THE LEAST PRIME WHOSE FRObenius IS AN $n$-CYCLE

BY

REN ZHU

A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
Graduate Department of Mathematics
University of Toronto

© 2020 Ren Zhu
ABSTRACT

The least prime whose Frobenius is an $n$-cycle

Ren Zhu
Doctor of Philosophy
Graduate Department of Mathematics
University of Toronto
2020

Let $L/K$ be a Galois extension of number fields. We consider the problem of bounding the least prime ideal of $K$ whose Frobenius lies in a fixed conjugacy class $C$. Under the assumption of Artin’s conjecture we work with Artin $L$-functions directly to obtain an upper bound in terms of irreducible characters which are nonvanishing at $C$. As a consequence we obtain stronger upper bounds for the least prime in $C$ when many irreducible characters vanish at $C$. We also prove a Deuring-Heilbronn phenomenon for Artin $L$-functions with nonnegative Dirichlet series coefficients as a key step.

We apply our results to the case when $\text{Gal}(L/K)$ is the symmetric group $S_n$. Using classical results on the representation theory of $S_n$ we give an upper bound for the least prime whose Frobenius is an $n$-cycle which is stronger than known bounds when the characters which are nonvanishing at $n$-cycles are unramified, as well a similar result for $(n - 1)$-cycles. We also give stronger bounds in the case of $S_n$-extensions over $\mathbb{Q}$ which are unramified over a quadratic field. We also consider other groups and conjugacy classes where unconditional improvements are obtained.
To my parents.
ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my supervisor Prof. Kumar Murty. I would like to thank him not only for suggesting the problem and the approach, but for his guidance, patience, and encouragement during my graduate studies. This thesis would not be possible without his support and generosity.

I would like to thank my supervisory committee members Prof. Stephen Kudla and Prof. Jacob Tsimerman for their helpful comments during the annual meetings.

I would also like to thank the staff at the Department of Mathematics for their administrative support during my time as a graduate student. In particular I would like to thank Ida Bulat for her help during my first years here. I also thank Jemima Merisca, Sonja Injac, Patrina Seepersaud, Marija Ignjatovic, Ashley Armogan, and Prof. Murty’s assistants for all their help. As well I thank the Department of Mathematics for its financial support during my studies.

I thank the members of the GANITA lab past and present as well as the attendees of the seminar for giving me a welcoming environment to explore my ideas and for the fruitful discussions. Big thanks to Hubert Dubé, Payman Eskandari, Samprit Ghosh, Debanjana Kundu, Gaurav Patil, Shuyang Shen, Jyothsnaa Sivaraman, Shenhui Liu, Vandita Patel, Anup Dixit, Abhishek Oswal, Asif Zaman, and many others that I failed to mention.

I also thank all my friends, both mathematical and otherwise, for their companionship. Special thanks to Adrian Keet and Wei Xi Fan for the conversations over the years.

Lastly I thank my parents for their support and encouragement in my pursuit of studying mathematics, and for believing in me even when I did not.
# CONTENTS

1 INTRODUCTION AND OVERVIEW 1
   1.1 The least prime ideal in a conjugacy class 1
   1.2 Statement of results 3
      1.2.1 General results 3
      1.2.2 $S_n$-extensions 5
      1.2.3 Unconditional results for $D_n$ and Camina elements 10

2 REPRESENTATION THEORY 13
   2.1 Basic representation theory of finite groups 13
   2.2 The symmetric group $S_n$ 15
      2.2.1 Conjugacy classes of $S_n$ 15
      2.2.2 Specht modules 17
      2.2.3 Murnaghan-Nakayama rule 18
      2.2.4 Branching rule 19
      2.2.5 Application to $n$-cycles 21
   2.3 Representation theory of the dihedral group $D_n$ 22
   2.4 Camina elements and Camina groups 24

3 ARTIN $L$-FUNCTIONS 27
   3.1 Frobenius of a prime 27
   3.2 Definition of Artin $L$-functions 28
   3.3 Properties of Artin $L$-functions and Artin’s conjecture 30
   3.4 The Artin conductor and discriminant 31
   3.5 Zeros of Artin $L$-functions 32
   3.6 Additional assumptions on zeros 34

4 THE LEAST PRIME IN A CONJUGACY CLASS: ESTIMATES 38
   4.1 The kernel function 38
   4.2 Preliminary estimates 40
   4.3 The contour integral 42
   4.4 Contribution of zeros 49
   4.5 Deuring-Heilbronn phenomenon 53
   4.6 Proofs of main results 55
      4.6.1 Proof of Theorem 1.4 55
      4.6.2 Proof of Theorem 1.1 60
   4.7 Estimates under additional assumptions 68
5 THE LEAST PRIME IN A CONJUGACY CLASS: APPLICATIONS 71
5.1 Application to $S_n$-extensions 71
   5.1.1 The least prime in an $n$-cycle 71
   5.1.2 The least prime in an $(n - 1)$-cycle 77
   5.1.3 Unramified $A_n$-extensions of quadratic fields 83
5.2 The least prime which is a reflection 86
5.3 The least prime which is a Camina element 87
INTRODUCTION AND OVERVIEW

Let $L/K$ be a Galois extension of number fields with Galois group $G$. Given a prime $p$ of $K$ which is unramified in $L$ there is a corresponding Frobenius conjugacy class $\sigma_p$ of $G$. The Chebotarev density theorem states that given a conjugacy class $C$ of $G$, the primes of $K$ whose Frobenius $\sigma_p$ is in $C$ has density $|C|/|G|$. In particular there are infinitely many such primes. An important problem therefore is to obtain effective bounds on primes whose Frobenius appears in $C$. In this chapter we will discuss known results including the effective bounds of Lagarias-Montgomery-Odlyzko. For special choices of groups and conjugacy classes we are able to obtain much improved bounds.

For notational convenience, if $f(x), g(x)$ are functions with $g(x) \geq 0$ we will write

$$f(x) \ll g(x)$$

whenever there is a positive constant $c$ such that

$$|f(x)| \leq cg(x).$$

The constant $c$ is to be effective and absolute unless stated otherwise. For two functions $f(x), g(x) \geq 0$ we will also write

$$f(x) \asymp g(x)$$

to mean $f(x) \ll g(x)$ and $g(x) \ll f(x)$.

1.1 THE LEAST PRIME IDEAL IN A CONJUGACY CLASS

Let $L/K$ be a Galois extension of number fields with Galois group $G$. For each prime $p$ of $K$ unramified in $L$ denote by $\sigma_p$ its Frobenius class in $G$. Let $C$ be a subset of $G$ which is stable under conjugation and define

$$\pi_C(x, L/K) = \# \{ p : \sigma_p \subseteq C, N_{K/Q}p \leq x \}. $$
The Chebotarev density theorem [Tsc26] states that
\[ \pi_C(x, L/K) \sim \frac{|C|}{|G|} \text{Li}(x). \]

In particular there are infinitely many \( p \) with \( \sigma_p \subseteq C \) so one may ask for bounds on such primes \( p \) with least norm.

In [LO77], under the assumption of the generalized Riemann hypothesis Lagarias and Odlyzko proved an effective version of the Chebotarev density theorem which states that if \( \zeta_L \) satisfies the GRH then for every \( x > 2 \)
\[ |\pi_C(x, L/K) - \frac{|C|}{|G|} \text{Li}(x)| \ll \frac{|C|}{|G|} x^{1/2} \log(d_L x^n) + \log d_L \]
where the implicit constant is absolute and effective. As a corollary they obtain that the least prime \( p \) with \( \sigma_p \subseteq C \) satisfies
\[ N_{K/Qp} \ll (\log d_L)^2. \]

Unconditionally one only has much weaker bounds. In [LMO79] Lagarias, Montgomery, and Odlyzko prove that unconditionally one has
\[ N_{K/Qp} \ll d_L^A \]
for an absolute and effective constant \( A \). In [Zam17a] Zaman shows that one may take \( A = 40 \), later improved in [Zam17b] to \( A = 35 \), for sufficiently large \( d_L \) with some improvements depending on the extension \( L/K \). This was improved to \( A = 16 \) by Kadiri-Ng-Wong in [KNW19]. Unconditionally, Ahn and Kwon showed in [AK19] that we may take \( A = 12577 \) for all \( d_L \).

We may contrast this with Linnik’s theorem [Lin44a, Lin44b] on the least prime in arithmetic progressions which states the following. Let \( k \) be a positive integer and \( a \) coprime to \( k \). Then the least prime \( p \) lying in the arithmetic progression \( a + nk \) satisfies
\[ p \ll k^L \]
for an absolute and effective constant \( L \), commonly called Linnik’s constant. Numerically, Heath-Brown showed in [HB92] that \( L = 5.5 \) is admissible which was improved by Xylouris in [Xyl11] to \( L = 5 \).

In terms of the splitting of primes, if we let \( k = q \) be prime then a prime \( p \) lying in the arithmetic progression \( a + nq \) means that the Frobenius of \( p \) in the group \( \text{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q}) \cong (\mathbb{Z}/q)^\times \) corresponds to the residue class \( a \).
statement of results 3

The discriminant of \( \mathbb{Q}(\zeta_q) \) has absolute value \( q^{q-2} \) so we see that the unconditional Linnik’s theorem provides a bound which is as good as the conditional bounds of Lagarias-Odlyzko and significantly better than the unconditional bound of Lagarias-Montgomery-Odlyzko. One may therefore ask for other cases when the LMO bound may be improved.

1.2 STATEMENT OF RESULTS

In this thesis we prove an effective bound for the least prime in certain conjugacy classes of particular Galois extensions \( L/K \) of number fields. We do this by following technique of [LMO79] and assuming Artin’s conjecture for the extension \( L/K \). By assuming Artin’s conjecture we are free to work with Artin \( L \)-functions instead of the usual reduction to Hecke \( L \)-functions. This allows us to keep track of information coming from each irreducible character \( \chi \) and track our estimates in terms of the individual Artin conductors \( A_{\chi} \) instead of the discriminant \( d_L \) which may be much larger.

By working with the Artin \( L \)-functions directly we are also able to pinpoint the characters which contribute nontrivially. This lets us improve the known bounds for the least prime whose Frobenius lies in the class \( C \) when \( C \) has the property that many characters of Gal\((L/K)\) vanish at \( C \). In some cases where Artin’s conjecture is known we obtain unconditional bounds which are sharper than previously known bounds. We will discuss unconditional results in Section 1.2.3.

1.2.1 General results

We first state the result for a general Galois extension \( L/K \) of number fields. Denote by \( \text{nv}(C) \) the set of irreducible characters of \( G \) which are nonvanishing at \( C \). As well, let \( P(L/K) \) be the set of all rational primes \( q \) below primes \( q \) of \( K \) which ramify in \( L \). We prove the following general bound for the least prime in a conjugacy class \( C \) under the assumption of Artin’s conjecture.

**Theorem 1.1.** Let \( L/K \) be a Galois extension of number fields with \( \text{Gal}(L/K) = G \). Let \( C \) be a conjugacy class in \( C \) and assume Artin’s conjecture for the \( L \)-functions \( L(s, \chi) \) for \( \chi \in \text{nv}(C) \).
1. If $\prod_{\chi \in \text{nv}(C)} L(s, \chi)$ has no exceptional zeros, then there is an absolute effective constant $C_1$ such that the least prime $p$ of $K$ whose Frobenius in $G$ is $C$ satisfies

$$N_{K/Q} \leq \left( \prod_{\chi \in \text{nv}(C)} A^{(1)}_{\chi} \right)^{C_1 \left( \sum_{\chi \in \text{nv}(C)} |\chi(C)| |\chi(1)|^2 \right)^{1/2}} \cdot \left( \frac{|G|}{|C|} n_K \sum_{p \in P(L/K)} \log p \right)^{C_1} \cdot (1.2)$$

2. Suppose that there is an exceptional zero $\beta_0$ of $\prod_{\chi \in \text{nv}(C)} L(s, \chi)$. Suppose that $\sum_{\chi \in \text{nv}(C)} \chi$ always takes nonnegative values. Then there is an absolute effective constant $C_2$ such that

$$N_{K/Q} \leq \left( \prod_{\chi \in \text{nv}(C)} A^{(1)}_{\chi} \right)^{C_2} \left( \frac{|G|}{|C|} n_K \sum_{p \in P(L/K)} \log p \right)^{C_2} \cdot (1.3)$$

Remark 1.2. In Theorem 1.1, an additional hypothesis that $\sum_{\chi \in \text{nv}} \chi$ always takes nonnegative values is required in the case of an exceptional zero. However, in all of our applications this will hold.

Remark 1.3. For Theorem 1.1 we only need Artin’s conjecture to hold for those $\chi \in \text{Irr} \left( \text{Gal}(L/K) \right)$ that are nonvanishing at $C$. In particular since Artin’s conjecture is known for linear characters we will see in Section 1.2.3 that we obtain unconditional bounds for classes $C$ where only linear characters are nonvanishing at $C$.

The proof of Theorem 1.1 relies on a zero-free region of Artin $L$-functions. Unfortunately it is only truly free of zeros up to one possible exceptional zero. The proof therefore divides into two cases: when there is no such zero, and when the zero exists. For the case of an exceptional zero we prove a version of the Deuring-Heilbronn phenomenon for $L$-functions with nonnegative Dirichlet series coefficients.

Theorem 1.4. Let $L/K$ be a Galois extension and $\varphi$ a (not necessarily irreducible) character of $G = \text{Gal}(L/K)$ which decomposes into a sum of irreducible characters as

$$\varphi = \sum_{i=1}^{m} a_i \chi_i$$
where \( a_i \) are positive integers and \( \langle \varphi, 1_G \rangle > 0 \). Suppose that for each \( g \in G \)

\[
\varphi(g) = \sum_{i=1}^{m} a_i \chi_i(g) \geq 0.
\]

Assume the Artin conjecture for the L-functions \( L(s, \chi_i) \) holds.

Then there are effective and absolute positive constants \( C_3, C_4 \) such that if \( \beta_0 \) is a real zero of \( L(s, \varphi) \) then there are no zeros in the region

\[
1 - C_4 \log \left( \frac{C_3 \langle \varphi, 1_G \rangle L(1)}{L(t)(1-\beta_0)} \right) < \sigma < 1
\]

where

\[
L(t) := \sum_{i=1}^{m} a_i \chi_i(1)(\log A_{\chi_i} + n_K \chi_i(1) \log(|t|+2)).
\]

### 1.2.2 \( S_n \)-extensions

As an application we prove upper bounds for the least prime \( p \) in \( C \) when \( \text{Gal}(L/K) = S_n \) and \( C \) is the class of \( n \)-cycles or \((n-1)\)-cycles. It will be in terms of hook characters which are characters corresponding to particular Young diagrams called hooks. The representation theory of \( S_n \) will be reviewed in Chapter 2 but briefly we recall here that there exist canonical bijections between the following three objects:

- Irreducible characters of \( S_n \)
- Young diagrams of size \( n \)
- Partitions of \( n \)

**Notation 1.5.** We write partitions in nonincreasing order. That is, if \( a_1 + a_2 + \cdots + a_m = n \) with \( a_1 \geq a_2 \geq \cdots \geq a_m \) we write \( (a_1, a_2, \ldots, a_m) \) for the corresponding partition. Each partition \( (a_1, a_2, \ldots, a_m) \vdash n \) defines a Young diagram with row lengths \( a_1, a_2, \ldots, a_m \). A partition of the form \((n-r, 1^r)\) and its corresponding Young diagram is called a hook. If \( \lambda \vdash n \) is a partition of \( n \) then we write \( \chi^\lambda \) for the corresponding irreducible character of \( S_n \). In particular if \( \lambda \) is a hook then \( \chi^\lambda \) is a hook character.

In Chapter 2 we will also see that the hook characters are exactly the characters which are nonvanishing at \( n \)-cycles and furthermore that they take values \( \pm 1 \) at \( n \)-cycles. In the setting of \( S_n \)-extensions and \( C \) the class of \( n \)-cycles we may phrase Theorem 1.1 as follows:
Theorem 1.6. Let \( L/K \) be a Galois extension of number fields with \( \text{Gal}(L/K) = S_n \). Suppose the Artin conjecture holds for \( L \)-functions attached to hook characters \( \chi^\Lambda \). Then the least prime \( p \) of \( K \) which is unramified in \( L \) and whose Frobenius in \( \text{Gal}(L/K) \) is an \( n \)-cycle satisfies the following:

1. If \( \prod_{\lambda = (n-r,1')} L(s, \chi^\Lambda) \) has no exceptional zeros, then there is an absolute effective constant \( C_5 \) such that
   \[
   N_{K/Q}p \leq \left( \prod_{\lambda = (n-r,1')} A_{\chi^\Lambda}^{\chi^\Lambda(1)} \right)^{C_5} \left( \frac{nn_{K}}{\sum_{p \in P(L/K)} \log p} \right)^{C_5}.
   \]

2. If \( \prod_{\lambda} L(s, \chi^\Lambda) \) admits an exceptional zero \( \beta_0 \) then there is an absolute effective constant \( C_6 \) such that
   \[
   N_{K/Q}p \leq \left( \prod_{\lambda = (n-r,1')} A_{\chi^\Lambda}^{\chi^\Lambda(1)} \right)^{C_6} \left( \frac{nn_{K}}{\sum_{p \in P(L/K)} \log p} \right)^{C_6}.
   \]  \hspace{1cm} (1.4)

Remark 1.7. Whereas the LMO bound depends on the discriminant of the extension field \( L \), our result depends on the product of Artin conductors \( A_{\chi^\Lambda}^{\chi^\Lambda(1)} \) attached to hook characters, as well as a small factor coming from ramification in the extension \( L/K \).

In terms of the extension \( L/K \) we may rewrite the bound as follows.

Corollary 1.8. Suppose \( L/K \) is a Galois extension of number fields such that \( \text{Gal}(L/K) = S_n \). Assume Artin’s conjecture holds for \( L \)-functions of hook characters. Then there are effective absolute constants \( C_7, C_8, C_9 \) such that the least prime \( p \) of \( K \) whose Frobenius in \( \text{Gal}(L/K) \) is an \( n \)-cycle satisfies

\[
N_{K/Q}p \leq d_L^{C_7/(n[L:K])} \left( \prod_{\lambda = (n-r,1')} N_{K/Q}A_{\chi^\Lambda}^{\chi^\Lambda(1)} \right)^{C_8} \left( \log d_L \right)^{C_9}.
\]

In particular,

\[
N_{K/Q}p \leq d_L^{C_7/(n[L:K])} \left( N_{K/Q}d_{L/K} \right)^{C_8} \left( \log d_L \right)^{C_9}.
\]  \hspace{1cm} (1.5)

If \( \prod_{\lambda = (n-r,1')} L(s, \chi^\Lambda) \) admits an exceptional zero then instead

\[
N_{K/Q}p \leq d_L^{C_7/(n[L:K])} \left( N_{K/Q}d_{L/K} \right)^{C_{10}} \left( \log d_L \right)^{C_9}.
\]

where \( C_{10} \) is also an effective and absolute constant.
Using similar techniques we also prove an upper bound for the least prime whose Frobenius is an \((n-1)\)-cycle. Here in place of hook characters we consider characters \(\chi^\mu\) corresponding to partitions \(\mu \vdash n\) of the form 
\[ \mu = (n-r, 2, 1^{r-2}) \] 
for \(2 \leq r \leq n-2\).

**Theorem 1.9.** Suppose \(L/K\) is a Galois extension of number fields such that \(\text{Gal}(L/K) = S_n\). Assume Artin’s conjecture holds for \(L\)-functions of characters \(\chi^\mu\) corresponding to partitions \(\mu \vdash n\) of the form 
\[ \mu = (n-r, 2, 1^{r-2}) \] 
Then the least prime \(p\) of \(K\) which is unramified in \(L\) and whose Frobenius in \(\text{Gal}(L/K)\) is an \((n-1)\)-cycle satisfies the following:

1. If \(\zeta_K(s)L(s, \text{sgn}) \prod_\mu L(s, \chi^\mu)\) has no exceptional zeros, then there is an effective constant \(C_{11}\) such that
\[
N_{K/Q} p \leq \left( d_KA_{\text{sgn}} \prod_{\mu=(n-r,2,1^{r-2})} A_{\chi^{\mu}}^{(1)} \right)^{C_{11} n^{1/2} 2^{5n}} \cdot \left( (n-1)n_K \sum_{p \in \mathcal{P}(L/K)} \log p \right)^{C_{11}}.
\]

2. Assume furthermore that Artin’s conjecture holds for \(L\)-functions of hook characters \(\chi^\lambda\). Then if \(\prod_\lambda L(s, \chi^\lambda) \prod_\mu L(s, \chi^\mu)\) admits an exceptional zero \(\beta_0\) then there is an effective constant \(C_{12}\) such that
\[
N_{K/Q} p \leq \left( \prod_{\lambda=(n-r,1^r)} A_{\chi^{\lambda}}^{(1)} \prod_{\mu=(n-r,2,1^{r-2})} A_{\chi^{\mu}}^{(1)} \right)^{C_{12}} \cdot \left( (n-1)n_K \sum_{p \in \mathcal{P}(L/K)} \log p \right)^{C_{12}}.
\]

As before we may rewrite in terms of \(d_L\).

**Corollary 1.10.** Suppose \(L/K\) is a Galois extension of number fields such that \(\text{Gal}(L/K) = S_n\). Assume Artin’s conjecture holds for \(L\)-functions of hook characters and characters corresponding to partitions of the form \((n-r, 2, 1^{r-2})\). Then there are effective absolute constants \(C_{13}, C_{14}, C_{15}\) such that the least prime \(p\) of \(K\) whose Frobenius in \(\text{Gal}(L/K)\) is an \((n-1)\)-cycle satisfies
\[
N_{K/Q} \leq d_L^{nC_{13}/[L:K]} (N_{K/Q} d_{L/K})^{C_{14}} (\log d_L)^{C_{15}}. \tag{1.6}
\]
If \( \prod_\lambda L(s, \chi^\lambda) \prod_\mu L(s, \chi^\mu) \) admits an exceptional zero then instead we have

\[
N_{K/Q} \leq d_L^{nC_{13}/[L:K]} (N_{K/Q} d_{L/K})^{C_{16}} (\log d_L)^{C_{15}}.
\]

where \( C_{16} \) is another effective absolute constant.

**Remark 1.11.** Since from Stirling’s approximation we know that

\[
[L : K] = n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n
\]

we see that the exponent of \( d_L \) in Equation 1.5 and Equation 1.6 is much smaller than in Equation 1.1.

In Corollary 1.8 unfortunately we have an exponential power of the relative discriminant \( N_{K/Q} d_{L/K} \) coming from the bound

\[
\prod_{\lambda = (n-r,1')} N_{K/Q} f_{\chi^\lambda}^{(1)} \leq N_{K/Q} d_{L/K}.
\]

The left-hand side is 1 if all \( f_{\chi^\lambda} \) is trivial for \( \lambda = (n-r,1') \), so in particular if \( L/K \) is unramified this will be the case. By a result of Fröhlich [Frö62] there exist infinitely many unramified extensions \( L/K \) with \( \text{Gal}(L/K) = S_n \).

When \( L \) is an \( S_n \)-extension over \( Q \) which is unramified over a quadratic field we still obtain an upper bound which is stronger than the LMO bound. Such extensions have been studied by Uchida [Uch70], Yamamoto [Yam70], Elstrodt-Grunewald-Mennicke [EGM85], Kondo [Kon95], and Kedlaya [Ked12] and it is known that there are infinitely many such extensions. For such a family we have the following upper bound:

**Theorem 1.12.** Let \( L/F/Q \) be a tower of number fields such that \( L/Q \) is an \( S_n \)-extension and \( F \) is a quadratic field of discriminant \( \Delta \) such that \( L/F \) is an unramified \( A_n \)-extension.

1. Suppose Artin’s conjecture holds for hook characters of \( \text{Gal}(L/Q) = S_n \). Then there exist constants \( C_{17}, C_{18} \) such that the least prime \( p \) whose Frobenius in \( L \) is an \( n \)-cycle satisfies

\[
p \leq \Delta^{C_{17}/n^3/2} (n \log \Delta)^{C_{18}}.
\]

In particular, we have

\[
p \leq d_L^{2C_{17}/(n^3/2)[L:Q]} \left( \frac{2}{(n-1)!} \log d_L \right)^{C_{18}}.
\]

2. Suppose furthermore that Artin’s conjecture holds for characters corresponding to partitions of the form \( \mu = (n-r,2,1^{r-2}) \). Then there exist
constants $C_{19}, C_{20}$ such that the least prime $p$ whose Frobenius in $L$ is an $(n-1)$-cycle satisfies

$$p \leq \Delta^{n^2 C_{19}} ((n-1) \log \Delta)^{C_{20}}.$$  \hfill (1.9)

In particular, we have

$$p \leq d_L^{2n^2 C_{19}/[L:Q]} \left( \frac{2(n-1)}{n!} \log d_L \right)^{C_{20}}.$$  \hfill (1.10)

Under some stronger assumptions we may ensure that the exponent of $N_{K/Q^{\text{L/k}}}$ in Corollary 1.8 is constant in all cases as follows.

