Instability of electroweak homogeneous vacua in strong magnetic fields

by

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Abstract

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We consider the classical (local) vacua of the Weinberg-Salam (WS) model of electroweak forces. These are defined as no-particle, static solutions to the WS equations minimizing the WS energy locally. In the absence of particles, the Weinberg-Salam model reduces to the Yang-Mills-Higgs (YMH) equations for the gauge group $U(2)$.

We consider the WS system in a constant external magnetic field, b, and prove that (i) there is a magnetic field threshold b_* such that for $b < b_*$, the vacua are translationally invariant, while, for $b > b_*$, they are not, (ii) for $b > b_*$, there are non-translationally invariant solutions with lower energy per unit volume and with the discrete translational symmetry of a 2D lattice in the plane transversal to the external magnetic field, and (iii) the lattice minimizing the energy per unit volume approaches the hexagonal one as the magnetic field strength approaches the threshold b_*.

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Contents

Chapter 1

Introduction

The Weinberg-Salam (WS) model of electroweak interactions was the first triumph of the program to unify the four fundamental forces of nature. It is a key part of the standard model of elementary particles. It unifies electromagnetic and weak interactions, two of the three forces dealt with in the standard model. It involves particle fields, gauge fields and the Higgs field.

While the gauge fields describe the electroweak interactions, the role of the Higgs field is to convert the original massless fields (zero masses are required by the relativistic invariance) to massive ones. This phenomenon is called the Higgs mechanism. This mechanism, together with the Goldstone theorem, leads to all gauge particles but one acquiring mass, resulting in two massive bosons – denoted W and Z – and a massless one – the photon. The W and Z particles where discovered experimentally 16 years after their theoretical prediction.

In this paper, we consider the vacuum solutions of the classical WS model. These are static, no-particle solutions minimizing the WS energy locally. For a constant external magnetic field, b, we prove that (i) there is a magnetic field threshold b_* such that for the magnetic fields $b < b_*$, the vacua are translationally invariant, while, for $b > b_*$, they are not, (ii) for $b > b_*$, there are non-translationally invariant solutions with lower energy per unit volume and with the discrete translational symmetry of a 2D lattice in the plane transversal to the external magnetic field, and (iii) the lattice minimizing the energy of the latter solutions per unit volume approaches the hexagonal one as the magnetic field strength approaches the threshold b_*. We expect that these solutions are stable under field fluctuations and, in fact, minimize the energy locally.

The phenomenon above was investigated extensively in the physics literature (see e.g. [12, 17, 23, 24] and the references therein). It is similar to one occurring in superconductivity and the solutions whose existence we establish are analogous to the superconducting Abrikosov vortex lattices ([1], see e.g. [32], for a review). It is estimated in [23] that the spontaneous symmetry breaking take place at the critical external magnetic field of approximately 10^{24} Gauss = 10^{20} Tesla. By comparison, the strongest magnetic field produced on Earth is 10^{14} Tesla (in particle accelerators, the strongest magnetic field is about 10^{11} Tesla).

Note that, in the absence of particles, the WS system reduces to the Yang-Mills-Higgs (YMH) one with the gauge group $U(2)$. So ultimately, these are the equations we deal with.

The only rigorous result ([35, 36]) on the classical WS model deals with the vortices in the self-dual regime, where the WS (or corresponding YMH) equations are equivalent to the first order equations, and it uses this equivalence in an essential way. (The self-dual regime in this context was discovered in [6, 7, 8], see also [33, 34].)

The paper is organized as follows. In Section 2, we formulate the problem and describe results. In Sections 3 - 4, we fix the gauge and pass from the original Yang-Mills fields to the W and Z (massive boson) and A (photon) fields and rescale the resulting equations. The proofs of the main results are given in Section 5 (Theorem 1), Sections 6 - 10 (Theorem 2) and Section 11 (Theorem 3). In Appendix A, we discuss various covariant derivatives used in the main text and in Appendix B, we review the time-dependent YMH equations and derive the expression for the conserved energy as well as the YMH equations used in the main text. In Appendix C, we write the YMH equations in the coordinate form and derive a convenient expression for the energy functional, and in Appendices D - E, we derive the WS equations in 3D and 2D, respectively, in terms of the fields W, Z, A and φ. In the remaining appendices, we carry out technical computations.

Chapter 2

The vacuum sector of Weinberg-Salam model

The vacuum sector of the Weinberg-Salam model involves the interacting, static Higgs and $SU(2)$ and $U(1)$ gauge fields, Φ and V and X, while the particle fields are set to zero. The field Φ is a vector-function defined on the physical space $\mathbb{R}^3$ with values in $\mathbb{C}^2$, and the fields V and X are one-forms on $\mathbb{R}^3$ with values in the algebras $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$, respectively. We write $Q = V + X$, which is a one-form with values in $\mathfrak{u}(2)$. We consider $SU(2)$ as a matrix group and $U(1)$ as multiples of the identity matrix $\mathbf{1}$ acting on $\mathbb{C}^2$. Geometrically, V, X and Q can be thought of as connections on the trivial bundles $\mathbb{R}^3 \times SU(2), \mathbb{R}^3 \times U(1)$ and $\mathbb{R}^3 \times U(2)$.

These fields satisfy the static YMH equations, which are the Euler-Lagrange equations for the energy functional[1]

$$E_T(\Phi, Q) := \|\nabla_Q \Phi\|^2_{\Omega^1_{\mathbb{C}^2}} + \frac{1}{2}\lambda(\|\Phi\|^2_{L^2} - \varphi_0^2)^2 + \frac{1}{2}\|F_Q\|^2_{\Omega^2_{\mathfrak{u}(2)}}, \tag{2.1}$$

where T is a bounded domain in $\mathbb{R}^3$ with appropriate boundary conditions (specified in (2.14) below), λ and φ_0 are positive parameters, and the remaining symbols are defined as follows:

∇_Q is the covariant derivative mapping $\mathbb{C}^2$-valued functions (sections) into $\mathbb{C}^2$-valued one-forms defined as

$$\nabla_Q = d + gV + g'X, \tag{2.2}$$

with the coupling constants g and g' and d, the exterior derivative;

F_Q is the curvature form of the one-form Q, viewed as a connection, i.e. the $\mathfrak{u}(2)$-valued two-form given by

$$F_Q = dQ + \frac{1}{2}[Q, Q], \tag{2.3}$$

where $[A, B]$ is defined in local coordinates $\{x^i\}$ as

$$[A, B] := [A_i, B_j]dx^i \wedge dx^j = [B, A], \tag{2.4}$$

with $A = A_i dx^i$ and $B = B_j dx^j$. Here and in what follows we use the convention of summing over repeated indices.

[1] For discussion of the the time-dependent theory and a derivation of the energy functional (2.1) see [21], [24], [30], [31] and Appendix B.

$\|\cdot\|_{L^2}$ is the standard norm on $L^2(T,\mathbb{C}^2)$ and $\|\cdot\|_{\Omega_U^p}$ is the norm on the space $\Omega_U^p := U\otimes\Omega^p(T)$ of U-valued p-forms on T (e.g. for $B = B_i dx^i \in \Omega_U^1$, we have $\|B\|_{\Omega_U^1} := (\int_T \sum_i \|B_i\|_U^2)^{1/2}$). [2]

Since $Q = V + X$ and X has the values in the center, $u(1)$, of the algebra $u(2)$, we have $F_Q = F_V + F_X$, where

$$F_V := dV + \frac{g}{2}[V,V] \quad \text{and} \quad F_X := dX \tag{2.5}$$

are the curvatures of the connections V and X.[3] Note also that $\|F_Q\|^2_{\Omega^2_{\mathfrak{u}(2)}} = \|F_V\|^2_{\Omega^2_{\mathfrak{u}(2)}} + \|F_X\|^2_{\Omega^2_{\mathfrak{u}(1)}}$.

We introduce the covariant derivative d_Q mapping $\mathfrak{u}(2)$-valued one-forms into $\mathfrak{u}(2)$-valued two-forms as[4]

$$d_Q B = d_V B := dB + g[V,B]. \tag{2.6}$$

The Euler-Lagrange equations for energy functional (2.1) are given by (see Appendix B)

$$\nabla_Q^*\nabla_Q\Phi = \lambda(\varphi_0^2 - ||\Phi||^2)\Phi, \tag{2.7}$$

$$d_Q^* F_Q = J(\Phi, Q), \tag{2.8}$$

where ∇_Q^* is the adjoint of ∇_Q and maps $\mathbb{C}^2$-valued one-forms into $\mathbb{C}^2$-valued functions, d_Q^* is the adjoint of d_Q and maps $\mathfrak{u}(2)$-valued two-forms into $\mathfrak{u}(2)$-valued one-forms, and $J(\Phi,Q)$ is the electroweak current, which is the $\mathfrak{u}(2)$-valued one-form given by

$$J(\Phi,Q) := -\frac{ig}{2}\tau_a \operatorname{Im}\langle\tau_a\Phi, \nabla_Q\Phi\rangle - \frac{ig'}{2}\tau_0\operatorname{Im}\langle\tau_0\Phi,\nabla_Q\Phi\rangle, \tag{2.9}$$

where summing over repeated indices is understood, $\tau_0 := \mathbf{1}$ and $\tau_a, a = 1,2,3$, are the Pauli matrices,

$$\tau_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \tau_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \tau_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.10}$$

(The Pauli matrices, multiplied by $-i/2$, form an orthonormal basis in $\mathfrak{su}(2)$ with the inner product $\langle g,h\rangle_{\mathfrak{su}(2)} := 2\operatorname{Tr}(g^*h) = -2\operatorname{Tr}(gh)$).

The energy functional (2.1) and Euler-Lagrange equations (2.7) - (2.8) are invariant under the group of rigid motions and the gauge transformations (gauge symmetry)

$$(\Phi(x), V(x), X(x)) \mapsto (h_1(x)h_2(x)\Phi(x), h_1(x)V(x)h_1^{-1}(x)$$

$$-i\frac{2}{g}h_1(x)dh_1^{-1}(x), X(x) - i\frac{2}{g'}h_2(x)dh_2^{-1}(x)), \tag{2.11}$$

$$\forall h_1(x)\in SU(2),\ h_2(x)\in U(1). \tag{2.12}$$

Equations (2.7) - (2.8) have the simple solution given (up to a gauge symmetry) by

$$U_*^b := (\Phi_0, -\frac{i}{2}\tau_3 A^b\sin\theta, -\frac{i}{2}\tau_0 A^b\cos\theta), \tag{2.13}$$

[2] The inner products in the vector spaces of $U-$valued differential forms (with $U = \mathbb{C}^2$ or $\mathfrak{u}(2)$) is given by $\langle A,B\rangle := \int_T\langle A_\alpha, B^\alpha\rangle_U$, where $A = A_\alpha dx^\alpha$ and $B_\alpha dx^\alpha$ are U-vauled n-forms, α is an n-index and $\langle\cdot,\cdot\rangle_U$ is an inner product in U (the summation over repeated indices is understood). For $\mathfrak{u}(2)$, the inner product is given by $\langle g,h\rangle_{\mathfrak{u}(2)} := 2\operatorname{Tr}(g^*h) = -2\operatorname{Tr}(gh)$.

[3] For more discussion of covariant derivatives and their curvatures, see Appendix A for the general case, or Appendix C for the case of the gauge group $G = U(2) = SU(2)\times U(1)$.

[4] This formula originates in the equation $(\delta_Q F_Q)(B) = d_Q B$, where δ_Q is the Gâteaux derivative with respect to Q.

where $\Phi_0 := (0, \varphi_0)$, $A^b(x)$ is a magnetic potential in the constant magnetic field of strength b in the x_3-direction, $dA^b(x) = bdx_1 \wedge dx_2$, and θ is *Weinberg's angle*, given by $\tan\theta = g'/g$.[5] This solution is gauge-translationally invariant, i.e. invariant under translations up to a gauge symmetry. It corresponds to the 'total vacuum' in the constant external magnetic field $bdx_1 \wedge dx_2$.

In this paper, we consider solutions for which the fields do not depend on the coordinate x^3 and therefore the problem is reduced to a 2D one.

We show that amongst 2D solutions, (i) (2.13) is linearly stable for $b < b_*$ and unstable for $b > b_*$, where $b_* := g^2\varphi_0^2/2e$, (ii) a new solution breaking the gauge-translational invariance bifurcates. This solution has the discrete translational symmetry of a lattice and has lower energy per unit area. Finally, we show also that (iii) the lattice shape minimizing the energy per unit area approaches the hexagonal lattice as b approaches b_*.

To formulate these results precisely, we introduce some definitions. We fix a lattice $\mathcal{L}$ in $\mathbb{R}^2$ and say a triple $(\Phi(x), V(x), X(x))$ is *$\mathcal{L}$-gauge-periodic*, or, *$\mathcal{L}$-equivariant*, if and only if it satisfies the equation

$$(T^{gauge}_{\gamma_s})^{-1} T^{trans}_s (\Phi, V, X) = (\Phi, V, X), \qquad \forall s \in \mathcal{L}, \tag{2.14}$$

for some $\gamma_s \in C^1(\mathbb{R}^2, SU(2) \times U(1))$, where T^{gauge}_γ is given by the right-hand side of (2.11), with $h_1(x)h_2(x) = \gamma(x)$, and T^{trans}_s is the group of translations, $T^{trans}_s f(x) = f(x+s)$. (When $\mathcal{L}$ is clear, we omit it from the definition above.)

We denote by $\mathcal{H}^s_{\mathcal{L}}$ the space of locally Sobolev class $\mathcal{L}$-equivariant triples (Φ, V, X) on $\mathbb{R}^2$ with the inner product given by the standard Sobolev inner product restricted to an arbitrary fundamental domain Ω of $\mathcal{L}$, and let $L^2_{\mathcal{L}} = \mathcal{H}^0_{\mathcal{L}}$.

We say a solution $U_* := (\Phi_*, V_*, X_*)$ of equation (2.7) - (2.8) is *linearly stable* (respectively *unstable*) if and only if the spectrum of the linearized operator for the Weinberg-Salam equations at $U_* := (\Phi_*, V_*, X_*)$ on $L^2_{\mathcal{L}}$ is positive (respectively has a negative piece). (This operator has the real spectrum.) A linearly stable solution is a local minimum of the energy functional (2.1).

For an $\mathcal{L}$-equivariant triple U and a fundamental domain Ω of $\mathcal{L}$, we define the energy per fundamental cell

$$E^{\mathcal{L}}(U) := \frac{1}{|\Omega|} E^{WS}_\Omega(U), \tag{2.15}$$

where $|\Omega|$ denotes the area of Ω. This energy is independent of Ω.

In what follows, Ω denotes an arbitrary (but fixed throughout) fundamental domain of $\mathcal{L}$, and $|\mathcal{L}|$, the area of a fundamental cell of $\mathcal{L}$, which is independent of the choice of the cell (and in particular, $|\mathcal{L}| = |\Omega|$).

Finally, let $M_W := \frac{1}{\sqrt{2}} g\varphi_0$, $M_Z := \frac{1}{\sqrt{2}\cos\theta} g\varphi_0$ and $M_H := \sqrt{2}\lambda\varphi_0$. These are the masses of the W, Z and Higgs bosons, respectively.[6] Let

$$b_* := \frac{g^2\varphi_0^2}{2e} = \frac{M_W^2}{e}, \qquad e := g\sin\theta. \tag{2.16}$$

With the above definitions, we will prove the following:

[5] Indeed, $d_Q\Phi_0 = (gV + g'X)\Phi_0 = (gA^b \sin\theta\tau_3 + g'A^b\cos\theta\tau_0)\Phi_0 = g'A^b\cos\theta(\tau_3+\tau_0)\Phi_0$. Since $(\tau_3+\tau_0)\Phi_0 = 0$, this implies $d_Q\Phi_0 = 0$. From $d_Q\Phi_0 = 0$, it is easy to see that (2.13) solves (2.7) - (2.8).

[6] This nomenclature will be explained in the discussion following equation (3.10).

Theorem 1. *The homogeneous vacuum solution* (2.13) *is linearly stable for* $b < b_*$ *and unstable for* $b > b_*$.

Theorem 2. *Let* $\mathcal{L}$ *be a lattice satisfying* $0 < 1 - \frac{M_W^2}{2\pi}|\mathcal{L}| \ll 1$ *and assume that* $M_Z < M_H$.[7] *Then the following holds:*

(a) Equations (2.7) - (2.8) *have a non-trivial solution* $U_{\mathcal{L}} \in \mathcal{H}^2_{\mathcal{L}}$ *in a neighbourhood of the vacuum solution* (2.13)*;*

(b) $U_{\mathcal{L}}$ *is unique, up to gauge symmetry, in a neighbourhood of the vacuum solution* (2.13)*;*

(c) $U_{\mathcal{L}}$ *has energy per unit area less than the vacuum solution* (2.13)*:* $E^{\mathcal{L}}(U_{\mathcal{L}}) < E^{\mathcal{L}}(U_*^b)$.

The solutions described in this theorem can be reinterpreted geometrically as representing a section ($\Phi(x)$) and a connection (($V(x), X(x)$)) on a vector bundle over a torus (cf. [18]). In the present situation of the gauge group $U(2)$, it is natural to consider vector bundles over a torus. However, vector bundles over a torus are topologically equivalent to direct sums of line bundles. In our case, this equivalence follows from equations (3.5) - (3.7) below.

For the next result, we introduce the standard parameterization of lattices in $\mathbb{R}^2$. Identifying $\mathbb{R}^2$ with $\mathbb{C}$ via $(x_1, x_2) \leftrightarrow x_1 + ix_2$, we can view a lattice $\mathcal{L} \subset \mathbb{R}^2$ as a subset of $\mathbb{C}$. It is a well-known fact (see e.g. [4]) that any lattice $\mathcal{L} \subset \mathbb{C}$ can be given a basis r, r' such that the ratio $\tau = \frac{r'}{r}$ belongs to the set

$$\{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0,\ |\tau| \geq 1,\ -\frac{1}{2} < \operatorname{Re}\tau \leq \frac{1}{2}\}, \tag{2.17}$$

which is the fundamental domain, $\mathbb{H}/SL(2,\mathbb{Z})$, of the modular group $SL(2,\mathbb{Z})$ acting on the Poincaré half-plane $\mathbb{H}$. For a given $\mathcal{L}$, the parameter τ is unique and is used as a parameterization (up to scaling) of the lattices. This gives the space of (normalized) lattices a topology.

Theorem 3. *For* $M_Z < M_H$*, the lattice shape for which the average energy per lattice fundamental domain is minimized approaches the hexagonal lattice as* $b \to b_*$ *in the sense that the shape parameter* τ *of the lattice* $\mathcal{L}$ *approaches* $\tau_{hexagonal} = e^{i\pi/3}$ *in* $\mathbb{C}$.

Our approached is based on a careful examination of the linearization of the WS equations on the homogeneous vacuum. The spectrum of the linearized problem determines the domains of the linear, or energetic, stability and the transition threshold. In the instability domain, we apply an equivariant bifurcation theory. Though main steps of this approach are fairly standard, there are many subtle points to be dealt with. This gives Theorem 2(a) and (b). For Theorems 2(c) and 3, we carefully study the asymptotic behaviour of the energy functions for small values of the bifurcation parameter.

[7] This assumption is justified experimentally since $M_Z = 91.1876 \pm 0.0021 GeV/c^2$ [15] and $M_H = 125.09 \pm 0.31 GeV/c^2$ [13]

Chapter 3

Gauge fixing and W and Z bosons

In this section, we choose a particular gauge and pass from the fields (one-forms) V and X to more suitable gauge fields. We eliminate a part of the gauge freedom by assuming that the Higgs field Φ is of the form

$$\Phi = (0, \varphi), \tag{3.1}$$

with φ real (this can be done using only the $SU(2)$ part of the gauge group). Then

$$\tau_a \Phi \neq 0, \ a = 0, 1, 2, 3, \tag{3.2}$$

where, recall, $\tau_a, \ a = 1, 2, 3$, are the Pauli matrices generating the Lie algebra $su(2)$, and $\tau_0 = \mathbf{1}$. However, there is one linear combination of τ_a's (unique up to a scalar multiple) which annihilates Φ:

$$(\tau_3 + \tau_0)\Phi = 0. \tag{3.3}$$

Thus, for the gauge $\Phi = (0, \varphi)$ the symmetries generated by $\tau_1, \tau_2, \tau_3 - \tau_0$ are broken and the $U(1)$ symmetry generated by $\tau_3 + \tau_0$ remains unbroken. The unbroken gauge symmetry is given by transformations (2.11) with

$$h_1(x) := e^{-\frac{i}{2}\gamma(x)\tau_3} \in SU(2), \ h_2(x) := e^{-\frac{i}{2}\gamma(x)\tau_0} \in U(1), \tag{3.4}$$

where $\gamma \in C^1(\mathbb{R}^3, \mathbb{R})$.

Continuing in the gauge $\Phi = (0, \varphi)$ and writing $V = -\frac{i}{2}\tau_a V^a$ and $X = -\frac{i}{2}\tau_0 X^0$, where X^0 and $V^a, a = 1, 2, 3$, are real fields (since V takes values in $su(2)$), we pass to the new fields corresponding to the broken and unbroken generators, $\tau_3 - \tau_0$ and $\tau_3 + \tau_0$, respectively:

$$Z = V^3 \cos\theta - X^0 \sin\theta \quad \text{and} \quad A = V^3 \sin\theta + X^0 \cos\theta, \tag{3.5}$$

where, recall, θ is Weinberg's angle, defined by $\tan\theta = g'/g$. Note that Z and A are real fields (real one-forms). Moreover, it is convenient to pass from the remaining two components, V^1, V^2, of V to a single complex field (complex one-form):

$$W = \frac{1}{\sqrt{2}}(V^1 - iV^2). \tag{3.6}$$

The gauge invariance under (2.11) of the original field equations with the unbroken gauge symmetry given by transformations (2.11) with (3.4) leads to the invariance under following gauge transformations:

$$\tilde{T}^{gauge}_{\gamma} : (W, A, Z, \varphi) \mapsto (e^{i\gamma}W, A - \frac{1}{e}d\gamma, Z, \varphi), \tag{3.7}$$

for $\gamma \in C^1(\mathbb{R}^3, \mathbb{R})$, where, as usual, $e^{i\gamma}W = \sum e^{i\gamma}W_i dx^i$ for $W = \sum W_i dx^i$, $e := g\sin\theta$ ($= g'\cos\theta = \frac{gg'}{\sqrt{g^2+g'^2}}$) is the electron charge.

In terms of W, Z and A fields, the vacua (2.13) of the Weinberg-Salam model become (up to a gauge symmetry):

$$(0, A^b(x), 0, \varphi_0), \tag{3.8}$$

where, recall, $A^b(x)$ is a magnetic potential for the constant magnetic field of strength b in the x^3-direction, $dA^b(x) = bdx_1 \wedge dx_2$, and φ_0 is a positive constant from (2.1) ($\Phi_0 := (0, \varphi_0)$). We choose the gauge so that $A^b(x)$ is of the form

$$A^b(x) = \frac{b}{2}(-x_2 dx_1 + x_1 dx_2). \tag{3.9}$$

We will show that for a large magnetic field b, these homogeneous vacua become unstable and new, inhomogeneous vacua emerge from them. This is a bifurcation problem from the branch of gauge-translationally invariant (homogeneous) solutions, (3.8).

From now on we consider the Weinberg-Salam (WS) model in $\mathbb{R}^2$ with the fields independent of the third dimension x_3, and correspondingly choose the gauge with $V_3 = X_3 = 0$ (and hence $W_3 = A_3 = Z_3 = 0$).

Also, we will work in a fixed coordinate system, $\{x_i\}_{i=1}^2$ and write the fields as $W = W_i dx^i$, $Z = Z_i dx^i$ and $A = A_i dx^i$. For ease of comparing our arguments with earlier results, and given that we use the standard Euclidean metric in $\mathbb{R}^2$, we identify (complex) one-forms W, Z and A with the (complex) vector fields $(W_1, W_2), (Z_1, Z_2)$ and (A_1, A_2). With this, we show in Appendix E that in this case Weinberg-Salam energy functional (2.1) can be written as

$$\begin{aligned} E^{WS}_{\Omega}(W, A, Z, \varphi) = \int_{\Omega} \Big[&|\operatorname{curl}_{gV^3} W|^2 + \frac{1}{2}|\operatorname{curl} Z|^2 + \frac{1}{2}|\operatorname{curl} A|^2 \\ &+ \frac{1}{2}g^2\varphi^2|W|^2 + \frac{1}{2}\kappa g^2\varphi^2|Z|^2 + \frac{g^2}{2}|\overline{W} \times W|^2 \\ &+ ig(\operatorname{curl} V^3)\overline{W} \times W + |\nabla\varphi|^2 + \frac{1}{2}\lambda(\varphi^2 - \varphi_0^2)^2 \Big], \end{aligned} \tag{3.10}$$

where $\kappa := \frac{g^2}{2\cos^2\theta}$, $\operatorname{curl}_U W := \nabla_1 W_2 - \nabla_2 W_1$, $\nabla_i := \partial_i - iU_i$ (for a $\mathfrak{u}(1)-$valued vector-field U), $\xi \times \eta := \xi_1\eta_2 - \xi_2\eta_1$, $\operatorname{curl} V^3 := \partial_1 V^3_2 - \partial_2 V^3_1$ and recall, $V^3 = Z\cos\theta + A\sin\theta$.

Expanding (3.10) in φ around φ_0, we see that the W, Z and ϕ (Higgs) fields have the masses $M_W := \frac{1}{\sqrt{2}}g\varphi_0$, $M_Z := \frac{1}{\sqrt{2}\cos\theta}g\varphi_0$ and $M_H = 2\sqrt{\lambda}\varphi_0$, respectively.

Using the relation $\xi \times \eta = J\xi \cdot \eta$, where J is the symplectic matrix,

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3.11}$$

we find the Euler-Lagrange equations for (3.10), which give the Weinberg-Salam equations in 2D in terms of the fields W, A, Z and φ

$$[\mathrm{curl}^*_{gV^3}\,\mathrm{curl}_{gV^3} + \frac{g^2}{2}\varphi^2 - ig(\mathrm{curl}\,V^3)J + g^2(\overline{W}\times W)J]W = 0, \tag{3.12}$$

$$\mathrm{curl}^*\,\mathrm{curl}\,A + 2e\,\mathrm{Im}[(\mathrm{curl}_{gV^3}\,W)J\overline{W} - \mathrm{curl}^*(\overline{W}_1 W_2)] = 0, \tag{3.13}$$

$$[\mathrm{curl}^*\,\mathrm{curl}{+}\kappa\varphi^2]Z + 2g\cos\theta\,\mathrm{Im}[(\mathrm{curl}_{gV^3}\,W)J\overline{W} - \mathrm{curl}^*(\overline{W}_1 W_2)] = 0, \tag{3.14}$$

$$[-\Delta + \lambda(\varphi^2 - \varphi_0^2) + \frac{g^2}{2}|W|^2 + \frac{1}{2}\kappa|Z|^2]\varphi = 0, \tag{3.15}$$

where, recall, $\kappa = \frac{g^2}{2\cos^2\theta}$, $V^3 = Z\cos\theta + A\sin\theta$ and Δ is the standard Laplacian. (For a derivation of (3.15) - (3.12) from (3.10), see Appendix E and also [23, 35].) Of course, (3.15) - (3.12) can also be derived directly from Equations (2.7) - (2.8).

