On Braidors: An Analogue of the Theory of Drinfel'd Associators for Braids in an Annulus

by

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Abstract

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We develop the theory of braidors, an analogue of Drinfel'd's theory of associators in which braids in an annulus are considered rather than braids in a disk. After defining braidors and showing they exist, we prove that a braidor is defined by a single equation, an analogue of a well-known theorem of Furusho [Furusho (2010)] in the case of associators. Next some progress towards an analogue of another key theorem, due to Drinfel'd [Drinfel'd (1991)] in the case of associators, is presented. The desired result in the annular case is that braidors can be constructed degree be degree. Integral to these results are annular versions $\mathbf{GT}_a$ and $\mathbf{GRT}_a$ of the Grothendieck-Teichmüller groups $\mathbf{GT}$ and $\mathbf{GRT}$ which act faithfully and transitively on the space of braidors. We conclude by providing surprising computational evidence that there is a bijection between the space of braidors and associators and that the annular versions of the Grothendieck-Teichmüller groups are in fact isomorphic to the usual versions potentially providing a new and in some ways simpler description of these important groups, although these computations rely on the unproven result to be meaningful.

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Chapter 1

Introduction

1.1 Motivation

Among the more powerful invariants for knots is the invariant coming from Drinfeld's theory of associators, sometimes also referred to as the Kontsevich integral. While at first glance invariants of knots may seem unrelated to Drinfel'd's theory of associators, it is implicit in the original work [Drinfel'd (1991)] and first explicitly explained in [Bar-Natan (1998)] and [Le & Murakami (1995)] that the data of a Drinfel'd associator is in fact equivalent to a well-behaved invariant of parenthesized braids, braids in which the distance between strands plays an important role rather than just the topology alone.

In this thesis ordinary braids, braids embedded in a disk cross an interval, are replaced by braids in an annulus cross an interval and an analogue of the theory of associators, which we call braidors, is developed in this topological setting.

One of the motivations for this work is the fact that the algebraic structure arising out of annular braids, which we construct in Section 3.1.2, does not not require introducing parenthesizations at all and furthermore all but one of the strand doubling operations (or partial composition operations in the operadic language often used) can be omitted. As such it appears the algebraic structure related to braidors is simpler in some respects than the algebraic structure in the case of associators. One of the main hindrances in practical applications of Drinfel'd associators (or the closely related Kontsevich integral) to knot theory is the extreme difficulty in evaluating this invariant for any nontrivial knot; even evaluating its value on the unknot is a difficult computation [Bar-Natan *et al.* (2000)]. Since the algebraic structure having to do with braidors is simpler, we hope in future work to be able to find braidors which are simpler to construct and compute than associators, however some evidence is given which seems to indicate thiw will require moving beyond the algebra coming from braids in the annulus. Since any invariant of annular braids yields an invariant of usual braids, the end goal here is to construct an easier to compute universal finite type invariant of knots.

One of the major theorems in the theory of associators due to [Furusho (2010)] is that an associator is essentially defined by a single equation, the pentagon equation and that the two hexagon equations are implied by the pentagon equation[1]. The analogue of this theorem for the case of associators is the major result of this work. To state it precisely let $\mathfrak{t}_{1,n}$ be the Drinfel'd-Kohno graded Lie algebra on $n+1$ strands. This is the graded Lie algebra generated by t_{ij} for $0 \leq i < j \leq n$[2] subject to the relations generated by $[t_{ij}, t_{kl}] = 0$ for $|\{i,j,k,l\}| = 4$ and $[t_{jk}, t_{ik} + t_{jk}] = 0$ when $|\{i,j,k\}| = 3$. Each t_{ij} has degree 1.

In order to define braidors some operations on the Drinfeld-Kohno algebras $\widehat{U\mathfrak{t}_{1,n}}$, the degree completed universal enveloping algebra of the Lie algebra $\mathfrak{t}_{1,n}$, need to be defined first. Let $f : \{0, \cdots, n\} \to \{0, \cdots, m\}$ be any set map. There is an induced map $T_f : \widehat{U\mathfrak{t}_{1,m}} \to \widehat{U\mathfrak{t}_{1,n}}$ which is defined on generaters by $t_{ij} \mapsto \sum_{p \in f^{-1}(i), q \in f^{-1}(j)} t_{pq}$ where $t_{ij} \mapsto 0$ if either $f^{-1}(i)$ or $f^{-1}(j)$ are empty. Given $A \in \widehat{U\mathfrak{t}_m}$ the notation $A^{f^{-1}(1), \cdots, f^{-1}(m)} = T_f(A)$ will frequently be used.

A braidor is an invertible, grouplike[3] element $B \in \widehat{U\mathfrak{t}_{1,2}}$ which satisfies the equations

$$B^{0,1,2}B^{02,1,3}B^{0,2,3} = B^{01,2,3}B^{0,1,3}B^{03,1,2} \qquad \text{(Braid Equation)}$$
$$R^{01,2} = BR^{0,2}B^{0,2,1} \qquad \text{(Mixed Equation)}$$

where $R = \exp(t_{01})$ (see also Definition 3.3.1 below.)

Theorem 1.1.1. *A braidor is an invertible, grouplike element $B \in \widehat{U\mathfrak{t}_{1,2}}$ such that the coefficient of the term t_{12} is $\frac{1}{2}$ and which satisfies the braid equation*

$$B^{0,1,2}B^{02,1,3}B^{0,2,3} = B^{01,2,3}B^{0,1,3}B^{03,1,2}.$$

In other words, the mixed equation is implied by the braid equation.

One of Drinfel'd's key results in [Drinfel'd (1991)] is that associators can be constructed degree by degree. We conjecture, with a sketch of a proof containing a major gap in Section 3.6, that the analogous

[1] See Section 2.5 for the detailed description of these equations.

