Fusion Product of D/G-valued Moment Maps

by

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Abstract

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A fusion product is defined for Hamiltonian spaces with moment maps valued in a Lie group D generalizing those of Alekseev-Malkin-Meinrenken. An analogous theory for these general Hamiltonian spaces is developed and, among other results, versions of symplectic reduction, duality and the shifting trick are derived. The Hamiltonian spaces with moment maps valued in a homogeneous space D/G of Alekseev-Kosmann-Schwarzbach are shown to be equivalent to certain Hamiltonian spaces with group-valued moment maps. The aforementioned theory is consequently brought to bear on that of D/G-valued moment maps, thereby defining a fusion product for these. This fusion product affords many new examples of D/G-valued moment maps, of which there was hitherto a paucity. Among said examples are moduli spaces of flat principal bundles over certain surfaces with boundary.

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Chapter 1

Introduction

1.1 Context and motivation

1.1.1 Moment map theories

In the classical theory of moment maps, one is given a function $J : X \to \mathfrak{g}^*$, called the *moment map*, from a manifold X to the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of a Lie group G. The manifold X carries a G-action and a 2-form $\omega \in \Omega^2(X)$ which, denoting the corresponding $\mathfrak{g}$-action by $\varrho : \mathfrak{g} \to \mathfrak{X}(X)$, are required to satisfy the following three axioms:

(a) ω is closed,
(b) ω is non-degenerate,
(c) $\iota_{\varrho(\gamma)}\omega = -d\langle\gamma, J\rangle$ for all $\gamma \in \mathfrak{g}$.

In that case the manifold X is called a *G-Hamiltonian space*. It was first observed by Sophus Lie himself in his seminal *Theorie der Transformationsgruppen* that the dual of a Lie algebra carries a canonical Poisson structure. The notion of a G-Hamiltonian space can be recast in terms of the canonical Poisson structure on $\mathfrak{g}^*$ and a Poisson structure on the manifold X that the moment map J is required to intertwine. On the other hand, the canonical Poisson structure on $\mathfrak{g}^*$ can be seen as dual to the trivial Poisson structure on the group G; putting

$$D = T^*G = \mathfrak{g}^* \rtimes G,$$

where G acts on $\mathfrak{g}^*$ via the coadjoint representation, the Lie group D is a Poisson Lie group, i.e. it carries a Poisson bivector field $\pi \in \mathfrak{X}^2(D)$ that is multiplicative in the sense that

$$(L_{d_1})_*\pi|_{d_2} + (R_{d_2})_*\pi|_{d_1} = \pi|_{d_1 d_2} \text{ for every } d_1, d_2 \in G,$$

In general, a triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ where $\mathfrak{d}$ is a Lie algebra equipped with an ad-invariant metric and $\mathfrak{g}, \mathfrak{h} \subseteq \mathfrak{d}$ are Lagrangian subalgebras is called a *Manin triple*. According to Drinfeld [25], the connected and simply connected Poisson Lie groups are in one-to-one correspondence with Manin triples. A group D integrating the Lie algebra $\mathfrak{d}$ inherits a Poisson Lie group structure and the connected subgroups $G, G^* \subseteq D$ integrating $\mathfrak{g}$ and $\mathfrak{h}$ respectively are Poisson Lie subgroups of D. The subgroup G^* may be regarded as the dual of the subgroup G (hence the notation) and this suggests a more general moment map theory, where moment maps may take value in the dual G^* arising from any Manin triple. This was undertaken by A. Weinstein's student J.-H. Lu [44]. In Lu's theory, the action on G^* is the so-called *dressing action* of G, which, assuming mult : $G \times G^* \to D$ is a diffeomorphism, is characterized by

$$gh = (g.h)g' \text{ for } g \in G \text{ and } h \in G^*,$$

where $g.h \in G^*$ and $g' \in G$ are the unique elements such that their product (in that order) is equal to gh. (In the case of the Manin triple $(\mathfrak{g}^* \rtimes \mathfrak{g}, \mathfrak{g}, \mathfrak{g}^*)$, the dressing action coincides with the coadjoint representation of course.)

Still more generally, one can consider instead of Manin triples so-called *Manin pair* $(\mathfrak{d}, \mathfrak{g})$, which consist of a Lie algebra $\mathfrak{d}$ equipped with an ad-invariant metric and a Lagrangian subalgebra $\mathfrak{g} \subseteq \mathfrak{d}$. Although in practice a Lagrangian subalgebra $\mathfrak{h} \subseteq \mathfrak{d}$ complementary to $\mathfrak{g}$ may often be found thus completing the Manin pair $(\mathfrak{d}, \mathfrak{g})$ to a Manin triple (see the work of P. Delorme [22] for far-reaching results on the matter), this is not always the case; specific counterexamples are given in this thesis in Chapter 4. This said, a Lagrangian (but not necessarily closed under the Lie bracket) complement $\mathfrak{h} \subseteq \mathfrak{d}$ of $\mathfrak{g}$ may always be found and the triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ in this case is called a *Manin quasi-triple*. Y. Kosmann-Schwarzbach first investigated Manin quasi-triples and showed, much like Drinfeld for the special case of Manin triples, that they classify the so-called *quasi-Poisson Lie groups* [33, 34]. A quasi-Poisson Lie group K carries a multiplicative bivector field whose Schouten-Nijenhuis bracket is no longer required to vanish and is instead required to satisfy conditions depending on a choice of tensor $\chi \in \bigwedge^3 \mathfrak{k}$ in the exterior algebra of the Lie algebra of K.

Suppose (D, G) is pair of Lie groups where D integrates the Lie algebra $\mathfrak{d}$ and $G \subseteq D$ is a closed and connected subgroup integrating the subalgebra $\mathfrak{g} \subseteq \mathfrak{d}$; the pair (D, G) is called a *group pair* for the Manin pair $(\mathfrak{d}, \mathfrak{g})$. Motivated by the theory of quasi-Poisson Lie groups, A. Alekseev and Y. Kosmann-Schwarzbach [2] have developed a theory with moment maps valued in the homogeneous space D/G. The action on D/G, also called the *dressing action*, is the action by G that descends from multiplication on the left, i.e.

$$g.[d] = [gd].$$

In this theory, after a choice of Lagrangian complement $\mathfrak{h} \subseteq \mathfrak{d}$ of $\mathfrak{g}$, a Hamiltonian space X is required to carry a bivector field $\pi^{\mathfrak{h}} \in \mathfrak{X}^2(X)$ satisfying a moment map condition as well as an integrability and an equivariance condition (this is made precise in Chapter 4). If $\mathfrak{h}' \subseteq \mathfrak{d}$ is another Lagrangian complement of $\mathfrak{g}$ then the bivector fields $\pi^{\mathfrak{h}}$ and $\pi^{\mathfrak{h}'}$ are related to each other in function of the "twist" $t \in \bigwedge^2 \mathfrak{g}$ relating $\mathfrak{h}$ and $\mathfrak{h}'$. The choice of Lagrangian complement is therefore essentially immaterial, but it must be made nonetheless in the framework adopted in [2]. The theory of D/G-valued moment maps as introduced by Alekseev and Kosmann-Schwarzbach thus leaves something to be desired: since the theory does not ultimately depend on the choice of a Lagrangian complement of $\mathfrak{g}$, one should like to find an invariant formulation of the theory. Exactly this was achieved in works of H. Bursztyn, M. Crainic, P. Ševera and D. Iglesias Ponte [16, 15, 14, 13], taking a Dirac-geometric approach. The discussion will turn to these ideas in the next section.

As a separate line of development of the idea of a moment map, is the theory of moment maps valued in a group G introduced by A. Alekseev, A. Malkin and E. Meinrenken. In this context, the group action on G is the adjoint action of G on itself and Hamiltonian spaces are G-spaces required to satisfy a list of three axioms, much like in the classical theory. The Dirac-geometric approach has also proven effective in recasting this theory in less computational and more abstract terms [1].

1.1.2 Initial problem

The theories of G^*- and G-valued moment maps have many of the familiar trappings of the classical theory of $\mathfrak{g}^*$-valued moment maps, with notions of fusion products (see [27] in the case of G^*-valued moment maps), duality, symplectic reduction etc. These notions have been understandably challenging to extend to the theory of D/G-valued moment maps due to the absence of a group structure on the target space. A fusion product was defined by P. Ševera [55] and results on symplectic reduction were derived by Bursztyn, Crainic, Iglesias Ponte and Ševera [13, 16]. The definitions and results in question are however ostensibly quite disparate and Ševera's fusion product in particular relies on structures far afield. It is therefore not clear how to bring them into a framework where they may be combined into e.g. a "shifting trick"-like result. Developing such a framework for the theory of D/G-valued moment maps wherein some of the most salient features of the classical theory of $\mathfrak{g}^*$-valued moment maps may be brought over has been the chief preoccupation of this author.

The initial impetus of this project came from a simple observation. Given a Hamiltonian space X with a moment map $J : X \to D/G$ valued in D/G, let X_{lift} be the pullback along J of the group D seen as a principal bundle over D/G. The map J lifts to a map $J_{\text{lift}} : X_{\text{lift}} \to D$ so that there is a commutative diagram

$$\begin{array}{ccc} X_{\text{lift}} & \xrightarrow{J_{\text{lift}}} & D \\ \Big\downarrow \cdot/G & & \Big\downarrow \cdot/G \\ X & \xrightarrow{J} & D/G \end{array} , \tag{1.1}$$

where the vertical arrows are the quotient maps. In this way the Hamiltonian space X is lifted to a kind of "Hamiltonian space" with moment map valued in the group D; note however that the dressing action on D/G lifts to the action of $G \times G$ on D via

$$(g_1, g_2).d = g_1 d g_2^{-1}$$

and not the usual action of D on itself by conjugation found in the prior theory of group-valued moment maps [4]. On the other hand, if X' is another Hamiltonian space with moment map $J' : X' \to D/G$ then there is an analogous lift X'_{lift} and diagram (1.1). The direct product $X_{\text{lift}} \times X'_{\text{lift}}$ of the lifts is a "Hamiltonian space" with moment map $J_{\text{lift}} \times J'_{\text{lift}}$ valued in $D \times D$. Composing the latter with the group multiplication mult : $D \times D$ gives a map

$$X_{\text{lift}} \times X'_{\text{lift}} \to D.$$

Quotienting the left-hand-side by the action of $e \times G$ on X'_{lift} and the right-hand-side by the action of G on D by inverse multiplication on the right gives a G-equivariant map

$$X_{\text{lift}} \times X' \to D/G.$$

The space $X_{\text{lift}} \times X'$ may thus be naively taken, as it were, to be the fusion product of X and X'. Actually, the space $X_{\text{lift}} \times X'$ is *not* a Hamiltonian space at all because its auxiliary 2-form does not satisfy the appropriate minimal degeneracy condition. Rather, it must be quotiented by the residual $e \times G_\Delta$-action (G_Δ the diagonal of $G \times G$). The space

$$X \circledast X' := \frac{X_{\text{lift}} \times X'}{e \times G_\Delta}$$

will reveal to be the correct definition of the fusion product of X and X'.

This picture also suggests a notion of duality. Namely, the two G-actions on the lift X_{lift}

are traded; equivalently the dual of X is the quotient of X_{lift} by the $G \times e$-action. Unlike the classical and Alekseev-Malkin-Meinrenken theories, dual Hamiltonian spaces need not be isomorphic as G-spaces.

1.2 Dirac-geometric approach

1.2.1 Courant algebroids and Dirac structures

The language of Dirac geometry has been remarkably effective in organizing, simplifying and elucidating the subjects of symplectic and Poisson geometry. The fundamental objects of Dirac geometry are *Courant algebroids*, the basic example of which is the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$ of a manifold M. The bundle $\mathbb{T}M$ possesses two natural structures of interest; aside from the metric $\langle\cdot,\cdot\rangle$ provided by the pairing of TM and T^*M, there is a bracket on the sections of $\mathbb{T}M$ given by

$$[\![v_1 + \mu_1, v_2 + \mu_2]\!] = [v_1, v_2] + \mathcal{L}_{v_1}\mu_2 - \iota_{v_2} d\mu_1.$$

for $v_i \in \mathfrak{X}(M)$ and $\mu_i \in \Omega^1(M)$. This bracket was essentially introduced by T. Courant [20] and was later revised by I. Dorfman [24], giving its current version. More general Courant algebroids, first defined by Z.-J. Liu, A. Weinstein, and P. Xu [41], generalize the structures found in the generalized tangent bundle $\mathbb{T}M$; a Courant algebroid is a vector bundle $\mathbb{A} \to M$ together with a metric $\langle\cdot,\cdot\rangle$, a *Courant bracket* $[\![\cdot,\cdot]\!]$ on its sections and a map $\mathbf{a} : \mathbb{A} \to TM$ called the *anchor* that satisfy the properties C1-C3 listed below for all sections $\sigma_i \in \Gamma(\mathbb{A})$

C1. $[\![\sigma_1, [\![\sigma_2, \sigma_3]\!]]\!] = [\![[\![\sigma_1, \sigma_2]\!], \sigma_3]\!] + [\![\sigma_2, [\![\sigma_1, \sigma_3]\!]]\!]$,
C2. $\mathbf{a}(\sigma_3)\langle\sigma_1, \sigma_2\rangle = \langle[\![\sigma_3, \sigma_1]\!], \sigma_2\rangle + \langle\sigma_1, [\![\sigma_3, \sigma_2]\!]\rangle$,
C3. $\mathbf{a}^* d\langle\sigma_1, \sigma_2\rangle = [\![\sigma_1, \sigma_2]\!] + [\![\sigma_2, \sigma_1]\!]$, where $\mathbb{A}^*$ has been identified with $\mathbb{A}$ via $\langle\cdot,\cdot\rangle$.

Most of the interesting geometric data in Dirac geometry is encoded by *Dirac structures* – involutive and fiberwise Lagrangian subbundles of Courant algebroids. Indeed, it is from the observation that a 2-form $\omega \in \Omega^2(M)$ or bivector field $\pi \in \mathfrak{X}^2(M)$ is closed (resp. Poisson) if and only if its graph is a Dirac structure of $\mathbb{T}M$ [20, 18] that the subject owes its inception.

The most ubiquitous class of Courant algebroids after generalized tangent bundles (and their "twists" [51, 56]) is that of the *action Courant algebroids* first defined by D. Li-Bland and E. Meinrenken [37]. Given an action of a Lie algebra $\mathfrak{d}$ equipped with an ad-invariant metric on a manifold M, the product $M \times \mathfrak{d}$ inherits the structure of a Courant algebroid provided the stabilizing subalgebras are all coisotropic. In that case the Courant bracket is the unique one which extends the Lie bracket of $\mathfrak{d}$ identified with the constant sections of $M \times \mathfrak{d}$ and the anchor is the $\mathfrak{d}$-action. If $\mathfrak{g} \subseteq \mathfrak{d}$ is a Lagrangian algebra, then the trivial vector subbundle $M \times \mathfrak{g}$ is a Dirac structure of the action Courant algebroid $M \times \mathfrak{d}$.

1.2.2 Moment maps in Dirac geometry

Let (D, G) be a group pair for a Manin pair $(\mathfrak{d}, \mathfrak{g})$. Of particular interest among action Courant algebroids are

$$D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}), \;\; D/G \times \mathfrak{d}$$

arising from the $D \times D$-action on D via $(d_1, d_2).d = d_1 d d_2^{-1}$ and the action of D on the homogeneous space D/G via $d.[d'] = [dd']$ respectively. The group-valued moment map theory of Alekseev-Malkin-Meinrenken turns out to be encoded by a relation between the Dirac structures $TM \subseteq \mathbb{T}M$, where M is a Hamiltonian space, and $D \times \mathfrak{d}_\Delta \subseteq D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$. Likewise, the D/G-moment map theory of Alekseev and Kosmann-Schwarzbach is encoded by the same type relation between the tangent bundle of a Hamiltonian space and the Dirac structure $D/G \times \mathfrak{g} \subseteq D/G \times \mathfrak{d}$ [16, 15, 14, 13].

The kind of relation between Dirac structures in question is called a *Dirac morphism* and will in general be denoted by

$$(\mathbb{A}_1, E_1) \dashrightarrow (\mathbb{A}_2, E_2)$$

for Dirac structures E_i of Courant algebroids $\mathbb{A}_i$ so that a Hamiltonian space with a moment map valued in D in the sense of Alekseev-Malkin-Meinrenken is the data of a Dirac morphism

$$(\mathbb{T}X, TX) \dashrightarrow (D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}), D \times \mathfrak{d}_\Delta) \tag{1.2}$$

and a Hamiltonian space with a moment map valued in D/G is the data of Dirac morphism

$$(\mathbb{T}X, TX) \dashrightarrow (D/G \times \mathfrak{d}, D/G \times \mathfrak{g}).$$

There is a diagram at the level of Dirac morphisms analogous to the diagram (1.1)

$$\begin{array}{ccc} (\mathbb{T}X_{\text{lift}}, TX_{\text{lift}}) & \dashrightarrow & (D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}), D \times (\mathfrak{g} \oplus \mathfrak{g})) \\ \downarrow & & \downarrow \\ (\mathbb{T}X, TX) & \dashrightarrow & (D/G \times \mathfrak{d}, D/G \times \mathfrak{g}) \end{array} . \tag{1.3}$$

The foregoing considerations behoove one to try to develop a theory of group-valued moment maps where the Dirac structure $D \times (\mathfrak{g} \oplus \mathfrak{g})$ is, so to speak, substituted for $D \times \mathfrak{d}_\Delta$ in (1.2). This theory could then be marshalled in the service of the programme of expanding the theory of D/G-valued moment maps.

1.3 Results and discussion

1.3.1 Overview

The following represents a quick summary of the constructions and results found in this thesis.

General Hamiltonian spaces. It should be stated that this thesis goes beyond its title, *Fusion product of D/G-valued moment maps*. Indeed, in wanting to develop a new theory of group-valued moment maps, it was realized that the subalgebra $\mathfrak{d}_\Delta \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ appearing in the prior theory of Alekseev-Malkin-Meinrenken [4] could be substituted not only by the subalgebra $\mathfrak{g} \oplus \mathfrak{g} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ but also by an arbitrary Lagrangian subalgebra $\mathfrak{l} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ so that general Hamiltonian spaces

$$(\mathbb{T}X, TX) \dashrightarrow (D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}), E \times \mathfrak{l}) \tag{1.4}$$

could be considered.

Fusion product. Given two Lagrangian subalgebras $\mathfrak{l}_1, \mathfrak{l}_2 \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$, their product $\mathfrak{l}_1 \circ \mathfrak{l}_2$ in the pair groupoid

$$\mathfrak{d} \oplus \overline{\mathfrak{d}} \rightrightarrows \mathfrak{d} \tag{1.5}$$

is also a Lagrangian subalgebra of $\mathfrak{d} \oplus \overline{\mathfrak{d}}$. Consequently one can consider Hamiltonian spaces

$$(\mathbb{T}X_i, TX_i) \dashrightarrow (D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}), D \times \mathfrak{l}_i) \tag{1.6}$$

and define a sensible notion of fusion whereby a new Hamiltonian space

$$(\mathbb{T}(X_1 \circledast X_2), T(X_1 \circledast X_2)) \dashrightarrow (D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}), D \times (\mathfrak{l}_1 \circ \mathfrak{l}_2)) \tag{1.7}$$

is constructed. Since

$$(\mathfrak{g} \oplus \mathfrak{g}) \circ (\mathfrak{g} \oplus \mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g},$$

this fusion product restricts to the category of D/G-valued moment maps as initially desired.

Duality. There is a natural notion of duality for a general Hamiltonian spaces (1.4) consisting in substituting $\mathfrak{l}$ by its image $\mathfrak{l}^{-1}$ under the inversion relation of the pair groupoid (1.5).

Synthesis. A number of results are proven regarding these general Hamiltonian spaces, notably a version of symplectic reduction and a shifting trick. The *multiplicative* Lagrangian subalgebras $\mathfrak{l} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$, i.e. with the property $\mathfrak{l} \circ \mathfrak{l} = \mathfrak{l}$, are classified: they are equivalent to pairs of coisotropic subalgebras of $\mathfrak{d}$.

D/G-valued moment maps. As regards D/G-valued moment maps, the fusion product herein defined is shown to coincide with that found in Lu's moment map theory in the presence of a Lagrangian subalgebra $\mathfrak{h} \subseteq \mathfrak{d}$ complementary to $\mathfrak{g}$. It is also argued that if $\mathfrak{g}$ admits an $\mathrm{ad}_{\mathfrak{g}}$-invariant Lagrangian complement $\mathfrak{h} \subseteq \mathfrak{d}$ then D/G-valued moment maps are classical $\mathfrak{g}^*$-valued moment maps near the coset $[e]$ of the group identity $e \in D$. In view of this, two examples of Manin pairs $(\mathfrak{d}, \mathfrak{g})$ are given for which there is no Lagrangian complement $\mathfrak{h} \subseteq \mathfrak{d}$ of $\mathfrak{g}$ that is either a subalgebra or $\mathrm{ad}_{\mathfrak{g}}$-invariant – thus showing that the D/G-valued moment map theory is not redundant.

Moduli space examples. Finally, inspired by work of Ševera [54], moduli spaces of flat D-bundles are given as examples of general Hamiltonian spaces (1.4). These include Hamiltonian spaces with moment maps valued in D/G; they correspond to 2-manifolds with corners where some boundary segments have been decorated with the Lagrangian subalgebra $\mathfrak{g} \subseteq \mathfrak{d}$ (e.g. Figure 1.1).

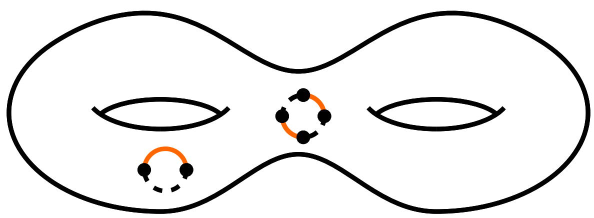

Figure 1.1: A boundary segment is coloured orange to indicate it is decorated with the Lagrangian subalgebra $\mathfrak{g} \subseteq \mathfrak{d}$; otherwise it is drawn with a dotted line.

1.3.2 Future directions

It is obviously impossible to pursue all lines of investigation that may present themselves in the course of a research project such as this one. The following are ideas regarding possible future endeavors.

Volume forms. The Hamiltonian spaces (1.4) carry canonical (up to a constant) volume forms, which follows from the machinery of Clifford modules and pure spinors developed by Alekseev and Meinrenken [1] provided some mild topological condition is satisfied. A question that remains to be answered entirely is that of the relationship between the volume forms of Hamiltonian spaces (1.6) and that of their fusion product (1.7). This is already known for the Lagrangian subalgebra $\mathfrak{l}_1 = \mathfrak{l}_2 = \mathfrak{d}_\Delta$ [6, Thm. 3.5] [1, Prop. 5.15], and partially for $\mathfrak{l}_1 = \mathfrak{l}_2 = \mathfrak{g} \oplus \mathfrak{g}$ to this author, but the general case remains unsettled.

Weil algebras and Duistermaat-Heckman measures. It is expected that the general Hamiltonian spaces (1.4) carry Liouville-like measures and that these define Duistermaat-Heckman measures on the target Lie group D. One motivation behind the search for a fusion product of D/G-valued moment maps is the expectation that a Weil algebra could be defined so that a version of equivariant cohomology could be used to frame results on such Duistermaat-Heckman measures in the spirit of [5, 6].

General $\mathcal{CA}$-groupoids. By virtue of the pair groupoid structure (1.5), the Courant algebroid $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$ is a $\mathcal{CA}$-groupoid [38]: it possesses a multiplicative structure compatible with the multiplication of their base Lie groups. More general $\mathcal{CA}$-groupoids exist [40] and since the fusion product (1.7) depends on the pair groupoid structure (1.5), the question would be whether an even more general moment map theory and auxiliary notions of fusion and duality could be developed. This would presumably not be as simple as the case considered here since morphisms of Courant algebroids do not generically take Dirac structures to Dirac structures; one has in mind the product $E_1 \times E_2$ of Dirac structures $E_1, E_2 \subseteq \mathbb{A}$ of a $\mathcal{CA}$-groupoid and its multiplication morphism $\mathrm{Mult} : \mathbb{A} \times \mathbb{A} \dashrightarrow \mathbb{A}$.

Chapter 2

Dirac geometry

A short but somewhat comprehensive introduction to the subject of Dirac geometry is given. If nothing else, this chapter serves to fix the notation for the remainder of the thesis.

2.1 Linear aspects

All vector spaces considered are real and finite-dimensional. A metric on a vector space V is understood to be a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$. The notation $\overline{V}$ will be used for V equipped with its opposite metric.

Given a subspace W of V, its orthogonal relative to the metric of V will be denoted by $W^\perp$. A subspace $W \subseteq V$ is called *isotropic* if $W \subseteq W^\perp$, *coisotropic* if $W \supseteq W^\perp$ and *Lagrangian* if $W = W^\perp$. Clearly, the orthogonal of a coisotropic subspace is isotropic and vice-versa. A Lagrangian subspace E is both isotropic and coisotropic, and is respectively maximal and minimal in those regards; if $C \subseteq E \subseteq I$ where C is coisotropic and I is isotropic then equality on either side holds. From the equality $\dim(W) + \dim(W^\perp) = \dim(V)$, it is seen that the dimension of a Lagrangian subspace is $\frac{1}{2}\dim(V)$. A pair (V, E), where $E \subseteq V$ is Lagrangian subspace, is called a *linear Manin pair*.

Proposition 2.1.1. *Let $C \subseteq V$ be a coisotropic subspace. Then there is an isotropic subspace $I \subseteq V$ such that $V = C \oplus I$.*

Proof. Let $D \subseteq V$ be an arbitrary complement of C, i.e $V = C \oplus D$. The orthogonal $D^\perp$ is a complement of $C^\perp$ in V. Let $\mathrm{pr}_{C^\perp} : V \to C^\perp$ be the projection of V onto $C^\perp$ along $D^\perp$ and put $I = \{v - \frac{1}{2}\,\mathrm{pr}_{C^\perp} v \ : \ v \in D\}$. Then $V = C + I$ as $C^\perp \subseteq C$. By a dimension count, this is a direct sum. Moreover, for $v, w \in D$ one has

$$\langle v - \tfrac{1}{2}\,\mathrm{pr}_{C^\perp} v, w - \tfrac{1}{2}\,\mathrm{pr}_{C^\perp} w \rangle = \langle v, w \rangle - \tfrac{1}{2}\langle v, \mathrm{pr}_{C^\perp} w \rangle - \tfrac{1}{2}\langle \mathrm{pr}_{C^\perp} v, w \rangle$$

$$= \langle v, w \rangle - \tfrac{1}{2} \langle v, w \rangle - \tfrac{1}{2} \langle v, w \rangle = 0,$$

in other words I is isotropic. This completes the proof. □

If $C \subseteq V$ is coisotropic then the quotient $V_{\text{red}} = C/C^{\perp}$ inherits the metric $\langle x + C^{\perp}, y + C^{\perp} \rangle = \langle x, y \rangle$. A Lagrangian subspace $E \subseteq V$ descends to the Lagrangian subspace

$$E_{\text{red}} = (E \cap C)/(E \cap C^{\perp})$$

of V_{red}. Conversely the preimage of a Lagrangian subspace of V_{red} by the quotient map $C \to C/C^{\perp}$ is a Lagrangian subspace of V. Note however that the preimage of E_{red} is not equal to E unless unless $C^{\perp} \subseteq E$.

Suppose now V_1 and V_2 are metrized vector spaces. A *Lagrangian relation* between V_1 and V_2 is a relation[1] $R \subseteq V_2 \times V_1$that is Lagrangian as a subspace of $V_2 \times \overline{V_1}$. One writes $v_1 \sim_R v_2$ to indicate that $(v_2, v_1) \in R$. Given subsets $S \subseteq V_1$ regarded as unitary relation, the composition of relations $R \circ S \subseteq V_2$ will be called the *forward image* of S by R. Likewise for a subset $S \subseteq V_2$, the composition of relations $S \circ R \subseteq V_1$ will be called the *backward image* of S by R. The following subspaces are also introduced

$$\begin{aligned} \ker(R) &= 0 \circ R, \\ \operatorname{ran}(R) &= R \circ V_1, \\ \ker^*(R) &= R \circ 0, \\ \operatorname{ran}^*(R) &= V_2 \circ R, \end{aligned}$$

where the notation was chosen for obvious reasons. Note that $\ker(R)$ is isotropic with coisotropic orthogonal $\operatorname{ran}^*(R)$, likewise for $\ker^*(R)$ and $\operatorname{ran}(R)$.

Definition 2.1.1. If $E_1 \subseteq R$ and $E_2 \subseteq V_2$ are Lagrangian subspaces, one will say that R is a Dirac relation from the linear Manin pair (V_1, E_1) to the linear Manin pair (V_2, E_2) if $E_1 \cap \ker(R) = 0$ and $R \circ E_1 = E_2$.

Note that a Dirac relation R induces a linear map $\varrho : E_2 \to E_1$ whose graph is contained in R.

Proposition 2.1.2 ([1]). *Let $E_1 \subseteq V_1$ and $E_2 \subseteq V_2$ be Lagrangian subspaces. Then*

(a) *The forward image $R \circ E_1 \subseteq V_2$ and backward image $E_2 \circ R \subseteq V_1$ are Lagrangian.*

[1]By convention a relation from A to B is a subset of $B \times A$. This order is chosen so that the composition of relations from left to right reads more naturally.

(b) *If R is a Dirac relation from (V_1, E_1) to (V_2, E_2) and F_2 is a Lagrangian complement of E_2, then $F_1 = F_2 \circ R$ is a Lagrangian complement of E_1. Furthermore, each element of F_1 is R-related to a unique element of F_2 and the induced map $F_1 \to F_2$ is dual to the induced map $\varrho : E_2 \to E_1$ after identifying $F_i = E_i^*$ via the metrics.*

Proof. (a) Regarding E_1 as a Lagrangian relation between 0 and V_1, then $R \circ E_1$ is the composition of Lagrangian relations and is therefore Lagrangian. Similarly for $E_1 \circ E$.

(b) The backward image $F_1 = F_2 \circ R$ is a Lagrangian subspace of V_1 by part (a). The intersection $E_1 \cap F_1$ must be trivial since an element of this intersection is at once only R-related to elements of E_2 and R-related to at least one element of F_2. Thus F_1 is a Lagrangian complement of E_1. Since $R \circ E_1 = E_2$, it follows that $\ker^*(R) \subseteq E_2$ and then that each element of F_1 is R-related to exactly one element of F_2. Finally, the last part of the statement follows from the fact that R is Lagrangian in $V_2 \times \overline{V_1}$. □

Say now, in the context of Proposition 2.1.2, that a Lagrangian complement $F \subseteq V_1$ (not necessarily the backward image of a Lagrangian subspace of V_2) of E_1 has been fixed, thereby identifying V with $E_1 \oplus E_1^*$ equipped with the metric induced from the natural pairing of E_1 and its dual. Then the backward image $F_1 = F_2 \circ R$ of a Lagrangian complement of E_2 can be regarded as a bivector $\pi^{F_2} \in \bigwedge^2 E_1$: its contraction with an element $\mu \in E_1^*$ is the necessarily unique element $v \in E_1$ such that[2] $v - \mu \in F_1$. Retain ϱ to denote its extension to a map of exterior algebras $\bigwedge^\bullet E_2 \to \bigwedge^\bullet E_1$.

Proposition 2.1.3 ([1]). *If F_2' is another Lagrangian complement of E_2, then*

$$\pi^{F_2'} = \pi^{F_2} - \varrho(t) \tag{2.1}$$

where $t \in \bigwedge^2 E_2$ is the twist defined by

$$t^\sharp : F_2 \simeq E_2^* \to E_2,\ v \mapsto \mathrm{pr}_{F_2'} v - v. \tag{2.2}$$

Proof. Let $\varrho^*_{F_2}$ and $\varrho^*_{F_2'}$ denote the maps $E_1^* \to E_2^*$ and $E_1^* \to E_2^*$ dual to ϱ, respectively. The duality of $\varrho^*_{F_2}$, alternatively $\varrho^*_{F_2'}$, and ϱ means that

$$\langle \varrho^*_{F_2}(\mu), v\rangle = \langle \varrho^*_{F_2'}(\mu), v\rangle = \langle \mu, \varrho(v)\rangle \implies \langle (\varrho^*_{F_2'} - \varrho^*_{F_2})(\mu), v\rangle = 0$$

for $\mu \in E_1^*$ and $v \in E_2$. As E_2 is Lagrangian, this implies that $(\varrho^*_{F_2'} - \varrho^*_{F_2})(\mu) \in E_2$, in other words $\varrho^*_{F_2'} = \mathrm{pr}_{F_2'} \circ \varrho^*_{F_2}$ so that one can write $\varrho^*_{F_2'}(\mu) = \varrho^*_{F_2}(\mu) + t^\sharp(\varrho^*_{F_2}(\mu))$. Note that

[2] The negative sign is due to the convention adopted in other literature, cf. [2].

$\varrho(t^\sharp(\varrho^*_{F_2}(\mu))) = \iota_\mu \varrho(t)$. Adding $(-\iota_\mu \pi^{F_2}, \mu) \sim_R \varrho^*_{F_2}(\mu)$ and $\iota_\mu \varrho(t) \sim_R t^\sharp(\varrho^*_{F_2}(\mu))$, one obtains

$$(-\iota_\mu(\pi^{F_2} - \varrho(t)), \mu) \sim_R \varrho^*_{F_2'}(\mu).$$

Finally, subtracting $(-\iota_\mu \pi^{F_2'}, \mu) \sim_R \varrho^*_{F_2'}(\mu)$ from the above gives

$$(-\iota_\mu(\pi^{F_2} - \varrho(t) - \pi^{F_2'}), 0) \sim_R 0.$$

However, the intersection $\ker(R) \cap E_1$, according to the definition of a Dirac structure, is trivial and therefore $\iota_\mu(\pi^{F_2} - \varrho(t) - \pi^{F_2'}) = 0$, i.e. (2.1) holds. □

2.2 Courant algebroids

2.2.1 Primitive notions

Consider the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$ of a finite-dimensional smooth manifold M. It comes with a natural fiber metric given by the pairing of TM and its dual, as well as the following bracket [24] on its sections:

$$[\![v_1 + \mu_1, v_2 + \mu_2]\!] = [v_1, v_2] + \mathcal{L}_{v_1}\mu_2 - \iota_{v_2} d\mu_1, \tag{2.3}$$

where $v_i \in \mathfrak{X}(M)$ and $\mu_i \in \Omega^1(M)$. Let $\mathbf{a} : \mathbb{T}M \to TM$ be the projection onto the first factor. Studying the properties of the structures on $\mathbb{T}M$ just introduced, the following definition is abstracted.

Definition 2.2.1 ([41])**.** A *Courant algebroid* is a vector bundle $\mathbb{A} \to M$ together with a non-degenerate fiber metric $\langle \cdot, \cdot \rangle$, a $\mathbb{R}$-bilinear bracket $[\![\cdot, \cdot]\!] : \Gamma(\mathbb{A}) \times \Gamma(\mathbb{A}) \to \Gamma(\mathbb{A})$, called the *Courant bracket*, and a bundle map $\mathbf{a} : \mathbb{A} \to TM$, called the *anchor*, such that for all sections $\sigma_1, \sigma_2, \sigma_3 \in \Gamma(\mathbb{A})$

C1. $[\![\sigma_1, [\![\sigma_2, \sigma_3]\!]]\!] = [\![[\![\sigma_1, \sigma_2]\!], \sigma_3]\!] + [\![\sigma_2, [\![\sigma_1, \sigma_3]\!]]\!]$,
C2. $\mathbf{a}(\sigma_3)\langle \sigma_1, \sigma_2 \rangle = \langle [\![\sigma_3, \sigma_1]\!], \sigma_2 \rangle + \langle \sigma_1, [\![\sigma_3, \sigma_2]\!] \rangle$,
C3. $\mathbf{a}^* d \langle \sigma_1, \sigma_2 \rangle = [\![\sigma_1, \sigma_2]\!] + [\![\sigma_2, \sigma_1]\!]$, where $\mathbb{A}^*$ has been identified with $\mathbb{A}$ via $\langle \cdot, \cdot \rangle$.

Properties 1-3 imply the further properties

C4. $\mathbf{a}([\![\sigma_1, \sigma_2]\!]) = [\mathbf{a}(\sigma_1), \mathbf{a}(\sigma_2)]$,
C5. $[\![\sigma_1, f\sigma_2]\!] = f[\![\sigma_1, \sigma_2]\!] + (\mathbf{a}(\sigma_1) f)\sigma_2$ where $f \in C^\infty(M)$.

The definition of a Courant algebroid first appeared in [41]. Properties C4 and C5 appear alongside C1-C3 in the definition, the redundancy seemingly not having been realized before some time [52].

Example 2.2.1. The first and quintessential example of a Courant algebroid is of course the generalized tangent bundle $\mathbb{T}M$ equipped with the above bracket and anchor. It will aptly be called the *standard* Courant algebroid of M from here on.

Example 2.2.2. More generally the bracket (2.3) can be twisted by a closed 3-form $\eta \in \Omega^3(M)$ to obtain a new bracket $[\![\cdot,\cdot]\!]_\eta$ on $\mathbb{T}M$:

$$[\![v_1 + \mu_1, v_2 + \mu_2]\!]_\eta = [v_1, v_2] + \mathcal{L}_{v_1}\mu_2 - \iota_{v_2} d\mu_1 + \iota_{v_2}\iota_{v_1}\eta \tag{2.4}$$

where $x_i \in \mathfrak{X}^1(M)$ and $\mu_i \in \Omega^1(M)$. With the fiber metric and anchor otherwise being the same, the bracket $[\![\cdot,\cdot]\!]_\eta$ determines a new Courant algebroid structure on $\mathbb{T}M$ [56]. It will be called the *η-twisted standard* Courant algebroid of M and denoted by $\mathbb{T}M_\eta$. Of course when $\eta = 0$ this corresponds to the previous example and η is omitted.

Conversely, if (2.4) is a Courant bracket for some 3-form $\eta \in \Omega^3(M)$ then η is necessarily closed by virtue of the equation

$$[\![\sigma_1, [\![\sigma_2, \sigma_3]\!]_\eta]\!]_\eta + \iota_{\mathbf{a}(\sigma_3)}\iota_{\mathbf{a}(\sigma_2)}\iota_{\mathbf{a}(\sigma_1)} d\eta = [\![[\![\sigma_1, \sigma_2]\!]_\eta, \sigma_3]\!]_\eta + [\![\sigma_2, [\![\sigma_1, \sigma_3]\!]_\eta]\!]_\eta. \tag{2.5}$$

Example 2.2.3. Let $\mathfrak{g}$ be a quadratic Lie algebra, i.e. $\mathfrak{g}$ is equipped with an Ad-invariant metric. Suppose $\varrho : M \times \mathfrak{g} \to TM$ is a $\mathfrak{g}$-action on the manifold M. If the stabilizer algebras $\mathfrak{g}_m = \{\gamma \in \mathfrak{g} : \varrho(x, \gamma) = 0\}$ are all coisotropic, then $M \times \mathfrak{g}$ possesses the structure of a Courant algebroid with anchor $\mathbf{a} = \varrho$ and Courant bracket the unique one extending the Lie bracket of $\mathfrak{g}$ identified with the constant sections [37]. The product $M \times \mathfrak{g}$ is called an *action Courant algebroid*.

Example 2.2.4 (Standard constructions). For a Courant algebroid $\mathbb{A}$, another Courant algebroid $\overline{\mathbb{A}}$ is obtained by reversing the sign of the metric. The direct product $\mathbb{A} = \mathbb{A}_1 \times \mathbb{A}_2$ of two Courant algebroids $\mathbb{A}_1 \to M_1$ and $\mathbb{A}_2 \to M_2$ is also a Courant algebroid with its anchor and metric being the respective direct products of those of $\mathbb{A}_1$ and $\mathbb{A}_2$. Its Courant bracket $[\![\cdot,\cdot]\!]$ is defined as the unique one which restricts to the Courant bracket of $\mathbb{A}_i$ on $\Gamma(\mathbb{A}_i)$ and such that $[\![\sigma, \tau]\!] = 0$ whenever $\sigma \in \Gamma(\mathbb{A}_1)$ and $\tau \in \Gamma(\mathbb{A}_2)$. Its existence can be established locally; given a local frame $\sigma_1, \ldots, \sigma_k$ of $\mathbb{A}_1$ and a local frame $\tau_1, \ldots, \tau_l$ of $\mathbb{A}_2$, let $\epsilon_1, \ldots, \epsilon_{kl}$ be the resulting local frame for $\mathbb{A}_1 \times \mathbb{A}_2$. The brackets $[\![\epsilon_i, \epsilon_j]\!]$ are all determined and Property C5 forces

$$[\![\sum_i f_i\epsilon_i, \sum_j g_j\epsilon_j]\!] = \sum_{i,j}(f_i(\mathbf{a}(\epsilon_i)g_j)\epsilon_j - g_j(\mathbf{a}(\epsilon_j)f_i)\epsilon_i + f_ig_j[\![\epsilon_i,\epsilon_j]\!])$$

for functions $f_i, g_j \in C^\infty(M_1 \times M_2)$ and this completely determines $[\![\cdot,\cdot]\!]$. Properties C1-C3 are then easily verified.

2.2.2 Involutive subbundles and Dirac structures

A subbundle $V \to S$ of a Courant algebroid $\mathbb{A} \to M$ (or more generally a collection $V \subseteq \mathbb{A}$ consisting of a choice of subspace in each fiber of $\mathbb{A}$ over a base manifold $S \subseteq M$) is called *isotropic*, *coisotropic* or *Lagrangian* provided its fibers (the subspaces constituting V) possess those respective properties. Let $\Gamma(\mathbb{A}, V)$ denote the space of sections $\sigma \in \Gamma(\mathbb{A})$ that take values in V on its base manifold S. Then V is called *involutive* if $\Gamma(\mathbb{A}, V)$ is closed under the Courant bracket $[\![\cdot,\cdot]\!]$ of $\mathbb{A}$ and $\mathbf{a}(V) \subseteq TS$. Note that, by C4, in this case $\mathbf{a}(V)$ is an involutive singular distribution of S in the sense of Stefan-Suessmann and thus V defines a singular foliation on S.

Example 2.2.5. Given a Lie algebra $\mathfrak{g}$ acting on a manifold M, one can form the action Lie algebroid $M \times \mathfrak{g}$ (not to be confused with an action Courant algebroid). The bracket $[\cdot,\cdot]_{\mathfrak{X}}$ on $C^\infty(M,\mathfrak{g})$ is the unique one extending the Lie bracket of $\mathfrak{g}$ identified with the constant sections of $M \times \mathfrak{g}$ and its anchor is the $\mathfrak{g}$-action $M \times \mathfrak{g} \to TM$.

Suppose now $\mathfrak{g}$ is quadratic and acts on M with coisotropic stabilizers. Then $\mathfrak{g}$ also defines a Courant bracket $[\![\cdot,\cdot]\!]$ on $C^\infty(M,\mathfrak{g})$. The brackets $[\cdot,\cdot]$ and $[\![\cdot,\cdot]\!]$ are related by [37, Lemma 4.1]

$$[\![\sigma_1,\sigma_2]\!] = [\sigma_1,\sigma_2]_{\mathfrak{X}} + \mathbf{a}^*\langle d\sigma_1,\sigma_2\rangle.$$

For a subspace $\mathfrak{s} \subseteq \mathfrak{g}$, let $E^{(\mathfrak{s})} = M \times \mathfrak{s}$. By virtue of the above equation, the subbundle $E^{(\mathfrak{s})}$ is involutive if $\mathfrak{s}$ is an isotropic subalgebra of $\mathfrak{g}$. Note that $E^{(\mathfrak{s})}$ is isotropic, coisotropic or Lagrangian if and only if $\mathfrak{s}$ possesses those respective properties.

Example 2.2.6. The kernel $\ker(\mathbf{a}) \subseteq \mathbb{A}$ of the anchor $\mathbf{a} : \mathbb{A} \to TM$ of a Courant algebroid $\mathbb{A}$ is involutive in view of C4. By C3 and C4, the composition $\mathbf{a} \circ \mathbf{a}^*$ is trivial, in other words $\mathrm{ran}(\mathbf{a}^*) \subseteq \ker(\mathbf{a})$. As $\langle \mathbf{a}^*\mu, x\rangle = \langle \mu, \mathbf{a}(x)\rangle$, one sees at once that $\ker(\mathbf{a})$ is coisotropic with $\mathrm{ran}(\mathbf{a}^*)$ as its isotropic complement. Thus $\ker(\mathbf{a})$ is an involutive coisotropic singular distribution of $\mathbb{A}$.

Remark 2.2.1. The definition of involutivity given here is more restrictive than the one already found in the literature [39, 57]; there it is not required that $\mathbf{a}(V) \subseteq TS$.

Proposition 2.2.1. *Let $V \subseteq \mathbb{A}$ be an involutive subbundle along $S \subseteq M$. Then:*

(a) [57] *If V is not isotropic then it contains* $\mathrm{ran}(\mathbf{a}^*|_S)$.

(b) *If V is coisotropic and $S' \hookrightarrow S$ is a submanifold of S that is tangent to the singular distribution $\mathbf{a}(V) \subseteq TS$, i.e. $\mathbf{a}(V|_{S'}) \subseteq TS'$, then $V' = V|_{S'}$ is also involutive.*

(c) *Suppose $V' \subseteq \mathbb{A}$ is an involutive subbundle with base manifold S' such that the intersection $V \cap V'$ is clean, i.e. $V \cap V'$ is a submanifold of $\mathbb{A}$ and $T(V \cap V') = TV \cap TV'$. Then the intersection $S \cap S'$ is clean and $V \cap V'$ is an involutive subbundle as well.*

(d) [37] *If V is isotropic and S is open then V inherits the structure of a Lie algebroid*[3].

(e) [12] *If V is coisotropic then $V^\perp$ is also involutive and is an "ideal" of V in the sense that the bracket of a section in $\Gamma(\mathbb{A}, V^\perp)$ and one in $\Gamma(\mathbb{A}, V)$ (in either order) is in $\Gamma(\mathbb{A}, V^\perp)$. Moreover, the quotient $V/V^\perp$ inherits the structure of a Courant algebroid and if $E \subseteq \mathbb{A}$ is a Lagrangian involutive subbundle of $\mathbb{A}$ transversal to V, then $(E \cap V)/(E \cap V^\perp)$ is a Lagrangian involutive singular distribution of $V/V^\perp$.*

Proof. (a) Suppose V is not isotropic. Then the restriction of the metric to V has non-trivial signature and thus, at least near any point in S, there is a section $\sigma \in \Gamma(\mathbb{A}, V)$ with $\langle \sigma, \sigma \rangle \neq 0$ along S. By C3 one has

$$f\mathbf{a}^* d\langle \sigma, \sigma \rangle + \langle \sigma, \sigma \rangle \mathbf{a}^* df = [\![f\sigma, \sigma]\!] + [\![\sigma, f\sigma]\!]$$

for an arbitrary function $f \in C^\infty(M)$. From the special case where f vanishes along S, one sees that $\langle \sigma, \sigma \rangle \mathbf{a}^* df \in \Gamma(\mathbb{A}, V)$, or equivalently $\mathbf{a}^* df \in \Gamma(\mathbb{A}, V)$. As f was arbitrary, this means that $\mathrm{ran}(\mathbf{a}^*)|_S \subseteq V$.

