THE LEAST PRIME WHOSE FROBENIUS IS AN n-CYCLE

by

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ABSTRACT

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Let L/K be a Galois extension of number fields. We consider the problem of bounding the least prime ideal of K whose Frobenius lies in a fixed conjugacy class C. Under the assumption of Artin's conjecture we work with Artin L-functions directly to obtain an upper bound in terms of irreducible characters which are nonvanishing at C. As a consequence we obtain stronger upper bounds for the least prime in C when many irreducible characters vanish at C. We also prove a Deuring-Heilbronn phenomenon for Artin L-functions with nonnegative Dirichlet series coefficients as a key step.

We apply our results to the case when $\mathrm{Gal}(L/K)$ is the symmetric group S_n. Using classical results on the representation theory of S_n we give an upper bound for the least prime whose Frobenius is an n-cycle which is stronger than known bounds when the characters which are nonvanishing at n-cycles are unramified, as well a similar result for $(n-1)$-cycles. We also give stronger bounds in the case of S_n-extensions over $\mathbb{Q}$ which are unramified over a quadratic field. We also consider other groups and conjugacy classes where unconditional improvements are obtained.

To my parents.

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CONTENTS

1

INTRODUCTION AND OVERVIEW

Let L/K be a Galois extension of number fields with Galois group G. Given a prime $\mathfrak{p}$ of K which is unramified in L there is a corresponding Frobenius conjugacy class $\sigma_\mathfrak{p}$ of G. The Chebotarev density theorem states that given a conjugacy class C of G, the primes of K whose Frobenius $\sigma_\mathfrak{p}$ is C has density $|C|/|G|$. In particular there are infinitely many such primes. An important problem therefore is to obtain effective bounds on primes whose Frobenius appears in C. In this chapter we will discuss known results including the effective bounds of Lagarias-Montgomery-Odlyzko. For special choices of groups and conjugacy classes we are able to obtain much improved bounds.

For notational convenience, if $f(x), g(x)$ are functions with $g(x) \geq 0$ we will write

$$f(x) \ll g(x)$$

whenever there is a positive constant c such that

$$|f(x)| \leq cg(x).$$

The constant c is to be effective and absolute unless stated otherwise. For two functions $f(x), g(x) \geq 0$ we will also write

$$f(x) \asymp g(x)$$

to mean $f(x) \ll g(x)$ and $g(x) \ll f(x)$.

1.1 THE LEAST PRIME IDEAL IN A CONJUGACY CLASS

Let L/K be a Galois extension of number fields with Galois group G. For each prime $\mathfrak{p}$ of K unramified in L denote by $\sigma_\mathfrak{p}$ its Frobenius class in G. Let C be a subset of G which is stable under conjugation and define

$$\pi_C(x, L/K) = \#\{\mathfrak{p} : \sigma_\mathfrak{p} \subseteq C,\ N_{K/\mathbb{Q}}\mathfrak{p} \leq x\}.$$

The Chebotarev density theorem [Tsc26] states that

$$\pi_C(x, L/K) \sim \frac{|C|}{|G|} \operatorname{Li}(x).$$

In particular there are infinitely many $\mathfrak{p}$ with $\sigma_{\mathfrak{p}} \subseteq C$ so one may ask for bounds on such primes $\mathfrak{p}$ with least norm.

In [LO77], under the assumption of the generalized Riemann hypothesis Lagarias and Odlyzko proved an effective version of the Chebotarev density theorem which states that if ζ_L satisfies the GRH then for every $x > 2$

$$|\pi_C(x, L/K) - \frac{|C|}{|G|} \operatorname{Li}(x)| \ll \frac{|C|}{|G|} x^{1/2} \log(d_L x^{n_L}) + \log d_L$$

where the implicit constant is absolute and effective. As a corollary they obtain that the least prime $\mathfrak{p}$ with $\sigma_{\mathfrak{p}} \subseteq C$ satisfies

$$N_{K/\mathbb{Q}}\mathfrak{p} \ll (\log d_L)^2.$$

Unconditionally one only has much weaker bounds. In [LMO79] Lagarias, Montgomery, and Odlyzko prove that unconditionally one has

$$N_{K/\mathbb{Q}}\mathfrak{p} \ll d_L^A \tag{1.1}$$

for an absolute and effective constant A. In [Zam17a] Zaman shows that one may take $A = 40$, later improved in [Zam17b] to $A = 35$, for sufficiently large d_L with some improvements depending on the extension L/K. This was improved to $A = 16$ by Kadiri-Ng-Wong in [KNW19]. Unconditionally, Ahn and Kwon showed in [AK19] that we may take $A = 12577$ for all d_L.

We may contrast this with Linnik's theorem [Lin44a, Lin44b] on the least prime in arithmetic progressions which states the following. Let k be a positive integer and a coprime to k. Then the least prime p lying in the arithmetic progression $a + nk$ satisfies

$$p \ll k^L$$

for an absolute and effective constant L, commonly called Linnik's constant. Numerically, Heath-Brown showed in [HB92] that $L = 5.5$ is admissible which was improved by Xylouris in [Xyl11] to $L = 5$.

In terms of the splitting of primes, if we let $k = q$ be prime then a prime p lying in the arithmetic progression $a + nq$ means that the Frobenius of p in the group $\operatorname{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q}) \cong (\mathbb{Z}/q)^\times$ corresponds to the residue class a

(mod q). The discriminant of $\mathbb{Q}(\zeta_q)$ has absolute value q^{q-2} so we see that the unconditional Linnik's theorem provides a bound which is as good as the conditional bounds of Lagarias-Odlyzko and significantly better than the unconditional bound of Lagarias-Montgomery-Odlyzko. One may therefore ask for other cases when the LMO bound may be improved.

1.2 STATEMENT OF RESULTS

In this thesis we prove an effective bound for the least prime in certain conjugacy classes of particular Galois extensions L/K of number fields. We do this by following technique of [LMO79] and assuming Artin's conjecture for the extension L/K. By assuming Artin's conjecture we are free to work with Artin L-functions instead of the usual reduction to Hecke L-functions. This allows us to keep track of information coming from each irreducible character χ and track our estimates in terms of the individual Artin conductors A_χ instead of the discriminant d_L which may be much larger.

By working with the Artin L-functions directly we are also able to pinpoint the characters which contribute nontrivially. This lets us improve the known bounds for the least prime whose Frobenius lies in the class C when C has the property that many characters of $\mathrm{Gal}(L/K)$ vanish at C. In some cases where Artin's conjecture is known we obtain unconditional bounds which are sharper than previously known bounds. We will discuss unconditional results in Section 1.2.3.

1.2.1 *General results*

We first state the result for a general Galois extension L/K of number fields. Denote by $\mathrm{nv}(C)$ the set of irreducible characters of G which are nonvanishing at C. As well, let $P(L/K)$ be the set of all rational primes q below primes $\mathfrak{q}$ of K which ramify in L. We prove the following general bound for the least prime in a conjugacy class C under the assumption of Artin's conjecture.

Theorem 1.1. *Let L/K be a Galois extension of number fields with $\mathrm{Gal}(L/K) = G$. Let C be a conjugacy class in C and assume Artin's conjecture for the L-functions $L(s,\chi)$ for $\chi \in \mathrm{nv}(C)$.*

1. *If $\prod_{\chi\in\mathrm{nv}(C)} L(s,\chi)$ has no exceptional zeros, then there is an absolute effective constant C_1 such that the least prime $\mathfrak{p}$ of K whose Frobenius in G is C satisfies*

$$N_{K/\mathbb{Q}}\mathfrak{p} \le \left(\prod_{\chi\in\mathrm{nv}(C)} A_\chi^{\chi(1)}\right)^{C_1\left(\sum_{\chi\in\mathrm{nv}(C)} |\chi(C)|\chi(1)^5\right)^{1/2}} \cdot \left(\frac{|G|}{|C|} n_K \sum_{p\in P(L/K)} \log p\right)^{C_1}. \tag{1.2}$$

2. *Suppose that there is an exceptional zero β_0 of $\prod_{\chi\in\mathrm{nv}(C)} L(s,\chi)$. Suppose that $\sum_{\chi\in\mathrm{nv}(C)} \chi$ always takes nonnegative values. Then there is an absolute effective constant C_2 such that*

$$N_{K/\mathbb{Q}}\mathfrak{p} \le \left(\prod_{\chi\in\mathrm{nv}(C)} A_\chi^{\chi(1)}\right)^{C_2} \left(\frac{|G|}{|C|} n_K \sum_{p\in P(L/K)} \log p\right)^{C_2}. \tag{1.3}$$

Remark 1.2. In Theorem 1.1, an additional hypothesis that $\sum_{\chi\in\mathrm{nv}} \chi$ always takes nonnegative values is required in the case of an exceptional zero. However, in all of our applications this will hold.

Remark 1.3. For Theorem 1.1 we only need Artin's conjecture to hold for those $\chi \in \mathrm{Irr}(\mathrm{Gal}(L/K))$ that are nonvanishing at C. In particular since Artin's conjecture is known for linear characters we will see in Section 1.2.3 that we obtain unconditional bounds for classes C where only linear characters are nonvanishing at C.

The proof of Theorem 1.1 relies on a zero-free region of Artin L-functions. Unfortunately it is only truly free of zeros up to one possible exceptional zero. The proof therefore divides into two cases: when there is no such zero, and when the zero exists. For the case of an exceptional zero we prove a version of the Deuring-Heilbronn phenomenon for L-functions with nonnegative Dirichlet series coefficients.

Theorem 1.4. *Let L/K be a Galois extension and φ a (not necessarily irreducible) character of $G = \mathrm{Gal}(L/K)$ which decomposes into a sum of irreducible characters as*

$$\varphi = \sum_{i=1}^{m} a_i \chi_i$$

where a_i are positive integers and $\langle \varphi, 1_G \rangle > 0$. Suppose that for each $g \in G$

$$\varphi(g) = \sum_{i=1}^{m} a_i \chi_i(g) \geq 0.$$

Assume the Artin conjecture for the L-functions $L(s, \chi_i)$ holds.

Then there are effective and absolute positive constants C_3, C_4 such that if β_0 is a real zero of $L(s, \varphi)$ then there are no zeros in the region

$$1 - C_4 \frac{\log\left(\frac{C_3}{\langle \varphi, 1_G \rangle \mathcal{L}(t)(1-\beta_0)}\right)}{\mathcal{L}(t)} < \sigma < 1$$

where

$$\mathcal{L}(t) := \sum_{i=1}^{m} a_i \chi_i(1)(\log A_{\chi_i} + n_K \chi_i(1) \log(|t| + 2)).$$

1.2.2 S_n-extensions

As an application we prove upper bounds for the least prime $\mathfrak{p}$ in C when $\mathrm{Gal}(L/K) = S_n$ and C is the class of n-cycles or $(n-1)$-cycles. It will be in terms of *hook characters* which are characters corresponding to particular Young diagrams called hooks. The representation theory of S_n will be reviewed in Chapter 2 but briefly we recall here that there exist canonical bijections between the following three objects:

- Irreducible characters of S_n
- Young diagrams of size n
- Partitions of n

Notation 1.5. We write partitions in nonincreasing order. That is, if $a_1 + a_2 + \cdots + a_m = n$ with $a_1 \geq a_2 \geq \cdots \geq a_m$ we write $(a_1, a_2, \ldots, a_m)$ for the corresponding partition. Each partition $(a_1, a_2, \ldots, a_m) \vdash n$ defines a Young diagram with row lengths $a_1, a_2, \ldots, a_m$. A partition of the form $(n-r, 1^r)$ and its corresponding Young diagram is called a *hook*. If $\lambda \vdash n$ is a partition of n then we write χ^λ for the corresponding irreducible character of S_n. In particular if λ is a hook then χ^λ is a *hook character*.

In Chapter 2 we will also see that the hook characters are exactly the characters which are nonvanishing at n-cycles and furthermore that they take values ± 1 at n-cycles. In the setting of S_n-extensions and C the class of n-cycles we may phrase Theorem 1.1 as follows:

Theorem 1.6. *Let L/K be a Galois extension of number fields with $\mathrm{Gal}(L/K) = S_n$. Suppose the Artin conjecture holds for L-functions attached to hook characters χ^λ. Then the least prime $\mathfrak{p}$ of K which is unramified in L and whose Frobenius in $\mathrm{Gal}(L/K)$ is an n-cycle satisfies the following:*

1. *If $\prod_{\lambda=(n-r,1^r)} L(s,\chi^\lambda)$ has no exceptional zeros, then there is an absolute effective constant C_5 such that*

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq \left(\prod_{\lambda=(n-r,1^r)} A_{\chi^\lambda}^{\chi^\lambda(1)} \right)^{C_5 2^{5n/2}/n} \left(nn_K \sum_{p \in P(L/K)} \log p \right)^{C_5}.$$

2. *If $\prod_\lambda L(s,\chi^\lambda)$ admits an exceptional zero β_0 then there is an absolute effective constant C_6 such that*

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq \left(\prod_{\lambda=(n-r,1^r)} A_{\chi^\lambda}^{\chi^\lambda(1)} \right)^{C_6} \left(nn_K \sum_{p \in P(L/K)} \log p \right)^{C_6}. \tag{1.4}$$

Remark 1.7. Whereas the LMO bound depends on the discriminant of the extension field L, our result depends on the product of Artin conductors $A_{\chi^\lambda}^{\chi^\lambda(1)}$ attached to hook characters, as well as a small factor coming from ramification in the extension L/K.

In terms of the extension L/K we may rewrite the bound as follows.

Corollary 1.8. *Suppose L/K is a Galois extension of number fields such that $\mathrm{Gal}(L/K) = S_n$. Assume Artin's conjecture holds for L-functions of hook characters. Then there are effective absolute constants C_7, C_8, C_9 such that the least prime $\mathfrak{p}$ of K whose Frobenius in $\mathrm{Gal}(L/K)$ is an n-cycle satisfies*

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq d_L^{C_7^n/(n[L:K])} \left(\prod_{\lambda=(n-r,1^r)} N_{K/\mathbb{Q}}\mathfrak{f}_{\chi^\lambda}^{\chi^\lambda(1)} \right)^{C_8^n} (\log d_L)^{C_9}.$$

In particular,

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq d_L^{C_7^n/(n[L:K])} (N_{K/\mathbb{Q}}\mathfrak{d}_{L/K})^{C_8^n} (\log d_L)^{C_9}. \tag{1.5}$$

If $\prod_{\lambda=(n-r,1^r)} L(s,\chi^\lambda)$ admits an exceptional zero then instead

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq d_L^{C_7^n/(n[L:K])} (N_{K/\mathbb{Q}}\mathfrak{d}_{L/K})^{C_{10}} (\log d_L)^{C_9}.$$

where C_{10} is also an effective and absolute constant.

Using similar techniques we also prove an upper bound for the least prime whose Frobenius is an $(n-1)$-cycle. Here in place of hook characters we consider characters χ^μ corresponding to partitions $\mu \vdash n$ of the form $\mu = (n-r, 2, 1^{r-2})$ for $2 \leq r \leq n-2$.

Theorem 1.9. *Suppose L/K is a Galois extension of number fields such that $\mathrm{Gal}(L/K) = S_n$. Assume Artin's conjecture holds for L-functions of characters χ^μ corresponding to partitions $\mu \vdash n$ of the form $\mu = (n-r, 2, 1^{r-2})$. Then the least prime $\mathfrak{p}$ of K which is unramified in L and whose Frobenius in $\mathrm{Gal}(L/K)$ is an $(n-1)$-cycle satisfies the following:*

1. *If $\zeta_K(s)L(s,\mathrm{sgn})\prod_\mu L(s,\chi^\mu)$ has no exceptional zeros, then there is an effective constant C_{11} such that*

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq \left(d_K A_{\mathrm{sgn}} \prod_{\mu=(n-r,2,1^{r-2})} A_{\chi^\mu}^{\chi^\mu(1)} \right)^{C_{11} n^{1/2} 2^{5n}} \cdot \left((n-1) n_K \sum_{p \in P(L/K)} \log p \right)^{C_{11}}.$$

2. *Assume furthermore that Artin's conjecture holds for L-functions of hook characters χ^λ. Then if $\prod_\lambda L(s,\chi^\lambda)\prod_\mu L(s,\chi^\mu)$ admits an exceptional zero β_0 then there is an effective constant C_{12} such that*

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq \left(\prod_{\lambda=(n-r,1^r)} A_{\chi^\lambda}^{\chi^\lambda(1)} \prod_{\mu=(n-r,2,1^{r-2})} A_{\chi^\mu}^{\chi^\mu(1)} \right)^{C_{12}} \cdot \left((n-1) n_K \sum_{p \in P(L/K)} \log p \right)^{C_{12}}.$$

As before we may rewrite in terms of d_L.

Corollary 1.10. *Suppose L/K is a Galois extension of number fields such that $\mathrm{Gal}(L/K) = S_n$. Assume Artin's conjecture holds for L-functions of hook characters and characters corresponding to partitions of the form $(n-r, 2, 1^{r-2})$. Then there are effective absolute constants C_{13}, C_{14}, C_{15} such that the least prime $\mathfrak{p}$ of K whose Frobenius in $\mathrm{Gal}(L/K)$ is an $(n-1)$-cycle satisfies*

$$N_{K/\mathbb{Q}} \leq d_L^{n C_{13}^n/[L:K]} (N_{K/\mathbb{Q}} \mathfrak{d}_{L/K})^{C_{14}^n} (\log d_L)^{C_{15}}. \tag{1.6}$$

If $\prod_\lambda L(s,\chi^\lambda)\prod_\mu L(s,\chi^\mu)$ *admits an exceptional zero then instead we have*

$$N_{K/\mathbb{Q}} \leq d_L^{nC_{13}^n/[L:K]}(N_{K/\mathbb{Q}}\mathfrak{d}_{L/K})^{C_{16}}(\log d_L)^{C_{15}}.$$

where C_{16} *is another effective absolute constant.*

Remark 1.11. Since from Stirling's approximation we know that

$$[L:K] = n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

we see that the exponent of d_L in Equation 1.5 and Equation 1.6 is much smaller than in Equation 1.1.

In Corollary 1.8 unfortunately we have an exponential power of the relative discriminant $N_{K/\mathbb{Q}}\mathfrak{d}_{L/K}$ coming from the bound

$$\prod_{\lambda=(n-r,1^r)} N_{K/\mathbb{Q}}\mathfrak{f}_{\chi^\lambda}^{\chi^\lambda(1)} \leq N_{K/\mathbb{Q}}\mathfrak{d}_{L/K}.$$

The left-hand side is 1 if all $\mathfrak{f}_{\chi^\lambda}$ is trivial for $\lambda = (n-r,1^r)$, so in particular if L/K is unramified this will be the case. By a result of Fröhlich [Frö62] there exist infinitely many unramified extensions L/K with $\mathrm{Gal}(L/K) = S_n$.

When L is an S_n-extension over $\mathbb{Q}$ which is unramified over a quadratic field we still obtain an upper bound which is stronger than the LMO bound. Such extensions have been studied by Uchida [Uch70], Yamamoto [Yam70], Elstrodt-Grunewald-Mennicke [EGM85], Kondo [Kon95], and Kedlaya [Ked12] and it is known that there are infinitely many such extensions. For such a family we have the following upper bound:

Theorem 1.12. *Let* $L/F/\mathbb{Q}$ *be a tower of number fields such that* $L/\mathbb{Q}$ *is an* S_n*-extension and* F *is a quadratic field of discriminant* Δ *such that* L/F *is an unramified* A_n*-extension.*

1. *Suppose Artin's conjecture holds for hook characters of* $\mathrm{Gal}(L/\mathbb{Q}) = S_n$. *Then there exist constants* C_{17}, C_{18} *such that the least prime* p *whose Frobenius in* L *is an* n*-cycle satisfies*

$$p \leq \Delta^{C_{17}^n/n^{3/2}}(n\log\Delta)^{C_{18}}. \tag{1.7}$$

In particular, we have

$$p \leq d_L^{2C_{17}^n/(n^{3/2}[L:\mathbb{Q}])}\left(\frac{2}{(n-1)!}\log d_L\right)^{C_{18}}. \tag{1.8}$$

2. *Suppose furthermore that Artin's conjecture holds for characters corresponding to partitions of the form* $\mu = (n-r,2,1^{r-2})$. *Then there exist*

constants C_{19}, C_{20} such that the least prime p whose Frobenius in L is an $(n-1)$-cycle satisfies

$$p \leq \Delta^{n^2 C_{19}^n}((n-1)\log\Delta)^{C_{20}}. \tag{1.9}$$

In particular, we have

$$p \leq d_L^{2n^2 C_{19}^n/[L:\mathbb{Q}]}\left(\frac{2(n-1)}{n!}\log d_L\right)^{C_{20}}. \tag{1.10}$$

Under some stronger assumptions we may ensure that the exponent of $N_{K/\mathbb{Q}}\mathfrak{d}_{L/K}$ in Corollary 1.8 is constant in all cases as follows.

Theorem 1.13. *Let L/K be a Galois extension with $\mathrm{Gal}(L/K) = S_n$. Suppose the following holds for all hook characters χ:*

1. *Artin's conjecture for $L(s,\chi)$*
2. *For $1 < \sigma < 3$ we have*
$$\left|\frac{L'}{L}(s,\chi)\right| \ll \frac{1}{s-1} + \log A_\chi$$
3. *There exists a constant c such that $L(s,\chi)$ has at most one zero in the region*
$$1 - \frac{c}{\chi(1)(\log A_\chi + \chi(1) n_K \log(|t|+2))} \leq \sigma \leq 1.$$

Then there exists an absolute effective constant C_{21} such that the least prime $\mathfrak{p}$ of K whose Frobenius in G is an n-cycle satisfies

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq \left(\prod_{\substack{\lambda\vdash n\\ \lambda=(n-r,1^r)}} A_{\chi^\lambda}^{\chi^\lambda(1)}\right)^{C_{21}}\left(n n_K \sum_{p\in P(L/K)} \log p\right)^{C_{21}}.$$

In particular, we have

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq d_L^{C_{21}2^n/[L:K]}(N_{K/\mathbb{Q}}\mathfrak{d}_{L/K})^{C_{21}}(\log d_L)^{C_{22}}$$

where C_{22} is another effective absolute constant.

As in the case of n-cycles, under stronger assumptions we obtain a constant exponent of $N_{K/\mathbb{Q}}\mathfrak{d}_{L/K}$ in the upper bound for the least prime whose Frobenius is an $(n-1)$-cycle.

Theorem 1.14. *Let L/K be a Galois extension with $\mathrm{Gal}(L/K) = S_n$. Suppose the following holds for characters χ corresponding to partitions $\lambda, \mu \vdash n$ where λ is a hook and μ is of the form $(n-r, 2, 1^{r-2})$:*

1. *Artin's conjecture for $L(s, \chi)$*
2. *For $1 < \sigma < 3$ we have*
$$\left|\frac{L'}{L}(s,\chi)\right| \ll \frac{1}{s-1} + \log A_\chi$$
3. *There exists a constant c such that $L(s,\chi)$ has at most one zero in the region*
$$1 - \frac{c}{\chi(1)(\log A_\chi + \chi(1) n_K \log(|t|+2))} \leq \sigma \leq 1.$$

Then there exists an absolute effective constant C_{23} such that the least prime $\mathfrak{p}$ of K whose Frobenius in G is an n-cycle satisfies

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq \left(\prod_{\lambda=(n-r,1^r)} A_{\chi^\lambda}^{\chi^\lambda(1)} \prod_{\mu=(n-r,2,1^{r-2})} A_{\chi^\mu}^{\chi^\mu(1)} \right)^{C_{23}} \cdot \left(n n_K \sum_{p \in P(L/K)} \log p \right)^{C_{23}}.$$

In particular, we have

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq d_L^{nC_{24}^n/[L:K]} (N_{K/\mathbb{Q}}\mathfrak{d}_{L/K})^{C_{25}} (\log d_L)^{C_{26}}$$

for absolute effective constants C_{24}, C_{25}, C_{26}.

1.2.3 *Unconditional results for D_n and Camina elements*

We can apply Theorem 1.1 to the case when the Galois group is either dihedral, or when the conjugacy class C has the property that only linear characters are nonvanishing at C. Elements of such a class are called *Camina elements* and will be discussed in Section 2.4. For these cases all relevant L-functions are known to satisfy Artin's conjecture and so all results in this subsection are unconditional.

For dihedral extensions we prove the following upper bound on the least prime whose Frobenius is a reflection. Denote by D_n the dihedral group of order $2n$. As well, for any group G we write G' for the commutator subgroup and $G^{ab} = G/G'$ for the abelianization of G.

