THE STABLE RANK OF DIAGONAL ASH ALGEBRAS

BY

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ABSTRACT

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Building on the work of Lutley in [Lut17b], we study a certain subclass of recursive subhomogeneous algebras, called DSH algebras, in which the pullback maps are all diagonal in a suitable sense. We examine inductive limits of DSH algebras, where each bonding map is itself diagonal in an appropriate way, and show that every simple algebra thus obtained has stable rank one. We are therefore able to show that every simple dynamical crossed product has stable rank one and that the Toms-Winter Conjecture holds for such algebras.

We also introduce the class of non-unital DSH algebras and make partial progress towards showing that inductive limits of such algebras with diagonal maps have stable rank one. Moreover, we investigate local notions of a diagonal map and matrix unit compatibility and show that in the full matrix algebra setting they agree with their global counterparts.

Forever for Julia,
my reason for everything.

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To be added.

PUBLICATIONS

Chapter 3 and parts of Chapter 2 have been submitted for publication and appear in [AL20].

CONTENTS

1

INTRODUCTION

1.1 A BRIEF HISTORY OF STABLE RANK ONE

With the aim of formulating a notion of dimension for a C*-algebra, in [Rie83], Rieffel introduced the concept of topological stable rank. The *topological stable rank* of a unital C*-algebra A is the least natural number n for which the set of all n-tuples of A that generate A as a left ideal is dense in A^n; if no such integer exists, the topological stable rank is said to be ∞. In [Rie83], Rieffel, together with Herman and Vaserstein in [HV84], showed that for unital C*-algebras, the topological stable rank coincides with the purely algebraic notion of Bass stable rank for rings. Therefore, we hereafter refer to the topological stable rank simply as stable rank. Of particular note is the instance when the stable rank is one. In [Rie83], Proposition 3.1, it is shown that a unital C*-algebra has stable rank one if and only if the set of invertible elements is dense within the algebra. An important problem in the field has been to determine when a C*-algebra has stable rank one.

In [Rø91], Rørdam supplied one of the first major results concerning stable rank. He showed that the tensor product of a simple unital stably finite C*-algebra and a UHF algebra (an inductive limit of matrix algebras) has stable rank one. This was followed by a result of Dădărlat, Nagy, Némethi, and Pasnicu, who proved in [DNNP92] that a simple unital inductive limit of full matrix algebras (those of the form $C(X, M_n(\mathbb{C}))$ for a compact Hausdorff space X) always has stable rank one assuming there is a uniform upper bound on the dimensions of the base spaces in the finite stage algebras. Later, in [Rø04], Rørdam also showed that every simple unital finite C*-algebra that absorbs the Jiang-Su algebra, $\mathcal{Z}$, tensorally has stable rank one.

Villadsen proved in [Vil98] that the converse to the result in [DNNP92] does not hold by constructing a unital simple limit of full matrix algebras, whose base space dimensions were not uniformly bounded above, yet which nonetheless still had stable rank one. He went on, in [Vil99], to construct a class of simple unital AH algebras—inductive limits of homogeneous C*-algebras (those whose irreducible representations all have the same dimension)—of arbitrary stable rank, thereby affirming the subtleties present in the problem of stable rank.

Given compact metric spaces X and Z, together with continuous functions $\lambda_1, \ldots, \lambda_k \colon Z \to X$, there is a naturally induced *-homomorphism from $C(X, M_n(\mathbb{C}))$ to $C(Z, M_{nk}(\mathbb{C}))$ given by

$$f \mapsto \operatorname{diag}(f \circ \lambda_1, \ldots, f \circ \lambda_k).$$

These induced maps between full matrix algebras are referred to as *diagonal*. They have been used to construct a rich class of examples in the field, including those of Goodearl in [Goo92] and Villadsen in [Vil98].

Just over a decade ago, another stable rank one result was obtained by Elliott, Ho, and Toms in [EHT08]. Their paper, which stemmed from Ho's work in [Ho06], showed that the condition of bounded dimension in [DNNP92] could be replaced with the assumption that all of the bonding maps in the inductive limit are diagonal. In [Lut17b] and the corresponding preprint [Lut17a], Lutley laid the foundation for how to generalize the result of Elliott, Ho, and Toms to a broader class of inductive limits.

In this thesis, we expand on the work of Lutley and extend the AH stable rank one result of Elliott, Ho, and Toms in [EHT08] to a suitable class of approximately subhomogeneous (ASH) algebras—inductive limits of subhomogeneous C*-algebras (those whose irreducible representations all have dimension at most some fixed integer).

The building-block algebras in the AH setting are full matrix algebras, whose primitive quotients are intrinsically matrix unit compatible. This internal compatibility is crucial to obtaining the stable rank one result in [EHT08]. To achieve this for the subhomogeneous building blocks in the ASH setting, it is necessary, therefore, to consider only subhomogeneous algebras whose primitive quotients fit together in a compatible (i.e. matrix unit compatible) way. We restrict our attention to the subclass of recursive subhomogeneous algebras introduced by Lutley in [Lut17b].

Recursive subhomogeneous algebras are a particularly tractable class of unital subhomogeneous algebras introduced by Phillips in [Phi07], which are iterative pullbacks of full matrix algebras. In order to ensure the aforementioned compatibility, it is necessary that all the pullback maps be diagonal in a suitable sense. Such algebras are called diagonal subhomogeneous (DSH) algebras.

DSH algebras arise naturally in the study of dynamical crossed products. The orbit-breaking subalgebras of crossed products introduced by Q. Lin in [Lin] (see also [LP98] and [LP04]) following the work of Putnam in [Put89] are examples of DSH algebras.

1.2 RESULTS AND ORGANIZATION OF THE THESIS

In Chapter 2, we formally define the class of DSH algebras and the notion of a diagonal map between them. We derive several structural properties of and results concerning DSH algebras, which are needed for the stable rank one result in the following chapter.

Section 3.1 and Section 3.2 of Chapter 3 are devoted to proving the extended result of [EHT08] in the ASH setting. We show that every simple inductive limit of DSH algebras with diagonal maps has stable rank one. This is the same result that Lutley claimed to have established in [Lut17b] and [Lut17a]. However, a careful scrutiny of the work there shows that this theorem can hereunto not be considered proved: there are several unidentified missing essential components of the argument, and the proofs of many of the results are incomplete and lack clarity. This can be seen, for instance, in Lemma 3.3 of the present thesis, the proof of which heavily relies on the novel results developed in Section 2.3.

In Section 3.3, we present two applications to dynamical crossed products. With our stable rank one theorem for inductive limits and results of Archey and Phillips developed in [AP15], we are able to prove a conjecture of Archey, Niu, and Phillips stated in the same paper (Conjecture 7.2); namely, that for an infinite compact metric space T and a minimal homeomorphism $h\colon T \to T$, the dynamical crossed product $C^*(\mathbb{Z}, T, h)$ has stable rank one. Using a result of Thiel in [Thi19], we are also able to show that, for such crossed products, classifiability is determined solely by strict comparison, thereby affirming the Toms-Winter Conjecture for simple dynamical crossed products.

Although the notion of stable rank one is intimately tied to the presence of a unit, it still makes sense in the non-unital setting. A non-unital C^*-algebra is said to have stable rank one, provided that its unitization does. In Chapter 4, we investigate the possibility of a non-unital stable rank one result analogous to the one proved in Section 3.2. We introduce the notion of a non-unital DSH algebra and adapt some of the lemmas in Section 3.1 to the non-unital setting, thereby making partial progress towards a stable rank one result. We also discuss the difficulty with generalising certain aspects of the proof from the unital case.

In Chapter 5, we return to the AH setting. In Section 5.1, we compare and contrast our stable rank one proof from Chapter 3 in the particular instance where all the DSH algebras are homogeneous with the one in [EHT08], showing in this case that our argument essentially reduces to the one of that paper. In Section 5.2, we formulate a purely local notion of diagonal map between two arbitrary subhomogeneous C^*-algebras and show that if both algebras are finite direct sums of full matrix algebras, then our local definition agrees with the global notion of diagonal map given in [EHT08] after conjugation by a piecewise-constant permutation unitary. In Section 5.3, we develop a local notion of matrix unit compatibility for a general unital homogeneous C*-algebra, and show that every such algebra satisfying this condition must be a full matrix algebra. We end by discussing potential analogous notions for subhomogeneous algebras that may help determine if the result in Chapter 3 is the best possible in the ASH setting.

1.3 NOTATION

We use $\mathbb{N}$ to denote the strictly positive integers and the symbol $\subset$ to denote non-strict set inclusion. Given a C*-algebra A, we let $\hat{A}$ denote the set of equivalence classes of non-zero irreducible representations of A equipped with the hull-kernel topology, and we let $\widetilde{A}$ denote the unitization of A. If A is unital, we use 1_A to denote the unit of A. For $n \in \mathbb{N}$, we use the shorthand M_n to refer to the matrix algebra $M_n(\mathbb{C})$. When speaking about a matrix $D \in M_n$, we denote the (i, j)-entry of D by $D_{i,j}$, and we let 1_n denote the identity matrix in M_n.

2 DSH ALGEBRAS

In this chapter we introduce the class of diagonal subhomogeneous algebras that we deal with in this thesis and examine their basic properties and structure. In Section 2.1, we define what a diagonal subhomogeneous algebra is and the notion of a diagonal map between two such algebras. We discuss some basic properties and notions concerning these algebras that are used throughout the remainder of the chapter and beyond. The chief purpose of Section 2.2 is to prove that given any inductive limit of diagonal subhomogeneous algebras with diagonal maps, one may always assume the maps in the sequence are injective. Section 2.3 explores the internal decomposition structure of a diagonal subhomogeneous algebra and establishes some key results used in Chapter 3. Finally, in Section 2.4, we show that all homogeneous diagonal subhomogeneous algebras are full matrix algebras.

2.1 INTRODUCTORY DEFINITIONS AND BASIC PROPERTIES

Let us start off by recalling what a recursive subhomogeneous algebra is.

Definition 2.1 ([Phi07], Definition 1.1)**.** A *recursive subhomogeneous algebra* is a C*-algebra given by the following recursive definition.

(1) If X is a compact metric space and $n \geq 1$, then $C(X, M_n)$ is a recursive subhomogeneous algebra.

(2) If A is a recursive subhomogeneous algebra, X is a compact metric space, $Y \subset X$ is closed, $\varphi\colon A \to C(Y, M_n)$ is any unital *-homomorphism, and $\rho\colon C(X, M_n) \to C(Y, M_n)$ is the restriction *-homomorphism, then the pullback

$$A \oplus_{C(Y,M_n)} C(X, M_n) := \{(a, f) \in A \oplus C(X, M_n) : \varphi(a) = \rho(f)\}$$

is a recursive subhomogeneous algebra.

Therefore, if A is a recursive subhomogeneous algebra, there are compact metric spaces $X_1, \ldots, X_l$ (the *base spaces* of A), closed subspaces $Y_1 := \varnothing$, $Y_2 \subset X_2, \ldots, Y_l \subset X_l$, positive integers $n_1, \ldots, n_l$, C*-algebras

$A^{(i)} \subset \bigoplus_{j=1}^{i} C(X_j, M_{n_j})$ for $1 \leq i \leq l$, and unital *-homomorphisms $\varphi_i \colon A^{(i)} \to C(Y_{i+1}, M_{n_{i+1}})$ for $1 \leq i \leq l-1$ such that:

(1) $A^{(1)} = C(X_1, M_{n_1})$;

(2) for all $1 \leq i \leq l-1$

$$A^{(i+1)} = \{(a, f) \in A^{(i)} \oplus C(X_{i+1}, M_{n_{i+1}}) : \varphi_i(a) = f|_{Y_{i+1}}\};$$

(3) $A = A^{(l)}$.

Simply put,

$$A = \left[\cdots \left[\left[C_1 \oplus_{C_2'} C_2 \right] \oplus_{C_3'} C_3 \right] \cdots \right] \oplus_{C_l'} C_l,$$

where $C_i := C(X_i, M_{n_i})$, $C_i' := C(Y_i, M_{n_i})$, and the maps $\varphi_1, \ldots, \varphi_{l-1}$ are used in the pullback. In this case, we say the length of the composition sequence is l. As shown in [Phi07], the decomposition of a recursive subhomogeneous is highly non-unique. We make the same tacit assumption adopted in that paper: unless otherwise specified, every recursive subhomogeneous algebra comes equipped with a decomposition of the form given above. In particular, we refer to the number l above as the *length of* A.

Since for all $1 \leq i \leq l$, we have $A^{(i)} \subset \bigoplus_{j=1}^{i} C(X_j, M_{n_j})$, we can view each element $f \in A^{(i)}$ as $(f_1, \ldots, f_i)$, where $f_j \in C(X_j, M_{n_j})$ for all $1 \leq j \leq i$. For $1 \leq i \leq l$ and $x \in X_i$, we have the usual *evaluation map* $\mathrm{ev}_x \colon A \to M_{n_i}$ given by $\mathrm{ev}_x(f) := f_i(x)$ for all $f \in A$. We let $\mathfrak{s}(A) := \min\{n_1, \ldots, n_l\}$ and $\mathfrak{S}(A) := \max\{n_1, \ldots, n_l\}$.

Let us now define what a diagonal subhomogeneous algebra is.

Definition 2.2 (DSH algebras; cf. [Lut17b], Definition 1.0.2)**.** A C*-algebra A is a *diagonal (recursive) subhomogeneous (DSH) algebra* (of length l) provided that it is a recursive subhomogeneous algebra (of length l) (with a decomposition as described above), and for all $1 \leq i \leq l-1$ and $y \in Y_{i+1}$, there is a list of points $x_1 \in X_{i_1} \setminus Y_{i_1}, \ldots, x_t \in X_{i_t} \setminus Y_{i_t}$ such that for all $f \in A^{(i)}$,

$$\varphi_i(f)(y) = \mathrm{diag}(f_{i_1}(x_1), \ldots, f_{i_t}(x_t)).$$

We say y *decomposes into* $x_1, \ldots, x_t$, that each x_j is a point *in the decomposition of* y, and that x_j *begins at index* $1 + n_{i_1} + \cdots + n_{i_{j-1}}$ *down the diagonal*. Given $1 \leq j \leq i$ and $y' \in Y_j$, we say that y' is *in the decomposition of* y if there exists a $1 \leq k \leq n_i$ with the property that for all $f \in A^{(i)}$ there are matrices $P \in M_{k-1}$ and $Q \in M_{n_i - n_j - (k-1)}$ such that $f_i(y) = \mathrm{diag}(P, f_j(y'), Q)$.

Whenever we work with a DSH algebra of length l we adopt, unless otherwise specified, the same notation for the decomposition used above. Thus, if A is a DSH algebra of length l, we can view A as the set of all $f := (f_1, \ldots, f_l) \in \bigoplus_{i=1}^{l} C(X_i, M_{n_i})$ such that for all $1 \leq i < l$ and $y \in Y_{i+1}$,

$$f_{i+1}(y) = \operatorname{diag}\left(f_{i_1}(x_1), \ldots, f_{i_t}(x_t)\right).$$

As is shown in Lemma 2.6 below, the decomposition of y is unique up to the re-indexing of identical points; that is, if y decomposes into $x_1, \ldots, x_t$ and $z_1, \ldots, z_s$, then $s = t$ and, for $1 \leq j \leq s$, $x_j = z_j$.

Definition 2.3 (Diagonal maps between DSH algebras; cf. [Lut17b], p. 29)**.** Given two DSH algebras A_1 and A_2 of lengths l_1 and l_2 and base spaces $X_1^1, \ldots, X_{l_1}^1$ and $X_1^2, \ldots, X_{l_2}^2$, respectively, we say that a *-homomorphism $\psi\colon A_1 \to A_2$ is *diagonal* provided that for all $1 \leq i \leq l_2$ and $x \in X_i^2$, there are points $x_1, \ldots, x_t$ with $x_j \in X_{i_j}^1$ such that for all $f \in A_1$,

$$\psi(f)_i(x) = \operatorname{diag}(f_{i_1}(x_1), \ldots, f_{i_t}(x_t)).$$

We say that x *decomposes into* $x_1, \ldots, x_t$.

Note that if $Y_1^1 \subset X_1^1, \ldots, Y_{l_1}^1 \subset X_{l_1}^1$ and $Y_1^2 \subset X_1^2, \ldots, Y_{l_2}^2 \subset X_{l_2}^2$ are the corresponding closed subsets of the base spaces in Definition 2.3, then, owing to the decomposition structure of A_1 and A_2, we get an equivalent definition by replacing X_i^2 and $X_{i_j}^1$ above with $X_i^2 \setminus Y_i^2$ and $X_{i_j}^1 \setminus Y_{i_j}^1$, respectively. Note, too, that by definition diagonal maps are automatically unital.

For the remainder of this section, assume that A is a DSH algebra of length l. We have the following description of the spectrum of A.

Lemma 2.4 ([Phi07], Lemma 2.1)**.** *The map $x \mapsto [\mathrm{ev}_x]$ defines a continuous bijection*

$$\bigsqcup_{i=1}^{\ell} (X_i \setminus Y_i) \to \hat{A},$$

(where, recall, $Y_1 := \varnothing$) whose restriction to each $X_i \setminus Y_i$ is a homeomorphism onto its image. In particular every irreducible representation of A is unitarily equivalent to ev_x for some $x \in \bigsqcup_{i=1}^{\ell}(X_i \setminus Y_i)$.

We often tacitly refer to a given irreducible representation ev_x simply as x, since we view such an element both as an irreducible representation and as a point in X_i.

Remark 2.5. A subset $D \subset X_i \setminus Y_i$ can be viewed as a subset both of X_i and of $\hat{A}$. We denote by $\overline{D}^{X_i}$ the closure of D with respect to the topology on

X_i. With one or two exceptions, when speaking about open and closed subsets of X_i in this paper, we mean with respect to the topology on X_i; such subsets could, in general, include points in Y_i, in which case they would not even be a subset of the spectrum. In any case, for subsets of $X_i \setminus Y_i$, the topology is always made explicit.

Lemma 2.6. *Suppose $2 \leq i \leq l$ and $y \in Y_i$. If y decomposes into $x_1, \dots, x_t$ and $z_1, \dots, z_s$, then $s = t$ and for $1 \leq j \leq s$, $x_j = z_j$.*

Proof. We see from Lemma 2.4 that A is liminary and that given distinct $w_1, \dots, w_k \in \bigsqcup_{i=1}^{l}(X_i \setminus Y_i)$, the representations $\mathrm{ev}_{w_1}, \dots, \mathrm{ev}_{w_k}$ of A are irreducible and pairwise inequivalent. It follows from Proposition 4.2.5 of [Dix77] that there is a function $f \in A$ such that $\mathrm{ev}_{w_1}(f) \neq 0$ and $\mathrm{ev}_{w_j}(f) = 0$ for $j = 2, \dots, k$. From this it is readily seen that $s = t$ and for $1 \leq j \leq s$, $x_j = z_j$. //

By Lemma 2.4 and Definition 2.2, given $y \in \bigsqcup_{i=1}^{l} X_i$, either ev_y is an irreducible representation of A or, if y is in some Y_i, ev_y splits up into irreducible representations of A.

Definition 2.7 (cf. [Lut17b], p. 4)**.** Given $1 \leq i \leq l$ and $1 \leq k \leq n_i$, we define $B_{i,k}$ to be the set of points in X_i at which an irreducible representation of A begins at index k down the diagonal. (See Definition 2.2 and Lemma 2.8 below.) For $k \leq 0$, we set $B_{i,k} := \varnothing$.

We make the following observations about the $B_{i,k}$'s defined above, all of which are direct consequences of Definition 2.2.

Lemma 2.8 (cf. [Lut17b] p. 4)**.**

(1) $B_{i,1} = X_i$ for all $1 \leq i \leq l$.

(2) If $1 \leq i \leq l$ and $k > 1$, then $B_{i,k} \subset Y_i$. In particular, $B_{1,k} = \varnothing$ for $k > 1$.

(3) If $2 \leq i \leq l$ and $y \in Y_i$ decomposes into $x_1 \in X_{i_1} \setminus Y_{i_1}, \dots, X_{i_t} \setminus Y_{i_t}$, then $y \in B_{i,k}$ if and only if $k = 1 + n_{i_1} + \cdots + n_{i_{j-1}}$ for some $1 \leq j \leq t$. In particular $B_{i,k} = \varnothing$ for all $n_i - (\mathfrak{s}(A) - 1) < k \leq n_i$, where, recall, $\mathfrak{s}(A) := \min\{n_j : 1 \leq j \leq l\}$.

Lemma 2.9 (cf. [Lut17b], p. 4)**.** *Suppose A is a DSH algebra of length l. Then for each $2 \leq i \leq l$, we may assume $\mathrm{int}(Y_i) = \varnothing$.*

Proof. Fix $1 \le i \le l-1$. Let $Y'_{i+1} := Y_{i+1} \setminus \operatorname{int}(Y_{i+1})$ and $X'_{i+1} := X_{i+1} \setminus \operatorname{int}(Y_{i+1})$. We have the following commutative diagram of restriction *-homomorphisms:

$$\begin{array}{ccc} C(X_{i+1}, M_{n_{i+1}}) & \xrightarrow{\rho} & C(Y_{i+1}, M_{n_{i+1}}) \\ \lambda\big\downarrow & & \big\downarrow\tau \\ C(X'_{i+1}, M_{n_{i+1}}) & \xrightarrow{\rho'} & C(Y'_{i+1}, M_{n_{i+1}}) \end{array}$$

Let $B^{(i+1)} := A^{(i)} \oplus_{C(Y'_{i+1}, M_{n_{i+1}})} C(X'_{i+1}, M_{n_{i+1}})$, where the connecting *-homomorphism is $\varphi'_i := \tau \circ \varphi_i \colon A^{(i)} \to C(Y'_{i+1}, M_{n_{i+1}})$. Let us show that $A^{(i+1)}$ is isomorphic to $B^{(i+1)}$. Given $a \in A^{(i)}$ and $f \in C(X_{i+1}, M_{n_{i+1}})$ with $(a, f) \in A^{(i+1)}$, define $\Gamma\colon A^{(i+1)} \to B^{(i+1)}$ by $\Gamma((a, f)) := (a, \lambda(f))$. Note that $\varphi'_i(a) = \tau(\varphi_i(a)) = \tau(\rho(f)) = \rho'(\lambda(f))$, so that Γ is well defined. It is easy to see that Γ is a *-homomorphism. To see that Γ is injective, suppose $(a, f) \in A^{(i+1)}$ with $(a, \lambda(f)) = \Gamma((a, f)) = (0, 0)$. Then $a = 0$ and so $f|_{Y_{i+1}} = \varphi_i(a) = 0$, which, together with the fact that $\lambda(f) = 0$, yields that $f = 0$. For surjectivity suppose $a \in A^{(i)}$ and $g \in C(X'_{i+1}, M_{n_{i+1}})$ with $(a, g) \in B^{(i+1)}$. Then $\varphi_i(a)|_{Y'_{i+1}} = g|_{Y'_{i+1}}$, so that the function $h\colon X_{i+1} \to M_{n_{i+1}}$ defined to be $\varphi_i(a)(x)$ for $x \in Y_{i+1}$ and $g(x)$ for $x \in X'_{i+1}$ is well defined and continuous. Moreover, $\varphi_i(a) = h|_{Y_{i+1}}$, which implies $(a, h) \in A^{(i+1)}$ and $\Gamma((a, h)) = (a, \lambda(h)) = (a, g)$, proving surjectivity. //

2.2 QUOTIENTS OF DSH ALGEBRAS

The aim of this section is to prove that one may assume the ψ_i's in Theorem 3.14 are injective (see Proposition 2.14). Let A be a DSH algebra of length l. Suppose we have a non-zero C*-algebra B and a surjective *-homomorphism $\rho\colon A \to B$. This yields an injective single-valued map $\hat{\rho}\colon \hat{B} \to \hat{A}$ given by $\hat{\rho}([\pi]) := [\pi \circ \rho]$. For $1 \le i \le l$, define $X'_i := \overline{X_i \cap \hat{\rho}(\hat{B})}^{X_i}$ and $Y'_i := X'_i \cap Y_i$. Recall that these definitions make sense by Lemma 2.4.

Lemma 2.10. *$\hat{\rho}(\hat{B})$ is closed in $\hat{A}$.*

Proof. Suppose $[\rho] \in \overline{\hat{\rho}(\hat{B})}$. Then

$$\ker \rho \supset \bigcap_{[\sigma] \in \hat{\rho}(\hat{B})} \ker \sigma = \bigcap_{[\tau] \in \hat{B}} \ker \hat{\rho}([\tau]) = \bigcap_{[\tau] \in \hat{B}} \ker(\tau \circ \rho).$$

Note that $a \in \bigcap_{[\tau] \in \hat{B}} \ker(\tau \circ \rho)$ if and only if $\rho(a) \in \bigcap_{[\tau] \in \hat{B}} \ker \tau$ if and only if $\rho(a) = 0$. Hence, $\ker \rho \supset \ker \rho$. Thus, the irreducible representation τ of

B given by $\tau(b) := \rho(a)$, where a is any lift of b under ρ is well defined. Therefore, $[\rho] = [\tau \circ \rho] = \hat{\rho}([\tau]) \in \hat{\rho}(\hat{B})$, so that $\overline{\hat{\rho}(\hat{B})} \subset \hat{\rho}(\hat{B})$. //

Lemma 2.11. *Suppose $1 \leq i \leq l$ and $y \in Y_i'$. If $1 \leq j < i$ and $x \in X_j \setminus Y_j$ is in the decomposition of y, then $x \in X_j \cap \hat{\rho}(\hat{B}) \subset X_j'$.*

Proof. Since $y \in Y_i'$, we have $y \in X_i' = \overline{X_i \cap \hat{\rho}(\hat{B})}^{X_i}$. Choose a sequence $(z_n)_n$ in $X_i \cap \hat{\rho}(\hat{B})$ such that $z_n \to y$ with respect to the topology on X_i.

Claim 2.11.1. $(\mathrm{ev}_{z_n})_n \to \mathrm{ev}_x$, *with respect to the hull-kernel topology on $\hat{A}$.*

Proof. Suppose U is an open set in $\hat{A}$ containing ev_x. Then there is a function $f \in A$ that is non-zero at x, but vanishes at each point in $\hat{A} \setminus U$. Since x is in the decomposition of y, this implies that $f_i(y) \neq 0$. Since $z_n \to y$ in X_i and since f_i is continuous, there is an n_0 such that for all $n \geq n_0$, $f_i(z_n) \neq 0$. In particular, this means that for all $n \geq n_0$, $\mathrm{ev}_{z_n} \in U$. Therefore, $\mathrm{ev}_{z_n} \to \mathrm{ev}_x$ in $\hat{A}$. //

By Lemma 2.10, $\hat{\rho}(\hat{B})$ is closed and, hence, this claim implies that $\mathrm{ev}_x \in \hat{\rho}(\hat{B})$. Thus, $x \in X_j \cap \hat{\rho}(\hat{B}) \subset X_j'$. //

In the following lemma, we construct a DSH algebra from A over the base spaces X_i', where the pullback maps are just restrictions of the pullback maps in the definition of A (the φ_i's). We show afterwards (see Proposition 2.13) that this new DSH algebra is isomorphic to the quotient B.

Lemma 2.12. *There is a DSH algebra D of length l with the following properties:*

(1) $D^{(1)} = C(X_1', M_{n_1})$;

(2) for all $1 \leq i \leq l$, $D^{(i)} \subset \bigoplus_{j=1}^{i} C(X_j', M_{n_j})$;

(3) for all $1 \leq i < l$, $f \in D^{(i)}$, *and* $y \in Y_{i+1}'$, *the pullback map* $\tau_i \colon D^{(i)} \to C(Y_{i+1}', M_{n_{i+1}})$ *is* $\tau_i(f)(y) := \mathrm{diag}(f_{i_1}(x_1), \ldots, f_{i_t}(x_t))$, *where* $x_1, \ldots, x_t$ *are the points in the decomposition of y coming from the definition of A;*

(4) for $1 \leq i < l$, $D^{(i+1)} = D^{(i)} \oplus_{C(Y_{i+1}', M_{n_{i+1}})} C(X_{i+1}', M_{n_{i+1}})$ *with pullback map* τ_i;

(5) for all $1 \leq i \leq l$, *if* $(f_1, \ldots, f_i) \in D^{(i)}$, *there is a* $(g_1, \ldots, g_i) \in A^{(i)}$ *such that for all* $1 \leq j \leq i$, $g_j|_{X_j'} = f_j$.

Proof. Let us proceed by induction on i. Define $D^{(1)} := C(X_1', M_{n_1})$ so that *(1)* holds. Since X_1' is closed in X_1, we may extend a function in $D^{(1)}$ to a function in $A^{(1)} = C(X_1, M_{n_1})$, so that *(5)* holds when $i = 1$.

Now, fix $1 \leq i \leq l-1$ and assume that we have defined $D^{(1)}, \ldots, D^{(i)}$ and $\tau_1, \ldots, \tau_{i-1}$ satisfying conditions *(1)* to *(5)*. Let us show how to define τ_i and $D^{(i+1)}$. Given $(f_1, \ldots, f_i) \in D^{(i)}$, use *(5)* to get $(g_1, \ldots, g_i) \in A^{(i)}$ such that $g_j|_{X'_j} = f_j$ for $1 \leq j \leq i$. Define $\tau_i \colon D^{(i)} \to C(Y'_{i+1}, M_{n_{i+1}})$ by $\tau_i((f_1, \ldots, f_i)) := \varphi_i((g_1, \ldots, g_i))|_{Y'_{i+1}}$.

Claim 2.12.1. *τ_i is a well-defined *-homomorphism that satisfies (3).*

Proof. Suppose that $(h_1, \ldots, h_i) \in A^{(i)}$ also restricts coordinate-wise to $(f_1, \ldots, f_i)$. If $y \in Y'_{i+1}$ decomposes into $x_1 \in X_{i_1} \setminus Y_{i_1}, \ldots, x_t \in X_{i_t} \setminus Y_{i_t}$, then by Lemma 2.11, we have $x_1 \in X'_{i_1}, \ldots, x_t \in X'_{i_t}$. Hence,

$$\begin{aligned} \varphi_i((g_1, \ldots, g_i))(y) &= \operatorname{diag}(g_{i_1}(x_1), \ldots, g_{i_t}(x_t)) \\ &= \operatorname{diag}(f_{i_1}(x_1), \ldots, f_{i_t}(x_t)) \\ &= \operatorname{diag}(h_{i_1}(x_1), \ldots, h_{i_t}(x_t)) \\ &= \varphi_i((h_1, \ldots, h_i))(y). \end{aligned}$$

Thus, τ_i satisfies *(3)* and is independent of the choice of extension. Moreover, $\tau_i((f_1, \ldots, f_i))$ is continuous, being the restriction of a continuous function. Therefore, τ_i is well defined and it is clearly a *-homomorphism since φ_i is. //

Now, define $D^{(i+1)} := D^{(i)} \oplus_{C(Y'_{i+1}, M_{n_{i+1}})} C(X'_{i+1}, M_{n_{i+1}})$, using τ_i as the pullback map. This ensures that *(2)* and *(4)* hold, and so we just need to verify *(5)*. Suppose $(d, f) \in D^{(i+1)}$, where $d \in D^{(i)}$ and $f \in C(X'_{i+1}, M_{n_{i+1}})$. By the inductive hypothesis, we may apply *(5)* to d to obtain a $b \in A^{(i)}$ such that $b_j|_{X'_j} = d_j$ for all $1 \leq j \leq i$. Let $g := \varphi_i(b) \in C(Y_{i+1}, M_{n_{i+1}})$. If $y \in X'_{i+1} \cap Y_{i+1} = Y'_{i+1}$, then $g(y) = \varphi_i(b)(y) = \tau_i(d)(y) = f(y)$. Thus, since X'_{i+1} and Y_{i+1} are both closed in X_{i+1} and since f and g agree on their intersection, they have a common extension $h \in C(X_{i+1}, M_{n_{i+1}})$. Since $\varphi_i(b) = g = h|_{Y_{i+1}}$, we have $(b, h) \in A^{(i+1)}$, and since $h|_{X'_{j+1}} = f$, it follows that *(5)* holds. //

Proposition 2.13 (cf. [Lut17b], Lemma 1.0.5)**.** *Let $D = D^{(l)}$ be the DSH algebra constructed in Lemma 2.12. There is a *-isomorphism $\Gamma \colon B \to D$ given coordinate-wise by $\Gamma(b)_i := a_i|_{X'_i}$ for $1 \leq i \leq l$, where $a \in A$ is any lift of b under ρ. In particular, the quotient B is a DSH algebra.*

Proof. Let us first show that $\Gamma(b)$ is independent of the choice of lift. Fix $1 \leq i \leq l$ and suppose $g, h \in A$ satisfy $\rho(g) = \rho(h)$. We must show that $g_i|_{X'_i} = h_i|_{X'_i}$. Note that $\overline{X'_i \setminus Y'_i}^{X_i} = X'_i$. Indeed, X'_i is closed with respect to the topology on X_i, and so the fact that $\overline{X'_i \setminus Y'_i}^{X_i}$ is a subset of X'_i is clear; for the reverse inclusion, if $z \in X'_i$, there is a sequence

$(z_n)_n \subset \hat{\rho}(\hat{B}) \cap X_i \subset X_i' \setminus Y_i \subset X_i' \setminus Y_i'$ that converges to z in X_i. Hence, by continuity, it suffices to show that $g_i|_{X_i' \setminus Y_i'} = h_i|_{X_i' \setminus Y_i'}$. To this end, suppose $x \in X_i' \setminus Y_i'$. Then $x \in \overline{X_i \cap \hat{\rho}(\hat{B})}^{X_i} = \overline{(X_i \setminus Y_i) \cap \hat{\rho}(\hat{B})}^{X_i}$ and $x \notin Y_i$. By Lemma 2.10 and Lemma 2.4, $(X_i \setminus Y_i) \cap \hat{\rho}(\hat{B})$ is closed in $X_i \setminus Y_i$ in the subspace topology coming from X_i. Thus,

$$\begin{aligned} x &\in \overline{(X_i \setminus Y_i) \cap \hat{\rho}(\hat{B})}^{X_i} \cap (X_i \setminus Y_i) \\ &= \overline{(X_i \setminus Y_i) \cap \hat{\rho}(\hat{B})}^{X_i \setminus Y_i} \\ &= (X_i \setminus Y_i) \cap \hat{\rho}(\hat{B}) \subset \hat{\rho}(\hat{B}). \end{aligned}$$

Therefore, there is a $[\pi] \in \hat{B}$ such that $[\pi \circ \rho] = \hat{\rho}([\pi]) = [\mathrm{ev}_x]$. But this implies that $g - h \in \ker \mathrm{ev}_x$ since $g - h \in \ker \rho$. Hence, $g_i(x) = h_i(x)$, as desired. Moreover, $\Gamma(b)_i$ belongs to $C(X_i', M_{n_i})$, being the restriction of a continuous function. To see that $\Gamma(b)$ respects the decomposition structure of D, suppose $y \in Y_i'$ decomposes into $x_1 \in X_{i_1}' \setminus Y_{i_1}', \ldots, x_t \in X_{i_t}' \setminus Y_{i_t}'$. Then,

$$\begin{aligned} \Gamma(b)_i(y) &= a_i(y) \\ &= \mathrm{diag}(a_{i_1}(x_1), \ldots, a_{i_t}(x_t)) \\ &= \mathrm{diag}(\Gamma(b)_{i_1}(x_1), \ldots, \Gamma(b)_{i_t}(x_t)). \end{aligned}$$

Therefore, Γ is well defined and it is straightforward to check that it is a *-homomorphism. We have left only to check that it is a bijection.

To see that Γ is injective, suppose $b \in B$ and $a \in A$ is such that $\rho(a) = b$. Assume that $\Gamma(b) = 0$. Let π be an arbitrary irreducible representation of B. To show that $b = 0$, it suffices to show that $\pi(b) = 0$. Note that $[\pi \circ \rho] = \hat{\rho}([\pi]) \in \hat{\rho}(\hat{B})$. Thus, for some $1 \leq i \leq l$, there is an $x \in (X_i \setminus Y_i) \cap \hat{\rho}(\hat{B}) \subset X_i'$ such that $[\pi \circ \rho] = [\mathrm{ev}_x]$. Since $\mathrm{ev}_x(a) = a_i(x) = \Gamma(b)_i(x) = 0$, it follows that $\pi(b) = \pi(\rho(a)) = 0$. Thus, Γ is injective.

To see that Γ is surjective, suppose $d \in D$. By property *(5)* in Lemma 2.12, there is a $g \in A$ such that $g_i|_{X_i'} = d_i$ for all $1 \leq i \leq l$. Let $h = \rho(g) \in B$ and observe that for all $1 \leq i \leq l$, we have $\Gamma(h)_i = g_i|_{X_i'} = d_i$. Thus, $\Gamma(h) = d$, so Γ is surjective.

