Borel Chain Conditions

by

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Abstract

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The subject matter of this Thesis is an instance of the Chain Condition Method of coarse classification of Boolean algebras and partially ordered sets. The first occurrence of the method can be traced back to the famous century-old problem of Mihail Souslin [21], asking if the countable chain condition and the σ-centeredness are equivalent conditions on a poset of intervals of a linearly ordered set. Another prominent classical problem where chain conditions play a crucial role is a problem from Von Neumann in 1937, asking if the countable chain condition is sufficient for the existence of a strictly positive measure on a complete weakly distributive algebras. The first systematic analysis of the chain condition method, appearing in a 1948 paper of Horn and Tarski[7], was also inspired by the problem of characterizing the Boolean algebras supporting strictly additive measures. After Paul Cohen's invention of forcing, the chain condition method was finding much more application in the construction of forcing notions and the analysis of its forcing extensions. For example, the forcing aspect of the chain condition method played an important role in the continuous effort for many decades through the work of Maharam[18], Jech, Balcar[2], Todorcevic[27] and Talarand[22], which has led to a complete solution to the von Neumann problem. Another application of the chain conditions is the Martin type axioms for classes of posets satisfying different chain conditions. It turns out that different chain conditions led to different corresponding Martin type forcing axioms, providing a rich array of consistent results in set theory (see, for example, [14] and [6]).

The descriptive combinatorics, especially the branch of descriptive graph theory starting with the discovery of G_0-dichotomy, the statement characterizing the analytic graphs of uncountable Borel chromatic number as analytic graphs G for which there is a continuous homomorphism from G_0 into G in [12], has been a rather active area of research for more than twenty years. The variation of Borel chromatic number from the classical chromatic number suggests that similar idea can be implemented in the context of chain condition, especially those requiring countable decomposition such as the σ-centeredness or the σ-finite chain condition. When the partially ordered set is a Borel structure on a Polish space, it becomes natural to require the pieces of the decomposition witnessing such chain condition are Borel as well. One of the purposes of this Thesis is to convince the readers that the corresponding new theory of

Borel chain conditions is an interesting and useful enrichment of the classical theory, with its own new phenomena and problems. For example, the new theory suggests the following interesting variation on the classical problem of Horn and Tarski [7]:

Problem 1. *Does there exist a Borel partially ordered set satisfying the Borel σ-finite chain condition but failing to satisfy the σ-bounded chain condition?*

It should be mentioned that there is a Borel poset $T(\pi(\mathbb{Q}))$ satisfying the σ-bounded chain condition but failing to satisfy the σ-bounded chain condition (see [28]). The following fact shows that, however, $T(\pi(\mathbb{Q}))$ does not solve the Problem 1.

Theorem 0.1. *For every countable decomposition $T(\pi(\mathbb{Q})) = \bigcup_{n<\omega} B_n$ of the poset $T(\pi(\mathbb{Q}))$ into countably many Borel sets, one of the sets B_n contains an infinite subset of pairwise incompatible elements.*

Thus we have an interesting phenomenon very much reminiscent of the phenomena in Borel graph combinatorics where Borel chromatic number could be larger than the classical chromatic number. For example, the graph G_0 of [12] is an acyclic and therefore a bipartite graph in the classical sense, but as shown in [12], G_0 cannot be decomposed into countably many Borel discrete sets. Moreover, there is a continuous homomorphism from G_0 into any other analytic graph G with uncountable Borel chromatic number (see [12]).

The Thesis is organized as follows. After necessary preliminaries, Chapter 2 introduces the classical chain conditions. In Chapter 3 a generalization of the G_0-dichotomy is given for a wider class of ideals. In the same chapter, we also provide a few general facts and present a survey of some recent activities in this area. These two chapters will serve as a basis for the last chapter of the Thesis which contains our main results.

In Chapter 4, we investigate the connections between the chain condition method as a coarse classification scheme of posets and a coarse classification scheme of graphs using the chromatic numbers. We then introduce the Borel version of some classical chain condition and show that the Borel poset $T(\pi\mathbb{Q})$, the Borel example Todorcevic used to distinguish σ-finite chain condition and σ-bounded chain condition in [28], cannot be decomposed into countably many Borel pieces witnessing the σ-finite chain condition, despite the fact that the such partition exists if they are allowed to be non-Borel (see [30]). From there on, we use the variations on the G_0-dichotomy analyzed in Chapter 3 to construct a number of examples of Borel posets of the form $\mathbb{D}(G)$ to show that the new hierarchy of Borel chain conditions is proper. In particular, we prove the following theorem:

Theorem 0.2. *For every pair of chain conditions from the following list there is a Bore poset that satisfies the formally weaker but not the stronger:*

1. *The countable chain condition*

2. *The Borel σ-finite- chain condition,*

3. *The Borel σ-bounded-chain-condition,*

4. *The Borel σ-n-linked for a particular $n \geq 2$,*

5. *The Borel σ-n-linked for all $n \geq 2$,*

6. *The Borel σ-centred.*

We finish the Chapter by providing a characterization of the Borel σ-linked posets in the class $\mathbb{D}(G)$. Moreover, we prove the following Borel version of an unpublished result of Galvin and Hajnal:

Theorem 0.3. *Let $P = (X, \leq)$ be a good Borel poset. If P satisfies the Borel σ-n-chain condition for some integer $n \geq 2$, then P is Borel σ-linked.*

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Contents

Chapter 1

Introduction

1.1 Chain Conditions on Boolean algebras and Posets

One of the main topics of this thesis traces back to an old question from the Scottish Book posed by Von Neumann in 1937: characterize the Boolean algebras supporting a strictly positive σ-additive measure (hereby called a measure algebra)[31]. Von Neumann noticed that the two conditions that every measure algebra has to satisfy are the countable chain condition and the σ-weak distributivity. He asked whether these two conditions together would be enough for B to support a strictly positive σ-additive measure.

The first major progress was made by Maharam. In [18], she introduced the notion of submeasure and the von Neumann problem splits into two questions:

1. Is it true that every Boolean algebra carrying a strictly positive exhaustive submeasure is a measure algebra?

2. Is it true that every σ-weakly distributive complete Boolean algebra with countable chain condition carries a strictly positive exhaustive submeasure?

The first question is later known as the *control measure problem* in the functional analysis. The second question is known as “Maharam problem”, and a Boolean algebra carrying a strictly positive exhaustive submeasure is called a *Maharam algebra*. In [18], a systematic analysis over Maharam algebras was carried out and the Maharam problem is transferred into a metrizability problem of certain topology induced by the algebraic operations. While in the same paper Maharam has already noticed that it is consistent to have a σ-weakly distributive complete Boolean algebra that is not a Maharam algebra thus the Maharam problem is consistently false, the full solution to the problem was given much later in [2] and [27]. In [2], Balcar, Jech and Pazák have shown that the Maharam problem consistently has a positive answer, and in [27] Todorcevic gave a full characterization of Maharam algebras in ZFC using a chain condition introduced by Horn and Tarski in [7].

In 1947, Horn and Tarski analyzed the measures on Boolean algebras (both finitely additive and σ-additive) and introduced, among others, two important new chain conditions refining the countable chain condition: the σ-bounded chain condition and the σ-finite chain condition. Every measure algebra must satisfy the σ-bounded chain condition and every Maharam algebra must satisfy the σ-finite chain condition. In [27] Todorcevic has shown that weakly distributive Boolean algebras satisfying the σ-finite chain condition are all Maharam algebras. This fully solved the ZFC version of the Maharam problem.

As mentioned above, the Horn and Tarski's [7] analysis led to the isolation of several chain conditions, many of which later became standard restrictions on forcing notions. They posed natural questions on whether these chain conditions are equivalent in a given class of Boolean algebras. Most of these problems were solved shortly after the invention of Forcing but one of them known as *The Horn Tarski Problem* remained unsolved until 2013: Are the σ-bounded chain condition and the σ-finite chain condition the same restriction on a given Boolean algebra?

This question was first solved in [24], by proposing a special poset that satisfies the σ-finite chain condition but fails the σ-bounded chain condition. The poset constructed in [24] belongs to a class of posets called Todorcevic ordering, which was first introduced in [25]. Then, inspired by Thumël's example, Todorcevic [28] constructed another example $T(\pi\mathbb{Q})$ that solves the Horn-Tarski problem that is also definable as a Borel subset of a Polish space. As both σ-bounded chain condition and the σ-finite chain condition are properties witnessed by countable fragmentation of the posets, the next natural question is: can the pieces of the fragmentation witnessing the σ-finite chain condition of $T(\pi\mathbb{Q})$ be chosen to be also Borel definable? In [30], Todorcevic and the author provided a negative answer to the problem above by using the absoluteness of a certain class of pointsets. This brings the analysis of chain conditions on posets to the area of descriptive combinatorics which in recent years has been developing rapidly. This descriptive aspect of the chain condition method is the subject matter of this Thesis. We shall exhibit the differences between the definable and undefinable chain conditions reminiscent of the differences between the definable and undefinable chromatic numbers of Borel graphs, exposed in the well-known paper [12] of Kechris, Solecki and Todorcevic. We shall also expose definable versions of the classical problems regarding chain conditions such as the Borel version of the Horn-Tarski problem, which asks for a Borel poset without the Borel version of σ-bounded chain condition but satisfies the Borel version of the σ-finite chain condition. In this Thesis, we shall also analyze basis problems for Borel posets satisfying a given chain condition bearing in mind the G_0-dichotomy of [12] which shows that the class of Borel graphs having uncountable Borel chromatic numbers have a one-element basis, the graph G_0. Another phenomenon investigated in this Thesis is that many standard chain conditions can be distinguished by Borel posets. This phenomenon leads to a natural general problem regarding the definable versions of these chain conditions: are they also distinguishable? If so, is there a finite basis for the class of Borel posets that distinguish them?

1.2 Background on Descriptive Combinatorics

Descriptive set theory is a subject focusing on the mathematical structures definable by certain effective constructions and has been one of the central topics of set theory for more than a century since its establishment by Baire, Borel and Lebesgue. Thenceforth, it resulted in numbers of other fields in mathematics, such as math logic, general set theory and dynamical systems.

Among the rich content of descriptive set theory, descriptive combinatorics studies the combinatorics of definable structures. This area has been actively developed since the discovery of the G_0-dichotomy in [12].

A Polish space is a separable completely metrizable topological space, and the collection of Borel sets is the smallest σ-algebra containing all of its open sets. The combinatorics considering only the Borel sets can be very different from the classic combinatorics. For instance, it is obvious that the (vertex) chromatic number of any acyclic graph is 2, but there is an acyclic Borel graph (which means that the vertices form a Polish space and the edge form a Borel set) with no countable chromatic number if we restrict the coloring to be Borel sets (see, for example, 3.1 in [12]). In Chapter 3, we investigate a notion generalizing the Borel chromatic number and formulate the G_0-dichotomy in this context.

On the other hand, many classical results have their counterparts in the context of descriptive combinatorics. For example, the Horn-Tarski problem in the Borel context still has a negative answer, as in the classical theory. This will be analyzed in Chapter 4.

1.3 Background on Descriptive Set Theory

We use the standard set theoretical notions in this thesis. See, for example, [9],[11] or [19].

Definition 1.1. *A Partial ordering $\leq$ on a set P is a binary transitive, reflexive and antisymmetric binary relation on P. A set P equipped with a partial ordering is called a partially ordered set, or poset.*

Definition 1.2. *Let A be a set and κ be an ordinal. the tree of height κ on A is the set $\bigcup_{\eta<\kappa} A^\eta$ of all sequences with elements from A of length $<\kappa$, ordered by end extension.*

A branch b of such a tree is a sequence with elements in A of length κ. Give a tree T, the set of all branches of T is denoted by $[T]$.

The elements from a tree are called nodes. For a node $t \in T$, the length of t is called the height of t and is denoted by $|t|$. For an ordinal $\eta < \kappa$, the η's level of T is the set of all nodes of height η, and is denoted by $T[\eta]$. The restriction of T at t, denoted by $T|t$, is the set of all nodes in T that are either end-extended by t, or end-extends t.

Definition 1.3. *A Boolean algebra is an algebra B with constants $0, 1$, commutative and associative binary operations $a \vee b$, $a \wedge b$ and an unary operation $\neg a$ satisfying the following conditions in addition:*

1. $\neg(a \vee b) = \neg a \wedge \neg b$. $\neg(a \wedge b) = \neg a \vee \neg b$.

2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

3. $a \vee \neg a = 1$, $a \wedge \neg a = 0$.

A Boolean algebra is κ-complete if for every subset $A \subset B$ of size $<\kappa$ both $\wedge A$ and $\vee A$ are well-defined. It is said to be complete if it is κ-complete for every κ. It is said to be σ-complete if it is $\aleph_1$-complete.

A Polish space is a separable completely metrizable topological space.