(b) For $i = 1, 2$, let $\sigma_i' \in \Gamma(\mathbb{A}, V')$ and let $\sigma_i \in \Gamma(\mathbb{A}, V)$ be sections coinciding with σ_i' on S'. One knows that $[\![\sigma_1, \sigma_2]\!] \subseteq \Gamma(\mathbb{A}, V)$ and would like to show $[\![\sigma_1', \sigma_2']\!] \in \Gamma(\mathbb{A}, V')$. Since $[\![\sigma_1, \sigma_2]\!] - [\![\sigma_1', \sigma_2']\!] = [\![\sigma_1, \sigma_2 - \sigma_2']\!] + [\![\sigma_1 - \sigma_1', \sigma_2']\!]$, it is sufficient to show that $[\![\sigma_1, f\tau]\!] \in \Gamma(\mathbb{A}, V')$ and $[\![f\tau, \sigma_2']\!] \in \Gamma(\mathbb{A}, V')$ for any section $\tau \in \Gamma(\mathbb{A})$ and function $f \in C^\infty(M)$ vanishing along S'. From C5 and the assumption $\mathbf{a}(V') \subseteq TS'$ it follows that $[\![\sigma_1, f\tau]\!]$ vanishes along S'. On the other hand, from C3 one has $[\![f\tau, \sigma_2']\!] = -[\![\sigma_2', f\tau]\!] + \mathbf{a}^* d\langle f\tau, \sigma_2' \rangle$, which is equal to $\langle \tau, \sigma_2' \rangle \mathbf{a}^* df$ along S'. Then for any $\sigma' \in \Gamma(\mathbb{A}, V')$ one has $\langle \mathbf{a}^* df, \sigma' \rangle = \langle df, \mathbf{a}(\sigma') \rangle = 0$ along S'. As V' is also coisotropic, it follows that $[\![f\tau, \sigma_2']\!] \in \Gamma(\mathbb{A}, V')$. One thus concludes that V' is indeed involutive.

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[3] Recall that a Lie algebroid is a vector bundle $A \to Q$ together with a Lie bracket on its space of sections $\Gamma(A)$ and a bundle map $\mathbf{a} : A \to TQ$ satisfying the properties analogous to C4 and C5. See for instance [46] for the general theory of Lie algebroids along with that of their integral counterparts, Lie groupoids.

(c) A result of Grabowski and Rotkievicz [28] states that a submanifold of a vector bundle is a vector subbundle if and only if it is closed under scalar multiplication. In particular, the intersection $V \cap V'$ is a vector subbundle of $\mathbb{A}$ and thus its base $S \cap S'$ is a submanifold of M. Seeing S, S' and M as the zero-sections of V, V' and $\mathbb{A}$ respectively, one has

$$T(S \cap S') = T(V \cap V') \cap TM = TV \cap TV' \cap TM = TS \cap TS',$$

and thus S and S' intersect cleanly. This shows the first part of the claim.

Now as V and V' intersect cleanly, the map $\Gamma(\mathbb{A}, V) \cap \Gamma(\mathbb{A}, V') \to \Gamma(\mathbb{A}, V \cap V')$ is surjective[4]. The involutivity of $V \cap V'$ then immediately follows from that of V and V'.

(d) Let $\sigma_1, \sigma_2 \in \Gamma(\mathbb{A}, V)$. Since V is isotropic and S is open, C3 implies that $[\![\sigma_1, \sigma_2]\!] = -[\![\sigma_2, \sigma_1]\!]$ along S. Now, similarly to the proof of (a), if $f \in C^\infty(M)$ vanishes along S then $[\![\sigma_1, f\sigma_2]\!]$ also vanishes along S. It follows that $[\![\cdot, \cdot]\!]$ restricts to an anti-symmetric bracket on $\Gamma(V)$. This last bracket along with the restriction of the anchor $\mathbf{a}|_V : V \to TS$ turn V into a Lie algebroid by virtue of C1, C4 and C5.

(e) If $V^\perp = V$ there is nothing to prove and it is therefore assumed that they are not equal. One knows that $V/V^\perp$ inherits a metric. Furthermore, by part (a), one has $\mathrm{ran}(\mathbf{a}^*)|_S \subseteq V$ or equivalently $V^\perp \subseteq \ker(\mathbf{a})|_S$. In particular, the anchor $\mathbf{a}$ descends to a map $\mathbf{a}_{\mathrm{red}} : V/V^\perp \to TS$.

As part of the proof of the statement, it will be shown that $[\![\cdot, \cdot]\!]$ descends to a bracket $[\![\cdot, \cdot]\!]_{\mathrm{red}}$ on $\Gamma(V/V^\perp)$. This entails showing that for $\sigma \in \Gamma(\mathbb{A}, V^\perp)$ and $\tau, \tau_1, \tau_2 \in \Gamma(\mathbb{A}, V)$ that (1) $[\![\sigma, \tau]\!], [\![\tau, \sigma]\!] \in \Gamma(\mathbb{A}, V^\perp)$ and in particular that $V^\perp$ is involutive and an ideal of V; and (2) $[\![\tau_1, f\tau_2]\!], [\![f\tau_1, \tau_2]\!] \in \Gamma(\mathbb{A}, V^\perp)$ whenever $f \in C^\infty(M)$ is a function vanishing on S.

(1) That $[\![\tau, \sigma]\!] \in \Gamma(\mathbb{A}, V^\perp)$ follows from C2. Now

$$\langle \mathbf{a}^* d\langle \tau, \sigma \rangle, \tau_1 \rangle = \langle d\langle \tau, \sigma\rangle, \mathbf{a}(\tau_1)\rangle = 0$$

along S since $\langle \tau, \sigma \rangle|_S = 0$. Thus $\mathbf{a}^* d\langle \tau, \sigma\rangle \in \Gamma(\mathbb{A}, V)$, so C3 gives $[\![\sigma, \tau]\!] \in \Gamma(\mathbb{A}, V^\perp)$. Thus $V^\perp$ is involutive and an ideal of V.

(2) Similarly as in part (a), the claim $[\![\tau_1, f\tau_2]\!] \in \Gamma(\mathbb{A}, V^\perp)$ is trivial and it suffices to argue that $\mathbf{a}^* df \in \Gamma(\mathbb{A}, V^\perp)$. This is obviously the case since $\mathbf{a}(\tau_3) f = 0$ for any $\tau_3 \in \Gamma(\mathbb{A}, V)$.

Taking $V/V^\perp$ with its metric, the map $\mathbf{a}_{\mathrm{red}}$ as its anchor and $[\![\cdot, \cdot]\!]_{\mathrm{red}}$ as its bracket, it is readily seen that C1-C3 are inherited from their counterparts for $\mathbb{A}$. Finally, suppose

[4] This follows from the fact that around any point in $V \cap V'$ local coordinates exist in which V and V' are linear subspaces, see e.g. [30].

E is a Lagrangian involutive subbundle of $\mathbb{A}$. Then the quotient $(E \cap V)/(E \cap V^{\perp})$ is fiberwise Lagrangian as observed in Section 2.1. It represents a subbundle of $V/V^{\perp}$ provided E is transversal to V and thus a fortiori to $V^{\perp}$. In this case, its involutivity is guaranteed by that of $E \cap V$, which is a subbundle of $\mathbb{A}$ by part (c). □

Example 2.2.7. Let $\mathfrak{g}$ and M be as in Example 2.2.3. Suppose $\mathfrak{s} \subseteq \mathfrak{g}$ is an isotropic subalgebra. Then $E^{(\mathfrak{s})} \subseteq M \times \mathfrak{g}$ inherits the structure of a Lie algebroid according to part (c) above. Its bracket is the restriction of the bracket Courant bracket $[\![\cdot,\cdot]\!]$ to $E^{(\mathfrak{s})}$. By definition $[\![\cdot,\cdot]\!]$ coincides with the Lie bracket of $\mathfrak{s}$ embedded in $E^{(\mathfrak{s})}$ as the constant sections. That is to say $[\![\cdot,\cdot]\!]$ extends the Lie bracket of $\mathfrak{s}$ to a bracket on $\Gamma(E^{(\mathfrak{s})})$, turning $E^{(\mathfrak{s})}$, along with the $\mathfrak{s}$-action $E^{(\mathfrak{s})} \to M$ as its anchor, into a Lie algebroid. In other words, the Lie algebroid structure of $E^{(\mathfrak{s})}$ is that of the action Lie algebroid $M \times \mathfrak{s}$.

On the other hand, suppose $\mathfrak{s} \subseteq \mathfrak{g}$ is coisotropic. Then $\mathfrak{s}^{\perp}$ is isotropic and $[\mathfrak{s}^{\perp}, \mathfrak{s}] \subseteq \mathfrak{s}^{\perp}$.

According to the previous proposition, involutive Lagrangian subbundles of $\mathbb{A}$ are maximal among the Lie algebroids naturally associated to $\mathbb{A}$. It stands to reason that they deserve special interest. As will be appreciated in the sequel, they constitute the foundation of Dirac geometry.

Definition 2.2.2 ([53, 59])**.** A *Dirac structure supported on* S is an involutive Lagrangian subbundle $E \subseteq \mathbb{A}|_S$. When $S = M$, it is simply called a *Dirac structure* and the pair $(\mathbb{A}, E)$ is called a *Manin pair*.

If $E \to S$ is a Lagrangian subbundle of a Courant algebroid $\mathbb{A}$ satisfying $\mathbf{a}(E) \subseteq TS$, the degree to which it fails to be involutive may be measured by its so-called *Courant tensor* $\Upsilon^E \in \Gamma(\bigwedge^3 E^*)$

$$\Upsilon^E(\sigma_1, \sigma_2, \sigma_3) = \langle [\![\widetilde{\sigma}_1, \widetilde{\sigma}_2]\!], \widetilde{\sigma}_3 \rangle|_S, \tag{2.6}$$

where $\sigma_i \in \Gamma(E)$ and $\widetilde{\sigma}_i \in \Gamma(\mathbb{A}, E)$ extends σ_i. One verifies with C1-C3 that Υ^E is well-defined and ipso facto tensorial in its arguments. Note that Υ^E vanishes if and only if E is a Dirac structure supported on S.

Example 2.2.8. Suppose $\mathbb{A} = \mathbb{T}M_\eta$. Given a 2-form $\omega \in \Omega^2(M)$, its graph $\mathrm{Gr}(\omega)$ is Lagrangian in $\mathbb{A}$ and a direct computation shows that $\Upsilon^{\mathrm{Gr}(\omega)} \in \Omega^3(M)$ (with the identification $\mathrm{Gr}(\omega) \simeq TM$) is $d\omega - \eta$. On the other hand, if $\pi \in \mathfrak{X}^2(M)$ is an anti-symmetric bivector field then $\Upsilon^{\mathrm{Gr}(\pi)} \in \mathfrak{X}^3(M)$ (with the identification $\mathrm{Gr}(\pi) \simeq T^*M$) is $\frac{1}{2}[\pi,\pi] - \bigwedge^3 \pi^\sharp(\eta)$ [56], where $[\cdot,\cdot]$ is the Schouten-Nijenhuis bracket.

Example 2.2.9. If $E \subseteq \mathbb{A}$ is a Dirac structure then its restriction to a submanifold $S \hookrightarrow M$ tangent to the generalized distribution $\mathbf{a}(E) \subseteq TM$ is a Dirac structure supported on S.

However, not all Dirac structures supported on a submanifold arise in this way. For instance, consider $\mathbb{A}$ included in $\mathbb{A} \times \overline{\mathbb{A}}$ diagonally. Then $\mathbb{A}$ is a Dirac structure of $\mathbb{A} \times \overline{\mathbb{A}}$ supported on the diagonal of $M \times M$ and the Courant bracket of $\mathbb{A} \times \overline{\mathbb{A}}$ has a well-defined restriction to $\Gamma(\mathbb{A})$, which is just the Courant bracket of $\mathbb{A}$. As such, the Courant algebroid $\mathbb{A}$ cannot be the restriction of a Dirac structure of $\mathbb{A} \times \overline{\mathbb{A}}$ as that would mean its bracket is anti-symmetric, which in view of C3 is never the case.

2.2.3 Exact Courant algebroids

A Courant algebroid is called *exact* [51, 56] if the sequence

$$0 \longrightarrow T^*M \xrightarrow{\mathbf{a}^*} \mathbb{A} \xrightarrow{\mathbf{a}} TM \longrightarrow 0 \tag{2.7}$$

is exact. In this case $\mathrm{ran}(\mathbf{a}^*) = \ker(\mathbf{a})$ is a Dirac structure of $\mathbb{A}$ (Example 2.2.6). According to Proposition 2.1.1, there exists a splitting $j : TM \to \mathbb{A}$ such that $\mathrm{ran}(j)$ is a Lagrangian complement of $\ker(\mathbf{a}) = \mathrm{ran}(\mathbf{a}^*)$. Such a splitting will be called *isotropic*. The map $\mathbf{a} \times j^*$ identifies $\mathbb{A} \simeq \mathbb{T}M$ as metrized vector bundles but *not* as Courant algebroids. The curvature tensor $\eta \in \Omega^3(M)$

$$\iota_{v_2}\iota_{v_1}\eta = [\![j(v_1), j(v_2)]\!] - [v_1, v_2],$$

where $v_1, v_2 \in \mathfrak{X}(M)$ and $[\![\cdot,\cdot]\!]$ is the Courant bracket of $\mathbb{A}$, is then introduced to measure the discrepancy between the Courant algebroid structures of $\mathbb{A}$ and $\mathbb{T}M$. One has

$$\iota_{v_3}\iota_{v_2}\iota_{v_1}\eta = \langle [\![j(v_1), j(v_2)]\!], j(v_3)\rangle \tag{2.8}$$

where $v_1, v_2, v_3 \in \mathfrak{X}(M)$, in other words η is the Courant tensor $\Upsilon^{\mathrm{ran}(j)}$ whence its tensoriality. As $[\![v_1, v_2]\!] = [v_1, v_2] + \iota_{v_2}\iota_{v_1}\eta$, it may be concluded that $[\![\cdot,\cdot]\!]$ has the form (2.4). Thus η is closed and $\mathbb{A} \simeq \mathbb{T}M_\eta$ as *Courant algebroids*.

Suppose now $j' : TM \to \mathbb{A}$ is another isotropic splitting with corresponding 3-form η'. With $\mathbb{A}$ identified with $\mathbb{T}M_{\eta'}$ via j', the image $\mathrm{ran}(j)$ is the graph of a 2-form $\varpi \in \Omega^2(M)$, i.e.

$$j(v) = j'(v) + \mathbf{a}^*(\iota_v\varpi). \tag{2.9}$$

Conversely, the image $\mathrm{ran}(j')$ in $\mathbb{A} \simeq \mathbb{T}M_\eta$ is the graph of $-\varpi$. In practice, the 2-form ϖ is computed by noticing that

$$\iota_{v_2}\iota_{v_1}\varpi = \langle \mathbf{a}^*(\iota_{v_1}\varpi), j(v_2)\rangle = -\langle j'(v_1), j(v_2)\rangle = \langle j(v_1), j'(v_2)\rangle. \tag{2.10}$$

The corresponding 3-forms on the other hand are related by

$$\eta = \eta' + d\varpi. \tag{2.11}$$

Conversely, a 2-form $\varpi \in \Omega^2(M)$ defines an isotropic splitting $j' : TM \to \mathbb{A}$ via (2.9). The space of isotropic splittings of $\mathbb{A}$ is therefore an affine space modelled on $\Omega^2(M)$ and the space of corresponding 3-forms η is a cohomology class in $H^3(M)$.

2.3 Morphisms of Courant algebroids

2.3.1 General morphisms

A morphism R of Courant algebroids is a fiberwise Lagrangian relation compatible with the Courant brackets.

Definition 2.3.1 ([1, 15])**.** A *Courant morphism* from $\mathbb{A}_1$ to $\mathbb{A}_2$, denoted $R : \mathbb{A}_1 \dashrightarrow \mathbb{A}_2$, is a Dirac structure of $\mathbb{A}_2 \times \overline{\mathbb{A}_1}$ supported on the graph $\mathrm{Gr}(f) \subseteq M_2 \times M_1$ of a smooth base map $f : M_1 \to M_2$.

Example 2.3.1. Let $f : M_1 \to M_2$ be a smooth map. Define a relation $R \subseteq \mathbb{T}M_2 \times \overline{\mathbb{T}M_1}$ by

$$v_1 + \mu_1 \sim_R v_2 + \mu_2 \iff v_2 = f_* v_1,\ \mu_1 = f^*\mu_2,$$

for $v_i \in TM_i$ and $\mu_i \in T^*M_i$. Then R is a Courant morphism $R : \mathbb{T}M_1 \dashrightarrow \mathbb{T}M_2$. This an example of an exact Courant morphism, see Section 2.3.3.

Example 2.3.2. Suppose $\mathfrak{g}_i$ ($i = 1, 2$) are quadratic Lie algebras acting on a manifolds M_i thus giving action Courant algebroids $\mathbb{A}_1 = M_1 \times \mathfrak{g}_1$ and $\mathbb{A}_2 = M_2 \times \mathfrak{g}_2$. Suppose further that $R \subseteq \mathfrak{g}_2 \times \overline{\mathfrak{g}_1}$ is a Lagrangian subalgebra and that $f : M_1 \to M_2$ is a smooth map such that $x_1 \sim_R x_2 \implies f_*\mathbf{a}(x_1) = \mathbf{a}(x_2)$ where $x_i \in \mathbb{A}_i$. Then $\mathrm{Gr}(f) \times R \subseteq \mathbb{A}_2 \times \overline{\mathbb{A}_1}$ is a Courant morphism $\mathbb{A}_1 \dashrightarrow \mathbb{A}_2$ [37, Prop. 4.7], which will also be denoted by R.

The choice of a dotted arrow as opposed to a solid one reflects the fact that a Courant morphism R is merely a relation and not generally a function. The notation introduced in Section 2.1 is reinterpreted in the obvious way so that one may speak of forward images $R \circ S_1 \subseteq f^*\mathbb{A}_2$ (in the pullback bundle of $\mathbb{A}_2$), backward images $S_2 \circ R \subseteq \mathbb{A}_1$ as well as singular distributions $\ker(R), \mathrm{ran}^*(R) \subseteq \mathbb{A}_1$ and $\ker^*(R), \mathrm{ran}(R) \subseteq f^*\mathbb{A}_2$.

If $R' : \mathbb{A}_2 \dashrightarrow \mathbb{A}_3$ is another morphism of Courant algebroids, by part (e) of Proposition 2.2.1 composing R and R' as relations gives a Courant morphism $R' \circ R : \mathbb{A}_1 \dashrightarrow \mathbb{A}_3$ provided

$R' \times R \subseteq \mathbb{A}_3 \times \overline{\mathbb{A}_2} \times \mathbb{A}_2 \times \overline{\mathbb{A}_1}$ is transversal to $0 \times \mathbb{A}_{2,\Delta} \times 0$ where $\mathbb{A}_{2,\Delta}$ is the diagonal of $\mathbb{A}_2$ since

$$R' \circ R = \frac{(R' \times R) \cap (\mathbb{A}_3 \times \mathbb{A}_{2,\Delta} \times \overline{\mathbb{A}_1})}{(R' \times R) \cap (0 \times \mathbb{A}_{2,\Delta} \times 0)}.$$

Example 2.3.3. A Dirac structure $E \subseteq \mathbb{A}$ is equivalent to a Courant morphism $E : \mathbb{A} \dashrightarrow 0$. In particular if $R : \mathbb{A}' \dashrightarrow \mathbb{A}$ is a Courant morphism then the backward image $E \circ R$ is a Dirac structure of $\mathbb{A}'$ provided the composition is transverse.

The Courant morphism $R : \mathbb{A}_1 \dashrightarrow \mathbb{A}_2$ will be called a *Courant isomorphism* if its base map f is a diffeomorphism. In that case, one can also consider R as a Dirac structure of $\mathbb{A}_1 \times \overline{\mathbb{A}_1}$ supported on $\mathrm{Gr}(f^{-1})$, in which case it is denoted by R^{-1} and called the *inverse* of R. Note that $R \circ R^{-1} = \mathrm{Id}_{\mathbb{A}_1}$ and $R^{-1} \circ R = \mathrm{Id}_{\mathbb{A}_2}$ where $\mathrm{Id}_{\mathbb{A}_i}$ are the identity morphisms.

2.3.2 Dirac morphisms

For Manin pairs $(\mathbb{A}_i, E_i)$, $i = 1, 2$, one is led to consider morphisms $\mathbb{A}_1 \dashrightarrow \mathbb{A}_2$ that relate E_1 and E_2 in some way.

Definition 2.3.2. A *Dirac morphism* from $(\mathbb{A}_1, E_1)$ to $(\mathbb{A}_2, E_2)$ is a Courant morphism $R : \mathbb{A}_1 \dashrightarrow \mathbb{A}_2$ such that, for every $m \in M_1$, each element in $E_2|_{f(m)}$ is R-related to a unique element of $E_1|_m$. The notation $R : (\mathbb{A}_1, E_1) \dashrightarrow (\mathbb{A}_2, E_2)$ is used.

In other words, the morphism $R : (\mathbb{A}_1, E_1) \dashrightarrow (\mathbb{A}_2, E_2)$ is a Dirac morphism provided the pullback bundle f^*E_2 is equal to the forward image $R \circ E_1$ and the intersection $E_1 \cap \ker(R)$ is trivial. Analogously to the case of linear Dirac morphisms, if $F_2 \subseteq \mathbb{A}_2$ is a Lagrangian complement of E_1 then the backward image $F_1 = F_2 \circ R$ is a Lagrangian complement of E_1. Furthermore, the Dirac morphism R induces a bundle map

$$\varrho : f^*E_2 \to E_1 \tag{2.12}$$

and, in the presence of the Lagrangian complement F_2, a bundle map

$$\varrho^*_{F_2} : F_1 \to f^*F_2 \tag{2.13}$$

dual to ϱ according to Proposition 2.1.2.

The composition of Dirac morphisms is categorical; for suppose $R : (\mathbb{A}_1, E_1) \dashrightarrow (\mathbb{A}_2, E_2)$ and $R' : (\mathbb{A}_2, E_2) \dashrightarrow (\mathbb{A}_3, E_3)$ are Dirac morphisms. The equality $(f' \circ f)^*E_3 = R' \circ R \circ E_1$ is immediate. If $x \in E_1 \cap \ker(R' \circ R)$, then $x \sim_R y \sim_{R'} 0$ for some $y \in E_2$ and since $E_2 \cap \ker(R') = 0$ and $E_1 \cap \ker(R) = 0$ it follows that $y = 0$ and in turn $x = 0$. Thus $R' \circ R : (\mathbb{A}_1, E_1) \dashrightarrow (\mathbb{A}_2, E_3)$ is a Dirac morphism.

Lemma 2.3.1 ([1]). *Suppose $R : (\mathbb{A}_1, E_1) \dashrightarrow (\mathbb{A}_2, E_2)$ is a Dirac morphism and that $F_2 \subseteq \mathbb{A}_2$ is a Lagrangian complement of E_2 so that the backward image $F_1 = F_2 \circ R$ is a Lagrangian complement according to Proposition 2.1.2. Then the Courant tensors $\Upsilon^{F_1} \in \Gamma(\bigwedge^3 E_1)$ and $\Upsilon^{F_2} \in \Gamma(\bigwedge^3 E_2)$ are related by*

$$\Upsilon^{F_1} = \varrho(f^* \Upsilon^{F_2}), \tag{2.14}$$

where ϱ has been retained to denote the extension of the bundle map (2.12) to a map of exterior algebras $\bigwedge^\bullet f^* E_2 \to \bigwedge^\bullet E_1$.

Proof. Let σ_i $(i = 1, 2, 3)$ be sections in $\Gamma(F_2 \times F_1) \cap \Gamma(\mathbb{A}_2 \times \overline{\mathbb{A}_1}, R)$. For $m \in M_1$, the value of σ_i at $(f(m), m)$ is $(\varrho^*_{F_2}(x_i), x_i)$ for some $x_i \in F_1|_m$. Now

$$\begin{aligned} \varrho(\Upsilon^{F_2})(x_1, x_2, x_3)|_m - \Upsilon^{F_1}(x_1, x_2, x_3)|_m &= \Upsilon^{F_2}(\varrho^*_{F_2}(x_1), \varrho^*_{F_2}(x_2), \varrho^*_{F_2}(x_3))|_{f(m)} - \\ &\quad \Upsilon^{F_1}(x_1, x_2, x_3)|_m \\ &= \Upsilon^{F_2 \times F_1}(\sigma_1, \sigma_2, \sigma_3)|_{(f(m),m)} \\ &= \langle [\![\sigma_1, \sigma_2]\!], \sigma_3 \rangle|_{(f(m),m)}, \end{aligned}$$

where the duality of ϱ and $\varrho^*_{F_2}$ was used in the first equality and the fact that $\Upsilon^{F_2 \times F_1}$ is the direct sum $\Upsilon^{F_2} \oplus \Upsilon^{F_1}$ in the second. On the other hand, the Courant tensor of R vanishes and thus

$$\begin{aligned} 0 &= \Upsilon^R(\sigma_1, \sigma_2, \sigma_3)|_{(f(m),m)} \\ &= \langle [\![\sigma_1, \sigma_2]\!], \sigma_3 \rangle|_{(f(m),m)}. \end{aligned}$$

The upshot is that

$$\varrho(\Upsilon^{F_2})(x_1, x_2, x_3) - \Upsilon^{F_1}(x_1, x_2, x_3) = 0,$$

which is what was needed. □

2.3.3 Exact morphisms

Suppose now $\mathbb{A}_1$ and $\mathbb{A}_2$ are exact and suppose $R : \mathbb{A}_1 \dashrightarrow \mathbb{A}_2$ is a morphism with base map $f : M_1 \to M_2$. Let $\mathrm{Gr}(f_*) \subseteq TM_2 \times TM_1$ and $\mathrm{Gr}(f^*) \subseteq T^*M_2 \times T^*M_1$ be the relations defined by the tangent map f_* and the cotangent map f^* of f respectively. The image $\mathbf{a}^*(\mathrm{Gr}(f^*))$, where $\mathbf{a} = \mathbf{a}_2 \times \mathbf{a}_1$, is orthogonal to the preimage $\mathbf{a}^{-1}(\mathrm{Gr}(f_*))$ in $\mathbb{A}_2 \times \overline{\mathbb{A}_1}$.

The former is thus contained in R. The morphism R will be called *exact* [51] if the sequence

$$0 \longrightarrow \mathrm{Gr}(f^*) \xrightarrow{\mathbf{a}^*} R \xrightarrow{\mathbf{a}} \mathrm{Gr}(f_*) \longrightarrow 0 \tag{2.15}$$

is exact. In fact, only exactness at $\mathrm{Gr}(f_*)$ must hold for the above sequence to be exact since $\mathbf{a}^*$ is injective and, by a dimension count, one has $\ker(\mathbf{a}) \cap R = \mathbf{a}^*(\mathrm{Gr}(f^*))$ if $\mathbf{a}(R) = \mathrm{Gr}(f_*)$. In particular, the morphism R is exact if and only if $\mathbf{a}_1(\mathrm{ran}^*(R)) = TM_1$.

Proposition 2.3.1. *The morphism R is exact if and only if it is a Dirac morphism*

$$R : (\mathbb{A}_1, \mathrm{ran}(\mathbf{a}_1^*)) \dashrightarrow (\mathbb{A}_2, \mathrm{ran}(\mathbf{a}_2^*)).$$

Proof. In one direction, suppose R is exact. Let $C = \mathbf{a}^{-1}(\mathrm{Gr}(f_*))$, which is coisotropic in $\mathbb{A}_2 \times \overline{\mathbb{A}_1}$ with isotropic orthogonal $C^\perp = \mathbf{a}^*(\mathrm{Gr}(f^*))$. Since $R \subseteq C$, it follows that $C^\perp \subseteq R$ and in particular R relates each element of $\mathrm{ran}(\mathbf{a}_2^*)$ to an element of $\mathrm{ran}(\mathbf{a}_1^*)$, thus establishing existence. Now if $\mu \sim_R 0$ for some non-zero $\mu \in \mathrm{ran}(\mathbf{a}_1^*)$ then by orthogonality the image under $\mathbf{a}_1$ of the projection of R onto $\mathbb{A}_1$ must be contained in $\ker(\mu) \subseteq TM_1$. Since $\mathbf{a}(R) = \mathrm{Gr}(f_*)$, this must mean that $\mu = 0$, establishing uniqueness.

In the other direction, suppose $R : (\mathbb{A}_1, \mathrm{ran}(\mathbf{a}_1^*)) \dashrightarrow (\mathbb{A}_2, \mathrm{ran}(\mathbf{a}_2^*))$ is a Dirac morphism. It must only be shown that $\mathbf{a}_1(\mathrm{ran}^*(R)) = TM_1$. The intersection $\mathrm{ran}(\mathbf{a}_1^*) \cap \ker(R)$ is trivial by definition, and taking its orthogonal one has

$$\ker(\mathbf{a}_1) + \mathrm{ran}^*(R) = \mathbb{A}_1 \implies \mathbf{a}_1(\mathrm{ran}^*(R)) = TM_1.$$

This completes the proof. □

Proposition 2.3.2 ([17])**.** *Suppose R is exact. Let $j_2 : TM_2 \to \mathbb{A}_2$ be an isotropic splitting of $\mathbb{A}_2$. Then there exists a unique isotropic splitting $j_1 : TM_1 \to \mathbb{A}_1$ of $\mathbb{A}_1$ such that $R \circ j_1 = j_2 \circ f_*$. Furthermore, the corresponding 3-forms are related by $\eta_1 = f^*\eta_2$.*

Proof. According to the previous proposition and Proposition 2.1.2, the backward image $\mathrm{ran}(j_2) \circ R$ is a Lagrangian complement of $\mathrm{ran}(\mathbf{a}_1^*)$. Defining $j_1 : TM_1 \to \mathbb{A}_1$ by $\mathrm{ran}(j_1) = \mathrm{ran}(j_2) \circ R$, it is clear the condition $R \circ j_1 = j_2 \circ f_*$ holds. Conversely, if $j_1 : TM_1 \to \mathbb{A}_1$ is an isotropic splitting satisfying this condition then $\mathrm{ran}(j_1) \subseteq \mathrm{ran}(j_2) \circ R$, which is actually an equality since both sides are Lagrangian. This shows the first part of the statement. The second part follows immediately from (2.8). □

Proposition 2.3.1 shows that composition of exact morphisms is also exact. On the other hand, Proposition 2.3.2 is key in giving a more explicit description of exact morphisms. Towards this, start by choosing arbitrary isotropic splittings $j_i : TM_i \to \mathbb{A}_i$ identifying $\mathbb{A}_i \simeq$

$\mathbb{T}M_{i,\eta_i}$ for $i = 1, 2$. If $R : \mathbb{A}_1 \dashrightarrow \mathbb{A}_2$ is exact then it determines an identification $\mathbb{A}_1 \simeq \mathbb{T}M_{1,f^*\eta_2}$. By (2.11) then, the image $\operatorname{ran}(j_1) \subseteq \mathbb{T}M_{1,f^*\eta_2}$ is the graph $\operatorname{Gr}(\omega)$ of a 2-form $\omega \in \Omega^2(M_1)$ satisfying

$$d\omega = \eta_1 - f^*\eta_2. \tag{2.16}$$

This leads to the explicit description

$$v_1 + \mu_1 \sim_R v_2 + \mu_2 \iff v_2 = f_* v_1,\ \mu_1 + \iota_{v_1}\omega = f^*\mu_2. \tag{2.17}$$

Conversely, suppose $f : M_1 \to M_2$ is a smooth map and $\omega \in \Omega^2(M_1)$ is a 2-form satisfying (2.16). Define a Lagrangian subbundle $R \subseteq \mathbb{A}_2 \times \overline{\mathbb{A}_1}$ along $\operatorname{Gr}(f)$ via (2.17). It is easily checked that R is a Dirac morphism[5], obviously an exact one. The upshot is that an exact morphism $\mathbb{T}M_{1,\eta_1} \dashrightarrow \mathbb{T}M_{2,\eta_2}$ is equivalent to a pair (f, ω) as above. The notation $\mathbb{T}f_\omega$, or $\mathbb{T}f$ if $\omega = 0$, will be used to denote the exact morphism corresponding to (f, ω) from here on.

If the composition of functions $f' \circ f$ is defined, the composition of exact Courant morphisms $\mathbb{T}f_\omega$ and $\mathbb{T}f'_{\omega'}$ with base maps f and f' is also defined with

$$\mathbb{T}f'_{\omega'} \circ \mathbb{T}f_\omega = \mathbb{T}(f' \circ f)_{\omega + f^*\omega'}. \tag{2.18}$$

Example 2.3.4. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and denote the group multiplication by $\operatorname{mult}_G : G \times G \to G$. Then the tangent bundle TG of G is also a Lie group where multiplication is given by the differential $(\operatorname{mult}_G)_* : TG \times TG \to TG$. On the other hand, the cotangent bundle carries a symplectic Lie groupoid structure $T^*G \rightrightarrows \mathfrak{g}^*$ where the source and target maps are respectively the left and right trivializations and multiplication is given by

$$(g_1, \mu_1) \circ (g_2, \mu_2) = (g, \mu) \iff g = g_1 g_2,\ \operatorname{mult}_G^* \mu = (\mu_1, \mu_2).$$

Multiplication in the direct sum of Lie groupoids $\mathbb{T}G = TG \oplus T^*G$ is therefore given by

$$(g_1, v_1, \mu_1) \circ (g_2, v_2, \mu_2) = (g, v, \mu) \iff$$
$$g = g_1 g_2,\ v = (\operatorname{mult}_G)_*(v_1, v_2),\ \operatorname{mult}_G^* \mu = (\mu_1, \mu_2).$$

This is precisely the exact morphism $\mathbb{T}\operatorname{mult}_G : \mathbb{T}G \times \mathbb{T}G \dashrightarrow \mathbb{T}G$.

[5]One may ipso facto assume that $\eta_1 = f^*\eta_2$, making this verification quite simple.

2.4 Coisotropic reduction

2.4.1 Reduction of Courant algebroids

Suppose $C \subseteq \mathbb{A}$ is a coisotropic subbundle along M. The quotient bundle $C/C^{\perp}$ inherits a fiber metric from $\mathbb{A}$, though it will rarely inherit a bracket. In an attempt to rectify this, the symmetries of a Courant algebroid are briefly considered.

The infinitesimal counterpart of an automorphism of vector bundles of $\mathbb{A}$ is a derivation, i.e. a $\mathbb{R}$-linear operator $\mathcal{D} : \Gamma(\mathbb{A}) \to \Gamma(\mathbb{A})$ together with a vector field $v \in \mathfrak{X}^1(M)$ satisfying the Leibniz rule: $\mathcal{D}(f\sigma) = \mathcal{D}\sigma + (vf)\sigma$ for all $f \in C^\infty(M)$ and $\sigma \in \Gamma(\mathbb{A})$. The space $\mathrm{Der}(\mathbb{A})$ of derivations is informally the Lie algebra of $\mathrm{Aut}(\mathbb{A})$, the space of vector bundle automorphisms. Its Lie bracket is the commutator of operators. A *Courant automorphism* is a Courant isomorphism $\mathbb{A} \dashrightarrow \mathbb{A}$. The infinitesimal counterpart of a Courant automorphism is a *Courant derivation*, i.e. a derivation $(\mathcal{D}, v) \in \mathrm{Der}(\mathbb{A})$ possessing the properties

$$\begin{aligned} v\langle\sigma_1, \sigma_2\rangle &= \langle\mathcal{D}\sigma_1, \sigma_2\rangle + \langle\sigma_1, \mathcal{D}\sigma_2\rangle, \\ \mathcal{D}[\![\sigma_1, \sigma_2]\!] &= [\![\mathcal{D}\sigma_1, \sigma_2]\!] + [\![\sigma_1, \mathcal{D}\sigma_2]\!], \\ \mathbf{a}(\mathcal{D}\sigma) &= [v, \mathbf{a}(\sigma)]\,, \end{aligned}$$

for all $\sigma, \sigma_1, \sigma_2 \in \Gamma(\mathbb{A})$. The space $\mathrm{Der}_{CA}(\mathbb{A}) \subseteq \mathrm{Der}(\mathbb{A})$ of Courant derivations is informally the Lie algebra of the group $\mathrm{Aut}_{CA}(\mathbb{A}) \subseteq \mathrm{Aut}(\mathbb{A})$ of Courant automorphisms. A section $\sigma \in \Gamma(\mathbb{A})$ determines a Courant derivation $\sigma \mapsto ([\![\sigma, \cdot]\!], \mathbf{a}(\sigma))$ and this assignment is bracket preserving.

Definition 2.4.1 ([17])**.** Suppose G is a Lie group, with Lie algebra $\mathfrak{g}$, acting on $\mathbb{A}$ by Courant automorphisms. Suppose moreover that $\varrho : \mathfrak{g} \to \Gamma(\mathbb{A})$ is a bracket-preserving linear map such that

$$\frac{\partial}{\partial t}\bigg|_{t=0} \exp(-t\gamma).\sigma = [\![\varrho(\gamma), \sigma]\!] \tag{2.19}$$

for all $\gamma \in \mathfrak{g}$ and $\sigma \in \Gamma(\mathbb{A})$. The elements $\varrho(\gamma) \in \Gamma(\mathbb{A})$ will be called *generators* for the G-action on $\mathbb{A}$.

In other words, the map $\varrho : \mathfrak{g} \to \Gamma(\mathbb{A})$ defines generators for a G-action on $\mathbb{A}$ if the infinitesimal $\mathfrak{g}$-action on $\mathbb{A}$ factors through the assignment $\Gamma(\mathbb{A}) \to \mathrm{Der}_{CA}(\mathbb{A})$ composed with ϱ. Note that G acts on M (embedded in $\mathbb{A}$ as the zero-section) as well, with $\mathbf{a} \circ \varrho : \mathfrak{g} \to \mathfrak{X}(M)$ as the infinitesimal action. The symbol ϱ will be retained to denote the corresponding bundle map $M \times \mathfrak{g} \to \mathbb{A}$.

Example 2.4.1. Suppose G is a Lie group acting on a manifold M with infinitesimal action $\varrho : \mathfrak{g} \to \mathfrak{X}(M)$ and that $\eta \in \Omega^3(M)$ is a G-invariant closed 3-form. Then ϱ defines isotropic generators for the action of G on $\mathbb{T}M_\eta$ by $g.(v + \mu) = g_* v + (g^{-1})^* \mu$ for $v \in TM$ and $\mu \in T^*M$.

Example 2.4.2. In the context of Example 2.2.3, suppose G is a Lie group integrating the Lie algebra $\mathfrak{g}$ and that the $\mathfrak{g}$-action $\mathbf{a} : \mathfrak{g} \to M \times \mathfrak{g}$ integrates to a G-action. Consider the action of G on $M \times \mathfrak{g}$ given by $g.(m, \gamma) = (gm, \mathrm{Ad}_{\mathfrak{g}}\, \gamma)$, which preserves the metric as well as the bracket of constant sections. Since the constant sections generate $\Gamma(M \times \mathfrak{g})$, one sees from C3 and C5 that the G-action on $M \times \mathfrak{g}$ is bracket-preserving for all pairs of sections in $\Gamma(M \times \mathfrak{g})$, i.e G acts on $M \times \mathfrak{g}$ by Courant automorphisms. The anchor $\mathbf{a} : \mathfrak{g} \to \Gamma(M)$ then defines generators for this G-action.

Suppose G acts on $\mathbb{A}$ with $\varrho : \mathfrak{g} \to \Gamma(\mathbb{A})$ defining generators for the G-action. If $E \to S$ is a Lagrangian subbundle of $\mathbb{A}$ satisfying $\mathbf{a}(E) \subseteq TS$, one may may in addition to the Courant tensor Υ^E consider the *Dirac tensor* $\Lambda^E \in \mathfrak{g}^* \otimes \Gamma(\bigwedge^2 E^*)$ defined by

$$\Lambda^E(\gamma, \sigma_1, \sigma_2) = \langle [\![\varrho(\gamma), \widetilde{\sigma}_1]\!], \widetilde{\sigma}_2 \rangle|_S$$

where $\gamma \in \mathfrak{g}$ and $\sigma_i \in \Gamma(E)$ with $\widetilde{\sigma}_i \in \Gamma(\mathbb{A}, E)$ extending σ_i. The Dirac tensor measures the degree to which E fails to be $\mathfrak{g}$-invariant, or G-invariant if G is connected. Here again one verifies from C1-C3 that Λ^E is well-defined and tensorial in σ_1 and σ_2.

Example 2.4.3. Suppose $\mathbb{A} = \mathbb{T}M$ and $\pi \in \mathfrak{X}^2(M)$ is a bivector field. Then

$$\begin{aligned}
\Lambda^{\mathrm{Gr}(\pi)}(\gamma, \mu_1, \mu_2) &= \langle [\![\varrho(\gamma), \iota_{\mu_1}\pi + \mu_1]\!], \iota_{\mu_2}\pi + \mu_2 \rangle \\
&= \langle [\varrho(\gamma), \iota_{\mu_1}\pi] + \mathcal{L}_{\varrho(\gamma)}\mu_1, \iota_{\mu_2}\pi + \mu_2 \rangle \\
&= \iota_{\mu_2}\mathcal{L}_{\varrho(\gamma)}\iota_{\mu_1}\pi - \iota_{\mu_2}\iota_{\mathcal{L}_{\varrho(\gamma)}\mu_1}\pi \\
&= \mathcal{L}_{\varrho(\gamma)}\iota_{\mu_2}\iota_{\mu_1}\pi - \iota_{\mathcal{L}_{\varrho(\gamma)}\mu_2}\iota_{\mu_1}\pi - \iota_{\mu_2}\iota_{\mathcal{L}_{\varrho(\gamma)}\mu_1}\pi \\
&= \iota_{\mu_2}\iota_{\mu_1}\mathcal{L}_{\varrho(\gamma)}\pi.
\end{aligned}$$

In other words $\Lambda^{\mathrm{Gr}(\pi)}(\gamma) = \mathcal{L}_{\varrho(\gamma)}\pi$. Similarly one finds that $\Lambda^{\mathrm{Gr}(\omega)}(\gamma) = \mathcal{L}_{\varrho(\gamma)}\omega$ for a 2-form $\omega \in \Omega^2(M)$.

Lemma 2.4.1. *Suppose G is a connected Lie group acting on $\mathbb{A}$ by Courant automorphisms and $\varrho : \mathfrak{g} \to \Gamma(\mathbb{A})$ defines generators for the G-action. Then the space $\Gamma(V)^G$ of G-invariant sections of V is closed under the Courant bracket.*

Proof. As G is connected, the G-invariant sections of V are those whose bracket with the generators $\varrho(\gamma)$ are trivial. Then the closure of $\Gamma(V)^G$ under the Courant bracket follows directly from C1. □

For the remainder of this subsection, suppose $\varrho : \mathfrak{g} \to \Gamma(\mathbb{A})$ defines generators for a G-action by Courant automorphisms on $\mathbb{A}$ where G is a connected Lie group, that $C^\perp = \mathrm{ran}(\varrho)$ is an isotropic subbundle and that the induced G-action on M is free and proper. Recall that $C^\perp$ is a Lie algebroid by part (d) of Proposition 2.2.1. In fact, since the G-action is free the map ϱ is injective and thus gives an isomorphism of Lie algebroids $M \times \mathfrak{g} \to C^\perp$.

Lemma 2.4.2. *The subbundle $C^\perp$ is involutive, its orthogonal C is G-invariant. Moreover, the Courant bracket descends to a bracket on $\Gamma(C/C^\perp)^G$.*

Proof. For $\gamma_1, \gamma_2 \in \mathfrak{g}$, one has $\varrho([\gamma_1, \gamma_2]) = [\![\varrho(\gamma_1), \varrho(\gamma_2)]\!]$ by (2.19). Thus $[\![\varrho(\gamma_1), \varrho(\gamma_2)]\!] \in \Gamma(C^\perp)$. For $f \in C^\infty(M)$, Property C5 gives $[\![\varrho(\gamma_1), f\gamma_2]\!] \in \Gamma(C^\perp)$ and C3 gives

$$[\![f\varrho(\gamma_1), \varrho(\gamma_2)]\!] = -[\![\varrho(\gamma_2), f\varrho(\gamma_1)]\!]$$

since $C^\perp$ is isotropic. Thus $[\![f\varrho(\gamma_1), \varrho(\gamma_2)]\!] \in \Gamma(C^\perp)$ as well. It follows that $C^\perp$ is involutive.

The G-invariance of C follows from the G-invariance of $C^\perp$ and the preservation of the metric by the G-action. Since $\Gamma(C^\perp)^G$ and $\Gamma(C)^G$ are closed under the Courant bracket by Lemma 2.4.1, the same argument regarding the descent of the Courant bracket given in the proof of part (e) of Proposition 2.2.1, mutatis mutandis, can be employed to show that the Courant bracket descends to $\Gamma(C/C^\perp)^G$. □

Consider the reduced space $\mathbb{A}_{\mathrm{red}} = \frac{C}{C^\perp}/G$, which is a vector bundle over $M_{\mathrm{red}} = M/G$.

Theorem 2.4.1 ([17])**.** *With everything as above, suppose M_{red} and $\mathbb{A}_{\mathrm{red}}$ are quotient manifolds so that $\mathbb{A}_{\mathrm{red}}$ is a smooth vector bundle over M_{red}. Then $\mathbb{A}_{\mathrm{red}}$ inherits the structure of a Courant algebroid.*

Proof. Since $\mathbb{A}_{\mathrm{red}}$ is a smooth vector bundle, the base manifold M_{red} is also smooth and the quotient map $M \to M_{\mathrm{red}}$ is a submersion. As $C/C^\perp$ inherits a metric so does $\mathbb{A}_{\mathrm{red}}$. The fundamental vector fields of the G-action on M span $\mathbf{a}(C^\perp)$ and, writing $TM_{\mathrm{red}} = \frac{TM}{\mathbf{a}(C^\perp)}/G$, let $\mathbf{a}_{\mathrm{red}} : \mathbb{A}_{\mathrm{red}} \to TM_{\mathrm{red}}$ be the map to which $\mathbf{a} : C \to TM$ descends.

The previous lemma implies that the Courant bracket $[\![\cdot,\cdot]\!]$ descends to a bracket $[\![\cdot,\cdot]\!]_{\mathrm{red}}$ on $\Gamma(\mathbb{A}_{\mathrm{red}})$. Equipping $\mathbb{A}_{\mathrm{red}}$ with the metric $\langle\cdot,\cdot\rangle_{\mathrm{red}}$, the anchor $\mathbf{a}_{\mathrm{red}}$ and the bracket $[\![\cdot,\cdot]\!]_{\mathrm{red}}$, it is readily seen that C1-C3 are inherited from $\mathbb{A}$. □

Example 2.4.4. In the context of Example 2.2.3, suppose $\mathfrak{h} \subseteq \mathfrak{g}$ is an isotropic subalgebra integrating to a Lie group H and that the $\mathfrak{h}$-action on M integrates to a free and proper H-action. Now H acts on $M \times \mathfrak{g}$ by Courant automorphisms via $s.(m, \gamma) = (s.m, \mathrm{Ad}_s \gamma)$ as in Example 2.4.2. Assuming the coisotropic orthogonal $\mathfrak{h}^\perp$ is also a subalgebra of $\mathfrak{g}$, the quotient $\mathfrak{h}^\perp/\mathfrak{h}$ is a Lie algebra (Proposition 2.2.1) and $(M \times \mathfrak{g})_{\mathrm{red}}$ and $M/H \times \mathfrak{h}^\perp/\mathfrak{h}$ are equal as vector bundles. Surely, they should also be equal as Courant algebroid.

To verify this, first note the anchors coincide since both are defined as the map $\mathbf{a}_{\mathrm{red}} : M/H \to \frac{TM}{\mathbf{a}(\mathfrak{h})}/G$ to which the anchor $\mathbf{a}$ of $M \times \mathfrak{g}$ descends. Next let $\sigma_1, \sigma_2 \in \Gamma(M/H \times \mathfrak{h}^\perp/\mathfrak{h})$ be constant sections. Then σ_1 and σ_2 lift to H-invariant sections of $M \times \mathfrak{g}$ of the form $\mathrm{Ad}_x \gamma_i$, $i = 1, 2$, respectively, for some fixed elements $\gamma_i \in \mathfrak{h}^\perp$. Then

$$[\![\mathrm{Ad}_x \gamma_1, \mathrm{Ad}_x \gamma_2]\!] = \mathrm{Ad}_x [\![\gamma_1, \gamma_2]\!] = \mathrm{Ad}_x [\gamma_1, \gamma_2],$$

from which one sees that

$$[\![\sigma_1, \sigma_2]\!]_{\mathrm{red}} = [\gamma_1, \gamma_2] + \mathfrak{h} = [\sigma_1, \sigma_2]_{\mathrm{red}},$$

where $[\cdot,\cdot]$ is the Lie bracket of $\mathfrak{h}^\perp/\mathfrak{h}$, i.e. the Courant bracket $[\![\cdot,\cdot]\!]$ agrees with the bracket of $\mathfrak{h}^\perp/\mathfrak{h}$ on constant sections. Since this property uniquely determines the bracket of the action Courant groupoid $M/H \times \mathfrak{h}^\perp/\mathfrak{h}$, it follows that $(M \times \mathfrak{g})_{\mathrm{red}}$ and $M/H \times \mathfrak{h}^\perp/\mathfrak{h}$ are equal as Courant algebroids.