We have shown that Γ is a *-isomorphism, from which it follows that B is a DSH algebra. //

Proposition 2.14. *Given an inductive limit*

$$A_1 \xrightarrow{\psi_1} A_2 \xrightarrow{\psi_2} A_3 \xrightarrow{\psi_3} \cdots \longrightarrow A := \varinjlim A_i$$

of DSH algebras with diagonal maps, there exist DSH algebras $D_1, D_2, \ldots$ and injective diagonal maps $\psi'_i \colon D_i \to D_{i+1}$ such that

$$D_1 \xrightarrow{\psi'_1} D_2 \xrightarrow{\psi'_2} D_3 \xrightarrow{\psi'_3} \cdots \longrightarrow A.$$

Proof. For $n \in \mathbb{N}$, let $\mu_n \colon A_n \to A$ denote the map in the construction of the inductive limit and consider the surjective map $\kappa_n \colon A_n \to A_n / \ker \mu_n =: B_n$. The induced map $\nu_n \colon B_n \to B_{n+1}$ given by $\nu_n(\kappa_n(a)) := \kappa_{n+1}(\psi_n(a))$ for all $a \in A_n$ is well defined and injective. Furthermore, $\varinjlim(B_n, \{\nu_n\}_n) = A$. Let $X_1^n, \ldots, X_{l(n)}^n$ denote the base spaces of A_n and let $Y_1^n, \ldots, Y_{l(n)}^n$ denote the corresponding closed subsets. Let D_n denote the DSH algebra given by Lemma 2.12 and isomorphic to B_n (with base spaces $\overline{X_i^n \cap \hat{\kappa}_n(\hat{B_n})}^{X_i^n} =: Z_i^n$ and corresponding closed subsets $Z_i^n \cap Y_i^n =: W_i^n$ for $1 \leq i \leq l(n)$). By Proposition 2.13, the injective map ν_n drops down to an injective map $\psi'_n \colon D_n \to D_{n+1}$ given by $\psi'_n(d)_i := \psi_n(a)_i|_{Z_i^{n+1}}$ for all $1 \leq i \leq l(n+1)$, where $a \in A_n$ is any coordinate-wise extension of d. Moreover, $\varinjlim(D_n, \{\psi'_n\}_n) = A$.

We need to check that ψ'_n is diagonal. Fix $1 \leq i \leq l(n+1)$ and suppose $x \in Z_i^{n+1} \setminus W_i^{n+1} \subset X_i^{n+1} \setminus Y_i^{n+1}$ decomposes into $x_1 \in X_{i_1}^n \setminus Y_{i_1}^n, \ldots, x_t \in X_{i_t}^n \setminus Y_{i_t}^n$ under the diagonal map ψ_n. We need to show that $x_j \in Z_{i_j}^n \setminus W_{i_j}^n$ for all $1 \leq j \leq t$. Since $\mathrm{ev}_x \circ \psi'_n$ is a *-representation of D_n, it is unitarily equivalent to a finite direct sum of irreducible representations $\mathrm{ev}_{z_1}, \ldots, \mathrm{ev}_{z_k} \in \bigsqcup_{s=1}^{l(n)} (Z_s^n \setminus W_s^n) \subset \hat{A}_n$. Fix $1 \leq j \leq t$. If $x_j \notin \{z_1, \ldots, z_k\}$, then by Proposition 4.2.5 of [Dix77], there is a function $a \in A_n$ such that $\mathrm{ev}_{z_s}(a) = 0$ for all $1 \leq s \leq k$, but $\mathrm{ev}_{x_j}(a) \neq 0$. Since x_j is in the decomposition of x under ψ_n, this implies that $\mathrm{ev}_x(\psi_n(a))$ is both zero and non-zero simultaneously. Therefore, it must be that $x_j \in \{z_1, \ldots, z_k\}$ and, thus, that $x_j \in Z_{i_j}^n \setminus W_{i_j}^n$, as desired. //

2.3 CHARACTERIZATION OF DECOMPOSITION POINTS

Let A be a fixed DSH Algebra of length l. In this section, we further explore the decomposition structure of a A. We provide a sufficient condition for an open set in $\bigsqcup_{i=1}^{l} (X_i \setminus Y_i)$ to be open in $\hat{A}$ (see Proposition 2.18), thereby proving a converse to Lemma 2.4; this is essential to the proof of Lemma 3.3. Moreover, we establish a necessary and sufficient condition for a point in the spectrum of $\hat{A}$ to be in the decomposition of a point in one of the base spaces of A (see Proposition 2.19).

Definition 2.15. Given $1 \le i < j \le l$ and $y \in Y_j$, let $\tau_{i,j}\colon Y_j \to \mathcal{P}(X_i)$ be the function (ignoring multiplicity) given by

$$\tau_{i,j}(y) := \{x \in X_i : x \text{ is in the decomposition of } y\}.$$

For $E \subset X_i$, let

$$\tau_{i,j}^{-1}(E) := \{y \in Y_j : \tau_{i,j}(y) \cap E \neq \varnothing\},$$

and, for $F \subset Y_j$, let

$$\tau_{i,j}(F) := \{x \in X_i : x \in \tau_{i,j}(y) \text{ for some } y \in F\}.$$

Lemma 2.16. *Suppose $1 \le i < j \le l$.*

(1) If E is a closed subset of X_i, then $\tau_{i,j}^{-1}(E)$ is closed in X_j.

(2) If F is a closed subset of X_j, then $\tau_{i,j}(F)$ is closed in X_i.

Proof. We prove *(1)* only; the proof of *(2)* is very similar. Suppose $(y_n)_n$ is a sequence of points in $\tau_{i,j}^{-1}(E)$ converging to $y \in Y_j$. For $n \in \mathbb{N}$, choose $z_n \in E$ such that $z_n \in \tau_{i,j}(y_n)$. Since E is closed, hence, compact, we may pass to a subsequence to conclude that $(z_n)_n$ converges to a point $x \in E$. By passing to a further subsequence, we may assume there is a $1 \le k \le n_j$ such that for all $n \in \mathbb{N}$, the representation z_n begins at index k down the diagonal in the decomposition of y_n. Let $f \in A$ be arbitrary. For each $n \in \mathbb{N}$, there are matrices $P_n \in M_{k-1}$ and $Q_n \in M_{n_j - n_i - (k-1)}$ such that $f_j(y_n) = \operatorname{diag}(P_n, f_i(z_n), Q_n)$. Since $\lim_{n\to\infty} f_j(y_n) = f_j(y)$, there are matrices $P \in M_{k-1}$ and $Q \in M_{n_j - n_i - (k-1)}$ such that

$$f_j(y) = \operatorname{diag}(P, f_i(x), Q). \tag{2.1}$$

If $x \in Y_i$, it follows by definition that x is in the decomposition of y. If $x \in X_i \setminus Y_i$ and $x \notin \tau_{i,j}(y)$, then we may use Proposition 4.2.5 of [Dix77] to find a function $g \in A$ that is non-zero at x, but vanishes on $\tau_{i,j}(y)$, which contradicts Equation 2.1. Thus, x is in the decomposition of y and, hence, $y \in \tau_{i,j}^{-1}(E)$, proving *(1)*. //

Lemma 2.17. *Suppose $2 \le k \le l$ and that $U_1 \subset X_1, \dots, U_k \subset X_k$ are such that:*

(1) U_k is open in X_k;

(2) $\tau_{j,k}^{-1}(U_j) \subset U_k$ for all $1 \le j < k$.

Suppose $g \in A^{(k-1)}$ is such that for all $1 \le r \le k-1$, $g_r|_{X_r \setminus U_r} \equiv 0$. Then, there is an $f \in C(X_k, M_{n_k})$ that vanishes outside of U_k and satisfies $g \oplus f \in A^{(k)}$.

Proof. We have $\varphi_{k-1}(g) \in C(Y_k, M_{n_k})$. If $\varphi_{k-1}(g)$ is non-zero at a point $y \in Y_k$, then there must be a $1 \leq j \leq k-1$ and a point $x \in X_j$ in the decomposition of y at which g_j is non-zero. By assumption, $x \in U_j$ and then by *(2)*, $y \in U_k$. Let $Y_k' := Y_k \cap U_k$ and note that $\varphi_{k-1}(g)|_{\overline{Y_k'} \setminus Y_k'} \equiv 0$. Hence, we may extend the continuous function $\varphi_{k-1}(g)|_{\overline{Y_k'}}$ to a function $f \in C(X_k, M_{n_k})$ that vanishes outside of U_k. Since $Y_k \setminus \overline{Y_k'} \subset Y_k \setminus Y_k' = Y_k \setminus U_k$, it follows that both f and $\varphi_{k-1}(g)$ vanish on $Y_k \setminus \overline{Y_k'}$. Thus, $\varphi_{k-1}(g)$ and f agree on all of Y_k, and therefore, $g \oplus f \in A^{(k)}$. //

Proposition 2.18. *Suppose $U_1 \subset X_1, \dots, U_l \subset X_l$ are defined so that:*

(1) for all $1 \leq i \leq l$, U_i is open in X_i;

(2) $\tau_{i,j}^{-1}(U_i) \subset U_j$ for all $1 \leq i < j \leq l$.

Then, $U := \bigsqcup_{i=1}^{l} U_i \cap (X_i \setminus Y_i)$ is open in $\hat{A}$.

Proof. Let $1 \leq i \leq l$ and $z \in U_i \cap (X_i \setminus Y_i)$. Put $g_1 = \cdots = g_{i-1} \equiv 0$ and choose $g_i \in C(X_i, M_{n_i})$ such that $g_i(x) \neq 0$ and $g|_{X_i \setminus U_i} \equiv 0$. Then $l - i$ applications of Lemma 2.17 (using the U_j's in this lemma) extend $(g_1, \dots, g_i) \in A^{(i)}$ to a function $g \in A$ that vanishes outside of U. This proves that U is open in $\hat{A}$. //

Proposition 2.19. *Suppose $1 \leq i < j \leq l$, $x \in X_i \setminus Y_i$, and $y \in Y_j$. Then, x is in the decomposition of y if and only if for all $U \subset \hat{A}$ open in the hull-kernel topology and containing x, we have $y \in \overline{U \cap (X_j \setminus Y_j)}^{X_j}$.*

Proof. Suppose first that x is in the decomposition of y, and assume on the contrary that for some open subset $U \ni x$ of the spectrum, $y \notin \overline{U \cap (X_j \setminus Y_j)}^{X_j}$. Then there is an $f \in A$ with $f_i(x) \neq 0$ and $f|_{\hat{A} \setminus U} \equiv 0$, and there is a set $V \subset X_j$ containing y and open in X_j such that $V \cap (U \cap (X_j \setminus Y_j)) = \varnothing$. Hence, $f_j|_{V \cap (X_j \setminus Y_j)} \equiv 0$. By Lemma 2.9, we may assume that $\operatorname{int}(Y_j) = \varnothing$, so that $V \subset \overline{V \cap (X_j \setminus Y_j)}^{X_j}$. Therefore, $f_j(y) = 0$, from which it follows that $f_i(x) = 0$ since x is in the decomposition of y. This yields the desired contradiction.

Conversely, suppose that x is not in the decomposition of y.

Claim 2.19.1. *There exist $U_i \subset X_i, \dots, U_j \subset X_j$ such that:*

(1) for all $i \leq k \leq j$, U_k is open in X_k;

(2) $\tau_{k,k'}^{-1}(U_k) \subset U_{k'}$ for all $i \leq k < k' \leq j$;

(3) $x \in U_i$;

(4) $y \notin \overline{U_j}^{X_j}$.

Proof. Applying the first part of Lemma 2.16 to $\{x\}$, we see that there is a neighbourhood $F \subset X_j$ of y, which we may take to be closed such that x is not in the decomposition of any point in F. Apply the second part of Lemma 2.16 to F to obtain an open set $U_i \subset X_i$ containing x with the property that no point in U_i is in the decomposition of any point in F. Let $U_j := X_j \setminus F$. By construction, *(3)* and *(4)* hold. We define the rest of the sets inductively. Fix $i < k < j$ and assume we have defined open $U_i \subset X_i, \ldots, U_{k-1} \subset X_{k-1}$ such that:

(a) $\tau_{k'',k'}^{-1}(U_{k''}) \subset U_{k'}$ for all $i \leq k'' < k' < k$;

(b) $\tau_{k',j}^{-1}(U_{k'}) \subset U_j$ for all $i \leq k' < k$.

We show how to define U_k. Let $E := \bigcup_{k'=i}^{k-1} \tau_{k',k}^{-1}(U_{k'}) \subset Y_k$ Then no point in E is in the decomposition of any point in F. Indeed, if $z \in E$ is in the decomposition of $z' \in Y_j$, then by the definition of E, for some $i \leq k' < k$ there is a point $w \in U_{k'}$, which is in the decomposition of z, and hence, of z'; thus, z' is not contained in F by (b). Therefore, we may use the second part of Lemma 2.16 to obtain an open subset $U_k \supset E$ of X_k such that no point in U_k is in the decomposition of any point in F. Hence, (b) holds for $k' = k$. To see that (a) holds for $k' = k$, observe that if $i \leq k'' < k$ and a point in $U_{k''}$ is in the decomposition of a point $z \in Y_k$, then by definition z belongs to $E \subset U_k$. This completes the inductive definition with (a) and (b) holding when $k = j$. It follows that *(1)* and *(2)* hold, proving the claim. //

For $1 \leq k < i$, put $U_k := \varnothing$ and for $j < k \leq l$, put $U_k := X_k$. Then $U_1, \ldots, U_l$ satisfy the conditions of Proposition 2.18, so that $U := \bigsqcup_{i=1}^{l} U_i \cap (X_i \setminus Y_i)$ is open in $\hat{A}$. By *(3)* and *(4)* in Claim 2.19.1, U contains x and $y \notin \overline{U \cap (X_j \setminus Y_j)}^{X_j}$, which proves the proposition. //

Corollary 2.20. *Suppose $1 \leq i < j \leq l$, $(x_n)_n$ is a sequence in $X_j \setminus Y_j$, and $x \in X_i \setminus Y_i$. Suppose $\mathrm{ev}_{x_n} \to \mathrm{ev}_x$, with respect to the hull-kernel topology on $\hat{A}$. Then there is a $y \in Y_j$, which is a cluster point in X_j of $(x_n)_n$. Moreover, every cluster point in Y_j of $(x_n)_n$ contains x in its decomposition.*

Proof. If no point in Y_j is a cluster point of $(x_n)_n$, then there is a set $U_j \subset X_j$ containing Y_j and open in X_j that has empty intersection with $(x_n)_n$. For $1 \leq k \leq l$ different from j, let $U_k = X_k$. By Proposition 2.18, $\mathcal{O} := \bigsqcup_{k=1}^{l} U_k \cap (X_k \setminus Y_k)$ is open in $\hat{A}$. Since $\mathcal{O}$ contains ev_x, this contradicts the fact that $\mathrm{ev}_{x_n} \to \mathrm{ev}_x$. If $y \in Y_j$ is a cluster point of $(x_n)_n$, then for all open $V \in \hat{A}$, the set $\overline{V \cap (X_j \setminus Y_j)}^{X_j}$ contains y. It follows by Proposition 2.19 that x is in the decomposition of y. //

2.4 HOMOGENEOUS DSH ALGEBRAS

Suppose A is a DSH algebra and that A is n-homogeneous. We show in this section that there is a compact Hausdorff space X such that A is isomorphic to $C(X, M_n)$. We start with the following proposition.

Proposition 2.21. *Let X_1, X_2 be compact Hausdorff spaces. Let Y_2 be a closed subset of X_2. Let $\varphi\colon C(X_1, M_n) \to C(Y_2, M_n)$ be a unital *-homomorphism and suppose the associated pullback $C(X_1, M_n) \oplus_{C(Y_2,M_n)} C(X_2, M_n)$ is a DSH algebra. Then, there is a compact Hausdorff space Z^* such that $C(X_1, M_n) \oplus_{C(Y_2,M_n)} C(X_2, M_n)$ is isomorphic to $C(Z^*, M_n)$.*

Proof. Let $\tau := \tau_{1,2}$ denote the map from Definition 2.15. For a given $y \in Y_2$, we know by Lemma 2.6 that the point it decomposes into is unique; alternatively, note that if there were two distinct points in the decomposition of y under φ, then these two points could not be separated by any function in $C(X_1, M_n)$. Therefore, τ is single-valued and we may view it as a map from Y_2 to X_1.

Claim 2.21.1. *$\tau\colon Y_2 \to X_1$ is a closed and continuous map.*

Proof. This is just a reinterpretation of Lemma 2.16. //

Let $Z := X_1 \sqcup X_2$. Then Z is both compact and Hausdorff. Given $z \in Z$, we define $[z]$ as follows:

$$[z] := \begin{cases} \{z\} & \text{if } z \in X_2 \setminus Y_2 \\ \{z\} \cup \tau^{-1}(z) & \text{if } z \in X_1 \\ \{\tau(z)\} \cup \tau^{-1}(\tau(z)) & \text{if } z \in Y_2. \end{cases}$$

Let $Z^* := \{[z] : z \in Z\}$ and let $p\colon Z \to Z^*$ denote the canonical surjection $p(z) := [z]$. Then Z^* is a collection of sets that partition Z. We equip it with the quotient topology induced by p; that is, a set $U \subset Z^*$ is open in Z^* if and only if $p^{-1}(U)$ is open in Z. Since Z is compact, so is Z^*.

Claim 2.21.2. *Z^* is Hausdorff.*

Proof. Suppose $z_1, z_2 \in Z$ with $[z_1] \neq [z_2]$. Let us show that $[z_1]$ and $[z_2]$ can be separated by open sets in Z^*. Without loss of generality, we must be in one of the following four cases.

Case one: $z_1, z_2 \in (X_2 \setminus Y_2) \cup (X_1 \setminus \tau(Y_2))$. In this case, it is easy to see (since Y_2 and $\tau(Y_2)$ are closed) that we may choose open sets $U_1 \ni z_1$ and $U_2 \ni z_2$ in Z that are disjoint and such that $U_i \subset X_2 \setminus Y_2$ if $z_i \in X_2 \setminus Y_2$ and $U_i \subset X_1 \setminus \tau(Y_2)$ if $z_i \in X_1 \setminus \tau(Y_2)$. Since $p|_{(X_2 \setminus Y_2) \cup (X_1 \setminus \tau(Y_2))}$ is a bijection, the sets $p(U_1)$ and $p(U_2)$ are open in Z^*, disjoint, and contain $[z_1]$ and $[z_2]$, respectively.

Case two: $z_1 \in X_1 \setminus \tau(Y_2)$ and $z_2 \in Y_2 \cup \tau(Y_2)$. Choose disjoint sets $U_1 \ni z_1$ and $U_2 \supset \tau(Y_2)$ that are open in X_1. Let $V_1 := p(U_1) \ni [z_1]$ and $V_2 := p(U_2 \cup X_2) \ni [z_2]$ and note that $V_1 \cap V_2 = \varnothing$. Since $p^{-1}(V_1) = U_1$ and $p^{-1}(V_2) = U_2 \cup X_2$ are both open in Z, it follows that V_1 and V_2 are open in Z^*.

Case three: $z_1 \in X_2 \setminus Y_2$ and $z_2 \in Y_2 \cup \tau(Y_2)$. Choose disjoint sets $U_1 \ni z_1$ and $U_2 \supset Y_2$ that are open in X_2. Let $V_1 := p(U_1) \ni [z_1]$ and $V_2 := p(X_1 \cup U_2) \ni [z_2]$ and note that $V_1 \cap V_2 = \varnothing$. Since $p^{-1}(V_1) = U_1$ and $p^{-1}(V_2) = X_1 \cup U_2$ are both open in Z, it follows that V_1 and V_2 are open in Z^*.

Case four: $z_1, z_2 \in Y_2 \cup \tau(Y_2)$. We may assume without loss of generality that $z_1, z_2 \in \tau(Y_2)$. Choose sets $U_1 \ni z_1$ and $U_2 \ni z_2$, which are open in X_1 and satisfy $\overline{U_1} \cap \overline{U_2} = \varnothing$. Since τ is continuous, there are open subsets V_1 and V_2 of X_2 such that $\tau^{-1}(U_1) = V_1 \cap Y_2$ and $\tau^{-1}(U_2) = V_2 \cap Y_2$. Choose disjoint open subsets W_1 and W_2 of X_2 containing $\tau^{-1}(\overline{U_1})$ and $\tau^{-1}(\overline{U_2})$, respectively. Put $\mathcal{O}_1 := V_1 \cap W_1$ and $\mathcal{O}_2 := V_2 \cap W_2$ and let $\mathcal{E}_1 := p(\mathcal{O}_1 \cup U_1)$ and $\mathcal{E}_2 := p(\mathcal{O}_2 \cup U_2)$. Note that $[z_1] \in \mathcal{E}_1$ and $[z_2] \in \mathcal{E}_2$. Let us show that $\mathcal{E}_1$ and $\mathcal{E}_2$ are disjoint and open in Z^*. Suppose $t_1 \in \mathcal{O}_1 \cup U_1$ and $t_2 \in \mathcal{O}_2 \cup U_2$. Assume first that $t_1 \in U_1$. If $t_2 \in U_2 \cup (X_2 \setminus Y_2)$, then $[t_1] \neq [t_2]$ since $U_1 \cap U_2 = \varnothing$. If instead $t_2 \in \mathcal{O}_2 \cap Y_2$, then $\tau(t_2) \in U_2$, so that we again have $[t_1] \neq [t_2]$. A symmetric analysis shows that $[t_1] \neq [t_2]$ if $t_2 \in U_2$. Thus, we may assume $t_1 \in \mathcal{O}_1$ and $t_2 \in \mathcal{O}_2$. If either t_1 or t_2 is in $X_2 \setminus Y_2$, then $[t_1] \neq [t_2]$ since $t_1 \neq t_2$, as $\mathcal{O}_1$ and $\mathcal{O}_2$ are disjoint. If instead $t_1 \in \mathcal{O}_1 \cap Y_2$ and $t_2 \in \mathcal{O}_2 \cap Y_2$, then $\tau(t_1) \in U_1$, $\tau(t_2) \in U_2$, and once again $[t_1] \neq [t_2]$. It follows that $\mathcal{E}_1 \cap \mathcal{E}_2 = \varnothing$. It remains to be shown that $\mathcal{E}_1$ and $\mathcal{E}_2$ are open in Z^*. Owing to the symmetry of the setup, we only show that $\mathcal{E}_1$ is open in Z^*, and to do this it is sufficient to prove that $p^{-1}(\mathcal{E}_1) \cap X_1 = U_1$ and $p^{-1}(\mathcal{E}_1) \cap X_2 = \mathcal{O}_1$. Assume we are given $t \in p^{-1}(\mathcal{E}_1) \cap X_1$. Thus $[t] \in p(\mathcal{O}_1) \cup p(U_1)$. If there is an $s \in U_1$ such that $[t] = [s]$, then, since both t and s lie in X_1, it must be that $t = s$. If instead there is an $s \in \mathcal{O}_1$ such that $[t] = [s]$, then it follows that $s \in Y_2$, and hence, that $\tau(s) \in U_1$. Thus, $[t] = [s] = [\tau(s)]$, from which we deduce as before that $t = \tau(s) \in U_1$. Therefore, we may conclude that $p^{-1}(\mathcal{E}_1) \cap X_1 \subset U_1$, and hence, $p^{-1}(\mathcal{E}_1) \cap X_1 = U_1$. Now, suppose that we are given $t \in p^{-1}(\mathcal{E}_1) \cap X_2$. As before, $[t] \in p(\mathcal{O}_1) \cup p(U_1)$. Suppose first that there is an $s \in \mathcal{O}_1$ such that $[t] = [s]$. If either t or s is in $X_2 \setminus Y_2$, then $t = s \in \mathcal{O}_1$; otherwise, it must be that $s, t \in Y_2$ and, in particular, $\tau(t) = \tau(s) \in U_1$. Therefore, $t \in V_1 \cap W_1 = \mathcal{O}_1$. If instead there is an $s \in U_1$ such that $[t] = [s]$, then it must be that $t \in Y_2$ and $\tau(t) = s \in U_1$, which implies (as above) that $t \in \mathcal{O}_1$. Thus, $p^{-1}(\mathcal{E}_1) \cap X_2 \subset \mathcal{O}_1$, and hence,

$p^{-1}(\mathcal{E}_1) \cap X_2 = \mathcal{O}_1$. Therefore, by our analysis, this implies that $\mathcal{E}_1$ and $\mathcal{E}_2$ are open in Z^*. //

Now, define $\Lambda\colon C(X_1, M_n) \oplus_{C(Y_2,M_n)} C(X_2, M_n) \to C(Z^*, M_n)$ by:

$$\Lambda((f,g))([z]) := \begin{cases} f(z) & \text{if } z \in X_1 \\ g(z) & \text{if } z \in X_2. \end{cases}$$

To conclude the proof of the proposition, we show that Λ is a well-defined *-isomorphism. To see that Λ is well defined, suppose $z_1, z_2 \in Z$ and that $[z_1] = [z_2]$. Unless $z_1 = z_2$, this implies that one of the two points is in the decomposition of the other. Assume without loss of generality that $\tau(z_2) = z_1$. Then for all $(f,g) \in C(X_1, M_n) \oplus_{C(Y_2,M_n)} C(X_2, M_n)$, we have $g(z_2) = \varphi(f)(z_2) = f(\tau(z_2)) = f(z_1)$. This shows that Λ is well defined. It is clear that Λ is an injective *-homomorphism. To see surjectivity, suppose $h \in C(Z^*, M_n)$ and define $f := h \circ p|_{X_1} \in C(X_1, M_n)$ and $g := h \circ p|_{X_2} \in C(X_2, M_n)$. Given $y \in Y_2$, we have

$$g(y) = h([y]) = h([\tau(y)]) = f(\tau(y)) = \varphi(f)(y),$$

so that $(f,g) \in C(X_1, M_n) \oplus_{C(Y_2,M_n)} C(X_2, M_n)$. Moreover, $\Lambda((f,g)) = h$, proving that Λ is surjective. This completes the proof of the proposition. //

Applying Proposition 2.21 inductively, we obtain the following corollary.

Corollary 2.22. *Every n-homogeneous DSH algebra is isomorphic to a full matrix algebra, i.e., isomorphic to $C(X, M_n)$ for some compact Hausdorff space X.*

Remark 2.23. If the base spaces X_1, X_2 in the given DSH algebra in Proposition 2.21 are metric spaces, then the resulting base space Z^* is also metrizable. To see this, let $w(Y)$ denote the smallest cardinality of a basis for a given topological space Y. If X_1 and X_2 are metric spaces, so too is the space Z, constructed in the proof of Proposition 2.21. By Theorem 3.1.22 of [Eng89], $w(Z^*) \leq w(Z)$. Since a compact Hausdorff space is metrizable if and only if it has a countable basis, this shows that Z^* is metrizable.

3

SIMPLE LIMITS OF DSH ALGEBRAS WITH DIAGONAL MAPS

This chapter focuses on simple inductive limits of DSH algebras with diagonal maps between them. The principal result is that every limit algebra of this type necessarily has stable rank one (see Theorem 3.14).

Section 3.1 consists of all the lemmas that are used in the proof of Theorem 3.14 in Section 3.2 with the following dependency diagram.

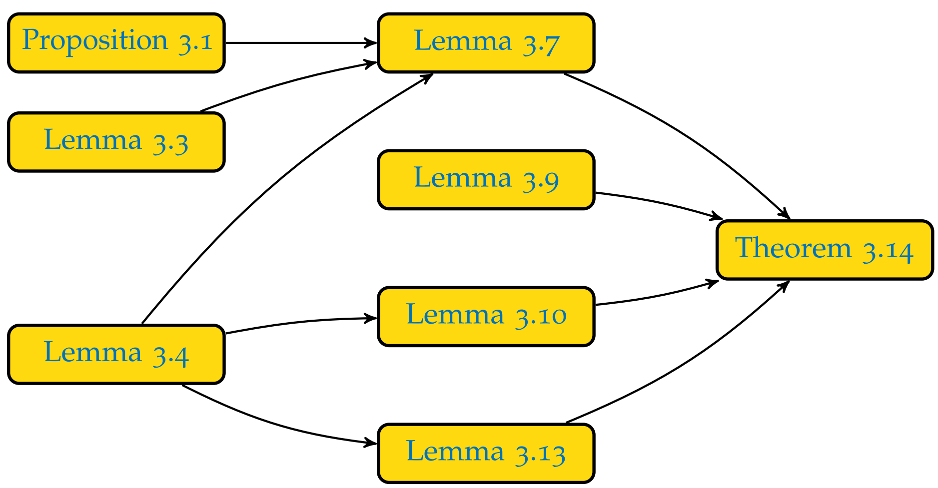

Figure 3.1: Dependency chart for the lemmas used in the proof of Theorem 3.14.

Let us now outline the importance of each of these lemmas and give a brief overview of how they are used to prove Theorem 3.14.

Our general strategy for proving that a simple inductive limit of DSH algebras with diagonal maps has stable rank one is essentially the one in [EHT08]. We start with a given element f in the limit algebra A, which may be assumed to lie in some finite-stage building block A_j. If f is invertible, then there is nothing to prove, and so we may assume that f is not invertible. The goal is then to show that the image $\psi_{j',j}(f)$ of f in a later stage algebra $A_{j'}$ is close to an invertible in $A_{j'}$.

If we approximate $\psi_{j',j}(f)$, multiply this approximation by unitaries, approximate again, multiply the new approximation by unitaries, and show that an element thus obtained is close to an invertible, then, upon unpacking the approximations, it follows that $\psi_{j',j}(f)$ is close to an invertible in $A_{j'}$. Finally, as Rørdam observed in [Rø91], every nilpotent element of a unital C*-algebra is close to an invertible. Therefore, it suffices to show that an element, obtained from $\psi_{j',j}(f)$ as above, is nilpotent.

To execute this strategy, one proceeds as follows. In Lemma 3.3, we show that there is a point x in one of the base spaces X_i of A_j at which $f_i(x)$ is a non-invertible matrix. After multiplying by unitary matrices on the left and right we obtain a new matrix whose first row and column contain only zeros (or one that has a *zero cross at index* 1 (see Definition 3.2)). We show that after perturbing f slightly, we may multiply this perturbation f' on the left and right by unitaries $w, v \in A_j$, so that $wf'v$ has a zero cross at index 1 not just at x, but at each point in a neighbourhood of x; this neighbourhood also turns out to be an open subset of the spectrum of A_j by Proposition 2.18.

By Proposition 2.14, we may assume the maps in the given sequence are injective. Hence, in Lemma 3.7, we may apply our simplicity criterion (Proposition 3.1) with the open subset of the spectrum obtained above to conclude that in some later stage algebra $A_{j'}$, the diagonal image $\psi_{j',j}(wf'v)$ has "many" (see the following paragraphs) zero crosses at every point in each base space of $A_{j'}$; because of simplicity and the fact that the maps in the sequence are diagonal, this "many" may be taken to be as large as desired. We are then able to construct unitaries $V, V' \in A_{j'}$ that organize the location of these zero crosses, so that the element $f'' = V\psi_{j',j}(f')V'$ has "many" zero crosses occurring at tractable locations at each point in every base space of $A_{j'}$.

We use Lemma 3.9 to approximate f'' by a function $g \in A_{j'}$ that preserves the zero crosses of f'' at each point, and, in addition, extends the block-diagonal structure of the algebra to neighbourhoods of the closed subsets of the base spaces (the Y_i's in the definition of $A_{j'}$). This allows us, in Lemma 3.10, to conjugate g by a unitary $W \in A_{j'}$, so that in the resulting conjugation $g' = WgW^*$, the zero crosses of g are grouped together into block zero crosses at every point in each of the base spaces of $A_{j'}$.

The unitaries V, V', and W above are constructed in such a way that at every point in each base space the *diagonal radius*, which measures how far a non-zero entry can occur from the diagonal in a matrix (see Definition 3.5), of g at that point is bounded above by a quantity independent of j'. Thus, by ensuring that the "many" above is at least as large as this upper bound, we are able to construct a unitary W' in Lemma 3.13 that shifts the block zero cross mentioned above so that $g'W'$ is strictly lower triangular at each point. This ensures that $g'W'$ is nilpotent and yields the desired result.

The unitaries V, V', W, and W' above are all defined using continuous paths of unitaries between permutation matrices (see Definition 3.6 and Definition 3.11). In Lemma 3.4, we construct certain indicator-function-like elements of DSH algebras, which help to define V, V', W, and W'. Their job is to keep track of the decomposition structure of their algebra and to tell the continuous paths used in defining V, V', W, and W' when to shift around the rows and columns.

In Section 3.3, we apply Theorem 3.14 to obtain two results about simple dynamical crossed products. Given an infinite compact metric space T and a minimal homeomorphism $h\colon T \to T$, we show that every Putnam subalgebra of the induced dynamical crossed product $C^*(\mathbb{Z}, T, h)$ associated to any singleton set is a simple inductive limit of DSH algebras with diagonal maps (see Theorem 3.17). Consequently, we are able to show that $C^*(\mathbb{Z}, T, h)$ has stable rank one (see Corollary 3.18), and that $\mathcal{Z}$-stability is determined for such an algebra by strict comparison of positive elements (see Corollary 3.19).

3.1 THE MAIN LEMMAS

The purpose of this section is to prove all of the lemmas listed in Figure 3.1, in the order that they are used in the proof of the Theorem 3.14 in the next section.

We start with a proposition that characterizes when a unital injective limit of subhomogeneous algebras is simple in terms of the corresponding maps between their spectra. This is essentially Proposition 2.1 in [DNNP92], except that ours discusses the general unital subhomogeneous case. The proof is very similar.

Given unital subhomogeneous C^*-algebras A and B and a unital homomorphism $\psi\colon A \to B$, an irreducible representation π of B yields a representation $\pi \circ \psi$ of A. The finite-dimensional representation $\pi \circ \psi$ is unitarily equivalent to a direct sum $\tau_1 \oplus \cdots \oplus \tau_s$ of irreducible representations of A. In this way, we get a map $\hat{\psi}\colon \hat{B} \to \mathcal{P}(\hat{A})$ given by $\hat{\psi}([\pi]) := \{[\tau_1], \ldots, [\tau_s]\}$, where multiplicities are ignored.

Proposition 3.1 (cf. [Lut17b], Theorem 2.3.1)**.** *Suppose we have an inductive limit of the form*

$$A_1 \xrightarrow{\psi_1} A_2 \xrightarrow{\psi_2} A_3 \xrightarrow{\psi_3} \cdots \longrightarrow A := \varinjlim A_i,$$

where A is unital and for each $i \in \mathbb{N}$, A_i is subhomogeneous and ψ_i is injective. (Let $\psi_{j,i} := \psi_{j-1} \circ \cdots \circ \psi_i$.) Then, the following statements are equivalent.

(1) A is simple.

(2) *For all $i \in \mathbb{N}$ and all non-empty open $U \subset \hat{A}_i$, there is a $j > i$ such that $\hat{\psi}_{j,i}([\pi]) \cap U \neq \varnothing$ for all $[\pi] \in \hat{A}_j$.*

(3) *For all $i \in \mathbb{N}$, if $f \in A_i$ is non-zero, then there is a $j > i$ such that $\pi(\psi_{j,i}(f)) \neq 0$ for every non-zero irreducible representation π of A_j.*

Proof. For $n \in \mathbb{N}$, let $\mu_n \colon A_n \to A$ denote the map in the construction of the inductive limit. Since the ψ_j's are injective and A is unital, we may assume that the A_j's are all unital and that the μ_j's are injective and unit-preserving.

Let us start by showing that *(1)* implies *(2)*. Suppose that *(2)* is false. To show *(1)* is false, let us construct a closed proper non-zero two-sided ideal of A. Let $i \in \mathbb{N}$ and let $U \subset \hat{A}_i$ be a non-empty open subset such that for all $j > i$ there is a $[\pi] \in \hat{A}_j$ with $\hat{\psi}_{j,i}([\pi]) \cap U = \varnothing$. For $j > i$, set $F_j := \{[\pi] \in \hat{A}_j : \hat{\psi}_{j,i}([\pi]) \cap U = \varnothing\}$ and set $I_j := \{f \in A_j : f \in \bigcap_{[\pi] \in F_j} \ker \pi\}$. We may assume that $U \neq \hat{A}_i$.