[2] When dealing with braidors, it is more convenient to index the generators of the Drinfeld-Kohno algebra by integers in the range $0 \leq i < j \leq n$ since the zeroth strand represents the core of an annulus rather than an actual strand. When considering associators however, we use the usual convention of indexing by integers in the range $1 \leq i < j \leq n$ and notate the resulting graded Lie algebra by $\mathfrak{t}_n$. Thus $\mathfrak{t}_{1,n}$ is isomorphic to $\mathfrak{t}_{n+1}$ via shifting indices up by one.

[3] The coproduct is the standard coproduct for universal enveloping algebras defined by setting $\Delta(t_{ij}) = 1 \otimes t_{ij} + t_{ij} \otimes 1$ for the generators $t_{ij} \in \mathfrak{t}_n$.

result for braidors holds as well. Any braidor $B \in \widehat{U\mathfrak{t}_3}$ decomposes as a sum $B = \sum_{m=0}^{\infty} B_m$ where B_m is homogeneous of degree m and the conjecture is

Conjecture 1.1.1. *Let $B \in \widehat{U\mathfrak{t}_3}$ be a braidor up to degree m. This means the braid and mixed equations hold in degrees 0 to m but possibly fail to hold in higher degrees. Then there exists $\beta \in \widehat{U\mathfrak{t}_3}$ homogeneous of degree $m+1$ such that $B + \beta$ is a braidor up to degree $m+1$.*

Despite the original motivation that there may be more braidors than associators and some of these may be easier to construct and compute, surprising computational evidence presented in Section 4.1 suggests the

Conjecture 1.1.2. *There is a bijection between the set of all braidors $\mathbf{BRAID}_0$ with degree one term equal to $\frac{1}{2}t_{12}$ and the set of Drinfel'd associators $\mathbf{ASSOC}$. Furthermore there are isomorphisms of affine group schemes $\mathbf{GRT}_{a,0} \cong \mathbf{GRT}$ and $\mathbf{GT}_{a,0} \cong \mathbf{GT}$ between the Grothendieck-Teichmüller groups $\mathbf{GT}$ and $\mathbf{GRT}$ and their annular counterparts.*

Precise definitions of associators as well as $\mathbf{GT}$ and $\mathbf{GRT}$, symmetry groups of the set of associators on which they act freely and transitively, are recalled in Section 2.5. Their annnular counterparts $\mathbf{GT}_a$ and $\mathbf{GRT}_a$ are symmetry groups of the set of braidors which we define in Section 3.3 and which act simply and transitively on $\mathbf{BRAID}$.

The computations in Chapter 4 unfortunately rely on Conjection 1.1.2 to be meaningful. This is because doing computations up to degree n is only meaningful if every braidor up to degree n actually comes from a full braidor, or in other words that every braidor up to degree n can be extended to a full braidor, so that information gained about braidors up to degree n yields information about full braidors.

Associators, the groups $\mathbf{GT}$ and $\mathbf{GRT}$, and the corresponding Lie algebras $\mathfrak{gt}$ and $\mathfrak{grt}$, show up in many seemingly unrelated areas of mathematics besides knot theory. For example they are closely related to the absolute Galois group [Grothendieck (1997); Drinfel'd (1991)] which is where the theory originates, the motivic Lie algebra and multiple zeta values [Furusho (2011)], the Kashiwara-Vergne problem in Lie theory [Alekseev & Torossian (2012)], formality of the little disks operad and to Kontsevich formality [Fresse (2017b); Tamarkin (2003, 1998)], and to quantization of Lie bialgebras [Etingof & Kazhdan (1996)] to give just a sampling. Having a new description of these important groups may lead to new information about the open conjectures involving these subjects and the relations between them.

While Conjecture 1.1.2 may appear to make the original idea of finding simpler braidors than there are associators impossible, the conjecture appears to only apply to braidors and associators in the Drinfel'd-Kohno algebra specifically while the notion of both associators and braidors makes sense in more general spaces. In fact, there are spaces in which only the notion of braidors can be defined but associators can't and furthermore the computations in Section 4.1 show that there are some spaces in which both braidors and associators exist but there are more braidors than associators. Thus there is still a possibility of finding simpler, more computable knot invariants using braidors rather than associators however finding these will require searching in algebras other than the Drinfel'd-Kohno algebra. An example of such an algebra is given in Section 4.2

Chapter 2

Algebraic Structures and Expansions

2.1 Algebraic Structures and Completions

A construction which appears often in low dimensional topology, especially when constructing invariants of some class of topological objects, is to begin with a set of topological objects together with some operations, for example the set of knots with strand insertion and deletion operations, extend this set into a filtered linear space, then construct the associated graded of this filtered linear space. Maps from the filtered linear space of topological objects to its associated graded which preserve the desired collection of operations yield powerful invariants of the topological objects in many cases.

The precise nature of the operations involved vary from example to example. The notion of *algebraic structures* developed in this chapter is intended to formalize this construction in general. An algebraic structure is a set of objects together with operations between them which are allowed to have both multiple inputs and multiple outputs as schematically represented in Figure 2.1.1. This is not a unique concept, being closely related to an algebra for a colored prop or functors between multicategories. A brief summary of the notion can also be found in [Bar-Natan & Dancso (2017)], although many of the details given here are not given in previous work, and some of the constructions required here are not covered in these previous expositions at all.

Definition 2.1.1. An *algebraic structure* is a pair $\mathcal{S} = (\mathcal{D}_\mathcal{S}, \mathcal{O}_\mathcal{S})$ where $\mathcal{D}_\mathcal{S} = \{B_\alpha\}_{\alpha\in\Theta}$ is a collection of sets indexed by a set Θ and $\mathcal{O}_\mathcal{S}$ is a collection of functions of the form $f : \prod_{i=1}^{I} B_{\alpha_i} \to \prod_{j=1}^{J} B_{\beta_j}$ for some B_{α_i} and B_{β_j} which are in $\mathcal{D}_\mathcal{S}$[1].

The elements of any of the sets in $\mathcal{D}_\mathcal{S}$ will be referred to as the objects in the algebraic structure.

[1] A function from the empty product is allowed, which amounts to choosing constants in the sets $B_\alpha \in \mathcal{D}_\mathcal{S}$.