**Theorem 1.13.** Let $L/K$ be a Galois extension with $\text{Gal}(L/K) = S_n$. Suppose the following holds for all hook characters $\chi$:

1. Artin’s conjecture for $L(s, \chi)$
2. For $1 < \sigma < 3$ we have
   $$\left| \frac{L'}{L}(s, \chi) \right| \ll \frac{1}{s-1} + \log A_{\chi}$$
3. There exists a constant $c$ such that $L(s, \chi)$ has at most one zero in the region
   $$1 - \frac{c}{\chi(1)(\log A_{\chi} + \chi(1)n_K \log(|t| + 2))} \leq \sigma \leq 1.$$

Then there exists an absolute effective constant $C_{21}$ such that the least prime $p$ of $K$ whose Frobenius in $G$ is an $n$-cycle satisfies

$$N_{K/Q^{\text{L/k}}} \leq \left( \prod_{\lambda=(n-r,1')} A_{\chi_{\lambda}}^{\chi_{\lambda}(1)} \right)^{C_{21}} \left( n_{K} \sum_{p \in \mathcal{P}(L/K)} \log p \right)^{C_{21}}.$$

In particular, we have

$$N_{K/Q^{\text{L/k}}} \leq d_L^{2n^{2}/[L:K]} (N_{K/Q^{\text{L/k}}})^{C_{21}} (\log d_L)^{C_{22}}$$

where $C_{22}$ is another effective absolute constant.

As in the case of $n$-cycles, under stronger assumptions we obtain a constant exponent of $N_{K/Q^{\text{L/k}}}$ in the upper bound for the least prime whose Frobenius is an $(n-1)$-cycle.
**Theorem 1.14.** Let $L/K$ be a Galois extension with $\text{Gal}(L/K) = S_n$. Suppose the following holds for characters $\chi$ corresponding to partitions $\lambda, \mu \vdash n$ where $\lambda$ is a hook and $\mu$ is of the form $(n - r, 2, 1^{r-2})$:

1. Artin’s conjecture for $L(s, \chi)$

2. For $1 < \sigma < 3$ we have
   \[
   \left| \frac{L'}{L}(s, \chi) \right| \ll \frac{1}{s-1} + \log A_{\chi}
   \]

3. There exists a constant $c$ such that $L(s, \chi)$ has at most one zero in the region
   \[
   1 - \frac{c}{\chi(1)(\log A_{\chi} + \chi(1)n_K \log(|t|+2))} \leq \sigma \leq 1.
   \]

Then there exists an absolute effective constant $C_{23}$ such that the least prime $p$ of $K$ whose Frobenius in $G$ is an $n$-cycle satisfies

\[
N_{K/Q}p \leq \left( \prod_{\lambda=(n-r,1')} A_{\chi_{\lambda}}^{\chi_{\lambda}(1)} \prod_{\mu=(n-r,2,1^{r-2})} A_{\chi_{\mu}}^{\chi_{\mu}(1)} \right)^{C_{25}} 
\]

\[
\cdot \left( nn_K \sum_{p \in \mathcal{P}(L/K)} \log p \right)^{C_{23}}.
\]

In particular, we have

\[
N_{K/Q}p \leq d_{L}^{n_{C_{24}}/[L:K]} (N_{K/Q}\mathfrak{d}_{L/K})^{C_{25}} (\log d_{L})^{C_{26}}
\]

for absolute effective constants $C_{24}, C_{25}, C_{26}$.

### 1.2.3 Unconditional results for $D_n$ and Camina elements

We can apply Theorem 1.11 to the case when the Galois group is either dihedral, or when the conjugacy class $C$ has the property that only linear characters are nonvanishing at $C$. Elements of such a class are called Camina elements and will be discussed in Section 2.4. For these cases all relevant $L$-functions are known to satisfy Artin’s conjecture and so all results in this subsection are unconditional.

For dihedral extensions we prove the following upper bound on the least prime whose Frobenius is a reflection. Denote by $D_n$ the dihedral group
statement of results

of order $2n$. As well, for any group $G$ we write $G'$ for the commutator subgroup and $G^{ab} = G/G'$ for the abelianization of $G$.

**Theorem 1.15.** Let $L/K$ be a Galois extension of number fields with $G = \text{Gal}(L/K) = D_n$. Let $F = L^{G'}$ denote the maximal abelian subextension of $L/K$. Then there are absolute effective constants $C_{27}, C_{28}$ such that the least prime $p$ whose Frobenius is a reflection in $G$ satisfies

$$N_{K/Q} p \leq d_F^{C_{27}} \left( n_K \sum_{p \in \mathcal{P}(L/K)} \log p \right)^{C_{28}}.$$

In terms of the discriminant of $L$ we may rewrite as follows:

**Corollary 1.16.** Let $L/K$ be a Galois extension of number fields with $G = \text{Gal}(L/K) = D_n$. Then there are absolute effective constants $C_{29}, C_{30}$ such that the least prime whose Frobenius in $G$ is a reflection satisfies

$$N_{K/Q} p \leq d_L^{C_{29}/[L:K]} (\log d_L)^{C_{30}}.$$

**Remark 1.17.** Corollary 1.16 is consistent with a conjecture of V.K. Murty [KMoo, Conjecture 2.1] which states that there are absolute constants $a, b$ such that for any conjugacy class $C$ the least prime $p$ whose Frobenius is $C$ satisfies

$$N_{K/Q} p \ll d_L^{a/[L:K]} (\log d_L)^b.$$

**Remark 1.18.** In [Wei83] Weiss proves an upper bound, later made explicit by Thorner-Zaman in [TZ17], for the least prime in a conjugacy class $C$ when there is a large abelian subgroup $A$ that has nontrivial intersection with $C$. This upper bound is in terms of the largest conductor of the abelian extension $L/L^A$. However, in our case the class of reflections in the dihedral group does not intersect a large abelian subgroup, and the upper bound for the least prime is in terms of the abelian extension $F/K$ over the base field $K$.

Similarly we obtain the following bound for the least prime whose Frobenius is a Camina element, that is an element $g$ such that the only irreducible characters that are nonvanishing at $g$ are linear. A group $G$ is a Camina group if all elements $g \in G - G'$ are Camina elements. We will review Camina elements and Camina groups in Section 2.4.

**Theorem 1.19.** Let $L/K$ be a Galois extension of number fields with Galois group $G = \text{Gal}(L/K)$. Suppose that $g \in G$ is a Camina element, and let $C$ denote its conjugacy class. Let $F = L^{G'}$ be the maximal abelian subextension of $L/K$. 
1. If $\zeta_F(s)$ has no exceptional zeros then there is an absolute effective constant $C_{31}$ such that the least prime $p$ of $K$ whose Frobenius in $G$ is $C$ satisfies
\[
N_{K/Q} \leq d_F^{C_{31}|L:F|^{1/2}} \left( \frac{|G|}{|C|} n_K \sum_{p \in P(L/K)} \log p \right)^{C_{31}}.
\]

2. If $\zeta_F(s)$ has an exceptional zero $\beta_0$ then there is an absolute effective constant $C_{32}$ such that
\[
N_{K/Q} \leq d_F^{C_{32}} \left( \frac{|G|}{|C|} n_K \sum_{p \in P(L/K)} \log p \right)^{C_{32}}.
\]

In particular if $G$ is a Camina group then the above holds for any $C$ which is the class of any element not in $G'$.

As before we can rephrase in terms of the discriminant $d_L$ of the extension field as follows.

**Corollary 1.20.** Let $L/K$ be a Galois extension of number fields with Galois group $G = \text{Gal}(L/K)$. Suppose that $g \in G$ is a Camina element and let $C$ denote its conjugacy class. Let $F$ be the maximal abelian subextension of $L/K$.

1. If $\zeta_F(s)$ has no exceptional zeros then there are absolute effective constants $C_{33}, C_{34}$ such that the least prime $p$ whose Frobenius in $G$ is $C$ satisfies
\[
N_{K/Q} \leq d_L^{C_{33}/|L:F|^{1/2}} (\log d_L)^{C_{34}}.
\] (1.11)

2. If $\zeta_F(s)$ has an exceptional zero $\beta_0$ then there are absolute effective constants $C_{35}, C_{36}$ such that
\[
N_{K/Q} \leq d_L^{C_{35}/|L:F|} (\log d_L)^{C_{36}}.
\]

In any case Equation 1.11 holds.

**Remark 1.21.** We observe that we obtain a power saving for the norm of the least prime $p$ over applying the Lagarias-Montgomery-Odlyzko bound directly.
In this chapter we recall the representation theory of finite groups. The material in the first section is standard and may be found in any text on representation theory, for example [Ser77] or [Isa76]. In the subsequent sections we will recall some facts about the symmetric group $S_n$ as well as its representation theory following the exposition in [Sag01]. We then review the representation theory of dihedral groups. Lastly we discuss Camina elements and Camina groups.

### 2.1 Basic Representation Theory of Finite Groups

Let $G$ be a finite group. Denote by $\text{Irr}(G)$ the set of characters of irreducible complex representations of $G$. Recall that there is an inner product defined on the set of all class functions of $G$ by

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}.$$  \hfill (2.1)

By Maschke’s theorem every complex representation of $G$ decomposes as a direct sum of irreducible representations. If $\psi$ is the character of a representation and $\chi$ is an irreducible representation then $\langle \psi, \chi \rangle$ gives the multiplicity of $\chi$ in $\psi$. In particular if $1_G$ is the trivial character then $\langle \psi, 1_G \rangle$ gives the dimension of the fixed subspace of $\psi$.

With respect to this pairing the irreducible characters of $G$ satisfy the orthogonality relation

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \chi = \psi \\ 0 & \chi \neq \psi. \end{cases}$$

commonly called the first orthogonality relation. The irreducible characters in fact form an orthonormal basis of the space of all class functions of $G$ and therefore there is a (non-canonical) bijection between the conjugacy classes of $G$ and the irreducible characters of $G$. 
The irreducible characters also satisfy a second orthogonality relation as follows: if $g, h \in G$ then

$$
\sum_{\chi \in \text{Irr}(G)} \overline{\chi(g)} \chi(h) = \begin{cases} |G|/|C| & \text{if } g, h \text{ are conjugate} \\ 0 & \text{otherwise} \end{cases}
$$

(2.2)

where $C$ denotes the conjugacy class of $g$.

If $H$ is a subgroup of $G$ and $\chi$ is a character of $G$ corresponding to a representation $\rho$ the restriction $\text{Res}_H^G \chi$ of $\chi$ is defined to be the character of the restriction $\rho|_H$. If $\psi$ is a character of $H$ then we define the induction of $H$ to $G$ as follows. Let $r = [G : H]$ and $g_1, \ldots, g_r$ be a complete system of coset representatives of $H$. Extend $\psi$ to all of $G$ by defining

$$
\tilde{\psi}(g) = \begin{cases} \psi(g) & g \in H \\ 0 & g \notin H \end{cases}.
$$

Then

$$
(\text{Ind}_H^G \psi)(g) = \sum_{i=1}^r \tilde{\psi}(g_1^{-1} g g_i) = \frac{1}{|H|} \sum_{s \in G} \tilde{\psi}(s^{-1} g s).
$$

If $\chi$ is a character of $G$ and $\psi$ is a character of $H$ then Frobenius reciprocity states that

$$
\langle \text{Res}_H^G \chi, \psi \rangle = \langle \chi, \text{Ind}_H^G \psi \rangle.
$$

**Example 2.1.** The regular representation

$$
\text{reg}_G = \sum_{\chi} \chi(1) \chi
$$

may be written as

$$
\text{reg}_G = \text{Ind}_{\{1\}}^G 1
$$

This is the induction of the trivial character from the trivial subgroup. From this one obtains the well-known formula

$$
|G| = \text{reg}_G(1) = \sum_{\chi} \chi(1)^2.
$$

**Definition 2.2.** A character $\chi$ is linear if it is of degree 1, that is if $\chi(1) = 1$. Denote by $\text{lc}(G)$ the set of linear characters of $G$.

Every linear character is a homomorphism $\chi : G \to \mathbb{C}^\times$ so in particular it is irreducible and nonvanishing for every $g \in G$. Since it is a homomorphism to an abelian group it factors through the abelianization $G^{ab} = G/G'$. Furthermore each irreducible character of $G^{ab}$ defines an
irreducible character of $G$ by inflation, so there is a bijection between the linear characters of $G$ and the irreducible characters of $G^{ab}$.

We also note the following simple observation when summing over linear characters.

**Proposition 2.3.** For any $g \in G$,

$$
\sum_{\chi \in \text{lc}(G)} \chi(g) = \begin{cases} 
|G^{ab}| & g \in G' \\
0 & g \notin G'.
\end{cases}
$$

In particular, $\sum_{\chi \in \text{lc}(G)} \chi(g) \geq 0$ for all $g \in G$.

**Proof.** Observe that $\sum_{\chi \in \text{lc}(G)} \chi$ is simply the regular character of $G/G'$ composed with the quotient map. \square

For each $g \in G$, it will be useful to give a name to the set of irreducible characters of $G$ which are nonvanishing at $g$.

**Definition 2.4.** Let $G$ be a group and $g \in G$. Define the set of nonvanishing characters of $G$ at $g$ to be

$$
\text{nv}(g) = \{\chi \in \text{Irr}(G) : \chi(g) \neq 0\}.
$$

It is clear that every linear character is nonvanishing for every $g \in G$ and that $\text{nv}(g)$ depends only on the conjugacy class of $g$. We will be paying special attention to classes $C$ of $G$ at which many characters vanish, that is $\text{nv}(C)$ is small.

## 2.2 The Symmetric Group $S_n$

In this section we review the representation theory of $S_n$. We first recall the canonical correspondence between conjugacy classes of $S_n$, partitions of $n$, and Young diagrams of size $n$, followed by a review of the correspondence between conjugacy classes and irreducible characters of $S_n$ given by Specht modules. Next we review the Murnaghan-Nakayama rule for computing character values and the branching rule for induction and restriction. Lastly we apply to the class of $n$-cycles which is the case of most interest to us. The material is standard and may be found for example in [Sag01].

### 2.2.1 Conjugacy Classes of $S_n$

The symmetric group $S_n$ is the group of permutations of the set $\{1, 2, \ldots, n\}$. There are $n!$ possible permutations so $|S_n| = n!$. The group $S_n$ is generated by transpositions and the set of all permutations that can be written as a
product of even number of transpositions forms the alternating group \( A_n \) which is the unique index 2 subgroup of \( S_n \). \( A_n \) is also the commutator subgroup of \( S_n \) so it follows that \( S_n \) has two linear characters. They are the trivial character and the sign character defined by \( \text{sgn}(\tau) = -1 \) for a transposition \( \tau \) and extended to all of \( S_n \) multiplicatively.

Each permutation \( \sigma \in S_n \) can also be written uniquely as a product of disjoint cycles and two permutations \( \sigma, \tau \) are conjugate if and only if they have the same cycle structure. In particular the set of permutations which consist of a single cycle of length \( n \) forms a conjugacy class which we will call the class of \( n \)-cycles. The conjugacy classes of \( S_n \) are also in bijection with partitions of \( n \).

**Definition 2.5.** A composition of \( n \) is an ordered sequence \((a_1, a_2, \ldots, a_k)\) of positive integers such that \( \sum_i a_i = n \). A partition of \( n \) is a composition which is nonincreasing. We write \( \lambda \vdash n \) to mean that \( \lambda \) is a partition of \( n \).

The conjugacy classes of \( S_n \) are easily seen to be in bijection with the partitions of \( n \) sending the class of permutations with cycle structure \((\lambda_1, \lambda_2, \ldots, \lambda_l)\), written in decreasing order, to the partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n \). Partitions \( \lambda \vdash n \) are also in bijection with Young diagrams of size \( n \) by sending a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \) to a Young diagram of shape \((\lambda_1, \lambda_2, \ldots, \lambda_l)\) which is an array of \( n \) boxes left-aligned where the \( i \)th row counting from the top has length \( \lambda_i \). We will adopt the convention that the box in row \( i \) and column \( j \) has coordinates \((i, j)\) and that row lengths are nonincreasing.

In summary, the following four objects are in bijection with one another:

- Conjugacy classes of \( S_n \)
- Partitions of \( n \)
- Young diagrams of size \( n \)
- Irreducible characters of \( S_n \)

\[ \text{Figure 2.1: A Young diagram of shape (5,3,3,1).} \]
2.2.2  Specht modules

The bijection between conjugacy classes and irreducible characters is in general not canonical, but in the case of $S_n$ it can be made canonical in the following way.

A *Young tableau* of shape $\lambda \vdash n$ is a Young diagram of shape $\lambda$ with the boxes filled with the numbers $1, 2, \ldots, n$ bijectively. Two Young tableaux $t_1, t_2$ of the same shape are *row equivalent* if corresponding rows have the same elements. A row equivalence class of Young tableaux $t$ of shape $\lambda$ is called a *tabloid* of shape $\lambda$ and written $\{t\}$. Graphically a tabloid will be denoted by an array of integers in shape $\lambda$ with horizontal lines between rows.

If $t$ is a tableau with rows $R_1, R_2, \ldots, R_l$ and columns $C_1, C_2, \ldots, C_k$ then the *row stabilizer* and *column stabilizer* of $t$ are defined respectively by

$$R_t = S_{R_1} \times S_{R_2} \times \cdots \times S_{R_l}$$

and

$$C_t = S_{C_1} \times S_{C_2} \times \cdots \times S_{C_k}$$

where $S_{R_i}$ and $S_{C_j}$ are the groups of permutations of $R_i$ and $C_j$ respectively. If $t$ is a tableau then the associated *polytabloid* is defined to be

$$e_t = \left( \sum_{\pi \in C_t} \text{sgn}(\pi) \pi \right) \{t\}.$$

Finally if $\lambda$ is any partition, the *Specht module* $S^\lambda$ is the module spanned by the polytabloids $e_t$ where $t$ ranges over tableau of shape $\lambda$. They are in fact cyclic so are spanned by any one polytabloid $e_t$. Over $\mathbb{C}$ the Specht modules are irreducible and form a complete list of irreducible representations of $S_n$. Denote by $\chi^\lambda$ by the character of $S^\lambda$.

A basis for $S^\lambda$ is given by the set of $e_t$ where $t$ range over standard Young tableaux of shape $\lambda$, that is Young tableaux whose rows and columns are increasing. Therefore the dimension of $S^\lambda$ and therefore the character degree $\chi^\lambda(1)$ is equal to the number $f^\lambda$ of standard Young tableaux of shape $\lambda$.

The number $f^\lambda$ may be computed using the *hook formula*. Given a Young diagram $\lambda$ and a node $(i, j)$, the *hook* $H_{i,j}$ is the subdiagram contained in
2.2 The symmetric group $S_n$

Figure 2.3: A Young diagram of shape $(6,4,4,2,1)$ with the hook at $(2,2)$.

$\lambda$ consisting of the node $(i, j)$ as well as all nodes to the right of $(i, j)$ and below $(i, j)$. The hooklength $h_{i,j}$ of a hook $H_{i,j}$ is the number of nodes in $H_{i,j}$.

Finally the hook formula states that if $\lambda \vdash n$ then

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}.$$

2.2.3 Murnaghan-Nakayama rule

Let $\chi$ be an irreducible character of $S_n$. From the previous section we know that $\chi = \chi^\lambda$ for some partition $\lambda$. If $\rho$ is another partition then we will write $\chi^\lambda_\rho$ for $\chi^\lambda(\rho)$. The character values of $S_n$ are completely described by the Murnaghan-Nakayama rule. We will now recall some preliminary notions before stating the rule.

A generalized tableau $T$ of shape $\lambda$ is obtained by filling a Young diagram of shape $\lambda$ by positive integers with possible repetition. The content of a generalized tableau is the composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ where $\alpha_i$ is the number of times the integer $i$ appears in $T$.

A skew diagram is the difference of two Young diagrams. That is, given $\mu \subseteq \lambda$ as two Young diagrams the skew diagram $\lambda/\mu$ is the difference

$$\lambda/\mu = \{ c : c \in \lambda \text{ and } c \notin \mu \}.$$

A rim hook is a connected skew diagram containing no $2 \times 2$ squares. Given a rim hook $\xi$, its leg length $ll(\xi)$ is defined to be the number of rows of $\xi$ minus 1.

Finally a rim hook tableau of shape $\lambda$ and content $\alpha = (a_1, a_2, \ldots, a_k)$ is a generalized tableau $T$ of shape $\lambda$ and content $\alpha$ satisfying the following:

- The rows and columns of $T$ are nondecreasing.
- For each $i$, the boxes with entries $a_i$ form a single rim hook $\xi^{(i)}$. 
The sign of a rim hook tableau with rim hooks $\xi^{(i)}$ is defined to be

$$(-1)^T := \prod_{\xi^{(i)} \in T} (-1)^{ll(\xi^{(i)})}.$$ 

We may now state the Murnaghan-Nakayama rule. Let $\lambda$ be a partition of $n$ and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be any composition of $n$. Then

$$\chi^{\lambda}_{\alpha} = \sum_T (-1)^T$$

where the sum is over all rim hook tableaux of shape $\lambda$ and content $\alpha$.

There is also a recursive formulation of the Murnaghan-Nakayama rule. With $\lambda$ and $\alpha$ as before, we have

$$\chi^{\lambda}_{\alpha} = \sum_{\xi} (-1)^{ll(\xi)} \chi^{\lambda \setminus \xi}_{\alpha \setminus \alpha_1}$$

where the sum is over all rim hooks $\xi$ of $\lambda$ with $\alpha_1$ cells such that its removal from $\lambda$ leaves a valid Young diagram. Here $\alpha \setminus \alpha_1$ is the composition obtained by removing the first element $\alpha_1$ from $\alpha$. The recursion terminates at the base case $\chi^{(i)}_{(i)} = 1$.

### 2.2.4 Branching rule

Consider a chain of subgroups $S_{n-1} \subseteq S_n \subseteq S_{n+1}$. The branching rule describes the decomposition of a character of $S_n$ induced to $S_{n+1}$, as well as the decomposition of a character of $S_n$ restricted to $S_{n-1}$.

We may describe the branching in terms of Young diagram as follows. Let $\lambda \vdash n$ with a corresponding Young diagram. Denote by $\lambda^+$ any Young diagram obtained by adding one box to $\lambda$ in such a way so that the resulting shape is a Young diagram. Similarly denote by $\lambda^-$ any Young diagram obtained by removing a box from $\lambda$ in such a way that results in a Young diagram.

**Theorem 2.6** (Branching rule). If $\lambda \vdash n$ then
2.2 THE SYMMETRIC GROUP $S_n$

1.

$$\text{Res}^{S_n}_{S_{n-1}} S^\lambda \cong \bigoplus_{\lambda^-} S^{\lambda^-}$$

and

2.

$$\text{Ind}^{S_{n+1}}_{S_n} S^\lambda \cong \bigoplus_{\lambda^+} S^{\lambda^+}.$$

Proof. See [Sag01, Theorem 2.8.3].

Example 2.7. Let $n = 6$ and $\lambda = (3,1^3) \vdash 6$. Then the Young diagram corresponding to $\lambda$ is given by

```
+ + +
+ +
+ + +
+ +
+ +
+ +
```

The Young diagrams $\lambda^+$ are obtained by adding one box in a way such that the resulting shape is a Young diagram. Hence $\lambda$ branches as follows

```
+ + + + + +
+ + + + + +
+ + +
+ +
+ +
+ +
```

and therefore

$$\text{Ind}^{S_7}_{S_6} \chi^{(3,1^3)} = \chi^{(3,1^4)} + \chi^{(3,2,1^2)} + \chi^{(4,1^3)}.$$  

Likewise the boxes that can be removed from $\lambda$ to obtain a Young diagram are given by

```
+ + + + + +
+ + + + + +
+ + +
+ +
+ +
+ +
```

Hence by removing one of the boxes we see that $\lambda^-$ are given by

```
+ + + + +
+ + + + +
+ + +
+ +
+ +
```

and therefore

$$\text{Res}^{S_6}_{S_5} \chi^{(3,1^3)} = \chi^{(3,1^2)} + \chi^{(2,1^3)}.$$
2.2.5 Application to $n$-cycles

Consider the class of $n$-cycles which corresponds to the partition $(n)$. We will now apply the results from the previous sections to obtain the character values $\chi^\lambda_{(n)}$ as well as the character degrees $\chi^\lambda(1)$.