Claim 3.1.1. *For all $j > i$, F_j is non-empty and closed in $\hat{A}_j$.*

Proof. Fix $j > i$. By our assumption F_j is non-empty. To show that F_j is closed, suppose $[\tau] \notin F_j$. Let us show that $[\tau] \notin \overline{F_j}$. Take $[\sigma] \in \hat{\psi}_{j,i}([\tau]) \cap U$. Since U is open, there must be an $f \in A_i$ such that $f \in \bigcap_{[\rho] \in \hat{A}_i \setminus U} \ker \rho$ and $\sigma(f) \neq 0$. Since $[\sigma] \in \hat{\psi}_{j,i}([\tau])$, this implies $\tau(\psi_{j,i}(f)) \neq 0$. If $[\pi] \in F_j$, then $\pi(\psi_{j,i}(f)) = 0$ since $\hat{\psi}_{j,i}([\pi]) \subset \hat{A}_i \setminus U$ and $f \in \bigcap_{[\rho] \in \hat{A}_i \setminus U} \ker \rho$. Hence, $\psi_{j,i}(f) \in \bigcap_{[\pi] \in F_j} \ker \pi$ but $\psi_{j,i}(f) \notin \ker \tau$. This proves that $[\tau] \notin \overline{F_j}$. //

Claim 3.1.2. *For all $j > i$, I_j is a closed proper non-zero two-sided ideal of A_j.*

Proof. Fix $j > i$. Verifying that I_j is a two-sided ideal of A_j is routine. To see that I_j is non-zero take $[\sigma] \in U$. There is an $f \in A_i$ such that $f \in \bigcap_{[\rho] \in \hat{A}_i \setminus U} \ker \rho$ but $\sigma(f) \neq 0$. Since $\psi_{j,i}$ is injective, $\psi_{j,i}(f) \neq 0$ and, just as in the proof of Claim 3.1.1, $\pi(\psi_{j,i}(f)) = 0$ for all $[\pi] \in F_j$. Hence, $\psi_{j,i}(f) \in I_j$. I_j is proper since it cannot contain the unit of A_j, as F_j is non-empty. A routine check shows that I_j is closed in A_j. //

It is straightforward to verify that for $k > j > i$ and $[\pi] \in \hat{A}_k$, we have $\hat{\psi}_{k,i}([\pi]) = \hat{\psi}_{j,i}(\hat{\psi}_{k,j}([\pi]))$. From this it follows that $\hat{\psi}_{k,j}(F_k) \subset F_j$. Thus, $\psi_{k,j}(I_j) \subset I_k$ for all $k > j > i$. Hence, $\{\mu_j(I_j)\}_{j>i}$ is an increasing sequence of C*-algebras, and so $I := \overline{\bigcup_{j>i} \mu_j(I_j)}$ is a sub-C*-algebra of A. It is not hard to see that I is a closed two-sided ideal in A. Since the μ_j's are injective, Claim 3.1.2 implies $I \neq \{0\}$. If $1_A \in I$, then for large enough j, I_j contains 1_{A_j}, contradicting Claim 3.1.2. Hence, $\{0\} \not\subset I \not\subset A$, which shows that A cannot be simple. This proves that *(1)* implies *(2)*.

Let us now show that *(2)* implies *(3)*. Fix $i \in \mathbb{N}$ and suppose $0 \neq f \in A_i$. Let $U := \{[\rho] \in \hat{A}_i : \rho(f) \neq 0\}$. U is a non-empty open subset of $\hat{A}_i$. By *(2)*, there is a $j > i$ such that $\hat{\psi}_{j,i}([\pi]) \cap U \neq \varnothing$ for all $[\pi] \in \hat{A}_j$. Thus, if π is any irreducible representation of A_j, $\pi(\psi_{j,i}(f)) \neq 0$, which proves *(3)*.

Finally, let us prove that *(3)* implies *(1)*. Suppose J is a non-zero closed two-sided ideal in A. For $j \in \mathbb{N}$, put $J_j := \mu_j^{-1}(J)$. Then for all $j \in \mathbb{N}$, J_j is a closed two-sided ideal in A_j.

Claim 3.1.3. *There is a $j \in \mathbb{N}$ such that $J_j = A_j$.*

Proof. Take $0 \neq a \in J$. It is well known that $J = \overline{\bigcup_{j=1}^{\infty}(\mu_j(A_j) \cap J)}$. Hence, there must be an i and an $a_i \in A_i$ such that $0 \neq \mu_i(a_i) \in J$. Thus, $a_i \neq 0$. By *(3)*, there is a $j > i$ such that for all irreducible representations π of A_j, $\pi(\psi_{j,i}(a_i)) \neq 0$. Since $\mu_j(\psi_{j,i}(a_i)) = \mu_i(a_i) \in J$, it follows that $\psi_{j,i}(a_i) \in J_j$. The bijective correspondence between closed two-sided ideals of A_j and closed subsets of $\hat{A}_j$ forces $J_j = A_j$. //

Thus, $1_A = \mu_j(1_{A_j}) \in J$, which shows that $J = A$. Therefore, A is simple, which proves *(1)*. //

Definition 3.2 (Zero Cross; cf. [Lut17b], Definition 2.1.1)**.** Given a matrix $D \in M_n$ and $1 \leq k \leq n$, we say that D has a *zero cross at index k* provided that each entry in the kth row and column of D is 0.

Lemma 3.3 (cf. [Lut17b], Lemma 2.3.3)**.** *Let A be a DSH algebra of length l. Let $\epsilon > 0$. Suppose that $f \in A$ is not invertible. Then, there is an $f' \in A$ with $\|f - f'\| \leq \epsilon$ and there are unitaries $w, v \in A$ such that for some $1 \leq i \leq l$, $(wf'v)_i$ has a zero cross at index 1 everywhere on some non-empty set $U \subset \hat{A} \cap (X_i \setminus Y_i)$, which is open with respect to the hull-kernel topology on $\hat{A}$. Moreover, there is a $\Delta \in A$ such that for every $1 \leq j \leq l$ and $x \in X_j$, $\Delta_j(x)$ is a diagonal matrix with entries in $[0,1]$, where $\Delta_j(x)_{k,k} > 0$ implies $(wf'v)_j(x)$ has a zero cross at index k; moreover, $\Delta_i(z)_{1,1} = 1$ for all $z \in U$.*

Proof. Let us first establish a quick claim.

Claim 3.3.1. *There is a $1 \leq i \leq l$ and a point $x \in X_i$ such that $f_i(x)$ is a non-invertible matrix.*

Proof. Suppose on the contrary that the assertion of the claim is false. Define $g \in \bigoplus_{j=1}^{l} C(X_j, M_{n_j})$ to be $(g_1, \ldots, g_l)$, where for $1 \leq j \leq l$ and $z \in X_j$, $g_j(z) := f_j(z)^{-1}$. One can readily check that g belongs to A, contradicting that f is not invertible in A. //

By the claim, we may choose $1 \leq i \leq l$ and $x \in X_i$ such that $f_i(x)$ is a non-invertible matrix. Owing to the diagonal decomposition at points in Y_i, we may assume that $x \in X_i \setminus Y_i$. We break the proof up into two cases.

Case one: x is not in the decomposition of any point in any Y_j for $j > i$. By the second part of Lemma 2.16, there is set $U_1 \subset X_i$ containing x, which is open in X_i and has the property that no point in it is in the decomposition of any point in Y_j for any $j > i$. Since Y_i is closed in X_i, the set $U_1 \cap (X_i \setminus Y_i)$ is open in X_i. By shrinking U_1, we may assume that $\|f_i(x) - f_i(z)\| \leq \epsilon$ for all $z \in U_1$. Choose a set U_2 that is open in X_i and satisfies $x \in U_2 \subset \overline{U_2}^{X_i} \subset U_1 \cap (X_i \setminus Y_i)$. Using Urysohn's Lemma, we can define a function $h \in C(X_i, M_{n_i})$ such that $h|_{\overline{U_2}^{X_i}} \equiv f_i(x)$, $h|_{X_i \setminus (U_1 \cap (X_i \setminus Y_i))} = f_i|_{X_i \setminus (U_1 \cap (X_i \setminus Y_i))}$, and $\|f_i - h\| \leq \epsilon$. Define f' coordinate-wise by $f' := (f_1, \dots, f_{i-1}, h, f_{i+1}, \dots, f_l)$. Since $h|_{Y_i} = f_i|_{Y_i}$, we have $(f_1, \dots, f_{i-1}, h) \in A^{(i)}$. Since no point in U_1 is in the decomposition of any point in Y_j for any $j > i$, and because h may only differ from f_i on $U_1 \cap (X_i \setminus Y_i) \subset U_1$, this perturbation does not violate the diagonal decomposition at any point. Thus, $f' \in A$ since $f \in A$ and $\|f - f'\| \leq \epsilon$ because $\|f_i - h\| \leq \epsilon$. Since $f_i(x)$ is a non-invertible matrix, there are unitary matrices W and V in M_{n_i} with the property that $Wf_i(x)V$ has a zero cross at index 1. Since the unitary group in M_{n_i} is connected we may, using the same reasoning as above, define unitaries $w, v \in A$ coordinate-wise with $w_j = v_j \equiv 1_{n_j}$ for all $j \neq i$ and $w_i, v_i \in C(X_i, M_{n_i})$ satisfying $w_i|_{\overline{U_2}^{X_i}} \equiv W$, $v_i|_{\overline{U_2}^{X_i}} \equiv V$, and $w_i|_{X_i \setminus (U_1 \cap (X_i \setminus Y_i))} = v_i|_{X_i \setminus (U_1 \cap (X_i \setminus Y_i))} \equiv 1_{n_i}$. Finally, choose a set U_3 that is open in X_i and satisfies $x \in U_3 \subset \overline{U_3}^{X_i} \subset U_2$. Define $\Delta \in A$ coordinate-wise as follows: $\Delta_j \equiv 0$ for $j \neq i$; let $g\colon X_i \to [0, 1]$ be any continuous function such that $g|_{\overline{U_3}^{X_i}} \equiv 1$ and $g|_{X_i \setminus U_2} \equiv 0$, and put $\Delta_i := \operatorname{diag}(g, 0, \dots, 0) \in C(X_i, M_{n_i})$. As argued above for f', we have $\Delta \in A$. Take $U := U_3$. Applying Proposition 2.18 with $U_i := U$ and $U_k = \varnothing$ for $k \neq i$, we conclude U is open in $\hat{A}$. Since $(wf'v)_i$ has a zero cross at index 1 everywhere on U_2 and since Δ vanishes outside U_2, the lemma holds in this case.

Case two: There is a $j > i$ such that x is in the decomposition of some point in Y_j. In this case, we cannot define f' as above, because we are not guaranteed a neighbourhood around x in which we may freely perturb f while remaining in A. Let i' denote the largest integer for which x is in the decomposition of some point in $Y_{i'}$. Choose $y \in Y_{i'}$ such that x is in the decomposition of y. Then $f_{i'}(y)$ is a non-invertible matrix. Since x is not in the decomposition of any point in any Y_j for any $j > i'$, neither is y. Hence, by the second part of Lemma 2.16, there is a set $U_1 \subset X_{i'}$ containing y that is open in $X_{i'}$ with the property that no point in U_1 is in the decomposition of any point in Y_j for any $j > i'$. Hence, as in case one, we are able to perturb f on $U_1 \cap (X_{i'} \setminus Y_{i'})$, while remaining in A. By shrinking U_1, we may assume that $\|f_{i'}(y) - f_{i'}(z)\| \leq \epsilon$ for all $z \in U_1$. By Lemma 2.9, we may assume that $Y_{i'}$ has empty interior, and thus, that there is a point $x' \in U_1 \cap (X_{i'} \setminus Y_{i'})$. Choose a set U_2 which is open in $X_{i'}$ and satisfies $x' \in U_2 \subset \overline{U_2}^{X_{i'}} \subset U_1 \cap (X_{i'} \setminus Y_{i'})$. As in case one, we may define $f' \in A$ with $\|f - f'\| \leq \epsilon$, $f'_j = f_j$ for $j \neq i'$, $f'_{i'}|_{\overline{U_2}^{X_{i'}}} \equiv f_{i'}(y)$, and $f'_{i'}|_{X_{i'} \setminus (U_1 \cap (X_{i'} \setminus Y_{i'}))} = f_{i'}|_{X_{i'} \setminus (U_1 \cap (X_{i'} \setminus Y_{i'}))}$. Choose unitary matrices $W, V \in M_{n_{i'}}$ such that $W f_{i'}(y) V$ has a zero cross at index 1. Then the rest of the proof proceeds verbatim as the proof of case one with i' in place of i and x' in place of x. //

Lemma 3.4 (cf. [Lut17b], Lemma 2.2.2). *Suppose A is a DSH algebra of length l. Suppose $M \in \mathbb{N}$ and $K := \{K_1 < K_2 < \cdots < K_m\}$ are such that $K_1 \geq 0$, $K_m < \mathfrak{s}(A) - M$, and $K_{t+1} - K_t \geq M$ for $1 \leq t < m$. Suppose that for each $1 \leq i \leq l$ and $1 \leq j \leq n_i$, we have a set $F_{i,j} \subset X_i$ that is closed in X_i and disjoint from each set $B_{i,j-K_t}$ (see Definition 2.7) for $1 \leq t \leq m$. Then, there is a function $\Theta \in A$ such that:*

(1) for all $1 \leq i \leq l$ and $x \in X_i$, $\Theta_i(x)$ is a diagonal matrix with entries in $[0,1]$ whose final M diagonal entries are all 0, and such that at most one of every M consecutive diagonal entries is non-zero;

(2) for all $1 \leq i \leq l$, $1 \leq j \leq n_i$, and $x \in F_{i,j}$, we have $\Theta_i(x)_{j,j} = 0$;

(3) for all $1 \leq i \leq l$ and $1 \leq j \leq n_i$, there is a (possibly empty) open subset $U_{i,j} \subset X_i$ containing $B_{i,j}$ with the property that if $x \in U_{i,j}$, then $\Theta_i(x)_{j+K_t,j+K_t} = 1$ for all $1 \leq t \leq m$.

Proof. Consider the following statement:

(4) for all $1 \leq i \leq \ell$ and $1 \leq j \leq n_i$: $\Theta_i(x)_{j,j} = 1$ if and only if there is a $1 \leq t \leq m$ such that $x \in B_{i,j-K_t}$.

Claim 3.4.1. *To prove the existence of function in A satisfying (1) to (3), it suffices to construct $\Theta \in A$ satisfying (1) and (4).*

Proof. Suppose there is a Θ satisfying (1) and (4). Given $\delta \in [0,1)$, define $g\colon [0,1] \to [0,1]$ by

$$g(x) := \begin{cases} 0 & \text{if } 0 \leq x \leq \delta \\ \text{linear} & \text{if } \delta \leq x \leq \frac{1+\delta}{2} \\ 1 & \text{if } \frac{1+\delta}{2} \leq x \leq 1. \end{cases}$$

For $1 \leq i \leq l$, define $\Phi_i\colon X_i \to M_{n_i}$ by

$$\Phi_i(x) := \operatorname{diag}(g(\Theta_i(x)_{1,1}), \ldots, g(\Theta_i(x)_{n_i,n_i})).$$

Then $\Phi := \bigoplus_{i=1}^{l} \Phi_i \in \bigoplus_{i=1}^{l} C(X_i, M_{n_i})$. Since each diagonal entry of Θ is modified in the same way in the definition of Φ, it is straightforward to check that Φ is compatible with the diagonal structure of A. Hence, $\Phi \in A$. Moreover, since $\Phi_i(x)_{j,j} = 0$ whenever $\Theta_i(x)_{j,j} = 0$, it is clear that Φ satisfies *(1)*.

To see that Φ satisfies *(2)*, fix $1 \leq i \leq l$ and $1 \leq j \leq n_i$. Since $F_{i,j}$ is disjoint from each $B_{i,j-K_t}$ (for $1 \leq t \leq m$), condition *(4)* guarantees that $\Theta_i(x)_{j,j} < 1$ for all $x \in F_{i,j}$. Since $F_{i,j}$ is compact, there is a $\delta_{i,j} \in [0,1)$ such that $\Theta_i(x)_{j,j} \leq \delta_{i,j}$ for all $x \in F_{i,j}$. On choosing $\delta := \max\{\delta_{i,j} : 1 \leq i \leq l, 1 \leq j \leq n_i\} \in [0,1)$ in our definition of g above, it follows that $\Phi_i(x)_{j,j} = 0$ whenever $1 \leq i \leq l$, $1 \leq j \leq n_i$, and $x \in F_{i,j}$, which proves *(2)*.

Finally, to see that Φ satisfies *(3)*, fix $1 \leq i \leq l$ and $1 \leq j \leq n_i$. If $j > n_i - (\mathfrak{s}(A) - 1)$, we may take $U_{i,j} = \varnothing$ since $B_{i,j} = \varnothing$ by Lemma 2.8 for such j. For $j \leq n_i - (\mathfrak{s}(A) - 1)$, note that if $x \in B_{i,j}$, then by *(4)*, $\Theta_i(x)_{j+K_t,j+K_t} = 1$ for all $1 \leq t \leq m$. Since g is 1 in a neighbourhood of 1, it follows that for each t, there is an open set $U_t \supset B_{i,j}$ on which the function $\Phi_i(\,\cdot\,)_{j+K_t,j+K_t} \colon X_i \to [0,1]$ is equal to 1. Taking $U_{i,j} := \bigcap_{1 \leq t \leq m} U_t$ yields *(3)* and proves the claim. //

In light of Claim 3.4.1, we now construct a function $\Theta \in A$ that satisfies *(1)* and *(4)*. We define Θ coordinate-wise inductively. Put $\Theta_1 \equiv \operatorname{diag}(\chi_K(0), \ldots, \chi_K(n_1 - 1))$, where χ_K is the indicator function of the set $K = \{K_1, \ldots, K_m\}$. By the assumption on the set K, condition *(1)* holds for Θ_1. To see that *(4)* holds, suppose $\Theta_1(x)_{j,j} = 1$. Then $\chi_K(j-1) = 1$, so there is a $1 \leq t \leq m$ such that $j = K_t + 1$. It follows that $x \in B_{1,1} = B_{1,j-K_t}$ (see Lemma 2.8). Conversely, if there is a $1 \leq t \leq m$ such that $x \in B_{1,j-K_t}$, then by Lemma 2.8, $j - K_t = 1$, so that $\Theta_1(x)_{j,j} = \chi_K(j-1) = 1$, which proves *(4)*.

Now suppose we have a fixed $1 < i \leq l$ and assume we have defined $(\Theta_1, \ldots, \Theta_{i-1}) \in A^{(i-1)}$ such that for all $i' < i$ and $x \in X_{i'}$:

(I) the matrix $\Theta_{i'}(x)$ satisfies the properties of conditions *(1)* and *(4)*;

(II) $\Theta_{i'}(x)_{j,j} = \chi_K(j-1)$ for all $1 \leq j \leq \mathfrak{s}(A)$.

Let $\Theta_i' := \varphi_{i-1}((\Theta_1, \ldots, \Theta_{i-1})) \in C(Y_i, M_{n_i})$. Fix $y \in Y_i$ and suppose y decomposes into $x_1 \in X_{i_1} \setminus Y_{i_1}, \ldots, x_r \in X_{i_r} \setminus Y_{i_r}$. Let us first check that conditions *(1)* and *(4)* hold for $\Theta_i'(y) = \operatorname{diag}(\Theta_{i_1}(x_1), \ldots, \Theta_{i_r}(x_r))$. By the inductive hypothesis, $\Theta_i(y)$ is a diagonal matrix with entries in $[0,1]$ and the last M diagonal entries of $\Theta_i'(y)$ are all 0. Given M consecutive entries down the diagonal of $\Theta_i'(y)$, if they are all contained in one of the diagonal blocks, then by the inductive hypothesis applied to that one block, at most one of these entries is non-zero. If instead the M consecutive entries span two blocks $\Theta_{i_q}(x_q)$ and $\Theta_{i_{q+1}}(x_{q+1})$, then by the inductive hypothesis, the last M diagonal entries of $\Theta_{i_q}(x_q)$ are 0 and at most 1 of the first M diagonal entries of $\Theta_{i_{q+1}}(x_{q+1})$ can be non-zero. This shows that *(1)* holds for $\Theta_i'(y)$. Let us now show that *(4)* holds for $\Theta_i'(y)$. Fix $1 \leq j \leq n_i$. Let

$1 \leq q \leq r$ and $1 \leq j' \leq n_{i_q}$ be such that $\Theta'_i(y)_{j,j} = \Theta_{i_q}(x_q)_{j',j'}$. Note that $j = n_{i_1} + \cdots + n_{i_{q-1}} + j'$. Given $1 \leq t \leq m$, we know by Lemma 2.8 that $y \in B_{i,j-K_t}$ if and only if there is a $1 \leq p \leq r$ such that

$$j' - K_t + n_{i_1} + \cdots + n_{i_{q-1}} = j - K_t = 1 + n_{i_1} + \cdots + n_{i_{p-1}} \tag{3.1}$$

(the right-hand side is 1 if $p = 1$). We claim that if Equation 3.1 holds, then $p = q$. Indeed, using the upper and lower bounds on j' and K_t, we have

$$1 - \mathfrak{s}(A) < 1 - (\mathfrak{s}(A) - M - 1) \leq j' - K_t \leq n_{i_q},$$

whence

$$1 + n_{i_1} + \cdots + n_{i_{q-1}} - \mathfrak{s}(A) < 1 + n_{i_1} + \cdots + n_{i_{p-1}} \leq n_{i_1} + \cdots + n_{i_{q-1}} + n_{i_q}.$$

The first inequality and the definition of $\mathfrak{s}(A)$ imply that $q \leq p$, while the second inequality forces $q \geq p$, so that $p = q$. Therefore, since $x_q \in X_{i_q} \setminus Y_{i_q}$, the above and Lemma 2.8 show that

$$\begin{aligned} y \in B_{i,j-K_t} &\iff j - K_t = 1 + n_{i_1} + \cdots + n_{i_{q-1}} \\ &\iff j' - K_t = 1 \\ &\iff x_q \in B_{i_q, j'-K_t}. \end{aligned}$$

Since the matrix $\Theta_{i_q}(x_q)$ satisfies *(4)* by the inductive hypothesis, it follows that there is a $1 \leq t \leq m$ with $y \in B_{i,j-K_t}$ if and only if $\Theta'_i(y)_{j,j} = \Theta_{i_q}(x_q)_{j',j'} = 1$, which proves that *(4)* holds for $\Theta'_i(y)$.

Let us now define $\Theta_i \in C(X_i, M_{n_i})$ to be a suitable extension of Θ'_i. Write $\Theta'_i = \operatorname{diag}(h'_1, \ldots, h'_{n_i})$, where $h'_j \in C(Y_i, [0,1])$ for $1 \leq j \leq n_i$. We define $\Theta_i = \operatorname{diag}(h_1, \ldots, h_{n_i})$ by specifying each h_j to be a continuous function $h_j \colon X_i \to [0,1]$ that extends h'_j. For $1 \leq j \leq \mathfrak{s}(A)$, put $h_j \equiv \chi_K(j-1)$ to insure that (II) in the inductive hypothesis is verified and set $h_j \equiv 0$ for $n_i - M + 1 \leq j \leq n_i$ (since (I) and (II) hold for $\Theta_1, \ldots, \Theta_{i-1}$, these h_j's do indeed extend the corresponding h'_j's). We define h_j for $\mathfrak{s}(A) + 1 \leq j \leq n_i - M$ inductively. Fix $\mathfrak{s}(A) + 1 \leq j \leq n_i - M$ and assume we have defined $h_1, \ldots, h_{j-1}$ so that the following property holds:

(♣) $\bigcup_{t=1}^{M-1} \overline{\operatorname{supp}(h_{j-t})} \subset X_i$ is disjoint from $\operatorname{supp}(h'_j) \subset Y_i$.

Note that $\bigcup_{t=1}^{M-1} \overline{\operatorname{supp}(h_{\mathfrak{s}(A)+1-t})} = \varnothing$, and so (♣) holds for the base case $j = \mathfrak{s}(A) + 1$. Since X_i is a metric space and, hence, perfectly normal, we may use (♣) to extend h'_j to a function f_j in $C(X_i, [0,1])$ that vanishes on $\bigcup_{t=1}^{M-1} \overline{\operatorname{supp}(h_{j-t})}$ and is strictly less than 1 on $X_i \setminus Y_i$. Define $g^0_j := h'_j - \sum_{t=1}^{M-1} h'_{j+t} \in C(Y_i)$. Then the range of g^0_j is contained in $[-1,1]$ since by (I) at most one of $h'_j, \ldots, h'_{j+M-1}$ is non-zero at any given point in Y_i.

Extend g_j^0 to a function g_j' in $C(X_i,[-1,1])$. Put $g_j := \max(g_j',0)$ and note that $g_j|_{Y_i} = h_j'$. Since $h_j'(y) = 0$ for each $y \in \bigcup_{t=1}^{M-1}\operatorname{supp}(h_{j+t}')$, we may choose an open subset $U \supset \bigcup_{t=1}^{M-1}\operatorname{supp}(h_{j+t}')$ of X_i on which g_j' is strictly negative, so that $g_j|_U \equiv 0$. Define $h_j := \min(f_j,g_j) \in C(X_i,[0,1])$ and note that $h_j|_{Y_i} = h_j'$. Since $h_j|_U \equiv 0$, we have $\operatorname{supp}(h_j)\cap U = \varnothing$, from which it follows that $\overline{\operatorname{supp}(h_j)} \cap \left(\bigcup_{t=1}^{M-1}\operatorname{supp}(h_{j+t}')\right) = \varnothing$. This ensures that (♣) holds with $j+1$ in place of j and, hence, that $\Theta_i := \operatorname{diag}(h_1,\ldots,h_{n_i})$ is well defined.

To conclude the proof, let us check that Θ_i satisfies *(1)* and *(4)*. In light of the analysis above, we may restrict ourselves to the diagonal entries $\mathfrak{s}(A)+1 \le j \le n_i - M$. By definition, the range of each h_j is contained in $[0,1]$. If $h_j(x) > 0$ for some $x \in X_i$, then $f_j(x) > 0$ and, hence, by the definition of f_j, $x \notin \bigcup_{t=1}^{M-1}\overline{\operatorname{supp}(h_{j-t})}$. This proves that at most one of any M consecutive entries down the diagonal of $\Theta_i(x)$ is non-zero. Hence, *(1)* is established. To prove *(4)*, suppose $x \in X_i$ satisfies $h_j(x) = 1$. Then $f_j(x) = 1$, which implies that $x \in Y_i$. Thus $h_j'(x) = 1$ and we already established that $x \in B_{i,j-K_t}$ for some t in this case. Conversely, suppose $x \in B_{i,j-K_t}$ for some t. If $j - K_t \neq 1$, then by Lemma 2.8, $x \in Y_i$ and we already concluded in this case that $h_j(x) = h_j'(x) = 1$. If instead $j - K_t = 1$, then it must be that $j \le \mathfrak{s}(A)$ and we previously defined $h_j \equiv 1$ in this case. Therefore, property *(4)* holds.

We verified that both (I) and (II) hold for $\Theta_i = \operatorname{diag}(h_1,\ldots,h_l)$, and since $\Theta_i|_{Y_i} = \Theta_i' = \varphi_{i-1}((\Theta_1,\ldots,\Theta_{i-1}))$, it follows that $(\Theta_1,\ldots,\Theta_i) \in A^{(i)}$. Thus, by induction, we obtain $\Theta := (\Theta_1,\ldots,\Theta_l) \in A$, which satisfies the requirements of the lemma. //

Before proceeding with the next few lemmas, we need some definitions.

Definition 3.5 (Diagonal Radius; cf. [Lut17b], p. 12)**.** Given a matrix $D \in M_n$, we let

$$\mathfrak{r}(D) := \min\{m \ge 0 : D_{i,j} = 0 \text{ whenever } |i-j| \ge m\}$$

if it exists, and $\mathfrak{r}(D) := n$ otherwise, and we call this number the *diagonal radius* of D.

Definition 3.6 (see [EHT08]; cf. [Lut17b], p. 9)**.** Given $n \in \mathbb{N}$ and a permutation $\pi \in S_n$, let $U[\pi]$ denote the permutation unitary in M_n obtained from the identity matrix by moving the ith row to the $\pi(i)$th row. If we are given a transposition $(i\ j) \in S_n$, let $u_{(i\ j)}\colon [0,1] \to \mathcal{U}(M_n)$ denote a continuous path of unitaries with the following properties:

(1) $u_{(i\ j)}(0) = 1_n$;

(2) $u_{(i\ j)}(1) = U[(i\ j)]$;

(3) for all $0 \le \theta \le 1$, $u_{(i\ j)}(\theta)$ may only differ from the identity matrix at entries (i,i), (i,j), (j,i), and (j,j).

Given a sequence of DSH algebras $A_1, A_2, \ldots$, we denote by $l(j)$ the length of the DSH algebra A_j. We denote the base spaces of A_j by $X_1^j, \ldots, X_{l(j)}^j$ and the corresponding closed subspaces by $Y_1^j, \ldots, Y_{l(j)}^j$. We denote the size of the matrix algebras in the pullback definition of A_j by $n_1^j, \ldots, n_{l(j)}^j$. Finally, we denote the sets defined in Definition 2.7 corresponding to A_j by $B_{i,k}^j$.

Lemma 3.7 (cf. [Lut17b], Lemma 2.3.4)**.** *Suppose* $\lim(A_j, \psi_j)$ *is a simple limit of infinite-dimensional DSH algebras with injective diagonal maps. Suppose that* f *is a non-invertible element belonging to some* A_j *and that* $\epsilon > 0$. *Then, there exist* $f' \in A_j$ *with* $\|f - f'\| \le \epsilon$ *and* $M \in \mathbb{N}$ *such that for all* $N \in \mathbb{N}$ *there exist* $j' > j$ *and unitaries* $V, V' \in A_{j'}$ *with the following properties:*

(1) for any $1 \le i \le l(j')$ *and* $1 \le k \le n_i^{j'}$, *there is a (possibly empty) open subset* $U_{i,k}$ *of* $X_i^{j'}$ *containing* $B_{i,k}^{j'}$ *such that for all* $x \in U_{i,k}$, $(V\psi_{j',j}(f')V')_i(x)$ *has zero crosses at indices* $k, k+M, k+2M, \ldots, k+(N-1)M$*;*

(2) for all $1 \le i \le l(j')$ *and* $x \in X_i^{j'}$, *we have* $\mathfrak{r}((V\psi_{j',j}(f')V')_i(x)) \le \mathfrak{S}(A_j) + M - 1$ *(where, recall,* $\mathfrak{S}(A_j) = \max\{n_t^j : 1 \le t \le l(j)\}$*).*

Proof. Let $f', w, v, \Delta \in A_j$ and $U \subset \hat{A}_j$ be given as in Lemma 3.3 (when applied to f and ϵ) and set $g := wf'v$. Then, at every point in U, g has a zero cross at index 1 and the $(1,1)$-entry of Δ is 1. By Proposition 3.1 and Lemma 2.4, there is a $j'' > j$ such that $\hat{\psi}_{j'',j}([\mathrm{ev}_x])$ contains a point in U for all $x \in \bigsqcup_{i=1}^{l(j'')}(X_i^{j''} \setminus Y_i^{j''})$. Since $\psi_{j'',j}$ is diagonal, this guarantees that for $1 \le i \le l(j'')$ and $x \in X_i^{j''} \setminus Y_i^{j''}$, at least one of the points x decomposes into under $\psi_{j'',j}$ lies in U, so that the matrices $\psi_{j'',j}(g)_i(x)$ and $\psi_{j'',j}(\Delta)_i(x)$ have a zero cross and a 1, respectively, at the some index along the diagonal. Owing to the decomposition structure of $A_{j''}$, these two results hold, in fact, for all $1 \le i \le l(j'')$ and $x \in X_i^{j''}$. Take $M := 2\mathfrak{S}(A_{j''})$ and let $N \in \mathbb{N}$ be arbitrary.

Claim 3.7.1. *There is a* $j' > j''$ *such that* $\mathfrak{s}(A_{j'}) > NM$.

Proof. Since $A_{j''}$ is infinite-dimensional, at least one of the base spaces must be infinite. Let $1 \le i \le l(j'')$ be the largest integer for which $X_i^{j''}$ is infinite. By Lemma 2.9, $X_i^{j''} \setminus Y_i^{j''}$ is also infinite and $Y_{i'}^{j''} = \varnothing$ for $i < i' \le l(j'')$. Choose pairwise-disjoint open in $X_i^{j''}$ sets $\mathcal{O}_1, \ldots, \mathcal{O}_{NM+1} \subset$

$X_i^{j''} \setminus Y_i^{j''}$. Proposition 2.18 guarantees that $\mathcal{O}_1, \ldots, \mathcal{O}_{NM+1}$ are all open with respect to the hull-kernel topology on $\hat{A}_{j''}$. By Proposition 3.1, there is a $j' > j''$ such that for all $x \in \hat{A}_{j'}$, $\hat{\psi}_{j',j''}(x)$ contains a point from each of $\mathcal{O}_1, \ldots, \mathcal{O}_{NM+1}$. Hence, $n_i^{j'} \geq NM + 1$ for all $1 \leq i \leq l(j')$, which proves the claim. //

Let $\Delta' := \psi_{j',j}(\Delta) = \psi_{j',j''}(\psi_{j'',j}(\Delta))$ and $g' := \psi_{j',j}(g) = \psi_{j',j''}(\psi_{j'',j}(g))$. Given $1 \leq i \leq l(j')$ and $x \in X_i^{j'}$ and regarding Δ' as a diagonal image under $\psi_{j',j''}$, it follows from the definition of M that any M consecutive entries down the diagonal of $\Delta'_i(x)$ must contain a 1. Moreover, regarding g' and Δ' as diagonal images under $\psi_{j',j}$ shows that $g'_i(x)$ has a zero cross at index k whenever $\Delta'_i(x)_{k,k} > 0$ (as a consequence of the conclusion of Lemma 3.3) and that $\mathfrak{r}(g'_i(x)) \leq \mathfrak{S}(A_j)$.

We now apply Lemma 3.4 with the natural number M, with $m = N$, $K_1 = 0, K_2 = M, \ldots, K_N = (N-1)M$, and $F_{i,k} = \varnothing$ for $1 \leq i \leq l(j')$ and $1 \leq k \leq n_i^{j'}$ (note that $K_N < \mathfrak{s}(A_{j'}) - M$ by Claim 3.7.1). This furnishes a function $\Theta \in A_{j'}$ with the following properties:

(I) for all $1 \leq i \leq l(j')$ and $x \in X_i^{j'}$, $\Theta_i(x)$ is a diagonal matrix with entries in $[0,1]$ whose final M diagonal entries are all 0, and such that at most one of every M consecutive diagonal entries is non-zero;

(II) for all $1 \leq i \leq l(j')$ and $1 \leq k \leq n_i^{j'}$, there is a (possibly empty) open subset $U_{i,k} \subset X_i^{j'}$ containing $B_{i,k}^{j'}$ with the property that if $x \in U_{i,k}$, then $\Theta_i(x)_{k+aM,k+aM} = 1$ for all $0 \leq a \leq N-1$.

Fix $1 \leq i \leq l(j')$. Given $x \in X_i^{j'}$ and $1 \leq k \leq n_i^{j'} - M$, let

$$u_k^i(x) := \prod_{t=1}^{M-1} u^i_{(k\ k+t)}\left(\Theta_i(x)_{k,k}\Delta'_i(x)_{k+t,k+t}\right) \in M_{n_i^{j'}}, \tag{3.2}$$

where each $u^i_{(k\ k+t)} \colon X_i^{j'} \to M_{n_i^{j'}}$ is a connecting path of unitaries as described in Definition 3.6. Let us now establish a sublemma, which will be helpful in facilitating the rest of the proof.

Sublemma 3.7.2 (cf. [Lut17b] Lemmas 2.1.3 and 2.1.4)**.**

(a) Suppose $D \in M_{n_i^{j'}}$, $\xi \in [0,1]$, and $(k_1\ k_2) \in S_{n_i^{j'}}$. If D has a zero cross at index $k \neq k_1, k_2$, then so does $u^i_{(k_1\ k_2)}(\xi) D u^i_{(k_1\ k_2)}(\xi)^*$.

(b) Suppose $D \in M_{n_i^{j'}}$ and $x \in X_i^{j'}$. If D has a zero cross at index $k' \in \{1, \ldots, n_i^{j'}\} \setminus \{k, \ldots, k+(M-1)\}$, then so does $u_k^i(x) D u_k^i(x)^*$.

(c) Suppose $x \in X_i^{j'}$ and $\Theta_i(x)_{k,k} = 1$ for some $1 \leq k \leq n_i^{j'} - (M-1)$. Suppose D is a matrix in $M_{n_i^{j'}}$ such that for all $k \leq k' \leq k+(M-1)$, D

has a zero cross at index k' whenever $\Delta_i'(x)_{k',k'} > 0$. Then $u_k^i(x)Du_k^i(x)^*$ has a zero cross at index k.

Proof. Let us start by proving *(a)*. Suppose D has a zero cross at index $k \neq k_1, k_2$. By property (3) of Definition 3.6, the k_1th and k_2th columns of $Du^i_{(k_1\ k_2)}(\xi)^*$ are linear combinations of the k_1th and k_2th columns of D, while every other column is identical to its corresponding column in D. Since $k \neq k_1, k_2$ and since every entry in the kth row of D is zero, it follows that $Du^i_{(k_1\ k_2)}(\xi)^*$ has a zero cross at index k. A similar analysis involving rows shows that $u^i_{(k_1\ k_2)}(\xi)Du^i_{(k_1\ k_2)}(\xi)^*$ has a zero cross at index k, which proves *(a)*. Looking at the definition of u_k^i, we see that *(b)* follows from $M-1$ applications of *(a)*.