In terms of the (W, A, Z, φ) fields, the gauge - periodicity (2.14) is expressed as

$$(\tilde{T}^{gauge}_{\gamma_s})^{-1} T^{trans}_s (W, A, Z, \varphi) = (W, A, Z, \varphi), \tag{3.16}$$

for all $s \in \mathcal{L}$, where $\gamma_s \in C^1(\mathbb{R}^2, \mathbb{R})$ for all $s \in \mathcal{L}$, and T^{trans}_s is the group of translations, $T^{trans}_s f(x) = f(x+s)$. We say that (W, A, Z, φ) is an *$\mathcal{L}$-equivariant* state. By evaluating the effect of translation by $s+t$ in two different ways, we see that the family of functions γ_s has the co-cycle property[1]

$$\gamma_{s+t}(x) - \gamma_s(x+t) - \gamma_t(x) \in 2\pi\mathbb{Z}, \qquad \forall s, t \in \mathcal{L}. \tag{3.17}$$

Since T^{trans}_s is an Abelian group, the co-cycle condition (3.17) implies that, for any basis $\{j_1, j_2\}$ in $\mathcal{L}$, the quantity

$$c(\gamma_s) = \frac{1}{2\pi}(\gamma_{j_2}(x+j_1) + \gamma_{j_1}(x) - \gamma_{j_1}(x+j_2) - \gamma_{j_2}(x)) \tag{3.18}$$

is independent of x and of the choice of the basis $\{j_1, j_2\}$, and is an integer. This topological invariant is equal to the degree of the corresponding line bundle.

One can show using Stokes' Theorem, for any A satisfying (3.16) - (3.18), the magnetic flux through any fundamental domain Ω of the lattice $\mathcal{L}$ is quantized:

$$\frac{e}{2\pi}\int_\Omega dA = n, \tag{3.19}$$

where e is defined after (3.7) and $n = c(\gamma_s) \in \mathbb{Z}$ defined in (3.18). The left-hand side of (3.19) is called the *Chern number* of the line bundle corresponding to γ_s. (We note that n is independent of the choice of Ω.)

The vacuum state (3.8) is $\mathcal{L}$-equivariant if and only if the magnetic field b is given by the relation

$$b = \frac{2\pi}{e|\mathcal{L}|}n, \tag{3.20}$$

where, by definition, $|\mathcal{L}| = |\Omega|$ for any fundamental cell Ω. In particular, b is quantized. For such b, the vector field $\frac{1}{e}A^b$ satisfies (3.19).

Furthermore, due to the reflection symmetry of the problem, we may assume that $b \geq 0$. Clearly, we

[1] A function $\gamma_s : \mathcal{L} \times \mathbb{R}^2 \to G$ satisfying the co-cycle property (3.17) is called the automorphy exponent and $e^{i\gamma_s}$, the automorphy factor.

have:

Lemma 4. *Equations* (2.7) - (2.8) *for* $\mathcal{L}$-equivariant *fields* (2.14) *in the gauge* $\Phi = (0, \varphi)$ *are equivalent to Equations* (3.12) - (3.15) *for* $\mathcal{L}$-equivariant *fields* (3.16)*, with the equivalence realized by the transformation* (3.5) - (3.6)*.*

Finally, we use the invariance of (3.12) - (3.15) under the gauge transformation (3.7) to choose a convenient gauge for the fields $W(x)$ and $A(x)$. We say that the fields (W, A, Z, φ) and (W', A', Z', φ') are *gauge-equivalent* if there is $\gamma \in C^1(\mathbb{R}^2, \mathbb{R})$ such that $(W', A', Z', \varphi') = \tilde{T}^{gauge}_\gamma(W, A, Z, \varphi)$. Clearly, if (W, A, Z, φ) and (W', A', Z', φ') are gauge-equivalent then (W, A, Z, φ) solves (3.12) - (3.15) if and only if (W', A', Z', φ') solves (3.12) - (3.15). The following proposition was first used in [27] and proven in [37] (an alternate proof is given in Appendix A of [38]):

Proposition 5. *Let* (W', A', Z', φ') *be an* $\mathcal{L}$*-equivariant state and let* b *be given by* (3.20)*. Then there is an* $\mathcal{L}$*-equivariant state* (W, A, Z, φ)*, gauge-equivalent to* (W', A', Z', φ')*, which satisfies* (3.16)*, with* $\chi_s(x) = \frac{eb}{2} s \wedge x + k_s$*, i.e. such that*

$$W(x+s) = e^{i(\frac{eb}{2} s \wedge x + k_s)} W(x), \tag{3.21}$$

$$A(x+s) = A(x) + \frac{b}{2} J s \ \forall s \in \mathcal{L}, \tag{3.22}$$

$$\operatorname{div} A = 0. \tag{3.23}$$

Here k_s *satisfies the condition* $k_{s+t} - k_s - k_t - \frac{eb}{2} s \wedge t \in 2\pi\mathbb{Z}$*, for all* $s, t \in \mathcal{L}$*, the matrix* J *is given in* (3.11)*.*

This gauge is consistent with the gauge chosen for the homogeneous vacua (3.9).

Our goal is to prove the instability of the vacuum state (3.8) and the existence of $\mathcal{L}-$equivariant (in the sense of (3.16)) solutions to Equations (3.12) - (3.15) having the properties described in Theorems 2 and 3.

Chapter 4

Rescaling

In this section, we rescale the Weinberg-Salam Equations (3.12) - (3.15) to keep the lattice size fixed. Specifically, we define the rescaled fields (w, z, a, ϕ) to be

$$(w(x), a(x), z(x), \phi(x)) := (rW(rx), rA(rx), rZ(rx), r\varphi(rx)), \tag{4.1}$$

$$r := \sqrt{\frac{n}{eb}} = \sqrt{\frac{|\Omega|}{2\pi}}. \tag{4.2}$$

where in the second equality (4.2), we used (3.20). Clearly, $(W(x), A(x), Z(x), \varphi(x))$ is $\mathcal{L}$-equivariant if and only if $(w(x), a(x), z(x), \phi(x))$ is $\mathcal{L}'$-equivariant, where $\mathcal{L}' := \frac{1}{r}\mathcal{L}$. Now, the size of a fundamental domain of the rescaled lattice $\mathcal{L}'$ is fixed as $|\Omega'| = 2\pi$.

Plugging the rescaled fields into (3.12) - (3.15) gives the rescaled Weinberg-Salem equations:

$$[\operatorname{curl}_\nu^* \operatorname{curl}_\nu + \frac{g^2}{2}\phi^2 - i(\operatorname{curl}\nu)J + g^2(\overline{w} \times w)J]w = 0, \tag{4.3}$$

$$\operatorname{curl}^* \operatorname{curl} a + 2e \operatorname{Im}[(\operatorname{curl}_\nu w)J\overline{w} - \operatorname{curl}^*(\overline{w}_1 w_2)] = 0, \tag{4.4}$$

$$[\operatorname{curl}^* \operatorname{curl} + \kappa\phi^2]z + 2g\cos\theta \operatorname{Im}[(\operatorname{curl}_\nu w)J\overline{w} - \operatorname{curl}^*(\overline{w}_1 w_2)] = 0, \tag{4.5}$$

$$[-\Delta + \lambda(\phi^2 - \xi^2) + \frac{g^2}{2}|w|^2 + \frac{1}{2}\kappa|z|^2]\phi = 0, \tag{4.6}$$

where $\xi := r\varphi_0$ (with r given in (4.2)), $\nu := g(a\sin\theta + z\cos\theta)$ and, recall, $\operatorname{curl}_q w = \nabla_1 w_2 - \nabla_2 w_1$, $\nabla_i := \partial_i - iq_i$ (for a $\mathfrak{u}(1)-$valued vector-field iq) and, recall, $\overline{w} \times w := \overline{w}_1 w_2 - \overline{w}_2 w_1$. We define the rescaled energy by

$$\mathcal{E}_{\Omega'}(w, a, z, \phi; r) := r^2 E_\Omega^{WS}(W, A, Z, \varphi). \tag{4.7}$$

with (W, A, Z, φ) related to (w, a, z, ϕ) by (4.1). Explicitly, we have

$$\begin{aligned}\mathcal{E}_{\Omega'}(w, a, z, \phi; r) = \int_{\Omega'} \big(&|\operatorname{curl}_\nu w|^2 + \frac{1}{2}|\operatorname{curl} a|^2 + \frac{1}{2}|\operatorname{curl} z|^2 \\ &+ \frac{1}{2}g^2\phi^2|w|^2 + \frac{1}{2}\kappa\phi^2|z|^2 + \frac{g^2}{2}|\overline{w} \times w|^2 \\ &+ i(\operatorname{curl}\nu)\overline{w} \times w + |\nabla\phi|^2 + \frac{1}{2}\lambda(\phi^2 - \xi^2)^2\big).\end{aligned} \tag{4.8}$$

We note that after rescaling, the vacuum solution (3.8) becomes

$$m^{n,r} := (0, \frac{1}{e}a^n, 0, \xi), \tag{4.9}$$

where $a^n(x) \equiv A^n(x) = \frac{n}{2}Jx$, and that the average magnetic flux per fundamental domain is now n/e. Furthermore, (3.16) and Proposition 5 imply that (w, a, z, ϕ) satisfy

$$w(x+s) = e^{i(\frac{n}{2}s\times x + c_s)}w(x) \text{ for all } s \in \mathcal{L}, \tag{4.10}$$

$$a(x+s) = a(x) + \frac{n}{2e}Js \text{ for all } s \in \mathcal{L}, \tag{4.11}$$

$$\operatorname{div} a = 0, \tag{4.12}$$

$$z(x+s) = z(x), \quad \phi(x+s) = \phi(x) \text{ for all } s \in \mathcal{L}, \tag{4.13}$$

where c_s satisfies the condition $c_{s+t} - c_s - c_t - \frac{n}{2}s \times t \in 2\pi\mathbb{Z}$, for all $s, t \in \mathcal{L}$.

Chapter 5

The linearized problem

In this section we prove Theorem 1, describing the linear stability/instability of the vacuum (3.8).

Let $m := (w, a, z, \phi)$ and denote by $G(b, m) \equiv G(m)$ the map given by the left-hand side of (4.3) - (4.6), given explicitly as

$$G(b, m) \equiv G(m) = (G_1(m), \dots, G_4(m)), \tag{5.1}$$

$$G_1(m) := [\operatorname{curl}_\nu^* \operatorname{curl}_\nu + \frac{g^2}{2}\phi^2 - i(\operatorname{curl}\nu)J + g^2(\overline{w} \times w)J]w, \tag{5.2}$$

$$G_2(m) := \operatorname{curl}^* \operatorname{curl} a + 2e \operatorname{Im}[(\operatorname{curl}_\nu w)J\overline{w} - \operatorname{curl}^*(\overline{w}_1 w_2)], \tag{5.3}$$

$$G_3(m) := [\operatorname{curl}^* \operatorname{curl} + \kappa\phi^2]z + 2g\cos\theta \operatorname{Im}[(\operatorname{curl}_\nu w)J\overline{w} - \operatorname{curl}^*(\overline{w}_1 w_2)], \tag{5.4}$$

$$G_4(m) := [-\Delta + \lambda(\phi^2 - \xi^2) + \frac{g^2}{2}|w|^2 + \frac{1}{2}\kappa|z|^2]\phi, \tag{5.5}$$

where, recall, J is the symplectic matrix given in (3.11), $\xi := r\varphi_0$ (with r given in (4.2)), $\nu := g(a\sin\theta + z\cos\theta)$, Δ is the standard Laplacian and the parameter b enters through periodicity conditions (4.10) - (4.13). Now, the Weinberg-Salam equations can be written as $G(m) = 0$.

Applied to the rescaled Weinberg-Salam equations (4.3) - (4.6), the definition of stability states that the vacuum solution (4.9) ($m^{n,r} := (0, \frac{1}{e}a^n, 0, \xi)$) is linearly stable (respectively unstable) if and only if the spectrum of the linearization of $G(m)$ at $m^{n,r}$ is non-negative (respectively, has a negative infimum).

In what follows we use the notation $\oplus_j A_j$ for diagonal operator-matrices with the operators A_j on the diagonal. Furthermore, we denote the total Gâteaux derivative by δ, and the partial (real) Gâteaux derivatives with respect to # by $\delta_\#$.

Consider the Gâteau derivative (linearization) $L_{n,\mu} := \delta G(m^{n,r})$ of $G(v)$ at the rescaled vacuum $m^{n,r}$. We compute it explicitly, while passing from the parameter $\xi = r\varphi_0$, or r, to the parameter

$\mu := g^2\xi^2/2$ and using that $\nu\big|_{a=a^n/e, z=0} = \frac{1}{e}a^n g\sin\theta = a^n$:

$$L_{n,\mu} = \oplus_{j=1}^4 H_j, \tag{5.6}$$
$$H_1(\mu) := \operatorname{curl}^*_{a^n}\operatorname{curl}_{a^n} + \mu - niJ, \tag{5.7}$$
$$H_2(\mu) := \operatorname{curl}^*\operatorname{curl}, \tag{5.8}$$
$$H_3(\mu) := \operatorname{curl}^*\operatorname{curl} + \frac{\mu}{\cos^2\theta}, \tag{5.9}$$
$$H_4(\mu) := -\Delta + \frac{4\lambda\mu}{g^2}, \tag{5.10}$$

where, recall, $\operatorname{curl}_q w = \nabla_1 w_2 - \nabla_2 w_1$, $\nabla_i := \partial_i - iq_i$. (Note that the matrix iJ is self-adjoint.)

The operator $L_{n,\mu}$ is the Hessian for the energy (4.8), considered as a functional of w, a, z, ϕ, or the Gâteaux derivative (linearization) of the left-hand side of (4.3) - (4.6), with $\xi = \sqrt{2\mu}/g$, at the rescaled vacuum solution $m^{n,r} := (0, \frac{1}{e}a^n, 0, \xi)$ of (4.9). Note that

$$L_{n,\mu=n}G_f = 0,\ G_f := (0, \nabla f, 0, 0). \tag{5.11}$$

We consider the operator $L_{n,\mu}$ on a space $\mathcal{X}$ tangent to the space of $\mathcal{H}^2_{\text{loc}}$ functions of the form (w, a, z, ϕ) satisfying the gauge - periodicity conditions (4.10) - (4.13). Explicitly,

$$\mathcal{X} := \mathcal{H}^2_n \times \mathcal{H}^2_0 \times \mathcal{H}^2_0 \times \mathcal{H}^2, \tag{5.12}$$

where $\mathcal{H}^s_n$, $\mathcal{H}^s_0$ and $\mathcal{H}^s$ are the respective Sobolev spaces for the L^2-spaces

$$L^2_n := \{w \in L^2_{loc}(\mathbb{R}^2, \mathbb{C}^2) : w(x+s) = e^{i(\frac{n}{2}s\times x + c_s)}w(x)\ \forall s \in \mathcal{L}\}, \tag{5.13}$$
$$L^2_0 := \{\alpha \in L^2_{loc}(\mathbb{R}^2, \mathbb{R}^2) : \alpha(x+s) = \alpha(x)\ \forall s \in \mathcal{L},\ \operatorname{div}\alpha = 0\}, \tag{5.14}$$
$$L^2 := \{\psi \in L^2_{loc}(\mathbb{R}^2, \mathbb{R}) : \psi(x+s) = \psi(x)\ \forall s \in \mathcal{L}\} \tag{5.15}$$

(see (4.10) - (4.12)), with inner products given (for $s \in \mathbb{Z}_{\geq 0}$) by

$$\langle w, w'\rangle_{\mathcal{H}^s_n} := \frac{1}{|\Omega'|}\sum_{i=1}^2\sum_{|\gamma|\leq s}\int_{\Omega'}\overline{(\nabla_{a^n})^\gamma w_i}(\nabla_{a^n})^\gamma w'_i, \tag{5.16}$$
$$\langle a, a'\rangle_{\mathcal{H}^s_0} := \frac{1}{|\Omega'|}\sum_{i=1}^2\sum_{|\gamma|\leq s}\int_{\Omega'}\partial^\gamma a_i\partial^\gamma a'_i, \tag{5.17}$$
$$\langle \psi, \psi'\rangle_{\mathcal{H}^s} := \frac{1}{|\Omega'|}\sum_{|\gamma|\leq s}\int_{\Omega'}\partial^\gamma\psi\partial^\gamma\psi', \tag{5.18}$$

where $w^\# = (w_1^\#, w_2^\#), a^\# = (a_1^\#, a_2^\#)$, Ω' is an arbitrary fundamental domain of the lattice $\mathcal{L}'$ and γ is a multi-index. The $\mathcal{L}'$-equivariance of the above functions implies that these inner products do not depend on the choice of fundamental domain Ω'.

For a null vector G_f defined in (5.11) to be in $\mathcal{X}$, f must satisfy $\operatorname{div}(df) = -\Delta f = 0$. This implies that f is a linear function, $f(x) = c\cdot x + d$ for some $c \in \mathbb{R}^2$ and $d \in \mathbb{R}$, and so

$$G_f \in \mathcal{X} \Longrightarrow G_f = (0, c, 0, 0). \tag{5.19}$$

In this section we shall prove the following result implying Theorem 1:

Theorem 6. *The operator $L_{n,\mu}$ on the space $\mathcal{X}$ has purely discrete spectrum. For $\mu \neq n$, $L_{\mu,n}$ has the simple eigenvalue 0 and corresponding eigenfuction $(0, c, 0, 0)$, $c \in \mathbb{R}^2$ (see (5.19)), and smallest non-zero eigenvalue given by $\mu - n$, having multiplicity n. For $\mu = n$, the eigenvalue 0 has the multiplicity $n+1$.*

Theorem 6 follows from Propositions 7 and 8 given below. □

Proposition 7. *The operators $H_2(\mu)$, $H_3(\mu)$ and $H_4(\mu)$ are non-negative on their respective domains with purely discrete spectra. Furthermore, $H_3(\mu)$ and $H_4(\mu)$ are strictly positive and $H_2(\mu)$ has a null space of dimension 1 consisting of the constant functions.*

Proof. The strict positivity of $H_3(\mu)$ and $H_4(\mu)$ and the non-negativity of $H_2(\mu)$ are obvious. The discreteness of the spectra and the form of the null space of $H_2(\mu)$ follow from the discreteness of the spectrum of the Laplacian on compact domains and the identity $\operatorname{curl}^* \operatorname{curl} v = -\Delta v$ when $\operatorname{div}(v) = 0$. □

Let $\nabla_q := \nabla - iq = (\nabla_1, \nabla_2)$, $\nabla_i := \partial_i - iq_i$. We have

Proposition 8. *(i) $H_1(\mu)$ is a self-adjoint operator on $\mathcal{H}^2_n$ and its spectrum is given by*

$$\sigma(H_1(\mu)) = \{(2m-1)n + \mu : m \in \mathbb{Z}_{\geq 0}\} \cup \{\mu\}, \tag{5.20}$$

where $n := eb|\mathcal{L}|/2\pi$.

(ii) The eigenspace of the eigenvalue $-n + \mu$ is n-dimensional and is spanned by functions of the form

$$\chi = (\eta, i\eta), \qquad \operatorname{curl}_{a^n} \chi = i\bar{\partial}_{a^n}\eta = 0, \tag{5.21}$$

i.e. $\operatorname{Null}(H_1(\mu) - \mu + n) = \{\chi = (\eta, i\eta) : \operatorname{curl}_{a^n} \chi = 0\}$, *and the eigenspace of the eigenvalue μ is of the form*

$$\operatorname{Null}(H_1(\mu) - \mu) = \{\nabla_{a^n} f : f \in \mathcal{H}^3_n\}. \tag{5.22}$$

Proof. Recall that the operator $H_1(\mu)$ acts on complex vectors $w = (w_1, w_2)$. We write it as check $2n$

$$H_1(\mu) := h_1 + \mu, \;\; h_1 := \operatorname{curl}^*_{a^n} \operatorname{curl}_{a^n} - niJ. \tag{5.23}$$

First, we will show that $\mathcal{H}^2_n = \mathcal{Y} \oplus \mathcal{Z}$, where

$$\mathcal{Y} := \{w \in \mathcal{H}^2_n : \operatorname{div}_{a^n} w = 0\}, \tag{5.24}$$

$$\mathcal{Z} := \{w \in \mathcal{H}^2_n : w = \nabla_{a^n} f \text{ for some } f \in \mathcal{H}^3_n\}, \tag{5.25}$$

with $\operatorname{div}_{a^n} w := (\nabla_{a^n})_1 w_1 + (\nabla_{a^n})_2 w_2$. Indeed, since $\Delta_{a^n} := \operatorname{div}_{a^n} \nabla_{a^n}$, then, for any $w \in \mathcal{H}^2_n$ we may write $w = w_0 + \nabla_{a^n} f$, where $\operatorname{div}_{a^n} w_0 = 0$ and $f \in \mathcal{H}^3_n$ solves $\Delta_{a^n} f = \operatorname{div}_{a^n} w$ (this solution exists and is unique, since by Proposition 36, 0 is not in the spectrum of Δ_{a^n}). The relations $\operatorname{curl}_{a^n} \nabla_{a^n} = [(\nabla_{a^n})_1, (\nabla_{a^n})_2] = -in$ and $\operatorname{curl}^*_{a^n} = -J\nabla_{a^n}$ yield that $h_1 \nabla_{a^n} f = 0$, which proves that the μ-eigenspace of $H_1(\mu)$ is of the form (5.22). Furthermore, since $\operatorname{div}_{a^n} w_0 = 0$, we have that $h_1 w_0 = (-\Delta_{a^n} - 2niJ)w_0$, for $w_0 \in \mathcal{Y}$. Hence, we may write $h_1 : \mathcal{Y} \oplus \mathcal{Z} \to \mathcal{Y} \oplus \mathcal{Z}$ as

$$h_1(w_0 \oplus \nabla_{a^n} f) = (h_{10} w_0) \oplus 0, \tag{5.26}$$

$$h_{10} := -\Delta_{a^n} - 2niJ. \tag{5.27}$$

(Here h_{10} sends $\mathcal{Y}$ to $\mathcal{Y}$ because $\mathrm{div}_{a^n} h_{10} w_0 = (-\Delta_{a^n}) \, \mathrm{div}_{a^n} w_0 = 0$.)

Identifying one-forms with vector-fields, we compute

$$U^*(iJ)U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \tag{5.28}$$

which gives

$$U^* h_{10} U = \begin{pmatrix} -\Delta_{a^n} + 2n & 0 \\ 0 & -\Delta_{a^n} - 2n \end{pmatrix}. \tag{5.29}$$

By Proposition 36, we know that

$$\sigma(-\Delta_{a^n}) = \{\, (2m+1)n : m \in \mathbb{Z}_{\geq 0} \} \tag{5.30}$$

and so the spectrum of $H_1(\mu) = h_1 + \mu$ is given by (5.20). Furthermore, by (5.29) and (5.30), any eigenvector χ of h_{10} corresponding to the eigenvalue 0 must be of the form

$$\chi = U(0, \eta) = \frac{1}{\sqrt{2}} (\eta, i\eta), \tag{5.31}$$

where η satisfies

$$-\Delta_{a^n} \eta = n\eta. \tag{5.32}$$

Since $\mathrm{curl}_{a^n} \chi = i \overline{\partial}_{a^n} \eta$, by (I.7) of Appendix I, this gives

$$\mathrm{curl}_{a^n} \chi = i \overline{\partial}_{a^n} \eta = 0. \tag{5.33}$$

Furthermore, by Proposition 36, the space of such functions is n-dimensional. Thus (after rescaling η by a factor of $\sqrt{2}$) χ is of the desired form. This together with (5.23) gives the desired result. □

We see that the operator $H_1(\mu)$ is non-negative for small magnetic fields ($b < b_* := g^2 \varphi_0^2 / 2e = M_W^2 / e$) and acquires a negative eigenvalue $\mu - n = (b_*/b - 1)\, n$ of multiplicity n as the magnetic field increases. Theorem 1 follows by undoing the rescaling (4.1) - (4.2).

Chapter 6

Setup of the bifurcation problem

We substitute $a = \frac{1}{e}a^n + \alpha$ (with $\operatorname{div}(\alpha) = 0$), $\phi = \xi + \psi$, $\nu = a^n + \tilde{\nu}$ and $\xi = \sqrt{2\mu}/g$ into (4.3) - (4.6) and relabel the unknowns w, α, z, ψ as u_1, u_2, u_3, u_4 to obtain the system

$$H_i u_i = -J_i(\mu, u), \; i = 1, \dots, 4, \tag{6.1}$$

where $u = (u_1, u_2, u_3, u_4) \equiv (w, \alpha, z, \psi)$, the operators H_i on the left-hand side are defined in (5.7) - (5.10), and

$$\begin{aligned} J_1(\mu, u) &:= Mw + \frac{g^2}{2}\psi^2 w + g\sqrt{2\mu}\psi w - i(\operatorname{curl}\tilde{\nu})Jw \\ &\qquad + g^2(\overline{w} \times w)Jw, &(6.2)\\ J_2(\mu, u) &:= 2e\operatorname{Im}[\operatorname{curl}_\nu wJ\overline{w} - \operatorname{curl}^*(\overline{w}_1 w_2)], &(6.3)\\ J_3(\mu, u) &:= 2g\cos\theta\operatorname{Im}[\operatorname{curl}_\nu wJ\overline{w} - \operatorname{curl}^*(\overline{w}_1 w_2)] \\ &\qquad + \frac{2\kappa}{g}\sqrt{2\mu}\psi z + \kappa\psi^2 z, &(6.4)\\ J_4(\mu, u) &:= 3\lambda\frac{\sqrt{2\mu}}{g}\psi^2 + \lambda\psi^3 + \frac{g^2}{2}|w|^2(\frac{\sqrt{2\mu}}{g} + \psi) + \frac{1}{2}\kappa|z|^2(\frac{\sqrt{2\mu}}{g} + \psi), &(6.5) \end{aligned}$$

with $\tilde{\nu} := g(\alpha\sin\theta + z\cos\theta)$, $\xi \times \eta := \xi_1\eta_2 - \xi_2\eta_1$, recall, $\operatorname{curl}_q w = \nabla_1 w_2 - \nabla_2 w_1$, $\nabla_i := \partial_i - iq_i$ and, recalling that $w : \mathbb{R}^2 \to \mathbb{C}^2$,

$$M := \operatorname{curl}^*_\nu \operatorname{curl}_\nu - \operatorname{curl}^*_{a^n} \operatorname{curl}_{a^n} = \begin{pmatrix} M_{22} & -M_{21} \\ -M_{12} & M_{11} \end{pmatrix}, \tag{6.6}$$

with $M_{ij} := i\tilde{\nu}_i(\nabla_{a^n})_j + i\tilde{\nu}_j(\nabla_{a^n})_i + i\partial_i\tilde{\nu}_j + \tilde{\nu}_i\nu_j$.

