounded away from 0 and π.

4.3.2. DEFINITIONS AND PRELIMINARIES

Let $\Omega^\delta_\Gamma \subset \Gamma$ be a simply connected domain with $\partial\Omega^\delta_\Gamma$ a simple closed curve. Let $V_0 := V(\Gamma) \cap \partial\Omega^\delta_\Gamma$ be the set of vertices in $\partial\Omega^\delta_\Gamma$. Let $\overline{v}_0, \overline{v}_{\text{end}}$ be the centers of two distinct edges of Γ on $\partial\Omega^\delta_\Gamma$ and A_0^+ (respectively A_0^-) be the corresponding positively (respectively negative) oriented arc on $\partial\Omega^\delta_\Gamma$ from $\overline{v}_0, \overline{v}_{\text{end}}$. Define $h_0 : V_0 \to \{0, 1\}$ to be 1 on $V_0 \cap A_0^+$ and 0 on $V_0 \cap A_0^-$. The following lemma shows we can generate the discrete harmonic extension of h which is the expected value of h at the point at with a simple random walk started at v hits V_0.

Lemma 4.3.2. *[10] For any planar graph H embedded in $\mathbb{C}$ (not necessarily isoradial), and A^+, A^- are two connected curves consisting of edges and vertices of H such that $A^+ \cup A^-$ is a simple closed curve and $A^+ \cap A^- \cap V(H) = \emptyset$. Define $h : (A^+ \cup A^-) \cap V(H) \to \{0, 1\}$ by $h|_{V(H)\cap A^+} \equiv 1$ and $h|_{V(H)\cap A^-} \equiv 0$. If we assign any probabilities to go from one vertex to one of its neighbors, the the harmonic extention $\overline{h}$ of h induced by such probabilities is defined on any $v \in V(H)$, starting from which the random walk induced by such probabilities a.s. hits $A^+ \cup A^-$ within finite steps. $\overline{h}(v)$ is defined to be the expected value of h at the point at which the random walk starting from v first hits $A^+ \cup A^-$. Then we have*

$$1 = \overline{h}(u_1) \geq \overline{h}(u_2) \geq \cdots \geq \overline{h}(u_n) = 0, \tag{4.3.1}$$

where $u_1 \in A^+, u_n \in A^-$ are connected by an edge, and $u_1, u_2, \cdots, u_n$ are the vertices, ordered clockwise or counterclockwise, of the face $f \in F(H)$ inside the domain bounded by the curve $A^+ \cup A^-$.

Proof. For any $1 \leq i \leq n-1$, we will show that $\overline{h}(u_i) \geq \overline{h}(u_{i+1})$ via coupling. For $k = i$ or $i+1$, let

$(\Omega_k, \mathbb{P}(\Omega_k), \mu_k)$ denote the probability space with Ω_k defined as the set of curves along $\bigcup E(H) \cup V(H)$ starting from u_k and ending at its first encounter with $A^+ \cup A^-$, and μ_k induced by the random walk. Let $T_k : \Omega_k \to \mathbb{R}$ be the random variable which gives 1 if the curve ends on A^+ and 0 if the curve ends on A^-. Then $\overline{h}(u_k) = \mathbb{E}(T_k)$. Now, consider the larger probability space $(\Omega, \mathbb{P}(\Omega), \mu)$ where we will perform the coupling. Let Ω be the pairs of curves $(\lambda_i, \lambda_{i+1}) \in \Omega_i \times \Omega_{i+1}$ where if λ_i and λ_{i+1} intersect at a vertex v (after v they will coincide on all steps). Then define $\mu(\lambda_i, \lambda_{i+1})$ to be the product of $\mu_i(\lambda_i)$ with the probability induced by the random walk of the part of λ_{i+1} from u_{i+1} to the first vertex at which λ_i and λ_{i+1} meet else $\mu_i(\lambda_i) \times \mu_{i+1}(\lambda_{i+1})$. Let $\widetilde{T}_i, \widetilde{T}_{i+1}$ be the random variables on $(\Omega, \mathbb{P}(\Omega), \mu)$ such that $\widetilde{T}_i(\lambda_i, \lambda_{i+1}) = T_i(\lambda_i), \widetilde{T}_{i+1}(\lambda_i, \lambda_{i+1}) = T_{i+1}(\lambda_{i+1})$.

Let $\pi_k : \Omega \to \Omega_k$ be the projections for $k = i, i+1$. Then we have that $\mu(\pi_k^{-1}(\lambda_k)) = \mu_k(\lambda_k)$ for $\lambda_k \in \Omega_k$. Hence, $\widetilde{T}_k$ and T_k have the same distributions. By topological considerations we can see that for each $(\lambda_i, \lambda_{i+1}) \in \Omega$, the event that λ_{i+1} ends on A^+ implies the event that λ_i ends on A^+. Therefore $\overline{h}(u_i) = \mathbb{E}[T_i] = \mathbb{E}[\widetilde{T}_i] \geq \mathbb{E}[\widetilde{T}_{i+1}] = \mathbb{E}[T_{i+1}] = \overline{h}(u_{i+1})$. □

The *Harmonic Explorer* is a random interface generated dynamically as follows: begin the path γ at an edge separating the left and right boundary components; when γ hits a black face, it turns right, and when it hits a white face, it turns left. Each time it hits a face f whose colour has yet to be determines, we perform a simple random walk on the space of faces beginning at f and let f assume the colour of the first black or white face hit by that walk. In other words, we colour black with probability equal to the value at f of the function which is equal to 1 on the black faces and 0 on the white faces and is discrete harmonic at the undetermined faces. That is, the harmonic explorer $H_{(\Omega_\Gamma^\delta, \overline{v}_0, \overline{v}_{\text{end}})}$ is a random simple path from $\overline{v}_0$ to $\overline{v}_{\text{end}}$ in $\overline{\Omega_\Gamma^\delta}$ defined as follows. Let $X_1, X_2, \cdots$ be i.i.d. random variables chosen uniformly in the interval $[0, 1]$. These are the coin flips we will use to define the model. Let $f_0 \subset \Omega_\Gamma^\delta$ be the face of the isoradial graph Γ whose boundary contains $\overline{v}_0$. By Lemma 4.3.2, there is a unique pair of consecutive vertices u_k and u_{k+1} of the face f_0 with $\overline{v}_0 \in \overline{f}_0$ such that

$$\begin{aligned} 1 = \overline{h}_0(u_n) \geq \cdots \geq \overline{h}_0(u_{k+2}) \geq \overline{h}_0(u_{k+1}) \\ \geq X_1 \geq \overline{h}_0(u_k) \geq \overline{h}_0(u_{k-1}) \geq \cdots \geq \overline{h}_0(u_1) = 0 \end{aligned}$$

Define $V_1 := V_0 \backslash \{u_1, \cdots, u_n\}$ and $h_1 : V_1 \to \mathbb{R}$ by $h_1|_{V_0} = h_0$ and $h_1(u_n) = \cdots = h_1(u_{k+1}) = 1$, $h_1(u_k) = \cdots = h_1(u_1) = 0$. We let $\overline{v}_1 := \frac{1}{2}(u_k + u_{k+1})$. This defines the first step of the harmonic explorer. If $\overline{v}_1 \neq \overline{v}_{\text{end}}$, define the subgraph Ω_1^δ of Ω_Γ^δ such that $\text{int}(\Omega_1^\delta)$ is the connected component whose boundary contains $\overline{v}_1$ and $\text{int}(\Omega_1^\delta) = \text{int}\left(\Omega_\Gamma^\delta \backslash \text{face } u_0 u_1 \cdots u_n\right)$. The process continues inductively.