Let us now prove *(c)*. Given an x as in *(c)*, we have

$$u_k^i(x) = \prod_{t=1}^{M-1} u^i_{(k\ k+t)}(\Delta_i'(x)_{k+t,k+t}).$$

Let $T := \{k+1 \leq q \leq k+M-1 : \Delta_i'(x)_{q,q} > 0\}$. If $\Delta_i'(x)_{k+t,k+t} = 0$, we have $u^i_{(k\ k+t)}(\Delta_i'(x)_{k+t,k+t}) = 1_{n_i^{j'}}$. Hence,

$$u_k^i(x) := \begin{cases} u^i_{(k\ k_1)}(\Delta_i'(x)_{k_1,k_1}) \cdots u^i_{(k\ k_r)}(\Delta_i'(x)_{k_r,k_r}) & \text{if } T = \{k_1 < \cdots < k_r\} \\ 1_{n_i^{j'}} & \text{if } T = \varnothing. \end{cases}$$

If $T = \varnothing$, then, since any M consecutive entries down the diagonal of $\Delta_i'(x)$ must contain a 1, it follows that $\Delta_i'(x)_{k,k} = 1$. Hence, $u_k^i(x)Du_k^i(x)^* = D$ has a zero cross at index k in this case by the assumption in the sublemma. Thus, we may assume $T \neq \varnothing$, so that D has zero crosses at indices $k_1, \ldots, k_r$. We consider two cases.

Case one: $\Delta_i'(x)_{k_s,k_s} < 1$ for all $1 \leq s \leq r$. In this case, as we argued above, it must be that D has a zero cross at index k. When conjugating D by $u^i_{(k\ k_r)}(\Delta_i'(x)_{k_r,k_r})$, we can see by property (3) of Definition 3.6 that $u^i_{(k\ k_r)}(\Delta_i'(x)_{k_r,k_r})$ is only acting on two zero crosses (the one at index k and the one at index k_r) of D and, hence,

$$u^i_{(k\ k_r)}(\Delta_i'(x)_{k_r,k_r})Du^i_{(k\ k_r)}(\Delta_i'(x)_{k_r,k_r})^* = D.$$

From this we can inductively see that $u_k^i(x)Du_k^i(x)^* = D$, which has a zero cross at index k.

Case two: $\Delta_i'(x)_{k_s,k_s} = 1$ for some $1 \le s \le r$. Let

$$D' := \left(\prod_{p=s+1}^{r} u^i_{(k\ k_p)}(\Delta_i'(x)_{k_p,k_p}) \right) D \left(\prod_{p=s+1}^{r} u^i_{(k\ k_p)}(\Delta_i'(x)_{k_p,k_p}) \right)^* .$$

Then $r - s$ applications of (a) show that D' has zero crosses at indices $k_1, \ldots, k_s$. Note that

$$\begin{aligned} u^i_k(x) D u^i_k(x)^* &= \left(\prod_{p=1}^{s} u^i_{(k\ k_p)}(\Delta_i'(x)_{k_p,k_p}) \right) D' \left(\prod_{p=1}^{s} u^i_{(k\ k_p)}(\Delta_i'(x)_{k_p,k_p}) \right)^* \\ &= \left(\prod_{p=1}^{s-1} u^i_{(k\ k_p)}(\Delta_i'(x)_{k_p,k_p}) \right) E \left(\prod_{p=1}^{s-1} u^i_{(k\ k_p)}(\Delta_i'(x)_{k_p,k_p}) \right)^* , \end{aligned}$$

where $E := U[(k\ k_s)]D'U[(k\ k_s)]^*$. Since D' has a zero cross at index k_s, conjugating it by $U[(k\ k_s)]$ brings this zero cross to index k. Thus, the matrix E has zero crosses at indices $k, k_1, \ldots, k_{s-1}$. Hence, as in the argument used in case one, the matrix E is unaltered when conjugated by $\prod_{p=1}^{s-1} u^i_{(k\ k_p)}(\Delta_i'(x)_{k_p,k_p})$. Therefore, $u^i_k(x) D u^i_k(x)^* = E$, which has a zero cross at index k. This proves *(c)* and establishes the sublemma. //

Returning to the proof of Lemma 3.7, define $W_i \in C(X_i^{j'}, M_{n_i^{j'}})$ to be the unitary

$$W_i(x) := \prod_{k=1}^{n_i^{j'} - M} u^i_k(x).$$

Set $W := (W_1, \ldots, W_{l(j')})$ and take $V := W\psi_{j',j}(w)$ and $V' := \psi_{j',j}(v)W^*$. Before showing that $W \in A_{j'}$, let us prove that statements *(1)* and *(2)* of Lemma 3.7 hold.

Fix $x \in X_i^{j'}$. Note that if $\Theta_i(x)_{k',k'} = 0$, then $u^i_{k'}(x) = 1_{n_i^{j'}}$. Let $\{k_1 < \cdots < k_s\}$ denote the set of indices r at which $\Theta_i(x)_{r,r} > 0$. By (I) above, we may write $W_i(x) = u^i_{k_1}(x) \cdots u^i_{k_s}(x)$, where $k_{p+1} - k_p \ge M$ for $1 \le p < s$ and $k_s \le n_i^{j'} - M$. Note that conjugating any matrix by $u^i_{k_p}(x)$ only affects the $k_p, \ldots, k_p + (M-1)$ rows and columns of that matrix. Thus, for $p \ne q$, the indices of the rows and columns affected when conjugating by $u^i_{k_p}(x)$ do not overlap with the indices of the rows and columns affected when conjugating by $u^i_{k_q}(x)$.

To prove *(1)*, fix $1 \le k \le n_i^{j'}$ and assume $x \in U_{i,k}$. By (II) and (I) above, we know that for all $0 \le p \le N-1$, $\Theta_i(x)_{k+pM,k+pM} = 1$ and $1 \le k \le n_i^{j'} - M$.

Therefore, we may apply (b) and (c) from Sublemma 3.7.2 above with $D = g_i'(x)$ and then inductively with

$$D = u^i_{k_t}(x) \cdots u^i_{k_s}(x) g_i'(x) u^i_{k_s}(x)^* \cdots u^i_{k_t}(x)^*,$$

for $t = s, s-1, \ldots, 1$ to conclude that $W_i(x) g_i'(x) W_i(x)^*$ has zero crosses at indices $k, k+M, k+2M, \ldots, k+(N-1)M$. This proves *(1)*.

Recall that g' is the diagonal image of g, which has diagonal radius at most $\mathfrak{S}(A_j)$ at every point. To prove *(2)*, therefore, it suffices to show that for any given matrix $D = (D_{q,t}) \in M_{n_i^{j'}}$, we have $\mathfrak{r}(W_i(x) D W_i(x)^*) \leq \mathfrak{r}(D) + M - 1$. This is most easily seen by drawing a picture and examining which rows and columns are potentially affected upon conjugation by the $u^i_{k_p}$'s:

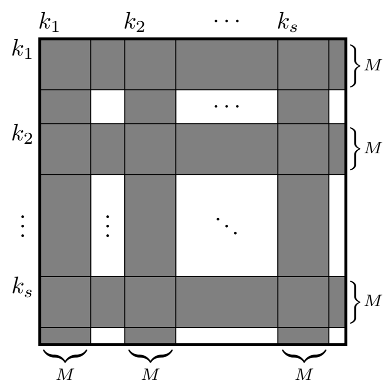

Figure 3.2: Affected Rows/Columns

Since the sets $\{k_p, \ldots, k_p + (M-1)\}$ for $1 \leq p \leq s$ are disjoint, the block rows and columns are disjoint. Suppose we are given an index (q,t) that lies in the shaded region, and suppose that λ is the number at entry (q,t) of $W_i(x) D W_i(x)^*$. The index (q,t) lies in one of the following three subregions:

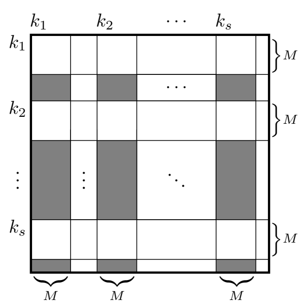

Figure 3.3: Region A

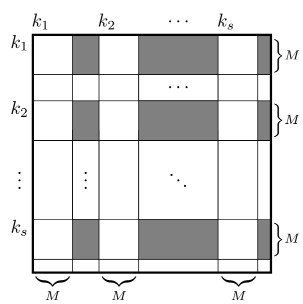

Figure 3.4: Region B

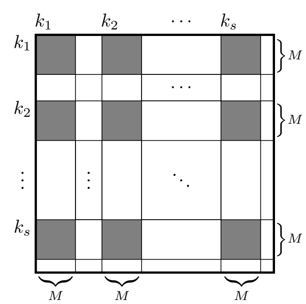

Figure 3.5: Region C

On Region A, the matrices $W_i(x) D W_i(x)^*$ and $D W_i(x)^*$ are equal. Hence, if (q,t) lies in Region A and p is such that $k_p \leq t \leq k_p + M - 1$, then λ is a linear combination of $D_{q,k_p}, \ldots, D_{q,k_p+M-1}$. Thus, λ can only be non-zero if one of $D_{q,k_p}, \ldots, D_{q,k_p+M-1}$ is non-zero. Hence, no non-zero entry in this region is more than $M-1$ indices away from a non-zero entry in D. On

Region B, the matrices $W_i(x)DW_i(x)^*$ and $W_i(x)D$ are equal, and so a symmetrical analysis shows that the same is also true for non-zero entries in this region. If (q,t) lies in one of the s^2 disjoint $M \times M$ blocks in Region C, then λ is a linear combination of the corresponding entries in D lying in that block. Hence, in this case λ is 0 unless that $M \times M$ block in D contains a non-zero entry. Thus, no non-zero entry of $W_i(x)DW_i(x)^*$ in Region C is more than $M-1$ units further away from the diagonal than a non-zero entry of D. This analysis proves that $\mathfrak{r}(W_i(x)DW_i(x)^*) \leq \mathfrak{r}(D) + M - 1$, yielding *(2)*.

To conclude, let us show that $W \in A_{j'}$. Fix $1 \leq i \leq l(j')$ and suppose that $y \in Y_i^{j'}$ decomposes into $x_1 \in X_{i_1}^{j'} \setminus Y_{i_1}^{j'}, \ldots, x_s \in X_{i_s}^{j'} \setminus Y_{i_s}^{j'}$. For $1 \leq k \leq s$, let $p_k := 1 + n_{i_1} + \cdots + n_{i_{k-1}}$. Note that by Claim 3.7.1, $p_s \leq n_i^{j'} - \mathfrak{s}(A_{j'}) + 1 \leq n_i^{j'} - M$. Thus, we may write

$$W_i(y) = \prod_{k=1}^{n_i^{j'}-M} u_k^i(y) = \prod_{m=1}^{s-1} \prod_{k=p_m}^{p_{m+1}-1} u_k^i(y) \times \prod_{k=p_s}^{n_i^{j'}-M} u_k^i(y). \tag{3.3}$$

Fix $1 \leq m < s$. Then,

$$\prod_{k=p_m}^{p_{m+1}-1} u_k^i(y) = \prod_{k=p_m}^{p_{m+1}-1} \prod_{t=1}^{M-1} u^i_{(k\ k+t)}(\Theta_i(y)_{k,k}\Delta_i'(y)_{k+t,k+t}).$$

By (I), the last M entries of $\Theta_{i_m}(x_m)$ are zero. Hence, on account of the diagonal decomposition of $\Theta_i(y)$, the quantity above is equal to

$$\prod_{k=p_m}^{(p_{m+1}-1)-M} \prod_{t=1}^{M-1} u^i_{(k\ k+t)}(\Theta_{i_m}(x_m)_{k-p_m+1,k-p_m+1}\Delta'_{i_m}(x_m)_{k-p_m+1+t,k-p_m+1+t}),$$

which, upon relabelling indices, becomes

$$\prod_{q=1}^{p_{m+1}-p_m-M} \prod_{t=1}^{M-1} u^i_{(q+p_m-1\ q+p_m-1+t)}(\Theta_{i_m}(x_m)_{q,q}\Delta'_{i_m}(x_m)_{q+t,q+t}). \tag{3.4}$$

For each $1 \leq q \leq p_{m+1} - p_m - M$ and $1 \leq t \leq M-1$, note that

$$u^i_{(q+p_m-1\ q+p_m-1+t)} = \operatorname{diag}(1_{p_m-1}, u^{i_m}_{(q\ q+t)}, 1_{n_i^{j'}-p_{m+1}+1}).$$

Hence, we may rewrite Equation 3.4 as

$$\begin{aligned}&\operatorname{diag}\left(1_{p_m-1}, \prod_{q=1}^{p_{m+1}-p_m-M} u_q^{i_m}(x_m), 1_{n_i^{j'}-p_{m+1}+1}\right)\\ &= \operatorname{diag}\left(1_{p_m-1}, W_{i_m}(x_m), 1_{n_i^{j'}-p_{m+1}+1}\right).\end{aligned}$$

Therefore, for all $1 \le m < s$,

$$\prod_{k=p_m}^{p_{m+1}-1} u_k^i(y) = \operatorname{diag}\left(1_{p_m-1}, W_{i_m}(x_m), 1_{n_i^{j'}-p_{m+1}+1}\right)$$

and, similarly,

$$\prod_{k=p_s}^{n_i^{j'}-M} u_k^i(y) = \operatorname{diag}(1_{p_s-1}, W_{i_s}(x_s)).$$

Plugging this into Equation 3.3 shows $W_i(y) = \operatorname{diag}(W_{i_1}(x_1), \ldots, W_{i_s}(x_s))$, which proves that $W \in A$. The proof of Lemma 3.7 is now complete. //

Definition 3.8 (Block Point; cf. [Lut17b], p. 5)**.** Given a matrix $D \in M_n$ and $1 \le k \le n$, we say that D has a *block point at index k* provided that $D_{i,j} = 0$ if either $i \ge k$ and $j < k$ or $i < k$ and $j \ge k$.

Lemma 3.9 (cf. [Lut17b], Lemma 2.2.1)**.** *Suppose A is a DSH algebra of length l. Suppose $f \in A$ and $\epsilon > 0$. Then there is a $g \in A$ with $\|g - f\| \le \epsilon$ and with the property that for all $1 \le i \le l$ and $1 \le k \le n_i$, there are (possible empty) open sets $\mathcal{O}_{i,k} \supset B_{i,k}$ in X_i such that $g_i(x)$ has a block point at index k whenever $x \in \mathcal{O}_{i,k}$. Moreover, g can be chosen so that for each $1 \le i \le l$ and $x \in X_i$, $g_i(x)$ has a zero cross at index k whenever $f_i(x)$ does, and $\mathfrak{r}(g_i(x)) \le \mathfrak{r}(f_i(x))$.*

Proof. Given $1 \le i \le l$ and $1 \le s,t \le n_i$, let $f_i(\,\cdot\,)_{s,t} \in C(X_i)$ denote the function taking x into $f_i(x)_{s,t}$. Let $\delta = \epsilon/\mathfrak{S}(A)^2$. Define $h \in C(\mathbb{C})$ by $h(z) := \frac{z}{|z|} \cdot \max(0, |z| - \delta)$, where it is understood that $h(0) = 0$. Note that for any $z \in \mathbb{C}$, if $|z| \le \delta$, then $|h(z) - z| = |z| \le \delta$, and if $|z| > \delta$, then $|h(z) - z| = \left|\frac{z}{|z|}(|z| - \delta) - z\right| = \left|\frac{z}{|z|}\delta\right| = \delta$. Thus, for all $z \in \mathbb{C}$, $|h(z) - z| \le \delta$ and, hence, $|f_i(x)_{s,t} - h(f_i(x)_{s,t})| \le \delta$ given any $1 \le i \le l$, $1 \le s,t \le n_i$, and $x \in X_i$. Define $g_i(x)_{s,t} := h(f_i(x)_{s,t})$ and denote by g_i the matrix-valued function in $C(X_i, M_{n_i})$ given by $(g_i(\,\cdot\,)_{s,t})_{s,t}$. Set $g := (g_1, \ldots, g_l) \in \bigoplus_{i=1}^l C(X_i, M_{n_i})$. For $x \in X_i$,

$$\|f_i(x) - g_i(x)\| \le \sum_{1 \le s,t \le n_i} \|f_i(x)_{s,t} - g_i(x)_{s,t}\| \le n_i^2 \delta \le \epsilon.$$

Hence, $\|f - g\| \le \epsilon$.

To see that $g \in A$, observe that if $y \in Y_i$ decomposes into $x_1 \in X_{i_1} \setminus Y_{i_1}, \ldots, x_t \in X_{i_t} \setminus Y_{i_t}$, then $f_i(y) = \operatorname{diag}(f_{i_1}(x_1), \ldots, f_{i_t}(x_t))$. Applying h to each coordinate yields that $g_i(y) = \operatorname{diag}(g_{i_1}(x_1), \ldots, g_{i_t}(x_t))$. Furthermore, since $h(0) = 0$, $g_i(x)$ must have a zero cross at any index that $f_i(x)$ does, and $\mathfrak{r}(g_i(x)) \leq \mathfrak{r}(f_i(x))$.

Lastly, fix $1 \leq i \leq l$ and $1 \leq k \leq n_i$. Let us show how to construct $\mathcal{O}_{i,k}$. If $B_{i,k} = \varnothing$, take $\mathcal{O}_{i,k} := \varnothing$. Otherwise, suppose $x \in B_{i,k}$. Then, $f_i(x)$ has a block point at index k. Let $I \subset \{1, \ldots, n_i\}^2$ denote the set of indices (s,t) such that $s \geq k$ and $t < k$ or such that $s < k$ and $t \geq k$. Given $(s,t) \in I$, it follows that $f_i(x)_{s,t} = 0$, and hence, that $g_i(\,\cdot\,)_{s,t}$ is 0 on an open set $U_{s,t}(x) \subset X_i$ containing x. Then $U_{s,t} := \bigcup_{x \in B_{i,k}} U_{s,t}(x)$ is an open set containing $B_{i,k}$ on which $g_i(\,\cdot\,)_{s,t}$ vanishes. Take $\mathcal{O}_{i,k} := \bigcap_{(s,t)\in I} U_{s,t}$. By construction, then, $g_i(x)_{s,t} = 0$ whenever $x \in \mathcal{O}_{i,k}$ and $(s,t) \in I$. Thus, $g_i(x)$ has a block point at index k provided that $x \in \mathcal{O}_{i,k}$, which completes the proof. //

Lemma 3.10 (cf. [Lut17b], Lemma 2.2.3)**.** *Suppose A is a DSH algebra of length l and that $M, N \in \mathbb{N}$ with $NM < \mathfrak{s}(A)$. Suppose f is an element of A with the property that for all $1 \leq i \leq l$ and $1 \leq j \leq n_i$, there is a (possibly empty) open set $U_{i,k} \supset B_{i,k}$ in X_i such that if $x \in U_{i,k}$, then $f_i(x)$ has zero crosses at indices $k, k+M, \ldots, k+(N-1)M$ and a block point at index k. Then, there exists a unitary $V \in A$ with the following properties:*

(1) for all $1 \leq i \leq l$ and $1 \leq k \leq n_i$, there are open sets $\mathcal{O}_{i,k} \supset B_{i,k}$ in X_i such that $V_i(x) f_i(x) V_i(x)^*$ has zero crosses at indices $k, k+1, \ldots, k+N-1$ whenever $x \in \mathcal{O}_{i,k}$;

(2) $\mathfrak{r}(V_i(x) f_i(x) V_i(x)^*) \leq \mathfrak{r}(f_i(x)) + 2$ for all $1 \leq i \leq l$ and $x \in X_i$.

Proof. Let us first establish the following sublemma, to facilitate the rest of the proof.

Sublemma 3.10.1 (cf. [Lut17b], Lemma 2.1.5)**.** *Suppose $n \in \mathbb{N}$ and that $T \in M_n$ has zero crosses at indices $z_1 < z_2 < \cdots < z_m$. Then there is a unitary $W \in C([0,1], M_n)$ (depending only on $z_1, \ldots, z_m$) with the following properties:*

(a) $W(1)TW(1)^*$ has zero crosses at indices $1, 2, \ldots, m$;
(b) $W(0) = 1_n$;
(c) $\mathfrak{r}(W(\xi)TW(\xi)^*) \leq \mathfrak{r}(T) + 2$ for all $\xi \in [0,1]$.

Proof. For $1 \leq i \leq j \leq n$, let $\delta_j^i \colon [0,1] \to [0,1]$ be given by the following definition:

$$\delta_j^i(\xi) := \begin{cases} 0 & \text{if } 0 \leq \xi \leq \frac{i-1}{j} \\ \text{linear} & \text{if } \frac{i-1}{j} \leq \xi \leq \frac{i}{j} \\ 1 & \text{if } \frac{i}{j} \leq \xi \leq 1. \end{cases}$$

For $1 \le i < j \le n$, let $w_j^i \in C([0,1], M_n)$ be the unitary defined by

$$w_j^i(\xi) := u_{(i\ i+1)}(\delta_{j-i}^{j-i}(\xi))u_{(i+1\ i+2)}(\delta_{j-i}^{j-i-1}(\xi)) \cdots u_{(j-1\ j)}(\delta_{j-i}^{1}(\xi)),$$

where the unitaries $u_{(k\ k+1)} \colon [0,1] \to M_n$ are those of Definition 3.6. Note that

$$w_j^i(1) = u_{(i\ i+1)}(1) \cdots u_{(j-1\ j)}(1) = U[(i\ \ i+1\ \ \cdots\ \ j)]. \qquad (3.5)$$

Set $w_i^i \equiv 1_n$. Define

$$W := (w_{z_m}^1 \circ \delta_m^m) \cdots (w_{z_1}^1 \circ \delta_m^1),$$

which is a unitary in $C([0,1], M_n)$.

By Definition 3.6, $W(0) = w_{z_m}^1(0) \cdots w_{z_1}^1(0) = 1_n$, so that *(b)* holds. Let σ denote the permutation $(1\ 2\ \cdots\ z_m)(1\ 2\ \cdots\ z_{m-1}) \cdots (1\ 2\ \cdots\ z_1) \in S_n$ and note that $\sigma(z_k) = m - k + 1$ for $1 \le k \le m$. Thus, $U[\sigma]TU[\sigma]^*$ has zero crosses at indices $1, \ldots, m$. Since $W(1)TW(1)^* = U[\sigma]TU[\sigma]^*$ by Equation 3.5, this proves *(a)*.

Finally, let us prove (c). This is proved using two short sublemmas, which we establish first.

Sublemma 3.10.1.1. *Suppose $D \in M_n$ has a zero cross at index j. Then, $\mathfrak{r}(w_j^1(1)Dw_j^1(1)^*) \le \mathfrak{r}(D)$ and, for $2 \le i \le j$, $\mathfrak{r}(w_j^i(1)Dw_j^i(1)^*) \le \mathfrak{r}(D) + 1$.*

Proof. By Equation 3.5, $w_j^1(1) = U[(1\ 2\ \cdots\ j)]$. Consider the matrix D broken up into the four regions created by the zero cross at j, together with the matrix $w_j^1(1)Dw_j^1(1)^*$:

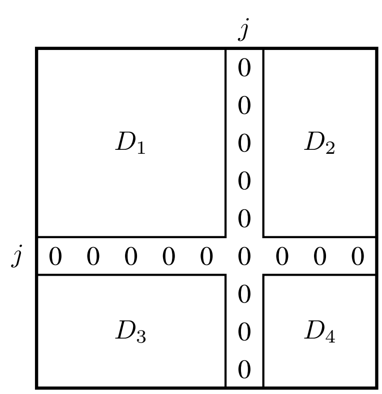

Figure 3.6: The Matrix D

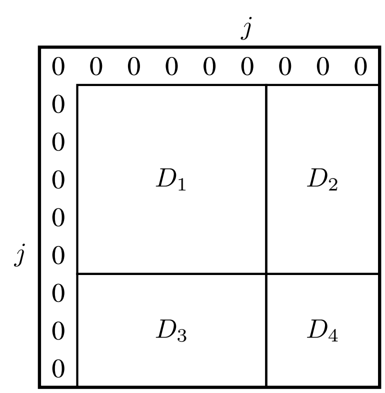

Figure 3.7: The Matrix $w_j^1(1)Dw_j^1(1)^*$

Since no non-zero entry gets shifted away from the diagonal, it follows that $\mathfrak{r}(w_j^1(1)Dw_j^1(1)^*) \le \mathfrak{r}(D)$.

Suppose now that $2 \leq i \leq j$. If $i = j$, then the desired inequality is trivial, so we may assume that $i < j$. By Equation 3.5, $w^i_j(1) = U[(i\ i+1\ \cdots\ j)]$. Consider the matrix D broken up into the following nine regions created by the zero cross at j and the ith row and column, together with the matrix $w^i_j(1)Dw^i_j(1)^*$:

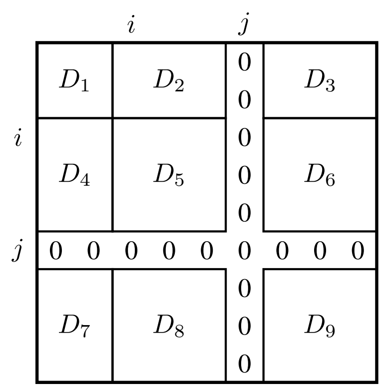

Figure 3.8: The Matrix D

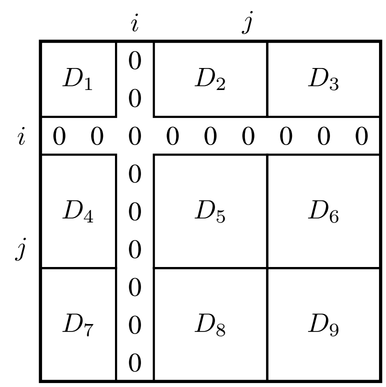

Figure 3.9: The Matrix $w^i_j(1)Dw^i_j(1)^*$

With the exception of D_2 and D_4, which get shifted one unit away from the diagonal, no entry in the other seven regions is moved away from the diagonal. Therefore, $\mathfrak{r}(w^i_j(1)Dw^i_j(1)^*) \leq \mathfrak{r}(D) + 1$, which proves Sublemma 3.10.1.1. //

Sublemma 3.10.1.2. *Suppose $D \in M_n$ has a zero cross at index j. Then, if $1 \leq i \leq j$ and $\xi \in [0,1]$, we have $\mathfrak{r}(w^i_j(\xi)Dw^i_j(\xi)^*) \leq \mathfrak{r}(D) + 2$.*

Proof. Fix $1 \leq i \leq j$ and $\xi \in [0,1]$. If $\xi = 0$ or if $i = j$, then $w^i_j(\xi) = 1_n$ and the result is trivial. Hence, we may assume $i < j$ and $\xi \in (0,1]$. Let $1 \leq k \leq j - i$ be the unique integer such that $\xi \in \left(\frac{k-1}{j-i}, \frac{k}{j-i}\right]$. Then,

$$\begin{aligned} w^i_j(\xi) &= u_{(i\ i+1)}(0) \cdots u_{(j-k-1\ j-k)}(0) \\ &\qquad \cdot u_{(j-k\ j-k+1)}(\delta^k_{j-i}(\xi)) u_{(j-k+1\ j-k+2)}(1) \cdots u_{(j-1\ j)}(1) \\ &= u_{(j-k\ j-k+1)}(\delta^k_{j-i}(\xi)) w^{j-k+1}_j(1). \end{aligned}$$

Let $D' := w^{j-k+1}_j(1)Dw^{j-k+1}_j(1)^*$. By Sublemma 3.10.1.1, $\mathfrak{r}(D') \leq \mathfrak{r}(D) + 1$. Now, consider the conjugation of D' by $u_{(j-k\ j-k+1)}(\delta^k_{j-i}(\xi))$, which we denote by E. The entries of D' affected by this conjugation lie in one of the following three regions:

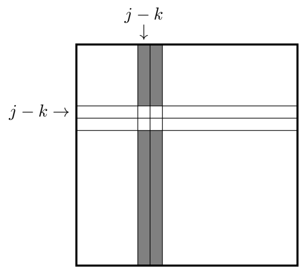

Figure 3.10: Region A

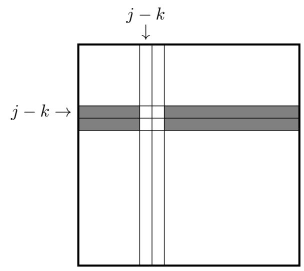

Figure 3.11: Region B

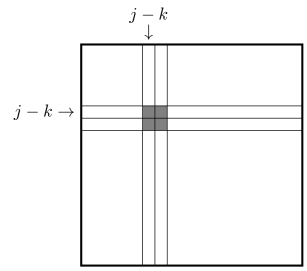

Figure 3.12: Region C

An analysis extremely similar to the one using Figure 3.2 yields the following observations:

- an entry in E lying in Region A can be non-zero only if at least one of the two corresponding shaded entries in D' lying in the same row is non-zero;
- an entry in E lying in Region B can be non-zero only if at least one of the two corresponding shaded entries in D' lying in the same column is non-zero;
- an entry in E lying in Region C can be non-zero only if at least one of the other four corresponding shaded entries in D' is non-zero.

We see that in all instances, a non-zero entry in E never appears more than one unit further away from the diagonal than a non-zero entry in D'. Thus,

$$\mathfrak{r}(u_{(j-k\ j-k+1)}(\delta^k_{j-i}(\xi))D'u_{(j-k\ j-k+1)}(\delta^k_{j-i}(\xi))^*) \leq \mathfrak{r}(D') + 1 \leq \mathfrak{r}(D) + 2,$$

which proves Sublemma 3.10.1.2. //

Let us now conclude the proof of Sublemma 3.10.1 by proving (c). Fix $\xi \in [0,1]$. If $\xi = 0$, the result is trivial, and so we may assume $\xi \in (0,1]$. Let $1 \leq k \leq m$ be the unique integer such that $\xi \in \left(\frac{k-1}{m}, \frac{k}{m}\right]$. Then we may write

$$\begin{aligned} W(\xi) &= w^1_{z_m}(0) \cdots w^1_{z_{k+1}}(0) w^1_{z_k}(\delta^k_m(\xi)) w^1_{z_{k-1}}(1) \cdots w^1_{z_1}(1) \\ &= w^1_{z_k}(\delta^k_m(\xi)) w^1_{z_{k-1}}(1) \cdots w^1_{z_1}(1). \end{aligned}$$

By Sublemma 3.10.1.1, $\mathfrak{r}(w^1_{z_1}(1)Tw^1_{z_1}(1)^*) \leq \mathfrak{r}(T)$ since T has a zero cross at index z_1. Moreover, $w^1_{z_1}(1)Tw^1_{z_1}(1)^*$ has a zero cross at index z_2 since z_2 is not among the indices affected by the conjugation (the reasoning for this is basically identically that used in the proof of Sublemma 3.7.2). Thus, we may apply Sublemma 3.10.1.1 again to conclude that

$$\mathfrak{r}(w^1_{z_2}(1)w^1_{z_1}(1)Tw^1_{z_1}(1)^*w^1_{z_2}(1)^*) \leq \mathfrak{r}(w^1_{z_1}(1)Tw^1_{z_1}(1)^*) \leq \mathfrak{r}(T).$$

Continuing inductively in this way, it follows that $\mathfrak{r}(D) \leq \mathfrak{r}(T)$, where

$$D = w^1_{z_{k-1}}(1)\cdots w^1_{z_1}(1)Tw^1_{z_1}(1)^*\cdots w^1_{z_{k-1}}(1)^*$$

and, moreover, D has a zero cross at index z_k. Thus, by Sublemma 3.10.1.2,

$$\mathfrak{r}(W(\xi)TW(\xi)^*) = \mathfrak{r}(w^1_{z_k}(\delta^k_m(\xi))Dw^1_{z_k}(\delta^k_m(\xi))^*) \leq \mathfrak{r}(D)+2 \leq \mathfrak{r}(T)+2,$$

which proves (c) and completes the proof of Sublemma 3.10.1. //

With Sublemma 3.10.1 in hand, let us now proceed with the proof of Lemma 3.10. Since $NM < \mathfrak{s}(A)$, we may apply Lemma 3.4 with the natural number NM, with $K = \{0\}$, and with $F_{i,k} := X_i \setminus U_{i,k}$ for $1 \leq i \leq l$ and $1 \leq k \leq n_i$. This yields a function $\Theta \in A$ with the following properties:

(I) for all $1 \leq i \leq l$ and $x \in X_i$, $\Theta_i(x)$ is a diagonal matrix with entries in $[0,1]$ whose final NM diagonal entries are all 0 and such that at most one of any NM consecutive diagonal entries is non-zero;

(II) for all $1 \leq i \leq l$ and $1 \leq k \leq n_i$, if $x \notin U_{i,k}$, then $\Theta_i(x)_{k,k} = 0$;

(III) for all $1 \leq i \leq l$ and $1 \leq k \leq n_i$, there is a (possibly empty) open subset $\mathcal{O}_{i,k} \subset X_i$ containing $B_{i,k}$ with the property that $\Theta_i(x)_{k,k} = 1$ whenever $x \in \mathcal{O}_{i,k}$.

Now, fix $1 \leq i \leq l$. For $1 \leq k \leq n_i - NM$, let $u_k \in C(X_i, M_{n_i})$ be the unitary

$$u_k(x) := \operatorname{diag}(1_{k-1}, W(\Theta_i(x)_{k,k}), 1_{n_i-(NM+k-1)}),$$

where W is the unitary in $C([0,1], M_{NM})$ given by Sublemma 3.10.1 with $z_1 := 1$, $z_2 := 1+M, \ldots, z_N := 1+(N-1)M$. For $n_i - NM < k \leq n_i$, set $u_k \equiv 1_{n_i}$. Define $V_i \in C(X_i, M_{n_i})$ to be the unitary

$$V_i := \prod_{k=1}^{n_i} u_k.$$

For $x \in X_i$, let $K(x) := \{1 \leq k \leq n_i : \Theta_i(x)_{k,k} > 0\}$ and write $K(x) = \{k_1, \ldots, k_s\}$, where $k_1 < \cdots < k_s$ and put $k_{s+1} := n_i + 1$. Note that $k_1 = 1$

by (III) above since $B_{i,1} = X_i$ by Lemma 2.8, and for $1 \leq t \leq s$, $k_{t+1} - k_t \geq NM$ by (I) above. If $k \notin K(x)$, then $u_k \equiv 1_{n_i}$. Hence, we may write

$$\begin{aligned} & V_i(x) \\ = & \prod_{t=1}^{s} u_{k_t}(x) \qquad (3.6) \\ = & \operatorname{diag}\left(W(\Theta_i(x)_{k_1,k_1}), 1_{d_1}, W(\Theta_i(x)_{k_2,k_2}), 1_{d_2}, \ldots, W(\Theta_i(x)_{k_s,k_s}), 1_{d_s}\right), \end{aligned}$$

where $d_t := k_{t+1} - (k_t + NM)$ for $1 \leq t \leq s$.

Let $V := (V_1, \ldots, V_l)$. In order to prove Lemma 3.10, let us show that *(1)* holds, then that *(2)* holds, and finally that $V \in A$.

To prove *(1)* and *(2)*, fix $1 \leq i \leq l$ and $x \in X_i$, and let $K(x) = \{k_1, \ldots, k_s\}$ and k_{s+1} be defined as above. For $1 \leq t \leq s$, we have $\Theta_i(x)_{k_t,k_t} > 0$. Hence, by (II) above, it must be that $x \in U_{i,k_t}$ and, thus, $f_i(x)$ has a block point at index k_t and zero crosses at indices $k_t, k_t + M, \ldots, k_t + (N-1)M$ by the assumption of the lemma. Thus, $f_i(x) = \operatorname{diag}(Q_1, Q_2, \ldots, Q_s)$, where Q_t is a $k_{t+1} - k_t$ block for $1 \leq t \leq s$ and has zero crosses at $1, 1 + M, \ldots, 1 + (N-1)M$. Therefore, in light of the decomposition of $V_i(x)$ in Equation 3.6, we may view $V_i(x) f_i(x) V_i(x)^*$ as a block-diagonal matrix $\operatorname{diag}(B_1, \ldots, B_s)$ with

$$B_t = \operatorname{diag}(W(\Theta_i(x)_{k_t,k_t}), 1_{d_t}) \cdot Q_t \cdot \operatorname{diag}(W(\Theta_i(x)_{k_t,k_t}), 1_{d_t})^*.$$

Thus, to prove *(2)*, it suffices to show that $\mathfrak{r}(B_t) \leq \mathfrak{r}(Q_t) + 2$. Furthermore, if $x \in \mathcal{O}_{i,k}$ for some $1 \leq k \leq n_i$, then by (III), $\Theta_i(x)_{k,k} = 1 > 0$, and so $k = k_t$ for some $1 \leq t \leq s$. Since the block B_t begins at index k_t down the diagonal of $V_i(x) f_i(x) V_i(x)^*$, to prove *(1)* it suffices to show that B_t has zero crosses at indices $1, 2, \ldots, N$ whenever $\Theta_i(x)_{k_t,k_t} = 1$.