Theorem 1.15. *Let L/K be a Galois extension of number fields with $G = \mathrm{Gal}(L/K) = D_n$. Let $F = L^{G'}$ denote the maximal abelian subextension of L/K. Then there are absolute effective constants C_{27}, C_{28} such that the least prime $\mathfrak{p}$ whose Frobenius is a reflection in G satisfies*

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq d_F^{C_{27}} \left(n_K \sum_{p \in P(L/K)} \log p \right)^{C_{28}}.$$

In terms of the discriminant of L we may rewrite as follows:

Corollary 1.16. *Let L/K be a Galois extension of number fields with $G = \mathrm{Gal}(L/K) = D_n$. Then there are absolute effective constants C_{29}, C_{30} such that the least prime whose Frobenius in G is a reflection satisfies*

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq d_L^{C_{29}/[L:K]} (\log d_L)^{C_{30}}$$

Remark 1.17. Corollary 1.16 is consistent with a conjecture of V.K. Murty [KM00, Conjecture 2.1] which states that there are absolute constants a, b such that for any conjugacy class C the least prime $\mathfrak{p}$ whose Frobenius is C satisfies

$$N_{K/\mathbb{Q}}\mathfrak{p} \ll d_L^{a/[L:K]} (\log d_L)^b.$$

Remark 1.18. In [Wei83] Weiss proves an upper bound, later made explicit by Thorner-Zaman in [TZ17], for the least prime in a conjugacy class C when there is a large abelian subgroup A that has nontrivial intersection with C. This upper bound is in terms of the largest conductor of the abelian extension L/L^A. However, in our case the class of reflections in the dihedral group does not intersect a large abelian subgroup, and the upper bound for the least prime is in terms of the abelian extension F/K over the base field K.

Similarly we obtain the following bound for the least prime whose Frobenius is a *Camina element*, that is an element g such that the only irreducible characters that are nonvanishing at g are linear. A group G is a *Camina group* if all elements $g \in G - G'$ are Camina elements. We will review Camina elements and Camina groups in Section 2.4.

Theorem 1.19. *Let L/K be a Galois extension of number fields with Galois group $G = \mathrm{Gal}(L/K)$. Suppose that $g \in G$ is a Camina element, and let C denote its conjugacy class. Let $F = L^{G'}$ be the maximal abelian subextension of L/K.*

1. *If $\zeta_F(s)$ has no exceptional zeros then there is an absolute effective constant C_{31} such that the least prime $\mathfrak{p}$ of K whose Frobenius in G is C satisfies*

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq d_F^{C_{31}[L:F]^{1/2}} \left(\frac{|G|}{|C|} n_K \sum_{p \in P(L/K)} \log p \right)^{C_{31}}.$$

2. *If $\zeta_F(s)$ has an exceptional zero β_0 then there is an absolute effective constant C_{32} such that*

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq d_F^{C_{32}} \left(\frac{|G|}{|C|} n_K \sum_{p \in P(L/K)} \log p \right)^{C_{32}}.$$

In particular if G is a Camina group then the above holds for any C which is the class of any element not in G'.

As before we can rephrase in terms of the discriminant d_L of the extension field as follows.

Corollary 1.20. *Let L/K be a Galois extension of number fields with Galois group $G = \mathrm{Gal}(L/K)$. Suppose that $g \in G$ is a Camina element and let C denote its conjugacy class. Let F be the maximal abelian subextension of L/K.*

1. *If $\zeta_F(s)$ has no exceptional zeros then there are absolute effective constants C_{33}, C_{34} such that the least prime $\mathfrak{p}$ whose Frobenius in G is C satisfies*

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq d_L^{C_{33}/[L:F]^{1/2}} (\log d_L)^{C_{34}}. \tag{1.11}$$

2. *If $\zeta_F(s)$ has an exceptional zero β_0 then there are absolute effective constants C_{35}, C_{36} such that*

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq d_L^{C_{35}/[L:F]} (\log d_L)^{C_{36}}.$$

In any case Equation 1.11 holds.

Remark 1.21. We observe that we obtain a power saving for the norm of the least prime $\mathfrak{p}$ over applying the Lagarias-Montgomery-Odlyzko bound directly.

2 REPRESENTATION THEORY

In this chapter we recall the representation theory of finite groups. The material in the first section is standard and may be found in any text on representation theory, for example [Ser77] or [Isa76]. In the subsequent sections we will recall some facts about the symmetric group S_n as well as its representation theory following the exposition in [Sag01]. We then review the representation theory of dihedral groups. Lastly we discuss Camina elements and Camina groups.

2.1 BASIC REPRESENTATION THEORY OF FINITE GROUPS

Let G be a finite group. Denote by $\mathrm{Irr}(G)$ the set of characters of irreducible complex representations of G. Recall that there is an inner product defined on the set of all class functions of G by

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}. \tag{2.1}$$

By Maschke's theorem every complex representation of G decomposes as a direct sum of irreducible representations. If ψ is the character of a representation and χ is an irreducible representation then $\langle \psi, \chi \rangle$ gives the multiplicity of χ in ψ. In particular if 1_G is the trivial character then $\langle \psi, 1_G \rangle$ gives the dimension of the fixed subspace of ψ.

With respect to this pairing the irreducible characters of G satisfy the orthogonality relation

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \chi = \psi \\ 0 & \chi \neq \psi. \end{cases}$$

commonly called the first orthogonality relation. The irreducible characters in fact form an orthonormal basis of the space of all class functions of G and therefore there is a (non-canonical) bijection between the conjugacy classes of G and the irreducible characters of G.

The irreducible characters also satisfy a second orthogonality relation as follows: if $g, h \in G$ then

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(g)\overline{\chi(h)} = \begin{cases} |G|/|C| & \text{if } g, h \text{ are conjugate} \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

where C denotes the conjugacy class of g.

If H is a subgroup of G and χ is a character of G corresponding to a representation ρ the *restriction* $\mathrm{Res}_H^G \chi$ of χ is defined to be the character of the restriction $\rho|_W$. If ψ is a character of H then we define the *induction* of H to G as follows. Let $r = [G : H]$ and $g_1, \ldots, g_r$ be a complete system of coset representatives of H. Extend ψ to all of G by defining

$$\tilde{\psi}(g) = \begin{cases} \psi(g) & g \in H \\ 0 & g \notin H \end{cases}.$$

Then

$$(\mathrm{Ind}_H^G \psi)(g) = \sum_{i=1}^{r} \tilde{\psi}(g_i^{-1} g g_i) = \frac{1}{|H|} \sum_{s \in G} \tilde{\psi}(s^{-1} g s).$$

If χ is a character of G and ψ is a character of H then *Frobenius reciprocity* states that

$$\langle \mathrm{Res}_H^G \chi, \psi \rangle = \langle \chi, \mathrm{Ind}_H^G \psi \rangle.$$

Example 2.1. The regular representation

$$\mathrm{reg}_G = \sum_{\chi} \chi(1)\chi$$

may be written as

$$\mathrm{reg}_G = \mathrm{Ind}_{\{1\}}^G 1$$

This is the induction of the trivial character from the trivial subgroup. From this one obtains the well-known formula

$$|G| = \mathrm{reg}_G(1) = \sum_{\chi} \chi(1)^2.$$

Definition 2.2. A character χ is *linear* if it is of degree 1, that is if $\chi(1) = 1$. Denote by $\mathrm{lc}(G)$ the set of linear characters of G.

Every linear character is a homomorphism $\chi : G \to \mathbb{C}^\times$ so in particular it is irreducible and nonvanishing for every $g \in G$. Since it is a homomorphism to an abelian group it factors through the abelianization $G^{ab} = G/G'$. Furthermore each irreducible character of G^{ab} defines an irreducible character of G by inflation, so there is a bijection between the linear characters of G and the irreducible characters of G^{ab}.

We also note the following simple observation when summing over linear characters.

Proposition 2.3. *For any $g \in G$,*

$$\sum_{\chi \in \mathrm{lc}(G)} \chi(g) = \begin{cases} |G^{ab}| & g \in G' \\ 0 & g \notin G' \end{cases}.$$

In particular, $\sum_{\chi \in \mathrm{lc}(G)} \chi(g) \geq 0$ for all $g \in G$.

Proof. Observe that $\sum_{\chi\, \mathrm{lc}(G)} \chi$ is simply the regular character of G/G' composed with the quotient map. □

For each $g \in G$, it will be useful to give a name to the set of irreducible characters of G which are nonvanishing at g.

Definition 2.4. Let G be a group and $g \in G$. Define the set of *nonvanishing characters* of G at g to be

$$\mathrm{nv}(g) = \{\chi \in \mathrm{Irr}(G) : \chi(g) \neq 0\}.$$

It is clear that every linear character is nonvanishing for every $g \in G$ and that $\mathrm{nv}(g)$ depends only on the conjugacy class of g. We will be paying special attention to classes C of G at which many characters vanish, that is $\mathrm{nv}(C)$ is small.

2.2 THE SYMMETRIC GROUP S_n

In this section we review the representation theory of S_n. We first recall the canonical correspondence between conjugacy classes of S_n, partitions of n, and Young diagrams of size n, followed by a review of the correspondence between conjugacy classes and irreducible characters of S_n given by Specht modules. Next we review the Murnaghan-Nakayama rule for computing character values and the branching rule for induction and restriction. Lastly we apply to the class of n-cycles which is the case of most interest to us. The material is standard and may be found for example in [Sag01].

2.2.1 *Conjugacy classes of S_n*

The symmetric group S_n is the group of permutations of the set $\{1, 2, \ldots, n\}$. There are $n!$ possible permutations so $|S_n| = n!$. The group S_n is generated by transpositions and the set of all permutations that can be written as a

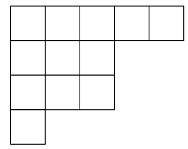


Figure 2.1: A Young diagram of shape $(5, 3, 3, 1)$.

product of even number of transpositions forms the alternating group A_n which is the unique index 2 subgroup of S_n. A_n is also the commutator subgroup of S_n so it follows that S_n has two linear characters. They are the trivial character and the *sign* character defined by $\operatorname{sgn}(\tau) = -1$ for a transposition τ and extended to all of S_n multiplicatively.

Each permutation $\sigma \in S_n$ can also be written uniquely as a product of disjoint cycles and two permutations σ, τ are conjugate if and only if they have the same cycle structure. In particular the set of permutations which consist of a single cycle of length n forms a conjugacy class which we will call the class of *n-cycles*. The conjugacy classes of S_n are also in bijection with *partitions* of n.

Definition 2.5. A *composition* of n is an ordered sequence $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers such that $\sum_i \alpha_i = n$. A *partition* of n is a composition which is nonincreasing. We write $\lambda \vdash n$ to mean that λ is a partition of n.

The conjugacy classes of S_n are easily seen to be in bijection with the partitions of n sending the class of permutations with cycle structure $(\lambda_1, \lambda_2, \ldots, \lambda_l)$, written in decreasing order, to the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$. Partitions $\lambda \vdash n$ are also in bijection with *Young diagrams* of size n by sending a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ to a Young diagram of shape $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ which is an array of n boxes left-aligned where the ith row counting from the top has length λ_i. We will adopt the convention that the box in row i and column j has coordinates (i, j) and that row lengths are nonincreasing.

In summary, the following four objects are in bijection with one another:

- Conjugacy classes of S_n
- Partitions of n
- Young diagrams of size n
- Irreducible characters of S_n

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \qquad\qquad \begin{array}{ccc}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \cline{1-2}\end{array}$$

Figure 2.2: A Young tableau, and tabloid of shape $(3, 2)$.

2.2.2 *Specht modules*

The bijection between conjugacy classes and irreducible characters is in general not canonical, but in the case of S_n it can be made canonical in the following way.

A *Young tableau* of shape $\lambda \vdash n$ is a Young diagram of shape λ with the boxes filled with the numbers $1, 2, \ldots, n$ bijectively. Two Young tableaux t_1, t_2 of the same shape are *row equivalent* if corresponding rows have the same elements. A row equivalence class of Young tableaux t of shape λ is called a *tabloid* of shape λ and written $\{t\}$. Graphically a tabloid will be denoted by an array of integers in shape λ with horizontal lines between rows.

If t is a tableau with rows $R_1, R_2, \ldots, R_l$ and columns $C_1, C_2, \ldots, C_k$ then the *row stabilizer* and *column stabilizer* of t are defined respectively by

$$R_t = S_{R_1} \times S_{R_2} \times \cdots \times S_{R_l}$$

and

$$C_t = S_{C_1} \times S_{C_2} \times \cdots \times S_{C_k}$$

where S_{R_i} and S_{C_j} are the groups of permutations of R_i and C_j respectively. If t is a tableau then the associated *polytabloid* is defined to be

$$e_t = \left(\sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi \right) \{t\}.$$

Finally if λ is any partition, the *Specht module* S^λ is the module spanned by the polytabloids e_t where t ranges over tableau of shape λ. They are in fact cyclic so are spanned by any one polytabloid e_t. Over $\mathbb{C}$ the Specht modules are irreducible and form a complete list of irreducible representations of S_n. Denote by χ^λ by the character of S^λ.

A basis for S^λ is given by the set of e_t where t range over *standard* Young tableaux of shape λ, that is Young tableaux whose rows and columns are increasing. Therefore the dimension of S^λ and therefore the character degree $\chi^\lambda(1)$ is equal to the number f^λ of standard Young tableaux of shape λ.

The number f^λ may be computed using the *hook formula*. Given a Young diagram λ and a node (i, j), the *hook* $H_{i,j}$ is the subdiagram contained in

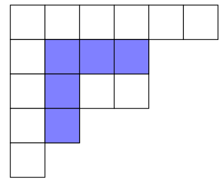


Figure 2.3: A Young diagram of shape $(6,4,4,2,1)$ with the hook at (2,2).

λ consisting of the node (i,j) as well as all nodes to the right of (i,j) and below (i,j). The *hooklength* $h_{i,j}$ of a hook $H_{i,j}$ is the number of nodes in $H_{i,j}$. Finally the hook formula states that if $\lambda \vdash n$ then

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}.$$

2.2.3 *Murnaghan-Nakayama rule*

Let χ be an irreducible character of S_n. From the previous section we know that $\chi = \chi^\lambda$ for some partition λ. If ρ is another partition then we will write χ^λ_ρ for $\chi^\lambda(\rho)$. The character values of S_n are completely described by the *Murnaghan-Nakayama rule*. We will now recall some preliminary notions before stating the rule.

A *generalized tableau* T of shape λ is obtained by filling a Young diagram of shape λ by positive integers with possible repetition. The *content* of a generalized tableau is the composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ where α_i is the number of times the integer i appears in T.

A *skew diagram* is the difference of two Young diagrams. That is, given $\mu \subseteq \lambda$ as two Young diagrams the skew diagram λ/μ is the difference

$$\lambda/\mu = \{c : c \in \lambda \text{ and } c \notin \mu\}.$$

A *rim hook* is a connected skew diagram containing no 2×2 squares. Given a rim hook ξ, its *leg length* $ll(\xi)$ is defined to be the number of rows of ξ minus 1.

Finally a *rim hook tableau* of shape λ and content $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a generalized tableau T of shape λ and content α satisfying the following:

- The rows and columns of T are nondecreasing.
- For each i, the boxes with entries α_i form a single rim hook $\xi^{(i)}$.

1	2	2	3
2	2	4	
2	4	4	
5			

Figure 2.4: A rim hook tableau of shape $(4,3,3,1)$ and content $(1,5,1,3,1)$.

The *sign* of a rim hook tableau with rim hooks $\xi^{(i)}$ is defined to be

$$(-1)^T := \prod_{\xi^{(i)} \in T} (-1)^{ll(\xi^{(i)})}.$$

We may now state the Murnaghan-Nakayama rule. Let λ be a partition of n and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be any composition of n. Then

$$\chi^\lambda_\alpha = \sum_T (-1)^T$$

where the sum is over all rim hook tableaux of shape λ and content α.

There is also a recursive formulation of the Murnaghan-Nakayama rule. With λ and α as before, we have

$$\chi^\lambda_\alpha = \sum_\xi (-1)^{ll(\xi)} \chi^{\lambda \setminus \xi}_{\alpha \setminus \alpha_1}$$

where the sum is over all rim hooks ξ of λ with α_1 cells such that its removal from λ leaves a valid Young diagram. Here $\alpha \setminus \alpha_1$ is the composition obtained by removing the first element α_1 from α. The recursion terminates at the base case $\chi^{()}_{()} = 1$.

2.2.4 *Branching rule*

Consider a chain of subgroups $S_{n-1} \subseteq S_n \subseteq S_{n+1}$. The *branching rule* describes the decomposition of a character of S_n induced to S_{n+1}, as well as the decomposition of a character of S_n restricted to S_{n-1}.

We may describe the branching in terms of Young diagram as follows. Let $\lambda \vdash n$ with a corresponding Young diagram. Denote by λ^+ any Young diagram obtained by adding one box to λ in such a way so that the resulting shape is a Young diagram. Similarly denote by λ^- any Young diagram obtained by removing a box from λ in such a way that results in a Young diagram.

Theorem 2.6 (Branching rule)**.** *If $\lambda \vdash n$ then*

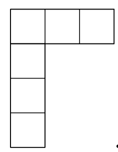
1.
$$\mathrm{Res}^{S_n}_{S_{n-1}} S^\lambda \cong \bigoplus_{\lambda^-} S^{\lambda^-}$$

and

2.
$$\mathrm{Ind}^{S_{n+1}}_{S_n} S^\lambda \cong \bigoplus_{\lambda^+} S^{\lambda^+}.$$

Proof. See [Sag01, Theorem 2.8.3]. □

Example 2.7. Let $n = 6$ and $\lambda = (3, 1^3) \vdash 6$. Then the Young diagram corresponding to λ is given by



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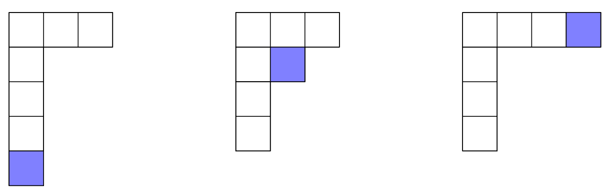
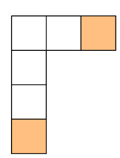
The Young diagrams λ^+ are obtained by adding one box in a way such that the resulting shape is a Young diagram. Hence λ branches as follows



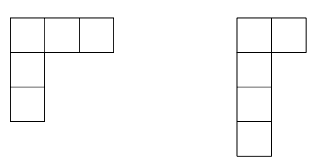
and therefore

$$\mathrm{Ind}^{S_7}_{S_6} \chi^{(3,1^3)} = \chi^{(3,1^4)} + \chi^{(3,2,1^2)} + \chi^{(4,1^3)}.$$

Likewise the boxes that can be removed from λ to obtain a Young diagram are given by



Hence by removing on of the boxes we see that λ^- are given by



therefore

$$\mathrm{Res}^{S_6}_{S_5} \chi^{(3,1^3)} = \chi^{(3,1^2)} + \chi^{(2,1^3)}.$$

2.2.5 *Application to n-cycles*

Consider the class of n-cycles which corresponds to the partition (n). We will now apply the results from the previous sections to obtain the character values $\chi^\lambda_{(n)}$ as well as the character degrees $\chi^\lambda(1)$.

Proposition 2.8. *We have*

$$\chi^\lambda_{(n)} = \begin{cases} (-1)^{n-r} & \lambda = (r, 1^{n-r}), \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. The formula in fact may be proven independently of the Murnaghan-Nakayama rule, for example [Sag01, Lemma 4.10.3]. We will give a proof as an application of the Murnaghan-Nakayama rule.

By the Murnaghan-Nakayama rule we have

$$\chi^\lambda_{(n)} = \sum_T (-1)^T$$

where the sum is over all rim hook tableaux T of shape λ and content given by the composition (n). That is, the content of T consists of n occurrences of 1. Since T is a rim hook tableaux, all occurrences of 1 inside T form a rim hook so therefore T itself must be a rim hook. As well, since T is also a Young diagram its shape and therefore λ must be of the form $(r, 1^{n-r})$.

Therefore the sum is nonempty if and only if λ is of the form $(r, 1^{n-r})$. In this case there is a unique rim hook tableau of content (n), namely each box filled with 1. The sign in this case is $(-1)^{n-r}$. □

Definition 2.9. A Young diagram is a *hook* if the corresponding partition is of the form $(n-r, 1^r)$. A character $\chi \in \mathrm{Irr}(S_n)$ is called a *hook character* if it is of the form χ^λ where λ is a Young diagram which is a hook.

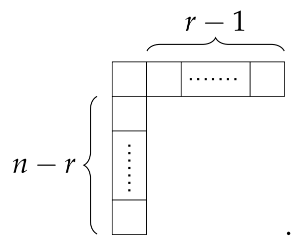
The hook characters are exactly the characters that do not vanish on n-cycles. We will now compute the degrees of hook characters.

Proposition 2.10. *If $\lambda = (r, 1^{n-r})$ is a hook then*

$$\chi^\lambda(1) = \binom{n-1}{r}.$$

Proof. The Young diagram corresponding to $\lambda = (r, 1^{n-r})$ is given by

The hook number for the top left square is simply the number of all boxes which is n. Any other box is either in the first row which has no boxes below, or first column which has no boxes to the right. Therefore the hook number of such a box in the first row is just the number of boxes to



the right and likewise the hook number of such a box in the first column is the number of boxes below. That is

$$h(i,j) = \begin{cases} n & i=j=1 \\ n-r-j & i=1,\, j>1 \\ r-i+2 & i>1,\, j=1 \end{cases}$$

and therefore by the hook-length formula

$$\chi^\lambda(1) = \frac{n!}{r!(n-r-1)!n} = \frac{(n-1)!}{r!(n-r-1)!} = \binom{n-1}{r}.$$

□

Remark 2.11. We could have avoided invoking the hook formula by noting that a standard Young tableaux of shape $(n-r, 1^r)$ must have n in the top-left corner and is determined by a choice of r integers from $\{1, \ldots, n-1\}$ used to fill the rest of the first column.

2.3 REPRESENTATION THEORY OF THE DIHEDRAL GROUP D_n

The dihedral group is the group of symmetries of the regular n-gon. It has order $2n$ and we will adopt the convention of writing it as D_n. It is generated by an element r corresponding to a rotation by an angle of $2\pi/n$ and an element s corresponding to a reflection subject to the following presentation

$$D_n = \langle r, s : r^n = s^2 = 1,\, srs = r^{-1} \rangle.$$

It has a cyclic normal subgroup generated by the rotations and may be written as a semidirect product

$$D_n \cong C_n \rtimes C_2$$

where C_k is the cyclic group of order k.

The group structure as well as its representation theory is well-known and is readily found in the literature, for example [Ser77]. We will collect them here. There are two cases depending on the parity of n.

If n is odd, then D_n has the following conjugacy classes:

- The class of the identity $\{1\}$,
- $(n-1)/2$ classes of rotations $\{r^{\pm 1}\}, \{r^{\pm 2}\}, \ldots, \{r^{\pm (n-1)/2}\}$, each of size 2, and
- The class of reflections $\{r^j s : 0 \leq j \leq n-1\}$ of size n.

In this case there are two linear characters giving by composing the quotient map $D_n \to C_2$ with the two irreducible linear characters of C_2. There are also $(n-1)/2$ irreducible two-dimensional characters induced from the cyclic subgroup of rotations. Explicitly they may be realized as 2×2 matrices as follows. Let $w = e^{2\pi i/n}$ and h an integer, and set

$$\rho^h(r^k) = \begin{pmatrix} w^{hk} & 0 \\ 0 & w^{-hk} \end{pmatrix}, \qquad \rho^h(sr^k) = \begin{pmatrix} 0 & w^{-hk} \\ w^{hk} & 0 \end{pmatrix}. \tag{2.3}$$

Taking $0 < h \leq (n-1)/2$ gives pairwise nonisomorphic irreducible two-dimensional representations. Counting all the representation we have $(n-1)/2 + 2$ irreducible representations which must be all of them since this is equal to the number of conjugacy classes.

If n is even, then D_n has the following conjugacy classes:

- The class of the identity $\{1\}$,
- The singleton class $\{r^{n/2}\}$,
- $n/2 - 1$ classes of rotations $\{r^{\pm 1}\}, \{r^{\pm 2}\}, \ldots, \{r^{\pm (n/2-1)}\}$, and
- Two classes of reflections, each of size $n/2$: $\{r^{2i}s : 0 \leq i \leq n/2 - 1\}$ and $\{r^{2i+1}s : 0 \leq i \leq n/2 - 1\}$.

Hence there must be $(n+6)/2$ irreducible characters. There are four linear characters, given by sending the generators r, s to ± 1 in all possible ways. For the other characters, taking the characters ρ^h as in Equation 2.3 for $0 < h < n/2$ gives $n/2 - 1$ pairwise nonisomorphic two-dimensional irreducible characters and we have found all of the irreducible characters of D_{2n}. Hence one easily observes the following

Proposition 2.12. *Let D_n be a dihedral group and let $\chi \in \mathrm{Irr}(D_n)$. Let s be a reflection. Then $\chi(s) = 0$ for all nonlinear χ and $\chi(s) = \pm 1$ otherwise.*

Proof. Clear from the character table of D_n and Equation 2.3. □

We also note that the commutator subgroup of D_n is $\langle r^2 \rangle$. If n is odd this is in fact just $\langle r \rangle$. It follows that the abelianization of D_n is

$$D_n^{ab} = D_n / \langle r^2 \rangle = \begin{cases} C_2 & n \text{ odd} \\ C_2 \times C_2 & n \text{ even} \end{cases} \tag{2.4}$$

2.4 CAMINA ELEMENTS AND CAMINA GROUPS

Let G be a finite group and N a normal subgroup. We say that (G, N) is a *Camina pair* if for every $g \in G - N$ is conjugate to all of gN. Such objects were first studied by Camina in [Cam78]. A *Camina group* is a finite nonabelian group G such that (G, G') is a Camina pair. The following proposition allows us to restate this in terms of vanishing of characters.