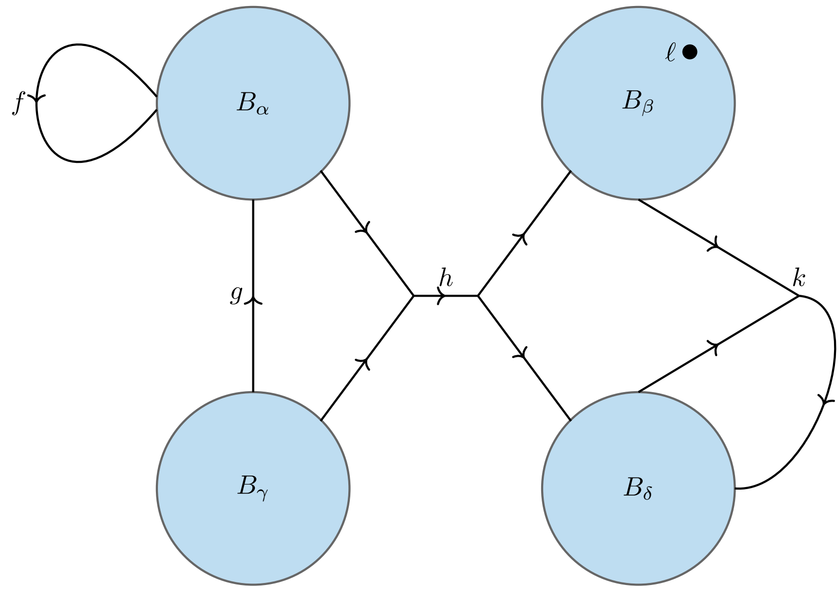


Figure 2.1.1: An example of an algebraic structure comprising four objects $B_\alpha, B_\beta, B_\gamma, B_\delta$, four operations $f : B_\alpha \to B_\alpha, g : B_\gamma \to B_\alpha, h : B_\alpha \times B_\gamma \to B_\beta \times B_\delta$ and $k : B_\beta \times B_\delta \to B_\delta$ between the various objects and a nullary operation $\ell : \star \to B_\beta$ which is a choice of constant in B_β.

The set Θ is called the set of types of the algebraic structure while objects in a set B_α will be referred to as objects of type α. An operation $f : \prod_{i=1}^{I} B_{\alpha_i} \to \prod_{j=1}^{J} B_{\beta_j}$ is called an I-ary operations with J outputs. A nullary operation is a choice of constants.

An example of an algebraic structure is depicted in Figure 2.1.1. This is an algebraic structure with four types of objects and five operations between the various types of objects of varying arity and with varying number of outputs including a choice of constant.

An algebraic structure has an underlying multigraph of types of objects and types of operations. The multigraph corresponding to the algebraic structure in Figure 2.1.1 is given in Figure 2.1.2. Given a multigraph labelled as described in the figure, an algebraic structure with this multigraph is obtained by choosing a set for each vertex labelled by a greek letter and choosing a set map with inputs and outputs as indicated by the edges for each collection of edges labelled with the same Latin letter (including the empty edge at a vertex which corresponds to a constant.) The underlying multigraph of an algebraic structure keeps track of what type of objects and what sorts of operations between objects are being encoded, while the algebraic structure itself is a choice of actual sets and set maps corresponding to the multigraph.

It generally does not make much sense to consider morphisms between algebraic structures of different

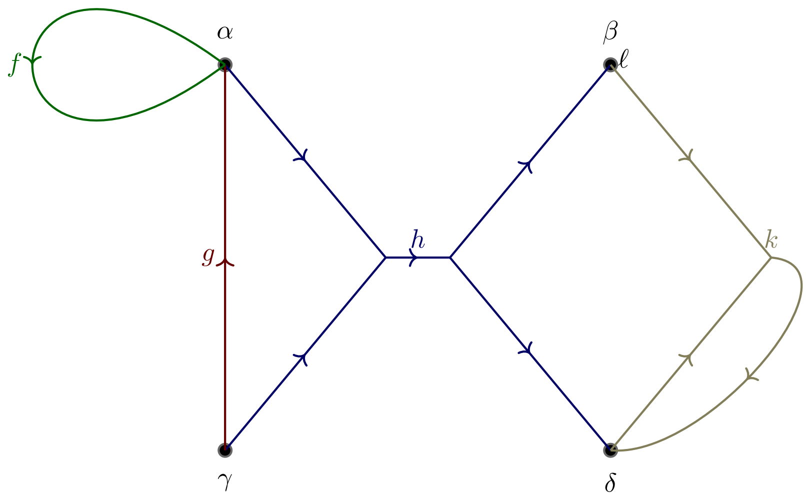


Figure 2.1.2: The multigraph underlying the algebraic structure in Figure 2.1.1. Some of the vertices are labelled in Greek letters by the types of the algebraic structure, although some vertices are unlabelled. The oriented edges of the graph come from the operations in the algebraic structure and each edge is labelled by the operation it comes from, indicated via colour coding. Constants are indicated by adding an extra label in the Latin alphabet to a vertex.

types, for example a morphism from a nonabelian group to a vector space. Hence morphisms between algebraic structures are only defined between algebraic structures with the same underlying multigraph.

While algebraic structures are defined for the specific purpose of invariants of topological objects, they are so general that many of the structures studied in mathematics are special cases.

Example 2.1.1. [Groups] Let G be a group with unit e. This can be viewed as an algebraic structure $\mathcal{G}$ by letting $\mathcal{D}_{\mathcal{G}} = \{G\}$ regarded just as a set. $\mathcal{O}_{\mathcal{G}}$ has three operations, a binary operation $m : G \times G \to G$ given by group multiplication, a unary operation $i : G \to G$ given by inversion and a nullary operation $u : \{\star\} \to G$ defined by $1_u(\star) = e$.