Assuming $i \geq 1$ and $\overline{v}_i \notin \partial\Omega_\Gamma^\delta$. Let f_i be the face of Γ containing $\overline{v}_i$ but not $\overline{v}_{i+1}$. Then by Lemma 4.3.2 again, there is a unique pair of consecutive vertices u_k and u_{k+1} of the face f_i with $\overline{v}_i \in \overline{f}_i$ such that

$$\begin{aligned} 1 = \overline{h}_i(u_n) \geq \cdots \geq \overline{h}_i(u_{k+2}) \geq \overline{h}_i(u_{k+1}) \\ \geq X_{i+1} \geq \overline{h}_i(u_k) \geq \overline{h}_i(u_{k-1}) \geq \cdots \geq \overline{h}_i(u_1) = 0 \end{aligned}$$

Then define $V_{i+1} := V_i \cup \{u_1, u_2, \cdots, u_n\}$ and $h_{i+1} : V_{i+1} \to \mathbb{R}$ by $h_{i+1}|_{V_i} = h_i$, $h_{i+1}(u_n) = \cdots h_{i+1}(u_{k+2}) = h_{i+1}(u_{k+1}) = 1$, $h_{i+1}(u_k) = \cdots h_{i+1}(u_2) = h_{i+1}(u_1) = 0$. Let $\overline{v}_{i+1} : \frac{1}{2}(u_k + u_{k+1})$. If $\overline{v}_{i+1} \neq \overline{v}_{\text{end}}$, define the subgraph Ω_{i+1}^δ of Ω_i^δ such that $\text{int}(\Omega_{i+1}^\delta)$ is the connected component, whose boundary contains $\overline{v}_{i+1}$, of $\text{int}(\Omega_i^\delta \setminus \text{face } u_0u_1\cdots u_n)$.

One can easily verify that this procedure terminates when $\overline{v}_n = \overline{v}_{\text{end}}$ for some $n < \infty$. Let N denote the termination time, i.e. $\overline{v}_N = \overline{v}_{\text{end}}$. We define the *Harmonic Explorer* on isoradial lattices to be the random curve from $\overline{v}_0$ to $\overline{v}_{\text{end}}$, connected by line segments from $\overline{v}_i$ and $\overline{v}_{i+1}$ to the center of the circumscribed circle of the face $f \in F(\Omega_\Gamma^\delta)$ with $\overline{v}_i, \overline{v}_{i+1} \in \overline{f}$, for $0 \leq i \leq N-1$. Thus, the Harmonic Explorer is a simple path from $\overline{v}_0$ to $\overline{v}_{\text{end}}$.

Lemma 4.3.3. *[10] Recall the harmonic extension $\overline{h}_n$ of h_n defined above. For any $v \in V(\Omega_\Gamma^\delta)$, $\overline{h}_n(v)$ is a martingale and $\overline{h}_N(v) \in \{0, 1\}$.*

Proof. Given $X_1, X_2, \cdots, X_n$. Let $u_1, u_2, \cdots u_m$ be the vertices of the face $f \in F(\Omega_\Gamma^\delta)$ such that $\overline{v}_n \in \overline{f}$. Then $\overline{h}_{n+1}(u_i) = 1$ with probability $\overline{h}_n(u_i)$ and $\overline{h}_{n+1}(u_i) = 0$ with probability $1 - \overline{h}_n(u_i)$, for $1 \leq i \leq m$. So

$$E[\overline{h}_{n+1}(u_i) | X_1, \cdots, X_n] = \overline{h}_n(u_i)$$

for $1 \leq i \leq m$. Thus by induction,

$$E[\overline{h}_{n+1}(v) | X_1, \cdots, X_n] = \overline{h}_n(v)$$

for $v \in V_{n+1}$. Since $\overline{h}_n$ is the harmonic extension of $\overline{h}_n|_{V_{n+1}}$, and $\overline{h}_{n+1}$ is the harmonic extension of $\overline{h}_{n+1}|_{V_{n+1}}$, and the fact that the value of the harmonic extension of any function on $v \in V(\Omega_\Gamma^\delta)$ is a linear combination, which only dependents on v, of the value of the function on $\partial\Omega_\Gamma^\delta \cap V(\Omega_\Gamma^\delta)$, we get that

$$E[\overline{h}_{n+1}(v) | X_1, \cdots, X_n] = \overline{h}_n(v)$$

for $v \in V(\Omega_\Gamma^\delta)$. Therefore $\overline{h}_n(v)$ is a martingale for $v \in V(\Omega_\Gamma^\delta)$. Then as A_N^+ and A_N^- are closed simple curves, we get $\overline{h}_N(v) \in \{0,1\}$. □

4.3.3. Harmonic explorer satisfies the KS condition

The result that the harmonic explorer satisfies the KS Condition for the hexagonal lattice appears in [53, Proposition 6.3]. Indeed, it is written in general provided the lattice has a weak Beurling-type estimate. We need the following estimate which is a discrete version of the classical Beurling estimate with the exponent $\frac{1}{2}$ replaced by some small positive constant β, see [14, Proposition 2.11].

Proposition 4.3.4 (Weak Beurling-type Estimates)**.** *There exists an absolute constant $\beta > 0$ such that for any simply connected discrete domain Ω_Γ^δ, point $u \in \operatorname{Int}\Omega_\Gamma^\delta$ and some part of the boundary $E \subset \Omega_\Gamma^\delta$ one has that the discrete harmonic measure $\omega^\delta(u; E; \Omega_\Gamma^\delta)$ satisfies the following bound*

$$\omega^\delta(u; E; \Omega_\Gamma^\delta) \leq \text{const}\left[\frac{\operatorname{dist}(u; \partial\Omega_\Gamma^\delta)}{\operatorname{dist}_{\Omega_\Gamma^\delta}(u; E)}\right]^\beta$$

where $\operatorname{dist}_{\Omega_\Gamma^\delta}$ *denotes the distance inside* Ω_Γ^δ.

Suppose that Ω_Γ^δ is an admissible domain and generate $\overline{h}_{\Omega_n^\delta}(v)$ via the harmonic explorer process defined previously. Recall that this model has the special property that the values of the harmonic functions $M_n = \overline{h}_{\Omega_n^\delta}(v)$ for fixed $v \in \Omega_n^\delta$ but varying $\Omega_n^\delta = \Omega_\Gamma^\delta \backslash \gamma(0, n]$ is a martingale with respect to the σ-algebra generated by the coin flips (equivalently by the domains (Ω_n^δ) or the curves), see Lemma 4.3.3. Due to this the harmonic observables

$$\left(\overline{h}_{\Omega_n^\delta}(v)\right)_{v\in\Omega_\Gamma^\delta, n=0,1,\cdots,N}$$

provide a method to check the KS Condition. Indeed, we will give an idea of the proof. Let Ω_Γ^δ be an admissible domain. Consider the annulus $A = B(z_0, R)\backslash B(z_0, r)$. Let V_+ be the vertices in $A_0^+ \cap B(z_0, 3r)$ that are disconnected from $\overline{v}_{\text{end}}$ by

$$A^{\Omega^\delta} = \left\{ z \in \Omega_\Gamma^\delta \cap A \;\middle|\; \begin{array}{l} \text{the connected component of } z \in \Omega_\Gamma^\delta \cap A \\ \text{does not disconnect } \overline{v}_0 \text{ from } \overline{v}_{\text{end}} \in \Omega_\Gamma^\delta \end{array} \right\}$$

and let the corresponding part of A^{Ω^δ} be A_+^δ. Let $\widetilde{M}_n = \sum_{v\in V_+} \widetilde{h}_{\Omega_n^\delta}(v)$ where $\widetilde{h}_{\Omega_n^\delta}$, $v \in V_+$ is the harmonic measure of A_0^- seen from v and is the average value of $\overline{h}_{\Omega_n^\delta}$ among the neighbours of v. By Lemma 4.3.3, the key observation is that $(\widetilde{M}_n)$ is a martingale. Using the weak-Beurling estimate stated above, we get that $\widetilde{M}_0 = O\left(\left(\frac{r}{R}\right)^\Delta\right)$ for some $\Delta > 0$ and on the event of crossing one of the $A_+^{\Omega^\delta}$, it increases to $O(1)$. Thus, a martingale stopping argument gives that the probability of the crossing event is $O\left(\left(\frac{r}{R}\right)^{\Delta}\right)$.

Proposition 4.3.5. *[53, Proposition 6.3]. The family of harmonic explorers on isoradial graphs staisfies the KS Condition.*

4.3.4. Harmonic explorer observable convergence

Recall that the harmonic explorer $\overline{h}$ is by definition a discrete harmonic function. Thus, we immediately deduce from Theorem 4.2.2 the polynomial convergence of the harmonic explorer uniformly in $\Omega^{\delta}_{\mathrm{int}(\eta)}$.