Proposition 2.13. *Let G be a finite group and $g \in G$. Then the following are equivalent:*

1. *The conjugacy class of g is gG'.*
2. $|C_G(g)| = |G : G'|$.
3. *For all $z \in G'$ there exists $y \in G$ such that $[g, y] = z$.*
4. $\chi(g) = 0$ *for every nonlinear $\chi \in \operatorname{Irr}(G)$.*

Proof. See [Lad08, Proposition 1.1] and [Lew09a, Lemma 2.1]. □

Following [Lew09b] we will also call an element $g \in G$ a *Camina element* if it satisfies the equivalent conditions of Proposition 2.13. In the literature Camina elements are also called *anticentral elements*. Hence Camina groups are exactly the finite groups such that every nonlinear irreducible character vanishes outside of its commutator subgroup, that is every element outside of the commutator subgroup is a Camina element.

Example 2.14. The commutator subgroup of the Dihedral group D_n is $\langle r^2 \rangle$ where r is a rotation generating D_n. By direct computation one sees that D_n is a Camina group if and only if n is odd.

In [Lad08] it is shown using the classification of finite simple groups that any group admitting a Camina element is solvable. In there a list of examples of groups admitting Camina elements is given, which we now reproduce for convenience.

Example 2.15. Let K be a group admitting a fixedpoint-free automorphism α. Then α is a Camina element in the semidirect product $G = K \rtimes \langle \alpha \rangle$. More generally we can take an abelian group A with a surjective homomorphism onto $\langle \alpha \rangle$ and take the semidirect product $G = K \rtimes A$.

Example 2.16. An *extraspecial* p-group is a p-group G such that $Z(G)$ has order p and $G/Z(G)$ is an elementary abelian p-group. In this case the centre coincides with the commutator subgroup. All elements outside the commutator subgroup of an extraspecial p-group are Camina elements. That is, every extraspecial p-group is a Camina group.

Example 2.17. If G is solvable and G' is a minimal normal subgroup of G then G admits Camina elements. There are two cases:

1. $G' \leq Z(G)$, $|G'| = p$, $G/Z(G)$ is an elementary abelian p-group, and every noncentral element is a Camina element.

2. $G' \cap Z(G) = 1$, then $G/Z(G)$ is a Frobenius group with kernel $(G' \times Z(G))/Z(G)$ and cyclic complement. In this case every $g \notin C_G(G')$ is a Camina element.

Example 2.18. If A is an abelian normal subgroup of G such that G/A is cyclic, then every element a such that $G/A = \langle aA \rangle$ is a Camina element.

Example 2.19. Any group of order p^n where p is a prime and $n \leq 4$ contains Camina elements.

Example 2.20. Let G be the group of all upper-triangular $n \times n$ matrices with entries from a finite field of order q and 1's on the main diagonal. Let $a \in G$ be an element with with minimal polynomial $(x-1)^n$. Then a is a Camina element.

Example 2.21. A p-group of order p^{n+1} for $n > 1$ is said to be of *maximal class* if it has nilpotency class n. For such a group, the elements not in $C_G(K_i(G)/K_{i+2}(G))$ for $i \geq 2$ are Camina elements, where $K_i(G)$ denotes the terms in the descending central series of G.

Camina groups admits the following characterization theorem as p-groups or Frobenius groups with particular structure. We recall that a *Frobenius group* is a finite group G which admits a nontrivial subgroup H (called the *Frobenius complement*) such that $H \cap gHg^{-1} = \{1\}$ for all $g \notin H$. The set K consisting of all elements of G not conjugate to any element of H along with the identity is called the *Frobenius kernel*. Frobenius's theorem states that K is a normal subgroup of G and G admits a decomposition $G = K \rtimes H$. We refer to [Isa76] for more information.

The following characterization theorem was first given by Dark and Scoppola in [DS96] with an alternate proof given by Lewis in [Lew14] and by Isaacs and Lewis in [IL15].

Theorem 2.22 (Dark-Scoppola)**.** *A finite group G is a Camina group if and only if one of the following holds:*

1. *G is a Camina p-group with nilpotence class* 2 *or* 3.
2. *G is a Frobenius group with cyclic complement.*
3. *G is a Frobenius group with complement isomorphic to the quaternion group.*

3 ARTIN L-FUNCTIONS

In this chapter we recall the required background on number theory. The material in the first section may be found in any standard text on number theory, for example [Neu99] or [Lan94]. The subsequent sections deal with Artin L-functions with exposition following [Mar77] and [MM97].

3.1 FROBENIUS OF A PRIME

Let K be a number field and L/K a Galois extension. Throughout the thesis we will be using the following notation:

$\mathcal{O}_K$	the ring of integers in K
n_K	the degree of K over $\mathbb{Q}$
d_K	the absolute discriminant of K
$\mathfrak{d}_{L/K}$	the relative discriminant of L/K
G	the Galois group of L/K
$N_{L/K}$	the norm to K of an ideal in L
$R(L/K)$	primes $\mathfrak{p}$ of K which ramify in L
$P(L/K)$	rational primes p below primes $\mathfrak{p}$ in $R(L/K)$
$S(L/K)$	primes $\mathfrak{p}$ of K unramified in L and degree 1 over $\mathbb{Q}$

We will frequently drop the reference to the field in the norm and simply write N when no confusion is possible.

Let $\mathfrak{p}$ be a prime ideal of K. Then $\mathfrak{p}$ decomposes into a product of prime ideals $\mathfrak{P}_1^{e_1}\mathfrak{P}_2^{e_2}\dots\mathfrak{P}_g^{e_g}$ in L. In this case we say the $\mathfrak{P}_i$ divide $\mathfrak{p}$ or is above $\mathfrak{p}$ and write $\mathfrak{P}_i \mid \mathfrak{p}$. The nonnegative integers e_i are the ramification indices. If $e_i > 0$ for any i we say $\mathfrak{p}$ is ramified in L. If $\mathfrak{P} \mid \mathfrak{p}$ then $\mathbb{F}_{\mathfrak{P}} := \mathcal{O}_L/\mathfrak{P}$ is a finite field which is an extension of $\mathbb{F}_{\mathfrak{p}} := \mathcal{O}_K/\mathfrak{p}$. The degree of the extension $[\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_{\mathfrak{p}}]$ is called the residual degree, written $f_{\mathfrak{P}/\mathfrak{p}}$.

For an extension of number fields L/K, not necessarily Galois, the residual degrees and the ramification indices satisfy the equation

$$\sum_{i=1}^{g} e_i f_i = [L : K].$$

In the case that L/K is Galois, each residual degree f_i is equal and each ramification indices e_i are equal and therefore the equation simplifies to

$$efg = [L : K].$$

Suppose henceforth that L/K is Galois. Let $\mathfrak{p}$ be a prime of K and let $\mathfrak{P}$ be a prime above $\mathfrak{p}$. The decomposition group of $\mathfrak{P}$ is defined as

$$D_{\mathfrak{P}/\mathfrak{p}} = \{\sigma \in G : \sigma\mathfrak{P} = \mathfrak{P}\}.$$

There is a natural short exact sequence

$$1 \longrightarrow I_{\mathfrak{P}/\mathfrak{p}} \longrightarrow D_{\mathfrak{P}/\mathfrak{p}} \longrightarrow \mathrm{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}) \longrightarrow 1$$

where the kernel $I_{\mathfrak{P}/\mathfrak{p}}$ is called the inertia group. If $\mathfrak{p}$ is unramified in L then the inertia group is trivial so there is a natural isomorphism $D_{\mathfrak{P}/\mathfrak{p}} \cong \mathrm{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}/\mathfrak{p})$. The latter group is cyclic and has the Frobenius automorphism as a canonical generator, and its inverse image in $D_{\mathfrak{P}/\mathfrak{p}}$ under the natural map is called the Frobenius element of $\mathfrak{P}$.

If $\mathfrak{P}, \mathfrak{Q}$ are two primes of L dividing $\mathfrak{p}$ then the decomposition groups $D_{\mathfrak{P}/\mathfrak{p}}$ and $D_{\mathfrak{Q}/\mathfrak{p}}$ are conjugate and conversely if $\sigma \in G$ then the group $\sigma D_{\mathfrak{P}/\mathfrak{p}}\sigma^{-1}$ is the decomposition group of the prime $\sigma\mathfrak{P}$. Hence the Frobenius elements of the primes above $\mathfrak{p}$ forms a conjugacy class called the *Frobenius class* of $\mathfrak{P}$ written $\sigma_{\mathfrak{p}}$. We will frequently refer to the Frobenius class of $\mathfrak{p}$ as simply the *Frobenius* of $\mathfrak{p}$.

3.2 DEFINITION OF ARTIN L-FUNCTIONS

In this section we will recall the basic facts about Artin L-functions. The material may be found in [Mar77] and [MM97]. We will assume throughout that L/K is a Galois extension of number fields with Galois group G. Let ρ be a complex representation of G, χ its character, and V the underlying complex vector space.

Let $\mathfrak{p}$ be a prime of K. If $\mathfrak{p}$ is unramified in L then there is a well-defined Frobenius conjugacy class $\sigma_{\mathfrak{p}}$. Define the local Euler factor at $\mathfrak{p}$ to be

$$L_{\mathfrak{p}}(s, \chi, K) = \det(I - \rho(\sigma_{\mathfrak{p}})(N\mathfrak{p})^{-s})^{-1}$$

which is well-defined since conjugate elements have the same characteristic polynomial.

If $\mathfrak{p}$ ramifies in L and $\mathfrak{P}$ is a prime above $\mathfrak{p}$, let $I_{\mathfrak{P}}$ be the inertial group at $\mathfrak{P}$. Define the Euler factor at $\mathfrak{p}$ to be

$$L_{\mathfrak{p}}(s,\chi,K) = \det(I - \rho(\sigma_{\mathfrak{p}})|_{V^{I_{\mathfrak{P}}}}(N\mathfrak{p})^{-s})^{-1}$$

where $V^{I_{\mathfrak{P}}}$ is the subspace of V invariant under the action of $I_{\mathfrak{P}}$. The definition of $L_{\mathfrak{p}}(s,\chi,K)$ is independent of the choice of $\mathfrak{p}$.

The Artin L-function is then defined to be the Euler product over finite primes

$$L(s,\chi,K) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(s,\chi,K).$$

The Artin L-functions satisfy a functional equation. We first define the gamma factor $\mathbf{\Gamma}(s,\chi,K)$ which incorporates the infinite places of K and then define the Artin conductor $\mathfrak{f}_\chi$ of χ which is defined in terms of inertia groups.

If v is an infinite place of K define

$$L_v(s,\chi,K) = \begin{cases} ((2\pi)^{-s}\Gamma(s))^{\chi(1)} & \text{if } v \text{ is complex} \\ ((\pi^{-s/2}\Gamma(s/2))^a(\pi^{-(s+1)/2}\Gamma((s+1)/2))^b & \text{if } v \text{ is real} \end{cases} \tag{3.1}$$

where a is the dimension of the $+1$ eigenspace of complex conjugation and similarly b is the dimension of the -1 eigenspace. The numbers a,b satisfy $a+b=\chi(1)$. Define

$$\mathbf{\Gamma}(s,\chi,K) = \prod_{v \text{ infinite}} L_v(s,\chi,K).$$

Next let v be a place of K (finite or infinite) and let w be a place of L dividing v and let G_0 be the inertia group I_w at w. There is a descending filtration of higher ramification groups

$$G_0 \supseteq G_1 \supseteq \ldots.$$

If V is the underlying space of ρ define

$$n(\chi,v) = \sum_{i=0}^{\infty} \frac{|G_i|}{|G_0|} \operatorname{codim}(V^{G_i}).$$

The summands are zero except for finitely many i so $n(\chi,v)$ is well-defined. It is also an integer and independent of the choice of w above v.

Define the *Artin conductor* of χ to be the ideal

$$\mathfrak{f}_\chi = \prod_v \mathfrak{p}_v^{n(\chi,v)}. \tag{3.2}$$

We also set

$$A_\chi = d_K^{\chi(1)} N_{K/\mathbb{Q}} \mathfrak{f}_\chi \tag{3.3}$$

and

$$\Lambda(s,\chi,K) = A_\chi^{s/2} \mathbf{\Gamma}(s,\chi,K) L(s,\chi,K). \tag{3.4}$$

The function $\Lambda(s,\chi,K)$ satisfies the functional equation

$$\Lambda(s,\chi,K) = W(\chi)\Lambda(1-s,\bar{\chi},K) \tag{3.5}$$

where $W(\chi)$ is a complex number of absolute value 1.

3.3 properties of artin L-functions and artin's conjecture

The Artin L-functions satisfy the following properties.

- For any $a_\chi \in \mathbb{Z}$ such that $\sum_\chi a_\chi \chi$ is a character of G,

$$L(s, \sum_\chi a_\chi \chi, K) = \prod_\chi L(s,\chi,K)^{a_\chi}.$$

- If H is a subgroup of G and L^H is its fixed subfield then

$$L(s, \mathrm{Ind}_H^G, K) = L(s,\chi,L^H).$$

In particular we see that

$$L(s, \mathrm{reg}_G, K) = L(s, \mathrm{Ind}_{\{1\}}^G 1, K) = L(s,1,L) = \zeta_L(s).$$

As well, since

$$\mathrm{reg}_G = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi$$

we see that

$$\zeta_L(s) = \prod_{\chi \in \mathrm{Irr}(G)} L(s,\chi,K)^{\chi(1)} = \zeta_K(s) \prod_{1 \neq \chi \in \mathrm{Irr}(G)} L(s,\chi,K)^{\chi(1)}.$$

By Brauer induction, given an irreducible character χ of G one can find subgroups H_i of G and linear characters ψ_i of H_i such that χ may be written as a $\mathbb{Z}$-linear combination of the induction of ψ_i, that is

$$\chi = \sum_i m_i \, \mathrm{Ind}_{H_i}^G \psi_i$$

for some integers m_i. Therefore any Artin L-function can be written as

$$L(s,\chi,K) = \prod_i L(s,\psi_i,L^{H_i})^{m_i}.$$

Each L-function on the right hand side comes from a linear character which by Artin reciprocity can be identified with a Hecke L-function whose analytic continuation is known. Therefore each Artin L-series admits a meromorphic continuation to all of $\mathbb{C}$. *Artin's conjecture* states that in fact each Artin L-function attached to characters which do not contain the trivial character admits a holomorphic continuation to all of $\mathbb{C}$.

Artin's conjecture is known in some specific cases. In the case when G is abelian each Artin L-function can be identified with a Hecke L-function whose analytic continuation is known. Likewise if each character of G is known to be induced from linear characters of subgroups then again each Artin L-function is a Hecke L-function and so Artin's conjecture holds. Such groups are called *monomial* or *M-groups*. Examples of monomial groups include supersolvable groups.

3.4 THE ARTIN CONDUCTOR AND DISCRIMINANT

The Artin conductor $\mathfrak{f}_\chi$ encodes information about the ramification of the character χ. For us it will appear in error terms analogous to the role of $\log d_L$ in [LMO79]. One can estimate the Artin conductor by the following

Proposition 3.1 (M.R. Murty-V.K. Murty-Saradha)**.** *Let L/K be a Galois extension with Galois group G and $n = [L:K]$. Let $P(L/K)$ be the set of rational primes p such that there exists a prime $\mathfrak{p}$ of K with $\mathfrak{p} \mid p$ which ramifies in L. Let $\chi \in \operatorname{Irr}(G)$ and $\mathfrak{f}_\chi$ be its Artin conductor. Then*

$$\log N_{K/\mathbb{Q}}\mathfrak{f}_\chi \leq 2\chi(1)n_K \left(\sum_{p \in P(L/K)} \log p + \log n \right).$$

Proof. This is [MMS88, Proposition 2.5]. □

Combining with the definition of A_χ (Equation 3.3) we see that

$$\log A_\chi = \chi(1)\log d_K + \log N\mathfrak{f}_\chi \ll \chi(1)\log d_K \left(\sum_{p \in P(L/K)} \log p + \log n \right).$$

The Artin conductors for irreducible characters of $\operatorname{Gal}(L/K)$ are related to the discriminant as follows.

Theorem 3.2 (Conductor-discriminant formula)**.** *If L/K is a Galois extension then*

$$\mathfrak{d}_{L/K} = \prod_{\chi \in \mathrm{Irr}(\mathrm{Gal}(L/K))} \mathfrak{f}_\chi^{\chi(1)}.$$

Proof. See [Neu99, 11.9]. $\square$

Lastly we recall the Minkowski bound.

Theorem 3.3 (Minkowski bound)**.** *Let K be a number field of degree n_K and let r_2 be the number of pairs of complex embeddings of K. Then every ideal class of K admits an integral ideal $\mathfrak{a}$ satisfying*

$$N_{K/\mathbb{Q}}\mathfrak{a} \leq d_K^{1/2} \left(\frac{4}{\pi}\right)^{r_2} \frac{n_K!}{n_K^{n_K}}.$$

Proof. See [Lan94, V, §4, Theorem 4]. $\square$

In particular since $N_{K/\mathbb{Q}}\mathfrak{a} \geq 1$ it follows that

$$d_K^{1/2} \geq \left(\frac{\pi}{4}\right)^{r_2} \frac{n_K^{n_K}}{n_K!}$$

and therefore

$$\log d_K \gg n_K. \tag{3.6}$$

3.5 zeros of artin L-functions

In this section we collect some results on zero density of Artin L-functions as well as a zero-free region of Artin L-functions assuming Artin's conjecture.

Let $n_\chi(T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$ inside the rectangle $0 \leq \beta \leq 1$, $|\gamma - T| \leq 1$, so

$$n_\chi(T) = \#\{\rho = \beta + i\gamma : L(\rho, \chi) = 0,\, 0 \leq \beta \leq 1,\, |\gamma - T| \leq 1\}.$$

Lemma 3.4. *Assume Artin's conjecture for L/K. Then*

$$n_\chi(T) \ll \log A_\chi + \chi(1) n_K \log(T+2).$$

Proof. This is a straightforward generalization of [LMO79, Lemma 2.1]. $\square$

There is also a second zero density estimate for Artin L-functions over a different region. Set

$$n_\chi(r; s) = \#\{\rho : L(\rho, \chi) = 0,\, |\rho - s| \leq r\}$$

so $n_\chi(r;s)$ counts the number of zeros of $L(s,\chi)$ with distance at most r from s. We will mainly be interested in the case $s=1$, that is, the density of zeros with distance at most r from $s=1$.

Lemma 3.5. *We have for* $0<r<2$

$$n_\chi(r;1) \ll \chi(1) + r(\log A_\chi + \chi(1) n_K \log(3)) \ll \chi(1) + r \log A_\chi. \tag{3.7}$$

Proof. This is a straightforward generalization of [LMO79, Lemma 2.2]. See [Mah99, Lemma 2.1 (ii)]. For the second inequality, we note that from the definition of A_χ Equation 3.3 we have

$$\log A_\chi = \chi(1)\log d_K + \log N_{K/\mathbb{Q}} \mathfrak{f}_\chi \ll \chi(1) n_K$$

since $\log d_K \gg n_K$ from the Minkowski bound (see Equation 3.6). □

The zero-density estimate given in Lemma 3.5 unfortunately has a dependence on the character degree $\chi(1)$ which is unaffected by the distance r from $s=1$. This comes from the estimate given in [Mah99, Equation 2.23] which states that

$$\left|\frac{L'}{L}(\sigma+it,\chi)\right| \ll \frac{\chi(1)}{\sigma-1} + \chi(1)\log d_K$$

for $1<\sigma<3$ obtained by comparing L'/L against the Dedekind zeta function ζ_K.

We will also require the following zero-free region of Artin L-functions which rules out the possible exceptional zero for nonlinear characters.

Proposition 3.6 (V.K. Murty)**.** *Assume Artin's conjecture. Let L/K be a Galois extension of number fields χ be an irreducible character of* $\mathrm{Gal}(L/K)$ *such that $\chi(1)\neq 1$. Then $L(\sigma+it,\chi)$ is zero-free for*

$$1-\frac{c}{\chi(1)^3(\log A_\chi + n_K\log(|t|+2))} \le \sigma \le 1$$

where c is an effective and absolute constant.

If $\chi(1)=1$, $L(\sigma+it,\chi)$ is zero-free in the above region with the possible exception of a single zero.

Proof. This is [Mur97, Proposition 3.1].

□

3.6 ADDITIONAL ASSUMPTIONS ON ZEROS

The estimates given in the previous section is enough to prove Theorem 1.1. In this section we discuss two further assumptions on the Artin L-functions $L(s,\chi)$ which together with Artin's conjecture will imply a stronger estimate in Theorem 1.1. Specifically they are as follows:

Assumption 1. *For $1 < \sigma < 3$, the logarithmic derivative of $L(\sigma+it,\chi)$ satisfies*

$$\left|\frac{L'}{L}(\sigma+it,\chi)\right| \ll \frac{1}{\sigma-1}+\log A_\chi.$$

Assumption 2. *There exists a constant c such that if χ is irreducible and $L(\sigma+it,\chi)$ satisfies Artin's conjecture then it has at most one zero in the region*

$$1-\frac{c}{\chi(1)(\log A_\chi+\chi(1)n_K\log(|t|+2))} \leq \sigma \leq 1.$$

The motivation for Assumption 1 comes from the following estimate found in [Mah99, Equation 2.23]

$$\left|\frac{L'}{L}(\sigma+it,\chi)\right| \leq \chi(1)\left(\frac{1}{\sigma-1}+\frac{1}{2}\log d_K+O(n_K)\right)$$

for $1<\sigma<3$, which is used in the proof of Lemma 3.5. However, if χ is nonprincipal then $L'/L(s,\chi)$ is holomorphic at $s=1$ so one can hope for an upper bound without the $1/(\sigma-1)$ term which leaves $\chi(1)\log d_K \ll \log A_\chi$.

Assumption 2 is a strengthening of the following zero-free region.

Proposition 3.7 ([Mur97, Corollary 3.2])**.** *There exists an absolute positive constant c such that if χ is irreducible then $L(\sigma+it,\chi)$ has at most one zero in the region*

$$1-\frac{c}{\chi(1)\log A_\chi} \leq \sigma \leq 1, \quad |t| \leq \frac{c}{\chi(1)\log A_\chi}.$$

For the rest of this thesis any dependence on the above assumptions will be made explicit. Assumption 1 implies a zero-density estimate for $L(s,\chi)$ which is stronger than Lemma 3.5 as follows.

Lemma 3.8. *Suppose χ is a nonprincipal character. Assume Artin's conjecture for $L(s,\chi)$ and suppose that Assumption 1 holds. Then*

$$n_\chi(r;1+it) \ll 1+r(\log A_\chi+n_K\chi(1)\log(|t|+2)). \tag{3.8}$$

Remark 3.9. Comparing with Equation 3.7, the constant term in the right-hand side of Equation 3.8 is 1 as opposed to $\chi(1)$. This will give us a sharper bound than the one stated in Theorem 1.1.