To this end, fix $1 \leq t \leq s$ and write

$$Q_t = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},$$

where $D_{11} \in M_{NM}$, $D_{22} \in M_{d_t}$, and D_{12} and D_{21} are $NM \times d_t$ and $d_t \times NM$ matrices, respectively. Note that D_{11} has zero crosses at indices $1, 1 + M, \ldots, 1 + (N-1)M$, while the rows of D_{12} and the columns of D_{21} at these same indices consist entirely of zeros. We may write

$$\begin{aligned} B_t &= \begin{pmatrix} W(\Theta_i(x)_{k_t,k_t}) & 0 \\ 0 & 1_{d_t} \end{pmatrix} \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} W(\Theta_i(x)_{k_t,k_t})^* & 0 \\ 0 & 1_{d_t} \end{pmatrix} \\ &= \begin{pmatrix} W(\Theta_i(x)_{k_t,k_t}) D_{11} W(\Theta_i(x)_{k_t,k_t})^* & W(\Theta_i(x)_{k_t,k_t}) D_{12} \\ D_{21} W(\Theta_i(x)_{k_t,k_t})^* & D_{22} \end{pmatrix}. \end{aligned} \qquad (3.7)$$

If $\Theta_i(x)_{k_t,k_t} = 1$, then

$$B_t = \begin{pmatrix} W(1)D_{11}W(1)^* & W(1)D_{12} \\ D_{21}W(1)^* & D_{22} \end{pmatrix}.$$

Therefore, by our definition of W, we may apply Sublemma 3.10.1 to conclude that $W(1)D_{11}W(1)^*$ has zero crosses at indices $1, 2, \ldots, N$. Inspecting the proof of that same part of the sublemma, we see that the first $1, 2, \ldots, N$ rows of $W(1)D_{12}$ and columns of $D_{21}W(1)^*$ consist of only zeros. It follows that B_t has zero crosses at indices $1, 2, \ldots, N$, which, based on the aforementioned analysis, proves *(1)*.

Let us now prove *(2)* by showing that $\mathfrak{r}(B_t) \leq \mathfrak{r}(Q_t) + 2$. By our definition of W, we may apply Sublemma 3.10.1 to conclude that

$$\mathfrak{r}(W(\Theta_i(x)_{k_t,k_t})D_{11}W(\Theta_i(x)_{k_t,k_t})^*) \leq \mathfrak{r}(D_{11}) + 2 \leq \mathfrak{r}(Q_t) + 2,$$

and, hence,

$$\mathfrak{r}\left(\begin{pmatrix} W(\Theta_i(x)_{k_t,k_t})D_{11}W(\Theta_i(x)_{k_t,k_t})^* & 0 \\ 0 & D_{22} \end{pmatrix}\right) \leq \mathfrak{r}(Q_t) + 2. \tag{3.8}$$

Next let us show that

$$\mathfrak{r}\left(\begin{pmatrix} 0 & W(\Theta_i(x)_{k_t,k_t})D_{12} \\ 0 & 0 \end{pmatrix}\right) \leq \mathfrak{r}(Q_t). \tag{3.9}$$

Just as in the proof of Sublemma 3.10.1, there exist $\zeta \in [0, 1]$ and $1 \leq k \leq N$ such that

$$W(\Theta_i(x)_{k_t,k_t}) = w^1_{z_k}(\zeta)w^1_{z_{k-1}}(1) \cdots w^1_{z_1}(1).$$

Hence, following the lines of the proof of Sublemma 3.10.1.1, we have

$$\mathfrak{r}\left(\begin{pmatrix} 0 & w^1_{z_{k-1}}(1) \cdots w^1_{z_1}(1)D_{12} \\ 0 & 0 \end{pmatrix}\right) \leq \mathfrak{r}\left(\begin{pmatrix} 0 & D_{12} \\ 0 & 0 \end{pmatrix}\right)$$

by Equation 3.5 since only rows of zeros are shifted up when multiplying D_{12} on the left by $w^1_{z_{k-1}}(1) \cdots w^1_{z_1}(1)$, while non-zero entries remain in place or are shifted down towards the diagonal. If $z_k = 1$, then $W(\Theta_i(x)_{k_t,k_t}) = 1_{NM}$ and Equation 3.9 follows. Otherwise, as in the proof of Sublemma 3.10.1.2, there exist $\beta \in [0, 1]$ and $1 \leq p \leq z_k - 1$ such that

$$w^1_{z_k}(\zeta) = u_{(z_k - p \;\; z_k - p + 1)}(\beta)U[(z_k - p + 1 \;\; z_k - p + 2 \;\; \cdots \;\; z_k)].$$

Since the z_kth row of a matrix remains unchanged when multiplying on the left by $w^1_{z_{k-1}}(1) \cdots w^1_{z_1}(1)$, the z_kth row of $w^1_{z_{k-1}}(1) \cdots w^1_{z_1}(1)D_{12}$

contains only zeros. Thus, multiplying this matrix on the left by $U[(z_k - p+1 \ \cdots \ z_k)]$ shifts the zero row from index z_k to index $z_k - p + 1$, while shifting the rows $z_k - p + 1, \ldots, z_k - 1$ down by one towards the diagonal. Thus,

$$\mathfrak{r}\left(\begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}\right) \leq \mathfrak{r}\left(\begin{pmatrix} 0 & w^1_{z_{k-1}}(1)\cdots w^1_{z_1}(1)D_{12} \\ 0 & 0 \end{pmatrix}\right) \leq \mathfrak{r}\left(\begin{pmatrix} 0 & D_{12} \\ 0 & 0 \end{pmatrix}\right),$$

where $E := U[(z_k - p + 1 \ \cdots \ z_k)]w^1_{z_{k-1}}(1)\cdots w^1_{z_1}(1)D_{12}$. Now, the matrices E and $u_{(z_k - p \ z_k - p+1)}(\beta)E$ may differ only on rows $z_k - p$ and $z_k - p + 1$, where these two rows of the latter matrix are linear combinations of the same two rows of E. From this and the fact that the $z_k - p + 1$ row of E consists only of zeros, it is clear that for a given column λ, the $(z_k - p, \lambda)$- or $(z_k - p + 1, \lambda)$-entry of $u_{(z_k - p \ z_k - p+1)}(\beta)E$ can be non-zero only if the $(z_k - p, \lambda)$ entry of E is non-zero. Hence,

$$\mathfrak{r}\left(\begin{pmatrix} 0 & u_{(z_k-p \ z_k-p+1)}(\beta)E \\ 0 & 0 \end{pmatrix}\right) \leq \mathfrak{r}\left(\begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}\right) \leq \mathfrak{r}\left(\begin{pmatrix} 0 & D_{12} \\ 0 & 0 \end{pmatrix}\right) \leq \mathfrak{r}(Q_t),$$

which proves Equation 3.9. Since the $z_1, \ldots, z_N$ columns of D_{21} consist only of zeros, it follows that the $z_1, \ldots, z_N$ rows of D^*_{21} consist of only zeros. The same is true of the matrix D_{12}, and this was the only fact we used about D_{12} in deriving Equation 3.9. Thus, since the diagonal radius is not changed when taking adjoints, we may use Equation 3.9 to conclude also that

$$\begin{aligned} \mathfrak{r}\left(\begin{pmatrix} 0 & 0 \\ D_{21}W(\Theta_i(x)_{k_t,k_t})^* & 0 \end{pmatrix}\right) &= \mathfrak{r}\left(\begin{pmatrix} 0 & W(\Theta_i(x)_{k_t,k_t})D^*_{21} \\ 0 & 0 \end{pmatrix}\right) \\ &\leq \mathfrak{r}(Q_t). \end{aligned} \tag{3.10}$$

On putting Equation 3.7, Equation 3.8, Equation 3.9, and Equation 3.10 together, it follows that

$$\mathfrak{r}(B_t) \leq \mathfrak{r}(Q_t) + 2,$$

as was required to prove *(2)*.

Finally, let us show that $V \in A$. Suppose $y \in Y_i$ decomposes into $x_1 \in X_{i_1} \setminus Y_{i_1}, \ldots, x_r \in X_{i_r} \setminus Y_{i_r}$. We need to show that V respects this decomposition, i.e., $V_i(y) = \operatorname{diag}(V_{i_1}(x_1), \ldots, V_{i_r}(x_r))$. Let $K(y) = \{1 \leq k \leq n_i : \Theta_i(y)_{k,k} > 0\}$, as defined above. Write $K(y) = \{k_1, \ldots, k_s\}$, where $1 = k_1 < \cdots < k_s$, and put $k_{s+1} := n_i + 1$. As before, let $d_t := k_{t+1} - (k_t + NM)$ for $1 \leq t \leq s$. Define $B(y) := \{1 \leq k \leq n_i : y \in B_{i,k}\}$. By (III) above, $B(y) \subset K(y)$. Hence by Lemma 2.8, for each $1 \leq j \leq r$, there is a

$t_j \in \{1, \dots, s\}$ such that $1 + n_{i_1} + \cdots + n_{i_{j-1}} = k_{t_j}$ (where $k_{t_1} = 1 = k_1$, so that $t_1 = 1$); set $t_{r+1} := s + 1$, so that $k_{t_{r+1}} = k_{s+1} = n_i + 1$.

Now, fix $1 \le j \le r$ and observe that $\Theta_{i_j}(x_j)_{k,k} = \Theta_i(y)_{k_{t_j}+k-1, k_{t_j}+k-1}$. Therefore,

$$\begin{aligned} K(x_j) &= \{1 \le k \le n_{i_j} : \Theta_{i_j}(x_j)_{k,k} > 0\} \\ &= \{k - k_{t_j} + 1 : k \in K(y) \text{ and } k_{t_j} \le k < k_{t_{j+1}}\} \\ &= \{k_t - k_{t_j} + 1 : t_j \le t < t_{j+1}\}. \end{aligned}$$

Moreover, if $t_j \le t < t_{j+1}$, then $(k_{t+1} - k_{t_j} + 1) - (k_t - k_{t_j} + 1 + NM) = d_t$. Given matrices $E_1, \dots, E_p$, let $\bigoplus_{q=1}^{p} E_q := \operatorname{diag}(E_1, \dots, E_p)$. Then, by the computation of $K(x_j)$ above and Equation 3.6, it follows that

$$\begin{aligned} &V_{i_j}(x_j) \\ &= \bigoplus_{t_j \le t < t_{j+1}} \operatorname{diag}\left(W(\Theta_{i_j}(x_j)_{k_t - k_{t_j}+1, k_t - k_{t_j}+1}), 1_{(k_{t+1}-k_{t_j}+1)-(k_t - k_{t_j}+1+NM)}\right) \\ &= \bigoplus_{t_j \le t < t_{j+1}} \operatorname{diag}\left(W(\Theta_i(y)_{k_t,k_t}), 1_{d_t}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \operatorname{diag}(V_{i_1}(x_1), \dots, V_{i_r}(x_r)) &= \bigoplus_{1 \le j \le r} \bigoplus_{t_j \le t < t_{j+1}} \operatorname{diag}\left(W(\Theta_i(y)_{k_t,k_t}), 1_{d_t}\right) \\ &= \bigoplus_{1 \le t \le s} \operatorname{diag}\left(W(\Theta_i(y)_{k_t,k_t}), 1_{d_t}\right) \\ &= V_i(y), \end{aligned}$$

where the last equality follows by Equation 3.6. This shows that $V \in A$. The proof of Lemma 3.10 is now complete. //

The final lemma in this section requires a new class of unitaries.

Definition 3.11 (cf. [Lut17b], p. 15)**.** For $1 \le j \le k \le n$, we define

$$\gamma^n_{j,k} := (j \;\; j+1 \;\; \cdots \;\; k) \in S_n.$$

Suppose $1 \le N \le n$. For $N \le j \le k \le n$, we define

$$\sigma^n_{j,k} := (j - N + 1 \;\; k - N + 1) \cdots (j \;\; k) \in S_n.$$

We define $u^n_{j,k} \colon [0,1] \to M_n$ to be the unitary

$$u^n_{j,k}(\xi) := u_{(j-N+1 \;\; k-N+1)}(\xi) \cdots u_{(j \;\; k)}(\xi),$$

where $u_{(i\ i')}(\xi)$ is the matrix in M_n defined as in Definition 3.6.

Remark 3.12 (cf. [Lut17b], p. 15). Note that in the definition above, if $N \leq k \leq n-N$, then $\sigma^n_{k,n}$ is the permutation in S_n that interchanges $k-N+1,\ldots,k$ and $n-N+1,\ldots,n$; moreover, in this case, all of the factors in the definition of $u^n_{k,n}(\xi)$ (for any $\xi \in [0,1]$) commute with each other by Definition 3.6.

Lemma 3.13 (cf. [Lut17b], Lemma 2.2.4)**.** *Suppose A is a DSH algebra of length l and that $1 \leq N < \mathfrak{s}(A)$. Suppose $f \in A$ is such that for all $1 \leq i \leq l$ and $1 \leq k \leq n_i$, there is an open subset $U_{i,k} \subset X_i$ containing $B_{i,k}$ with the property that if $x \in U_{i,k}$, then $f_i(x)$ has zero crosses at indices $k, k+1, \ldots, k+N-1$, and such that $\mathfrak{r}(f_i(x)) \leq N$ for all $x \in X_i$. Then, there is a unitary $V \in A$ such that for all $1 \leq i \leq l$ and $x \in X_i$, the matrix $(fV)_i(x)$ is strictly lower triangular.*

Proof. Given an integer $n \geq N$, let $W_n \in C([0,1]^n, M_n)$ be the unitary

$$W_n(\xi_1,\ldots,\xi_n) := U[\gamma^n_{1,n}]^N \left(\prod_{k=N}^{n-1} u^n_{k,n}(\xi_{k+1})\right), \tag{3.11}$$

where $u^n_{k,n}$ is the product of N factors as in Definition 3.11. Note that $W_n = 1_n$ if $n = N$.

To facilitate the proof of Lemma 3.13, let us first establish the following sublemma.

Sublemma 3.13.1 (cf. [Lut17b], Lemma 2.1.8)**.** *Let N be as in Lemma 3.13 and suppose $n > N$. Suppose $\vec{\xi} := (\xi_1,\ldots,\xi_n)$ is a vector in $[0,1]^n$ with the property that $\xi_1 = 1$, the final N entries are all zero, and for any consecutive N entries, at most one is non-zero. Suppose $K = \{1 = k_1 < k_2 < \cdots < k_m\}$ is any set of indices, containing 1, at which $\vec{\xi}$ is 1; put $k_{m+1} := n+1$. Then,*

$$W_n(\vec{\xi}) = \operatorname{diag}\left(W_{k_2-k_1}(\xi_{k_1},\ldots,\xi_{k_2-1}),\ldots,W_{k_{m+1}-k_m}(\xi_{k_m},\ldots,\xi_{k_{m+1}-1})\right). \tag{3.12}$$

Proof. Fix $n > N$, a vector $\vec{\xi}$, and an associated set K, satisfying the hypotheses of the sublemma. Let us proceed by induction on the size m of K. If $m = 1$, there is nothing to show. Fix $m \geq 2$ and suppose that Sublemma 3.13.1 holds for every natural number $n' > N$, vector ζ, and associated set K' of size $m-1$, provided they satisfy the required hypotheses. Assume that $|K| = m$. Let us show Equation 3.12 holds in this case.

Note that by assumption

$$N < k_2 \leq n - N. \tag{3.13}$$

The following calculation is needed.

Claim 3.13.1.1.

$$\begin{aligned}&U[\gamma^n_{1,n}]^N\left(\prod_{k=N}^{k_2-2}u^n_{k,n}(\xi_{k+1})\right)U[\sigma^n_{k_2-1,n}]\\=&U[\gamma^n_{1,k_2-1}]^N\left(\prod_{k=N}^{k_2-2}u^n_{k,k_2-1}(\xi_{k+1})\right)U[\gamma^n_{k_2,n}]^N.\end{aligned}$$

Proof. If $k_2-1=N$, then the products on either side of the equality above are empty and the equation reduces to

$$U[\gamma^n_{1,n}]^N U[\sigma^n_{k_2-1,n}] = U[\gamma^n_{1,k_2-1}]^N U[\gamma^n_{k_2,n}]^N. \tag{3.14}$$

By Equation 3.13 and Remark 3.12, it is elementary to see that Equation 3.14 holds. Therefore, for the remainder of the proof of the claim, we may assume that $k_2-2\geq N$.

To establish the claim, we'll first show that for all $N\leq k\leq k_2-2$,

$$u^n_{k,n}(\xi_{k+1}) = U[\sigma^n_{k_2-1,n}]u^n_{k,k_2-1}(\xi_{k+1})U[\sigma^n_{k_2-1,n}]. \tag{3.15}$$

Suppose first that $k_2-N\leq k\leq k_2-2$. Since $\xi_{k_2}=1$, it must be that $\xi_{k+1}=0$ because at most one of any N consecutive entries of $\vec{\xi}$ can be non-zero. Hence, Equation 3.15 reduces to $1_n = U[\sigma^n_{k_2-1,n}]^2$, and, using Equation 3.13, this equality holds by Remark 3.12.

Suppose instead that $N\leq k\leq k_2-N-1$. By definition,

$$\begin{aligned}&U[\sigma^n_{k_2-1,n}]u^n_{k,k_2-1}(\xi_{k+1})\\=&\left(\prod_{j=-(N-1)}^{0}U[(k_2-1+j\ \ n+j)]\right)\left(\prod_{j=-(N-1)}^{0}u_{(k+j\ \ k_2-1+j)}(\xi_{k+1})\right).\end{aligned}$$

Note that for any $-(N-1)\leq j,j'\leq 0$, Equation 3.13 yields that

$$k_2-1+j\leq k_2-1\leq n-N-1<n-(N-1)\leq n+j'$$

and

$$k+j\leq k_2-N-1+j\leq k_2-N-1<k_2-(N-1)-1\leq k_2-1+j'.$$

Thus, whenever $0\geq j',j\geq -(N-1)$ with $j\neq j'$, the permutations $(k_2-1+j'\ \ n+j')$ and $(k+j\ \ k_2-1+j)$ are disjoint, and hence, $U[(k_2-$

$1+j' \ \ n+j')]$ and $u_{(k+j \ \ k_2-1+j)}(\xi_{k+1})$ commute. Thus, the expression above is equal to

$$\prod_{j=-(N-1)}^{0} U[(k_2-1+j \ \ n+j)]u_{(k+j \ \ k_2-1+j)}(\xi_{k+1}).$$

Therefore, by the same reasoning,

$$\begin{aligned}
&U[\sigma^n_{k_2-1,n}]u^n_{k,k_2-1}(\xi_{k+1})U[\sigma^n_{k_2-1,n}] \\
=&\prod_{j=-(N-1)}^{0} U[(k_2-1+j \ \ n+j)]u_{(k+j \ \ k_2-1+j)}(\xi_{k+1}) \\
&\qquad \cdot \prod_{j=-(N-1)}^{0} U[(k_2-1+j \ \ n+j)] \\
=&\prod_{j=-(N-1)}^{0} U[(k_2-1+j \ \ n+j)]u_{(k+j \ \ k_2-1+j)}(\xi_{k+1})U[(k_2-1+j \ \ n+j)].
\end{aligned}$$

It is elementary to see that $U[(a \ b)]u_{(c \ b)}(\zeta)U[(a \ b)] = u_{(c \ a)}(\zeta)$ whenever $c \leq b \leq a$ and $\zeta \in [0,1]$. Hence,

$$\begin{aligned}
&U[\sigma^n_{k_2-1,n}]u^n_{k,k_2-1}(\xi_{k+1})U[\sigma^n_{k_2-1,n}] \\
=&\prod_{j=-(N-1)}^{0} u_{(k+j \ n+j)}(\xi_{k+1}) = u^n_{k,n}(\xi_{k+1}),
\end{aligned}$$

which proves that Equation 3.15 holds.

Let us now use Equation 3.14 and Equation 3.15 to complete the proof of Claim 3.13.1.1. We argued earlier that $U[\sigma^n_{k_2-1,n}]^2 = 1_n$. It follows by Equation 3.15 that

$$\begin{aligned}
\prod_{k=N}^{k_2-2} u^n_{k,n}(\xi_{k+1}) &= \prod_{k=N}^{k_2-2} U[\sigma^n_{k_2-1,n}]u^n_{k,k_2-1}(\xi_{k+1})U[\sigma^n_{k_2-1,n}] \\
&= U[\sigma^n_{k_2-1,n}] \left(\prod_{k=N}^{k_2-2} u^n_{k,k_2-1}(\xi_{k+1}) \right) U[\sigma^n_{k_2-1,n}].
\end{aligned}$$

Therefore, by Equation 3.14,

$$\begin{aligned}
&U[\gamma^n_{1,n}]^N \prod_{k=N}^{k_2-2} u^n_{k,n}(\xi_{k+1}) \\
=&U[\gamma^n_{1,k_2-1}]^N U[\gamma^n_{k_2,n}]^N \left(\prod_{k=N}^{k_2-2} u^n_{k,k_2-1}(\xi_{k+1}) \right) U[\sigma^n_{k_2-1,n}].
\end{aligned} \tag{3.16}$$

Moreover, for each $k = N, \ldots, k_2 - 2$, the indices in each transposition-like unitary factor in $u^n_{k,k_2-1}(\xi_{k+1})$ are distinct from $k_2, \ldots, n$. Hence, $U[\gamma^n_{k_2,n}]^N$ and $\prod_{k=N}^{k_2-2} u^n_{k,k_2-1}(\xi_{k+1})$ commute, and so, using Equation 3.16, it follows that

$$\begin{aligned}&U[\gamma^n_{1,n}]^N \prod_{k=N}^{k_2-2} u^n_{k,n}(\xi_{k+1})\\=&U[\gamma^n_{1,k_2-1}]^N \left(\prod_{k=N}^{k_2-2} u^n_{k,k_2-1}(\xi_{k+1})\right) U[\gamma^n_{k_2,n}]^N U[\sigma^n_{k_2-1,n}].\end{aligned}$$

Multiplying on the right of both sides by $U[\sigma^n_{k_2-1,n}]$ yields the result asserted in Claim 3.13.1.1. //

With the calculation established by Claim 3.13.1.1 in hand, let us now continue with the induction proof of Sublemma 3.13.1. By Equation 3.11 and Equation 3.13,

$$\begin{aligned}W_n(\vec{\xi}) &= U[\gamma^n_{1,n}]^N \left(\prod_{k=N}^{n-1} u^n_{k,n}(\xi_{k+1})\right)\\&= U[\gamma^n_{1,n}]^N \left(\prod_{k=N}^{k_2-2} u^n_{k,n}(\xi_{k+1})\right) u^n_{k_2-1,n}(\xi_{k_2}) \left(\prod_{k=k_2}^{n-1} u^n_{k,n}(\xi_{k+1})\right).\end{aligned}$$

Since $u^n_{k_2-1,n}(\xi_{k_2}) = u^n_{k_2-1,n}(1) = U[\sigma^n_{k_2-1,n}]$, we apply Claim 3.13.1.1 to obtain

$$W_n(\vec{\xi}) = U[\gamma^n_{1,k_2-1}]^N \left(\prod_{k=N}^{k_2-2} u^n_{k,k_2-1}(\xi_{k+1})\right) U[\gamma^n_{k_2,n}]^N \left(\prod_{k=k_2}^{n-1} u^n_{k,n}(\xi_{k+1})\right). \tag{3.17}$$

Let $\vec{\xi'} := (\xi_1, \ldots, \xi_{k_2-1})$ and $\vec{\xi''} := (\xi_{k_2}, \ldots, \xi_n)$. By Equation 3.13, $|\vec{\xi'}| \geq N$, so that

$$W_{k_2-k_1}(\vec{\xi'}) = U[\gamma^{k_2-1}_{1,k_2-1}]^N \left(\prod_{k=N}^{(k_2-1)-1} u^{k_2-1}_{k,k_2-1}(\xi_{k+1})\right) \in M_{k_2-k_1}. \tag{3.18}$$

Furthermore, $|\vec{\xi}''| > N$, so that

$$\begin{aligned} W_{n+1-k_2}(\vec{\xi}'') &= W_{n+1-k_2}(\xi_{k_2}, \ldots, \xi_n) \\ &= U[\gamma^{n+1-k_2}_{1,n+1-k_2}]^N \left(\prod_{k=N}^{n+1-k_2-1} u^{n+1-k_2}_{k,n+1-k_2}(\xi_{k_2+k}) \right) \\ &= U[\gamma^{n+1-k_2}_{1,n+1-k_2}]^N \left(\prod_{k=1}^{n-k_2} u^{n+1-k_2}_{k,n+1-k_2}(\xi_{k_2+k}) \right) \\ &= U[\gamma^{n+1-k_2}_{1,n+1-k_2}]^N \left(\prod_{k=k_2}^{n-1} u^{n+1-k_2}_{k+1-k_2,n+1-k_2}(\xi_{k+1}) \right) \in M_{n+1-k_2}, \end{aligned} \tag{3.19}$$

where the penultimate equality follows since $\xi_{k_2} = 1$ and at most one of any consecutive entries of $\vec{\xi}$ is non-zero. Therefore,

$$\begin{aligned} &\operatorname{diag}(W_{k_2-k_1}(\vec{\xi}'), W_{n+1-k_2}(\vec{\xi}'')) \\ =& \operatorname{diag}(W_{k_2-k_1}(\vec{\xi}'), 1_{n+1-k_2}) \operatorname{diag}(1_{k_2-k_1}, W_{n+1-k_2}(\vec{\xi}'')) \\ =& U[\gamma^n_{1,k_2-1}]^N \left(\prod_{k=N}^{(k_2-1)-1} u^n_{k,k_2-1}(\xi_{k+1}) \right) U[\gamma^n_{k_2,n}]^N \left(\prod_{k=k_2}^{n-1} u^n_{k,n}(\xi_{k+1}) \right), \end{aligned}$$

where in the last equality the indices in the γ's and u's have been altered appropriately from the ones in Equation 3.18 and Equation 3.19 to accommodate for the identity factors in the diagonal. Combining this with Equation 3.17 yields that

$$W_n(\vec{\xi}) = \operatorname{diag}(W_{k_2-k_1}(\vec{\xi}'), W_{n+1-k_2}(\vec{\xi}'')). \tag{3.20}$$

We may apply the inductive hypothesis to $n' = |\vec{\xi}''| > N$, vector $\vec{\xi}''$, and associated set $K' = \{k_2, \ldots, k_m\}$ of size $m-1$ to conclude that

$$\begin{aligned} &W_{n+1-k_2}(\vec{\xi}'') \\ =& \operatorname{diag}\left(W_{k_3-k_2}(\xi_{k_2}, \ldots, \xi_{k_3-1}), \ldots, W_{k_{m+1}-k_m}(\xi_{k_m}, \ldots, \xi_{k_{m+1}-1})\right). \end{aligned}$$

Substituting this into Equation 3.20 yields that

$$W_n(\vec{\xi}) = \operatorname{diag}\left(W_{k_2-k_1}(\xi_{k_1}, \ldots, \xi_{k_2-1}), \ldots, W_{k_{m+1}-k_m}(\xi_{k_m}, \ldots, \xi_{k_{m+1}-1})\right),$$

which proves Sublemma 3.13.1. //

With Sublemma 3.13.1 in hand, let us continue with the proof of Lemma 3.13. Apply Lemma 3.4 with the natural number N, the set $K = \{0\}$, and with the closed sets $F_{i,k} := X_i \setminus U_{i,k}$ for $1 \leq i \leq l$ and $1 \leq k \leq n_i$. This yields a function $\Theta \in A$ with the following properties:

(I) for all $1 \le i \le l$ and $x \in X_i$, $\Theta_i(x)$ is a diagonal matrix with entries in $[0,1]$ whose final N entries are all 0 and such that at most one of every N consecutive diagonal entries is non-zero;
(II) for all $1 \le i \le l$ and $1 \le k \le n_i$, if $x \notin U_{i,k}$, then $\Theta_i(x)_{k,k} = 0$;
(III) for all $1 \le i \le l$ and $1 \le k \le n_i$, if $x \in B_{i,k}$, then $\Theta_i(x)_{k,k} = 1$.

Since $N < \mathfrak{s}(A)$, we may, for each $1 \le i \le l$, define $W_{n_i} \in C([0,1]^{n_i}, M_{n_i})$ as in Equation 3.11. For $1 \le i \le l$, let $\vec{\xi^i}\colon X_{n_i} \to [0,1]^{n_i}$ denote the vector-valued function constituting the diagonal of Θ_i. That is, $\vec{\xi^i}(x) = (\xi^i(x)_1, \ldots, \xi^i(x)_{n_i})$ for all $x \in X_i$, where $\xi^i(x)_k = \Theta_i(x)_{k,k}$ for $1 \le k \le n_i$. Define the unitary $V \in \bigoplus_{i=1}^{l} C(X_i, M_{n_i})$ by $V := (W_{n_1} \circ \vec{\xi^1}, \ldots, W_{n_l} \circ \vec{\xi^l})$. To conclude the proof of Lemma 3.13, let us first argue that $V \in A$ and then show that $(fV)_i(x)$ is strictly lower triangular for all $1 \le i \le l$ and $x \in X_i$.

To see that $V \in A$, suppose that $2 \le i \le l$ and that $y \in Y_i$ decomposes into $x_1 \in X_{i_1} \setminus Y_{i_1}, \ldots, x_s \in X_{i_s} \setminus Y_{i_s}$. Then,

$$\begin{aligned}\vec{\xi^i}(y) &= (\xi^i(y)_1, \ldots, \xi^i(y)_{n_i}) \\ &= (\xi^{i_1}(x_1)_1, \ldots, \xi^{i_1}(x_1)_{n_{i_1}}, \ldots\ldots\ldots, \xi^{i_s}(x_s)_1, \ldots, \xi^{i_s}(x_s)_{n_{i_s}}).\end{aligned}$$

Define $B(y) := \{1 \le k \le n_i : y \in B_{i,k}\}$. By Lemma 2.8, $B(y) = \{1 = k_1 < \cdots < k_s\}$, where $k_t = 1 + n_{i_1} + \cdots + n_{i_{t-1}}$ for $1 \le t \le s$. Set $k_{s+1} := n_i + 1$ and note that $k_{t+1} - k_t = n_{i_t}$ for all $1 \le t \le s$. By assumption $n_i \ge \mathfrak{s}(A) > N$, and so, in light of (I) and (III) above, we may apply Sublemma 3.13.1 with the vector $\vec{\xi^i}(y)$ and the set $B(y)$, to obtain

$$\begin{aligned}V_i(y) &= W_{n_i}(\vec{\xi^i}(y)) \\ &= \operatorname{diag}\left(W_{k_2-k_1}(\vec{\xi^{i_1}}(x_1)), \ldots, W_{k_{s+1}-k_s}(\vec{\xi^{i_s}}(x_s))\right) \\ &= \operatorname{diag}(V_{i_1}(x_1), \ldots, V_{i_s}(x_s)).\end{aligned}$$

Therefore, $V \in A$.

To conclude the proof, fix $1 \le i \le l$ and $x \in X_i$. Let us show that $(fV)_i(x)$ is strictly lower triangular. From Equation 3.11, we have

$$(fV)_i(x) = f_i(x)W_{n_i}(\vec{\xi^i}(x)) = f_i(x)U[\gamma_{1,n_i}^{n_i}]^N \prod_{k=N}^{n_i-1} u_{k,n_i}^{n_i}(\xi^i(x)_{k+1}). \tag{3.21}$$

If we write $f_i(x) = \left[\, C_1 \,|\, \cdots \,|\, C_{n_i} \,\right]$, where C_j is the jth column of $f_i(x)$, then we have $f_i(x)U[\gamma_{1,n_i}^{n_i}]^N = \left[\, C_{N+1} \,|\, \cdots \,|\, C_{n_i} \,|\, C_1 \,|\, \cdots \,|\, C_N \,\right]$. By the assumption of the lemma, $\mathfrak{r}(f_i(x)) \le N$. Hence, all non-zero entries in the first $n_i - N$ columns of the matrix $f_i(x)U[\gamma_{1,n_i}^{n_i}]^N$ must lie strictly below the diagonal. But by Lemma 2.8, $x \in B_{i,1}$ and, hence, by the assumptions of the lemma, $f_i(x)$ has zero crosses at indices $1,\dots,N$. In particular, the columns $C_1,\dots,C_N$ consist entirely of zeros. Therefore, $f_i(x)U[\gamma_{1,n_i}^{n_i}]^N$ is strictly lower triangular. To complete the proof, it suffices to show that $(fV)_i(x) = f_i(x)U[\gamma_{1,n_i}^{n_i}]^N$. To do this, it is enough, by Equation 3.21, to check that for each $N \leq k \leq n_i - 1$,

$$f_i(x)U[\gamma_{1,n_i}^{n_i}]^N u_{k,n_i}^{n_i}(\xi^i(x)_{k+1}) = f_i(x)U[\gamma_{1,n_i}^{n_i}]^N. \tag{3.22}$$

To this end, fix $N \leq k \leq n_i - 1$. If $\xi^i(x)_{k+1} = 0$, then $u_{k,n_i}^{n_i}(\xi^i(x)_{k+1}) = 1_{n_i}$ and there is nothing to show, and so we may assume $\xi^i(x)_{k+1} > 0$. Then, by (II) above, necessarily $x \in U_{i,k+1}$. Hence, by the assumption of the lemma, $f_i(x)$ has zero crosses at indices $k+1,\dots,k+N$, from which it follows that the columns $C_{k+1},\dots,C_{k+N}$ consist entirely of zeros. As noted above, these correspond to the columns of $f_i(x)U[\gamma_{1,n_i}^{n_i}]^N$ at indices $k+1-N,\dots,k$. By definition, the columns of $f_i(x)U[\gamma_{1,n_i}^{n_i}]^N u_{k,n_i}^{n_i}(\xi^i(x)_{k+1})$ at indices $k-N+1,\dots,k$ and $n_i - N+1,\dots,n_i$ are linear combinations of the same set of columns of $f_i(x)U[\gamma_{1,n_i}^{n_i}]^N$ (i.e., of $C_{k+1},\dots,C_{k+N},C_1,\dots,C_N$). But every column in this latter set consists entirely of zeros. Hence, since multiplying by the unitary $u_{k,n_i}^{n_i}(\xi^i(x)_{k+1})$ on the right only alters columns at indices $k-N+1,\dots,k$ and $n_i-N+1,\dots,n_i$, Equation 3.22 holds. This completes the proof of Lemma 3.13. //

3.2 STABLE RANK ONE

Theorem 3.14 (cf. [Lut17b], Theorem 2.4.1)**.** *Suppose*

$$A_1 \xrightarrow{\psi_1} A_2 \xrightarrow{\psi_2} A_3 \xrightarrow{\psi_3} \cdots \longrightarrow A := \varinjlim A_i$$

is a simple inductive limit of DSH algebras with diagonal maps. Then A has stable rank one.

Proof. For $n \in \mathbb{N}$, let $\mu_n \colon A_n \to A$ denote the map in the construction of the inductive limit, which is unital (since the ψ's are) and, by Proposition 2.14 we may assume, injective. Furthermore, we may assume that the A_j's are infinite-dimensional.

Fix $\epsilon > 0$ and $a \in A$. Our goal is to find an invertible element $a' \in A$ with $\|a - a'\| \leq \epsilon$. To start, choose $j \in \mathbb{N}$ and $f \in A_j$ such that $\|a - \mu_j(f)\| \leq \epsilon/4$. If f is invertible in A_j, then $\mu_j(f)$ is invertible in A, in which case we are finished. Thus, we may assume that f is not invertible in A_j.

Since A_j is infinite-dimensional, we may apply Lemma 3.7 with f, $\epsilon/4$, and $N = \mathfrak{S}(A_j) + M + 1$, where M is the natural number depending on f and ϵ, coming from the statement of Lemma 3.7. This yields a function $f' \in A_j$ with $\|f - f'\| \le \epsilon/4$, a $j' > j$, and unitaries $V, V' \in A_{j'}$ with the following two properties (we adopt the same notation for the decomposition of $A_{j'}$ discussed in Section 2.1):

(1) for any $1 \le i \le l$ and $1 \le k \le n_i$, there is a (possibly empty) open subset $U_{i,k}$ of X_i containing $B_{i,k}$ such that $(V\psi_{j',j}(f')V')_i(x)$ has zero crosses at indices $k, k+M, k+2M, \ldots, k+(N-1)M$;

(2) for all $1 \le i \le l$ and $x \in X_i$, we have $\mathfrak{r}((V\psi_{j',j}(f')V')_i(x)) \le \mathfrak{S}(A_j) + M - 1$.

Let $f'' := V\psi_{j',j}(f')V' \in A_{j'}$.