Corollary 4.3.6. *Let $h^{\delta}_{\mathrm{int}(\eta)}$ be the solution to the continuous Dirichlet boundary value problem in $\Omega^{\delta}_{\mathrm{int}(\eta)}$ such that $h^{\delta} = \overline{h}_{\Omega^{\delta}}$ at $\partial\Omega^{\delta}_{\mathrm{int}(\eta)}$ where $\overline{h}$ is the harmonic function defined by the harmonic explorer process given above. Then there exists $\alpha(\eta) > 0$ such that, provided δ is small enough (depending only on η) so that the functions are uniformly (with respect to both Ω^{δ} and δ) close:*

$$|\overline{h}_{\Omega^{\delta}} - h^{\delta}_{\mathrm{int}(\eta)}| = O(\delta^{\alpha(\eta)}) \quad \textit{uniformly in } \Omega^{\delta}_{\mathrm{int}(\eta)}$$

where the implicit constant only depends on the diameter of Ω^{δ}.

Let $\Omega^{\delta}_{\Gamma} \subset \Gamma$ be a simply connected domain with $\partial\Omega^{\delta}$ a simple curve. Assume the setup and notation of Section 4.3.2. Let $\gamma : [0, N] \to \Omega^{\delta}_{\Gamma} \cup \{\overline{v}_0, \overline{v}_{\mathrm{end}}\}$ be the harmonic explorer path parameterized in such a way that $\gamma(n) = \overline{v}_n$ and proportional to arc length between $\gamma(n)$ and $\gamma(n+1)$ for $n \in \{0, 1, \cdots, N-1\}$. Let $\phi : \overline{\Omega^{\delta}} \to \mathbb{H} \cup \{\infty\}$ be a conformal map onto the upper half plane $\mathbb{H}$ such that $\phi(A_0^+) = (0, \infty)$, $\phi(A_0^-) = (-\infty, 0)$, $\phi(\overline{v}_0) = 0$, $\phi(\overline{v}_{\mathrm{end}}) = \infty$. Note that ϕ is unique only up to positive rescaling. Let $p_0 = \phi^{-1}(i)$.

As in [53], when considering the limiting properties, instead of letting $\delta \to 0$, we consider $\rho = \rho(\Omega^{\delta}_{\Gamma}, \phi) := \mathrm{dist}(p_0, \partial\Omega^{\delta}) \to \infty$. For $j \in [0, N]$, let $\widetilde{\Omega}_j := \Omega^{\delta}_{\Gamma} \backslash \gamma[0, n]$ and $\phi_j : \widetilde{\Omega}_j \to \mathbb{H}$ be the conformal map normalized by $\phi_j \circ \phi^{-1}(z) - z \to 0$ as $z \to \infty$ in $\mathbb{H}$. Now, we have uniform polynomial closeness of the approximation of $\overline{h}_j$ by a continuous harmonic function.

Lemma 4.3.7. *Given any $\epsilon \in (0, 1)$, there is an $r_0(\epsilon) > 1$ such that for every $u \in \Omega^{\delta}_{\Gamma}$, any $j < N$ and any $r > r_0(\epsilon)$ if $\mathrm{dist}(v, \partial\widetilde{\Omega}_j) > r > \epsilon\,\mathrm{diam}(\Omega^{\delta}_{\Gamma})$ then*

$$|\overline{h}_j(v) - \widetilde{h}(\phi_j(v) - W(t_j))| < O(r^{-\alpha}) \tag{4.3.2}$$

where $\widetilde{h}(z) = 1 - (1/\pi)\arg(z)$ and the implied constant only depends on the diameter of Ω^{δ}.

Proof. Observe that $\widetilde{h} : \mathbb{H} \to (0,1)$ is a harmonic function with boundary values 0 on $(-\infty, 0)$ and 1 on $(0, \infty)$. Since $W(t_j) = \phi_j(\gamma(j))$, $z \mapsto \widetilde{h}(\phi_j(z) - W(t_j))$ is harmonic in $\widetilde{\Omega}_j$, and has boundary values 0 on A_0^- and the "right side" of $\gamma[0, j]$ and 1 on A_0^+ and the "left side" of $\gamma[0, j]$.

Without loss of generality, we can assume $0 \in \Omega_\Gamma^\delta$. Assume $\text{dist}(v, \partial\widetilde{\Omega}_j) > r > \epsilon\text{diam}(\Omega_\Gamma^\delta)$. Then by scaling the coordinate system by a factor of r', we manage to make sure that in the new coordinate system, $\text{dist}(v, \partial\widetilde{\Omega}_j) > \delta^{1-\eta}$, $\delta = \frac{1}{r'}$ and $\Omega_\Gamma^\delta \subset B(0, R)$ for some constant R. So we can apply Corollary 4.3.6 to get that the function $\overline{h}_{\Omega^\delta}$ and h^δ are uniformly close to each other (with respect to Ω^δ and δ) in $\Omega^\delta_{\text{int}(\eta)}$. By the a priori Hölder estimates for the gradient of $\overline{h}_{\Omega^\delta}$, we have the following uniform estimate on $\partial\Omega^\delta_{\text{int}(\eta)}$:

$$h^\delta(u) \;=\; \overline{h}_{\Omega^\delta}(u) \;=\; O\left(\frac{\delta^\eta}{(\text{dist}_{\Omega^\delta}(u, A_0^-))^\eta}\right) \quad \text{if} \quad \underset{\Omega^\delta}{\text{dist}}(u, A_0^+) = \delta^{1-\eta}$$

and similarly

$$h^\delta(u) \;=\; \overline{h}_{\Omega^\delta}(u) \;=\; 1 - O\left(\frac{\delta^\eta}{(\text{dist}_{\Omega^\delta}(u, A_0^+))^\eta}\right) \quad \text{if} \quad \underset{\Omega^\delta}{\text{dist}}(u, A_0^-) = \delta^{1-\eta}.$$

Thus, by applying the Beurling estimate for *continuous* harmonic functions, one can deduce that h^δ and $\text{hm}_\Omega(\cdot, A_0^-)$ must be uniformly (in δ) close to 0 near the boundary arc A_0^+ and close to 1 near the complementary arc A_0^-. Hence we have convergence of h^δ to $\text{hm}_\Omega(\cdot, A_0^-)$ as $\delta \to 0$. We can apply this to $\widetilde{h}$ to get (4.3.2). □

4.3.5. Harmonic explorer convergence rate for paths

Recall the notation of the last section. Let $\Omega_\Gamma^\delta \subset \Gamma$ be a simply connected domain with $\partial\Omega^\delta$ a simple curve. Assume the setup and notation of Section 4.3.2. Let $\gamma : [0, N] \to \Omega_\Gamma^\delta \cup \{\overline{v}_0, \overline{v}_{\text{end}}\}$ be the harmonic explorer path parameterized in such a way that $\gamma(n) = \overline{v}_n$ and proportional to arc length between $\gamma(n)$ and $\gamma(n+1)$ for $n \in \{0, 1, \cdots, N-1\}$. Let $\phi : \overline{\Omega^\delta} \to \mathbb{H} \cup \{\infty\}$ be a conformal map onto the upper half plane $\mathbb{H}$ such that $\phi(A_0^+) = (0, \infty)$, $\phi(A_0^-) = (-\infty, 0)$, $\phi(\overline{v}_0) = 0$, $\phi(\overline{v}_{\text{end}}) = \infty$. Note that ϕ is unique only up to positive rescaling. Let $p_0 = \phi^{-1}(i)$. Let γ^ϕ be the path $\phi \circ \gamma$, parameterized by capacity from ∞ in $\overline{\mathbb{H}}$, and $\widetilde{\gamma}$ be the SLE_4 path in $\overline{\mathbb{H}}$.

Let $d_*(\cdot, \cdot)$ be the metric on $\overline{\mathbb{H}} \cup \{\infty\}$ defined by $d_*(z, w) = |\Psi(z) - \Psi(w)|$ where $\Psi(z) = \frac{z-i}{z+i}$ maps $\overline{\mathbb{H}} \cup \{\infty\}$ onto $\overline{\mathbb{D}}$. If $z \in \overline{\mathbb{H}}$ then $d_*(z_n, z) \to 0$ is equivalent to $|z_n - z| \to 0$ and $d_*(z_n, \infty) \to 0$ is equivalent to $|z_n| \to \infty$. This is a metric corresponding to mapping $(\mathbb{H}, 0, \infty)$ to $(\mathbb{D}, -1, 1)$ which is convenient because it is compact. By Proposition 4.3.5, Lemma 4.3.7, and Lemma 3.4.1, we can apply Theorem 1.5.3 to conclude that: As $\rho \to \infty$, the law of γ^ϕ converges polynomially to the law of the

SLE_4 path $\widetilde{\gamma}$, with respect to uniform convergence in the metric d_*.