Conversely, any algebraic structure $\mathcal{G}$ with $\mathcal{D}_{\mathcal{G}} = \{G\}$ consisting of a single set and with three operations $\mathcal{O}_{\mathcal{G}} = \{m : G \times G \to G, i : G \to G, u : \{1\} \to G\}$ which satisfy the associativity relation $m \circ (\mathrm{id}_G \times m) = m \circ (m \times \mathrm{id}_G)$, the unit relation $m \circ (u \times \mathrm{id}_G) = \mathrm{id}_G = m \circ (\mathrm{id}_G \times u)$ and the inverse relation $m \circ (\mathrm{id}_G, i) = u \circ \epsilon = (\mathrm{id}_G, i) \circ m$ where $\epsilon : G \to \{1\}$ is the uniqe map, defines a group.

Example 2.1.2 (Categories). Let $\mathcal{C}$ be a category. To view it as an algebraic structure, let $\mathcal{S}$ be the algebraic structure with $\mathcal{D}_{\mathcal{S}} = \{\mathrm{Mor}_{\mathcal{C}}(A, B) : A, B \in \mathrm{Obj}_{\mathcal{C}}\}$ and $\mathcal{O}_{\mathcal{S}} = \{\circ_{A,B,C} : \mathrm{Mor}_{\mathcal{C}}(A, B) \times \mathrm{Mor}_C\, C(B, C) \to \mathrm{Mor}_{\mathcal{C}}(A, C) : A, B, C \in \mathrm{Obj}_{\mathcal{C}}\}$ where $\circ_{A,B,C}$ is composition.

Example 2.1.3 (Operads). Let $\mathcal{O}$ be a nonsymmetric operad in a symmetric monoidal category, that is a sequence $\{\mathcal{O}(n)\}_{n\in\mathbb{N}}$ of objects in the category together with composition operations $\circ_{n_1,\cdots,n_k} : \mathcal{O}(k) \otimes \mathcal{O}(n_1)\otimes\cdots\otimes\mathcal{O}(n_k) \to \mathcal{O}(n_1+\cdots+n_k)$. This yields an algebraic structure $\mathcal{S}$ with $\mathcal{D}_{\mathcal{S}} = \{\mathcal{O}(n)\}_{n\in\mathbb{N}}\}$ and $\mathcal{O}_{\mathcal{S}} = \{\circ_{n_1,\cdots,n_k} : n_1,\cdots,n_k \in \mathbb{N}\}$. A symmetric symmetric operad can be modelled by also including the operations $\{\rho_\sigma : \mathcal{O}(n) \to \mathcal{O}(n) : \sigma \in S_n, n \in \mathbb{N}\}$ in the algebraic structure where ρ_σ is given by the action of the symmetric group on $\mathcal{O}(n)$.

The notion of operads will be used in describing the theory of Drinfel'd associators, however very little of the general theory of operads will be used. In particular, thinking of an operad as comprising a collection of n-ary operations $\mathcal{O}(n)$ for every n together with the composition operations $\circ_{n_1,\cdots,n_k}$ obtained by plugging the output of an n_i-ary operation is the ith input of a k-ary operation. A detailed exposition can however be found in [Fresse (2017a)].

The above list of examples could be extended to include many more of the common structures in mathematics, for example group homomorphisms, group actions on sets, and so on, however not everything is amenable to description as an algebraic structure. For the example of categories and groups, the exact notion can be characterized using the language of algebraic structures as was done for groups by enforcing certain relations to hold between the operations, but this is not possible for all examples.

Example 2.1.4 (Fields). A field is not an algebraic structure because inversion is only a partially defined operation.

Example 2.1.5 (Bialgebras). Let A be a bialgebra over the field $\mathbb{K}$ with product $m : A \otimes A \to A$, coproduct $\Delta : A \to A \otimes A$, unit $\iota : \mathbb{K} \to A$ and counit $\epsilon : A \to \mathbb{K}$. A map $m : A \otimes A \to A$ is equivalent to a map $A \times A \to A$ however a map $\Delta : A \to A \otimes A$ is not equivalent to a map $A \to A \times A$ so a bialgebra is not an algebraic structure according to our definitions.

The definition of an algebraic structure could be extended to an algebraic structure in a monoidal category rather than the monoidal category of sets and then a bialgebra would be an algebraic structure in the monoidal category of vector spaces with tensor products as the monoidal structure.

Definition 2.1.2. An algebraic structure $\mathcal{S}$ is a *substructure* of an algebraic structure $\mathcal{T}$, denoted $\mathcal{S} \subseteq \mathcal{T}$, if both have the same underlying multigraph, each $B_\alpha \in \mathcal{D}_{\mathcal{S}}$ is a subset of the corresponding set $C_\alpha \in \mathcal{D}_{\mathcal{T}}$ and each operation in $\mathcal{O}_{\mathcal{S}}$ is a restriction of the corresponding operation in $\mathcal{O}_{\mathcal{T}}$.

Definition 2.1.3. Let $\mathcal{S} = (\mathcal{D}_{\mathcal{S}} = \{B_\alpha\}_{\alpha\in\Theta}, \mathcal{O}_{\mathcal{S}})$ and $\mathcal{T} = (\mathcal{D}_{\mathcal{T}} = \{C_\alpha\}_{\alpha\in\Theta}, \mathcal{O}_{\mathcal{T}})$ be two algebraic structures with the same underlying multigraph. A *map* $\Xi : \mathcal{S} \to \mathcal{T}$ is a pair $\left(\{\xi_\alpha\}_{\alpha\in\Theta}, \chi\right)$ where

$\xi_\alpha : B_\alpha \to C_\alpha$ and χ is a bijection from $\mathcal{O}_\mathcal{S} \to \mathcal{O}_\mathcal{T}$ subject to the following compatibility condition. For any operation $f : \prod_{i=1}^{I} B_{\alpha_i} \to \prod_{j=1}^{J} B_{\beta_j}$, $\chi(f) : \prod_{i=1}^{I} C_{\alpha_i} \to \prod_{j=1}^{J} C_{\beta_j}$.