**Proposition 2.8.** We have

$$
\chi^\lambda_{(n)} = \begin{cases} 
(-1)^{n-r} \quad &\lambda = (r, 1^{n-r}), \\
0 \quad &\text{otherwise}
\end{cases}
$$

**Proof.** The formula in fact may be proven independently of the Murnaghan-Nakayama rule, for example [Sagan, Lemma 4.10.3]. We will give a proof as an application of the Murnaghan-Nakayama rule.

By the Murnaghan-Nakayama rule we have

$$
\chi^\lambda_{(n)} = \sum_T (-1)^T
$$

where the sum is over all rim hook tableaux $T$ of shape $\lambda$ and content given by the composition $(n)$. That is, the content of $T$ consists of $n$ occurrences of 1. Since $T$ is a rim hook tableau, all occurrences of 1 inside $T$ form a rim hook so therefore $T$ itself must be a rim hook. As well, since $T$ is also a Young diagram its shape and therefore $\lambda$ must be of the form $(r, 1^{n-r})$.

Therefore the sum is nonempty if and only if $\lambda$ is of the form $(r, 1^{n-r})$. In this case there is a unique rim hook tableau of content $(n)$, namely each box filled with 1. The sign in this case is $(-1)^{n-r}$.

**Definition 2.9.** A Young diagram is a **hook** if the corresponding partition is of the form $(n-r, 1^r)$. A character $\chi \in \text{Irr}(S_n)$ is called a **hook character** if it is of the form $\chi^\lambda$ where $\lambda$ is a Young diagram which is a hook.

The hook characters are exactly the characters that do not vanish on $n$-cycles. We will now compute the degrees of hook characters.

**Proposition 2.10.** If $\lambda = (r, 1^{n-r})$ is a hook then

$$
\chi^\lambda(1) = \binom{n-1}{r}.
$$

**Proof.** The Young diagram corresponding to $\lambda = (r, 1^{n-r})$ is given by

The hook number for the top left square is simply the number of all boxes which is $n$. Any other box is either in the first row which has no boxes below, or first column which has no boxes to the right. Therefore the hook number of such a box in the first row is just the number of boxes to
the right and likewise the hook number of such a box in the first column is the number of boxes below. That is

\[ h(i,j) = \begin{cases} 
  n & i = j = 1 \\
  n - r - j & i = 1, j > 1 \\
  r - i + 2 & i > 1, j = 1 
\end{cases} \]

and therefore by the hook-length formula

\[ \chi^{\lambda}(1) = \frac{n!}{r!(n-r-1)!n} = \frac{(n-1)!}{r!(n-r-1)!} = \binom{n-1}{r}. \]

Remark 2.11. We could have avoided invoking the hook formula by noting that a standard Young tableaux of shape \((n-r, 1^r)\) must have \(n\) in the top-left corner and is determined by a choice of \(r\) integers from \(\{1, \ldots, n-1\}\) used to fill the rest of the first column.

2.3 Representation Theory of the Dihedral Group \(D_n\)

The dihedral group is the group of symmetries of the regular \(n\)-gon. It has order \(2n\) and we will adopt the convention of writing it as \(D_n\). It is generated by an element \(r\) corresponding to a rotation by an angle of \(2\pi/n\) and an element \(s\) corresponding to a reflection subject to the following presentation

\[ D_n = \langle r, s : r^n = s^2 = 1, srs = r^{-1} \rangle. \]

It has a cyclic normal subgroup generated by the rotations and may be written as a semidirect product

\[ D_n \cong C_n \rtimes C_2 \]

where \(C_k\) is the cyclic group of order \(k\).
The group structure as well as its representation theory is well-known and is readily found in the literature, for example [Ser77]. We will collect them here. There are two cases depending on the parity of \( n \).

If \( n \) is odd, then \( D_n \) has the following conjugacy classes:

- The class of the identity \( \{1\} \),
- \( (n - 1)/2 \) classes of rotations \( \{r^{\pm 1}\}, \{r^{\pm 2}\}, \ldots, \{r^{\pm (n-1)/2}\} \), each of size 2, and
- The class of reflections \( \{r^js : 0 \leq j \leq n - 1\} \) of size \( n \).

In this case there are two linear characters giving by composing the quotient map \( D_n \to C_2 \) with the two irreducible linear characters of \( C_2 \). There are also \( (n - 1)/2 \) irreducible two-dimensional characters induced from the cyclic subgroup of rotations. Explicitly they may be realized as \( 2 \times 2 \) matrices as follows. Let \( w = e^{2\pi i/n} \) and \( h \) an integer, and set

\[
\rho^h(r^k) = \begin{pmatrix} w^{hk} & 0 \\ 0 & w^{-hk} \end{pmatrix}, \quad \rho^h(sr^k) = \begin{pmatrix} 0 & w^{hk} \\ w^{-hk} & 0 \end{pmatrix}. \tag{2.3}
\]

Taking \( 0 < h \leq (n - 1)/2 \) gives pairwise nonisomorphic irreducible two-dimensional representations. Counting all the representation we have \( (n - 1)/2 + 2 \) irreducible representations which must be all of them since this is equal to the number of conjugacy classes.

If \( n \) is even, then \( D_n \) has the following conjugacy classes:

- The class of the identity \( \{1\} \),
- The singleton class \( \{r^{n/2}\} \),
- \( n/2 - 1 \) classes of rotations \( \{r^{\pm 1}\}, \{r^{\pm 2}\}, \ldots, \{r^{\pm (n/2-1)}\} \), and
- Two classes of reflections, each of size \( n/2 \): \( \{r^{2i}s : 0 \leq i \leq n/2 - 1\} \) and \( \{r^{2i+1}s : 0 \leq i \leq n/2 - 1\} \).

Hence there must be \( (n + 6)/2 \) irreducible characters. There are four linear characters, given by sending the generators \( r, s \) to \( \pm 1 \) in all possible ways. For the other characters, taking the characters \( \rho^h \) as in Equation 2.3 for \( 0 < h < n/2 \) gives \( n/2 - 1 \) pairwise nonisomorphic two-dimensional irreducible characters and we have found all of the irreducible characters of \( D_{2n} \). Hence one easily observes the following

**Proposition 2.12.** Let \( D_n \) be a dihedral group and let \( \chi \in \text{Irr}(D_n) \). Let \( s \) be a reflection. Then \( \chi(s) = 0 \) for all nonlinear \( \chi \) and \( \chi(s) = \pm 1 \) otherwise.

**Proof.** Clear from the character table of \( D_n \) and Equation 2.3. \qed
We also note that the commutator subgroup of $D_n$ is $\langle r^2 \rangle$. If $n$ is odd this is in fact just $\langle r \rangle$. It follows that the abelianization of $D_n$ is

$$D_n^{ab} = D_n / \langle r^2 \rangle = \begin{cases} C_2 & n \text{ odd} \\ C_2 \times C_2 & n \text{ even} \end{cases} \quad (2.4)$$

### 2.4 Camina Elements and Camina Groups

Let $G$ be a finite group and $N$ a normal subgroup. We say that $(G, N)$ is a Camina pair if for every $g \in G - N$ is conjugate to all of $gN$. Such objects were first studied by Camina in [Cam78]. A Camina group is a finite nonabelian group $G$ such that $(G, G')$ is a Camina pair. The following proposition allows us to restate this in terms of vanishing of characters.

**Proposition 2.13.** Let $G$ be a finite group and $g \in G$. Then the following are equivalent:

1. The conjugacy class of $g$ is $gG'$.
2. $|C_G(g)| = |G : G'|.$
3. For all $z \in G'$ there exists $y \in G$ such that $[g, y] = z$.
4. $\chi(g) = 0$ for every nonlinear $\chi \in \text{Irr}(G)$.

**Proof.** See [Lado8, Proposition 1.1] and [Lew09a, Lemma 2.1]. \qed

Following [Lew09b] we will also call an element $g \in G$ a Camina element if it satisfies the equivalent conditions of Proposition 2.13. In the literature Camina elements are also called anticentral elements. Hence Camina groups are exactly the finite groups such that every nonlinear irreducible character vanishes outside of its commutator subgroup, that is every element outside of the commutator subgroup is a Camina element.

**Example 2.14.** The commutator subgroup of the Dihedral group $D_n$ is $\langle r^2 \rangle$ where $r$ is a rotation generating $D_n$. By direct computation one sees that $D_n$ is a Camina group if and only if $n$ is odd.

In [Lado8] it is shown using the classification of finite simple groups that any group admitting a Camina element is solvable. In there a list of examples of groups admitting Camina elements is given, which we now reproduce for convenience.

**Example 2.15.** Let $K$ be a group admitting a fixedpoint-free automorphism $\alpha$. Then $\alpha$ is a Camina element in the semidirect product $G = K \rtimes \langle \alpha \rangle$. More generally we can take an abelian group $A$ with a surjective homomorphism onto $\langle \alpha \rangle$ and take the semidirect product $G = K \rtimes A$. 
Example 2.16. An extraspecial $p$-group is a $p$-group $G$ such that $Z(G)$ has
order $p$ and $G/Z(G)$ is an elementary abelian $p$-group. In this case the
centre coincides with the commutator subgroup. All elements outside the
commutator subgroup of an extraspecial $p$-group are Camina elements.
That is, every extraspecial $p$-group is a Camina group.

Example 2.17. If $G$ is solvable and $G'$ is a minimal normal subgroup of $G$
then $G$ admits Camina elements. There are two cases:

1. $G' \leq Z(G)$, $|G'| = p$, $G/Z(G)$ is an elementary abelian $p$-group, and
every noncentral element is a Camina element.

2. $G' \cap Z(G) = 1$, then $G/Z(G)$ is a Frobenius group with kernel
$(G' \times Z(G))/Z(G)$ and cyclic complement. In this case every $g \notin
C_G(G')$ is a Camina element.

Example 2.18. If $A$ is an abelian normal subgroup of $G$ such that $G/A$ is
cyclic, then every element $a$ such that $G/A = \langle aA \rangle$ is a Camina element.

Example 2.19. Any group of order $p^n$ where $p$ is a prime and $n \leq 4$
contains Camina elements.

Example 2.20. Let $G$ be the group of all upper-triangular $n \times n$ matrices
with entries from a finite field of order $q$ and 1's on the main diagonal. Let
$a \in G$ be an element with with minimal polynomial $(x - 1)^n$. Then $a$ is a
Camina element.

Example 2.21. A $p$-group of order $p^{n+1}$ for $n > 1$ is said to be of maximal
class if it has nilpotency class $n$. For such a group, the elements not in
$C_G(K_i(G)/K_{i+2}(G))$ for $i \geq 2$ are Camina elements, where $K_i(G)$ denotes
the terms in the descending central series of $G$.

Camina groups admits the following characterization theorem as $p$-
groups or Frobenius groups with particular structure. We recall that a
Frobenius group is a finite group $G$ which admits a nontrivial subgroup $H$
(called the Frobenius complement) such that $H \cap gHg^{-1} = \{1\}$ for all $g \notin H$.
The set $K$ consisting of all elements of $G$ not conjugate to any element of $H$
along with the identity is called the Frobenius kernel. Frobenius’s theorem
states that $K$ is a normal subgroup of $G$ and $G$ admits a decomposition
$G = K \rtimes H$. We refer to [Isa76] for more information.

The following characterization theorem was first given by Dark and
Scoppola in [DS96] with an alternate proof given by Lewis in [Lew14] and
by Isaacs and Lewis in [IL15].

Theorem 2.22 (Dark-Scoppola). A finite group $G$ is a Camina group if and
only if one of the following holds:
1. \(G\) is a Camina \(p\)-group with nilpotence class 2 or 3.

2. \(G\) is a Frobenius group with cyclic complement.

3. \(G\) is a Frobenius group with complement isomorphic to the quaternion group.
ARTIN $L$-FUNCTIONS

In this chapter we recall the required background on number theory. The material in the first section may be found in any standard text on number theory, for example [Neu99] or [Lan94]. The subsequent sections deal with Artin $L$-functions with exposition following [Mar77] and [MM97].

3.1 Frobenius of a Prime

Let $K$ be a number field and $L/K$ a Galois extension. Throughout the thesis we will be using the following notation:

- $\mathcal{O}_K$ the ring of integers in $K$
- $n_K$ the degree of $K$ over $\mathbb{Q}$
- $d_K$ the absolute discriminant of $K$
- $\mathfrak{d}_{L/K}$ the relative discriminant of $L/K$
- $G$ the Galois group of $L/K$
- $N_{L/K}$ the norm to $K$ of an ideal in $L$
- $R(L/K)$ primes $p$ of $K$ which ramify in $L$
- $P(L/K)$ rational primes $p$ below primes $p$ in $R(L/K)$
- $S(L/K)$ primes $p$ of $K$ unramified in $L$ and degree 1 over $\mathbb{Q}$

We will frequently drop the reference to the field in the norm and simply write $N$ when no confusion is possible.

Let $\mathfrak{p}$ be a prime ideal of $K$. Then $\mathfrak{p}$ decomposes into a product of prime ideals $\mathfrak{p}_1^{e_1} \mathfrak{p}_2^{e_2} \ldots \mathfrak{p}_g^{e_g}$ in $L$. In this case we say the $\mathfrak{p}_i$ divide $\mathfrak{p}$ or is above $\mathfrak{p}$ and write $\mathfrak{p}_i | \mathfrak{p}$. The nonnegative integers $e_i$ are the ramification indices. If $e_i > 0$ for any $i$ we say $\mathfrak{p}$ is ramified in $L$. If $\mathfrak{p} | \mathfrak{p}$ then $\mathbb{F}_\mathfrak{p} := \mathcal{O}_L/\mathfrak{p}$ is a finite field which is an extension of $\mathbb{F}_p := \mathcal{O}_K/\mathfrak{p}$. The degree of the extension $[\mathbb{F}_\mathfrak{p} : \mathbb{F}_p]$ is called the residual degree, written $f_{\mathfrak{p}/p}$.

For an extension of number fields $L/K$, not necessarily Galois, the residual degrees and the ramification indices satisfy the equation

$$\sum_{i=1}^{g} e_i f_i = [L : K].$$
In the case that \(L/K\) is Galois, each residual degree \(f_i\) is equal and each ramification indices \(e_i\) are equal and therefore the equation simplifies to \[efg = [L : K].\]

Suppose henceforth that \(L/K\) is Galois. Let \(p\) be a prime of \(K\) and let \(\mathfrak{P}\) be a prime above \(p\). The decomposition group of \(\mathfrak{P}\) is defined as \[D_{\mathfrak{P}/p} = \{\sigma \in G : \sigma \mathfrak{P} = \mathfrak{P}\} \]

There is a natural short exact sequence \[1 \rightarrow I_{\mathfrak{P}/p} \rightarrow D_{\mathfrak{P}/p} \rightarrow \text{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_p) \rightarrow 1\]

where the kernel \(I_{\mathfrak{P}/p}\) is called the inertia group. If \(p\) is unramified in \(L\) then the inertia group is trivial so there is a natural isomorphism \(D_{\mathfrak{P}/p} \cong \text{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}/p)\). The latter group is cyclic and has the Frobenius automorphism as a canonical generator, and its inverse image in \(D_{\mathfrak{P}/p}\) under the natural map is called the Frobenius element of \(\mathfrak{P}\).

If \(\mathfrak{P}, \mathfrak{Q}\) are two primes of \(L\) dividing \(p\) then the decomposition groups \(D_{\mathfrak{P}/p}\) and \(D_{\mathfrak{Q}/p}\) are conjugate and conversely if \(\sigma \in G\) then the group \(\sigma D_{\mathfrak{P}/p} \sigma^{-1}\) is the decomposition group of the prime \(\sigma \mathfrak{P}\). Hence the Frobenius elements of the primes above \(p\) forms a conjugacy class called the Frobenius class of \(\mathfrak{P}\) written \(\sigma_p\). We will frequently refer to the Frobenius class of \(p\) as simply the Frobenius of \(p\).

### 3.2 Definition of Artin \(L\)-functions

In this section we will recall the basic facts about Artin \(L\)-functions. The material may be found in [Mar77] and [MM97]. We will assume throughout that \(L/K\) is a Galois extension of number fields with Galois group \(G\). Let \(\rho\) be a complex representation of \(G\), \(\chi\) its character, and \(V\) the underlying complex vector space.

Let \(p\) be a prime of \(K\). If \(p\) is unramified in \(L\) then there is a well-defined Frobenius conjugacy class \(\sigma_p\). Define the local Euler factor at \(p\) to be \[L_p(s, \chi, K) = \det(I - \rho(\sigma_p)(Np)^{-s})^{-1}\]

which is well-defined since conjugate elements have the same characteristic polynomial.
If \( p \) ramifies in \( L \) and \( \mathfrak{p} \) is a prime above \( p \), let \( I_{\mathfrak{p}} \) be the inertial group at \( \mathfrak{p} \). Define the Euler factor at \( p \) to be

\[
L_p(s, \chi, K) = \det(I - \rho(\sigma_p)|_{V^{\mathfrak{p}}}) (Np)^{-s}^{-1}
\]

where \( V^{\mathfrak{p}} \) is the subspace of \( V \) invariant under the action of \( I_{\mathfrak{p}} \). The definition of \( L_p(s, \chi, K) \) is independent of the choice of \( p \).

The Artin \( L \)-function is then defined to be the Euler product over finite primes

\[
L(s, \chi, K) = \prod_p L_p(s, \chi, K).
\]

The Artin \( L \)-functions satisfy a functional equation. We first define the gamma factor \( \Gamma(s, \chi, K) \) which incorporates the infinite places of \( K \) and then define the Artin conductor \( f_\chi \) of \( \chi \) which is defined in terms of inertia groups.

If \( v \) is an infinite place of \( K \) define

\[
L_v(s, \chi, K) = \begin{cases} 
((2\pi)^{-s}\Gamma(s))^{a(1)} & \text{if } v \text{ is complex} \\
((\pi^{-s/2}\Gamma(s/2))^{a(\pi^{-(s+1)/2}\Gamma((s+1)/2)})^b & \text{if } v \text{ is real}
\end{cases}
\]

(3.1)

where \( a \) is the dimension of the \( +1 \) eigenspace of complex conjugation and similarly \( b \) is the dimension of the \( -1 \) eigenspace. The numbers \( a, b \) satisfy \( a + b = \chi(1) \). Define

\[
\Gamma(s, \chi, K) = \prod_{v \text{ infinite}} L_v(s, \chi, K).
\]

Next let \( v \) be a place of \( K \) (finite or infinite) and let \( w \) be a place of \( L \) dividing \( v \) and let \( G_0 \) be the inertia group \( I_w \) at \( w \). There is a descending filtration of higher ramification groups

\[
G_0 \supseteq G_1 \supseteq \ldots
\]

If \( V \) is the underlying space of \( \rho \) define

\[
n(\chi, v) = \sum_{i=0}^{\infty} \frac{|G_i|}{|G_0|} \codim(V^{G_i}).
\]

The summands are zero except for finitely many \( i \) so \( n(\chi, v) \) is well-defined. It is also an integer and independent of the choice of \( w \) above \( v \).

Define the \textit{Artin conductor} of \( \chi \) to be the ideal

\[
f_\chi = \prod_v P_v^{n(\chi, v)}. \tag{3.2}
\]
We also set
\[ A_\chi = d_\chi^{(1)} N_{K/Q} f_\chi \]  
and
\[ \Lambda(s, \chi, K) = A_\chi^{s/2} \Gamma(s, \chi, K) L(s, \chi, K). \]  
The function \( \Lambda(s, \chi, K) \) satisfies the functional equation
\[ \Lambda(s, \chi, K) = W(\chi) \Lambda(1 - s, \overline{\chi}, K) \]  
where \( W(\chi) \) is a complex number of absolute value 1.

### 3.3 Properties of Artin L-functions and Artin’s Conjecture

The Artin L-functions satisfy the following properties.

- For any \( a_\chi \in \mathbb{Z} \) such that \( \sum_\chi a_\chi \chi \) is a character of \( G \),
  \[ L(s, \sum_\chi a_\chi \chi, K) = \prod_\chi L(s, \chi, K)^{a_\chi}. \]

- If \( H \) is a subgroup of \( G \) and \( L^H \) is its fixed subfield then
  \[ L(s, \text{Ind}^G_H, K) = L(s, \chi, L^H). \]

In particular we see that
\[ L(s, \text{reg}_G, K) = L(s, \text{Ind}^G_{\{1\}} 1, K) = L(s, 1, L) = \zeta_L(s). \]

As well, since
\[ \text{reg}_G = \sum_{\chi \in \text{Irr}(G)} \chi(1) \chi \]
we see that
\[ \zeta_L(s) = \prod_{\chi \in \text{Irr}(G)} L(s, \chi, K)^{\chi(1)} = \zeta_K(s) \prod_{1 \neq \chi \in \text{Irr}(G)} L(s, \chi, K)^{\chi(1)}. \]

By Brauer induction, given an irreducible character \( \chi \) of \( G \) one can find subgroups \( H_i \) of \( G \) and linear characters \( \psi_i \) of \( H_i \) such that \( \chi \) may be written as a \( \mathbb{Z} \)-linear combination of the induction of \( \psi_i \), that is
\[ \chi = \sum_i m_i \text{Ind}^G_{H_i} \psi_i. \]
for some integers $m_i$. Therefore any Artin $L$-function can be written as

$$L(s, \chi, K) = \prod_i L(s, \psi_i, L^{H_i})^{m_i}.$$ 

Each $L$-function on the right hand side comes from a linear character which by Artin reciprocity can be identified with a Hecke $L$-function whose analytic continuation is known. Therefore each Artin $L$-series admits a meromorphic continuation to all of $\mathbb{C}$. Artin’s conjecture states that in fact each Artin $L$-function attached to characters which do not contain the trivial character admits a holomorphic continuation to all of $\mathbb{C}$.

Artin’s conjecture is known in some specific cases. In the case when $G$ is abelian each Artin $L$-function can be identified with a Hecke $L$-function whose analytic continuation is known. Likewise if each character of $G$ is known to be induced from linear characters of subgroups then again each Artin $L$-function is a Hecke $L$-function and so Artin’s conjecture holds. Such groups are called monomial or $M$-groups. Examples of monomial groups include supersolvable groups.

## 3.4 The Artin Conductor and Discriminant

The Artin conductor $f_\chi$ encodes information about the ramification of the character $\chi$. For us it will appear in error terms analogous to the role of $\log d_L$ in [LMO79]. One can estimate the Artin conductor by the following

**Proposition 3.1** (M.R. Murty-V.K. Murty-Saradha). Let $L/K$ be a Galois extension with Galois group $G$ and $n = [L : K]$. Let $P(L/K)$ be the set of rational primes $p$ such that there exists a prime $p$ of $K$ with $p | p$ which ramifies in $L$. Let $\chi \in \text{Irr}(G)$ and $f_\chi$ be its Artin conductor. Then

$$\log N_{K/Q} f_\chi \leq 2\chi(1)n_K \left( \sum_{p \in P(L/K)} \log p + \log n \right).$$

**Proof.** This is [MMS88, Proposition 2.5].

Combining with the definition of $A_\chi$ (Equation 3.3) we see that

$$\log A_\chi = \chi(1) \log d_K + \log N f_\chi \ll \chi(1) \log d_K \left( \sum_{p \in P(L/K)} \log p + \log n \right).$$

The Artin conductors for irreducible characters of $\text{Gal}(L/K)$ are related to the discriminant as follows.
Theorem 3.2 (Conductor-discriminant formula). If $L/K$ is a Galois extension then

$$d_{L/K} = \prod_{\chi \in \text{Irr} \left( \text{Gal}(L/K) \right)} f_\chi^{(1)}.$$

Proof. See [Neu99, 11.9].

Lastly we recall the Minkowski bound.

Theorem 3.3 (Minkowski bound). Let $K$ be a number field of degree $n_K$ and let $r_2$ be the number of pairs of complex embeddings of $K$. Then every ideal class of $K$ admits an integral ideal $a$ satisfying

$$N_{K/Q} a \leq d_{K}^{1/2} \left( \frac{4}{\pi} \right)^{r_2} \frac{n_K!}{n_K^2}.$$

Proof. See [Lan94, V, §4, Theorem 4].

In particular since $N_{K/Q} a \geq 1$ it follows that

$$d_{K}^{1/2} \geq \left( \frac{\pi}{4} \right)^{r_2} \frac{n_K!}{n_K^2}.$$

and therefore

$$\log d_K \gg n_K. \quad (3.6)$$

3.5 Zeros of Artin $L$-functions

In this section we collect some results on zero density of Artin $L$-functions as well as a zero-free region of Artin $L$-functions assuming Artin’s conjecture.

Let $n_\chi(T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$ inside the rectangle $0 \leq \beta \leq 1, |\gamma - T| \leq 1$, so

$$n_\chi(T) = \# \{ \rho = \beta + i\gamma : L(\rho, \chi) = 0, 0 \leq \beta \leq 1, |\gamma - T| \leq 1 \}.$$

Lemma 3.4. Assume Artin’s conjecture for $L/K$. Then

$$n_\chi(T) \ll \log A_\chi + \chi(1)n_K \log(T + 2).$$

Proof. This is a straightforward generalization of [LMO79, Lemma 2.1].