Note that system (6.1) can be also written as $G(m^{n,r} + u)|_{\xi=\sqrt{2\mu}/g} = 0$, where G is defined in (5.1) and $m^{n,r} := (0, \frac{1}{e}a^n, 0, \xi)$.

Applying div to the second equation in (6.1), we find a solution (μ, u) should satisfy $\operatorname{div} J_2(\mu, u) = 0$. To prove that a solution (μ, u) satisfies this constraint, we consider the following auxiliary problem

$$F(\mu, u) = 0, \quad \text{where} \quad F(\mu, u) := L_{n,\mu}u + P'J(\mu, u), \tag{6.7}$$

where $P' = \mathbf{1} \otimes P_0 \otimes \mathbf{1} \otimes \mathbf{1}$, with P_0 the orthogonal projection onto the divergence-free vector fields ($P_0 = \frac{1}{-\Delta} \operatorname{curl}^* \operatorname{curl}$), and, recall, $L_{n,\mu} = \oplus H_i$ and $J(\mu, u)$ given in (5.6) and

$$J(\mu, u) := (J_1(\mu, u), \ldots, J_4(\mu, u)). \tag{6.8}$$

We consider $F(\mu, u)$ as a map from the space $\mathbb{R}_{>0} \times \mathcal{X}$, where $\mathcal{X} := \mathcal{H}_n^2 \oplus \mathcal{H}_0^2 \oplus \mathcal{H}_0^2 \oplus \mathcal{H}^2$, to the space $\mathcal{Y} := L_n^2 \oplus L_0^2 \oplus L_0^2 \oplus L^2$, and let $F = (F_1, \ldots, F_4)$, where

$$F_i(\mu, u) = H_i u + \delta_{i,2} P_0 J_i(\mu, u), \quad i = 1, \ldots, 4. \tag{6.9}$$

Proposition 9. *Assume (μ, u) is a solution of the system* (6.7) *satisfying the gauge - periodicity conditions* (4.10) - (4.13)*. Then* $\operatorname{div} J(\mu, u) = 0$ *and therefore (μ, u) solves the original system* (6.1)*.*

Proof. We follow [38]. Assume $\chi \in H^1_{\text{loc}}$ and is $\mathcal{L}-$periodic (we say, $\chi \in H^1_{\text{per}}$). The gauge invariance implies that

$$E_{\Omega'}(e^{is\chi} w, a + s\nabla\chi, z, \phi) = E_{\Omega'}(w, a, z, \phi), \tag{6.10}$$

where $E_{\Omega'}(w, a, z, \phi)$ is given in (4.8). Differentiating this equation with respect to s at $s = 0$ gives $\delta_w E_{\Omega'}(w, a, z, \phi)(i\chi w) + \delta_a E_{\Omega'}(w, a, z, \phi)(\nabla\chi) = 0$. Now, we use the fact that the partial Gâteaux derivative with respect to w vanishes, $\delta_w E_{\Omega'}(w, a, z, \phi) = 0$, and that $\operatorname{curl} \nabla\chi = 0$, and integrate by parts, to obtain

$$\langle J(\mu, u), \nabla\chi \rangle = 0. \tag{6.11}$$

(Due to conditions (4.10) - (4.13) and the $\mathcal{L}-$periodicity of χ, there are no boundary terms.) Since the last equation holds for any $\chi \in H^1_{\text{per}}$, we conclude that $\operatorname{div} J(\mu, u) = 0$. □

In Sections 7 - 8 we solve equation (6.7), subject to conditions (4.10) - (4.13).

In conclusion of this section, we investigate properties of the map $F(\mu, u)$. For $f = (f_1, f_2, f_3, f_4)$ and $\delta \in \mathbb{R}$, define the global transformation

$$T_\delta f = (e^{i\delta} f_1, f_2, f_3, f_4). \tag{6.12}$$

Proposition 10. *$F(\mu, u)$ has the following properties:*

(i) $F : \mathbb{R}_{>0} \times \mathcal{X} \to \mathcal{Y}$ *is continuously differentiable of all orders;*

(ii) $F(\mu, 0) = 0$ *for all* $\mu \in \mathbb{R}_{>0}$*;*

(iii) $\delta_u F(\mu, 0) = L_{n,\mu}$ *for all* $\mu \in \mathbb{R}_{>0}$*;*

(iv) $F(\mu, T_\delta u) = T_\delta F(\mu, u)$ *for all* $\delta \in \mathbb{R}$*;*

(v) $\langle u, F(\mu, u) \rangle_{\mathcal{Y}} \in \mathbb{R}$ *(respectively* $\langle w, F_1(\mu, u) \rangle_{L_n^2} \in \mathbb{R}$*) for all* $u \in \mathcal{X}$ *(respectively* $w \in \mathcal{H}_n^2$*).*

Proof. (i) follows because F is a polynomial in the components of u and their first- and second-order (covariant) derivatives. (ii), (iii) and (iv) follow from an easy calculation (in fact, u and $L_{n,\mu}$ were defined so that (ii) and (iii) hold). For (v), it suffices to show that $\langle w, F_1(\mu, u) \rangle_{L_n^2} \in \mathbb{R}$. To simplify notation we return to the coordinates $(w, a, z, \phi) = (w, \frac{1}{e}a^n + \alpha, z, \frac{\sqrt{2\mu}}{g} + \psi)$. Then

$$\begin{aligned}\langle w, F_1(\mu, u)\rangle_{L^2_n} = \frac{1}{|\Omega'|}\int_{\Omega'} |\operatorname{curl}_\nu w|^2 + \frac{1}{|\Omega'|}\int_{\Omega'} \frac{g^2}{2}\phi^2|w^2| \\ + \frac{1}{|\Omega'|}\int_{\Omega'} i(\operatorname{curl}\nu)(\overline{w}\times w) + \frac{1}{|\Omega'|}\int_{\Omega'} g^2|\overline{w}\times w|^2.\end{aligned} \tag{6.13}$$

The first, second and fourth terms are clearly real, while the third term is real because ν is real and $\overline{w}\times w$ is imaginary. □

Chapter 7

Reduction to a Finite-Dimensional Problem

In this section we shall reduce solving equation (6.7), i.e. $F(\mu, u) = 0$, with $u = (u_1, u_2, u_3, u_4) \equiv (w, \alpha, z, \psi)$ and $F : \mathbb{R}_{>0} \times \mathcal{X} \to \mathcal{Y}$ defined in (6.7) - (6.8)), to a finite-dimensional problem.

Recall that $L_{n,\mu}$ is defined in (5.6). Let P be the orthogonal projection onto $\mathcal{K} := \mathrm{Null}(L_{n,\mu=n})$, which can be written explicitly as

$$P = P_1 \oplus P_2 \oplus 0 \oplus 0, \tag{7.1}$$

$$P_1 w := -\frac{1}{2\pi i}\oint_{\gamma_n} (H_1(n) - z)^{-1} w \, dz, \tag{7.2}$$

$$P_2 \alpha := \langle \alpha \rangle, \tag{7.3}$$

where $H_1(n)$ is defined in (5.7), γ_n is any simple closed curve in $\mathbb{C}$ containing the eigenvalue 0 and no other eigenvalues of $H_1(n)$ (see Proposition 8), and $\langle \alpha \rangle$ is the mean value of α in Ω', $\langle \alpha \rangle := \frac{1}{|\Omega'|}\int_{\Omega'} \alpha$. P_1 is a projection onto $\mathrm{Null}(H_1(n))$ (spanned by vectors of the form (5.21)). Since $H_1(n)$ is self-adjoint, P_1 is an orthogonal projection (relative to the inner product of L^2_n). By Theorem 6, $\mathcal{K} := \mathrm{Null}(L_{n,\mu=n})$ is $(n+1)$-dimensional.

Let $P^\perp = 1 - P$ be the projection onto the orthogonal complement of $\mathcal{K}$. Then we may rewrite the equation $F(\mu, u) = 0$ (see (6.7)) as

$$PF(\mu, v + u') = 0, \tag{7.4}$$

$$P^\perp F(\mu, v + u') = 0, \tag{7.5}$$

where $v := Pu,\ u' := P^\perp u$.

Our next goal is to solve (7.5) for u' in terms of μ and v. First, we shall need the following proposition to bound the polynomials of functions appearing below:

Proposition 11. *Let X be one of the spaces $\mathcal{H}^2_n$, $\mathcal{H}_0$ or $\mathcal{H}^2$ defined before equation (5.13). Let $p(x_1, ..., x_n)$ be a polynomial with positive coefficients and let $f_1, ..., f_n \in X$. Then $||p(f_1, ..., f_n)||_X \lesssim p(||f_1||_X, ..., ||f_n||_X)$.*

Proof. Write $p(x_1, ..., x_n) = \sum_{|\alpha| \le N} p_\alpha x^\alpha$, where $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index and $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$.

Since by the Sobolev Embedding Theorem (see e.g. [2]), X is a Banach algebra, we have

$$||p(f_1, ..., f_n)||_X \le \sum_{|\alpha|\le N} p_\alpha ||f^\alpha||_X \tag{7.6}$$

$$\lesssim \sum_{|\alpha|\le N} p_\alpha \prod_{i=1}^{n} ||f_i||_X^{\alpha_i} \tag{7.7}$$

$$= p(||f_1||_X, ..., ||f_n||_X), \tag{7.8}$$

which implies the desired result. □

Recall that we denote the partial (real) Gâteaux derivatives with respect to # by $\delta_\#$. Let $\mathcal{X}^\perp := P^\perp \mathcal{X} = \mathcal{X} \ominus \mathcal{K}$ and $\mathcal{Y}^\perp := P^\perp \mathcal{Y} = \mathcal{Y} \ominus \mathcal{K}$, and let $\partial_i \equiv \partial_{x_i}$.

Lemma 12. *There is a neighbourhood $U \subset \mathbb{R}_{>0} \times \mathcal{K}$ of $(n, 0)$ such that for every $(\mu, v) \in U$, equation (7.5) for u' has a unique solution $u' = u'(\mu, v)$. Furthermore, this solution $u' = (u'_1, u'_2, u'_3, u'_4)$ has the following properties:*

$$u' : \mathbb{R}_{>0} \times \mathcal{K} \to \mathcal{X}^\perp \text{ is continuously differentiable of all orders;} \tag{7.9}$$

$$\|(\nabla_{a^n})_j^m u'_1\|_{\mathcal{H}_n^2} \lesssim \|v\|_{\mathcal{X}}^2; \tag{7.10}$$

$$\|\partial_j^m u'_k\|_{\mathcal{H}_k^2} \lesssim \|v\|_{\mathcal{X}}^2; \tag{7.11}$$

$$||\delta_{v_i} (\nabla_{a^n})_j^m u'_1(\mu, v_{\hat{i}})||_{\mathcal{H}_n^2} \lesssim ||v_{\hat{i}}||_{\mathcal{X}}; \tag{7.12}$$

$$||\delta_{v_i} \partial_j^m u'_k(\mu, v_{\hat{i}})||_{\mathcal{H}_k^2} \lesssim ||v_{\hat{i}}||_{\mathcal{X}}; \tag{7.13}$$

$$||\partial_\mu u'(\mu, v)||_{\mathcal{X}} \lesssim ||v||_{\mathcal{X}}^2, \tag{7.14}$$

where $i = 1, ...4,\ m = 0, 1,\ j = 1, 2,\ k = 2, 3, 4,\ v_{\hat{i}} \equiv v|_{v_i=0}$, for $v = (v_1, v_2, v_3, v_4)$ and $i = 1, ..., 4$, and $\mathcal{H}_k^2 = \mathcal{H}_0^2,\ \mathcal{H}_0^2,\ \mathcal{H}^2$ for $k = 2, 3, 4$.

Proof. Define $F^\perp : \mathbb{R}_{>0} \times \mathcal{K} \times \mathcal{X}^\perp \to \mathcal{Y}^\perp$ by

$$F^\perp(\mu, v, u') := P^\perp F(\mu, v + u'). \tag{7.15}$$

By Proposition 10 (i) and (ii), $F^\perp$ is continuously differentiable of all orders as a map between Banach spaces and $F^\perp(\mu, 0, 0) = 0$ for all $\mu \in \mathbb{R}_{>0}$. Furthermore,

$$\delta_{u'} F^\perp(\mu, 0, 0) = P^\perp L_{n,\mu} P^\perp|_{\mathcal{X}^\perp}, \tag{7.16}$$

which is invertible for $\mu = n$ because $P^\perp$ is the projection onto the orthogonal complement of $\mathcal{K} = \mathrm{Null}(L_{n,\mu=n})$. By the Implicit Function Theorem (see e.g. [14]), there exists a function $u'(\mu, v)$ with continuous derivatives of all orders such that for (μ, v) in a sufficiently small neighbourhood $U \subset \mathbb{R}_{>0} \times \mathcal{K}$ of $(n, 0)$, (μ, v, u') solves (7.5) if and only if $u' = u'(\mu, v)$. This proves the first statement and property (7.9).

We define the operator

$$L_{n,\mu}^\perp := P^\perp L_{n,\mu} P^\perp|_{\mathcal{X}^\perp} : \mathcal{X}^\perp \to \mathcal{Y}^\perp. \tag{7.17}$$

Then by (6.7) and (7.16), we can write equation (7.5) as $L^{\perp}_{n,\mu}u' = -P^{\perp}P'J(\mu, u)$. By Theorem 6 and the relation $\mathcal{K} := \text{Null}(L_{n,\mu=n}) = \text{Null}(L_{n,\mu} - \mu + n)$, for μ in a neighbourhood of n, the operator $L^{\perp}_{n,\mu}$ has a uniformly bounded inverse $(L^{\perp}_{n,\mu})^{-1} : \mathcal{Y}^{\perp} \to \mathcal{X}^{\perp}$. Hence equation $L^{\perp}_{n,\mu}u' = -P^{\perp}P'J(\mu, u)$, with $(\mu, v) \in U$ (replacing U with a smaller neighbourhood if necessary), is equivalent to

$$u' = -(L^{\perp}_{n,\mu})^{-1}P^{\perp}P'J(\mu, u); \tag{7.18}$$

hence

$$\|u'\|_{\mathcal{X}} \lesssim \|J(\mu, u)\|_{\mathcal{Y}}, \tag{7.19}$$

uniformly in μ. Recall that $\mathcal{X} = \mathcal{H}^2_n \oplus \mathcal{H}^2_0 \oplus \mathcal{H}^2_0 \oplus \mathcal{H}^2$ and $\mathcal{Y} = L^2_n \oplus L^2_0 \oplus L^2_0 \oplus L^2$. $J(\mu, u)$ is a polynomial in the components of u and their first-order (covariant) derivatives consisting of terms of degree at least 2, so the left-hand side of (7.19) can be bounded above by a sum of products of one $\mathcal{L}^2$-norm and at least one $\mathcal{L}^{\infty}$-norm of these terms. $\mathcal{H}^1$ is trivially continously embedded in $\mathcal{L}^2$, and by the Sobolev Embedding Theorem, $\mathcal{H}^1$ is continuously embedded in $\mathcal{L}^{\infty}$. Therefore,

$$\|J(\mu, u)\|_{\mathcal{Y}} \lesssim \|u\|^2_{\mathcal{X}}. \tag{7.20}$$

Recalling that $u = v + u'$, this proves (7.10) and (7.11) when $m = 0$. The other case is proven similarly.

For $v = (v_1, \dots, v_4)$, we let $v_{\hat{i}} \equiv v|_{v_i=0}$, $i = 1, \dots, 4$. By the Taylor theorem for Banach spaces (see e.g. [14]), we have

$$u'(\mu, v) = u'(\mu, v_{\hat{i}}) + \delta_{v_i}u'(\mu, v_{\hat{i}})v_i + R_2(\mu, v_{\hat{i}})(v_i), \tag{7.21}$$

$$R_2(\mu, v_{\hat{i}})(v_i) := \int_0^1 (1-t)\delta^2_{v_i}u'(\mu, v_{\hat{i}} + tv_i)(v_i, v_i)dt. \tag{7.22}$$

Let $(\mu, v) \in U$ with $||v_{\hat{i}}|| = ||v_i|| = 1$, and let $\epsilon > 0$. Then

$$\begin{aligned} ||\delta_{v_i}u'(\mu, \epsilon v_{\hat{i}})\epsilon v_i||_{\mathcal{X}} &= ||u'(\mu, \epsilon v) - u'(\mu, \epsilon v_{\hat{i}}) - R_2(\mu, \epsilon v_{\hat{i}})(\epsilon v_i)||_{\mathcal{X}} \\ &\leq ||u'(\mu, \epsilon v)||_{\mathcal{X}} + ||u'(\mu, \epsilon v_{\hat{i}})||_{\mathcal{X}} \\ &+ \epsilon^2||v_i||^2 \sup_{0\leq t\leq 1}(1-t)||\delta^2_{v_i}u'(\mu, \epsilon v_{\hat{i}} + t\epsilon v_i)||^2_{\mathcal{X}^*\otimes\mathcal{X}^*\otimes\mathcal{X}} \\ &\lesssim \epsilon^2. \end{aligned} \tag{7.23}$$

with the norm taken in the appropriate space for v_i. Taking the supremum over all v_i with $||v_i|| = 1$ gives

$$||\delta_{v_i}u'(\mu, \epsilon v_{\hat{i}})||_{\mathcal{X}} \lesssim \epsilon, \ \|v_{\hat{i}}\|_{\mathcal{X}} = 1, \tag{7.24}$$

proving (7.12) - (7.13) for $m = 0$. The other cases are proven in exactly the same way.

Again by Taylor's Theorem,

$$\partial_\mu u'(\mu,v) = \partial_\mu u'(\mu,0) + \partial_\mu \delta_v u'(\mu,0)v + \tilde{R}_2(\mu,0)(v), \tag{7.25}$$

$$\tilde{R}_2(\mu,0)(v) := \int_0^1 (1-t)\partial_\mu \delta_v^2 u'(\mu,tv)(v,v)dt. \tag{7.26}$$

By Equations (7.11) and (7.12) - (7.13) with $m=0$, we have $u'(\mu,0)=0$ and $\delta_v u'(\mu,0)=0$, so

$$||\partial_\mu u'(\mu,v)||_{\mathcal{X}} = ||\tilde{R}_2(\mu,0)(v)||_{\mathcal{X}} \tag{7.27}$$

$$\leq ||v||_{\mathcal{X}}^2 \sup_{0\leq t\leq 1}(1-t)||\partial_\mu \delta_v^2 u'(\mu,tv)||^2_{\mathcal{X}^*\otimes\mathcal{X}^*\otimes\mathcal{X}} \tag{7.28}$$

$$\lesssim ||v||_{\mathcal{X}}^2, \tag{7.29}$$

proving (7.14). □

We plug the solution $u' = u'(\mu,v)$ into equation (7.4) to get the *bifurcation equation*

$$\gamma(\mu,v) := PF(\mu, v+u'(\mu,v)) = 0. \tag{7.30}$$

Corollary 13. *In a neighbourhood of* $(n,0)$ *in* $\mathbb{R}_{>0}\times\mathcal{X}$*, the pair* (μ,u) *solves* (6.7) *if and only if* (μ,v) *solves the finite-dimensional equation* (7.30)*. Moreover, a solution of* (6.7) *can be constructed from a solution* (μ,v) *of* (7.30) *by setting* $u = v+u'(\mu,v)$*, where* $u'(\mu,v)$ *is given by Lemma 12.*

Since $F:\mathbb{R}_{>0}\times\mathcal{X}\to\mathcal{Y}$ and $u':\mathbb{R}_{>0}\times\mathcal{K}\to\mathcal{Y}^\perp$ have been shown to be continuously differentiable of all orders, we conclude:

Corollary 14. $\gamma:\mathbb{R}\times\mathcal{K}\to\mathcal{K}$ *is continuously differentiable of all orders.*

Furthermore, $\gamma(\mu,v)$ inherits the following symmetry of $F(\mu,u)$, which we will use to find a solution of (7.30):

Lemma 15. *Let* T_δ *be given by* (6.12)*. For every* $\delta\in\mathbb{R}$ *and* (μ,v) *in a neighbourhood of* $(n,0)$*, we have*

$$u'(\mu,T_\delta v) = T_\delta u'(\mu,v), \tag{7.31}$$

$$\gamma(\mu,T_\delta v) = T_\delta\gamma(\mu,v). \tag{7.32}$$

Proof. For equation (7.31), we note that by Proposition 10 (*iv*)

$$\begin{aligned}P^\perp F(\mu,T_\delta v + T_\delta u'(\mu,v)) &= P^\perp T_\delta F(\mu, v+u'(\mu,v)) \\ &= T_\delta P^\perp F(\mu,v+u'(\mu,v)) = 0.\end{aligned} \tag{7.33}$$

(Here we used $P^\perp T_\delta = T_\delta P^\perp$, which follows because $T_\delta = e^{i\delta}\oplus 1\oplus 1\oplus 1$ and $P^\perp = 1-P$ where P is defined in (7.1).) Since $u' = u'(\mu,T_\delta v)$ is the unique solution to $P^\perp F(\mu,T_\delta v+u')=0$ for (μ,v) in a neighbourhood $U\subset\mathbb{R}\times\mathcal{K}$ of $(n,0)$, we conlcude that $u'(\mu,T_\delta v) = T_\delta u'(\mu,v)$.

For equation (7.32), we note that by (7.31) and Proposition 10 (*iv*),

$$\begin{aligned}\gamma(\mu,T_\delta v) &= PF(\mu,T_\delta v+u'(\mu,T_\delta v)) = PF(\mu,T_\delta(v+u'(\mu,v))) \\ &= T_\delta PF(\mu,v+u'(\mu,v)) = T_\delta\gamma(\mu,v)\end{aligned} \tag{7.34}$$

(where again we used $PT_\delta = T_\delta P$). $\square$

Chapter 8

The bifurcation result when $n = 1$

Theorem 16. *Assume that $n = 1$ and $|1 - b_*/b| \ll 1$, $b_* := M_W^2/e$. Then there exists $\epsilon > 0$ and a branch $(\mu_s, u_s) := (\mu_s, w_s, \alpha_s, z_s, \psi_s)$, with $s \in [0, \sqrt{\epsilon})$, of non-trivial solutions of equation (6.1), unique modulo a gauge symmetry in a sufficiently small neighbourhood of the rescaled vacuum solution (4.9) in $\mathbb{R}_{>0} \times \mathcal{X}$, such that*

$$\begin{cases} w_s = s\chi + sg_1(s^2), \\ \alpha_s = g_2(s^2), \\ z_s = g_3(s^2), \\ \psi_s = g_4(s^2), \\ \mu_s = n + g_5(s^2), \end{cases} \tag{8.1}$$

where χ solves the eigenvalue problem $H_1(n)\chi = 0$ (see Proposition 8), $\mu := g^2\xi^2/2 = g^2 r^2 \varphi_0^2/2$, $g_1 : [0, \epsilon) \to \mathcal{H}_n^2$ and is orthogonal to $\mathrm{Null}(H_1(n))$, $g_2 : [0, \epsilon) \to \mathcal{H}_0^2$, $g_3 : [0, \epsilon) \to \mathcal{H}_0^2$, $g_4 : [0, \epsilon) \to \mathcal{H}^2$, $g_5 : [0, \epsilon) \to \mathbb{R}_{>0}$, and g_j for $j = 1, \cdots, 5$ are functions, continuously differentiable of all orders in s, such that $g_j(0) = 0$.

Proof of Theorem 16. For the proof below, recall that we denote the partial (real) Gâteaux derivatives with respect to $\#$ by $\delta_\#$, and let $\partial_i \equiv \partial_{x_i}$.

By Proposition 9, solving equation (6.1) is equivalent to solving (6.7). By Corollary 13, solving (6.7) is equivalent to solving the bifurcation equation (7.30). Hence, we address the latter equation.