Ξ is a *morphism* if it interwines the operations. This means for any operation $f : \prod_{i=1}^{I} B_{\alpha_i} \to \prod_{j=1}^{J} B_{\beta_j} \in \mathcal{O}_S$, the following diagram commutes

$$\begin{tikzcd}
B_{\alpha_1} \times \cdots \times B_{\alpha_I} \arrow[r, "f"] \arrow[d, "\xi_{\alpha_1}\times\cdots\times\xi_{\alpha_I}"'] & B_{\beta_1} \times \cdots \times B_{\beta_J} \arrow[d, "\xi_{\beta_1}\times\cdots\times\xi_{\beta_J}"] \\
C_{\alpha_1} \times \cdots \times C_{\alpha_I} \arrow[r, "\chi(f)"'] & C_{\beta_1} \times \cdots \times C_{\beta_J}
\end{tikzcd}.$$

Ξ is an *isomorphism* if it is a morphism and each map ξ_α is a bijection.

2.1.1 $\mathcal{S}$-Algebras and the Unipotent Filtration

The following construction is a generalization of the group algebra construction to general algebraic structures. The basic idea throughout is to apply the usual construction for groups to each set B_α in an algebraic structure individually and extend operations via multilinearity.

Definition 2.1.4. Let A be a $\mathbb{Q}$-algebra and let $\mathcal{S}$ be an algebraic structure. The $\mathcal{S}$-*algebra* $A[\mathcal{S}]$ is the algebraic structure with the same underlying multicategory as $\mathcal{S}$ and with

$$\mathcal{D}_{A[\mathcal{S}]} = \{A[B_\alpha]\}_{\alpha\in\Theta}$$

where $A[B_\alpha]$ is the A-module of formal A-linear combinations of elements of B_α. Operations in $A[\mathcal{S}]$ are multilinear extensions of the operations in $\mathcal{O}_\mathcal{S}$ to module homomorphisms.

Remark 1. For many of the later definitions, an algebraic structure $\mathcal{S} = (\{B_\alpha\}_{\alpha\in\Theta}, \mathcal{O}_\mathcal{S})$ will be given, which leads to $A[\mathcal{S}]$ as defined in the previous definition and then further constructions will be applied to the sets $A[B_\alpha]$. Strictly speaking we should distinguish between the original set B_α and the A-module of formal linear combinations $A[B_\alpha]$ however to avoid cluttered and unreadable notation we will use B_α to refer to both sets and hope it is clear from context which of the two is meant.

Definition 2.1.5. The *augmentation ideal* of the $\mathcal{S}$-algebra $A[\mathcal{S}]$ is the substructure $\mathcal{I}_\mathcal{S} \subseteq A[\mathcal{S}]$ whose objects are formal A-linear combinations $\sum_{i=1}^{n} a_i B_i$ where $a_i \in A$ satisfy $\sum_{i=1}^{n} a_i = 0$ and whose operations are restrictions of operations in $A[\mathcal{S}]$ to objects in $\mathcal{I}_\mathcal{S}$.

Definition 2.1.6. The m^{th} *power of the augmentation ideal*, denoted $\mathcal{I}_{\mathcal{S}}^m$, is the substructure of $\mathcal{I}_{\mathcal{S}}$ with $\mathcal{D}_{\mathcal{I}_{\mathcal{S}}^m} = \{B_\alpha^m\}_{\alpha\in\Theta}$ where

$$B_\alpha^m = \left\{\pi_{B_\alpha} \circ f_1 \circ \cdots \circ f_n(x_1^1, \cdots, x_1^p) : \text{ at least } m \text{ inputs are objects in } \mathcal{I}_{\mathcal{S}} \right\},$$

$f_1, \cdots f_n$ are arbitrary operations in $A[\mathcal{S}]$ such that one of the outputs of f_1 is B_α, and π_{B_α} is the projection from the product of output sets of f_1 to B_α. In other words, objects in $\mathcal{I}_{\mathcal{S}}^m$ are outputs of arbitrarily many compositions of operations in $A[\mathcal{S}]$ with at least m inputs being objects in $\mathcal{I}_{\mathcal{S}}$. The operations in $\mathcal{I}_{\mathcal{S}}^m$ are restrictions of operations in $\mathcal{I}_{\mathcal{S}}$.

Powers of the augmentation ideal form a decreasing sequence of substructures $A[\mathcal{S}] \supset \mathcal{I}_{\mathcal{S}} \supset \mathcal{I}_{\mathcal{S}}^2 \supset \cdots$ of $A[\mathcal{S}]$. For notational convenience define $\mathcal{I}_{\mathcal{S}}^0 = A[\mathcal{S}]$.

Definition 2.1.7. The *unipotent filtration* $\mathcal{F}_\bullet A[\mathcal{S}]$ for an algebraic structure $\mathcal{S}$ is the decreasing filtration with $\mathcal{F}_m A[\mathcal{S}] = \mathcal{I}^{m+1}$ for $m \in \mathbb{N}$.

2.1.2 Coproducts

The algebraic structures appearing later will include a coproduct which requires the definition of a tensor product $\mathcal{S} \boxtimes \mathcal{S}$ of algebraic structures. To avoid overloading notation later, the tensor product of algebraic structures will be notated using $\boxtimes$ rather than $\otimes$.

Definition 2.1.8. Let $\mathcal{S}$ be an algebraic structure. The algebraic structure $A[\mathcal{S}] \boxtimes A[\mathcal{S}]$ has the same underlying multigraph as $\mathcal{S}$ and has objects

$$\mathcal{D}_{A[\mathcal{S}]\boxtimes A[\mathcal{S}]} = \{A[B_\alpha] \otimes_A A[B_\alpha]\}_{\alpha\in\Theta}.$$

To define operations in the tensor product, let $f : B_{\alpha_1} \times \cdots \times B_{\alpha_m} \to B_{\beta_1} \times \cdots \times B_{\beta_n}$ be an operation in $A[\mathcal{S}]$ and define an operation $f \boxtimes f : (B_{\alpha_1} \otimes B_{\alpha_1}) \times \cdots \times (B_{\alpha_m} \otimes B_{\alpha_m}) \to (B_{\beta_1} \otimes B_{\beta_1}) \times \cdots \times (B_{\beta_n} \otimes B_{\beta_n})$ on simple tensors by

$$f \boxtimes f(c_1 \otimes b_1, \cdots, c_m \otimes b_m) = \Big(f_1(c_1, \cdots, c_m) \otimes f_1(b_1, \cdots, b_m), \cdots, f_n(c_1, \cdots, c_m) \otimes f_n(b_1, \cdots, b_m)\Big)$$

and extend by linearity. Then $\mathcal{O}_{A[\mathcal{S}]\boxtimes A[\mathcal{S}]} = \left\{f \boxtimes f : f \in \mathcal{O}_{A[\mathcal{S}]}\right\}$.