Remark 2.4.1. With the same assumptions as Theorem 2.4.1, if S is a G-invariant submanifold of M and $\mathbf{a}(C)|_S \subseteq TS$ then $(\mathbb{A}_{\mathrm{red}})|_{S/G} = \frac{C|_S}{C^\perp|_S}/G$ is also a Courant algebroid, i.e. a "Courant subalgebroid" of $\mathbb{A}_{\mathrm{red}}$.

The process of passing from $\mathbb{A}$ to $\mathbb{A}_{\mathrm{red}}$ will be called the *coisotropic reduction of* $\mathbb{A}$ *by* C. The *reduction morphism* $q : \mathbb{A} \dashrightarrow \mathbb{A}_{\mathrm{red}}$ is defined by $x \sim_q [x]$ for all $x \in C$. Its involutivity is manifest from the definition of $[\![\cdot,\cdot]\!]_{\mathrm{red}}$ in the proof of the preceding theorem.

Proposition 2.4.1. *Suppose $E \subseteq \mathbb{A}$ is a G-invariant Dirac structure and that $E \cap C^\perp$, or equivalently $E \cap C$, has constant rank. Then its forward image $q \circ E$, which can be taken as a subset of $\mathbb{A}_{\mathrm{red}}$ as E is G-invariant, is a Dirac structure of $\mathbb{A}_{\mathrm{red}}$. Conversely, if F is a Dirac structure of $\mathbb{A}_{\mathrm{red}}$ then its backward image $F \circ q \subseteq \mathbb{A}_1$ is a G-invariant Dirac structure of $\mathbb{A}_1$ and $E \cap C^\perp$ has constant rank.*

Proof. In the first direction, note first that $q \circ E$ is subbundle of $\mathbb{A}_{\mathrm{red}}$ since its fiber over $[m] \in M_{\mathrm{red}}$ does not depend on the representative $m \in M$ (smoothness follows from the constant rank of $E \cap C^\perp$). By Proposition 2.1.2, it is Lagrangian. As $\Gamma(q \circ E) = \Gamma(E)^G \cap \Gamma(C)^G$, the involutivity of $q \circ E$ follows from Lemma 2.4.1.

The second direction is trivial as the backward image $F \circ q$ is obviously G-invariant and contains $C^\perp$, as well as being a Dirac structure by Example 2.3.3. □

The previous proposition can be extended to encompass G-invariant Dirac structures supported on a G-invariant submanifold $S \hookrightarrow P$ in one direction and its quotient S/G in the other provided $\mathbf{a}(C)|_S \subseteq TS$; indeed, in view of Remark 2.4.1, this is in essence the same result.

Proposition 2.4.2. *Suppose, under the assumptions of Theorem 2.4.1, that $\mathbb{A}$ is exact and* $\mathrm{ran}(\mathbf{a}^*) \cap C^\perp = 0$. *Then $\mathbb{A}_{\mathrm{red}}$ is also exact and the reduction morphism $q : \mathbb{A} \dashrightarrow \mathbb{A}_{\mathrm{red}}$ is exact.*

Proof. Clearly, one has $\mathrm{rank}(\mathbb{A}_{\mathrm{red}}) = 2\dim(M_{\mathrm{red}})$ since $\mathrm{rank}(\mathbb{A}) = 2\dim(M)$. It thus suffices to show that $\mathbf{a}^*_{\mathrm{red}} : TM_{\mathrm{red}} \to \mathbb{A}_{\mathrm{red}}$ is injective to establish the exactness of $\mathbb{A}_{\mathrm{red}}$. The reduced cotangent bundle T^*M_{red} canonically identifies with $\mathrm{Ann}(\mathbf{a}(C^\perp))/G$, where $\mathrm{Ann}(\mathbf{a}(C^\perp)) \subseteq T^*M$ is the annihilator of $\mathbf{a}(C^\perp)$. Then since $\mathrm{ran}(\mathbf{a}^*) \cap C^\perp = 0$ and $\mathbf{a}^* : TM \to \mathbb{A}$ is injective, the map $T^*M_{\mathrm{red}} \to \mathbb{A}_{\mathrm{red}}$ to which $\mathbf{a}^*$ descends, which is none other than $\mathbf{a}^*_{\mathrm{red}}$, must be injective as well.

From the foregoing and Proposition 2.3.1 the exactness of the reduction morphism $q : \mathbb{A} \dashrightarrow \mathbb{A}_{\mathrm{red}}$ is immediate. □

With $\mathbb{A}$ exact and $\mathrm{ran}(\mathbf{a}^*) \cap C^\perp = 0$, an isotropic splitting $j_{\mathrm{red}} : TM_{\mathrm{red}} \to \mathbb{A}_{\mathrm{red}}$ determines a G-invariant isotropic splitting $j : TM \to \mathbb{A}$ with $C^\perp \subseteq \mathrm{ran}(j)$ according to Proposition 2.3.2. The corresponding 3-forms are then related by $\eta = (\cdot/G)^*\eta_{\mathrm{red}}$, where $\cdot/G : M \to M_{\mathrm{red}}$ is the quotient map. Conversely, one recovers j_{red} from j by observing that

$$\mathrm{ran}(j_{\mathrm{red}}) = q \circ \mathrm{ran}(j) = \frac{\mathrm{ran}(j)}{C^\perp}/G. \tag{2.20}$$

On the other hand, if $j : TM \to \mathbb{A}$ is an arbitrary G-equivariant isotropic splitting with $C^\perp \subseteq \mathrm{ran}(j)$, in which case it is called a *G-basic* splitting, then (2.20) defines an isotropic splitting j_{red} of $\mathbb{A}_{\mathrm{red}}$.

Example 2.4.5. Suppose $j : TM \to \mathbb{A}$ is a G-basic splitting with corresponding 3-form η. One has $C^\perp \subseteq TM$ under the identification $\mathbb{A} \simeq \mathbb{T}M_\eta$. Then (Example (2.4.1)) the G-action on $\mathbb{T}M_\eta$ in terms of the G-action on M is $g.(v + \mu) = g_*v + (g^{-1})^*\mu$, for $v \in TM$ and $\mu \in T^*M$. The identification $\mathbb{A}_{\mathrm{red}} \simeq \mathbb{T}M_{\eta_{\mathrm{red}}}$ has a simple interpretation: it descends from the identification $\mathbb{A} \simeq \mathbb{T}M_\eta$ by virtue of

$$\mathbb{A}_{\mathrm{red}} = \tfrac{C}{C^\perp}/G \simeq (\tfrac{TM}{C^\perp} \oplus \mathrm{Ann}(C^\perp))/G = TM_{\mathrm{red}} \oplus T^*M_{\mathrm{red}}.$$

2.4.2 Reduction of Courant morphisms

Having described a process for reducing Courant algebroids, the question of the reduction of Courant morphisms naturally arises. Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be Courant algebroids. For $i = 1, 2$, assume that $\varrho_i : \mathfrak{g}_i \to \Gamma(\mathbb{A}_i)$ define generators for G_i-actions on $\mathbb{A}_i$ where G_i are connected Lie groups. Assume furthermore that $C_i = \operatorname{ran}(\varrho_i)^\perp$ are coisotropic so that $C_i^\perp = \operatorname{ran}(\varrho_i)$ are isotropic, and that the G_i-actions on the base manifolds M_i are free and proper. Let

$$R : \mathbb{A}_1 \dashrightarrow \mathbb{A}_2$$

be a Courant morphism with base map $f : M_1 \to M_2$. Fix a homomorphism of Lie algebras $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$. The symbol ϕ will also be used for the corresponding bundle map

$$C_1^\perp \simeq M_1 \times \mathfrak{g}_1 \to M_2 \times \mathfrak{g}_2 \simeq C_2^\perp$$

where the identifications have been made through the infinitesimal actions ϱ_i.

Definition 2.4.2 ([17])**.** The morphism R is said to *intertwine the generators* (with respect to ϕ) if

$$\varrho_1(\gamma) \sim_R \varrho_2(\phi(\gamma)) \tag{2.21}$$

for all Lie algebra elements $\gamma \in \mathfrak{g}_1$.

A few remarks are in order if R intertwines the generators. First, one has the useful containment

$$C_1^\perp \subseteq C_2^\perp \circ R$$

or equivalently

$$C_2 \circ R \subseteq C_1.$$

Moreover, if $\varrho_1(\gamma_1) \sim_R \varrho_2(\gamma_2)$ for elements $\gamma_i \in \mathfrak{g}_i$ then $0 \sim_R \varrho_2(\phi(\gamma_1) - \gamma_2)$ and thus $\mathbf{a}_2(\varrho_2(\phi(\gamma_1) - \gamma_2)) = 0$. As the G_2-action on M_2 is free and $\mathbf{a}_2 \circ \varrho_2$ is its infinitesimal action, the elements γ_2 and $\varrho_2(\phi(\gamma_1))$ must be equal. Thus the intersection $R \cap (C_2^\perp \times C_1^\perp)$ is, taking $\varrho_i : M_i \times \mathfrak{g}_i \to \mathbb{A}_i$ and $\phi : M_1 \times \mathfrak{g}_1 \to M_2 \times \mathfrak{g}_2$ as bundle maps, the image $(\varrho_2 \times \varrho_1)(\operatorname{Gr}(\phi))$.

Assume now that ϕ integrates to homomorphism of Lie groups $\Phi : G_1 \to G_2$.

Definition 2.4.3. The Courant morphism R will be called *equivariant* (with respect to Φ, or simply G if $G_1 = G_2 = G$ and Φ is the identity map) if for all group elements $g \in G_1$ and elements $x_i \in \mathbb{A}_i$ one has

$$x_1 \sim_R x_2 \implies g.x_1 \sim_R \Phi(g).x_2. \tag{2.22}$$

Equivalently, R is equivariant if it is invariant under the action of $\mathrm{Gr}(\Phi)$ on $\mathbb{A}_2 \times \overline{\mathbb{A}_1}$.

Note that if R is equivariant then the base map $f : M_1 \to M_2$ is necessarily Φ-equivariant.

Lemma 2.4.3. *Suppose R intertwines the generators. Then:*

(a) *It is equivariant.*[6]

(b) *If R is also a Dirac morphism $(\mathbb{A}_1, E_1) \dashrightarrow (\mathbb{A}_2, E_2)$ where $E_i \subseteq \mathbb{A}_i$ are Dirac structures then, given a Lagrangian complement $F_2 \subseteq \mathbb{A}_2$ of E_2, the Dirac tensor $\Lambda^{F_1} \in \mathfrak{g}^* \times \bigwedge^2 E_1$ of the backward image $F_1 = F_2 \circ R$ is related to the Dirac tensor $\Lambda^{F_2} \in \mathfrak{g}^* \times \bigwedge^2 E_2$ of F_2 by*

$$\Lambda^{F_1} = (\phi^* \times \varrho)(f^* \Lambda^{F_2}).$$

Proof. (a) As seen above, the Courant morphism R contains $(\varrho_2 \times \varrho_1)(\mathrm{Gr}(\phi))$. In particular, the contraction of the Courant tensor

$$\Upsilon^R((\varrho_2 \times \varrho_1)(\phi(\gamma), \gamma), \sigma_1, \sigma_2)$$

vanishes for all $\gamma \in \mathfrak{g}_1$ and $\sigma_1, \sigma_2 \in \Gamma(\mathbb{R})$. But this expression is equal to $\Lambda^R(\gamma, \sigma_1, \sigma_2)$, so the Dirac tensor of R vanishes. As G_1 is connected, this means that R is equivariant.

(b) One proceeds much like in the proof of Lemma 2.3.1. Let $\sigma_1, \sigma_2 \in \Gamma(F_2 \times F_1) \cap \Gamma(\mathbb{A}_2 \times \overline{\mathbb{A}_1}, R)$ and $\gamma \in \mathfrak{g}$. For $m \in M_1$, the value of σ_i at $(f(m), m)$ is $(\varrho^*_{F_2}(x_i), x_i)$ for some $x_i \in F_1|_m$. Then

$$\begin{aligned}
(\phi^* \times \varrho)(f^* \Lambda^{F_2})(\gamma, x_1, x_2)|_m -{} \\
\Lambda^{F_1}(\gamma, x_1, x_2)|_m &= \Lambda^{F_2}(\phi(\gamma), \varrho^*_{F_2}(x_1), \varrho^*_{F_2}(x_2)) - \\
&\quad \Lambda^{F_1}(\gamma, x_1, x_2) \\
&= \Lambda^{F_2 \times F_1}(\phi(\gamma), \gamma, \sigma_1, \sigma_2)|_{(f(m),m)} \\
&= \langle [\![(\varrho_2 \times \varrho_1)(\phi(\gamma), \gamma), \sigma_1]\!], \sigma_2 \rangle|_{(f(m),m)}.
\end{aligned}$$

On the other hand

$$\iota_{(\phi(\gamma),\gamma)} \Lambda^R(\sigma_1, \sigma_2) = \langle [\![(\varrho_2 \times \varrho_1)(\phi(\gamma), \gamma), \sigma_1]\!], \sigma_2 \rangle|_{\mathrm{Gr}(f)}$$

[6] It must be emphasized that this is not true in the more general case where the group G_1 is not connected.

vanishes as seen in part (a) and thus

$$(\phi^* \times \varrho)(f^*\Lambda^{F_2})(\gamma, x_1, x_2) - \Lambda^{F_1}(\gamma, x_1, x_2) = 0$$

as needed. □

This chapter culminates with the following theorem, appearing in [17] except for parts (b) and (d) which are strengthened here.

Theorem 2.4.2. *Suppose $R : \mathbb{A}_1 \dashrightarrow \mathbb{A}_2$ is a that intertwines the generators (and is therefore equivariant). Suppose furthermore that $\mathbb{A}_i$ $(i = 1, 2)$ satisfy the conditions of Theorem 2.4.1. Then:*

(a) *The Courant morphism R descends to a Courant morphism $R_{\text{red}} : \mathbb{A}_{1,\text{red}} \dashrightarrow \mathbb{A}_{2,\text{red}}$ such that the diagram*

$$\begin{array}{ccc} \mathbb{A}_1 & \overset{R}{\dashrightarrow} & \mathbb{A}_2 \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_2} \\ \mathbb{A}_{1,\text{red}} & \overset{R_{\text{red}}}{\dashrightarrow} & \mathbb{A}_{2,\text{red}} \end{array} \tag{2.23}$$

commutes, where $q_i : \mathbb{A}_i \dashrightarrow \mathbb{A}_{i,\text{red}}$ are the reduction morphisms, in the sense that $R \circ q_1 = q_2 \circ R$. Its base map is $f_{\text{red}} : M_{1,\text{red}} \to M_{2,\text{red}}$, the map which descends from $f : M_1 \to M_2$.

(b) *Let $E_1 \subseteq \mathbb{A}_1$ and $E_2 \subseteq \mathbb{A}_2$ be Dirac structures such that $E_{1,\text{red}} = q_1 \circ E_1$ and $E_{2,\text{red}} = q_2 \circ E_2$ are Dirac structures of $\mathbb{A}_{1,\text{red}}$ and $\mathbb{A}_{2,\text{red}}$ respectively. Suppose $\phi : C_1^\perp \to C_2^\perp$ restricts to a bijective map*

$$\phi : E_1 \cap C_1^\perp \to E_2 \cap C_2^\perp. \tag{2.24}$$

If R is a Dirac morphism $R : (\mathbb{A}_1, E_1) \dashrightarrow (\mathbb{A}_2, E_2)$ then R_{red} is a Dirac morphism $R_{\text{red}} : (\mathbb{A}_{1,\text{red}}, E_{1,\text{red}}) \dashrightarrow (\mathbb{A}_{2,\text{red}}, E_{2,\text{red}})$.

(c) *Suppose $\mathbb{A}_i$ are exact Courant algebroids and that $\operatorname{ran}(\mathbf{a}_i^*) \cap C_i^\perp = 0$ so that $\mathbb{A}_{i,\text{red}}$ are exact by Proposition 2.4.2. Let $f_{\text{red}} : M_{1,\text{red}} \to M_{2,\text{red}}$ be the map which descends from the base map $f : M_1 \to M_2$. Then the reduction procedure gives a one-to-one correspondence between Courant morphisms $\mathbb{A}_1 \dashrightarrow \mathbb{A}_2$ intertwining the generators with base map f and Courant morphisms $\mathbb{A}_{1,\text{red}} \dashrightarrow \mathbb{A}_{2,\text{red}}$ with base map f_{red}.*

(d) *The correspondence in (c) preserves the exactness of Courant morphisms. If $j_i : TM_i \to \mathbb{A}_i$ are G_i-basic splittings with corresponding 3-forms $\eta_i \in \Omega^3(M_i)$ then, after identify-*

ing $\mathbb{A}_i \simeq \mathbb{T}\mathbb{A}_{i,\eta_i}$*, the* 2*-forms* $\omega \in \Omega^2(M_1)$ *and* $\omega_{\mathrm{red}} \in \Omega^2(M_{1,\mathrm{red}})$ *such that* $R = \mathbb{T} f_\omega$ *and* $R_{\mathrm{red}} = \mathbb{T}(f_{\mathrm{red}})_{\omega_{\mathrm{red}}}$ *are related by* $(\cdot/G_1)^*\omega_{\mathrm{red}} = \omega$ *.*

Proof. (a) Let $\widetilde{R}$ be the flow-out of R by $G = G_2 \times G_1$. Then $\widetilde{R} = \cup_{g\in G} g.R$ and each translation $g.R$ is a Dirac structure of $\mathbb{A}_2 \times \overline{\mathbb{A}_1}$ supported on $g.\mathrm{Gr}(f)$. From this it follows that $\widetilde{R}$ is a G-invariant Dirac structure of $\mathbb{A} = \mathbb{A}_2 \times \overline{\mathbb{A}_1}$ supported on $\widetilde{\mathrm{Gr}}(f)$, the flow-out of $\mathrm{Gr}(f)$ by G. As the intersection $R \cap (C_2^\perp \cap C_1^\perp$ has constant rank as previously observed, so does the intersection $\widetilde{R} \cap (C_2^\perp \times C_1^\perp)$. Thus $\widetilde{R}$ descends to a Dirac structure R_{red} of $\mathbb{A}_{2,\mathrm{red}} \times \overline{\mathbb{A}_{1,\mathrm{red}}}$ supported on $\mathrm{Gr}(\widetilde{f})/G$ according to the discussion following Proposition 2.4.1. The submanifold $\mathrm{Gr}(\widetilde{f})/G \hookrightarrow M_{2,\mathrm{red}} \times M_{1,\mathrm{red}}$ is the graph of f_{red}, in other words R_{red} is a Courant morphism $\mathbb{A}_{1,\mathrm{red}} \dashrightarrow \mathbb{A}_{2,\mathrm{red}}$ with base map f_{red}.

Now say $x_1 \sim_R x_2 \sim_{q_2} y_2$. Then $x_2 \in C_2$ and, since R intertwines the generators, one has $x_1 \in C_1$. So $x_1 \sim_{q_1} y_1 \sim_{R_{\mathrm{red}}} y_2$ where $y_1 = [x_1]$. Conversely, if $x_1 \sim_{q_1} y_1 \sim_{R_{\mathrm{red}}} y_2$, then $x_1 \sim_R x_2$ for some $x_2 \in C_2$ with $x_2 \sim_{q_2} y_2$. It follows that $x_1 \sim_{q_2\circ R} y_2$. This shows the diagram (2.23) commutes.

(b) It is claimed is that every element of $(E_{2,\mathrm{red}})|_{f_{\mathrm{red}}([m])}$ is R_{red}-related to a unique element of $(E_{1,\mathrm{red}})|_{[m]}$ for any $m \in M_1$,. Now suppose $y_2 \in (E_{2,\mathrm{red}})|_{f_{\mathrm{red}}([m])}$ and let $x_2 \in (E_2 \cap C_2)|_{f(m)}$ be a lift $x_2 \sim_{q_2} y_2$. Let x_1 be the unique element in $E_1|_m$ that is R-related to x_2. Since R intertwines generators, one has $x_1 \in C_1$. So $y_1 = [x_1]$ is R_{red}-related to y_2. This shows existence.

Say now $y_1 \sim_{R_{\mathrm{red}}} 0$ for some $y_1 \in E_{1,\mathrm{red}}$ and let $x_1 \in E_1 \cap C_1$ be a lift $x_1 \sim_{q_1} y_1$. Then $x_1 \sim_R x_2$ for some $x_2 \in E_2 \cap C_2^\perp$ since $R \circ E_1 = f^*E_2$. As x_1 is the unique element of $E_1|_m$ that is R-related to x_2 and the restriction (2.24) is a bijection, on concludes that $x_1 \in E_1 \cap C_1^\perp$. Thus $y_1 = 0$, showing uniqueness.

(c) It is shown how, starting with $R_{\mathrm{red}} : \mathbb{A}_{1,\mathrm{red}} \dashrightarrow \mathbb{A}_{2,\mathrm{red}}$, to obtain a Courant morphism $\mathbb{A}_1 \dashrightarrow \mathbb{A}_2$ intertwining generators, whereby R is recovered. Let $G = G_2 \times G_1$ and $M = M_2 \times M_1$. The preimage $\widetilde{R}$ of R_{red} under the quotient map $C \to \frac{C}{C^\perp}/G$, where $C = C_2 \times C_1$, is a G-invariant Lagrangian subbundle of $\mathbb{A} = \mathbb{A}_2 \times \overline{\mathbb{A}_1}$ supported on the flow-out of $\mathrm{Gr}(f)$ according to the discussion following Proposition 2.4.1.

Recall that, as the G_i-actions on M_i are free, the spaces C_i and $M_i \times \mathfrak{g}_i$ are equal as Lie algebroids. Let $D = \mathbf{a}^{-1}(\mathrm{Gr}(f_*))$, which is an involutive coisotropic subbundle along $\mathrm{Gr}(f)$ with isotropic orthogonal $D^\perp = \mathbf{a}^*(\mathrm{Gr}(f^*))$. As $\mathrm{Gr}(\phi) \subseteq D$, the subbundle D is invariant under the action of the diagonal $G_\Delta = \{(g, \Phi(g)) : g \in G_1\}$. Consider the Lagrangian subbundle

$$\widetilde{R} \cap D + D^\perp. \tag{2.25}$$

As $[\![D, D^\perp]\!] \subseteq D^\perp$ (Proposition 2.2.1), the space (2.25) is a Dirac structure supported on $\mathrm{Gr}(f)$, i.e. a Courant morphism $\mathbb{A}_1 \dashrightarrow \mathbb{A}_2$. Furthermore, clearly (2.25) contains $\mathrm{Gr}(\phi)$ and thus intertwines the generators.

Conversely, if the Courant morphism $R : \mathbb{A}_1 \dashrightarrow \mathbb{A}_2$ is the starting point, then $\widetilde{R}$ is the flow-out of R by G and, as $R \subseteq D$, the morphism R contains both summands of (2.25) and equality follows.

(d) Suppose R is exact. One knows that R is a Dirac morphism

$$R : (\mathbb{A}_1, \mathrm{ran}(\mathbf{a}_1^*)) \dashrightarrow (\mathbb{A}_2, \mathrm{ran}(\mathbf{a}_2^*))$$

by Proposition 2.3.1. The condition (2.24) then trivially holds and thus R_{red} is a Dirac morphism

$$R_{\mathrm{red}} : (\mathbb{A}_{1,\mathrm{red}}, \mathrm{ran}(\mathbf{a}_{1,\mathrm{red}}^*)) \dashrightarrow (\mathbb{A}_{2,\mathrm{red}}, \mathrm{ran}(\mathbf{a}_{2,\mathrm{red}}^*)),$$

meaning that R_{red} is exact.

Conversely, suppose R_{red} is exact. The morphism R is exact provided every element in $f^*\,\mathrm{ran}(\mathbf{a}_2^*)$ is R-related to a unique element in $\mathrm{ran}(\mathbf{a}_1^*)$. Existence is guaranteed since $\mathbf{a}^*(\mathrm{Gr}(f^*)) \subseteq R$ in all case. For uniqueness, suppose $x \sim_R 0$ where $x \in \mathrm{ran}(\mathbf{a}_1^*)$. Then $x \in C_1$ as R intertwines generators and consequently $[x] \in \mathrm{ran}(\mathbf{a}_{1,\mathrm{red}}^*)$ is R_{red}-related to 0. Since R_{red} is exact, $[x] = 0$, i.e. $x \in C_1^\perp$. But then $x \in \mathrm{ran}(\mathbf{a}_1^*) \cap C_1^\perp$, meaning that $x = 0$. Thus R is also exact.

Next recall that j_2 induces an isotropic splitting $j_1' : TM_1 \to \mathbb{A}_1$ of $\mathbb{A}_1$ whose range is $\mathrm{ran}(j_2) \circ R$. Since R is equivariant and intertwines the generators, it follows that j_1' is G_1-basic. Now

$$q_1 \circ \mathrm{ran}(j_1') = q_1 \circ (\mathrm{ran}(j_2) \circ R) = (q_2 \circ \mathrm{ran}(j_2)) \circ R_{\mathrm{red}},$$

where the second equality follows from the commutativity of (2.23). In other words, the isotropic splitting of $\mathbb{A}_{1,\mathrm{red}}$ to which j_1' descends coincides with the isotropic splitting of $\mathbb{A}_{1,\mathrm{red}}$ induced by $j_{2,\mathrm{red}}$. This means that, after identifying $\mathbb{A}_1 \simeq \mathbb{T}M_{1,f^*\eta_2}$ and $\mathbb{A}_{1,\mathrm{red}} \simeq \mathbb{T}(M_{1,\mathrm{red}})_{f_{\mathrm{red}}^*\eta_{2,\mathrm{red}}}$, the containment

$$\mathrm{ran}(\omega) = \mathrm{ran}(j_1) \subseteq \mathbb{T}M_{1,f^*\eta_2}$$

descends to the containment

$$\mathrm{ran}(\omega_{\mathrm{red}}) = \mathrm{ran}(j_{1,\mathrm{red}}) \subseteq \mathbb{T}(M_{1,\mathrm{red}})_{f_{\mathrm{red}}^*\eta_{2,\mathrm{red}}},$$

showing that $\omega = (\cdot/G_1)^*\omega_{\mathrm{red}}$. □

Chapter 3

L-Hamiltonian spaces

A theory of general group-valued moment maps is developed. Unlike the prior theory of Alekseev-Malkin-Meinrenken [4], the Hamiltonian action comes from an arbitrary Lagrangian subalgebra of the double $\mathfrak{d} \oplus \overline{\mathfrak{d}}$, where $\mathfrak{d}$ is a quadratic Lie algebra. Notions of fusion, duality and symplectic reduction are defined and various results are derived regarding these.

3.1 Hamiltonian spaces

Suppose $\mathbb{A}$ is a Courant algebroid over M and $E \subseteq \mathbb{A}$ is a Dirac structure. A *Hamiltonian space* of $(\mathbb{A}, E)$ is a manifold X together with a Dirac morphism $R : (\mathbb{T}X, TX) \dashrightarrow (\mathbb{A}, E)$. The base map $J : X \to M$ is then called the *moment map* of the Hamiltonian space X. Since $J^*E = R \circ TX$ and $TX \cap \ker(R) = 0$ by definition, the morphism R defines a vector bundle map $J^*E \to TX$. Consider the induced map

$$\Gamma(E) \to \mathfrak{X}(X), \;\; \sigma \mapsto \sigma_X. \tag{3.1}$$

Note that $\sigma_X \sim_R \sigma$ and, with R being a Courant morphism, this means that $\sigma \mapsto \sigma_X$ is a Lie algebra homomorphism. Moreover, the section σ_X is J-related to $\mathbf{a}(\sigma)$, where $\mathbf{a}$ is the anchor of $\mathbb{A}$. Thus R defines a comorphism of Lie algebroids $TX \dashrightarrow E$, in other words a Lie algebroid action of E on X along J.

Suppose now $\mathbb{A}$ is exact. A Hamiltonian space of $\mathbb{A}$ will be called *exact* if the Courant morphism $R : \mathbb{T}X \dashrightarrow \mathbb{A}$ is exact. Choosing an isotropic splitting $j : TM \to \mathbb{A}$ identifying $\mathbb{A} \simeq \mathbb{T}M_\eta$, exact Hamiltonian spaces are characterized as follows.

Proposition 3.1.1 ([17])**.** *An exact Hamiltonian space X of $(\mathbb{A}, E)$ is equivalently a manifold X together with an L-equivariant map $J : X \to M$, a 2-form $\omega \in \Omega^2(X)$ and a E-action $TX \dashrightarrow E$ on X along J such that*

(a) $d\omega = -J^*\eta$,

(b) $\ker(\omega) \cap \ker(J_*) = 0$,

(c) $\iota_{\sigma_X}\omega = J^*(j^*\sigma)$, *where* $\Gamma(E) \to \mathfrak{X}(X)$, $\sigma \mapsto \sigma_X$ *is the* E*-action.*

Proof. Condition (a) holds if and only if (J, ω) defines an exact Courant morphism $\mathbb{T}J_\omega : \mathbb{T}X \dashrightarrow \mathbb{A}$. It must simply be argued, with (a) holding, that (b) and (c) are equivalent to $\mathbb{T}J_\omega$ being a Dirac morphism $(\mathbb{T}X, TX) \dashrightarrow (\mathbb{A}, E)$, i.e. each element of $E|_{J(m)}$ is $\mathbb{T}J_\omega$-related to a unique element of T_mX for every $m \in X$.

Suppose now that (b) and (c) hold. From (2.17), one has the expression

$$\ker(R) = \{v - \iota_v\omega \ : \ v \in \ker(J_*)\}.$$

Uniqueness then immediately follows. On the other hand, the section σ_X being J-related to $\mathbf{a}(\sigma)$ together with (c) imply that $(\mathbf{a} \times j^*)(\sigma)$ is $\mathbb{T}J_\omega$-related to σ_X, establishing existence. Conversely if $\mathbb{T}J_\omega : (\mathbb{T}X, TX) \dashrightarrow (\mathbb{A}, E)$ is a Dirac morphism, then a symmetrical argument shows that (b) and (c) hold. □

Example 3.1.1. Let $\iota : \mathcal{O} \hookrightarrow M$ be a leaf of the singular distribution $\mathbf{a}(E) \subseteq TM$. Then for each tangent vector $v \in T\mathcal{O}$ there exists some $\mu \in T^*M$ such that $v + \mu \in E$. Define a 2-form on $\mathcal{O}$ by putting

$$\iota_v\omega_{\mathcal{O}} = \iota^*\mu. \tag{3.2}$$

To see that $\omega_{\mathcal{O}}$ is well-defined, suppose $\mu_1, \mu_2 \in T^*M$ are such that $(v, \mu_i) \in E$ for $i = 1, 2$. Then $(0, \mu_2 - \mu_1) \in E$ and it follows that $\mu_2 - \mu_1 \in \mathrm{Ann}(T\mathcal{O})$, i.e. $\iota^*(\mu_2 - \mu_1) = 0$. According to Proposition 2.2.1, the restriction of E to $\mathcal{O} \subseteq M$ is involutive and from this it follows that $d\omega_{\mathcal{O}} = -d\iota^*\eta$. Thus $\omega_{\mathcal{O}}$ defines an exact morphism $\mathbb{T}\iota_{\omega_{\mathcal{O}}} : \mathbb{T}\mathcal{O} \dashrightarrow \mathbb{T}M_\eta$. It is obvious from the definition of $\omega_{\mathcal{O}}$ and (2.17) that each element $(v, \mu) \in E|_{\mathcal{O}}$ is $\mathbb{T}\iota_{\omega_{\mathcal{O}}}$-related to a unique element of $T\mathcal{O}$, namely v. Thus $\mathcal{O}$ is an exact Hamiltonian space of $(\mathbb{A}, E)$.

Remark 3.1.1. In the context Proposition 3.1.1, if the E-action on X is transitive, that is to say the assignment (3.1) is surjective onto $\mathfrak{X}(X)$, then the 2-form $\omega \in \Omega^2(X)$ is necessarily unique for this particular choice of E-action and moment map since it is then completely determined by (c). In particular, the 2-form $\omega_{\mathcal{O}}$ defined in Example 3.1.1 is the unique one for which the leaf $\mathcal{O}$ together with the natural E-action and inclusion as the moment map is an exact Hamiltonian space for $\mathbb{T}M_\eta$.

3.2 L-Hamiltonian spaces

3.2.1 Manin pairs determined by quadratic Lie algebras

Of particular interest in the category of Hamiltonian spaces are those where the target Manin pair $(\mathbb{A}, E)$ arises from a quadratic Lie algebra $\mathfrak{d}$ and Lagrangian subalgebra $\mathfrak{l}$ of $\mathfrak{d} \oplus \overline{\mathfrak{d}}$. Let D be a connected Lie group integrating $\mathfrak{d}$. Consider the quadratic Lie algebra of split signature $\mathfrak{d} \oplus \overline{\mathfrak{d}}$, which integrates to the Lie group $D \times D$. The group $D \times D$ acts transitively on D as $(d_1, d_2) \cdot d = d_1 d d_2^{-1}$. The corresponding $\mathfrak{d} \oplus \overline{\mathfrak{d}}$-action is given by $\mathbf{a}_D : \mathfrak{d} \oplus \overline{\mathfrak{d}} \to \mathfrak{X}(D)$ with $\mathbf{a}_D(\xi_1, \xi_2) = \xi_2^L - \xi_1^R$ where ξ_i^L and ξ_i^R are respectively the left- and right-invariant vector fields corresponding to ξ_i. The stabilizer subalgebra at the identity $e \in D$ is the diagonal $\mathfrak{d}_\Delta \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$, which is Lagrangian. Since $D \times D$ acts transitively, it follows that the stabilizer subalgebras at every group element $d \in D$ is Lagrangian and in fact given explicitly by

$$\ker(\mathbf{a}_D|_d) = \{(\mathrm{Ad}_d\, \xi, \xi) \;:\; \xi \in \mathfrak{d}\}. \tag{3.3}$$

The above thus determines an action Courant algebroid $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$ with anchor $\mathbf{a}_D$. It is exact, admitting as a $D \times D$-invariant isotropic splitting

$$j_D(v) = \frac{1}{2}(-\iota_v \theta^R, \iota_v \theta^L). \tag{3.4}$$

where $v \in \mathfrak{X}(D)$ and $\theta^L, \theta^R : \mathfrak{X}(D) \to \mathfrak{d}$ are respectively the left- and right-invariant Maurer-Cartan forms . The corresponding 3-form is the Cartan 3-form

$$\eta_D = \frac{1}{12}\langle [\theta^L, \theta^L], \theta^L \rangle, \tag{3.5}$$

where $\langle \cdot, \cdot \rangle$ is the metric of $\mathfrak{d}$.

Consider now a Lagrangian subalgebra $\mathfrak{l} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ integrating to the connected subgroup $L \subseteq D \times D$. The subbundle $E^{(\mathfrak{l})} = D \times \mathfrak{l}$ is a Dirac structure of $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$ as well as an action Lie algebroid (Examples 2.2.5 and 2.2.7).If X is an exact Hamiltonian space of $(D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}), E^{(\mathfrak{l})})$ with moment map $J : X \to D$ then restricting the assignment (3.1) to $\mathfrak{l}$ (identified with the constant sections) gives a Lie algebra action $\varrho : \mathfrak{l} \to \mathfrak{X}^1(X)$. As $E^{(\mathfrak{l})}$ is an action Lie algebroid, the pullback bundle $J^* E^{(\mathfrak{l})} = X \times \mathfrak{l}$ is the action Lie algebroid corresponding to ϱ and J is $\mathfrak{l}$-equivariant by construction.

Remark 3.2.1. The Dirac structure $E^{(\mathfrak{d}_\Delta)}$ has a special name in the literature: it is the so-called *Cartan-Dirac structure* [1, §3.4].

Definition 3.2.1. Let $\mathfrak{l} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ be a Lagrangian subalgebra integrating to the closed and connected subgroup $L \subseteq D^2$. An *L-Hamiltonian space* (for $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$) is an exact Hamiltonian space

$$R : (\mathbb{T}X, TX) \dashrightarrow (D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}), E^{(\mathfrak{l})})$$

such that the induced $\mathfrak{l}$-action on X integrates to an L-action.

Note that, since the Lie group L is connected and R intertwines the generators of the $\mathfrak{l}$-actions on X and D by definition, the morphism R must be L-equivariant by Lemma 2.4.3.

Proposition 3.2.1 ([17]). *A L-Hamiltonian space is equivalently a triple (X, J, ω) where X is a L-manifold, the moment map $J : X \to D$ is an L-equivariant map and $\omega \in \Omega^2(X)$ is an L-invariant 2-form such that*

(a) $d\omega = -J^*\eta_D$,

(b) $\ker(\omega) \cap \ker(J_*) = 0$,

(c) $\iota_{\varrho(\xi_1,\xi_2)}\omega = -\frac{1}{2}J^*(\langle \xi_2, \theta^L\rangle + \langle \xi_1, \theta^R\rangle)$ *for all* $(\xi_1, \xi_2) \in \mathfrak{l}$, *where* $\varrho : \mathfrak{l} \to \mathfrak{X}(X)$ *is the infinitesimal action.*

Proof. This is simply a rewording of Proposition 3.1.1, save for the L-invariance of ω. But the latter simply follows from the fact that the isotropic splitting (3.4) is $D \times D$-invariant. Thus it must only be shown that (c) here is equivalent to (c) in Proposition 3.1.1. Consider dual $j_D^* : D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}) \to T^*D$ of the isotropic splitting (3.4). Then for $(\xi_1, \xi_2) \in \mathfrak{d} \oplus \overline{\mathfrak{d}}$ and $v \in \mathfrak{X}(D)$ one has

$$\langle j_D^*(\xi_1, \xi_2), v\rangle = \langle (\xi_1, \xi_2), j_D(v)\rangle = -\frac{1}{2}(\langle \xi_2, \iota_v\theta^L\rangle + \langle \xi_1, \iota_v\theta^R\rangle),$$

which is what needed to be shown. □

Example 3.2.1. Adapting Example 3.1.1, an $\mathfrak{l}$-orbit $\mathcal{O} \hookrightarrow D$ is a L-Hamiltonian space.

Example 3.2.2. With $\mathfrak{l} = \mathfrak{d}_\Delta$, the previous proposition gives precisely the axioms defining a quasi-Hamiltonian D-space [4]. In other words, q-Hamiltonian D-spaces and D_Δ-Hamiltonian spaces are the same objects.

3.2.2 Multiplicative structures

Multiplicative structure of $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$. In [1], a natural multiplication morphism[1] Mult : $D^2 \times (\mathfrak{d} \oplus \overline{\mathfrak{d}} \oplus \mathfrak{d} \oplus \overline{\mathfrak{d}}) \to D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$ is defined over the group multiplication map mult :

[1] By convention $D^2 \times (\mathfrak{d} \oplus \overline{\mathfrak{d}} \oplus \mathfrak{d} \oplus \overline{\mathfrak{d}}) = D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}) \times D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$ where the i^{th} $\mathfrak{d}$-factor on the left-hand-side corresponds to the i^{th} $\mathfrak{d}$-factor on the right-hand-side.

$D \times D \to D$. In terms of the procedure for manufacturing Courant morphisms between action Courant algebroids given in Example 2.3.2, the morphism Mult is determined by the graph in $\mathfrak{d} \times \bar{\mathfrak{d}} \times \mathfrak{d}$ of the multiplication of pair groupoid $\mathfrak{d} \oplus \bar{\mathfrak{d}} \rightrightarrows \mathfrak{d}$. It may also be seen as the quotient morphism corresponding to a coisotropic reduction of $D^2 \times (\mathfrak{d} \oplus \bar{\mathfrak{d}} \oplus \mathfrak{d} \oplus \bar{\mathfrak{d}})$ and this approach will prove quite profitable.

Let $\mathfrak{d}_{\Delta(2,3)} = 0 \oplus \mathfrak{d}_\Delta \oplus 0$, which is an isotropic subalgebra of $\mathfrak{d} \oplus \bar{\mathfrak{d}} \oplus \mathfrak{d} \oplus \bar{\mathfrak{d}}$ with coisotropic orthogonal $\mathfrak{d}^\perp_{\Delta(2,3)} = \mathfrak{d} \oplus \mathfrak{d}_\Delta \oplus \mathfrak{d}$. Then the elements $E^{(\mathfrak{d}_{\Delta(2,3)})}$ are generators for the action of $D_{\Delta(2,3)} = e \times D_\Delta \times e$ on $D^2 \times (\mathfrak{d} \oplus \bar{\mathfrak{d}} \oplus \mathfrak{d} \oplus \bar{\mathfrak{d}})$ The coisotropic reduction of $D^2 \times (\mathfrak{d} \oplus \bar{\mathfrak{d}} \oplus \mathfrak{d} \oplus \bar{\mathfrak{d}})$ by $E^{(\mathfrak{d}_{\Delta(2,3)})}$ gives

$$D^2/D_{\Delta(2,3)} \times \mathfrak{d}_{\Delta(2,3)}{}^\perp/\mathfrak{d}_{\Delta(2,3)} \simeq D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}}), \tag{3.6}$$

where D has been identified with the second factor of D^2. Define

$$\mathrm{Mult} : D^2 \times (\mathfrak{d} \oplus \bar{\mathfrak{d}} \oplus \mathfrak{d} \oplus \bar{\mathfrak{d}}) \dashrightarrow D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}})$$

as the quotient morphism corresponding to the coisotropic reduction (3.6). Since $(d_1, d_2) \sim_{D_{\Delta(2,3)}} (d_1 d_2, e)$ for $d_1, d_2 \in D$, the base map of Mult is simply the group multiplication, denoted by $\mathrm{mult} : D \times D \to D$.

Note that, omitting base points,

$$(\xi_1, \xi_2, \xi_3, \xi_4) \sim_{\mathrm{Mult}} (\xi_1', \xi_2') \iff \xi_1 = \xi_1', \quad \xi_2 = \xi_3, \quad \xi_4 = \xi_2'$$

for $\xi_i, \xi_i' \in \mathfrak{d}$ considered as constant sections of $D \times \mathfrak{d}$. Thus Mult can alternatively be described as the product $\mathrm{mult} \times \mathrm{Gr}(\circ)$, where $\circ$ is the multiplication in the pair groupoid

$$\mathfrak{d} \oplus \bar{\mathfrak{d}} \rightrightarrows \mathfrak{d}. \tag{3.7}$$

In the sequel, the groupoid multiplication $\circ$ will also be denoted by $\mathrm{Mult} : \mathfrak{d} \oplus \bar{\mathfrak{d}} \times \mathfrak{d} \oplus \bar{\mathfrak{d}} \dashrightarrow \mathfrak{d} \oplus \bar{\mathfrak{d}}$ when one wishes to see it as a Courant morphism.

Proposition 3.2.2 ([1]). *The Courant morphism* $\mathrm{Mult} : D^2 \times (\mathfrak{d} \oplus \bar{\mathfrak{d}} \oplus \mathfrak{d} \oplus \bar{\mathfrak{d}}) \dashrightarrow D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}})$ *is exact. Furthermore, the* 2*-form* $\varsigma \in \Omega^2(D^2)$

$$\varsigma = \tfrac{1}{2}\langle \theta^{L,1}, \theta^{R,2} \rangle \tag{3.8}$$

is the 2*-form* (2.9)*. In particular, it relates* $\eta_D^1 + \eta_D^2$ *and* $\mathrm{mult}^*\eta_D$ *via*

$$d\varsigma = \eta_D^1 + \eta_D^2 - \mathrm{mult}^*\eta_D. \tag{3.9}$$

Proof. In view of Proposition 2.4.2, to show that Mult is exact it suffices to show that the intersection of $\mathrm{ran}(\mathbf{a}^*)\oplus\mathrm{ran}(\mathbf{a}^*)$ and $E^{(\mathfrak{d}_{\Delta(2,3)})}$ is trivial. Recall that the space $\mathrm{ran}(\mathbf{a}^*) = \ker(\mathbf{a})$ at $d \in D$ is given by

$$\mathrm{ran}(\mathbf{a}^*|_d) = \{(\mathrm{Ad}_d\,\xi, \xi)\ :\ \xi \in \mathfrak{d}\}.$$

For $d_1, d_2 \in D$ and $\xi, \xi', \xi'' \in \mathfrak{d}$, the equation

$$(\mathrm{Ad}_{d_1}\,\xi, \xi, \mathrm{Ad}_{d_2}\,\xi', \xi') = (0, \xi'', \xi'', 0)$$

has the unique solution $\xi = \xi' = \xi'' = 0$, whence Mult is exact.

Next, let $j_{\mathrm{mult}} : TD^2 \to D^2 \times (\mathfrak{d} \oplus \bar{\mathfrak{d}} \oplus \mathfrak{d} \oplus \bar{\mathfrak{d}})$ be the induced isotropic splitting as in Proposition 2.3.2. Identify $D\times(\mathfrak{d}\oplus\bar{\mathfrak{d}}) \simeq \mathbb{T}\,D_{\eta_D}$ via j_D and $D^2\times(\mathfrak{d}\oplus\bar{\mathfrak{d}}\oplus\mathfrak{d}\oplus\bar{\mathfrak{d}}) \simeq \mathbb{T}\,D^2_{\mathrm{mult}^*\eta_D}$ via j_{Mult}. Consider the image $\mathrm{ran}(j_D \times j_D)$ of the isotropic splitting $j_D \times j_D : TD^2 \to D^2 \times (\mathfrak{d} \oplus \bar{\mathfrak{d}} \oplus \mathfrak{d} \oplus \bar{\mathfrak{d}})$ in $\mathbb{T}\,D^2_{\mathrm{mult}^*\eta_D}$, which is the graph of a 2-form $\varsigma \in \Omega^2(D^2)$ satisfying (3.9). It will be shown that ς is given by (3.8). To compute ς, notice that it is invariant under the action of $D \times D$ on D^2 via

$$(d_1, d_2).(c_1, c_2) = (d_1 c_1, c_2 d_2^{-1})$$

since both $\mathrm{mult}^*\eta_D$ and $\eta_D^1 + \eta_D^2$ are $D \times D$-invariant. From Proposition 2.3.2 and its proof,

$$\mathrm{Gr}(-\varsigma|_{(d_1,d_2)}) = \mathrm{Mult}|_{(d_1,d_2)} \circ \mathrm{ran}(j_D|_{d_1 d_2}) = \{(-\mathrm{Ad}_{d_1}\,\xi, \xi', \xi', \mathrm{Ad}_{d_2^{-1}}\,\xi)\ :\ \xi, \xi' \in \mathfrak{d}\},$$

which, together with (3.4) and $D \times D$-invariance, allows one to compute (recall (2.10))

$$\iota_{v'}\iota_v\varsigma = -\langle j_{\mathrm{mult}}(v), (j_D \times j_D)(v')\rangle = \tfrac{1}{2}\langle \iota_v\theta^{L,1}, \iota_{v'}\theta^{R,2}\rangle - \tfrac{1}{2}\langle \iota_{v'}\theta^{L,1}, \iota_v\theta^{R,2}\rangle$$

for $v, v' \in \mathfrak{X}(D^2)$, which is to say ς is given by (3.8). □

Multiplicative structure of the Lagrangian subalgebras of $\mathfrak{d}\oplus\bar{\mathfrak{d}}$. For subspaces $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{d}\oplus\bar{\mathfrak{d}}$, let $\mathfrak{a} \circ \mathfrak{b}$ denote set of all elements of $\mathfrak{d} \oplus \bar{\mathfrak{d}}$ resulting from the product of elements in $\mathfrak{a}$ and $\mathfrak{b}$, in that order. Note that

$$\mathfrak{a} \circ \mathfrak{b} = \frac{(\mathfrak{a} \times \mathfrak{b}) \cap \mathfrak{d}^{\perp}_{\Delta(2,3)}}{\mathfrak{d}_{\Delta(2,3)}}.$$

In particular, if $\mathfrak{a}$ and $\mathfrak{b}$ are Lagrangian subalgebras of $\mathfrak{d}\oplus\bar{\mathfrak{d}}$ then so is $\mathfrak{a}\circ\mathfrak{b}$ according to Section 2.1.