Definition 1.4. *The Borel hierarchy of a Polish space X is the hierarchy of subsets inductively defined as:*

1. $\mathbf{\Sigma^0_1}$ =*open sets.*

2. *For every η,* $\mathbf{\Pi^0_\eta} = \{X \setminus S : S \in \mathbf{\Sigma^0_\eta}\}$.

3. *For every η,* $\mathbf{\Sigma^0_{\eta+1}} = \{\bigcup_{n<\omega} S_n : S_n \in \mathbf{\Pi^0_\eta}\}$.

4. *For limit η,* $\mathbf{\Sigma^0_\eta} = \bigcup_{\zeta<\eta} \mathbf{\Sigma^0_\zeta}$.

5. *For each η,* $\mathbf{\Delta^0_\eta} = \mathbf{\Sigma^0_\eta} \cap \mathbf{\Pi^0_\eta}$.

It is well-known that the hierarchy becomes constant after ω_1 steps. The union of Borel hierarchy $\bigcup_{\eta<\omega_1} \mathbf{\Sigma^0_\eta} = \bigcup_{\eta<\omega_1} \mathbf{\Pi^0_\eta}$ is the collection of Borel sets of X.

Definition 1.5. *The analytical hierarchy is the hierarchy of subsets of a Polish space X inductively defined as:*

1. $\mathbf{\Sigma^1_1} = \mathbf{\Pi^0_1} = \{f(S) : f$ *is a continuous function and* $S \in \mathbf{\Pi^0_1}\}$.
2. *For each* η, $\mathbf{\Pi^1_\eta} = \{X \setminus S : S \in \mathbf{\Sigma^1_\eta}\}$.
3. *For each* η, $\mathbf{\Sigma^1_\eta} = \{f(S) : f$ *is a continuous function and* $S \in \mathbf{\Pi^1_\eta}\}$.
4. *For each limit* η, $\mathbf{\Sigma^1_\eta} = \bigcup_{\zeta<\eta} \mathbf{\Sigma^1_\zeta}$.
5. *For each* η, $\mathbf{\Delta^1_\eta} = \mathbf{\Sigma^1_\eta} \cap \mathbf{\Pi^1_\eta}$.

The famous Souslin's theorem states that $\mathbf{\Delta^1_1}$ is precisely the collection of all Borel sets. The sets in the class $\mathbf{\Sigma^1_1}$ are called analytic sets and the sets in the class $\mathbf{\Pi^1_1}$ are called coanalytic sets.

Also, recall Kleene's (effective) hierarchy. Let X be a Polish space. Given a class of first or second order formulas Γ ($\Gamma = \Sigma^0_1, \Pi^0_1, ..., \Sigma^1_1, \Pi^1_1 ...$, etc.) over recursive predicates, denote the class of subsets definable by Γ-formulas in X using the same symbol Γ . For $a \in X$, denote the relativization of Γ in a by $\Gamma(a)$. The power of recursion theory depends on the following fact:

Theorem 1.1. *Let* Γ *be one of the classes in the Kleene's hierarchy. For every set* $A \in \mathbf{\Gamma}$ *for the corresponding boldface* $\mathbf{\Gamma}$, *there is a real* a *such that* $A \in \Gamma(a)$.

Definition 1.6. *A class* $\Phi \subset Pow(X)$ *of subsets of* X *is said to be* Π^1_1 *over* Σ^1_1 *if for any Polish space* Y *and* $\mathbf{\Sigma^1_1}$ *set* $G \subset Y \times X$, *the set* $\{y \in Y : G_y \in \Phi\}$ *is* Π^1_1.

We are going to use the following facts:

Theorem 1.2. *The class of sets having the property of Baire form a* σ*-algebra containing all open sets. In particular, all Borel sets have the property of Baire.*

Theorem 1.3 (First Reflection Theorem)**.** *Let* X *be a Polish space and let* $\Phi \subset Pow(X)$. *If* Φ *is* $\mathbf{\Pi^1_1}$ *on* $\mathbf{\Sigma^1_1}$, *then for every* $A \in \Phi$ *being* $\mathbf{\Sigma^1_1}$, *there is a Borel set* $A' \in \Phi$ *such that* $A \subset A'$.

Theorem 1.4 (Shoenfiled's Absoluteness Theorem)**.** *Every* $\Sigma^1_2(a)$ *and* $\Pi^1_2(a)$ *statement is absolute for inner models* M *of* $ZF + DC$ *if* $a \in M$.

Chapter 2

The Chain Condition Method

2.1 Chain Conditions on Posets

Let $(P, \leq)$ be a poset. For $x, y \in P$ we say that x and y are incompatible if there is not $z \in P$ such that $z \leq x$ and $z \leq y$ and denote by $x \perp y$. We list the related notions as follows. Given a Boolean algebra $(B, 0, 1, \vee, \wedge, -)$ we consider it as a poset with the ordering $x \leq y$ iff $x \vee y = y$. Note that 0 is the minimal and 1 is the maximal element of B. When considering chain conditions on a Boolean algebra $(B, 0, 1, \vee \wedge, -)$ we will really be only working with the positive part $B^+ = B \setminus \{0\}$ of the algebra. Note that in this case, the incompatibility relation $\perp$ becomes the disjointness relation, i.e. $x \perp y$ iff $x \wedge y = 0$. If a given poset $(P, \leq)$ is considered as a topological space with the topology generated by sets of the form

$$\{y : y \leq x\}, (x \in P)$$

we get a topological space that has only T_0 separation property but it could be used to generate a Boolean algebra B which satisfies essentially all chain conditions which P satisfies. We put B to be simply the Boolean algebra $ro(P)$ of regular open subsets of P. When P is a *separative* poset then $x \mapsto \{y : y \leq x\}$ is an isomorphic embedding of P into a dense subset of $ro(P)$, which explains why P and $ro(P)$ share the same chain conditions.

Definition 2.1. *A subset $X \subset P$ is called:*

1. *disjoint, or an antichain, if for any $x \neq y \in X$, $x \perp y$.*
2. *κ-chain condition (κ-cc) if every disjoint subset of A is of size $< \kappa$.*
3. *countable chain condition (ccc) if it is ω_1-cc.*
4. *finite chain condition (fcc) if it is ω-cc.*
5. *bounded chain condition (bcc) if it is n-cc for some finite n.*
6. *n-linked if for any nonempty subset $\{x_i\}_{0 \leq i < k}$ of size $k \leq n$, there is a $y \in P$ so that $y \leq x_i$ for all $i < k$.*
7. *linked if it is 2-linked.*

8. *centred if it is ω-linked.*

Definition 2.2. *Given a poset, it is said to be:*

1. *σ-finite chain condition (σ-fcc) if it is a union of countably many fcc subsets.*

2. *σ-bounded chain condition (σ-bcc) if it is a union of countably many bcc subsets.*

3. *σ-n-chain condition (σ-n-cc) if it is union of countably many n-cc subsets.*

4. *σ-n-linked if it is a union of countably many n-linked subsets.*

5. *σ-linked if it is σ-2-linked.*

6. *σ-centred if it is a union of countable many centred subsets.*

It should be noted that some of these chain conditions impose cardinality restrictions in the class of separative posets such as Boolean algebras. For example, note the following fact.

Theorem 2.1. *Every σ-linked Boolean algebra has cardinality at most the continuum.*

Proof. Suppose a Boolean algebra B has cardinality bigger than the continuum and let x_α ($\alpha < \mathfrak{c}^+$) be a one-to-one sequence of positive elements of B. Let $B^+ = \bigcup_{n<\omega} B_n$ be a countable partition of B^+. Consider the mapping $f : [\mathfrak{c}^+]^2 \to \mathbb{Z}$ where for $\alpha < \beta < \mathfrak{c}^+$, we set

$$f(\alpha,\beta) = \begin{cases} -2 & \text{if } x_\alpha \setminus x_\beta = 0 \\ -1 & \text{if } x_\beta \setminus x_\alpha = 0 \\ n. & \text{if } x_\beta \setminus x_\alpha \in B_n \end{cases} \tag{2.1}$$

By the Erdős-Rado theorem, there is uncountable $\Gamma \subseteq \mathfrak{c}^+$ and $n \in \mathbb{Z}$ such that $f(\alpha,\beta) = n$ for all $\alpha < \beta$ in Γ. Note that $n = -1$ or $n = -2$ would imply that B fails to satisfy the countable chain condition, so in particular it is not σ-linked. So let us consider the case that $n \geq 0$. Fix $\alpha < \beta < \gamma$ in Γ, then $x_\beta \setminus x_\alpha$ and $x_\gamma \setminus x_\beta$ are two disjoint elements of the subset B_n, so in particular this set is not linked. This shows that the decomposition $B^+ = \bigcup_{n<\omega} B_n$ is not a decomposition of B^+ into countably many linked subsets.

□

Definition 2.3 ([25], [3])**.** *Given a topological space X, the Todorcevic ordering $(T(X), \leq)$ is the set of all compact sets containing only finitely many limit points, equipped with a partial ordering $\leq$ defined as: $p \leq q$ iff $p \subset q$ and $q^{(1)} \cap p = p^{(1)}$.*

Example 2.1. *$T(X)$ is σ-centred for $X \subset \omega_1$ non-stationary.*

Proof. Pick a closed unbounded set $C \subset \omega_1$ missing X. For every $\alpha \in C$, let $\alpha^+ = min\{\beta \in C : \alpha < \beta\}$. Then for every $p \in T(X)$, there is a finite subset $F_p = \{\alpha_{p,i}\}_{i<k} \subset C$ such that $p \subset \bigcup_{i<k}[\alpha_{p,i}, \alpha_{p,i}+)$. Otherwise one of the limit point in p is the same as a limit point of these infinitely many α's, contradicting our choice of C.

Then note that each $[\alpha, \alpha+)$ is countable thus $T([\alpha, \alpha^+))$ is σ-centred (for every finite subset S, all p such that $p^{(1)} = S$ form a centred family). Let $T([\alpha, \alpha^+)) = \bigcup_{i<\omega} T_{\alpha,i}$ be a decomposition witnessing the σ-centredness. Then consider the topological space $\Pi_{\alpha\in C}\omega$ equipped with product topology (where each ω is discrete). By the Hewitt-Marczewski-Pondiczery theorem it is separable. Take a countable dense subset D. For each $f \in D$, let $P_f = \{p : p \subset \bigcup_{\alpha \in F_p} T_{\alpha,f(\alpha)}\}$. Clearly P_f is centred and $\bigcup_{f \in D} P_f = T(X)$. Thus $T(X)$ is σ-centred.

□

Example 2.2. *The localization forcing P is the poset of all functions f from ω to $[\omega]^{<\omega}$ such that for $i < j$ $f(i) \subset f(j)$, $|f(i)| < i$ and is eventually constant. The localization forcing is σ-n-linked for all n but not σ-centred.*

Example 2.3. *Let $T = \omega^{<\omega}$. For each $f \neq g \in [T]$ denote by $\Delta(f,g) = min\{i : f(i) \neq g(i)\}$ and for a set $A \subset |T|$, let $\Delta(A) = \{\Delta(f,g) : f \neq g \in A\}$. A subset $A \subset [T]$ is said to be $\leq n$-ary if for every subset $B \subset A$, $|B| \leq n$ whenever $|\Delta(B)| = 1$. Let $P = \{p \subset [T] : p$ is finite and $\leq n$-ary$\}$. Order P by reversed inclusion. Then P_n is σ-n-linked but not σ-$(n+1)$-linked for every $n \geq 2$.*

Proof. Take an uncountable sequence $A = \{f_\lambda\}_{\lambda<\omega_1}$ such that for $\lambda < \mu$, $f_\lambda \leq^* f_\mu$ and is $\leq^*$ unbounded. If $P = \bigcap_i P_i$ is a partition into countably many pieces, there is one piece P_n containing an uncountable (thus unbounded) subsequence $A' \subset A$. Then there is an $t \in T$ so that $\{f \in A' : t \sqsubset f\}$ is unbounded and $\{f(|t|+1) : f \in A'$ and $f \sqsupset t\}$ is infinite. This set is then obviously not $n+1$-linked. Therefore P is not σ-$n+1$-linked.

To see that P is σ-n-linked, let $Q = \{a \subset T$ finite and $\leq n$-ary$\}$. For each $a \in Q$, let $P_a = \{p \in P$: for each $t \in a$ there is exactly one $x \in p$ so that $t \sqsubset x$ and for each $x \in p$ there is one $t \in a$ so that $t \sqsubset x\}$. Then clearly each P_a is n-linked and as Q is countable, P is σ-n-linked. □

Example 2.4. *The measure algebra of character bigger than the continuum is σ-bcc but not σ-linked.*

Theorem 2.2 (Thümmel [24])**.** *There is a poset of the form $T(X)$ which is σ-fcc but not σ-bcc.*

This gives a solution to the Horn-Tarski problem. See also section 2.2.