There is also a second zero density estimate for Artin $L$-functions over a different region. Set

$$n_\chi(r; s) = \# \{ \rho : L(\rho, \chi) = 0, |\rho - s| \leq r \}.$$
so \( n_\chi(r; s) \) counts the number of zeros of \( L(s, \chi) \) with distance at most \( r \) from \( s \). We will mainly be interested in the case \( s = 1 \), that is, the density of zeros with distance at most \( r \) from \( s = 1 \).

**Lemma 3.5.** We have for \( 0 < r < 2 \)

\[
n_\chi(r; 1) \ll \chi(1) + r(\log A_\chi + \chi(1)n_K \log(3)) \ll \chi(1) + r \log A_\chi.
\] (3.7)

**Proof.** This is a straightforward generalization of [LMO79, Lemma 2.2]. See [Mah99, Lemma 2.1 (ii)]. For the second inequality, we note that from the definition of \( A_\chi \) Equation 3.3 we have

\[
\log A_\chi = \chi(1) \log d_K + \log N_{K/Q}\chi \ll \chi(1)n_K
\]

since \( \log d_K \gg n_K \) from the Minkowski bound (see Equation 3.6). \( \square \)

The zero-density estimate given in Lemma 3.5 unfortunately has a dependence on the character degree \( \chi(1) \) which is unaffected by the distance \( r \) from \( s = 1 \). This comes from the estimate given in [Mah99, Equation 2.23] which states that

\[
\left| \frac{L'}{L}(\sigma + it, \chi) \right| \ll \frac{\chi(1)}{\sigma - 1} + \chi(1) \log d_K
\]

for \( 1 < \sigma < 3 \) obtained by comparing \( L'/L \) against the Dedekind zeta function \( \zeta_K \).

We will also require the following zero-free region of Artin \( L \)-functions which rules out the possible exceptional zero for nonlinear characters.

**Proposition 3.6** (V.K. Murty). Assume Artin’s conjecture. Let \( L/K \) be a Galois extension of number fields \( \chi \) be an irreducible character of \( \text{Gal}(L/K) \) such that \( \chi(1) \neq 1 \). Then \( L(\sigma + it, \chi) \) is zero-free for

\[
1 - \frac{c}{\chi(1)^3(\log A_\chi + n_K \log(|t| + 2))} \leq \sigma \leq 1
\]

where \( c \) is an effective and absolute constant.

If \( \chi(1) = 1 \), \( L(\sigma + it, \chi) \) is zero-free in the above region with the possible exception of a single zero.

**Proof.** This is [Mur97, Proposition 3.1]. \( \square \)
3.6 ADDITIONAL ASSUMPTIONS ON ZEROS

The estimates given in the previous section is enough to prove Theorem 1.1. In this section we discuss two further assumptions on the Artin $L$-functions $L(s, \chi)$ which together with Artin’s conjecture will imply a stronger estimate in Theorem 1.1. Specifically they are as follows:

**Assumption 1.** For $1 < \sigma < 3$, the logarithmic derivative of $L(\sigma + it, \chi)$ satisfies

$$\left| \frac{L'}{L}(\sigma + it, \chi) \right| \ll \frac{1}{\sigma - 1} + \log A \chi.$$

**Assumption 2.** There exists a constant $c$ such that if $\chi$ is irreducible and $L(\sigma + it, \chi)$ satisfies Artin’s conjecture then it has at most one zero in the region

$$1 - \frac{c}{\chi(1)(\log A \chi + \chi(1)n \log(|t| + 2))} \leq \sigma \leq 1.$$

The motivation for **Assumption 1** comes from the following estimate found in [Mah99, Equation 2.23]

$$\left| \frac{L'}{L}(\sigma + it, \chi) \right| \leq \chi(1) \left( \frac{1}{\sigma - 1} + \frac{1}{2} \log d \chi + O(n \chi) \right)$$

for $1 < \sigma < 3$, which is used in the proof of Lemma 3.5. However, if $\chi$ is nonprincipal then $L'/L(s, \chi)$ is holomorphic at $s = 1$ so one can hope for an upper bound without the $1/(\sigma - 1)$ term which leaves $\chi(1) \log d \chi \ll \log A \chi$.

**Assumption 2** is a strengthening of the following zero-free region.

**Proposition 3.7** ([Mur97, Corollary 3.2]). There exists an absolute positive constant $c$ such that if $\chi$ is irreducible then $L(\sigma + it, \chi)$ has at most one zero in the region

$$1 - \frac{c}{\chi(1) \log A \chi} \leq \sigma \leq 1, \quad |t| \leq \frac{c}{\chi(1) \log A \chi}.$$

For the rest of this thesis any dependence on the above assumptions will be made explicit. **Assumption 1** implies a zero-density estimate for $L(s, \chi)$ which is stronger than Lemma 3.5 as follows.

**Lemma 3.8.** Suppose $\chi$ is a nonprincipal character. Assume Artin’s conjecture for $L(s, \chi)$ and suppose that **Assumption 1** holds. Then

$$n_\chi(r; 1 + it) \ll 1 + r(\log A \chi + n \chi(1) \log(|t| + 2)). \quad (3.8)$$

**Remark 3.9.** Comparing with Equation 3.7, the constant term in the right-hand side of Equation 3.8 is 1 as opposed to $\chi(1)$. This will give us a sharper bound than the one stated in Theorem 1.1.
Proof. The proof is very similar to the one for Lemma 3.5, given in [Mah99, Lemma 2.1 (ii)]. We record the details for completeness. Since $\chi$ is non-principal, $\Lambda$ is entire and has a Hadamard factorization

$$\Lambda(s, \chi) = e^{a(\chi)s + \beta(\chi)s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right)^{e^{s/\rho}}$$

(3.9)

where $a(\chi), \beta(\chi)$ are constants and $\rho$ runs over all nontrivial zeros of $L(s, \chi)$. Logarithmically differentiating Equation 3.9 we obtain

$$-\frac{\Lambda'(s, \chi)}{\Lambda(s, \chi)} = -\beta(\chi) - \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right)$$

and therefore using Equation 3.4 we obtain

$$-\frac{L'}{L}(s, \chi) = -\beta(\chi) - \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right) + \frac{1}{2} \log A_\chi + \frac{\Gamma'}{\Gamma}(s, \chi).$$

Therefore evaluating at $s = \sigma + it$ and at $s = 3 + it$ and subtracting we have

$$\frac{L'}{L}(s, \chi) - \frac{L'}{L}(3 + it, \chi) = -\sum_{\rho} \left( \frac{1}{s - \rho} - \frac{1}{(3 + it) - \rho} \right)$$

$$- \frac{\Gamma'}{\Gamma}(s, \chi) + \frac{\Gamma'}{\Gamma}(3 + it, \chi) \beta(\chi).$$

We know (see Lemma 4.9) that if $\Re s > -1/2$ and $|s| \geq 1/8$ we have

$$\left| \frac{\Gamma'}{\Gamma}(s, \chi) \right| \ll nK\chi(1) \log(|s| + 2)$$

and therefore

$$\left| \frac{\Gamma'}{\Gamma}(3 + it, \chi) \right| \ll nK\chi(1) \log(|t| + 2).$$

Hence it follows that for $1/2 \leq \sigma \leq 3$ we have

$$\left| \frac{L'}{L}(s, \chi) - \sum_{\rho \leq 1} \frac{1}{s - \rho} \right| \ll nK\chi(1) \log(|t| + 2)$$

$$+ \sum_{|\gamma - t| > 1} \left| \frac{1}{s - \rho} - \frac{1}{(3 + it) - \rho} \right| + \sum_{|\gamma - t| \leq 1} \left| \frac{1}{(3 + it) - \rho} \right|.$$
Since for $\rho = \beta + i\gamma$ we have
\[
\left| \frac{1}{(3+it) - \rho} \right| \leq \frac{1}{3-\beta} \ll 1
\]
using Lemma 3.4 we see that
\[
\sum_{|\gamma-t| \leq 1} \left| \frac{1}{(3+it) - \rho} \right| \ll n_\chi(t) \ll \log A_\chi + \chi(1)n_\chi \log(|t|+2).
\]
Similarly
\[
\sum_{|\gamma-t| > 1} \left| \frac{1}{s-\rho} - \frac{1}{(3+it) - \rho} \right| = \sum_{|\gamma-t| > 1} \frac{3-\sigma}{|s-\rho| |(3+it) - \rho|} \ll \sum_{|\gamma-t| > 1} \frac{1}{|\gamma-t|^2} \ll \sum_{j=2}^{\infty} \frac{n_\chi(t+j) + n_\chi(t-j)}{(j-1)^2} \ll \log A_\chi + n_\chi(1) \log(|t|+2).
\]
Lastly using the bound for $|L'/L(s,\chi)|$ in Assumption 1 we obtain that for $1 < \sigma < 3$
\[
\left| \sum_{|\gamma-t| \leq 1} \frac{1}{s-\rho} \right| \ll \frac{1}{\sigma-1} + \log A_\chi + n_\chi(1) \log(|t|+2)
\]
which we sum over $t+j$, $-3 \leq j \leq 3$ to obtain
\[
\left| \sum_{|\gamma-t| \leq 4} \frac{1}{s-\rho} \right| \ll \frac{1}{\sigma-1} + \log A_\chi + n_\chi(1) \log(|t|+2). \tag{3.10}
\]
On the other hand,
\[
\left| \sum_{|\gamma-t| \leq 4} \frac{1}{s-\rho} \right| \geq \sum_{|\gamma-t| \leq 4} \Re \frac{1}{s-\rho} = \sum_{|\gamma-t| \leq 4} \frac{\sigma - \beta}{|s-\rho|^2} \geq \sum_{|\gamma-t| \leq 4} \frac{\sigma - 1}{|s-\rho|^2}.
\]
Let $0 < r < 2$ and choose $\sigma = 1 + r$ and let $k = \#\{\rho : s - \rho \leq 2r\}$. Then $\chi(r, t) \leq k$. Therefore

$$\left| \sum_{\rho} \frac{1}{s - \rho} \right| \geq \frac{kr}{4r^2} \geq \frac{\chi(r, t)}{4r}.$$ 

Combining with Equation 3.10 gives

$$\chi(r, t) \ll 1 + r(\log A + n_K \chi(1) \log(|t| + 2)).$$
THE LEAST PRIME IN A CONJUGACY CLASS: ESTIMATES

In this chapter we adapt the methods of [LMO79] to obtain estimates on the least prime whose Frobenius is in a fixed conjugacy class \(C\). Throughout this section we will be assuming Artin’s conjecture and work with Artin \(L\)-functions directly.

4.1 THE KERNEL FUNCTION

We will consider the kernel functions of [LMO79], defined by

\[
k_1(s) = k_1(s; x) = \left( \frac{x^{2(s-1)} - x^{s-1}}{s-1} \right)^2
\]

and

\[
k_2(s) = k_2(s; x) = x^{s^2 + s}.
\]

The kernel functions \(k_1, k_2\) depend on a parameter \(x > 1\) whose value will be chosen later.

For any function \(f(s)\) and real number \(\sigma\) we write

\[
\int_{(\sigma)} f(s) \, ds := \int_{\sigma - i\infty}^{\sigma + i\infty} f(s) \, ds.
\]

We consider the inverse Mellin transforms of the kernel functions \(k_j\), given by

\[
\hat{k}_j(u) = \frac{1}{2\pi i} \int_{(2)} k_j(s) u^{-s} \, ds
\]

and compute them explicitly in the following lemmas.

Lemma 4.1. The function

\[
k(s) = k(s; x, y) = \left( \frac{y^{s-1} - x^{s-1}}{s-1} \right)^2
\]
has inverse Mellin transform

\[ \hat{k}(u) = \begin{cases} \frac{1}{u} \log \frac{y^2}{u} & xy \leq u \leq y^2 \\ \frac{1}{u} \log \frac{u}{x^2} & x^2 \leq u \leq xy \\ 0 & \text{otherwise} \end{cases} \]

By Lemma 4.1 the function \( k_1(s) \) has the inverse Mellin transform

\[ \hat{k}_1(u) = \frac{1}{2\pi i} \int_{(2)} k_1(s) u^{-s} \, ds = \begin{cases} \frac{1}{u} \log \frac{x^4}{u} & x^3 \leq u \leq x^4 \\ \frac{1}{u} \log \frac{u}{x^2} & x^2 \leq u \leq x^3 \\ 0 & \text{otherwise} \end{cases} \]

For \( k_2(s) \) we have the following.

**Lemma 4.2.** The function \( k_2(s) \) has the inverse Mellin transform

\[ \hat{k}_2(u) = \frac{1}{\sqrt{4\pi \log x}} \exp \left[ -\frac{(\log u/x)^2}{4 \log x} \right]. \]

**Proof.** Change of variables gives a Gaussian integral which can be evaluated to give the result. \( \square \)

To prove Lemma 4.1 we will need the following

**Lemma 4.3.** For \( \alpha \geq 0 \),

\[ \int_{(1)} \alpha^s \frac{ds}{s^2} = \begin{cases} 0 & 0 \leq \alpha \leq 1 \\ \log \alpha & \alpha > 1 \end{cases}. \]

**Proof.** Suppose \( 0 \leq \alpha \leq 1 \). We claim that for any \( \sigma > 1 \), we have

\[ \int_{(1)} \alpha^s \frac{ds}{s^2} = \int_{(\sigma)} \alpha^s \frac{ds}{s^2}. \]

Indeed, consider the rectangle with corners \( 1 \pm iT \) and \( \sigma \pm iT \) oriented counterclockwise. Then since the integrand has no poles in the interior the integral around the rectangle vanishes. Over the horizontal edges we have

\[ \left| \int_{1 \pm iT}^{\sigma \pm iT} \alpha^s \frac{ds}{s^2} \right| \leq \frac{\sigma - 1}{1 + T^2} \xrightarrow{T \to \infty} 0 \]

which establishes the claim.
Now let $\gamma(t) = \sigma + it$ so that

$$\left| \int_{\sigma-iT}^{\sigma+iT} \frac{\alpha^s}{s^2} \frac{ds}{s^2} \right| = \left| \int_{-T}^{T} \frac{\alpha^{\gamma(t)}}{\gamma(t)^2} \frac{dt}{\gamma(t)^2} \right|$$

$$\leq \int_{-T}^{T} \frac{dt}{\gamma(t)^2} = \int_{-T}^{T} \frac{dt}{\sigma^2 + t^2} = \frac{2}{\sigma} \arctan \frac{T}{\sigma}.$$ 

Therefore

$$\left| \int_{(1)} \frac{\alpha^s}{s^2} \frac{ds}{s^2} \right| = \left| \int_{(\sigma)} \frac{\alpha^s}{s^2} \frac{ds}{s^2} \right| \leq \frac{\pi}{\sigma} \sigma \rightarrow \infty \rightarrow 0.$$ 

Next suppose that $\alpha > 1$. This time move the line of integration to the left, and consider a rectangle with vertices $1 \pm iT$ and $\sigma \pm iT$ for some $\sigma < 0$. Now there is a pole at $s = 0$ with residue $\log \alpha$ which then equals the value of the integral around the rectangle. As before the integral along the horizontal edges can be shown to vanish as $T \rightarrow \infty$ so

$$\frac{1}{2\pi i} \int_{(1)} \frac{\alpha^s}{s} \frac{ds}{s} = \log \alpha + \frac{1}{2\pi i} \int_{(\sigma)} \frac{\alpha^s}{s^2} \frac{ds}{s^2} \sigma \rightarrow -\infty \rightarrow \log \alpha$$

where as before the integral along $\text{Re}s = \sigma$ can be shown to vanish as $\sigma \rightarrow -\infty$. \qed

Proof of Lemma 4.1. Perform a change of variables and expand the integral to obtain

$$\hat{k}(u) = \frac{1}{2\pi i} \int_{(1)} \left( \frac{y^2 - x^2}{u} \right)^2 \frac{ds}{s^{s-1}}$$

$$= \frac{1}{u} \left( \frac{1}{2\pi i} \int_{(1)} \left( \frac{y^2}{u} \right)^s \frac{ds}{s^2} - \frac{2}{2\pi i} \int_{(1)} \left( \frac{xy}{u} \right)^s \frac{ds}{s^2} + \frac{1}{2\pi i} \int_{(1)} \left( \frac{x^2}{u} \right)^s \frac{ds}{s^2} \right).$$

Using Lemma 4.3 to compute the integrals for each case proves the lemma. \qed

4.2 PRELIMINARY ESTIMATES

Let $L/K$ be a Galois extension of number fields with Galois group $G = \text{Gal}(L/K)$. For each irreducible character $\chi$ of $G$ we have the Artin $L$-function

$$L(s, \chi) = L(s, \chi, L/K)$$

which are known to admit a meromorphic extension to the complex plane. The dependence on the field extension $L/K$ will be suppressed for the sake of notation. We will assume Artin’s conjecture which states that the
Artin $L$-functions corresponding to nontrivial irreducible characters are holomorphic.

Logarithmically differentiating the Euler product gives

$$\frac{L'(s, \chi)}{L(s, \chi)} = -\sum_p \log(Np) \sum_{m=1}^{\infty} \frac{\chi(\sigma_p^m)}{Np^{ms}}$$

where $\sigma_p$ is the Frobenius above $p$ for $p$ unramified in $L$, and if $p$ is ramified it is defined by

$$\chi(\sigma_p^m) = \frac{1}{e} \sum_{\tau \sim \sigma_p^m} \chi(\tau)$$

where the sum is over all preimages of $\sigma_p^m$.

Let $C$ be a conjugacy class in $G$ and define

$$F_C(s) = -\frac{|C|}{|G|} \sum_{\chi \in \text{Irr}(G)} \frac{\chi(C) L'(s, \chi)}{L(s, \chi)}.$$

Using the character orthogonality relations Equation 2.2 this is

$$F_C(s) = \sum_p \sum_{m=1}^{\infty} \theta(p^m) (\log Np)(Np)^{-ms}$$

where for $p$ unramified and $\sigma_p$ its Frobenius we have

$$\theta(p^m) = \begin{cases} 1 & \text{if } \sigma_p^m = C \\ 0 & \text{otherwise} \end{cases}$$

and for $p$ ramified we have $0 \leq \theta(p^m) \leq 1$.

Then

$$I_j := \frac{1}{2\pi i} \int(2) F_C(s) k_j(s) \, ds = \sum_{p^m} \theta(p^m) (\log Np) \hat{k}_j(Np^m). \quad (4.1)$$

We will break this into three sums. The first will be over all $p$ which are unramified in $L$ and of degree 1 over $\mathbb{Q}$, the second over the ramified primes, and the third over primes of higher degree.

**Lemma 4.4.** Let $R(L/K)$ denote the set of primes of $K$ which ramify in $L$ and $P(L/K)$ the set of rational primes below primes of $R(L/K)$. Then

$$\sum_{p \in R(L/K)} \sum_{m=1}^{\infty} \theta(p^m) (\log Np) \hat{k}_1(Np^m) \ll n_K \frac{\log x}{x^2} \sum_{p \in P(L/K)} \log p$$
and
\[ \sum_{p \in \mathcal{R}(L/K)} \sum_{m=1}^{\infty} \theta(p^m) (\log Np) \hat{k}_2(Np^m) \ll n_K (\log x)^{1/2} \sum_{p \in \mathcal{P}(L/K)} \log p. \]

Proof. This is [LMO79, Lemma 3.1] without the final step of bounding the contribution of the ramified primes by \( \log d_L \) and noting that
\[ \sum_{p \mid \mathcal{P}} \log Np = \sum_{p \mid \mathcal{P}} f_{p/p} \log p \leq n_K \log p. \]

Lemma 4.5. The sum over prime powers \( p^m \) where \( Np^m \) is not a rational prime satisfies
\[ \sum_{\substack{p^m \mid Np^m \text{ not a rational prime}}} \theta(p^m) (\log Np) \hat{k}_1(Np^m) \ll n_K \frac{(\log x)^2}{x} \]
and
\[ \sum_{\substack{p^m \mid Np^m \text{ not a rational prime}}} \theta(p^m) (\log Np) \hat{k}_2(Np^m) \ll n_K x^{7/4}. \]

Proof. This is [LMO79, Lemma 3.2].

For the kernel function \( k_2 \) we also require the following estimate over prime powers \( p^m \) with large norm.

Lemma 4.6. We have
\[ \sum_{\substack{Np^m \geq x^{10}}} \theta(p^m) (\log Np) \hat{k}_2(Np^m) \ll n_K x^{-10}. \]

Proof. This is [LMO79, Lemma 3.3].

4.3 THE CONTOUR INTEGRAL

We will now evaluate the integral \( I \) in Equation 4.1 by moving the line of integration to the left to get a sum over the zeros of the Artin \( L \)-functions in the critical strip. We first collect a few lemmas.

Lemma 4.7. Let \( \chi \) be a character of \( G \) and let \( \bar{\chi} \) be the conjugate character. Then
\[ A_\chi = A_{\bar{\chi}}. \]
Proof. From the definition of $A_\chi$ and $f_\chi$ (Equation 3.2 and Equation 3.3) it suffices to show that $n(\chi, v) = n(\bar{\chi}, v)$ for each place $v$. Let $V$ be the underlying space for $\chi$. Since $\bar{\chi}$ is the character of the contragredient representation with underlying space $V^*$ we have $\dim V = \dim V^*$. Thus it suffices to show that $\dim V_{G_i} = \dim (V^*)_{G_i}$ for each $i$.

From the character inner product (Equation 2.4) we see that
\[
\dim V_{G_i} = \langle \text{Res}_{G_i}^G \chi, 1_{G_i} \rangle = \frac{1}{|G_i|} \sum_{g \in G_i} \chi(g)
\]

where $1_{G_i}$ denotes the trivial character on $G_i$. Taking the complex conjugate gives
\[
\dim V_{G_i} = \dim V_{G_i} = \frac{1}{|G_i|} \sum_{g \in G_i} \bar{\chi}(g) = \langle \bar{\chi}, 1_{G_i} \rangle = \dim (V^*)_{G_i}.
\]

We now collect a few elementary estimates which are easy generalizations to Artin $L$-functions of Lemmas in [LO77] for $\zeta_L(s)$ and Hecke $L$-function estimates.

Lemma 4.8. If $\sigma = \Re s > 1$ then
\[
\left| \frac{L'(s, \chi)}{L(s, \chi)} \right| \ll \frac{\chi(1) n_K}{\sigma - 1}.
\]

Proof. Comparing Dirichlet series gives
\[
\left| \frac{L'(s, \chi)}{L(s, \chi)} \right| \leq -\chi(1) \frac{\zeta_K'}{\zeta_K}(\sigma) \leq -\chi(1) n_K \frac{\zeta'}{\zeta}(\sigma) \ll \frac{\chi(1) n_K}{\sigma - 1}.
\]

Lemma 4.9. If $\sigma = \Re s > -1/2$ and $|s| \geq 1/8$ then
\[
\left| \frac{\Gamma'(s, \chi)}{\Gamma(s, \chi)} \right| \ll \chi(1) n_K \log(|s| + 2).
\]

Proof. From the definition of $L_v(s, \chi)$ for an infinite place $v$ (Equation 3.4), we see that since $a + b = \chi(1)$ we have
\[
\frac{L'(s, \chi)}{L_v(s, \chi)} \ll \chi(1) \frac{\Gamma'(s)}{\Gamma(s)} \ll \chi(1) \log(|s| + 2).
\]

Taking the sum over all infinite places $v$ gives the lemma.
Lemma 4.10. If $s = \sigma + it$ with $\sigma \leq -1/4$ and $|s + m| \geq 1/4$ for all nonnegative integers $m$, then
\[
\frac{L'(s, \chi)}{L}(s, \chi) \ll \log A_{\chi} + \chi(1)nK\log(|s| + 2).
\]

Proof. Taking the logarithmic derivative of the functional equation (Equation 3.4 and Equation 3.5) and noting that $A_{\bar{\chi}} = A_{\chi}$ (Lemma 4.7) and $\gamma(s, \chi) = \gamma(s, \bar{\chi})$ gives
\[
\frac{L'(s, \chi)}{L}(s, \chi) = -\frac{L'}{L}(1 - s, \bar{\chi}) - \log A_{\chi} - \frac{\Gamma'}{\Gamma}(1 - s, \bar{\chi}) - \frac{\Gamma'}{\Gamma}(s, \chi).
\]

Using the estimates from Lemma 4.8 and Lemma 4.9 gives the result. \(\square\)

Set
\[
I_j(\chi) := \frac{1}{2\pi i} \int_{(2)} -\frac{L'}{L}(s, \chi)k_j(s)\, ds
\]
and
\[
I_j(\chi, T) := \frac{1}{2\pi i} \int_{B(T)} -\frac{L'}{L}(s, \chi)k_j(s)\, ds
\]
where $B(T)$ is the positively oriented rectangle with vertices $2 \pm iT$ and $-1/2 \pm iT$, see Figure 4.1.