The variety $\mathcal{L}(\mathfrak{d} \oplus \bar{\mathfrak{d}})$ of Lagrangian subalgebras of $\mathfrak{d} \oplus \bar{\mathfrak{d}}$ may thus be said to be "closed under multiplication". Expanding on this idea, one has

$$\mathfrak{d}_\Delta \circ \mathfrak{l} = \mathfrak{l} \circ \mathfrak{d}_\Delta = \mathfrak{l}, \quad \mathfrak{l}_1 \circ (\mathfrak{l}_2 \circ \mathfrak{l}_3) = (\mathfrak{l}_1 \circ \mathfrak{l}_2) \circ \mathfrak{l}_3 \tag{3.10}$$

for any Lagrangian subalgebras $\mathfrak{l}, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3 \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$, which means that $\mathcal{L}(\mathfrak{d} \oplus \overline{\mathfrak{d}})$ is a monoid. The most obvious examples of multiplicative Lagrangian subalgebras are the diagonal $\mathfrak{d}_\Delta \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ and the direct sum $\mathfrak{g} \oplus \mathfrak{h}$ of Lagrangian subalgebras $\mathfrak{g}, \mathfrak{h} \subseteq \mathfrak{d}$. In general, multiplicative Lagrangian subalgebras are classified by pairs of coisotropic subalgebras of $\mathfrak{d}$.

Theorem 3.2.1 (Classification of multiplicative Lagrangian subalgebras)**.** *A subset $\mathfrak{l} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ is a multiplicative Lagrangian subalgebra if and only if*

$$\mathfrak{l} = ((\mathfrak{c}_1 \oplus \mathfrak{c}_2) \cap \mathfrak{d}_\Delta) + (\mathfrak{c}_1^\perp \oplus \mathfrak{c}_2^\perp), \tag{3.11}$$

where $\mathfrak{c}_1, \mathfrak{c}_2 \subseteq \mathfrak{d}$ are coisotropic subalgebras of $\mathfrak{d}$.

Proof. In one direction, assume $\mathfrak{c}_1, \mathfrak{c}_2 \subseteq \mathfrak{d}$ are coisotropic subalgebras. Then the orthogonal of (3.11) is

$$((\mathfrak{c}_1^\perp \oplus \mathfrak{c}_2^\perp) + \mathfrak{d}_\Delta) \cap (\mathfrak{c}_1 \oplus \mathfrak{c}_2).$$

Since $\mathfrak{c}_i^\perp \subseteq \mathfrak{c}_i$ ($i = 1, 2$), the above is actually equal to (3.11), which is ipso facto Lagrangian. By part (d) of Proposition 2.2.1, the isotropic orthogonals $\mathfrak{c}_1^\perp, \mathfrak{c}_2^\perp \subseteq \mathfrak{d}$ are subalgebras of $\mathfrak{d}$ and are ideals in their respective coisotropic orthogonals, i.e. $\left[\mathfrak{c}_i^\perp, \mathfrak{c}_i\right] \subseteq \mathfrak{c}_i^\perp$. From this it follows that (3.11) is a (Lagrangian) subalgebra of $\mathfrak{d} \oplus \overline{\mathfrak{d}}$. Finally, as $\mathfrak{d}_\Delta$ is multiplicative and

$$(\mathfrak{c}_1^\perp \oplus \mathfrak{c}_2^\perp) \circ (\mathfrak{c}_1^\perp \oplus \mathfrak{c}_2^\perp) = \mathfrak{c}_1^\perp \oplus \mathfrak{c}_2^\perp,$$

one concludes that (3.11) is multiplicative as well.

In the other direction suppose $\mathfrak{l} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ is a multiplicative Lagrangian subalgebra. Let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be the projections of $\mathfrak{l}$ onto the first and second factors of $\mathfrak{d} \oplus \overline{\mathfrak{d}}$ respectively. Then $\mathfrak{c}_1^\perp \oplus 0$ is orthogonal to $\mathfrak{l}$ and thus contained in $\mathfrak{l}$; the same goes for $0 \oplus \mathfrak{c}_2^\perp$. From this it follows that $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are coisotropic in $\mathfrak{d}$. Now suppose $(\xi, \xi) \in (\mathfrak{c}_1 \oplus \mathfrak{c}_2) \cap \mathfrak{d}_\Delta$. Then there exist elements $\xi', \xi'' \in \mathfrak{d}$ such that $(\xi, \xi'), (\xi'', \xi) \in \mathfrak{l}$. As $\mathfrak{l}$ is multiplicative, it contains the pair

$$(\xi'', \xi) \circ (\xi, \xi') = (\xi'', \xi').$$

But it also contains the pair

$$(\xi, \xi') + (\xi'', \xi) - (\xi'', \xi') = (\xi, \xi),$$

and thus $(\mathfrak{c}_1 \oplus \mathfrak{c}_2) \cap \mathfrak{d}_\Delta \subseteq \mathfrak{l}$. So $\mathfrak{l}$ contains the right-hand-side of (3.11) and, as both are Lagrangian, equality must hold. □

Example 3.2.3. The example $\mathfrak{l} = \mathfrak{d}_\Delta$ corresponds to the choices $\mathfrak{c}_1 = \mathfrak{c}_2 = \mathfrak{d}$ in (3.11), i.e. $\mathfrak{c}_1$ and $\mathfrak{c}_2$ have maximal dimensions. Incidentally, this shows that $\mathfrak{d}_\Delta$ is the only multiplicative Lagrangian subalgebra of $\mathfrak{d} \oplus \overline{\mathfrak{d}}$ if the metric of $\mathfrak{d}$ is positive- or negative-definite, which is the case when $\mathfrak{d}$ is simple and compact for instance. The other extreme, where $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are of minimal dimension, corresponds to $\mathfrak{l} = \mathfrak{g} \oplus \mathfrak{h}$ where $\mathfrak{g}, \mathfrak{h} \subseteq \mathfrak{d}$ are Lagrangian subalgebras of $\mathfrak{d}$. (Of course, the metric of $\mathfrak{d}$ must be of split signature in that case.) There are a number of known coisotropic subalgebras and therefore ways of generating multiplicative Lagrangian subalgebras, see [60].

Corollary 3.2.1. *Let $\mathfrak{l} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ be a multiplicative Lagrangian subalgebra and let $\iota : \mathcal{O} \hookrightarrow D$ be the leaf through the group identity $e \in D$ of the singular distribution $\mathbf{a}(E^{(\mathfrak{l})}) \subseteq TD$. Then the* 2*-form* (3.2) *defined in Example 3.1.1 vanishes. In particular $\mathcal{O}$, provided the connected subgroup $L \subseteq D \times D$ integrating $\mathfrak{l}$ is closed, is a L-Hamiltonian space with corresponding Courant morphism $\mathbb{T}\iota : \mathbb{T}\mathcal{O} \dashrightarrow \mathbb{T}D_\eta$.*

Proof. It is sufficient to show that the 2-form (3.2) vanishes at the group identity $e \in \mathcal{O}$. By the previous theorem, the subalgebra $\mathfrak{l}$ is equal to $(\mathfrak{c}_1 \oplus \mathfrak{c}_2) \cap \mathfrak{d}_\Delta + \mathfrak{c}_1^\perp \oplus \mathfrak{c}_2^\perp$ where $\mathfrak{c}_1, \mathfrak{c}_2 \subseteq \mathfrak{d}$ are coisotropic subalgebras. Now suppose $(\xi_i, \xi_i) + (\xi_i', \xi_i'') \in \mathfrak{l}\ (i = 1, 2)$ are elements of $\mathfrak{l}$ where $\xi_i \in \mathfrak{c}_1 \cap \mathfrak{c}_2$ and $\xi_i' \in \mathfrak{c}_1^\perp$ and $\xi_i'' \in \mathfrak{c}_2^\perp$. Then the anchor $\mathbf{a}_D : \mathfrak{d} \oplus \overline{\mathfrak{d}} \to \mathfrak{d}$ sends these elements to the vectors $\xi_i'' - \xi_i' \in \mathfrak{d}$ and the dual $j_D^* : \mathfrak{d} \oplus \overline{\mathfrak{d}} \to \mathfrak{d}^*$ of the splitting (3.4) sends them to the 1-forms $-\frac{1}{2}\langle 2\xi_i + \xi_i' + \xi_i'', \cdot\rangle$, all respectively. But it is clear that

$$-\frac{1}{2}\langle 2\xi_1 + \xi_1' + \xi_1'', \xi_2'' - \xi_2'\rangle = 0,$$

which implies that the 2-form (3.2), by way of its definition, vanishes. □

At the level of Lie groups, if $\mathfrak{l}_1, \mathfrak{l}_2 \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ are Lagrangian subalgebras integrating to the connected subgroups $L_1, L_2 \subseteq D^2$ respectively then the connected subgroup $\mathrm{Lie}(\mathfrak{l}_1 \circ \mathfrak{l}_2)$ is the identity component of the quotient group

$$L_1 \circ L_2 = \frac{(L_1 \times L_2) \cap (D \times D_\Delta \times D)}{(L_1 \times L_2) \cap (e \times D_\Delta \times e)} \subseteq \frac{D \times D_\Delta \times D}{e \times D_\Delta \times e} = D^2, \tag{3.12}$$

where $D_\Delta \subseteq D^2$ is the diagonal of D. In particular, if L_1 and L_2 are closed then so is $\mathrm{Lie}(\mathfrak{l}_1 \circ \mathfrak{l}_2)$.

3.2.3 Fusion product of L-Hamiltonian spaces

The existence of a monoidal structure on the space of Lagrangian subalgebras of $\mathfrak{d}\oplus\overline{\mathfrak{d}}$ (and consequently on the space of connected subgroups of D^2 integrating these) motivates the search for an analogous monoidal structure on the category of L-Hamiltonian spaces. Suppose, for $i = 1, 2$, that $\mathfrak{l}_i \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ are Lagrangian subalgebras integrating to closed and connected subgroups $L_i \subseteq D$ and that L_i-Hamiltonian spaces

$$R_i : (\mathbb{T}X_i, TX_i) \dashrightarrow (D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}), E^{(\mathfrak{l}_i)})$$

with moment maps $J_i : X_i \to D$ are given. A reasonable multiplication procedure for L-Hamiltonian spaces should produce a Lie($\mathfrak{l}_1 \circ \mathfrak{l}_2$)-Hamiltonian space from X_1 and X_2. As a first attempt, consider the direct product $X_1 \times X_2$ together with the composition

$$\mathrm{Mult} \circ (R_1 \times R_2) : \mathbb{T}X_1 \times \mathbb{T}X_2 \dashrightarrow D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}).$$

Unfortunately, this last Courant morphism may fail to be a Dirac morphism (see Definition 2.3.2)

$$(\mathbb{T}X_1 \times \mathbb{T}X_2, TX_1 \times TX_2) \dashrightarrow (D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}), E^{(\mathfrak{l}_1 \circ \mathfrak{l}_2)}).$$

Indeed, although $R \circ (TX_1 \times TX_2) = E^{(\mathfrak{l}_1 \oplus \mathfrak{l}_2)}$, the intersection of $E^{(\mathfrak{l}_1 \oplus \mathfrak{l}_2)}$ and $\ker(\mathrm{Mult}) = E^{(\mathfrak{d}_{\Delta(2,3)})}$ may not be trivial. This intersection is of course $E^{((\mathfrak{l}_1 \oplus \mathfrak{l}_2) \cap \mathfrak{d}_{\Delta(2,3)})}$ and this pathology therefore rests on the size of the subalgebra

$$(\mathfrak{l}_1 \oplus \mathfrak{l}_2) \cap \mathfrak{d}_{\Delta(2,3)}.$$

For $\mathfrak{l}_1 = \mathfrak{l}_2 = \mathfrak{d}_\Delta$, for instance, the subalgebra (3.2.3) is trivial and $X_1 \times X_2$ is indeed a D_Δ-Hamiltonian space, equivalently a quasi-Hamiltonian D-space [4]. For $\mathfrak{l}_1 = \mathfrak{l}_2 = \mathfrak{g} \oplus \mathfrak{g}$ on the other hand, where $\mathfrak{g} \subseteq \mathfrak{d}$ is a Lagrangian subalgebra, it is equal to $0 \oplus \mathfrak{g}_\Delta \oplus 0$. The following construction, which is central to this thesis, deals with this pathology.

Definition 3.2.2 (Fusion product)**.** Let $\mathrm{Lie}((\mathfrak{l}_1 \oplus \mathfrak{l}_2) \cap \mathfrak{d}_{\Delta(2,3)})$ be the connected (and necessarily closed) subgroup of D^4 integrating $(\mathfrak{l}_1 \oplus \mathfrak{l}_2) \cap \mathfrak{d}_{\Delta(2,3)}$. The *fusion product* $X_1 \circledast X_2$ of X_1 and X_2 is the quotient

$$X_1 \circledast X_2 = \frac{X_1 \times X_2}{\mathrm{Lie}((\mathfrak{l}_1 \oplus \mathfrak{l}_2) \cap \mathfrak{d}_{\Delta(2,3)})}. \tag{3.13}$$

Lemma 3.2.1. *The space $X_1 \circledast X_2$ is a smooth manifold.*

Proof. The $\mathrm{Lie}((\mathfrak{l}_1 \oplus \mathfrak{l}_2) \cap \mathfrak{d}_{\Delta(2,3)})$-action on $D \times D$ is clearly free and proper. As $J_1 \times J_2$:

$X_1 \times X_2 \to D \times D$ is $L_1 \times L_2$-equivariant, it follows that the $\mathrm{Lie}((\mathfrak{l}_1 \oplus \mathfrak{l}_2) \cap \mathfrak{d}_{\Delta(2,3)})$-action on $X_1 \times X_2$ is also be free and proper, whence $X_1 \circledast X_2$ is a smooth manifold. □

Theorem 3.2.2. *Let X_1 and X_2 as the previous definition. Then the fusion product $X_1 \circledast X_2$ is a $\mathrm{Lie}(\mathfrak{l}_1 \circ \mathfrak{l}_2)$-Hamiltonian space.*

Proof. Let $\varrho : X_1 \times X_2 \times (\mathfrak{l}_1 \oplus \mathfrak{l}_2) \to TX_1 \oplus TX_2$ be the $\mathfrak{l}_1 \oplus \mathfrak{l}_2$-action on $X_1 \times X_2$, and let $C^\perp$ be the image of $X_1 \times X_2 \times ((\mathfrak{l}_1 \oplus \mathfrak{l}_2) \cap \mathfrak{d}_{\Delta(2,3)})$ under ϱ. Then by definition

$$R \circ E^{(\mathfrak{l}_1 \oplus \mathfrak{l}_2)} = J^* \operatorname{ran}(\varrho) \implies R \circ E^{((\mathfrak{l}_1 \oplus \mathfrak{l}_2) \cap \mathfrak{d}_{\Delta(2,3)})} = J^* C^\perp \implies J^* C^\perp \subseteq R \circ E^{(\mathfrak{d}_{\Delta(2,3)})}$$

where $R = R_1 \times R_2$ and $J = J_1 \times J_2$. Thus R intertwines the generators of the $(L_1 \times L_2) \cap D_{\Delta(2,3)}$-action on $\mathbb{T}X_1 \times \mathbb{T}X_2$ and the $D_{\Delta(2,3)}$-action on $D^2 \times (\mathfrak{d} \oplus \bar{\mathfrak{d}} \oplus \mathfrak{d} \oplus \bar{\mathfrak{d}})$. Since R is $L_1 \times L_2$-equivariant, it is also equivariant with respect to the inclusion of $\mathrm{Lie}((\mathfrak{l}_1 \oplus \mathfrak{l}_2) \cap \mathfrak{d}_{\Delta(2,3)})$ into $D_{\Delta(2,3)}$.

Now part (a) of Theorem 2.4.2 gives a commutative diagram

$$\begin{array}{ccc} \mathbb{T}X_1 \times \mathbb{T}X_2 & \overset{R}{\dashrightarrow} & D^2 \times (\mathfrak{d} \oplus \bar{\mathfrak{d}} \oplus \mathfrak{d} \oplus \bar{\mathfrak{d}}) \\ \downarrow q & & \downarrow \mathrm{Mult} \\ \mathbb{T}(X_1 \circledast X_2) & \overset{R_{\mathrm{red}}}{\dashrightarrow} & D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}}) \end{array} \tag{3.14}$$

where the vertical arrows are the reduction morphisms. Since $E^{((\mathfrak{l}_1 \oplus \mathfrak{l}_2) \cap \mathfrak{d}_{\Delta(2,3)})} \subseteq E^{(\mathfrak{l}_1 \oplus \mathfrak{l}_2)}$ and $C^\perp \subseteq TX$, parts (b)-(d) of Theorem 2.4.2 imply that R_{red} is exact in addition to being a Dirac morphism

$$R_{\mathrm{red}} : (\mathbb{T}(X_1 \circledast X_2), T(X_1 \circledast X_2)) \dashrightarrow (D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}}), E^{(\mathfrak{l}_1 \circ \mathfrak{l}_2)}),$$

where the fact that $E^{(\mathfrak{l}_1 \circ \mathfrak{l}_2)} = E^{(\mathfrak{l}_1 \oplus \mathfrak{l}_2)}_{\mathrm{red}}$ was used. Thus $X_1 \circledast X_2$ is an exact Hamiltonian space for $(D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}}), E^{(\mathfrak{l}_1 \circ \mathfrak{l}_2)})$.

The $\mathfrak{l}_1 \circ \mathfrak{l}_2$-action on $X_1 \circledast X_2$ coincides with the map $\varrho_{\mathrm{red}} : \mathfrak{l}_1 \circ \mathfrak{l}_2 \to \mathfrak{X}(X_1 \circledast X_2)$ to which the restriction of ϱ to $(\mathfrak{l}_1 \oplus \mathfrak{l}_2) \cap \mathfrak{d}^\perp_{\Delta(2,3)}$ descends (now seeing ϱ as a map $\mathfrak{l}_1 \oplus \mathfrak{l}_2 \to \mathfrak{X}(X_1 \times X_2)$). Thus the $\mathrm{Lie}(\mathfrak{l}_1 \circ \mathfrak{l}_2)$-action on $X_1 \circledast X_2$ to which the $\mathrm{Lie}((\mathfrak{l}_1 \times \mathfrak{l}_2) \cap (\mathfrak{d} \times \mathfrak{d}_\Delta \times \mathfrak{d}))$-action on $X_1 \times X_2$ descends integrates the $\mathfrak{l}_1 \circ \mathfrak{l}_2$-action on $X_1 \circledast X_2$. The claim is thus proven. □

Associativity. Suppose X_1, X_2 and X_2 are L_1-, L_2- and L_3-Hamiltonian spaces, respectively.

Theorem 3.2.3. *The fusion product is associative, that is* $(X_1 \circledast X_2) \circledast X_3$ *and* $X_1 \circledast (X_2 \circledast X_3)$ *are canonically isomorphic as* $\mathrm{Lie}(\mathfrak{l}_1 \circ \mathfrak{l}_2 \circ \mathfrak{l}_2)$*-Hamiltonian spaces.*

Proof. Let $D_{\Delta(2,3)} = e \times D_\Delta \times e \times e \times e$ and $D_{\Delta(4,5)} = e \times e \times e \times D_\Delta \times e$. Rewrite

$$\begin{aligned}(X_1 \circledast X_2) \circledast X_3 &= \frac{\frac{X_1 \times X_2}{\mathrm{Lie}((\mathfrak{l}_1 \times \mathfrak{l}_2) \cap \mathfrak{d}_{\Delta(2,3)})} \times X_3}{\mathrm{Lie}(((\mathfrak{l}_1 \circ \mathfrak{l}_2) \times \mathfrak{l}_3) \cap \mathfrak{d}_{\Delta(2,3)})} \\ &= \frac{X_1 \times X_2 \times X_3}{\mathrm{Lie}((\mathfrak{l}_1 \times \mathfrak{l}_2 \times \mathfrak{l}_3) \cap \mathfrak{d}_{\Delta(2,3)})} \Bigg/ \frac{\mathrm{Lie}((\mathfrak{l}_1 \times \mathfrak{l}_2 \times \mathfrak{l}_3) \cap (0 \times \mathfrak{d}_\Delta \times \mathfrak{d}_\Delta \times 0))}{\mathrm{Lie}((\mathfrak{l}_1 \times \mathfrak{l}_2 \times \mathfrak{l}_3) \cap \mathfrak{d}_{\Delta(2,3)})} \\ &= \frac{X_1 \times X_2 \times X_3}{\mathrm{Lie}((\mathfrak{l}_1 \times \mathfrak{l}_2 \times \mathfrak{l}_3) \cap (e \times \mathfrak{d}_\Delta \times \mathfrak{d}_\Delta \times e))}.\end{aligned}$$

Similarly, one finds that

$$\begin{aligned}X_1 \circledast (X_2 \circledast X_3) &= \frac{X_1 \times X_2 \times X_3}{\mathrm{Lie}((\mathfrak{l}_1 \times \mathfrak{l}_2 \times \mathfrak{l}_3) \cap \mathfrak{d}_{\Delta(4,5)})} \Bigg/ \frac{\mathrm{Lie}((\mathfrak{l}_1 \times \mathfrak{l}_2 \times \mathfrak{l}_3) \cap (0 \times \mathfrak{d}_\Delta \times \mathfrak{d}_\Delta \times 0))}{\mathrm{Lie}((\mathfrak{l}_1 \times \mathfrak{l}_2 \times \mathfrak{l}_3) \cap \mathfrak{d}_{\Delta(4,5)})} \\ &= \frac{X_1 \times X_2 \times X_3}{\mathrm{Lie}((\mathfrak{l}_1 \times \mathfrak{l}_2 \times \mathfrak{l}_3) \cap (0 \times \mathfrak{d}_\Delta \times \mathfrak{d}_\Delta \times 0))}.\end{aligned}$$

Since

$$\mathfrak{l}_1 \circ \mathfrak{l}_2 \circ \mathfrak{l}_3 = \frac{\mathfrak{l}_1 \times \mathfrak{l}_2 \times \mathfrak{l}_3}{(\mathfrak{l}_1 \times \mathfrak{l}_2 \times \mathfrak{l}_3) \cap (0 \times \mathfrak{d}_\Delta \times \mathfrak{d}_\Delta \times 0)},$$

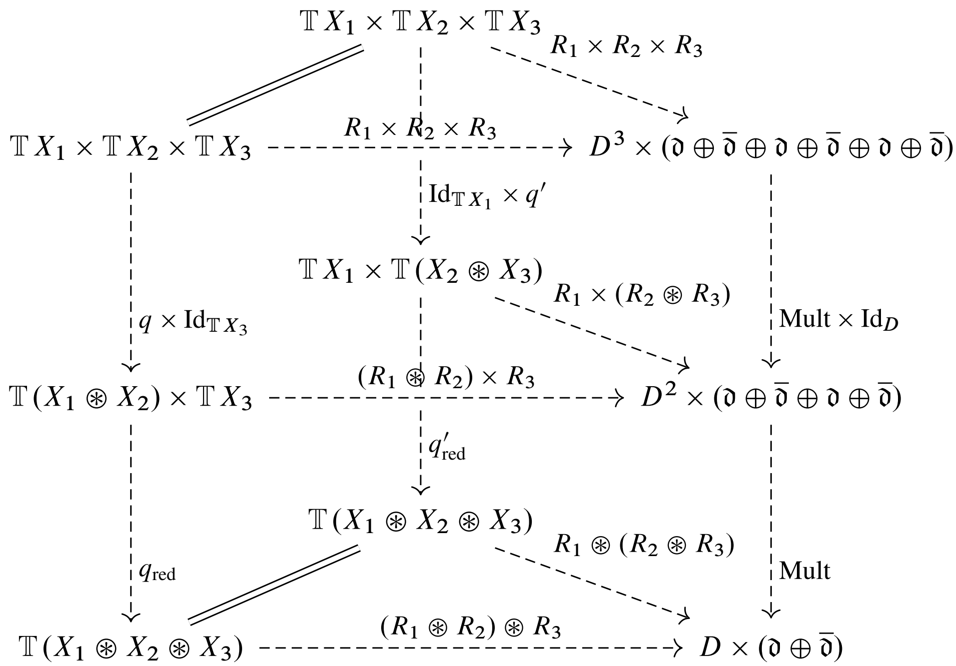
it follows that $(X_1 \circledast X_2) \circledast X_3 = X_1 \circledast (X_2 \circledast X_3)$ as $\mathrm{Lie}(\mathfrak{l}_1 \circ \mathfrak{l}_2 \circ \mathfrak{l}_3)$-manifolds.

Next, let

$$\begin{aligned} q &: \mathbb{T}X_1 \times \mathbb{T}X_2 \dashrightarrow \mathbb{T}(X_1 \circledast X_2), \\ q' &: \mathbb{T}X_2 \times \mathbb{T}X_3 \dashrightarrow \mathbb{T}(X_2 \circledast X_3), \\ q_{\mathrm{red}} &: \mathbb{T}(X_1 \circledast X_2) \times \mathbb{T}X_3 \dashrightarrow \mathbb{T}(X_1 \circledast X_2 \circledast X_3), \\ q'_{\mathrm{red}} &: \mathbb{T}X_1 \times \mathbb{T}(X_2 \circledast X_3) \dashrightarrow \mathbb{T}(X_1 \circledast X_2 \circledast X_3), \end{aligned}$$

be the reduction morphisms. In view of the above, one has $q_{\mathrm{red}} \circ (q \times \mathrm{Id}_{\mathbb{T}X_3}) = q'_{\mathrm{red}} \circ (\mathrm{Id}_{\mathbb{T}X_1} \times q)$, where $\mathrm{Id}_{\mathbb{T}X_i} : \mathbb{T}X_i \dashrightarrow \mathbb{T}X_i$ are the identity isomorphisms. The diagram (3.14) then gives a diagram in the shape of a triangular prism

(3.15)



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known to commute along all faces (and their subdivisions) except the bottom face. A simple diagram chase then shows that it commutes along the latter as well. This completes the proof. □

Commutativity. The fusion product of L-Hamiltonian spaces cannot be commutative in general. For instance, one has $(\mathfrak{g} \oplus \mathfrak{h}) \circ (\mathfrak{h} \oplus \mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}$, whereas $(\mathfrak{h} \oplus \mathfrak{g}) \circ (\mathfrak{g} \oplus \mathfrak{h}) = \mathfrak{h} \oplus \mathfrak{h}$ for Lagrangian subalgebras $\mathfrak{g}, \mathfrak{h} \subseteq \mathfrak{d}$, equality thus not holding if $\mathfrak{g}$ and $\mathfrak{h}$ are distinct. This said, the fusion product is indeed commutative in the subcategory of q-Hamiltonian spaces [4, Thm. 6.2], cf. Example 3.2.2. One might wonder if commutativity also holds in the subcategory of $G \times G$-Hamiltonian spaces. As will be seen in the next chapter, the answer is negative; in that subcategory, the fusion products $X_1 \circledast X_2$ and $X_2 \circledast X_1$ are generally not isomorphic as $G \times G$-spaces.

Monoidal structure. The singleton $\{e\}$ where $e \in D$ is the identity element is a q-Hamiltonian space, alternatively a D_Δ-Hamiltonian space. As should be clear from (3.10), one has

$$X \circledast \{e\} = \{e\} \circledast X = X, \tag{3.16}$$

for any L-Hamiltonian space X. Thus the category of L-Hamiltonian spaces carries a monoidal structure.

Fusion product without Dirac geometry. In view of part (d) of Theorem 2.4.2, the fusion product $X_1 \circledast X_2$ is described at the level of 2-forms. For $i = 1, 2$, let (X_i, J_i, ω_i) be a L_i-

Hamiltonian space. Denote by $J_1 \circledast J_2$ the moment map of $X_1 \circledast X_2$ and by $\omega_1 \circledast \omega_2$ the 2-form in $\Omega^2(X_1 \circledast X_2)$ such that, in the notation of the proof of Theorem 3.2.2, one has

$$R_{\mathrm{red}} = \mathbb{T}(J_1 \circledast J_2)_{\omega_1 \circledast \omega_2}.$$

By changing isotropic splittings $\mathbb{T} D^2_{\eta_D^1+\eta_D^2} \simeq \mathbb{T} D^2_{\mathrm{mult}^* \eta_D}$, the morphism $R = \mathbb{T}(J_1 \times J_2)_{\omega_1^1+\omega_2^2}$ changes to

$$\mathbb{T}(J_1 \times J_2)_{\omega_1^1+\omega_2^2+(J_1\times J_2)^*\varsigma},$$

where $\varsigma \in \Omega^2(D^2)$ is the 2-form (3.8). According to part (d) of Theorem 2.4.2 then

$$(\cdot/\,\mathrm{Lie}((\mathfrak{l}_1 \oplus \mathfrak{l}_2) \cap \mathfrak{d}_{\Delta(2,3)}))^*(\omega_1 \circledast \omega_2) = \omega_1^1 + \omega_2^2 + (J_1 \times J_2)^*\varsigma. \tag{3.17}$$

Internal fusion product. In addition to the fusion product of L-Hamiltonian spaces, one may also consider the "internal" fusion product of a L-Hamiltonian space (X, J, ω) for $D^2 \times (\mathfrak{d} \oplus \overline{\mathfrak{d}} \oplus \mathfrak{d} \oplus \overline{\mathfrak{d}})$, where $L \subseteq D^4$ is a closed and connected subgroup integrating a Lagrangian subalgebra $\mathfrak{l} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}} \oplus \mathfrak{d} \oplus \overline{\mathfrak{d}}$. The object in question is the quotient

$$\boldsymbol{X} = \frac{X}{\mathrm{Lie}(\mathfrak{l} \cap \mathfrak{d}_{\Delta(2,3)})},$$

which is seen to be a $\boldsymbol{L}$-Hamiltonian space by adjusting the proof of Theorem 3.2.2, where $\boldsymbol{L}$ is the closed and connected subgroup of D integrating the Lagrangian subalgebra of $\mathfrak{d} \oplus \overline{\mathfrak{d}}$

$$\boldsymbol{l} = \frac{\mathfrak{l} \cap (\mathfrak{d} \cap \mathfrak{d}_{\Delta(2,3)})^{\perp}}{\mathfrak{d} \cap \mathfrak{d}_{\Delta(2,3)}}.$$

(The fusion product considered above corresponds to the special case where $\mathfrak{l}$ is the direct sum of two Lagrangian subalgebras of $\mathfrak{d} \oplus \overline{\mathfrak{d}}$.) Its moment map $\boldsymbol{J} : \mathbf{X} \to D$ descends from $\mathrm{mult} \circ J$ and the corresponding 2-form $\boldsymbol{\omega}$ descends from $\omega + J^*\varsigma$.

It will turn out to be of great usefulness to recast the relationship between $\boldsymbol{\omega}$ and ω in terms of a certain group structure introduced by P. Ševera [54]. For any smooth manifold M, the space $C^\infty(M, D) \times \Omega^2(M)$ possesses a group structure given by

$$(d_1, \kappa_1) \cdot (d_2, \kappa_2) = (d_1 d_2, \kappa_1 + \kappa_2 + (d_1 \times d_2)^*\varsigma)$$

and when equipped with this group structure will be denoted by $C^\infty(M, D) \bowtie \Omega^2(M)$. If N is another manifold and $f : M \to N$ is a smooth map, define the *pullback by f* of an element

$(d,\kappa) \in C^\infty(M,D) \bowtie \Omega^2(M)$ by f, denoted by $f^*(d,\kappa)$, to be the element

$$(d \circ f, f^*\kappa) \in C^\infty(N,D) \bowtie \Omega^2(N).$$

This next lemma is a simple observation, it is recorded here for later use.

Lemma 3.2.2.

(a) *The pullback map* $f^* : C^\infty(M,D) \bowtie \Omega^2(M) \to C^\infty(N,D) \bowtie \Omega^2(N)$ *is a group homomorphism, i.e.*

$$f^*((d_1,\kappa_1)\cdot(d_2,\kappa_2)) = f^*(d_1,\kappa_1)\cdot f^*(d_2,\kappa_2).$$

(b) *One has*

$$(d,\kappa_1)(d^{-1},\kappa_2) = (d^{-1},\kappa_2)(d,\kappa_1) = (e,\kappa_1+\kappa_2).$$

Returning now to the L-Hamiltonian space X, let J_1 and J_2 denote the composition of the moment map $J : X \to D \times D$ with the projections on the first and second D-factors respectively. Let $\pi : X \to \boldsymbol{X}$ be the quotient map. Then the relationship between $\boldsymbol{\omega}$ and ω takes the form

$$(J_1J_2, \pi^*\boldsymbol{\omega}) = (J_1,\omega)\cdot(J_2,0), \tag{3.18}$$

where the product is taken in the group $C^\infty(X,D) \bowtie \Omega^2(X)$. Note that (3.18) amounts to (3.17) in the special where X is the direct product of two L-Hamiltonian spaces.

In full generality, one can consider an L-Hamiltonian space (X,J,ω) for a power[2] of $D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}})$

$$\underbrace{D\times(\mathfrak{d}\oplus\bar{\mathfrak{d}})\times\cdots\times D\times(\mathfrak{d}\oplus\bar{\mathfrak{d}})}_{n \text{ times}} = D^n \times \underbrace{\mathfrak{d}\oplus\bar{\mathfrak{d}}\oplus\cdots\oplus\mathfrak{d}\oplus\bar{\mathfrak{d}}}_{n\text{ times}}$$

and perform the fusion product along any two $D\times(\mathfrak{d}\oplus\bar{\mathfrak{d}})$-factors. This may be repeated indefinitely until an $\boldsymbol{L}$-Hamiltonian space $(\boldsymbol{X},\boldsymbol{J},\boldsymbol{\omega})$ for $(D\times(\mathfrak{d}\oplus\bar{\mathfrak{d}}))^m$, where $m < n$, is obtained. The particular order in which this "fusion product in stages" is carried out among a fixed list of $D\times(\mathfrak{d}\oplus\bar{\mathfrak{d}})$-factors is inconsequential following the same line of argument given for the associativity of the fusion product. Denote by J_i the composition of the moment map $J : X \to D^n$ with the projection onto the i^{th} factor of D^n. Let $\pi : X \to \boldsymbol{X}$ be again the quotient map. The following result generalizes (3.18).

[2]Here again the convention is the natural one: the i^{th} $\mathfrak{d}$-factor appearing on the left-hand-side corresponds to the i^{th} factor appearing on the right-hand-side.

Theorem 3.2.4 (Fusion product in stages)**.** *Suppose $\boldsymbol{X}$ is obtained from X by fusing the latter along the i_1^{th}, i_2^{th}, ..., i_{k+1}^{th} factors of $(D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}}))^n$, where $k = n - m$. Then*

$$(J_{i_1} J_{i_2} \cdots J_{i_{k+1}}, \pi^* \boldsymbol{\omega}) = (J_{i_1}, \omega)(J_{i_2}, 0) \cdots (J_{i_{k+1}}, 0). \tag{3.19}$$

Proof. Without loss of generality the indices can be taken to be $i_1 = 1$, $i_1 = 2$, ..., $i_{k+1} = k + 1$. The proof proceeds by induction on k. The base case $k = 1$ is essentially (3.18). Now suppose the statement holds for $k \leq l < n - 1$; one wishes to show it holds for $k = l + 1$ as well. Let (X', J', ω') be the L'-Hamiltonian space obtained by taking the internal fusion product of X along the first $l + 1$ factors (the subgroup $L' \subseteq D^{m+1}$ is implicitly defined), let $\pi' : X \to X'$ be the quotient map. Then by the induction hypothesis

$$(J_1, \omega)(J_2, 0) \cdots (J_{l+1}, 0)(J_{l+2}, 0) = (J_1 J_2 \cdots J_{l+1}, (\pi')^* \omega)(J_{l+2}, 0)$$

and one is thus left with the task of showing that the right-hand-side of the equation above is equal to the left-hand-side of (3.19). Let $\boldsymbol{\omega}' \in \Omega^2(X')$ be the pullback of $\boldsymbol{\omega}$ to X'. According to (3.18) again,

$$(J_1' J_2', \boldsymbol{\omega}') = (J_1', \omega')(J_2', 0),$$

Note that $J_1 \cdots J_{l+1} = J_1' \circ \pi$ and $J_{l+2} = J_2' \circ \pi$ and thus part (a) of Lemma 3.2.2 in combination with the above equation give

$$(J_1 \cdots J_{l+2}, (\pi')^* \boldsymbol{\omega}') = (J_1 \ldots J_{l+1}, (\pi')^* \omega')(J_{l+2}, 0).$$

Since $(\pi')^* \boldsymbol{\omega}' = \pi^* \boldsymbol{\omega}$, this completes the proof. □

Remark 3.2.2. Equation (3.19) appears in [54] (although only the case $\omega = 0$ is considered) under quite different considerations; there it is interpreted as a form of Stokes theorem relating a flat connection on a polygon to its pullback onto the boundary. More will be said on this matter in the ultimate chapter.

Example 3.2.4 ([4]). In [4], the double $D(D)$ and the fused double $\boldsymbol{D}(D)$ were introduced as fundamental examples of quasi-Hamiltonian spaces. To illustrate the usefulness of Theorem 3.2.4, these constructions are recovered by means of internal fusion products of suitable L-Hamiltonian spaces.

Let $\mathfrak{l} \subseteq (\mathfrak{d} \oplus \bar{\mathfrak{d}})^4$ be the Lagrangian subalgebra

$$\mathfrak{l} = \{(\xi_1, \xi_2, \xi_1', \xi_2', \xi_2, \xi_1, \xi_2', \xi_1') \ : \ \xi_i, \xi_i' \in \mathfrak{d}\}. \tag{3.20}$$

Its orbit through the group identity in D^4 is

$$\mathcal{O} = \{(a, b, a^{-1}, b^{-1}) \ : \ a, b \in D\} \simeq D^2,$$

which is a L-Hamiltonian space for $(D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}}))^4$, where

$$L = \{(d_1, d_2, d'_1, d'_2, d_2^{-1}, d_1^{-1}, (d'_2)^{-1}, (d'_1)^{-1}) \ : \ d_i, d'_i \in D\} \subseteq D^8$$

is the closed and connected subgroup integrating $\mathfrak{l}$. The 2-form $\omega_{\mathcal{O}}$ defined in Example 3.1.1 is easily seen to be trivial (this is also argued in Proposition 5.1.1 in Chapter 4). Let $D(D)$ be its internal fusion product along the first three factors of $(D\times(\mathfrak{d}\oplus\bar{\mathfrak{d}}))^4$, which is a $D_\Delta \times D_\Delta \simeq D^2$-Hamiltonian space for $D^2 \times (\mathfrak{d} \oplus \bar{\mathfrak{d}} \oplus \mathfrak{d} \oplus \bar{\mathfrak{d}})$. The space $D(D)$ is the quotient of $\mathcal{O}$ by the action of the closed and connected subgroup of D^8 integrating

$$\mathfrak{l} \cap \{(0, \xi_1, \xi_1, \xi_2, \xi_2, 0, 0, 0) \ : \ \xi_i \in \mathfrak{d}\},$$

which is, in view of (3.20), trivial. Thus $D(D) = \mathcal{O} \simeq D^2$ as manifolds. Its moment map is $J(a, b) = (aba^{-1}, b^{-1})$ and the $\boldsymbol{L} \simeq D^2$-action is

$$(d_1, d_2).(a, b) = (d_1 a d_2^{-1}, d_2 b d_2^{-1}).$$

According to Theorem 3.2.4, the 2-form $\omega \in \Omega^2(D(D))$ is given by

$$\begin{aligned}(aba^{-1}, \omega) &= (a, 0)(b, 0)(a^{-1}, 0)\\ &= (ab, (a \times b)^*\varsigma)(a^{-1}, 0)\\ &= (aba^{-1}, (a \times b)^*\varsigma + (ab \times a^{-1})^*\varsigma)\end{aligned}$$

which can then be expanded to

$$\begin{aligned}\omega &= (a \times b)^*\varsigma + (ab \times a^{-1})^*\varsigma\\ &= \frac{1}{2}\big(\langle \theta^{L,1}, \theta^{R,2}\rangle + \langle \theta^{L,1}, \theta^{L,2}\rangle - \langle \mathrm{Ad}_{b^{-1}} \theta^{L,1}, \theta^{L,1}\rangle\big)\end{aligned}$$

Let $\boldsymbol{D}(D)$ be the internal fusion product of $D(D)$; equivalently it is the internal fusion product of $\mathcal{O}$ along all four factors of $(D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}}))^4$. Here once again $\boldsymbol{D}(D) = \mathcal{O}$ as manifolds and the 2-form $\boldsymbol{\omega} \in \Omega^2(\boldsymbol{D}(D))$ is characterized by

$$(aba^{-1}b^{-1}, \boldsymbol{\omega}) = (a, 0)(b, 0)(a^{-1}, 0)(b^{-1}, 0)$$

$$= (ab, \varsigma)(a^{-1}b^{-1}, \frac{1}{2}\langle\theta^{R,1}, \theta^{L,2}\rangle)$$

and so

$$\boldsymbol{\omega} = \frac{1}{2}(\langle\theta^{L,1}, \theta^{R,2}\rangle + \langle\theta^{R,1}, \theta^{L,2}\rangle + \langle(ab)^*\theta^{L,1}, (a^{-1}b^{-1})^*\theta^{R,2}\rangle.$$

The spaces $D(D)$ and $\boldsymbol{D}(D)$ are called the *double* and *fused double* of D, respectively. They are first defined in [4]. As shown there, and in this thesis in Chapter 4, when taken as Hamiltonian spaces for $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$ and $(D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}))^2$, respectively, they correspond to spaces lifting the moduli spaces of flat connections on a cylinder and on a 1-holed torus, respectively.

3.3 Dual, symplectic reduction and shifting trick

3.3.1 Dualization

Let $\mathrm{inv} : D \to D$ be the group inversion. The inverse map of the pair groupoid (3.7) is

$$(\xi_1, \xi_2)^{-1} = (\xi_2, \xi_1), \tag{3.21}$$

which is an isometry $\mathfrak{d} \oplus \overline{\mathfrak{d}} \to \overline{\mathfrak{d}} \oplus \mathfrak{d}$. Viewing $\mathfrak{d} \oplus \overline{\mathfrak{d}}$ as the constant sections of $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$, one has

$$\mathrm{inv}_* \mathbf{a}_D(\xi_1, \xi_2) = \mathrm{inv}_*(\xi_2^L - \xi_1^R) = \xi_1^L - \xi_2^R = \mathbf{a}_D(\xi_2, \xi_1),$$

and thus according to Example 2.3.2 the product of Gr(inv) and the graph of $(\cdot)^{-1}$ is a Courant morphism $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}) \dashrightarrow \overline{D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})}$, which will be denoted by Inv. Note that Inv is equivariant with respect to the group homomorphism

$$D \times D \to D \times D, \quad (a, b) \mapsto (b, a). \tag{3.22}$$

Proposition 3.3.1. *The morphism Inv* $: D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}) \dashrightarrow \overline{D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})}$ *is exact. With the identifications* $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}) \simeq \mathbb{T} D_{\eta_D}$ *and* $\overline{D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})} \simeq \mathbb{T} D_{-\eta_D}$*, the morphism Inv is* $\mathbb{T}\,\mathrm{inv}$.

Proof. An element of $\mathrm{ran}(\mathbf{a}_D^*) = \ker(\mathbf{a}_D)$ is of the form $(d, \mathrm{Ad}_d\, \xi, \xi)$ for some $\xi \in \mathfrak{d}$. This element is Inv-related to and only to $(d^{-1}, \xi, \mathrm{Ad}_d\, \xi) = (d^{-1}, \mathrm{Ad}_{d^{-1}}\, \xi', \xi')$ where $\xi' = \mathrm{Ad}_d\, \xi$. This shows that an element of $\mathrm{ran}(\mathbf{a}_D^*) \subseteq \overline{D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})}$ is Inv-related to a unique element of $\mathrm{ran}(\mathbf{a}_D^*) \subseteq D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$, which establishes the exactness of Inv.

For the second part of the claim, it suffices to show that $\mathrm{Inv} \circ \mathrm{ran}(j_D) = \mathrm{ran}(j_D)$ (recall that j_D is the splitting (3.4)). This is clear since

$$\begin{aligned}(\mathrm{Inv}|_d) \circ \mathrm{ran}(j_D) &= \{(d, \xi, -\mathrm{Ad}_{d^{-1}} \xi) \;:\; \xi \in \mathfrak{d}\} \\ &= \{(d, -\mathrm{Ad}_d \xi, \xi) \;:\; \xi \in \mathfrak{d}\} \\ &= \mathrm{ran}(j_D|_d).\end{aligned}$$

□

Given a subspace $\mathfrak{a} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$, let $\mathfrak{a}^{-1}$ denote the subspace of elements obtained from elements of $\mathfrak{a}$ by inversion. If $\mathfrak{a}^{-1} = \mathfrak{a}$, one will say that $\mathfrak{a}$ is *symmetric*. For composable elements $\zeta_1, \zeta_2 \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ one has $(\zeta_1 \circ \zeta_2)^{-1} = \zeta_2^{-1} \circ \zeta_1^{-1}$ and thus for another subspace $\mathfrak{b} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ there is a relationship

$$(\mathfrak{a} \circ \mathfrak{b})^{-1} = \mathfrak{b}^{-1} \circ \mathfrak{a}^{-1}. \tag{3.23}$$

For a Lagrangian subalgebra $\mathfrak{l} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$, one has a Dirac morphism

$$\mathrm{Inv} : (D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}), E^{(\mathfrak{l})}) \dashrightarrow (\overline{D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})}, E^{(\mathfrak{l}^{-1})}).$$

If $\mathfrak{l}$ is multiplicative and symmetric then, by Proposition 3.2.1, it is a subset of $\mathfrak{d} \oplus \overline{\mathfrak{d}}$ of the form

$$\mathfrak{l} = \mathfrak{c}_\Delta + (\mathfrak{c}^\perp \oplus \mathfrak{c}^\perp), \tag{3.24}$$

where $\mathfrak{c} \subseteq \mathfrak{d}$ is a coisotropic subalgebra. For example, the diagonal $\mathfrak{d}_\Delta \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ and the direct sum $\mathfrak{g} \oplus \mathfrak{g}$, where $\mathfrak{g} \subseteq \mathfrak{d}$ is a Lagrangian subalgebra of $\mathfrak{d}$, are multiplicative and symmetric Lagrangian subalgebras.