Theorem 4.3.8. *For any $T \geq 1$ and $\epsilon \in (0,1)$ there is some $R = R(\epsilon, T)$ such that if $\rho > R$ then there is a coupling of γ^ϕ and $\widetilde{\gamma}$ such that*

$$\mathbb{P}\left\{\sup_{0\leq t\leq T} d_*(\gamma^\phi(t), \widetilde{\gamma}(t)) > R^{-u}\right\} < R^{-u}$$

for some $u \in (0,1)$ under the assumption $\rho > \epsilon \,\mathrm{diam}(\Omega_\Gamma^\delta)$.

Moreover, if Ω is an α-Hölder domain, then under the same coupling the SLE curve in the image is polynomially close to the original discrete curve:

$$\mathbb{P}\left\{\sup_{t\in[0,T]} d_*\left(\gamma(t), \phi^{-1}(\tilde{\gamma}(t))\right) > R^{-v}\right\} < R^{-v}$$

where v depends only on α and u.

4.4. Ising Model

4.4.1. Setup and main results

Throughout this paper we will work with the *isoradial lattice* or equivalently *rhombic lattices.* In this section, we review some of the basics of discrete complex analysis on isoradial graphs for more details see [14] and [15].

Definition 4.4.1. A planar graph Γ embedded in $\mathbb{C}$ is called **δ-isoradial** if each face is inscribed into a circle of a common radius δ. If all the circle centers are inside the corresponding faces, then the dual graph Γ^* can be embedded in $\mathbb{C}$ isoradially with the same δ by taking the circle centers as vertices of Γ^*.

The corresponding bipartite graph $\Lambda = (V(\Lambda) = \Gamma \cup \Gamma^*, E(\Lambda) = \text{radii of the circles})$, called **rhombic lattice**, have rhombi faces with sides of length δ. The set of rhombi centers is denoted by $\Diamond$ with the mild assumption that the rhombi angles are uniformly bounded away from 0 and π: that is, all these angles belong to $[\epsilon, \pi - \epsilon]$ for some fixed $\epsilon > 0$.

For a function H defined on vertices of Λ, the *discrete Laplacian* is defined as follows

$$[\Delta^\delta H](u); = \frac{1}{\mu_\Gamma^\delta(u)} \sum_{u_s\sim u} \tan\theta_s[H(u_s) - H(u)]$$

where $\mu_\Gamma^\delta(u) = \frac{1}{2}\delta^2 \sum_{u_s\sim u} \sin 2\theta_s$. For any $u \in \Gamma$, we enumerate counterclockwise the neighbours by

$1, 2, \cdots, s, s+1, \cdots, n$ and the corresponding dual neighbours $w_1, \cdots, w_s, w_{s+1}, \cdots, w_n$. Then $\mu_\Gamma^\delta(u)$ is the area of the polygon formed by $w_1 w_2 \cdots w_n$ with u as the center and θ_s is the angle formed by uw_s and uu_s. H is *discrete harmonic* in Ω_Γ^δ if $\Delta^\delta H = 0$ for all interior vertices of Ω_Γ^δ. As with the continuous case, the discrete harmonic functions satisfy (uniformly with respect to δ and the structure $\Diamond$) a version of the Harnack Lemma.

Proposition 4.4.2. *[15, Proposition A.4] Let $u_0 \in \Gamma$ and $H : \mathcal{B}_\Gamma^\delta(u_0, R) \to \mathbb{R}$ be a nonnegative discrete harmonic function. Then*

1. *for $u_2, u_2 \in \mathcal{B}_\Gamma^\delta(u_0, r) \subset \operatorname{Int} \mathcal{B}_\Gamma^\delta(u_0, R)$*

$$\exp\left[-\text{const} \cdot \frac{r}{R-r}\right] \leq \frac{H(u_2)}{H(u_1)} \leq \exp\left[\text{const} \cdot \frac{r}{R-r}\right]$$

2. *and for any $u_1 \sim u_0$*

$$|H(u_1) - H(u_0)| \leq \text{const} \cdot \delta \frac{H(u_0)}{R}.$$

The notion of s-holomorphicity appears naturally for holomorphic fermions in the Ising model, see [15, §2].

Definition 4.4.3. Let $\Omega_\Diamond^\delta \subset \Diamond$ be a discrete domain. The function $F : \Omega_\Diamond^\delta \to \mathbb{C}$ is *spin-holomorphic* or *s-holomorphic* if for each pair of neighbours $z_0, z_1 \in \Omega_\Diamond^\delta,\ z_0 \sim z$, the following projections of two values of F are equal:

$$\text{Proj}[F(z_0); [i(w_1 - u)]^{-\frac{1}{2}}] = \text{Proj}[F(z_1); [i(w_1 - u)]^{-\frac{1}{2}}]$$

or equivalently,

$$\overline{F(z_1)} - \overline{F(z_0)} = -i(w_1 - u)\delta^{-1} \cdot (F(z_1) - F(z_0))$$

where $(w_1 u),\ u \in \Gamma,\ w_1 \in \Gamma^*$ is the common edge of rhombi z_0, z_1.

Remark 4.4.4. The notion of s-holomorphic is stronger than the usual discrete holomorphicity, see [15, §3].

Given an s-holomorphic function F, there is an associated primitive

$$H_F = \int^\delta (F(z))^2 d^\delta z : \Omega_\Lambda^\delta \to \mathbb{C}$$

defined simultaneously on Γ and Γ^* (up to an additive constant) by:

$$H(u) - H(w_1) := 2\delta \cdot |\text{Proj}[F(z_j); [i(w_1 - u)]^{-\frac{1}{2}}]|^2,\ u \sim w_1$$

where $(uw_1),\ u \in \Gamma,\ w_1 \in \Gamma^*$ is the common edge of two neighbouring rhombi $z_0, z_1 \in \diamondsuit$ and for $j = 0, 1$ gives the same value.

Proposition 4.4.5. *[15, Proposition 3.6 and 3.11] Let $\Omega^\delta_\diamondsuit$ be a simply connected discrete domain. If $F : \Omega^\delta_\diamondsuit \to \mathbb{C}$ is an s-holomorphic function then*

1. *The function $H_F := \int^\delta (F(z))^2 d^\delta z : \Omega^\delta_\Lambda \to \mathbb{C}$ is well-defined up to an additive constant. H_F is discrete subharmonic on Γ and superharmonic on Γ^*.*

2. *For any neighbourhood $v_1, v_2 \in \Omega^\delta_\Gamma \subset \Gamma$ or $v_1, v_2 \in \Omega^\delta_{\Gamma^*} \subset \Gamma^*$ the identity*

$$H_F(v_2) - H_F(v_1) = \operatorname{Im}\left[(v_2 - v_1)\left(F(\frac{1}{2}(v_1 + v_2))\right)^2\right]$$

   holds.

3. *(Harnack Lemma.) Take $v_0 \in \Lambda = \Gamma \cup \Gamma^*$. If $H_F \geq 0$ in $\mathcal{B}^\delta_\Lambda(v_0, R)$ then*

$$H_F(v_1) \leq \text{const.} \cdot H_F(v_0) \quad \text{for any } v_1 \in \mathcal{B}^\delta_\Lambda\left(v_0, \frac{1}{2}r\right).$$

There is the following regularity of s-holomorphic functions, see [15, Theorem 3.12].

Theorem 4.4.6. *For a simply connected domain $\Omega^\delta_\diamondsuit$, s-holomorphic function F and associated primitive $H_F = \int^\delta (F(z))^2 d^\delta z : \Omega^\delta_\Lambda \to \mathbb{C}$ defined as above. Let $z_0 \in \operatorname{Int}\Omega^\delta_\diamondsuit$ be further than δ from the boundary: $d = \operatorname{dist}(z_0; \partial\Omega^\delta_\diamondsuit) \geq \text{constant} \cdot \delta$ and set $M = \max_{v \in \Omega^\delta_\diamondsuit} |H(v)|$. Then*

$$|F(z_0)| \leq \text{const} \cdot \frac{M^{\frac{1}{2}}}{d^{\frac{3}{2}}} \cdot \delta$$

and for neighbourhood $z_1 \sim z_0$

$$|F(z_1) - F(z_0)| \leq \text{const} \cdot \frac{M^{\frac{1}{2}}}{d^{\frac{3}{2}}} \cdot \delta.$$

Further, the subharmonic function $H_F|_\Gamma$ and superharmonic function $H_F|_{\Gamma^*}$ are uniformly close to each other in Ω^δ, $H_F|_\Gamma - H_F|_{\Gamma^*} = O\left(\delta \frac{M}{d}\right)$ and so $|\Delta^\delta H_F| = O\left(\delta \frac{M}{d^3}\right)$.