Proof. The proof is very similar to the one for Lemma 3.5, given in [Mah99, Lemma 2.1 (ii)]. We record the details for completeness. Since χ is non-principal, Λ is entire and has a Hadamard factorization

$$\Lambda(s,\chi) = e^{\alpha(\chi)+\beta(\chi)s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \tag{3.9}$$

where $\alpha(\chi), \beta(\chi)$ are constants and ρ runs over all nontrivial zeros of $L(s,\chi)$. Logarithmically differentiating Equation 3.9 we obtain

$$-\frac{\Lambda'}{\Lambda}(s,\chi) = -\beta(\chi) - \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

and therefore using Equation 3.4 we obtain

$$-\frac{L'}{L}(s,\chi) = -\beta(\chi) - \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + \frac{1}{2}\log A_{\chi} + \frac{\mathbf{\Gamma}'}{\mathbf{\Gamma}}(s,\chi).$$

Therefore evaluating at $s = \sigma + it$ and at $s = 3 + it$ and subtracting we have

$$\frac{L'}{L}(s,\chi) - \frac{L'}{L}(3+it,\chi) = -\sum_{\rho} \left(\frac{1}{s-\rho} - \frac{1}{(3+it)-\rho}\right) - \frac{\mathbf{\Gamma}'}{\mathbf{\Gamma}}(s,\chi) + \frac{\mathbf{\Gamma}'}{\mathbf{\Gamma}}(3+it,\chi)\beta(\chi).$$

We know (see Lemma 4.9) that if $\operatorname{Re} s > -1/2$ and $|s| \geq 1/8$ we have

$$\left|\frac{\mathbf{\Gamma}'}{\mathbf{\Gamma}}(s,\chi)\right| \ll n_K \chi(1) \log(|s|+2)$$

and therefore

$$\left|\frac{\mathbf{\Gamma}'}{\mathbf{\Gamma}}(3+it,\chi)\right| \ll n_K \chi(1) \log(|t|+2).$$

Hence it follows that for $1/2 \leq \sigma \leq 3$ we have

$$\left|\frac{L'}{L}(s,\chi) - \sum_{\substack{\rho \\ |\gamma-t|\leq 1}} \frac{1}{s-\rho}\right| \ll n_K \chi(1) \log(|t|+2) + \sum_{\substack{\rho \\ |\gamma-t|>1}} \left|\frac{1}{s-\rho} - \frac{1}{(3+it)-\rho}\right| + \sum_{\substack{\rho \\ |\gamma-t|\leq 1}} \left|\frac{1}{(3+it)-\rho}\right|.$$

Since for $\rho = \beta + i\gamma$ we have

$$\left|\frac{1}{(3+it)-\rho}\right| \le \frac{1}{3-\beta} \ll 1$$

using Lemma 3.4 we see that

$$\sum_{\substack{\rho \\ |\gamma - t| \le 1}} \left|\frac{1}{(3+it)-\rho}\right| \ll n_\chi(t) \ll \log A_\chi + \chi(1) n_K \log(|t|+2).$$

Similarly

$$\begin{aligned}\sum_{\substack{\rho \\ |\gamma - t| > 1}} \left|\frac{1}{s-\rho} - \frac{1}{(3+it)-\rho}\right| &= \sum_{\substack{\rho \\ |\gamma - t| > 1}} \frac{3-\sigma}{|s-\rho||(3+it)-\rho|} \\ &\ll \sum_{\substack{\rho \\ |\gamma - t| > 1}} \frac{1}{|\gamma - t|^2} \\ &\ll \sum_{j=2}^{\infty} \frac{n_\chi(t+j) + n_\chi(t-j)}{(j-1)^2} \\ &\ll \log A_\chi + n_K \chi(1) \log(|t|+2).\end{aligned}$$

Lastly using the bound for $|L'/L(s,\chi)|$ in Assumption 1 we obtain that for $1 < \sigma < 3$

$$\left|\sum_{\substack{\rho \\ |\gamma - t| \le 1}} \frac{1}{s-\rho}\right| \ll \frac{1}{\sigma - 1} + \log A_\chi + n_K \chi(1) \log(|t|+2)$$

which we sum over $t+j$, $-3 \le j \le 3$ to obtain

$$\left|\sum_{\substack{\rho \\ |\gamma - t| \le 4}} \frac{1}{s-\rho}\right| \ll \frac{1}{\sigma - 1} + \log A_\chi + n_K \chi(1) \log(|t|+2). \tag{3.10}$$

On the other hand,

$$\left|\sum_{\substack{\rho \\ |\gamma - t| \le 4}} \frac{1}{s-\rho}\right| \ge \sum_{\substack{\rho \\ |\gamma - t| \le 4}} \operatorname{Re} \frac{1}{s-\rho} = \sum_{\substack{\rho \\ |\gamma - t| \le 4}} \frac{\sigma - \beta}{|s-\rho|^2} \ge \sum_{\substack{\rho \\ |\gamma - t| \le 4}} \frac{\sigma - 1}{|s-\rho|^2}.$$

Let $0 < r < 2$ and choose $\sigma = 1 + r$ and let $k = \#\{\rho : |s - \rho| \leq 2r\}$. Then $n_\chi(r, t) \leq k$. Therefore

$$\left| \sum_{\substack{\rho \\ |\gamma - t| \leq 4}} \frac{1}{s - \rho} \right| \geq \frac{kr}{4r^2} \geq \frac{n_\chi(r; t)}{4r}.$$

Combining with Equation 3.10 gives

$$n_\chi(r; t) \ll 1 + r(\log A_\chi + n_K \chi(1) \log(|t| + 2)).$$

$\square$

4

THE LEAST PRIME IN A CONJUGACY CLASS: ESTIMATES

In this chapter we adapt the methods of [LMO79] to obtain estimates on the least prime whose Frobenius is in a fixed conjugacy class C. Throughout this section we will be assuming Artin's conjecture and work with Artin L-functions directly.

4.1 THE KERNEL FUNCTION

We will consider the kernel functions of [LMO79], defined by

$$k_1(s) = k_1(s;x) = \left(\frac{x^{2(s-1)} - x^{s-1}}{s-1}\right)^2$$

and

$$k_2(s) = k_2(s;x) = x^{s^2+s}.$$

The kernel functions k_1, k_2 depend on a parameter $x > 1$ whose value will be chosen later.

For any function $f(s)$ and real number σ we write

$$\int_{(\sigma)} f(s)\, ds := \int_{\sigma-i\infty}^{\sigma+i\infty} f(s)\, ds.$$

We consider the inverse Mellin transforms of the kernel functions k_j, given by

$$\hat{k}_j(u) = \frac{1}{2\pi i}\int_{(2)} k_j(s)u^{-s}\, ds$$

and compute them explicitly in the following lemmas.

Lemma 4.1. *The function*

$$k(s) = k(s;x,y) = \left(\frac{y^{s-1} - x^{s-1}}{s-1}\right)^2$$

has inverse Mellin transform

$$\hat{k}(u) = \begin{cases} \frac{1}{u}\log\frac{y^2}{u} & xy \le u \le y^2 \\ \frac{1}{u}\log\frac{u}{x^2} & x^2 \le u \le xy \\ 0 & \textit{otherwise} \end{cases}.$$

By Lemma 4.1 the function $k_1(s)$ has the inverse Mellin transform

$$\hat{k}_1(u) = \frac{1}{2\pi i}\int_{(2)} k_1(s)u^{-s}\,ds = \begin{cases} \frac{1}{u}\log\frac{x^4}{u} & x^3 \le u \le x^4 \\ \frac{1}{u}\log\frac{u}{x^2} & x^2 \le u \le x^3 \\ 0 & \text{otherwise} \end{cases}.$$

For $k_2(s)$ we have the following.

Lemma 4.2. *The function $k_2(s)$ has the inverse Mellin transform*

$$\hat{k}_2(u) = \frac{1}{\sqrt{4\pi\log x}}\exp\left[-\frac{(\log u/x)^2}{4\log x}\right].$$

Proof. Change of variables gives a Gaussian integral which can be evaluated to give the result. □

To prove Lemma 4.1 we will need the following

Lemma 4.3. *For $\alpha \ge 0$,*

$$\int_{(1)} \alpha^s\,\frac{ds}{s^2} = \begin{cases} 0 & 0 \le \alpha \le 1 \\ \log\alpha & \alpha > 1 \end{cases}.$$

Proof. Suppose $0 \le \alpha \le 1$. We claim that for any $\sigma > 1$, we have

$$\int_{(1)} \alpha^s\,\frac{ds}{s^2} = \int_{(\sigma)} \alpha^s\,\frac{ds}{s^2}.$$

Indeed, consider the rectangle with corners $1 \pm iT$ and $\sigma \pm iT$ oriented counterclockwise. Then since the integrand has no poles in the interior the integral around the rectangle vanishes. Over the horizontal edges we have

$$\left|\int_{1\pm iT}^{\sigma\pm iT} \alpha^s\,\frac{ds}{s^2}\right| \le \frac{\sigma-1}{1+T^2} \xrightarrow{T\to\infty} 0$$

which establishes the claim.

Now let $\gamma(t) = \sigma + it$ so that

$$\left| \int_{\sigma - iT}^{\sigma + iT} \alpha^s \frac{ds}{s^2} \right| = \left| \int_{-T}^{T} \alpha^{\gamma(t)} \frac{i\, dt}{\gamma(t)^2} \right|$$
$$\leq \int_{-T}^{T} \frac{dt}{|\gamma(t)|^2} = \int_{-T}^{T} \frac{dt}{\sigma^2 + t^2} = \frac{2}{\sigma} \arctan \frac{T}{\sigma}.$$

Therefore

$$\left| \int_{(1)} \alpha^s \frac{ds}{s^2} \right| = \left| \int_{(\sigma)} \alpha^s \frac{ds}{s^2} \right| \leq \frac{\pi}{\sigma} \xrightarrow{\sigma \to \infty} 0.$$

Next suppose that $\alpha > 1$. This time move the line of integration to the left, and consider a rectangle with vertices $1 \pm iT$ and $\sigma \pm iT$ for some $\sigma < 0$. Now there is a pole at $s = 0$ with residue $\log \alpha$ which then equals the value of the integral around the rectangle. As before the integral along the horizontal edges can be shown to vanish as $T \to \infty$ so

$$\frac{1}{2\pi i} \int_{(1)} \alpha^s \frac{ds}{s} = \log \alpha + \frac{1}{2\pi i} \int_{(\sigma)} \alpha^s \frac{ds}{s^2} \xrightarrow{\sigma \to -\infty} \log \alpha$$

where as before the integral along $\operatorname{Re} s = \sigma$ can be shown to vanish as $\sigma \to -\infty$. □

Proof of Lemma 4.1. Perform a change of variables and expand the integral to obtain

$$\hat{k}(u) = \frac{1}{2\pi i} \int_{(1)} \left(\frac{y^s - x^s}{s} \right)^2 u^{-s-1}\, ds$$
$$= \frac{1}{u} \left(\frac{1}{2\pi i} \int_{(1)} \left(\frac{y^2}{u} \right)^s \frac{ds}{s^2} - \frac{2}{2\pi i} \int_{(1)} \left(\frac{xy}{u} \right)^s \frac{ds}{s^2} + \frac{1}{2\pi i} \int_{(1)} \left(\frac{x^2}{u} \right)^s \frac{ds}{s^2} \right).$$

Using Lemma 4.3 to compute the integrals for each case proves the lemma. □

4.2 PRELIMINARY ESTIMATES

Let L/K be a Galois extension of number fields with Galois group $G = \mathrm{Gal}(L/K)$. For each irreducible character χ of G we have the Artin L-function

$$L(s, \chi) = L(s, \chi, L/K)$$

which are known to admit a meromorphic extension to the complex plane. The dependence on the field extension L/K will be suppressed for the sake of notation. We will assume Artin's conjecture which states that the

Artin L-functions corresponding to nontrivial irreducible characters are holomorphic.

Logarithmically differentiating the Euler product gives

$$\frac{L'}{L}(s,\chi) = -\sum_{\mathfrak{p}} \log(N\mathfrak{p}) \sum_{m=1}^{\infty} \frac{\chi(\sigma_{\mathfrak{p}}^m)}{N\mathfrak{p}^{ms}}$$

where $\sigma_{\mathfrak{p}}$ is the Frobenius above $\mathfrak{p}$ for $\mathfrak{p}$ unramified in L, and if $\mathfrak{p}$ is ramified it is defined by

$$\chi(\sigma_{\mathfrak{p}}^m) = \frac{1}{e} \sum_{\tau \mapsto \sigma_{\mathfrak{p}}^m} \chi(\tau)$$

where the sum is over all preimages of $\sigma_{\mathfrak{p}}^m$.

Let C be a conjugacy class in G and define

$$F_C(s) = -\frac{|C|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \overline{\chi(C)} \frac{L'}{L}(s,\chi).$$

Using the character orthogonality relations Equation 2.2 this is

$$F_C(s) = \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \theta(\mathfrak{p}^m)(\log N\mathfrak{p})(N\mathfrak{p})^{-ms}$$

where for $\mathfrak{p}$ unramified and $\sigma_{\mathfrak{p}}$ its Frobenius we have

$$\theta(\mathfrak{p}^m) = \begin{cases} 1 & \text{if } \sigma_{\mathfrak{p}}^m = C \\ 0 & \text{otherwise} \end{cases}$$

and for $\mathfrak{p}$ ramified we have $0 \leq \theta(\mathfrak{p}^m) \leq 1$.

Then

$$I_j := \frac{1}{2\pi i} \int_{(2)} F_C(s) k_j(s)\, ds = \sum_{\mathfrak{p}^m} \theta(\mathfrak{p}^m)(\log N\mathfrak{p}) \hat{k}_j(N\mathfrak{p}^m). \tag{4.1}$$

We will break this into three sums. The first will be over all $\mathfrak{p}$ which are unramified in L and of degree 1 over $\mathbb{Q}$, the second over the ramified primes, and the third over primes of higher degree.

Lemma 4.4. *Let $R(L/K)$ denote the set of primes of K which ramify in L and $P(L/K)$ the set of rational primes below primes of $R(L/K)$. Then*

$$\sum_{\mathfrak{p} \in R(L/K)} \sum_{m=1}^{\infty} \theta(\mathfrak{p}^m)(\log N\mathfrak{p}) \hat{k}_1(N\mathfrak{p}^m) \ll n_K \frac{\log x}{x^2} \sum_{p \in P(L/K)} \log p$$

and

$$\sum_{\mathfrak{p}\in R(L/K)} \sum_{m=1}^{\infty} \theta(\mathfrak{p}^m)(\log N\mathfrak{p})\hat{k}_2(N\mathfrak{p}^m) \ll n_K(\log x)^{1/2} \sum_{p\in P(L/K)} \log p.$$

Proof. This is [LMO79, Lemma 3.1] without the final step of bounding the contribution of the ramified primes by $\log d_L$ and noting that

$$\sum_{\mathfrak{p}|p} \log N\mathfrak{p} = \sum_{\mathfrak{p}|p} f_{\mathfrak{p}/p} \log p \leq n_K \log p.$$

□

Lemma 4.5. *The sum over prime powers $\mathfrak{p}^m$ where $N\mathfrak{p}^m$ is not a rational prime satisfies*

$$\sum_{\substack{\mathfrak{p}^m \\ N\mathfrak{p}^m \text{ not a rational prime}}} \theta(\mathfrak{p}^m)(\log N\mathfrak{p})\hat{k}_1(N\mathfrak{p}^m) \ll n_K \frac{(\log x)^2}{x}$$

and

$$\sum_{\substack{\mathfrak{p}^m \\ N\mathfrak{p}^m \text{ not a rational prime}}} \theta(\mathfrak{p}^m)(\log N\mathfrak{p})\hat{k}_2(N\mathfrak{p}^m) \ll n_K x^{7/4}.$$

Proof. This is [LMO79, Lemma 3.2]. □

For the kernel function k_2 we also require the following estimate over prime powers $\mathfrak{p}^m$ with large norm.

Lemma 4.6. *We have*

$$\sum_{N\mathfrak{p}^m \geq x^{10}} \theta(\mathfrak{p}^m)(\log N\mathfrak{p})\hat{k}_2(N\mathfrak{p}^m) \ll n_K x^{-10}.$$

Proof. This is [LMO79, Lemma 3.3]. □

4.3 THE CONTOUR INTEGRAL

We will now evaluate the integral I in Equation 4.1 by moving the line of integration to the left to get a sum over the zeros of the Artin L-functions in the critical strip. We first collect a few lemmas.

Lemma 4.7. *Let χ be a character of G and let $\bar{\chi}$ be the conjugate character. Then*

$$A_\chi = A_{\bar{\chi}}.$$

Proof. From the definition of A_χ and $\mathfrak{f}_\chi$ (Equation 3.2 and Equation 3.3) it suffices to show that $n(\chi, v) = n(\bar{\chi}, v)$ for each place v. Let V be the underlying space for χ. Since $\bar{\chi}$ is the character of the contragredient representation with underlying space V^* we have $\dim V = \dim V^*$. Thus it suffices to show that $\dim V^{G_i} = \dim (V^*)^{G_i}$ for each i.

From the character inner product (Equation 2.1) we see that

$$\dim V^{G_i} = \langle \operatorname{Res}^G_{G_i} \chi, 1_{G_i} \rangle = \frac{1}{|G_i|} \sum_{g \in G_i} \chi(g)$$

where 1_{G_i} denotes the trivial character on G_i. Taking the complex conjugate gives

$$\dim V^{G_i} = \overline{\dim V^{G_i}} = \frac{1}{|G_i|} \sum_{g \in G_i} \bar{\chi}(g) = \langle \bar{\chi}, 1_{G_i} \rangle = \dim (V^*)^{G_i}.$$

□

We now collect a few elementary estimates which are easy generalizations to Artin L-functions of Lemmas in [LO77] for $\zeta_L(s)$ and Hecke L-function estimates.

Lemma 4.8. *If $\sigma = \operatorname{Re} s > 1$ then*

$$\left| \frac{L'}{L}(s, \chi) \right| \ll \frac{\chi(1) n_K}{\sigma - 1}.$$

Proof. Comparing Dirichlet series gives

$$\left| \frac{L'}{L}(s, \chi) \right| \leq -\chi(1) \frac{\zeta_K'}{\zeta_K}(\sigma) \leq -\chi(1) n_K \frac{\zeta'}{\zeta}(\sigma) \ll \frac{\chi(1) n_K}{\sigma - 1}.$$

□

Lemma 4.9. *If $\sigma = \operatorname{Re} s > -1/2$ and $|s| \geq 1/8$ then*

$$\left| \frac{\mathbf{\Gamma}'}{\mathbf{\Gamma}}(s, \chi) \right| \ll \chi(1) n_K \log(|s| + 2).$$

Proof. From the definition of $L_v(s, \chi)$ for an infinite place v (Equation 3.1), we see that since $a + b = \chi(1)$ we have

$$\frac{L_v'}{L_v}(s, \chi) \ll \chi(1) \frac{\Gamma'}{\Gamma}(s) \ll \chi(1) \log(|s| + 2).$$

Taking the sum over all infinite places v gives the lemma. □

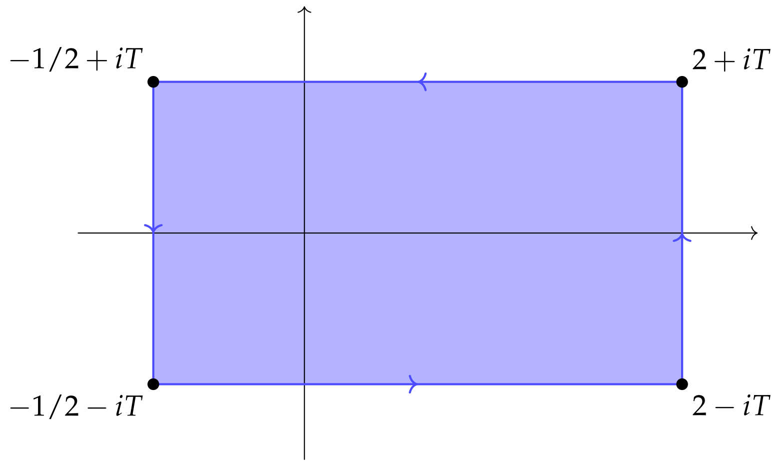


Figure 4.1: $B(T)$

Lemma 4.10. *If $s = \sigma + it$ with $\sigma \leq -1/4$ and $|s+m| \geq 1/4$ for all nonnegative integers m, then*

$$\frac{L'}{L}(s,\chi) \ll \log A_\chi + \chi(1) n_K \log(|s|+2).$$

Proof. Taking the logarithmic derivative of the functional equation (Equation 3.4 and Equation 3.5) and noting that $A_{\bar{\chi}} = A_\chi$ (Lemma 4.7) and $\gamma(s,\chi) = \gamma(s,\bar{\chi})$ gives

$$\frac{L'}{L}(s,\chi) = -\frac{L'}{L}(1-s,\bar{\chi}) - \log A_\chi - \frac{\Gamma'}{\Gamma}(1-s,\chi) - \frac{\Gamma'}{\Gamma}(s,\chi).$$

Using the estimates from Lemma 4.8 and Lemma 4.9 gives the result. □

Set

$$I_j(\chi) := \frac{1}{2\pi i}\int_{(2)} -\frac{L'}{L}(s,\chi) k_j(s)\, ds$$

and

$$I_j(\chi,T) := \frac{1}{2\pi i}\int_{B(T)} -\frac{L'}{L}(s,\chi) k_j(s)\, ds$$

where $B(T)$ is the positively oriented rectangle with vertices $2 \pm iT$ and $-1/2 \pm iT$, see Figure 4.1.

Lemma 4.11. *The vertical integral satisfies*

$$\left|\frac{1}{2\pi i}\int_{-1/2+iT}^{-1/2-iT} \frac{L'}{L}(s,\chi) k_j(s)\, ds\right| \ll k_j(-1/2)(\log A_\chi + \chi(1) n_K)$$

where the implicit constant is absolute and effective.

Proof. From Lemma 4.10 we have

$$\left|\frac{L'}{L}(s,\chi)\right| \ll \log A_\chi + \chi(1)n_K \log(|s|+2)$$

and from the definition of $k_1(s)$ we have

$$k_1(-1/2) \asymp x^{-3}$$

where x is the parameter for $k_1(s) = k_1(s;x)$. As well,

$$k_1(-1/2+it) \ll \frac{x^{-3}}{9/4+t^2} \ll \frac{k_1(-1/2)}{9/4+t^2}.$$

Therefore

$$\begin{aligned}
&\left|\int_{-1/2+iT}^{-1/2-iT} \frac{L'}{L}(s,\chi)k_1(s)\,ds\right| \\
&\qquad \ll \int_{-1/2-iT}^{-1/2+iT} \frac{x^{-3}(\log A_\chi + \chi(1)n_K \log(|s|+2))}{9/4+(\operatorname{Im} s)^2}\,ds \\
&\qquad \ll \int_{-T}^{T} \frac{k_1(-1/2)(\log A_\chi + \chi(1)n_K \log(|t|+2))}{9/4+t^2}\,dt \\
&\qquad \ll k_1(-1/2)(\log A_\chi + \chi(1)n_K).
\end{aligned}$$

Similarly, from the definition of $k_2(s)$ we have

$$k_2(-1/2+it) = x^{-1/4-t^2} = k_2(-1/2)x^{-t^2}.$$

Therefore

$$\begin{aligned}
&\left|\int_{-1/2+iT}^{-1/2-iT} \frac{L'}{L}(s,\chi)k_2(s)\,ds\right| \\
&\qquad \ll \int_{-T}^{T} k_2(-1/2)x^{-t^2}(\log A_\chi + \chi(1)n_K \log(|t|+2))\,dt \\
&\qquad \ll k_2(-1/2)(\log A_\chi + \chi(1)n_K).
\end{aligned}$$

□

Lemma 4.12. *The horizontal integrals satisfy*

$$\left|\frac{1}{2\pi i}\int_{-1/2\pm iT}^{2\pm iT} \frac{L'}{L}(s,\chi)k_1(s)\,ds\right| \ll \frac{x^4}{1+T^2}(\log A_\chi + \chi(1)n_K \log T)$$

and

$$\left| \frac{1}{2\pi i} \int_{-1/2 \pm iT}^{2 \pm iT} \frac{L'}{L}(s,\chi) k_2(s)\, ds \right| \quad \ll \quad x^{6-T^2} (\log A_\chi \; + \; \chi(1) n_K \log T)$$

where the implied constants are absolute and effective.

Proof. From Lemma 4.10 we have

$$\begin{aligned} \frac{L'}{L}(\sigma + it, \chi) &\ll \log A_\chi + \chi(1) n_K \log(|\sigma + it| + 2) \\ &\ll \log A_\chi + \chi(1) n_K (\log(|\sigma| + 2) + \log(|t|)). \end{aligned}$$

As well, for σ with $-1/2 \le \sigma \le 2$ we have

$$k_1(\sigma + iT) \ll \frac{x^4}{1 + T^2}$$

and

$$k_2(\sigma + iT) \ll x^{6-T^2}.$$

Therefore

$$\begin{aligned} &\left| \frac{1}{2\pi i} \int_{-1/2 \pm iT}^{2 \pm iT} \frac{L'}{L}(s,\chi) k_1(s)\, ds \right| \\ &\qquad \ll \int_{-1/2}^{2} \frac{x^4}{1+T^2} (\log A_\chi + \chi(1) n_K (\log(|\sigma| + 2) + \log T))\, d\sigma \\ &\qquad \ll \frac{x^4}{1+T^2} (\log A_\chi + \chi(1) n_K \log T) \end{aligned}$$

and

$$\begin{aligned} &\left| \frac{1}{2\pi i} \int_{-1/2 \pm iT}^{2 \pm iT} \frac{L'}{L}(s,\chi) k_2(s)\, ds \right| \\ &\qquad \ll \int_{-1/2}^{2} x^{6-T^2} (\log A_\chi + \chi(1) n_K (\log(|\sigma| + 2) + \log T))\, d\sigma \\ &\qquad \ll x^{6-T^2} (\log A_\chi + \chi(1) n_K \log T). \end{aligned}$$

□

In particular we see that the horizontal integrals vanish as we take $T \to \infty$.