Next, apply Lemma 3.9 with $A_{j'}$, f'', and $\epsilon/4$. This yields a function $g \in A_{j'}$ with $\|g - f''\| \le \epsilon/4$ and, for $1 \le i \le l$ and $1 \le k \le n_i$, open sets $\mathcal{O}_{i,k} \subset X_i$ containing $B_{i,k}$ on which g_i always has a block point at index k; moreover, for all $1 \le i \le l$ and $x \in X_i$, the matrix $g_i(x)$ has a zero cross at every index that $f''_i(x)$ does, and $\mathfrak{r}(g_i(x)) \le \mathfrak{r}(f''_i(x)) \le \mathfrak{S}(A_j) + M - 1$. Thus, intersecting the $\mathcal{O}_{i,k}$'s with the $U_{i,k}$'s, we may assume that $g_i(x)$ has zero crosses at indices $k, k+M, \ldots, k+(N-1)M$ whenever $x \in \mathcal{O}_{i,k}$.

Now note that Claim 3.7.1 in the proof of Lemma 3.7 guarantees that $\mathfrak{s}(A_{j'}) > NM$. As a result of this and the paragraph above, we may apply Lemma 3.10 on $A_{j'}$ with g and the $\mathcal{O}_{i,k}$'s to obtain a unitary $W \in A_{j'}$ with the following properties:

(I) for all $1 \le i \le l$ and $1 \le k \le n_i$, there are open sets $\mathcal{O}'_{i,k} \supset B_{i,k}$ in X_i such that $W_i(x)g_i(x)W_i(x)^*$ has zero crosses at indices $k, k+1, \ldots, k+N-1$ whenever $x \in \mathcal{O}'_{i,k}$;

(II) $\mathfrak{r}(W_i(x)g_i(x)W_i(x)^*) \le \mathfrak{r}(g_i(x)) + 2 \le \mathfrak{S}(A_j) + M + 1 = N$ for all $1 \le i \le l$ and $x \in X_i$.

Let $g' := WgW^* \in A_{j'}$.

Using these properties and the fact that $\mathfrak{s}(A_{j'}) > NM \ge N$, we may apply Lemma 3.13 on $A_{j'}$ with g' and the $\mathcal{O}'_{i,k}$'s to conclude that there is a unitary $W' \in A_{j'}$ such that for all $1 \le i \le l$ and $x \in X_i$, the matrix $(g'W')_i(x)$ is strictly lower triangular. Thus, $g'W'$ is a nilpotent element. As observed in [Rø91], every nilpotent element of a unital C*-algebra is arbitrarily close to an invertible element. Thus, there is an invertible element $h \in A_{j'}$ such that $\|g'W' - h\| \le \epsilon/4$.

Take $a' := \mu_{j'}(V^*W^*h(W')^*W(V')^*)$ and observe that a' is invertible in A. Then, since the μ_n's are injective,

$$\begin{aligned}
\|\mu_j(f') - a'\| &= \|\psi_{j',j}(f') - V^*W^*h(W')^*W(V')^*\| \\
&= \|V^*W^*[WV\psi_{j',j}(f')V'W^*W' - h](W')^*W(V')^*\| \\
&\leq \|V^*W^*\|\|Wf''W^*W' - h\|\|(W')^*W(V')^*\| \\
&= \|Wf''W^*W' - h\| \\
&\leq \|Wf''W^*W' - WgW^*W'\| + \|WgW^*W' - h\| \\
&\leq \|W\|\|f'' - g\|\|W^*W'\| + \|g'W' - h\| \\
&\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} \\
&= \frac{\epsilon}{2},
\end{aligned}$$

and

$$\begin{aligned}
\|a - \mu_j(f')\| &\leq \|a - \mu_j(f)\| + \|\mu_j(f) - \mu_j(f')\| \\
&\leq \frac{\epsilon}{4} + \|f - f'\| \\
&\leq \frac{\epsilon}{2}.
\end{aligned}$$

Therefore,

$$\|a - a'\| \leq \|a - \mu_j(f')\| + \|\mu_j(f') - a'\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as desired. //

3.3 APPLICATIONS TO MINIMAL DYNAMICAL CROSSED PRODUCTS

Let T be an infinite compact metric space and let $h\colon T \to T$ be a minimal homeomorphism. In this section, we present two applications of Theorem 3.14, both concerning the dynamical crossed product $A := C^*(\mathbb{Z}, T, h)$. We show that A has stable rank one (see Corollary 3.18), thereby affirming a conjecture of Archey, Niu, and Phillips (see Conjecture 7.2 of [AP15]). We also apply a result of Thiel from [Thi19] to conclude that classification for A is determined by strict comparison (see Corollary 3.19), which establishes the Toms-Winter Conjecture for minimal dynamical crossed products.

The Toms-Winter Conjecture dates back to 2008 and stipulates that for separable, unital, simple, non-elementary, nuclear C*-algebras three different notions of regularity are equivalent. Below is a precise statement of the conjecture.

Conjecture 3.15 (Toms-Winter; [ET08], [WZ10])**.** *Let A be a separable, unital, simple, non-elementary, nuclear* C*-*algebra. The following statements are equivalent.*

(1) A has finite nuclear dimension.

(2) A is $\mathcal{Z}$-stable; that is, $A \otimes \mathcal{Z} \cong A$.

(3) A has strict comparison of positive elements.

At the time, part of this conjecture had already been established by Rørdam, who, in Theorem 4.5 and Corollary 4.6 of [Rø04], proved that *(2)* implies *(3)*. Winter showed in Corollary 7.3 of [Win11] that *(1)* implies *(2)*. The work of various hands established that *(2)* implies *(1)* in special cases, but very recently, in Theorem A of [CET⁺19], this implication was shown to hold in full generality. Therefore, to establish the conjecture for a given C*-algebra, one needs only to check that strict comparison of positive elements yields $\mathcal{Z}$-stability.

Let $\sigma\colon C(T) \to C(T)$ denote the automorphism arising from h given by $\sigma(f) := f \circ h^{-1}$. Let u denote the unitary in the associated crossed product A implementing the σ action, i.e., $ufu^* = \sigma(f)$ for all $f \in C(T)$. Then A is the C*-algebra generated by $C(T)$ and u. Given a closed set $S \subset T$ with non-empty interior, let A_S denote the orbit-breaking Putnam sub-C*-algebra of A associated to S (see [Put89]); that is, A_S is the C*-algebra generated by $\{f, ug : f \in C(T), g \in C_0(T \setminus S)\}$, where we adopt the shorthand $C_0(T \setminus S) := \{g \in C(T) : g|_S \equiv 0\}$. In [Lin], [LP98], and [LP04], Q. Lin and Phillips showed that A_S is a recursive subhomogeneous algebra, and in fact a DSH algebra. We outline this below. For a more in-depth discussion, see Theorems 3.1-3.3. of [LP98].

Given $s \in S$, let $\lambda_S(s) := \min\{n > 0 : h^n(s) \in S\}$ (the first return time of s to S). Since T is compact, it follows that $\sup_{s\in S} \lambda_S(s)$ is finite (see also Lemma 2.2 of [LP04]). Thus there exist $1 \leq n_1^S < n_2^S < \cdots < n_{l(S)}^S$ such that $\{\lambda_S(s) : s \in S\} = \{n_i^S : 1 \leq i \leq l(S)\}$. For $1 \leq i \leq l(S)$, let $X_i^S := \overline{\lambda_S^{-1}(n_i^S)}$ and $Y_i^S := X_i^S \setminus \lambda_S^{-1}(n_i^S)$. Then, for given $1 \leq i \leq l(S)$ and $y \in Y_i^S$, there are indices $1 \leq i_1, \ldots, i_p < i$ and a point $x \in X_{i_1}^S \setminus Y_{i_1}^S$, such that $n_{i_1}^S + \cdots + n_{i_p}^S = n_i^S$ and such that $h^k(x) \in S$ if and only if $k = n_{i_1}^S + \cdots + n_{i_j}^S$ for some $1 \leq j \leq p$. Note, too, that $h^{n_{i_1}^S + \cdots + n_{i_{j-1}}^S}(x) \in X_{i_j}^S \setminus Y_{i_j}^S$ for all $2 \leq j \leq p$. Then, A_S is isomorphic to a sub-C*-algebra of $\bigoplus_{i=1}^{l(S)} C(X_i^S, M_{n_i^S})$, where an element $(f_1, \ldots, f_{l(S)})$ of $\bigoplus_{i=1}^{l(S)} C(X_i^S, M_{n_i^S})$ is in A_S if and only if, for given $1 \leq i \leq l(S)$ and $y \in Y_i^S$,

$$f_i(y) = \operatorname{diag}\left(f_{i_1}(x), f_{i_2}(h^{n_{i_1}^S}(x)), \ldots, f_{i_p}(h^{n_{i_1}^S + \cdots + n_{i_{p-1}}^S}(x))\right),$$

where $i_1, \ldots, i_p$ and x are as described above. It follows that A_S is a DSH algebra.

By [LP04], Proposition 2.4, there is a unique homomorphism $\gamma_S \colon A_S \to \bigoplus_{i=1}^{l(S)} C(X_i^S, M_{n_i^S})$ with the property that for $f \in C(T)$ and $g \in C_0(T \setminus S)$,

$$\gamma_S(f)_i = \operatorname{diag}\left(f \circ h|_{X_i^S}, f \circ h^2|_{X_i^S}, \ldots, f \circ h^{n_i^S}|_{X_i^S}\right) \tag{3.23}$$

and

$$\gamma_S(ug)_i = \begin{pmatrix} 0 & & & & \\ g \circ h|_{X_i^S} & 0 & & & \\ & g \circ h^2|_{X_i^S} & \ddots & & \\ & & \ddots & 0 & \\ & & & g \circ h^{n_i^S - 1}|_{X_i^S} & 0 \end{pmatrix} \tag{3.24}$$

for $1 \leq i \leq l(S)$.

Now, suppose $R \subset S \subset T$. By examining the generating sets, it follows that A_S is contained in A_R. Let $\psi \colon A_S \to A_R$ denote the inclusion map.

Lemma 3.16 (cf. [Lut17b], Lemma 2.5.1)**.** *ψ is a diagonal map (see Definition 2.3) between DSH algebras.*

Proof. Fix $1 \leq i \leq l(R)$ and $x \in X_i^R \setminus Y_i^R = \lambda_R^{-1}(n_i^R)$.

Claim 3.16.1. *There are indices $1 \leq i_1, \ldots, i_q \leq l(S)$ such that:*

(1) $n_{i_1}^S + \cdots + n_{i_q}^S = n_i^R$;

(2) for all $1 \leq k \leq n_i^R$, $h^k(x) \in S$ if and only if $k = \beta_j$ for some $1 \leq j \leq q$, where $\beta_j := n_{i_1}^S + \cdots + n_{i_j}^S$;

(3) for all $1 \leq j \leq q$, $h^{\beta_{j-1}}(x) \in X_{i_j}^S \setminus Y_{i_j}^S$ (here $\beta_0 := 0$, so that $x \in X_{i_1}^S \setminus Y_{i_1}^S$).

Proof. Since $R \subset S$ and the sets $X_j^S \setminus Y_j^S$ for $1 \leq j \leq l(S)$ partition S, there is a unique $1 \leq i_1 \leq l(S)$ such that $x \in X_{i_1}^S \setminus Y_{i_1}^S$. Moreover, $n_{i_1}^S = \lambda_S(x) \leq \lambda_R(x) = n_i^R$. If $n_{i_1}^S = n_i^R$, then there is nothing to show. Otherwise, there is an i_2 such that $h^{n_{i_1}^S}(x) \in X_{i_2}^S \setminus Y_{i_2}^S$. Note that $n_{i_2}^S = \lambda_S(h^{n_{i_1}^S}(x)) \leq n_i^R - n_{i_1}^S$. If $n_{i_2}^S = n_i^R - n_{i_1}^S$, the desired result follows. Otherwise, we let i_3 be such that $h^{n_{i_1}^S + n_{i_2}^S}(x) \in X_{i_3}^S \setminus Y_{i_3}^S$ and proceed as before. Eventually, this process terminates (when $n_{i_1}^S + \cdots + n_{i_q}^S = n_i^R$) and yields indices with the desired properties. This proves the claim. //

Let us now show that x decomposes into $h^{\beta_0}(x) = x, h^{\beta_1}(x), \ldots, h^{\beta_{q-1}}(x)$ under ψ.

Suppose that $f \in C(T)$. Let us show that

$$\begin{aligned} &\gamma_R(\psi(f))_i(x) \\ &= \operatorname{diag}\left(\gamma_S(f)_{i_1}(x), \gamma_S(f)_{i_2}(h^{\beta_1}(x)), \ldots, \gamma_S(f)_{i_q}(h^{\beta_{q-1}}(x))\right). \end{aligned} \tag{3.25}$$

Fix $1 \le j \le q$. By Equation 3.23,

$$\begin{aligned} \gamma_S(f)_{i_j}(h^{\beta_{j-1}}(x)) &= \operatorname{diag}\left(f(h(h^{\beta_{j-1}}(x))), \ldots, f(h^{n^S_{i_j}}(h^{\beta_{j-1}}(x)))\right) \\ &= \operatorname{diag}\left(f(h^{\beta_{j-1}+1}(x)), \ldots, f(h^{\beta_j}(x))\right). \end{aligned}$$

Hence,

$$\begin{aligned} &\operatorname{diag}\left(\gamma_S(f)_{i_1}(x), \gamma_S(f)_{i_2}(h^{\beta_1}(x)), \ldots, \gamma_S(f)_{i_q}(h^{\beta_{q-1}}(x))\right) \\ =&\operatorname{diag}\left(f(h^{\beta_0+1}(x)), \ldots, f(h^{\beta_1}(x)), \ldots\ldots, f(h^{\beta_{q-1}+1}(x)), \ldots, f(h^{\beta_q}(x))\right) \\ =&\operatorname{diag}\left(f(h(x)), \ldots, f(h^{n_i^R}(x))\right) \\ =&\gamma_R(f)_i(x), \end{aligned}$$

which yields Equation 3.25.

Next, suppose $g \in C_0(T \setminus S)$. Let us show that

$$\begin{aligned} &\gamma_R(\psi(ug))_i(x) \\ &= \operatorname{diag}\left(\gamma_S(ug)_{i_1}(x), \gamma_S(ug)_{i_2}(h^{\beta_1}(x)), \ldots, \gamma_S(ug)_{i_q}(h^{\beta_{q-1}}(x))\right). \end{aligned} \tag{3.26}$$

By Equation 3.24,

$$\gamma_R(ug)_i(x) = \begin{pmatrix} 0 & & & & \\ g(h(x)) & 0 & & & \\ & g(h^2(x)) & \ddots & & \\ & & \ddots & 0 & \\ & & & g(h^{n_i^R-1}(x)) & 0 \end{pmatrix}.$$

For $1 \le j \le q$, we have $h^{\beta_j}(x) \in S$ by Claim 3.16.1, so that $g(h^{\beta_j}(x)) = 0$. Hence, partitioning $\{1, 2, \ldots, n_i^R\}$ into the sets $\{\beta_{j-1}+1, \ldots, \beta_j\}$ for $1 \le$

$j \leq q$, we may view $\gamma_R(ug)_i(x)$ as a block-diagonal matrix $\mathrm{diag}(B_1, \ldots, B_q)$, where

$$\begin{aligned} B_j &:= \begin{pmatrix} 0 & & & & \\ g(h^{\beta_{j-1}+1}(x)) & 0 & & & \\ & g(h^{\beta_{j-1}+2}(x)) & \ddots & & \\ & & \ddots & 0 & \\ & & & g(h^{\beta_j-1}(x)) & 0 \end{pmatrix} \\ &= \gamma_S(ug)_{i_j}(h^{\beta_{j-1}}(x)), \end{aligned}$$

which yields Equation 3.26.

We have shown that x decomposes into $h^{\beta_0}(x) = x, h^{\beta_1}(x), \ldots, h^{\beta_{q-1}}(x)$ under ψ on the generators of A_S. Let us now use continuity to prove that this decomposition is maintained for all elements of A_S. Let $a \in A_S$ be arbitrary. By definition, we may write $a = \lim_{n\to\infty} w_n$, where for each $n \in \mathbb{N}$, w_n is a word in $C(T) \cup uC_0(T \setminus S) \cup C_0(T \setminus S)u^*$. By Equation 3.25 and Equation 3.26,

$$\begin{aligned} &\gamma_R(\psi(w_n))_i(x) \\ &= \mathrm{diag}\left(\gamma_S(w_n)_{i_1}(x), \gamma_S(w_n)_{i_2}(h^{\beta_1}(x)), \ldots, \gamma_S(w_n)_{i_q}(h^{\beta_{q-1}}(x))\right) \end{aligned}$$

for all $n \in \mathbb{N}$. Hence, by continuity of *-homomorphisms,

$$\begin{aligned} &\gamma_R(\psi(a))_i(x) \\ &= \lim_{n\to\infty} \gamma_R(\psi(w_n))_i(x) \\ &= \lim_{n\to\infty} \mathrm{diag}\left(\gamma_S(w_n)_{i_1}(x), \gamma_S(w_n)_{i_2}(h^{\beta_1}(x)), \ldots, \gamma_S(w_n)_{i_q}(h^{\beta_{q-1}}(x))\right) \\ &= \mathrm{diag}\left(\gamma_S(a)_{i_1}(x), \gamma_S(a)_{i_2}(h^{\beta_1}(x)), \ldots, \gamma_S(a)_{i_q}(h^{\beta_{q-1}}(x))\right), \end{aligned}$$

which yields the desired diagonal decomposition and completes the proof of Lemma 3.16. //

Theorem 3.17. *Let T be an infinite compact metric space and let $h\colon T \to T$ be a minimal homeomorphism. Given $x \in T$, the Putnam subalgebra $A_{\{x\}}$ of $A := \mathrm{C}^*(\mathbb{Z}, T, h)$ is a simple inductive limit of DSH algebras with diagonal maps. In particular, $A_{\{x\}}$ has stable rank one.*

Proof. Let $x \in T$ be arbitrary. Choose a sequence $S_1 \supset S_2 \supset \cdots$ of closed sets with non-empty interior such that $\bigcap_{n=1}^\infty S_n = \{x\}$. For each $n \in \mathbb{N}$, let $A_{S_n} \subset A$ denote the subalgebra as described above and let $\psi_n\colon A_{S_n} \to A_{S_{n+1}}$ denote the canonical inclusion. Since $\overline{\bigcup_{n=1}^\infty A_{S_n}} = A_{\{x\}}$, it follows by Lemma 3.16 that $A_{\{x\}}$ is an inductive limit of DSH algebras with diagonal maps. Moreover, by Theorem 1.2 of [LP98], $A_{\{x\}}$ is simple (see Proposition 2.5 of [LP10] for a proof). Therefore, by Theorem 3.14, $A_{\{x\}}$ has stable rank one. //

Corollary 3.18 (cf. [AP15], Conjecture 7.2; [Lut17b], Corollary 2.5.2)**.** *Let T be an infinite compact metric space and let $h\colon T \to T$ be a minimal homeomorphism. The dynamical crossed product $A := \mathrm{C}^*(\mathbb{Z}, T, h)$ has stable rank one.*

Proof. Let $x \in T$ be arbitrary. By Theorem 3.17, $A_{\{x\}}$ has stable rank one. Since h is minimal and T is infinite, $h^n(x) \neq x$ for all $n \in \mathbb{N}$. Thus, on combining Theorem 7.10 of [Phi14] with Theorem 4.6 of [AP15], it follows that $A_{\{x\}}$ is a centrally large subalgebra of A. But by Theorem 6.3 of [AP15], any infinite-dimensional unital simple separable C*-algebra containing a centrally large subalgebra with stable rank one must itself have stable rank one. //

Corollary 3.19. *Let T be an infinite compact metric space and let $h\colon T \to T$ be a minimal homeomorphism. The dynamical crossed product $A := \mathrm{C}^*(\mathbb{Z}, T, h)$ is $\mathcal{Z}$-stable if and only if it has strict comparison of positive elements.*

Proof. Let $x \in T$ be arbitrary. By Theorem 3.17, $A_{\{x\}}$ has stable rank one. Thus, by Theorem 9.6 of [Thi19], the Toms-Winter Conjecture (Conjecture 3.15) holds for $A_{\{x\}}$. In particular, $A_{\{x\}}$ is $\mathcal{Z}$-stable if and only if it has strict comparison of positive elements. But by Theorem 3.3 and Corollary 3.5 of [ABP17], A is $\mathcal{Z}$-stable if and only if $A_{\{x\}}$ is. Furthermore, by Theorem 6.14 of [Phi14], A has strict comparison if and only if $A_{\{x\}}$ does. Therefore, Corollary 3.19 follows. //

4

NON-UNITAL DSH ALGEBRAS (DSH_0 ALGEBRAS)

The purpose of this chapter is to investigate a possible non-unital analogue to Theorem 3.14. We develop non-unital versions of some of the lemmas from Chapter 3 and put these together to form the beginning of a potential stable rank one argument.

In Section 4.1, we introduce the notion of a DSH_0 algebra (see Definition 4.3), which serves as a non-unital DSH algebra counterpart, and we define the notion of a diagonal map between DSH_0 algebras. These algebras are constructed using the extended class of non-unital recursive subhomogeneous algebras introduced by Tikuisis in [Tik11].

In Section 4.2, we prove that a C^*-algebra is a DSH_0 algebra if and only if its unitization is a DSH algebra (see Corollary 4.10). We show, furthermore, that the class of DSH_0 algebras is preserved when taking ideals (closed and two-sided) and quotients (see Proposition 4.9 and Proposition 4.11). This allows us to apply a result of Tikuisis to obtain a characterisation of simplicity for an inductive limit of DSH_0 algebras (see Proposition 4.12) similar to the one in Proposition 3.1 used in the unital setting.

Lastly, in Section 4.3, we prove a non-unital analogue of Lemma 3.3 (see Lemma 4.16). We show how this can be combined with the simplicity criterion from the previous section to make partial progress towards a potential stable rank one result in the non-unital setting. We close the chapter by examining the difficulties in generalising Lemma 3.4 and by discussing a possible remedy, which might serve as a direction for future work.

4.1 INTRODUCTORY DEFINITIONS AND PROPERTIES

Let us start off by defining what a non-unital recursive subhomogeneous algebra is. This is essentially the same as the definition of the usual unital recursive subhomogeneous algebras, except that we do not insist that the pullback maps be unital or that the base spaces be compact (though, as observed in Remark 4.2, this latter condition does not change the class in question). Non-unital recursive subhomogeneous algebras were first introduced by Tikuisis in [Tik11], where he showed that every separable ASH algebra can be written as an inductive limit of non-unital recursive subhomogeneous algebras, whose base spaces all have finite topological dimension. Ng and Winter had previously shown this for unital ASH and RSH algebras in [NW06].

Definition 4.1 (RSH_0 algebras)**.** A *non-unital recursive subhomogeneous (RSH_0) algebra* is a C*-algebra given by the following recursive definition.

(1) If X is a locally compact Hausdorff space and $n \geq 1$, then $C_0(X, M_n)$ is an RSH_0 algebra.

(2) If A is an RSH_0 algebra, X is a locally compact Hausdorff space, $Y \subset X$ is closed, $\varphi\colon A \to C_0(Y, M_n)$ is a *-homomorphism, and $\rho\colon C_0(X, M_n) \to C_0(Y, M_n)$ is the restriction *-homomorphism, then the pullback

$$A \oplus_{C_0(Y,M_n)} C_0(X, M_n) := \{(a, f) \in A \oplus C_0(X, M_n) : \varphi(a) = \rho(f)\}$$

is an RSH_0 algebra.

Remark 4.2. As Tikuisis observed in [Tik12], we may replace local compactness with compactness in the definition above without changing the class of algebras under consideration since the pullback

$$\{(a, f) \in A \oplus C_0(X, M_n) : \varphi(a) = f|_Y\}$$

is isomorphic to the pullback

$$\{(a, f) \in A \oplus C(X^+, M_n) : \varphi'(a) = f|_{Y^+}\},$$

where X^+ and Y^+ are the respective one-point compactifications of X and Y and

$$\varphi'(a)(y) := \begin{cases} \varphi(a)(y) & \text{if } y \in Y \\ 0 & \text{if } y = \infty. \end{cases}$$

Thus, local compactness in (2) above is redundant. Moreover, we may always regard the full matrix algebra $C_0(X, M_n)$ as the pullback

$$\{0\} \oplus_{C(\{\infty\},M_n)} C(X^+, M_n),$$

where $\{0\} =: C(\varnothing)$ and the pullback *-homomorphism is just the zero map.

For the remainder of the chapter, we assume that the one-point compactifications are metrizable. Therefore, if A is an RSH_0 algebra, there are compact metric spaces $X_1 := \varnothing, X_2, \ldots, X_l$, closed subspaces $Y_1 := \varnothing, Y_2 \subset X_2, \ldots, Y_l \subset X_l$, positive integers $n_1 := 1, n_2, \ldots, n_l$, C*-algebras $A^{(i)} \subset \bigoplus_{j=1}^{i} C(X_j, M_{n_j})$ for $1 \leq i \leq l$, and *-homomorphisms $\varphi_i \colon A^{(i)} \to C(Y_{i+1}, M_{n_{i+1}})$ for $1 \leq i \leq l-1$, where $\varphi_1 := 0$ such that:

(1) $A^{(1)} = C(\varnothing) = \{0\}$;

(2) for all $1 \leq i \leq l-1$

$$A^{(i+1)} = \{(a, f) \in A^{(i)} \oplus C(X_{i+1}, M_{n_{i+1}}) : \varphi_i(a) = f|_{Y_{i+1}}\};$$

(3) $A = A^{(l)}$.

As in the unital case, let us use l to denote the *length* of the decomposition. We may view an element f of A coordinate-wide as $(f_1, f_2, \ldots, f_l)$, where in this case we always have $f_1 = 0$.

Let us now introduce the notion of a non-unital DSH algebra. This is done just as in the unital case except that we allow for zero blocks to occur in the block-diagonal decompositions. We carry forward all of the notation and conventions from the unital setting.

Definition 4.3 (DSH_0 algebras)**.** A C*-algebra A is a *non-unital diagonal (recursive) subhomogeneous (DSH_0) algebra* (of length l) provided that it is an RSH_0 algebra (of length l) with a decomposition as given above, and for all $1 \leq i \leq l-1$ and $y \in Y_{i+1}$, there are points $x_1 \in X_{i_1} \setminus Y_{i_1}, \ldots, x_t \in X_{i_t} \setminus Y_{i_t}$ (where this list may be empty) and non-negative integers $m_0, \ldots, m_t$, such that for all $f \in A^{(i)}$,

$$\varphi_i(f)(y) = \mathrm{diag}(0_{m_0}, f_{i_1}(x_1), 0_{m_1}, \ldots, 0_{m_{t-1}}, f_{i_t}(x_t), 0_{m_t}). \qquad (4.1)$$

We say y *decomposes into* $x_1, \ldots, x_t$ and that each x_j is a point *in the decomposition of* y. Given $1 \leq j \leq i$ and $y' \in Y_j$, we say that y' is *in the decomposition of* y if there exists a $1 \leq k \leq n_i$ with the property that for all $f \in A^{(i)}$ there are matrices $P \in M_{k-1}$ and $Q \in M_{n_i - n_j - (k-1)}$ such that $f_i(y) = \mathrm{diag}(P, f_j(y'), Q)$.

Remark 4.4. As in the unital case, we have that the map $x \mapsto [\mathrm{ev}_x]$ defines a continuous bijection

$$\bigsqcup_{i=2}^{l} (X_i \setminus Y_i) \to \hat{A},$$

(recall $X_1 := \varnothing$) whose restriction to each $X_i \setminus Y_i$ is a homeomorphism onto its image. In particular every irreducible representation of A is unitarily equivalent to ev_x for some $x \in \bigsqcup_{i=2}^{l}(X_i \setminus Y_i)$. Moreover, as in Lemma 2.9, we may assume that Y_i has empty interior for $2 \leq i \leq l$.

When defining diagonal maps between DSH$_0$ algebras, we need to allow for zero blocks just as in Definition 4.3.

Definition 4.5 (Diagonal maps between DSH$_0$ algebras)**.** Given two DSH$_0$ algebras A_1 and A_2 of lengths l_1 and l_2 and base spaces $X_1^1, \ldots, X_{l_1}^1$ and $X_1^2, \ldots, X_{l_2}^2$, respectively, we say that a *-homomorphism $\psi\colon A_1 \to A_2$ is *diagonal* provided that for all $1 \leq i \leq l_2$ and $x \in X_i^2$, there are points $x_1 \in X_{i_1}^1, \ldots, x_t \in X_{i_t}^1$ (where this list may be empty) and non-negative integers $m_0, \ldots, m_t$ such that for all $f \in A_1$,

$$\psi(f)_i(x) = \mathrm{diag}(0_{m_0}, f_{i_1}(x_1), 0_{m_1}, \ldots, 0_{m_{t-1}}, f_{i_t}(x_t), 0_{m_t}).$$

We say x *decomposes into* $x_1, \ldots, x_t$ and that each x_j is a point *in the decomposition of* x.

4.2 UNITIZATIONS, IDEALS, AND QUOTIENTS OF DSH$_0$ ALGEBRAS

In this section it is shown that a C*-algebra is a DSH$_0$ algebra if and only if its unitization is a DSH algebra (see Corollary 4.10). Moreover, it is proved that closed two-sided ideals and quotients of DSH$_0$ algebras are again DSH$_0$ algebras (see Proposition 4.9 and Proposition 4.11). These results are needed to apply Tikuisis's simplicity criterion for non-unital ASH algebras when obtaining Proposition 4.12 at the end of the section. All ideals in this section are assumed to be closed and two-sided.

Suppose A is a DSH$_0$ algebra of length l. If we fix a one point space $\{\infty\}$, then A is contained in the unital C*-algebra $C(\{\infty\}) \oplus \bigoplus_{i=2}^{l} C(X_i, M_{n_i})$, whose unit is not contained in A. It follows that $\widetilde{A}$ is isomorphic to $A + \mathbb{C}(1, 1_{n_2}, \ldots, 1_{n_l})$. From this we can see that $\widetilde{A}$ is a recursive subhomogeneous algebra of length l with the following properties:

(1) $\widetilde{A}^{(1)} = C(\{\infty\})$ ($\cong \mathbb{C}$);

(2) for all $1 \leq i \leq l$, $\widetilde{A}^{(i)} \subset C(\{\infty\}) \oplus \bigoplus_{j=2}^{i} C(X_j, M_{n_j})$;

(3) for all $1 \leq i \leq l-1$, $\widetilde{A}^{(i+1)} = \widetilde{A}^{(i)} \oplus_{C(Y_i, M_{n_i})} C(X_i, M_{n_i})$, where the pullback *-homomorphism is $\widetilde{\varphi}_i$, the unique unital *-homomorphism extending φ_i.

In particular, from (3) we see that for $1 \leq i \leq l-1$, $f = (0, f_2, \ldots, f_i) \in A^{(i)}$, $y \in Y_{i+1}$, and $\lambda \in \mathbb{C}$,

$$\begin{aligned}
&\widetilde{\varphi}_i(f + \lambda(1, 1_{n_2}, \ldots, 1_{n_i}))(y) \\
=&\varphi_i(f)(y) + \lambda 1_{n_{i+1}} \\
=&\operatorname{diag}(\lambda 1_{m_0}, f_{i_1}(x_1) + \lambda 1_{n_{i_1}}, \lambda 1_{m_1}, \ldots, \lambda 1_{m_{t-1}}, f_{i_t}(x_t) + \lambda 1_{n_{i_t}}, 1\lambda_{m_t}).
\end{aligned} \tag{4.2}$$

Hence, $\widetilde{\varphi}_i$ is diagonal and y decomposes into a list consisting of $x_1, \ldots, x_t$ and $m_0 + \cdots + m_t$ copies of ∞. Therefore, the unitization of a DSH_0 algebra of length l is a DSH algebra of length l.

An analogous computation to that done in Equation 4.2 shows that if A_1 and A_2 are DSH_0 algebras and $\psi\colon A_1 \to A_2$ is a diagonal map, then the unique unital extension $\widetilde{\psi}\colon \widetilde{A}_1 \to \widetilde{A}_2$ is diagonal in the sense of Definition 2.3. Hence, we obtain the following proposition.

Proposition 4.6. *If*

$$A_1 \xrightarrow{\psi_1} A_2 \xrightarrow{\psi_2} A_3 \xrightarrow{\psi_3} \cdots \longrightarrow A := \varinjlim A_i$$

is an inductive system of DSH_0 algebras with diagonal maps, then

$$\widetilde{A}_1 \xrightarrow{\widetilde{\psi}_1} \widetilde{A}_2 \xrightarrow{\widetilde{\psi}_2} \widetilde{A}_3 \xrightarrow{\widetilde{\psi}_3} \cdots \longrightarrow \widetilde{A} = \varinjlim \widetilde{A}_i$$

is an inductive system of DSH algebras with (unital) diagonal maps.

In [Tik11], Proposition 3.2.1, Tikuisis showed that ideals of RSH_0 algebras are again RSH_0 algebras. Let us now verify that ideals of DSH_0 algebras are DSH_0 algebras.

Suppose A is a DSH_0 algebra of length l. For any $1 \leq i \leq l-1$, we have canonical projections $\delta_i^{i+1}\colon A^{(i+1)} \to A^{(i)}$ and $\sigma_{i+1}\colon A^{(i+1)} \to C(X_{i+1}, M_{n_{i+1}})$ yielding the following commutative diagram:

$$\begin{array}{ccc}
A^{(i+1)} & \xrightarrow{\sigma_{i+1}} & C(X_{i+1}, M_{n_{i+1}}) \\
\big\downarrow \delta_i^{i+1} & & \big\downarrow \rho_{i+1} \\
A^{(i)} & \xrightarrow{\varphi_i} & C(Y_{i+1}, M_{n_{i+1}})
\end{array} \tag{4.3}$$

where ρ_{i+1} denotes the restriction mapping. Note that δ_i^{i+1} is surjective. For $1 \leq j \leq i$, put $\delta_j^i := \delta_j^{j+1} \circ \cdots \circ \delta_{i-1}^i$, where $\delta_j^j := \mathrm{id}$.

Suppose that $J \subset A$ is an ideal of A. Then, $\delta_i^l(J)$ is an ideal in $A^{(i)}$ for each $1 \leq i \leq l$ since δ_i^l is surjective. For $1 \leq i \leq l-1$, $\sigma_{i+1}(\delta_{i+1}^l(J))$ is a sub-C*-algebra of $C(X_{i+1}, M_{n_{i+1}})$. Hence, there is a unique open set $U_{i+1} \subset$

X_{i+1} such that the ideal generated by $\sigma_{i+1}(\delta^l_{i+1}(J))$ is $\{f \in C(X_{i+1}, M_{n_{i+1}}) : f|_{U^c_{i+1}} \equiv 0\}$, which is isomorphic to $C(X_{i+1} \setminus U_{i+1}, M_{n_{i+1}})$.

The proofs of the following two results are contained in the proof of Proposition 3.2.1 of [Tik11].

Lemma 4.7 (see Proposition 3.2.1 of [Tik11])**.** *If $1 \le i \le l-1$ and $a \in \delta^l_i(J)$, then $\varphi_i(a)|_{Y_{i+1} \setminus U_{i+1}} \equiv 0$.*

Lemma 4.8 (see Proposition 3.2.1 of [Tik11])**.** *Suppose $1 \le i \le l-1$ and $(r,s) \in A^{(i+1)}$ with $r \in A^{(i)}$ and $s \in C(X_{i+1}, M_{n_{i+1}})$. If $s|_{X_{i+1}\setminus U_{i+1}} \equiv 0$ and $r \in \delta^l_i(J)$, then $(r,s) \in \delta^l_{i+1}(J)$.*

For $1 \le i \le l$, let $J^{(i)} := \delta^l_i(J)$ and, for $1 \le i \le l-1$, define $\nu_i \colon J^{(i)} \to C(Y_{i+1} \cup (X_{i+1} \setminus U_{i+1}), M_{n_{i+1}})$ by

$$\nu_i(g)(y) := \begin{cases} \varphi_i(g)(y) & \text{if } y \in Y_{i+1} \\ 0 & \text{if } y \in X_{i+1} \setminus U_{i+1}. \end{cases}$$

By Lemma 4.7, ν_i is well defined and Lemma 4.8 shows that the pullback $J^{(i)} \oplus_{C(Y_{i+1} \cup (X_{i+1}\setminus U_{i+1}), M_{n_{i+1}})} C(X_{i+1}, M_{n_{i+1}})$ associated to ν_i is equal to $J^{(i+1)}$. Moreover, given $1 \le i \le l-1$ we see that ν_i is a diagonal map. Indeed, if $y \in Y_{i+1}$ decomposes into $x_1 \in X_{i_1} \setminus Y_{i_1}, \ldots, x_t \in X_{i_t} \setminus Y_{i_t}$ under φ_i, then, under ν_i, y decomposes into the sublist consisting of those x_j belonging to U_{i_j}. It follows that J is a DSH_0 algebra of length l.