Theorem 2.3 (Todorcevic [25])**.** *If X is a polish space then $T(X)$ is ccc but not σ-fcc.*

Thus we have the following:

Theorem 2.4. *The conditions σ-finite chain condition, σ-bounded chain condition, σ-linked, σ-n-linked for all n, and σ-centred are all distinct.*

On the other hand, we have the following unpublished result of Galvin and Hajnal.

Theorem 2.5 (Galvin-Hajnal)**.** *If for some positive integer n, a poset P satisfies the σ-n-chain condition then, in fact, it is σ-linked.*

Proof. Let P be a poset satisfying σ-n-chain condition for some $n > 2$. We shall show that it satisfies the stronger σ-$(n-1)$-chain condition. This will finish the proof.

Suppose not. Let $P = \bigcup_{i<\omega} P_i$ where each P_i satisfies n chain condition. Fix k such that P_k cannot be decomposed into countably many sets of $n-1$ chain condition.

For each $p \in P_k$, let $L(p) = \{q \in P_k$ so that p and q are incompatible$\}$ and $R_i(p) = \{q \in P_k$: there is a $r \in P_i$ extending both p and $q\}$. Then for every p, $L(p) \cup (\bigcup_i R_i(p) = P_k$. Moreover, for each p there are integers $i(p)$ so that $R_{i(p)}(p)$ contains an antichain of size $n-1$. For each i, let $Q_i = \{p \in P_k : i(p) = i\}$. Clearly $\bigcup_i Q_i = P_k$, thus there is an l so that Q_l contains an antichain $p_1, ..., p_{n-1}$ of size $n-1$. Then for each $i = 1, 2, ...n-1$ we can find antichain $q_{i1}, ..., q_{i(n-1)}$ in R_l of size $n-1$. For each $i = 1, ..., n-1$ and $j = 1, 2, ...n-1$, we fix r_{ij} in P_l extending both p_i and q_{ij}. Then $\{r_{ij} : i, j = 1, 2, ..., n-1\}$ is an antichain and it is a subset of P_l. Since $(n-1)^2 > n$, we have a contradiction.

□

2.2 The Horn-Tarski Problem Solved by a Borel Poset

The question that whether there is a poset as stated in the example 2.2 was originated in the study of the existence of strictly positive measures on Boolean algebra in [7]. The first such example was given in [24] by using Todorcevic ordering over uncountable trees. The following example is the one given in [28].

Definition 2.4. *The topological space $\pi\mathbb{Q}$ is the set of all bounded subsets of $\mathbb{Q}$ that contain their lower bound. Order it by $r_0 \sqsubseteq r_1$ if r_0 is an initial segment of r_1 and $r_1 \setminus r_0$ contains the lower bound. A basic open neighbourhood of $r \in \pi\mathbb{Q}$ bounded by $q \in \mathbb{Q}$ is defined as $B_q(r) = \{t \in \pi\mathbb{Q} : r \sqsubseteq t \text{ and } sup(t) < q\}$. This gives us a first countable topology on $\pi(\mathbb{Q})$. We let $T(\pi\mathbb{Q})$ denote the Todorcevic ordering generated by this topology.*

Theorem 2.6 (Todorcevic[28])**.** *$T(\pi\mathbb{Q})$ is not σ-bounded cc.*

Proof. Suppose it is. Let $T(\pi\mathbb{Q}) = T_i$ where each T_i has no antichains of size more than i. For every $t \in \pi\mathbb{Q}$, let $T_i(t, q) = \{p \in T_i : t^{(1)} \cap B_q(t) \neq \emptyset\}$. Let $\phi_n(t, q)$ be the largest integer k of which size an antichain can be found in $T_i(t, q)$.

Note that $\phi(t_0, q_0) \leq \phi(t_1, q_1)$ for $t_0 \sqsupseteq t_1$ and $q_0 \leq q_1$. Thus for every $n \in \omega$, $t \in \pi\mathbb{Q}$ and $q > \sup(t)$, there are $t' \sqsupseteq t$ and $q' \in (\sup(t'), q]$ such that for any $t'' \sqsupseteq t'$ with $sup(t'') < q'$ and $q'' \in (\sup(t''), q']$, $\phi_n(t', q') = \phi_n(t'', q'')$.

Using this fact, starting from arbitrary $t_0 \in \pi\mathbb{Q}$ and $q_0 > \sup(t_0)$, we construct a *sqsubseteq*-increasing sequence t_i and a decreasing sequence of rationals q_i such that:

1. $\sup(t_i) < q_i$.
2. $\phi_i(u, q_i) = \phi_i(t_i, q_i)$ for all $u \in B_{q_i}(t_i)$. Call this number $\phi_i(t_i)$.
3. $\sup_i(\sup(t_i)) = \inf_i(q_i)$.

Let $t_\omega = \bigcup_i t_i$. Let $s_i = t_\omega \cup \{q_{i+1}\}$. By (2), $\phi_i(s_i, q_i) = \phi_i(t_i)$. Let $A_i \subset T_i(s_i, q_i)$ be an antichain of size $\phi_i(t_i)$. Let $F_i = \{p^{(1)} \cap B_{s_i}(q_i) : p \in A_i\}$. Then F_i is a finite subset of $B_{s_i}(q_i)$. Since s_i decreases to t_ω and $sup(s_i) = q_{i+1}$, $\bigcup_i F_i$ has exactly one limit point, t_ω. Let $S = \bigcup_i F_i \cup \{t_\omega\}$. Then $S \in B_{t_i}(q_i)$, so there is a $T_n(t_n, q_n)$ containing it. Also, S is incompatible with every element in A_n since it includes F_n as its discrete points, so $A_n \cup \{S\}$ is an antichain in $T_n(t_n, q_n)$ of size $\phi_n(t_n) + 1$. This contradicts our construction of ϕ_n and choice of t_n and q_n.

As a result, $T(\pi\mathbb{Q})$ is not σ-bounded cc.

□

Theorem 2.7 (Todorcevic[28])**.** *$T(\pi\mathbb{Q})$ is σ-finite cc.*

Proof. Let $\leq_w$ be a well-ordering of $\pi\mathbb{Q}$. Enumerate $\mathbb{Q} = \{q_k\}_{k<\omega}$. For each $t \in \pi\mathbb{Q}$, Then let $\leq_l$ be the lexicographical order on $\pi\mathbb{Q}$ induced by this enumeration of $\mathbb{Q}$. For each $t \in \pi\mathbb{Q}$, define $v(t) = \min_{\leq_w}\{v : t \sqsubset v\}$.

Let $\pi_k\mathbb{Q} = \{t \in \pi\mathbb{Q} : \min(v(t) \setminus t) = q_k\}$. Then for every t there is a k so that $t \in \pi_k\mathbb{Q}$.

For each $p \in T(\pi\mathbb{Q})$, let $b(p)$ be the smallest integer b such that $p^{(1)} \subseteq \bigcup_{i<b} \pi_i\mathbb{Q}$, and $l(p) = |p^{(1)}|$. List $p^{(1)} = \{x^p_1 \leq_l x^p_2 \leq_l ... \leq_l x_{l(p)^p}\}$. For each p and every $0 < i \leq l(p)$, fix a rational $q^p_i > \sup(x^p_i)$ such that all $B_{q^p_i}(x^p_i)$ are mutually disjoint for $0 < i \leq l(p)$. Then since $D(p) = p \setminus \bigcup_{0<i\leq l(p)} B_{q^p_i}(x^p_i)$ has no limit points and is compact, it is finite. Denote its cardinality by $d(p)$.

Without loss of generality, assume that for $0 < i, j \leq l(p)$ such that $\sup(x^p_i) < \sup(x^p_j)$ implies $q^p_i < \sup(x^p_j)$, and that $\sup(x^p_i) = \sup(x^p_j)$ implies $q^p_i = q^p_j$.

Then for each integer b, d, l and finite sequence $S = \langle s_i : 0 < i \leq l\rangle \subset \mathbb{Q}$, let $P(b, d, l, S) = \{p \in T(\pi\mathbb{Q}) : b(p) = b, d(p) = d, l(p) = l$ and $q^p_i = s_i$ for all $0 < i \leq l\}$. We claim that each such $P(b, d, l, S)$ includes no infinite antichains.

Fix such b, d, l, S. Suppose $A = \{p_n\}_{n<\omega}$ is an infinite antichain in $P(b, d, l, S)$. Since $|D(p_n) \cup p^{(1)}| = d + l$, there is an infinite subset of ω so that $D(p_n) \cup p^{(1)}$ form a Δ-system on that infinite set. By using Ramsey's theorem, shrink A so that $\{D(p_n)\}_{n<\omega}$ is a Δ-system with root D, $\{p^{(1)}_n\}_{n<\omega}$ is a Δ-system with root K, and $\{D(p_n) \cup p^{(1)}_n\}_{n<\omega}$ is a Δ-system with root $D \cup K$. Then clearly $D \cap K = \emptyset$ and For every $n, m < \omega$, $D(p_n) \cap p^{(1)}_m = \emptyset$. So without loss of generality, we can assume that $d = 0$ as $D(p_n)$ does not contribute to the incompatibility in A.

Now for each $n < \omega$ and $0 < i \leq l$, let $p_n(i) = p_n \cap B_{s_i}(x^{p_n}_i)$. Using Ramsey's theorem, refine A again so that the for each $0 < i \leq l$, the sequence $x^{p_n}_i{}_{n<\omega}$ is either strictly ordered as $\sqsubset$-chains or is a $\sqsubseteq$-antichains. Since $\sqsubseteq$ is well-founded on $\pi_k\mathbb{Q}$, any such $\sqsubset$-chain is increasing in n.

Use Ramsey's theorem again, we can assume that either:

1. There are $0 < i, j \leq l$ so that $x^{p_m}_i \in p_n(j)$ for all $m < n < \omega$, or

2. There are $0 < i, j \leq l$ so that $x^{p_n}_i \in p_m(j)$ for all $m < n < \omega$.

If case 1 happens, $x^{p_n}_j \sqsubset x^{p_m}_i$ for $n < m$ since $p_n(j)$ is a subset of a basic neighbourhood of $x^{p_n}_j$. Then for arbitrary $n_0 > n_1 > n_2$, we have $x^{p_{n_0}}_j \sqsubset x^{p_{n_2}}_i$ and $x^{p_{n_1}}_j \sqsubset x^{p_{n_2}}_i$ and this implies that $x^{p_{n_0}}_j$ and $x^{p_{n_1}}_j$ are $\sqsubseteq$-comparable. In particular, $x^{p_n}_j$ is strictly $\sqsubset$-increasing in n. Therefore, for each m, $x^{p_m}_j \sqsubset x^{p_{m+1}}_j \sqsubset x^{p_m}_i$, thus $\sup(x^{p_m}_j) < \sup(x^{p_m}_i)$. By our choice of S, we have $s_j < \sup(x^{p_m}_i)$ for every m. This rules out the possibility that $x^{p_m}_i \in B_{s_j}(x^{p_n}_j)$, contraicting 1 since $p_n(j) \subset B_{s_j}(x^{p_n}_j)$.

In case 2, by the similar argument as above, $x^{p_n}_j$ is strictly $\sqsubset$-increasing in n. Let $q = min(x^{p_1}_j \setminus x^{p_0}_j$. Since $x^{p_n}_j$ is strictly $\sqsubset$-increasing in n, we have that $min(x^{p_n}_j \setminus x^{p_0}_j) = q$ for all $n \geq 1$. Therefore $\sup(x^{p_m}_j) > q$ for all $m \geq 1$ while $\sup(x^{p_0}_j) < q$, thus $x^{p_0}_j$ is not a limit point of $\{x^{p_m}_j\}_{m\geq 1}$. On the other hand, $\{x^{p_m}_i\}_{m\geq 1}$ form an infinite subset of $p_0(j)$ and therefore needs to converge to $x^{p_0}_j$, a contradiction.

As neither of the cases works, there cannot be infinite antichains in each $P(b, d, l, S)$, thus this partition witnesses the σ-finite chain condition of $T(\pi\mathbb{Q})$.

□

Note that the set $T(\pi\mathbb{Q})$ can be defined as a Borel set (in some Polish space). So it would be interesting if the fragmentation witnessing the σ-finite cc can be also chosen to be Borel sets. This turns out to be impossible and it will be further analyzed in section 4.2.

2.3 Fragmentation Properties and Measures

The fragmentation properties were first raised in [7] to categorize the Boolean algebras, in a hope to find out an algebraic (combinatoric) characterization of the Boolean algebras on which a measure (or a submeasure) can be defined.

Also, note that the chain conditions are preserved on dense subsets. So the examples in section 2.1 induces different Boolean algebras satisfying different chain conditions.

Definition 2.5. *Let κ be a cardinal. A Boolean algebra B is said to be κ-distributive if for every double sequence $\{a_{\mu,\nu}\}_{\mu,\nu<\kappa}$,*

$$\wedge_\mu \vee_\nu a_{\mu,\nu} = \vee_{f\in\kappa^\kappa} \wedge_\mu a_{\mu,f(\mu)}.$$

B is said to be σ-distributive if it is ω-distributive. B is said to be distributive if it is κ-distributive for every κ.