Proposition 3.3.2. *Let $\mathfrak{l} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ be a Lagrangian subalgebra integrating to the connected subgroup $L \subseteq D \times D$. Suppose the triple (X, J, ω) is a L-Hamiltonian space for $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$ as per Proposition 3.2.1. Then the triple $(X, \mathrm{inv} \circ J, -\omega)$ is a L^{-1}-Hamiltonian space for $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$.*

Proof. By definition, the function J and the 2-form ω determine the L-equivariant Dirac morphism $\mathbb{T} J_\omega : (\mathbb{T} X, TX) \dashrightarrow (\mathbb{T} D_{\eta_D}, E^{(\mathfrak{l})})$. The composition $\mathbb{T}\mathrm{inv} \circ \mathbb{T} J_\omega = \mathbb{T}(\mathrm{inv} \circ J)_\omega$ is a Dirac morphism

$$\mathbb{T}(\mathrm{inv} \circ J)_\omega : (\mathbb{T} X, TX) \dashrightarrow (\mathbb{T} D_{-\eta_D}, E^{(\mathfrak{l}^{-1})}).$$

The resulting $\mathfrak{l}^{-1}$-action $\varrho : \mathfrak{l} \to \mathfrak{X}(X)$ is the $\mathfrak{l}$-action on X composed with the isometry (3.21). Since Inv is equivariant with respect to the homomorphism (3.22), it follows that $\mathbb{T}(\mathrm{inv} \circ J)_\omega$ is equivariant with respect to L^{-1}. This shows that $(X, \mathrm{inv} \circ J, \omega)$ is a L^{-1}-Hamiltonian space for $\overline{D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}})}$. According to Proposition 3.2.1 one has (a) $d\omega = J^*\mathrm{inv}^*\eta_D$, (b) $\ker(\omega) \cap \ker(\mathrm{inv}_* J_*) = 0$ and (c) $\iota_{\varrho(\xi_1,\xi_2)}\omega = \frac{1}{2}J^*\mathrm{inv}^*(\langle \xi_2, \theta^L\rangle + \langle \xi_1, \theta^R\rangle)$ for all $(\xi_1, \xi_2) \in \mathfrak{l}^{-1}$. The conditions may be rewritten (a) $d(-\omega) = -J^*\eta_D$, (b) $\ker(-\omega) \cap \ker(\mathrm{inv}_* J_*) = 0$ and (c) $\iota_{\varrho(\xi_1,\xi_2)}(-\omega) = -\frac{1}{2}J^*\mathrm{inv}^*(\langle \xi_2, \theta^L\rangle + \langle \xi_1, \theta^R\rangle)$. These are precisely the conditions for $(X, \mathrm{inv} \circ J, -\omega)$ to be a L^{-1}-Hamiltonian space according to Proposition 3.2.1 again. □

Definition 3.3.1 (Dualization)**.** Given a L-Hamiltonian space (X, J, ω), the L^{-1}-Hamiltonian space $(X, \mathrm{inv} \circ J, -\omega)$ is called the *dual* of X and is denoted by X^*.

Remark 3.3.1. Note that $(X^*)^* = X$. If X is a L-Hamiltonian space for $D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}})$. In fact, the proof of the preceding proposition makes it clear that in the special case $\mathfrak{l} = \mathfrak{d}_\Delta$ the q-Hamiltonian space X and its dual X^* are isomorphic as D_Δ-space. (The space X^* coincides with the space X^- constructed in [4, Prop. 4.4].) It will be seen in the next chapter that this does not hold for $\mathfrak{l} = \mathfrak{g} \oplus \mathfrak{g}$.

The next result is the L-Hamiltonian analogue of the relationship (3.23).

Proposition 3.3.3. *Suppose the Lagrangian subalgebra $\mathfrak{l}_i \subseteq \mathfrak{d} \oplus \bar{\mathfrak{d}}$ ($i = 1, 2$) integrates to a closed and connected subgroup L_i. Let (X_i, J_i, ω_i) be a L_i-Hamiltonian space. Then*

$$(X_1 \circledast X_2)^* \simeq X_2^* \circledast X_1^* \tag{3.25}$$

as $\mathrm{Lie}(\mathfrak{l}_2^{-1} \circ \mathfrak{l}_1^{-1})$*-Hamiltonian spaces.*

Proof. The $\mathfrak{l}_i$-action on X_i^* is obtained by composing the $\mathfrak{l}_i$-action on X_i with the group isomorphism $(d_1, d_2) \mapsto (d_2, d_1)$. It is clear that the $(\mathfrak{l}_1 \times \mathfrak{l}_2) \cap D_{\Delta(2,3)}$-action on $X_1 \times X_2$ and the $(\mathfrak{l}_2^{-1} \times \mathfrak{l}_1^{-1}) \cap D_{\Delta(2,3)}$-action on $X_2^* \times X_1^*$ are identical under the obvious identification $X_1 \times X_2 = X_2^* \times X_1^*$. Since $(\mathfrak{l}_1 \circ \mathfrak{l}_2)^{-1} = \mathfrak{l}_2^{-1} \circ \mathfrak{l}_1^{-1}$, the spaces $X_1 \times X_2$ and $X_2^* \times X_1^*$ are moreover also equal as L-spaces, where $\mathfrak{l} = \mathrm{Lie}((\mathfrak{l}_1 \circ \mathfrak{l}_2)^{-1}) = \mathrm{Lie}(\mathfrak{l}_2^{-1} \circ \mathfrak{l}_1^{-1})$.

Recall that the maps $J_1 \times J_2$ and $J_1 \circledast J_2$ are related by the following commutative diagram

$$\begin{array}{ccc} X_1 \times X_2 & \xrightarrow{\quad J_1 \times J_2 \quad} & D \times D \\ \Big\downarrow {\cdot / \mathrm{Lie}((\mathfrak{l}_1 \times \mathfrak{l}_2) \cap \mathfrak{d}_{\Delta(2,3)})} & & \Big\downarrow \mathrm{mult}\,\cdot \\ X_1 \circledast X_2 & \xrightarrow{\quad J_1 \circledast J_2 \quad} & D \end{array} \tag{3.26}$$

The moment map of $(X_1 \circledast X_2)^*$ is $\mathrm{inv} \circ (J_1 \circledast J_2)$ and, as mult intertwines $\mathrm{inv} \times \mathrm{inv}$ and inv, one also has the commutative diagram

$$
\begin{array}{ccc}
X_1 \times X_2 & \xrightarrow{\quad (\mathrm{inv} \times \mathrm{inv}) \circ (J_1 \times J_2) \quad} & D \times D \\
\Big\downarrow {\scriptstyle \cdot/\,\mathrm{Lie}((\mathfrak{l}_1 \times \mathfrak{l}_2) \cap \mathfrak{d}_{\Delta(2,3)})} & & \Big\downarrow {\scriptstyle \mathrm{mult}\,\cdot} \\
(X_1 \circledast X_2)^* & \xrightarrow{\quad \mathrm{inv} \circ (J_1 \circledast J_2) \quad} & D
\end{array}
\tag{3.27}
$$

On the other hand, the moment map $(\mathrm{inv} \circ J_2) \circledast (\mathrm{inv} \circ J_1)$ of $X_2^* \circledast X_1^*$ fits in the diagram

$$
\begin{array}{ccc}
X_2^* \times X_1^* & \xrightarrow{\quad (\mathrm{inv} \circ J_2) \times (\mathrm{inv} \circ J_1) \quad} & D \times D \\
\Big\downarrow {\scriptstyle \cdot/\,\mathrm{Lie}((\mathfrak{l}_2^{-1} \times \mathfrak{l}_1^{-1}) \cap \mathfrak{d}_{\Delta(2,3)})} & & \Big\downarrow {\scriptstyle \mathrm{mult}\,\cdot} \\
X_2^* \circledast X_1^* & \xrightarrow{\quad (\mathrm{inv} \circ J_2) \circledast (\mathrm{inv} \circ J_1) \quad} & D
\end{array}
\tag{3.28}
$$

The top, left and right maps in the diagrams (3.27) and (3.28) all agree, and thus the moment maps of $(X_1 \circledast X_2)^*$ and $X_2^* \circledast X_1^*$ also agree.

It remains to be shown that the 2-forms carried by $(X_1 \circledast X_2)^*$ and $X_2^* \circledast X_1^*$ coincide. In the same notation as in (3.17), the 2-form corresponding to $(X_1 \circledast X_2)^*$ is $-(\omega_1 \circledast \omega_2)$ and one has

$$(\cdot/\,\mathrm{Lie}((\mathfrak{l} \times \mathfrak{l}) \cap \mathfrak{d}_{\Delta(2,3)}))^*(-(\omega_1 \circledast \omega_2)) = -\omega_1^1 - \omega_2^2 - (J_1 \times J_2)^* \varsigma.$$

On the other hand, the 2-form corresponding to $X_2^* \circledast X_1^*$ is $(-\omega_2) \circledast (-\omega_1)$ and one has

$$
\begin{aligned}
(\cdot/\,\mathrm{Lie}((\mathfrak{l} \times \mathfrak{l}) \cap \mathfrak{d}_{\Delta(2,3)}))^*((-\omega_2) \circledast (-\omega_1)) &= -\omega_1^1 - \omega_2^2 + (J_1 \times J_2)^* (\mathrm{inv} \times \mathrm{inv})^* \varsigma \\
&= -\omega_1^1 - \omega_2^2 - (J_1 \times J_2)^* \varsigma.
\end{aligned}
$$

This completes proof. □

3.3.2 Symplectic reduction

Let $\mathfrak{l} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ be a Lagrangian subalgebra integrating to a closed and connected subgroup $L \subseteq D \times D$. Suppose $R : \mathbb{T}X \dashrightarrow D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$ is a L-Hamiltonian space with moment map J and corresponding 2-form $\omega \in \Omega^2(X)$ as per Proposition 3.2.1. Let $\varrho : X \times \mathfrak{l} \to TX$ be the $\mathfrak{l}$-action on X.

Lemma 3.3.1. *The* 2*-form* ω *sends* $\ker(J_*)$ *to the annihilator*[3] $\mathrm{Ann}(\mathrm{ran}(\varrho)) \subseteq T^*X$ *of* $\mathrm{ran}(\varrho)$

[3]The subscript indicates the space in whose cotangent bundle the annihilator is taken (one must distinguish between the annihilator in T^*X and the annihilator in T^*S).

bijectively.

Proof. Recall from the proof of Proposition 3.1.1 that

$$\ker(R) = \{v - \iota_v\omega \ : \ v \in \ker(J_*)\}.$$

As R intertwines the generators, it relates the fundamental vector field $\varrho(\zeta) \in \mathfrak{X}(X)$ and the constant section $\zeta \in \Gamma(D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}))$ for all $\zeta \in \mathfrak{l}$ by definition. Thus $\mathrm{ran}(\varrho) \subseteq \mathrm{ran}^*(R)$ and taking the orthogonal on either side gives the containment $\mathrm{ran}(\varrho)^\perp \supseteq \ker(R)$. The differential form $\iota_v\omega$, for $v \in \ker(J_*)$, must thus be in the annihilator $\mathrm{Ann}(\mathrm{ran}(\varrho)) \subseteq T^*X$ of $\mathrm{ran}(\varrho)$. On the other hand the intersection $\ker(\omega) \cap \ker(J_*)$ is trivial by part (b) of Proposition 3.2.1. The upshot is that ω sends $\ker(J_*)$ to $\mathrm{Ann}(\mathrm{ran}(\varrho))$ injectively and it must only be shown that the dimensions of $\ker(J_*|_m)$ and $\mathrm{Ann}(\mathrm{ran}(\varrho)|_m)$ are equal for all $m \in X$. Now the backward image

$$E^{(\mathfrak{l})} \circ R = \mathrm{ran}(\varrho) + \ker(R)$$

is a Dirac structure of $\mathbb{T}X$ (Example 2.3.3). As $TX \cap \ker(R) = 0$ (by the definition of a Dirac morphism), the dimension of $\mathrm{ran}(\varrho)|_m$ is $\dim(X) - \dim(\ker(J_*|_m))$. The dimension of $\mathrm{Ann}(\mathrm{ran}(\varrho)|_m)$ is the codimension of $\mathrm{ran}(\varrho)|_m$ in T_mX, which is $\dim(\ker(J_*|_m))$. This is what needed to be proven. □

Theorem 3.3.1 (Symplectic reduction)**.** *Suppose $\iota_{\mathcal{O}} : \mathcal{O} \hookrightarrow D$ is an orbit of L and that J is transversal to $\mathcal{O}$ so that $\mathcal{S} = J^{-1}(\mathcal{O})$ is a submanifold $\iota_{\mathcal{S}} : \mathcal{S} \hookrightarrow X$. Then $X_{\mathcal{O}} = \mathcal{S}/L$ is a symplectic orbifold.*

To clarify, it is claimed that there is a closed L-invariant 2-form $\omega' \in \Omega^2(\mathcal{S})$ whose kernel consists of the L-orbit directions. In particular, if $X_{\mathcal{O}}$ is a quotient manifold [4] then ω' descends to a symplectic form $\omega'_{\mathrm{red}} \in \Omega^2(X_{\mathcal{O}})$.

Proof. Let $\omega_{\mathcal{O}} \in \Omega^2(\mathcal{O})$ be the form (3.2). Define a 2-form $\omega' \in \Omega^2(\mathcal{S})$ by putting

$$\omega' = \iota_{\mathcal{S}}^*(\omega - J^*\omega_{\mathcal{O}}) \tag{3.29}$$

The 2-form ω' is closed since $\iota_{\mathcal{S}}^* d\omega = \iota_{\mathcal{S}}^* J^* d\omega_{\mathcal{O}} = -\iota_{\mathcal{S}}^* J^*\eta$ according to part (a) of Proposition 3.2.1. It is in addition L-invariant since J is L-equivariant and ω and ω' are L-invariant. It must be shown that $\ker(\omega') = \mathrm{ran}(\varrho)|_{\mathcal{S}}$, which will be argued by double containment.

[4] Given a Lie group G acting on a manifold M, the quotient M/G will be called a quotient manifold provided M/G has a (necessarily unique) smooth structure compatible with the quotient topology such that the quotient map $M \to M/G$ is a smooth submersion. For more details see [10, 5.9.5].

Since the fundamental vector field $\varrho(\zeta) \in \mathfrak{X}(X)$ and the constant section $\zeta \in \Gamma(D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}}))$ are R-related for all $\zeta \in \mathfrak{l}$, one has $J_*(\mathrm{ran}(\varrho)|_{\mathcal{S}}) = T\mathcal{O}$. Consequently

$$T\mathcal{S} = \ker(J_*|_{\mathcal{S}}) + \mathrm{ran}(\varrho)|_{\mathcal{S}}.$$

As X and $\mathcal{O}$ are L-Hamiltonian spaces with corresponding Courant morphisms $R = \mathbb{T}J_\omega$ and $\mathbb{T}(\iota_{\mathcal{O}})_{\omega_{\mathcal{O}}}$ respectively, the differential forms $\iota_{\varrho(\zeta)}\iota_{\mathcal{S}}^*\omega$ and $\iota_{\varrho(\zeta)}\iota_{\mathcal{S}}^* J^*\omega_{\mathcal{O}}$ are equal by part (c) of Proposition 3.2.1. The 2-form ω' therefore vanishes on $\mathrm{ran}(\varrho)|_{\mathcal{S}}$, i.e. $\mathrm{ran}(\varrho)|_{\mathcal{S}} \subseteq \ker(\omega')$.

For the other containment, it will be argued that the kernel of the restriction of ω' to $\ker(J^*)$ is $\ker(J_*|_{\mathcal{S}}) \cap \mathrm{ran}(\varrho)|_{\mathcal{S}}$. First note that ω' coincides with $\iota_{\mathcal{S}}^*\omega$ on $\ker(J_*)$. According to Lemma 3.3.1, the 2-form ω sends $\ker(J_*)$ to the annihilator[5] $\mathrm{Ann}_X(\mathrm{ran}(\varrho)) \subseteq T^*X$ of $\mathrm{ran}(\varrho)$. One thus has a diagram

$$\begin{array}{ccc} \ker(J_*|_{\mathcal{S}}) & \xrightarrow{\iota_\bullet\omega} & \mathrm{Ann}_X(\mathrm{ran}(\varrho)|_{\mathcal{S}}) \\ & \searrow^{\iota_\bullet\omega'} & \downarrow \iota_{\mathcal{S}}^* \\ & & \mathrm{Ann}_{\mathcal{S}}(\mathrm{ran}(\varrho)|_{\mathcal{S}}). \end{array}$$

where the top map $\iota_\bullet\omega$ is an isomorphism. The kernel of $\iota_\bullet\omega'$ is the preimage of the intersection $\mathrm{Ann}_X(\ker(J_*|_{\mathcal{S}})) \cap \mathrm{Ann}_X(\mathrm{ran}(\varrho)|_{\mathcal{S}})$ by $\iota_\bullet\omega$. The rank of this last intersection is

$$\begin{aligned} \dim(X) - \mathrm{rank}(\mathrm{Ann}_X(\ker(J_*|_{\mathcal{S}})) + \mathrm{Ann}_X(\mathrm{ran}(\varrho)|_{\mathcal{S}})) &= \dim(X) - (\dim(X) + \\ &\qquad \mathrm{rank}(\ker(J_*|_{\mathcal{S}}) \cap \mathrm{ran}(\varrho)|_{\mathcal{S}})) \\ &= \mathrm{rank}(\ker(J_*|_{\mathcal{S}}) \cap \mathrm{ran}(\varrho)|_{\mathcal{S}}), \end{aligned}$$

which shows that the kernel of the restriction of ω' to $\ker(J_*)$ is indeed $\ker(J_*|_{\mathcal{S}}) \cap \mathrm{ran}(\varrho)|_{\mathcal{S}}$. The claim is thus proven. □

Remark 3.3.2. In the context of Theorem 3.3.1, the map J_* sends $\mathrm{ran}(\varrho|_m)$ surjectively onto $T_{J(m)}\mathcal{O}$ for $m \in \mathcal{S}$ as J is L-equivariant. Thus J is transversal to $\mathcal{O}$ if and only if some, and therefore every, point of $\mathcal{O}$ is a regular value of J. In particular, one may instead of $\mathcal{S}$ consider the preimage $J^{-1}(o)$ of a point $o \in \mathcal{O}$. Then $X_{\mathcal{O}} = J^{-1}(o)/Z_o$, where Z_o is the stabilizer of the L-action on $\mathcal{O}$ at o.

Remark 3.3.3. For q-Hamiltonian spaces, this theorem is a special case of [4, Thm. 5.1].

Via Poisson structures. The 2-form (3.29) may be characterized by a Poisson structure it induces, which is the approach taken by Bursztyn-Crainic [13, § 4.4] and Bursztyn-Iglesias Ponte-Ševera [16, § 3.4]. This will now be explained.

[5] The subscript indicates the space in whose cotangent bundle the annihilator is taken (one must distinguish between the annihilator in T^*X and the annihilator in $T^*\mathcal{S}$).

Consider the algebra of L-invariant functions $C^\infty(X)^L$. For $f \in C^\infty(X)^L$, one has $\iota_{\varrho(\zeta)} df = 0$ whenever $\zeta \in \mathfrak{l}$ and thus $df \in \mathrm{Ann}_X(\mathrm{ran}(\varrho))$. Since ω sends $\ker(J_*)$ to $\mathrm{Ann}_X(\mathrm{ran}(\varrho))$ bijectively by Lemma 3.3.1 , there exists a unique (rough) vector field v_f of X with values in $\ker(J_*)$ such that

$$\iota_{v_f}\omega = -df, \tag{3.30}$$

i.e. v_f is the "Hamiltonian" vector field corresponding to f. Actually, the vector field v_f is smooth.

Proposition 3.3.4 ([16, Prop. 3.14])**.** *The vector field v_f is smooth.*

Proof. Let $F \subseteq D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$ be an arbitrary isotropic complement of $E^{(\mathfrak{l})}$. This defines a projection $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}) \to E^{(\mathfrak{l})}$. Considering R as a subset of $J^*(D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})) \times \mathbb{T}X$, there is a projection $\mathrm{pr} : R \to J^*E^{(\mathfrak{l})} \times T^*X$. The map pr is injective. Indeed, if $\mathrm{pr}(x_2, x_1) = 0$ for $x_1 \in \mathbb{T}X$ and $x_2 \in J^*(D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}))$ then $x_1 \in TX$. As $x_1 \sim_R x_2$, one has $x_2 \in J^*E^{(\mathfrak{l})}$ according to the definition of a Dirac morphism, implying that $x_2 = 0$. But the definition of a Dirac morphism then also implies that x_1 is trivial.

Now the pair (v_f, df) is a rough section of $\ker(R)$ and thus the triple $(vf, df, 0)$ is a rough section of $R \subseteq J^*(D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})) \times \mathbb{T}X$. The image of this triple under pr is $(df, 0)$, meaning that $(v_f, df, 0)$ is the section $\mathrm{pr}^{-1}(df, 0)$. So $(v_f, df, 0)$ is in fact smooth and therefore v_f is smooth. □

Next, define an L-invariant bracket $\{\cdot,\cdot\}$ on $C^\infty(X)^L$ by putting

$$\{f, \cdot\} = v_f. \tag{3.31}$$

Proposition 3.3.5 ([16, Lem. 3.12])**.** *The bracket $\{\cdot,\cdot\}$ is Poisson.*

Proof. That $\{\cdot,\cdot\}$ is $\mathbb{R}$-bilinear and Leibniz is standard. Let $f, g, h \in C^\infty(X)^L$. Since $\ker(R) = \{v - \iota_v\omega \,:\, v \in \ker(J_*)\}$, one has $v_f + df \in \ker(R)$. In particular $\langle v_f + df, v_g + dg\rangle$ vanishes and thus

$$\{g, f\} = \iota_{v_g} df = -\iota_{v_f} dg = -\{f, g\},$$

i.e. $\{\cdot,\cdot\}$ is skew-symmetric.

Next, observe that since $d\{f, g\} = \mathcal{L}_{v_f} dg - \iota_{v_f} ddg = \mathcal{L}_{v_f} dg$ one has

$$[\![v_f + df, v_g + dg]\!] = [v_f, v_g] + \mathcal{L}_{v_f} dg - \iota_{v_g} dg = [v_f, v_g] + d\{f, g\},$$

where the equality $\iota_{v_g} dg = 0$, which is equivalent to $\langle v_g + dg, v_g + dg \rangle = 0$, was used. Since $\ker(R)$ is involutive and $\omega^\sharp$ is injective on $\ker(J_*)$, it follows that $v_{\{f,g\}} = [v_f, v_g]$. Property C1 then gives

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\},$$

which is the Jacobi identity. □

Let $C^\infty(\mathcal{S})^L$ be the space of L-invariant smooth functions on $\mathcal{S}$. If the conditions of Theorem 3.3.1 are met then the 2-form (3.29), by virtue of descending to a symplectic form on $X_\mathcal{O} = \mathcal{S}/L$, defines a Poisson bracket $\{\cdot, \cdot\}'$ on $C^\infty(\mathcal{S})^L$ via

$$\{f, \cdot\}' = w_f,$$

where $w_f \in \mathfrak{X}(\mathcal{S})$ is the unique L-invariant vector field such that

$$\iota_{w_f} \omega' = -df. \tag{3.32}$$

Proposition 3.3.6 ([16, Prop. 3.16])**.** *If the conditions of Theorem 3.3.1 are met, then $\{\cdot, \cdot\}'$ is the unique Poisson bracket on $C^\infty(\mathcal{S})^L$ such that the restriction map $\iota_\mathcal{S}^* : C^\infty(X)^L \to C^\infty(\mathcal{S})^L$, with $C^\infty(X)^L$ equipped with the Poisson bracket $\{\cdot, \cdot\}$, is Poisson.*

Proof. Let $f \in C^\infty(X)^L$. Then the vector field $v_f \in \mathfrak{X}(X)$ with values in $\ker(J_*)$ defined by (3.30) is parallel to $\mathcal{S}$ since $\mathcal{S} = \ker(J_*) + \operatorname{ran}(\varrho)$. Furthermore, as ω' and $\iota_\mathcal{S}^* \omega$ coincide on $\ker(J_*)$, one may substitute ω' for ω in (3.30), i.e $\iota_{v_f|_\mathcal{S}} \omega' = -\iota_\mathcal{S}^* df$. From this it follows that $v_f|_\mathcal{S}$ is equal to $w_{f \circ \iota_\mathcal{S}}$ as defined by (3.32). In particular this means, if g is another function in $C^\infty(X)^L$, that

$$\{f \circ \iota_\mathcal{S}, g \circ \iota_\mathcal{S}\}' = \iota_\mathcal{S} \circ \{f, g\},$$

meaning that the restriction map $\iota_\mathcal{S}^*$ is indeed Poisson. On the other hand, the above equation determines $\{\cdot, \cdot\}'$, hence its uniqueness. □

If $X_{\text{red}} = X/L$ and $X_\mathcal{O} = \mathcal{S}/L$ are quotient manifolds, then Proposition 3.3.6 equivalently says that $X_\mathcal{O}$ is a symplectic leaf of X_{red}.

Shifting trick. The final result of this chapter relates the fusion product, dualization and symplectic reduction. Let $\mathfrak{l} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ be a multiplicative and symmetric Lagrangian subalgebra integrating to the closed and connected subgroup $L \subseteq D \times D$.

Theorem 3.3.2 (Shifting trick)**.** *Let (X, J, ω) be a L-Hamiltonian space for $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$. Given an L-orbit $\iota_\mathcal{O} : \mathcal{O} \hookrightarrow D$, then (X, J, ω) and $\mathcal{O}$ meet the conditions of Theorem 3.3.1 if and* *only if* $X \circledast \mathcal{O}^*$ *and the* L*-orbit* $\iota_{\mathcal{O}_e} : \mathcal{O}_e \hookrightarrow D$ *through the group identity* $e \in D$ *do*[6] *In that case there is a canonical symplectomorphism*

$$X_{\mathcal{O}} \simeq (X \circledast \mathcal{O}^*)_{\mathcal{O}_e}. \tag{3.33}$$

The second part of the statement must be clarified. Let $\mathcal{S}_X$ and $\mathcal{S}_{X\circledast\mathcal{O}^*}$ be the preimages of $\mathcal{O}$ and $\mathcal{O}_e$ under the moment maps of X and $X \circledast \mathcal{O}^*$ respectively. Then the claim is that there is a commutative diagram

$$\begin{array}{ccc} \mathcal{S}_X & \xrightarrow{\;f\;} & \mathcal{S}_{X\circledast\mathcal{O}^*} \\ \Big\downarrow{\cdot/L} & & \Big\downarrow{\cdot/L} \\ X_{\mathcal{O}} & \xrightarrow{\;f_{\text{red}}\;} & X_{\mathcal{O}_e} \end{array} \tag{3.34}$$

where 1. the bottom horizontal arrow f_{red} is a bijection, and 2. the top horizontal arrow f is a smooth map under which the 2-form $\omega'_X \in \Omega^2(\mathcal{S}_X)$ defined in (3.29) is the pullback of its counterpart $\omega'_{X\circledast\mathcal{O}^*} \in \Omega^2(\mathcal{S}_{X\circledast\mathcal{O}^*})$.

Proof. The moment map of $X \circledast \mathcal{O}^*$ is $J \circledast (\text{inv} \circ \iota_{\mathcal{O}})$, which is related to the map $J \times (\text{inv} \circ \iota_{\mathcal{O}}) : X \times \mathcal{O}^* \to D \times D$ via the diagram

$$\begin{array}{ccc} X \times \mathcal{O}^* & \xrightarrow{\;J \times (\text{inv} \circ \iota_{\mathcal{O}})\;} & D \times D \\ \Big\downarrow{\cdot/\,\text{Lie}((\mathfrak{l} \times \mathfrak{l}) \cap \mathfrak{d}_{\Delta(2,3)})} & & \Big\downarrow{\text{mult}} \\ X \circledast \mathcal{O}^* & \xrightarrow{\;J \circledast (\text{inv} \circ \iota_{\mathcal{O}})\;} & D \end{array}, \tag{3.35}$$

Now $J \circledast (\text{inv} \circ \iota_{\mathcal{O}})$ is transversal to $\mathcal{O}_e$ if and only if the group identity is a regular value for it (Remark 3.3.2), or equivalently (according to the diagram (3.35)) if and only if $J \times (\text{inv} \circ \iota_{\mathcal{O}})$ is transversal to the anti-diagonal $D_{\overline{\Delta}} = \{(d, d^{-1}) \ : \ d \in D\}$. The preimage of $D_{\overline{\Delta}}$ by $J \times (\text{inv} \circ \iota_{\mathcal{O}})$ is

$$\{(m, J(m)) \ : \ m \in \mathcal{S}\} \simeq \mathcal{S}.$$

For $m \in \mathcal{S}$, consider the sum of subspaces

$$\text{ran}((J \times (\text{inv} \circ \iota_{\mathcal{O}}))_*|_{(m,J(m))}) + T_{(J(m),J(m)^{-1})} D_{\overline{\Delta}}. \tag{3.36}$$

Since $T_{(J(m),J(m)^{-1})} D_{\overline{\Delta}} = \{(v, \text{inv}_* v) \ : \ v \in T_{J(m)}\}$, the dimension of the intersection of the two summands in (3.36) is equal to that of the intersection $\text{ran}(J_*|_m) \cap T_{J(m)}\mathcal{O}$. The dimension

[6] Note that in general $\mathcal{O}_e$ is not a singleton like in other symplectic reduction theorems.

of (3.36) is therefore equal to

$$\operatorname{rank}(J) + \dim(\mathcal{O}) + \dim(D) - \dim(\operatorname{ran}(J_*|_m) \cap T_{J(m)}\mathcal{O}).$$

So the sum (3.36) is equal to $T_{(J(m),J(m)^{-1})}D \times D$ if and only if the sum $\operatorname{ran}(J_*|_m) + T\mathcal{O}_{J(m)}$ is equal to $T_{J(m)}D$, that is to say $J \times (\operatorname{inv} \circ \iota_{\mathcal{O}})$ is transversal to $D_{\overline{\Delta}}$ if and only if J is transversal to $\mathcal{O}$. This establishes the first part of the statement.

Suppose now that X and $\mathcal{O}$, and hence $X \circledast \mathcal{O}^*$ and $\mathcal{O}_e$, satisfy the conditions of Theorem 3.3.1. Let $\widehat{\mathcal{S}}_{X\circledast\mathcal{O}^*}$ be the lift of $\mathcal{S}_{X\circledast\mathcal{O}^*}$ to $X \times \mathcal{O}^*$. Explicitly,

$$\widehat{\mathcal{S}}_{X\circledast\mathcal{O}^*} = \{(m, o) \in X \times \mathcal{O}^* \ : \ J(m)o^{-1} \in \mathcal{O}_e\}. \tag{3.37}$$

In particular there is a natural embedding of $\mathcal{S}$ into $\widehat{\mathcal{S}}_{X\circledast\mathcal{O}^*}$ sending $m \in \mathcal{S}_X$ to $(m, J(m))$. Define $f : \mathcal{S}_X \to \mathcal{S}_{X\circledast\mathcal{O}^*}$ to be the composition this embedding and the quotient map $\cdot/((L \times L) \cap D_{\Delta(2,3)})$. It is now claimed that

$$\omega'_X = f^*\omega'_{X\circledast\mathcal{O}^*} \tag{3.38}$$

or equivalently that $\omega'_X = (\mathrm{Id} \times J)^*\widehat{\omega}'_{X\circledast\mathcal{O}^*}$, where $\widehat{\omega}'_{X\circledast\mathcal{O}^*} \in \Omega^2(\widehat{\mathcal{S}}_{X\circledast\mathcal{O}^*})$ is the pullback of $\omega'_{X\circledast\mathcal{O}^*}$ to $\widehat{\mathcal{S}}_{X\circledast\mathcal{O}^*}$. First, recall Corollary 3.2.1: the 2-form (3.2) corresponding to $\mathcal{O}_e$ is trivial since $\mathfrak{l}$ is multiplicative. In particular the 2-form $\omega'_{X\circledast\mathcal{O}^*}$ is simply equal to the pullback to $\mathcal{S}_{X\circledast\mathcal{O}^*}$ of the 2-form of the L-Hamiltonian space $X \circledast \mathcal{O}^*$, which is $\omega \circledast (-\omega_{\mathcal{O}}) \in \Omega^2(X \circledast \mathcal{O}^*)$ where $\omega_{\mathcal{O}} \in \Omega^2(\mathcal{O})$ is the 2-form (3.2) corresponding to $\mathcal{O}$. On the other hand, the pullback of $\omega \circledast (-\omega_{\mathcal{O}})$ to $X \times \mathcal{O}^*$ is given by (3.17). The upshot is that

$$\widehat{\omega}'_{X\circledast\mathcal{O}^*} = \iota^*_{\widehat{\mathcal{S}}_{X\circledast\mathcal{O}^*}}(\omega^1 - \omega^2_{\mathcal{O}} + (J \times \operatorname{inv})^*\varsigma).$$

Now,

$$\begin{aligned}
(\mathrm{Id} \times J)^*\widehat{\omega}'_{X\circledast\mathcal{O}^*} &= \iota^*_{\mathcal{S}_X}\omega - \iota^*_{\mathcal{S}_X}J^*\omega_{\mathcal{O}} + \frac{1}{2}\langle J^*\theta^L, J^*\operatorname{inv}^*\theta^R\rangle \\
&= \iota^*_{\mathcal{S}_X}\omega - \iota^*_{\mathcal{S}_X}J^*\omega_{\mathcal{O}} - \cancel{\frac{1}{2}\langle J^*\theta^L, J^*\theta^L\rangle} \\
&= \iota^*_{\mathcal{S}_X}(\omega - J^*\omega_{\mathcal{O}}) \\
&= \omega'_X,
\end{aligned}$$

proving the equality (3.38).

Finally, it will be shown that the map $f : \mathcal{S}_X \to \mathcal{S}_{X\circledast\mathcal{O}^*}$, $m \mapsto [m, J(m)]$ descends to a bijection $X_{\mathcal{O}} \to X_{\mathcal{O}_e}$. First notice that $(d_1, d_2).m$, where $(d_1, d_2) \in L$, is sent by f to

$[(d_1, d_2).m, (d_2, d_1).J(m)]$ (recall that the L-action on $\mathcal{O}^*$ is $(d_1, d_2).o = d_2 o d_1^{-1}$). Since $(d_1, d_2) \circ (d_2, d_1) = (d_1, d_1)$, one has

$$[(d_1, d_2).m, (d_2, d_1).J(m)] = (d_1, d_1).[m, J(m)].$$

according to the definition of the L-action on $X \circledast \mathcal{O}^*$. This shows that f descends to a map $f_{\mathrm{red}} : X_{\mathcal{O}} \to X_{X \circledast \mathcal{O}^*}$ as in the diagram (3.34).

Suppose now $f(m) = (d_1, d_2).f(m')$ for $m, m' \in \mathcal{S}_{\mathcal{O}}$ and $(d_1, d_2) \in L$. Then

$$\begin{aligned} f(m) = (d_1, d_2).f(m') &\implies [m, J(m)] = [(d_1, d).m', (d, d_2).m'] \\ &\implies m = (e, d')(d_1, d)m' \end{aligned}$$

where $d, d' \in D$ are elements such that $(d_1, d), (d, d_2), (e, d') \in L$. This shows that $[m] = [m']$ in $X_{\mathcal{O}}$ and therefore that f_{red} is injective.

For surjectivity, suppose an element $[m, o] \in \mathcal{S}_{X \circledast \mathcal{O}^*}$ is given. One has $J(m)o^{-1} \in \mathcal{O}_e$ and thus $J(m)o^{-1} = d_1 d_2^{-1}$ for some $(d_1, d_2) \in L$. Now

$$d_1^{-1} J(m) d_2 (d_2^{-1} o d_2)^{-1} = d_1^{-1} J(m) o^{-1} d_2 = e. \tag{3.39}$$

The pairs (d_1^{-1}, d_2^{-1}) and (d_2^{-1}, d_2^{-1}) are in L since the first is the inverse of (d_1, d_2) and the second is equal to $(d_2^{-1}, d_1^{-1}) \circ (d_1^{-1}, d_2^{-1})$. One may thus rewrite (3.39) in terms of the L-actions on X and $\mathcal{O}^*$ as

$$J((d_1^{-1}, d_2^{-1}).m)((d_2^{-1}, d_2^{-1}).o)^{-1} = e \iff (d_2^{-1}, d_2^{-1}).o = J((d_1^{-1}, d_2^{-1}).m).$$

This leads to

$$(d_1^{-1}, d_2^{-1})[m, o] = [(d_1^{-1}, d_2^{-1}).m, (d_2^{-1}, d_2^{-1}).o] = f(m')$$

where $m' = (d_1^{-1}, d_2^{-1}).m$. So f sends m' to an element of $\mathcal{S}_{X \circledast \mathcal{O}^*}$ that is L-equivalent to $[m, o]$ and f_{red} is therefore surjective. The proof is now complete. □

Chapter 4

D/G-valued moment maps

The results proven for L-Hamiltonian spaces in the preceding chapter can now be brought to bear on the theory of D/G-valued moment maps. Of particular interest, fusion and duality behave differently in this category than in other familiar settings: the fusion product is not commutative and duals are not isomorphic as G-spaces. Explicit counterexamples are provided.

4.1 Hamiltonian spaces for action Courant algebroids

Suppose $\mathfrak{d}$ is a quadratic Lie algebra and $\mathfrak{g} \subseteq \mathfrak{d}$ is a Lagrangian subalgebra. Let $\mathfrak{h} \subseteq \mathfrak{d}$ be a Lagrangian complement of $\mathfrak{g}$, which is guaranteed to exist by Proposition 2.1.1. The triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is called a *Manin quasi-triple*. If $\mathfrak{h}$ is a Lagrangian subalgebra of $\mathfrak{d}$, the triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is simply called a *Manin triple*.

Example 4.1.1. Let $\mathfrak{g}$ be a real semisimple Lie algebra and consider its complexification $\mathfrak{g}^{\mathbb{C}}$ viewed as a real Lie algebra. The imaginary part of the Killing form of $\mathfrak{g}^{\mathbb{C}}$ is an Ad-invariant metric under which $\mathfrak{g}$ is a Lagrangian subalgebra of $\mathfrak{g}^{\mathbb{C}}$. The triple $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}, \sqrt{-1}\mathfrak{g})$ is an example of a Manin quasi-triple. According to results by P. Delorme [22], there is at least one Lagrangian subalgebra $\mathfrak{h} \subseteq \mathfrak{g}^{\mathbb{C}}$ complementary to $\mathfrak{g}$. If $\mathfrak{g}$ is compact then there is a canonical (up to conjugation) such complementary Lagrangian subalgebra; it is the solvable Lie algebra $\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{n}$ appearing in the Iwasawa decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ with $\mathfrak{k} = \mathfrak{g}$ [45].

Let $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ be a Manin quasi-triple. Suppose S is a manifold on which $\mathfrak{d}$ acts with coisotropic stabilizers thus defining an action Courant algebroid $S \times \mathfrak{d}$. The anchor will be denoted by $\mathbf{a}_S : S \times \mathfrak{d} \to TS$. Consider a Hamiltonian space

$$R : (\mathbb{T}X, TX) \dashrightarrow (S \times \mathfrak{d}, E^{(\mathfrak{g})})$$

with moment map $J : X \to S$. The same considerations as in the case of L-Hamiltonian spaces reveal that the Lie algebroid action (3.1) is a Lie algebra action

$$\varrho : \mathfrak{g} \to \mathfrak{X}^1(X).$$

The extension of ϱ to a morphism of exterior algebras $\bigwedge^\bullet \mathfrak{g} \to \mathfrak{X}^\bullet(X)$ will also be denoted by ϱ. Recall Propositions 2.1.2 and 2.1.3, and the intervening discussion: the backward image $E^{(\mathfrak{h})} \circ R$ is the graph $\mathrm{Gr}(-\pi^{\mathfrak{h}})$ of a bivector field $-\pi^{\mathfrak{h}} \in \mathfrak{X}^2(X)$. Moreover, R induces to a bundle map

$$\varrho^*_{\mathfrak{h}} : \mathrm{Gr}(-\pi^{\mathfrak{h}}) \simeq T^*X \to J^*E^{(\mathfrak{h})}$$

dual to ϱ (taken as a bundle map $J^*E^{(\mathfrak{g})} \to TX$).

Recall Lemmas 2.3.1 and 2.4.3: the Courant tensor $\Upsilon^{\mathrm{Gr}(-\pi^{\mathfrak{h}})} \in \Gamma(\bigwedge^3 TX)$ and the Dirac tensor $\Lambda^{\mathrm{Gr}(-\pi^{\mathfrak{h}})} \in \mathfrak{g}^* \times \Gamma(\bigwedge^2 E)$ defined in Chapter 2 are related to the Courant and Dirac tensors of $E^{(\mathfrak{h})}$ via

$$\Upsilon^{\mathrm{Gr}(-\pi^{\mathfrak{h}})} = \varrho(J^*\Upsilon^{E^{(\mathfrak{h})}}) = \varrho(\Upsilon^{\mathfrak{h}})$$

and

$$\Lambda^{\mathrm{Gr}(-\pi^{\mathfrak{h}})} = (\mathrm{Id}_{\mathfrak{g}} \times \varrho)(J^*\Lambda^{E^{(\mathfrak{h})}}) = (\mathrm{Id}_{\mathfrak{g}} \times \varrho)(\Lambda^{\mathfrak{h}})$$

where ϱ has been extended to map $\bigwedge^\bullet E^{(\mathfrak{g})} \to \bigwedge^\bullet TX$ in the second terms of either equation. The tensors

$$\Upsilon^{E^{(\mathfrak{h})}} \in \Gamma(\bigwedge^3 E^{(\mathfrak{g})}) = \Gamma(X \times \bigwedge^3 \mathfrak{g})$$
$$\Lambda^{E^{(\mathfrak{h})}} \in \Gamma(\mathfrak{g}^* \times \bigwedge^2 E^{(\mathfrak{g})}) = \Gamma(X \times \mathfrak{g}^* \times \bigwedge^2 \mathfrak{g})$$

are of course the Courant and Dirac tensors of $E^{(\mathfrak{h})}$, respectively.

According to Examples 2.2.8 and 2.4.3, the tensors $\Upsilon^{\mathrm{Gr}(-\pi^{\mathfrak{h}})}$ and $\Lambda^{\mathrm{Gr}(-\pi^{\mathfrak{h}})}$ are $\frac{1}{2}[\pi^{\mathfrak{h}}, \pi^{\mathfrak{h}}]$ and $\mathcal{L}_{\varrho(\gamma)}\pi^{\mathfrak{h}}$ respectively. The upshot is that

$$\frac{1}{2}[\pi^{\mathfrak{h}}, \pi^{\mathfrak{h}}] = \varrho(\Upsilon^{\mathfrak{h}}), \quad \mathcal{L}_{\varrho(\gamma)}\pi^{\mathfrak{h}} = -\varrho(\Lambda^{\mathfrak{h}}(\gamma)). \tag{4.1}$$

These conditions are precisely the ones defining a quasi-Poisson action of $\mathfrak{g}$ on a manifold X found in [2] equipped with a bivector field $\pi^{\mathfrak{h}}$ in [2].

Definition 4.1.1 ([2])**.** Let X be a manifold equipped with a bivector field $\pi^{\mathfrak{h}} \in \mathfrak{X}^2(X)$. A quasi-Poisson action of $\mathfrak{g}$ on X is a $\mathfrak{g}$-action $\varrho : \mathfrak{g} \to \mathfrak{X}(X)$ such that the conditions (4.1) are satisfied. In this case X is called a *quasi-Poisson* $\mathfrak{g}$*-space* (G*-space* if the $\mathfrak{g}$-action integrates to a G-action). The quasi-Poisson $\mathfrak{g}$-space X is called *non-degenerate* if

$$\mathrm{ran}(\varrho) + \mathrm{ran}((\pi^{\mathfrak{h}})^{\sharp}) = TX.$$

As shown in [2], the choice of Lagrangian complement $\mathfrak{h}$ in the previous definition is immaterial; the prescription

$$\pi^{\mathfrak{h}'} = \pi^{\mathfrak{h}} - \varrho(t),$$

where $t \in \bigwedge^2 \mathfrak{g}$ is the twist defined by

$$t^{\sharp} : \mathfrak{h} \simeq \mathfrak{g}^* \to \mathfrak{g},\ \zeta \mapsto \mathrm{pr}_{\mathfrak{h}'}\zeta - \zeta$$

(Proposition 2.1.3) shows how to modify $\pi^{\mathfrak{h}}$ given another choice of Lagrangian complement $\mathfrak{h}' \subseteq \mathfrak{d} \oplus \bar{\mathfrak{d}}$ of $\mathfrak{g}$. The next proposition is essentially due to Bursztyn and Crainic [14]. There the corresponding results are presented as a categorical equivalence between two moment map theories, one stated in terms of Dirac geometry and the other in terms of quasi-Poisson actions.

Proposition 4.1.1. *A Hamiltonian space for $(S \times \mathfrak{d}, E^{(\mathfrak{g})})$ is equivalently a quasi-Poisson $\mathfrak{g}$-space X together with a $\mathfrak{g}$-equivariant map $J : X \to S$ such that the moment map condition*

$$(\pi^{\mathfrak{h}})^{\sharp} \circ J^* = \varrho \circ (\mathbf{a}_S|_{E^{(\mathfrak{h})}})^*, \tag{4.2}$$

where $\varrho : \mathfrak{g} \to X$ is the $\mathfrak{g}$-action and $\pi^{\mathfrak{h}} \in \mathfrak{X}^2(X)$ is the corresponding $\mathfrak{l}$-Hamiltonian quasi-Poisson bivector field, is satisfied[1]*. If $S \times \mathfrak{d}$ is exact then X is an exact Hamiltonian space if and only if it is non-degenerate as a quasi-Poisson $\mathfrak{g}$-space.*

Proof. In one direction, suppose X is a Hamiltonian space for $(S \times \mathfrak{d}, E^{(\mathfrak{g})})$. It was seen above that $\mathfrak{h}$ determines a bivector field $\pi^{\mathfrak{h}} \in \mathfrak{X}^2(X)$ satisfying (4.1). One knows that

$$(\mathbf{a}_S \times \mathbf{a})^*(\mathrm{Gr}(J^*)) \subseteq R$$

and thus for $\mu \in T^*X$ and $\alpha \in T^*S$

$$\begin{aligned}\langle -\iota_{\mu}\pi^{\mathfrak{h}} + \mu, J^*\alpha\rangle &= \langle \varrho^*_{\mathfrak{h}}(\mu), \mathbf{a}^*_S\alpha\rangle \\ &= \langle \mu, \varrho \circ \mathrm{pr}_{E^{(\mathfrak{g})}} \circ \mathbf{a}^*_S(\alpha)\rangle \\ &= \langle \mu, \varrho \circ (\mathbf{a}_S|_{E^{(\mathfrak{h})}})^*(\alpha)\rangle,\end{aligned}$$

where the duality of $\varrho^*_{\mathfrak{h}}$ and ϱ was used. Since $\pi^{\mathfrak{h}}$ is anti-symmetric, the above equation is

[1] It is emphasized that this characterization depends on $\mathfrak{h}$.

precisely (4.2).