Recall that P is the projection onto $\mathcal{K} = \mathrm{Null}\, L_{n,\mu=n} = \mathrm{Null}(H_1(n)) \times \{\text{constants}\} \times \{0\} \times \{0\}$. The projection onto constant vector fields in $\mathcal{H}_0^2$ can be written as the mean value $\langle \alpha \rangle := \frac{1}{|\Omega'|} \int_{\Omega'} \alpha$. Since $\dim \mathrm{Null}(H_1(n)) = 1$ for $n = 1$, we may choose $\chi \in \mathrm{Null}(H_1(n))$ such that

$$P(w, \alpha, z, \psi) = (s\chi, c, 0, 0), \tag{8.2}$$

$$s := \langle \chi, w \rangle_{L_n^2} \in \mathbb{C}, \ c := \langle \alpha \rangle \in \mathbb{R}^2, \tag{8.3}$$

and χ satisfies $\|\chi\|_{L_n^2}^2 = \langle |\chi|^2 \rangle = 1$ (see (5.16)), where, recall, χ is described in (5.21). Hence we may write the γ from the bifurcation equation (7.30) as $\gamma = (\tilde{\gamma}_1 \chi, \tilde{\gamma}_2, 0, 0)$, where $\tilde{\gamma}_1, \tilde{\gamma}_2 : \mathbb{R}_{>0} \times \mathbb{C} \times \mathbb{R}^2 \to \mathbb{C}$

are given by

$$\tilde{\gamma}_1(\mu, s, c) := \langle \chi, F_1(\mu, v(s,c) + u'(\mu, v(s,c)) \rangle_{L^2_n}, \tag{8.4}$$

$$\tilde{\gamma}_2(\mu, s, c) := \langle F_2(\mu, v(s,c) + u'(\mu, v(s,c)) \rangle, \tag{8.5}$$

where, recall, $F_j,\ j = 1, \ldots, 4$ are defined by (6.9), $s \in \mathbb{C},\ c \in \mathbb{R}^2$ and (see (8.2))

$$v(s,c) := (s\chi, c, 0, 0). \tag{8.6}$$

Note that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are continuously differentiable of all orders in μ, s and c by Corollary 14. ($\tilde{\gamma}_2$ is independent of μ.) The bifurcation equation (7.30) is then equivalent to the equations

$$\tilde{\gamma}_1(\mu, s, c) = 0, \tag{8.7}$$

$$\tilde{\gamma}_2(\mu, s, c) = 0. \tag{8.8}$$

Lemma 17. *There exists a neighbourhood $U \subset \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ of $(n, 0)$ and a unique function $c : U \to \mathbb{R}^2$ with continuous derivatives of all orders such that*

$$\tilde{\gamma}_2(\mu, s, c(\mu, s^2)) = 0 \tag{8.9}$$

and

$$||\partial^l_\mu c(\mu, s^2)||_{\mathbb{R}^2} = \mathcal{O}(|s|^2),\ \ l = 0, 1. \tag{8.10}$$

Proof. Recall that $F_2(\mu, u) = H_2(\mu)\alpha + P_0 J_2(\mu, u)$ (see Equation (6.7)), with P_0 the projection onto the divergence-free vector fields and

$$u = (w, \alpha, z, \psi) = v + u', \tag{8.11}$$

where $v = v(s,c)$ and $u' = u'(\mu, v)$ solves (7.5). By definition, $(\mathbf{1} - P_0)f = \Delta^{-1}\nabla \operatorname{div} f$ and therefore $\langle(\mathbf{1} - P_0)f\rangle = 0$. Hence $\langle P_0 f\rangle = \langle f\rangle$. This and the relation $\langle H_2(\mu)\alpha\rangle = \frac{1}{|\Omega'|}\int_{\Omega'} \operatorname{curl}^* \operatorname{curl}\alpha = 0$ give

$$\tilde{\gamma}_2(\mu, s, c) = \langle J_2(\mu, v(s,c) + u'(\mu, v(s,c)))\rangle. \tag{8.12}$$

Using (6.3), $\nu = a^n + \tilde{\nu}$, $\operatorname{curl}_{a^n} w = \operatorname{curl}_{a^n} w - i\tilde{\nu} \times w$ and that the final term in (6.3) vanishes after taking the mean, we find

$$\langle J_2(\mu, u)\rangle = 2e \operatorname{Im}\langle(\operatorname{curl}_{a^n} w - i\tilde{\nu} \times w) J\overline{w}\rangle. \tag{8.13}$$

Recall $u' = (w', \alpha', z', \psi')$. Then (8.6) and (8.11) give $w = s\chi + w'$ and (using that $e = g\sin\theta$) $\tilde{\nu} = ec + \nu'$. Using these relations and $\operatorname{curl}_{a^n}\chi = 0$ (by (5.21)) and (8.12) and (8.13), we find for

$\overline{\gamma}_2(\mu, s, c) := (2e)^{-1}|s|^{-2}\tilde{\gamma}_2(\mu, s, c)$

$$\overline{\gamma}_2(\mu, s, c) := -e\langle \mathrm{Re}[(c \times \chi)J\overline{\chi}]\rangle + \mathrm{Im}\, s^{-1}\langle (\mathrm{curl}_{a^n} w')J\overline{\chi}\rangle \tag{8.14}$$

$$+ \mathrm{Im}\langle \bar{R}_2(\mu, s, c)\rangle, \tag{8.15}$$

$$\bar{R}_2(\mu, s, c) := |s|^{-2}[-i(ec \times s\chi)J\overline{w'} - i(ec \times w')J\overline{w'} \tag{8.16}$$

$$- i(ec \times w')J\overline{s\chi} - i(\nu' \times w')J\overline{s\chi} - i(\nu' \times s\chi)J\overline{w'} \tag{8.17}$$

$$- i(\nu' \times s\chi)J\overline{s\chi} - i(\nu' \times w')J\overline{w'} + (\mathrm{curl}_{a^n} w')J\overline{w'}]. \tag{8.18}$$

Note that we expect (8.14) $= \mathcal{O}(|s|^2)$ and (8.15) $= \mathcal{O}(|s|^4)$. We now simplify (8.14). For the first term on the right-hand side, we use (5.21) and the condition $\langle |\chi|^2\rangle = 1$ to compute

$$\langle \mathrm{Re}[(c \times \chi)J\overline{\chi}]\rangle = -\frac{1}{2}c. \tag{8.19}$$

For the second term on the right-hand side of (8.14), we use $\langle fJ\overline{\chi}\rangle = \langle f(i\overline{\eta}, \overline{\eta})\rangle = \langle f\overline{\eta}\rangle(i, 1) = \langle \eta, f\rangle(i, 1)$ and integrate by parts to compute

$$\langle (\mathrm{curl}_{a^n} w')J\overline{\chi}\rangle = \langle \eta, \mathrm{curl}_{a^n} w'\rangle(i, 1) = \langle \mathrm{curl}^*_{a^n} \eta, w'\rangle(i, 1). \tag{8.20}$$

Abusing notation, we write in what follows $w(\mu, s, c) \equiv w(\mu, v(s, c))$. Then (8.14) becomes

$$\overline{\gamma}_2(\mu, s, c) = \frac{1}{2}ec + \mathrm{Im}\, s^{-1}\langle \mathrm{curl}^*_{a^n} \eta, w'(\mu, s, c)\rangle\rangle(i, 1) + \mathrm{Im}\langle \tilde{R}_2(\mu, s, c)\rangle. \tag{8.21}$$

Now, Equation (7.10), with $m = 0$, implies that

$$|\,\mathrm{Im}\langle \mathrm{curl}^*_{a^n} \eta, w'(\mu, s, c)\rangle\rangle| = \mathcal{O}(|s|^2). \tag{8.22}$$

Furthermore, we show below the following estimate on the remainder:

$$||\,\mathrm{Im}\langle \partial^l_c \tilde{R}_2(\mu, s, c)\rangle||_{\mathbb{R}^2} = \mathcal{O}(|s|^{2-l}),\ l = 0, 1. \tag{8.23}$$

Hence $\overline{\gamma}_2(\mu, 0, 0) = 0$. To apply the Implicit Function Theorem to solve for c as a function of μ and s, we have to estimate the derivative

$$\partial_c\overline{\gamma}_2(\mu, s, c) = \frac{1}{2}e\mathbf{1} + \mathrm{Im}\, s^{-1}\langle \mathrm{curl}^*_{a^n} \eta, \partial_c w'(\mu, s, c)\rangle(i, 1)$$

$$+ \mathrm{Im}\langle \partial_c \tilde{R}_2(\mu, s, c)\rangle. \tag{8.24}$$

at $(n, s, 0)$. At the first step, we use the following

Lemma 18. *Using Dirac's bra-ket notation, we have*

$$(\partial_c w')(n, s, 0) = -n^{-1}es|\,\mathrm{curl}^*_{a^n} \eta\rangle\langle(1, i)| + \mathcal{O}(|s|^2). \tag{8.25}$$

Proof of Lemma 18. By definition (7.2), $P_1^{\perp}$ projects onto the orthogonal complement of the eigenspace of $H_1(n)$ corresponding to the eigenvalue 0 and therefore the operator $H_1^{\perp}(n)$ is invertible on $\mathrm{Ran}\, P_1^{\perp}$. Hence (6.1) with $i = 1$ can be rewritten as $w' = -(H_1^{\perp}(n))^{-1}P_1^{\perp}J_1(n, u)$ (which is the first component of (7.18)), which gives

$$\partial_c w' = -(H_1^{\perp}(n))^{-1} P_1^{\perp} \partial_c J_1(n, u), \tag{8.26}$$

where $u \equiv u(s, c) := v(s, c) + u'(\mu, v(s, c))$. By (6.2) and (6.6), we have

$$\partial_c J_1(n, u) = \partial_c[\operatorname{curl}_\nu^* \operatorname{curl}_\nu w]. \tag{8.27}$$

Using $w = s\chi + w'$, $\nu = a^n + ec + \nu'$ and $\operatorname{curl}_\nu = \operatorname{curl}_{a^n} + iJ(ec + \nu')\cdot$, $\operatorname{curl}_\nu^* = \operatorname{curl}_{a^n}^* - iJ(ec + \nu')$ and that $\nu' = \mathcal{O}(|s|^2)$, we compute

$$\begin{aligned}
\partial_c J_1(n, u)c' &= s\partial_c[\operatorname{curl}_\nu^* \operatorname{curl}_\nu]\chi c' + \mathcal{O}(|s|^2) \\
&= sie[-Jc' \operatorname{curl}_\nu + \operatorname{curl}_\nu^* Jc'\cdot]\chi + \mathcal{O}(|s|^2) && (8.28)\\
&= sie[-Jc' \operatorname{curl}_{a^n+ec} + \operatorname{curl}_{a^n+ec}^* Jc'\cdot]\chi + \mathcal{O}(|s|^2). && (8.29)
\end{aligned}$$

Since $\operatorname{curl}_{a^n} \chi = \nabla_1 i\eta - \nabla_2 \eta = i\bar{\partial}_{a^n}\eta = 0$ and $Jc' \cdot \chi = (-c_2', c_1') \cdot (\eta, i\eta) = -c_2'\eta + c_1' i\eta = i(c_1' + ic_2')\eta$ and therefore $\operatorname{curl}_{a^n}^* Jc' \cdot \chi = i \operatorname{curl}_{a^n}^* \eta(c_1' + ic_2')$, this yields

$$\partial_c J_1(n, u)c'\Big|_{c=0} = -se \operatorname{curl}_{a^n}^* \eta(c_1' + ic_2') + \mathcal{O}(|s|^2). \tag{8.30}$$

By Proposition 8(ii), $\operatorname{Null}(H_1(\mu) - \mu + n) = \{\chi = (\eta, i\eta) : \operatorname{curl}_{a^n} \chi = i\bar{\partial}_{a^n}\eta = 0\}$. The relation $\operatorname{curl}_{a^n} \chi = 0$ implies also $\langle \chi, \operatorname{curl}_{a^n}^* \chi \rangle = \langle \operatorname{curl}_{a^n} \chi, \chi \rangle = 0$, which, for $n = 1$, gives that $P_1^{\perp} \partial_c J_1(n, u)c' = \partial_c J_1(n, u)c'$ and therefore

$$P_1^{\perp} \partial_c J_1(n, u)c' = -se \operatorname{curl}_{a^n}^* \eta(c_1' + ic_2') + \mathcal{O}(|s|^2). \tag{8.31}$$

By (5.21), we have $\operatorname{curl}_{a^n}^* \eta = i\nabla_{a^n}\eta$, and by (5.22), we have $H_1(n)\nabla_{a^n}\eta = n\nabla_{a^n}\eta$; hence $(H_1^{\perp}(n))^{-1} \operatorname{curl}_{a^n}^* \eta = n^{-1} \operatorname{curl}_{a^n}^* \eta$. This relation, together with (8.26) and (8.31), yields

$$\partial_c w' c' = sen^{-1} \operatorname{curl}_{a^n}^* \eta(c_1' + ic_2') + \mathcal{O}(|s|^2), \tag{8.32}$$

which gives (8.25). □

Using Equation (8.25), we calculate the second term on the right-hand side of (8.24) at $(n, s, 0)$:

$$\begin{aligned}
&\operatorname{Im} s^{-1} \langle \operatorname{curl}_{a^n}^* \eta, \partial_c w'(\mu, s, c)c' \rangle (i, 1) \\
&= en^{-1} \operatorname{Im} \langle \operatorname{curl}_{a^n}^* \eta, \operatorname{curl}_{a^n}^* \eta \rangle (c_1' + ic_2')(i, 1). && (8.33)
\end{aligned}$$

The inner product term is real. Integrating it by parts and using that, by Equation (5.32), η satisfies $\operatorname{curl}_{a^n} \operatorname{curl}_{a^n}^* \eta = -\Delta_{a^n}\eta = n\eta$ and using that $\|\eta\|_{L_n^2}^2 = \frac{1}{2}\|\chi\|_{L_n^2}^2 = \frac{1}{2}$, gives

$$\langle \operatorname{curl}_{a^n}^* \eta, \operatorname{curl}_{a^n}^* \eta \rangle = \langle \eta, -\Delta_{a^n}\eta \rangle_{L_n^2} = \frac{1}{2}n. \tag{8.34}$$

The last two equations and the relation $\operatorname{Im}(c_1' + ic_2')(i,1) = \operatorname{Im}\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} c' = \mathbf{1}c'$ imply

$$\operatorname{Im} s^{-1} \langle \operatorname{curl}^*_{a^n} \eta, \partial_c w'(\mu, s, c) \rangle (i, 1) = \frac{1}{2} e\mathbf{1}. \tag{8.35}$$

This, together with (8.24), gives

$$\partial_c \overline{\gamma}_2(n, s, 0) = \frac{1}{2} e\mathbf{1} + \frac{1}{2} e\mathbf{1} + \operatorname{Im}\langle \partial_c \tilde{R}_2(n, s, 0) \rangle. \tag{8.36}$$

Therefore, (8.36) and (8.23) (with $l = 1$) imply

$$\partial_c \overline{\gamma}_2(n, 0, 0) = e\mathbf{1}, \tag{8.37}$$

proving that $\partial_c \overline{\gamma}_2(n, 0, 0)$ is invertible, as required.

Recall that, by (8.21), (8.22) and (8.23) (with $l = 0$), we have

$$\overline{\gamma}_2(n, 0, 0) = 0. \tag{8.38}$$

Since $\partial_c \overline{\gamma}_2(n, 0, 0)$ is invertible, by the Implicit Function Theorem there exists a unique function $\tilde{c} : \mathbb{R}_{>0} \times \mathbb{C} \to \mathbb{R}^2$ with continuous derivatives of all orders such that $\overline{\gamma}_2(\mu, s, \tilde{c}(\mu, s)) = 0$ for (μ, s) in a sufficiently small neighbourhood of $(n, 0)$. Furthermore, the symmetry (7.32) implies that $\overline{\gamma}_2(\mu, |s|, \tilde{c}(\mu, s)) = \overline{\gamma}_2(\mu, e^{i \arg s}|s|, \tilde{c}(\mu, s)) = \overline{\gamma}_2(\mu, s, \tilde{c}(\mu, s)) = 0$, so by the uniqueness of the branch $\tilde{c}(\mu, s)$ we have

$$\tilde{c}(\mu, s) = \tilde{c}(\mu, |s|). \tag{8.39}$$

In particular, $\partial_\mu^l \tilde{c}(\mu, s)$, $l = 0, 1$, restricted to $s \in \mathbb{R}$ are even functions with continuous derivatives of all orders; thus $\partial_s \partial_\mu^l \tilde{c}(\mu, 0) = 0$ and hence $\partial_\mu^l \tilde{c}(\mu, s) = \mathcal{O}(|s|^2)$, since the first two terms of the Taylor expansion are 0. We define $c : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}^2$ by $c(\mu, s) := \tilde{c}(\mu, \sqrt{s})$, which is a function with continuous derivatives of all orders satisfying $||\partial_\mu^l c(\mu, s^2)||_{\mathbb{R}^2} = \mathcal{O}(|s|^2)$, $l = 0, 1$, and $\tilde{\gamma}_2(\mu, s, c(\mu, s^2)) = |s|^2 \overline{\gamma}_2(\mu, s, c(\mu, s^2)) = 0$, as required. □

Lemma 19. *For $\epsilon > 0$ sufficiently small, there exists a unique function $\mu : [0, \epsilon) \to \mathbb{R}_{>0}$ with continuous derivatives of all orders such that*

$$\tilde{\gamma}_1(\mu(s^2), s, c(\mu(s^2), s^2)) = 0. \tag{8.40}$$

Proof. To simplify notation for this lemma, we set $u = v_s + u_s'$, with $v \equiv v_s \equiv (s\chi, c(\mu, s^2), 0, 0)$, $u' \equiv u_s' \equiv u'(\mu, v_s)$, $c \equiv c(\mu, s^2)$.

We first show that $\tilde{\gamma}_1(\mu, s, c) \in \mathbb{R}$ for $s \in \mathbb{R}$. Since u' by definition solves $P_1^\perp F_1(\mu, v + u') = 0$, where $P_1^\perp w' = w'$ and $P_1^\perp$ is self-adjoint, we have

$$\langle w', F_1(\mu, v + u') \rangle_{L^2_n} = \langle w', P_1^\perp F_1(\mu, v + u') \rangle_{L^2_n} = 0. \tag{8.41}$$

Therefore, for $s \neq 0$, we find

$$\begin{aligned} \tilde{\gamma}_1(\mu, s, c) &= s^{-1}\langle s\chi, F_1(\mu, v + u'\rangle_{L^2_n} \\ &= s^{-1}\langle s\chi + w', F_1(\mu, v + u')\rangle_{L^2_n}, \end{aligned} \tag{8.42}$$

which is real by Proposition 10 (v). Furthermore, by equations (7.32) and (8.39), we have $\tilde{\gamma}_1(\mu, s, c(\mu, s^2)) = e^{i\arg(s)}\tilde{\gamma}_1(\mu, |s|, c(\mu, |s|^2))$, so we may restrict s to be real.

Next, we show that

$$\tilde{\gamma}_1(n, s, c(n, s^2)) = \mathcal{O}(|s|^2) \tag{8.43}$$

Indeed,

$$\begin{aligned} |\tilde{\gamma}_1(n, s, c(n, s^2))| &\leq ||\chi||_{L^2_n}||F_1(n, v + u')||_{L^2_n} \\ &\leq ||\chi||_{L^2_n}\big[||H_1(n)(s\chi + w')||_{L^2_n} \\ &\quad + ||J_1(n, v + u')||_{L^2_n})\big]. \end{aligned} \tag{8.44}$$

Recall that $H_1(n)\chi = 0$, so that

$$\begin{aligned} |\tilde{\gamma}_1(n, s, c(n, s^2))| \leq ||\chi||_{L^2_n}\big[&||H_1(n)||_{L^2_n \otimes (L^2_n)^*}||w'||_{L^2_n} \\ &+ ||J_1(n, v + u')||_{L^2_n}\big]. \end{aligned} \tag{8.45}$$

By the definition $v \equiv v_s \equiv (s\chi, c(\mu, s^2), 0, 0)$ and equation (8.10), $||v||_{\mathcal{X}} = \mathcal{O}(|s|)$; hence by Lemma 12,

$$||w'||_{L^2_n} \leq ||w'||_{\mathcal{H}^2_n} = \mathcal{O}(|s|^2). \tag{8.46}$$

Furthermore, by equation (7.20) and recalling that $H_1(n)\chi = 0$,

$$||J_1(n, v + u')||_{L^2_n} \leq ||J_1(n, v + u')||_{\mathcal{H}^2_n} \lesssim ||v + u'||^2_{\mathcal{X}} = \mathcal{O}(|s|^2). \tag{8.47}$$

This proves that (8.45) is $\mathcal{O}(|s|^2)$, as required.

In light of equation (8.43), we can define a function $\overline{\gamma}_1 : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}$ with continuous derivatives of all orders by

$$\overline{\gamma}_1(\mu, s) \equiv \begin{cases} s^{-1}\tilde{\gamma}_1(\mu, s, c(\mu, s^2)), & s \neq 0, \\ 0, & s = 0. \end{cases} \tag{8.48}$$

We now find a non-trivial branch of solutions $(\mu, s) = (\tilde{\mu}(s), s)$ by applying the Implicit Function Theorem to $\overline{\gamma}_1$.

Lemma 20. *There exists $\epsilon > 0$ and a unique function $\tilde{\mu} : (-\sqrt{\epsilon}, \sqrt{\epsilon}) \to \mathbb{R}_{>0}$ with continuous derivatives of all orders such that $\tilde{\mu}(0) = n$ and $\mu = \tilde{\mu}(s)$ solves $\overline{\gamma}_1(\mu, s) = 0$ for $s \in (-\sqrt{\epsilon}, \sqrt{\epsilon})$. Moreover, $\tilde{\mu}$ is an even function: $\tilde{\mu}(s) = \tilde{\mu}(-s)$.*

Proof. Recall that $F_1(\mu, u) = H_1(\mu)w + J_1(\mu, u)$ (where $H_1(\mu)$ and $J_1(\mu, u)$ are defined in (5.7) and (6.2)). Using that $\partial_\mu F_1(\mu, u) = (1 + \frac{g}{2\sqrt{2\mu}}\psi)w$ and setting $u = v_s + u'_s$, with $v \equiv v_s \equiv (s\chi, c(\mu, s^2), 0, 0)$, $u' \equiv$

$u'_s \equiv u'(\mu, v_s),\ c = c(\mu, s^2)$, we compute

$$\partial_\mu[s^{-1}F_1(\mu, v + u')] = s^{-1}(1 + \frac{g}{2\sqrt{2\mu}}\psi')(s\chi + w')$$
$$+ s^{-1}\sum_{i=1}^{4}\delta_{u_i}F_1 w(\mu, v + u')(\partial_\mu v_i + \partial_\mu u'_i)$$
$$= s^{-1}(1 + \frac{g}{2\sqrt{2\mu}}\psi')(s\chi + w') + s^{-1}\delta_\alpha F_1(\mu, v + u')\partial_\mu c$$
$$+ s^{-1}\sum_{i=1}^{5}\delta_{u_i}F_1(\mu, v + u')\partial_\mu u'_i. \tag{8.49}$$

By Lemma 17, $\|\partial^l_\mu c\|_{\mathbb{R}^2} = \mathcal{O}(|s|^2)$, $l = 0, 1$. Since $||v||_{\mathcal{X}}$ is $\mathcal{O}(|s|)$, by Lemma 12, the terms $||\partial^l_\mu u'_i||$ ($l = 0, 1$, $i = 1, \dots, 4$, with the norms taken in the appropriate spaces), are $\mathcal{O}(|s|^2)$. By Proposition 11, this implies that all terms in (8.49) containing $c, w', \alpha', z', \psi'$ or their μ-derivatives vanish at $(\mu, s) = (n, 0)$. Therefore

$$\partial_\mu[s^{-1}F_1(\mu, v + u')]|_{(\mu,s)=(n,0)} = \chi \tag{8.50}$$

and hence

$$\partial_\mu\overline{\gamma}_1(n, 0) = \langle\chi, \partial_\mu[s^{-1}F_1(\mu, s)]\,|_{(\mu,s)=(n,0)}\rangle_{L^2_n} = \|\chi\|^2_{L^2_n} \neq 0. \tag{8.51}$$

Since $\overline{\gamma}_1(\mu, s)$ is continuously differentiable of all orders in μ and s, by the Implicit Function Theorem, we obtain the first statement of the lemma.

By the symmetry $\overline{\gamma}_1(\mu, -s) = -\overline{\gamma}_1(\mu, s)$ of $\overline{\gamma}_1$ and the uniqueness of the branch $\tilde{\mu}(s)$, we have $\tilde{\mu}(s) = \tilde{\mu}(-s)$, which gives the second statement. □

We define $\mu(s) \equiv \tilde{\mu}(\sqrt{s})$, which is a function with continuous derivatives of all orders for $s \in [0, \epsilon)$ for the same reasons that $c(\mu, s) := \tilde{c}(\mu, \sqrt{s})$ was shown to be continuously differentiable of all orders in Lemma 17. Furthermore, μ satisfies $\tilde{\gamma}_1(\mu(s^2), s, c(\mu(s^2), s^2)) = s\overline{\gamma}_1(\mu(s^2), s, c(\mu(s^2), s^2)) = 0$, as required. □

We will now use the branch of solutions to (8.7) - (8.8), provided by Lemmas 17 and 19, and Corollary 13 to obtain the corresponding unique branch, (μ_s, u_s), of solutions to (6.7), with

$$\mu_s \equiv \mu(s^2), \quad u_s \equiv v_s + u'_s, \tag{8.52}$$
$$v_s \equiv (s\chi, c_s, 0, 0), \quad c_s \equiv c(\mu_s, s^2), \tag{8.53}$$
$$u'_s \equiv u'(\mu, v_s). \tag{8.54}$$

(8.52) - (8.54) have continuous s-derivatives of all orders because each component function has continuous derivatives of all orders. Symmetry (7.31) with $\delta = \pi$ and the relation $T_\pi(f_1, f_2, f_3, f_4) = (-f_1, f_2, f_3, f_4)$ imply that $(u'_s)_1$ is an odd function of s and $(u'_s)_2$, $(u'_s)_3$ and $(u'_s)_4$ are even functions of s. Arguing as in the case of Lemma 17 above shows that the functions:

$$g_1(s) := \begin{cases} \frac{1}{\sqrt{s}}(u'_{\sqrt{s}})_1, & s \neq 0, \\ 0, & s = 0, \end{cases} \quad g_2(s) := c_{\sqrt{s}} + (u'_{\sqrt{s}})_2, \tag{8.55}$$

$$g_3(s) := (u'_{\sqrt{s}})_3, \quad g_4(s) := (u'_{\sqrt{s}})_4, \quad g_5(s) := \mu_{\sqrt{s}} - n, \tag{8.56}$$

are well-defined for $s \geq 0$ and have continuous derivatives of all orders. By Lemma 12, these functions have the properties listed in Theorem 16. The above definitions and equations (8.52) - (8.54) imply $u_s = (s\chi, \frac{1}{e}a^n, 0, 0) + (g_1(s), \ldots, g_4(s))$. Hence, this solution is of the form (8.1). Now, by Proposition 9, this also solves system (4.3) - (4.6), completing the proof. $\square$

Chapter 9

Proof of Theorem 2(a), (b)

Recall that M_W, M_Z, M_H are the masses of the W, Z and Higgs bosons, respectively, and that τ is the shape parameter of the lattice $\mathcal{L}$ (see the paragraph before Theorem 3 of Section 2). We will need the following function:

$$\alpha(M_Z, M_H; \tau) := \langle U_{M_Z,M_H}(|X_r|^2)|X_r|^2\rangle/\langle |X_r|^2\rangle^2, \tag{9.1}$$

where

$$X_r(x) := r^{-1}\chi(r^{-1}x), \tag{9.2}$$

with χ given in (5.21), and for $f \in \mathcal{L}^2_{loc}$,

$$U_{M_1,M_2}(f)(\rho) := \frac{1}{2\pi}\int_{M_1}^{M_2}\int |\rho-\rho'|K_1(M|\rho-\rho'|)f(\rho')d^2\rho'\, dM, \tag{9.3}$$

with K_1 a modified Bessel function of the third kind. α is related to the η function appearing in [23] by

$$\eta(M_Z, M_H; \tau) = [M_W^2\alpha(M_Z, M_H; \tau) + \sin^2\theta]^{-1}.^1 \tag{9.4}$$

Theorem 21. *If $M_Z < M_H$, the parameter s of the branch* (8.1) *is related to the magnetic field strength by*

$$s^2 = \frac{eb}{g^2\langle |X_r|^2\rangle}\eta(M_Z, M_H; \tau)[1-\frac{M_W^2}{eb}] + R_s([1-\frac{M_W^2}{eb}]), \tag{9.5}$$

where $R_s(\lambda)$ is a real function with continuous derivatives of all orders satisfying $R_s(\lambda) = \mathcal{O}(|\lambda|^2)$, so that

$$R_s([1-\frac{M_W^2}{eb}]) = \mathcal{O}(|1-\frac{M_W^2}{eb}|^2). \tag{9.6}$$

[1]The authors of [23] used the notation $\eta(M_Z/M_H)$, remarking that η only depends on the ratio M_Z/M_H to the order of magnitude they were calculating.

Before proving Theorem 21, we shall see how it implies statements (a) and (b) of Theorem 2. Since K_1 is positive, when $M_Z < M_H$, the function
$U_{M_Z,M_H}(|X_r|^2)(\rho)$ is a positive function; in particular, $\alpha(M_Z, M_H; \tau)$ (and hence $\eta(M_Z, M_H, \tau)$) is positive. Furthermore, when the right-hand side of (9.5) is positive, we may take the square root, solving for s as a function of b, $s = s(b)$, having continuous derivatives of all orders. When $|1 - \frac{M_W^2}{eb}| \ll 1$, the right-hand side of (9.5) is positive if and only if $1 - \frac{M_W^2}{eb} > 0$.[2] Plugging $s = s(b)$ into (8.1) (i.e. passing from the bifurcation parameter s to the physical parameter b), undoing the rescaling (4.1), and recalling that $b_* = \frac{M_W^2}{e}$, we arrive at the branch, $U_{\mathcal{L}} \equiv (W_b, A_b, Z_b, \varphi_b)$, of solutions of (3.12) - (3.15), which has the properties listed in statements (a) and (b) of Theorem 2. □

The following statement follows from the proof above:

Proposition 22. *$U_{\mathcal{L}}$ is continuously differentiable of all orders in b for b in an open right half-interval of b_**

In the proof below, we will use the following result:

Proposition 23. *Let L^2_{per} denote any of the spaces (5.13) - (5.15), and let $\mathcal{H}^2_{per}$ denote the corresponding Sobolev space. Suppose that $f_s, g_s : \mathbb{R} \to \mathcal{H}^2_{per}$ satisfy $||f_s||_{\mathcal{H}^2_{per}} = \mathcal{O}(|s|^k)$ and $||g_s||_{\mathcal{H}^2_{per}} = \mathcal{O}(|s|^l)$ for some $k, l \in \mathbb{Z}$. Then for $i, j = 1, 2$ and $p, q = 0, 1$,*

$$|\int_{\Omega'} \partial_i^p f_s \partial_j^q g_s| = \mathcal{O}(|s|^{k+l}). \tag{9.7}$$

Furthermore, if f_s and g_s have continuous derivatives of all orders in s, then so does the above integral.