Definition 2.6. *Let κ be a cardinal. A Boolean algebra B is said to be κ-weakly distributive if for every double sequence $\{a_{\mu,\nu}\}_{\mu,\nu<\kappa}$,*

$$\wedge_\mu \vee_\nu a_{\mu,\nu} = \vee_{F\in([\kappa]^{<\omega})^\kappa} \wedge_\mu a_{\mu,F(\mu)}$$

where every $a_{\mu,F(\mu)} = \vee_{\nu\in F(\mu)} a_{\mu,\nu}$. B is said to be σ-weakly distributive if it is ω-weakly distributive.

Definition 2.7. *Let B be a Boolean algebra. A function $f : B \to \mathbb{R}$ is said to be:*

1. *strictly positive if $f(a) > 0$ for all $a \neq 0$.*
2. *exhaustive if for every countable antichain $\{a_n\}_{n<\omega} \subset B$, $lim_n f(a_n) = 0$.*
3. *uniform exhaustive if for every $\epsilon > 0$ there is a $k \in \omega$ such that for every antichain $\{a_n\}_{n<\omega} \subset B$, $|\{n : |f(a_n)| \geq \epsilon\} \leq k$.*
4. *monotone if $f(x) \leq f(y)$ for $x \leq y$.*
5. *subadditive if $a \perp b$ implies $f(a \vee b) \leq f(a) + f(b)$.*
6. *additive if $a \perp b$ implies $f(a \vee b) = f(a) + f(b)$.*
7. *submeasure if f is monotone, subadditive and f(0)=0.*
8. *measure if it is submeasure and additive.*
9. *σ-additive if for every sequence of pair-wise disjoint $\langle a_n\rangle_{n<\omega}$, $f(\vee_n a_n) = \sigma_{n<\omega} f(a_n)$.*
10. *σ-measure if it is a measure and σ-additive.*

Problem 2 (Von Neumann [31])**.** *Are the following conditions on a complete Boolean algebra B sufficient for the existence of a strictly positive countably additive measure on B:*

1. *B satisfies the countable chain condition.*
2. *B is weakly distributive.*

Definition 2.8. *Let B be a Boolean algebra and non-empty $A \subset B$. For each finite $F \subset B$ let $i(F)$ be the maximal size of centred subset of F. The intersection number $I(A) = inf_{F\in[A]^{<\omega}}\{i(F)/|F|\}$ (F ranges over all non-empty finite subsets of A).*

By some basic combinatorics it can be seen that once the Boolean algebra permits a strictly positive measure, it can be decomposed into countably many pieces with positive intersection numbers. In [13] Kelley showed the reverse is also true:

Theorem 2.8 (Kelley[13]). *Let B be a Boolean algebra. The following are equivalent:*

1. *There is a strictly positive measure $m : B \to \mathbb{R}^+$.*
2. *$B = \bigcup_n B_n$ where each B_n satisfying $I(B_n) > 0$.*

Additionally, the measure can be taken to be a σ-additive measure if and only if B is σ-weakly distributive.

Maharam's version of Von Neumann's problem replaces the existence of a countably additive strictly positive measure with the formally weaker condition of the existence of strictly positive exhaustive submeasure (see [18]). More precisely, she posed the following version of Von Neumann's problem:

Problem 3 (Maharam [18]). *Are the following conditions on a complete Boolean algebra B sufficient for the existence of a strictly positive exhaustive submeasure on B:*

1. *B satisfies the countable chain condition.*
2. *B is weakly distributive.*

In the same 1947 paper [18] she noticed that Souslin algebra is a weakly distributive ccc algebra supporting no strictly positive exhaustive submeasure and therefore that a negative answer to Souslin's problem also gives negative answer to Von Neumann's problem as well as her version of the problem. Soon after the invention of Forcing the independence of Souslin's Hypothesis from ZFC was established in the work of Jech [8], Tennenbaum [23] and Solovay-Tennenbaum [20]. In particular, it has been shown that Martin's axiom $MA_{\aleph_1}$ implies Souslin's Hypothesis. It turns out that $MA_{\aleph_1}$ is not sufficient for the positive answer to Maharam's version of Von Neumann's problem and that it has taken more than 30 years of research before the consistency of this problem was fully established. To state this result, recall the P-ideal dichotomy, PID, introduced in [26] that is a consequence of the Proper Forcing Axiom as well as consistent with GCH:

PID: *For every P-ideal*[1] *I of countable subsets of some set S, either S can be decomposed into countably many subsets S_n which have finite intersections with every element of I, or S contains an uncountable subset T all of whose countable sets are in I.*

The following result shows that PID eliminates the Souslin algebra as a potential counterexample to Maharam's problem and its proof points out that it might be relevant to the positive answer to its full positive answer

Theorem 2.9 (Abraham, Todorcevic [1]). *PID implies the Souslin Hypothesis.*

Proof. Recall that (see [15]), a positive answer to Soulsin's problem is equivalent to the non-existence of a Souslin tree T, an uncountable tree with no uncountable chains nor antichains. Taking $ro(T)$ for T a Souslin tree, we get a Souslin algebra, a σ-distributive complete ccc algebra. Conversely, every nonatomic ccc σ-distributive algebra contains a Souslin subtree, so the existence of a Souslin tree is equivalent to the existence of a Souslin algebra.

[1] An ideal I of countable subsets of some set S is a P-ideal if for every sequence (a_n) of elements of I there is $b \in I$ such that $a_n \setminus b$ is finite for all n.

Let T be a given tree of height ω_1 with countable levels. Let I be the ideal of countable subsets a of T such that every infinite subset of a is unbounded in T. It is easily checked that I is a P-ideal, so assuming PID, we may consider the following two cases:

Case 1. There is an uncountable subset X of T such that $[X]^{\aleph_0} \subseteq I$. It follows that every infinite subset of X is unbounded in T. In particular, X contains an uncountable antichain, and therefore the tree T is not Souslin.

Case 2. There is a decomposition $t = \bigcup_{i<\omega} T_n$ such that no T_n contains an infinite subset in I and so in particular no T_n could contain an infinite antichain. Note that a subset of a tree that contains no infinite antichain must be the union of finitely many chains. So case 2 gives us that our tree T is covered by countably many chains, so in particular, it can't be Souslin. □

Theorem 2.10 (Balcar, Jech and Pazák[2]). *Assume PID. Let B be a complete Boolean algebra. The following are equivalent:*

1. *B is ccc and σ-weakly distributive.*
2. *There is a strictly positive exhaustive submeasure on B.*

To obtain a ZFC characterization, one needs to strengthen the countable chain condition:

Theorem 2.11 (Todorcevic[28]). *Let B be a complete Boolean algebra. The following are equivalent:*

1. *B satisfies σ-finite chain condition and is σ-weakly distributive.*
2. *There is a strictly positive exhaustive submeasure on B.*

2.4 Iterated Forcing

The preservation of the properties of a notion of forcing is a central problem in the context of forcing. By separately considering the iteration of the Boolean algebras allowing a countable fragmentation with pieces having positive intersection numbers and the iteration of Boolean algebras of σ-weakly distributivity, one can reach the following conclusion as a corollary of theorem.2.8:

Theorem 2.12 (Kamburelis[10]). *Let B be a Boolean algebra and $\dot{D}$ be a B-name of a Boolean algebra. Then:*

1. *If B permits a measure and $B \vdash$ "$\dot{D}$ permits a measure", then the two-step iteration $B * \dot{D}$ also permits a measure.*
2. *If B permits a σ-additive measure and $B \vdash$ "$\dot{D}$ permits a σ-additive measure", then the two-step iteration $B * \dot{D}$ also permits a σ additive measure.*

Clearly, the theorem above can be extended to the finite support iterations. The same method also works when $\dot{D}$ is a name of σ-bounded cc Boolean algebra:

Theorem 2.13. *Let B be a Boolean algebra with a measure and $\dot{D}$ be a B-name of a poset so that $B \vdash$ "$\dot{D}$ satisfies σ-bounded cc". Then $B * \dot{D}$ also satisfies σ-bounded cc.*

Proof. Let $f : B \to \omega$ be a partition of B into countably many pieces, each having a positive intersection number $\epsilon_n = I(f^{-1}(n)) > 0$. Since $B \vdash$"$\dot{D}$ is σ-bounded cc", we can find a name $\dot{g} : \dot{D} \to \check{\omega}$ so that $B \vdash$"$\dot{g}^{-1}(n)$ includes no antichain of size $\geq n$". Let $P \subset B * \dot{D}$ be a subset of pairs $(b, \dot{d})$ so that b decides $\dot{g}(\dot{d})$. Clearly P is dense in $B * \dot{D}$. For each pair of integers n, m, let $P_{n,m} = \{(b, d) : f(b) = n, b \vdash$"$\dot{g}(\dot{d}) = m$"$\}$. Then for every n, m and finite subset $A \subset P_{n,m}$ of size $> m/\epsilon_n$. By our choice of f, there is a finite subset $A' \subset A$ with first coordinate centred. Let $b \in B$ extending the first coordinates in A'. Let $\dot{F}$ denote the name of the finite set $\{\dot{d}$:there is a b so that $(b, \dot{d} \in A')\}$. Then either $b \vdash$"$|\dot{F}| < n$" or $b \vdash$"$\dot{F}$ is not an antichain". In the first case, there are $\dot{d}_0 \neq \dot{d}_1 \in \dot{F}$ so that $b \vdash$"$\dot{d}_0 = \dot{d}_1$". In the second case there are $\dot{d}_0, \dot{d}_1 \in \dot{F}$ so that $b \vdash$"$\dot{d}_0$ and $\dot{d}_1$ are compatible", so there is a $\dot{d}$ such that $(b, \dot{d}) \leq (b, \dot{d}_0), (b, \dot{d}_1)$. In either case, A' is not an antichain, thus A is not an antichain. Therefore $P_{n,m}$ includes no antichain of size $> m/\epsilon_n$.

□

Chapter 3

Borel Hypergraphs

3.1 Borel Hypergraph

Definition 3.1. *Given an integer k and a set X, the pair $G = (X, R)$ is called a k-dimensional hypergraph over X when $R \subset [X]^k$, a collection of subsets of X of size k. Moreover, it is a Borel hypergraph if X is Polish and R is Borel as in the product topology.*

A graph is a 2-dimensional hypergraph.

Definition 3.2. *A k-dimensional hypergraph $G = (X, R)$ is called:*

1. *the k-dimensional discrete hypergraphs when R is empty.*
2. *the k-dimensional complete hypergraphs of size κ when $R = [X]^k$ and $|X| = \kappa$. In this case the hypergraph is written as $G = K^k_\kappa$.*

A simple but important observation is that if a k-dimensional hypergraph is of size $< k$, then it is automatically discrete.

For each finite hypergraph H, let T_H be the tree $|H|^{<\omega}$. In the following content, we denote by D_H a subset of T_H such that D_H intersects each level of T_H at exactly one point.

Definition 3.3. *Let $H = (|H|, R_H)$ be a finite non-discrete k-dimensional hypergraph. $G_0(H, D_H) = (|H|^\omega, R)$ is a k-dimensional hypergraph defined over the branches of the tree $|H|^{<\omega}$. R is obtained as the following:*

$\{x_1, ...x_k\} \in R$ if and only if there is a $d \in D_H$,$\{h_1, ...h_k\} \in R_H$ and a $z \in |H|^\omega$ so that $x_i = d \frown h_i \frown z$ for all $1 \leq i \leq k$.

Definition 3.4. *Given two k-dimensional hypergraphs $G = (X, R_1)$ and $H = (Y, R_2)$, a map $f : G \to H$ is called a homomorphism if for every $\{x_i\}_{i<k} \in R_1$, $\{f(x_i)\}_{i<k} \in R_2$. Similar to the graph case, we write $G \leq H$ if there is a homomorphism from G to H. When the homomorphism can be chosen to be continuous, we write $G \leq_c H$, and when the homomorphism can be chosen to be Borel, we write $G \leq_B H$.*

3.2 Borel generated σ-ideal of sets forbidding a finite hypergraph

Definition 3.5. *Given a finite hypergraph H. Let $G = (X, E)$ be a Borel hypergraph. The ideal of Borel sets σ-forbidding H over G is the collection*

$$I_B^H(G) = \{Y \subset X : Y = \bigcup_{i<\omega} Y_i \text{ where each } Y_i \text{ is Borel and } H \nleq G \restriction Y_i\}.$$

Definition 3.6. *For an ideal I of Borel subsets of X, a subset $Y \subset X$ is said to be I-small if there is a $Y' \in I$ such that $Y \subset Y'$, and is said to be I-large if it is not I-small.*

Lemma 3.1. *Given a finite hypergraph $H = (V_H, R_H)$ and a Borel hypergraph $G = (X, E)$. An analytic subset $Y \subset X$ is $I_B^H(G)$-small if and only if there is are countably many analytic sets Y_i such that $Y = \bigcup_{i<\omega} Y_i$ and $H \nleq Y_i$.*

Proof. Let n be the size of H. Let $\sigma_H(x_1, ..., x_n, R)$ be a quantifier-free second order formula on free first order variables $x_1, ..., x_n$ and free second order variable R stating that $x_1, ..., x_n$ with relation R is a homomorphic copy of H.