In the other direction, suppose a bivector field $\pi^{\mathfrak{h}} \in \mathfrak{X}^2(X)$ satisfying (4.1) and (4.2) is given. One wishes to recover a Dirac morphism

$$R : (\mathbb{T}X, TX) \dashrightarrow (S \times \mathfrak{d}, E^{(\mathfrak{g})})$$

with base map J and corresponding bivector field $\pi_{\mathfrak{h}}$. Since R must define a bundle map $\varrho_{\mathfrak{h}}^* : T^*X \to J^*E^{(\mathfrak{h})}$ dual to $\varrho : J^*E^{(\mathfrak{g})} \to TX$, one is forced to write

$$R = \{f(\gamma) \ : \ \gamma \in \mathfrak{g}\} + \{g(\mu) \ : \ \mu \in T^*X\} \tag{4.3}$$

where

$$f : \mathfrak{g} \to S \times \mathfrak{d} \times \overline{\mathbb{T}X}, \ \ \gamma \mapsto (\gamma, \varrho(\gamma))$$

and

$$g : T^*X \to S \times \mathfrak{d} \times \overline{\mathbb{T}X}, \ \ \mu \mapsto (\varrho_{\mathfrak{h}}^*(\mu), -\iota_\mu \pi^{\mathfrak{h}} + \mu).$$

The two summands on the right-hand-side of (4.3) intersect trivially and a dimension count establishes that R is indeed Lagrangian in $S \times \mathfrak{d} \times \overline{\mathbb{T}X}$. Dualizing the $\mathfrak{g}$-equivariance condition $J_* \circ \varrho = \mathbf{a}_S|_{E^{(\mathfrak{g})}}$, one has

$$\varrho_{\mathfrak{h}}^* \circ J^* = (\mathbf{a}_S|_{E^{(\mathfrak{g})}})^* = \mathrm{pr}_{E^{(\mathfrak{h})}} \circ \mathbf{a}_S^*.$$

For $\alpha \in T^*S$, let $\gamma = (\mathbf{a}_S|_{E^{(\mathfrak{h})}})^*(\alpha)$. Then

$$\varrho_{\mathfrak{h}}^*(J^*\alpha) + \gamma = (\mathbf{a}_S|_{E^{(\mathfrak{g})}})^*(\alpha) + (\mathbf{a}_S|_{E^{(\mathfrak{h})}})^*(\alpha) = \mathbf{a}_S^*(\alpha).$$

Moreover (4.2) gives $\iota_{J^*\alpha}\pi^{\mathfrak{h}} = \varrho_{\mathfrak{h}}^*(\gamma)$ and thus

$$(J^*\alpha, \alpha) = f(\gamma) + g(J^*\alpha),$$

and the containment $(\mathbf{a}_S \times \mathbf{a})^*(\mathrm{Gr}(J^*)) \subseteq R$, equivalently $(\mathbf{a}_S \times \mathbf{a})(R) \subseteq \mathrm{Gr}(J_*)$, follows. The Courant tensor $\Upsilon^R \in \Gamma(\bigwedge^3 R^*)$ of R is therefore defined and one must show that it vanishes to verify that R is a Dirac structure supported on $\mathrm{Gr}(J)$. Consider sections $\sigma_i, \tau_j \in \Gamma(E^{(\mathfrak{h})} \times \mathrm{Gr}(-\pi^{\mathfrak{h}}))$ ($i = 1, 2, 3$ and $j = 1, 2$) such that σ_i restricts to $g(\mu_i)$ on $\mathrm{Gr}(J)$ for a differential form $\mu_i \in \Omega^1(X)$ and τ_j restricts to $f(\gamma_i)$ on $\mathrm{Gr}(J)$ for some $\gamma_i \in \mathfrak{g}$. Then for $m \in X$ one has

$$\Upsilon^R(g(\mu_1), g(\mu_2), g(\mu_3))|_{(J(m),m)} = \langle [\![\sigma_1, \sigma_2]\!], \sigma_3 \rangle|_{(J(m),m)}$$

$$
\begin{aligned}
&= \Upsilon^{E^{(\mathfrak{h})}\times \mathrm{Gr}(-\pi^{\mathfrak{h}})}(\sigma_1,\sigma_2,\sigma_3)|_{(J(m),m)} \\
&= \Upsilon^{E^{(\mathfrak{h})}}(\varrho_{\mathfrak{h}}^*(\mu_1),\varrho_{\mathfrak{h}}^*(\mu_2),\varrho_{\mathfrak{h}}^*(\mu_3))|_{J(m)} - \\
&\quad \Upsilon^{\mathrm{Gr}(-\pi^{\mathfrak{h}})}(\mu_1,\mu_2,\mu_3)|_m \\
&= \varrho(\Upsilon^{\mathfrak{h}})(\mu_1,\mu_2,\mu_3)|_m - \frac{1}{2}[\pi^{\mathfrak{h}},\pi^{\mathfrak{h}}](\mu_1,\mu_2,\mu_3)|_m \\
&= 0.
\end{aligned}
$$

Also

$$
\begin{aligned}
\Upsilon^R(f(\gamma_1), g(\mu_1), g(\mu_2))|_{(J(m),m)} &= \langle [\![\tau_1,\sigma_1]\!],\sigma_2\rangle|_{(J(m),m)} \\
&= \Upsilon^{E^{(\mathfrak{h})}\times \mathrm{Gr}(-\pi^{\mathfrak{h}})}((\gamma_1,\varrho(\gamma_1)),\sigma_1,\sigma_2)|_{(J(m),m)} \\
&= \Lambda^{E^{(\mathfrak{h})}\times \mathrm{Gr}(-\pi^{\mathfrak{h}})}(\gamma_1,\sigma_1,\sigma_2)|_{(J(m),m)} \\
&= \Lambda^{E^{(\mathfrak{h})}}(\gamma_1,\varrho_{\mathfrak{h}}^*(\mu_1),\varrho_{\mathfrak{h}}^*(\mu_2))|_{J(m)} - \\
&\quad \Lambda^{\mathrm{Gr}(-\pi^{\mathfrak{h}})}(\gamma_1,\mu_1,\mu_2)|_m \\
&= \varrho(\Lambda^{\mathfrak{h}})(\gamma_1,\mu_1,\mu_2)|_m + \mathcal{L}_{\gamma_1}\pi^{\mathfrak{h}}(\mu_1,\mu_2)|_m \\
&= 0.
\end{aligned}
$$

Finally for any section $\sigma \in \Gamma(R)$ one has

$$
\begin{aligned}
\Upsilon^R(f(\gamma_1), f(\gamma_2),\sigma) &= \langle [\![\tau_1,\tau_2]\!]|_{\mathrm{Gr}(J)},\sigma\rangle \\
&= \langle f([\gamma_1,\gamma_2]),\sigma\rangle \\
&= 0.
\end{aligned}
$$

These calculations show that Υ^R vanishes identically and thereby prove the first part of the statement.

The second part follows immediately from the fact that the Courant morphism $R : \mathbb{T}X \dashrightarrow S \times \mathfrak{d}$ is exact if and only if

$$
\mathbf{a}(\mathrm{ran}^*(R)) = \mathrm{ran}(\varrho) + \mathrm{ran}((\pi^{\mathfrak{h}})^\sharp) = TX. \qquad \square
$$

A quasi-Poisson $\mathfrak{g}$-space X is called a *Hamiltonian quasi-Poisson* $\mathfrak{g}$*-space (with moment map valued in* S*)* if a moment map $J : X \to S$ satisfying the moment map condition 4.2 has been chosen. In this case the bivector field $\pi^{\mathfrak{h}} \in \mathfrak{X}^2(X)$ will be called a $\mathfrak{g}$-*Hamiltonian bivector field* for X.

4.1.1 Fusion product of L-Hamiltonian spaces

The L-Hamiltonian spaces considered in Chapter 3 are special cases of the kind of Hamiltonian spaces considered above. One could expect the bivector fields associated to the fusion product of two L-Hamiltonian spaces to be related to their respective associated bivector fields in some way. This section culminates in Theorem 4.1.1, which gives this relation and generalizes a quasi-Poisson geometric characterization of the fusion product of quasi-Hamiltonian spaces appearing in [3]. Specifically, given Lagrangian subalgebras $\mathfrak{l}_i \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ $(i = 1, 2)$ integrating to closed and connected subgroups $L_i \subseteq D$ and L_i-Hamiltonian spaces

$$R_i : \mathbb{T} X_i \dashrightarrow D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}),$$

Theorem 4.1.1 exhibits a $\mathfrak{l}_1 \circ \mathfrak{l}_2$-Hamiltonian bivector field for the fusion product $X_1 \circledast X_2$ in terms of $\mathfrak{l}_i$-Hamiltonian bivector fields for X_i. This result relies on the existence of ad_{L_i}-invariant Lagrangian complements of the Lagrangian subalgebras $\mathfrak{l}_i \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$. Such complements do not always exist (see Examples 4.4.1 and 4.4.2), but presumably do in most cases encountered in practice according the following proposition.

Proposition 4.1.2. *Suppose the Lagrangian subalgebra $\mathfrak{l} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ integrates to the closed and connected subgroup $L \subseteq D \times D$. Then $\mathfrak{l}$ admits an Ad_L-invariant Lagrangian complement if it is semisimple or if L is compact.*

Proof. If $\mathfrak{l}$ is semisimple then any real finite-dimensional representation of it is completely reducible (this is the real analogue of Weyl's theorem on complete reducibility, see e.g. [32, Appx. B]). Applying this to the adjoint representation of $\mathfrak{l}$ in $\mathfrak{d} \oplus \overline{\mathfrak{d}}$, it follows that $\mathfrak{l} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ admits an ad-invariant complement. On the other hand, if L is compact then one can define an Ad_L-invariant positive definite metric on $\mathfrak{d} \oplus \overline{\mathfrak{d}}$. So $\mathfrak{l}$ admits an Ad_L-invariant complement in $\mathfrak{d} \oplus \overline{\mathfrak{d}}$ in either case. One can thus start with such a complement in the proof of Proposition 2.1.1; the constructions derived from this complement given there are also Ad_L-invariant and therefore produce an Ad_L-invariant Lagrangian complement of $\mathfrak{l}$ in the end. □

Suppose now the Lagrangian subalgebras $\mathfrak{l}_i \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ admit Ad_{L_i}-invariant Lagrangian complements $\mathfrak{k}_i \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$. Choose a (not necessarily $\mathrm{Ad}_{\mathrm{Lie}(\mathfrak{l}_1 \circ \mathfrak{l}_2)}$-invariant) Lagrangian complement $\mathfrak{k}$ of $\mathfrak{l}_1 \circ \mathfrak{l}_2$. Denote the moment maps of the L_i-Hamiltonian spaces X_i by $J_i : X_i \to D$ and the corresponding $\mathfrak{l}_i$-actions by $\varrho_i : \mathfrak{l}_i \mapsto \mathfrak{X}(X_i)$. Let

$$C^\perp = (\mathfrak{l}_1 \oplus \mathfrak{l}_2) \cap \mathfrak{d}_{\Delta(2,3)}$$

where $\mathfrak{d}_{\Delta(2,3)} = 0 \times \mathfrak{d}_\Delta \times 0$. Denote by $(D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}))_{\mathrm{red}}$ the coisotropic reduction of $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$

by $E^{(C)}$, i.e.

$$(D\times(\mathfrak{d}\oplus\bar{\mathfrak{d}}))_{\mathrm{red}} = \frac{E^{(C)}}{E^{(C^\perp)}}\Big/\,\mathrm{Lie}((\mathfrak{l}_1\times\mathfrak{l}_2)\cap\mathfrak{d}_{\Delta(2,3)}).$$

Let $R=R_1\times R_2$ and $J=J_1\times J_2$ and $J_{\mathrm{red}}=J_1\circledast J_2$ and $\varrho=\varrho_1\times\varrho_2$. Since $C^\perp\subseteq\mathfrak{d}_{\Delta(2,3)}$, part (a) of Theorem 2.4.2 gives a commutative diagram

$$\begin{array}{ccccc}
\mathbb{T}X_1\times\mathbb{T}X_2 & \overset{R}{\dashrightarrow} & D^2\times(\mathfrak{d}\oplus\bar{\mathfrak{d}}\oplus\mathfrak{d}\oplus\bar{\mathfrak{d}}) & & \\
\Big\downarrow q & & \Big\downarrow q' & \searrow\ \mathrm{Mult} & \\
\mathbb{T}(X_1\circledast X_2) & \overset{R_{\mathrm{red}}}{\dashrightarrow} & (D\times(\mathfrak{d}\oplus\bar{\mathfrak{d}}))_{\mathrm{red}} & \overset{\mathrm{Mult}_{\mathrm{red}}}{\dashrightarrow} & D\times(\mathfrak{d}\oplus\bar{\mathfrak{d}})
\end{array}$$

where q and q' (and Mult) are the reduction morphisms and $\mathrm{Mult}_{\mathrm{red}}$ descends from the identity automorphism of $D^2\times(\mathfrak{d}\oplus\bar{\mathfrak{d}}\oplus\mathfrak{d}\oplus\bar{\mathfrak{d}})$. According to Proposition 2.4.1, the Dirac structure $E^{(\mathfrak{l}_1\oplus\mathfrak{l}_2)}\subseteq D^2\times(\mathfrak{d}\oplus\bar{\mathfrak{d}}\oplus\mathfrak{d}\oplus\bar{\mathfrak{d}})$ descends to a Dirac structure $E^{(\mathfrak{l}_1\oplus\mathfrak{l}_2)}_{\mathrm{red}}$ of $(D\times(\mathfrak{d}\oplus\bar{\mathfrak{d}}))_{\mathrm{red}}$. Moreover, (b) of Theorem 2.4.2 implies that R_{red} is a Dirac morphism

$$R_{\mathrm{red}}:(\mathbb{T}(X_1\circledast X_2),T(X_1\circledast X_2))\dashrightarrow((D\times(\mathfrak{d}\oplus\bar{\mathfrak{d}}))_{\mathrm{red}},E^{(\mathfrak{l}_1\times\mathfrak{l}_2)}_{\mathrm{red}}).$$

Denote by ϱ_{red} the bundle map $J^*_{\mathrm{red}}E^{(\mathfrak{l}_1\times\mathfrak{l}_2)}_{\mathrm{red}}\to T(X_1\circledast X_2)$ that R_{red} induces.

Recall from the previous section that the backward image of $E^{(\mathfrak{k}_i)}$ by R_i is the graph of a bivector field $-\pi^{\mathfrak{k}_i}$. Since $\mathfrak{k}_i$ is Ad_{L_i}-invariant, the bivector field $-\pi^{\mathfrak{k}_i}$ is L_i-invariant. In particular the bivector field $-\pi^{\mathfrak{k}_1\oplus\mathfrak{k}_2}=(-\pi^{\mathfrak{k}_1})\times(-\pi^{\mathfrak{k}_2})$ is $L_1\times L_2$-invariant and its forward image under the reduction morphism q is a Lagrangian subbundle of $\mathbb{T}(X_1\circledast X_2)$. On the other hand, the Lagrangian subbundle $E^{(\mathfrak{k}_1\oplus\mathfrak{k}_2)}$ descends to a Lagrangian subbundle $E^{(\mathfrak{k}_1\oplus\mathfrak{k}_2)}_{\mathrm{red}}$.

Lemma 4.1.1. *The forward image of the graph of $-\pi^{\mathfrak{k}_1\oplus\mathfrak{k}_2}$ under q coincides backward image $R_{\mathrm{red}}\circ E^{(\mathfrak{k}_1\oplus\mathfrak{k}_2)}_{\mathrm{red}}$. In particular it is the graph of a bivector field $-\pi^{\mathfrak{k}_1\oplus\mathfrak{k}_2}_{\mathrm{red}}\in\mathfrak{X}^2(X_1\circledast X_2)$.*

Proof. Given an element $x\in q\circ(\mathrm{Gr}(-\pi^{\mathfrak{k}_1\oplus\mathfrak{k}_2}))$, choose $x'\in\mathrm{Gr}(-\pi^{\mathfrak{k}_1\oplus\mathfrak{k}_2})\cap\mathrm{ran}(\varrho)^\perp$ such that $x'\sim_q x$. Then, as $\mathrm{Gr}(-\pi^{\mathfrak{k}_1\oplus\mathfrak{k}_2})$ is the backward image of $E^{(\mathfrak{k}_1\oplus\mathfrak{k}_2)}$ by R, there exists an element $y'\in E^{(\mathfrak{k}_1\oplus\mathfrak{k}_2)}$ such that $x'\sim_R y'$. Dualizing the containment $J^*E^{(C^\perp)}\subseteq R\circ\mathrm{ran}(\varrho)$ gives

$$J^*(R\circ\mathrm{ran}(\varrho)^\perp)\subseteq E^{(C)},$$

and thus $y'\in E^{(C)}$. So there is an element $y\in E^{(\mathfrak{k}_1\oplus\mathfrak{k}_2)}_{\mathrm{red}}$ such that $y'\sim_{q'}y$. This implies that $x\sim_{R_{\mathrm{red}}}y$ and consequently

$$q\circ(\mathrm{Gr}(-\pi^{\mathfrak{k}_1\oplus\mathfrak{k}_2}))\subseteq E^{(\mathfrak{k}_1\oplus\mathfrak{k}_2)}_{\mathrm{red}}\circ R_{\mathrm{red}}.$$

As both sides are Lagrangian, equality ensues. The second part of the claim follows from the fact that $E^{(\mathfrak{k}_1\oplus\mathfrak{k}_2)}_{\mathrm{red}} \circ R_{\mathrm{red}}$, being a Lagrangian complement of $T(X_1 \circledast X_2)$ according to Proposition 2.1.2, is the graph of a bivector field. □

Finally, consider the backward image $E^{(\mathfrak{k})} \circ \mathrm{Mult}_{\mathrm{red}}$ of $E^{(\mathfrak{k})}$ by $\mathrm{Mult}_{\mathrm{red}}$. As both it and $E^{(\mathfrak{k}_1\oplus\mathfrak{l}_2)}_{\mathrm{red}}$ are Lagrangian complements of $E^{(\mathfrak{l}_1\oplus\mathfrak{l}_2)}_{\mathrm{red}}$ one can define a twist $t \in \bigwedge^2 E^{(\mathfrak{l}_1\oplus\mathfrak{l}_2)}_{\mathrm{red}}$ by putting

$$t^\sharp : E^{(\mathfrak{k}_1\oplus\mathfrak{k}_2)}_{\mathrm{red}} \simeq (E^{(\mathfrak{l}_1\oplus\mathfrak{l}_2)}_{\mathrm{red}})^* \to E^{(\mathfrak{l}_1\oplus\mathfrak{l}_2)}_{\mathrm{red}}, \quad x \mapsto \mathrm{pr}_{E^{(\mathfrak{k})}\circ \mathrm{Mult}_{\mathrm{red}}} x - x.$$

Theorem 4.1.1. *The bivector field*

$$\pi^{\mathfrak{k}_1\oplus\mathfrak{k}_2}_{\mathrm{red}} - \varrho_{\mathrm{red}}(t) \tag{4.4}$$

is a $\mathfrak{l}_1 \circ \mathfrak{l}_2$-Hamiltonian bivector field for $X_1 \circledast X_2$.

Proof. The graph of $-\pi^{\mathfrak{k}_1\oplus\mathfrak{k}_2}_{\mathrm{red}}$ is the backward image of $E^{(\mathfrak{k}_1\oplus\mathfrak{k}_2)}_{\mathrm{red}}$ as seen in the previous lemma. The claim then follows immediately from Proposition 2.1.3. □

4.2 D/G-valued moment maps

4.2.1 Background

Let $(\mathfrak{d}, \mathfrak{g})$ be a Manin pair and suppose D is a Lie group integrating $\mathfrak{d}$ such that its connected subgroup G integrating $\mathfrak{g}$ is closed. The pair (D, G) is called a *group pair* for the Manin pair $(\mathfrak{d}, \mathfrak{g})$. There is a natural action of the group D on the homogeneous space D/G

$$d.d'G = dd'G. \tag{4.5}$$

The restriction of this action to the acting group G is called the *dressing action*. The stabilizer of the action (4.5) at a coset $dG \in D/G$ is $\mathrm{Ad}_d\, G$ and so the stabilizer algebras of the corresponding $\mathfrak{d}$-action are all Lagrangian. In particular it defines a Courant algebroid $D/G \times \mathfrak{d}$. As a consequence of Proposition 4.1.1, a Hamiltonian space for $(D/G \times \mathfrak{d}, D/G \times \mathfrak{g})$ is equivalently a quasi-Poisson $\mathfrak{g}$-space (G-space if the induced $\mathfrak{g}$-action integrates) with a moment map valued in D/G as defined by Alekseev and Kosmann-Schwarzbach [2].

The following observation is central to this chapter.

Theorem 4.2.1. *There is a one-to-one correspondence between non-degenerate quasi-Poisson G-spaces with moment maps valued in D/G and L-Hamiltonian spaces where $L = G \times G$.*

Proof. One may regard $D/G \times \mathfrak{d}$ as the Courant algebroid reduced from $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$ with $C^\perp = E^{(0\oplus\mathfrak{g})}$. Since the corresponding G-action $g.d = dg^{-1}$ is free and proper, Proposition

2.4.2 ensures that $D/G \times \mathfrak{d}$ is exact. According to Proposition 4.1.1 (in this special case due Bursztyn-Crainic [14] building on work of Bursztyn-Ševera-Iglesias Ponte [16]), a non-degenerate quasi-Poisson G-space X is equivalently an exact Hamiltonian space

$$R : (\mathbb{T}X, TX) \dashrightarrow (D/G \times \mathfrak{d}, D/G \times \mathfrak{g}),$$

with base map $J : X \to D/G$, such that the induced $\mathfrak{g}$-action $\varrho : \mathfrak{g} \to \mathfrak{X}(X)$ integrates to a G-action. Let $X_{\text{lift}} \hookrightarrow X \times D$ be the pullback of $J : X \to D/G$ and the principal G-bundle $D \to D/G$ so that one has a commutative diagram

$$\begin{array}{ccc} X_{\text{lift}} & \xrightarrow{J_{\text{lift}}} & D \\ \big\downarrow \cdot/G & & \big\downarrow \cdot/G \\ X & \xrightarrow{J} & D/G \end{array},$$

Note that X_{lift} is a $G \times G$-space. Denote the corresponding $\mathfrak{g} \oplus \mathfrak{g}$-action by $\varrho_{\text{lift}} : \mathfrak{g} \oplus \mathfrak{g} \dashrightarrow \mathfrak{X}(\mathfrak{X}_{\text{lift}})$. The projection onto the D-factor is a $G \times G$-equivariant map $J_{\text{lift}} : X_{\text{lift}} \to D$. Since $X = X_{\text{lift}}/(e \times G)$ and J is the map to which J_{lift} descends, part (c) and (d) of Theorem 2.4.2 imply that R lifts to a G-equivariant exact Courant morphism

$$R_{\text{lift}} : \mathbb{T}X_{\text{lift}} \dashrightarrow D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$$

intertwining the generators of the $e \times G$-actions and with base map J_{lift} such that the diagram

$$\begin{array}{ccc} \mathbb{T}X_{\text{lift}} & \overset{R_{\text{lift}}}{\dashrightarrow} & D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}) \\ \Big\downarrow q_1 & & \Big\downarrow q_2 \\ \mathbb{T}X & \overset{R}{\dashrightarrow} & D/G \times \mathfrak{d} \end{array} \tag{4.6}$$

commutes, where q_1 and q_2 are the reduction morphisms. It will be shown that R_{lift} is a Dirac morphism with respect to TX_{lift} and $E^{(\mathfrak{g}\oplus\mathfrak{g})}$. The element $(d, \gamma_1, \gamma_2) \in E^{(\mathfrak{g}\oplus\mathfrak{g})}$ is q_2-related to (dG, γ_1), which in turn is R-related to a unique element of TX. It follows that (d, γ_1, γ_2) is R_{lift}-related to an element of TX_{lift}. This shows existence. Now if $v \in TX_{\text{lift}} \cap \ker(R_{\text{lift}})$ then $[v]$ belongs to $TX \cap \ker(R) = 0$ and so $v = \varrho_{\text{lift}}(0, \gamma)$ for some $\gamma \in \mathfrak{g}$. Since $v \sim_R 0$ and $(J_{\text{lift}})_* v = (0, \gamma)$ by equivariance, it follows that $\gamma = 0$ and thus $v = 0$. This establishes uniqueness. Note that the $\mathfrak{g} \oplus \mathfrak{g}$-action on X_{lift} induced by R_{lift} coincides with ϱ_{lift}.

Conversely, if the $G \times G$-Hamiltonian space X_{lift} was the starting point then coisotropic reduction gives the diagram (4.6). The two constructions being inverse to each other, the claim is proven. □

4.2.2 Fusion product and dual

Suppose $(\mathfrak{d}, \mathfrak{g})$ is a fixed Manin pair admitting the group pair (D, G). Since $\mathfrak{g} \oplus \mathfrak{g} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ is multiplicative and invertible, Theorem 4.2.1 gives a way to bring over the notions of fusion product and duality developed in the context of L-Hamiltonian spaces in Chapter 3 as well as their attendant results to the category of non-degenerate quasi-Poisson G-spaces with moment maps valued in D/G. The notation introduced for the category of L-Hamiltonian spaces is retained.

The picture for the fusion product in this category is as follows. Given two non-degenerate quasi-Poisson G-spaces

$$R_i : (\mathbb{T}X_i, TX_i) \dashrightarrow (D/G \times \mathfrak{d}, D/G \times \mathfrak{g}) \ (i = 1, 2)$$

with moment map $J_i : X_i \to D/G$, form the product

$$J_1 \times J_2 : X_{1,\text{lift}} \times X_{2,\text{lift}} \to D^2 \times (\mathfrak{d} \oplus \overline{\mathfrak{d}} \oplus \mathfrak{d} \oplus \overline{\mathfrak{d}}).$$

As a manifold, the fusion product $X_1 \circledast X_2$ is equal to the quotient

$$\frac{X_{1,\text{lift}} \times X_2}{e \times G_\Delta}$$

and one has a commutative diagram

$$\begin{array}{ccc}
(\mathbb{T}(X_{1,\text{lift}} \times X_{2,\text{lift}}), T(X_{1,\text{lift}} \times X_{2,\text{lift}})) & \xrightarrow{R_{1,\text{lift}} \times R_{2,\text{lift}}} & (D^2 \times (\mathfrak{d} \oplus \overline{\mathfrak{d}} \oplus \mathfrak{d} \oplus \overline{\mathfrak{d}}), E^{(\mathfrak{g}\oplus\mathfrak{g})}) \\
\downarrow & & \downarrow \\
(\mathbb{T}(X_1 \circledast X_2), T(X_1 \circledast X_2)) & \xrightarrow{R_1 \circledast R_2} & (D/G \times \mathfrak{d}, E^{(\mathfrak{g})})
\end{array}$$

where the vertical arrows are the reduction morphisms corresponding to the appropriate coisotropic reductions.

On the other hand, the dual X^* of a Hamiltonian space X in this category is obtained by quotient the lift X_{lift} of X by the left G-action as opposed to the right G-action. In other words, the spaces X^*_{lift} and X_{lift} are equal as manifolds and the $G \times G$-action of the former is obtained from the $G \times G$-action of the latter by trading the two G-factors.

Example 4.2.1. To illustrate the novelty of the notion of fusion product just given, an example of a target homogeneous space D/G admitting no Lie group structure is given. Let $\mathfrak{d} = \mathfrak{sl}_2(\mathbb{C})$

and $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ with $D = \mathrm{SL}(2,\mathbb{C})$ and $G = \mathrm{SL}(2,\mathbb{R})$. One can identify the homogeneous space $D/G = \mathrm{SL}(2,\mathbb{C})/\mathrm{SL}(2,\mathbb{R})$ with the 3-dimensional de Sitter space dS_3, i.e. the hyperboloid of one sheet

$$\mathrm{dS}_3 = \{(t, x_1, x_2, x_3) \in \mathbb{R}^{1,3} \ : \ t^2 - x_1^2 - x_2^2 - x_3^2 = -1\},$$

where $\mathbb{R}^{1,3}$ is Minkowski spacetime. One may identify $\mathbb{R}^{1,3}$ with the space of 2×2 complex Hermitian matrices with the assignment

$$(t, x_1, x_2, x_3) \mapsto \begin{pmatrix} t + x_3 & x_1 - \sqrt{-1}x_2 \\ x_1 + \sqrt{-1}x_2 & t - x_3 \end{pmatrix},$$

whereby the metric of $\mathbb{R}^{1,3}$ becomes the determinant. Then dS_3 is identified with the set of Hermitian matrices of determinant -1 and $\mathrm{SL}(2,\mathbb{C})$ acts transitively on dS_3 via $C.H = CHC^*$ where $C \in \mathrm{SL}(2,\mathbb{C})$ and H is a Hermitian matrix. The stabilizer at $\left(\begin{smallmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{smallmatrix}\right)$ is $\mathrm{SL}(2,\mathbb{R})$ and therefore $\mathrm{SL}(2,\mathbb{C})/\mathrm{SL}(2,\mathbb{R}) = \mathrm{dS}_3$. Now since dS_3 is topologically the product $\mathbb{R} \times S^2$ of the real line and the 2-sphere, it does not admit a Lie group structure. Indeed, the second homotopy group of a Lie group must vanish [11] whereas $\pi_2(\mathrm{dS}_3) = \mathbb{Z}$.

Non-isomorphic dual and non-commutativity. In the context of Example 4.2.1, let $\mathcal{O} \hookrightarrow D/G$ through $dG \in D/G$, where $d = \left(\begin{smallmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{smallmatrix}\right)$. Then its dual $\mathcal{O}^* = \mathcal{O}^*_{\mathrm{lift}}/(e \times G)$ is the G-orbit of $d^{-1}G$ in D/G. Consider now the matrices

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Note that $dbd^{-1} = a$, from which it follows that $a \in \mathrm{Stab}(xG)$ and $b \in \mathrm{Stab}(d^{-1}G)$. The matrices a and b are not conjugate in $\mathrm{SL}(2,\mathbb{R})$: any matrix conjugate to a in $\mathrm{SL}(2,R)$ has a non-negative upper-right entry, which is seen by a simple computation. So $\mathrm{Stab}(dG)$ and $\mathrm{Stab}(d^{-1}G)$ are not conjugate subgroups of G and consequently $\mathcal{O}$ and $\mathcal{O}^*$ are of different orbit types [26, Lemma 2.6.2], i.e. they are not isomorphic as G-spaces.

Now by definition

$$\mathcal{O} \circledast \mathcal{O}^* = \mathcal{O}_{\mathrm{lift}} \times_G \mathcal{O}^*, \ \mathcal{O}^* \circledast \mathcal{O} = \mathcal{O}^*_{\mathrm{lift}} \times_G \mathcal{O}.$$

Consider the element $[(d, d^{-1}G)] \in \mathcal{O} \circledast \mathcal{O}^*$. It is clear that $a \in \mathrm{Stab}([(d, d^{-1}G)])$ since

$$a.[(d, d^{-1}G)] = [(ad, d^{-1}G)] = [(db, d^{-1}G)] = [(d, b^{-1}d^{-1}G)] = [(d, d^{-1}G)].$$

Now consider the element $[(d^{-1}, dG)] \in \mathcal{O}^* \circledast \mathcal{O}$. As similar argument shows that $b \in \mathrm{Stab}([(d^{-1}, dG)])$. One thus concludes as before that $\mathcal{O} \circledast \mathcal{O}^*$ and $\mathcal{O}^* \circledast \mathcal{O}$ are not isomorphic as G-spaces. **Therefore the fusion product in the category of non-degenerate quasi-Poisson G-spaces with moment maps in D/G is in general non-commutative and duals are generally not isomorphic as G-spaces.**

4.3 Comparison with Poisson $\mathfrak{g}$-spaces

4.3.1 Poisson $\mathfrak{g}$-spaces

Suppose (D, G) is a Lie group pair for $(\mathfrak{d}, \mathfrak{g})$ and that $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Manin triple. Let $H \subseteq D$ be the connected subgroup integrating $\mathfrak{h}$. In J.-H. Lu's moment map theory [44], moment maps take value in H (which may be substituted by its universal cover with minor adjustments), which she denotes by G^*. Lu views G^* as the Lie group dual to G; the Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ defines a multiplicative Poisson structure on G [25] and the linearization of such a Poisson structure at the group identity gives a Lie bracket on $\mathfrak{g}^*$ which coincides with the Lie algebra structure of $\mathfrak{h}$ from the point view just adopted. Consider the maps $G \times H \to D$ and $H \times G \to D$ given by multiplication. Since they are local diffeomorphisms, one can find neighbourhood of the group identities $U_G \subseteq G$ and $U_H \subseteq H$ on whose products $U_G \times U_H$ and $U_H \times U_G$ the multiplication map is diffeomorphic with $U_G U_H = U_H U_G$. For this reason, the decomposition $D = GH = HG$ will be assumed to be global in the sequel with the understanding that the discussion can be adapted to the general case by considering suitable neighbourhoods of the group identities in the groups D, G and H.

Now one can define an action of D on H extending multiplication on the left in H by putting, for $d \in D$ and $h \in H$,

$$dh = (d.h)g, \tag{4.7}$$

where $d.h \in U_H$ and $g \in U_G$ are the unique elements such that (4.7) holds. This action coincides with the action (4.5), and likewise its restriction to the acting group G is called the *left dressing action* on H. The induced $\mathfrak{d}$-action is

$$-L_h \operatorname{pr}_{\mathfrak{h}} \operatorname{Ad}_{h^{-1}} \xi \tag{4.8}$$

for $\xi \in \mathfrak{d}$. Likewise, one can define a right D-action on H extending right multiplication in H

by putting

$$hd = g(d.h) \tag{4.9}$$

where $g \in G$ and $d.h \in H$ are the unique elements such that this equality holds. Naturally, its restriction to the acting group G is called the *right dressing action* on H.

Similarly, there are D-actions on G with their restrictions to the acting group H called the left and right dressing actions on G.

Definition 4.3.1 ([44])**.** A Poisson $\mathfrak{g}$-space (G-space) is a $\mathfrak{g}$- (resp. G)-manifold P together with a 2-form $\omega \in \Omega^2(P)$ and a $\mathfrak{g}$-equivariant map $J : P \to H$ such that

(a) ω is closed,
(b) ω is non-degenerate,
(c) $\iota_{\varrho(\gamma)}\omega = -J^*\langle\theta^R, \gamma\rangle$, where $\varrho : \mathfrak{g} \to P$ is the $\mathfrak{g}$-action.

Remark 4.3.1. As a consequence of (4.1) and this last proposition, a Poisson $\mathfrak{g}$-space carries a Poisson structure, hence the terminology.

The kernel at $h \in H$ of the Courant algebroid $H \times \mathfrak{d} = D/G \times \mathfrak{d}$ is $\mathrm{Ad}_h\, \mathfrak{g}$ and there is thus a natural isotropic splitting

$$j_H(v) = -\iota_v\theta^R. \tag{4.10}$$

Since $\mathrm{ran}(j_H) = E^{(\mathfrak{h})}$ is a Dirac structure of $H \times \mathfrak{d}$, the corresponding 3-form is trivial and so $H \times \mathfrak{d} \simeq \mathbb{T}H$.

Proposition 4.3.1. *A Poisson $\mathfrak{g}$-space is equivalently an exact Hamiltonian space for* $(H \times \mathfrak{d}, E^{(\mathfrak{g})})$.

Proof. Identifying $H \times \mathfrak{d} \simeq \mathbb{T}H$ via the splitting j_H, (a) and (c) follow directly from Proposition 3.1.1. Since (b) here is stronger than its counterpart in Proposition 3.1.1, it must only be only that if $\mathbb{T}J_\omega : (\mathbb{T}P, TP) \dashrightarrow (H \times \mathfrak{d}, E^{(\mathfrak{g})})$ is an exact Hamiltonian space then ω is non-degenerate. Recall that $\mathrm{Gr}(-\omega) = \mathrm{ran}(j_H) \circ \mathbb{T}J_\omega$. However, since $\mathrm{ran}(j_H) = E^{(\mathfrak{h})}$ is transversal to $E^{(\mathfrak{g})}$, its backward image $\mathrm{ran}(j_H) \circ \mathbb{T}J_\omega$ is also the graph of a bivector field on P. It follows that ω is non-degenerate. □

Example 4.3.1 (Classical moment map theory). Given an arbitrary Lie algebra $\mathfrak{g}$ integrating to a connected Lie group G, let $\mathfrak{d} = \mathfrak{h} \rtimes \mathfrak{g}$ be the semi-direct product of the Lie algebra $\mathfrak{g}$ and the abelian Lie algebra $\mathfrak{h} = \mathfrak{g}^*$ where the underlying map $\mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ is $\gamma \mapsto \mathrm{ad}^*_\gamma$. As a metrized vector space, the semi-direct product $\mathfrak{h} \rtimes \mathfrak{g}$ is equal to $\mathfrak{g}^* \oplus \mathfrak{g}$. As subalgebras of $\mathfrak{d}$, the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are Lagrangian and $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Manin triple. The Lie group $D = \mathfrak{h} \rtimes G$ integrates $\mathfrak{d}$ and its connected subgroups integrating $\mathfrak{g}$ and $\mathfrak{h}$ are respectively G and $H = \mathfrak{h}$. The left dressing action on H is simply the coadjoint representation. As $\mathfrak{h}$ is G-invariant in this case, the above proposition then specializes as follows: an exact Hamiltonian space for $(\mathfrak{h} \times \mathfrak{d}, \mathfrak{h} \times \mathfrak{g})$ is equivalently a $\mathfrak{g}$-manifold P together with a $\mathfrak{g}$-equivariant map $J : P \to \mathfrak{g}^*$ and a 2-form ω such that

(a) ω is closed,

(b) ω is non-degenerate,

(c) $\iota_{\varrho(\gamma)}\omega = -d\langle \gamma, J\rangle$ for all $\gamma \in \mathfrak{g}$, where $\varrho : \mathfrak{g} \to P$ is the $\mathfrak{g}$-action.

This is the classical definition of a moment map valued in the dual of a Lie algebra.

Let $j'_D : TD \to D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$ be the isotropic splitting induced by j_H as in Proposition 2.3.2, in other the isotropic splitting with image $E^{(\mathfrak{h}\oplus\mathfrak{g})}$. It identifies $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}) \simeq \mathbb{T}D$ and is given explicitly by

$$j'_D(v) = (-\operatorname{Ad}_h \operatorname{pr}_{\mathfrak{h}} \theta^{HG}, \operatorname{Ad}_{g^{-1}} \operatorname{pr}_{\mathfrak{g}} \theta^{HG}), \tag{4.11}$$

where θ^{HG} is defined by $\iota_v\theta^{HG} = L_{h^{-1}} R_{g^{-1}} v$ for $v \in T_{hg}D$ where $h \in H$ and $g \in G$. The 2-form corresponding to the change of isotropic splitting $\mathbb{T}D \simeq \mathbb{T}D_{\eta_D}$ is (according to (2.10))

$$\varpi = \frac{1}{2}\langle\theta^{HG}, \operatorname{pr}_{\mathfrak{g}} \theta^{HG}\rangle. \tag{4.12}$$

and the 2-form corresponding to the morphism $\operatorname{Mult}_D : D^2 \times (\mathfrak{d} \oplus \overline{\mathfrak{d}} \oplus \mathfrak{d} \oplus \overline{\mathfrak{d}}) \dashrightarrow D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})$ is

$$\varsigma' = \varsigma + \varpi^1 + \varpi^2 - \operatorname{mult}^*\varpi. \tag{4.13}$$

The group G (or H) is itself a $G \times G$-Hamiltonian space since it is the orbit of $E^{(\mathfrak{g}\oplus\mathfrak{g})}$ through the identity. In terms of the identification $D \times \mathfrak{d} \oplus \overline{\mathfrak{d}} \simeq \mathbb{T}D$, the corresponding Courant morphism is $\mathbb{T}\iota_G$ where $\iota_G : G \hookrightarrow D$ is the inclusion, according to Corollary 3.2.1. Consequently, the group G carries a natural non-trivial Poisson structure $\pi_G^{\mathfrak{h}\oplus\mathfrak{h}} \in \mathfrak{X}^2(G)$ corresponding to the Lagrangian complement $\mathfrak{h} \oplus \mathfrak{h}$ of $\mathfrak{g} \oplus \mathfrak{g}$ in $\mathfrak{d} \oplus \overline{\mathfrak{d}}$. Now

$$v + \mu \sim_{\mathbb{T}\iota_G} (\xi_1, \xi_2) \iff \xi_2 = \operatorname{pr}_{\mathfrak{h}}(\operatorname{Ad}_{g^{-1}} \xi_1),\ v = -L_g \operatorname{pr}_{\mathfrak{g}} \operatorname{Ad}_{g^{-1}} \xi_1,\ \mu = \langle \xi_1, \theta^L\rangle$$

for $v \in T_g G$ and $\mu \in T_g^* G$. Since $\mathrm{Gr}(-\pi_G^{\mathfrak{h}\oplus\mathfrak{h}})$ is the backward image of $E^{(\mathfrak{h}\oplus\mathfrak{h})}$ under $\mathbb{T}\iota_G$, the above expression shows that the symplectic leaves of $\pi_G^{\mathfrak{h}\oplus\mathfrak{h}}$ are precisely the orbits of the dressing actions on G.

4.3.2 Fusion product

Recall Example 2.3.4: The generalized tangent bundle $\mathbb{T}H$ carries a natural Lie groupoid structure $\mathbb{T}H \rightrightarrows \mathfrak{h}^* \simeq \mathfrak{g}$ where the source and target map are respectively the left and right trivializations and multiplication is the exact morphism $\mathbb{T}\mathrm{mult}_H : \mathbb{T}H \times \mathbb{T}H \dashrightarrow \mathbb{T}H$. Through the identification $H \times \mathfrak{d} \simeq \mathbb{T}H$, the source and target maps become

$$\mathbf{s}(h,\xi) = \mathrm{pr}_{\mathfrak{g}} \mathrm{Ad}_{h^{-1}} \xi, \;\; \mathbf{t}(h,\xi) = \mathrm{pr}_{\mathfrak{g}} \xi, \tag{4.14}$$

and multiplication becomes

$$(h_1,\xi_1)\circ(h_2,\xi_2) = (h,\xi) \iff h = h_1h_2,\; -\iota_{\mathbf{a}_H(\xi_1)}\theta^R + \mathrm{Ad}_{h_1}\xi_2 = \xi \tag{4.15}$$

for composable pairs (h_1,ξ_1) and (h_2,ξ_2).

Proposition 4.3.2. *The morphism* Mult_H *is a Dirac morphism*

$$\mathrm{Mult}_H : (H\times\mathfrak{d}\times H\times\mathfrak{d}, E^{(\mathfrak{g}\oplus\mathfrak{g})}) \dashrightarrow (H\times\mathfrak{d}, E^{(\mathfrak{g})}).$$

Proof. Suppose $h_1, h_2 \in H$ and say $\gamma \in \mathfrak{g}$. Writing $h = h_1h_2$, one wishes to solve

$$(h_1,\gamma_1)\circ(h_2,\gamma_2) = (h,\gamma).$$

By (4.14), one is forced to take $\gamma_1 = \gamma$ and $\gamma_2 = \mathrm{pr}_{\mathfrak{g}}\mathrm{Ad}_{h_1^{-1}}\gamma$. Then

$$-\iota_{\mathbf{a}_H(\gamma_1)}\theta^R + \mathrm{Ad}_{h_1}\gamma_2 = \mathrm{Ad}_{h_1}\mathrm{pr}_{\mathfrak{h}}\mathrm{Ad}_{h_1^{-1}}\gamma + \mathrm{Ad}_{h_1}\mathrm{pr}_{\mathfrak{g}}\mathrm{Ad}_{h_1^{-1}}\gamma = \gamma,$$

i.e. this choice of γ_1 and γ_2 satisfies (4.15). This proves the claim. □

Remark 4.3.2. Note that

$$g.(h_1h_2) = (g.h_1)((h_1.g).h_2)$$

where H acts on G by the right dressing action. Putting $g = \exp(-t\gamma)$ for $\gamma \in G$ and differentiating $(g.h_1, (h_1.g).h_2)$ at $t = 0$, one obtains $(\mathbf{a}_H(\gamma)|_{h_1}, \mathbf{a}_H(\mathrm{pr}_{\mathfrak{g}}\mathrm{Ad}_{h_1^{-1}}\gamma)|_{h_2})$ in accordance with the proof above.

The upshot of Proposition 4.3.2 is that a fusion product in the category of Poisson $\mathfrak{g}$-spaces can be defined: Given Poisson $\mathfrak{g}$-spaces

$$R_i : \mathbb{T}P_1 \dashrightarrow H \times \mathfrak{d} \; (i = 1, 2),$$

their fusion product is

$$P_1 \circledast P_2 = P_1 \times P_2$$

with corresponding morphism

$$\mathrm{Mult}_H \circ (R_1 \times R_2).$$

In terms of Definition 4.3.1, the corresponding 2-form and moment map are, respectively, the sum and product of those of P_1 and P_2, and the $\mathfrak{g}$-action on $P_1 \circledast P_2$ is given by

$$\varrho(\gamma)|_{(p_1,p_2)} = (\varrho_1(\gamma)|_{p_1}, \varrho_2(\mathrm{pr}_{\mathfrak{g}} \mathrm{Ad}_{J_1(p_1)^{-1}} \gamma)|_{p_2}), \tag{4.16}$$

where $\varrho_i : \mathfrak{g} \to \mathfrak{X}(P_i)$ $(i = 1, 2)$ is the $\mathfrak{g}$-action on P_i and $J_i : P_i \to H$ the moment map. This definition of the fusion product of Poisson $\mathfrak{g}$-spaces agrees with that derived by H. Flaschka and T. Ratiu [27, Lem. 2.19]. Following Remark 4.3.2, if the $\mathfrak{g}$-actions on P_1 and P_2 integrate to G-actions, then the $\mathfrak{g}$-action on $P_1 \circledast P_2$ integrates to the G-action

$$g.(p_1, p_2) = (g.p_1, (J(p_1).g).p_2). \tag{4.17}$$

Now if P_i $(i = 1, 2)$ are Poisson G-spaces with moment maps $J_i : P \to H$ then they are exact Hamiltonian spaces for

$$(H \times \mathfrak{d}, E^{(\mathfrak{g})}) = (D/G \times \mathfrak{d}, E^{(\mathfrak{g})}).$$

There are therefore a priori two ways of taking the fusion product $P_1 \circledast P_2$: one in the category of Poisson $\mathfrak{g}$-spaces and one in the category of non-degenerate quasi-Poisson G-spaces. The next results show that these two fusion products agree. In the following, the subscript H will be appended to the fusion product symbol $\circledast$ to indicate it is interpreted in the category of Poisson $\mathfrak{g}$-space. Otherwise it is interpreted in the category of quasi-Poisson G-spaces.

The group G is the orbit $\iota_G : G \hookrightarrow D$ through the group identity $e \in D$ of the Dirac structure $E^{(\mathfrak{g}\oplus\mathfrak{g})}$ and is therefore a $G \times G$-Hamiltonian space. According to Corollary 3.2.1, its 2-form with respect to the identification $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}) \simeq TD_{\eta_D}$ is trivial.