4.4.2. REVIEW OF THE MODEL

Suppose that $G = (V(G), E(G))$ is a finite graph, possibly a multigraph. For any $q > 0$ and $p \in (0, 1)$, define a probability measure on $\{0, 1\}^{E(G)}$ by

$$\mu_G^{p,q}(\omega) = \frac{1}{Z_G^{p,q}} \left(\frac{p}{1-p}\right)^{o(\omega)} q^{c(\omega)}$$

where $o(\omega)$ is the number of edges, $c(\omega)$ is the number of connected components in ω, and $Z_G^{p,q}$ is the normalizing constant which makes the measure a probability measure. This random edge configuration is called *the Fortuin–Kasteleyn model (FK).*

Let Ω be a bounded simply connected domain and $a, b \in \partial\Omega$ be two distinct boundary points (e.g. two degenerate prime ends). Approximate Ω by subgraphs of the δ-isoradial lattice successively refined as $\delta \to 0$. Let Ω^δ be such a simply connected approximation and a^δ, b^δ be two vertices near a, b on the bounardy $\partial\Omega^\delta$. Following in clockwise order, there are two arcs $(a^\delta b^\delta)$ and $(b^\delta a^\delta)$. Consider Dobrushin boundary conditions on $(\Omega^\delta; a^\delta, b^\delta)$: that is, free boundary conditions on $(a^\delta b^\delta)$ and wired on $(b^\delta a^\delta)$. It is easiest to view configurations together with their dual counterparts defined on dual graph G^*. The dual model is again the critical FK-Ising model with boundary conditions dual-wired on $(a^\delta b^\delta)$ and dual-free on $(b^\delta a^\delta)$. The γ^δ be the unique interface that separates the FK cluster on G connected with $(b^\delta a^\delta)$ and the FK cluster on G^* connected with $(a^\delta b^\delta)$.

LOOP REPRESENTATION OF THE MODEL ON ISORADIAL GRAPHS AND HOLOMORPHIC FERMION

Let $\Omega_\diamondsuit^\delta \subset \diamondsuit$ be a simply connected discrete domain composed of inner rhombi with $z \in \operatorname{Int}\Omega_\diamondsuit^\delta$ and boundary half rhombi $\zeta \in \partial\Omega_\diamondsuit^\delta$ with two marked boundary points a^δ, b^δ and Dobrushin boundary conditions: $\partial\Omega_\diamondsuit^\delta$ consists of a "white" arc $a_w^\delta b_w^\delta$, a "black" arc $b_b^\delta a_b^\delta$ and two edges $[a_b^\delta a_w^\delta][b_b^\delta b_w^\delta]$ of Λ. Without loss of generality, we can assume that $b_b^\delta - b_w^\delta = i\delta$. The set of configurations is obtained in the following way: For each innter rhombus $z \in \operatorname{Int}\Omega_\diamondsuit^\delta$, choose two possibilities to connect its sides (across the white vertices or across the black vertices). There is only one choice for boundary half-rhombic. Due to the boundary conditions, each configuration consists of loops and one interface γ^δ running from a^δ to b^δ. The *partition function* of the critical FK-Ising model is given by

$$Z = \sum_{\text{config.}} \sqrt{2}^{\#(\text{loops})} \prod_{z \in \operatorname{Int}\Omega_\diamondsuit^\delta} \sin \frac{1}{2}\theta_{\text{config}}(z)$$

where $\theta_{\text{config}}(z)$ is either θ if the connectors inside the rhombus z connect across the white vertices and $\theta^* = \frac{\pi}{2} - \theta$ else.

Definition 4.4.7. Let $\xi = [\xi_b \xi_w]$ be some inner edge of $\Omega^\delta_\diamondsuit$ where $\xi_b \in \Gamma, \xi_w \in \Gamma^*$. The *holomorphic fermion* is defined as

$$F^\delta(\xi) = F^\delta_{(\Omega^\delta;a^\delta,b^\delta)}(\xi) := (2\delta)^{-\frac{1}{2}}\mathbb{E}[\chi(\xi \in \gamma^\delta) \cdot e^{-\frac{i}{2}(\text{winding}(\gamma^\delta;b^\delta \rightsquigarrow \xi))}]$$

where $\chi(\xi \in \gamma^\delta)$ is the indicator function of the event that the interface intersects ξ and winding$(\gamma^\delta; b^\delta \rightsquigarrow \xi) =$ winding$(\gamma^\delta; a^\delta \rightsquigarrow \xi) -$ winding$(\gamma^\delta; a^\delta \rightsquigarrow b^\delta)$ is the total turn of γ^δ measured in radians from b^δ to ξ.

Since the winding winding$(\gamma^\delta; b^\delta \rightsquigarrow \xi)$ is independent of the beginning of the interface, we can immediately deduce the martingale property for F^δ.

Lemma 4.4.8 (Martingale Property)**.** *For each ξ, $F^\delta_{(\Omega^\delta\setminus[a^\delta\gamma^\delta_1,\cdots\gamma^\delta_j];\gamma^\delta_j,b^\delta)}(\xi)$ is a martingale with respect to the growing interface $(a^\delta = \gamma^\delta_0, \gamma^\delta_1, \cdots, \gamma^\delta_j, \cdots)$ up to stopping time when γ^δ hits ξ or ξ becomes separated from b^δ by the interface.*

Further, F^δ can be extended to the centers of rhombi $z \in \Omega^\delta_\diamondsuit$. The following proposition essentially gives that F^δ are *spin holomorphic.*

Proposition 4.4.9. *[15, Proposition 2.2] Let $z \in \operatorname{Int}\Omega^\delta_\diamondsuit$ be the center of some inner rhombus $u_1w_1u_2w_2$. Then, there exists a complex number $F^\delta(z)$ such that*

$$F^\delta([u_j w_k]) = \operatorname{Proj}\left[F^\delta(z); [i(w_k - u_j)] - \frac{1}{2}\right],\ j,k = 1,2.$$

4.4.3. FK Ising satisfies KS Condition

For each admissible domain $(\Omega; a, b)$ define a conformal and onto map $\phi_\Omega : (\Omega; a, b) \to (\mathbb{D}; -1, 1)$ where $\phi_\Omega(a) = -1$ and $\phi_\Omega(b) = 1$.

Proposition 4.4.10. *Let $\mathbb{P}_{\Omega^\delta}$ be the law of the critical FK model interface in Ω^δ, i.e. $\mathbb{P}_{\Omega^\delta}$ is the law of γ under $\mu^{p_{sd},2}_{\Omega^\delta}$. Then the family*

$$\Sigma_{\text{FKIsing}} = \{(\phi_{\Omega^\delta}, \mathbb{P}_{\Omega^\delta})\ :\ \delta \searrow 0\}$$

satisfies KS Condition.

Remark 4.4.11. In [28, Proposition 4.3 and 4.7], it is shown for the square lattice the critical FK model interface with parameter $q \geq 1$ satisfies KS Condition provided the statement in Corollary 4.4.13 holds.

In fact, KS Condition is verified for $1 \leq q \leq 4$ in [17] based on estimates on crossing probabilities. The proof follows similarly for isoradial graphs which we include for completeness of the exposition.

The following result on crossing probabilities is the main tool in showing that the FK model satisfies the KS Condition.

Theorem 4.4.12. *[15, Theorem C] Let discrete domains $(\Omega^\delta; a^\delta, b^\delta, c^\delta, d^\delta)$ with alternating (wired/free/wired/free) boundary conditions on four sides approximate some continuous topological quadrilateral $(\Omega; a, b, c, d)$ as $\delta \to 0$. Then the probability of an FK cluster crossing between two wired sides has a scaling limit, which depends only on the conformal modulus of the limiting quadrilateral and is given for the half-plane by*

$$p(\mathbb{H}; 0, 1-u, 1, \infty) = \frac{\sqrt{1-\sqrt{1-u}}}{\sqrt{1-\sqrt{u}} + \sqrt{1-\sqrt{1-u}}}, \quad u \in [0, 1].$$

Thus, we have the following consequence.