Lemma 4.11. The vertical integral satisfies
\[
\left| \frac{1}{2\pi i} \int_{-1/2 + iT}^{1/2 - iT} -\frac{L'}{L}(s, \chi)k_j(s)\, ds \right| \ll k_j(-1/2)(\log A_{\chi} + \chi(1)nK)
\]
where the implicit constant is absolute and effective.
Proof. From Lemma 4.10 we have
\[ \left| \frac{L'(s, \chi)}{L(s, \chi)} \right| \ll \log A_\chi + \chi(1)n_K \log(|s| + 2) \]
and from the definition of \( k_1(s) \) we have
\[ k_1(-1/2) \asymp x^{-3} \]
where \( x \) is the parameter for \( k_1(s) = k_1(s;x) \). As well,
\[ k_1(-1/2 + it) \ll \frac{x^{-3}}{9/4 + t^2} \ll \frac{k_1(-1/2)}{9/4 + t^2}. \]
Therefore
\[
\left| \int_{-1/2-iT}^{-1/2+iT} \frac{L'(s, \chi)k_1(s)}{L(s, \chi)} \, ds \right| 
\ll \int_{-1/2-iT}^{-1/2+iT} \frac{x^{-3}(\log A_\chi + \chi(1)n_K \log(|s| + 2))}{9/4 + (\operatorname{Im} s)^2} \, ds 
\ll \int_{-T}^{T} k_1(-1/2)(\log A_\chi + \chi(1)n_K \log(|t| + 2)) \, dt 
\ll k_1(-1/2)(\log A_\chi + \chi(1)n_K). 
\]
Similarly, from the definition of \( k_2(s) \) we have
\[ k_2(-1/2 + it) = x^{-1/4-t^2} = k_2(-1/2)x^{-t^2}. \]
Therefore
\[
\left| \int_{-1/2-iT}^{-1/2+iT} \frac{L'(s, \chi)k_2(s)}{L(s, \chi)} \, ds \right| 
\ll \int_{-T}^{T} k_2(-1/2)x^{-t^2}(\log A_\chi + \chi(1)n_K \log(|t| + 2)) \, dt 
\ll k_2(-1/2)(\log A_\chi + \chi(1)n_K). 
\]

Lemma 4.12. The horizontal integrals satisfy
\[
\frac{1}{2\pi i} \int_{-1/2\pm iT}^{2\pm iT} \frac{L'(s, \chi)k_1(s)}{L(s, \chi)} \, ds 
\ll \frac{x^4}{1 + T^2}(\log A_\chi + \chi(1)n_K \log T) 
\]
and
\[
\left| \frac{1}{2\pi i} \int_{-1/2\pm iT}^{1/2\pm iT} \frac{L'}{L}(s, \chi)k_2(s) \, ds \right| \ll x^{6-T^2}(\log A_X + \chi(1)n_K \log T)
\]
where the implied constants are absolute and effective.

Proof. From Lemma 4.10 we have
\[
\frac{L'}{L}(\sigma + it, \chi) \ll \log A_X + \chi(1)n_K \log(|\sigma + it| + 2)
\ll \log A_X + \chi(1)n_K(\log(|\sigma| + 2) + \log(|t|)).
\]

As well, for \(\sigma\) with \(-1/2 \leq \sigma \leq 2\) we have
\[
k_1(\sigma + iT) \ll \frac{x^4}{1 + T^2}
\]
and
\[
k_2(\sigma + iT) \ll x^{6-T^2}.
\]

Therefore
\[
\left| \frac{1}{2\pi i} \int_{-1/2\pm iT}^{1/2\pm iT} \frac{L'}{L}(s, \chi)k_1(s) \, ds \right|
\ll \int_{-1/2}^{1/2} \frac{x^4}{1 + T^2}(\log A_X + \chi(1)n_K(\log(|\sigma| + 2) + \log T)) \, d\sigma
\ll \frac{x^4}{1 + T^2}(\log A_X + \chi(1)n_K \log T)
\]
and
\[
\left| \frac{1}{2\pi i} \int_{-1/2\pm iT}^{1/2\pm iT} \frac{L'}{L}(s, \chi)k_2(s) \, ds \right|
\ll \int_{-1/2}^{1/2} x^{6-T^2}(\log A_X + \chi(1)n_K(\log(|\sigma| + 2) + \log T)) \, d\sigma
\ll x^{6-T^2}(\log A_X + \chi(1)n_K \log T).
\]

In particular we see that the horizontal integrals vanish as we take \(T \to \infty\).
Proposition 4.13. Let $S(L/K)$ denote the set of primes $p$ of $K$ that do not ramify in $L$ and also have degree 1 over $Q$. There exist absolute and effective positive constants $c_1, c_2, c_3$ such that

$$\sum_{p \in S(L/K)} \theta(p)(\log Np)\hat{k}_1(Np) \geq \frac{|C|}{|G|} k_1(1) - \frac{|C|}{|G|} \sum_{\chi \in \text{Irr}(G)} |\chi(C)| \sum_{\rho_X} |k_1(\rho_X)|$$

$$- c_1 \frac{|C|}{|G|} \sum_{\chi \in \text{Irr}(G)} |\chi(C)| (\log A_X + \chi(1)n_K)$$

$$- c_2 n_K \frac{\log x}{x^2} \sum_{p \in P(L/K)} \log p - c_3 n_K \frac{(\log x)^2}{x}$$

and constants $c_4, c_5, c_6$ such that

$$\sum_{p \in S(L/K)} \theta(p)(\log Np)\hat{k}_2(Np) \geq \frac{|C|}{|G|} k_2(1) - \frac{|C|}{|G|} \sum_{\chi \in \text{Irr}(G)} |\chi(C)| \sum_{\rho_X} |k_2(\rho_X)|$$

$$- c_4 \frac{|C|}{|G|} \sum_{\chi \in \text{Irr}(G)} |\chi(C)| (\log A_X + \chi(1)n_K)$$

$$- c_5 n_K (\log x)^{1/2} \sum_{p \in P(L/K)} \log p - c_6 n_K x^{7/4}$$

(4.3)

where the inner sum in the second term of the right hand side in both equations is over all zeros $\rho_X$ of $L(s, \chi)$ with $-1/2 \leq \Re \rho_X \leq 1$.

Proof. Let $\delta(\chi) = 1$ if $\chi$ is the trivial character and $\delta(\chi) = 0$ otherwise. By Cauchy’s theorem we have

$$I_j(\chi, T) = \frac{1}{2\pi i} \int_{B(T)} \frac{L'}{L}(s, \chi)k_j(s) \, ds = \delta(\chi)k_j(1) - \sum_{-1/2 \leq \Re \rho \leq 1 \atop |\Im \rho| \leq T} k_j(\rho)$$

where the sum is over zeros $\rho$ of $L(s, \chi)$ lying inside $B(T)$. Thus

$$I_j(\chi) = \lim_{T \to \infty} I_j(\chi, T)$$

$$= \lim_{T \to \infty} \frac{1}{2\pi i} \left( \int_{2+iT}^{2+iT} \frac{L'}{L}(s, \chi)k_j(s) \, ds + \int_{2+iT}^{-1/2+iT} \frac{L'}{L}(s, \chi)k_j(s) \, ds + \int_{-1/2-iT}^{-1/2+iT} \frac{L'}{L}(s, \chi)k_j(s) \, ds + \int_{-1/2-iT}^{2+iT} \frac{L'}{L}(s, \chi)k_j(s) \, ds \right)$$

$$= \frac{1}{2\pi i} \int_{(2)} \frac{L'}{L}(s, \chi)k_j(s) \, ds + O(k_j(-1/2)(\log A_X + \chi(1)n_K)).$$
That is, there is an effective and absolute constant $c_7$ with

$$\frac{1}{2\pi i} \int_{(2)} -\frac{L'}{L}(s, \chi) k_j(s) \, dx \geq \delta(\chi) k_j(1) - \sum_{-1/2 \leq \text{Re} \rho \leq 1} |k_j(\rho \chi)|$$

$$- c_7 k_j(-1/2) (\log A_\chi + \chi(1)n_K).$$

Multiplying by $(|C|/|G|) \overline{\chi(C)}$ and taking the sum over all $\chi$ we obtain

$$I_j = \frac{1}{2\pi i} \int_{(2)} F_C(s) k_j(s) \, ds \geq \frac{|C|}{|G|} k_j(1) - \sum_{\chi \in \text{Irr}(G)} |\chi| |\sum_{\rho \chi} k_j(\rho \chi)|$$

$$- c_7 k_j(-1/2) \frac{|C|}{|G|} \sum_{\chi \in \text{Irr}(G)} |\chi| (\log A_\chi + \chi(1)n_K). \quad (4.4)$$

From Lemma 4.4 and Lemma 4.5 we have

$$I_1 = \sum_{p \in S(L/K)} \theta(p) (\log N_p) \hat{k}_1(N_p)$$

$$+ O \left( n_k \frac{\log x}{x^2} \sum_{p \in P(L/K)} \log p \right) + O \left( n_k \frac{\log x^2}{x} \right).$$

From Lemma 4.6 we also have

$$\sum_{N p^m \geq x^{10}} \theta(p^m) (\log N_p) \hat{k}_2(N_p^m) \ll n_k x^{-10} \ll n_k x^{7/4}$$

so we obtain

$$I_2 = \sum_{p \in S(L/K)} \theta(p) (\log N_p) \hat{k}_2(N_p)$$

$$+ O \left( n_k (\log x)^{1/2} \sum_{p \in P(L/K)} \log p \right) + O \left( n_k x^{7/4} \right)$$

where the implicit constants are effective and absolute. Lastly, a direct computation shows that since $x > 1$ we have $k_j(0), k_j(-1/2) \ll 1$. Combining with Equation 4.4 and rearranging the terms gives Equation 4.2.
4.4 Contribution of Zeros

We now use the results of Section 3.5 and Section 3.4 to estimate the contribution in Equation 4.2 coming from zeros of \( L \)-functions. The first step is to show that the contribution from zeros far from 1 is small.

**Lemma 4.14.** The sum over zeros \( \rho_\chi \) with \( |\rho_\chi - 1| \geq 1 \) satisfies

\[
\sum_{|\rho_\chi - 1| \geq 1} |k_1(\rho_\chi)| \ll \log A_\chi + \chi(1)n_K.
\]

**Proof.** Since \( x > 1 \), we have

\[
|k_1(\rho_\chi)| = \frac{|x^{2(\rho_\chi - 1)} - x^{\rho_\chi - 1}|^2}{|\rho_\chi - 1|^2} \ll \frac{1}{|\rho_\chi - 1|^2}
\]

and furthermore if \( \rho_\chi = \beta + i\gamma \) then

\[
\frac{1}{|\rho_\chi - 1|^2} = \frac{1}{(\beta - 1)^2 + \gamma^2} \leq \frac{1}{\gamma^2}.
\]

Therefore using Lemma 3.4 and Lemma 3.5 we have

\[
\sum_{|\rho_\chi - 1| \geq 1} |k_1(\rho_\chi)| \leq \sum_{T=1}^{\infty} \sum_{T \leq |\gamma| \leq T+1} |k_1(\rho_\chi)| + n_\chi(3/2, 1)
\]

\[
\ll \sum_{T=1}^{\infty} \sum_{T \leq |\gamma| \leq T+1} \frac{1}{\gamma^2} + \log A_\chi
\]

\[
\ll \sum_{T=1}^{\infty} \frac{1}{T^2} n_\chi(T) + \log A_\chi
\]

\[
\ll \sum_{T=1}^{\infty} \frac{1}{T^2} (\log A_\chi + \chi(1)n_K \log(T + 2)) + \log A_\chi
\]

\[
\ll \log A_\chi + \chi(1)n_K.
\]

\[\square\]

**Lemma 4.15.** The sum over zeros \( \rho_\chi = \beta + i\gamma \) with \( |\gamma| \geq 1 \) satisfies

\[
\sum_{|\gamma| \geq 1} |k_2(\rho_\chi)| \ll x \log A_\chi.
\]

**Proof.** For \( \rho_\chi = \beta + i\gamma \) with \( |\gamma| \geq 1 \) we have

\[
|k_2(\rho_\chi)| \leq x^{2-\gamma^2}
\]
so using Lemma 3.4 we have

\[ \sum_{|\gamma| \geq 1} |k_2(\rho_\chi)| \leq \sum_{|\gamma| \geq 1} x^{2-\gamma^2} \]

\[ = x^2 \sum_{T=1}^{\infty} \sum_{1 \leq |\gamma| \leq T+1} x^{-\gamma^2} \]

\[ \leq x^2 \sum_{T=1}^{\infty} x^{-(T-1)^2} n_\chi(T) \]

\[ = x \log A_\chi + x \sum_{T=2}^{\infty} x^{-(T-1)^2} n_\chi(T) \]

\[ \ll x \log A_\chi \]

\[ + x \int_2^{\infty} x^{-T^2+2T} (\log A_\chi + n_\chi(1) \log(|T|+2)) \, dT \]

\[ \ll x \log A_\chi. \]

Let us now consider the contribution from zeros close to 1, that is

\[ \sum_{\chi \in \text{Irr}(G)} |\chi(C)| \sum_{|\rho_\chi - 1| \leq 1} |k_j(\rho_\chi)|. \]

**Lemma 4.16.** Let \( L/K \) be a Galois extension of number fields of degree \( n \) and \( \chi \) an irreducible character of \( \operatorname{Gal}(L/K) \).

If \( \chi(1) > 1 \) then

\[ \sum_{|\rho_\chi - 1| < 1} |k_1(\rho_\chi)| \ll \chi(1)^7 (\log A_\chi)^2. \]

If \( \chi(1) = 1 \) then the same conclusion holds with the left-hand sum replaced by a sum over non-exceptional zeros:

\[ \sum_{|\rho_\chi - 1| < 1, \rho_\chi \neq \rho_0} |k_1(\rho_\chi)| \ll (\log A_\chi)^2. \]

**Proof.** We may assume \( x > 1 \). Write each \( \rho_\chi \) as \( \rho_\chi = \sigma + it \). Since \( |\rho_\chi - 1| < 1 \) we can estimate each term as

\[ |k_1(\rho_\chi)| = \frac{|x^{2(\rho_\chi - 1) - x^{\rho_\chi - 1}|^2}{|\rho_\chi - 1|^2} \ll \frac{x^{-2(1-\sigma)}}{|\rho_\chi - 1|^2}. \]
The number of such terms can be estimated using the zero-free region of Proposition 3.6. Set

\[ B_\chi := \frac{c_8}{\chi(1)^3(\log A_\chi + n_K)} \]

where \( c_8 \) is some fixed absolute constant so that whenever \( \chi(1) > 1 \), any zero \( \rho_\chi \) in the critical strip with \( |\rho_\chi - 1| \leq 1 \) satisfies

\[ \sigma < 1 - B_\chi \]

and if \( \chi(1) = 1 \) then there is at most one exception \( \beta_0 \). Therefore

\[ \frac{x^{-2(1-\sigma)}}{|\rho_\chi - 1|^2} < \frac{x^{-2B_\chi}}{|\rho_\chi - 1|^2} \]

and hence by partial summation

\[ \sum_{|\rho_\chi - 1| < 1} |k_1(\rho_\chi)| \ll x^{-2B_\chi} \sum_{|\rho_\chi - 1| < 1} \frac{1}{|\rho_\chi - 1|^2} \ll x^{-2B_\chi} \int_{B_\chi}^1 \frac{1}{t^2} \, dn_\chi(t; 1). \]
We integrate over the region $B_\chi \leq |\rho_\chi - 1| \leq 1$ as shown in Figure 4.2 so in particular $n_\chi(B_\chi; 1) = 0$ if $L(s, \chi)$ has no exceptional zeros. From Lemma 3.5 we see that

$$n_\chi(t; 1) \ll \chi(1) + t(\log A_\chi + \chi(1)n_K)$$

and therefore

$$\int_{B_\chi} \frac{1}{t^2} dn_\chi(t; 1) = \left. \frac{n_\chi(t; 1)}{t^2} \right|_{B_\chi}^{1} + \int_{B_\chi} \frac{n_\chi(t; 1)}{t^3} dt \ll \left. \frac{n_\chi(t; 1)}{t^2} \right|_{B_\chi}^{1} + \int_{B_\chi} \frac{\chi(1) + t(\log A_\chi + \chi(1)n_K)}{t^3} dt \ll n_\chi(1; 1) + \chi(1) \int_{B_\chi} \frac{dt}{t^3} + (\log A_\chi + \chi(1)n_K) \int_{B_\chi} \frac{dt}{t^2} \ll \frac{\chi(1)}{B_\chi} + \frac{\log A_\chi + \chi(1)n_K}{B_\chi} \ll \chi(1)^7(\log A_\chi + \chi(1)n_K)^2 \ll \chi(1)^7(\log A_\chi)^2.$$

Hence

$$\sum_{|\rho_\chi - 1| < 1} |k_1(\rho_\chi)| \ll x^{-2B_\chi} \int_{B_\chi} \frac{1}{t^2} dn_\chi(t; 1) \ll x^{-2B_\chi}\chi(1)^7(\log A_\chi)^2 \ll \chi(1)^7(\log A_\chi)^2$$

where we’ve used the fact that $x > 1$.

Combining the results of this section we arrive at the following refinement of Equation 4.2.
Proposition 4.17. There exist absolute and effective positive constants $c_9$, $c_{10}$, $c_{11}$, $c_{12}$ such that
\[
\sum_{p \in \mathcal{S}(L/K)} \theta(p)(\log Np)\hat{k}_1(Np) \geq \frac{|C|}{|G|}k_1(1) - \frac{|C|}{|G|}k_1(\beta_0)
- c_9 \frac{|C|}{|G|} \sum_{\chi \in \text{Irr}(G)} |\chi(C)||\chi(1)^7(\log A_{\chi})^2
- c_{10} \frac{|C|}{|G|} \sum_{\chi \in \text{Irr}(G)} |\chi(C)||\chi(1)|\left(\log A_{\chi} + \chi(1)n_K\right)
- c_{11} n_K \frac{\log x}{x^2} \sum_{p \in \mathcal{P}(L/K)} \log p - c_{12} n_K \frac{(\log x)^2}{x},
\]
where the $k_1(\beta_0)$ term appears only if the exceptional zero $\beta_0$ exists.

Proof. We break the term coming from the zeros $\rho_{\chi}$ into two parts: zeros with $|\rho_{\chi} - 1| \geq 1$ and zeros with $|\rho_{\chi} - 1| < 1$.

From Lemma 4.14 we have
\[
\sum_{\chi \in \text{Irr}(G)} |\chi(C)| \sum_{|\rho_{\chi} - 1| \geq 1} |k_1(\rho_{\chi})| \ll \sum_{\chi \in \text{Irr}(G)} |\chi(C)\left(\log A_{\chi} + \chi(1)n_K\right).\]

Therefore we may take $c_1$ sufficiently large in Equation 4.2 and combine the contribution from zeros $\rho_{\chi}$ with $|\rho_{\chi} - 1| \geq 1$ into the third term $\sum_{\chi} |\chi(C)||\chi(1)|\left(\log A_{\chi} + \chi(1)n_K\right)$.

For contribution from zeros $\rho_{\chi}$ close to 1, Lemma 4.16 gives the estimate
\[
\sum_{\chi \in \text{Irr}(G)} |\chi(C)| \sum_{|\rho_{\chi} - 1| < 1} |k_1(\rho_{\chi})| = k_1(\beta_0) + O\left(\sum_{\chi \in \text{Irr}(G)} |\chi(C)|\chi(1)^7(\log A_{\chi})^2\right)
\]

where the term corresponding to the exceptional zero $\beta_0$ appears if and only if $\beta_0$ exists. Combining the estimates gives Equation 4.5.

\hfill $\Box$

4.5 Deuring-Heilbronn Phenomenon

In general zero-free regions for $L$-functions are only guaranteed to be free of zeros up to one possible exception. In particular Proposition 3.6 guarantees a true zero-free region for nonlinear characters.
As can be seen in Equation 4.5, the existence of an exceptional zero diminishes the contribution $k_j(1)$ from the pole at $s = 1$ in the main term. However as a general phenomenon the existence of a zero close to 1 will push the other zeros away, resulting in an enlarged zero-free region. This is typically called the Deuring-Heilbronn phenomenon. A version for $\zeta_L$ is given in [LMO79] whose statement we will recall here.

**Theorem 4.18** (Lagarias-Montgomery-Odlyzko). There are effective positive absolute constants $c_{13}, c_{14}$ such that if $\zeta_L$ has a real zero $\beta_0 > 0$ then it is zero-free with $s = \sigma + it$ in the region

$$\sigma \geq 1 - c_{14} \frac{\log \left( \frac{c_{13}}{(1-\beta_0) \log(d_L(|t|+2)^{n_L})} \right)}{\log(d_L(|t|+2)^{n_L})}$$

with the single exception $\beta_0$.

**Proof.** This is [LMO79, Theorem 5.1].

Using this one obtains an unconditional lower bound for the exceptional zero as follows.

**Corollary 4.19.** There is an effective positive absolute constant $c_{15}$ such that any exceptional zero $\beta_0$ of $\zeta_L(s)$ satisfies

$$1 - \beta_0 \geq d_L^{-c_{15}}.$$

**Proof.** This is [LMO79, Corollary 5.2].

In this section we exhibit a Deuring-Heilbronn phenomenon for Artin $L$-functions proven by Mahmoudian in [Mah99]. The proof follows that of [LMO79, Theorem 5.1] using the function $L(s, (1 + \chi) \otimes (1 + \bar{\chi}))$ instead of $\zeta_L$. We recall the version of Turán’s power sum theorem as stated in [LMO79].

**Theorem 4.20** (Turán, Lagarias-Montgomery-Odlyzko). Let

$$s_m = \sum_{n=1}^{\infty} b_n z_n^m$$

with

1. $|z_n| \leq |z_1|$ for all $n \geq 1$,

2. the $b_n$ are real, and

3. $b_n \geq 0$ for $n$ with $\frac{1}{3}|z_1| \leq |z_n| \leq |z_1|$.
Set
\[ L = \frac{1}{b_1 |z_1|} \sum_{n=1}^{\infty} |b_n z_n|. \]

Then there exists \( j_0 \) with \( 1 \leq j_0 \leq 24L \) such that
\[ \text{Re} s_{j_0} \geq \frac{b_1}{8} |z_1|^j_0. \]

Proof. This is [LMO79, Theorem 4.2]. \( \square \)

**Theorem 4.21** (Mahmoudian). Let \( \chi \) be a nontrivial irreducible character of \( G \) and suppose \( \beta_0 \) is a real zero of \( L(s, \chi) \). Assume Artin’s conjecture for \( L(s, \chi) \) and \( L(s, \chi \otimes \overline{\chi}) \). Then there exist effective and absolute constants \( c_{16}, c_{17} \) such that \( L(s, \chi) \) has no other zeros in the region
\[ 1 - c_{17} \frac{\log \left( \frac{L(s, \chi) (1 - \beta_0)}{L(s, \chi)} \right)}{L(s, \chi)} < \sigma < 1 \]
where
\[ L(s, \chi) = \chi(1) \left( \log A_{\chi} + \chi(1)nK \log(|t| + 2) \right). \]

Proof. This is [Mah99, Theorem 3.1]. \( \square \)

As in **Corollary 4.19** we obtain a lower bound for how close the exceptional zero can be to 1.

**Corollary 4.22.** There exists an absolute effective positive constant \( c_{18} \) such that any exceptional zero \( \beta_0 \) for \( L(s, \chi) \) satisfies
\[ 1 - \beta_0 \ll A_{\chi}^{-c_{18}}. \]

### 4.6 Proofs of Main Results

We now use the estimates of the previous sections to prove the main results. First we will prove **Theorem 1.4** which will allow us to handle the case of exceptional zeros when the nonvanishing characters satisfy a nonnegativity condition. We then use **Theorem 1.4** to conclude the proof of **Theorem 1.1**.