Lemma 4.3.1. *Let X be a $G \times G$-Hamiltonian space. Then $G \circledast X = X$.*

Proof. Let J be the moment map of X and ω its 2-form with respect to the identification

$D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}) \simeq \mathbb{T}D$. Its 2-form with respect to the identification $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}) \simeq \mathbb{T}D_{\eta_D}$ on the other hand is $\omega + J^*\varpi$ and, as the latter is $G \times G$-invariant and $J \circ (g, e) = L_g \circ J$ (the map (g, e) here stands for the action by $(g, e) \in G \times G$ on X), one has

$$(g, e)^*\omega = \omega + J^*\varpi - J^*L_g^*\varpi.$$

Furthermore, by Proposition 3.2.1, one has

$$\iota_{\varrho(\gamma,0)}\omega = -\frac{1}{2}J^*\langle \xi_1, \theta^R\rangle + J^*\iota_{\gamma^R}\varpi.$$

As manifolds, one has $G \circledast X = X$ with the identification descending from the action map

$$\mathcal{A} : G \times X \to X$$

of the $G \times e$-action on X. As such, the $G \times G$-action on $G \circledast X$ coincides with that on X and the moment maps of those spaces are equal. Now, the 2-form of $G \circledast X$ with respect to $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}) \simeq \mathbb{T}D$ is

$$\omega + (\iota_G^* \times J^*)(\zeta') = \omega + (\iota_G^* \times J^*)(\zeta + \varpi^1 + \varpi^2 - \mathrm{mult}^*\varpi),$$

which must be shown to be equal to $\mathcal{A}^*\omega$ to complete the proof. Note that, since $\iota_G^*\varpi = 0$, the term ϖ^1 may be omitted in the above expression. For $\gamma_i \in \mathfrak{g}$ and $v_i \in \mathfrak{X}(X)$, one has

$$\begin{aligned}
\mathcal{A}^*\omega(\iota_G^*\gamma_1^R + v_1, \iota_G^*\gamma_2^R + v_2) &= \omega((g,e)_*v_1 - \varrho(\gamma_1, 0), (g,e)_*v_2 - \varrho(\gamma_2, 0)) \\
&= (g,e)^*\omega(v_1, v_2) - \omega(\varrho(\gamma_1, 0), (g,e)_*v_2) + \\
&\quad\ \omega(\varrho(\gamma_2, 0), (g,e)_*v_1) + \omega(\varrho(\gamma_1, 0), \varrho(\gamma_2, 0)) \\
&= (\omega + J^*\varpi)(v_1, v_2) + \frac{1}{2}\langle \gamma_1, \theta^R L_g J_* v_2\rangle - \\
&\quad\ \frac{1}{2}\langle \gamma_2, \theta^R L_g J_* v_1\rangle - J^*L_g\varpi(v_1, v_2) - \varpi(\gamma_1^R, L_g J_* v_2) + \\
&\quad\ \varpi(\gamma_2^R, L_g J_* v_1) - \varpi(\gamma_1^R, \gamma_2^R) \\
&= \omega(\iota_G^*\gamma_1^R + v_1, \iota_G^*\gamma_2^R + v_2) + \\
&\quad\ (\mathrm{Id} \times J)^*(\zeta + \varpi^2 - \mathrm{mult}^*\varpi)(\iota_G^*\gamma_1^R + v_1, \iota_G^*\gamma_2^R + v_2),
\end{aligned}$$

as claimed. □

Theorem 4.3.1. *Let $R_i : \mathbb{T}P_i \dashrightarrow H \times \mathfrak{d}$ $(i = 1, 2)$ be Poisson $\mathfrak{g}$-spaces with moment maps J_i. Suppose the actions ϱ_i integrate to G-actions. Then the fusion product $P_1 \circledast P_2$ interpreted in* *the category of Poisson $\mathfrak{g}$-spaces agrees with its interpretation in the category of quasi-Poisson G-spaces with moment maps valued in D/G.*

Proof. In the notation of Theorem 4.2.1,

$$P_{i,\text{lift}} = \{(p, J_i(p)g) \ : \ p \in P_i,\ g \in G\} = P_i \times G$$

and $R_{i,\text{lift}} = \mathbb{T}\text{mult}_D \circ (R_i \times \text{Id})$ in terms of the identifications $H \times \mathfrak{d} \simeq \mathbb{T}H$ and $D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}) \simeq \mathbb{T}D$. Likewise, $(P_1 \circledast_H P_2)_{\text{lift}} = P_1 \times P_{2,\text{lift}}$ and its corresponding Courant morphism is $\mathbb{T}\text{mult}_D \circ (R_1 \times R_{2,\text{lift}})$. Furthermore, as manifolds,

$$P_{1,\text{lift}} \circledast P_{2,\text{lift}} = P_1 \times (G \circledast P_{2,\text{lift}}) = (P_1 \circledast_H P_2)_{\text{lift}}.$$

Now denote by mult_{HG}, mult_{HD} and mult_{GD} the restrictions of the multiplication map mult_D to $H \times G$, $H \times D$ and $G \times D$ respectively. Consider the diagram

$$\begin{array}{ccccc}
(\mathbb{T}P_1 \times \mathbb{T}G) \times \mathbb{T}P_{2,\text{lift}} & \xrightarrow{R_1 \times \text{Id} \times R_{2,\text{lift}}} & \mathbb{T}H \times \mathbb{T}G \times \mathbb{T}D & \xrightarrow{\mathbb{T}\text{mult}_{HG} \times \text{Id}} & \mathbb{T}(D \times D) \\
\downarrow q & & \downarrow \text{Id} \times \mathbb{T}(\text{mult}_{GD})_{(\iota_G \times \text{Id})^* \varsigma'} & & \downarrow \text{Mult}_D \\
\mathbb{T}(P_1 \circledast_H P_2)_{\text{lift}} & \xrightarrow{R_1 \times R_{2,\text{lift}}} & \mathbb{T}H \times \mathbb{T}D & \xrightarrow{\mathbb{T}\text{mult}_{HD}} & \mathbb{T}D
\end{array}$$

where q is the morphism labelled q_1 in (4.6). This diagram is commutative; commutativity in the left square follows from Lemma 4.3.1 and commutativity in the right square amounts to the equality

$$(\text{mult}_{HG} \times \text{Id})^* \varsigma' = ((\iota_G \times \text{Id})^* \varsigma')^2,$$

which can be verified by direct computation. Composing the top horizontal morphisms gives $R_{1,\text{lift}} \times R_{2,\text{lift}}$, which therefore descends to $\mathbb{T}\text{mult}_D \circ (R_1 \times R_{2,\text{lift}})$ in the diagram (3.14). That is to say $P_{1,\text{lift}} \circledast P_{2,\text{lift}} = (P_1 \circledast_H P_2)_{\text{lift}}$ as $G \times G$-Hamiltonian spaces, which is what needed to be argued. □

4.4 Some examples

Let $(\mathfrak{d}, \mathfrak{g})$ be a Manin pair. In the case where it can be completed to a Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$, the D/G-valued moment map theory might be argued to be redundant since one already has Lu's G^*-valued moment map theory. Likewise, in the case where $\mathfrak{g}$ admits an $\text{ad}_{\mathfrak{g}}$-invariant complement then the theory reduces to the classical theory near the coset of the group identity (Appendix 4.A). Following are examples of Manin pairs $(\mathfrak{d}, \mathfrak{g})$ where no such complement $\mathfrak{h} \subseteq \mathfrak{d}$ may found. In the case of the first example, the subalgebra $\mathfrak{g}$ is an ideal of $\mathfrak{d}$ and therefore D/G is a Lie group for any group pair (D, G) and thus an objection may still be raised. The second example does away with any such objection: the subalgebra $\mathfrak{g}$ given there does not admit a Lagrangian complement that is a subalgebra or $\mathrm{ad}_{\mathfrak{g}}$-invariant and there is a group pair (D, G) for $(\mathfrak{d}, \mathfrak{g})$ for which D/G cannot be made into a Lie group.

Example 4.4.1. Let $\mathfrak{h}$ be the Heisenberg algebra, i.e. the 3-dimensional Lie algebra admitting the presentation

$$\langle A, B, E \mid [A, B] = E \rangle .$$

Let A, B, E be a basis as above and let α, β, ε be the basis dual, with the obvious pairing convention. Let $\theta = \alpha \wedge \beta \wedge \varepsilon$ and define the following bracket on $\mathfrak{h} \oplus \mathfrak{h}^*$:

$$[x + \varphi, y + \psi] = [x, y]_{\mathfrak{h}} + \mathrm{ad}_x \psi - \mathrm{ad}_y \phi + \theta(x, y).$$

This bracket is Lie and moreover the natural metric on $\mathfrak{h} \oplus \mathfrak{h}^*$ is ad-invariant with respect to it; see [9, Ex. 4.2]. Denote this metrized Lie algebra by $T^*_\theta\mathfrak{h}$. The non-zero commutation relations in $T^*_\theta\mathfrak{h}$ with respect to these bases are

$$[A, B] = E + \varepsilon, [A, E] = -\beta, [B, E] = \alpha, [A, \varepsilon] = -\beta, [B, \varepsilon] = \alpha. \tag{4.18}$$

It is readily seen from the above that $T^*_\theta\mathfrak{h}$ is nilpotent with nilindex 3. Clearly $\mathfrak{h}^*$, identified with the second factor, is a Lagrangian subalgebra (in fact an abelian ideal) of $T^*_\theta\mathfrak{h}$.

Now Suppose that there were a 3-dimensional subalgebra $\mathfrak{g} \subseteq T^*_\theta\mathfrak{h}$ such that $T^*_\theta\mathfrak{h} = \mathfrak{h}^* + \mathfrak{g}$. Since the only nilpotent Lie algebras of dimension 3 are the abelian and Heisenberg Lie algebras [21], the subalgebra $\mathfrak{g}$ would be nilpotent of nilindex 1 or 2. Note that $[T^*_\theta\mathfrak{h}, \mathfrak{h}^*]$ is contained in the center of $T^*_\theta\mathfrak{h}$. Therefore given $p, q, r \in T^*_\theta\mathfrak{h}$, one has $[p, [q, r]] = 0$ if any of p, q, or r is in $T^*_\theta\mathfrak{h}$. It follows that

$$[T^*_\theta\mathfrak{h}, [T^*_\theta\mathfrak{h}, T^*_\theta\mathfrak{h}]] \subseteq [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0,$$

meaning $T^*_\theta\mathfrak{h}$ has nilindex 2, a contradiction.

Note that $\mathfrak{h}^*$ does not admit an $\mathrm{ad}_{\mathfrak{h}^*}$-invariant Lagrangian complement since that would contradict the commutations (4.18). **Therefore $(T^*_\theta\mathfrak{h}, \mathfrak{h}^*)$ is an example of a Manin pair whose Lagrangian subalgebra $\mathfrak{h}^*$ admits no Lie algebra complement nor any $\mathrm{ad}_{\mathfrak{h}^*}$-invariant complement.** Since $\mathfrak{h}^*$ is an ideal of $T^*_\theta\mathfrak{h}$, the connected subgroup integrating the former is closed in the simply connected Lie group integrating the latter, **meaning $(T^*_\theta\mathfrak{h}, \mathfrak{h})$ admits a group pair.**

Example 4.4.2. Retaining the notation in the previous example, consider the Manin pair

$$(T^*_\theta\mathfrak{h},\mathfrak{h}^*)\oplus(\mathfrak{sl}_2(\mathbb{C}),\mathfrak{sl}_2(\mathbb{R}))=(T^*_\theta\mathfrak{h}\oplus\mathfrak{sl}_2(\mathbb{C}),\mathfrak{h}^*\oplus\mathfrak{sl}_2(\mathbb{R})).$$

Suppose $\mathfrak{g}\subseteq T^*_\theta\mathfrak{h}\oplus\mathfrak{sl}_2(\mathbb{C})$ were an Lagrangian complement of $\mathfrak{h}\oplus\mathfrak{sl}_2(\mathbb{R})$. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be the images of $\mathfrak{g}$ in the first and second factors respectively. Then $\mathfrak{h}^*+\mathfrak{g}_1=T^*_\theta\mathfrak{h}$ and $\mathfrak{sl}_2(\mathbb{R})+\mathfrak{g}_2=\mathfrak{sl}_2(\mathbb{C})$, hence $\dim(\mathfrak{g}_1)\geq 4$ and $\dim(\mathfrak{g}_2)\geq 3$. Let $\mathfrak{k}_1=(\mathfrak{g}_1\oplus 0)\cap\mathfrak{g}$ and $\mathfrak{k}_2=(0\oplus\mathfrak{g}_2)\cap\mathfrak{g}$. According to Goursat's lemma, the image of $\mathfrak{g}$ in $\mathfrak{g}_1/\mathfrak{k}_1\oplus\mathfrak{g}_2/\mathfrak{k}_2$ is the graph of an isomorphism of vector spaces

$$\mathfrak{g}_1/\mathfrak{k}_1\simeq\mathfrak{g}_2/\mathfrak{k}_2. \tag{4.19}$$

Note that $\dim(\mathfrak{g}_i)\geq 3$ and $\dim(\mathfrak{k}_i)\leq 3$ and

$$\dim(\mathfrak{g}_1)+\dim(\mathfrak{k}_2)=\dim(\mathfrak{g}_2)+\dim(\mathfrak{k}_1)=6. \tag{4.20}$$

Case where $\mathfrak{g}$ is a Lie algebra. In this case $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Lie algebras with $\mathfrak{k}_1$ and $\mathfrak{k}_2$ respectively as ideals. Furthermore (4.19) is an isomorphism of Lie algebras. From the previous example it is known that $\dim(\mathfrak{g}_1)\geq 4$. Suppose first that $\dim(\mathfrak{g}_1)=4$. Arguing similarly as in the previous example, one has $[T^*_\theta\mathfrak{h},[T^*_\theta\mathfrak{h},T^*_\theta\mathfrak{h}]]\subseteq[\mathfrak{g}_1,[\mathfrak{g}_1,\mathfrak{g}_1]]$ and the nilindex of $\mathfrak{g}_1$ must therefore be 3. There is only one 4-dimensional nilpotent Lie algebra with nilindex 3 [21], it admits the presentation

$$\langle X_1,X_2,X_3,X_4\mid[X_1,X_2]=X_3,[X_1,X_3]=X_4\rangle\,.$$

Since $\mathfrak{h}^*$ is an abelian ideal in $T^*_\theta\mathfrak{h}$, $\mathfrak{h}^*\cap\mathfrak{g}_1$ is a 1-dimensional ideal of $T^*_\theta\mathfrak{h}$. It is readily seen that the only 1-dimensional ideals of $T^*_\theta\mathfrak{h}$ contained in $\mathfrak{h}^*$ are spanned by linear combinations of the basis elements α and β. Therefore $\mathfrak{h}^*\cap\mathfrak{g}_1$ is in the center of $T^*_\theta\mathfrak{h}$ and a fortiori in that of $\mathfrak{g}_1$. Taking a basis X_i as in the above presentation, it is clear that the center of $\mathfrak{g}_1$ is the span of X_4. Thus $X_4=a\alpha+b\beta$ for $a,b\in\mathbb{R}$ not both 0. Thus

$$[T^*_\theta\mathfrak{h},T^*_\theta\mathfrak{h}]=\langle\alpha,\beta,E+\varepsilon\rangle$$

but also $\langle\alpha,\beta,X_3\rangle$. So without loss of generality one has $X_3=E+\varepsilon+k\alpha+l\beta$ for some $k,l\in\mathbb{R}$. But then

$$\ker\operatorname{ad}_{X_3}=\langle X,Y,Z,z\rangle=T^*_\theta\mathfrak{h}+\langle X_3\rangle\,,$$

which does not include X_2, a contradiction. One concludes that $\dim(\mathfrak{g}_1)\geq 5$.

Now consider $\mathfrak{g}_2$. There are no 5-dimensional Lie subalgebras of $\mathfrak{sl}_2(\mathbb{C})^2$. Up to conjugation there is only one 4-dimensional Lie subalgebra, the Borel subalgebra, which is in fact a complex subalgebra of complex dimension 2 with real form $\mathcal{B}$, the Borel subalgebra of $\mathfrak{sl}_2(\mathbb{R})$. Suppose then that $\mathfrak{g}_2 = \mathcal{B} + \sqrt{-1}\mathcal{B}$. As it is a centerless complex Lie algebra, it does not admit a real 1-dimensional ideal. Also $\dim(\mathfrak{k}_2) \neq 0$ since $\mathcal{B} + \sqrt{-1}\mathcal{B}$ is not nilpotent. Thus $\dim(\mathfrak{k}_2) \geq 2$ and $\dim(\mathfrak{g}_1) + \dim(\mathfrak{k}_2) \geq 7$, contradicting (4.20). One concludes that $\dim(\mathfrak{g}_2) = 3$.

There are up to conjugation five non-isomorphic 3-dimensional subalgebras of $\mathfrak{sl}_2(\mathbb{C})$, none of which are abelian or nilpotent. So $\dim(\mathfrak{k}_2) \neq 0$, forcing $\dim(\mathfrak{k}_2) = 1$. Since the quotient algebra $\mathfrak{g}_2/\mathfrak{k}_2$ is nilpotent of dimension 2, it is abelian. It follows that the derived algebra $[\mathfrak{g}_2, \mathfrak{g}_2]$ is just $\mathfrak{k}_2$ and in particular 1-dimensional. Now the real forms of $\mathfrak{sl}_2(\mathbb{C})$ are up to conjugation $\mathfrak{su}(2)$ and $\mathfrak{sl}_2(\mathbb{R})$, both of which are semisimple, excluding them as possibilities for $\mathfrak{g}_2$. Therefore the complex Lie algebra $\mathfrak{g}_2 + \sqrt{-1}\mathfrak{g}_2$ has complex dimension 2 and the complex Lie algebra $\mathfrak{g}_2 \cap \sqrt{-1}\mathfrak{g}_2$ has complex dimension 1. As $\mathfrak{g}_2 \cap \sqrt{-1}\mathfrak{g}_2$ is abelian but $\mathfrak{g}_2$ is not, it follows that $\dim([\mathfrak{g}_2 \cap \sqrt{-1}\mathfrak{g}_2, \mathfrak{g}_2]) > 0$. But that means $[\mathfrak{g}_2 \cap \sqrt{-1}\mathfrak{g}_2, \mathfrak{g}_2]$ is a non-trivial complex vector space contained in the real 1-dimensional space $[\mathfrak{g}_2, \mathfrak{g}_2]$, a contradiction.

Case where $\mathfrak{g}$ is $\mathrm{ad}_{\mathfrak{h}\oplus\mathfrak{sl}_2(\mathbb{R})}$-invariant. In this case $\mathfrak{g}_1$ and $\mathfrak{k}_1$ are $\mathrm{ad}_{\mathfrak{h}}$-invariant and $\mathfrak{g}_2$ and $\mathfrak{k}_2$ are $\mathrm{ad}_{\mathfrak{sl}_2(\mathbb{R})}$-invariant. As was argued in the previous example, the subalgebra $\mathfrak{h}$ does not admit an $\mathrm{ad}_{\mathfrak{h}}$-invariant Lagrangian complement and thus $\dim(\mathfrak{k}_1) \leq 2$. As $\mathfrak{sl}_2(\mathbb{R})$ is a simple Lie algebra, the $\mathrm{ad}_{\mathfrak{sl}_2(\mathbb{R})}$-invariant subspace $\mathfrak{g}_2 \cap \mathfrak{sl}_2(\mathbb{R})$ must be trivial and thus $\dim(\mathfrak{g}_2) = 3$. However, (4.20) then forces $\dim(\mathfrak{k}_1) = 3$, a contradiction.

Therefore $(T^*_\theta\mathfrak{h} \oplus \mathfrak{sl}_2(\mathbb{C}), \mathfrak{h}^* \oplus \mathfrak{sl}_2(\mathbb{R}))$ is an example of a Manin pair that does not admit a Lie algebra complement nor an $\mathrm{ad}_{\mathfrak{h}\oplus\mathfrak{sl}_2(\mathbb{R})}$-invariant Lagrangian complement, and whose Lagrangian subalgebra is not an ideal. Furthermore it admits a group pair (D, G) obtained as the componentwise product of $(\mathrm{SL}(2,\mathbb{C}), \mathrm{SL}(2,\mathbb{R}))$ and a group pair (K, H) for $(T^*_\theta\mathfrak{h}, \mathfrak{h}^*)$. Since K/H is a Lie group, the second homotopy group of the quotient D/G is isomorphic to that of $\mathrm{dS}_3 = \mathrm{SL}(2,\mathbb{C})/\mathrm{SL}(2,\mathbb{R})$ which is non-trivial as seen above. **Therefore D/G does not admit a Lie group structure.**

[2] For a list of the subalgebras of $\mathfrak{sl}_2(\mathbb{C})$, identified as the Lie algebra of the Lorentz group $\mathrm{SO}(1,3)$, see [29, Chapter 6].

4.A Appendix: Classicism near the coset of the group identity

Suppose $(\mathfrak{d}, \mathfrak{g})$ is a Manin pair admitting the group pair (D, G). In some circumstances, for instance if $\mathfrak{g}$ is semisimple or if G is compact according to Proposition 4.1.2, an Ad_G-invariant Lagrangian complement $\mathfrak{h} \subseteq \mathfrak{d}$ of $\mathfrak{g}$ can be found. In that case the exponential map $\exp : \mathfrak{d} \to D$ sends a small neighbourhood of $0 \in \mathfrak{h}$ diffeomorphically onto a neighbourhood of the coset $[e] \in D/G$ of the group identity $e \in D$. The exponential map furthermore intertwines the coadjoint representation of G (identifying $\mathfrak{h} \simeq \mathfrak{g}^*$) and the dressing action on D/G.

On the other hand, in the classical theory of $\mathfrak{g}$-Hamiltonian spaces, moment maps are valued in the dual $\mathfrak{h} \simeq \mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ and intertwine a G-action on their domain and the coadjoint representation of G. It would therefore seem that the D/G-valued moment map theory might converge towards the classical moment map theory near the coset $[e]$. This indeed will be argued here. There is however an important caveat: the discussion requires a passage through infinite-dimensional manifolds. These are handled loosely so as to not get bogged down in auxiliary questions of regularity that would interfere with the geometric aspects of the arguments given. License has been taken in extending results presented in Chapter 2 to this context. This, however, is not entirely unjustified; the results in question can also be derived in the formalism of Hilbert manifolds [17], barring additional topological closure conditions. See also [4, §8.1, §9.1].

4.A.1 Gauge-theoretic preliminaries

Let PD be the group of paths $\gamma : [0,1] \to D$ into the Lie group D and let $P\mathfrak{d}$ be the Lie algebra of paths $[0,1] \to \mathfrak{d}$ in the Lie algebra $\mathfrak{d}$. Consider the space of connections

$$P\mathfrak{d}^* = \Omega^1(D \times [0,1], \mathfrak{d})^D \tag{4.21}$$

of the trivial principal bundle[3] $[0,1] \times D$. A connection $A \in P\mathfrak{d}^*$ is completely determined by its pullback onto $[0,1]$, embedded in $[0,1] \times D$ in the natural way. Alternatively, it is determined by its corresponding horizontal path $P \in PD$ with initial condition $P(0) = e$. The space of connections of $[0,1] \times D$ may thus be regarded as the group of paths in D starting at the group identity e.

Consider the Lie algebra $P\mathfrak{d}^* \rtimes P\mathfrak{d}$ of the semi-direct product $P\mathfrak{d}^* \rtimes PD$ where PD acts on $P\mathfrak{d}^*$ via $d.A = \mathrm{Ad}_d\, A$. It carries a natural Ad-invariant metric given by the pairing

[3]Contrary to the usual convention, the group D acts on $[0,1] \times D$ on the left via $d.(t, d') = (t, dd')$. The space $P\mathfrak{d}^*$ consists of $\mathfrak{d}$-valued 1-forms ω that are equivariant in the sense that $d^*\omega = \mathrm{Ad}_d$.

$$\int_0^1 \langle A, \xi \rangle, \tag{4.22}$$

for $A \in P\mathfrak{d}^*$ and $\xi \in P\mathfrak{d}$. Consider then the natural action of $P\mathfrak{d}^* \rtimes PD$ on $P\mathfrak{d}^*$. As this action is transitive, the stabilizer algebras of the action of $P\mathfrak{d}^* \rtimes Pd$ on $P\mathfrak{d}^*$ are translations of the stabilizer algebra at the identity, which is the Lagrangian subalgebra $P\mathfrak{d}$ of $P\mathfrak{d}^* \rtimes P\mathfrak{d}$. These considerations determine an action Courant algebroid

$$\mathbb{P}\mathfrak{d} = P\mathfrak{d}^* \times (P\mathfrak{d}^* \rtimes PD). \tag{4.23}$$

Its anchor will be denoted by $\mathbf{a}_{P\mathfrak{d}^*}$. It admits the natural $P\mathfrak{d}^* \rtimes PD$-invariant isotropic splitting

$$j_{P\mathfrak{d}^*}(A) = (A, 0), \tag{4.24}$$

with corresponding identification $\mathbb{P}\mathfrak{d} \simeq \mathbb{T}P\mathfrak{d}^*$.

Consider next the action of PD on $P\mathfrak{d}^*$ by gauge transformations

$$d.A = \mathrm{Ad}_d\, A - d^*\theta^R. \tag{4.25}$$

Identifying $A \in P\mathfrak{d}^*$ with its corresponding horizontal path $P \in PD$ with initial condition $P(0) = e$, the action (4.25) admits the alternative description

$$(d.P)(t) = d(0)^{-1}P(t)d(t), \tag{4.26}$$

from which one sees that the action (4.25) by gauge transformations is transitive. Define the *holonomy map* $\mathrm{hol} : P\mathfrak{d}^* \to D$ as follows: if $A = d.0$ then $\mathrm{hol}(A) = d(0)d(1)^{-1}$. This map is well-defined since if d is in the stabilizer of 0 then so is d^{-1} and therefore $d(0)d(1)^{-1} = e$ according to the description (4.26). Note that

$$\mathrm{hol}(d.A) = d(0)\mathrm{hol}_t(A)d(1)^{-1}. \tag{4.27}$$

The infinitesimal counterpart of the action (4.25) by gauge transformations is the covariant derivative $\partial_A\xi = d\xi + \mathrm{ad}_A\,\xi$. For a subspace $\mathfrak{s} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$, define

$$\mathcal{E}^{(\mathfrak{s})} = \{(d\xi, -\xi) \in P\mathfrak{d}^* \rtimes P\mathfrak{d} \;:\; \xi \in P\mathfrak{d}^*,\ (\xi(0), \xi(1)) \in \mathfrak{s}\}, \tag{4.28}$$

which is a subspace of $P\mathfrak{d}^* \rtimes P\mathfrak{d}$ and will also be regarded as a (trivial) subbundle of $\mathbb{P}\mathfrak{d}$. For a subgroup $K \subseteq D \times D$ define

$$\mathcal{G}_K = \{(-d^*\theta^R, d) \in P\mathfrak{d}^* \rtimes PD \ :\ d \in PD,\ (d(0), d(1)) \in K\} \tag{4.29}$$

which is a subgroup of $P\mathfrak{d}^* \rtimes PD$. If $\mathfrak{s} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ is a subalgebra, then in fact $\mathcal{E}^{(\mathfrak{s})}$ is a subalgebra of $P\mathfrak{d}^* \rtimes P\mathfrak{d}$ (see Proposition 4.A.1 below) integrating to the subgroup $\mathcal{G}_S$, where $S \subseteq D \times D$ is the connected subgroup integrating $\mathfrak{s}$. For the special case $\mathfrak{s} = \mathfrak{d} \oplus \overline{\mathfrak{d}}$ and $S = D \times D$, the supercript in (4.28) and the subscript in (4.29) will be omitted and these will be called respectively the *gauge Lie algebra* and the *gauge group*. The gauge group $\mathcal{G}$ implements the action (4.25), hence the terminology.

Proposition 4.A.1. *Let $\mathfrak{s} \subseteq \mathfrak{d} \oplus \overline{\mathfrak{d}}$ be a subspace. The subbunbdle $\mathcal{E}^{(\mathfrak{s})}$, seen as a subbundle of $\mathbb{P}\mathfrak{d}$, is closed in $\mathbb{P}\mathfrak{d}$ and its orthogonal is $\mathcal{E}^{(\mathfrak{s}^\perp)}$. Furthermore $\mathcal{E}^{(\mathfrak{s})}$ is,*

(a) *isotropic if $\mathfrak{s}$ is isotropic,*
(b) *coisotropic if $\mathfrak{s}$ is coisotropic,*
(c) *involutive if $\mathfrak{s}$ is a Lie subalgebra.*

Proof. For $(d\xi_1, -\xi_1), (d\xi_2, -\xi_2) \in \mathcal{E}^{(\mathfrak{d}\oplus\overline{\mathfrak{d}})}$, observe that

$$\begin{aligned}\langle (d\xi_1, -\xi_1), (d\xi_2, -\xi_2)\rangle &= -\int_0^1 \langle d\xi_1, \xi_2\rangle - \int_0^1 \langle \xi_1, d\xi_2\rangle \\ &= \langle \xi_1(0)\xi_2(0)\rangle - \langle \xi_1(1)\xi_2(1)\rangle,\end{aligned} \tag{4.30}$$

where integration by parts was used. It is immediate from (4.30) that $(\mathcal{E}^{(\mathfrak{s})})^\perp = \mathcal{E}^{(\mathfrak{s}^\perp)}$. In particular $(\mathcal{E}^{(\mathfrak{s}^\perp)})^\perp = \mathcal{E}^{(\mathfrak{s})}$ and it follows that $E^{(\mathfrak{s})}$ is closed. If $\mathfrak{s}$ is (co)isotropic in $\mathfrak{d} \oplus \overline{\mathfrak{d}}$ then it also follows from (4.30) that $\mathcal{E}^{(\mathfrak{s})}$ is (co)isotropic in $\mathbb{P}\mathfrak{d}$. Finally, if $\mathfrak{s}$ is a subalgebra of $\mathfrak{d} \oplus \overline{\mathfrak{d}}$ then, regarding $\mathcal{E}^{(\mathfrak{s})}$ as a subspace of $P\mathfrak{d}^* \rtimes P\mathfrak{d}$, one has

$$[(-d\xi_1, \xi_1), (-d\xi_2, \xi_2)] = (\mathrm{ad}_{\xi_2} d\xi_1 - \mathrm{ad}_{\xi_1} d\xi_2, [\xi_1, \xi_2]) = (-d[\xi_1, \xi_2], [\xi_1, \xi_2])$$

and thus $\mathcal{E}^{(\mathfrak{s})}$ is a subalgebra of $P\mathfrak{d}^* \rtimes P\mathfrak{d}$. Arguing as in Example 2.2.5 then, one concludes that $\mathcal{E}^{(\mathfrak{s})}$ is an involutive subbundle of $\mathbb{P}\mathfrak{d}$. □

The gauge Lie algebra $\mathcal{E}$ is coisotropic with isotropic complement $\mathcal{E}^{(0)}$. From the description (4.26), it follows that the action of $\mathcal{G}_e$ on $P\mathfrak{d}^*$ is free and, as $P\mathfrak{d}^*/\mathcal{G}_0 = D$ is finite-dimensional, it is also proper. This defines a Courant algebroid

$$\frac{\mathcal{E}^{(\mathfrak{d}\oplus\overline{\mathfrak{d}})}}{\mathcal{E}^{(0)}}/\mathcal{G}_0 \tag{4.31}$$

obtained by coisotropic reduction. As $[\mathcal{E}^{(0)}, \mathcal{E}] \subseteq \mathcal{E}^{(0)}$, according to Example 2.4.4 the reduced Courant algebroid (4.31) is an action Courant algebroid with base manifold D and acting Lie algebra

$$\frac{\mathcal{E}^{(\mathfrak{d}\oplus\bar{\mathfrak{d}})}}{\mathcal{E}^{(0)}} = \mathfrak{d} \oplus \bar{\mathfrak{d}},$$

where $\mathcal{E}^{(\mathfrak{d}\oplus\bar{\mathfrak{d}})}$ and $\mathcal{E}^{(0)}$ are interpreted as subspaces of $P\mathfrak{d}^* \rtimes P\mathfrak{d}$ this time. The quotient map will identify two connections $A_1, A_2 \in P\mathfrak{d}^*$ if only if there is an element $d \in PD$ with $d(0) = d(1) = e$ such that $A_2 = d.A_1$. According to (4.26) and (4.27), this means that A_1 and A_2 are identified if and only if $\mathrm{hol}(A_1) = \mathrm{hol}(A_2)$. One can thus take the view that the quotient map is the holonomy map $\mathrm{hol} : P\mathfrak{d}^* \to D$ and, by (4.27) again, the reduced Courant algebroid (4.31) is non other than the familiar Courant algebroid $D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}})$. The reduction morphism $\mathbb{P}\mathfrak{d} \dashrightarrow D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}})$ will be denoted by

$$\mathrm{Hol} : \mathbb{P}\mathfrak{d} \dashrightarrow D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}}).$$

Note that Hol is equivariant with respect to the quotient of Lie groups $\cdot/\mathcal{G}_0 : \mathcal{G} \to D \times D$ and intertwines the generators.

Proposition 4.A.2. *The reduction morphism* $\mathrm{Hol} : \mathbb{P}\mathfrak{d} \dashrightarrow D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}})$ *is exact.*

Proof. Since $[\mathcal{E}, \mathcal{E}^{(0)}] \subseteq \mathcal{E}^{(0)}$, it follows from C5 that the subbundle $\mathcal{E}^{(0)} \subseteq \mathbb{P}\mathfrak{d}$ is invariant under the action of the gauge group $\mathcal{G}$ on $\mathbb{P}\mathfrak{d}$. Now the kernel of $\mathbf{a}_{P\mathfrak{d}^*}$ at $A = 0$ is $\{(0, \xi) : \xi \in P\mathfrak{d}\}$, which intersects $\mathcal{E}^{(0)}|_{A=0}$ trivially. As the action of $\mathcal{G}$ on $P\mathfrak{d}^*$ is transitive and Hol is equivariant with respect to the quotient $\cdot/\mathcal{G}_0 : \mathcal{G} \to D \times D$, it follows that the intersection $\mathcal{E}^{(0)} \cap \ker(\mathbf{a}_{P\mathfrak{d}^*})$ is everywhere trivial and the claim then follows from Proposition 2.4.2. □

Let $j'_{P\mathfrak{d}^*}$ be the isotropic splitting of $\mathbb{P}\mathfrak{d}$ defined by the isotropic splitting j_D of $D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}})$ as in Proposition 2.3.2. Note that $j'_{P\mathfrak{d}^*}$ is $\mathcal{G}$-invariant. It is therefore determined by its restriction $j'_{P\mathfrak{d}^*}|_0$. Explicitly,

$$j'_{P\mathfrak{d}^*}(A)|_0 = (A, -\xi_A)$$

where $\xi_A \in P\mathfrak{d}$ solves $d\xi_A = A$ with initial condition $(d(0), d(1)) \in \mathfrak{d}_{\overline{\Delta}}$ (recall that $\mathfrak{d}_{\overline{\Delta}}$ is the anti-diagonal of $\mathfrak{d} \oplus \bar{\mathfrak{d}}$). In fact,

$$\xi_A(t) = \int_0^t A - \frac{1}{2}\int_0^1 A.$$

For $B \in P\mathfrak{d}^*$ and let $d_B : [0, 1] \to D$ be the (necessarily unique) solution to the differential equation $-d^*\theta^R = B$ with initial condition $d(0) = e$ (if B is a constant connection then

d is an integral curve of the fundamental vector field $-\nu^R \in \mathfrak{X}(D)$). Then according to the description (4.25), one has $d.0 = \zeta$. In particular

$$j'_{P\mathfrak{d}^*}|_B(A) = j'_{P\mathfrak{d}^*}|_0(\mathrm{Ad}^{-1}_{(B,d_B)} A) = j'_{P\mathfrak{d}^*}|_0(\mathrm{Ad}^{-1}_{d_B} A) = (d\xi_{\mathrm{Ad}^{-1}_{d_B} A}, -\xi_{\mathrm{Ad}^{-1}_{d_B} A}).$$

Let $\omega \in \Omega^2(P\mathfrak{d}^*)$ be the 2-form corresponding to the Courant morphism Hol under the identifications $D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}}) \simeq \mathbb{T}D$ and $\mathbb{P}\mathfrak{d} \simeq \mathbb{T}\mathfrak{d}$. This 2-form is $\mathcal{G}$-invariant and according to (2.10), it is given by

$$\omega(A_1, A_2)|_B = \langle j_{P\mathfrak{d}^*}(A_1), j'_{P\mathfrak{d}^*}(A_2)\rangle|_B = -\int_0^1 \langle A_1, \xi_{\mathrm{Ad}^{-1}_{d_B} A_2}\rangle. \tag{4.32}$$

4.A.2 Classicism of D/G-valued moment maps

Consider the action Courant algebroid

$$\mathbb{A}_{\mathfrak{h}} = \mathfrak{g}^* \times (\mathfrak{g}^* \rtimes \mathfrak{g}) \simeq \mathfrak{h} \times (\mathfrak{h} \rtimes \mathfrak{g})$$

introduced in Example 4.3.1. Denote its anchor by $\mathbf{a}_{\mathfrak{h}}$. It is exact, admitting the isotropic splitting $j_{\mathfrak{h}}(A) = (A, 0)$ thereby identifying it with $\mathbb{T}\mathfrak{h}$. (This is completely analogous to the construction of $\mathbb{P}\mathfrak{d}$ above.) The inclusion map $\iota_{\mathfrak{h}} : \mathfrak{h} \to P\mathfrak{d}^*$, where $\mathfrak{h}$ is identified with the constant $\mathfrak{h}$-valued connections, thus defines a Courant morphism

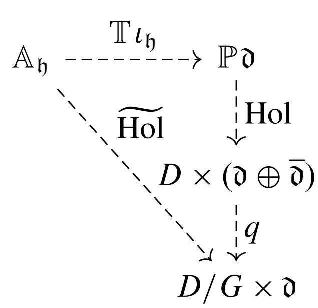
$$\mathbb{T}\iota_{\mathfrak{h}} : \mathbb{A}_{\mathfrak{h}} \simeq \mathbb{T}\mathfrak{h} \dashrightarrow \mathbb{T}P\mathfrak{d}^* \simeq \mathbb{P}\mathfrak{d}^*.$$

Note that the inclusion map $\iota_{\mathfrak{h}}$ is equivariant with respect to the inclusion

$$G \hookrightarrow \mathcal{G}_{G\times G},\ g \mapsto (0, g). \tag{4.33}$$

and thus the Courant morphism $\mathbb{T}\iota_{\mathfrak{h}}$ is also equivariant with respect to (4.33). Now the coisotropic reduction of $D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}})$ by $E^{(0\oplus\mathfrak{g})}$ is $D/G \times \mathfrak{d}$. The upshot is the diagram

(4.34)



where q is the reduction morphism and $\widetilde{\mathrm{Hol}} = q \circ \mathrm{Hol} \circ \mathbb{T}\iota_{\mathfrak{h}}$. As q is equivariant with respect to the projection $D \times G \to D$, the morphism $\widetilde{\mathrm{Hol}}$ is G-equivariant. The holonomy of a constant connection $\zeta \in \mathfrak{d} \subseteq P\mathfrak{d}^*$ is the exponential $\exp(tA)$ and the base map of the Courant morphism $\widetilde{\mathrm{Hol}}$ is therefore

$$\zeta \mapsto [\exp(\zeta)],$$

and for this reason it will be denoted by $\exp_{\mathfrak{h}} : \mathfrak{h} \to D/G$. The base map $\exp_{\mathfrak{h}}$ is thus invertible near 0 in the sense that there is an open neighbourhood $U \subseteq \mathfrak{h}$ of 0 sent diffeomorphically onto an open neighbourhood V of $[e] \in D/G$. Let $(D/G \times \mathfrak{d})|_V$ denote the restriction of $D/G \times \mathfrak{d}$ to the base manifold V, made into a Courant algebroid in the obvious way, similarly for $(\mathbb{A}_{\mathfrak{h}})|_U$. Retain $\widetilde{\mathrm{Hol}}$ to denote the restriction of $\widetilde{\mathrm{Hol}} : \mathbb{A}_{\mathfrak{h}} \dashrightarrow D/G \times \mathfrak{d}$ to $(V \times U) \cap \mathrm{Gr}(\exp_{\mathfrak{h}})$.

Theorem 4.A.1 (Classicism near the coset of the group identity)**.** *The Courant morphism* $\widetilde{\mathrm{Hol}} : (\mathbb{A}_{\mathfrak{h}})|_U \dashrightarrow (D/G \times \mathfrak{d})|_V$ *is a Dirac morphism*

$$((\mathbb{A}_{\mathfrak{h}})|_U, (\mathfrak{h} \times \mathfrak{g})|_U) \dashrightarrow ((D/G \times \mathfrak{d})|_V, E^{(\mathfrak{g})}|_V).$$

Proof. Since $\widetilde{\mathrm{Hol}} : (\mathbb{A}_{\mathfrak{h}})|_U \dashrightarrow (D/G \times \mathfrak{d})|_V$ is a Courant isomorphism, it suffices to show that

$$\widetilde{\mathrm{Hol}} \circ (\mathfrak{h} \times \mathfrak{g})|_U \subseteq E^{(\mathfrak{g})}|_V.$$

Actually, as $\widetilde{\mathrm{Hol}} : \mathbb{A}_{\mathfrak{h}} \to D/G \times \mathfrak{d}$ is G-equivariant, it suffices to verify that

$$\widetilde{\mathrm{Hol}} \circ (\mathfrak{h} \times \mathfrak{g})|_0 \subseteq E^{(\mathfrak{g})}|_{[e]}. \tag{4.35}$$

Now $E^{(\mathfrak{g})}|_0 = \mathfrak{g}$ is the kernel of the anchor of $\mathbb{A}_{\mathfrak{h}}$ at 0. Its forward image by $\mathbb{T}\iota_{\mathfrak{h}}$ is therefore the kernel of $\mathbf{a}_{P\mathfrak{d}^*}$ at 0, which is $(P\mathfrak{d}^* \times P\mathfrak{d}^*)|_0 = P\mathfrak{d}^*$. On the other hand, the intersection of $(P\mathfrak{d}^* \times P\mathfrak{d}^*)|_0$ and $\mathcal{E}^{(\mathfrak{d}\oplus\overline{\mathfrak{d}})}$ (seen as a subbdundle of $\mathbb{P}\mathfrak{d}$) is $(P^*\mathfrak{d} \times \mathfrak{d})|_0 = \mathfrak{d}$, where $\mathfrak{d}$ is interpreted as the space of constant connections. The upshot is that

$$\mathrm{pr}_{\mathfrak{g}}(\zeta) \sim_{\mathbb{T}\iota_{\mathfrak{h}}} \zeta \sim_{\mathrm{Hol}} (\zeta, \zeta) \in (D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}))|_e$$

for any $\zeta \in \mathfrak{d}$. In particular, for $\gamma \in \mathfrak{g}$ one has

$$\gamma \sim_{\mathbb{T}\iota_{\mathfrak{h}}} \gamma \sim_{\mathrm{Hol}} (\gamma, \gamma) \sim_q \gamma \in (D/G \times \mathfrak{d})|_{[e]}.$$

This establishes the containment (4.35) and the proof is therefore complete. □

Theorem 4.A.1 is significant in that it shows that if X is a Hamiltonian space for $(D/G \times \mathfrak{d}, E^{(\mathfrak{g})})$, in other words a quasi-Poisson $\mathfrak{g}$-space with a moment, say J, valued in D/G, then

$X|_{J^{-1}(V)}$ is a $\mathfrak{g}$-Hamiltonian space in the classical sense. Thus, if $\mathfrak{g}$ admits an Ad_G-invariant Lagrangian complement $\mathfrak{h} \subseteq \mathfrak{d}$, then the theory of quasi-Poisson $\mathfrak{g}$-space with D/G-valued moment maps becomes the classical moment map theory near the coset of the group identity. It should be noted however that their respective notions of the fusion product are different.

The morphism $\widetilde{\mathrm{Hol}}$ is exact since it is the composition of exact morphisms. Consider the subbundle $E^{(\mathfrak{h})}$ of $D/G \times \mathfrak{d}$. Its intersection with the kernel of the anchor of $D/G \times \mathfrak{d}$ at $[e]$, which is $[e] \times \mathfrak{g} = \mathfrak{g}$, is trivial. Thus, making U (and V) smaller if need be, the subbundle $E^{(\mathfrak{h})}|_V$ is a Lagrangian complement of the kernel of the anchor of $D/G \times \mathfrak{d}$ and thus defines an isotropic splitting $j_{D/G} : T(D/G)|_V \to (D/G \times \mathfrak{d})|_V$. Let $\eta_{D/G} \in \Omega^3(V)$ be its corresponding 3-form. Now the quotient morphism q present in the diagram (4.34) restricts to a Courant morphism $(D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}}))|_{\exp(U)G} \dashrightarrow (D/G \times \mathfrak{d})|_V$ where $\exp(U)G$ is the flow out by the $\{e\} \times G$-action of $\exp(U) \subseteq D$. Let $j'_D : TD|_{\exp(U)G} \to (D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}}))|_{\exp(U)G}$ be the isotropic splitting induced by $j_{D/G}$ as per Proposition 2.3.2, let $\eta'_D \in \Omega^3(\exp(U)G)^3$ be its corresponding curvature tensor and let ϖ be the 2-form corresponding to the change of isotropic splitting $\mathbb{T}D_{\eta'_D} \simeq \mathbb{T}D_{\eta_D}$, where η_D is the Cartan 3-form defined in (3.5). The calculations carried out to obtain the expressions (4.11) and (4.12), substituting $\exp(U)$ for G, are still valid and thus

$$j'_D(v) = (-\mathrm{Ad}_{\exp(\zeta)} \mathrm{pr}_{\mathfrak{h}} \theta^{\exp(U)G}, \mathrm{Ad}_{g^{-1}} \mathrm{pr}_{\mathfrak{g}} \theta^{\exp(U)G}),$$

and

$$\varpi = \frac{1}{2} \langle \theta^{\exp(U)G}, \mathrm{pr}_{\mathfrak{g}} \theta^{\exp(U)G} \rangle, \tag{4.36}$$

where $\iota_v \theta^{\exp(U)G} = L_{\exp(\zeta)^{-1}} R_{g^{-1}} v$ for $v \in T_{\exp(\zeta)g}$ and $\zeta \in U$. The 2-form ϖ can also be interpreted as the 2-form corresponding to quotient morphism $q : (D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}}))|_{\exp(U)G} \dashrightarrow (D/G \times \mathfrak{d})|_V$.

Finally, let $\omega_{\mathfrak{h}} \in \Omega^2(U)$ be the 2-form such that $\widetilde{\mathrm{Hol}} = \mathbb{T}_{\omega_{\mathfrak{h}}}$. Putting (4.36) and (4.32) together gives

$$\omega_{\mathfrak{h}} = \iota_{\mathfrak{h}}^*(\mathrm{hol}^* \varpi + \omega). \tag{4.37}$$

Theorem 4.A.1 may be recast as follows.

Theorem 4.A.2. *Let $\omega_{\mathfrak{h}} \in \Omega^2(U)$ defined by be the 2-form defined by (4.37). Then*

1. $d\omega_{\mathfrak{h}} = -\exp_{\mathfrak{h}}^* \eta_{D/G}$,
2. *$\omega_{\mathfrak{h}}$ is G-invariant*[4],

[4]Here G-invariance is considered only insofar the G-action on $\mathfrak{h} \simeq \mathfrak{g}^*$ preserves V.

3. *and* $\iota_{\mathbf{a}_{\mathfrak{h}}(\gamma)}\omega = -d\langle\gamma,\cdot\rangle + \exp_{\mathfrak{h}}^* j_{D/G}^*\gamma$ *for* $\gamma \in \mathfrak{g}$.

Proof. Since $\widetilde{\mathrm{Hol}}$ is equal to $\mathbb{T}(\exp_{\mathfrak{h}})_{\omega_{\mathfrak{h}}}$ under the identifications afforded by the splitting $j_{\mathfrak{h}}$ and $j_{D/G}$, the first property follows. As $j_{\mathfrak{h}}$ and $j_{D/G}$ are G-invariant and $\widetilde{\mathrm{Hol}}$ is G-equivariant, the second property follows as well. Finally, by Theorem 4.A.1 one has $\gamma \sim_{\widetilde{\mathrm{Hol}}} \gamma$ which then implies

$$\mathbf{a}_{D/G}(\gamma) = (\exp_{\mathfrak{h}})_*\mathbf{a}_{\mathfrak{h}}(\gamma), \quad j_{\mathfrak{h}}^*(\gamma) + \iota_{\mathbf{a}_{\mathfrak{h}}(\gamma)}\omega = \exp_{\mathfrak{h}}^* j_{D/G}^*\gamma.$$

Since $j_{\mathfrak{h}}^*(\gamma) = d\langle\gamma,\xi\rangle$, the claim follows. □

Chapter 5

Moduli space examples

Moduli spaces of flat principal bundles over surfaces carrying certain decorations provide a rich class of examples of L-Hamiltonian spaces. The *coloured surfaces* introduced below generalize those of Ševera [54] and it is indeed with an eye towards his work that this chapter proceeds. In particular, a class of examples of D/G-valued moment maps is provided.

5.1 Moduli spaces of flat D-bundles on coloured surfaces

5.1.1 Coloured surfaces

Let Σ be a compact oriented 2-manifold with corners. It is assumed that Σ has at least one boundary component and that each boundary component has at least one corner (the case of a single corner corresponds to a boundary circle with a base point). Let $\mathfrak{d}$ be a quadratic Lie algebra integrating to a connected group D. A boundary segment a of Σ may be decorated with a Lagrangian subalgebra $\mathfrak{g}_a \subseteq \mathfrak{d}$ integrating to a closed and connected subgroup $G_a \subseteq D$, in which case a is called *coloured*, or left undecorated in which case a is called *free*. Let $\mathcal{V}$ and $\mathcal{E}$ denote respectively the set of vertices (corners) and the set of boundary segments of Σ. The subsets of coloured and free boundary segments will be denoted by $\mathcal{E}_{\text{col}}, \mathcal{E}_{\text{free}} \subseteq \mathcal{E}$ respectively. It is required that there be at least one free boundary segment and that no two free boundary segments be adjacent[1]. With these decorations, the surface Σ is called a *coloured surface*.

[1]The second requirement is to simplify the characterization of the subalgebra $\mathfrak{l}(\Sigma)$ given below; it can be lifted at the cost of somewhat complicating the exposition.