Proposition 4.13. *Let $S(L/K)$ denote the set of primes $\mathfrak{p}$ of K that do not ramify in L and also have degree 1 over $\mathbb{Q}$. There exist absolute and effective positive constants c_1, c_2, c_3 such that*

$$\begin{aligned}\sum_{\mathfrak{p}\in S(L/K)} \theta(\mathfrak{p})(\log N\mathfrak{p})\hat{k}_1(N\mathfrak{p}) \geq \frac{|C|}{|G|}k_1(1) &- \frac{|C|}{|G|}\sum_{\chi\in \mathrm{Irr}(G)} |\chi(C)| \sum_{\rho_\chi} |k_1(\rho_\chi)| \\ &- c_1\frac{|C|}{|G|}\sum_{\chi\in \mathrm{Irr}(G)} |\chi(C)|(\log A_\chi + \chi(1)n_K) \\ &- c_2 n_K \frac{\log x}{x^2}\sum_{p\in P(L/K)} \log p - c_3 n_K \frac{(\log x)^2}{x}\end{aligned} \tag{4.2}$$

and constants c_4, c_5, c_6 such that

$$\begin{aligned}\sum_{\substack{\mathfrak{p}\in S(L/K)\\ N\mathfrak{p}<x^{10}}} \theta(\mathfrak{p})(\log N\mathfrak{p})\hat{k}_2(N\mathfrak{p}) \geq \frac{|C|}{|G|}k_2(1) &- \frac{|C|}{|G|}\sum_{\chi\in \mathrm{Irr}(G)} |\chi(C)| \sum_{\rho_\chi} |k_2(\rho_\chi)| \\ &- c_4\frac{|C|}{|G|}\sum_{\chi\in \mathrm{Irr}(G)} |\chi(C)|(\log A_\chi + \chi(1)n_K) \\ &- c_5 n_K (\log x)^{1/2}\sum_{p\in P(L/K)} \log p - c_6 n_K x^{7/4}\end{aligned} \tag{4.3}$$

where the inner sum in the second term of the right hand side in both equations is over all zeros ρ_χ of $L(s,\chi)$ with $-1/2 \leq \operatorname{Re}\rho_\chi \leq 1$.

Proof. Let $\delta(\chi) = 1$ if χ is the trivial character and $\delta(\chi) = 0$ otherwise. By Cauchy's theorem we have

$$I_j(\chi, T) = \frac{1}{2\pi i}\int_{B(T)} -\frac{L'}{L}(s,\chi)k_j(s)\,ds = \delta(\chi)k_j(1) - \sum_{\substack{-1/2\leq \operatorname{Re}\rho\leq 1\\ |\operatorname{Im}\rho|\leq T}} k_j(\rho)$$

where the sum is over zeros ρ of $L(s,\chi)$ lying inside $B(T)$. Thus

$$\begin{aligned}I_j(\chi) &= \lim_{T\to\infty} I_j(\chi, T) \\ &= \lim_{T\to\infty}\frac{1}{2\pi i}\left(\int_{2-iT}^{2+iT} -\frac{L'}{L}(s,\chi)k_j(s)\,ds + \int_{2+iT}^{-1/2+iT} -\frac{L'}{L}(s,\chi)k_j(s)\,ds\right. \\ &\quad\left. + \int_{-1/2+iT}^{-1/2-iT} -\frac{L'}{L}(s,\chi)k_j(s)\,ds + \int_{-1/2-iT}^{2+iT} -\frac{L'}{L}(s,\chi)k_j(s)\,ds\right) \\ &= \frac{1}{2\pi i}\int_{(2)} -\frac{L'}{L}(s,\chi)k_j(s)\,ds + O(k_j(-1/2)(\log A_\chi + \chi(1)n_K)).\end{aligned}$$

That is, there is an effective and absolute constant c_7 with

$$\frac{1}{2\pi i}\int_{(2)} -\frac{L'}{L}(s,\chi)k_j(s)\,dx \geq \delta(\chi)k_j(1) - \sum_{-1/2\leq \operatorname{Re}\rho_\chi\leq 1}|k_j(\rho_\chi)| \\ - c_7 k_j(-1/2)(\log A_\chi + \chi(1)n_K).$$

Multiplying by $(|C|/|G|)\overline{\chi(C)}$ and taking the sum over all χ we obtain

$$I_j = \frac{1}{2\pi i}\int_{(2)} F_C(s)k_j(s)\,ds \geq \frac{|C|}{|G|}k_j(1) - \sum_{\chi\in\operatorname{Irr}(G)}|\chi(C)|\sum_{\rho_\chi}|k_j(\rho_\chi)| \\ - c_7 k_j(-1/2)\frac{|C|}{|G|}\sum_{\chi\in\operatorname{Irr}(G)}|\chi(C)|(\log A_\chi + \chi(1)n_K). \quad (4.4)$$

From Lemma 4.4 and Lemma 4.5 we have

$$I_1 = \sum_{\mathfrak{p}\in S(L/K)}\theta(\mathfrak{p})(\log N\mathfrak{p})\hat{k}_1(N\mathfrak{p}) \\ + O\left(n_K\frac{\log x}{x^2}\sum_{p\in P(L/K)}\log p\right) + O\left(n_K\frac{(\log x)^2}{x}\right).$$

From Lemma 4.6 we also have

$$\sum_{N\mathfrak{p}^m\geq x^{10}}\theta(\mathfrak{p}^m)(\log N\mathfrak{p})\hat{k}_2(N\mathfrak{p}^m) \ll n_K x^{-10} \ll n_K x^{7/4}$$

so we obtain

$$I_2 = \sum_{\substack{\mathfrak{p}\in S(L/K)\\ N\mathfrak{p}<x^{10}}}\theta(\mathfrak{p})(\log N\mathfrak{p})\hat{k}_2(N\mathfrak{p}) \\ + O\left(n_K(\log x)^{1/2}\sum_{p\in P(L/K)}\log p\right) + O\left(n_K x^{7/4}\right)$$

where the implicit constants are effective and absolute. Lastly, a direct computation shows that since $x > 1$ we have $k_j(0), k_j(-1/2) \ll 1$. Combining with Equation 4.4 and rearranging the terms gives Equation 4.2.

□

4.4 CONTRIBUTION OF ZEROS

We now use the results of Section 3.5 and Section 3.4 to estimate the contribution in Equation 4.2 coming from zeros of L-functions. The first step is to show that the contribution from zeros far from 1 is small.

Lemma 4.14. *The sum over zeros ρ_χ with $|\rho_\chi - 1| \geq 1$ satisfies*

$$\sum_{|\rho_\chi - 1| \geq 1} |k_1(\rho_\chi)| \ll \log A_\chi + \chi(1) n_K.$$

Proof. Since $x > 1$, we have

$$|k_1(\rho_\chi)| = \frac{|x^{2(\rho_\chi - 1)} - x^{\rho_\chi - 1}|^2}{|\rho_\chi - 1|^2} \ll \frac{1}{|\rho_\chi - 1|^2}$$

and furthermore if $\rho_\chi = \beta + i\gamma$ then

$$\frac{1}{|\rho_\chi - 1|^2} = \frac{1}{(\beta - 1)^2 + \gamma^2} \leq \frac{1}{\gamma^2}.$$

Therefore using Lemma 3.4 and Lemma 3.5 we have

$$\begin{aligned}
\sum_{|\rho_\chi - 1| \geq 1} |k_1(\rho_\chi)| &\leq \sum_{T=1}^{\infty} \sum_{T \leq |\gamma| \leq T+1} |k_1(\rho_\chi)| + n_\chi(3/2, 1) \\
&\ll \sum_{T=1}^{\infty} \sum_{T \leq |\gamma| \leq T+1} \frac{1}{\gamma^2} + \log A_\chi \\
&\ll \sum_{T=1}^{\infty} \frac{1}{T^2} n_\chi(T) + \log A_\chi \\
&\ll \sum_{T=1}^{\infty} \frac{1}{T^2} (\log A_\chi + \chi(1) n_K \log(T+2)) + \log A_\chi \\
&\ll \log A_\chi + \chi(1) n_K.
\end{aligned}$$

□

Lemma 4.15. *The sum over zeros $\rho_\chi = \beta + i\gamma$ with $|\gamma| \geq 1$ satisfies*

$$\sum_{|\gamma| \geq 1} |k_2(\rho_\chi)| \ll x \log A_\chi.$$

Proof. For $\rho_\chi = \beta + i\gamma$ with $|\gamma| \geq 1$ we have

$$|k_2(\rho_\chi)| \leq x^{2-\gamma^2}$$

so using Lemma 3.4 we have

$$\begin{aligned}
\sum_{|\gamma|\geq 1} |k_2(\rho_\chi)| &\leq \sum_{|\gamma|\geq 1} x^{2-\gamma^2} \\
&= x^2 \sum_{T=1}^{\infty} \sum_{T-1\leq|\gamma|\leq T+1} x^{-\gamma^2} \\
&\leq x^2 \sum_{T=1}^{\infty} x^{-(T-1)^2} n_\chi(T) \\
&= x \log A_\chi + x \sum_{T=2}^{\infty} x^{-(T-1)^2} n_\chi(T) \\
&\ll x \log A_\chi \\
&\quad + x \int_2^\infty x^{-T^2+2T} (\log A_\chi + n_K \chi(1) \log(|T|+2))\, dT \\
&\ll x \log A_\chi.
\end{aligned}$$

□

Let us now consider the contribution from zeros close to 1, that is

$$\sum_{\chi\in\mathrm{Irr}(G)} |\chi(C)| \sum_{|\rho_\chi-1|\leq 1} |k_j(\rho_\chi)|.$$

Lemma 4.16. *Let L/K be a Galois extension of number fields of degree n and χ an irreducible character of* $\mathrm{Gal}(L/K)$.

If $\chi(1) > 1$ then

$$\sum_{|\rho_\chi-1|<1} |k_1(\rho_\chi)| \ll \chi(1)^7 (\log A_\chi)^2.$$

If $\chi(1) = 1$ then the same conclusion holds with the left-hand sum replaced by a sum over non-exceptional zeros:

$$\sum_{\substack{|\rho_\chi-1|<1 \\ \rho_\chi\neq\beta_0}} |k_1(\rho_\chi)| \ll (\log A_\chi)^2.$$

Proof. We may assume $x > 1$. Write each ρ_χ as $\rho_\chi = \sigma + it$. Since $|\rho_\chi - 1| < 1$ we can estimate each term as

$$|k_1(\rho_\chi)| = \frac{|x^{2(\rho_\chi-1)} - x^{\rho_\chi-1}|^2}{|\rho_\chi-1|^2} \ll \frac{x^{-2(1-\sigma)}}{|\rho_\chi-1|^2}.$$

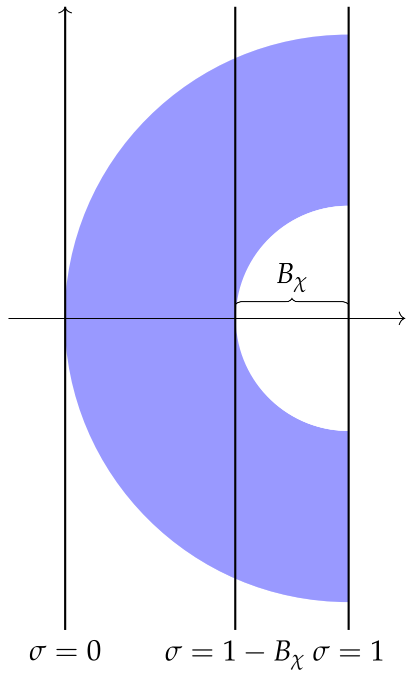


Figure 4.2: Region of integration

The number of such terms can be estimated using the zero-free region of Proposition 3.6. Set

$$B_\chi := \frac{c_8}{\chi(1)^3(\log A_\chi + n_K)}$$

where c_8 is some fixed absolute constant so that whenever $\chi(1) > 1$, any zero ρ_χ in the critical strip with $|\rho_\chi - 1| \le 1$ satisfies

$$\sigma < 1 - B_\chi$$

and if $\chi(1) = 1$ then there is at most one exception β_0. Therefore

$$\frac{x^{-2(1-\sigma)}}{|\rho_\chi - 1|^2} < \frac{x^{-2B_\chi}}{|\rho_\chi - 1|^2}$$

and hence by partial summation

$$\sum_{|\rho_\chi - 1| < 1} |k_1(\rho_\chi)| \ll x^{-2B_\chi} \sum_{|\rho_\chi - 1| < 1} \frac{1}{|\rho_\chi - 1|^2} \ll x^{-2B_\chi} \int_{B_\chi}^{1} \frac{1}{t^2} \, dn_\chi(t; 1).$$

We integrate over the region $B_\chi \leq |\rho_\chi - 1| \leq 1$ as shown in Figure 4.2 so in particular $n_\chi(B_\chi; 1) = 0$ if $L(s, \chi)$ has no exceptional zeros. From Lemma 3.5 we see that

$$n_\chi(t; 1) \ll \chi(1) + t(\log A_\chi + \chi(1) n_K)$$

and therefore

$$\begin{aligned}
\int_{B_\chi}^{1} \frac{1}{t^2}\, dn_\chi(t;1) &= \frac{n_\chi(t;1)}{t^2}\Big|_{B_\chi}^{1} + \int_{B_\chi}^{1} \frac{n_\chi(t;1)}{t^3}\, dt \\
&\ll \frac{n_\chi(t;1)}{t^2}\Big|_{B_\chi}^{1} + \int_{B_\chi}^{1} \frac{\chi(1) + t(\log A_\chi + \chi(1) n_K)}{t^3}\, dt \\
&\ll n_\chi(1;1) + \chi(1) \int_{B_\chi}^{1} \frac{dt}{t^3} + (\log A_\chi + \chi(1) n_K) \int_{B_\chi}^{1} \frac{dt}{t^2} \\
&\ll \frac{\chi(1)}{B_\chi^2} + \frac{\log A_\chi + \chi(1) n_K}{B_\chi} \\
&\ll \chi(1)^7 (\log A_\chi + \chi(1) n_K)^2 \\
&\ll \chi(1)^7 (\log A_\chi)^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{|\rho_\chi - 1| < 1} |k_1(\rho_\chi)| &\ll x^{-2B_\chi} \int_{B_\chi}^{1} \frac{1}{t^2}\, dn_\chi(t;1) \\
&\ll x^{-2B_\chi} \chi(1)^7 (\log A_\chi)^2 \\
&\ll \chi(1)^7 (\log A_\chi)^2
\end{aligned}$$

where we've used the fact that $x > 1$. □

Combining the results of this section we arrive at the following refinement of Equation 4.2.

Proposition 4.17. *There exist absolute and effective positive constants c_9, c_{10}, c_{11}, c_{12} such that*

$$\begin{aligned}\sum_{\mathfrak{p}\in S(L/K)} \theta(\mathfrak{p})(\log N\mathfrak{p})\hat{k}_1(N\mathfrak{p}) \geq{}& \frac{|C|}{|G|}k_1(1) - \frac{|C|}{|G|}k_1(\beta_0) \\ & - c_9\frac{|C|}{|G|}\sum_{\chi\in\mathrm{Irr}(G)} |\chi(C)|\chi(1)^7(\log A_\chi)^2 \\ & - c_{10}\frac{|C|}{|G|}\sum_{\chi\in\mathrm{Irr}(G)} |\chi(C)|(\log A_\chi + \chi(1)n_K) \\ & - c_{11}n_K\frac{\log x}{x^2}\sum_{p\in P(L/K)} \log p - c_{12}n_K\frac{(\log x)^2}{x}. \end{aligned} \tag{4.5}$$

where the $k_1(\beta_0)$ term appears only if the exceptional zero β_0 exists.

Proof. We break the term coming from the zeros ρ_χ into two parts: zeros with $|\rho_\chi - 1| \geq 1$ and zeros with $|\rho_\chi - 1| < 1$.

From Lemma 4.14 we have

$$\sum_{\chi\in\mathrm{Irr}(G)} |\chi(C)| \sum_{|\rho_\chi-1|\geq 1} |k_1(\rho_\chi)| \ll \sum_{\chi\in\mathrm{Irr}(G)} |\chi(C)|(\log A_\chi + \chi(1)n_K).$$

Therefore we may take c_1 sufficiently large in Equation 4.2 and combine the contribution from zeros ρ_χ with $|\rho_\chi - 1| \geq 1$ into the third term $\sum_\chi |\chi(C)|(\log A_\chi + \chi(1)n_K)$.

For contribution from zeros ρ_χ close to 1, Lemma 4.16 gives the estimate

$$\begin{aligned}\sum_{\chi\in\mathrm{Irr}(G)} |\chi(C)| \sum_{|\rho_\chi-1|<1} |k_1(\rho_\chi)| \\ = k_1(\beta_0) + O\left(\sum_{\chi\in\mathrm{Irr}(G)} |\chi(C)|\chi(1)^7(\log A_\chi)^2\right)\end{aligned}$$

where the term corresponding to the exceptional zero β_0 appears if and only if β_0 exists. Combining the estimates gives Equation 4.5.

□

4.5 DEURING-HEILBRONN PHENOMENON

In general zero-free regions for L-functions are only guaranteed to be free of zeros up to one possible exception. In particular Proposition 3.6 guarantees a true zero-free region for nonlinear characters.

As can be seen in Equation 4.5, the existence of an exceptional zero diminishes the contribution $k_j(1)$ from the pole at $s = 1$ in the main term. However as a general phenomenon the existence of a zero close to 1 will push the other zeros away, resulting in an enlarged zero-free region. This is typically called the *Deuring-Heilbronn phenomenon*. A version for ζ_L is given in [LMO79] whose statement we will recall here.

Theorem 4.18 (Lagarias-Montgomery-Odlyzko)**.** *There are effective positive absolute constants c_{13}, c_{14} such that if ζ_L has a real zero $\beta_0 > 0$ then it is zero-free with $s = \sigma + it$ in the region*

$$\sigma \geq 1 - c_{14} \frac{\log\left(\frac{c_{13}}{(1-\beta_0)\log(d_L(|t|+2)^{n_L})}\right)}{\log(d_L(|t|+2)^{n_L})}$$

with the single exception β_0.

Proof. This is [LMO79, Theorem 5.1]. □

Using this one obtains an unconditional lower bound for the exceptional zero as follows.

Corollary 4.19. *There is an effective positive absolute constant c_{15} such that any exceptional zero β_0 of $\zeta_L(s)$ satisfies*

$$1 - \beta_0 \geq d_L^{-c_{15}}.$$

Proof. This is [LMO79, Corollary 5.2]. □

In this section we exhibit a Deuring-Heilbronn phenomenon for Artin L-functions proven by Mahmoudian in [Mah99]. The proof follows that of [LMO79, Theorem 5.1] using the function $L(s, (1+\chi) \otimes (1+\bar{\chi}))$ instead of ζ_L. We recall the version of Turán's power sum theorem as stated in [LMO79].

Theorem 4.20 (Turán, Lagarias-Montgomery-Odlyzko)**.** *Let*

$$s_m = \sum_{n=1}^{\infty} b_n z_n^m$$

with

1. $|z_n| \leq |z_1|$ *for all* $n \geq 1$,
2. *the* b_n *are real, and*
3. $b_n \geq 0$ *for* n *with* $\frac{1}{3}|z_1| \leq |z_n| \leq |z_1|$.

Set

$$L = \frac{1}{b_1|z_1|} \sum_{n=1}^{\infty} |b_n z_n|.$$

Then there exists j_0 with $1 \le j_0 \le 24L$ such that

$$\operatorname{Re} s_{j_0} \ge \frac{b_1}{8}|z_1|^{j_0}.$$

Proof. This is [LMO79, Theorem 4.2]. □

Theorem 4.21 (Mahmoudian)**.** *Let χ be a nontrivial irreducible character of G and suppose β_0 is a real zero of $L(s,\chi)$. Assume Artin's conjecture for $L(s,\chi)$ and $L(s,\chi\otimes\bar{\chi})$. Then there exist effective and absolute constants c_{16}, c_{17} such that $L(s,\chi)$ has no other zeros in the region*

$$1 - c_{17}\frac{\log\left(\frac{c_{16}}{L_\chi(t)(1-\beta_0)}\right)}{L_\chi(t)} < \sigma < 1$$

where

$$L_\chi(t) = \chi(1)(\log A_\chi + \chi(1) n_K \log(|t|+2)).$$

Proof. This is [Mah99, Theorem 3.1]. □

As in Corollary 4.19 we obtain a lower bound for how close the exceptional zero can be to 1.

Corollary 4.22. *There exists an absolute effective positive constant c_{18} such that any exceptional zero β_0 for $L(s,\chi)$ satisfies*

$$1 - \beta_0 \ll A_\chi^{-c_{18}}.$$

4.6 PROOFS OF MAIN RESULTS

We now use the estimates of the previous sections to prove the main results. First we will prove Theorem 1.4 which will allow us to handle the case of exceptional zeros when the nonvanishing characters satisfy a nonnegativity condition. We then use Theorem 1.4 to conclude the proof of Theorem 1.1.

4.6.1 *Proof of Theorem 1.4*

We see that the key step in the proof of Theorem 4.21 is to note that the Dirichlet series of the function $F_\chi(s)$ has nonnegative coefficients. Hence we may adapt the proof to prove Theorem 1.4.

Proof of Theorem 1.4. Let φ be a (not necessarily irreducible) character of G which decompose into a sum of irreducible characters as

$$\varphi = \sum_{i=1}^{m} a_i \chi_i.$$

Set $F(s) = L(s, \varphi) = L(s, \sum_{i=1}^{m} a_i \chi_i)$. By the functoriality of L-functions

$$F(s) = \prod_{i=1}^{m} L(s, \chi_i)^{a_i}.$$

The order of the pole at $s = 1$ is given by $\langle \varphi, 1_G \rangle$ where 1_G denotes the trivial character. Hence $(s-1)^{\langle \varphi, 1_G \rangle} F(s)$ is entire and has a Hadamard product factorization

$$(s-1)^{\langle \varphi, 1_G \rangle} F(s) = s^r e^{\alpha_1 + \alpha_2 s} \prod_{\substack{\omega \neq 0 \\ F_\chi(\omega)=0}} \left(1 - \frac{s}{\omega}\right) e^{s/\omega} \tag{4.6}$$

where the ω run over the zeros of $F(s)$, possibly with multiplicity and the dependence of the quantities r, α_1, α_2 on φ will be suppressed for sake of notation.

Logarithmically differentiating both sides of Equation 4.6 gives

$$\frac{\langle \varphi, 1_G \rangle}{s-1} + \frac{F'}{F}(s) = \frac{r}{s} + \alpha_2 + \sum_{\omega \neq 0} \left(\frac{1}{s-\omega} + \frac{1}{\omega} \right)$$

hence

$$-\frac{F'}{F}(s) = \frac{\langle \varphi, 1_G \rangle}{s-1} - \frac{r}{s} - \alpha_2 - \sum_{\omega \neq 0} \left(\frac{1}{s-\omega} + \frac{1}{\omega} \right).$$

On the other hand, we know from logarithmically differentiating the Euler product that

$$-\frac{F'}{F}(s) = \sum_{\mathfrak{p}^m} \frac{\varphi(\mathfrak{p}^m)(\log N\mathfrak{p})}{(N\mathfrak{p})^{ms}}$$

so

$$\sum_{\mathfrak{p}^m} \varphi(\mathfrak{p}^m)(\log N\mathfrak{p})(N\mathfrak{p})^{-ms} = \frac{r}{s} + \alpha_2 + \sum_{\omega} \left(\frac{1}{s-\omega} + \frac{1}{\omega} \right). \tag{4.7}$$

Differentiating both sides of Equation 4.7 $2j-1$ times gives

$$\frac{1}{(2j-1)!}\sum_{\mathfrak{p}^m}\varphi(\mathfrak{p}^m)(\log N\mathfrak{p})(\log N\mathfrak{p}^m)^{2j-1}(N\mathfrak{p}^m)^{-s} = \frac{\langle\varphi, 1_G\rangle}{(s-1)^{2j}} - \sum_{\omega}\frac{1}{(s-\omega)^{2j}} \tag{4.8}$$

which is valid for $\sigma > 1$ and $j \geq 1$. Evaluating Equation 4.8 at $s = \sigma$ and $s = \sigma + it$ and adding the result gives

$$\frac{1}{(2j-1)!}\sum_{\mathfrak{p}^m}\varphi(\mathfrak{p}^m)(\log N\mathfrak{p})(\log N\mathfrak{p}^m)^{2j-1}(N\mathfrak{p}^m)^{-\sigma}(1+(N\mathfrak{p}^m)^{-it}) = \frac{\langle\varphi, 1_G\rangle}{(\sigma-1)^{2j}} + \frac{\langle\varphi, 1_G\rangle}{(s-1)^{2j}} - \sum_{\omega}\left(\frac{1}{(\sigma-\omega)^{2j}} + \frac{1}{(s-\omega)^{2j}}\right).$$

Let β_0 be a real zero of $F(s)$. Separating the contribution from β_0 in Equation 4.8 we have

$$\frac{1}{(2j-1)!}\sum_{\mathfrak{p}^m}\varphi(\mathfrak{p}^m)(\log N\mathfrak{p})(\log N\mathfrak{p}^m)^{2j-1}(N\mathfrak{p}^m)^{-\sigma}(1+(N\mathfrak{p}^m)^{-it}) = \frac{\langle\varphi, 1_G\rangle}{(\sigma-1)^{2j}} + \frac{\langle\varphi, 1_G\rangle}{(s-1)^{2j}} - \frac{1}{(\sigma-\beta_0)^{2j}} - \frac{1}{(s-\beta_0)^{2j}} - \sum_n z_n^j$$

where each z_n is of the form $(\sigma-\omega)^{-2}$ or $(s-\omega)^{-2}$ with ω a zero of F.