Let $T = \{M \subset X : H \nleq M\}$, then $T = \{M : \forall x_1, ...x_n \in M(\neg\sigma(x_1, ..., x_n, E))\}$. Since E is Borel, T is $\mathbf{\Pi_1^1}$ on $\mathbf{\Sigma_1^1}$. Then by the first reflection theorem, every analytic $M \in T$ is included in a Borel $M' \in T$. Since each $Y_i \in T$, we can pick the corresponding Borel $Y_i' \in T$ including them. Thus $Y \subset \bigcup_i Y_i'$, thus $I_B^H(G)$-small. □

Definition 3.7. *A Borel hypergraph A is called Borel basic for a class of ideals $I(G)$ over Borel hypergraphs if for every Borel hypergraph G being $I(G)$ large, $A \leq_B G$ and A is $I(A)$ large.*

Fact 3.1. *For a Borel hypergraph $G = (X, E)$, if H is a discrete finite hypergraph, then $I_B^H(G)$ is empty.*

Fact 3.2. *For a Borel hypergraph G, if H is a disconnected finite hypergraph with $H = \bigcup_{i<k} H_i$ where each H_i is a connected component, then $I_B^H(G) = \bigcap_{i<k} I_B^{H_i}(G)$. Thus G is $I_B^H(G)$ large if and only if G is $I_B^{H_i}(G)$ large for some $i < k$.*

Fact 3.3. *For a finite hypergraph $H = (V_H, R_H)$, the hypergraph $G_0(H, D_H)$ is $I_B^H(G_0(H, D_H))$-large if and only if D_H is somewhere dense.*

Proof. Without loss of generality, we assume that D_H is dense since $G_0(H, D_H)$ is identical to any of its basic open set. If $|H|^\omega = \bigcup_n B_n$ where each B_n is Borel, then there is a $t \in D_H$ and an integer n such that B_n is comeager in the set $|H|^\omega | t = \{b \in |H|^\omega, t \sqsubseteq b\}$.

For each $i \in V_H$, let $C_i = \{z : z \frown \{i\} \frown z \in B_n\}$. Let $C = \bigcap_{i<|V_H|} C_i$. C is non-empty since each C_i is comeager. Pick a $z \in C$, $\{t \frown \{i\} \frown z\}_{i<|V_H|}$ is then an isomorphic copy of H in B_n. □

For each finite hypergraph H, fix a hypergraph on the (finite) cardinal number $|V_H|$ isomorphic to H. Without loss of generality, we still call this new hypergraph H.

Theorem 3.1. *If H is a finite non-discrete k-dimensional hypergraph and G is a Borel hypergraph, then either:*

1. *G is $I_B^H(G)$-small, or*

2. $G_0(H, D_H) \leq_B G$.

Moreover, if D_H is dense, then exactly one of the above holds, thus in this case, $G_0(H, D_H)$ are Borel basic in the class of ideals $I_B^H(G)$.

It's worth noting that the notion $I_B^H(G)$ is a natural extension of the traditional notion "having countable chromatic number".

Definition 3.8. *Given a hypergraph G, the Borel chromatic number $\chi_B(G)$ is the smallest cardinal number κ such that $G \leq_B K_\kappa$.*

Then it is routine to observe the following:

Fact 3.4. *Given a hypergraph G, $\chi_B(G) \leq \aleph_0$ if and only if G is $I_B^{K_k^k}(G)$-small.*

Then by taking $k = 2$, and H to be the 2-dimensional complete graph of size 2, i.e. K_2^2, we can reach one of the fundamental results by A. S. Kechris, S. Solecki and S. Todorcevic:

Corollary 3.1 (G_0 dichotomy[12])**.** *Given a Borel graph G, either one of the following holds:*

1. $\chi_B(G) \leq \aleph_0$

2. $G_0(K_2^2, D_{K_2^2}) \leq_B G$.

Moreover, if $D_{K_2^2}$ is dense, then exactly one of the above holds.

3.3 Proof of Theorem 3.1

Here we employ Bernshteyn's technique used to prove the G_0 dichotomy.

Proof. Fix a finite hypergraph $H = (V_H, R_H)$ and a Borel hypergraph $G = (X, R_G)$ as in the statement of the theorem. Let Y be a Polish space and $\psi : Y \to [X]^2$ be a continuous map with range R_G. Suppose G is $I_B^H(H)$-large.

Definition 3.9. *Let $H_0 = (V_0, R_0)$ and $H_1 = (V_1, R_1)$ be hypergraphs of the same dimension k. Let $u \in V_1$ be a vertex of H_1. The rooted product of H_0 and H_1 with the root u is a k-dimensional hypergraph $H_0 \circ_u H_1 = (V_0 \times V_1, R)$ with R defined as: $\{(v_i, u_i)\}_{i<k} \in R$ if and only if:*

1. *u_i=u for all $i < k$ and $\{v_i\}_{i<k} \in R_0$, or*

2. *$v_i = v_j$ for all pairs $i, j < k$ and $\{u_i\}_{i<k} \in R_1$.*

Given a finite hypergraph $H_0 = (V_0, R_0)$. In this section, a homomorphism from H_0 to G is a map $f : V_0 \cup R_0 \to X \cup Y$, such that $f(V_0) \subset X$, $f(R_0) \subset Y$, and for each $\{v_i\}_{i<k} \in R_0$, we have $\psi(f(\{v_i\}_{i<k})) = \{f(v_i)\}_{i<k}$,

Denote by $Hom(H_0, G)$ the set of all homomorphisms from H_0 to G, and equip it with the pointwise convergence topology. Clearly, $Hom(H_0, G)$ is Borel in $X^{V_0} \cup Y^{R_0}$ since R_G is Borel in X^k and ψ is continuous.

Given a subset $S \subset Hom(H_0, G)$ and $u \in H_0$, denote by $S(u) = \{f(u) : f \in S\}$ the range of u under S. Note that if S is Borel then $S(u)$ is analytic.

Definition 3.10. *Let I be an σ-ideal over X and $H_0 = (V_0, R_0)$ be a hypergraph. Let $S \subset Hom(H_0, G)$. S is said to be:*

1. *I-tiny if $S(u)$ is I-small for some $u \in V_0$.*

2. *I-small if it is in the σ-ideal generated by the I-tiny sets.*

3. *I-large if it is not I-small.*

Note that when I is a σ-ideal over G, then the I-small subsets of $Hom(H_0, G)$ also form a σ-ideal.

Definition 3.11. *Let $H_0 = (V_0, R_0)$ and $H_1 = (V_1, R_1)$ be finite hypergraphs. Let $S \subset Hom(H_1, G)$ and $u \in V_1$. The rooted product of H_0 and S rooted at u is a subset $H_0 \circ_u S$ of all homomorphisms $f in Hom(H_0 \circ_u H_1, G)$ such that for ever $v \in V_0$, the left restriction $f|_v(x) = f(v, x) \in S$.*

Lemma 3.2. *Let $H_0 = (V_0, R_0)$ be a finite hypergraph. If $S \subset Hom(H_0, G)$ is $I_B^H(G)$-large, then $H \circ_u S$ is $I_B^H(G)$-large for all $u \in V_0$.*

Proof. Suppose $H \circ_u S$ is $I_B^H(G)$-small for some $u \in V_0$. Then there is a $h \in V_H$ such that $(H \circ_u S)(h)$ is $I_B^H(G)$-small. Let $(H \circ_u S) = B_n$ where each B_n is I-tiny witness by a $v_n \in V_0$. For each n let $S_n = \{f \in S : f(v_n) \in B_n(v_n)\}$. Then S_n is $I_B^H(G)$-tiny so $\bigcup S_n$ is $I_B^H(G)$-small. Let $S' = S \setminus \bigcup S_n$. Since S is $I_B^H(G)$-large, so is S'. Thus $S'(u')$ is $I_B^H(G)$-large for every $u' \in V_0$, thus in particular $H \leq_B S(u)$, and therefore $H \circ_u S'$ is non-empty. However $(H \circ_u S') \cap (\bigcup S_n) = \emptyset$, which is a contradiction with the fact $H \circ_u S' \subset H \circ_u S$.

□

Since $I_B^H(G)$ is a σ-ideal, for every finite hypergraph $H_0 = (V_0, R_0)$, for every $\epsilon > 0$ and every large $S \subset Hom(H_0, G)$, there is a $I_B^H(G)$-large $S' \subset S$ so that all $S'(u)$ and $S'(r)$ have radius $< \epsilon$ for every $u \in V_0$ and $r \in R_0$.

Recall that $G_0(H, D_H) = (|H|^\omega, R)$ is the hypergraph defined on the branches of the tree $|H|^{<\omega}$. Let $F_n = (|H|^n, E_n)$ be the hypergraph defined on the n'th level of $|H|^\omega$ such that $\{t_1, ...t_k\} \in E_n$ if and only if there are branches $\{x_1, ...x_k\} \in R$ so that each x_i extends t_i for $1 \leq i \leq k$.

Note that $F_0 \cong \{p\}$, a singleton, and $F_{n+1} \cong H \circ_{d_n} F_n$. Let S_n be a sequence of $I_B^H(G)$-large sets of homomorphisms so that $S_{n+1} \subset H \circ_{d_n} S_n$ and for all $u \in |H|^{n+1}$ and $r \in E_{n+1}$, $S_{n+1}(u)$ and $S_{n+1}(r)$ have radius $< 1/(n+1)$. This sequence can be constructed due to the fact that G is $I_B^H(G)$-large so that $S_0 = Hom(\{p\}, G)$ is $I_B^H(G)$-large.

For each $b \in |H|^\omega$, let $f(b)$ be the only point in $\bigcap_n \overline{S_n(b|n)}$. For each $r = \{b_1, ...b_k\} \in R$, let $f(r)$ be the only point in $\bigcap_n \overline{S_n(\{b_1|n, ..., b_k|n\})}$. Clearly, $f(r) = \{f(b_1), ...f(b_k)\}$ thus f is a homomorphism from $G_0(H, D_H)$ to G. It is also clear that f is continuous.

□

3.4 Quasi-Order of Borel Graphs

In this section, we will by custom use term G_0 for $G_0(K_2^2, D_{K_2^2})$, for any $D_{K_2^2}$ dense. It should be noted that G_0 refers to a class of graphs (generated by different $D_{K_2^2}$'s), and the statements "...G_0..." should be read as "...every graph from G_0 for a dense $D_{K_2^2}$...".

Definition 3.12. *A quasi-ordering is a transitive reflexive relation. Let $(A, \leq)$ be a quasi-ordered set, a basis $B \subset A$ is a set such that for every $a \in A$ there is a $b \in B$ such that $b \leq a$. A point $x_0 \in A$ is basic if $\{x_0\}$ form a singleton basis.*

Every result in this chapter so far is formulated based on the homomorphisms in between graphs. In particular, the G_0-dichotomy can be formulated as the following: G_0 is $\leq_c$-basic in the class of Borel graphs with uncountable chromatic numbers. Several studies were done intending to generalize or find analogues of the G_0-dichotomy in this direction.

The most natural thought might be to extend the ideal to the $\mathbf{\Pi_1^1}$ graphs: can we find a basic graph in the class of $\mathbf{\Pi_1^1}$ graphs with an uncountable chromatic number?

In [12] the question is answered negatively as the following:

Theorem 3.2 ([12])**.** *If $\aleph_1^{\mathrm{L}[a]} = \aleph_1$ for some $a \subseteq \omega$ then there are two $\mathbf{\Pi_1^1}$ graphs H_1 and H_2 so that:*

1. *H_0 and H_1 are both uncountably chromatic, but*
2. *Every graph H such that $H \leq H_0$ and $H \leq H_1$ must be countably chromatic.*

This suggests the non-existence of any reasonable ZFC extension of G_0-dichotomy to $\mathbf{\Pi_1^1}$ graphs.

Every function $f : X \to X$ induces a natural graph on X in which two points are connected if and only if f sends one of them to the other. Let S be the map on $[\mathbb{N}]^{\mathbb{N}}$(the set of strictly increasing infinite sequences of integers) to itself by shifting every x to the left(i.e. $f(\langle x_0, ..., x_n, ...\rangle) = \langle x_1, ..., x_{n+1}, ...\rangle$). S is called the shift map and the induced graph G_S is called a shift mapping. G_S is a Borel graph and $\chi_B(G_S) = \aleph_0$. (Taking the first number in the sequence gives a countable clopen coloring. Galvin-Prikry theorem denies any finite colorings). In [12] a question asking whether G_S is $\leq_B$ basic in the class of Borel graphs with infinite Borel chromatic numbers was raised. A few counterexamples were found after (see, e.g. [5]), and later in [29] it was shown that the class of Borel graphs with infinite Borel chromatic numbers has no countable basis.