Corollary 4.4.13. *We say that $(\Omega; a, b, c, d)$ is* crossed by an open path *if there is an open path which connects the wired arcs, denote this event by $O(\Omega)$. If $(\Omega; a, b, c, d)$ is nondegenerate, then there are $\epsilon > 0$ and $0 < \delta_0 < \infty$ such that*

$$\epsilon < \mathbb{P}^\delta(O(\Omega^\delta)) < 1 - \epsilon$$

for any $0 < \delta < \delta_0$ where $\mathbb{P}^\delta$ is the probability measure $\mu_{\Omega^\delta}^{p_{sd}, 2}$.

Proof of KS Condition. Fix a Dobrushin domain $(\Omega_\Gamma^\delta; a^\delta, b^\delta)$ and consider the exploration path γ^δ in the loop representation on $\Omega_\diamondsuit^\delta$. Let $A = A(z, r, R)$ be an annulus. For stopping time τ, fix a realization of $\gamma^\delta[0, \tau]$ and consider slit Dobrushin domain $(\Omega_\Gamma^\delta \backslash \gamma^\delta[0, \tau]; c^\delta, b^\delta)$ where c^δ is the vertex of Γ bordered by the last edge of $\gamma^\delta[0, \tau]$. Let $A_\tau \subset \Omega_\diamondsuit^\delta$ be such that $A_\tau :=$ if $\partial B(z, r) \cap \partial(\Omega_\diamondsuit^\delta \backslash \gamma^\delta[0, \tau])$ and

$$A_\tau = \begin{Bmatrix} z \in A \cap (\Omega_\diamondsuit^\delta \backslash \gamma^\delta[0, \tau]) \text{ such that the connected component of } z \\ \text{in } A \cap (\Omega_\diamondsuit^\delta \backslash \gamma^\delta[0, \tau]) \text{ does not disconnect } \gamma^\tau \text{ from } b \in \Omega_\diamondsuit^\delta \backslash \gamma^\delta[0, \tau] \end{Bmatrix}.$$

Notice that there are three options for the connected component $\mathcal{C}$: (i) $\partial\mathcal{C}$ intersects both boundary arcs $\partial^\diamondsuit_{c^\delta b^\delta}$ and $\partial^\diamondsuit_{b^\delta c^\delta}$, (ii) $\partial\mathcal{C}$ intersects $\partial^\diamondsuit_{b^\delta c^\delta}$ but not $\partial^\diamondsuit_{c^\delta b^\delta}$, and (iii) $\partial\mathcal{C}$ intersects $\partial^\diamondsuit_{c^\delta b^\delta}$ but not $\partial^\diamondsuit_{b^\delta c^\delta}$. It is topologically impossible to have component $\mathcal{C}$ of type (i). Indeed, assume $\mathcal{C}$ exists. Let P be a self-avoiding path in $\mathcal{C}$ going from $\partial^\diamondsuit_{b^\delta c^\delta}$ to $\partial^\diamondsuit_{c^\delta b^\delta}$. Then $c_\diamondsuit^\delta$ and $b_\diamondsuit^\delta$ must be on two different sides of P in $(\Omega_\diamondsuit^\delta \backslash \gamma^\delta[0, \tau]) \backslash P$. Thus, $\mathcal{C}$ is not a part of A_τ which is a contradiction. So, it is safe to assume that $\mathcal{C}$ is either type (ii) or (iii). Now, consider the interpretation in terms of graphs.

Let S be a subgraph of $\Omega_\Gamma^\delta \backslash \gamma^\delta[0, \tau]$ composed of the union of connected components (viewed on isoradial graph) of type (ii). This is a subset of A_τ. Also, conditioned on $\gamma^\delta[0,\tau]$, the boundary conditions are wired on $\partial S \backslash \partial A_\tau$. Thus, conditioned on $\gamma^\delta[0,\tau]$ and the configuration outside $A(z,r,R)$, the configuration ω in S dominates the configuration $\omega'|_S$ where ω' follows the law of a random cluster model in $A(z,r,R)$ with free boundary conditions. So, if there is an open circuit in ω' surrounding $B(z,r)$ in $A(z,r,R)$ then the restriction of this path to S is also an open path in ω which disconnected $B(z,r)$ from $B(z,R)$ in S. Hence, since the exploration path γ^δ slides between open edges and dual-open dual-edges, for $\gamma^\delta[\tau,\infty]$ to cross A_τ inside S there would be a dual-open dual-path from outer to inner part of A^*_τ. Thus, $\gamma^\delta[\tau,\infty]$ cannot cross A_τ inside S.

Theorem 4.4.12 says that this open circuit in ω' is bounded below by constant $c > 0$ (independent of δ). Thus, $\gamma^\delta[\tau,\infty]$ cannot cross A_τ inside S with probability larger than c uniformly on the configuration outside A_τ.

Let S^* be the subgraph of $\Omega^\delta_{\Gamma^*}\backslash\gamma^\delta[0,\tau]$ given by the union of the connected components (seen as dual graph Γ^*) of type (iii). Similarly as above, the exploration path $\gamma^\delta[\tau,\infty]$ cannot cross A^*_τ inside S^* with probability $c > 0$.

Combining it altogether, $\gamma^\delta[\tau,\infty]$ cannot cross A_τ with probability c^2. By Corollary 4.4.13 for fixed ratio R/r and since using this in several concentric annuli gives the result for larger annulus, c can be taken to be equal to $1-(1-\epsilon)^{\lfloor \log_2(R/r)\rfloor}$. Since $R/r \geq C$, it is possible to guarantee that $c^2 \geq 1/2$ by choosing C large enough.

□

4.4.4. FK Ising observable convergence

In this section, we rephrase the approximate harmonicity up to a thin strip from the boundary as in the context of [12, [Theorem 4.1]. This theorem in [12] is written for S-graphs but is even new in the isoradial context.

Proposition 4.4.14. *Let $\Omega^\delta \subset \mathbb{C}$ be a bounded simply connected discrete domain drawn on isoradial graph. Assume that F is an s-holomorphic function in (the bulk of) Ω^δ and that $|H_F| \leq 1$ in Ω^δ where H_F is constructed from F as in Section 4.4.1.*

Let $\eta \in (0,1)$ and $\Omega^\delta_{\mathrm{int}(\eta)} \subset \mathbb{C}$ be (one of the connected components of) the $\delta^{1-\eta}$-interior of Ω^δ. Denote by $h_{\mathrm{int}(\eta)}$ the harmonic continuation of the function H_F from the boundary to the bulk of the domain $\Omega^\delta_{\mathrm{int}(\eta)}$ (i.e. $h_{\mathrm{int}(\eta)}$ is the solution of the continuous Dirichlet problem in $\Omega^\delta_{\mathrm{int}(\eta)}$ with boundary values given by H_F). Then there exists an exponent $\alpha > 0$ such that provided δ is small enough (depending *only on η), the following estimate holds:*

$$|H_F - h_{\text{int}(\eta)}| = O(\delta^{\alpha(\eta)}) \quad \text{uniformly in } \Omega^\delta_{\text{int}(\eta)}$$

Furthermore, there exists an exponent $\beta(\eta) > 0$ such that

$$|F(z) - \sqrt{\Phi'(z)}| = O(\delta^{\beta(\eta)}) \quad \text{uniformly in } \Omega^\delta_{\text{int}(\eta)}$$

where Φ denotes the conformal mapping from Ω onto the strip $\mathbb{R} \times (0,1)$ such that a, b are mapped to $\mp\infty$ and the explicit constants in the O-estimates depends only on the diameter of Ω^δ.