Observe that the left-hand side has nonnegative real part since $\varphi(\mathfrak{p}^m)$ is nonnegative for each $\mathfrak{p}^m$ by assumption. Setting $\sigma = 2$, we obtain

$$\begin{aligned}\operatorname{Re}\sum z_n^j &\leq \langle\varphi, 1_G\rangle - \frac{1}{(2-\beta_0)^{2j}} + \operatorname{Re}\left(\frac{\langle\varphi, 1_G\rangle}{(1+it)^{2j}} - \frac{2}{(2-\beta_0+it)^2}\right)\\ &\leq c_{19}j\langle\varphi, 1_G\rangle(1-\beta_0)\end{aligned} \tag{4.9}$$

where c_{19} is an absolute and effective constant and the second inequality follows from [LMO79, Equation 5.5].

Suppose $\rho = \beta + i\gamma \neq \beta_0$ is a zero of F. Evaluate Equation 4.9 at $t = \gamma$.

Suppose z_1 has the largest absolute value of all the z_n. Then

$$|z_1| \geq \frac{1}{(2-\beta)^2}$$

since if $s = 2 + i\gamma$ then $(s-\rho)^{-2} = (2-\beta)^{-2}$.

As well, if $\rho' = \beta' + i\gamma'$ is another zero then

$$\begin{aligned}|(s-\rho')^{-2}| &= |(2-\beta+\beta-\beta'+i(\gamma-\gamma'))^{-2}| \\ &= \frac{1}{(2-\beta+\beta-\beta')^2+(\gamma-\gamma')^2} \\ &\le \frac{1}{(2-\beta)^2} \\ &= \frac{1}{|s-\rho|^2}\end{aligned}$$

and similarly

$$|(\sigma-\rho')^{-2}| \le |s-\rho|^{-2}.$$

Set

$$L = \frac{1}{|z_1|}\sum_{n=1}^{\infty}|z_n|$$

and

$$s_m = \sum_{n=1}^{\infty} z_n^m.$$

Applying the Turán power sum theorem Theorem 4.20, there exists j_0, $1 \le j_0 \le 24L$, with

$$\operatorname{Re} s_{j_0} \ge \frac{1}{8}|z_1|^{j_0}$$

and therefore

$$L \ll \sum_{\omega}\left(\frac{1}{|z-\omega|^2}+\frac{1}{|2+i\gamma-\omega|^2}\right)$$

where the $(2-\beta)^2$ has been absorbed in the implicit constant as it is at most 4.

Applying partial summation and Lemma 3.4, we obtain

$$\begin{aligned}L &\ll \int_0^{\infty}\frac{1}{u^2+1}\,dn_F(u)+\int_0^{\infty}\frac{1}{u^2+1}\,dn_F(u+\gamma) \\ &\ll \sum_{i=1}^{m} a_i\chi_i(1)(\log A_\chi+\chi_i(1)n_K\log(|\gamma|+2)) = \mathcal{L}(|\gamma|).\end{aligned}$$

Hence for some $j_0 \ll L \ll \mathcal{L}(|\gamma|)$ we have

$$\begin{aligned}\operatorname{Re}\sum_{n=1}^{\infty} z_n^{j_0} &\ge \frac{1}{8}(2-\beta)^{-2j_0} \\ &= \frac{1}{8}\exp\left(-2j_0\log(1+1-\beta)\right) \\ &\ge \frac{1}{8}\exp\left(-2j_0(1-\beta)\right).\end{aligned}$$

Putting the inequalities together we obtain

$$\begin{aligned}\frac{1}{8}\exp\left(-2j_0(1-\beta)\right) &\le \mathrm{Re}\sum_{n=1}^{\infty} z_n^{j_0} \\ &\le c_{19} j_0 \langle \varphi, 1_G\rangle (1-\beta_0) \\ &\ll \langle \varphi, 1_G\rangle \mathcal{L}(|\gamma|)(1-\beta_0)\end{aligned}$$

from which we deduce that there are absolute positive constants c_{20}, c_{21} with

$$\beta \le 1 - c_{21}\frac{\log\left(\frac{c_{20}}{\langle\varphi,1_G\rangle\mathcal{L}(|\gamma|)(1-\beta_0)}\right)}{\mathcal{L}(|\gamma|)}$$

as required. $\square$

Remark 4.23. Notice that for the character of the regular representation of any finite G one has

$$\mathrm{reg}_G(g) = \sum_{\chi\in\mathrm{Irr}(G)} \chi(1)\chi(g) = \begin{cases} |G| & a = 1 \\ 0 & a \neq 0 \end{cases}$$

so taking $\varphi = \mathrm{reg}_G$ for $G = \mathrm{Gal}(L/K)$ satisfies the hypotheses of Theorem 1.4. In fact applying Theorem 1.4 with $\varphi = \mathrm{reg}_G$ recovers Theorem 4.18 since

$$\sum_{\chi\in\mathrm{Irr}(G)} \chi(1)(\log A_\chi + n_K\chi(1)\log(|t|+2)) = \log(d_L \log(|t|+2)^{n_L}).$$

As before we obtain a bound on the exceptional zero similar to Corollary 4.19 and Corollary 4.22.

Corollary 4.24. *Let $L(s,\varphi)$ satisfy the hypothesis of Theorem 1.4. Then there is a positive, absolute, effective constant c_{22} such that any real zero β_0 of $L(s,\varphi)$ satisfies*

$$1-\beta_0 \ge \left(\prod_{i=1}^{n} A_{\chi_i}^{a_i\chi_i(1)}\right)^{-c_{22}} \tag{4.10}$$

Proof. Choose c_{22} large enough so that

$$c_{21}\log\left(\frac{c_{20}\left(\prod_i A_{\chi_i}^{a_i\chi_i(1)}\right)^{c_{22}}}{\langle\varphi,1_G\rangle\sum_i a_i\chi_i(1)\log A_{\chi_i}}\right) > 3\sum_{i=1}^{n} a_i\chi_i(1)\log A_{\chi_i}. \tag{4.11}$$

Then if Equation 4.10 fails, substituting Equation 4.11 into Theorem 1.4 gives that $L(s,\varphi)$ has no zeros for $\sigma > -2$ except at β_0. But this is a contradiction since there are trivial zeros at $s=-1$ or $s=0$.

□

4.6.2 *Proof of Theorem 1.1*

We now use the results of Section 4.5 to give a refinement of Proposition 4.13 which will be used in the next chapter to give application to specific choices of G and C. The main strategy is to observe that in Proposition 4.13 the only L-functions that has nonzero contribution comes from characters which are nonvanishing at C, that is $\chi \in \mathrm{nv}(C)$ (see Definition 2.4). Hence we obtain a saving when $\mathrm{nv}(C)$ is small compared to $\mathrm{Irr}(G)$.

Proposition 4.25. *Let L/K be a Galois extension of number fields with $G = \mathrm{Gal}(L/K)$. Let C be a conjugacy class of G. Denote by $\mathrm{nv}(C)$ the set of characters which are nonvanishing at C, as in Definition 2.4. Let k_j, θ, and $S(L/K)$ be defined as before in Section 4.1 and Section 4.2.*

Suppose Artin's holomorphy conjecture holds for each $L(s,\chi)$ with $\chi \in \mathrm{nv}(C)$. Then

1. *If $\prod_{\chi\in\mathrm{nv}(C)} L(s,\chi)$ has no exceptional zero, then there are effective constants $c_{23}, c_{24}, c_{25}, c_{26}$ such that*

$$\begin{aligned}\sum_{\mathfrak{p}\in S(L/K)} &\theta(\mathfrak{p})(\log N\mathfrak{p})\hat{k}_1(N\mathfrak{p}) \\ &\geq \frac{|C|}{|G|}(\log x)^2 - c_{23}\frac{|C|}{|G|}\sum_{\chi\in\mathrm{nv}(C)} |\chi(C)|\chi(1)^7(\log A_\chi)^2 \\ &\quad - c_{24}\frac{|C|}{|G|}\sum_{\chi\in\mathrm{nv}(C)} |\chi(C)|\log A_\chi \\ &\quad - c_{25}n_K\frac{\log x}{x^2}\sum_{p\in P(L/K)}\log p - c_{26}n_K\frac{(\log x)^2}{x}.\end{aligned} \tag{4.12}$$

2. *If $\prod_{\chi\in\mathrm{nv}(C)} L(s,\chi)$ has an exceptional zero β_0 assume furthermore that*

$$\sum_{\chi\in\mathrm{nv}(C)} \chi(g) \geq 0 \tag{4.13}$$

for each $g \in G$. Set

$$\mathcal{L} := \sum_{\chi\in\mathrm{nv}(C)} \chi(1)\log A_\chi.$$

Then there are effective absolute constants $c_{27}, c_{28}, c_{29}, c_{30}, c_{31}, c_{32}$ such that

$$\begin{aligned}
\sum_{\mathfrak{p} \in S(L/K)} & \theta(\mathfrak{p})(\log N\mathfrak{p})\hat{k}_1(N\mathfrak{p}) \\
& \geq \frac{|C|}{|G|}(\log x)^2 \min\{1, (1-\beta_0)\log x\} \\
& \quad - c_{27}\frac{|C|}{|G|}\left(\sum_{\chi \in \mathrm{nv}(C)} |\chi(C)|\chi(1)\right) \\
& \qquad \cdot \mathcal{L}^2[(1/c_{28})(1-\beta_0)\mathcal{L}]^{2c_{29}\log x/\mathcal{L}} \\
& \quad - c_{30}\frac{|C|}{|G|}\sum_{\chi \in \mathrm{nv}(C)} |\chi(C)| \log A_\chi \\
& \quad - c_{31} n_K \frac{\log x}{x^2} \sum_{p \in P(L/K)} \log p - c_{32} n_K \frac{(\log x)^2}{x}.
\end{aligned} \tag{4.14}$$

Furthermore we may take $c_{28} > 1$.

3. *Suppose furthermore that the exceptional zero satisfies*

$$1 - \beta_0 \ll \frac{1}{\mathcal{L}^2}$$

for an effective and absolute implicit constant. Then there are effective absolute constants $c_{33}, c_{34}, c_{35}, c_{36}, c_{37}$ such that

$$\begin{aligned}
\sum_{\substack{\mathfrak{p} \in S(L/K) \\ N\mathfrak{p} < x^{10}}} & \theta(\mathfrak{p})(\log N\mathfrak{p})\hat{k}_2(N\mathfrak{p}) \\
& \geq \frac{|C|}{|G|}x^2 \log x(1-\beta_0) - c_{38}\frac{|C|}{|G|}x^2(1-\beta_0)^{c_{34}\log x/\mathcal{L}}\mathcal{L} \\
& \quad - c_{39}\frac{|C|}{|G|}x\mathcal{L} - c_{40} n_K (\log x)^{1/2} \sum_{p \in P(L/K)} \log p \\
& \quad - c_{41} n_K x^{7/4}
\end{aligned} \tag{4.15}$$

Proof. We first observe that $k_1(1) = (\log x)^2$ and furthermore that

$$\begin{aligned}
k_1(1) - k_1(\beta_0) &= (\log x)^2 - \left(\frac{x^{2(\beta_0 - 1)} - x^{\beta_0 - 1}}{\beta_0 - 1}\right)^2 \\
&\gg (\log x)^2 \min\{1, (1-\beta_0)\log x\}
\end{aligned}$$

and

$$k_2(1) - k_2(\beta_0) = x^2 - x^{\beta_0+\beta_0^2} \gg x^2 \min\{1, (1-\beta_0)\log x\}.$$

Since $\log A_\chi \gg \chi(1)n_K$ we see easily that

$$\sum_{\chi\in\mathrm{Irr}(G)} |\chi(C)|(\log A_\chi + \chi(1)n_K) \ll \sum_{\chi\in\mathrm{nv}(C)} |\chi(C)|\log A_\chi. \tag{4.16}$$

As well, from Lemma 4.16 we have

$$\sum_{\chi\in\mathrm{Irr}(G)} |\chi(C)| \sum_{\substack{|\rho_\chi-1|<1 \\ \rho_\chi\neq\beta_0}} |k_1(\rho_\chi)| \ll \sum_{\chi\in\mathrm{nv}(C)} |\chi(C)|\chi(1)^7(\log A_\chi)^2.$$

Combining with Equation 4.2 yields Equation 4.12

Now suppose that there is an exceptional zero β_0 of $\prod_{\chi\in\mathrm{nv}(C)} L(s,\chi)$. By Equation 4.13 in the hypothesis we may use Theorem 1.4 with

$$\varphi = \sum_{\chi\in\mathrm{nv}(C)} \chi.$$

We note that

$$\mathcal{L}(t) \ll \mathcal{L}(1) \ll \mathcal{L} := \sum_{\chi\in\mathrm{nv}(C)} \chi(1)\log A_\chi$$

and so by Theorem 1.4 any nonexceptional zero $\rho = \beta + it$ of

$$L(s,\varphi) = \prod_{\chi\in\mathrm{nv}(C)} L(s,\chi)$$

with $|\rho - 1| < 1$ satisfies

$$\beta \le 1 - c_{29}\frac{\log\left(\frac{c_{28}}{\mathcal{L}(1-\beta_0)}\right)}{\mathcal{L}}.$$

By taking c_{29} smaller if necessary we may take $c_{28} \ge 1$.

Set

$$\mathcal{B} := c_{29}\frac{\log\left(\frac{c_{28}}{\mathcal{L}(1-\beta_0)}\right)}{\mathcal{L}}$$

so that any zero $\rho = \sigma + i\gamma \neq \beta_0$ satisfies

$$\sigma \le 1 - \mathcal{B}.$$

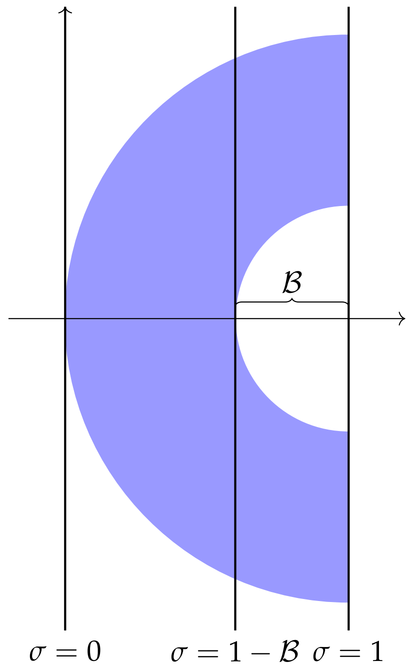


Figure 4.3: Region of integration

Thus

$$|k_1(\rho)| = \left| \frac{x^{2(\rho-1)} - x^{\rho-1}}{\rho - 1} \right|^2 \leq x^{-2\mathcal{B}} \frac{1}{|\rho - 1|^2}.$$

Furthermore such zeros with $|\rho - 1| < 1$ will lie in the shaded region of Figure 4.3 to the right of the line $\sigma = 1 - \mathcal{B}$. Thus if $t = |\rho - 1|$ then in particular $\mathcal{B} \leq t \leq 1$ and so

$$\sum_{\chi \in \mathrm{nv}(C)} |\chi(C)| \sum_{\substack{|\rho_\chi - 1| < 1 \\ \rho_\chi \neq \beta_0}} |k_1(\rho_\chi)| \leq x^{-2\mathcal{B}} \sum_{\chi \in \mathrm{nv}(C)} |\chi(C)| \int_{\mathcal{B}}^1 \frac{1}{t^2} dn_\chi(t; 1).$$

Then using Lemma 3.5 we have

$$\begin{aligned}
\sum_{\chi \in \mathrm{nv}(C)} |\chi(C)| \int_{\mathcal{B}}^{1} \frac{1}{t^2} dn_\chi(t;1) \\
&\ll \sum_{\chi \in \mathrm{nv}(C)} |\chi(C)| \left(\frac{n_\chi(t;1)}{t^2}\bigg|_{\mathcal{B}}^{1} + \int_{\mathcal{B}}^{1} \frac{n_\chi(t;1)}{t^3}\, dt \right) \\
&\ll \sum_{\chi \in \mathrm{nv}(C)} |\chi(C)| \left(\frac{\chi(1)}{\mathcal{B}^2} + \frac{\log A_\chi}{\mathcal{B}} \right) \\
&\ll \left(\sum_{\chi \in \mathrm{nv}(C)} |\chi(C)|\chi(1) \right) \left(\frac{1}{\mathcal{B}^2} + \frac{\mathcal{L}}{\mathcal{B}} \right) \\
&\ll \left(\sum_{\chi \in \mathrm{nv}(C)} |\chi(C)|\chi(1) \right) \frac{\mathcal{L}}{\mathcal{B}}.
\end{aligned}$$

Since $\mathcal{B}^{-1} \ll \mathcal{L}$ we have

$$\begin{aligned}
\sum_{\chi \in \mathrm{nv}(C)} |\chi(C)| \sum_{\substack{|\rho_\chi - 1| < 1 \\ \rho_\chi \neq \beta_0}} |k_1(\rho_\chi)| \\
&\ll \left(\sum_{\chi \in \mathrm{nv}(C)} |\chi(C)|\chi(1) \right) x^{-2\mathcal{B}} \mathcal{B}^{-1} \mathcal{L} \\
&\ll \left(\sum_{\chi \in \mathrm{nv}(C)} |\chi(C)|\chi(1) \right) \mathcal{L}^2 \left[(1/c_{28})(1-\beta_0)\mathcal{L} \right]^{2c_{29} \log x/\mathcal{L}}.
\end{aligned}$$

Once again combining with Equation 4.2 gives Equation 4.14.

Now suppose that

$$1 - \beta_0 \le \frac{{c_{42}}^2}{\mathcal{L}^2} \tag{4.17}$$

for an absolute effective c_{42} which we choose to satisfy

$$c_{42}^2 < c_{28}$$

so that we have

$$\frac{1}{2} \log(1-\beta_0)^{-1} \le \log \frac{c_{42}}{(1-\beta_0)\mathcal{L}}.$$

Therefore by Theorem 1.4 there is an absolute constant c_{34} such that if $\rho = \beta + i\gamma$ is any zero with $|\gamma| \le 1$ and $\rho \neq \beta_0$ then

$$\beta \le 1 - c_{34} \frac{\log(1-\beta_0)^{-1}}{\mathcal{L}}.$$

Now

$$|k_2(\rho)| \le x^{\beta^2+\beta} \le x^{1+\beta} = x^2 x^{\beta-1}$$

and so

$$|k_2(\beta)| \le x^2 \exp\left(-c_{34}\frac{\log x \log(1-\beta_0)^{-1}}{\mathcal{L}}\right) = x^2(1-\beta_0)^{c_{34}\log x/\mathcal{L}}.$$

Using Lemma 3.4 we obtain

$$\sum_{\chi\in\mathrm{nv}(C)} |\chi(C)| \sum_{|\gamma|\le 1} |k_2(\rho_\chi)| \ll x^2(1-\beta_0)^{c_{34}\log x/\mathcal{L}} \sum_{\chi\in\mathrm{nv}(C)} |\chi(C)| \log A_\chi.$$

Combining the above with Equation 4.3, Equation 4.16, and Lemma 4.15 we obtain

$$\begin{aligned}
&\sum_{\substack{\mathfrak{p}\in S(L/K)\\ N\mathfrak{p}<x^{10}}} \theta(\mathfrak{p})(\log N\mathfrak{p})\hat{k}_2(N\mathfrak{p}) \\
&\qquad \ge \frac{|C|}{|G|}x^2\log x(1-\beta_0) \\
&\qquad\quad - c_{43}\frac{|C|}{|G|}x^2(1-\beta_0)^{c_{16}\log x/\mathcal{L}} \sum_{\chi\in\mathrm{nv}(C)} |\chi(C)|\log A_\chi \\
&\qquad\quad - c_{44}\frac{|C|}{|G|}x \sum_{\chi\in\mathrm{nv}(C)} |\chi(C)|\log A_\chi \\
&\qquad\quad - c_{45}\frac{|C|}{|G|} \sum_{\chi\in\mathrm{nv}(C)} |\chi(C)|\log A_\chi \\
&\qquad\quad - c_{46}n_K(\log x)^{1/2} \sum_{p\in P(L/K)} \log p - c_{47}n_K x^{7/4} \\
&\qquad \ge \frac{|C|}{|G|}x^2\log x(1-\beta_0) - c_{43}\frac{|C|}{|G|}x^2(1-\beta_0)^{c_{16}\log x/\mathcal{L}}\mathcal{L} \\
&\qquad\quad - c_{48}\frac{|C|}{|G|}x\mathcal{L} - c_{46}n_K(\log x)^{1/2} \sum_{p\in P(L/K)} \log p - c_{47}n_K x^{7/4}
\end{aligned}$$

□

Proof of Theorem 1.1. We see that trivially

$$\sum_{\chi\in\mathrm{nv}(C)} |\chi(C)|\chi(1)^7(\log A_\chi)^2 \le \sum_{\chi\in\mathrm{nv}(C)} |\chi(C)|\chi(1)^5 \left(\sum_{\chi\in\mathrm{nv}(C)} \chi(1)\log A_\chi\right)^2$$

so we see that the left-hand side of Equation 4.12 is positive if we set

$$\log x \gg \left(\sum_{\chi\in \mathrm{nv}(C)} |\chi(C)|\chi(1)^5\right)^{1/2} \sum_{\chi\in\mathrm{nv}(C)} \chi(1)\log A_\chi + \log\left(\frac{|G|}{|C|} n_K \sum_{p\in P(L/K)} \log p\right).$$

Suppose there is an exceptional zero β_0 which satisfies

$$1-\beta_0 \geq \frac{c_{42}^2}{\left(\sum_{\chi\in\mathrm{nv}(C)}\chi(1)\log A_\chi\right)^2}.$$

with c_{42} as in Equation 4.17. Then noting that $c_{42}^2/c_{28} < 1$ by choice of c_{28}, c_{42} we have

$$\begin{aligned}
&\left(\sum_{\chi\in\mathrm{nv}(C)} |\chi(C)|\chi(1)\right)\mathcal{L}^2[(1/c_{28})(1-\beta_0)\mathcal{L}]^{2c_{29}\log x/\mathcal{L}}\\
&\qquad\leq \left(\sum_{\chi\in\mathrm{nv}(C)} |\chi(C)|\chi(1)\right)\mathcal{L}^2[(c_{42}^2/c_{28})\mathcal{L}^{-1}]^{2c_{29}\log x/\mathcal{L}}\\
&\qquad\leq \left(\sum_{\chi\in\mathrm{nv}(C)} \chi(1)^2\right)\mathcal{L}^{2(1-c_{29}\log x/\mathcal{L})}\\
&\qquad\leq \frac{\mathcal{L}^3}{\mathcal{L}^{2c_{29}\log x/\mathcal{L}}}.
\end{aligned}$$

Then any choice of x satisfying

$$c_{29}\log x \geq \mathcal{L}$$

will make

$$\left(\sum_{\chi\in\mathrm{nv}(C)} |\chi(C)|\chi(1)\right)\mathcal{L}^2[(1/c_{28})(1-\beta_0)\mathcal{L}]^{2c_{29}\log x/\mathcal{L}} \leq \mathcal{L}.$$

Then noting that

$$\sum_{\chi\in\mathrm{nv}(C)} |\chi(C)|\log A_\chi \leq \mathcal{L}$$

we have that for any such choice of x Equation 4.14 becomes

$$\begin{aligned}\frac{|G|}{|C|}&\sum_{\mathfrak{p}\in S(L/K)}\theta(\mathfrak{p})(\log N\mathfrak{p})\hat{k}_1(N\mathfrak{p})\\ &\geq c_{49}(\log x)^3\mathcal{L}^{-2}-c_{50}\mathcal{L}-c_{51}\frac{|G|}{|C|}n_K\frac{\log x}{x^2}\sum_{p\in P(L/K)}\log p\\ &\quad -c_{52}\frac{|G|}{|C|}n_K\frac{(\log x)^2}{x}.\end{aligned}$$

Then from Equation 4.14 we see that we can take

$$\log x \gg \sum_{\chi\in\mathrm{nv}(C)}\chi(1)\log A_\chi+\log\left(\frac{|G|}{|C|}n_K\sum_{p\in P(L/K)}\log p\right).$$

Lastly, if

$$1-\beta_0\leq\frac{c_{42}^2}{\left(\sum_{\chi\in\mathrm{nv}(C)}\chi(1)\log A_\chi\right)^2}$$

then we use Equation 4.15. From Corollary 4.24 we have that $1-\beta_0\geq\exp(-c_{22}\mathcal{L})$. Hence if we take

$$x=\exp\left[c_{53}\mathcal{L}+c_{54}\log\left(\frac{|G|}{|C|}n_K\sum_{p\in P(L/K)}\log p\right)\right] \tag{4.18}$$

for a sufficiently large absolute constant c_{53}, c_{54} then Equation 4.15 yields

$$\begin{aligned}\frac{|G|}{|C|}&\sum_{\substack{\mathfrak{p}\in S(L/K)\\ N\mathfrak{p}<x^{10}}}\theta(\mathfrak{p})(\log N\mathfrak{p})\hat{k}_2(N\mathfrak{p})\\ &\geq x^2\log x(1-\beta_0)-c_{55}\frac{|G|}{|C|}n_K(\log x)^{1/2}\sum_{p\in P(L/K)}\log p>0.\end{aligned}$$

Therefore in this case we see that the least prime $\mathfrak{p}$ whose Frobenius in G is C satisfies

$$N_{K/\mathbb{Q}}\mathfrak{p}\leq\left(\prod_{\chi\in\mathrm{nv}(C)}A_\chi^{\chi(1)}\right)^{c_{53}}\left(\frac{|G|}{|C|}n_K\sum_{p\in P(L/K)}\log p\right)^{c_{54}}.$$

□

4.7 ESTIMATES UNDER ADDITIONAL ASSUMPTIONS

In this section we will show stronger estimates for the least prime whose Frobenius is C under Assumption 1 and Assumption 2 of Section 3.6. Write $s = \sigma + it$. We recall the assumptions as follows.