On the other hand, in [4], a $\leq_B$-basic graph for the class of graphs with Borel chromatic number ≥ 3 was constructed.

One may seek other directions of generalizations. The idea of extending the notion of chromatic number in section 3.2 is another one.

Also, there are a lot of works done to obtain results regarding the basis of different quasi-orders.

Definition 3.13. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs.*

1. *When there is an injective homomorphism from G to H, we write $G \preceq H$. When this injective homomorphism can be taken to be continuous or Borel, we write $G \preceq_c H$ and $G \preceq_B H$, respectively.*
2. *A reduction from G to H is a function $f : V_G \to V_H$ such that $(v, u) \in E_G$ if and only if $(f(v), f(u)) \in E_H$.*

3. *An embedding of G in H is an one-to-one reduction from G to H. When there is a embedding from G to H, we write $G \sqsubseteq H$. When the embedding can be taken to be Borel or continuous, we write $G \sqsubseteq_B H$ and $G \sqsubseteq_c H$, respectively.*

In [12], a basic theorem regarding $\preceq$ in the class of graphs that are being "almost acyclic" is provided, with the same basic point G_0:

Definition 3.14. *An analytic graph $G = (V_G, E_G)$ is almost acyclic if $E_G = \bigcup_{n<\omega} E_n$, with each E_n analytic, symmetric and the following holds: For every pair $(v_0, v_1) \in E_G$, every sequence $n_1, ...n_k$ of natural numbers and every sequence $x_1, y_1, x_2, y_2, ..., x_k, y_k$, if $(x, x_1) \in E_{n_1}$, $(x_1, x_2) \in E_{n_2}$,...,$(x_{k-1}, x_k) \in E_{n_k}$, $(y, y_1) \in E_{n_1}$, $(y_1, y_2) \in E_{n_2}$,..., $(y_{k-1}, y_k) \in E_{n_k}$, then $x_k \neq y_k$.*

It's worth noting that when G is actually acyclic, then taking every E_n to be E_G gives a witnessing of the definition.

Also, when G is locally countable and Borel, there is a sequence of functions F_n whose orbits generating G, let $Y_{i,n} = \{y \in F_n(x) : F_n^{-1}(y)$ has cardinality $i\}$ for each $n < \omega$ and $1 \leq i \leq \omega$. Enumerate each $F_n^{-1}(y)$ as $\{f_{n,i}^j(y) : j < i\}$, then well order all such $f_{n,i}^j$'s as a sequence of functions $\{f_k\}_{k<\omega}$. Then $E_k = \{(x, y) : f_k(x) = y\}$ become a decomposition of E_G witnessing the almost acyclicness of G.

With this notion, the following theorem was proved in [12] by explicitly making the homomorphism in the G_0-dichotomy to be injective by using the decomposition of E_G:

Theorem 3.3 ([12])**.** *: Let G be an analytic almost acyclic graph. Then exactly one of the following holds:*

1. *$\chi_B(G) \leq \aleph_0$;*

2. *$G_0 \preceq_c G$.*

In a Polish space X, a subset $A \subset X$ is said to be universally Baire if for every compact Hausdorff space Y and every continuous function $f : Y \to X$, $f^{-1}(A)$ has the property of Baire. A function $f : X \to Y$ between Polish spaces is said to be universally Baire measurable.

In [17], a result developing in the same stream considering the class of directed graphs that can be reduced to a locally countable graph by universally Baire measurable functions, and regarding the stronger $\sqsubseteq$ instead of $\preceq$ is proved:

Theorem 3.4 ([17])**.** *Let G be an analytic graph that admits a universally Baire measurable reduction to a locally countable analytic graph, and also admits a universally Baire measurable reduction to an acyclic analytic graph, then exactly one of the following holds:*

1. *$\chi_B(G) \leq \aleph_0$;*

2. *$G_0 \sqsubseteq_c G$.*

However, this fact heavily depends on the graphs to be "close enough" to a locally countable analytic graph. While it was conjectured in [12] that the theorem 3.3 can be extended to arbitrary analytic graphs, it was shown in [16] that this conjecture fails:

Theorem 3.5 ([16])**.** *There is a Borel graph G generated by countably many Borel functions such that $\chi_B(G) > \aleph_0$, but every locally countable Borel subgraph of G has countable Borel chromatic number.*

and thus,

Theorem 3.6 ([16]). *There does not exist any analytic graph H_0 such that for every Borel graph G exactly one of the following fails:*

1. $\chi_B(G) < \aleph_0$;
2. $H_0 \preceq G$.

One then turns to search for a "reasonably simple" $\sqsubseteq$-basis in the class of graphs with uncountable Borel chromatic numbers. In this direction, the following example is constructed in [17]:

For each pair $S \in P(\bigcup_{n<\omega} 2^n \times 2^n) \times P(\bigcup_{n<\omega} 2^n \times 2^n)$, written as $S = (S^0, S^1)$, we associate a directed graph $G(S) = (V, E)$ on $[\bigcup_{n<\omega} 2^n]$ by $(x, y) \in E$ if and only if there is a $z \in [\bigcup n < \omega 2^n]$, an $i \in \{0, 1\}$ and a pair $(s, t) \in S^i$ such that $x = s \frown \{i\} \frown z$ and $y = s \frown \{1 - i\} \frown z$.

A pair S is said to be dense if it is dense in the first coordinate, i.e., for every $r \in \bigcup_{n<\omega} 2^n$, there is a $(s, t) \in S^0$ such that $r \subset s, t$. Note that when S is dense, by the similar reasoning for G_0, $\chi_B(G(S)) > \aleph_0$.

Theorem 3.7 ([17]). *Let G be an analytic locally countable graph on a Polish space which admits a universally Baire measurable reduction to a locally countable analytic graph, then exactly one of the following holds:*

1. $\chi_B(G) \leq \aleph_0$;
2. *There is a dense pair S such that* $G(S) \sqsubseteq_c G$.

By taking different choices of S, it provides a basis of size continuum in the class of locally countable Borel graphs. This left the following question to end this section:

Question 3.1. *Is there any $\sqsubseteq$-basis of size smaller than continuum in the class of graphs with Borel chromatic number $> \aleph_0$?*

Chapter 4

Borel Posets

4.1 Borel Posets and Fragmentation Properties

Definition 4.1. *A poset $P = (X, \leq)$ is called a Borel poset if X is a Polish space and the relation $\leq$ is a Borel subset of X^2 in the product topology.*

Definition 4.2. *A Borel poset $P = (X, \leq)$ is said to satisfy :*

1. *The Borel σ-finite chain condition (Borel σ-fcc) if it is a union of countably many fcc Borel subsets.*
2. *The Borel σ-bounded chain condition (Borel σ-bcc) if it is a union of countably many bcc Borel subsets.*
3. *The Borel σ–n-chain condition (Borel σ-n-cc) if it is union of countably many n-cc Borel subsets.*
4. *The Borel σ-linked if it is Borel σ-2-linked.*
5. *The Borel σ-n-linked if it is a union of countably many n-linked Borel subsets.*
6. *The Borel σ-centred if it is a union of countable many Borel centred subsets.*

Definition 4.3. *Let P be a poset. Denote by $G^k_\perp(P) = (P, \perp_k)$ the k-dimensional hypergraph defined over P, and here $\perp_k$ is the collection of all non-centred subsets of size k in P.*

Definition 4.4. *A Borel poset is said to be good if $G_\perp(P)$ is a Borel graph.*

Clearly, when a poset P is a good Borel poset, then $G^k_\perp(P)$ are Borel hypergraphs for all k.

Lemma 4.1. *For a good Borel poset P, it is equivalent to require the fragmentation in definition 4.2(3) and (4) to be analytic, instead of Borel.*

Proof. This follows from Lemma 3.1. To see this, note that for a subset of P, being included in a Borel n-cc subset is equivalent to being $I_B^{K^2_n}(G^2_\perp(P))$-small and being included in a Borel n-linked subset is equivalent to being $I_B^{K^n_n}(G^n_\perp)$-small. □

With this lemma in mind, we can see that each step of construction in the proof of theorem.2.5 can be done in a definable way if the poset is good Borel. Namely, we have the following theorem which is simply the Borel version of the result of Galvin and Hajnal mentioned above (see Theorem 2.5).

Theorem 4.1. *Let $P = (X, \leq)$ be a good Borel poset. If P satisfies the Borel σ-n-chain condition for some integer $n \geq 2$, then P is Borel σ-linked.*

Proof. Let $n > 2$. We show that if a poset P satisfies the Borel σ-n-chain condition then it satisfies the Borel σ-$(n-1)$-chain condition. Since Borel σ-2-chain condition is equivalent to being Borel σ-link, the proof will be finished.

Suppose that P satisfies the Borel σ-n-chain condition but fails to satisfy the Borel σ-$(n-1)$-chain condition. Let $P = \bigcup_{i<\omega} P_i$ where each P_i is analytic and satisfies the n-chain-condition. Fix k such that P_k cannot be decomposed into countably many analytic sets of $(n-1)$-chain-condition. We work towards the contradiction.

For each $p \in P_k$, let $L(p) = \{q \in P_K$ so that p and q are incompatible$\}$ and $R_i(p) = \{q \in P_k$: there is a $r \in P_i$ extending both p and $q\}$. Then for every p, $L(p) \cup (\bigcup_i R_i(p) = P_k$. Moreover, $L(p)$ and $R_i(p)$ are analytic thus for each p there are integers $I(p)$ so that $R_{I(p)}(p)$ contains an antichain of size $n-1$. For each i, let $Q_i = \{p : R_i(p)$ contains an antichain of size $n-1\}$. Note that Q_i is actually the sets of all $p \in P_i$ such that $\exists q_1, ..., q_{n-1} \in P_k \exists r_1, ..., r_{n-1} \in P_i (r_1 < q_1 \wedge ... \wedge r_{n-1} < q_{n-1} \wedge r_1 < p \wedge ... \wedge r_{n-1} < p \wedge (\wedge_{i \neq j} q_i \perp q_j))$, thus analytic. Clearly, $\bigcup_i Q_i = P_k$, thus there is an l so that Q_l contains an antichain $p_1, ..., p_{n-1}$of size $n-1$, Then for each $i = 1, 2, ...n-1$ we can find antichain $q_{i1}, ..., q_{i(n-1)}$ in R_l of size $n-1$. For each $i = 1, ..., n-1$ and $j = 1, 2, ...n-1$, we fix r_{ij} in P_l extending both p_i and q_{ij}. Then $\{r_{ij} : i, j = 1, 2, ..., n-1\}$ is an antichain and it is a subset of P_l. Since $(n-1)^2 \geq n$, we have a contradiction.

□

4.2 Borel fragmentation of $T(\pi\mathbb{Q})$

Recall the Todorcevic ordering $T(\pi\mathbb{Q})$ from Section 2.2. In [28], Todorcevic showed that $T(\pi\mathbb{Q})$ is a Borel poset satisfying the σ-finite chain condition. In this section, we show that $T(\pi\mathbb{Q})$ dos not satisfy the Borel σ-finite chain condition. This follows from the following general fact:

Theorem 4.2 (Todorcevic and Xiao[30])**.** *Suppose for a Borel definable topological space X there is a collection of analytic subsets $C = \{X_t : t \in 2^{<\omega}\}$ such that:*

1. *If $t \sqsubseteq s$ then $X_s \subset X_t$.*
2. *For each $b \in 2^\omega$, $\cap_{n<\omega} X_{b|n}$ is a singleton. For each branch b, call the only element in this singleton x_b.*
3. *For any sequence $\{b_k \in 2^\omega\}_{k<\omega}$ and $b \in 2^\omega$ such that $\lim_{k\to\infty} |b \vee b_k| = \omega$ and $b(|b \vee b_k| + 1) = 0$ for all k, then $x_{b_k} \to x_b$ in X for any $x_{b_k} \in \cap_{n<\omega} X_{b_k|n}$ and $x_b \in \cap_{n<\omega} X_{b|n}$. (Here $b_0 \vee b_1$ denote the maximum node contained in both b_0 and b_1 for two branches b_0 and b_1 of the tree $2^{<\omega}$).*

Then $T(X)$ is not Borel σ-finite cc.

To see that $T(\pi\mathbb{Q})$ satisfies this condition, for each $r \in 2^{<\omega}$, let $t_r = \{\Sigma_{i<j, r(i)=1} 2^{-i} : j \leq |r|\}$, $q_r = max(t_r) + 2^{-|r|}$ and $X_r = B_{t_r}(q_r)$. Clearly $\{X_r\}$ is a collection satisfying the conditions in theorem 4.2 and the conclusion follows.

For each real α, denote by A_α the Borel set coded by it.