#### 4.6.1 Proof of Theorem 1.4

We see that the key step in the proof of **Theorem 4.21** is to note that the Dirichlet series of the function \( F_{\chi}(s) \) has nonnegative coefficients. Hence we may adapt the proof to prove **Theorem 1.4**.
Proof of Theorem 1.4. Let \( \varphi \) be a (not necessarily irreducible) character of \( G \) which decompose into a sum of irreducible characters as

\[
\varphi = \sum_{i=1}^{m} a_i \chi_i.
\]

Set \( F(s) = L(s, \varphi) = L(s, \sum_{i=1}^{m} a_i \chi_i) \). By the functoriality of \( L \)-functions

\[
F(s) = \prod_{i=1}^{m} L(s, \chi_i)^{a_i}.
\]

The order of the pole at \( s = 1 \) is given by \( \langle \varphi, 1_G \rangle \) where \( 1_G \) denotes the trivial character. Hence \((s - 1)\langle \varphi, 1_G \rangle F(s)\) is entire and has a Hadamard product factorization

\[
(s - 1)\langle \varphi, 1_G \rangle F(s) = s^r e^{\alpha_1 + \alpha_2 s} \prod_{\omega \neq 0, F_{\chi}(\omega) = 0} \left( 1 - \frac{s}{\omega} \right) e^{s/\omega}
\]

(4.6)

where the \( \omega \) run over the zeros of \( F(s) \), possibly with multiplicity and the dependence of the quantities \( r, \alpha_1, \alpha_2 \) on \( \varphi \) will be suppressed for sake of notation.

Logarithmically differentiating both sides of Equation 4.6 gives

\[
\frac{\langle \varphi, 1_G \rangle}{s - 1} + \frac{F'}{F}(s) = \frac{r}{s} + \alpha_2 + \sum_{\omega \neq 0} \left( \frac{1}{s - \omega} + \frac{1}{\omega} \right)
\]

hence

\[
-\frac{F'}{F}(s) = \frac{\langle \varphi, 1_G \rangle}{s - 1} - \frac{r}{s} - \alpha_2 - \sum_{\omega \neq 0} \left( \frac{1}{s - \omega} + \frac{1}{\omega} \right).
\]

On the other hand, we know from logarithmically differentiating the Euler product that

\[
-\frac{F'}{F}(s) = \sum_{p^m} \frac{\varphi(p^m) (\log Np) (Np)^{ms}}{(Np)^{ms}}
\]

so

\[
\sum_{p^m} \varphi(p^m) (\log Np) (Np)^{-ms} = \frac{r}{s} + \alpha_2 + \sum_{\omega} \left( \frac{1}{s - \omega} + \frac{1}{\omega} \right).
\]

(4.7)
Differentiating both sides of Equation 4.7 $2j - 1$ times gives
\[
\frac{1}{(2j-1)!} \sum_{p^m} \phi(p^m) (\log Np)(\log Np^m)^{2j-1}(Np^m)^{-s} = \frac{\langle \phi, 1_G \rangle}{(s-1)^{2j}} - \sum_{\omega} \frac{1}{(s - \omega)^{2j}} \tag{4.8}
\]
which is valid for $\sigma > 1$ and $j \geq 1$. Evaluating Equation 4.8 at $s = \sigma$ and $s = \sigma + it$ and adding the result gives
\[
\frac{1}{(2j-1)!} \sum_{p^m} \phi(p^m) (\log Np)(\log Np^m)^{2j-1}(Np^m)^{-\sigma}(1 + (Np^m)^{-it})
= \frac{\langle \phi, 1_G \rangle}{(\sigma - 1)^{2j}} + \frac{\langle \phi, 1_G \rangle}{(s - 1)^{2j}} - \sum_{\omega} \left( \frac{1}{(\sigma - \omega)^{2j}} + \frac{1}{(s - \omega)^{2j}} \right).
\]

Let $\beta_0$ be a real zero of $F(s)$. Separating the contribution from $\beta_0$ in Equation 4.8 we have
\[
\frac{1}{(2j-1)!} \sum_{p^m} \phi(p^m) (\log Np)(\log Np^m)^{2j-1}(Np^m)^{-\sigma}(1 + (Np^m)^{-it})
= \frac{\langle \phi, 1_G \rangle}{(\sigma - 1)^{2j}} + \frac{\langle \phi, 1_G \rangle}{(s - 1)^{2j}} - \frac{1}{(\sigma - \beta_0)^{2j}} - \frac{1}{(s - \beta_0)^{2j}} - \sum_{n} z_n^j
\]
where each $z_n$ is of the form $(\sigma - \omega)^{-2}$ or $(s - \omega)^{-2}$ with $\omega$ a zero of $F$.

Observe that the left-hand side has nonnegative real part since $\phi(p^m)$ is nonnegative for each $p^m$ by assumption. Setting $\sigma = 2$, we obtain
\[
\text{Re} \sum z_n^j \leq \langle \phi, 1_G \rangle - \frac{1}{(2 - \beta_0)^{2j}} + \text{Re} \left( \frac{\langle \phi, 1_G \rangle}{(1 + it)^{2j}} - \frac{2}{(2 - \beta_0 + it)^2} \right)
\leq c_{19} j \langle \phi, 1_G \rangle (1 - \beta_0) \tag{4.9}
\]
where $c_{19}$ is an absolute and effective constant and the second inequality follows from [LMO79, Equation 5.5].

Suppose $\rho = \beta + i\gamma \neq \beta_0$ is a zero of $F$. Evaluate Equation 4.9 at $t = \gamma$.

Suppose $z_1$ has the largest absolute value of all the $z_n$. Then
\[
|z_1| \geq \frac{1}{(2 - \beta)^2}
\]
since if $s = 2 + i\gamma$ then $(s - \rho)^{-2} = (2 - \beta)^{-2}$. 

As well, if $\rho' = \beta' + i\gamma'$ is another zero then
\[
|s - \rho'|^{-2} = \frac{1}{(2 - \beta + \beta' + i(\gamma - \gamma'))^2 + (\gamma - \gamma')^2} \leq \frac{1}{(2 - \beta)^2} = \frac{1}{|s - \rho|^2}
\]
and similarly
\[
|(\sigma - \rho')^{-2}| \leq |s - \rho|^{-2}.
\]
Set
\[
L = \frac{1}{|z_1|} \sum_{n=1}^{\infty} |z_n|
\]
and
\[
s_m = \sum_{n=1}^{\infty} z_n^m.
\]
Applying the Turán power sum theorem Theorem 4.20, there exists $j_0$, $1 \leq j_0 \leq 24L$, with
\[
\text{Re } s_{j_0} \geq \frac{1}{8} |z_1|^{j_0}
\]
and therefore
\[
L \ll \sum_{\omega} \left( \frac{1}{|z - \omega|^2} + \frac{1}{|2 + i\gamma - \omega|^2} \right)
\]
where the $(2 - \beta)^2$ has been absorbed in the implicit constant as it is at most 4.

Applying partial summation and Lemma 3.4, we obtain
\[
L \ll \int_{0}^{\infty} \frac{1}{u^2 + 1} \, dn_F(u) + \int_{0}^{\infty} \frac{1}{u^2 + 1} \, dn_F(u + \gamma) \\
\ll \sum_{i=1}^{m} a_i \chi_i(1) (\log A_\chi + \chi_i(1)n_\chi \log(\gamma + 2)) = \mathcal{L}(|\gamma|).
\]
Hence for some $j_0 \ll L \ll \mathcal{L}(|\gamma|)$ we have
\[
\text{Re } \sum_{n=1}^{\infty} z_n^{j_0} \geq \frac{1}{8} (2 - \beta)^{-2j_0} \geq \frac{1}{8} \exp \left( -2j_0 \log(1 + 1 - \beta) \right) \geq \frac{1}{8} \exp \left( -2j_0 (1 - \beta) \right).
\]
Putting the inequalities together we obtain
\[
\frac{1}{8} \exp(-2j_0(1 - \beta)) \leq \text{Re} \sum_{n=1}^{\infty} z_n^{j_0} \leq c_{19} j_0 \langle \varphi, 1_G \rangle (1 - \beta_0)
\ll \langle \varphi, 1_G \rangle L(|\gamma|) (1 - \beta_0)
\]
from which we deduce that there are absolute positive constants $c_{20}, c_{21}$ with
\[
\beta \leq 1 - c_{21} \frac{\log \left( \frac{c_{20}}{\langle \varphi, 1_G \rangle L(|\gamma|)(1 - \beta_0)} \right)}{L(|\gamma|)}
\]
as required.

\[ \square \]

Remark 4.23. Notice that for the character of the regular representation of any finite $G$ one has
\[
\text{reg}_G(g) = \sum_{\chi \in \text{Irr}(G)} \chi(1) \chi(g) = \begin{cases} |G| & a = 1 \\\(0 & a \neq 0 \end{cases}
\]
so taking $\varphi = \text{reg}_G$ for $G = \text{Gal}(L/K)$ satisfies the hypotheses of Theorem 1.4. In fact applying Theorem 1.4 with $\varphi = \text{reg}_G$ recovers Theorem 4.18 since
\[
\sum_{\chi \in \text{Irr}(G)} \chi(1) (\log A_\chi + n_k \chi(1) \log(||t| + 2)) = \log(d_L \log(||t| + 2)^{n_k}).
\]

As before we obtain a bound on the exceptional zero similar to Corollary 4.19 and Corollary 4.22.

**Corollary 4.24.** Let $L(s, \varphi)$ satisfy the hypothesis of Theorem 1.4. Then there is a positive, absolute, effective constant $c_{22}$ such that any real zero $\beta_0$ of $L(s, \varphi)$ satisfies
\[
1 - \beta_0 \geq \left( \prod_{i=1}^{n} A_{\chi_i}^{a_i \chi_i(1)} \right)^{-c_{22}} \quad (4.10)
\]

**Proof.** Choose $c_{22}$ large enough so that
\[
c_{21} \log \left( \frac{c_{20} \left( \prod_i A_{\chi_i}^{a_i \chi_i(1)} \right)^{c_{22}}}{\langle \varphi, 1_G \rangle \sum_i a_i \chi_i(1) \log A_{\chi_i}} \right) > 3 \sum_{i=1}^{n} a_i \chi_i(1) \log A_{\chi_i}. \quad (4.11)
\]
Then if Equation 4.10 fails, substituting Equation 4.11 into Theorem 1.4 gives that $L(s, \varphi)$ has no zeros for $\sigma > -2$ except at $\beta_0$. But this is a contradiction since there are trivial zeros at $s = -1$ or $s = 0$. 
4.6.2 Proof of Theorem 1.1

We now use the results of Section 4.5 to give a refinement of Proposition 4.13 which will be used in the next chapter to give application to specific choices of $G$ and $C$. The main strategy is to observe that in Proposition 4.13 the only $L$-functions that has nonzero contribution comes from characters which are nonvanishing at $C$, that is $\chi \in \text{nv}(C)$ (see Definition 2.4). Hence we obtain a saving when $\text{nv}(C)$ is small compared to $\text{Irr}(G)$.

**Proposition 4.25.** Let $L/K$ be a Galois extension of number fields with $G = \text{Gal}(L/K)$. Let $C$ be a conjugacy class of $G$. Denote by $\text{nv}(C)$ the set of characters which are nonvanishing at $C$, as in Definition 2.4. Let $k_j$, $\theta$, and $S(L/K)$ be defined as before in Section 4.1 and Section 4.2.

Suppose Artin’s holomorphy conjecture holds for each $L(s, \chi)$ with $\chi \in \text{nv}(C)$. Then

1. If $\prod_{\chi \in \text{nv}(C)} L(s, \chi)$ has no exceptional zero, then there are effective constants $c_{23}, c_{24}, c_{25}, c_{26}$ such that

$$\sum_{p \in S(L/K)} \theta(p) \log Np \hat{k}_1(Np) \geq \frac{|C|}{|G|} (\log x)^2 - c_{23} \left( \frac{|C|}{|G|} \sum_{\chi \in \text{nv}(C)} |\chi(C)| \chi(1)^7 (\log A_\chi)^2 \right)$$

$$- c_{24} \left( \frac{|C|}{|G|} \sum_{\chi \in \text{nv}(C)} |\chi(C)| \log A_\chi \right) - c_{25} n_K \log x \frac{\log x}{x^2} \sum_{p \in P(L/K)} \log p - c_{26} n_K \log \frac{x}{x^2}.$$  \hspace{1cm} (4.12)

2. If $\prod_{\chi \in \text{nv}(C)} L(s, \chi)$ has an exceptional zero $\beta_0$ assume furthermore that

$$\sum_{\chi \in \text{nv}(C)} \chi(g) \geq 0$$  \hspace{1cm} (4.13)

for each $g \in G$. Set

$$L := \sum_{\chi \in \text{nv}(C)} \chi(1) \log A_\chi.$$
Then there are effective absolute constants \( c_{27}, c_{28}, c_{29}, c_{30}, c_{31}, c_{32} \) such that

\[
\sum_{p \in S(L/K)} \theta(p) (\log Np) k_1(Np) \\
\geq \frac{|C|}{|G|} (\log x)^2 \min\{1, (1 - \beta_0) \log x\} \\
- c_{27} \frac{|C|}{|G|} \sum_{\chi \in \text{inv}(C)} |\chi(C)| |\chi(1)| \cdot L^2 [(1/c_{28})(1 - \beta_0) L]^{2c_{29}\log x/L} \\
- c_{30} \frac{|C|}{|G|} \sum_{\chi \in \text{inv}(C)} |\chi(C)| \log A_x \\
- c_{31} n_K \frac{\log x}{x^2} \sum_{p \in P(L/K)} \log p - c_{32} n_K \frac{(\log x)^2}{x}.
\]  

(4.14)

Furthermore we may take \( c_{28} > 1 \).

3. Suppose furthermore that the exceptional zero satisfies

\[
1 - \beta_0 \ll \frac{1}{L^2}
\]

for an effective and absolute implicit constant. Then there are effective absolute constants \( c_{33}, c_{34}, c_{35}, c_{36}, c_{37} \) such that

\[
\sum_{p \in S(L/K)} \theta(p) (\log Np) k_2(Np) \\
\geq \frac{|C|}{|G|} x^2 \log x (1 - \beta_0) - c_{38} \frac{|C|}{|G|} x^2 (1 - \beta_0) c_{34} \log x/L \cdot L \\
- c_{39} \frac{|C|}{|G|} x L - c_{40} n_K (\log x)^{1/2} \sum_{p \in P(L/K)} \log p \\
- c_{41} n_K x^{7/4}
\]  

(4.15)

Proof. We first observe that \( k_1(1) = (\log x)^2 \) and furthermore that

\[
k_1(1) - k_1(\beta_0) = (\log x)^2 - \left( \frac{x^{\beta_0(1-\beta_0)}}{\beta_0 - 1} \right)^2 \\
\gg (\log x)^2 \min\{1, (1 - \beta_0) \log x\}
\]
$$k_2(1) - k_2(\beta_0) = x^2 - x^{\beta_0 + \beta_0^3} \gg x^2 \min\{1, (1 - \beta_0) \log x\}.$$ 

Since $\log A_{\chi} \gg (1) nk$ we see easily that

$$\sum_{\chi \in \text{Irr}(G)} |\chi(C)| (\log A_{\chi} + k(1)nk) \ll \sum_{\chi \in \text{nv}(C)} |\chi(C)| \log A_{\chi}. \quad (4.16)$$

As well, from Lemma 4.16 we have

$$\sum_{\chi \in \text{Irr}(G)} |\chi(C)| \sum_{|\rho_{\chi} - 1| < 1} |k_1(\rho_{\chi})| \ll \sum_{\chi \in \text{nv}(C)} |\chi(C)| \chi(1)^7 (\log A_{\chi})^2.$$ 

Combining with Equation 4.2 yields Equation 4.12.

Now suppose that there is an exceptional zero $\beta_0$ of $\prod_{\chi \in \text{nv}(C)} L(s, \chi)$. By Equation 4.13 in the hypothesis we may use Theorem 1.4 with

$$\varphi = \sum_{\chi \in \text{nv}(C)} \chi.$$ 

We note that

$$\mathcal{L}(t) \ll \mathcal{L}(1) \ll \mathcal{L} := \sum_{\chi \in \text{nv}(C)} \chi(1) \log A_{\chi}$$

and so by Theorem 1.4 any nonexceptional zero $\rho = \beta + it$ of

$$L(s, \varphi) = \prod_{\chi \in \text{nv}(C)} L(s, \chi)$$

with $|\rho - 1| < 1$ satisfies

$$\beta \leq 1 - c_{29} \frac{\log \left( \frac{c_{28}}{\mathcal{L}(1-\beta_0)} \right)}{\mathcal{L}}.$$ 

By taking $c_{29}$ smaller if necessary we may take $c_{28} \geq 1$.

Set

$$B := c_{29} \frac{\log \left( \frac{c_{28}}{\mathcal{L}(1-\beta_0)} \right)}{\mathcal{L}}$$

so that any zero $\rho = \sigma + i\gamma \neq \beta_0$ satisfies

$$\sigma \leq 1 - B.$$
Thus

$$|k_1(\rho)| = \left| \frac{x^{2(\rho-1)} - x^{\rho-1}}{\rho - 1} \right|^2 \leq x^{-2B} \frac{1}{|\rho - 1|^2}.$$ 

Furthermore such zeros with $|\rho - 1| < 1$ will lie in the shaded region of Figure 4.3 to the right of the line $\sigma = 1 - B$. Thus if $t = |\rho - 1|$ then in particular $B \leq t \leq 1$ and so

$$\sum_{\chi \in \text{nv}(C)} |\chi(C)| \sum_{\substack{|\rho - 1| < 1 \ \rho \neq \rho_0 \ \rho_\chi \neq \beta_0}} |k_1(\rho_\chi)| \leq x^{-2B} \sum_{\chi \in \text{nv}(C)} |\chi(C)| \int_{B}^{1} \frac{1}{t^2} dn_{\chi}(t;1).$$
Then using Lemma 3.5 we have

\[
\sum_{\chi \in \text{nv}(C)} |\chi(C)| \int_B^1 \frac{1}{t^2} dn_\chi(t; 1) \ll \sum_{\chi \in \text{nv}(C)} |\chi(C)| \left( \left| \frac{n_\chi(t; 1)}{t^2} \right|_B + \int_B^1 \frac{n_\chi(t; 1)}{t^3} \, dt \right)
\]

\[
\ll \sum_{\chi \in \text{nv}(C)} |\chi(C)| \left( \frac{\chi(1)}{B^2} + \frac{\log A_\chi}{B} \right)
\]

\[
\ll \left( \sum_{\chi \in \text{nv}(C)} |\chi(C)|\chi(1) \right) \left( \frac{1}{B^2} + \frac{L}{B} \right)
\]

\[
\ll \left( \sum_{\chi \in \text{nv}(C)} |\chi(C)|\chi(1) \right) \frac{L}{B}.
\]

Since \(B^{-1} \ll L\) we have

\[
\sum_{\chi \in \text{nv}(C)} |\chi(C)| \sum_{|\rho_\chi - 1| < 1 \atop \rho_\chi \neq \beta_0} |k_1(\rho_\chi)| \ll \left( \sum_{\chi \in \text{nv}(C)} |\chi(C)|\chi(1) \right) x^{-2B} B^{-1} L
\]

\[
\ll \left( \sum_{\chi \in \text{nv}(C)} |\chi(C)|\chi(1) \right) L^2 [(1/c_{28})(1 - \beta_0)L]^{2c_{29} \log x / L}.
\]


Now suppose that

\[
1 - \beta_0 \leq \frac{c_{42}^2}{L^2}
\]

for an absolute effective \(c_{42}\) which we choose to satisfy

\[
c_{42}^2 < c_{28}
\]

so that we have

\[
\frac{1}{2} \log(1 - \beta_0)^{-1} \leq \log \frac{c_{42}}{(1 - \beta_0)L}.
\]

Therefore by Theorem 1.4 there is an absolute constant \(c_{34}\) such that if \(\rho = \beta + i\gamma\) is any zero with \(|\gamma| \leq 1\) and \(\rho \neq \beta_0\) then

\[
\beta \leq 1 - c_{34} \frac{\log(1 - \beta_0)^{-1}}{L}.
\]
Now

\[ |k_2(\rho)| \leq x^{\beta^2 + \beta} \leq x^{1 + \beta} = x^2 \beta^{-1} \]

and so

\[ |k_2(\beta)| \leq x^2 \exp \left( -c_{34} \frac{\log x \log (1 - \beta_0)^{-1}}{L} \right) = x^2 (1 - \beta_0)^{c_{34} \log x/L}. \]

Using Lemma 3.4 we obtain

\[
\sum_{\chi \in \text{nv}(C)} |\chi(C)| \sum_{|\gamma| \leq 1} |k_2(\rho \chi)| \ll x^2 (1 - \beta_0)^{c_{34} \log x/L} \sum_{\chi \in \text{nv}(C)} |\chi(C)| \log A_{\chi}.
\]

Combining the above with Equation 4.3, Equation 4.16, and Lemma 4.15 we obtain

\[
\sum_{p \in S(L/K)} \theta(p) (\log Np) \hat{k}_2(Np)
\]

\[ \geq \frac{|C|}{|G|} x^2 \log x (1 - \beta_0) - c_{43} \frac{|C|}{|G|} x^2 (1 - \beta_0)^{c_{16} \log x/L} \sum_{\chi \in \text{nv}(C)} |\chi(C)| \log A_{\chi}
\]

\[ - c_{44} \frac{|C|}{|G|} x \sum_{\chi \in \text{nv}(C)} |\chi(C)| \log A_{\chi}
\]

\[ - c_{45} \frac{|C|}{|G|} \sum_{\chi \in \text{nv}(C)} |\chi(C)| \log A_{\chi}
\]

\[ - c_{46} n_K (\log x)^{1/2} \sum_{p \in P(L/K)} \log p - c_{47} n_K x^{7/4}
\]

\[ \geq \frac{|C|}{|G|} x^2 \log x (1 - \beta_0) - c_{43} \frac{|C|}{|G|} x^2 (1 - \beta_0)^{c_{16} \log x/L} L
\]

\[ - c_{48} \frac{|C|}{|G|} xL - c_{46} n_K (\log x)^{1/2} \sum_{p \in P(L/K)} \log p - c_{47} n_K x^{7/4}
\]

\[ \square
\]

Proof of Theorem 1.1. We see that trivially

\[
\sum_{\chi \in \text{nv}(C)} |\chi(C)| \chi(1)^7 (\log A_{\chi})^2 \leq \sum_{\chi \in \text{nv}(C)} |\chi(C)| \chi(1)^5 \left( \sum_{\chi \in \text{nv}(C)} \chi(1) \log A_{\chi} \right)^2
\]
so we see that the left-hand side of Equation 4.12 is positive if we set

\[ \log x \gg \left( \sum_{\chi \in \text{nv}(C)} |\chi(C)|\chi(1)^{5} \right)^{1/2} \sum_{\chi \in \text{nv}(C)} \chi(1) \log A_{\chi} \]
\[ + \log \left( \frac{|G|}{|C|} \sum_{p \in P(L/K)} \log p \right). \]

Suppose there is an exceptional zero \( \beta_0 \) which satisfies

\[ 1 - \beta_0 \geq \frac{c_{42}^2}{\left( \sum_{\chi \in \text{nv}(C)} \chi(1) \log A_{\chi} \right)^{2}}. \]

with \( c_{42} \) as in Equation 4.17. Then noting that \( c_{42}^2 / c_{28} < 1 \) by choice of \( c_{28}, c_{42} \) we have

\[ \left( \sum_{\chi \in \text{nv}(C)} |\chi(C)|\chi(1) \right) L^2 [(1/c_{28})(1 - \beta_0)L]^2c_{29}\log x/L \]
\[ \leq \left( \sum_{\chi \in \text{nv}(C)} |\chi(C)|\chi(1) \right) L^2 [(c_{42}^2 / c_{28})L^{-1}]^2c_{29}\log x/L \]
\[ \leq \left( \sum_{\chi \in \text{nv}(C)} \chi(1)^2 \right) L^2 (1 - c_{29}\log x/L) \]
\[ \leq \frac{L^3}{L^{2c_{29}\log x/L}}. \]

Then any choice of \( x \) satisfying

\[ c_{29}\log x \geq L \]

will make

\[ \left( \sum_{\chi \in \text{nv}(C)} |\chi(C)|\chi(1) \right) L^2 [(1/c_{28})(1 - \beta_0)L]^2c_{29}\log x/L \leq L. \]

Then noting that

\[ \sum_{\chi \in \text{nv}(C)} |\chi(C)| \log A_{\chi} \leq L. \]
we have that for any such choice of $x$ Equation 4.14 becomes

\[
\frac{|G|}{|C|} \sum_{p \in S(L/K)} \theta(p) \log(Np) \hat{k}_1(Np)
\geq c_{49}(\log x)^3 \mathcal{L}^{-2} - c_{50} \mathcal{L} - c_{51} \frac{|G|}{|C|} n_K \frac{\log x}{x^2} \sum_{p \in P(L/K)} \log p
- c_{52} \frac{|G|}{|C|} n_K \frac{(\log x)^2}{x}.
\]

Then from Equation 4.14 we see that we can take

\[
\log x \gg \sum_{\chi \in \text{nv}(C)} \chi(1) \log A_{\chi} + \log \left( \frac{|G|}{|C|} n_K \sum_{p \in P(L/K)} \log p \right).
\]

Lastly, if

\[
1 - \beta_0 \leq \frac{c_{42}^2}{(\sum_{\chi \in \text{nv}(C)} \chi(1) \log A_{\chi})^2}
\]

then we use Equation 4.15. From Corollary 4.24 we have that $1 - \beta_0 \geq \exp(-c_{22} \mathcal{L})$. Hence if we take

\[
x = \exp \left[ c_{53} \mathcal{L} + c_{54} \log \left( \frac{|G|}{|C|} n_K \sum_{p \in P(L/K)} \log p \right) \right]
\]

for a sufficiently large absolute constant $c_{53}, c_{54}$ then Equation 4.15 yields

\[
\frac{|G|}{|C|} \sum_{p \in S(L/K), Np < x^{10}} \theta(p) \log(Np) \hat{k}_2(Np)
\geq x^2 \log x (1 - \beta_0) - c_{55} \frac{|G|}{|C|} n_K (\log x)^{1/2} \sum_{p \in P(L/K)} \log p > 0.
\]

Therefore in this case we see that the least prime $p$ whose Frobenius in $G$ is $C$ satisfies

\[
N_K/Qp \leq \left( \prod_{\chi \in \text{nv}(C)} A_{\chi}^{\chi(1)} \right)^{c_{53}} \left( \frac{|G|}{|C|} n_K \sum_{p \in P(L/K)} \log p \right)^{c_{54}}.
\]

\qed
4.7 ESTIMATES UNDER ADDITIONAL ASSUMPTIONS

In this section we will show stronger estimates for the least prime whose Frobenius is \( C \) under Assumption 1 and Assumption 2 of Section 3.6. Write \( s = \sigma + it \). We recall the assumptions as follows.