Proof. Equation (9.7) follows from the following chain of inequalities:

$$\begin{aligned} |\int_{\Omega'} \partial_i^p f_s \partial_j^q g_s| &\lesssim ||\partial_i^p f_s||_{\mathcal{L}^2_{per}} ||\partial_j^q g_s||_{\mathcal{L}^2_{per}} \\ &\lesssim ||f_s||_{\mathcal{H}^2_{per}} ||g_s||_{\mathcal{H}^2_{per}} = \mathcal{O}(|s|^{k+l}). \end{aligned} \tag{9.8}$$

If f_s and g_s have continuous derivatives of all orders in s, then their s-derivatives of all orders are in $\mathcal{H}^2_{per}$. In particular, this means that $\partial_s^k(f_s g_s)$, $k \in \mathbb{Z}_{\geq 0}$, remains integrable, so the $s-$derivatives of the above integral (obtained by differentiation under the integral sign) are well-defined. □

Proof of Theorem 21. Consider the solution branch (μ_s, w_s, a_s, z_s) given in equation (8.1) and described in Theorem 16. Using Taylor's Theorem for Banach spaces (see e.g. [14]) and recalling the relation $\xi = \sqrt{2\mu}/g$, we may expand this branch in s as follows:

$$\begin{cases} w_s = s\chi + s^3 w' + \mathcal{O}(|s|^5), \\ a_s = \frac{1}{e} a^n + s^2 a' + s^4 a'' + \mathcal{O}(|s|^6), \\ z_s = s^2 z' + \mathcal{O}(|s|^4), \\ \psi_s := \phi_s - \xi_s = s^2 \psi' + \mathcal{O}(|s|^4), \\ \xi_s := \sqrt{2\mu_s}/g = \sqrt{2n}/g + s^2 \xi' + \mathcal{O}(|s|^4), \end{cases} \tag{9.9}$$

[2]The condition $0 < 1 - \frac{M_W^2}{eb} \ll 1$ is equivalent to the condition $0 < 1 - \frac{M_W^2}{2\pi}|\mathcal{L}| \ll 1$ of Theorem 2.

where w', a', z', ψ', ξ' and a'' are the coefficients of s^2 and s^4, respectively, in the Taylor expansion of $g_j(s^2)$, $j = 0, ..., 5$, in (8.1). Here $\mathcal{O}(|s|^p)$ stand for various error terms which, together with their (covariant) derivatives, have norms of order $\mathcal{O}(|s|^p)$ when taken in the appropriate spaces.

To rewrite the asymptotics in terms of the parameter b, we analyze how s depends on b. For this, we use the definitions $\xi_s = \sqrt{2\mu_s}/g$ and $\mu := \frac{1}{2}(g\xi)^2 = \frac{1}{2}(gr\varphi_0)^2$, with $r := \sqrt{\frac{n}{eb}}$ (see (4.2)) to find the following equation for s^2:

$$\xi_s = \sqrt{\frac{n}{eb}}\varphi_0. \tag{9.10}$$

To solve this equation for s^2, we use the Implicit Function Theorem. By (9.9), we can write $\xi_s = \sqrt{2n}/g + g_\xi(s^2)$, where recall, $g_\xi(0) = 0$ and $g'_\xi(0) = \xi'$. Hence, we have to show that $\xi' \neq 0$.

Lemma 24. *We have $\xi' \neq 0$.*

Proof. We find relations between ψ', a' and z' entering (9.9). Plugging (9.9) into Equations (4.4) - (4.6), we obtain at order s^4

$$\begin{cases} -\Delta a' - e\,\mathrm{curl}^*\,|\chi|^2 = 0 \\ (-\Delta + \frac{n}{\cos^2\theta})z' - g\cos\theta\,\mathrm{curl}^*\,|\chi|^2 = 0 \\ (-\Delta + \frac{4\lambda n}{g^2})\psi' + \frac{g}{2}\sqrt{2n}|\chi|^2 = 0, \end{cases} \tag{9.11}$$

which have the solutions

$$\begin{cases} a' = e\,\mathrm{curl}^*\,U_0(|\chi|^2) - \frac{1}{2}e\langle|\chi|^2\rangle x^\perp, \\ z' = g\cos\theta\,\mathrm{curl}^*\,U_{m_z}(|\chi|^2), \\ \psi' = -\frac{g}{2}\sqrt{2n}\,U_{m_h}(|\chi|^2), \end{cases} \tag{9.12}$$

where, recall, $x^\perp := -x_2dx_1 + x_1dx_2$, $m_z := \frac{\sqrt{n}}{\cos\theta}$ and $m_h := \frac{\sqrt{4\lambda n}}{g}$ are the masses of the rescaled Z and Higgs boson (Φ) fields, z and ϕ, respectively, and for $f \in \mathcal{L}^2_{loc}$,

$$U_M(f)(\rho) := \begin{cases} \frac{1}{2\pi}\int K_0(M|\rho-\rho'|)f(\rho')d^2\rho', & M > 0 \\ \frac{1}{2\pi}\int -\ln(|\rho-\rho'|)f(\rho')d^2\rho', & M = 0 \end{cases}$$

with K_0 a modified Bessel function of the third kind. Note that U_M satisfies

$$(-\Delta + M^2)U_M(f) = f \quad\text{and}\quad M^2\langle U_M(f)\rangle = \langle f\rangle \quad (M \neq 0). \tag{9.13}$$

Plugging equations (9.12) into the relation

$$\int_{\Omega'} g\sqrt{2n}\xi'|\chi|^2 = \int_{\Omega'} -g\sqrt{2n}\psi'|\chi|^2 + \mathrm{curl}\,\nu'|\chi|^2 - g^2|\chi|^4, \tag{9.14}$$

proven in Appendix F, and taking $\nu' := g(a'\sin\theta + z'\cos\theta)$, gives

$$g\sqrt{2n}\xi'\langle|\chi|^2\rangle = -g^2[m_w^2\langle U_{m_z,m_h}(|\chi|^2)|\chi|^2\rangle + \sin^2\theta\langle|\chi|^2\rangle^2], \tag{9.15}$$

where $m_w := \sqrt{n}$ is the mass of the rescaled W boson field w, and for $f \in \mathcal{L}^2_{loc}$,

$$U_{M_1,M_2}(f)(\rho) := \frac{1}{2\pi}\int_{M_2}^{M_1}\int |\rho-\rho'|K_1(M|\rho-\rho'|)f(\rho')d^2\rho' dM \\ = U_{M_1}(f)(\rho) - U_{M_2}(f)(\rho), \tag{9.16}$$

with K_1 a modified Bessel function of the third kind. We solve for ξ' and simplify the solution as follows:

$$\xi' = -\frac{g}{\sqrt{2n}}\langle|\chi|^2\rangle\eta'^{-1}, \tag{9.17}$$

where

$$\eta' \equiv \eta'(m_z, m_h; \tau, r) := [m_w^2\alpha'(m_z, m_h; \tau, r) + \sin^2\theta]^{-1}, \tag{9.18}$$

with, recall, $m_w := \sqrt{n}$, $m_z := \frac{\sqrt{n}}{\cos\theta}$ and $m_h := \frac{\sqrt{4\lambda n}}{g}$ the masses of the rescaled W, Z and Higgs boson fields, w, z and ϕ, respectively, and

$$\alpha'(m_z, m_h; \tau, r) := \langle U_{m_z,m_h}(|\chi|^2)|\chi|^2\rangle / \langle|\chi|^2\rangle^2. \tag{9.19}$$

Since K_1 is a positive function, $\alpha'(m_z, m_h; \tau, r)$ (and hence η') is positive if and only if $m_z < m_h$ (equivalently, $M_Z < M_H$), in which case $\xi' < 0$. □

We now derive the estimate (9.5) - (9.6) for s^2. Equations (9.9) and (9.10) give ξ_s as a function of s and b respectively, yielding

$$\xi_s^2 = [\frac{\sqrt{2n}}{g} + g_\xi(s^2)]^2 = \frac{n}{eb}\varphi_0^2, \tag{9.20}$$

which can be rearranged to give

$$\frac{2\sqrt{2n}}{g}g_\xi(s^2) + g_\xi(s^2)^2 = \frac{2n}{g^2}[1 - \frac{M_W^2}{eb}], \tag{9.21}$$

where, recall, $M_W = \frac{1}{\sqrt{2}}g\varphi_0$. Recall that $g_\xi(0) = 0$ and $g_\xi'(0) = \xi'$. We have

$$\frac{d}{ds^2}|_{s^2=0}[\frac{2\sqrt{2n}}{g}g_\xi(s^2) + g_\xi(s^2)^2] = \frac{2\sqrt{2n}}{g}\xi'. \tag{9.22}$$

Since $\xi' \neq 0$ and $g_\xi(s^2)$ is continuously differentiable of all orders (see Theorem 16), by the Implicit Function Theorem, we may solve (9.21) for s^2, with the solution, $s^2 = \sigma(1 - \frac{M_W^2}{eb})$, $\sigma : \mathbb{R} \to \mathbb{R}$, having continuous derivatives of all orders. Explicitly, (9.21) - (9.22) give

$$s^2 = \frac{g}{2\sqrt{2n}}\xi'^{-1}\frac{2n}{g^2}[1 - \frac{M_W^2}{eb}] + \mathcal{O}(|1 - \frac{M_W^2}{eb}|^2). \tag{9.23}$$

Plugging (9.17) into (9.23) gives

$$s^2 = \frac{n}{g^2}\frac{[1 - \frac{M_W^2}{eb}]}{\langle|\chi|^2\rangle}\eta' + \tilde{R}_s(1 - \frac{M_W^2}{eb}), \tag{9.24}$$

where $\tilde{R}_s(\lambda)$ satisfies $\tilde{R}_s(\lambda) = \mathcal{O}(|\lambda|^2)$. Furthermore, since the solution $s^2 = \sigma(1 - \frac{M_W^2}{eb})$ is continuously differentiable of all orders in b, so is the remainder term $\tilde{R}_s(1 - \frac{M_W^2}{eb})$.

To derive (9.5) from (9.24), we prove the following lemma in Appendix G:

Lemma 25. *$\eta(M_Z, M_H; \tau)$ is related to $\eta'(m_z, m_h; \tau, r)$ by*

$$\eta'(m_z, m_h; \tau, r) = \eta(M_Z, M_H; \tau) + R_\eta(1 - \frac{M_W^2}{eb}), \tag{9.25}$$

where $R_\eta(\lambda)$ is a real function with continuous derivatives of all orders satisfying $R_\eta(\lambda) = \mathcal{O}(|\lambda|)$, so that

$$R_\eta(1 - \frac{M_W^2}{eb}) = \mathcal{O}(|1 - \frac{M_W^2}{eb}|). \tag{9.26}$$

Equation (9.5) follows by plugging $\langle|\chi|^2\rangle = r^2\langle|X_r|^2\rangle = \frac{n}{eb}\langle|X_r|^2\rangle$ and (9.25) into (9.24). □

Chapter 10

Asymptotics of the Weinberg-Salam energy near $b = M_W^2/e$

The main result of this section is the following:

Theorem 26. *If $M_Z < M_H$, then the WS energy* (D.1) *of the branch of solutions* (8.1) *has the following expansion:*

$$\frac{1}{|\Omega|}E_\Omega^{WS}(W_b, A_b, Z_b, \varphi_b) = \frac{1}{2}b^2 - \frac{1}{2}b^2 \sin^2\theta\ \eta(M_Z, M_H, \tau)[1 - \frac{M_W^2}{eb}]^2 + R_E(1 - \frac{M_W^2}{eb}), \tag{10.1}$$

where $R_E(\lambda)$ is a real function with continuous derivatives of all orders satisfying $R_E(\lambda) = \mathcal{O}(|\lambda|^3)$ so that

$$R_E(1 - \frac{M_W^2}{eb}) = \mathcal{O}(|1 - \frac{M_W^2}{eb}|^3). \tag{10.2}$$

Before proving Theorem 26, we shall see how it implies Theorem 2 (c). Since $\eta(M_Z, M_H, \tau)$ is positive,[1] the second term in Equation (10.1) is negative, and so for $0 < 1 - \frac{M_W^2}{eb} << 1$, E_Ω^{WS} is less than the vacuum energy $\frac{1}{2}b^2|\Omega|$. This proves Theorem 2 (c).

Proof. Let $\mathcal{E}'(w_s, a_s, z_s, \psi_s + \xi_s; r) := \frac{1}{|\Omega'|}\mathcal{E}_{\Omega'}(w_s, a_s, z_s, \psi_s + \xi_s; r)$, where $\mathcal{E}_{\Omega'}$ is the rescaled WS energy given in (4.8). In Appendix H, we derive the following expansion of $\mathcal{E}'$ evaluated at family (9.9) of solutions, up to order s^4:

[1] See the discussion following Theorem 21 for details.

$$\mathcal{E}'(w_s, a_s, z_s, \psi_s + \xi_s; r) = \frac{1}{2}\frac{n^2}{e^2} + s^4 \frac{1}{|\Omega'|} \int_{\Omega'} \frac{1}{2}|\operatorname{curl} z'|^2$$
$$+\frac{1}{2}|\operatorname{curl} a'|^2 + \ g\sqrt{2n}(\psi' + \xi')|\chi|^2 + \frac{n}{2\cos^2\theta}|z'|^2$$
$$+|\nabla\psi'|^2 + \frac{4\lambda n}{g^2}\psi'^2 - |\chi|^2 \operatorname{curl} \nu' + \frac{g^2}{2}|\chi|^4$$
$$+R_\varepsilon(s), \tag{10.3}$$

where $R_\varepsilon(s) = \mathcal{O}(|s|^6)$ and has continuous derivatives of all orders, $\nu' := g(a' \sin\theta + z' \cos\theta)$ and, recall, $\xi_s = \sqrt{2\mu_s}/g$.

To simplify notation, in what follows, we shall suppress the arguments $(w_s, a_s, z_s, \psi_s + \xi_s; r)$ of $\mathcal{E}'$. We claim the following relation:

$$\mathcal{E}' = \frac{1}{2}\frac{n^2}{e^2} - s^4 \frac{g^2}{2}\langle|\chi|^2\rangle^2 \eta'^{-1} + R_\varepsilon(s), \tag{10.4}$$

where, recall, $\eta' = \eta'(m_z, m_h; \tau, r) := \left[m_w^2 \alpha(m_z, m_h, \tau) + \sin^2\theta\right]^{-1}$ and $\alpha'(m_z, m_h; \tau, r)$ is given in (9.19).

Proof of (10.4). We simplify the integral at order s^4 in (10.3) by applying equations (9.11) for a', z' and ψ' to convenient groupings of terms.

First, we address the z' terms. Integrating by parts and factoring out z' gives

$$\frac{1}{|\Omega'|}\int_{\Omega'} \frac{1}{2}|\operatorname{curl} z'|^2 + \frac{n}{\cos^2\theta}|z'|^2 = \frac{1}{|\Omega'|}\int_{\Omega'} \frac{1}{2} z' \cdot (-\Delta + \frac{n}{\cos^2\theta}) z'. \tag{10.5}$$

Applying (9.11) for z' gives

$$\frac{1}{|\Omega'|}\int_{\Omega'} \frac{1}{2}|\operatorname{curl} z'|^2 + \frac{n}{\cos^2\theta}|z'|^2 = \frac{1}{|\Omega'|}\int_{\Omega'} \frac{1}{2} z' \cdot g\cos\theta \operatorname{curl}^* |\chi|^2. \tag{10.6}$$

Integrating by parts again gives

$$\frac{1}{|\Omega'|}\int_{\Omega'} \frac{1}{2}|\operatorname{curl} z'|^2 + \frac{n}{\cos^2\theta}|z'|^2 = \frac{1}{|\Omega'|}\int_{\Omega'} \frac{1}{2} g\cos\theta(\operatorname{curl} z')|\chi|^2. \tag{10.7}$$

Next, we address the a' term. Integrating by parts gives

$$\frac{1}{|\Omega'|}\int_{\Omega'} \frac{1}{2}|\operatorname{curl} a'|^2 = \frac{1}{|\Omega'|}\int_{\Omega'} \frac{1}{2} a' \cdot (-\Delta) a'. \tag{10.8}$$

Applying (9.11) for a' gives

$$\frac{1}{|\Omega'|}\int_{\Omega'} \frac{1}{2}|\operatorname{curl} a'|^2 = \frac{1}{|\Omega'|}\int_{\Omega'} \frac{1}{2} a' \cdot e \operatorname{curl}^* |\chi|^2. \tag{10.9}$$

Integrating by parts again gives

$$\frac{1}{|\Omega'|}\int_{\Omega'} \frac{1}{2}|\operatorname{curl} a'|^2 = \frac{1}{|\Omega'|}\int_{\Omega'} \frac{1}{2} g\sin\theta(\operatorname{curl} a')|\chi|^2. \tag{10.10}$$

Next, we address the ψ' terms. Integrating by parts and factoring out ψ' gives

$$\frac{1}{|\Omega'|}\int_{\Omega'} |\nabla\psi'|^2 + \frac{4\lambda n}{g^2}\psi'^2 + g\sqrt{2n}\psi'|\chi|^2 = \frac{1}{|\Omega'|}\int_{\Omega'} \psi'(-\Delta + \frac{4\lambda n}{g^2} + g\sqrt{2n}|\chi|^2)\psi'. \tag{10.11}$$

Applying (9.11) for ψ' gives

$$\frac{1}{|\Omega'|}\int_{\Omega'} |\nabla\psi'|^2 + \frac{4\lambda n}{g^2}\psi'^2 + g\sqrt{2n}\psi'|\chi|^2 = \frac{1}{|\Omega'|}\int_{\Omega'} \frac{g}{2}\sqrt{2n}\psi'|\chi|^2. \tag{10.12}$$

The ξ' term is addressed by (9.14).

Finally, there are two remaining terms of the integral at order s^4 in (10.3), which we will not presently simplify:

$$\frac{1}{|\Omega'|}\int_{\Omega'} -|\chi|^2 \operatorname{curl}\nu' + \frac{g^2}{2}|\chi|^4. \tag{10.13}$$

Adding equations (9.14), (10.7), (10.10), (10.12) and (10.13) gives

$$\mathcal{E}' = \frac{1}{2}\frac{n^2}{e^2} + s^4\frac{1}{|\Omega'|}\int_{\Omega'} \frac{1}{2}g\sqrt{2n}\psi'|\chi|^2 - \frac{1}{2}\operatorname{curl}\nu'|\chi|^2 + \frac{1}{2}g^2|\chi|^4. \tag{10.14}$$

Equations (9.12) and (9.13) imply that

$$\operatorname{curl}\nu' = g^2|\chi|^2 - eg\sin\theta\langle|\chi|^2\rangle - g^2 m_z^2\cos^2\theta\, U_{m_z}(|\chi|^2). \tag{10.15}$$

Recall that $e = g\sin\theta$ and $m_w^2 = m_z^2\cos^2\theta = n$. Plugging Equation (10.15) for $\operatorname{curl}\nu'$ and (9.12) for ψ' into (10.14) gives Equation (10.4), as required. □

Plugging (9.24) into (10.4) gives

$$\mathcal{E}' = \frac{1}{2}\frac{n^2}{e^2} - \frac{1}{2}\frac{n^2}{g^2}[1 - \frac{M_W^2}{eb}]^2\eta'(m_z, m_h; \tau, r) + \tilde{R}_\varepsilon(1 - \frac{M_W^2}{eb}), \tag{10.16}$$

where $\tilde{R}_\varepsilon(\lambda)$ has continuous derivatives of all orders and satisfies $\tilde{R}_\varepsilon(\lambda) = \mathcal{O}(|s|^3)$.

To compute the WS energy (3.10), evaluated at $(W_b, A_b, Z_b, \varphi_b)$, we recall that $E_\Omega^{WS} = \frac{1}{r^2}\mathcal{E}_{\Omega'}$, which implies

$$\frac{1}{|\Omega|}E_\Omega^{WS} = \frac{|\Omega'|}{r^2|\Omega|}\mathcal{E}' = \frac{e^2b^2}{n^2}\mathcal{E}', \quad r = \sqrt{\frac{|\Omega|}{|\Omega'|}} = \sqrt{\frac{n}{eb}}. \tag{10.17}$$

Equation (10.1) follows by plugging (10.16) and (9.25) into (10.17). Since the remainder terms $\tilde{R}_s$, R_α and $\tilde{R}_\varepsilon$ (of (9.24), (9.25) and (10.16), respectively) have continuous derivatives of all orders, so does the remainder term R_E. □

Chapter 11

Shape of lattice solutions

In this section we shall prove Theorem 3. Recall the shape parameter τ described in the paragraph preceding (2.17). We return briefly to working with the rescaled fields to prove that $\mathcal{E}_{\Omega'}(u;r)$ (and hence $E_\Omega^{WS}(U)$), given in (4.8), is continuously differentiable of all orders in the shape parameter τ (restricted to domain (2.17)), which enters through Ω' and Ω, as well as the spaces containing u and U. Below, we write

$$u_{\tau,b}(x) \equiv (w_{\tau,b}(x), a_{\tau,b}(x), z_{\tau,b}(x), \phi_{\tau,b}(x)), \tag{11.1}$$

$$\mathcal{E}(\tau, b, u) \equiv \mathcal{E}_{\Omega'}(u; r), \tag{11.2}$$

$$U_{\tau,b}(x) \equiv (W_{\tau,b}(x), A_{\tau,b}(x), Z_{\tau,b}(x), \varphi_{\tau,b}(x)), \tag{11.3}$$

$$E(\tau, b, U) \equiv E_\Omega^{WS}(U), \tag{11.4}$$

$$\mathcal{X}_\tau \equiv \mathcal{X}, \tag{11.5}$$

to emphasize the dependence of the family of solutions (9.9), the corresponding energy (4.8) (respectively (3.10)) and the space (5.12) containing these solutions on the shape parameter τ, the magnetic field strength b and the position in space $x \in \mathbb{R}^2$. Also, recall the notation $r := \sqrt{n/eb}$.

To get rid of the dependency of the space $\mathcal{X}_\tau$ containing $u_{\tau,b}$, on the shape parameter τ, we make the change of coordinates $y = m_\tau^{-1} x$ with $m_\tau = \frac{1}{\sqrt{\mathrm{Im}(\tau)}} \begin{pmatrix} 1 & \mathrm{Re}(\tau) \\ 0 & \mathrm{Im}(\tau) \end{pmatrix}$. This defines a function

$$M_\tau : \mathcal{X}_\tau \to \mathcal{X}_1,$$
$$(M_\tau u)(x) = u(m_\tau x), \tag{11.6}$$

that is linear in u (this change of coordinates transforms Ω' into a square of area 2π). This in turn allows us to define the following functions on the fixed space $\mathcal{X}_1$:

$$G' : \mathbb{C} \times \mathbb{R} \times \mathcal{X}_1 \to \mathbb{C} \times \mathbb{R} \times \mathcal{Y}_1$$
$$G'(\tau, b, v) = M_\tau G(b, M_\tau^{-1} v), \tag{11.7}$$

and

$$\begin{aligned}\varSigma &: \mathbb{C} \times \mathbb{R} \times \mathcal{X}_1 \to \mathbb{C} \times \mathbb{R} \times \mathcal{Y}_1 \\ \varSigma(\tau, b, v) &= M_\tau \varepsilon_{WS}(b, M_\tau^{-1} v),\end{aligned} \tag{11.8}$$

where, recall, $G(b,v)$ is the map given by the left-hand side of (4.3) - (4.6), given explicitly in (5.1), and $\varepsilon_{WS}(b,u) := \varepsilon_{WS}(u;r)$ is the rescaled energy density given by the integrand in (4.8) (ε_{WS} depends on the magnetic field strength b but does not directly depend on the shape parameter τ).

Lemma 27. *$G'(\tau,b,v)$ and $\varSigma(\tau,b,v)$ are continuously differentiable of all orders in* $\mathrm{Re}(\tau)$*,* $\mathrm{Im}(\tau)$*, b and v.*

Proof. Since $G(b,u)$ and $\varepsilon_{WS}(b,u)$ have continuous b and u derivatives of all orders, and M_τ is a linear map independent of b and v, it follows that $G'(\tau,b,v)$ and $\varSigma(\tau,b,v)$ have continuous b- and v-derivatives of all orders.