Proof. Suppose X satisfies the conditions as required in the theorem and $T(X) = \bigcup T_k$ where each T_k is Borel. Let $a_k \in \omega^\omega$ be the code of T_k and a be the code of $T(X)$.

Let $P = \{\langle r_0, r_1, ..., r_n\rangle$: for all $0 < k \leq n, r_k \in 2^{<\omega}$, there is an strictly increasing sequence $\{h_k\}_{0<k\leq n}$ of natural numbers so that $|r_n| > h_n, r_0(h_k) = 0$ for all $0 < k \leq n$ and $r_k \sqsupseteq r_0|_{(h_k-1)} \frown 1\}$ and order it by $p_0 < p_1$ if p_0 coordinate-wisely extends a $p' \in P$ that end extends p_1 as a sequence.

Regard C as a subset of $X \times 2^{<\omega}$, so that $x \in X_t$ if and only if $(x, t) \in C$ for every $x \in X$ and $t \in 2^{<\omega}$. Note that the class C is countable, so we can assume that C is Σ^1_1.

Consider a large enough $H(\theta)$ containing C, X, $T(X)$, a, all a_k, T_k and P. Let M be a countable elementary substructure of $H(\theta)$ containing all these objects.

For $x \in X$ and $b \in 2^\omega$, consider the formula with two variables $\sigma(x, b) \leftrightarrow (\forall n \in \omega)((x, b|n) \in C)$ and the formula $\tau(b) \leftrightarrow \exists x(\sigma(x, b))$. Then $\sigma(x, b)$ is a Σ^1_1 statement since C is Σ^1_1, and therefore $\tau(b)$ is a Σ^1_1 statement. By Shoenfield's absoluteness theorem $\tau(b)$ is preserved by forcing extensions.

Let $\dot{G}$ be the canonical name for P-generic ultrafilters, denote by $\langle \dot{b_k}\rangle$ the name for the sequence of branches in 2^ω induced by $\dot{G}$. By condition (3) in the theorem, we know that for every P-generic ultrafilter G, $\bigcap_{n<\omega} X_{\dot{b}^G|n}$ is non-empty, i.e. $M[G] \models \tau(\dot{b}^G)$. Pick the unique $x \in \bigcap_{n<\omega} X_{\dot{b}^G|n}$, this x must also be in $M[G]$ and $M[G] \models \sigma(x, \dot{b}^G)$. Denote by $\dot{x_k}$ the P-name for this x induced by $\dot{b_k}$. Since our choice of G is arbitrary P-generic, $P \vdash$ "For each $k < \omega(\sigma(\dot{x_k}, \dot{b_k}))$". Then since for every Borel code α the relativization A^N_α for any substructure N of $H(\theta)$ is a subset of $A_\alpha \cup N$, every P-generic ultrafilter over M induces a sequence $\{x^G_k\}_{1\leq k<\omega}$ converges to x^G_0.

Let $\dot{S} = \{\dot{x_k}\}_{k<\omega}$. Since for every P-generic G, $H(\theta) \models \dot{S}^G \in T(X) = A_a$, thus $P \vdash \dot{S} \in (A_{\hat{a}})$. Then since the statement $A_a = \bigcup_k A_{a_k}$ is absolute, there must be a $p_0 \in P$ and $l \in \mathbb{N}$ such that $p_0 \vdash \dot{S} \in A_{\hat{a_l}}$.

Next, we build a sequence of generic filters $\{G_k\}_{k>|p_0|}$ containing p_0 so $\dot{S}^{G_k}$ are mutually incompatible, and thus produces an infinite antichain inside of $A_{a_l} = T_l$. Actually, if we write $\dot{b_k}^{G_m}$ as b_{mk}, the sequence G_k will satisfy that $b_{m0} = b_{0m}$ and $b_{mk} = b_{0k}$ whenever $0 < k < m$.

For $p = \langle r_0, ..., r_n\rangle \in P$ and an integer $m \leq n$, let $\pi_m(p) = \langle r_0, ..., r_m\rangle$ and $\pi'_m(p) = \langle r_m, r_1, ..., r_{m-1}\rangle$. Note that both π_m and π'_m are well-defined on an open dense subset of P.

To construct such G_m for $m > |p_0|$, enumerate the dense subsets of P in M by $\{E_i\}_{i<\omega}$. Fix $i < \omega$. Starting from p_0. Extend it into $q'_0 \in G_0$ and find some $q'_0 < p_0$ so that $\pi'_m(q'_0) = \pi_{m-1}(q_0)$. This can be done since $|p_0| < m$.

Suppose $q_n \in P$ and $q'_n \in G_0$ so that π'_m is defined on both of them and $\pi'_m(q'_n) = \pi_{m-1}(q_n)$. Let $D_n = \{r \in E_n : r < q_n\}$. Let $F_n = \pi_{m-1}(D_n)$ and $D'_n = \pi'^{-1}_m(F_n)$. Clearly D'_n is dense below q'_n. Since $q'_n \in G_0$, there is a $q'_{n+1} \in D'_n \cap G_0$ and $q_{n+1} \in D_n$ such that $\pi_m(q_{n+1}) = \pi'_m(q'_{n+1})$.

So inductively, we have constructed a decreasing sequence q_n in P intersecting every dense set in M. Let G_m be the ultrafilter including this sequence. G_m is thus P-generic over M and contains p_0. By our construction $\dot{x_m}^{G_0} = \dot{x_0}^{G_m}$ and $\{\dot{S}^{G_m}\}_{|p_0|<m<\omega} \subset A_{a_l} = T_l$ form an infinite antichain.

Since our choice of $\{T_k\}_{k<\omega}$ is arbitrary Borel, T cannot be Borel σ-finite cc.

□

Corollary 4.1. *The Borel poset $T(\pi\mathbb{Q})$ has the following properties;*

1. *$T(\pi\mathbb{Q})$ satisfies the σ-finite chain condition,*

2. *$T(\pi\mathbb{Q})$ does not satisfy the Borel σ-finite chain condition.*

3. *$T(\pi\mathbb{Q})$ does not satisfy the σ-bounded chain condition.*

Question 4.1 (Borel Horn-Tarski Problem). *Does there exist a Borel poset P satisfying the Borel σ-finite chain condition but not the Borel σ-bounded chain condition?*

4.3 Borel Posets Generated by Borel Hypergraphs

Definition 4.5. *Let $G = (X, R)$ be a hypergraph. The poset $\mathbb{D}(G)$ is the collection of all finite discrete subsets of G ordered by reverse inclusion.*

Fact 4.1. *$\mathbb{D}(G)$ is a good Borel poset if G is Borel.*

Fact 4.2. *Every k-dimensional hypergraph G is isomorphic to a subset of $G^k_\perp(\mathbb{D}(G))$*

Theorem 4.3. *For every $n > 2$, the poset $\mathbb{D}(G_0(K^n_n, D_{K^n_n}))$ is Borel σ-$n-1$-linked. However they are not Borel σ-n-linked if $D_{K^n_n}$ are dense.*

Proof. It is clear from the fact 3.3 that it is not Borel σ-n-linked.

Let $H = K^n_n$ be the n-dimensional complete hypergraph of size H and $G = G_0(H, D_H) = (n^\omega, R)$.

Fix n and $P = \mathbb{D}(G)$. For each integer i, let U_i be the collection of size i subsets $\{t_1, ...t_i\} \subset n^{<\omega}$ such that for any $b_1 \sqsupset t_1, ...b_i \sqsupset t_i$, $\{b_1, ...b_i\} \notin R$.

For each tuple $\tau = \{t_1, ...t_i\} \in U_i$, let Q_τ be the collection of all size i subsets $\{b_1, ...b_i\} \subset n^\omega$ such that $b_1 \sqsupset t_1, ..., b_i \sqsupset t_i$. Then for any size $n-1$ subset A of Q_τ, there are at most $n-1$ branches in $\bigcup A$ extending each t_i, thus for each t_i, the subset of $\bigcup A$ extending t_i is R-discrete. Thus by our choice of τ, $\bigcup A$ is R-discrete. In another word, A is centred.

As there are only countably many choices of such τ, there are only countably many Q_τ. Also, note that each discrete subset of G is contained in some Q_τ and Q_τ are Borel (indeed, open). Thus $P = \bigcup_i \bigcup_{\tau \in U_i} Q_\tau$ is a countable partition witnessing the Borel σ-$(n-1)$-linkedness.

□

Theorem 4.4. *For every integer $n \geq 2$, $\mathbb{D}(G_0(K^2_n, D_{K^2_n})$ is Borel σ-bounded cc.*

Note also that by fact 3.3, when $D_{K^2_n}$ is somewhere dense then this poset can not Borel σ-linked.

Proof. As in the proof of last example. Let $H = K^2_n$ and $G = G_0(H, D_H) = (n^\omega, R)$. Fix n and $P = \mathbb{D}(G)$. For each integer i let U_i be the collection of size i subsets $\{t_0, t_1, ..., t_i\} \subset n^{<\omega}$ such that for any $b_1 \sqsupset t_0, b_1 \sqsupset t_1, ..., b_i \sqsupset t_i$, $\{b_m, b_n\} \notin R$ for $1 \leq m, n \leq i$.

For each tuple $\tau = \{t_1, ...t_i\} \in U_i$, let Q_τ be the collection of all size i subsets $\{b_1, ...b_i\} \subset n^\omega$ such that $b_1 \sqsupset t_1, ..., b_i \sqsupset t_i$. Then for every pair of incompatible $p_1 \perp p_2 \in Q_\tau$, there are l, $b_1 \in p_1$ and $b_2 \in p_2$ so that $b_1, b_2 \sqsupset t_l$ and $\{b_1, b_2\} \in R$. Then for any large enough (precisely, greater than the 2-dimensional Ramsey number for a size n monochromatic subset with i colors) finite antichain $A \subset Q_\tau$, there is subset $A' = \{a_1, ..., a_{n+1}\} \subset A$ of size $n+1$ so that there is an l and for every pair a_j, a_m, the only $b_j \in a_j$ and $b_m \in a_m$ end extending t_l are connected in G. For each a_j fix this b_j. Then $B' = \{b_1, ..., b_{n+1}\}$ form a complete graph of size $n+1$ in G, which should not exist. Thus Q_τ is bounded cc.

Since every finite G-discrete subset is contained in some Q_τ and there are only countably many such τ, and clearly each Q_τ is Borel, P is Borel σ-bounded cc.

□

Besides $T(\pi\mathbb{Q})$ mentioned above, we here provide another *ccc* Borel poset that is not Borel σ-finite cc:

Theorem 4.5. *$\mathbb{D}(G_0(K^2_\omega, D_{K^2_\omega}))$ is not Borel σ-finite cc if $D_{k^2_\omega}$ is somewhere dense.*

Proof. Similarly to other examples, we show this by showing that when partition $G_0 K^2_\omega, D_{K^2_\omega} = \bigcup T_i$ where each T_i is Borel, there is an n so that T_n includes an infinite complete subgraph. Fix $t \in \omega^{<\omega}$ and n so that $D_{K^2_\omega}$ is dense above t and T_i is comeager in $\omega^\omega | t$. Then $\bigcap_i \{z : t \frown i \frown z \in T_i\}$ is comeager in ω^ω, and in particular non-empty. Then $\{t \frown i \frown z\}$ is an infinite complete subgraph in T_i as wished. □

Using a similar construction, consider the following graph:

Let T be the tree of all functions $f : n \to n$ for some integer n and satisfying $f(k) \leq k$ for all $k < n$, ordered by the initial segment relation. Let $[T]$ be the set of all branches of T with the usual topology. Let $D \subset T$ be a dense subset of T intersecting each level of T at exactly one node. Define a graph $G = ([T], E)$ similarly to G_0 as following:

For $b_0, b_1 \in [T]$, $\{b_0, b_1\} \in E$ if and only if there is $d \in D$, $i \neq j \leq |d| + 1$ so that $b_0|_{|d|} = b_1|_{|d|} = d$, $b_0(|d| + 1) = i$, $b_1(|d| + 1) = j$ and for all $k > |d| + 1$, $b_0(k) = b_1(k)$.

Theorem 4.6. *$P = \mathbb{D}(G)$ is Borel σ finite cc but not Borel σ bounded cc.*

Proof. First, suppose $T = \bigcup_i T_i$ for each T_i Borel. Then there is an i and a $d \in D$ so that P_i is comeager in $[T[d]]$. Let $n \geq i$ be an arbitrary integer, we show that there is a complete subgraph of size $> n$ in T_i, which is then an antichain in P of size $> n$. Thus P is not Borel σ bounded cc.

Let $d' \sqsupset d$ be in D with level $> n$. Then T_i is still comeager in $T[d']$, thus is comeager in each $T[d' \frown j]$, $j \leq |d'| + 1$. Pick a z so that $d' \frown j \frown z \in T_i$ for all $j \leq |d'| + 1$. Clearly $d' \frown j \frown z$ form a complete graph of size $> n$, as wanted.