$A[\mathcal{S}] \boxtimes A[\mathcal{S}]$ is again filtered with $\mathcal{F}_k(A[\mathcal{S}] \boxtimes A[\mathcal{S}]) = (\mathcal{I} \boxtimes \mathcal{I})^k$ where $(\mathcal{I} \boxtimes \mathcal{I})^k$ is the algebraic structure with types

$$D_{(\mathcal{I}\boxtimes\mathcal{I})^k} = \left\{ \sum_{i+j=k} B_\alpha^i \otimes_A B_\alpha^j \right\}_{\alpha\in\Theta}.$$

The operations are restriction of operations in $A[\mathcal{S}] \boxtimes A[\mathcal{S}]$.

The coproduct is defined by making all objects in $\mathcal{S}$ grouplike.

Definition 2.1.9. The coproduct $\square = (\{\xi_\alpha\}_{\alpha\in\Theta}, \chi) : A[\mathcal{S}] \to A[\mathcal{S}] \boxtimes A[\mathcal{S}]$ is the morphism of algebraic structures with $\xi_\alpha(b) = b \otimes b$ for any object $b \in B_\alpha$ and extended via linearity to $A[B_\alpha]$. The map χ is defined by $\chi(f) = f \boxtimes f$.

2.1.3 The Associated Graded Construction

A filtered algebraic structure $\mathcal{F}_\bullet A[\mathcal{S}]$ has been constructed, and given a filtered object it is often useful to construct its associated graded as is the case with algebraic structures.

Definition 2.1.10. Let $A[\mathcal{S}] = \mathcal{I}^0 \supset \mathcal{I}^1 \supset \mathcal{I}^2 \supset \cdots$ be the augmentation ideal of an algebraic structure $\mathcal{S}$ together with the induced unipotent filtration[2]. The *associated graded* $\operatorname{gr}\mathcal{S}$ of $\mathcal{S}$ is the algebraic structure with the same underlying multigraph as $A[\mathcal{S}]$ and with objects defined by

$$\mathcal{D}_{\operatorname{gr}\mathcal{S}} = \left\{ \bigoplus_m B_\alpha^m / B_\alpha^{m+1} \right\}_{\alpha\in\Theta}.$$

Let $f : B_{\alpha_1} \times \cdots \times B_{\alpha_n} \to B_{\beta_1} \times \cdots \times B_{\beta_p}$ be an operation in $A[\mathcal{S}]$. This induces an operation $B^m_{\alpha_1} \times \cdots \times B^m_{\alpha_n} \to B^m_{\beta_1} \times \cdots \times B^m_{\beta_p}$ in the algebraic structure $\mathcal{I}^m$ as has been described previously which further descends to a quotient map $f_m : B^m_{\alpha_1}/B^{m+1}_{\alpha_1} \times \cdots \times B^m_{\alpha_n}/B^{m+1}_{\alpha_n} \to B^m_{\beta_1}/B^{m+1}_{\beta_1} \times \cdots \times B^m_{\beta_p}/B^{m+1}_{\beta_p}$ The maps f_m form the components of a graded map

$$\operatorname{gr} f : \left(\bigoplus_m B^m_{\alpha_1}/B^{m+1}_{\alpha_1} \right) \times \cdots \times \left(\bigoplus_m B^m_{\alpha_n}/B^{m+1}_{\alpha_n} \right) \to \left(\bigoplus_m B^m_{\beta_1}/B^{m+1}_{\beta_1} \right) \times \cdots \times \left(\bigoplus_m B^m_{\beta_p}/B^{m+1}_{\beta_p} \right).$$

$\mathcal{O}_{\operatorname{gr}\widehat{\mathcal{S}}}$ is defined to be the set of all maps of the form $\operatorname{gr} f$ for $f \in A[\mathcal{S}]$.

The functorial nature of the associated graded construction extends to algebraic structures as well since the associated graded morphism can be constructed for any morphism of algebraic structures in the following way.

Definition 2.1.11. Let $\Xi = (\{\xi_\alpha\}_{\alpha\in\Theta}, \chi) : \mathcal{A} \to \mathcal{B}$ be a morphism between the algebraic structures $\mathcal{A} = (\{B_\alpha\}_{\alpha\in\Theta}, \mathcal{O}_\mathcal{A})$ and $\mathcal{B} = (\{C_\alpha\}_{\alpha\in\Theta}, \mathcal{O}_\mathcal{B})$ which have the same underlying multigraph and have

[2]The associated graded could of course be defined for any filtered algebraic structure rather than just this particular filtration.

types Θ. The associated graded morphism $\operatorname{gr}\Xi = (\{\operatorname{gr}\xi_\alpha\}, \operatorname{gr}\chi) : \operatorname{gr}\mathcal{A} \to \operatorname{gr}\mathcal{B}$ can be constructed as follows. Given $B_\alpha \in \mathcal{D}_\mathcal{A}$, the map $\xi_\alpha : B_\alpha \to C_\alpha$ induces a map $B_\alpha^m/B_\alpha^{m+1} \to C_\alpha^m/C_\alpha^{m+1}$ for each m and hence a map $\operatorname{gr}\xi_\alpha : \bigoplus_m B_\alpha^m/B_\alpha^{m+1} \to \bigoplus_m C_\alpha^m/C_\alpha^{m+1}$.

χ maps an operation $f \in \mathcal{O}_\mathcal{A}$ to an operation $\chi(f) \in \mathcal{O}_\mathcal{B}$. The associated graded map $\operatorname{gr}\chi$ maps $\operatorname{gr} f \to \operatorname{gr}\chi(f)$ where the associated graded of an operation was defined in Definition 2.1.14.