Now, suppose that for $k = 1, 2$ we have a DSH_0 algebra A_k with base spaces $X^k_1, \ldots, X^k_{l_k}$ and corresponding closed subspaces $Y^k_1, \ldots, Y^k_{l_k}$, and an ideal J_k of A_k; for $2 \le i \le l_k$, let $U^k_i \subset X^k_i$ denote the open set determined by J_k, as defined above. Suppose we are given a diagonal map $\psi \colon A_1 \to A_2$ with the property that $\psi(J_1) \subset J_2$. Given $1 \le i \le l_2$ and $x \in X^2_i$, if x decomposes into $x_1 \in X^1_{i_1} \setminus Y^1_{i_1}, \ldots, x_t \in X^1_{i_t} \setminus Y^1_{i_t}$ under ψ, then, under $\psi|_{J_1}$, x decomposes into the sublist consisting of those x_j belonging to $U^1_{i_j}$. Hence, we obtain the following result.

Proposition 4.9. *Every closed two-sided ideal of a DSH_0 algebra is itself a DSH_0 algebra. If A_1 and A_2 are DSH_0 algebras with ideals J_1 and J_2, respectively, and if $\psi \colon A_1 \to A_2$ is a diagonal map with the property that $\psi(J_1) \subset J_2$, then $\psi|_{J_1} \colon J_1 \to J_2$ is a diagonal map.*

Corollary 4.10. *Let A be a C^*-algebra. Then, A is a DSH_0 algebra if and only if $\widetilde{A}$ is a DSH algebra.*

Proof. The forward direction of the implication is shown in Proposition 4.6. For the converse, note that any DSH algebra is seen to be a (unital) DSH_0 algebra by simply appending the direct summand $\{0\}$ to the beginning of its decomposition. Hence, if $\widetilde{A}$ is a DSH algebra, then A is a DSH$_0$ algebra by Proposition 4.9, being an ideal of $\widetilde{A}$. //

It is straightforward to check that the results of Section 2.2 carry over to the non-unital section without any changes. Thus, we obtain the following analogue.

Proposition 4.11. *Every quotient of a DSH$_0$ algebra is itself a DSH$_0$ algebra. Moreover, given an inductive limit*

$$A_1 \xrightarrow{\psi_1} A_2 \xrightarrow{\psi_2} A_3 \xrightarrow{\psi_3} \cdots \longrightarrow A := \varinjlim A_i$$

of DSH$_0$ algebras with diagonal maps, there exist DSH$_0$ algebras $D_1, D_2, \ldots$ and injective diagonal maps $\psi_i' \colon D_i \to D_{i+1}$ such that

$$D_1 \xrightarrow{\psi_1'} D_2 \xrightarrow{\psi_2'} D_3 \xrightarrow{\psi_3'} \cdots \longrightarrow A.$$

With Proposition 4.9 and Proposition 4.11 in hand, let us now apply two of Tikuisis's results to obtain a simplicity criterion analogous to Proposition 3.1 in the non-unital case.

Suppose

$$A_1 \xrightarrow{\psi_1} A_2 \xrightarrow{\psi_2} A_3 \xrightarrow{\psi_3} \cdots \longrightarrow A := \varinjlim A_i$$

is a simple limit of DSH$_0$ algebras with diagonal maps. By Proposition 4.11, we lose nothing by assuming the ψ_i's are all injective. As in the proof of Proposition 3.6 of [Tik12], for all $i \in \mathbb{N}$, there are non-zero ideals J_i of A_i with compact spectrum such that for all $j > i$, the ideal generated by $\psi_{j,i}(J_i)$ is J_j. By Proposition 4.9, each J_i is a DSH$_0$ and $\psi_i|_{J_i}$ is a diagonal map for all $i \in \mathbb{N}$. Moreover, since A is simple,

$$J_1 \xrightarrow{\psi_1|_{J_1}} J_2 \xrightarrow{\psi_2|_{J_2}} J_3 \xrightarrow{\psi_3|_{J_3}} \cdots \longrightarrow A.$$

Thus, we may apply Proposition 3.7 of [Tik12] to obtain the following result.

Proposition 4.12 (cf. [Tik12], Propositions 3.6 & 3.7)**.** *Suppose A is a simple inductive limit of DSH$_0$ algebras with diagonal maps. Then, there are DSH$_0$ algebras $A_1, A_2, \ldots$ and, for $i \in \mathbb{N}$, diagonal maps $\psi_i \colon A_i \to A_{i+1}$ with the following properties:*

(1) $\varinjlim(A_i, \psi_i) = A$*;*

(2) for $i \in \mathbb{N}$, ψ_i is injective and A_i has compact spectrum;

(3) *for* $j > i$, $\psi_{j,i}(A_i)$ *generates* A_j *as a closed two-sided ideal;*

(4) *for* $0 \neq f \in A_i$, *there is a* $j > i$ *such that* $\pi(\psi_{j,i}(f)) \neq 0$ *for every non-zero irreducible representation* π *of* A_j.

Remark 4.13. As in the unital case, *(3)* above is equivalent to saying that given any $i \in \mathbb{N}$ and non-empty open subset $U \subset \hat{A}_i$, there is a $j > i$ such that every $x \in \hat{A}_j$, contains a point from U in its decomposition under the diagonal map $\psi_{j,i}$.

4.3 TOWARDS A STABLE RANK ONE RESULT

In this section, we discuss a possible non-unital version of Theorem 3.14. Let us start by proving a couple of lemmas, including an analogue to Lemma 3.3. Combining these with Proposition 4.12, we show how this makes partial progress towards a potential stable rank one result for a simple inductive limit of DSH_0 algebras with diagonal maps. We end the section by pointing out the difficulties in generalising the remaining argument used in the unital case.

Lemma 4.14. *Suppose* A *is a subhomogeneous* C^**-algebra with compact spectrum. Suppose* $(\pi_\alpha)_\alpha$ *is a net in* $\hat{A}$. *Then, there exists an* $a \in A$ *such that* $\|\pi_\alpha(a)\|$ *does not converge to zero.*

Proof. Suppose the conclusion is false and let $a \in A$ be arbitrary. Since $\hat{A}$ is compact, there is a subnet $(\pi_{\alpha_\beta})_\beta$ of $(\pi_\alpha)_\alpha$ converging to a point $\pi \in \hat{A}$. Let $\epsilon > 0$ be arbitrary. By 3.3.2 of [Dix77], $Z := \{\tau \in \hat{A} : \|\tau(a)\| \leq \epsilon\}$ is a closed subset of $\hat{A}$. Hence, $\pi \in Z$ since $\lim_\beta \|\pi_{\alpha_\beta}(a)\| = 0$ by assumption. Therefore, $\|\pi(a)\| \leq \epsilon$. Since ϵ was arbitrary, this implies that $\pi(a) = 0$. Since a was arbitrary, this means that $\pi = 0$, which is not possible. //

Corollary 4.15. *Suppose* A *is a* DSH_0 *algebra of length* l *with compact spectrum. If* $2 \leq i \leq l$ *and* $y \in Y_i$, *then* $f_i(y) \neq 0$ *for some* $f \in A$*; in particular, the list of points* y *decomposes into is non-empty.*

Proof. Fix $2 \leq i \leq l$ and $y \in Y_i$. Since we may assume Y_i has empty interior, there is a sequence $(x_n)_n$ of points in $X_i \setminus Y_i$ that converges to y. The corollary then follows by Lemma 4.14. //

Lemma 4.16. *Let* A *be a non-unital* DSH_0 *algebra of length* l. *Let* $\epsilon > 0$ *and* $f \in A$. *Then, there exist* $f' \in A$ *with* $\|f - f'\| \leq \epsilon$ *and unitaries* $w, v \in \widetilde{A}$ *such that for some* $1 \leq i \leq l$, $(wf'v)_i$ *has a zero cross at index* 1 *everywhere on some non-empty set* $U \subset \hat{A} \cap (X_i \setminus Y_i)$, *which is open with respect to the hull-kernel topology on* $\hat{A}$. *Moreover, there is a* $\Delta \in A$ *such that for every* $1 \leq j \leq l$ *and*

$x \in X_j$, $\Delta_j(x)$ is a diagonal matrix with entries in $[0,1]$, $\Delta_j(x)_{k,k} > 0$ implies $(wf'v)_j(x)$ has a zero cross at index k, and $\Delta_i(z)_{1,1} = 1$ for all $z \in U$.

Proof. Since A is not unital, it cannot be a DSH algebra. By Definition 4.3, then, there must be a $2 \leq i \leq l$ and a point in Y_i, which has a zero block in its decomposition. Let i' denote the largest such integer and choose $y \in Y_{i'}$ having a zero block in its decomposition. Then, y is not in the decomposition of any point in Y_j for $i' < j \leq l$.

Claim 4.16.1. *There is a set $U_1 \subset X_{i'}$, containing y and open in $X_{i'}$, such that no point in U_1 is in the decomposition of any point in Y_j for any $i' < j \leq l$.*

Proof. Suppose on the contrary that there is a sequence of points $(z_n)_n$ in $X_{i'}$ converging to y, each of which is in the decomposition of some point in some Y_j. By passing to a subsequence, we may assume that there exist $i' < j \leq l$ and $1 \leq k \leq n_j$ such that for each $n \in \mathbb{N}$, z_n occurs in the decomposition of a point $p_n \in Y_j$ at index k down the diagonal. Let $g \in A$ be arbitrary. Then, for all $n \in \mathbb{N}$, there are matrices $P_n \in M_{k-1}$ and $Q_n \in M_{n_j - n_{i'} - (k-1)}$ such that

$$g_j(p_n) = \operatorname{diag}(P_n, g_{i'}(z_n), Q_n).$$

Since Y_j is compact, we may, passing to a further subsequence, choose a point $p \in Y_j$ such that $p_n \to p$. Thus, there are matrices $P \in M_{k-1}$ and $Q \in M_{n_j - n_{i'} - (k-1)}$ such that

$$g_j(p) = \lim_n g_j(p_n) = \lim_n \operatorname{diag}(P_n, g_{i'}(z_n), Q_n) = \operatorname{diag}(P, g_{i'}(y), Q).$$

Since g was arbitrary, this implies that y is in the decomposition of p, which is a contradiction. //

By shrinking U_1, we may assume that $\|f_{i'}(y) - f_{i'}(z)\| \leq \epsilon$ for all $z \in U_1$. Moreover, since $Y_{i'}$ may be assumed to have empty interior, there must be a point $x' \in U_1 \cap (X_{i'} \setminus Y_{i'})$. Choose a set U_2, which is open in $X_{i'}$ and satisfies $x' \in U_2 \subset \overline{U_2}^{X_{i'}} \subset U_1 \cap (X_{i'} \setminus Y_{i'})$. As in the proof of Lemma 3.3, we may define a function $f' \in A$ such that $\|f - f'\| \leq \epsilon$, $f'_j = f_j$ for $j \neq i'$, $f'_{i'}|_{\overline{U_2}^{X_{i'}}} \equiv f_{i'}(y)$, and $f'_{i'}|_{X_{i'} \setminus (U_1 \cap (X_{i'} \setminus Y_{i'}))} = f_{i'}|_{X_{i'} \setminus (U_1 \cap (X_{i'} \setminus Y_{i'}))}$. Since y has a zero block in its decomposition, there are (permutation) unitary matrices $W, V \in M_{n_{i'}}$ such that $W f_{i'}(y) V$ has a zero cross at index 1. Define unitaries $w, v \in \widetilde{A}$ coordinate-wise with $w_j = v_j \equiv 1_{n_j}$ for $j \neq i'$ and $w_{i'}, v_{i'} \in C(X_{i'}, M_{n_{i'}})$ satisfying $w_i|_{\overline{U_2}^{X_{i'}}} \equiv W$, $v_i|_{\overline{U_2}^{X_{i'}}} \equiv V$, and $w_{i'}|_{X_{i'} \setminus (U_1 \cap (X_{i'} \setminus Y_{i'}))} = v_{i'}|_{X_{i'} \setminus (U_1 \cap (X_{i'} \setminus Y_{i'}))} \equiv 1_{n_{i'}}$. Lastly, choose a set U, which is open in $X_{i'}$ and satisfies $x' \in U \subset \overline{U}^{X_{i'}} \subset U_2$ and define Δ as in the proof of Lemma 3.3. Given $z \in U$, there exists a $g \in A$ that vanishes outside of U, but not at z. Therefore, U is open in $\hat{A}$. //

For the remainder of the section, suppose

$$A_1 \xrightarrow{\psi_1} A_2 \xrightarrow{\psi_2} A_3 \xrightarrow{\psi_3} \cdots \longrightarrow A := \varinjlim A_i \tag{4.4}$$

is a simple inductive limit of non-unital DSH_0 algebras with diagonal maps. We denote the base spaces of A_j by $X_1^j, \ldots, X_{l(j)}^j$ and the corresponding closed subspaces by $Y_1^j, \ldots, Y_{l(j)}^j$.

A non-unital result analogous to Theorem 3.14 would be that A has stable rank one. Since A is not unital, this amounts (by definition) to showing that $\widetilde{A}$ has stable rank one. A natural first step in this direction is to determine whether every element of A can be approximated by an invertible in $\widetilde{A}$. This is the problem we consider in the remainder of this section.

We may assume that the system in Equation 4.4 satisfies all of the properties listed in Proposition 4.12. Starting with an element $a \in A$, we approximate it by an element f at some finite stage A_j. The first step in the proof of Theorem 3.14 is to invoke Lemma 3.7, which appeals to Lemma 3.3. Let us proceed the same way here.

Applying Lemma 4.16, we obtain an approximation $f' \in A_j$ of f, unitaries $w, v \in \widetilde{A_j}$, and an open set $U \subset \hat{A}_j$ such that $wf'v$ has a zero cross at index 1 at all points in U. Moreover, the lemma yields an element $\Delta \in A_j$, which is a diagonal matrix at every point with entries in $[0, 1]$ having a 1 in the first index at all points in U, and with the property that $(wf'v)_i(x)$ has a zero cross at index k whenever $\Delta_i(x)_{k,k} > 0$.

By Remark 4.13, there is a $j'' > j$ such that every $x \in \hat{A}_{j''}$ contains a point from U in its decomposition under the diagonal map $\psi_{j'',j}$. By Corollary 4.15, this holds, in fact, for all $2 \leq i \leq l(j'')$ and $x \in X_i^{j''}$ (where the same notation is being used as outline above Lemma 3.7).

Continuing as in the proof of Lemma 3.7 in the unital setting, we may find a $j' > j''$ such that every point in each base space of $A_{j'}$ contains an arbitrarily large number of points in its decomposition under the diagonal map $\psi_{j',j''}$. Thus, we arrive at the following conclusion.

Conclusion. *Given $N \in \mathbb{N}$, there is a $j' > j$ such that for each $2 \leq i \leq l(j')$ and $x \in X_i^{j'}$, the matrix $\psi_{j',j}(wf'v)_i(x)$ has N zero crosses and, moreover, that $\psi_{j',j}(wf'v)_i(x)$ has a zero cross at index k whenever $\psi_{j',j}(\Delta)_i(x)_{k,k} > 0$.*

The next step in the stable rank one argument is to put the zero crosses into uniform locations. In the unital setting, this is done with the unitaries defined in Equation 3.2, which depend on the indicator-function-like elements constructed using Lemma 3.4. However, as we now outline, there does not seem to be a natural non-unital analogue to these functions.

The main obstruction is that in the non-unital setting, there may be zero blocks arising in the block decomposition of the pullback maps in the DSH_0 algebra definition (see Equation 4.1). In particular, we do not necessarily have a non-zero representation beginning at the first index down the diagonal at all points in each base space of a given DSH_0 algebra, as we do for a unital DSH algebra (see Lemma 2.8). To generalize Lemma 3.4 to the non-unital DSH_0 algebra $A_{j'}$ above, we would need to construct a function $\Theta \in A_{j'}$ such that for all $1 \leq i \leq l(j')$ and $x \in X_i^{j'} \setminus Y_i^{j'}$, $\Theta_i(x)_{1,1} = 1$. Since the $Y_i^{j'}$'s have non-empty interior, this implies by continuity that the first diagonal coordinate of Θ is 1 at all points (including the $Y_i^{j'}$'s!). But then every point, in each $Y_i^{j'}$, must have a non-zero representation beginning at the first index down the diagonal, which need not be true. One might attempt to remedy this by defining Θ in the unitization $\widetilde{A_j}$. However, $\widetilde{A_{j'}}$ has an irreducible representation of dimension 1, while the functions in Lemma 3.4 can only be constructed in a DSH algebra in which the size of the smallest irreducible representation is suitably large. Otherwise, a problem similar to the one outlined above arises.

Let us illustrate the difficulty with constructing Θ in a non-unital DSH_0 algebra in the simplest possible setting. Let B be the point-line algebra

$$B := \{g \in C([0,1], M_4) : g(1) = \mathrm{diag}(0, c, 0) \text{ for some } c \in M_2\}.$$

It is straightforward to verify that B is a DSH_0 algebra of length 2. In this example, a desired function Θ must satisfy:

(1) $\Theta(x)_{1,1} = 1$ and $\Theta(x)_{2,2} = 0$ for all $x \in [0, 1)$;

(2) $\Theta(1)_{1,1} = 0$ and $\Theta(1)_{2,2} = 1$.

Due to the continuity restrictions, such a function clearly could not belong to B. This leads naturally to the following questions.

Question 4.17. Is there a way to augment the unitaries constructed in Lemma 3.4 in the non-unital setting, perhaps by permuting the matrix units in a suitable way or by working in the unitization?

Permuting matrix units is possible in the above example, but it is not immediately clear how to do this in more general DSH_0 algebras. Moreover, the arguments used throughout Chapter 3 assume fixed matrix units. The relation of matrix units to the ASH stable rank problem will be discussed in detail in Chapter 5.

Question 4.18. Assuming we are given a limit as in Equation 4.4, may we conclude that $\widetilde{A}$ has stable rank one? May we at least say that elements of A can be approximated by invertibles in $\widetilde{A}$?

5 THE HOMOGENEOUS CASE REVISITED

In this chapter, we further analyse the structure of simple AH systems with diagonal maps. In Section 5.1, we inspect our stable rank one proof from Chapter 3 in the special context where each algebra in the inductive limit is homogeneous, comparing it with the original proof of Elliott, Ho, and Toms in [EHT08].

In Section 5.2, we start off by recalling the definition of a system of matrix units for a finite-dimensional C^*-algebra. We use the notion of a matrix unit compatible map between two such algebras to obtain a formulation of a compatible map between subhomogeneous C^*-algebras. These maps aim to reformulate the global notion of a diagonal map, which doesn't exist for a general subhomogeneous algebra, in a local way. We show that in the case where each of the subhomogeneous C^*-algebras is a finite direct sum of full matrix algebras, any compatible map is diagonal (up to a piecewise-constant permutation unitary) in the global sense (see Theorem 5.11).

Finally, in Section 5.3, we investigate the intrinsic matrix unit compatibility of a full matrix algebra, which plays a crucial role in the proof of stable rank one. We introduce, for a unital homogeneous C^*-algebra, a local notion of internal matrix unit compatibility and show that any homogeneous algebra satisfying it is, in fact, a full matrix algebra (see Proposition 5.15). We end by discussing analogous notions for a subhomogeneous algebra and by suggesting possible further avenues of investigation.

5.1 STABLE RANK ONE IN THE DIAGONAL AH SETTING

In this section we examine our proof of stable rank one from Chapter 3 in the very special case where each of the DSH algebras in the inductive limit of Theorem 3.14 are homogeneous. As shown in Corollary 2.22, such algebras are necessarily full matrix algebras. The following proposition reinterprets the main result of Chapter 3 in this particular case.

Proposition 5.1 (cf. [EHT08], Theorem 4.1). *For each $i \in \mathbb{N}$, suppose X_i is a compact Hausdorff space and $n_i \in \mathbb{N}$. If*

$$C(X_1, M_{n_1}) \xrightarrow{\psi_1} C(X_2, M_{n_2}) \xrightarrow{\psi_2} \cdots \longrightarrow A := \varinjlim C(X_i, M_{n_i})$$

is a simple inductive limit with diagonal maps, then A has stable rank one.

Proof. As in the proof of Theorem 3.14, we lose nothing by assuming the X_i's are infinite-dimensional and that the ψ_i's are injective.

Suppose $a \in A$ and $\epsilon > 0$. There exist a $j \in \mathbb{N}$ and an $f \in A_j$ that approximates a to within $\epsilon/3$. We may assume without loss of generality that f is not invertible. Hence, there is an $x_0 \in X_j$ such that $f(x_0)$ is a non-invertible matrix. Choose unitary matrices W and V in M_{n_j} with the property that $Wf(x_0)V$ has a zero cross at index 1. Choose open sets $x_0 \in U_2 \subset \overline{U_2} \subset U_1 \subset X_j$ such that f differs from $f(x_0)$ by no more that $\epsilon/3$ on U_1. Then, we may define a function $f' \in C(X_j, M_{n_j})$ with $\|f - f'\| \leq \epsilon/3$ such that $f'|_{\overline{U_2}} \equiv f(x_0)$ and $f'|_{X_j \setminus U_1} = f|_{X_j \setminus U_1}$. We may also choose unitaries $w, v \in C(X_j, M_{n_j})$ such that $w|_{\overline{U_2}} \equiv W$, $v|_{\overline{U_2}} \equiv V$, and $w|_{X_j \setminus U_1} = v|_{X_j \setminus U_1} = 1_{n_j}$. Choose an open set $x_0 \in U \subset \overline{U} \subset U_2$ and define $\Delta := \operatorname{diag}(h, 0, \ldots, 0) \in C(X_j, M_{n_j})$, where $h|_{\overline{U}} \equiv 1$ and $h|_{X_j \setminus U_2} \equiv 0$. Let $g := wf'v$. Then, for all $x \in X_j$, the matrix $g(x)$ has a zero cross at index 1 whenever $\Delta(x)_{1,1} > 0$ and, moreover, $\Delta(x)_{1,1} = 1$ provided $x \in U$.

Since A is simple, Proposition 3.1 yields a $j'' > j$ such that for all $x \in X_{j''}$, at least one of the points x decomposes into under the diagonal map $\psi_{j'',j}$ lies in U. It follows that for all $x \in X_{j''}$, the matrices $\psi_{j'',j}(g)(x)$ and $\psi_{j'',j}(\Delta)(x)$ have a zero cross and a 1, respectively, at some index down the diagonal, and, moreover, $\psi_{j'',j}(g)(x)$ has a zero cross at index k whenever $\psi_{j'',j}(\Delta)(x)_{k,k} > 0$. For $1 \leq k_1 < k_2 \leq n_{j''}$, let $u_{(k_1\ k_2)} \colon [0,1] \to M_{n_{j''}}$ denote a continuous path of unitaries as defined in Definition 3.6. Let $u \in C(X_{j''}, M_{n_{j''}})$ be the unitary

$$u(x) := u_{(1\ 2)}(\psi_{j'',j}(\Delta)(x)_{2,2}) \cdots u_{(1\ n_{j''})}(\psi_{j'',j}(\Delta)(x)_{n_{j''},n_{j''}}). \tag{5.1}$$

Then, just as in part (c) of Sublemma 3.7.2, for each $x \in X_{j''}$, the matrix $(u\psi_{j'',j}(g)u^*)(x)$ has zero cross at index 1.

Since the X_i's are infinite-dimensional, we may use simplicity again (just as in Claim 3.7.1), to obtain a $j' > j''$ such that $m := n_{j'}/n_{j''} \geq n_{j''} - 1$. Letting $g' := \psi_{j',j''}(u\psi_{j'',j}(g)u^*)$, it follows that for all $x \in X_{j'}$, the matrix $g'(x)$ has a zero cross at indices $z_1, \ldots, z_m$, where $z_k := 1 + (k-1)n_{j''}$ for $1 \leq k \leq m$. This is illustrated in Figure 5.1.

Let $\sigma := (1\ 2\ \cdots\ z_m)(1\ 2\ \cdots\ z_{m-1}) \cdots (1\ 2\ \cdots\ z_1) \in S_{n_{j'}}$. We see that $\sigma(z_k) = m - k + 1$ for all $1 \leq k \leq m$. On letting $U[\sigma] \in M_{n_{j'}}$ denote the unitary matrix as defined in Definition 3.6, it follows that for all

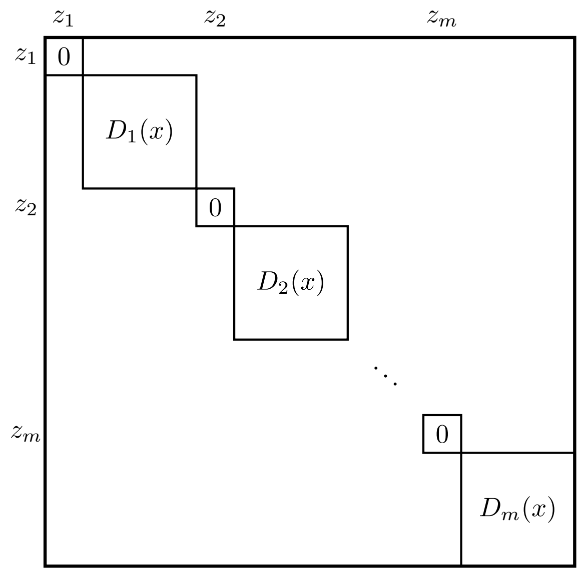

Figure 5.1: At each point $x \in X_{j'}$, the matrix $g'(x)$ has the block structure above, where blank entries are zero. For $1 \leq k \leq m$, $D_k(x)$ is a block of size $n_{j''} - 1$ by $n_{j''} - 1$.

$x \in X_{j'}$, $U[\sigma]g'(x)U[\sigma]^*$ has zero crosses at indices $1, 2, \ldots, m$ with the block structure shown in Figure 5.2.

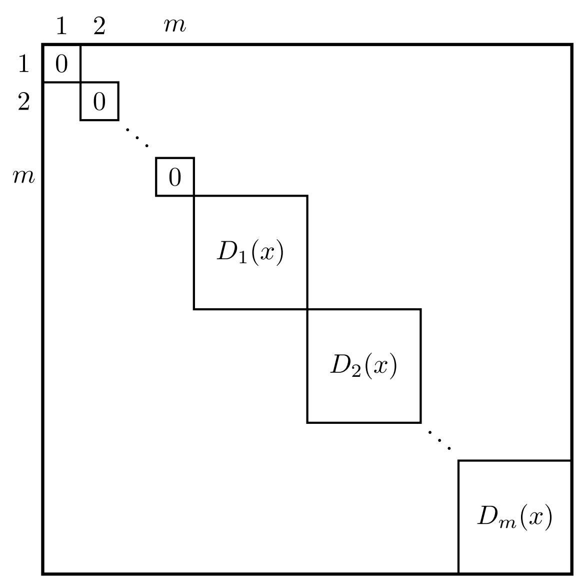

Figure 5.2: The block structure of $U[\sigma]g'(x)U[\sigma]^*$ for a given $x \in X_{j'}$ (blank entries are zero).

Therefore, multiplying $U[\sigma]g'U[\sigma]^*$ on the right by $U[(1\ 2\ \cdots\ n_{j'})]^m$, we obtain the following block structure shown in Figure 5.3 at each point $x \in X_{j'}$.

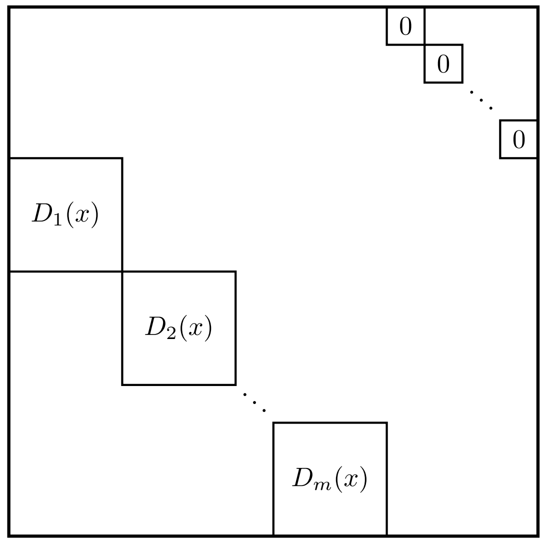

Figure 5.3: The block structure of $U[\sigma]g'(x)U[\sigma]^*U[(1\ 2\ \cdots\ n_{j'})]^m$ for a given $x \in X_{j'}$ (blank entries are zero).

Since $m \geq n_{j''} - 1$, we see that any matrix of the form in Figure 5.3 is strictly lower triangular and, hence, nilpotent. Therefore, as noted in the proof of Theorem 3.14, $U[\sigma]g'U[\sigma]^*U[(1\ 2\ \cdots\ n_{j'})]^m$ is with $\epsilon/3$ of an invertible element. A calculation much like the one done there then shows that our original given element a is within ϵ of an invertible element of A. //

The chief reason that the proof in the AH setting is much simpler than its ASH counterpart is that the blocks in Figure 5.1 are all the same size at each point in $X_{j'}$. This effectively removes the need for the indicator-function-like elements constructed in Lemma 3.4, which, in the ASH case, keep track of the different sizes and locations of the blocks. Consequently, the unitaries defined in Lemma 3.10 and Lemma 3.13 reduce to the constant permutation unitaries $U[\sigma]$ and $U[(1\ 2\ \cdots\ n_{j'})]^m$, respectively, in the proof above. Moreover, this AH proof does not require the full matrix algebra building blocks to be separable. In the ASH setting, however, separability is necessary since the indicator-function-like elements constructed in Lemma 3.4 rely on the assumption that the base spaces of any given DSH algebra are metrizable.

The only non-trivial unitary used in the proof above is u, defined in Equation 5.1. This is a simplified version of the unitary defined in

Equation 3.2 that is used in proof of Lemma 3.7 in the ASH setting. Initially, the element $\psi_{j'',j}(g)$ above has a zero cross at some index at each point in $X_{j''}$. But this index may vary with the point in $X_{j''}$. The role of u is to conjugate the aforementioned element so that $u\psi_{j'',j}(g)u^*$ has a zero cross at the same index, namely 1, across all points in $X_{j''}$. As shown in Figure 5.1, this also yields fixed locations for the zero crosses at each point in $X_{j'}$. In the ASH setting, this would not follow, as the blocks in Figure 5.1 may be of different sizes at each point in $X_{j'}$. Thus, in the proof of Lemma 3.7, the unitaries only act on the given element of $A_{j'}$, where enough zero crosses have been furnished.

Finally, it should be noted that the proof above shares many similarities with the original AH proof found in [EHT08]. In particular, the unitary u constructed above is the one constructed in Theorem 3.1 of that paper, while $U[\sigma]$ and $U[(1\ 2\ \cdots\ n_{j'})]^m$ are those used in Proposition 3.2 and Lemma 4.2, respectively. Thus, in the AH setting, our argument reduces to the one of [EHT08].

5.2 MATRIX UNITS AND MATRIX UNIT COMPATIBLE MAPS

Definition 5.2 (System of Matrix Units)**.** Let $r \in \mathbb{N}$, $n_1, \ldots, n_r \in \mathbb{N}$, and $A = M_{n_1} \oplus \cdots \oplus M_{n_r}$. We say $E := \{e^k_{ij} : 1 \le k \le r,\ 1 \le i,j \le n_k\}$ is a *system of matrix units for* A provided:

(1) $e^k_{ij}e^k_{jl} = e^k_{il}$ if $j = l$;

(2) $e^k_{ij}e^l_{mn} = 0$ if $k \ne l$ or $j \ne m$;

(3) $(e^k_{ij})^* = e^k_{ji}$;

(4) $A = \operatorname{span} E$.

Recall (Proposition 7.1.5 of [RLL00], for instance) that every non-zero finite-dimensional C*-algebra is isomorphic to $M_{n_1} \oplus \cdots \oplus M_{n_r}$ for some $r \in \mathbb{N}$ and $n_1, \ldots, n_r \in \mathbb{N}$.

Definition 5.3 (Matrix Unit Compatible Map)**.** Let A and B be finite-dimensional C*-algebras. Let E and F be sets of systems of matrix units for A and B, respectively. Let $\varphi\colon A \to B$ be a *-homomorphism. We say φ is *matrix unit compatible* provided it sends matrix units in E to finite sums of matrix units in F; that is, for all $e \in E$, there is a set $\{\beta^e_f : f \in F\}$ of non-negative integers such that $\varphi(e) = \sum_{f\in F} \beta^e_f f$.

Proposition 5.4. *Suppose A and B are finite-dimensional C*-algebras with systems of matrix units E and F, respectively. Assume that $\varphi\colon A \to B$ is a matrix* *unit compatible map and adopt the notation from Definition 5.3. Then, for all $e \in E$ and $f \in F$:*

(1) β^e_f is either 0 or 1;

(2) if e is a projection, then $\beta^e_f = 0$ unless f is a projection.

Proof. By projecting onto each individual matrix component in B, we may assume that $B \cong M_n$ for some $n \in \mathbb{N}$. Let $F = \{f_{ij} : 1 \leq i,j \leq n\}$ be any numbering of the elements of F so that they satisfy the properties in Definition 5.2. If $e \in E$, then

$$1 = \|e\| \geq \|\varphi(e)\| = \left\| \sum_{i,j=1}^{n} \beta^e_{f_{ij}} f_{ij} \right\| \geq \max_{1 \leq i,j \leq n} |\beta^e_{f_{ij}}|,$$

which proves *(1)*. To prove *(2)*, suppose e is a projection. Then,

$$\sum_{i,j=1}^{n} \beta^e_{f_{ij}} f_{ij} = \varphi(e) = \left(\sum_{i,k=1}^{n} \beta^e_{f_{ik}} f_{ik} \right) \left(\sum_{l,j=1}^{n} \beta^e_{f_{lj}} f_{lj} \right) = \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} \beta^e_{f_{ik}} \beta^e_{f_{kj}} \right) f_{ij}$$

and

$$\sum_{i,j=1}^{n} \beta^e_{f_{ij}} f_{ij} = \varphi(e) = \left(\sum_{i,j=1}^{n} \beta^e_{f_{ij}} f_{ij} \right)^* = \sum_{i,j=1}^{n} \beta^e_{f_{ij}} f_{ji} = \sum_{i,j=1}^{n} \beta^e_{f_{ji}} f_{ij},$$

from which it follows that

$$\beta^e_{f_{ij}} = \beta^e_{f_{ji}} = \sum_{k=1}^{n} \beta^e_{f_{ik}} \beta^e_{f_{kj}} \tag{5.2}$$

for all $1 \leq i,j \leq n$. If $i \neq j$ and $\beta^e_{f_{ij}} = 1$, then Equation 5.2 implies that

$$\beta^e_{f_{ii}} = \beta^e_{f_{ii}} \beta^e_{f_{ii}} + \beta^e_{f_{ij}} \beta^e_{f_{ji}} + \sum_{\substack{1 \leq k \leq n \\ k \neq i,j}} \beta^e_{f_{ik}} \beta^e_{f_{ki}} \geq (\beta^e_{f_{ii}})^2 + 1,$$

which is impossible. Therefore, $\beta^e_{f_{ij}} = 0$ whenever e is a projection and $i \neq j$, which proves *(2)*. //

Suppose A and B are subhomogeneous C*-algebras and that $\varphi\colon A \to B$ is a *-homomorphism. Fix $\pi \in \hat{B}$. Then, the representation $\pi \circ \varphi\colon A \to M_{\dim \tau}$ is unitarily equivalent to a direct sum of irreducible representations of A, provided it is non-zero. Let $\{\tau_1, \ldots, \tau_r\} \subset \hat{A}$ denote the unique set of distinct non-zero irreducible classes in this decomposition. For $1 \leq k \leq r$, put $I_k := \ker \tau_k$ and set $I := \bigcap_{k=1}^{r} I_k$. Since $A/I_k \cong \tau_k(A) = M_{\dim \tau_k}$, which is simple, it follows that $\{I_k\}$ is closed in $\mathrm{Prim}(A)$ for all $1 \leq k \leq r$. Thus, Prim (A/I) is homeomorphic to $\text{hull}(I) = \{I_1, \ldots, I_r\}$ and, hence, discrete. It follows (see Theorem 8.1 of [Kap49], for example) that A/I is isomorphic to the direct sum of its primitive quotients. Therefore,

$$A/I \cong \bigoplus_{k=1}^{r} (A/I)/(I_k/I) \cong \bigoplus_{k=1}^{r} A/I_k \cong \bigoplus_{k=1}^{r} \tau_k(A) = \bigoplus_{k=1}^{r} M_{\dim \tau_k}.$$

Let $\rho\colon A \to \bigoplus_{k=1}^{r} M_{\dim \tau_k}$ denote the induced canonical surjection. This induces a canonical injective homomorphism $\Lambda\colon \bigoplus_{k=1}^{r} M_{\dim \tau_k} \to M_{\dim \pi}$ given by $\Lambda(c) := \pi(\varphi(a))$, where a is any lift of c under ρ, yielding the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\quad \varphi \quad} & B \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \pi} \\ \bigoplus_{k=1}^{r} M_{\dim \tau_k} & \xrightarrow{\ \Lambda\ } & M_{\dim \pi} \end{array} \tag{5.3}$$

Moreover, if for each $1 \le k \le r$, $M_{\dim \tau_k}$ comes equipped with a system of matrix units $\{e_{ij}^k : 1 \le i,j \le \dim \tau_k\}$, then there is a natural choice of matrix units for $\bigoplus_{k=1}^{r} M_{\dim \tau_k}$. Indeed, on letting $\iota_k\colon M_{\dim \tau_k} \to \bigoplus_{l=1}^{r} M_{\dim \tau_l}$ denote the canonical inclusion for $1 \le k \le r$, it follows $\{\iota_k(e_{ij}^k) : 1 \le k \le r, 1 \le i,j \le \dim \tau_k\}$ is a system of matrix units for $\bigoplus_{k=1}^{r} M_{\dim \tau_k}$. When speaking about a system of matrix units for $\bigoplus_{k=1}^{r} M_{\dim \tau_k}$, this is the one we have in mind.