1. For \( 1 < \sigma < 3 \) the logarithmic derivative of \( L(s, \chi) \) satisfies
   \[
   \left| \frac{L'}{L}(s, \chi) \right| \ll \frac{1}{\sigma - 1} + \log A_\chi.
   \]

2. There exists a constant \( c \) such that if \( \chi \) is irreducible and \( L(s, \chi) \) satisfies Artin’s conjecture then it has at most one zero in the region
   \[
   1 - \frac{c}{\chi(1)(\log A_\chi + \chi(1)n_K \log(|t| + 2))} \leq \sigma \leq 1.
   \]

Under Assumption 1 by Lemma 3.8 one has the stronger zero-density estimate
\[
n_\chi(r; 1) \ll 1 + r(\log A_\chi + n_K \chi(1) \log(3)) \ll 1 + r \log A_\chi.
\]

We now use this in addition to the expanded zero-free region of Assumption 2 to obtain an estimate for the sum over zeros close to \( s = 1 \).

**Lemma 4.26.** Let \( L/K \) be a Galois extension of number fields of degree \( n \) and \( \chi \) an irreducible character of \( \text{Gal}(L/K) \). Assume that Assumption 1 and Assumption 2 hold. Then
\[
\sum_{\begin{subarray}{c} |\rho_\chi - 1| < 1 \\ \rho \not= \beta_0 \end{subarray}} |k_1(\rho_\chi)| \ll (\chi(1) \log A_\chi)^2.
\]

**Proof.** The proof is similar to that of Lemma 4.16 while applying the stronger zero-density estimates and zero-free regions. We may assume \( x > 1 \). As in Lemma 4.16 we see that
\[
|k_1(\rho_\chi)| \ll \frac{x^{-2(\sigma - 1)}}{|\rho_\chi - 1|^2}.
\]

It follows from Assumption 2 that for all \( \rho_\chi = \sigma + it \) with \( |\rho_\chi - 1| < 1 \) and \( \rho_\chi \not= \beta_0 \) we have for some effective absolute constant \( c_{20} \) that
\[
\sigma < 1 - \frac{c_{20}}{\chi(1) \log A_\chi}.
\]

Set
\[
B'_\chi := \frac{c_{20}}{\chi(1) \log A_\chi}.
\]
so that $B'_X < 1 - \sigma$ and therefore

$$x^{-2(\sigma - 1)} \frac{|\rho_X - 1|}{|\rho_X - 1|^2} \leq x^{-2B'_X} \frac{|\rho_X - 1|}{|\rho_X - 1|^2}.$$  

Thus as in Lemma 4.16 we have by partial summation

$$\sum_{|\rho_X - 1| < 1} |k_1(\rho_X)| \ll x^{-2B'_X} \int_{B'_X}^1 \frac{1}{t^2} dn_X(t; 1)$$

where the region of integration is similar to the one given in Figure 4.2.

Now using Lemma 3.8 we have

$$\int_{B'_X}^1 \frac{1}{t^2} dn_X(t; 1) = \frac{n_X(t; 1)}{t^2} \bigg|_{B'_X}^1 + \int_{B'_X}^1 \frac{n_X(t; 1)}{t^3} dt$$

$$\ll n_X(1; 1) + \int_{B'_X}^1 \frac{1 + t(\log A_X + \chi(1)n_K)}{t^3} dt$$

$$\ll \frac{1}{B'_X^2} + \frac{\log A_X + \chi(1)n_K}{B'_X}$$

$$\ll (\chi(1) \log A_X)^2.$$

Thus

$$\sum_{|\rho_X - 1| < 1} |k_1(\rho_X)| \ll x^{-2B'_X} \int_{B'_X}^1 \frac{1}{t^2} dn_X(t; 1) \ll (\chi(1) \log A_X)^2.$$  

where $x^{-2B'_X} < 1$ since $x > 1$.  

**Proposition 4.27.** Let $L/K$ be a Galois extension of number fields with $G = \text{Gal}(L/K)$ and let $P(L/K)$ be the set of rational primes below primes of $K$ which ramify in $L$. Let $C$ be a conjugacy class of $G$. Suppose that for each $L(s, \chi)$ with $\chi \in \text{nv}(C)$ Artin’s conjecture, Assumption 1, and Assumption 2 hold. If $L(s, \chi)$ does not have an exceptional zero for each $\chi \in \text{nv}(C)$, then there exists an absolute effective constant $c_{56}$ such that the least prime $p$ of $K$ whose Frobenius in $G$ is $C$ satisfies

$$N_{K/Q} \leq \left( \prod_{\chi \in \text{nv}(C)} A_{\chi}^{\chi(1)|\chi(C)|^{1/2}} \right)^{c_{56}} \left( \frac{|G|}{|C|} \sum_{p \in P(L/K)} \log p \right)^{c_{56}}.$$
Proof. The estimate in Lemma 4.26 allows us to replace the second term in Equation 4.12 to obtain

\[
\frac{|G|}{|C|} \sum_{p \in \mathbb{P}(L/K)} \theta(p)(\log Np)\hat{k}_1(Np) \\ \geq (\log x)^2 - c_{57} \sum_{\chi \in \text{nv}(C)} |\chi(C)|(\chi(1) \log A_\chi)^2 \\
- c_{58} \sum_{\chi \in \text{nv}(C)} |\chi(C)| \log |A_\chi| - c_{59} \frac{|G|}{|C|} n_K \frac{\log x}{x^2} \sum_{p \in \mathbb{P}(L/K)} \log p \\
- c_{60} \frac{|G|}{|C|} n_K \frac{(\log x)^2}{x}.
\]

We see that

\[
\sum_{\chi \in \text{nv}(C)} |\chi(C)|(\chi(1) \log A_\chi)^2 \leq \left( \sum_{\chi \in \text{nv}(C)} |\chi(C)|^{1/2} \chi(1) \log A_\chi \right)^2
\]

so we can take

\[
\log x \gg \sum_{\chi \in \text{nv}(C)} |\chi(C)|^{1/2} \chi(1) \log A_\chi + \log \left( \frac{|G|}{|C|} n_K \sum_{p \in \mathbb{P}(L/K)} \log p \right).
\]

\[\square\]
THE LEAST PRIME IN A CONJUGACY CLASS: APPLICATIONS

In this chapter we apply the results in Chapter 4 and in particular Theorem 1.1 to obtain bounds for the least prime in Frobenius classes that are vanishing for many characters. We first consider the case of Gal($L/K$) = $S_n$ and the class of $n$-cycles and $(n - 1)$-cycles. Next we consider applications to the class of reflections in $D_n$-extensions and to the general case of classes of Camina elements.

5.1 APPLICATION TO $S_n$-EXTENSIONS

We first apply Theorem 1.1 for a general $S_n$-extension to obtain upper bounds for the least prime in whose Frobenius is an $n$-cycle or an $(n - 1)$-cycle. Following that we consider the special case of an $S_n$-extension of $\mathbb{Q}$ which is unramified over a quadratic extension where we prove strengthened estimates for the least prime in an $n$-cycle or $(n - 1)$-cycle.

5.1.1 The least prime in an $n$-cycle

Throughout this section let $L/K$ be a Galois extension of number fields with Gal($L/K$) = $S_n$ and let $C$ denote the conjugacy class of $n$-cycles. For this class we have $|C| = n$ and $|G| = n!$. From Section 2.2.5 we recall Proposition 2.8 and Proposition 2.10:

- $|\chi(C)| = 1$ if $\chi$ is a hook character, that is $\chi = \chi^\lambda$ where $\lambda$ is a hook; otherwise $\chi(C) = 0$. In particular $nv(C)$ is the set of hook characters.

- If $\chi^\lambda$ is a hook character with corresponding hook $\lambda = (r, 1^{n-r})$ then it has character degree $\binom{n-1}{r}$.

We will use these in conjunction with Theorem 1.1 to obtain estimates for the special case when $C$ is the class of $n$-cycles. We will need an asymptotic on power sums of binomial coefficients from [PS72, Pt. II, 40]. We collect the details here for completeness.
Lemma 5.1 ([PS72, Pt. II, 54]). Suppose $f(x)$ is integrable over $[a, b]$. Set

$$\delta_n = \frac{b - a}{n}$$

and

$$f_{vn} = f(a + v\delta_n).$$

Then

$$\lim_{n \to \infty} (1 + f_{1n}\delta_n)(1 + f_{2n}\delta_n) \cdots (1 + f_{nn}\delta_n) = \exp \left( \int_a^b f(x) \, dx \right).$$

Proof. Taking the logarithm of the left-hand side and using the power series expansion of the logarithm we obtain

$$\sum_{v=1}^n \log(1 + f_{vn}\delta_n) = \sum_{v=1}^n \sum_{m=1}^{\infty} -\frac{(f_{vn}\delta_n)^m}{m} = \sum_{v=1}^n f_{vn}\delta_n + O\left(\delta_n \sum_{v=1}^n f_{vn}^2 \delta_n\right).$$

Finally we observe that

$$\sum_{v=1}^n f_{vn}\delta_n \xrightarrow{n \to \infty} \int_a^b f(x) \, dx$$

and similarly

$$\sum_{v=1}^n f_{vn}^2\delta_n \xrightarrow{n \to \infty} \int_a^b f(x)^2 \, dx$$

and $\delta_n \to 0$ as $n \to \infty$. 

Lemma 5.2 ([PS72, Pt. II, 58]). Let $n, v$ be integers with $0 < v < n$ and furthermore suppose that $n, v$ increase to infinity in such a way that

$$\lim_{n \to \infty} \frac{v - n/2}{\sqrt{n}} = \lambda.$$ 

Then

$$\lim_{n \to \infty} \sqrt{n} \binom{n}{v} = \sqrt{\frac{2}{\pi}} e^{-2\lambda^2}.$$ 

Proof. Suppose first that $n$ is even and set $n = 2m$. Then

$$\frac{v - m}{\sqrt{m}} \to \lambda \sqrt{2}.$$
We may further assume that \( \lambda \geq 0 \) and \( \nu > m \). As well, we recall for example from Stirling’s formula that
\[
\binom{2m}{m} \sim \frac{2^{2m}}{\sqrt{m\pi}}.
\]

Therefore it suffices to show that
\[
\lim_{m \to \infty} \frac{\binom{2m}{\nu}}{\binom{2m}{m}} = e^{-2\lambda^2}.
\]

Now
\[
\frac{\binom{2m}{\nu}}{\binom{2m}{m}} = \frac{m \cdot m - 1 \cdots m - (\nu - m - 1)}{m + 1 \cdot m + 2 \cdots m + (\nu - m)}
\]
\[
= \frac{1}{1 + \frac{1}{\sqrt{m}} \frac{1}{\sqrt{m}}} \frac{1 - \frac{1}{\sqrt{m}} \frac{1}{\sqrt{m}}}{1 + \frac{2}{\sqrt{m}} \frac{1}{\sqrt{m}}} \frac{1 - \frac{\nu - m - 1}{\sqrt{m}} \frac{1}{\sqrt{m}}}{1 + \frac{\nu - m}{\sqrt{m}} \frac{1}{\sqrt{m}}}
\]

By adapting the argument in Lemma 5.1 we see that
\[
\left( 1 - \frac{1}{\sqrt{m}} \frac{1}{\sqrt{m}} \right) \left( 1 - \frac{2}{\sqrt{m}} \frac{1}{\sqrt{m}} \right) \cdots \left( 1 - \frac{\nu - m - 1}{\sqrt{m}} \frac{1}{\sqrt{m}} \right) \xrightarrow{m \to \infty} \exp \left( - \int_{0}^{\lambda \sqrt{2}} x \, dx \right) = e^{-\lambda^2}
\]
and similarly
\[
\left( 1 + \frac{1}{\sqrt{m}} \frac{1}{\sqrt{m}} \right) \left( 1 + \frac{2}{\sqrt{m}} \frac{1}{\sqrt{m}} \right) \cdots \left( 1 + \frac{\nu - m}{\sqrt{m}} \frac{1}{\sqrt{m}} \right) \xrightarrow{m \to \infty} \exp \left( \int_{0}^{\lambda \sqrt{2}} x \, dx \right) = e^{\lambda^2}
\]
which proves the case when \( n \) is even.

Now if \( n \) is odd, set \( n = 2m + 1 \). We note that
\[
\frac{\binom{2m+1}{\nu}}{\binom{2m+1}{m+1}} = \frac{m+1}{\nu} \binom{2m+1}{\nu-1} \binom{2m}{m}.
\]

Since
\[
\lim_{m \to \infty} \frac{\nu - m}{\sqrt{2m}} = \lim_{m \to \infty} \frac{\nu - \frac{2m+1}{2}}{\sqrt{2m+1}} = \lambda
\]
it suffices to show that \((m + 1)/v \to 1\). We note that
\[
\lim_{m \to \infty} \frac{1 - m/v}{\sqrt{m}/v} = \lim_{m \to \infty} \frac{v - m}{\sqrt{m}} = \lambda \sqrt{2}.
\]
Since \(v > m\) we have \(1 > m/v > \sqrt{m}/v > 0\) so \(\lim \inf \sqrt{m}/v = 0\) since otherwise \(m/v = \sqrt{m}/v\) will be unbounded. By taking a subsequence if necessary we see that \(\sqrt{m}/v \to 0\) so \(1 - m/v \to 0\).

\[\square\]

**Proposition 5.3 ([PS72, Pt. II, 40]).** For \(k\) fixed, we have
\[
\sum_{v=0}^{n} \left( \frac{n}{v} \right)^k \sim 2^{kn} \left( \frac{2}{\pi n} \right)^{k/2} \frac{1}{\sqrt{k}}.
\]

**Proof.** From Lemma 5.2 we see that
\[
\sum_{v=0}^{n} \left( \frac{n}{v} \right)^k \sim 2^{kn} \left( \frac{2}{\pi n} \right)^{k/2} \frac{1}{\sqrt{k}} \left( \sum_{v=0}^{n} \exp \left( -2k \left( \frac{v - n/2}{\sqrt{n}} \right)^2 \right) \frac{1}{\sqrt{n}} \right)
\]
Now
\[
\lim_{n \to \infty} \sum_{v=0}^{n} \exp \left( -2k \left( \frac{v - n/2}{\sqrt{n}} \right)^2 \right) \frac{1}{\sqrt{n}} = \int_{-\infty}^{\infty} e^{-2kx^2} \, dx = \sqrt{\frac{\pi}{2k}}
\]
so
\[
\sum_{v=0}^{n} \left( \frac{n}{v} \right)^k \sim 2^{kn} \left( \frac{2}{\pi n} \right)^{k/2} \frac{1}{\sqrt{k}} \left( \sum_{v=0}^{n} \exp \left( -2k \left( \frac{v - n/2}{\sqrt{n}} \right)^2 \right) \frac{1}{\sqrt{n}} \right)
\]
as required. \[\square\]

In the case that there is an exceptional zero \(\beta_0\) for \(\prod_{\lambda=\lambda_0} \lambda \lambda^\Lambda s L(s, \chi^\lambda)\) we use Theorem 1.4 to obtain a larger zero-free region for the nonexceptional zeros. First we note the following proposition which allows us to use Theorem 1.4.

**Proposition 5.4 (Regev).** Let \(\mu \vdash n\) be a partition of length \(\ell\). Then
\[
\sum_{r=0}^{n-1} \chi^{(n-r,1)^r}(\mu) = \begin{cases} 2^{\ell-1} & \text{if all parts of } \mu \text{ are odd} \\ 0 & \text{otherwise} \end{cases}
\]

**Proof.** This is [Reg13, Proposition 1.1]. \[\square\]

**Remark 5.5.** Proposition 5.4 was proven in [Reg13] using representations of Lie superalgebras. A proof using only \(S_n\) characters was given in [Tay17].

We are now ready to prove Theorem 1.6.
Proof of Theorem 1.6. Let $C$ be the class of $n$-cycles. Then
\[ \frac{|C|}{|G|} = \frac{n!}{n} = \frac{1}{n}. \]

We know from Proposition 2.8 that
\[ n^v(C) = \{ \chi^\lambda : \lambda = (n - r, 1') \}. \]

As well, from Proposition 2.10 we have for $\lambda = (n - r, 1')$
\[ \chi^\lambda(1) = \binom{n - 1}{r} \]
and therefore
\[ \sum_{\lambda = (n - r, 1')} \chi^\lambda(1) = 2^{n-1} \ll 2^n. \]

Therefore Theorem 1.1 gives the following upper bounds for the least prime $p$ whose Frobenius in $G = \text{Gal}(L/K)$ is an $n$-cycle depending on whether $\beta_0$ exists.

If there is no $\beta_0$ then from Equation 1.2
\[
\log N_{K/Q}^p \ll \left( \sum_{r=0}^{n-1} \binom{n - 1}{r}^5 \right)^{1/2} \sum_{\lambda = (n - r, 1')} \chi^\lambda(1) \log A_{\chi^\lambda} + \log \left( n n_K \sum_{p \in P(L/K)} \log p \right).
\]

From Proposition 5.3 we obtain
\[
\left( \sum_{r=0}^{n-1} \binom{n - 1}{r}^5 \right)^{1/2} \ll \frac{2^{5n/2}}{n}
\]

and therefore
\[
\log N_{K/Q}^p \ll \frac{2^{5n/2}}{n} \sum_{\lambda = (n - r, 1')} \chi^\lambda(1) \log A_{\chi^\lambda} + \log \left( n n_K \sum_{p \in P(L/K)} \log p \right).
\]

For the case of the exceptional zero $\beta_0$ it remains to verify that $\sum_{\lambda} \chi^\lambda$ is always nonnegative where the sum is over all hooks $\lambda = (n - r, 1')$. This
follows from Proposition 5.4. Hence Equation 1.4 follows directly from Equation 1.3.

At the cost of worsening the bound, we may write the above in terms of the discriminant of $L$. First we estimate the terms from ramified primes.

**Lemma 5.6.** Let $L/K$ be a Galois extension of number fields with Galois group $G$ and $C$ a conjugacy class in $C$. Then

$$\frac{|G|}{|C|} n_K \sum_{p \in \mathcal{P}(L/K)} \log p \leq (\log d_L)^2.$$

**Proof.** We observe that since

$$d_L = d_{[L:K]}^{K/Q} N_{K/Q} \alpha_{L/K}$$

we have

$$\log d_L = |G| \log d_K + \log N_{K/Q} \alpha_{L/K}.$$ 

Therefore

$$(\log d_L)^2 \geq |G| \log d_K \log N_{K/Q} \alpha_{L/K}$$

from which it follows that

$$\frac{|G|}{|C|} n_K \sum_{p \in \mathcal{P}(L/K)} \log p \leq |G| \log d_K \log N_{K/Q} \alpha_{L/K} \leq (\log d_L)^2.$$ 

**Proof of Corollary 1.8.** From Lemma 5.6 we obtain

$$nn_K \sum_{p \in \mathcal{P}(L/K)} \log p \leq (\log d_L)^2.$$ 

As well, recall that

$$A_\chi = d_\chi^{(1)} K/Q \alpha_\chi.$$ 

Hence by Proposition 5.3

$$\prod_{\lambda=(n-r,\nu)} d_{\lambda}^{(1)^2} = d_{\nu}^{(2s^{\nu}/n}.$$ 

Furthermore by the conductor-discriminant formula Theorem 3.2 we see that

$$\prod_{\lambda=(n-r,\nu)} N_{\lambda} K/Q \alpha_\lambda^{(1)} \leq N_{K/Q} \alpha_{L/K}.$$
Therefore we see from the two cases in Theorem 1.6 that the bound for the least prime in an $n$-cycle in the case of no exceptional zeros can be written as

$$N_{K/QP} \leq d_K^{\epsilon_1/n} N_{K/Q0}^{d_{L/K}} (\log d_L)^{c_{26}}$$

and the for the case of exceptional zeros the bound becomes

$$N_{K/QP} \leq d_K^{\epsilon_1/n} N_{K/Q0}^{d_{L/K}} (\log d_L)^{c_{26}}$$

for effective absolute constants $c_{24}, c_{25}, c_{26}, c_{27}$.

Finally, observing that $|\chi^\lambda(C)| = 1$ allows us to deduce Theorem 1.13 from Proposition 4.27.

5.1.2 The least prime in an $(n-1)$-cycle

By using similar techniques as the $n$-cycle case we can also obtain an upper bound for the least prime whose Frobenius is an $(n-1)$-cycle. We first establish the nonvanishing characters of $(n-1)$-cycles as well as the character values at $(n-1)$-cycles.

Proposition 5.7. Let $C$ be the class of $(n-1)$-cycles and let $\chi^\lambda \in \text{Irr}(S_n)$ with $\lambda \vdash n$ the corresponding partition. Then

$$\chi^\lambda(C) = \begin{cases} 1 & \text{if } \lambda = (n) \\ \text{sgn}(C) & \text{if } \lambda = (1^n) \\ (-1)^r & \text{if } \lambda = (n-r, 2, 1^{r-2}) \\ 0 & \text{otherwise} \end{cases}$$

Proof. By the Murnaghan-Nakayama rule, we have

$$\chi_{(n-1,1)}^\lambda = \sum_T (-1)^T$$
where the sum is over all rim hook tableaux \( T \) of shape \( \mu \) and content \((n - 1, 1)\). That is, \( T \) is a Young tableaux with weakly increasing rows and columns with \( n - 1 \) entries of 1s arranged in a rim hook and a single entry of 2.

Since the entries are weakly increasing the rim hook corresponding to 1 must be a Young diagram, so in particular a hook diagram of size \( n - 1 \), say \( \mu \). Thus \( \lambda \) is obtained by adding a box to \( \mu \) in such a way that the resulting shape is a Young diagram. Then \( \lambda \) is either itself a hook or a hook with a square added at position \((2,2)\), where \((i,j)\) denotes the \(i\)-th row and \(j\)-th column. See Figure 5.1 for the possible shapes, along with \((n)\) and \((1^n)\).

If \( \lambda = (n) \) then \( \chi^\lambda = 1 \) and if \( \lambda = (1^n) \) then \( \chi^\lambda = \text{sgn} \). Now suppose \( \lambda \) is a hook of shape \((n - r - 1, 1^r)\). Then since the entries are weakly increasing any hook with content \((n - 1, 1)\) must have entry 2 at either the bottommost box at \((r + 1, 1)\) or at the rightmost box at \((1, n - r)\). In any case the singleton 2 has leg length 0 but the hook with entry 1 will have leg length \( r - 1 \) and \( r \). Thus

\[
\chi^{(n-r,1^r)}(C) = (-1)^{r-1} + (-1)^r = 0.
\]

Next suppose \( \lambda \) is obtained by taking a hook of shape \((n - r - 1, 1^r)\) and adding a box at position \((2,2)\). Again since the entries are weakly increasing and the boxes containing 1 must form a skew diagram the entry at \((2,2)\) is forced to be 2 and the boxes containing 1 must be the hook of shape \((n - r - 1, 1^r)\). Once again the single 2 has leg length 0 and the hook has leg length \( r \), so

\[
\chi^{(n-r,2,1^{r-2})}(C) = (-1)^r.
\]

Next we determine the character degrees.