For the τ-derivatives, note that

$$M_\tau \circ \partial_{x_1} \circ M_\tau^{-1}(v_j)(x) = \frac{1}{\sqrt{\mathrm{Im}(\tau)}} \partial_{x_1} v_j(x), \quad j = 1, ..., 4, \tag{11.9}$$

and

$$\begin{aligned}M_\tau \circ &\partial_{x_2} \circ M_\tau^{-1}(v_j)(x) \\ &= \frac{1}{\sqrt{\mathrm{Im}(\tau)}} (\mathrm{Re}(\tau)\partial_{x_1} v_j(x) + \mathrm{Im}(\tau)\partial_{x_2} v_j(x)), \quad j = 1, ..., 4,\end{aligned} \tag{11.10}$$

are continuously differentiable of all orders in $\mathrm{Re}(\tau)$ and $\mathrm{Im}(\tau)$. Since $G(b,u)$ and $\varepsilon_{WS}(b,u)$ are polynomials in the components of u and their (covariant) derivatives, G' and $\varSigma$ are simply G and ε_{WS} with the coefficients of the derivative-containing terms multiplied by smooth functions of $\mathrm{Re}(\tau)$ and $\mathrm{Im}(\tau)$. Therefore $G'(\tau,b,v)$ and $\varSigma(\tau,b,v)$ have continuous $\mathrm{Re}(\tau)$- and $\mathrm{Im}(\tau)$-derivatives of all orders. □

Lemma 28. *$v_{\tau,b} := M_\tau u_{\tau,b}$ is continuously differentiable of all orders in $Re(\tau)$ and $Im(\tau)$.*

Proof. Let τ_0 be an arbitrary shape parameter, and recall that $\delta_\#$ denotes the partial (real) Gâteaux derivative with respect to #. Then $G'(\tau_0, b, v_{\tau_0,b}) =$
$M_{\tau_0} G(b, u_{\tau_0,b}) = 0$, $\delta_v G(\tau_0, b, v_{\tau_0,b}) = M_{\tau_0} \circ \delta_u G(b, u_{\tau_0,b}) \circ M_{\tau_0}^{-1}$ is invertible, and by Lemma 27, G' is continuously differentiable of all orders in τ, b and v. Therefore, by the Implicit Function Theorem, the unique solution $v_{\tau,b}$ to the equation $G(\tau,b,v) = 0$ is continuously differentiable of all orders in $\mathrm{Re}(\tau)$ and $\mathrm{Im}(\tau)$ near $(\mathrm{Re}(\tau), \mathrm{Im}(\tau)) = (\mathrm{Re}(\tau_0), \mathrm{Im}(\tau_0))$. Since τ_0 was arbitrary, this proves the result. □

Proposition 29. *$E(\tau, b, U_{\tau,b})$ is continuously differentiable of all orders in $Re(\tau)$ and $Im(\tau)$.*

Proof. To get rid of the dependency of $\mathcal{E}(\tau, b, u_{\tau,b})$ on the domain of integration Ω', we again make the change of coordinates $y = m_\tau^{-1} x$. Then

$$\begin{aligned}\mathcal{E}(\tau, b, u_{\tau,b}) &= \int_{\Omega'} \varepsilon_{WS}(b, u_{\tau,b})(x)\, d^2x \\ &= \int_0^{\sqrt{2\pi}} \int_0^{\sqrt{2\pi}} \varSigma(\tau, b, v_{\tau,b})(y) d^2y.\end{aligned} \tag{11.11}$$

By Lemma 28, $v_{\tau,b}$ has continuous Re(τ)- and Im(τ)-derivatives of all orders, and by Lemma 27, Σ has continuous derivatives of all orders mapping $\mathbb{C} \times \mathbb{R} \times \mathcal{X}_1$ to $\mathbb{C} \times \mathbb{R} \times \mathcal{Y}_1$. In particular, the Re($\tau$)- and Im($\tau$)-derivatives of $\Sigma(\tau, b, v_{\tau,b})$ remain integrable, so we conclude that $\mathcal{E}(\tau, b, u_{\tau,b})$ (and hence $E(\tau, b, U_{\tau,b})$) is continuously differentiable of all orders in Re(τ) and Im(τ). □

Theorem 30. *When $M_Z < M_H$, the minimizers τ_b of $E_{WS}(\tau, b, U_{\tau,b})$ are related to the maximizers τ_* of $\eta(M_Z, M_H; \tau)$ as $\tau_b - \tau_* = \mathcal{O}(|1 - \frac{M_W^2}{eb}|^{\frac{1}{2}})$. In particular, $\tau_b \to \tau_*$ as $b \to b_* = M_W^2/e$.*

Proof. By Theorem 26, the minimizers of $E(\tau, b, U_{\tau,b})$ are equivalent to the minimizers of the energy functional

$$\begin{aligned}\tilde{E}(\tau, U_{\tau,b}) &:= [1 - \frac{M_W^2}{eb}]^{-2}(E(\tau, b, U_{\tau,b}) - \frac{1}{2}b^2) \\ &= -\frac{1}{2}b^2 \sin^2\theta \ \eta(M_Z, M_H; \tau) + \mathcal{O}(|1 - \frac{M_W^2}{eb}|).\end{aligned}$$

Since $\partial_\tau \tilde{E}(\tau, U_{\tau,b})|_{\tau=\tau_b} = 0$, we have the expansion

$$\begin{aligned}\tilde{E}(\tau_*, U_{\tau_*,b}) - \tilde{E}(\tau_b, U_{\tau_b,b}) &= \frac{1}{2}\partial_\tau^2 \tilde{E}(\tau, U_{\tau,b})|_{\tau=\tau_b}[\tau_* - \tau_b]^2 + \mathcal{O}([\tau_* - \tau_b]^3) \\ &= -\frac{1}{4}b^2 \sin^2\theta \ (\partial_\tau^2 \eta)|_{(M_Z, M_H, \tau_b)} \ [\tau_* - \tau_b]^2 \\ &\quad + \mathcal{O}([\tau_* - \tau_b]^3) + \mathcal{O}(|1 - \frac{M_W^2}{eb}|).\end{aligned} \tag{11.12}$$

For both expansions to hold, we must have $\tau_b - \tau_* = \mathcal{O}(|1 - \frac{M_W^2}{eb}|^{\frac{1}{2}})$, as required. □

The maximizer of $\eta(M_Z, M_H; \tau)$, defined in (9.4), was found numerically in [23] (cf. [26, 3]):

Theorem 31 ([23])**.** *When $M_Z < M_H$, $\eta(M_Z, M_H; \tau)$ has a maximum at $\tau_* = e^{i\pi/3}$.*

Theorem 3 follows from Theorems 30 and 31.

Appendix A

Covariant derivatives and curvature

In this appendix, we briefly review some basic definitions from gauge theory. In what follows, we use the Einstein summation convention of summing over repeated indices.

Let V be an inner product vector space, G a Lie group acting transitively on V via a unitary representation $\rho : g \mapsto \rho_g$, and let $\mathfrak{g}$ be the Lie algebra of G acting on V via the representation $\tilde{\rho} : A \mapsto \tilde{\rho}_A$ induced by ρ.

To simplify notation below, we take $V = \mathbb{C}^m$ and G a matrix group, acting on V by matrix rules (and similarly for $\mathfrak{g}$) and write $\rho_g \Psi = g\Psi$ and $\tilde{\rho}_A \Psi = A\Psi$. Moreover, we assume that G is either $U(m)$ or a Lie subgroup of $U(m)$.

Let M be an open subset in a finite-dimensional vector space, with local coordinates $\{x^i\}$, and let $\partial_i \equiv \partial_{x^i}$.

For a $\mathfrak{g}$-valued connection (one-form) $A \equiv A_i dx^i$ on M, we define the covariant derivatives:

- ∇_A, mapping functions (sections), $\Psi : M \to V$, into $\mathfrak{g}$-valued one-forms, as

$$\nabla_A \Psi := d\Psi + A\Psi \equiv (\partial_i \Psi + A_i \Psi) dx^i; \tag{A.1}$$

- d_A, mapping $\mathfrak{g}$-valued functions f into $\mathfrak{g}$-valued one-forms

$$d_A f := df + [A, f] \equiv (\partial_i f + [A_i, f]) dx^i; \tag{A.2}$$

- d_A, mapping $\mathfrak{g}$-valued one-forms into $\mathfrak{g}$-valued two-forms

$$d_A B := dB + [A, B], \tag{A.3}$$

with $[A, B]$ defined in local coordinates $\{x^i\}$ as

$$[A, B] := [A_i, B_j] dx^i \wedge dx^j \equiv [B, A], \tag{A.4}$$

for $A = A_i dx^i$ and $B = B_j dx^j$.[1]

[1] More generally, if A is a $\mathfrak{g}-$valued $p-$form and B is a $\mathfrak{g}-$valued $q-$form, written as $A = A^a \otimes \gamma_a$ and $B = B^b \otimes \gamma_b$, where A^a and B^b are $p-$ and $q-$forms and $\{\gamma_a\}$ is a basis in $\mathfrak{g}$, then

$$[A, B] := (A^a \wedge B^b) \otimes [\gamma_a, \gamma_b] = (-1)^{pq+1} [B, A]. \tag{A.5}$$

The curvature form of the connection A is the $\mathfrak{g}$-valued two-form given by the formula

$$F_A = dA + \frac{1}{2}[A, A]. \tag{A.6}$$

It is related to the curvature operator (denoted by the same symbol) $F_A := d_A \circ d_A$. As a simple computation shows, this operator is a matrix-multiplication operator given by the matrix-valued 2-form (A.6).

Let U be a vector space (V or $\mathfrak{g}$ in our case) and let $\Omega_U^p \equiv U \otimes \Omega^p$ denote the space of U-valued p-forms. On Ω_U^p, one defined the inner product, $\langle \cdot, \cdot \rangle_{\Omega_U^p}$ as

$$\langle A, B \rangle_{\Omega_U^p} := \langle A_\alpha, B^\alpha \rangle_U, \tag{A.7}$$

where $A = A_\alpha dx^\alpha$ and $B_\alpha dx^\alpha$ are U-valued p-forms, α is an p-index and $\langle \cdot, \cdot \rangle_U$ is the inner product on U.

Above, we did not display the coupling constants. Doing so would change the covariant derivative to $d_A \Psi = (d + gA)\Psi$, if G is simple. If G is not simple, then each simple and $U(1)$ component of G gets its own coupling constant, as was done in the main text for $G = SU(2) \times U(1)$.

Appendix B

The time-dependent Yang-Mills-Higgs system

In this appendix, we briefly review the Yang-Mills-Higgs theory, including the derivation of the energy functional (2.1). In what follows, we use the Einstein summation convention of summing over repeated indices. Furthermore, we use the convention of raising or lowering an index by contracting a tensor T with the metric tensor:

$$T^{\alpha}_{i,\beta} = \eta_{ij} T^{j,\alpha}_{\beta} \tag{B.1}$$

where η is the Minkowski metric of signature $(+, -, ..., -)$ on $M \subset \mathbb{R}^{d+1}$ and α, β are multi-indices.

Lagrangian. Let $M = \Omega \times [0, T] \subset \mathbb{R}^{d+1}$ be spacetime equipped with the Minkowski metric η of signature $(+, -, ..., -)$ and V and G be as in Appendix A. The theory involves a Higgs field $\Psi : M \to V$ interacting with the gauge field A, a connection (one-form) on M with values in the algebra $\mathfrak{g}$. The dynamics are given by the Lagrangian

$$\mathcal{L}(\Psi, A) := \int_{\Omega} \langle \nabla_A \Psi, \nabla_A \Psi \rangle_{\Omega^1_V} - U(\Psi) + \langle F_A, F_A \rangle_{\Omega^2_{\mathfrak{g}}}, \tag{B.2}$$

with corresponding action $\mathcal{S} := \int_0^T \mathcal{L}(\Psi, A) dt$, $T > 0$, given explicitly by

$$\mathcal{S}(\Psi, A) = \int_{M} \langle \nabla_A \Psi, \nabla_A \Psi \rangle_{\Omega^1_V} - U(\Psi) + \langle F_A, F_A \rangle_{\Omega^2_{\mathfrak{g}}}, \tag{B.3}$$

where $U : V \to \mathbb{R}^+$ is a self-interaction potential, which is assumed to be gauge invariant: $U(\rho_g \Psi) = U(\Psi)$ (a typical example of $U(\Psi)$ is $U(\Psi) = \frac{1}{2}\lambda(1 - \|\Psi\|^2)^2$). For convenience, we assume that Ψ and A are T-periodic in t.

Euler-Lagrange equations. The Euler-Lagrange equations (called Yang-Mills-Higgs equations) for the fields Ψ and A are

$$\nabla_A^* \nabla_A \Psi = U'(\Psi), \tag{B.4}$$

$$d_A^* F_A = J(\Psi, A), \tag{B.5}$$

where ∇_A^* and d_A^* are the adjoints of ∇_A and d_A in the appropriate inner products and $J(\Psi, A)$ is the YMH current given by

$$J(\Psi, A) := \mathrm{Re}\langle \gamma_a \Psi, \nabla \Psi \rangle_V \gamma_a \tag{B.6}$$

$$= \mathrm{Re}\langle \gamma_a \Psi, \nabla_i \Psi \rangle_V \gamma_a \otimes dx^i, \tag{B.7}$$

where γ_a is an orthonormal basis of $\mathfrak{g}$ and $\nabla_i := \partial_i + A_i$, so that $\nabla_A \Psi = \nabla_i \Psi dx^i$. (B.5) is the Yang-Mills equation.

Proof of (B.4) - (B.5). We *assume periodic or Dirichlet boundary conditions* and calculate the Gâteaux derivatives *formally*.

Recall that $\delta_\#$ denotes the partial (real) Gâteaux derivative with respect to $\#$. First we calculate the (complex) Gâteaux derivative of (B.3) in the Ψ-direction. Define $\partial_z \equiv \frac{1}{2}(\partial_{\mathrm{Re}\, z} - i\partial_{\mathrm{Im}\, z})$ and $\delta_\Psi \equiv \frac{1}{2}(\delta_{\mathrm{Re}\,\Psi} - i\delta_{\mathrm{Im}\,\Psi})$. Then $\delta_\Psi \mathcal{S}(\Psi, A)\Psi' = \partial_z \mathcal{S}(\Psi_z, A)|_{z=0}$, where $\Psi_z = \Psi + z\Psi'$, $z \in \mathbb{C}$. Using this, we find

$$\delta_\Psi \mathcal{S}(\Psi, A)\Psi' = \int_M \langle \nabla_A \Psi, \nabla_A \Psi' \rangle_{\Omega^1_V} - \langle U'(\Psi), \Psi' \rangle_V. \tag{B.8}$$

Integrating the first term by parts and factoring out Ψ' gives

$$\delta_\Psi \mathcal{S}(\Psi, A)\Psi' = \int_M \langle \nabla_A^* \nabla_A \Psi - U'(\Psi), \Psi' \rangle_V. \tag{B.9}$$

For this derivative to be zero for every variation Ψ', (B.4) must hold.

Next we calculate the Gâteaux derivative of (B.3) in the A-direction. Using the definition $\delta_A f(A)B = \partial_s f(A_s)|_{s=0}$, where $A_s = A + sA'$, $s \in \mathbb{R}$, we find

$$\delta_A \mathcal{S}(\Psi, A)B = \int_M \langle B\Psi, \nabla_A \Psi \rangle_{\Omega^1_V} + c.c. + 2\langle d_A B, F_A \rangle_{\Omega^2_{\mathfrak{g}}} \tag{B.10}$$

$$= I + II. \tag{B.11}$$

Writing $B = B^a \gamma_a = B^a_i dx^i \otimes \gamma_a$ (with B^a_i real) and $\nabla_A \Psi = \nabla_i \Psi dx^i$, so that

$$\langle B\Psi, \nabla_A \Psi \rangle_{\Omega^1_V} = \langle B^a, \langle \gamma_a \Psi, \nabla_i \Psi \rangle_V dx^i \rangle_{\Omega^1}, \tag{B.12}$$

and using that $B^a C^a = -\mathrm{Tr}[(B^c \gamma_c)(C^a \gamma_a)]$ (since $\mathrm{Tr}(\gamma_c^* \gamma_a) = -\mathrm{Tr}(\gamma_c \gamma_a) = \delta_{ca}$), gives

$$I = -\int_M \langle B, \langle \gamma_a \Psi, \nabla_i \Psi \rangle_V \gamma_a \otimes dx^i \rangle_{\Omega^1_{\mathfrak{g}}} + c.c.. \tag{B.13}$$

which gives $I = \int_M \langle B, J(\Psi, A) \rangle_{\Omega^1_{\mathfrak{g}}}$. For the second term in (B.10), integrating the last term by parts yields $II = \int_M \langle B, d_A^* F_A\rangle_{\Omega^1_{\mathfrak{g}}}$. Collecting the last two equations gives

$$\delta_A \mathcal{S}(\Psi, A)B = 2\int_M \langle B, -J(\Psi, A) + d_A^* F_A\rangle_{\Omega^1_{\mathfrak{g}}}. \tag{B.14}$$

For this derivative to be zero for every variation B, (B.5) must hold. □

Conserved energy. Again, the Gâteaux derivative calculations in the following subsection are formal. Recall that $M := \Omega \times [0, T] \subset \mathbb{R}^{d+1}$.

Proposition 32. *The Legendre transform of* (B.2) *yields the conserved energy*

$$E(\Psi, A) := \int_\Omega \|\nabla_A \Psi\|^2_{\Omega^1_V} + U(\Psi) + \|F_A\|^2_{\Omega^2_{\mathfrak{g}}}, \tag{B.15}$$

where the norms are taken using the Euclidean metric on $\mathbb{R}^{d+1}$ *(rather than the Minkowski metric).*

Note that for static (time-indepent) fields, $E(\Psi, A) = -\mathcal{L}(\Psi, A)$.

Proof. Let $\partial_\#$ denote the partial derivative with respect to the symbol #, and recall that $\delta_\#$ denotes the partial (real) Gâteaux derivative with respect to #. The Legendre transform of (B.2) is given by

$$\begin{aligned} E(\Psi, A) = {} & \partial_{\nabla_0\Psi}\mathcal{L}(\Psi, A)\nabla_0\Psi + \partial_{\overline{\nabla_0\Psi}}\mathcal{L}(\Psi, A)\overline{\nabla_0\Psi} \\ & + \sum_{i=1}^d \partial_{F_{0i}}\mathcal{L}(\Psi, A)F_{0i} - \mathcal{L}(\Psi, A). \end{aligned} \tag{B.16}$$

We calculate

$$\partial_{\nabla_0\Psi}\mathcal{L}(\Psi, A)\nabla_0\Psi = \int_\Omega \|\nabla_0\Psi\|^2_V = \partial_{\overline{\nabla_0\Psi}}\mathcal{L}(\Psi, A)\overline{\nabla_0\Psi} \tag{B.17}$$

and

$$\sum_{i=1}^d \partial_{F_{0i}}\mathcal{L}(\Psi, A)F_{0i} = 2\sum_{i=1}^d |F_{0i}|^2. \tag{B.18}$$

(B.15) results.

It remains to show that (B.15) is conserved by the YMH equations (B.4) - (B.5). Applying the chain rule gives

$$\frac{d}{dt}E(\Psi, A) = \delta_\Psi E(\Psi, A)\partial_0\Psi + \delta_{\overline{\Psi}}E(\Psi, A)\partial_0\overline{\Psi} + \delta_A E(\Psi, A)\partial_0 A. \tag{B.19}$$

We now calculate the first term using (B.4).

$$\begin{aligned} \delta_\Psi E(\Psi, A)\partial_0\Psi = {} & \int_\Omega \langle\nabla_0\Psi, \nabla_0\partial_0\Psi\rangle_V + \sum_{k=1}^d \langle\nabla_k\Psi, \nabla_k\partial_0\Psi\rangle_V \\ & + \langle U'(\Psi), \partial_0\Psi\rangle_V. \end{aligned} \tag{B.20}$$

Integrating the second term by parts gives

$$\delta_\Psi E(\Psi, A)\partial_0\Psi = \int_\Omega \langle \nabla_0\Psi, \nabla_0\partial_0\Psi\rangle_V + \sum_{k=1}^{d} \langle \nabla_k^*\nabla_k\Psi, \partial_0\Psi\rangle_V + \langle U'(\Psi), \partial_0\Psi\rangle_V. \tag{B.21}$$

By (B.4), we have

$$\nabla_0^*\nabla_0\Psi - \sum_{k=1}^{d} \nabla_k^*\nabla_k\Psi = U'(\Psi), \tag{B.22}$$

so (B.21) becomes

$$\delta_\Psi E(\Psi, A)\partial_0\Psi = \int_\Omega \langle \nabla_0\Psi, \nabla_0\partial_0\Psi\rangle_V + \langle \nabla_0^*\nabla_0\Psi, \partial_0\Psi\rangle_V. \tag{B.23}$$

Here $\nabla_0^* = -\partial_0 + A_0^\dagger = -\partial_0 - A_0$, where the second equality follows because the representation of $\mathfrak{g}$ is unitary. Therefore,

$$\begin{aligned}\delta_\Psi E(\Psi, A)\partial_0\Psi &= \int_\Omega \langle (\partial_0 + A_0)\Psi, (\partial_0 + A_0)\partial_0\Psi\rangle_V \\ &\quad + \langle (-\partial_0 - A_0)(\partial_0 + A_0)\Psi, \partial_0\Psi\rangle_V \\ &= \int_\Omega \partial_0\langle \Psi, A_0\partial_0\Psi\rangle_V.\end{aligned} \tag{B.24}$$

Similarly,

$$\delta_{\overline{\Psi}} E(\Psi, A)\partial_0\overline{\Psi} = \int_\Omega \partial_0\langle A_0\partial_0\Psi, \Psi\rangle_V, \tag{B.25}$$

and so

$$\delta_\Psi E(\Psi, A)\partial_0\Psi + \delta_{\overline{\Psi}} E(\Psi, A)\partial_0\overline{\Psi} = \int_\Omega \partial_0 J_0(\Psi, A), \tag{B.26}$$

where $J_0(\Psi, A)$ is the time component of the YMH current (B.6).

One may show using (B.5) that

$$\delta_A E(\Psi, A)\partial_0 A = -\int_\Omega \partial_0 J_0(\Psi, A). \tag{B.27}$$

Hence, by (B.19) we have $\frac{d}{dt}E(\Psi, A) = 0$, as required. □

Gauge symmetries. We define the local action, $\rho_g A$, of the group G on A, by the equation $d_{\rho_g A} = g d_A g^{-1}$, for all $g \in C^1(N, G)$, where N is either M or Ω. We compute

$$\rho_g A = gAg^{-1} + gdg^{-1}. \tag{B.28}$$

Proposition 33. *The Lagrangian* (B.2) *is invariant under the Poincaré group and the gauge transformations*

$$T_g^{gauge} : (\Psi, A) \mapsto (g\Psi, \rho_g A), \qquad \forall g \in C^1(M, G). \tag{B.29}$$

Proof. The invariance under the Poincaré group follows easily from the definition of this group and the choice of the Minkowski metric on $M \subset \mathbb{R}^{d+1}$.

Recall that $U(\Psi)$ is $\mathfrak{g}$-invariant, and that the representations $g \mapsto \rho_g$ (on V) and the adjoint representation $g \mapsto \mathrm{ad}_g$ (on $\mathfrak{g}$) are unitary. Therefore, to prove invariance under the gauge transformation (B.29), it suffices therefore to show that

$$\nabla_{\rho_g A} g\Psi = g\nabla_A \Psi, \tag{B.30}$$

$$F_{\rho_g A} = gF_A g^{-1}. \tag{B.31}$$

We shall use the equation

$$hdh^{-1} = -dhh^{-1}, \quad \forall h \in G \tag{B.32}$$

which follows from $d(hh^{-1}) = 0$. For (B.30) we compute

$$\nabla_{\rho_g A} g\Psi = d(g\Psi) + (gAg^{-1} + gdg^{-1})(g\Psi) \tag{B.33}$$

$$= (dg)\Psi + gd\Psi + gA\Psi + gdg^{-1}g\Psi. \tag{B.34}$$

Since $gdg^{-1}g = -gg^{-1}dg = -dg$, this gives $\nabla_{\rho_g A} g\Psi = g\nabla_A \Psi$.

For (B.31), computing in coordinates $\{x^i\}$ and writting $F_{\rho_g A} := (F_{\rho_g A})_{ij} dx^i \wedge dx^j$ and $F_A := (F_A)_{ij} dx^i \wedge dx^j$, we find

$$\begin{aligned}(F_{\rho_g A})_{ij} = {}& \frac{1}{2}[\partial_i(gA_j g^{-1} + g\partial_j g^{-1}) - \partial_j(gA_i g^{-1} + g\partial_i g^{-1})] \\ & + \frac{1}{2}[gA_i g^{-1} + g\partial_i g^{-1}, gA_j g^{-1} + g\partial_j g^{-1}].\end{aligned} \tag{B.35}$$

Expanding the partial derivative and commutators gives

$$\begin{aligned}(F_{\rho_g A})_{ij} = {}& \frac{1}{2}[\partial_i g A_j g^{-1} + g\partial_i A g^{-1} + gA_j \partial_i g^{-1} + \partial_i g \partial_j g^{-1} + g\partial_i \partial_j g^{-1} \\ & + (gA_i g^{-1} + \partial_i g g^{-1})(gA_j g^{-1} + g\partial_j g^{-1}) \\ & - (i \leftrightarrow j)].\end{aligned} \tag{B.36}$$

Expanding the product on the second line gives

$$\begin{aligned}(F_{\rho_g A})_{ij} = {}& \frac{1}{2}[\partial_i g A_j g^{-1} + g\partial_i A g^{-1} + gA_j \partial_i g^{-1} + \partial_i g \partial_j g^{-1} + g\partial_i \partial_j g^{-1} \\ & + gA_i A_j g^{-1} + \partial_i g A_j g^{-1} + gA_i \partial_j g^{-1} + \partial_i g \partial_j g^{-1} \\ & - (i \leftrightarrow j)].\end{aligned} \tag{B.37}$$

Cancelling terms symmetrical in i and j and simplifying gives

$$(F_{\rho_g A})_{ij} = g(\frac{1}{2}[\partial_i A_j - \partial_j A_i] + \frac{1}{2}[A_i A_j - A_j A_i])g^{-1} \tag{B.38}$$

$$= g(F_A)_{ij}g^{-1}, \tag{B.39}$$

as required. □

Specifying (B.15) to the WS model gives (2.1).

Appendix C

The YMH equations in coordinate form

In what follows, we use the Einstein summation convention of summing over repeated indices. Furthermore, we use the convention of raising or lowering an index by contracting a tensor T with the metric tensor:

$$T^{\alpha}_{i,\beta} = \eta_{ij} T^{j,\alpha}_{\beta} \tag{C.1}$$

where η is the Minkowski metric of signature $(+,-,...,-)$ on $\mathbb{R}^{d+1}$ and α, β are multi-indices. The same equations could be *reinterpreted as stationary* equations by taking the *Euclidean metric* δ_{ij}*, instead of* η_{ij}*, and letting the indices range over* $1, \dots, d$*, rather than* $1, \dots, d+1$. In this case, $T^{\alpha}_{i,\beta} = T^{i,\alpha}_{\beta}$.

As above, Ω is either a bounded domain in $\mathbb{R}^d$ or $\mathbb{R}^{d+1}$. In the former case, we assume either periodic or Dirichlet boundary conditions.

In coordinate form, the differential form (gauge field) entering the YMH Lagrangian (B.2) is written as $A = A_i dx^i$. The local coordinate expression for the curvature is $F_A = F_{ij} dx^i \wedge dx^j$, where $F_{ij} := \frac{1}{2}(\partial_i A_j - \partial_j A_i) + \frac{1}{2}[A_i, A_j]$. Furthermore, for the covariant derivatives ∇_A and d_A, we have $\nabla_A \Psi = \nabla_i \Psi dx^i$ and $d_A^* F_A = -\nabla^i F_{ij} dx^j$, where $\nabla_i \Psi := (\partial_i + A_i)\Psi$ and $\nabla^i F_{ij} := \partial^i F_{ij} + [A^i, F_{ij}]$.

For an arbitrary $\mathfrak{g}$-valued one-form $B = B_i dx^i$, we have $d_A B = \nabla_i B_j dx^i \wedge dx^j$ and $d_A^* B = -\nabla^i B_i$, where

$$\nabla^i B_j := \partial^i B_j + [A^i, B_j]. \tag{C.2}$$

We write $F_{ij} = F^a_{ij} \gamma_a$ for an orthonormal basis γ_a of $\mathfrak{g}$ and the lower case roman indices run over the spatial components $1, 2, \dots, d$. Note that $F_{ij} = [\nabla_i, \nabla_j]$, but $F_{ij} \neq \frac{1}{2}(\nabla_i A_j - \nabla_j A_i)$.