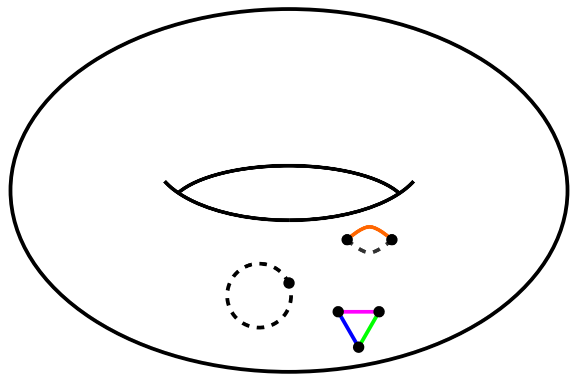


Figure 5.1: A coloured surface with 1-sided, 2-sided and 3-sided boundary components. Free boundary segments are indicated by dotted lines. Colours correspond to Lagrangian subalgebras of $\mathfrak{d}$.

Consider the restricted fundamental groupoid of Σ

$$\Pi_1(\Sigma, \mathcal{V}) \rightrightarrows \mathcal{V}$$

consisting of all homotopy classes of paths with end points in $\mathcal{V}$. Choose an orientation on Σ so that each boundary segment is now directed. One will be interested in the space of homomorphisms of groupoids

$$M_\Sigma := \{F \in \mathrm{Hom}(\Pi_1(\Sigma, \mathcal{V}), D) \ : \ F([a]) \in G_a \text{ for } a \in \mathcal{E}_{\mathrm{col}}\}.$$

There is an action of $\mathrm{Map}(\mathcal{V}, D)$ on $\mathrm{Hom}(\Pi_1(\Sigma, \mathcal{V}), D)$ given by

$$(d.\kappa)([p]) = d_{p(0)}\kappa([p])d_{p(1)}^{-1} \tag{5.1}$$

for $\kappa \in \mathrm{Hom}(\Pi_1(\Sigma, \mathcal{V}), D)$ and $[p] \in \Pi_1(\Sigma, \mathcal{V})$. The acting group is restricted so that this action preserves M_Σ. Namely let

$$\mathrm{Map}_{\mathrm{col}}(\mathcal{V}, D) \subseteq \mathrm{Map}(\mathcal{V}, D)$$

be the subgroup defined by the following rules on its elements $f \in \mathrm{Map}_{\mathrm{col}}(\mathcal{V}, D)$:

Rules 5.1.1. For $v \in \mathcal{V}$, let $a, b \in \mathcal{E}$ be the (possibly identical) boundary segments incident to v.

1. If $a \in \mathcal{E}_{\mathrm{col}}$ and $b \in \mathcal{E}_{\mathrm{free}}$ then $f(v) \in G_a$;
2. if $a, b \in \mathcal{E}_{\mathrm{col}}$ then $f(v) \in \mathrm{Lie}(\mathfrak{g}_a \cap \mathfrak{g}_b)$;
3. if $a, b \in \mathcal{E}_{\mathrm{free}}$ then $f(v)$ is unconstrained.

Let $\mathrm{Map}_{\mathrm{col},0}(\mathcal{V}, D)$ be the subgroup of $\mathrm{Map}_{\mathrm{col}}(\mathcal{V}, D)$ of elements $f \in \mathrm{Map}_{\mathrm{col}}(\mathcal{V}, D)$ such that $f(v) = e$ whenever v is incident to a free boundary segment. Consider the quotient

$$\boldsymbol{M}_\Sigma := M_\Sigma / \mathrm{Map}_{\mathrm{col},0}(\mathcal{V}, D). \tag{5.2}$$

The quotient $\boldsymbol{M}_\Sigma$ is interpreted as the space of flat D-bundles over Σ (determined by their holonomies) up to a suitable notion of gauge transformation. It inherits an action by the group

$$L(\Sigma) := \frac{\mathrm{Map}_{\mathrm{col}}(\mathcal{V}, D)}{\mathrm{Map}_{\mathrm{col},0}(\mathcal{V}, D)}$$

which can also be seen as the subgroup of elements $f \in \mathrm{Map}_{\mathrm{col}}(\mathcal{V}, D)$ such that $f(v) = e$ whenever v is not incident to a free boundary segment. Let

$$\boldsymbol{J} : \boldsymbol{M}_\Sigma \to \mathrm{Map}(\mathcal{E}_{\mathrm{free}}, D) \tag{5.3}$$

be the pullback map. There is a natural action of $L(\Sigma)$ on $\mathcal{E}_{\mathrm{free}}$ and $\boldsymbol{J}$ is $\mathrm{Map}_{\mathrm{col}}(\mathcal{V}, D)$-equivariant. Consider the action Courant algebroid

$$\mathbb{A} = \mathrm{Map}(\mathcal{E}_{\mathrm{free}}, D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}})) \tag{5.4}$$

defined analogously to $(D \times (\mathfrak{d} \oplus \overline{\mathfrak{d}}))^r$ where $r = |\mathcal{E}_{\mathrm{free}}|$. The group $L(\Sigma)$ may be regarded as the closed and connected subgroup of $\mathrm{Map}(\mathcal{E}_{\mathrm{free}}, D \times D)$ integrating the Lagrangian subalgebra

$$\mathfrak{l}(\Sigma) \subseteq \mathrm{Map}(\mathcal{E}_{\mathrm{free}}, \mathfrak{d} \oplus \overline{\mathfrak{d}}) \tag{5.5}$$

defined by the following rules on its elements $\phi \in \mathfrak{l}(\Sigma)$:

Rules 5.1.2. For $a \in \mathcal{E}_{\mathrm{free}}$, let $b, c \in \mathcal{E}$ be the (possibly identical) boundary segments adjacent to a in the order dictated by the orientation of Σ (i.e. the boundary segments appear in the order $b - a - c$).

1. If $b, c \in \mathcal{E}_{\mathrm{col}}$ then $\phi(a) \in (\mathfrak{g}_b, \mathfrak{g}_c)$,
2. if $b = c$ so that a is a boundary circle, then $\phi(a) \in \mathfrak{d}_\Delta$.

The central result of this chapter can now be stated. Its proof is discussed peacemeal in what follows it.

Theorem 5.1.1. *The space $\boldsymbol{M}_\Sigma$ possesses a canonical smooth structure and is canonically a $L(\Sigma)$-Hamiltonian space for the Courant algebroid $\mathbb{A}$.*

Smooth structure of M_Σ. To endow M_Σ with a smooth structure, start by cutting the surface Σ until a glueing polygon P is obtained. Specifically choose a vertex v incident to a free boundary segment in one of the boundary polygons of Σ. Then, capping off all boundary components of Σ, introduce loops at v corresponding to a fundamental polygon for the resulting closed orientable surface in the usual way, i.e. with loops going around and loops going along "handles". Next, for each boundary polygon other the one on which v lies choose a vertex and join it to v via a path. This is done in such a way that no two loops or paths intersect other than at v. The surface Σ is then cut along the paths and loops introduced in this way to give a polygon P.

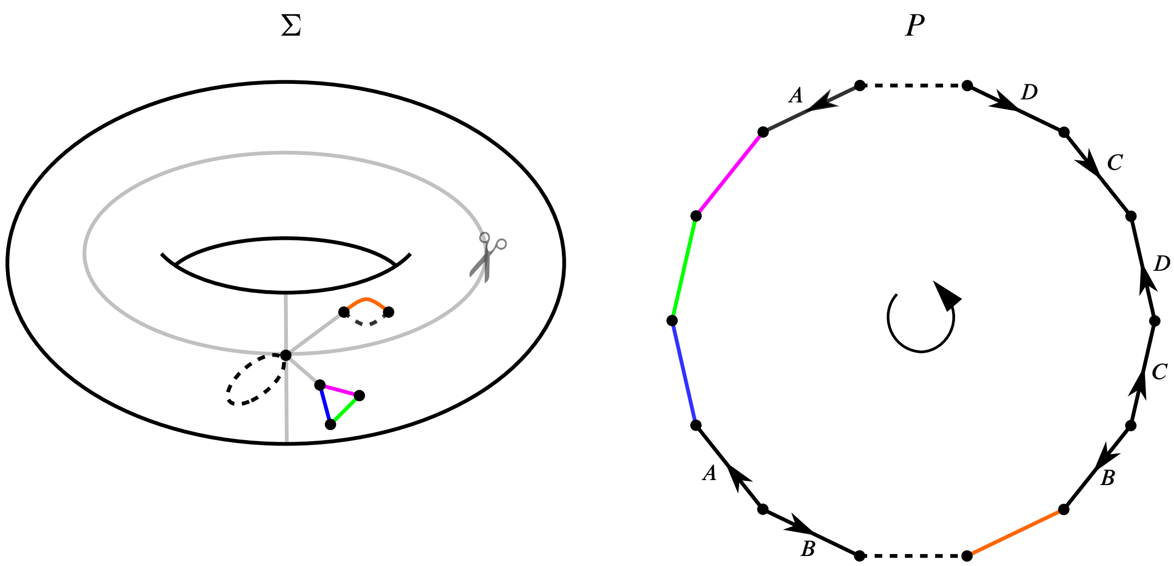


Figure 5.2: The coloured surface Σ is cut into a glueing polygon P.

By convention, it will be assumed that the orientation on Σ corresponds to the CCW orientation of P. Denote the set of edges of P by $\hat{\mathcal{E}}$. Its elements will be listed as, in CCW order,

$$a_0, \ldots, a_{|\hat{\mathcal{E}}|-1} \tag{5.6}$$

where a_0 is by convention a free boundary segment (now an edge of P) of Σ The free boundary segments of Σ will be listed as $a_{f_0} = a_0, \ldots, a_{f_{r-1}}$ in the order in which they appear in (5.6).

The edges of P, together with the identifications corresponding to the cutting of Σ, may be regarded as a set of generators for M_Σ and consequently there is an inclusion

$$M_\Sigma \hookrightarrow \mathrm{Map}(\hat{\mathcal{E}}, D).$$

As such, the space M_Σ inherits a smooth structure as an embedded submanifold. To show the canonicity of this smooth structure, suppose P' is another polygon obtained by cutting Σ and denote its edges by $\hat{\mathcal{E}}'$. Then the respective embeddings of M_Σ into $\mathrm{Map}(\hat{\mathcal{E}}, D)$ and into $\mathrm{Map}(\hat{\mathcal{E}}', D)$ both lift to the same embedding

$$M_\Sigma \hookrightarrow \mathrm{Map}(\hat{\mathcal{E}} \cup \hat{\mathcal{E}}', D)$$

end must therefore induce the same smooth structure. Since $\boldsymbol{M}_\Sigma$ is the quotient of M_Σ by a free and proper Lie group action, it follows that $\boldsymbol{M}_\Sigma$ possesses a canonical smooth structure as well.

Hamiltonian space structure. Having cut Σ into a polygon P, it will be shown how to endow Σ with the structure of a $L(\Sigma)$-Hamiltonian space for $\mathbb{A}$. Consider the quadratic Lie algebra

$$\mathrm{Map}(\hat{\mathcal{E}} \setminus \{a_0\}, \mathfrak{d} \oplus \bar{\mathfrak{d}}) \times \mathrm{Map}(\mathcal{E}_{\mathrm{free}} \setminus \{a_0\}, \mathfrak{d} \oplus \bar{\mathfrak{d}}). \tag{5.7}$$

Define a Lagrangian subalgebra $\mathfrak{l}(P)$ of (5.7) according to the following rules on its elements $(\phi, \psi) \in \mathfrak{l}(P)$, where $\phi \in \mathrm{Map}(\hat{\mathcal{E}} \setminus \{a_0\}, \mathfrak{d} \oplus \bar{\mathfrak{d}})$ and $\psi \in \mathrm{Map}(\mathcal{E}_{\mathrm{free}} \setminus \{a_0\}, \mathfrak{d} \oplus \bar{\mathfrak{d}})$:

Rules 5.1.3.

1. If $a \in \mathcal{E}_{\mathrm{free}}$ then[2] $\phi(a) = \psi(a)^{-1}$;
2. if $a, a' \in \hat{\mathcal{E}}$ are edges corresponding (as a pair) to a cut introduced on Σ then $\phi(a) = \phi(a')^{-1}$;
3. if $a \in \mathcal{E}_{\mathrm{col}}$ then $\phi(a) \in (\mathfrak{g}_a, \mathfrak{g}_a)$.

Consider now the Courant algebroid

$$\hat{\mathbb{A}} = \mathrm{Map}(\hat{\mathcal{E}} \setminus \{a_0\}, D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}})) \times \mathrm{Map}(\mathcal{E}_{\mathrm{free}} \setminus \{a_0\}, D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}})). \tag{5.8}$$

By way of its definition, the orbit $\mathcal{O}(P)$ of $\mathfrak{l}(P)$ through the group identity of the group

$$\mathrm{Map}(\hat{\mathcal{E}} \setminus \{a_0\}, D) \times \mathrm{Map}(\mathcal{E}_{\mathrm{free}} \setminus \{a_0\}, D) \tag{5.9}$$

can be identified with M_Σ; under this identification, regarding the edges of P as generators for M_Σ, the inclusion of M_Σ into the group (5.9) becomes the map

$$J : F \mapsto F([a_1]^{-1}), F([a_2]^{-1}), \ldots, F([a_{|\hat{\mathcal{E}}|-1}]^{-1}), F([a_{f_0}]), \ldots, F([a_{f_{r-1}}]). \tag{5.10}$$

for $F \in M_\Sigma$. Let $L(P)$ denote the closed subgroup of the group (5.9) integrating $\mathfrak{l}(P)$. As per

[2] Recall that in the pair groupoid $\mathfrak{d} \oplus \bar{\mathfrak{d}} \rightrightarrows \mathfrak{d}$, inversion consists in substituting (β, α) for (α, β).

Proposition 3.2.1, there is a 2-form $\omega \in \Omega^2(M_\Sigma)$ for which (M_Σ, J, ω) is a $L(\Sigma)$-Hamiltonian space for $\hat{\mathbb{A}}$.

Proposition 5.1.1. *The two form $\omega \in \Omega^2(M_\Sigma)$ is trivial.*

Proof. The Lagrangian subalgebra $\mathfrak{l}(P)$ is a product of Lagrangian subalgebras of $\mathfrak{d} \oplus \bar{\mathfrak{d}}$ of the form $\mathfrak{g}_a \oplus \mathfrak{g}_a$ and a number of copies of the Lagrangian subalgebra

$$\mathfrak{d}_{\Delta(1,4)} + \mathfrak{d}_{\Delta(2,3)} = \{(\alpha, \beta, \beta, \alpha) \ : \ \alpha, \beta \in \mathfrak{d}\}.$$

of $(\mathfrak{d} \oplus \bar{\mathfrak{d}})^2$. The $\mathfrak{l}(P)$-orbit $\mathcal{O}(P)$ is therefore the product of the orbits through the group identity of D (or D^2) of those various Lagrangian subalgebras of $\mathfrak{d}\oplus\bar{\mathfrak{d}}$ (resp. $\mathfrak{d}\oplus\bar{\mathfrak{d}}\oplus\mathfrak{d}\oplus\bar{\mathfrak{d}}$) and ω is the product of the 2-forms thereof. As the subalgebras $\mathfrak{g}_a \oplus \mathfrak{g}_a$ are multiplicative, the 2-forms they contribute are trivial according to Corollary 3.2.1. It must thus only be shown that the 2-form associated to the orbit through the group identity $e \in D^2$ of the Lagrangian subalgebra $\mathfrak{d}_{\Delta(1,4)} + \mathfrak{d}_{\Delta(2,3)}$ is trivial. This is a matter of direct computation; it suffices to compute this 2-form at the group identity and for elements $(\alpha, \beta, \beta, \alpha)$, $(\alpha', \beta', \beta', \alpha')$ of $(\mathfrak{d} \oplus \bar{\mathfrak{d}})^2$ one has

$$\begin{aligned}
\langle j_{D^2}^*(\alpha, \beta, \beta, \alpha), \mathbf{a}_{D^2}(\alpha', \beta', \beta', \alpha')\rangle &= \langle(\alpha, \beta, \beta, \alpha), j_{D^2}\, \mathbf{a}_{D^2}(\alpha', \beta', \beta', \alpha')\rangle \\
&= \frac{1}{2}\langle(\alpha, \beta, \beta, \alpha), (\alpha' - \beta', \beta' - \alpha', \beta' - \alpha', \alpha' - \beta')\rangle \\
&= 0,
\end{aligned}$$

which completes the proof. □

Take fusion product of M_Σ is taken along the factors

$$\mathrm{Map}(a_{|\hat{\mathcal{E}}|-1}, D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}})), \mathrm{Map}(a_{|\hat{\mathcal{E}}|-2}, D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}})), \ldots, \mathrm{Map}(a_1, D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}})),$$

in that order. Let C be the coisotropic subalgebra of (5.7) defined by the condition[3] on its elements $(\phi, *) \in C$.

$$\mathbf{s}(\mathrm{pr}_{\mathrm{Map}(a_i, \mathfrak{d}\oplus\bar{\mathfrak{d}})}\, \phi) = \mathbf{t}(\mathrm{pr}_{\mathrm{Map}(a_{i+1}, \mathfrak{d}\oplus\bar{\mathfrak{d}})}\, \phi) \text{ for } i = 1, \ldots, |\hat{\mathcal{E}}| - 2.$$

At the level of the Courant algebroids $\mathbb{T} M_\Sigma$ and $\hat{\mathbb{A}}$, the fusion product just described amounts to the coisotropic reductions by $\varrho(C^\perp)^\perp$ and C respectively, where $\varrho : \mathfrak{l}(P) \to \mathfrak{X}(M_\Sigma)$ is the $\mathfrak{l}(P)$-action. In fact, the reduction of $\mathbb{T} M_\Sigma$ is precisely $\mathbb{T} \boldsymbol{M}_\Sigma$ and the reduction of $\hat{\mathbb{A}}$ is the

[3] Here $\mathbf{s}$ and $\mathbf{t}$ are the source and target maps in the pair groupoid $\mathfrak{d} \oplus \bar{\mathfrak{d}} \rightrightarrows \mathfrak{d}$, respectively.

Courant algebroid $\mathbb{A}$. Since one has

$$F([a_0]) = F([a_{|\hat{\mathcal{E}}|-1}]^{-1}) \cdot F([a_{|\hat{\mathcal{E}}|-2}]^{-1}) \cdots F([a_1])^{-1}$$

for $F \in \boldsymbol{M}_\Sigma$, the reduced moment map of $\boldsymbol{M}_\Sigma$ is the map $\boldsymbol{J}$ defined earlier. On the other hand the reduced Lagrangian subalgebra

$$\frac{\mathfrak{l}(P) \cap C}{\mathfrak{l}(P) \cap C^\perp}$$

coincides with the Lagrangian subalgebra $\mathfrak{l}(\Sigma)$. Denoting the 2-form reduced from ω by $\boldsymbol{\omega} \in \Omega^2(\boldsymbol{M}_\Sigma)$, the upshot is that $(\boldsymbol{M}_\Sigma, \boldsymbol{J}, \boldsymbol{\omega})$ is a $L(\Sigma)$-Hamiltonian space for $\mathbb{A}$. According to Theorem 3.2.4 this 2-form is characterized by

(5.11) $$(F([a_0]), \pi^*\boldsymbol{\omega}) = (F([a_{|\hat{\mathcal{E}}|-1}]^{-1}), 0) \cdot (F([a_{|\hat{\mathcal{E}}|-2}]^{-1}), 0) \cdots (F([a_1]^{-1}), 0),$$

where $\pi : M_\Sigma \to \boldsymbol{M}_\Sigma$ is the quotient map.

Canonicity of the 2-form. Next, it is argued that the 2-form $\boldsymbol{\omega} \in \Omega^2(\boldsymbol{M}_\Sigma)$ does not depend on the cutting of Σ. The case where every boundary component is a boundary circle corresponds to that considered by M. Atiyah [7] in the spirit of his famous paper with R. Bott on the Yang-Mills equations on Riemann surfaces [8]; see also [4]. The more general case considered here can be reduced to the preceding one; this is now explained.

To simplify matters, it is assumed that every boundary segment of Σ is free. Although this contradicts the requirement that no two free segments be adjacent, there is actually no harm in doing so as far as the foregoing constructions are concerned. The Lie algebra $\mathfrak{l}(\Sigma)$ may then be understood as follows. For any boundary component, suppose $a_{i_0}, a_{i_1}, \ldots, a_{i_{l-1}}$ are its segments, listed in CCW order. Then for $\phi \in \mathfrak{l}(\Sigma)$,

$$\mathbf{t}(\mathrm{pr}_{\mathrm{Map}(a_{i_j}, \mathfrak{d}\oplus\bar{\mathfrak{d}})}\, \phi) = \mathbf{s}(\mathrm{pr}_{\mathrm{Map}(a_{i_{j+1}}, \mathfrak{d}\oplus\bar{\mathfrak{d}})}\, \phi) \text{ for } j = 0, \ldots, l-1,$$

where the subscript j is interpreted modulo l. One has $M_\Sigma = \mathrm{Hom}(\Pi_1(\Sigma, \mathcal{V}), D)$. If Σ is an arbitrary coloured surface on the other hand, it may be substituted by a surface Σ' whose boundary segments are all free and as such $M_\Sigma \subseteq M_{\Sigma'}$. A cutting of Σ is also a cutting of Σ' and in view of part (a) of Lemma 3.2.2, the formula (5.11) for Σ is the "pullback" of its counterpart for Σ', meaning it suffices to deal with Σ'.

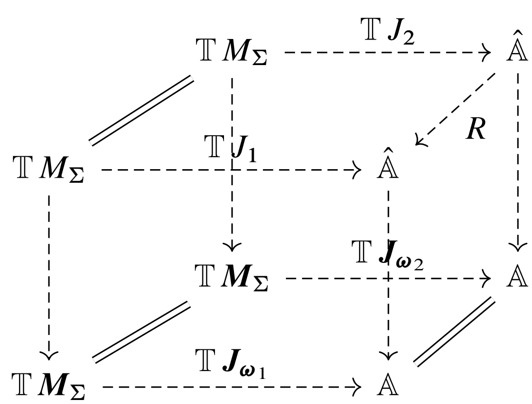
Let $\tilde{\Sigma}$ be the surface obtained by contracting all but one boundary segment in each boundary component of Σ so that $\tilde{\Sigma}$ is a surface where all boundary components are boundary circles with the same genus and number of boundary components as Σ. Denote by $B \subseteq \mathcal{E}$ the set of boundary segments of Σ chosen for contraction. There is a natural embedding of

$$\iota : M_{\tilde{\Sigma}} \hookrightarrow M_{\Sigma} \tag{5.12}$$

whereby the segments in B map identically to the group identity. In terms of (5.12), one has

$$M_{\Sigma} = \mathrm{Map}(B, D) \times M_{\tilde{\Sigma}}. \tag{5.13}$$

Now suppose P_1 and P_2 are two polygons obtained by cutting the surface Σ in possibly different ways. Their respective edges a_0 may be a priori different. However, it follows immediately from part (b) of Lemma 3.2.2 that the right-hand-side of (5.11) augmented with the factor $(F([a_0], 0)$ is invariant under cyclic permutations of its factors, i.e. under admissible rotations of the polygon P. As a consequence, one can assume that a_0 is the same in either case. Subscripts will be added to the objects defined in the previous subsections in function of whether they correspond to P_1 or to P_2 when there is a need to distinguish them. Each edge of P_2 can be expressed as a concatenation of edges of P_1 (and their inverses). According to Theorem 2.4.2, there is an exact Courant morphism $R : \hat{\mathbb{A}} \to \hat{\mathbb{A}}$ lifting the identity morphism $\mathbb{A} \dashrightarrow \mathbb{A}$ whose base map sends $\mathrm{Map}(a, D)$, where $a \in \hat{\mathcal{E}}_2 \setminus \{a_0\}$, to its expression in terms of the edges in of P_1 in $\hat{\mathcal{E}}_2 \setminus \{a_0\}$. The upshot is a commutative diagram



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where the vertical arrows are reduction morphisms. The composition of R and $\mathbb{T} J_2$ is a Courant morphism

$$\mathbb{T} J_{1,\omega} : \mathbb{T} M_\Sigma \dashrightarrow \hat{\mathbb{A}}$$

for some 2-form $\omega \in \Omega^2(M_\Sigma)$. To complete the argument, it must be argued that ω is trivial. By the very fact that $\mathbb{T} J_1$ is a Dirac morphism $(\mathbb{T} M_\Sigma, TM_\Sigma) \dashrightarrow (\hat{\mathbb{A}}, E^{(\iota(P_1))})$ one can conclude that

$$E^{(\iota(P_1))} \circ TJ_{1,\omega} = \mathrm{Gr}(-\omega).$$

An element of $E^{(\iota(P_2))}$ where all the entries corresponding to edges in $\hat{\mathcal{E}}_2 \setminus \mathcal{E}$ (i.e. cuts) are set to 0 is also an element of $E^{(\iota(P_1))}$ where all the entries corresponding to cuts are set to 0. In fact, such elements must be R-related to themselves. In particular, in terms of the identification (5.13), one has $\iota_v \omega = 0$ whenever $v \in T\,\mathrm{Map}(B, D)$.

Finally, a cutting of Σ can be interpreted as a cutting of $\tilde{\Sigma}$ in the obvious way. Let $\tilde{P}_i$ denote the polygon for $\tilde{\Sigma}$ under the cutting of Σ that gives the polygon P_i; the polygon $\tilde{P}_i$ is obtained from P_i by contracting some of its edges. Provided $a_0 \notin B$, the inclusion $\mathcal{O}(\tilde{P}_i) \hookrightarrow \mathcal{O}(P_i)$ is the inclusion (5.12) after the identifications

$$M_\Sigma \simeq \mathcal{O}(P_i), \; M_{\tilde{\Sigma}} \simeq \mathcal{O}(\tilde{P}_i).$$

As a consequence, the pullback of ω to $M_{\tilde{\Sigma}}$ is precisely its counterpart for the two cuttings of Σ interpreted as cuttings for $\tilde{\Sigma}$. Since $\tilde{\Sigma}$ is a surface whose boundary components are all boundary circles, this last 2-form is known to be trivial. The upshot is that ω is indeed trivial and the proof that the 2-form $\boldsymbol{\omega} \in \Omega^2(\boldsymbol{M}_\Sigma)$ is independent of the cutting of Σ is thus complete.

5.1.2 Fusion of coloured surfaces

Let Σ_1 and Σ_2 be two coloured surfacers each having at least one boundary circle. Define their *fusion product* $\Sigma_1 \circledast \Sigma_2$ to be the coloured surface obtained by joining the two surfaces along boundary circles by a "pair of pants", i.e. a 3-holed sphere (Figure 5.3).

Let P_i be glueing polygons for the coloured surfaces Σ_i. Denote the edges of P_1 by $a_0, \dots, a_{m_1}$ and those of P_2 by $b_0, \dots, b_{m_2}$, with the usual convention: a_0 and b_0 are free boundary segments and the edges are listed in CCW order. Form a new polygon $P_1 \circledast P_2$ by joining a_0 and b_0 along a triangle (Figure 5.4). Then $P_1 \circledast P_2$ is a polygon for $\Sigma_1 \circledast \Sigma_2$ corresponding to a certain cutting of the latter.

Let $\boldsymbol{\omega}_1 \in \Omega^2(\boldsymbol{M}_{\Sigma_1})$ and $\boldsymbol{\omega}_2 \in \Omega^2(\boldsymbol{M}_{\Sigma_2})$ be the 2-forms constructed above. Then as a consequence of formula (5.11) one has

Theorem 5.1.2. *The $L(\Sigma_1 \circledast \Sigma_2)$-Hamiltonian space $\boldsymbol{M}_{\Sigma_1 \circledast \Sigma_2}$ is the fusion product $\boldsymbol{M}_{\Sigma_1} \circledast \boldsymbol{M}_{\Sigma_2}$ along the respective first factors of* $\mathrm{Map}(\mathcal{E}_{1,free}, D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}}))$ *and* $\mathrm{Map}(\mathcal{E}_{2,free}, D \times (\mathfrak{d} \oplus \bar{\mathfrak{d}}))$.

Remark 5.1.1. Since any coloured surface can be obtained by joining via pairs of pants number of 1-holed tori equal to the genus of the surface and cylinders each with one boundary circle and one boundary having possibly multiple segments, the computation of the 2-form $\boldsymbol{\omega}$ reduces to computing the corresponding 2-forms for a number of "building blocks", analogously to [4, Thm. 9.3].

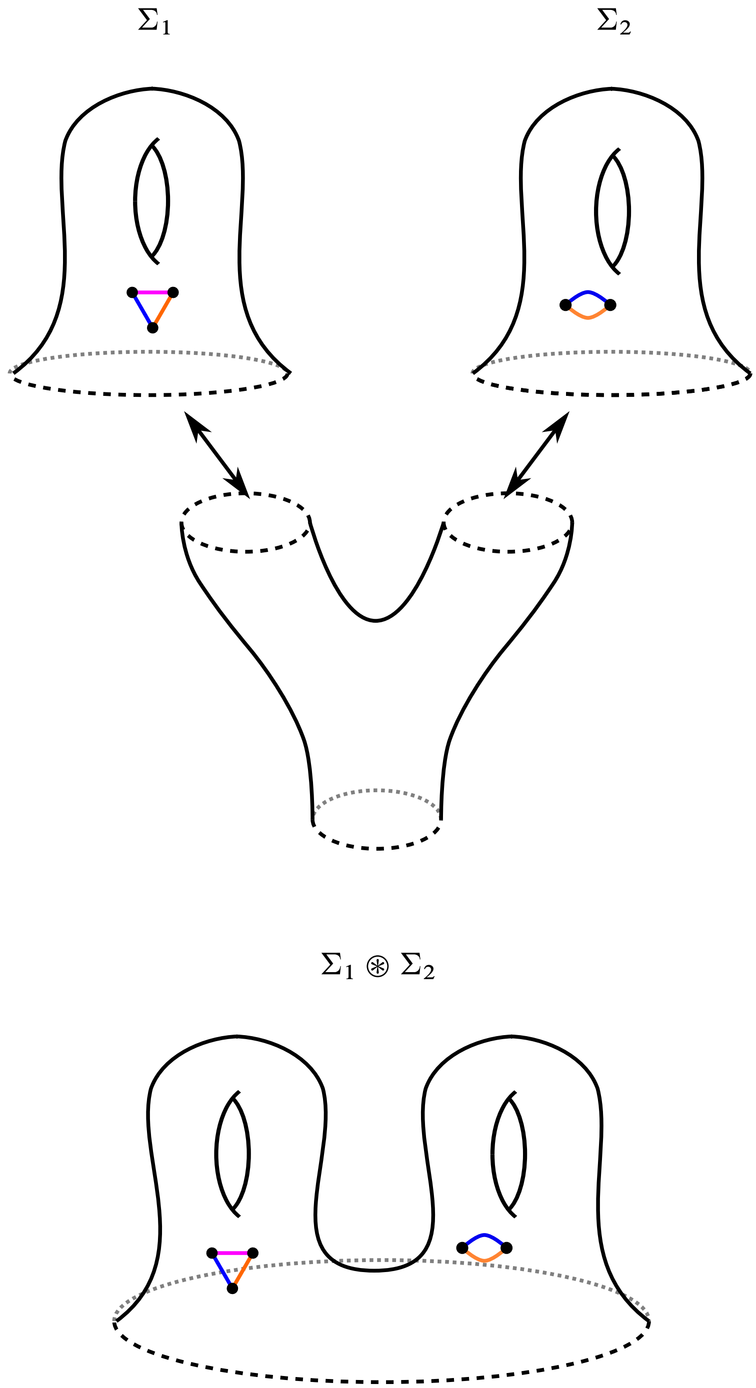

Figure 5.3: Illustration of the fusion of two coloured surfaces.

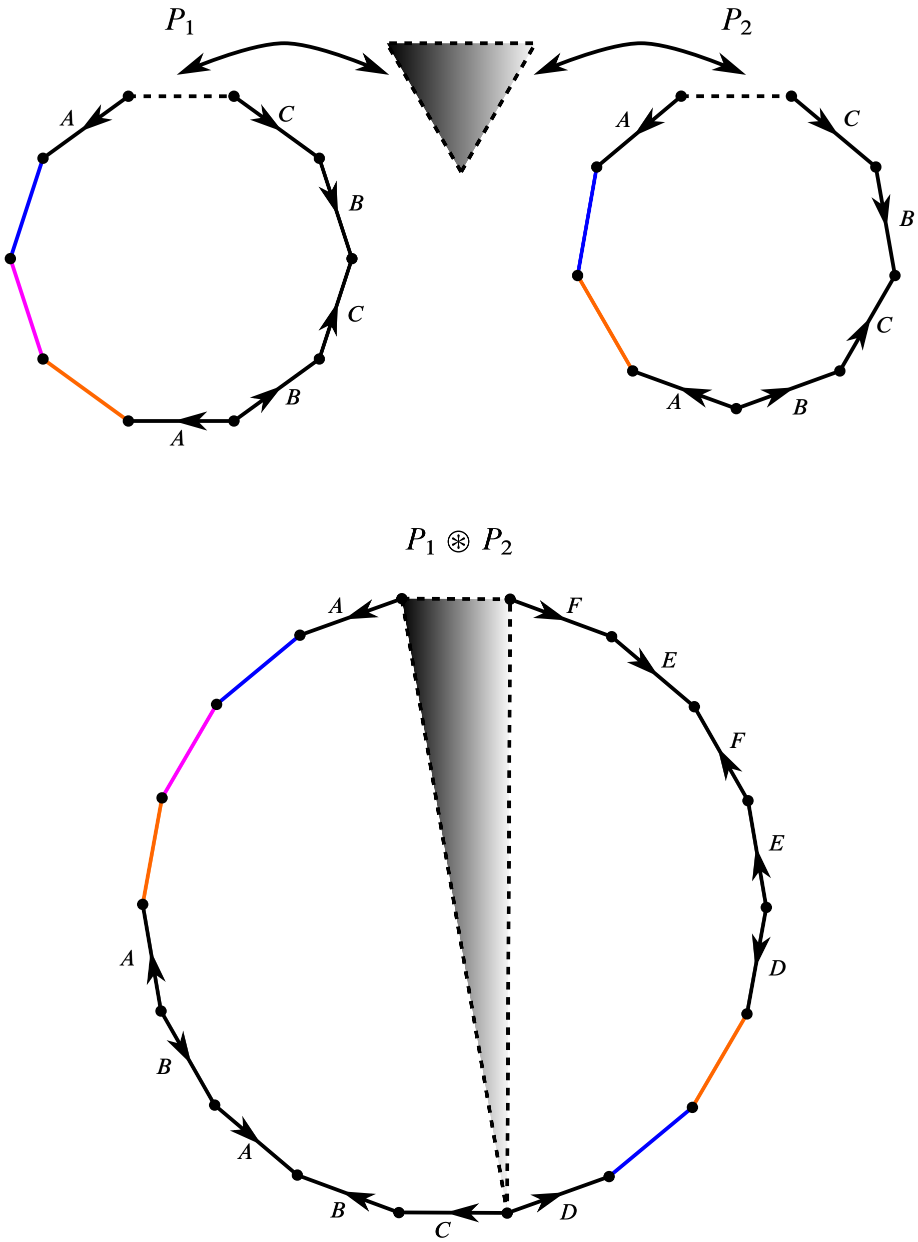

Figure 5.4: The polygon $P_1 \circledast P_2$.

5.1.3 Duality

The dual of the $L(\Sigma)$-Hamiltonian structure on $\boldsymbol{M}(\Sigma)$ corresponds to the Hamiltonian structure obtained via the opposite orientation of Σ. Indeed, according to (5.11) one has for $F \in M_\Sigma$

$$(F([a_0^{-1}]), -\pi^*\boldsymbol{\omega}) = (F([a_1]), 0) \cdot (F([a_2], 0),)) \cdots (F([a_{|\hat{\mathcal{E}}|-1}]), 0).$$

This is formula (5.11) for the glueing polygon P oriented CW; equivalently it is formula (5.11) for the horizontal reflection of P, which is a glueing polygon for Σ with the opposite orientation.

5.1.4 Symplectic structure

Let $\mathcal{B} \hookrightarrow \mathrm{Map}(\mathcal{E}_{\mathrm{free}}, D)$ be an $L(\Sigma)$-orbit. Then, if $\boldsymbol{J}$ is transversal to $\mathcal{B}$, the quotient space $\boldsymbol{J}^{-1}(\mathcal{B})/L(\Sigma)$ inherits a symplectic form according to Theorem 3.3.1. Define

$$\mathcal{M}_\Sigma := \boldsymbol{M}_\Sigma / L(\Sigma).$$

The quotient $\mathcal{M}_\Sigma$ is interpreted as the space of flat D-bundles with prescribed holonomies along the free boundary segments, again up to a suitable notion of gauge transformation.

The connected components of the preimages $\boldsymbol{J}^{-1}(\mathcal{B}_i)$, where i indexes the family of L-orbits in $\mathrm{Map}(\mathcal{E}_{\mathrm{free}}, D)$, foliate $\boldsymbol{M}_\Sigma$. The quotient map $\cdot/L(\Sigma) : \boldsymbol{M}_\Sigma \to \mathcal{M}_\Sigma$ sends the members of this foliation bijectively onto the connected components of $\mathcal{M}_\Sigma$. Thus each connected component of the orbifold $\mathcal{M}_\Sigma$ carries a symplectic form.

5.2 Ševera's coloured surfaces

The surfaces Ševera considers correspond to the coloured surfaces discussed above with the exception that there are no free boundary segment [54] and if a and b are adjacent edges then the Lagrangian subalgebras decorating them must intersect trivial. Such a surfaces can be brought into the fold of the coloured surfaces considered here by removing a disc thereby creating a free boundary segment. The resulting surface will be denoted by Σ'. The moduli space of interest will then correspond to that of coloured flat connections whose holonomy along the free boundary segment (i.e. the boundary of the disc removed) is trivial.

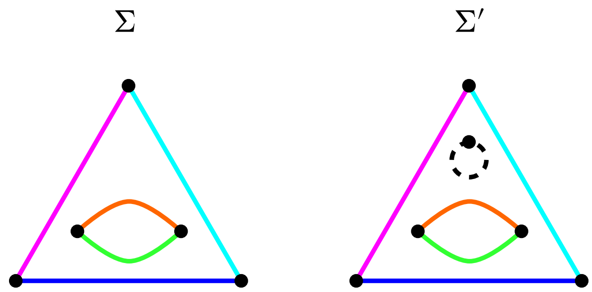

Figure 5.5: As before, colours correspond to Lagrangian subalgebras of $\mathfrak{d}$.

Since the only free boundary segment is a boundary circle, the space $\boldsymbol{M}'_\Sigma$ is a quasi-Hamiltonian space for $D\times(\mathfrak{d}\oplus\overline{\mathfrak{d}})$ according to Theorem 5.1.1. In fact since adjacent coloured edges are decorated with transverse Lagrangian subalgebras, it follows that $M_{\Sigma'} = \boldsymbol{M}_{\Sigma'}$. One is interested in the symplectic reduction

$$\boldsymbol{M}_\Sigma := \boldsymbol{J}^{-1}(e)/D_\Delta.$$

Now cut the coloured surface Σ' into a glueing polygon P' as before (Figure 5.6). Call its edges $a_0,\dots,a_{m-1}$ with the same convention as before, i.e. a_0 is the free boundary circle of Σ' and the edges are listed in CCW order. In terms of the identification

$$\mathcal{O}(P') \simeq M_{\Sigma'} = \boldsymbol{M}_{\Sigma'},$$

the space $\boldsymbol{M}_\Sigma$ admits a section

$$\boldsymbol{M}_\Sigma \hookrightarrow \boldsymbol{J}^{-1}(e) \subseteq \mathcal{O}(P') \simeq \boldsymbol{M}_{\Sigma'} \tag{5.14}$$

whereby an element of $\boldsymbol{M}_\Sigma$ is sent to the element of $\mathcal{O}(P) \simeq \boldsymbol{M}_{\Sigma'}$ whose holonomy along the free boundary segment as well as the holonomies along its two adjacent edges are trivial. The symplectic form $\omega_{\text{symp}} \in \boldsymbol{M}_\Sigma$ descends from the pullback of the 2-form $\boldsymbol{\omega} \in \Omega^2(\boldsymbol{M}_{\Sigma'})$ to $\boldsymbol{J}^{-1}(e)$ according to the proof of Theorem 3.3.1. In terms of the section (5.14), the symplectic form ω_{symp} is the pullback of $\boldsymbol{\omega}$ to $\boldsymbol{M}_\Sigma \hookrightarrow \boldsymbol{J}^{-1}(e)$. According to part (a) of Lemma 3.2.2 and the equation (5.11), one has

$$(e,\omega_{\text{symp}}) = \cancel{(F([a_1]),0)}(F([a_2]),0)\cdots(F([a_{m-2}]),0)\cancel{(F([a_{m-1}]),0)}. \tag{5.15}$$

In this way [54, Thm. 3.1] is rederived.

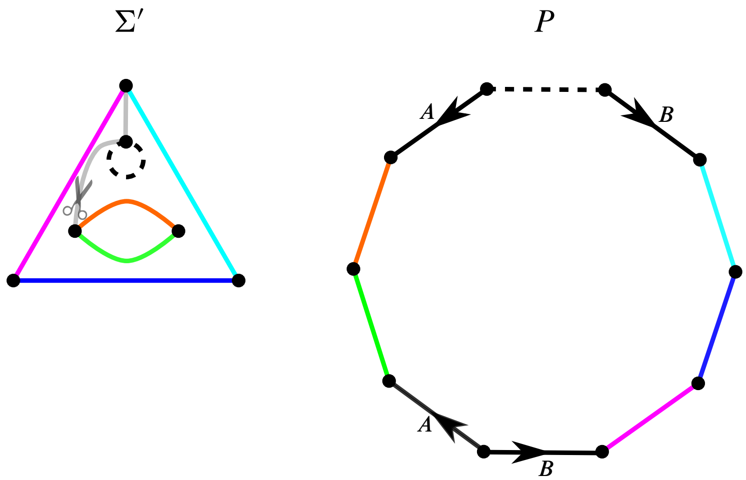

Figure 5.6: Cutting the coloured surface Σ' to obtain a coloured polygon P.

5.3 D/G-valued moment maps examples

Let $\mathfrak{g} \subseteq \mathfrak{d}$ be a Lagrangian subalgebra integrating to the closed and connected subgroup $G \subseteq D$. Consider a coloured surface Σ whose coloured boundary segments are decorated with $\mathfrak{g}$ exclusively and which does not possess a boundary circle (i.e. each free segment is adjacent to a coloured one). Since the boundary segments adjacent to free segments are decorated with $\mathfrak{g}$, the Lagrangian subalgebra $\mathfrak{l}(\Sigma) \subseteq \mathrm{Map}(\mathcal{E}_{\mathrm{free}}, \mathfrak{d} \oplus \overline{\mathfrak{d}})$ is

$$\mathfrak{l}(\Sigma) = \mathrm{Map}(\mathcal{E}_{\mathrm{free}}, \mathfrak{g} \oplus \mathfrak{g}).$$

In particular, the space $\boldsymbol{M}_\Sigma$ is a $G \times G$-Hamiltonian space (or power thereof) and thus provides, after quotienting by the $e \times G$-actions, an example of a Hamiltonian space for $(D/G \times \mathfrak{d}, E^{(\mathfrak{g})})$ (or power thereof). Any such surface may be obtained by joining a number equal to the genus of Σ of 1-holed tori and as many cylinders as Σ has boundary components.

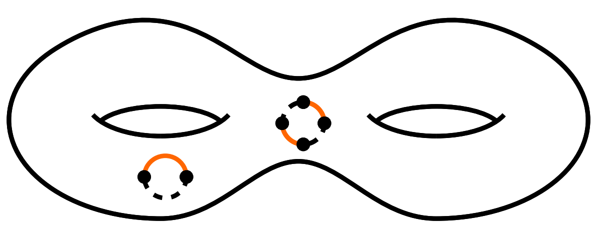

Figure 5.7: Surface where all free boundary segments are adjacent to coloured segments, which are decorated with the same Lagrangian subalgebra.

Suppose some given boundary component of Σ has m segments. The decorations present on the boundary can be encoded by a string of length m

$$\{(s_0, s_1, \ldots, s_{m-1}) \in \{0,1\}^* \ : \ (s_i, s_{i+1}) \neq (1,1) \text{ for } i = 0, \ldots, m-1\},$$

where the subscripts are interpreted modulo m. Entries with 0 correspond to segments decorated by $\mathfrak{g}$ and entries with 1 to free segments. In general, the notation

$$(\Sigma_g^r, s_1, s_2, \ldots, s_r)$$

will be used to denote the coloured surface of genus g and r boundary components with decorations corresponding to the maps $s_1, \ldots, s_r$. Denote by $\boldsymbol{D}(D)$ and $D_s(D)$ respectively the space (5.2) for the 1-holed torus and the cylinder with a free boundary circle and the other boundary component decorated in accordance to the map s (cf. Example 3.2.4). With this notation, the foregoing discussion amounts to the following corollary of Theorems 5.1.1 and 5.1.2:

Theorem 5.3.1. *Let $(\Sigma_g^r, s_1, \ldots, s_r)$ be a coloured surface of the kind considered above. Then the space (5.2) for this surface is a* $\mathrm{Map}(\mathcal{E}_{free}, G \times G)$*-Hamiltonian space and as such is equal to the fusion product*

$$\underbrace{\boldsymbol{D}(D) \circledast \cdots \circledast \boldsymbol{D}(D)}_{g \text{ times}} \circledast D_{s_1}(D) \circledast D_{s_2}(D) \circledast \cdots \circledast D_{s_r}(D),$$

where the fusion product is always taken along factors corresponding to boundary circles.

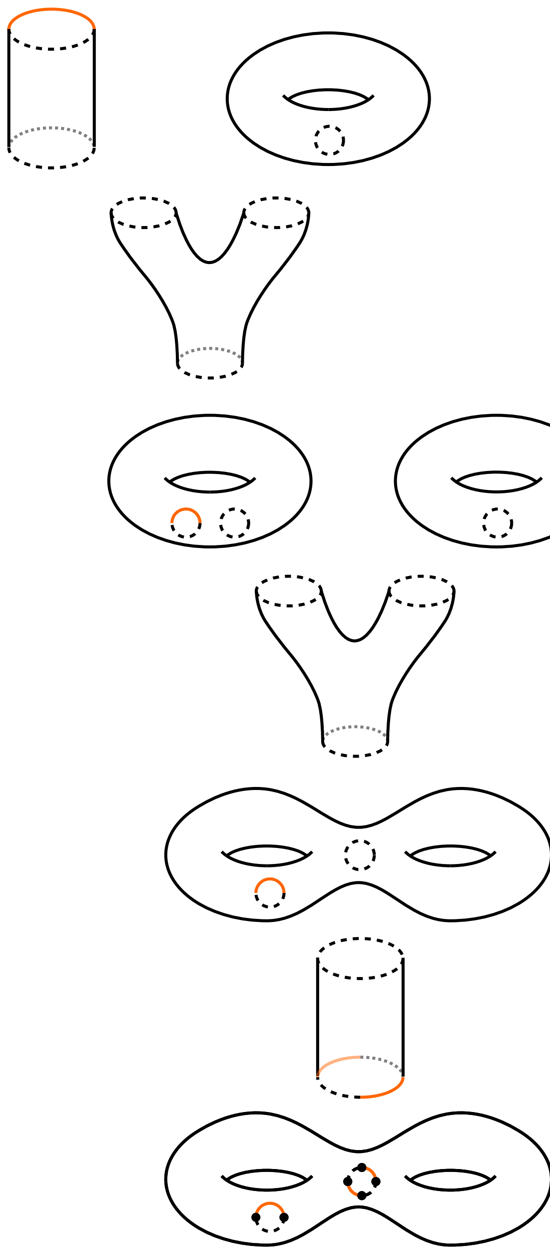

Figure 5.8: Joining 1-holed tori and cylinders to obtain $(\Sigma_2^2, 01, 0101)$.

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