The coproduct defined on algebraic structures induces one on the associated graded algebraic structure as well. The morphism $\Box : \operatorname{gr}\mathcal{S} \to \operatorname{gr}\mathcal{S} \boxtimes \operatorname{gr}\mathcal{S}$ is defined for an object $\bar{x} \in B_\alpha^m/B_\alpha^{m+1}$ by letting $\Box(\bar{x}) = \overline{\Box(x)}$ where the overlines indicate equivalence classes in the relevant quotient.

2.1.4 Prounipotent Completions of Algebraic Structures

The construction in this section is an analogue of the prounipotent completion of a group generalized to arbitrary algebraic structures. As before, the idea throughout is to apply the usual construction for groups to each set $B_\alpha \in \mathcal{D}_\mathcal{S}$ for some algebraic structure and extend operations via multilinearity.

In Section 2.1.1, a filtration $\mathcal{F}_\bullet A[\mathcal{S}]$ was constructed for any algebraic structure $\mathcal{S}$ where $\mathcal{F}_m A[\mathcal{S}] = \mathcal{I}^{m+1}$. The algebraic structure can be completed with respect to this filtration to obtain the prounipotent completion of $\mathcal{S}$.

Definition 2.1.12. The m^{th} *unipotent quotient of* $A[\mathcal{S}]$ is the algebraic structure $\mathcal{S}^{(m)} = \mathcal{S}/\mathcal{I}^{m+1}$ with the same underlying multigraph as $A[\mathcal{S}]$ and with objects defined by

$$\mathcal{D}_{\mathcal{S}^{(m)}} = \{B_\alpha/\sim\}_{\alpha\in\Theta}$$

where $\sim$ is the equivalence relation $x \sim y$ if and only if $x - y \in B_\alpha^{m+1}$ [3]. The operations in $\mathcal{S}^{(m)}$ are quotients of operations in $A[\mathcal{S}]$.

The collection of unipotent quotients $A[\mathcal{S}] = \mathcal{S}^{(0)} \leftarrow \mathcal{S}^{(1)} \leftarrow \mathcal{S}^{(2)} \leftarrow \cdots$ form an inverse system, and the completion of the algebraic structure $A[\mathcal{S}]$ is defined to be the inverse limit of this system. More precisely, for every type $\alpha \in \Theta$ we have an inverse system $B_\alpha \leftarrow B_\alpha/B_\alpha^1 \leftarrow B_\alpha/B_\alpha^2 \leftarrow \cdots$ and an algebraic structure can be constructed whose set of objects of type α is the inverse limit of the corresponding system.

[3]That this actually is an equivalence relation can be shown from the definition of the augmentation ideal. Calling $\mathcal{I}$ an ideal is justified by the fact that it induces such an equivalence relation and so an algebraic structure can be quotiented by $\mathcal{I}$.

Definition 2.1.13. The *prounipotent completion* $\widehat{\mathcal{S}}$ of the algebraic structure $\mathcal{S}$ is the $\mathcal{I}_{\mathcal{S}}$-adic completion $\widehat{\mathcal{S}} = \varprojlim \mathcal{S}^{(m)}$ of $A[\mathcal{S}]$. $\widehat{\mathcal{S}}$ has the same multigraph as $\mathcal{S}$ and the objects of $\widehat{\mathcal{S}}$ are defined by

$$\mathcal{D}_{\widehat{\mathcal{S}}} = \{\widehat{B_\alpha} = \varprojlim B_\alpha / B_\alpha^{m+1}\}_{\alpha \in \Theta}$$

The operations in the algebraic structure $\widehat{\mathcal{S}}$ are defined using the universal properties of inverse limits and products as follows. If $f : B_{\alpha_1} \times \cdots \times B_{\alpha_m} \to B_{\beta_1} \times \cdots \times B_{\beta_n}$ is an operation in $A[\mathcal{S}]$, an operation $\widehat{f} : \widehat{B_{\alpha_1}} \times \cdots \widehat{B_{\alpha_m}} \to \widehat{B_{\beta_1}} \times \cdots \times \widehat{B_{\beta_n}}$ must be defined. Let $\pi_i : B_{\beta_1} \times \cdots \times B_{\beta_m} \to B_{\beta_i}$ be the projections associated to a product and let $p_i^k : \widehat{B_{\alpha_i}} \to B_{\alpha_i}/B_{\alpha_i}^{k+1}$ be the projections associated to inverse limit. For each k and i define the map $f_k^i : \widehat{B_{\alpha_1}} \times \cdots \times \widehat{B_{\alpha_m}} \to B_{\beta_i}/B_{\beta_i}^{k+1}$ as in the diagram

$$\begin{tikzcd}
\widehat{B}_{\alpha_1} \times \cdots \times \widehat{B}_{\alpha_m} \arrow[d, "p_1^k \times \cdots \times p_m^k"] \arrow[ddddd, bend right, "f_k^i"'] \\
B_{\alpha_1}/B_{\alpha_1}^{k+1} \times \cdots \times B_{\alpha_m}/B_{\alpha_m}^{k+1} \arrow[d, equal] \\
\frac{B_{\alpha_1} \times \cdots \times B_{\alpha_m}}{B_{\alpha_1}^{k+1} \times \cdots \times B_{\alpha_m}^{k+1}} \arrow[d, "\overline{f}"] \\
\frac{B_{\beta_1} \times \cdots \times B_{\beta_n}}{B_{\beta_1}^{k+1} \times \cdots \times B_{\beta_n}^{k+1}} \arrow[d, equal] \\
B_{\beta_1}/B_{\beta_1}^{k+1} \times \cdots \times B_{\beta_n}/B_{\beta_n}^{k+1} \arrow[d, "\pi_i"] \\
B_{\beta_i}/B_{\beta_i}^{k+1}
\end{tikzcd}.$$

For each i, the maps f_k^i are compatible with the inverse system and so induce a map $\widehat{f^i} : \widehat{B_{\alpha_1}} \times \cdots \times \widehat{B_{\alpha_m}} \to \widehat{B_{\beta_i}}$. Finally, the maps $\widehat{f^i}$ are compatible with the projection maps for the product and so yield the required map $\widehat{f} : \widehat{B_{\alpha_1}} \times \cdots \widehat{B_{\alpha_m}} \to \widehat{B_{\beta_1}} \times \cdots \times \widehat{B_{\beta_n}}$. Using this construction, operations in $\widehat{S}$ are defined to be $\mathcal{O}_{\widehat{\mathcal{S}}} = \{\widehat{f} : f \in \mathcal{O}_{A[\mathcal{S}]}\}$.