Proof. Let $(\Omega^\delta; a^\delta, b^\delta)$ be a discrete domain with two marked boundary points a^δ, b^δ. Let $0 < \epsilon \ll \eta$ be a small parameter as chosen in the proof of Theorem 4.2.2. Define

$$d_u := \text{dist}(u, \partial\Omega^\delta) \text{ and } \rho_u := \delta^\epsilon \,\text{crad}(u, \Omega^\delta) \asymp \delta^\epsilon d_u \gg \delta \text{ for } u \in \Omega^\delta_{\text{int}(\eta)}$$

where crad is the conformal radius of u in Ω^δ. Then by Theorem 4.4.6, we get the estimate:

$$|\Delta H_F|(u) = O\left(\frac{\delta^2}{d_u^3}\right) \quad \text{uniformly for } u \in \Omega^\delta_{\text{int}(\eta)}$$

Recall that H_F satisfies the 0/1 Dirichlet boundary conditions and $H_F \in [0,1]$ everywhere in Ω^δ due to the maximum principle. Thus, we can apply Theorem 4.2.2 to deduce that

$$|H_F - h_{\text{int}(\eta)}| = O(\delta^{\alpha(\eta)})$$

for some $\alpha(\eta) > 0$ and constant depending only on $\text{diam}\,\Omega^\delta$.

By construction of the function H_F, its gradient is bounded by $|F|^2$. From uniform crossing estimates of annulus, one obtains

$$|F(z)| \le O\left(\delta^{-1/2}\left[\frac{\delta}{\text{dist}(z, (b^\delta a^\delta)}\right]^\eta\right).$$

Thus, there are the following uniform estimates on $\partial\Omega^\delta_{\text{int}(\eta)}$:

$$h(u) \;=\; H_F(u) \;=\; O\biggl(\frac{\delta^\eta}{(\text{dist}_{\Omega^\delta}(u, (b^\delta a^\delta))^\eta}\biggr) \quad \text{if } \operatorname*{dist}_{\Omega^\delta}(u, (a^\delta b^\delta)) = \delta^{1-\eta}$$

and similarly

$$h(u) \;=\; H_F(u) \;=\; 1 - O\bigg(\frac{\delta^\eta}{(\mathrm{dist}_{\Omega^\delta}(u,(a^\delta b^\delta))^\eta}\bigg) \quad \text{if} \quad \operatorname*{dist}_{\Omega^\delta}(u,(b^\delta a^\delta)) = \delta^{1-\eta}.$$

Now that we have a polynomial $O(\delta^{\alpha(\eta)})$ convergence of H_F to harmonic function h_m in the bulk of Ω^δ. It is easy to derive the same for F since

$$H_F = \int \mathrm{Im}[F^2 dz] \quad , \quad h_m = \int \mathrm{Im}[f^2 dz]$$

where f is a holomorphic function in Ω and $\|\nabla F\| = O(1)$ in the bulk.

Indeed, suppose that there is a direction where $\mathrm{Im}[(F^2(z) - f^2(z))dz]$ is large. That is, there is some $z_e \in \Omega^\delta_\diamondsuit$ such that $|F^2(z_e) - f^2(z_e)| > C\delta^{\alpha(\eta)/4}$. Since $\|\nabla F\| = O(1)$ in the bulk, this difference cannot jump too much. In particular, it will still be large on the edge containing z_e: there is some $\epsilon \in (0,1)$ such that

$$|F^2(w) - f^2(w)| > C\delta^{\alpha/4} \quad \text{for } |w - z_e| < \delta^{\alpha/2+\epsilon}.$$

Then as functions H_F can be obtained from F by integrating a piecewise constant differential form directly in $\mathbb{C}$,

$$\int \mathrm{Im}[(F^2(z) - f^2(z))dz] > C\delta^{\alpha/2+\epsilon}$$

over a segment of length $\delta^{\alpha/2+\epsilon}$ which is a contradiction. By taking the square root which is 1/2- Hölder we get that $|F(z) - f(z)| \le C\delta^{\alpha/2}$ uniformly in $\Omega^\delta_{\mathrm{int}(\eta)}$.

□

4.4.5. FK Ising convergence rate for paths

Recall the setup and notation from Section 4.4.2. By Lemma 4.4.8, Proposition 4.4.10, and Proposition 4.4.14, we can apply the framework outlined in Theorem 1.5.3 to conclude that: the interface of the critical FK-Ising model on δ-isoradial graph converges weakly, as $\delta \to 0$, to the chordal $SLE_{16/3}$ interface.

Theorem 4.4.15. *Let Ω be a bounded simply connected domain with two distinct boundary points (degenerate prime ends) a.b. Let $(\Omega^\delta; a^\delta, b^\delta)$ be a discrete domain with two marked boundary points a^δ, b^δ that converges to $(\Omega; a, b)$ in the Caratheodory sense as $\delta \to 0$. Consider the interface γ^δ in the critical FK-Ising model with Dobrushin boundary conditions on $(\Omega^\delta; a^\delta, b^\delta)$. Let $\phi : \overline{\Omega^\delta} \to \mathbb{H} \cup \{\infty\}$ be a conformal map onto the upper half plane $\mathbb{H}$ such that $\phi((ab)) = (0, \infty)$, $\phi((ba)) = (-\infty, 0)$, $\phi(a) = 0$, $\phi(b) = \infty$. Note that ϕ is unique only up to positive rescaling. Let γ^ϕ be the path $\phi \circ \gamma^\delta$ parameterized* *by capacity from ∞ in $\overline{\mathbb{H}}$ and $\widetilde{\gamma}$ be the $SLE_{16/3}$ path in $\overline{\mathbb{H}}$. Let $d_*(\cdot,\cdot)$ be the metric on $\overline{\mathbb{H}} \cup \{\infty\}$ defined above.*

For any $T > 0$, there is some $\delta_0(T) < \infty$ such that if $0 < \delta < \delta_0$ then there is a coupling of γ^ϕ and $\widetilde{\gamma}$ such that

$$\mathbb{P}\left\{\sup_{0\le t\le T} d_*(\gamma^\phi(t), \widetilde{\gamma}(t)) > \delta^u\right\} < \delta^u$$

for some $u \in (0,1)$.

Moreover, if Ω is an α-Hölder domain, then under the same coupling the SLE curve in the image is polynomially close to the original discrete curve:

$$\mathbb{P}\left\{\sup_{t\in[0,T]} d_*\left(\gamma(t), \phi^{-1}(\tilde{\gamma}(t))\right) > \delta^v\right\} < \delta^v$$

where v depends only on α and u.

Bibliography

[1] M. Aizenman and A. Burchard. Hölder regularity and dimension bounds for random curves. *Duke Math. J.*, 99(3):419–453, 1999. 2, 3, 50, 51

[2] Vincent Beffara. Cardy's formula on the triangular lattice, the easy way. 02 2007. 66

[3] Vincent Beffara. The dimension of the sle curves. *Ann. Probab.*, 36(4):1421–1452, 07 2008. 18

[4] D. Beliaev. Conformal maps and geometry. 2019. 12

[5] Christian Beneš, Fredrik Johansson Viklund, and Michael J. Kozdron. On the rate of convergence of loop-erased random walk to SLE_2. *Comm. Math. Phys.*, 318(2):307–354, 2013. 32, 36, 37, 40, 41, 42, 43, 45

[6] Olivier Bernardi. Percolation on triangulations, and a bijective path to liouville quantum gravity. *Notices of the American Mathematical Society*, 66(04):1, Apr 2019. 31

[7] I. Binder, L. Chayes, and H. K. Lei. On convergence to SLE_6 I: Conformal invariance for certain models of the bond-triangular type. *J. Stat. Phys.*, 141(2):359–390, 2010. 63, 64

[8] I. Binder, L. Chayes, and H. K. Lei. On convergence to SLE_6 II: Discrete approximations and extraction of Cardy's formula for general domains. *J. Stat. Phys.*, 141(2):391–408, 2010. 64, 65, 66

[9] I. Binder, L. Chayes, and H. K. Lei. On the rate of convergence for critical crossing probabilities. *Ann. Inst. Henri Poincaré Probab. Stat.*, 51(2):672–715, 2015. 1, 65, 66, 67

[10] Ilia Binder and Zhiqiang Li. The harmonic explorer on isoradial lattices and it's convergence to SLE_4. *Privately communicated by authors.* 72, 74

[11] L. Chayes and H. Lei. Cardy's formula for certain models of the bond-triangular type. *Reviews in Mathematical Physics*, 19, 01 2006. 60, 61, 63, 66

[12] Dmitry Chelkak. Ising model and s-embeddings of planar graphs, 2020. 68, 69, 84

[13] Dmitry Chelkak, Hugo Duminil-Copin, Clément Hongler, Antti Kemppainen, and Stanislav Smirnov. Convergence of Ising interfaces to Schramm's SLE curves. *C. R. Math. Acad. Sci. Paris*, 352(2):157–161, 2014. 1