Proposition 34. *The Lagrangian and energy for the YMH model are given in coordinates by*

$$\mathcal{L}(\Psi, A) = \int_\Omega \langle \nabla_k \Psi, \nabla^k \Psi \rangle_V - U(\Psi) + \frac{1}{2} F^a_{ij} F^{a,ij}, \tag{C.3}$$

$$E_\Omega(\Psi, A) = \int_\Omega \langle \nabla_k \Psi, \nabla_k \Psi \rangle_V + U(\Psi) + \frac{1}{2} F^a_{ij} F^a_{ij} \tag{C.4}$$

(with different ranges of indices as mentioned above). The YMH equations are given in coordinates by

$$-\nabla^i\nabla_i\Psi = U'(\Psi), \tag{C.5}$$
$$-\nabla^i F_{ij} = \mathrm{Re}\langle\gamma_a\Psi, \nabla_j\Psi\rangle_V\gamma_a. \tag{C.6}$$

Proof. Equations (C.3) and (C.4) follow from the coordinate expressions $d_A\Psi = \nabla_k\Psi dx^k$ and $F_A = F^a_{ij}\gamma_a\otimes dx^i\wedge dx^j$, together with the fact that dx^k and $\gamma_a\otimes dx^i\wedge dx^j$ form orthonormal bases for Ω^1 and $\Omega^2_{\mathfrak{g}}$, respectively.

Equations (C.5) - (C.6) follow from equations (B.4) - (B.6) and the coordinate expressions for d_A and d_A^* above. □

We specify equation (C.3) - (C.6) for to the Weinberg-Salam (WS) model, which has the gauge group $G = U(2) = SU(2)\times U(1)$. In this case, there is a slight discrepancy in the definition of the covariant derivative due to the fact that $U(2)$ is not simple, but a (semi-)direct product of the simple group $SU(2)$ and $U(1)$, with each component having a coupling constant. We choose the standard inner product

$$\langle\gamma,\delta\rangle_{\mathfrak{u}(2)} := 2\,\mathrm{Tr}\,\gamma^*\delta = -2\,\mathrm{Tr}\,\gamma\delta \tag{C.7}$$

on $\mathfrak{u}(2)$, for which $-\frac{i}{2}\tau_a$, $a = 0,1,2,3$, (where τ_a, $a = 1,2,3$, are the the Pauli matrices together with $\tau_0 := \mathbf{1}$) form an orthonormal basis. It is customary to factor out the coefficient of $-\frac{i}{2}$. In coordinates, we write

$$\nabla_Q\Phi = \nabla_i\Phi dx^i, \quad Q = -\frac{i}{2}Q_i dx^i \;\text{ and }\; F_Q = -\frac{i}{2}Q_{ij}dx^i\wedge dx^j, \tag{C.8}$$

with $Q_i(x)$, $Q_{ij}(x) \in i\mathfrak{u}(2)$. Using equation (2.3), we compute $Q_{ij} = \frac{1}{2}(\partial_i Q_j - \partial_j Q_i) - \frac{i}{4}[Q_i, Q_j]$. Furthermore, we write $Q = V + X$ and

$$V = -\frac{i}{2}V_i dx^i \;\text{ and }\; X = -\frac{i}{2}X_i dx^i, \tag{C.9}$$

with $V_i(x) \in i\mathfrak{su}(2)$ and $X_i(x) \in i\mathfrak{u}(1)$. Then $Q_{ij} = V_{ij} + X_{ij}$ and

$$\nabla_i\Phi := (\partial_i - \frac{ig}{2}V_i - \frac{ig'}{2}X_i)\Phi, \tag{C.10}$$
$$V_{ij} := \frac{1}{2}(\partial_i V_j - \partial_j V_i) - \frac{ig}{4}[V_i, V_j], \tag{C.11}$$
$$X_{ij} := \frac{1}{2}(\partial_i X_j - \partial_j X_i). \tag{C.12}$$

Using the formulae above, we express the Lagrangian and the energy in coordinates as

$$\mathcal{L}(\Phi, Q) := \int_\Omega \langle\nabla_i\Phi, \nabla^i\Phi\rangle_{\mathbb{C}^2} - U(\Phi) + \frac{1}{2}\,\mathrm{Tr}\,Q_{ij}Q^{ij}, \tag{C.13}$$
$$E(\Phi, Q) := \int_\Omega \langle\nabla_i\Phi, \nabla_i\Phi\rangle_{\mathbb{C}^2} + U(\Phi) + \frac{1}{2}\,\mathrm{Tr}\,Q_{ij}Q_{ij}, \tag{C.14}$$

(with indices ranging from 0 to d and 1 to d, respectively, as mentioned above), and the Euler-Lagrange equations are written in coordinates as

$$-\nabla^i \nabla_i \Phi = U'(\Phi), \tag{C.15}$$

$$\nabla^i Q_{ij} = \frac{1}{2} g \operatorname{Im}\langle \tau_a \Phi, \nabla_j \Phi \rangle_{\mathbb{C}^2} \tau_a + \frac{1}{2} g' \operatorname{Im}\langle \tau_0 \Phi, \nabla_j \Phi \rangle_{\mathbb{C}^2} \tau_0. \tag{C.16}$$

In Section 3, we expressed equations (C.14) - (C.16) in their standard form (D.1) - (3.15), involving the W, Z, Higgs and electromagnetic fields defined therein.

Appendix D

The Weinberg-Salam energy in 3D in terms of the fields W, A, Z and φ

We work in a fixed coordinate system, $\{x_i\}_{i=1}^3$ and write the fields as $W = W_i dx^i$, $Z = -\frac{i}{2} Z_i dx^i$ and $A = -\frac{i}{2} A_i dx^i$. We show

Lemma 35. *Energy* (2.1)*, written in terms of the fields* W, A, Z *and* φ *and coordinates* $\{x_i\}_{i=1}^3$*, is given by (see also [35]):*

$$E_\Omega^{WS}(W, A, Z, \varphi) := \int_\Omega \big[\sum_{ij} (\frac{1}{2}|W_{ij}|^2 + \frac{1}{4}|Z_{ij}|^2 + \frac{1}{4}|A_{ij}|^2) \\ + \frac{1}{2} g^2 \varphi^2 |W|^2 + \frac{1}{4\cos^2\theta} g^2 \varphi^2 |Z|^2 + T(W, A, Z) \\ + |\nabla\varphi|^2 + \frac{1}{2}\lambda(\varphi^2 - \varphi_0^2)^2 \big], \tag{D.1}$$

where $W_{ij} := \nabla_i W_j - \nabla_j W_i$*, with* $\nabla_k := \partial_k - igV_k^3$*,* $Z_{ij} := \partial_i Z_j - \partial_j Z_i$*,* $A_{ij} := \partial_i A_j - \partial_j A_i$ *and* $T(W, A, Z)$ *is the sum of super-quadratic terms,*

$$T(W, A, Z) := \frac{g^2}{2} \sum_{ij} (|W_i W_j|^2 - W_i^2 \overline{W}_j^2) - ig \sum_{ij} V_{ij}^3 W_i \overline{W}_j, \tag{D.2}$$

where $V^3 := Z\cos\theta + A\sin\theta$ *and* $V_{ij}^3 := \partial_i V_j - \partial_j V_i$*, with the important property that* $T(W, A, Z)$ *is invariant under the gauge transformation* (3.7).

Proof of (D.1). We proceed by rewriting the terms in the coordinate expression of the WS energy (C.14), in terms of the fields $W = W_i dx^i$, $Z = -\frac{i}{2} Z_i dx^i$, $A = -\frac{i}{2} A_i dx^i$ and φ.

For the first term, first we calculate $\nabla_i \Phi$. Recall the definition $\nabla_i \Phi := (\partial_i - \frac{ig}{2} V_i - \frac{ig'}{2} X_i)\Phi$. We simplify the matrix representing the connection's action on Φ:

$$\begin{aligned}
-\frac{ig}{2}V_i - \frac{ig'}{2}X_i &= -\frac{ig}{2}V_i^a\tau_a - \frac{ig'}{2}X_i\tau_0 \\
&= -\frac{ig}{2}\begin{pmatrix} 0 & V_i^1 \\ V_i^1 & 0 \end{pmatrix} - \frac{ig}{2}\begin{pmatrix} 0 & -iV_i^2 \\ iV_i^2 & 0 \end{pmatrix} \\
&- \frac{ig}{2}\begin{pmatrix} V_i^3 & 0 \\ 0 & -V_i^3 \end{pmatrix} - \frac{ig}{2}\tan\theta\begin{pmatrix} X_i & 0 \\ 0 & X_i \end{pmatrix} \\
&= -\frac{ig}{2\cos\theta}\begin{pmatrix} V_i^3\cos\theta + X_i\sin\theta & V_i^1\cos\theta - iV_i^2\cos\theta \\ V_i^1\cos\theta + iV_i^2\cos\theta & -V_i^3\cos\theta + X_i\sin\theta \end{pmatrix}.
\end{aligned} \tag{D.3}$$

In terms of the fields Z, A and W (see equations (3.5) - (3.6) for the definitions of these fields), (D.3) becomes

$$-\frac{ig}{2}V_i - \frac{ig'}{2}X_i = -\frac{ig}{2\cos\theta}\begin{pmatrix} Z_i\cos 2\theta + A_i\sin 2\theta & \sqrt{2}\,W_i\cos\theta \\ \sqrt{2}\,\overline{W}_i\cos\theta & -Z_i \end{pmatrix}. \tag{D.4}$$

Hence, for $\Phi = (0, \varphi)$,

$$\nabla_i\Phi = \begin{pmatrix} -\frac{ig}{\sqrt{2}}W_i\varphi \\ \partial_i\varphi + \frac{ig}{2\cos\theta}Z_i\varphi \end{pmatrix}. \tag{D.5}$$

Therefore, the first term of (C.14), written in terms of the fields W, A, Z and φ, becomes

$$\begin{aligned}
\langle\nabla_i\Phi, \nabla^i\Phi\rangle_{\mathbb{C}^2} &= \overline{\frac{ig}{\sqrt{2}}W_i}\frac{ig}{\sqrt{2}}W^i \\
&+ \overline{(\partial_i\varphi + \frac{ig}{2\cos\theta}Z_i\varphi)}(\partial^i\varphi + \frac{ig}{2\cos\theta}Z^i\varphi) \\
&= \frac{g^2}{2}\varphi^2|W|^2 + |\nabla\varphi|^2 + \frac{g^2}{4\cos^2\theta}\varphi^2|Z|^2.
\end{aligned} \tag{D.6}$$

The second term of (C.14) becomes

$$U(\Phi) = \frac{1}{2}\lambda(\|\Phi\|^2 - \varphi_0^2)^2 = \frac{1}{2}\lambda(\varphi^2 - \varphi_0^2)^2. \tag{D.7}$$

For the third term of (C.14), we will use the fact that $\operatorname{Tr} Q_{ij}Q^{ij} = \operatorname{Tr} V_{ij}V^{ij} + \operatorname{Tr} X_{ij}X^{ij}$, where

$$U_{ij} := \frac{1}{2}(\partial_i U_j - \partial_j U_i) - \frac{ig}{4}[U_i, U_j] \tag{D.8}$$

with

$$V_i := V_i^a\tau_a = \begin{pmatrix} V_i^3 & \sqrt{2}\,W_i \\ \sqrt{2}\,\overline{W}_i & -V_i^3 \end{pmatrix}, \tag{D.9}$$

and

$$X_{ij} := \frac{1}{2}(\partial_i X_j - \partial_j X_i). \tag{D.10}$$

To simplify (D.8), we use (D.9), let $U_{ij} := \partial_i U_j - \partial_j U_i$, and recall $V^3_{ij} = \partial_i V^3_j - \partial_j V^3_i$ and $W^0_{ij} = \partial_i W_j - \partial_j W_i$, to calculate

$$\frac{1}{2}V_{ij} = \frac{1}{2}\begin{pmatrix} V^3_{ij} & \sqrt{2}\, W_{ij} \\ \sqrt{2}\, \overline{W}^0_{ji} & -V^3_{ij} \end{pmatrix}, \tag{D.11}$$

and, with $K_{ij} := V^3_i W_i - V^3_j W_i$,

$$\begin{aligned} -\frac{ig}{4}[V_i, V_j] &= -\frac{ig}{4}\begin{pmatrix} V^3_i & \sqrt{2}\, W_i \\ \sqrt{2}\, \overline{W}_i & -V^3_i \end{pmatrix}\begin{pmatrix} V^3_j & \sqrt{2}\, W_j \\ \sqrt{2}\, \overline{W}_j & -V^3_j \end{pmatrix} - (i \leftrightarrow j) \\ &= -\frac{ig}{4}\begin{pmatrix} V^3_i V^3_j + 2W_i\overline{W}_j & \sqrt{2}\, K_{ij} \\ \sqrt{2}\, \overline{K}_{ij} & -V^3_i V^3_j - 2W_i\overline{W}_j \end{pmatrix} - (i \leftrightarrow j) \\ &= -\frac{ig}{2}\begin{pmatrix} W_i\overline{W}_j - \overline{W}_i W_j & \sqrt{2}\, K_{ij} \\ \sqrt{2}\, \overline{K}_{ji} & -W_i\overline{W}_j + \overline{W}_i W_j \end{pmatrix}. \end{aligned} \tag{D.12}$$

Adding (D.11) and (D.12), using that $W_{ij} = W^0_{ij} + K_{ij}$ and denoting $L_{ij} := V^3_{ij} - ig(W_i\overline{W}_j - \overline{W}_i W_j)$ gives

$$U_{ij} = \frac{1}{2}\begin{pmatrix} L_{ij} & \sqrt{2}\, W_{ij} \\ -\sqrt{2}\, \overline{W}_{ij} & -L_{ij} \end{pmatrix}. \tag{D.13}$$

Since U_{ij} and X_{ij} are Hermitian, $\operatorname{Tr} U_{ij}U^{ij}$ and $\operatorname{Tr} X_{ij}X^{ij}$ are the sum of the squared absolute values of the matrix coefficients of U_{ij} and X_{ij}, respectively. Thus

$$\begin{aligned} \frac{1}{2}\operatorname{Tr} Q_{ij}Q^{ij} &= \frac{1}{2}\operatorname{Tr} U_{ij}U^{ij} + \frac{1}{2}\operatorname{Tr} X_{ij}X^{ij} \\ &= \frac{1}{8}\sum_{ij} 2|L_{ij}|^2 + 4|W_{ij}|^2 + 2|X_{ij}|^2. \end{aligned} \tag{D.14}$$

Using $L_{ij} = V^3_{ij} - ig(W_i\overline{W}_j - \overline{W}_i W_j)$ and expanding the first term gives

$$\begin{aligned} \frac{1}{2}\operatorname{Tr} Q_{ij}Q^{ij} &= \sum_{ij}\frac{1}{2}|W_{ij}|^2 + \frac{1}{4}|V^3_{ij}|^2 + \frac{1}{4}|X_{ij}|^2 \\ &+ \frac{g^2}{4}\sum_{ij}|W_i\overline{W}_j - \overline{W}_i W_j|^2 - \frac{ig}{4}\sum_{ij} 2V^3_{ij}(W_i\overline{W}_j - \overline{W}_i W_j). \end{aligned} \tag{D.15}$$

Recall that $A_{ij} = V^3_{ij}\sin\theta + X_{ij}\cos\theta$ and $Z_{ij} = V^3_{ij}\cos\theta - X_{ij}\sin\theta$. Writing the first line of (D.15) in terms of these fields gives

$$\begin{aligned} \frac{1}{2}\operatorname{Tr} Q_{ij}Q^{ij} &= \sum_{ij}\frac{1}{2}|W_{ij}|^2 + \frac{1}{4}|Z_{ij}|^2 + \frac{1}{4}|A_{ij}|^2 \\ &+ \frac{g^2}{4}\sum_{ij}|W_i\overline{W}_j - \overline{W}_i W_j|^2 - \frac{ig}{2}\sum_{ij} V^3_{ij}(W_i\overline{W}_j - \overline{W}_i W_j). \end{aligned} \tag{D.16}$$

Expanding the first term of the second line, and using $V^3_{ij} = -V^3_{ij}$ in the second term, (D.16) becomes

$$\begin{aligned}\frac{1}{2}\operatorname{Tr} Q_{ij}Q^{ij} &= \sum_{ij}\frac{1}{2}|W_{ij}|^2 + \frac{1}{4}|Z_{ij}|^2 + \frac{1}{4}|A_{ij}|^2 \\ &+ \frac{g^2}{4}\sum_{ij}(|W_i|^2|\overline{W}_j|^2 - W_i^2\overline{W}_j^2 + (i \leftrightarrow j)) \\ &- \frac{ig}{2}\sum_{ij}(V^3_{ij}W_i\overline{W}_j + (i \leftrightarrow j)).\end{aligned} \tag{D.17}$$

Recalling the definition (D.2) of $T(W, A, Z)$ gives

$$\frac{1}{2}\operatorname{Tr} Q_{ij}Q^{ij} = \sum_{ij}\frac{1}{2}|W_{ij}|^2 + \frac{1}{4}|A_{ij}|^2 + \frac{1}{4}|Z_{ij}|^2 + T(W, A, Z). \tag{D.18}$$

Adding (D.6), (D.7) and (D.18) gives (D.1). □

Appendix E

The Weinberg-Salam equations in 2D in terms of the fields W, A, Z and φ

Proof of (3.10). Now, we consider the Weinberg-Salam (WS) model in $\mathbb{R}^2$ with fields independent of the third dimension x_3, and correspondingly choose the gauge with $V_3 = X_3 = 0$ (and hence $W_3 = A_3 = Z_3 = 0$). In this case the summation in (D.1) contains only two terms, $(ij) = (12)$ and $(ij) = (21)$, and we use this to simplify (D.1).

We proceed by simplifying the terms of (D.2) and the first line of (D.1); the remaining terms are unchanged.

$$\begin{aligned}\sum_{ij}(\frac{1}{2}|W_{ij}|^2 + \frac{1}{4}|Z_{ij}|^2 + \frac{1}{4}|A_{ij}|^2) &= \sum_{i<j}(|W_{ij}|^2 + \frac{1}{2}|Z_{ij}|^2 + \frac{1}{2}|A_{ij}|^2) \\ &= |\operatorname{curl}_{gV^3} W|^2 + \frac{1}{2}|\operatorname{curl} Z|^2 + \frac{1}{2}|\operatorname{curl} A|^2; \end{aligned} \tag{E.1}$$

$$\begin{aligned}&\sum_{ij}(|W_iW_j|^2 - W_i^2\overline{W}_j^2) \\ &\qquad = W_1W_2\overline{W}_1\overline{W}_2 - W_1^2\overline{W}_2^2 + W_2W_1\overline{W}_2\overline{W}_1 - W_2^2\overline{W}_1^2 \\ &\qquad = \overline{(\overline{W}_1W_2 - W_1\overline{W}_2)}(\overline{W}_1W_2 - W_1\overline{W}_2) \\ &\qquad = |\overline{W} \times W|^2; \end{aligned} \tag{E.2}$$

$$\begin{aligned}-\sum_{ij} V_{ij}^3 W_i\overline{W}_j &= \sum_{i<j} V_{ij}^3(-W_i\overline{W}_j + W_j\overline{W}_i) \\ &= (\operatorname{curl} V^3)\overline{W} \times W. \end{aligned} \tag{E.3}$$

Replacing corresponding terms in (D.1) - (D.2) with (E.1) - (E.3) proves (3.10). □

Proof of (3.12) - (3.15). We proceed by calculating the (complex) Gâteaux derivatives of (3.10).

Let $\delta_{\#}$ denote the partial (real) Gâteaux derivative with respect to #. Let $W_z = W + zW'$, $z \in \mathbb{C}$, and define $\partial_{\overline{z}} \equiv \frac{1}{2}(\partial_{\operatorname{Re} z} + i\partial_{\operatorname{Im} z})$ and $\delta_{\overline{W}} \equiv \frac{1}{2}(\delta_{\operatorname{Re} W} + i\delta_{\operatorname{Im} W})$. Then

$$\begin{aligned}\delta_{\overline{W}} E_\Omega^{WS}(W, A, Z, \varphi)\overline{W'} &= \partial_{\overline{z}} E_\Omega^{WS}(W_z, A, Z, \varphi)|_{z=0} \\ &= \int_\Omega \operatorname{curl}_{gV^3} W \cdot \overline{\operatorname{curl}_{gV^3} W'} + \frac{1}{2} g^2 \varphi^2 W \cdot \overline{W'} \\ &\quad - ig(\operatorname{curl} V^3) JW \cdot \overline{W'} + g^2(\overline{W} \times W) JW \cdot \overline{W'}. \end{aligned} \tag{E.4}$$

Integrating the first term by parts and factoring out W and $\overline{W}'$ gives

$$\begin{aligned}\delta_{\overline{W}} E_\Omega^{WS}(W, A, Z, \varphi)\overline{W'} = \int_\Omega &[\operatorname{curl}^*_{gV^3} \operatorname{curl}_{gV^3} + \frac{g^2}{2}\varphi^2 - ig(\operatorname{curl} V^3)J \\ &+ g^2(\overline{W} \times W)J] W \cdot \overline{W'}. \end{aligned} \tag{E.5}$$

For the derivative to be zero for every variation W', (3.12) must hold.

Let $A_s = A + sA'$, $s \in \mathbb{R}$. Then

$$\begin{aligned}\delta_A E_\Omega^{WS}(W, A, Z, \varphi)A' &= \partial_s E_\Omega^{WS}(W, A_s, Z, \varphi)|_{s=0} \\ &= \int_\Omega \operatorname{curl}_{gV^3} W \overline{(-ieA' \times W)} + \overline{\operatorname{curl}_{gV^3} W}(-ieA' \times W) \\ &\quad + (\operatorname{curl} A)(\operatorname{curl} A') + ie(\operatorname{curl} A')\overline{W} \times W. \end{aligned} \tag{E.6}$$

Using $A' \times W = -JW \cdot A'$ in the first two terms, and integrating the last two terms by parts, gives

$$\begin{aligned}\delta_A E_\Omega^{WS}(W, A, Z, \varphi)A' = \int_\Omega &[-ie(\operatorname{curl}_{gV^3} W)J\overline{W} + ie\overline{(\operatorname{curl}_{gV^3} W)J\overline{W}} \\ &+ \operatorname{curl}^* \operatorname{curl} A + ie \operatorname{curl}^*(\overline{W} \times W)] \cdot A', \end{aligned} \tag{E.7}$$

which simplifies to

$$\begin{aligned}\delta_A E_\Omega^{WS}(W, A, Z, \varphi)A' = \int_\Omega &[\operatorname{curl}^* \operatorname{curl} A + 2e \operatorname{Im}[(\operatorname{curl}_{gV^3} W)J\overline{W} \\ &- \operatorname{curl}^*(\overline{W}_1 W_2)]] \cdot A'. \end{aligned} \tag{E.8}$$

For the derivative to be zero for every variation A', (3.13) must hold.

The proof of (3.14) is essentially the same as the proof of (3.13), so we omit it.

Let $\varphi_s = \varphi + s\varphi'$, $s \in \mathbb{R}$. Then

$$\begin{aligned}\delta_\varphi E_\Omega^{WS}(W, A, Z, \varphi)\varphi' &= \partial_s E_\Omega^{WS}(W, A, Z, \varphi_s)|_{s=0} \\ &\int_\Omega g^2 \varphi\varphi' |W|^2 + \frac{g^2}{2\cos^2\theta}\varphi\varphi'|Z|^2 \\ &\quad + 2\nabla\varphi' \cdot \nabla\varphi + 2\lambda(\varphi^2 - \varphi_0^2)\varphi\varphi' \end{aligned} \tag{E.9}$$

Integrating the third term by parts and factoring out $2\varphi'$ gives

$$= \int_\Omega [\frac{g^2}{2}|W|^2 + \frac{1}{2}\kappa|Z|^2 \\ - \Delta + \lambda(\varphi^2 - \varphi_0^2)]\varphi \cdot 2\varphi'. \tag{E.10}$$

For the derivative to be zero for every variation φ', (3.15) must hold. □

Appendix F

Proof of (9.14)

Proof of (9.14). To prove (9.14), we use the w-field Equation (4.3), and $\nu_s := g(a_s \sin\theta + z_s \cos\theta)$, to get

$$\int_{\Omega'} \overline{\chi} \cdot [\operatorname{curl}^*_{\nu_s} \operatorname{curl}_{\nu_s} + \frac{g^2}{2}(\psi_s + \xi_s)^2 \\ - i(\operatorname{curl}\nu_s)J + g^2(\overline{w_s} \times w_s)J]w_s = 0. \tag{F.1}$$

We shall calculate each term of the integral (F.1) up to order s^3 using Proposition 23 and the Taylor expansions (9.9).