Now we give a partition of P into countably many Borel sets with no infinite antichains.

Construct for each i, U_i and for each $\tau \in U_i$, Q_τ as in the proof of previous two theorems. Then suppose that there is an infinite antichain $A \subset Q_\tau$. By the infinite Ramsey theorem, there is an infinite subset $A' \subset A$ and $l \leq i$ such that the set $\{b \sqsupset t_l : b \in p \in A\}$ for an infinite complete graph in G, which does not exist. A contradiction and thus each Q_τ satisfies finite chain condition.

This finishes the proof of the theorem. □

With the same T and D, define a collection E' of finite subsets of size > 1 in $[T]$ as:

$\{b_0, ..., b_{n+1}\} \in E'$ if and only if for the only $d \in D$ at level n, $b_i|_{|d|} = d$ for all $i \leq n+1$, $b_i(|d|+1) \neq b_j(|d| + 1)$ for $i \neq j$ and $b_i(l) = b_j(l)$ for all $l > |d| + 1$ and $i, j \leq n + 1$.

Still, let $G' = ([T], E')$, although it is no longer a hypergraph. Also, let $\mathbb{D}(G')$ be the poset of all finite subsets of $[T]$ not including any elements of E' as a subset (the collection of all finite "discrete" subsets), ordered by reverse inclusion. Note that this is a good Borel poset.

Theorem 4.7. *The poset $P = \mathbb{D}(G')$ is Borel σ-n-linked for every n but not Borel σ-centred.*

Proof. First we show that it is not Borel σ-centred. Similar to the previous theorems, it is enough to show this on G'. Suppose $[T] = \bigcup_i T_i$ where each i is Borel. There is a $d \in D$ so that T_i is comeager in $[T[d]]$. Then by the similar argument as before, we can find $b_0, ..., b_{|d|+1}$ extending d in T_i so that $\{b_0, ..., b_{|d|+1}\} \in E'$.

With the same sets U_i and Q_τ for $\tau \in U_i$ as in the proof of the previous theorem, we only focus on those τ satisfying that for all $t \in tau$, $|t| > n$. Suppose $\tau = \{t_1, ..., t_m\} \in U_m$. For a subset

$S = \{p_1, ..., p_n\} \subset Q_\tau$ of size n, and let each $p_i = \{b_{i1}, ..., b_{in}\}$ so that b_{ij} end extends t_i for every $0 < j \leq n$. Then since every t_i has more than n successors in T, the sets $\{b_{ij}\}_{0<j\leq n}$ are discrete for all $0 < i \leq m$. Thus by our choice of τ, S is centred.

□

Based on the examples above, we can conclude the following theorem:

Theorem 4.8. *The conditions ccc, Borel σ finite cc, Borel σ bounded cc, Borel σ-n-linked for each n, Borel σ-n-linked for all n and Borel σ-centred, are all distinct even for good Borel posets.*

Although the posets of discrete subsets mentioned in this section satisfy different Borel chain conditions, they are in general all σ-centred.

Theorem 4.9. *If a hypergraph $H = (X_H, R_H)$ has no more than continuum many connected components and each component is countable(or, in another word, a locally countable hypergraph of size no larger than $\mathfrak{c}$), then $\mathbb{D}(H)$ is σ-centred.*

Proof. Let $X_H = \bigcup_{\iota<\mathfrak{c}} X_\iota$ decomposes X_H into its connected components. Denote by $\mathbb{D}(X_\iota)$ the set of finite discrete subsets of X_ι and give it discrete topology. Then by the Hewitt-Marczewski-Pondiczery theorem, the space $\Pi_{\iota<mathfrakc} X_\iota$ is separable. Let $D = \{d_i\}_{i<\omega}$ be a countable dense subset of it. Let $P_{d_i} = \{p \subset d_i\}$. Clearly every $p \in P_{d_i}$ is discrete and P_{d_i} is centred. Also every $p \in \mathbb{D}(H)$ is a finite union of discrete subsets from each connected component, thus induces a basic open set in the product space $\Pi_{\iota<mathfrakc} X_\iota$, which contains some d_i. In another word, $p \in P_{d_i}$. Therefore $\mathbb{D}(H) = \bigcup_{i<\omega} P_{d_i}$ is a fragmentation witness the σ-centredness. □

Note that the proof above also extends to $\mathbb{D}(G)$ and $\mathbb{D}(G')$.

Although the aforementioned posets of finite discrete subsets are all σ-centred, this is not always the case. Note that the example 2.3 is a Borel poset and the fragmentation witnessing σ-n-linkedness in the first chapter is actually Borel. So for each n there is a poset that is Borel σ-n-linked but not σ-$n+1$-linked.

4.4 Borel σ-centredness of $\mathbb{D}(H)$

In the examples above, one may notice that the denying a fragmentation condition in $P = \mathbb{D}(H)$, for a Borel hypergraph H, actually relies on the denying certain fragmentation condition of H(e.g., $\mathbb{D}(G_0(K_2^2, D_{K_2^2}))$ is not Borel σ-linked because $G_0(K_2^2, D_{K_2^2})$ has uncountable Borel chromatic number). A natural question is that, is it enough for $\mathbb{D}(H)$ to be Borel σ-linked if H is a Borel graph with countable Borel chromatic number?

The answer is no.

Example 4.1. *Let $H = (V_H, E_H)$ be a Borel graph with uncountable Borel chromatic number. Consider the graph $H' = (V_{H'}, E_{H'})$ with:*

1. $V_{H'} = V_H \times \{0, 1\}$.
2. $E_{H'} = \{\{(v, 0), (u, 1)\} : \{u, v\} \in E_H\}$.

Since $V_{H'} = V_H \times \{0\} \cup V_H \times \{1\}$, each part discrete, we have $\chi_B(H') = 2$.

On the other hand, consider the subset of $\mathbb{D}(H')$ consists of discrete pairs (denote it by $\mathbb{D}_2(H')$): $\{\{(u,0),(u,1)\} : uinV_H\}$ is a subgraph of $\mathbb{D}_2(H')$ that is Borel-isomorphic to H, which thus denies the Borel σ-linked-ness.

Now, a further question is: how about $\mathbb{D}_2(H)$ being Borel σ-linked? Is it enough for $\mathbb{D}(H)$ to be Borel σ-linked?

Denote by $\mathbb{D}_n(H)$ the subset of $\mathbb{D}(H)$ which consists of discrete subsets of H of size n.

Theorem 4.10. *Let H be a k-dimensional Borel hypergraph. Let $P = \mathbb{D}(H)$. The following are equivalent:*

1. *P is Borel σ-k-linked (thus σ-centred).*
2. *$\mathbb{D}_k(H)$ is Borel σ-k-linked as a subset of P.*

Proof. Let $H = (V_H, E_H)$. It is obvious that (1) implies (2). To see (2) implies (1), fix Borel fragmentation $\mathbb{D}_k(H) = \bigcup_{n<\omega} D_n$.

Without loss of generality, assume that each $p \in \mathbb{D}(H)$ is linearly ordered (so it is a subset of $\bigcup_{i<\omega} V_H^i$). For a subset $a \subset l$ and $p \in \mathbb{D}_l(H)$, denote by $p|a$ the restriction of p on a. So $p|a \in \mathbb{D}_{|a|}(H)$.

First note that the condition (2) implies that $\mathbb{D}_i(H)$ has countable Borel chromatic number for all $i < k$, so it is enough to show that $\mathbb{D}_l(H)$ also has countable Borel chromatic number for each $l > k$.

Now, fix $l > k$. Let s be a function from $[l]^k \to \omega$. Let $D(s) = \{p$:For each $a \in [l]^k, p|a \in D_{s(a)}\}$. Clearly $D(s)$ is Borel and there are only countably many such s.

If $\{p_1, p_2, ..., p_k\} \subset D(s)$ is not centred, there are k (possibly empty) subsets $a_1, ..., a_k \subset l$ such that $|\bigcup_{0<i\leq k} a_i| = k$ and $\bigcup_{0<i\leq k} p_i|a_i \in E_H$. However, this implies that, by letting $c = \bigcup_{0<i\leq k} a_i$, $\{p_i|c\}_{0<i\leq k}$ is not centred, contradicting the fact that $p_i|c \in D_{s(c)}$ for all $0 < i \leq k$.

So $\mathbb{D}_l(H)$ are Borel σ-k-linked for all $l < \omega$, thus $\mathbb{D}(H)$ is Borel σ-k-linked.

□

Now the question is: what is the relationship between H and $\mathbb{D}_k(H)$? Here we generalize a result of Lecomte in [16]. To state this result, we need the following terminology:

Definition 4.6. *Let $H = (V_H, E_H)$ be a Borel k-dimensional hypergraph. H is said to be potentially closed if there is a Polish topology on V_H refining the original one with the same Borel sets such that E_H is closed in V_H^2 with the product topology of the new topology.*

Theorem 4.11. *Let H be a Borel k-dimensional hypergraph. The following are equivalent:*

1. *H is potentially closed.*
2. *$\mathbb{D}(H)$ is σ-centred.*

To see this, first let $H = (V_H, E_H)$ be a k-dimensional hypergraph. Let σ be a permutation of k. Let $\mathbb{S}_\sigma(H) = (V_H^k \setminus E_H, R)$ be a k-dimensional relation defined as: $\{v_1, ..., v_k\} \in R$ if and only if $\{v_i(\sigma)(i)\}_{0<i\leq k} \in E_H$.

Lemma 4.2. *Let H be a k-dimensional Borel hypergraph. Then the following are equivalent:*

1. *H is potentially closed.*

2. *For every σ being a permutation of k, $\mathbb{S}_\sigma(H)$ is Borel countable chromatic. (i.e. There is a countable partition $V_H = \bigcup_{n<\omega} V_n$ such that each V_n is Borel and R-discrete).*

Proof. (1) → (2): Refine the topology on V_H so that E_H is closed in the product topology. Since the new topology is still Polish, $V_H^k \setminus E_H = \bigcup_{n<\omega} A_n$ where each A_n is open in V_H(hence Borel in the original topology). One can readily check that each A_n is R-discrete.

(2) → (1): Let $c : V_H^k \setminus E_H = \bigcup_{n<\omega} A_n$ be a partition into countably many R-discrete sets. For each $0 < i \leq k$ and n, let $A_{ni} = proj_i(A_n)$ be the projection of A_n down to i'th coordinate. Fix $0 < i \leq k$ and $n < \omega$, pick a point $(v_1, ..., v_k) \in \Pi_{0<i\leq k} A_{ni}$. For each v_i, let $x_i \in A_n$ be an element such that $proj_i(x_i) = v_i$. Since the tuple $(x_{\sigma^{-1}(1)}, ..., x_{\sigma^{-1}(k)}) \notin R$, we have that $\{proj_{\sigma(\sigma^{-1}(1))}(x_{\sigma^{-1}(1)}), ..., proj_{\sigma(\sigma^{-1}(k))}(x_{\sigma^{-1}(k)})\} \notin E_H$. In other words, $\{v_1, ..., , v_k\} \notin E_H$.

By using the first reflection theorem, find Borel $B_{ni} \supseteq A_{ni}$ such that for each $n < \omega$, $\Pi_{0<i\leq k} B_{ni} \cap E_H = \emptyset$. Since there are only countably many such B_{ni}, there is a Polish topology refining the original one with the same Borel sets in which all B_{ni} are open, and therefore H is potentially closed.

□

Now we get into the proof of theorem.4.11.

Proof. (2) → (1) directly follows from the lemma.

(1) → (2): It is enough to show that $\mathbb{D}_k(H)$ is Borel σ-k-linked. Without loss of generality we assume that every $p \in \mathbb{D}_k(H)$ is well-ordered. Refine to the Polish topology so that E_H is closed. Let $V_H^k \setminus E_H = \bigcup_{n<\omega} S_n$ for countably many $S_n = \Pi_{0<i\leq k} S_{ni}$ open basic.

For each function $\sigma : k \to k$, if σ is one-to-one(i.e. a permutation), fix a Borel coloring $\mathbb{S}_\sigma(H) = S_n^\sigma$ obtained with the lemma. For each x, let $f(\sigma, x)$ be the n such that $x \in S_n^\sigma$.

If σ is not one-to-one, by the definition of a k-dimensional hypergraph, for every tuple $x = \{x_1, ..., x_k\} \in \mathbb{D}_k(H)$, $\sigma(x) = \{x_{\sigma(1)}, ..., x_{\sigma(k)}\} \in V_H^k \setminus E_H$. Let $f_(\sigma, x)$ be the n so that $\sigma(x) \in U_n$.

Let Φ be the collection of all the functions $\phi : k^k \to \omega$. For each ϕ, let $P_\phi = \{x \in \mathbb{D}_k(H) : f(\sigma, x) = \phi(\sigma)$ for all $\sigma \in k^k\}$. Clearly, each P_ϕ is k-linked and Borel. There are also only countably many ϕ.

This finishes the proof.

□

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