[14] Dmitry Chelkak and Stanislav Smirnov. Discrete complex analysis on isoradial graphs. *Advances in Mathematics*, 228(3):1590–1630, Oct 2011. 75, 78

[15] Dmitry Chelkak and Stanislav Smirnov. Universality in the 2d ising model and conformal invariance of fermionic observables. *Inventiones mathematicae*, 189(3):515–580, Jan 2012. 78, 79, 80, 82, 83

[16] Richard M. Dudley. *Real analysis and probability.* Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1989. 40

[17] Hugo Duminil-Copin, Vladas Sidoravicius, and Vincent Tassion. Continuity of the phase transition for planar random-cluster and potts models with $1 \leq q \leq 4$. *Communications in Mathematical Physics*, 349(1):47–107, Oct 2016. 83

[18] Bertrand Duplantier, Jason Miller, and Scott Sheffield. Liouville quantum gravity as a mating of trees, 2014. 31

[19] Ewain Gwynne, Nina Holden, and Xin Sun. Joint scaling limit of site percolation on random triangulations in the metric and peanosphere sense, 2019. 31

[20] Ewain Gwynne, Adrien Kassel, Jason Miller, and David B. Wilson. Active spanning trees with bending energy on planar maps and sle-decorated liouville quantum gravity for $\kappa > 8$. *Communications in Mathematical Physics*, 358(3):1065–1115, Feb 2018. 31

[21] Ewain Gwynne, Cheng Mao, and Xin Sun. Scaling limits for the critical fortuin–kasteleyn model on a random planar map i: Cone times. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 55(1):1–60, Feb 2019. 31

[22] Ewain Gwynne and Jason Miller. Convergence of the self-avoiding walk on random quadrangulations to $\mathrm{sle}_{8/3}$ on $\sqrt{8/3}$-liouville quantum gravity, 2016. 31

[23] Ewain Gwynne and Jason Miller. Convergence of percolation on uniform quadrangulations with boundary to sle_6 on $\sqrt{8/3}$-liouville quantum gravity, 2017. 31

[24] Ewain Gwynne and Xin Sun. Scaling limits for the critical fortuin-kastelyn model on a random planar map iii: finite volume case. 10 2015. 31

[25] Nina Holden, Xinyi Li, and Xin Sun. Natural parametrization of percolation interface and pivotal points. 04 2018. 30

[26] Nina Holden and Xin Sun. Convergence of uniform triangulations under the cardy embedding, 2019. 32

[27] Fredrik Johansson Viklund. Convergence rates for loop-erased random walk and other Loewner curves. *Ann. Probab.*, 43(1):119–165, 2015. 2, 3, 14, 17, 24, 25, 26, 27, 33, 34, 48, 57

[28] Antti Kemppainen and Stanislav Smirnov. Random curves, scaling limits and Loewner evolutions. *Ann. Probab.*, 45(2):698–779, 2017. 2, 3, 4, 21, 22, 23, 27, 30, 52, 63, 82

[29] Richard Kenyon, Jason Miller, Scott Sheffield, and David B. Wilson. Bipolar orientations on planar maps and SLE_{12}. *Ann. Probab.*, 47(3):1240–1269, 05 2019. 31

[30] G. Kirchhoff. Ueber die auflösung der gleichungen, auf welche man bei der untersuchung der linearen vertheilung galvanischer ströme geführt wird. *Annalen der Physik*, 148:497–508. 10

[31] Michael J. Kozdron, Larissa M. Richards, and Daniel W. Stroock. Determinants, their applications to markov processes, and a random walk proof of kirchhoff's matrix tree theorem, 2013. 10

[32] G. Lawler and V. Limic. Random walk: A modern introduction. 2010. 9

[33] Gregory Lawler. *Multifractal Analysis of the Reverse Flow for the Schramm-Loewner Evolution*, volume 61, pages 73–107. 01 2009. 18

[34] Gregory Lawler, Oded Schramm, and Wendelin Werner. Conformal restriction: The chordal case. *Journal of the American Mathematical Society*, 16(4):917–955, Jun 2003. 18

[35] Gregory F. Lawler. *Conformally invariant processes in the plane*, volume 114 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. 12, 13, 16, 18, 19, 20

[36] Gregory F. Lawler and Mohammad A. Rezaei. Minkowski content and natural parameterization for the schramm–loewner evolution. *Ann. Probab.*, 43(3):1082–1120, 05 2015. 30

[37] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Values of brownian intersection exponents, i: Half-plane exponents. *Acta Mathematica*, 187(2):237–273, 2001. 13, 18

[38] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees [mr2044671]. In *Selected works of Oded Schramm.*

Volume 1, 2, Sel. Works Probab. Stat., pages 931–987. Springer, New York, 2011. 1, 10, 14, 15, 23, 37

[39] Gregory F. Lawler and Scott Sheffield. A natural parametrization for the schramm–loewner evolution. *Ann. Probab.*, 39(5):1896–1937, 09 2011. 30

[40] Gregory F. Lawler and Fredrik Viklund. Convergence of loop-erased random walk in the natural parametrization, 2016. 1

[41] Gregory F. Lawler and Fredrik Viklund. Convergence of loop-erased random walk in the natural parametrization, 2016. 30

[42] Gregory F. Lawler and Fredrik Viklund. Convergence of radial loop-erased random walk in the natural parametrization, 2017. 30

[43] Gregory F. Lawler and Wang Zhou. Sle curves and natural parametrization. *Ann. Probab.*, 41(3A):1556–1584, 05 2013. 30

[44] Yiting Li, Xin Sun, and Samuel S. Watson. Schnyder woods, sle(16), and liouville quantum gravity, 2017. 31

[45] Donald Marshall and Steffen Rohde. The loewner differential equation and slit mappings. *Journal of the American Mathematical Society*, 18, 07 2005. 14

[46] Jason Miller and Scott Sheffield. Liouville quantum gravity and the brownian map iii: the conformal structure is determined, 2016. 31

[47] Robin Pemantle. Choosing a spanning tree for the integer lattice uniformly. *Ann. Probab.*, 19(4):1559–1574, 10 1991. 9

[48] Ch. Pommerenke. *Boundary behaviour of conformal maps*, volume 299 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992. 4, 5, 50

[49] Ch. Pommerenke. Conformal maps at the boundary. In *Handbook of complex analysis: geometric function theory, Vol. 1*, pages 37–74. North-Holland, Amsterdam, 2002. 5, 13

[50] Steffen Rohde and Oded Schramm. Basic properties of SLE [mr2153402]. In *Selected works of Oded Schramm. Volume 1, 2*, Sel. Works Probab. Stat., pages 989–1030. Springer, New York, 2011. 14, 15

[51] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees [mr1776084]. In *Selected works of Oded Schramm. Volume 1, 2*, Sel. Works Probab. Stat., pages 791–858. Springer, New York, 2011. 1, 9, 15, 16

[52] Oded Schramm and Scott Sheffield. Harmonic explorer and its convergence to SLE_4. *Ann. Probab.*, 33(6):2127–2148, 2005. 1

[53] Oded Schramm and Scott Sheffield. Harmonic explorer and its convergence to sle 4. *Ann. Probab.*, 33(6):2127–2148, 11 2005. 75, 76

[54] Oded Schramm and Scott Sheffield. Contour lines of the two-dimensional discrete Gaussian free field. *Acta Math.*, 202(1):21–137, 2009. 1

[55] Stanislav Smirnov. Critical percolation in the plane: conformal invariance, cardy's formula, scaling limits. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, 333(3):239 – 244, 2001. 60, 65, 66

[56] Stanislav Smirnov. Critical percolation and conformal invariance. In *XIVth International Congress on Mathematical Physics*, pages 99–112. World Sci. Publ., Hackensack, NJ, 2005. 1, 20

[57] Stanislav Smirnov. Towards conformal invariance of 2D lattice models. In *International Congress of Mathematicians. Vol. II*, pages 1421–1451. Eur. Math. Soc., Zürich, 2006. 16

[58] S. E. Warschawski. On the degree of variation in conformal mapping of variable regions. *Trans. Amer. Math. Soc.*, 69:335–356, 1950. 56

[59] David Bruce Wilson. Generating random spanning trees more quickly than the cover time. In *Proceedings of the Twenty-Eighth Annual ACM Symposium on Theory of Computing*, STOC '96, page 296–303, New York, NY, USA, 1996. Association for Computing Machinery. 9