Definition 5.5 (Compatible map)**.** Let A and B be subhomogeneous C*-algebras and $\varphi\colon A \to B$ be a *-homomorphism. Suppose that for each $\tau \in \hat{A}$, we are given a system of matrix units for $\tau(A) \cong M_{\dim \tau}$, and similarly for each $\pi \in \hat{B}$. We say that φ is *compatible* provided that for each $\pi \in \hat{B}$, the induced map Λ in Equation 5.3 is matrix unit compatible with respect to these systems.

Definition 5.6 (see [EHT08])**.** Suppose we are given compact Hausdorff spaces X and Y, $m, n \in \mathbb{N}$, and systems of matrix units $E = \{e_{ij} : 1 \le i,j \le m\}$ and $F = \{f_{ij} : 1 \le i,j \le mn\}$ for M_m and M_{mn}, respectively. A unital *-homomorphism $\varphi\colon M_m(C(X)) \to M_{mn}(C(Y))$ is said to be *diagonal* provided there exist continuous functions $\lambda_1, \ldots, \lambda_n\colon Y \to X$, such that for all $g \in M_m(C(X))$,

$$\varphi(g) = \text{diag}(g \circ \lambda_1, \ldots, g \circ \lambda_n)$$

with respect to E and F. That is, for all $g = \sum_{i,j=1}^{m} g_{ij}e_{ij} \in M_m(C(X))$ and $y \in Y$,

$$\varphi(g)(y) = \sum_{l=1}^{n} \sum_{i,j=1}^{m} g_{ij}(\lambda_l(y))f_{(l-1)m+i\ (l-1)m+j}.$$

Remark 5.7.

- It is not too hard to show using Gelfand's Theorem for commutative C*-algebras that the continuity of the maps $\lambda_1, \dots, \lambda_n$ in the definition above is automatic. In particular, our definition of diagonal maps between DSH algebras (Definition 2.3) is identical to the one above in the case where the DSH algebras are full matrix algebras, or, equivalently by Corollary 2.22, homogeneous.
- The definition above extends naturally to *-homomorphisms going from $\bigoplus_{i=1}^{n} M_{n_i}(C(X_i))$ to $\bigoplus_{j=1}^{m} M_{m_j}(C(Y_j))$ obtained by requiring that each partial map $M_{n_i}(C(X_i)) \to M_{m_j}(C(X_j))$ be diagonal. See Definition 2.1 of [EHT08] for complete details.

When given a direct sum of full matrix algebras $\bigoplus_{i=1}^{n} M_{n_i}(C(X_i))$ and a system of matrix units for $\bigoplus_{i=1}^{n} M_{n_i}$, it is automatically assumed that each primitive quotient of $\bigoplus_{i=1}^{n} M_{n_i}(C(X_i))$ comes equipped with this given system.

Lemma 5.8. *Let X and Y be compact Hausdorff spaces and $n \in \mathbb{N}$. Suppose $E = \{1\}$ and $F = \{f_{ij} : 1 \leq i, j \leq n\}$ are systems of matrix units for $\mathbb{C}$ and M_n, respectively. If $\varphi\colon C(X) \to M_n(C(Y))$ is a unital *-homomorphism, which is compatible with respect to these systems, then φ is diagonal with respect to these systems.*

Proof. Given $y \in Y$, let $X^y \subset X$ denote the finite set of distinct points that the representation $\mathrm{ev}_y \circ \varphi$ decomposes into. Let $\rho_y\colon C(X) \to C(X^y) \cong \mathbb{C}^{|X^y|}$ denote the restriction map and let $\Lambda_y\colon \mathbb{C}^{|X^y|} \to M_n$ be the induced injective map arising as in Equation 5.3. By assumption, Λ_y is a matrix unit compatible map.

Now, fix $y \in Y$ and $g \in C(X)$. Since $\rho_y(g)$ is a linear combination of projections, it follows by *(2)* of Proposition 5.4 that there are numbers $\beta_j^{g,y} \in \mathbb{C}$ for $1 \leq j \leq n$ such that

$$\varphi(g)(y) = \Lambda_y(\rho_y(g)) = \sum_{j=1}^{n} \beta_j^{g,y} f_{jj}.$$

In particular, $\varphi(g)(y)$ is a diagonal matrix. Now, fix $1 \leq j \leq n$ and consider the map $\varphi_j\colon C(X) \to f_{jj}M_n(C(Y))f_{jj} \cong C(Y)$ taking g into $f_{jj}\varphi(g)f_{jj}$.

By Gelfand's Theorem for commutative C*-algebras, there is a unique continuous function $\lambda_j \colon Y \to X$ such that $\varphi_j(g) = (g \circ \lambda_j) f_{jj}$ for all $g \in C(X)$. Therefore,

$$\varphi(g) = \sum_{j=1}^{n} \varphi_j(g) = \sum_{j=1}^{n} (g \circ \lambda_j) f_{jj} = \mathrm{diag}(g \circ \lambda_1, \ldots, g \circ \lambda_n)$$

for all $g \in C(X)$. //

Lemma 5.9. *Let $X_1, \ldots, X_r, Y$ be compact Hausdorff spaces and $N_1, \ldots, N_r, N \in \mathbb{N}$. Suppose $E = \{e^k_{ij} : 1 \leq k \leq r,\ 1 \leq i,j \leq N_k\}$ and $F = \{f_{ij} : 1 \leq i,j \leq N\}$ are systems of matrix units for $\bigoplus_{k=1}^r M_{N_k}$ and M_N, respectively. Suppose $\varphi \colon \bigoplus_{k=1}^r M_{N_k}(C(X_k)) \to M_N(C(Y))$ is a unital *-homomorphism such that $\varphi(e^k_{ij})$ is a constant function for all $1 \leq k \leq r$ and $1 \leq i,j \leq N_k$. Then, if φ is compatible with respect to E and F, it is also diagonal with respect to E and F after conjugation by a constant permutation unitary.*

Proof. Given $1 \leq s \leq r$, we obtain the following commutative diagram, where the unlabelled arrows denote the canonical inclusion and restriction.

$$\begin{array}{ccc} M_{N_s}(C(X_s)) & \xrightarrow{\varphi_s} & \varphi(1_{N_s}) M_N(C(Y)) \varphi(1_{N_s}) \\ \big\downarrow & & \big\uparrow \\ \bigoplus_{k=1}^r M_{N_k}(C(X_k)) & \xrightarrow{\varphi} & M_N(C(Y)) \end{array}$$

It is straightforward to check that the resulting composition map φ_s is compatible with respect to $\{e^s_{ij} : 1 \leq i,j \leq N_s\}$ and $\{\varphi(1_{N_s}) f_{ij} \varphi(1_{N_s}) : 1 \leq i,j \leq N\}$. Therefore, we may assume that $r = 1$ and we may write $X := X_1$, $N_1 = m$, $N = mn$, and $E = \{e_{ij} : 1 \leq i,j \leq m\}$.

With this reduction established, let us introduce some notation, which is needed in the proof. Let $\pi \colon C(X) \otimes M_m \to M_m(C(X))$ be the isomorphism $\pi(g \otimes \sum_{i,j=1}^m \xi_{ij} e_{ij}) := \sum_{i,j=1}^m g \xi_{ij} e_{ij}$ and let $\psi \colon C(X) \otimes M_m \to M_{mn}(C(y))$ be $\psi := \varphi \circ \pi$. For $1 \leq i,j \leq m$, let $p_{ij} := \psi(1 \otimes e_{ij}) = \varphi(e_{ij})$ and set $B := p_{11} M_{mn}(C(Y)) p_{11}$. Define $\psi' \colon C(X) \to B$ by $\psi'(g) := \psi(g \otimes e_{11}) = \varphi(g e_{11})$. Then, it is straightforward to check that the map $\Theta \colon B \otimes M_m \to M_{mn}(C(Y))$ given by $\Theta\left(g \otimes \sum_{i,j=1}^m \xi_{ij} e_{ij}\right) := \sum_{i,j=1}^m \xi_{ij} p_{i1} g p_{1j}$ is a *-isomorphism (see 1.3.4 of [EG96] for details).

Fix $y \in Y$ and let $X^y \subset X$ denote the finite set of distinct points that the representation $\mathrm{ev}_y \circ \varphi$ decomposes into. Let $\rho_y \colon M_m(C(X)) \to M_m(C(X^y)) \cong M_m^{|X^y|}$ denote the restriction map and let $\Lambda_y \colon M_m^{|X^y|} \to M_{mn}$ be the induced injective map arising as in Equation 5.3. By assumption, Λ_y is a matrix unit compatible map and we obtain the following commutative diagram:

$$
\begin{array}{ccc}
C(X) \otimes M_m & \xrightarrow{\psi' \otimes \mathrm{id}_m} & B \otimes M_m \\
\downarrow \pi & \searrow^{\psi} & \downarrow \Theta \\
M_m(C(X)) & \xrightarrow{\varphi} & M_{mn}(C(Y)) \\
\downarrow \rho_y & & \downarrow \mathrm{ev}_y \\
M_m(C(X^y)) \cong M_m^{|X^y|} & \xrightarrow{\Lambda_y} & M_{mn}
\end{array}
\tag{5.4}
$$

Now, given $1 \leq k, l \leq n$, the projections $\varphi(e_{kk})(y)$ and $\varphi(e_{ll})(y)$ are Murray-von Neumann equivalent and, thus, have the same rank. Since φ is unital, this rank must be $mn/m = n$. Moreover, $\varphi(e_{ll})(y) = \Lambda_y(\rho_y(e_{ll}))$, and so, by Proposition 5.4, $\varphi(e_{ll})$ is the sum of n distinct diagonal matrix units. As $\varphi(e_{ll})$ is assumed to be constant, this implies that there is a partition $\{J(l) : 1 \leq l \leq m\}$ of $\{1, 2, \ldots, mn\}$ into sets of size n such that $\varphi(e_{ll}) = \sum_{j \in J(l)} f_{jj}$. Similarly, for fixed $1 \leq i, j \leq m$, $\varphi(e_{ij})$ is the sum of distinct matrix units in F. Let J_{ij} denote the unique subset of $J \times J$ such that $\varphi(e_{ij}) = \sum_{(k,l) \in J_{ij}} f_{kl}$.

Claim 5.9.1.

(1) $|J_{ij}| = n$;

(2) $J^1_{ij} := \{1 \leq k \leq mn : \exists 1 \leq l \leq mn \text{ such that } (k,l) \in J_{ij}\} = J(i)$;

(3) $J^2_{ij} := \{1 \leq l \leq mn : \exists 1 \leq k \leq mn \text{ such that } (k,l) \in J_{ij}\} = J(j)$.

Proof. Observe:

$$\sum_{k \in J(j)} f_{kk} = \varphi(e_{jj}) = \varphi(e_{ij})^* \varphi(e_{ij}) = \sum_{(k,l),(s,t) \in J_{ij}} f^*_{kl} f_{st} = \sum_{(k,l),(s,t) \in J_{ij}} \delta_{ks} f_{lt}, \tag{5.5}$$

where $\delta_{ks} = 1$ if $k = s$ and 0 otherwise. Therefore, given $(k,l), (s,t) \in J_{ij}$, if $k = s$, then $l = t$. Similarly, we obtain

$$\sum_{k \in J(i)} f_{kk} = \varphi(e_{ij}) \varphi(e_{ij})^* = \sum_{(k,l),(s,t) \in J_{ij}} \delta_{lt} f_{sk}, \tag{5.6}$$

from which we deduce that for given $(k,l), (s,t) \in J_{ij}$, if $l = t$, then $s = k$. Hence, for all $(k,l), (s,t) \in J_{ij}$, we have $k = s$ if and only if $l = t$. Thus, $|J_{ij}| = |J^1_{ij}| = |J^2_{ij}|$ and, reinterpreting Equation 5.5, we obtain

$$\sum_{k \in J(j)} f_{kk} = \sum_{(k,l) \in J_{ij}} f_{ll} = \sum_{l \in J^2_{ij}} f_{ll}.$$

It follows that *(3)* and, hence, *(1)* hold. Similarly reinterpreting Equation 5.6 shows that *(2)* holds, which proves the claim. //

By the claim, there is, for each $1 \leq s \leq m$, a bijection $\sigma_s \colon J(1) \to J(s)$, such that $\sigma_s(r) = t$ if and only if $(r,t) \in J_{1s}$ (hence, if and only if $(t,r) \in J_{s1}$).

Now, if $g = \sum_{i,j=1}^{mn} g_{ij} f_{ij} \in M_{mn}(C(Y))$, then

$$p_{11} g p_{11} = \varphi(e_{11}) g \varphi(e_{11}) = \sum_{i,j \in J(1)} g_{ij} f_{ij},$$

from which we deduce that B is canonically isomorphic to $M_n(C(Y))$ and that $G := \{f_{ij} : i,j \in J(1)\}$ is a system of matrix units for M_n. We compute further:

$$\begin{aligned}
&\Theta\left(\sum_{i,j\in J(1)} g_{ij} f_{ij} \otimes \sum_{k,l=1}^{m} \xi_{kl} e_{kl}\right) \\
&= \sum_{k,l=1}^{m} \xi_{kl} p_{k1} \left(\sum_{i,j\in J(1)} g_{ij} f_{ij}\right) p_{1l} \\
&= \sum_{k,l=1}^{m} \xi_{kl} \left(\sum_{(p,q)\in J_{k1}} f_{pq}\right)\left(\sum_{i,j\in J(1)} g_{ij} f_{ij}\right)\left(\sum_{(r,s)\in J_{1l}} f_{rs}\right) \\
&= \sum_{k,l=1}^{m} \xi_{kl} \left(\sum_{q\in J(1)} f_{\sigma_k(q)q}\right)\left(\sum_{i,j\in J(1)} g_{ij} f_{ij}\right)\left(\sum_{r\in J(1)} f_{r\sigma_l(r)}\right) \\
&= \sum_{k,l=1}^{m} \xi_{kl} \sum_{i,j\in J(1)} g_{ij} f_{\sigma_k(i)\sigma_l(j)}.
\end{aligned} \tag{5.7}$$

Therefore, defining $\kappa \colon p_{11} M_{mn} p_{11} \otimes M_m \to M_{mn}$ by

$$\kappa\left(\sum_{i,j\in J(1)} \zeta_{ij} f_{ij} \otimes \sum_{k,l=1}^{m} \xi_{kl} e_{kl}\right) := \sum_{k,l=1}^{m} \xi_{kl} \sum_{i,j\in J(1)} \zeta_{ij} f_{\sigma_k(i)\sigma_l(j)}$$

and $\alpha \colon C(X^y) \otimes M_m \to M_m(C(X^y))$ by

$$\alpha\left(g \otimes \sum_{k,l=1}^{m} \xi_{kl} e_{kl}\right) := \sum_{k,l=1}^{m} g \xi_{kl} e_{kl},$$

we obtain an expanded version of the commutative diagram given in Equation 5.4:

$$\begin{array}{ccc} \mathbb{C}^{|X^y|} \otimes M_m & \xrightarrow{\Gamma_y \otimes \mathrm{id}_m} & p_{11} M_{mn} p_{11} \otimes M_m \\ \uparrow{\scriptstyle \rho_y \otimes \mathrm{id}_m} & & \uparrow{\scriptstyle \mathrm{ev}_y \otimes \mathrm{id}_m} \\ C(X) \otimes M_m & \xrightarrow{\psi' \otimes \mathrm{id}_m} & B \otimes M_m \\ \downarrow{\scriptstyle \pi} & \searrow^{\psi} & \downarrow{\scriptstyle \Theta} \\ M_m(C(X)) & \xrightarrow{\varphi} & M_{mn}(C(Y)) \\ \downarrow{\scriptstyle \rho_y} & & \downarrow{\scriptstyle \mathrm{ev}_y} \\ M_m^{|X^y|} & \xrightarrow{\Lambda_y} & M_{mn} \end{array} \qquad \alpha: \mathbb{C}^{|X^y|} \otimes M_m \to M_m^{|X^y|}, \quad \kappa: p_{11} M_{mn} p_{11} \otimes M_m \to M_{mn} \tag{5.8}$$

Here, the map Γ_y is the induced map arising as in Equation 5.3 with respect to $\mathrm{ev}_y \circ \psi'$ and system of matrix units G. There is a slight abuse of notation here as we are using ρ_y to refer both to one map and its matrix extension.

Claim 5.9.2. *Γ_y is a matrix unit compatible map.*

Proof. We need to show that Γ_y takes matrix units in $C(X^y)$ into finite sums of elements of G. Fix a matrix unit $h \in C(X^y)$. To show that $\Gamma_y(h) = \sum_{i,j \in J(1)} \Gamma_y(h)_{ij} f_{ij}$ is a sum of matrix units in G, it suffices to prove that $\Gamma_y(h)_{ij} \in \{0, 1\}$ for all $i, j \in J(1)$.

To this end, note that the commutativity of Equation 5.8 yields that

$$\begin{aligned} \Lambda_y(he_{11}) &= \Lambda_y(\alpha(h \otimes e_{11})) \\ &= \kappa\left(\sum_{i,j \in J(1)} \Gamma_y(h)_{ij} f_{ij} \otimes e_{11}\right) \\ &= \sum_{i,j \in J(1)} \Gamma_y(h)_{ij} f_{\sigma_1(i)\sigma_1(j)}. \end{aligned} \tag{5.9}$$

Since h is a matrix unit in $C(X^y)$, he_{11} is a matrix unit in $M_m(C(X^y))$. Hence, since Λ_y is assumed to be matrix unit compatible, it follows that $\Lambda_y(he_{11})$ is a finite sum of matrix units in F. By Equation 5.9, this implies that $\Gamma_y(h)_{ij} \in \{0, 1\}$ for all $i, j \in J(1)$, which proves the claim. //

Finally, for $1 \leq s \leq m$, write $J(s) = \{j_1^s, \ldots, j_n^s\}$, where $j_1^s < \cdots < j_n^s$, so that $J = \{j_t^s : 1 \leq s \leq m,\ 1 \leq t \leq n\}$. By Claim 5.9.2 and Lemma 5.8, there are continuous maps $\lambda_1, \ldots, \lambda_n \colon Y \to X$ such that for all $g \in C(X)$ we have $\psi'(g) = \sum_{t=1}^n (g \circ \lambda_t) f_{j_t^1 j_t^1}$. Since $J = \{\sigma_s(j_t^1) : 1 \leq s \leq m,\ 1 \leq t \leq n\}$, we may define a permutation $\tau \in S_{mn}$ by $\tau(\sigma_s(j_t^1)) := (t-1)m + s$ for $1 \leq s \leq m$ and $1 \leq t \leq n$. Let $U[\tau] \in M_{mn}$ denote the unitary as defined in Definition 3.6. Let us conclude the proof by showing that φ is diagonal with respect to E and F upon conjugation by $U[\tau]$.

Let $g = \sum_{k,l=1}^{m} g_{kl}e_{kl} \in M_m(C(X))$ be arbitrary. Then, by Equation 5.7,

$$\begin{aligned}\varphi(g) &= \Theta\left((\psi' \otimes \mathrm{id}_m)\left(\sum_{k,l=1}^{m} g_{kl} \otimes e_{kl}\right)\right)\\ &= \Theta\left(\sum_{k,l=1}^{m}\left(\sum_{t=1}^{n}(g_{kl}\circ\lambda_t)f_{j_t^1 j_t^1}\right)\otimes e_{kl}\right)\\ &= \sum_{k,l=1}^{m}\sum_{t=1}^{n}(g_{kl}\circ\lambda_t)f_{\sigma_k(j_t^1)\sigma_l(j_t^1)}.\end{aligned}$$

Therefore,

$$\begin{aligned}U[\tau]\varphi(g)U[\tau]^* &= \sum_{k,l=1}^{m}\sum_{t=1}^{n}(g_{kl}\circ\lambda_t)f_{\tau(\sigma_k(j_t^1))\tau(\sigma_l(j_t^1))}\\ &= \sum_{k,l=1}^{m}\sum_{t=1}^{n}(g_{kl}\circ\lambda_t)f_{(t-1)m+k\ (t-1)m+l}\\ &= \mathrm{diag}(g\circ\lambda_1,\ldots,g\circ\lambda_n)\end{aligned}$$

with respect to E and F, which establishes the assertion and, therefore, the lemma. //

Lemma 5.10. *Let X be a compact space. Suppose $\mathcal{G}$ is a finite set of locally constant functions in $C(X)$. Then, there is a partition of X into finitely many clopen sets on each of which every function in $\mathcal{G}$ is constant.*

Proof. For each $x \in X$ and $g \in \mathcal{G}$, there is by assumption an open subset U_x^g of X containing x on which g is constant. Let $U_x := \bigcap_{g\in\mathcal{G}} U_x^g$. The family of sets $\{U_x : x \in X\}$ forms an open cover of X, which, by compactness, reduces to a finite subcover $U_{x_1}, \ldots, U_{x_r}$. If $1 \le i, j \le r$ and $U_{x_i} \cap U_{x_j} \neq \varnothing$, then each $g \in \mathcal{G}$ is constant on $U_{x_i} \cup U_{x_j}$. By taking unions, then, we may assume the U_{x_i}'s are disjoint and, therefore, closed. //

Theorem 5.11. *Suppose $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_p$ are compact Hausdorff spaces and E and F are systems of matrix units for $\bigoplus_{k=1}^{r} M_{m_k}$ and $\bigoplus_{k=1}^{p} M_{n_k}$, respectively. Suppose $\varphi\colon \bigoplus_{k=1}^{r} M_{m_k}(C(X_k)) \to \bigoplus_{k=1}^{p} M_{n_k}(C(Y_k))$ is a unital *-homomorphism, which is compatible with respect to E and F. Then, there is a partition of each Y_k into clopen subsets together with a unitary W in $\bigoplus_{k=1}^{p} M_{n_k}(C(Y_k))$, which is a constant permutation matrix on each clopen subset, such that φ is diagonal with respect to E and F after conjugation by W.*

Proof. We may assume that $p = 1$ and write $Y := Y_1$ and $n := n_1$, since φ is diagonal if and only if for all $1 \le j \le p$, each restriction

$\bigoplus_{k=1}^{r} M_{m_k}(C(X_k)) \to M_{n_j}(C(Y_j))$ of φ is. Suppose $e \in E$. Let ρ_y and Λ_y denote the canonical surjection and decomposition map induced by $\mathrm{ev}_y \circ \varphi$ as in Equation 5.3. The fact that φ is compatible implies that $\varphi(e)(y) = \Lambda_y(\rho_y(e))$ is a finite sum of matrix units in F. Since this holds for all $y \in Y$, it follows by the continuity of $\varphi(e)$ that $\varphi(e)$ is locally constant. By Lemma 5.10, we may partition Y into finitely many clopen subsets $U_1, \dots, U_s$ such that for any given $1 \le t \le s$, the collection of functions $\{\varphi(e) : e \in E\}$ is constant on U_t. On letting $\pi_t \colon M_n(C(Y)) \to M_n(C(U_t))$ denote the restriction map for each $1 \le t \le s$, it is immediate that $\pi_t \circ \varphi$ is compatible with respect to the systems E and F. Since $\pi_t(\varphi(e))$ is constant for each $e \in E$, we may apply Lemma 5.9 to obtain a constant permutation unitary matrix $W_t \in M_n$, such that $\mathrm{Ad}_{W_t} \circ \pi_t \circ \varphi$ is diagonal with respect to E and F. That is, there are unitaries $V_{1,t}, \dots, V_{r,t}$ satisfying $\bigoplus_{l=1}^{r} V_{l,t} = W_t$ with the property that given any $1 \le k \le r$, there are continuous functions $\lambda_1^{k,t}, \dots, \lambda_{q_k}^{k,t} \colon U_t \to X_k$, such that the partial map $M_{m_k}(C(X_k)) \to M_n(U_t)$ of φ is

$$g \mapsto V_{k,t} \operatorname{diag}(g \circ \lambda_1^{k,t}, \dots, g \circ \lambda_{q_k}^{k,t}) V_{k,t}^*.$$

For $1 \le l \le q_k$, define $\lambda_l^k := \sum_{t=1}^{s} \chi_t \lambda_l^{k,t}$, where $\chi_t \colon Y \to \{0,1\}$ is the indicator function of the set U_t, and put $W := \sum_{t=1}^{s} \chi_t W_t$. It follows that $\mathrm{Ad}_W \circ \varphi$ is diagonal with respect to E and F, thereby proving the theorem. //

Remark 5.12. Given a compact Hausdorff space Y, a clopen partition $U_1, \dots, U_s$ of Y, and $n \in \mathbb{N}$, it is elementary to check that $M_n(C(Y))$ is isomorphic to $\bigoplus_{t=1}^{s} M_n(C(U_t))$. Moreover, conjugating a system of matrix units by a permutation unitary results in a renumbering of the system. Hence, in the theorem above, we may renumber the given matrix units of F accordingly over each subset in each clopen partition to deduce that φ is diagonal with respect to this renumbering.

Corollary 5.13. *For each $k \in \mathbb{N}$, suppose A_k is a finite direct sum of full matrix algebras, equipped with a system of matrix units E_k. If*

$$A_1 \xrightarrow{\psi_1} A_2 \xrightarrow{\psi_2} A_3 \xrightarrow{\psi_3} \cdots \longrightarrow A := \varinjlim A_i$$

is a simple unital inductive limit where each ψ_k is compatible with respect to E_k and E_{k+1}, then A has stable rank one.

Proof. This follows from Theorem 5.11, Remark 5.12, and the result in [EHT08]. //

Corollary 5.13 effectively shows that in the AH setting, the numbering of the elements within the systems of matrix units for the finite-stage algebras is irrelevant, provided that the bonding maps are compatible with respect to these systems. Consequently, the a priori stronger notion of a diagonal map, which presupposes an ordering of the matrix units, is not needed. In the ASH setting studied in Chapter 3, all systems of matrix units are numbered (by the definition of a DSH algebra). This numbering is used when constructing the unitaries in Lemma 3.7, Lemma 3.10, and Lemma 3.13. It is not clear to the author whether a compatible map between DSH algebras must be diagonal in the sense of Definition 2.3 after a suitable renumbering of the matrix units (see Question 5.20). The more general question of whether an ASH stable rank one result is possible without numbered matrix units is discussed in the next section.

5.3 INTERNAL MATRIX UNIT COMPATIBILITY

The notion of matrix unit compatibility plays a prominent role in the proof of stable rank one of inductive limits. The building block algebras in the inductive limit themselves must have an internal compatibility among the systems of matrix units for each primitive quotient; this is used in the construction of all of the unitaries and in Lemma 3.3 to extend from non-invertibility at a point to a whole neighbourhood of the spectrum.

This naturally leads to the question of possible formulations of internal matrix unit compatibility for a homogeneous or a subhomogeneous C*-algebra. One possible formulation in the homogeneous setting, which is strictly local in nature, is the following definition.

Definition 5.14 (Homogeneous Compatibility Condition)**.** Suppose A is a unital n-homogeneous C*-algebra. Suppose that for each $\tau \in \hat{A}$, we are given a system $\{e^{\tau}_{ij}\}_{i,j=1}^{n}$ of matrix units for $\tau(A) = M_n$. If for each $\tau \in \hat{A}$, there is a closed neighbourhood K_τ of τ in $\hat{A}$ together with a set of elements $\{a^{\tau}_{ij}\}_{i,j=1}^{n}$ of A such that for all $\sigma \in K_\tau$ and $1 \leq i, j \leq n$, we have $\sigma(a^{\tau}_{ij}) = e^{\sigma}_{ij}$, then we say A is *internally matrix unit compatible*.

Since the spectrum of every homogeneous C*-algebra is Hausdorff (see, for instance, 4.5.3 of [Dix77]) and in the unital case, compact, we are able to recover global matrix units from the local compatibility condition above in this case, as the following proposition shows.

Proposition 5.15. *Suppose A is a unital n-homogeneous* C**-algebra and that for each $\tau \in \hat{A}$, we are given a system $\{e^{\tau}_{ij}\}_{i,j=1}^{n}$ of matrix units for $\tau(A) = M_n$. If A is internally matrix unit compatible, then $A \cong C(\hat{A}, M_n)$.*

Proof. Let us first prove the following claim.

Claim 5.15.1. *There is a set of elements* $\{a_{ij}\}_{i,j=1}^n$ *of* A *such that for all* $\sigma \in \hat{A}$ *and* $1 \le i,j \le n$*, we have* $\sigma(a_{ij}) = e_{ij}^\sigma$*.*

Proof. Since A is unital, $\hat{A}$ is compact. Hence, there is a finite collection of closed neighbourhoods $K_{\tau_1}, \ldots, K_{\tau_m} \subset \hat{A}$ as described in Definition 5.14 whose interiors cover $\hat{A}$. Moreover, $\hat{A}$ is Hausdorff since A is homogeneous. Let $\{f_l\}_{l=1}^m$ be a partition of unity subordinate to the open cover $\{\text{int}(K_{\tau_l})\}_{l=1}^m$. By the Dauns-Hofmann Theorem, for each $1 \le l \le m$ and $1 \le i,j \le n$, there is an element $f_l a_{ij}^{\tau_l} \in A$ such that for each $\sigma \in \hat{A}$, $\sigma(f_l a_{ij}^{\tau_l}) = f_l(\sigma)\sigma(a_{ij}^{\tau_l})$. Now, fix $1 \le i,j \le n$ and set $a_{ij} := f_1 a_{ij}^{\tau_1} + \cdots + f_m a_{ij}^{\tau_m}$. Given $\sigma \in \hat{A}$, let $J := \{1 \le l \le m : \sigma \in \text{int}(K_{\tau_l})\}$. Then,

$$\sigma(a_{ij}) = \sum_{l=1}^m f_l(\sigma)\sigma(a_{ij}^{\tau_l}) = \sum_{l \in J} f_l(\sigma)\sigma(a_{ij}^{\tau_l}) = \sum_{l \in J} f_l(\sigma) e_{ij}^\sigma = e_{ij}^\sigma,$$

which proves the claim. //

With the claim established, the rest of the argument proceeds much like the one on page 524 of [Spr76]. Given $b \in A$ and $\tau \in \hat{A}$, there is a unique set $\{\beta_{ij}^b(\tau)\}_{i,j=1}^n$ of complex numbers such that $\tau(b) = \sum_{i,j=1}^n \beta_{ij}^b(\tau) e_{ij}^\tau$. It is straightforward to check that the map $\lambda_\tau \colon M_n \to \tau(A)$ given by $\lambda_\tau((\beta_{ij})_{ij}) := \sum_{i,j=1}^n \beta_{ij} e_{ij}^\tau$ is a *-isomorphism.

Now, define $\Gamma \colon A \to C(\hat{A}, M_n)$ by $\Gamma(b)(\tau) := \lambda_\tau^{-1}(\tau(b)) = (\beta_{ij}^b(\tau))_{ij}$. To show that Γ is well defined, it suffices to show that for fixed $1 \le i,j \le n$, the function $\tau \mapsto \beta_{ij}^b(\tau)$ is continuous on $\hat{A}$. Fix $\sigma \in \hat{A}$. Since $\hat{A}$ is Hausdorff, the map $\tau \mapsto \|\tau(a)\|$ is continuous at σ for all $a \in A$. In particular, the map

$$\tau \mapsto \|\tau(a_{ii} b a_{jj} - \beta_{ij}^b(\sigma) a_{ij})\| = \|\beta_{ij}^b(\tau) e_{ij}^\tau - \beta_{ij}^b(\sigma) e_{ij}^\tau\| = |\beta_{ij}^b(\tau) - \beta_{ij}^b(\sigma)|$$

is continuous at σ, so that Γ is well defined. A routine check shows that Γ is a *-homomorphism. For injectivity, note that if $\Gamma(b) = 0$, then $\tau(b) = 0$ for all $\tau \in \hat{A}$, which implies that $a = 0$. For surjectivity, suppose we have an arbitrary map in $C(\hat{A}, M_n)$ taking τ into $(g_{ij}(\tau))_{ij}$, where each g_{ij} is a function in $C(\hat{A})$. Since $\hat{A}$ is compact, each g_{ij} is bounded. Hence, by the Dauns-Hofmann Theorem, for each $1 \le i,j \le n$, there is an element $g_{ij} a_{ij} \in A$ such that $(g_{ij} a_{ij})(\tau) = g_{ij}(\tau) a_{ij}(\tau) = g_{ij}(\tau) e_{ij}^\tau$ for all $\tau \in \hat{A}$. Therefore, $\Gamma\left(\sum_{i,j=1}^n g_{ij} a_{ij}\right)(\tau) = \lambda_\tau^{-1}\left(\sum_{i,j=1}^n g_{ij}(\tau) e_{ij}^\tau\right) = (g_{ij}(\tau))_{ij}$ for all $\tau \in \hat{A}$, proving surjectivity. //

Proposition 5.15 invites one to consider different formulations for internal matrix unit compatibility for a subhomogeneous C*-algebra. The definition of a DSH algebra (Definition 2.2) is one such formulation. It is noteworthy since it leads to a stable rank one result, but one potential drawback is that it requires a recursive subhomogeneous decomposition. Below is an analogue to Definition 5.14 in the subhomogeneous setting.

Definition 5.16 (Subhomogeneous Compatibility Condition I)**.** Suppose that A is a separable unital subhomogeneous C*-algebra and that for each $\tau \in \hat{A}$, we are given a system $\{e^{\tau}_{ij} : 1 \leq i,j \leq \dim \tau\}$ of matrix units for $\tau(A) = M_{\dim \tau}$. Suppose that, whenever $(\tau_n)_n$ is a sequence of equidimensional representations in $\hat{A}$ converging to τ in $\hat{A}$, there is a set $\{a^{\tau}_{ij} : 1 \leq i,j \leq \dim \tau\}$ of elements of A with $\tau(a^{\tau}_{ij}) = e^{\tau}_{ij}$ for all $1 \leq i,j \leq \dim \tau$, together with an index $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\{\tau_n(a^{\tau}_{ij}) : 1 \leq i,j \leq \dim \tau\}$ is a system of matrix units for $M_{\dim \tau} \subset M_{\dim \tau_n}$, where for all $1 \leq i,j \leq \dim \tau$, $\tau_n(a^{\tau}_{ij})$ can be written as a sum $\sum_{k=1}^{l} e^{\tau_n}_{i_k j_k}$ with the $e^{\tau_n}_{i_k j_k}$'s having pairwise orthogonal support and range projections. Then, we say A is *internally matrix unit compatible*.

By IV.1.4.3 of [Bla06], every subhomogeneous C*-algebra is isomorphic to a sub-C*-algebra of $C(X, M_n)$ for some compact Hausdorff space X and $n \in \mathbb{N}$. Therefore, another reasonable definition for internal matrix unit compatibility for a subhomogeneous C*-algebra might be the following definition.

Definition 5.17 (Subhomogeneous Compatibility Condition II)**.** Suppose that A is a subhomogeneous C*-algebra and that for each $\tau \in \hat{A}$, we are given a system of matrix units for $\tau(A) = M_{\dim \tau}$. Suppose that there are a compact Hausdorff space X, an $n \in \mathbb{N}$, a system of matrix units F of M_n, and an injective *-homomorphism $\iota \colon A \to C(X, M_n)$ such that the induced decomposition map given by Equation 5.3 is compatible with respect to the given systems of matrix units. Then, we say A is *internally matrix unit embeddable*.

We end with a few questions.

Question 5.18. Are Definition 5.16 and Definition 5.17 equivalent (possibly under suitable assumptions)?

Question 5.19. If we assume that A is a separable subhomogeneous algebra satisfying either of the two compatibility conditions above, may it be expressed as a DSH algebra? What if we further assume that $\hat{A}$ is Hausdorff?

Question 5.20. Is every compatible map (see Definition 5.5) between DSH algebras necessarily diagonal (see Definition 2.3) after a suitable renumbering of the matrix units?

Question 5.21. Is it possible to obtain an ASH stable rank one result without using the recursive subhomogeneous algebra framework (with this replaced with another notion of internal matrix unit compatibility)?

Question 5.22. Is there a converse to the ASH stable rank one result? That is, if we are given a simple separable ASH algebra with stable rank one, can it be realized as an inductive limit of internally matrix unit compatible subhomogeneous algebras with compatible bonding maps?

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COLOPHON

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