1. For $1 < \sigma < 3$ the logarithmic derivative of $L(s, \chi)$ satisfies
$$\left|\frac{L'}{L}(s, \chi)\right| \ll \frac{1}{\sigma - 1} + \log A_\chi.$$

2. There exists a constant c such that if χ is irreducible and $L(s, \chi)$ satisfies Artin's conjecture then it has at most one zero in the region
$$1 - \frac{c}{\chi(1)(\log A_\chi + \chi(1) n_K \log(|t| + 2))} \leq \sigma \leq 1.$$

Under Assumption 1 by Lemma 3.8 one has the stronger zero-density estimate
$$n_\chi(r; 1) \ll 1 + r(\log A_\chi + n_K \chi(1) \log(3)) \ll 1 + r \log A_\chi.$$

We now use this in addition to the expanded zero-free region of Assumption 2 to obtain an estimate for the sum over zeros close to $s = 1$.

Lemma 4.26. *Let L/K be a Galois extension of number fields of degree n and χ an irreducible character of* $\mathrm{Gal}(L/K)$. *Assume that Assumption 1 and Assumption 2 hold. Then*
$$\sum_{\substack{|\rho_\chi - 1| < 1 \\ \rho \neq \beta_0}} |k_1(\rho_\chi)| \ll (\chi(1) \log A_\chi)^2.$$

Proof. The proof is similar to that of Lemma 4.16 while applying the stronger zero-density estimates and zero-free regions. We may assume $x > 1$. As in Lemma 4.16 we see that
$$|k_1(\rho_\chi)| \ll \frac{x^{-2(\sigma-1)}}{|\rho_\chi - 1|^2}.$$

It follows from Assumption 2 that for all $\rho_\chi = \sigma + it$ with $|\rho_\chi - 1| < 1$ and $\rho_\chi \neq \beta_0$ we have for some effective absolute constant c_{20} that
$$\sigma < 1 - \frac{c_{20}}{\chi(1) \log A_\chi}.$$

Set
$$B'_\chi := \frac{c_{20}}{\chi(1) \log A_\chi}$$

so that $B'_\chi < 1 - \sigma$ and therefore

$$\frac{x^{-2(\sigma-1)}}{|\rho_\chi - 1|^2} \leq \frac{x^{-2B'_\chi}}{|\rho_\chi - 1|^2}.$$

Thus as in Lemma 4.16 we have by partial summation

$$\sum_{|\rho_\chi - 1| < 1} |k_1(\rho_\chi)| \ll x^{-2B'_\chi} \int_{B'_\chi}^{1} \frac{1}{t^2}\, dn_\chi(t;1)$$

where the region of integration is similar to the one given in Figure 4.2.

Now using Lemma 3.8 we have

$$\begin{aligned}
\int_{B'_\chi}^{1} \frac{1}{t^2}\, dn_\chi(t;1) &= \frac{n_\chi(t;1)}{t^2}\bigg|_{B'_\chi}^{1} + \int_{B'_\chi}^{1} \frac{n_\chi(t;1)}{t^3}\, dt \\
&\ll n_\chi(1;1) + \int_{B'_\chi}^{1} \frac{1 + t(\log A_\chi + \chi(1) n_K)}{t^3}\, dt \\
&\ll \frac{1}{B'^2_\chi} + \frac{\log A_\chi + \chi(1) n_K}{B'_\chi} \\
&\ll (\chi(1) \log A_\chi)^2.
\end{aligned}$$

Thus

$$\sum_{|\rho_\chi - 1| < 1} |k_1(\rho_\chi)| \ll x^{-2B'_\chi} \int_{B'_\chi}^{1} \frac{1}{t^2}\, dn_\chi(t;1) \ll (\chi(1) \log A_\chi)^2.$$

where $x^{-2B'_\chi} < 1$ since $x > 1$. $\square$

Proposition 4.27. *Let L/K be a Galois extension of number fields with $G = \mathrm{Gal}(L/K)$ and let $P(L/K)$ be the set of rational primes below primes of K which ramify in L. Let C be a conjugacy class of G. Suppose that for each $L(s,\chi)$ with $\chi \in \mathrm{nv}(C)$ Artin's conjecture, Assumption 1, and Assumption 2 hold. If $L(s,\chi)$ does not have an exceptional zero for each $\chi \in \mathrm{nv}(C)$, then there exists an absolute effective constant c_{56} such that the least prime $\mathfrak{p}$ of K whose Frobenius in G is C satisfies*

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq \left(\prod_{\chi \in \mathrm{nv}(C)} A_\chi^{\chi(1)|\chi(C)|^{1/2}} \right)^{c_{56}} \left(\frac{|G|}{|C|} n_K \sum_{p \in P(L/K)} \log p \right)^{c_{56}}.$$

Proof. The estimate in Lemma 4.26 allows us to replace the second term in Equation 4.12 to obtain

$$\begin{aligned}\frac{|G|}{|C|}\sum_{\mathfrak{p}\in S(L/K)}\theta(\mathfrak{p})(\log N\mathfrak{p})\hat{k}_1(N\mathfrak{p})\\ &\geq (\log x)^2 - c_{57}\sum_{\chi\in\mathrm{nv}(C)}|\chi(C)|(\chi(1)\log A_\chi)^2\\ &\quad - c_{58}\sum_{\chi\in\mathrm{nv}(C)}|\chi(C)|\log A_\chi - c_{59}\frac{|G|}{|C|}n_K\frac{\log x}{x^2}\sum_{p\in P(L/K)}\log p\\ &\quad - c_{60}\frac{|G|}{|C|}n_K\frac{(\log x)^2}{x}.\end{aligned}$$

We see that

$$\sum_{\chi\in\mathrm{nv}(C)}|\chi(C)|(\chi(1)\log A_\chi)^2 \leq \left(\sum_{\chi\in\mathrm{nv}(C)}|\chi(C)|^{1/2}\chi(1)\log A_\chi\right)^2$$

so we can take

$$\log x \gg \sum_{\chi\in\mathrm{nv}(C)}|\chi(C)|^{1/2}\chi(1)\log A_\chi + \log\left(\frac{|G|}{|C|}n_K\sum_{p\in P(L/K)}\log p\right).$$

□

5 THE LEAST PRIME IN A CONJUGACY CLASS: APPLICATIONS

In this chapter we apply the results in Chapter 4 and in particular Theorem 1.1 to obtain bounds for the least prime in Frobenius classes that are vanishing for many characters. We first consider the case of $\mathrm{Gal}(L/K) = S_n$ and the class of n-cycles and $(n-1)$-cycles. Next we consider applications to the class of reflections in D_n-extensions and to the general case of classes of Camina elements.

5.1 APPLICATION TO S_n-EXTENSIONS

We first apply Theorem 1.1 for a general S_n-extension to obtain upper bounds for the least prime in whose Frobenius is an n-cycle or an $(n-1)$-cycle. Following that we consider the special case of an S_n-extension of $\mathbb{Q}$ which is unramified over a quadratic extension where we prove strengthened estimates for the least prime in an n-cycle or $(n-1)$-cycle.

5.1.1 *The least prime in an n-cycle*

Throughout this section let L/K be a Galois extension of number fields with $\mathrm{Gal}(L/K) = S_n$ and let C denote the conjugacy class of n-cycles. For this class we have $|C| = n$ and $|G| = n!$. From Section 2.2.5 we recall Proposition 2.8 and Proposition 2.10:

- $|\chi(C)| = 1$ if χ is a *hook character*, that is $\chi = \chi^\lambda$ where λ is a hook; otherwise $\chi(C) = 0$. In particular $\mathrm{nv}(C)$ is the set of hook characters.
- If χ^λ is a hook character with corresponding hook $\lambda = (r, 1^{n-r})$ then it has character degree $\binom{n-1}{r}$.

We will use these in conjunction with Theorem 1.1 to obtain estimates for the special case when C is the class of n-cycles. We will need an asymptotic on power sums of binomial coefficients from [PS72, Pt. II, 40]. We collect the details here for completeness.

Lemma 5.1 ([PS72, Pt. II, 54])**.** *Suppose $f(x)$ is integrable over $[a, b]$. Set*

$$\delta_n = \frac{b-a}{n}$$

and

$$f_{vn} = f(a + v\delta_n).$$

Then

$$\lim_{n\to\infty} (1 + f_{1n}\delta_n)(1 + f_{2n}\delta_n)\dots(1 + f_{nn}\delta_n) = \exp\left(\int_a^b f(x)\,dx\right).$$

Proof. Taking the logarithm of the left-hand side and using the power series expansion of the logarithm we obtain

$$\sum_{v=1}^{n} \log(1 + f_{vn}\delta_n) = \sum_{v=1}^{n}\sum_{m=1}^{\infty} -\frac{(-f_{vn}\delta_n)^m}{m} = \sum_{v=1}^{n} f_{vn}\delta_n + O\left(\delta_n \sum_{v=1}^{n} f_{vn}^2\delta_n\right).$$

Finally we observe that

$$\sum_{v=1}^{n} f_{vn}\delta_n \xrightarrow{n\to\infty} \int_a^b f(x)\,dx$$

and similarly

$$\sum_{v=1}^{n} f_{vn}^2\delta_n \xrightarrow{n\to\infty} \int_a^b f(x)^2\,dx$$

and $\delta_n \to 0$ as $n \to \infty$. □

Lemma 5.2 ([PS72, Pt. II, 58])**.** *Let n, v be integers with $0 < v < n$ and furthermore suppose that n, v increase to infinity in such a way that*

$$\lim_{n\to\infty} \frac{v - n/2}{\sqrt{n}} = \lambda.$$

Then

$$\lim_{n\to\infty} \frac{\sqrt{n}}{2^n}\binom{n}{v} = \sqrt{\frac{2}{\pi}}e^{-2\lambda^2}.$$

Proof. Suppose first that n is even and set $n = 2m$. Then

$$\frac{v-m}{\sqrt{m}} \to \lambda\sqrt{2}.$$

We may further assume that $\lambda \geq 0$ and $v > m$. As well, we recall for example from Stirling's formula that

$$\binom{2m}{m} \sim \frac{2^{2m}}{\sqrt{m\pi}}.$$

Therefore it suffices to show that

$$\lim_{m\to\infty} \frac{\binom{2m}{v}}{\binom{2m}{m}} = e^{-2\lambda^2}.$$

Now

$$\begin{aligned} \frac{\binom{2m}{v}}{\binom{2m}{m}} &= \frac{m}{m+1}\frac{m-1}{m+2}\cdots\frac{m-(v-m-1)}{m+(v-m)} \\ &= \frac{1}{1+\frac{1}{\sqrt{m}}\frac{1}{\sqrt{m}}}\frac{1-\frac{1}{\sqrt{m}}\frac{1}{\sqrt{m}}}{1+\frac{2}{\sqrt{m}}\frac{1}{\sqrt{m}}}\cdots\frac{1-\frac{v-m-1}{\sqrt{m}}\frac{1}{\sqrt{m}}}{1+\frac{v-m}{\sqrt{m}}\frac{1}{\sqrt{m}}} \end{aligned}$$

By adapting the argument in Lemma 5.1 we see that

$$\left(1-\frac{1}{\sqrt{m}}\frac{1}{\sqrt{m}}\right)\left(1-\frac{2}{\sqrt{m}}\frac{1}{\sqrt{m}}\right)\cdots\left(1-\frac{v-m-1}{\sqrt{m}}\frac{1}{\sqrt{m}}\right) \xrightarrow{m\to\infty} \exp\left(-\int_0^{\lambda\sqrt{2}} x\,dx\right) = e^{-\lambda^2}$$

and similarly

$$\left(1+\frac{1}{\sqrt{m}}\frac{1}{\sqrt{m}}\right)\left(1+\frac{2}{\sqrt{m}}\frac{1}{\sqrt{m}}\right)\cdots\left(1+\frac{v-m}{\sqrt{m}}\frac{1}{\sqrt{m}}\right) \xrightarrow{m\to\infty} \exp\left(\int_0^{\lambda\sqrt{2}} x\,dx\right) = e^{\lambda^2}$$

which proves the case when n is even.

Now if n is odd, set $n = 2m+1$. We note that

$$\frac{\binom{2m+1}{v}}{\binom{2m+1}{m+1}} = \frac{m+1}{v}\frac{\binom{2m}{v-1}}{\binom{2m}{m}}.$$

Since

$$\lim_{m\to\infty}\frac{v-m}{\sqrt{2m}} = \lim_{m\to\infty}\frac{v-\frac{2m+1}{2}}{\sqrt{2m+1}} = \lambda$$

it suffices to show that $(m+1)/v \to 1$. We note that

$$\lim_{m\to\infty} \frac{1-m/v}{\sqrt{m}/v} = \lim_{m\to\infty} \frac{v-m}{\sqrt{m}} = \lambda\sqrt{2}.$$

Since $v > m$ we have $1 > m/v > \sqrt{m}/v > 0$ so $\liminf \sqrt{m}/v = 0$ since otherwise $m/v = \sqrt{m}\sqrt{m}/v$ will be unbounded. By taking a subsequence if necessary we see that $\sqrt{m}/v \to 0$ so $1 - m/v \to 0$.

□

Proposition 5.3 ([PS72, Pt. II, 40])**.** *For k fixed, we have*

$$\sum_{v=0}^{n} \binom{n}{v}^k \sim \frac{2^{kn}}{\sqrt{k}} \left(\frac{2}{\pi n}\right)^{(k-1)/2}.$$

Proof. From Lemma 5.2 we see that

$$\sum_{v=0}^{n} \binom{n}{v}^k \sim 2^{kn} \left(\frac{2}{\pi}\right)^{k/2} n^{-(k-1)/2} \sum_{v=0}^{n} \exp\left(-2k\left(\frac{v-n/2}{\sqrt{n}}\right)^2\right) \frac{1}{\sqrt{n}}$$

Now

$$\lim_{n\to\infty} \sum_{v=0}^{n} \exp\left(-2k\left(\frac{v-n/2}{\sqrt{n}}\right)^2\right) \frac{1}{\sqrt{n}} = \int_{-\infty}^{\infty} e^{-2kx^2}\, dx = \sqrt{\frac{\pi}{2k}}$$

so

$$\sum_{v=0}^{n} \binom{n}{v}^k \sim \frac{2^{kn}}{\sqrt{k}} \left(\frac{2}{kn}\right)^{(k-1)/2}$$

as required. □

In the case that there is an exceptional zero β_0 for $\prod_{\lambda=(n-r,1^r)} L(s,\chi^\lambda)$ we use Theorem 1.4 to obtain a larger zero-free region for the nonexceptional zeros. First we note the following proposition which allows us to use Theorem 1.4.

Proposition 5.4 (Regev)**.** *Let $\mu \vdash n$ be a partition of length ℓ. Then*

$$\sum_{r=0}^{n-1} \chi^{(n-r,1^r)}(\mu) = \begin{cases} 2^{\ell-1} & \textit{if all parts of } \mu \textit{ are odd} \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. This is [Reg13, Proposition 1.1]. □

Remark 5.5. Proposition 5.4 was proven in [Reg13] using representations of Lie superalgebras. A proof using only S_n characters was given in [Tay17].

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. Let C be the class of n-cycles. Then

$$\frac{|C|}{|G|} = \frac{n!/n}{n!} = \frac{1}{n}.$$

We know from Proposition 2.8 that

$$\mathrm{nv}(C) = \{\chi^\lambda : \lambda = (n-r, 1^r)\}.$$

As well, from Proposition 2.10 we have for $\lambda = (n-r, 1^r)$

$$\chi^\lambda(1) = \binom{n-1}{r}$$

and therefore

$$\sum_{\lambda=(n-r,1^r)} \chi^\lambda(1) = 2^{n-1} \ll 2^n.$$

Therefore Theorem 1.1 gives the following upper bounds for the least prime $\mathfrak{p}$ whose Frobenius in $G = \mathrm{Gal}(L/K)$ is an n-cycle depending on whether β_0 exists.

If there is no β_0 then from Equation 1.2

$$\log N_{K/\mathbb{Q}}\mathfrak{p} \ll \left(\sum_{r=0}^{n-1}\binom{n-1}{r}^5\right)^{1/2} \sum_{\lambda=(n-r,1^r)} \chi^\lambda(1)\log A_{\chi^\lambda} + \log\left(n n_K \sum_{p\in P(L/K)} \log p\right).$$

From Proposition 5.3 we obtain

$$\left(\sum_{r=0}^{n-1}\binom{n-1}{r}^5\right)^{1/2} \ll \frac{2^{5n/2}}{n}$$

and therefore

$$\log N_{K/\mathbb{Q}}\mathfrak{p} \ll \frac{2^{5n/2}}{n} \sum_{\lambda=(n-r,1^r)} \chi^\lambda(1)\log A_{\chi^\lambda} + \log\left(n n_K \sum_{p\in P(L/K)} \log p\right).$$

For the case of the exceptional zero β_0 it remains to verify that $\sum_\lambda \chi^\lambda$ is always nonnegative where the sum is over all hooks $\lambda = (n-r, 1^r)$. This follows from Proposition 5.4. Hence Equation 1.4 follows directly from Equation 1.3.

□

At the cost of worsening the bound, we may write the above in terms of the discriminant of L. First we estimate the terms from ramified primes.

Lemma 5.6. *Let L/K be a Galois extension of number fields with Galois group G and C a conjugacy class in C. Then*

$$\frac{|G|}{|C|} n_K \sum_{p \in P(L/K)} \log p \leq (\log d_L)^2.$$

Proof. We observe that since

$$d_L = d_K^{[L:K]} N_{K/\mathbb{Q}} \mathfrak{d}_{L/K}$$

we have

$$\log d_L = |G| \log d_K + \log N_{K/\mathbb{Q}} \mathfrak{d}_{L/K}.$$

Therefore

$$(\log d_L)^2 \geq |G| \log d_K \log N_{K/\mathbb{Q}} \mathfrak{d}_{L/K}$$

from which it follows that

$$\frac{|G|}{|C|} n_K \sum_{p \in P(L/K)} \log p \leq |G| \log d_K \log N_{K/\mathbb{Q}} \mathfrak{d}_{L/K} \leq (\log d_L)^2.$$

□

Proof of Corollary 1.8. From Lemma 5.6 we obtain

$$n n_K \sum_{p \in P(L/K)} \log p \leq (\log d_L)^2.$$

As well, recall that

$$A_\chi = d_K^{\chi(1)} N_{K/\mathbb{Q}} \mathfrak{f}_\chi.$$

Hence by Proposition 5.3

$$\prod_{\lambda = (n-r, 1^r)} d_K^{\chi^\lambda(1)^2} = d_K^{c_{23} 4^n / n}.$$

Furthermore by the conductor-discriminant formula Theorem 3.2 we see that

$$\prod_{\lambda = (n-r, 1^r)} N_{K/\mathbb{Q}} \mathfrak{f}_{\chi^\lambda}^{\chi^\lambda(1)} \leq N_{K/\mathbb{Q}} \mathfrak{d}_{L/K}.$$

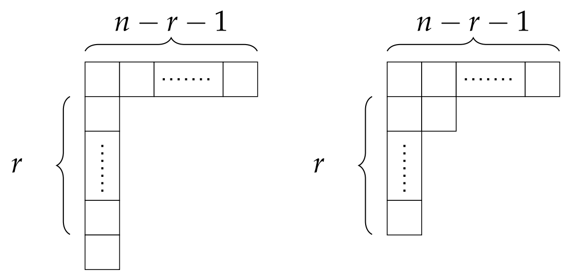


Figure 5.1: The possible shapes of λ

Therefore we see from the two cases in Theorem 1.6 that the bound for the least prime in an n-cycle in the case of no exceptional zeros can be written as

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq d_K^{c_{24}^n/n} N_{K/\mathbb{Q}}\mathfrak{d}_{L/K}^{c_{25}^n}(\log d_L)^{c_{26}}$$

and the for the case of exceptional zeros the bound becomes

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq d_K^{c_{24}^n/n} N_{K/\mathbb{Q}}\mathfrak{d}_{L/K}^{c_{27}}(\log d_L)^{c_{26}}$$

for effective absolute constants $c_{24}, c_{25}, c_{26}, c_{27}$. □

Finally, observing that $|\chi^\lambda(C)| = 1$ allows us to deduce Theorem 1.13 from Proposition 4.27.

5.1.2 *The least prime in an $(n-1)$-cycle*

By using similar techniques as the n-cycle case we can also obtain an upper bound for the least prime whose Frobenius is an $(n-1)$-cycle. We first establish the nonvanishing characters of $(n-1)$-cycles as well as the character values at $(n-1)$-cycles.

Proposition 5.7. *Let C be the class of $(n-1)$-cycles and let $\chi^\lambda \in \mathrm{Irr}(S_n)$ with $\lambda \vdash n$ the corresponding partition. Then*

$$\chi^\lambda(C) = \begin{cases} 1 & \text{if } \lambda = (n) \\ \mathrm{sgn}(C) & \text{if } \lambda = (1^n) \\ (-1)^r & \text{if } \lambda = (n-r, 2, 1^{r-2}) \\ 0 & \text{otherwise} \end{cases}.$$

Proof. By the Murnaghan-Nakayama rule, we have

$$\chi^\lambda_{(n-1,1)} = \sum_T (-1)^T$$

where the sum is over all rim hook tableaux T of shape μ and content $(n-1,1)$. That is, T is a Young tableaux with weakly increasing rows and columns with $n-1$ entries of 1s arranged in a rim hook and a single entry of 2.

Since the entries are weakly increasing the rim hook corresponding to 1 must be a Young diagram, so in particular a hook diagram of size $n-1$, say μ. Thus λ is obtained by adding a box to μ in such a way that the resulting shape is a Young diagram. Then λ is either itself a hook or a hook with a square added at position $(2,2)$, where (i,j) denotes the i-th row and j-th column. See Figure 5.1 for the possible shapes, along with (n) and (1^n).

If $\lambda = (n)$ then $\chi^\lambda = 1$ and if $\lambda = (1^n)$ then $\chi^\lambda = \mathrm{sgn}$. Now suppose λ is a hook of shape $(n-r-1, 1^{r+1})$. Then since the entries are weakly increasing any hook with content $(n-1,1)$ must have entry 2 at either the bottommost box at $(r+1,1)$ or at the rightmost box at $(1, n-r)$. In any case the singleton 2 has leg length 0 but the hook with entry 1 will have leg length $r-1$ and r. Thus

$$\chi^{(n-r,1^r)}(C) = (-1)^{r-1} + (-1)^r = 0.$$

Next suppose λ is obtained by taking a hook of shape $(n-r-1, 1^r)$ and adding a box at position $(2,2)$. Again since the entries are weakly increasing and the boxes containing 1 must form a skew diagram the entry at $(2,2)$ is forced to be 2 and the boxes containing 1 must be the hook of shape $(n-r-1, 1^r)$. Once again the single 2 has leg length 0 and the hook has leg length r, so

$$\chi^{(n-r,2,1^{r-2})}(C) = (-1)^r.$$

$\square$

Next we determine the character degrees.