**Proposition 5.8.** Let \( \mu = (n - r, 2, 1^{r-2}) \) with \( 1 < r < n - 1 \). Then

\[
\chi^\mu(1) = \frac{n(r-1)(n-r-1)}{(n-1)(n-r)} \binom{n-1}{r} \leq n \binom{n-1}{r}.
\]

**Proof.** Applying the hook length formula we obtain

\[
\chi^\mu(1) = \frac{n!}{r(n-r)(n-1)(r-2)!(n-r-2)!} = \frac{n(r-1)(n-r-1)\cdot(n-1)!}{(n-r)(n-1)! \cdot r!(n-1-r)!} = \frac{n(r-1)(n-r-1)}{(n-r)(n-1)} \binom{n-1}{r}.
\]
and we observe that
\[
\frac{(r - 1)(n - r - 1)}{(n - 1)(n - r)} \leq 1. 
\]

**Proposition 5.9.** Let \( L/K \) be a Galois extension of number fields with \( G = \text{Gal}(L/K) = S_n \). Assume Artin’s conjecture for \( L \)-functions of characters \( \chi^\mu \) corresponding to partitions of the form \( \mu = (n - r, 2, 1^{r-2}) \) for \( 1 < r < n - 1 \). If \( \zeta_K(s)L(s, \text{sgn}) \) has no exceptional zeros, then there exist absolute effective constants \( c_{61}, c_{62} \) such that the least prime \( p \) of \( K \) whose Frobenius in \( G \) is an \((n-1)\)-cycle satisfies

\[
N_{K/Qp} \leq \left( d_K A_{\text{sgn}} \prod_{\mu=(n-r,2,1^{r-2})} A_{\chi^\mu}^{2}(1) \right)^{c_{61} n^{1/2} 2^{5n}} \cdot \left( (n-1)n_K \sum_{p \in \mathcal{P}(L/K)} \log p \right)^{c_{62}} \tag{5.1}
\]

**Proof.** We will apply Theorem 1.1. First we note that \(|C| = n(n-2)!\) so \(|G|/|C| = n - 1\). From Proposition 5.7 we see that

\[
\text{nv}(C) = \{\chi^\mu : \mu = (n-r,2,1^{r-2}), 1 < r < n - 1\} \cup \{1, \text{sgn}\}
\]

so applying Proposition 5.8 and Proposition 5.3 we obtain

\[
\sum_{\chi \in \text{nv}(C)} \chi(1)^5 \ll \sum_{\mu=(n-r,2,1^{r-2})} n^5 \binom{n-1}{r}^5 \ll n^3 2^{5n}
\]

which combined with Theorem 1.1 yields Equation 5.1. \( \Box \)

In the case of the exceptional zero we will need to invoke Theorem 1.4. However it is not true in general that \( \sum_{\chi \in \text{nv}(C)} \chi \) is nonnegative as the following proposition shows. The inclusion \( \{1, \ldots, n-1\} \subseteq \{1, \ldots, n\} \) induces an embedding \( S_{n-1} \hookrightarrow S_n \). For any \( \sigma \in S_n \), define \( \epsilon(\sigma) = 1 \) if the cycle decomposition of \( \sigma \) contains only odd cycles and \( \epsilon(\sigma) = 0 \) otherwise. Let \( \text{Fix}(\sigma) \) denote the set of fixed points of \( \sigma \) acting on \( \{1, \ldots, n\} \).

**Proposition 5.10.** Let \( \sigma \in S_n \) corresponding to a partition of length \( \ell \). Then

\[
\sum_{\mu=(n-r,2,1^{r-2})} \chi^\mu(\sigma) = \epsilon(\sigma)(\# \text{Fix}(\sigma)2^{\ell-2} - 2^\ell) + 1 + \text{sgn}(\sigma).
\]
Proof. By the branching rule (Theorem 2.6) we have
\[
\sum_{\nu \vdash n} \text{Ind}_{S_{n-1}}^{S_n} \nu^\nu = 1_G + \text{sgn} + 2 \sum_{\lambda = (n-r,1^r)} \chi^\lambda + \sum_{\mu = (n-r,2,1^{r-2})} \chi^\mu
\]
\[
= 2 \sum_{\lambda \vdash n} \chi^\lambda + \sum_{\mu = (n-r,2,1^{r-2})} \chi^\mu - 1_G - \text{sgn}.
\]

From Proposition 5.4 we have
\[
2 \sum_{\lambda \vdash n} \chi^\lambda(\sigma) = \varepsilon(\sigma)2^\ell.
\]

Next we use the induced character formula given in Equation 2.1 and Equation 2.1. A transversal for $S_{n-1}$ in $S$ is given by the transpositions $(1,n), (2,n), \ldots, (n,n) = 1$. Hence
\[
\sum_{\nu \vdash n} \text{Ind}_{S_{n-1}}^{S_n} \nu^\nu(\sigma) = \sum_{i=1}^n \sum_{\nu \vdash n} \tilde{\chi}^\nu((i,n)\sigma(i,n))
\]

Note that $\tilde{\chi}^\nu$ is nonvanishing only on $S_{n-1} = \{\sigma \in S_n : \sigma(n) = n\}$ and that $(i,n)\sigma(i,n)$ fixes $n$ if and only if $\sigma$ fixes $i$. As well, since conjugation does not change the cycle structure so if $\sigma$ has cycle length $\ell$ so will all its conjugates. Hence changing the order of summation and invoking Proposition 5.4 we obtain
\[
\sum_{\nu \vdash n} \text{Ind}_{S_{n-1}}^{S_n} \nu^\nu(\sigma) = \sum_{i=1}^n \sum_{\nu \vdash n} \tilde{\chi}^\nu((i,n)\sigma(i,n)) = \# \text{Fix}(\sigma)\varepsilon(\sigma)2^{\ell-2}
\]

where the exponent is $\ell - 2$ since we truncate the 1-cycle $(n)$ in $\sigma$ when considered as an element of $S_{n-1}$. We also note that if $\sigma'$ is obtained from $\sigma$ by truncating a 1-cycle then $\varepsilon(\sigma') = \varepsilon(\sigma)$.

Proposition 5.11. Let $L/K$ be a Galois extension of number fields with $G = \text{Gal}(L/K) = S_n$. Assume Artin’s conjecture for L-functions of characters $\chi^\mu$ corresponding to partitions of the form $\mu = (n-r,2,1^{r-2})$ for $1 < r < n-1$ as well as L-functions of hook characters. Let
\[
\varphi := \sum_{\lambda \vdash n} \text{Ind}_{S_{n-1}}^{S_n} \lambda^\lambda = 1_G + \text{sgn} + 2 \sum_{\nu = (n-r,1^r)} \chi^\nu + \sum_{\mu = (n-r,2,1^{r-2})} \chi^\mu.
\]
Suppose \( L(s, \varphi) \) admits an exceptional zero \( \beta_0 \). Then there is an effective absolute constant \( c_{63} \) such that the least prime \( p \) whose Frobenius is an \( (n - 1) \)-cycle in \( S_n \) satisfies

\[
Np \leq \left( \prod_{\lambda = (n - r, 1')} A_{\lambda}^{(1)} \right) \left( \prod_{\mu = (n - r, 2, 1^{r - 2})} A_{\mu}^{(1)} \right)^{c_{63}} \cdot \left( \frac{1}{n - 1} n_K \sum_{p \in \mathcal{P}(L/K)} \log p \right)^{c_{63}}. \tag{5.2}
\]

**Proof.** Observe that \( \varphi \) contains all the characters \( \chi^\mu \in n \nu(C) \) and is also nonnegative by Proposition 5.1.2 so we may apply Theorem 1.4. We note that if \( C \) is the class of \( (n - 1) \)-cycles then \( |C| = n(n - 2)! \) so \( |C|/|G| = 1/(n - 1) \).

By Equation 4.14 of Proposition 4.25 we have

\[
(n - 1) \sum_{p \in S(L/K)} \theta(p)(\log Np)k_1(Np) \geq (\log x)^3(1 - \beta_0) \]

\[
- c_{64} \sum_{\mu = (n - r, 2, 1^{r - 2})} \chi^\mu(1) \mathcal{L}^2[(1/c_{65})(1 - \beta_0)\mathcal{L}]^{2c_{66}} \log x/\mathcal{L}
\]

\[
- c_{67} \sum_{\mu = (n - r, 2, 1^{r - 2})} \log A_{\chi^\mu}
\]

\[
- c_{68}(n - 1)n_K \frac{\log x}{x^2} \sum_{p \in \mathcal{P}(L/K)} \log p
\]

\[
- c_{69}(n - 1)n_K \frac{(\log x)^2}{x}
\]

where

\[
\mathcal{L} = \log A_{1_G} + \log A_{\text{sgn}} + 2 \sum_{\nu = (n - r, 1')} \chi^\nu(1) \log A_{\chi^\nu} + \sum_{\mu = (n - r, 2, 1^{r - 2})} \chi^\mu(1) \log A_{\chi^\mu}
\]

and \( c_{65} > 1 \). Using an argument similar to Equation 4.17 in the proof of Theorem 1.1 we see that if

\[
1 - \beta_0 \gg \frac{1}{\mathcal{L}^2}
\]
then we can take
\[ \log x \gg L + \log \left( (n - 1)n_k \sum_{p \in P(L/K)} \log p \right). \]

If instead
\[ 1 - \beta_0 \ll \frac{1}{L^2} \]
then again as before we apply Equation 4.18 to obtain
\[ x = \exp \left[ c_{70}L + c_{71} \log \left( (n - 1)n_k \sum_{p \in P(L/K)} \log p \right) \right] \]
so that in any case Equation 5.2 holds.

Combining Proposition 5.9 and Proposition 5.11 yields Theorem 1.9.

Proof of Corollary 1.10. As in the proof of Corollary 1.8, from Lemma 5.6 we have
\[ (n - 1)n_k \sum_{p \in P(L/K)} \log p \leq (\log d_L)^2 \]
and we know that
\[ \prod_{\lambda = (n-r,1^r)} A_{\lambda}^{\chi_{\lambda}(1)} = d_K^n \chi_{\lambda}(1)^2 \prod_{\lambda = (n-r,1^r)} N_{K/Q} \chi_{\lambda}(1) \leq d_K^{c_{72}4^n/n} N_{K/Q} d_L / K. \]

As well, from Proposition 5.8 and Proposition 5.3 we see that
\[ \sum_{\mu = (n-r,2,1^{r-2})} \chi_{\mu}(1)^2 \leq n^2 \sum_{r=0}^{n-1} \binom{n-1}{r}^2 \ll n4^n. \]
Therefore
\[ \prod_{\mu = (n-r,2,1^{r-2})} A_{\mu}^{\chi_{\mu}(1)} = d_K^{\sum_{\mu} \chi_{\mu}(1)^2} \prod_{\mu = (n-r,2,1^{r-2})} N_{K/Q} \chi_{\mu}(1) \leq d_K^{c_{73}4^n} N_{K/Q} d_L / K \]
for some constant \( c_{73} \). Hence
\[ \prod_{\lambda = (n-r,1^r)} A_{\lambda}^{\chi_{\lambda}(1)} \prod_{\mu = (n-r,2,1^{r-2})} A_{\mu}^{\chi_{\mu}(1)} \leq d_K^{c_{74}4^n} N_{K/Q} d_L^2 / K \]
for some constant \( c_{74} \).
As in the case of \( n \)-cycles, combining with Proposition 4.27 yields Theorem 1.14.

5.1.3 Unramified \( A_n \)-extensions of quadratic fields

In this section consider a tower of number fields \( L/F/Q \) where \( L/Q \) is an \( S_n \)-extension and \( L/F \) is an unramified \( A_n \)-extension. Unramified \( A_n \)-extensions of quadratic fields have been studied by Uchida [Uch70], Yamamoto [Yam70], Elstrodt-Grunewald-Mennicke [EGM85], Kondo [Kon95], and Kedlaya [Ked12]. In particular it is known that there are infinitely many such extensions. We can apply the results of the previous sections to such an extensions to obtain an upper bound for the least prime in an \( n \)-cycle. First we adapt the proof of the upper bound for the Artin conductor Proposition 3.1 given by Murty-Murty-Saradha in [MMS88, Proposition 2.5] to obtain a stronger bound for the current case.

**Proposition 5.12.** Let \( L/F/Q \) be a tower of extensions where \( L/F \) is unramified and \( F/Q \) is a quadratic extension with discriminant \( \Delta \). Let \( \chi \in \text{Irr(Gal}(L/Q)) \). Then

\[
\log Nf_\chi \leq 4\chi(1) \log \Delta.
\]

**Proof.** Since the base field is \( Q \) we identify the Artin conductor \( f \) with the integer

\[
f_\chi = \prod_p p^{n(\chi, p)}.
\]

Proceeding as in the proof of [MMS88, Proposition 2.5] we obtain that

\[
n(\chi, p) \leq \frac{2\chi(1)w(\mathfrak{D}_{L/Q})}{e_{w/p}}
\]

where \( w \) is a place of \( L \) dividing \( p \) and \( \mathfrak{D}_{L/Q} \) is the different of \( L/Q \). By an estimate of Hensel we have

\[
w(\mathfrak{D}_{L/Q}) = e_{w/p} - 1 + s_{w/p}
\]
for some $s_{w/p}$ satisfying $0 \leq s_{w/p} \leq w(e_{w/p})$. Therefore

$$n(\chi, p) \leq \frac{2\chi(1) (e_{w/p} - 1 + s_{w/p})}{e_{w/p}} \leq 2\chi(1) \left( 1 - \frac{1}{e_{w/p}} + \frac{w(e_{w/p})}{e_{w/p}} \right).$$

Since $L/F$ is unramified, we have $e_{w/v} = 1$ for all places $v$ of $F$ below $w$ and above $p$. Thus if $p$ is ramified in $L$ it must be ramified in $F$ and we have $e_{v/p} = 2$ from which it follows that $e_{w/p}$ is 2 if $p$ is ramified in $L$ and is 1 if $p$ is unramified. Thus

$$w(e_{w/p}) = \begin{cases} 2 & \text{if } p = 2 \\ 0 & \text{otherwise} \end{cases}$$

so in any case we obtain

$$n(\chi, p) \leq 4\chi(1)$$

for all primes $p$ that ramify in $L$. Therefore

$$\log Nf_\chi = \sum_{p \in P(L/Q)} n(\chi, p) \log p \leq 4\chi(1) \sum_{p \in P(L/Q)} \log p \leq 4\chi(1) \log \Delta.$$

**Proof of Theorem 1.12.** Suppose the Frobenius of $p$ in $\text{Gal}(L/Q)$ is an $n$-cycle. From Theorem 1.6 we see that $p$ satisfies

$$p \leq \left( \prod_{\lambda=(n-r,1)} A^\lambda_{\chi(1)} \right)^{c_752^{n/2}/n} \left( n \sum_{p \in P(L/Q)} \log p \right)^{c_76}.$$ 

Since $L/F$ is unramified, any prime $p$ which ramifies in $L$ must divide $\Delta$. Hence

$$\sum_{p \in P(L/Q)} \log p \leq \log \Delta.$$ 

As well, the base field is $Q$ so we see that

$$A_{\chi} = Nf_{\chi} = \prod_p P^{n(\chi,p)}.$$
Using Proposition 5.12, Proposition 2.10, and Proposition 5.3 we obtain

\[
\prod_{\lambda=(n-r,1')} A_{\lambda}^{\chi_{\lambda}(1)} \leq \exp \left( 4 \log \Delta \sum_{\lambda=(n-r,1')} \chi_{\lambda}(1)^2 \right)
\leq \exp \left( 4 \log \Delta \sum_{r=0}^{n-1} \left( \frac{n-1}{r} \right)^2 \right)
\leq \Delta^{4n+1/\sqrt{n}}.
\] (5.3)

Combining with Theorem 1.6 yields Equation 1.7.

If the Frobenius of \( p \) in Gal(\( L/\mathbb{Q} \)) is an \((n-1)\)-cycle, then from Theorem 1.9 we have

\[
p \leq \left( \Delta \prod_{\mu=(n-r,2,1'^{-2})} A_{\mu}^{\chi_{\mu}(1)} \right)^{c_{77}n^{1/2}2^{5n}} \left( n-1 \sum_{p \in \mathcal{P}(L/K)} \log p \right)^{c_{78}}
\] (5.4)

if there are no exceptional zeros. Using Proposition 5.12 and Proposition 5.8 we have

\[
\log A_{\lambda^p} = \log f_{\lambda^p} \leq 4\chi_{\mu}(1) \log \Delta \leq 4n \left( \frac{n-1}{r} \right) \log \Delta
\]

and so using Proposition 5.3 we have

\[
\prod_{\mu=(n-r,2,1'^{-2})} A_{\mu}^{\chi_{\mu}(1)} \leq \exp \left( 4 \log \Delta \sum_{\mu=(n-r,2,1'^{-2})} \chi_{\mu}(1)^2 \right)
\leq \exp \left( 4 \log \Delta \sum_{r=0}^{n-1} n^2 \left( \frac{n-1}{r} \right)^2 \right)
\leq \Delta^{n/24^{n+1}}.
\]

Combining with Equation 5.4 we obtain

\[
p \leq \Delta^{77n^{n}} \left( (n-1) \log \Delta \right)^{c_{80}}.
\]

If there are exceptional zeros, then instead Theorem 1.9 gives

\[
p \leq \left( \prod_{\lambda=(n-r,1')} A_{\lambda}^{\chi_{\lambda}(1)} \prod_{\mu=(n-r,2,1'^{-2})} A_{\mu}^{\chi_{\mu}(1)} \right)^{c_{81}} \left( n-1 \sum_{p \in \mathcal{P}(L/K)} \log p \right)^{c_{81}}.
\]
Hence combining with Equation 5.3 we obtain
\[ p \leq \Delta^{\text{reg}} ((n - 1) \log \Delta)^{c_{\text{reg}}}. \]
In any case we see that Equation 1.9 holds.

To obtain Equation 1.8 and Equation 1.10, observe that since \( L/F \) is unramified we have
\[ d_L = \Delta^{[L:F]} = \Delta^{n/2}. \]

\[ \square \]

Remark 5.13. The assumptions of Section 3.6 can only improve the base of the exponent in the exponent of the upper bound given in Theorem 1.12. In particular, Equation 1.8 and Equation 1.10 essentially remain the same.

5.2 THE LEAST PRIME WHICH IS A REFLECTION

Let \( L/K \) be a dihedral extension, so \( G = \text{Gal}(L/K) = D_n \). In this case the Artin conjecture is known since \( D_n \) is monomial. Let \( C \) denote a class of reflections. As in Section 2.3 there is a unique such class if \( n \) is odd, and if \( n \) is even there are two. We see that
\[ \frac{|D_n|}{|C|} = \begin{cases} 
2 & n \text{ odd} \\
4 & n \text{ even}
\end{cases} \]

so that in any case it is absolutely bounded.

As well, from Proposition 2.12 we find that for \( \chi \in \text{Irr}(D_n) \),
\[ |\chi(C)| = \begin{cases} 
1 & \chi(1) = 1 \\
0 & \chi(1) > 1
\end{cases}. \]

Therefore \( \text{nv}(C) \) is the set of linear characters of \( D_n \). Hence just as in the case when \( G = S_n \) and \( C \) is the class of \( n \)-cycles we are in a situation where \( C \) is a conjugacy class at which most irreducible characters vanish.

Proof of Theorem 1.15. Since \( D_n \) is monomial we know Artin’s conjecture holds for \( L/K \). Applying Theorem 1.1 we obtain the following. If there are no exceptional zeros, then the least prime \( p \) in \( K \) whose Frobenius is a reflection in \( G \) satisfies
\[ N_{K/Q}p \ll \left( \prod_{\chi \in \text{lc}(D_n)} A_{\chi} \right)^{c_{\text{reg}}} \left( n_K \sum_{p \in \mathcal{P}(L/K)} \log p \right)^{c_{\text{reg}}}. \]  
(5.5)
5.3 The least prime which is a Camina element

If there is an exceptional zero $\beta_0$, then we note that since $\text{nve}(C)$ is the set of linear characters of $D_n$ by Proposition 2.3 we have that for all $g \in D_n$,

$$\sum_{\chi \in \text{nve}(C)} \chi(g) \geq 0$$

so we are in a position to apply Theorem 1.4 and therefore the exceptional zero case of Theorem 1.1. Hence in any case Equation 5.5 holds by taking $c_{84}, c_{85}$ to be larger if necessary.

Finally we note that there is a canonical correspondence between $\text{lc}(G)$ and $\text{Irr}(G^{ab})$ and invoke the conductor-discriminant formula.

Proof of Corollary 1.16. Let $F$ be the fixed field of the commutator subgroup $G'$. Then from Equation 2.4 we have

$$d_F \leq d_L^{1/[L:F]} \leq d_L^{4/[L:K]}$$

and therefore

$$d_F^{c_{86}} \leq d_L^{4c_{86}/[L:K]}.$$

As well, from Lemma 5.6 we obtain

$$\left( n_K \sum_{p \in P(L/K)} \log p \right)^{c_{87}} \leq (\log d_L)^{2c_{87}}.$$

$\square$

5.3 The least prime which is a Camina element

Recall from Section 2.4 that a Camina element $g$ of a finite group $G$ is an element at which every nonlinear irreducible character vanishes. Hence applying Theorem 1.1 to the class of Camina elements will give the strongest estimates. In particular the results of this section will apply to the groups and elements listed in Section 2.4

Proof of Theorem 1.19. From Section 2.4 we know that the characters that are nonvanishing at $C$ are exactly the linear characters. Therefore Artin’s conjecture holds and by Proposition 2.3 we have

$$\sum_{\chi \in \text{nve}(C)} \chi(C) \geq 0.$$
Furthermore
\[ \sum_{\chi \in \text{Irr}(C)} \chi(1) = \#\text{lc}(G) = |G^{ab}| = [L : F]. \]

Likewise we recall the correspondence between \( \text{lc}(G) \) and \( \text{Irr}(G^{ab}) \). Invoking the conductor-discriminant formula we obtain
\[ \prod_{\chi \in \text{lc}(G)} A_{\chi}^{(1)} = d_F. \]

Proof of Corollary 1.20. Combine the estimate
\[ d_F \leq d_L^{1/[L:F]} \]
with Lemma 5.6.

For some of the examples listed in Section 2.4 we may rephrase the estimate to make it more explicit.

Corollary 5.14. Let \( L/K \) be a Galois extension of number fields with \( \text{Gal}(L/K) = G \). Then there exist constants \( c_{88}, c_{89} \) such that for the following choices of \( G \) and conjugacy class \( C \), the least prime \( p \) of \( K \) whose Frobenius in \( G \) is \( C \) unconditionally satisfies the following:

1. Let \( G \) be an extraspecial \( p \)-group of order \( p^n \) and \( C \) the class of any element outside the centre. Then
   \[ N_{K/Q} p \leq d_L^{c_{88}/p^{1/2}} (\log d_L)^{c_{89}}. \]

2. Let \( G \) be a group of order \( p^4 \) and \( C \) the class of any Camina element. If \( \#G' = p \) then
   \[ N_{K/Q} p \leq d_L^{c_{88}/p^{1/2}} (\log d_L)^{c_{89}}. \]
   If instead \( \#G' = p^2 \) then
   \[ N_{K/Q} p \leq d_L^{c_{88}/p} (\log d_L)^{c_{89}}. \]

3. Let \( G \) be a \( p \)-group of maximal class of order \( p^n \) and \( C \) the class of any element not in \( C_G(K_i(G)/K_{i+2}(G)) \) for \( i \geq 2 \), where \( K_i(G) \) denotes the terms in the descending central series of \( G \). Then
   \[ N_{K/Q} p \leq d_L^{c_{88}/p^{(n-2)/2}} (\log d_L)^{c_{89}}. \]
4. Let $G$ be the group of upper-triangular matrices over a finite field of order $q$ with diagonal entries 1, and $C$ the class of any element with minimal polynomial $(x - 1)^n$. Then

$$N_{K/Q} \leq d_L^{c_8/q^{1/2}} (\log d_L)^{c_9}.$$  

If there are exceptional zeros then the denominator of the exponent of $d_L$ may be replaced by its square.

Proof. This follows from Corollary 1.20 by substituting the order of the commutator subgroup in $G$. If $G$ is an extraspecial $p$-group then its commutator subgroup has size $p$. If $G$ is a $p$-group of maximal class of order $p^n$, then its commutator subgroup has size $p^{n-2}$. If $G$ is the group of upper-triangular matrices over a field of order $q$ with diagonal entries 1 then the commutator subgroup has size $q$. 

\qed
BIBLIOGRAPHY


COLOPHON

This thesis was typeset using the typographical look-and-feel classicthesis developed by André Miede and Ivo Pletikosić.

The style was inspired by Robert Bringhurst’s seminal book on typography “The Elements of Typographic Style”.

Figures were created with TikZ. Young diagrams were created using the genyoungtabtikz package by Matthew Fayers.