Integrating the first term of (F.1) by parts gives

$$\int_{\Omega'} \overline{\chi} \cdot \operatorname{curl}^*_{\nu_s} \operatorname{curl}_{\nu_s} w_s = \int_{\Omega'} \overline{\operatorname{curl}_{\nu_s} \chi} \cdot \operatorname{curl}_{\nu_s} w. \tag{F.2}$$

Plugging in the Taylor expansions (9.9) gives

$$\int_{\Omega'} \overline{\chi} \cdot \operatorname{curl}^*_{\nu_s} \operatorname{curl}_{\nu_s} w_s = \int_{\Omega'} [\overline{\operatorname{curl}_{a^n} \chi} + \mathcal{O}(|s|^2)] \\ \cdot [s \operatorname{curl}_{a^n} \chi - s^3 i\nu' w' + \mathcal{O}(|s|^5)], \tag{F.3}$$

where, recall, $\nu' := g(a' \sin\theta + z' \cos\theta)$. Recall from Equation (5.21) that $\operatorname{curl}_{a^n} \chi = 0$. Therefore, applying Proposition 23 gives

$$\int_{\Omega'} \overline{\chi} \cdot \operatorname{curl}^*_{\nu_s} \operatorname{curl}_{\nu_s} w_s = \mathcal{O}(|s|^5). \tag{F.4}$$

Plugging the Taylor expansions (9.9) into the second term of (F.1) gives

$$\int_{\Omega'} \overline{\chi} \cdot \frac{g^2}{2}(\psi_s + \xi_s)^2 w_s = \int_{\Omega'} \overline{\chi} \cdot \frac{g^2}{2}(\frac{\sqrt{2n}}{g} + s^2(\psi' + \xi') + \mathcal{O}(|s|^4))^2 \\ \times (s\chi + \mathcal{O}(|s|^5)). \tag{F.5}$$

Expanding this product and applying Proposition 23 gives

$$\int_{\Omega'} \overline{\chi} \cdot \frac{g^2}{2}(\psi_s + \xi_s)^2 w_s = s \int_{\Omega'} n|\chi|^2 + s^3 \int_{\Omega'} g\sqrt{2n}(\psi' + \xi')|\chi|^2 + s^3 \int_{\Omega'} n\overline{\chi} \cdot w' + \mathcal{O}(|s|^5). \tag{F.6}$$

Recall that $\chi \in \text{Null}(H_1(n))$ and that w' is orthogonal to Null $H_1(n)$. Therefore the third term vanishes:

$$\int_{\Omega'} \overline{\chi} \cdot \frac{g^2}{2}(\psi_s + \xi_s)^2 w_s = s \int_{\Omega'} n|\chi|^2 + s^3 \int_{\Omega'} g\sqrt{2n}(\psi' + \xi')|\chi|^2 + \mathcal{O}(|s|^5). \tag{F.7}$$

Plugging the Taylor expansions (9.9) into the third term of (F.1) gives

$$\int_{\Omega'} \overline{\chi} \cdot (-i(\operatorname{curl} \nu_s)Jw_s) = \int_{\Omega'} \overline{\chi} \cdot (-in - s^2 i(\operatorname{curl} \nu') + \mathcal{O}(|s|^4)) \times (sJ\chi + s^3 Jw' + \mathcal{O}(|s|^5)). \tag{F.8}$$

Recall from Equation (5.21) that χ is of the form $\chi = (\eta, i\eta)^T$, so
$\overline{\chi} \cdot J\chi = -i|\chi|^2$ and $\overline{\chi} \cdot Jw' = -i\overline{\chi} \cdot w'$. Therefore (F.8) simplifies to

$$\int_{\Omega'} \overline{\chi} \cdot (-i(\operatorname{curl} \nu_s)Jw_s) = \int_{\Omega'} (-in - s^2 i(\operatorname{curl} \nu') + \mathcal{O}(|s|^4)) \times (-si|\chi|^2 - s^3 i\overline{\chi} \cdot w' + \mathcal{O}(|s|^5)). \tag{F.9}$$

Expanding this product and applying Proposition 23 gives

$$\int_{\Omega'} \overline{\chi} \cdot (-i(\operatorname{curl} \nu_s)Jw_s) = -s \int_{\Omega'} n|\chi|^2 - s^3 \int_{\Omega'} (\operatorname{curl} \nu')|\chi|^2 - s^3 \int_{\Omega'} n\overline{\chi} \cdot w' + \mathcal{O}(|s|^5). \tag{F.10}$$

Recall that $\chi \in \text{Null}(H_1(n))$ and that w' is orthogonal to Null $H_1(n)$. Therefore the third term vanishes:

$$\int_{\Omega'} \overline{\chi} \cdot (-i(\operatorname{curl} \nu_s)Jw_s) = -s \int_{\Omega'} n|\chi|^2 - s^3 \int_{\Omega'} (\operatorname{curl} \nu')|\chi|^2 + \mathcal{O}(|s|^5). \tag{F.11}$$

Using $\overline{\chi} \cdot Jw_s = -\overline{\chi} \times w_s$, the fourth term of (F.1) becomes

$$\int_{\Omega'} \overline{\chi} \cdot (g^2 \overline{w}_s \times w_s)Jw_s = \int_{\Omega'} -g^2(\overline{\chi} \times w_s) \times (\overline{w}_s \times w_s). \tag{F.12}$$

Plugging in the Taylor expansions (9.9) gives

$$\int_{\Omega'} \overline{\chi} \cdot (g^2 \overline{w}_s \times w_s)Jw_s = \int_{\Omega'} -g^2(s\overline{\chi} \times \chi + \mathcal{O}(|s|^3)) \times (s^2 \overline{\chi} \times \chi + \mathcal{O}(|s|^4)). \tag{F.13}$$

Recall from Equation (5.21) that χ is of the form $\chi = (\eta, i\eta)^T$, so $\overline{\chi} \times \chi = i|\chi|^2$. This fact and Proposition 23 gives

$$\int_{\Omega'} \overline{\chi} \cdot (g^2 \overline{w}_s \times w_s) J w_s = s^3 \int_{\Omega'} g^2 |\chi|^4 + \mathcal{O}(|s|^5). \tag{F.14}$$

The s^3 terms of (F.4), (F.7), (F.11) and (F.14) must sum to 0, and so (9.14) results. □

Appendix G

Proof of Lemma 25

Proof of Lemma 25. Recall the definitions (9.4) and (9.18) of η and η', respectively. We proceed by expressing the term $\alpha'(m_z, m_h; \tau, r)$ appearing in the denominator of (9.18) in terms of the non-rescaled masses and fields. First, from $r = \xi_s/\varphi_0$ and the Taylor expansions (9.9), we have

$$m_p/r = M_P + R_P(s^2),\ p \in \{w, z, h\}, \tag{G.1}$$
$$R_P(s^2) = \mathcal{O}(|s|^2), \tag{G.2}$$

By (9.24) and $\alpha'(m_z, m_h; \tau, r) > 0$, $s^2 = \mathcal{O}(|1 - \frac{M_W^2}{eb}|)$, so (G.1) - (G.2) become

$$m_p/r = M_P + \tilde{R}_P(1 - \frac{M_W^2}{eb}),\ p \in \{w, z, h\}, \tag{G.3}$$
$$\tilde{R}_P(1 - \frac{M_W^2}{eb}) = \mathcal{O}(|1 - \frac{M_W^2}{eb}|). \tag{G.4}$$

Next, we find an asymptotic relation between the numerators of $\alpha(M_Z, M_H; \tau)$ and $\alpha'(m_z, m_h; \tau, r)$ (see Equations (9.1) and (9.19), respectively, for the definitions of these terms). Using the definition (9.3) of U and changing the order of integration, we obtain

$$\begin{aligned}&\langle U_{m_z,m_h}(|\chi|^2)|\chi|^2\rangle\\ =&\frac{1}{2\pi|\Omega'|}\int_{m_h}^{m_z}\int_{\Omega'}\int |\chi(\rho)|^2|\rho-\rho'|K_1(M|\rho-\rho'|)|\chi(\rho')|^2 d^2\rho'\ d^2\rho\ dM.\end{aligned} \tag{G.5}$$

Recall that $\chi(\rho) = rX_r(r\rho)$. Making the change of coordinates $\varsigma \equiv r\rho$, $\varsigma' \equiv r\rho'$ and $M' \equiv M/r$ gives

$$\begin{aligned}&\langle U_{m_z,m_h}(|\chi|^2)|\chi|^2\rangle\\ &= \frac{1}{2\pi|\Omega'|}\int_{m_h/r}^{m_z/r}\int_{\Omega'}\int |X_r(\varsigma)|^2|\varsigma-\varsigma'|K_1(M'|\varsigma-\varsigma'|)|X_r(\varsigma')|^2 d^2\varsigma'\ d^2\varsigma\ dM'\\ &= \frac{|\Omega|}{|\Omega'|}\langle U_{m_z/r,m_h/r}(|X_r|^2)|X_r|^2\rangle.\end{aligned} \tag{G.6}$$

Define the remainder term

$$R_U(1-\frac{M_W^2}{eb}) := \langle U_{m_z/r,m_h/r}(|X_r|^2)|X_r|^2\rangle - \langle U_{M_Z,M_H}(|X_r|^2)|X_r|^2\rangle. \tag{G.7}$$

We will now estimate R_U. By Equation (G.3),

$$\begin{aligned} &R_U(1-\frac{M_W^2}{eb}) \\ &= \frac{1}{2\pi|\Omega|}\int_T |X_r(\rho)|^2|\rho-\rho'|K_1(M|\rho-\rho'|)|X_r(\rho')|^2 d^2\rho'\ d^2\rho\ dM \\ &- \frac{1}{2\pi|\Omega|}\int_T |X_r(\rho)|^2|\rho-\rho'|K_1(M|\rho-\rho'|)|X_r(\rho')|^2 d^2\rho'\ d^2\rho\ dM, \end{aligned} \tag{G.8}$$

where $T := [M_Z, M_Z+\tilde{R}_Z]\times\Omega\times\mathbb{R}$. The integral kernel $|\rho-\rho'|K_1(M|\rho-\rho'|)$ defines a continuous linear map on the space of locally square-integrable functions, hence it is bounded. Therefore, there exists $C>0$ such that

$$\begin{aligned} |R_U(1-\frac{M_W^2}{eb})| &\le \frac{1}{2\pi}|\int_{M_Z}^{M_Z+\tilde{R}_Z} C\ \langle|X_r|^4\rangle dM| + \frac{1}{2\pi}|\int_{M_H}^{M_H+\tilde{R}_H} C\ \langle|X_r|^4\rangle dM| \\ &= \frac{C}{2\pi}\langle|X_r|^4\rangle|\tilde{R}_Z(1-\frac{M_W^2}{eb})| + \frac{C}{2\pi}\langle|X_r|^4\rangle|\tilde{R}_H(1-\frac{M_W^2}{eb})| \\ &= \mathcal{O}(|1-\frac{M_W^2}{eb}|). \end{aligned} \tag{G.9}$$

Collecting (G.6) and (G.9) into one expression, and using $|\Omega|/|\Omega'| = r^2$, we obtain:

$$\langle U_{m_z,m_h}(|\chi|^2)|\chi|^2\rangle = r^2\langle U_{M_Z,M_H}(|X_r|^2)|X_r|^2\rangle + r^2 R_U(1-\frac{M_W^2}{eb}), \tag{G.10}$$

$$R_U(1-\frac{M_W^2}{eb}) = \mathcal{O}(|1-\frac{M_W^2}{eb}|). \tag{G.11}$$

Similarly,

$$\langle|\chi|^2\rangle = r^2\langle|X_r|^2\rangle. \tag{G.12}$$

Recall the definitions (9.1) and (9.19) of α and α', respectively. (G.10) - (G.12) imply

$$m_w^2\alpha'(m_z,m_h;\tau,r) = (\frac{m_w}{r})^2\alpha(M_Z,M_H;\tau) + \frac{1}{r^2\langle|X_r|^2\rangle^2}R_U(1-\frac{M_W^2}{eb}). \tag{G.13}$$

Using Equations (G.3) - (G.4), we obtain

$$m_w^2\alpha'(m_z,m_h;\tau,r) = M_W^2\alpha(M_Z,M_H;\tau) + R_\alpha(1-\frac{M_W^2}{eb}), \tag{G.14}$$

$$R_\alpha(1-\frac{M_W^2}{eb}) = \mathcal{O}(|1-\frac{M_W^2}{eb}|). \tag{G.15}$$

Plugging (G.14) - (G.15) into (9.18) gives (9.25) - (9.26).

It remains to show that R_α has continuous derivatives of all orders. We proceed by first showing the remainder terms R_P, $\tilde{R}_P$ and R_U have continuous derivatives of all orders. The continous differentiabilty of R_P follows because ξ_s has continuous derivatives of all orders in s. The continuous differentiability of $\tilde{R}_P$ follows from the continuous differentiability of R_P because $s^2 = \sigma(1 - \frac{M_W^2}{eb})$ and σ has continuoous derivatives of all orders in b (see Equation (9.24) and the surrounding discussion). The continous differentiabilty of R_U follows because the right-hand side of (G.8) is an integral of a continuously differentiable function whose limits have continuous derivatives of all orders in b. The continous differentiabilty of R_α follows from (G.3) - (G.4), (G.13) and the continuous differentiability of $\tilde{R}_P$ and R_U. The continuous differentiability of R_η follows from the continuous differentiability of R_α and the definition (9.18) of η' (since $x \mapsto [x + \sin^2\theta]^{-1}$ is analytic in x for $x > 0$). $\square$

Appendix H

Proof of (10.3)

Proof of (10.3). We shall calculate each term in the integral (4.8) up to order s^6 using Proposition 23 and the Taylor expansions (9.9).

Plugging the Taylor expansions (9.9) into the first term of (4.8) gives

$$\int_{\Omega'} |\operatorname{curl}_\nu w_s|^2 = \int_{\Omega'} |s \operatorname{curl}_{a^n} \chi + \mathcal{O}(|s|^3)|^2. \tag{H.1}$$

Recall from Equation (5.21) that $\operatorname{curl}_{a^n} \chi = 0$. Therefore, applying Proposition 23 gives

$$\int_{\Omega'} |\operatorname{curl}_\nu w_s|^2 = \mathcal{O}(|s|^6). \tag{H.2}$$

Plugging the Taylor expansions (9.9) into the second term of (4.8) gives

$$\int_{\Omega'} \frac{1}{2} |\operatorname{curl} z_s|^2 = \int_{\Omega'} \frac{1}{2} |s^2 \operatorname{curl} z' + \mathcal{O}(|s|^4)|^2. \tag{H.3}$$

Expanding the square and applying Proposition 23 gives

$$\int_{\Omega'} \frac{1}{2} |\operatorname{curl} z_s|^2 = s^4 \int_{\Omega'} \frac{1}{2} |\operatorname{curl} z'|^2 + \mathcal{O}(|s|^6). \tag{H.4}$$

Plugging the Taylor expansions (9.9) into the third term of (4.8) gives

$$\int_{\Omega'} \frac{1}{2} |\operatorname{curl} a_s|^2 = \int_{\Omega'} \frac{1}{2} |\operatorname{curl} \frac{1}{e} a^n + s^2 \operatorname{curl} a' + s^4 \operatorname{curl} a'' + \mathcal{O}(|s|^6)|^2. \tag{H.5}$$

Recall that $\operatorname{curl} a^n = n$. Expanding the square gives

$$\begin{aligned} \int_{\Omega'} \frac{1}{2} |\operatorname{curl} a_s|^2 = \int_{\Omega'} [\frac{1}{2} \frac{n^2}{e^2} + s^2 \frac{n}{e} \operatorname{curl} a' + s^4 \frac{n}{e} \operatorname{curl} a'' \\ + s^4 \frac{1}{2} |\operatorname{curl} a'|^2 + \mathcal{O}(|s|^6)]. \end{aligned} \tag{H.6}$$

The second and third terms vanish because a' and a'' are $\mathcal{L}'$-periodic. Therefore, applying Proposition

23 gives

$$\int_{\Omega'} \frac{1}{2}|\operatorname{curl} a_s|^2 = \frac{1}{2}\frac{n^2}{e^2}|\Omega'| + s^4 \int_{\Omega'} \frac{1}{2}|\operatorname{curl} a'|^2 + \mathcal{O}(|s|^6). \tag{H.7}$$

Plugging the Taylor expansions (9.9) into the fourth term of (4.8) gives

$$\int_{\Omega'} \frac{1}{2} g^2 \phi_s^2 |w_s|^2 = \int_{\Omega'} \frac{1}{2} g^2 [\frac{\sqrt{2n}}{g} + s^2(\xi' + \psi') + \mathcal{O}(|s|^4)]^2 \\ \times |s\chi + s^3 w' + \mathcal{O}(|s|^6)|^2. \tag{H.8}$$

Expanding the square terms gives

$$\int_{\Omega'} \frac{1}{2} g^2 \phi_s^2 |w_s|^2 = \int_{\Omega'} \frac{1}{2} g^2 [\frac{2n}{g^2} + s^2 2\frac{\sqrt{2n}}{g}(\xi' + \psi') + \mathcal{O}(|s|^4)] \\ \times [s^2|\chi|^2 + s^4 2\operatorname{Re}(\overline{\chi}\cdot w') + \mathcal{O}(|s|^6)]. \tag{H.9}$$

Expanding this product and applying Proposition 23 gives

$$\int_{\Omega'} \frac{1}{2} g^2 \phi_s^2 |w_s|^2 = s^2 \int_{\Omega'} n|\chi|^2 \\ + s^4 \int_{\Omega'} [g\sqrt{2n}(\xi' + \psi')|\chi|^2 + 2n\operatorname{Re}(\overline{\chi}\cdot w')] + \mathcal{O}(|s|^6). \tag{H.10}$$

Recall that $\chi \in \operatorname{Null}(H_1(n))$ and that w' is orthogonal to $\operatorname{Null}(H_1(n))$. Therefore the third term vanishes:

$$\int_{\Omega'} \frac{1}{2} g^2 \phi_s^2 |w_s|^2 = s^2 \int_{\Omega'} n|\chi|^2 + s^4 \int_{\Omega'} g\sqrt{2n}(\xi' + \psi)|\chi|^2 + \mathcal{O}(|s|^6). \tag{H.11}$$

Plugging the Taylor expansions (9.9) into the fifth term of (4.8) and expanding the square terms gives

$$\int_{\Omega'} \frac{1}{4\cos^2\theta} g^2 \phi_s^2 |z_s|^2 = \int_{\Omega'} \frac{1}{4\cos^2\theta} g^2 \\ \times [\frac{2n}{g^2} + s^2 2\frac{\sqrt{2n}}{g}(\xi' + \psi') + \mathcal{O}(|s|^4)][s^4|z'|^2 + \mathcal{O}(|s|^6)]. \tag{H.12}$$

Expanding this product and applying Proposition 23 gives

$$\int_{\Omega'} \frac{1}{4\cos^2\theta} g^2 \phi_s^2 |z_s|^2 = s^4 \int_{\Omega'} \frac{n}{2\cos^2\theta}|z'|^2 + \mathcal{O}(|s|^6). \tag{H.13}$$

Plugging the Taylor expansions (9.9) into the sixth term of (4.8) gives

$$\int_{\Omega'} |\overline{w}_s \times w_s|^2 = \int_{\Omega'} |s^2 \overline{\chi}\times\chi + \mathcal{O}(|s|^4)|^2, \tag{H.14}$$

Recall from Equation (5.21) that χ is of the form $\chi = (\eta, i\eta)^T$, so $\overline{\chi}\times\chi = i|\chi|^2$. Therefore, applying

Proposition 23 gives

$$\int_{\Omega'} |\overline{w}_s \times w_s|^2 = s^4 \int_{\Omega'} |\chi|^4 + \mathcal{O}(|s|^6). \tag{H.15}$$

Plugging the Taylor expansions (9.9) into the seventh term of (4.8) gives

$$\int_{\Omega'} i(\operatorname{curl}\nu_s)\overline{w}_s \times w_s = \int_{\Omega'} i[g\sin\theta \operatorname{curl}\frac{1}{e}a^n + s^2\operatorname{curl}\nu' + \mathcal{O}(|s|^4)] \\ \times [s\overline{\chi} + s^3\overline{w'} + \mathcal{O}(|s|^5)] \times [s\chi + s^3 w' + \mathcal{O}(|s|^5)]. \tag{H.16}$$

where, recall, $\nu' := g(a'\sin\theta + z'\cos\theta)$. Recall that $\operatorname{curl} a^n = n$ and $e = g\sin\theta$. Expanding the wedge product of the second and third terms gives

$$\int_{\Omega'} i(\operatorname{curl}\nu_s)\overline{w}_s \times w_s = \int_{\Omega'} i[\frac{n}{g}2 + s^2\operatorname{curl}\nu' + \mathcal{O}(|s|^4)] \\ \times [s^2\overline{\chi}\times\chi + s^4(\overline{\chi}\times w' + \overline{w'}\times\chi) + \mathcal{O}(|s|^6)]. \tag{H.17}$$

Recall from Equation (5.21) that χ is of the form $\chi = (\eta, i\eta)^T$, so $\overline{\chi}\times\chi = i|\chi|^2$ and $\overline{\chi}\times w' = i\overline{\chi}\cdot w'$. Therefore

$$\int_{\Omega'} i(\operatorname{curl}\nu_s)\overline{w}_s \times w_s = \int_{\Omega'} [in + s^2 i\operatorname{curl}\nu' + \mathcal{O}(|s|^4)] \\ \times [s^2 i|\chi|^2 + s^4 2\operatorname{Re}(i\overline{\chi}\cdot w') + \mathcal{O}(|s|^6)]. \tag{H.18}$$

Expanding this product and using Proposition 23 gives

$$\int_{\Omega'} i(\operatorname{curl}\nu_s)\overline{w}_s \times w_s = -s^2\int_{\Omega'} n|\chi|^2 - s^4\int_{\Omega'} [2in\operatorname{Im}(\overline{\chi}\cdot w') \\ - s^4\int_{\Omega'} (\operatorname{curl}\nu')|\chi|^2 + \mathcal{O}(|s|^6). \tag{H.19}$$

Recall that $\chi \in \operatorname{Null}(H_1(n))$ and w' is orthogonal to $\operatorname{Null}(H_1(n))$. Therefore the second term vanishes:

$$\int_{\Omega'} i(\operatorname{curl}\nu_s)\overline{w}_s \times w_s = -s^2\int_{\Omega'} n|\chi|^2 - s^4\int_{\Omega'} (\operatorname{curl}\nu')|\chi|^2 \\ + \mathcal{O}(|s|^6). \tag{H.20}$$

Plugging the Taylor expansions (9.9) into the eigth term of (4.8) gives

$$\int_{\Omega'} |\nabla\phi_s|^2 = \int_{\Omega'} |s^2\nabla\psi' + \mathcal{O}(|s|^4)|^2. \tag{H.21}$$

Expanding the square and using Proposition 23 gives

$$\int_{\Omega'} |\nabla\phi_s|^2 = s^4\int_{\Omega'} |\nabla\psi'|^2 + \mathcal{O}(|s|^6). \tag{H.22}$$

Plugging the Taylor expansions (9.9) into the ninth term of (4.8) and expanding the inner squares gives

$$\begin{aligned}
&\int_{\Omega'} \frac{1}{2}\lambda(\phi_s^2 - \xi_s^2) \\
&\quad = \int_{\Omega'} \frac{1}{2}\lambda[\frac{2n}{g^2} + s^2 2\frac{\sqrt{2n}}{g}(\xi' + \psi') - \frac{2n}{g^2} - s^2 2\frac{\sqrt{2n}}{g}\xi' + \mathcal{O}(|s|^4)]^2 \\
&\quad = \int_{\Omega'} \frac{1}{2}\lambda[s^2 2\frac{\sqrt{2n}}{g}\psi' + \mathcal{O}(|s|^4)]^2.
\end{aligned} \tag{H.23}$$

Expanding the outer square gives and using Proposition 23 gives

$$\int_{\Omega'} \frac{1}{2}\lambda(\phi_s^2 - \xi_s^2) = s^4 \int_{\Omega'} \frac{4\lambda n}{g^2}\psi'^2 + \mathcal{O}(|s|^6). \tag{H.24}$$

Adding (H.2) - (H.24) and dividing by $|\Omega'|$ gives (10.3), where R_ε collects the $\mathcal{O}(|s|^6)$ remainder terms. R_ε has continuous derivatives of all orders because it is a sum of integrals of the form (9.7) with f_s and g_s coming from the continuously differentiable remainder terms $\mathcal{O}(|s|^p)$ of (9.9). □

Appendix I

Spectral analysis of the operator $-\Delta_{a^n}$

In this appendix we shall verify the properties of $-\Delta_{a^n}$ used in the main text of this paper, following Section 5 of [16]. Recall from the main text, but in vector notation, that $a^n := \frac{n}{2}x^\perp$, where $(x_1, x_2)^\perp = (-x_2, x_1)$.

Proposition 36. *The operator $-\Delta_{a^n}$ is self-adjoint on its natural domain and its spectrum is given by*

$$\sigma(-\Delta_{a^n}) = \{\, (2m+1)n : m \in \mathbb{Z}_{\geq 0} \,\}, \tag{I.1}$$

with each eigenvalue is of the multiplicity n. Moreover,

$$\mathrm{Null}(-\Delta_{a^n} - n) = e^{\frac{in}{2}x_2(x_1+ix_2)} V_n, \tag{I.2}$$

where V_n is spanned by functions of the form (below $z = (x_1 + ix_2)/\sqrt{\frac{2\pi}{\operatorname{Im}\tau}}$)

$$\theta(z,\tau) := \sum_{m=-\infty}^{\infty} c_m e^{i2\pi m z},\ c_{m+n} = e^{-in\pi z} e^{i2m\pi\tau} c_m. \tag{I.3}$$

Such functions are determined entirely by the values of $c_0, \ldots, c_{n-1}$ and therefore form an n-dimensional vector space.

Proof. The self-adjointness of the operator $-\Delta_{a^n}$ is well-known. To find its spectrum, we introduce the complexified covariant derivatives (harmonic oscillator annihilation and creation operators), $\bar\partial_{a^n}$ and $\bar\partial^*_{a^n} = -\partial_{a^n}$, with

$$\bar\partial_{a^n} := (\nabla_{a^n})_1 + i(\nabla_{a^n})_2 = \partial_{x_1} + i\partial_{x_2} + \frac{1}{2}n(x_1 + ix_2). \tag{I.4}$$

One can verify that these operators satisfy the following relations:

$$[\bar\partial_{a^n}, (\bar\partial_{a^n})^*] = 2\,\mathrm{curl}\, a^n = 2n; \tag{I.5}$$

$$-\Delta_{a^n} - n = (\bar\partial_{a^n})^* \bar\partial_{a^n}. \tag{I.6}$$

As for the harmonic oscillator (see e.g. [20]), this gives explicit information about the spectrum of $-\Delta_{a^n}$, namely (I.1), with each eigenvalue is of the same multiplicity. Furthermore, the above properties imply

$$\text{Null}(-\Delta_{a^n} - n) = \text{Null}\,\bar{\partial}_{a^n}. \tag{I.7}$$

We find $\text{Null}\,\bar{\partial}_{a^n}$. A simple calculation gives the following operator equation

$$e^{-\frac{n}{2}(ix_1x_2 - x_2^2)}\bar{\partial}_{a^n} e^{\frac{n}{2}(ix_1x_2 - x_2^2)} = \partial_{x_1} + i\partial_{x_2}.$$

(The transformation on the left-hand side is highly non-unique.) This immediately proves that

$$\bar{\partial}_{a^n}\psi = 0, \tag{I.8}$$

if and only if $\theta = e^{-\frac{n}{2}(ix_1x_2 - x_2^2)}\psi$ satisfies $(\partial_{x_1} + i\partial_{x_2})\theta = 0$. We now identify $x \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$ and see that this means that θ is analytic and

$$\psi(x) = e^{-\frac{\pi n}{2\,\text{Im}\,\tau}(|z|^2 - z^2)}\theta(z,\tau),\ z = (x_1 + ix_2)/\sqrt{\frac{2\pi}{\text{Im}\,\tau}}. \tag{I.9}$$

where we display the dependence of θ on τ. The quasiperiodicity of ψ transfers to θ as follows:

$$\theta(z+1,\tau) = \theta(z,\tau), \qquad \theta(z+\tau,\tau) = e^{-2\pi i n z}e^{-in\pi\tau}\theta(z,\tau).$$

The first relation ensures that θ have a absolutely convergent Fourier expansion of the form $\theta(z,\tau) = \sum_{m=-\infty}^{\infty} c_m e^{2\pi m i z}$. The second relation, on the other hand, leads to relation for the coefficients of the expansion: $c_{m+n} = e^{-in\pi z}e^{i2m\pi\tau}c_m$, which together with the previous statement implies (I.3). □

Next, we claim that the solution (I.9) satisfies

$$\psi(x) = \psi(-x). \tag{I.10}$$

By (I.9), it suffices to show that $\theta(z) = \theta(-z)$. We show this for $n = 1$. Denote the corresponding θ by $\theta(z,\tau)$. Iterating the recursive relation for the coefficients in (I.3), we obtain the following standard representation for the theta function

$$\theta(z,\tau) = \sum_{m=-\infty}^{\infty} e^{2\pi i(\frac{1}{2}m^2\tau + mz)}. \tag{I.11}$$

We observe that $\theta(-z,\tau) = \theta(z,\tau)$ and therefore $\psi_0(-x) = \psi_0(x)$. Indeed, using the expression (I.11), we find, after changing m to $-m'$, we find

$$\theta(-z,\tau) = \sum_{m=-\infty}^{\infty} e^{2\pi i(\frac{1}{2}m^2\tau - mz)} = \sum_{m'=-\infty}^{\infty} e^{2\pi i(\frac{1}{2}m'^2\tau + m'z)} = \theta(z,\tau). \tag{I.12}$$

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