Proposition 5.8. *Let $\mu = (n-r, 2, 1^{r-2})$ with $1 < r < n-1$. Then*

$$\chi^\mu(1) = \frac{n(r-1)(n-r-1)}{(n-1)(n-r)} \binom{n-1}{r} \leq n \binom{n-1}{r}.$$

Proof. Applying the hook length formula we obtain

$$\begin{aligned}
\chi^\mu(1) &= \frac{n!}{r(n-r)(n-1)(r-2)!(n-r-2)!} \\
&= \frac{n(r-1)(n-r-1)}{(n-r)(n-1)} \cdot \frac{(n-1)!}{r!(n-1-r)!} \\
&= \frac{n(r-1)(n-r-1)}{(n-r)(n-1)} \binom{n-1}{r}
\end{aligned}$$

and we observe that
$$\frac{(r-1)(n-r-1)}{(n-1)(n-r)} \le 1.$$
□

Proposition 5.9. *Let L/K be a Galois extension of number fields with $G = \mathrm{Gal}(L/K) = S_n$. Assume Artin's conjecture for L-functions of characters χ^μ corresponding to partitions of the form $\mu = (n-r, 2, 1^{r-2})$ for $1 < r < n-1$. If $\zeta_K(s)L(s, \mathrm{sgn})$ has no exceptional zeros, then there exist absolute effective constants c_{61}, c_{62} such that the least prime $\mathfrak{p}$ of K whose Frobenius in G is an $(n-1)$-cycle satisfies*

$$N_{K/\mathbb{Q}}\mathfrak{p} \le \left(d_K A_{\mathrm{sgn}} \prod_{\mu=(n-r,2,1^{r-2})} A_{\chi^\mu}^{\chi^\mu(1)} \right)^{c_{61} n^{1/2} 2^{5n}} \cdot \left((n-1) n_K \sum_{p \in P(L/K)} \log p \right)^{c_{62}} \tag{5.1}$$

Proof. We will apply Theorem 1.1 First we note that $|C| = n(n-2)!$ so $|G|/|C| = n-1$. From Proposition 5.7 we see that

$$\mathrm{nv}(C) = \{\chi^\mu : \mu = (n-r, 2, 1^{r-2}), 1 < r < n-1\} \cup \{1, \mathrm{sgn}\}$$

so applying Proposition 5.8 and Proposition 5.3 we obtain

$$\sum_{\chi \in \mathrm{nv}(C)} \chi(1)^5 \ll \sum_{\mu=(n-r,2,1^{r-2})} n^5 \binom{n-1}{r}^5 \ll n^3 2^{5n}$$

which combined with Theorem 1.1 yields Equation 5.1. □

In the case of the exceptional zero we will need to invoke Theorem 1.4. However it is not true in general that $\sum_{\chi \in \mathrm{nv}(C)} \chi$ is nonnegative as the following proposition shows. The inclusion $\{1, \dots, n-1\} \subseteq \{1, \dots, n\}$ induces an embedding $S_{n-1} \hookrightarrow S_n$. For any $\sigma \in S_n$, define $\varepsilon(\sigma) = 1$ if the cycle decomposition of σ contains only odd cycles and $\varepsilon(\sigma) = 0$ otherwise. Let $\mathrm{Fix}(\sigma)$ denote the set of fixed points of σ acting on $\{1, \dots, n\}$.

Proposition 5.10. *Let $\sigma \in S_n$ corresponding to a partition of length ℓ. Then*

$$\sum_{\mu=(n-r,2,1^{r-2})} \chi^\mu(\sigma) = \epsilon(\sigma)(\#\,\mathrm{Fix}(\sigma) 2^{\ell-2} - 2^\ell) + 1 + \mathrm{sgn}(\sigma).$$

Proof. By the branching rule (Theorem 2.6) we have

$$\begin{aligned}\sum_{\substack{\nu\vdash n-1\\ \nu \text{ hook}}} \operatorname{Ind}_{S_{n-1}}^{S_n} \chi^\nu &= 1_G + \operatorname{sgn} + 2 \sum_{\substack{\lambda=(n-r,1^r)\\ 1\le r\le n-2}} \chi^\lambda + \sum_{\mu=(n-r,2,1^{r-2})} \chi^\mu \\ &= 2 \sum_{\substack{\lambda\vdash n\\ \lambda \text{ hook}}} \chi^\lambda + \sum_{\mu=(n-r,2,1^{r-2})} \chi^\mu - 1_G - \operatorname{sgn}.\end{aligned}$$

From Proposition 5.4 we have

$$2 \sum_{\substack{\lambda\vdash n\\ \lambda \text{ hook}}} \chi^\lambda(\sigma) = \varepsilon(\sigma) 2^\ell.$$

Next we use the induced character formula given in Equation 2.1 and Equation 2.1. A transversal for S_{n-1} in S is given by the transpositions $(1,n),(2,n),\ldots,(n,n)=1$. Hence

$$\sum_{\substack{\nu\vdash n-1\\ \nu \text{ hook}}} \operatorname{Ind}_{S_{n-1}}^{S_n} \chi^\nu(\sigma) = \sum_{\substack{\nu\vdash n-1\\ \nu \text{ hook}}} \sum_{i=1}^{n} \widetilde{\chi^\nu}((i,n)\sigma(i,n))$$

Note that $\widetilde{\chi^\nu}$ is nonvanishing only on $S_{n-1} = \{\sigma \in S_n : \sigma(n) = n\}$ and that $(i,n)\sigma(i,n)$ fixes n if and only if σ fixes i. As well, since conjugation does not change the cycle structure so if σ has cycle length ℓ so will all its conjugates. Hence changing the order of summation and invoking Proposition 5.4 we obtain

$$\sum_{\substack{\nu\vdash n-1\\ \nu \text{ hook}}} \operatorname{Ind}_{S_{n-1}}^{S_n} \chi^\nu(\sigma) = \sum_{i=1}^{n} \sum_{\substack{\nu\vdash n-1\\ \nu \text{ hook}}} \widetilde{\chi^\nu}((i,n)\sigma(i,n)) = \#\operatorname{Fix}(\sigma)\varepsilon(\sigma)2^{\ell-2}$$

where the exponent is $\ell-2$ since we truncate the 1-cycle (n) in σ when considered as an element of S_{n-1}. We also note that if σ' is obtained from σ by truncating a 1-cycle then $\varepsilon(\sigma') = \varepsilon(\sigma)$.

$\square$

Proposition 5.11. *Let L/K be a Galois extension of number fields with $G = \operatorname{Gal}(L/K) = S_n$. Assume Artin's conjecture for L-functions of characters χ^μ corresponding to partitions of the form $\mu = (n-r,2,1^{r-2})$ for $1 < r < n-1$ as well as L-functions of hook characters. Let*

$$\varphi := \sum_{\substack{\lambda\vdash n-1\\ \lambda \text{ hook}}} \operatorname{Ind}_{S_{n-1}}^{S_n} \chi^\lambda = 1_G + \operatorname{sgn} + 2 \sum_{\substack{\nu=(n-r,1^r)\\ 1\le r\le n-2}} \chi^\nu + \sum_{\mu=(n-r,2,1^{r-2})} \chi^\mu.$$

Suppose $L(s,\varphi)$ admits an exceptional zero β_0. Then there is an effective absolute constant c_{63} such that the least prime $\mathfrak{p}$ whose Frobenius is an $(n-1)$-cycle in S_n satisfies

$$N\mathfrak{p} \leq \left(\prod_{\lambda=(n-r,1^r)} A_{\chi^\lambda}^{\chi^\lambda(1)} \prod_{\mu=(n-r,2,1^{r-2})} A_{\chi^\mu}^{\chi^\mu(1)} \right)^{c_{63}} \cdot \left((n-1)n_K \sum_{p\in P(L/K)} \log p \right)^{c_{63}}. \quad (5.2)$$

Proof. Observe that φ contains all the characters $\chi^\mu \in \mathrm{nv}(C)$ and is also nonnegative by Proposition 5.1.2 so we may apply Theorem 1.4. We note that if C is the class of $(n-1)$-cycles then $|C| = n(n-2)!$ so $|C|/|G| = 1/(n-1)$.

By Equation 4.14 of Proposition 4.25 we have

$$\begin{aligned}(n-1) &\sum_{\mathfrak{p}\in S(L/K)} \theta(\mathfrak{p})(\log N\mathfrak{p})\hat{k}_1(N\mathfrak{p}) \\ &\geq (\log x)^3(1-\beta_0) \\ &\quad - c_{64} \sum_{\mu=(n-r,2,1^{r-2})} \chi^\mu(1)\mathcal{L}^2[(1/c_{65})(1-\beta_0)\mathcal{L}]^{2c_{66}\log x/\mathcal{L}} \\ &\quad - c_{67} \sum_{\mu=(n-r,2,1^{r-2})} \log A_{\chi^\mu} \\ &\quad - c_{68}(n-1)n_K \frac{\log x}{x^2} \sum_{p\in P(L/K)} \log p \\ &\quad - c_{69}(n-1)n_K \frac{(\log x)^2}{x}\end{aligned}$$

where

$$\begin{aligned}\mathcal{L} = \log A_{1_G} + \log A_{\mathrm{sgn}} \\ + 2 \sum_{\substack{\nu=(n-r,1^r) \\ 1\leq r\leq n-2}} \chi^\nu(1)\log A_{\chi^\nu} + \sum_{\mu=(n-r,2,1^{r-2})} \chi^\mu(1)\log A_{\chi^\mu}\end{aligned}$$

and $c_{65} > 1$. Using an argument similar to Equation 4.17 in the proof of Theorem 1.1 we see that if

$$1-\beta_0 \gg \frac{1}{\mathcal{L}^2}$$

then we can take

$$\log x \gg \mathcal{L} + \log\left((n-1)n_K \sum_{p \in P(L/K)} \log p\right).$$

If instead

$$1 - \beta_0 \ll \frac{1}{\mathcal{L}^2}$$

then again as before we apply Equation 4.18 to obtain

$$x = \exp\left[c_{70}\mathcal{L} + c_{71}\log\left((n-1)n_K \sum_{p \in P(L/K)} \log p\right)\right]$$

so that in any case Equation 5.2 holds.

□

Combining Proposition 5.9 and Proposition 5.11 yields Theorem 1.9.

Proof of Corollary 1.10. As in the proof of Corollary 1.8, from Lemma 5.6 we have

$$(n-1)n_K \sum_{p \in P(L/K)} \log p \leq (\log d_L)^2$$

and we know that

$$\prod_{\lambda=(n-r,1^r)} A_{\chi^\lambda}^{\chi^\lambda(1)} = d_K^{\sum_\lambda \chi^\lambda(1)^2} \prod_{\lambda=(n-r,1^r)} N_{K/\mathbb{Q}} \mathfrak{f}_{\chi^\lambda}^{\chi^\lambda(1)} \leq d_K^{c_{72}4^n/n} N_{K/\mathbb{Q}} \mathfrak{d}_{L/K}.$$

As well, from Proposition 5.8 and Proposition 5.3 we see that

$$\sum_{\mu=(n-r,2,1^{r-2})} \chi^\mu(1)^2 \leq n^2 \sum_{r=0}^{n-1} \binom{n-1}{r}^2 \ll n4^n.$$

Therefore

$$\begin{aligned}\prod_{\mu=(n-r,2,1^{r-2})} A_{\chi^\mu}^{\chi^\mu(1)} &= d_K^{\sum_\mu \chi^\mu(1)^2} \prod_{\mu=(n-r,2,1^{r-2})} N_{K/\mathbb{Q}} \mathfrak{f}_{\chi^\mu}^{\chi^\mu(1)} \\ &\leq d_K^{c_{73}n4^n} N_{K/\mathbb{Q}} \mathfrak{d}_{L/K}\end{aligned}$$

for some constant c_{73}. Hence

$$\prod_{\lambda=(n-r,1^r)} A_{\chi^\lambda}^{\chi^\lambda(1)} \prod_{\mu=(n-r,2,1^{r-2})} A_{\chi^\mu}^{\chi^\mu(1)} \leq d_K^{c_{74}4^n} N_{K/\mathbb{Q}} \mathfrak{d}_{L/K}^2$$

for some constant c_{74}. □

$$
\begin{tikzcd}
L \arrow[dd, dash, "S_n"'] \arrow[dr, dash, "A_n"] & \\
& F \\
\mathbb{Q} \arrow[ur, dash, "2"'] &
\end{tikzcd}
$$

Figure 5.2: Tower of extensions

As in the case of n-cycles, combining with Proposition 4.27 yields Theorem 1.14.

5.1.3 *Unramified A_n-extensions of quadratic fields*

In this section consider a tower of number fields $L/F/\mathbb{Q}$ where $L/\mathbb{Q}$ is an S_n-extension and L/F is an unramified A_n-extension. Unramified A_n-extensions of quadratic fields have been studied by Uchida [Uch70], Yamamoto [Yam70], Elstrodt-Grunewald-Mennicke [EGM85], Kondo [Kon95], and Kedlaya [Ked12]. In particular it is known that there are infinitely many such extensions. We can apply the results of the previous sections to such an extensions to obtain an upper bound for the least prime in an n-cycle. First we adapt the proof of the upper bound for the Artin conductor Proposition 3.1 given by Murty-Murty-Saradha in [MMS88, Proposition 2.5] to obtain a stronger bound for the current case.

Proposition 5.12. *Let $L/F/\mathbb{Q}$ be a tower of extensions where L/F is unramified and $F/\mathbb{Q}$ is a quadratic extension with discriminant Δ. Let $\chi \in \mathrm{Irr}(\mathrm{Gal}(L/\mathbb{Q}))$. Then*

$$\log N\mathfrak{f}_\chi \leq 4\chi(1)\log\Delta.$$

Proof. Since the base field is $\mathbb{Q}$ we identify the Artin conductor $\mathfrak{f}$ with the integer

$$\mathfrak{f}_\chi = \prod_p p^{n(\chi,p)}.$$

Proceeding as in the proof of [MMS88, Proposition 2.5] we obtain that

$$n(\chi,p) \leq \frac{2\chi(1)w(\mathfrak{D}_{L/\mathbb{Q}})}{e_{w/p}}$$

where w is a place of L dividing p and $\mathfrak{D}_{L/\mathbb{Q}}$ is the different of $L/\mathbb{Q}$. By an estimate of Hensel we have

$$w(\mathfrak{D}_{L/\mathbb{Q}}) = e_{w/p} - 1 + s_{w/p}$$

for some $s_{w/p}$ satisfying $0 \leq s_{w/p} \leq w(e_{w/p})$. Therefore

$$n(\chi, p) \leq \frac{2\chi(1)(e_{w/p} - 1 + s_{w/p})}{e_{w/p}} \leq 2\chi(1)\left(1 - \frac{1}{e_{w/p}} + \frac{w(e_{w/p})}{e_{w/p}}\right).$$

Since L/F is unramified, we have $e_{w/v} = 1$ for all places v of F below w and above p. Thus if p is ramified in L it must be ramified in F and we have $e_{v/p} = 2$ from which it follows that $e_{w/p}$ is 2 if p is ramified in L and is 1 if p is unramified. Thus

$$w(e_{w/p}) = \begin{cases} 2 & \text{if } p = 2 \\ 0 & \text{otherwise} \end{cases}$$

so in any case we obtain

$$n(\chi, p) \leq 4\chi(1)$$

for all primes p that ramify in L. Therefore

$$\log N\mathfrak{f}_\chi = \sum_{p \in P(L/\mathbb{Q})} n(\chi, p) \log p \leq 4\chi(1) \sum_{p \in P(L/\mathbb{Q})} \log p \leq 4\chi(1) \log \Delta.$$

$\square$

Proof of Theorem 1.12. Suppose the Frobenius of p in $\mathrm{Gal}(L/\mathbb{Q})$ is an n-cycle. From Theorem 1.6 we see that p satisfies

$$p \leq \left(\prod_{\lambda=(n-r,1^r)} A_{\chi^\lambda}^{\chi^\lambda(1)}\right)^{c_{75}2^{5n/2}/n} \left(n \sum_{p \in P(L/\mathbb{Q})} \log p\right)^{c_{76}}.$$

Since L/F is unramified, any prime p which ramifies in L must divide Δ. Hence

$$\sum_{p \in P(L/\mathbb{Q})} \log p \leq \log \Delta.$$

As well, the base field is $\mathbb{Q}$ so we see that

$$A_\chi = N\mathfrak{f}_\chi = \prod_p p^{n(\chi,p)}.$$

Using Proposition 5.12, Proposition 2.10, and Proposition 5.3 we obtain

$$\begin{aligned}\prod_{\lambda=(n-r,1^r)} A_{\chi^\lambda}^{\chi^\lambda(1)} &\leq \exp\left(4\log\Delta \sum_{\lambda=(n-r,1^r)} \chi^\lambda(1)^2\right)\\ &\leq \exp\left(4\log\Delta \sum_{r=0}^{n-1} \binom{n-1}{r}^2\right)\\ &\leq \Delta^{4^{n+1}/\sqrt{n}}. \end{aligned} \tag{5.3}$$

Combining with Theorem 1.6 yields Equation 1.7.

If the Frobenius of p in $\mathrm{Gal}(L/\mathbb{Q})$ is an $(n-1)$-cycle, then from Theorem 1.9 we have

$$p \leq \left(\Delta \prod_{\mu=(n-r,2,1^{r-2})} A_{\chi^\mu}^{\chi^\mu(1)}\right)^{c_{77} n^{1/2} 2^{5n}} \left((n-1) \sum_{p\in P(L/K)} \log p\right)^{c_{78}} \tag{5.4}$$

if there are no exceptional zeros. Using Proposition 5.12 and Proposition 5.8 we have

$$\log A_{\chi^\mu} = \log \mathfrak{f}_{\chi^\mu} \leq 4\chi^\mu(1)\log\Delta \leq 4n\binom{n-1}{r}\log\Delta$$

and so using Proposition 5.3 we have

$$\begin{aligned}\prod_{\mu=(n-r,2,1^{r-2})} A_{\chi^\mu}^{\chi^\mu(1)} &\leq \exp\left(4\log\Delta \sum_{\mu=(n-r,2,1^{r-2})} \chi^\mu(1)^2\right)\\ &\leq \exp\left(4\log\Delta \sum_{r=0}^{n-1} n^2\binom{n-1}{r}^2\right)\\ &\leq \Delta^{n^{3/2}4^{n+1}}. \end{aligned}$$

Combining with Equation 5.4 we obtain

$$p \leq \Delta^{n^2 c_{79}^n} \left((n-1)\log\Delta\right)^{c_{80}}.$$

If there are exceptional zeros, then instead Theorem 1.9 gives

$$p \leq \left(\prod_{\lambda=(n-r,1^r)} A_{\chi^\lambda}^{\chi^\lambda(1)} \prod_{\mu=(n-r,2,1^{r-2})} A_{\chi^\mu}^{\chi^\mu(1)}\right)^{c_{81}} \left((n-1)\sum_{p\in P(L/K)} \log p\right)^{c_{81}}.$$

Hence combining with Equation 5.3 we obtain

$$p \leq \Delta^{nc_{82}^{n}}((n-1)\log\Delta)^{c_{83}}.$$

In any case we see that Equation 1.9 holds.

To obtain Equation 1.8 and Equation 1.10, observe that since L/F is unramified we have

$$d_L = \Delta^{[L:F]} = \Delta^{n!/2}.$$

□

Remark 5.13. The assumptions of Section 3.6 can only improve the base of the exponent in the exponent of the upper bound given in Theorem 1.12. In particular, Equation 1.8 and Equation 1.10 essentially remain the same.

5.2 THE LEAST PRIME WHICH IS A REFLECTION

Let L/K be a dihedral extension, so $G = \mathrm{Gal}(L/K) = D_n$. In this case the Artin conjecture is known since D_n is monomial. Let C denote a class of reflections. As in Section 2.3 there is a unique such class if n is odd, and if n is even there are two. We see that

$$\frac{|D_n|}{|C|} = \begin{cases} 2 & n \text{ odd} \\ 4 & n \text{ even} \end{cases}$$

so that in any case it is absolutely bounded.

As well, from Proposition 2.12 we find that for $\chi \in \mathrm{Irr}(D_n)$,

$$|\chi(C)| = \begin{cases} 1 & \chi(1) = 1 \\ 0 & \chi(1) > 1 \end{cases}.$$

Therefore $\mathrm{nv}(C)$ is the set of linear characters of D_n. Hence just as in the case when $G = S_n$ and C is the class of n-cycles we are in a situation where C is a conjugacy class at which most irreducible characters vanish.

Proof of Theorem 1.15. Since D_n is monomial we know Artin's conjecture holds for L/K. Applying Theorem 1.1 we obtain the following. If there are no exceptional zeros, then the least prime $\mathfrak{p}$ in K whose Frobenius is a reflection in G satisfies

$$N_{K/\mathbb{Q}}\mathfrak{p} \ll \left(\prod_{\chi \in \mathrm{lc}(D_n)} A_\chi\right)^{c_{84}} \left(n_K \sum_{p \in P(L/K)} \log p\right)^{c_{85}}. \tag{5.5}$$

If there is an exceptional zero β_0, then we note that since $\mathrm{nv}(C)$ is the set of linear characters of D_n by Proposition 2.3 we have that for all $g \in D_n$,

$$\sum_{\chi \in \mathrm{nv}(C)} \chi(g) \geq 0$$

so we are in a position to apply Theorem 1.4 and therefore the exceptional zero case of Theorem 1.1. Hence in any case Equation 5.5 holds by taking c_{84}, c_{85} to be larger if necessary.

Finally we note that there is a canonical correspondence between $\mathrm{lc}(G)$ and $\mathrm{Irr}(G^{ab})$ and invoke the conductor-discriminant formula. □

Proof of Corollary 1.16. Let F be the fixed field of the commutator subgroup G'. Then from Equation 2.4 we have

$$d_F \leq d_L^{1/[L:F]} \leq d_L^{4/[L:K]}$$

and therefore

$$d_F^{c_{86}} \leq d_L^{4c_{86}/[L:K]}.$$

As well, from Lemma 5.6 we obtain

$$\left(n_K \sum_{p \in P(L/K)} \log p \right)^{c_{87}} \leq (\log d_L)^{2c_{87}}.$$

□

5.3 THE LEAST PRIME WHICH IS A CAMINA ELEMENT

Recall from Section 2.4 that a Camina element g of a finite group G is an element at which every nonlinear irreducible character vanishes. Hence applying Theorem 1.1 to the class of Camina elements will give the strongest estimates. In particular the results of this section will apply to the groups and elements listed in Section 2.4

Proof of Theorem 1.19. From Section 2.4 we know that the characters that are nonvanishing at C are exactly the linear characters. Therefore Artin's conjecture holds and by Proposition 2.3 we have

$$\sum_{\chi \in \mathrm{nv}(C)} \chi(C) \geq 0.$$

Furthermore

$$\sum_{\chi \in \mathrm{nv}(C)} \chi(1)^5 = \#\,\mathrm{lc}(G) = |G^{ab}| = [L:F].$$

Likewise we recall the correspondence between $\mathrm{lc}(G)$ and $\mathrm{Irr}(G^{ab})$. Invoking the conductor-discriminant formula we obtain

$$\prod_{\chi \in \mathrm{lc}(G)} A_\chi^{\chi(1)} = d_F.$$

□

Proof of Corollary 1.20. Combine the estimate

$$d_F \leq d_L^{1/[L:F]}$$

with Lemma 5.6. □

For some of the examples listed in Section 2.4 we may rephrase the estimate to make it more explicit.

Corollary 5.14. *Let L/K be a Galois extension of number fields with $\mathrm{Gal}(L/K) = G$. Then there exist constants c_{88}, c_{89} such that for the following choices of G and conjugacy class C, the least prime $\mathfrak{p}$ of K whose Frobenius in G is C unconditionally satisfies the following:*

1. *Let G be an extraspecial p-group of order p^n and C the class of any element outside the centre. Then*

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq d_L^{c_{88}/p^{1/2}} (\log d_L)^{c_{89}}.$$

2. *Let G be a group of order p^4 and C the class of any Camina element. If $\#G' = p$ then*

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq d_L^{c_{88}/p^{1/2}} (\log d_L)^{c_{89}}.$$

   If instead $\#G' = p^2$ then

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq d_L^{c_{88}/p} (\log d_L)^{c_{89}}.$$

3. *Let G be a p-group of maximal class of order p^n and C the class of any element not in $C_G(K_i(G)/K_{i+2}(G))$ for $i \geq 2$, where $K_i(G)$ denotes the terms in the descending central series of G. Then*

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq d_L^{c_{88}/p^{(n-2)/2}} (\log d_L)^{c_{89}}.$$

4. *Let G be the group of upper-triangular matrices over a finite field of order q with diagonal entries 1, and C the class of any element with minimal polynomial $(x-1)^n$. Then*

$$N_{K/\mathbb{Q}}\mathfrak{p} \leq d_L^{c_{88}/q^{1/2}} (\log d_L)^{c_{89}}.$$

If there are exceptional zeros then the denominator of the exponent of d_L may be replaced by its square.

Proof. This follows from Corollary 1.20 by substituting the order of the commutator subgroup in G. If G is an extraspecial p-group then its commutator subgroup has size p. If G is a p-group of maximal class of order p^n, then its commutator subgroup has size p^{n-2}. If G is the group of upper-triangular matrices over a field of order q with diagonal entries 1 then the commutator subgroup has size q.

□

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