More concretely, an element of $\widehat{B_{\alpha_i}}$ is of the form $(b_i^k)_{k \in \mathbb{N}}$ where each b_i^k as a representative of an equivalence class in $B_{\alpha_i}/B_{\alpha_i}^{k+1}$ and such that b_i^k is congruent to b_i^{k-1} modulo $B_{\alpha_i}^k$. Then $\widehat{f}$ has the explicit formula

$$\widehat{f}\big((b_1^k)_k, \cdots, (b_m^k)_k\big) = \Big(\big(f_1(b_1^k, \cdots, b_m^k) \mod B_{\beta_1}^{k+1}\big)_k, \cdots, \big(f_n(b_1^k, \cdots, b_m^k) \mod B_{\beta_n}^{k+1}\big)_k\Big).$$

The inverse system $A[\mathcal{S}] = \mathcal{S}^{(0)} \leftarrow \mathcal{S}^{(1)} \leftarrow \mathcal{S}^{(2)} \leftarrow \cdots$ induces a filtration $\widehat{\mathcal{S}} = \widehat{\mathcal{S}}^{(0)} \supset \widehat{\mathcal{S}}^{(1)} \supset \widehat{\mathcal{S}}^{(2)} \supset$

$\cdots$ on $\widehat{\mathcal{S}}$ where each $\widehat{\mathcal{S}}^{(m)}$ is the inverse image in $\widehat{\mathcal{S}}$ of $\mathcal{S}^{(m)}$ via the canonical projection. In more detail, let $B_\alpha \in \mathcal{D}_{\mathcal{S}}$. We've defined $\widehat{B_\alpha} = \varprojlim B_\alpha/B_\alpha^m \in \mathcal{D}_{\widehat{\mathcal{S}}}$. Let $\widehat{B_\alpha}^{(m)} = \pi_m^{-1}(B_\alpha/B_\alpha^m)$ where $\pi_m : \widehat{B_\alpha} \to B_\alpha/B_\alpha^m$ is the canonical projection associated to the inverse limit. The algebraic structure $\widehat{S}^{(m)}$ has objects defined by $\mathcal{D}_{\widehat{S}^{(m)}} = \{\widehat{B_\alpha}^{(m)}\}_{\alpha\in\Theta}$ and has operations which are restrictions of the operations in $\widehat{\mathcal{S}}$.

Tensor products and the associated coproduct also extend to completions, although some care must be taken when dealing with filtrations for completed tensor products. Let $\mathcal{S}$ be an algebraic structure and let $\widehat{\mathcal{S}}$ be the prounipotent completion. $\widehat{\mathcal{S}} \boxtimes \widehat{\mathcal{S}}$ can be defined as in Definition 2.1.8. There is an induced unipotent filtration $(\widehat{\mathcal{S}} \boxtimes \widehat{\mathcal{S}})^{(m)}$ defined to have objects

$$\mathcal{D}_{(\widehat{\mathcal{S}}\boxtimes\widehat{\mathcal{S}})^{(m)}} = \left\{ \sum_{i+j=m} B_\alpha/B_\alpha^{i+1} \otimes B_\alpha/B_\alpha^{j+1} \right\}_{\alpha\in\Theta}$$

and with operations being the restriction of operations in $\widehat{\mathcal{S}} \boxtimes \widehat{\mathcal{S}}$.

The construction in Definition 2.1.13 can now be applied to $\widehat{\mathcal{S}} \boxtimes \widehat{\mathcal{S}}$ with this filtration to obtain the completed tensor product $\widehat{\mathcal{S}}\widehat{\boxtimes}\widehat{\mathcal{S}}$. The coproduct in Definition 2.1.9 naturally extends to a coproduct $\square : \widehat{\mathcal{S}} \to \widehat{\mathcal{S}}\widehat{\boxtimes}\widehat{\mathcal{S}}$.

Since $\widehat{\mathcal{S}}$ is a filtered algebraic structure, there is a corresponding associated graded, which in the completed case amounts to using infinite sums rather than finite sums.

Definition 2.1.14. Let $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^{(0)}} \leftarrow \widehat{\mathcal{S}}^{(1)} \leftarrow \widehat{\mathcal{S}}^{(2)} \leftarrow \cdots$ be the prounipotent completion of an algebraic structure $\mathcal{S}$ together with the induced unipotent filtration. The *completed associated graded* $\widehat{\mathrm{gr}}\widehat{\mathcal{S}}$ of $\mathcal{S}$ is the algebraic structure with the same underlying multigraph and with objects defined by

$$\mathcal{D}_{\widehat{\mathrm{gr}}\widehat{\mathcal{S}}} = \left\{ \prod_m \widehat{B_\alpha}^{(m)} / \widehat{B_\alpha}^{(m+1)} \right\}_{\alpha\in\Theta}.$$

Let $f : B_{\alpha_1} \times \cdots \times B_{\alpha_n} \to B_{\beta_1} \times \cdots \times B_{\beta_p}$ be an operation in $A[\mathcal{S}]$. This induces an operation $\widehat{B_{\alpha_1}}^m \times \cdots \times \widehat{B_{\alpha_n}}^m \to \widehat{B_{\beta_1}}^m \times \cdots \times \widehat{B_{\beta_p}}^m$ in the algebraic structure $\widehat{\mathcal{S}}^{(m)}$ as has been described previously which further descends to a quotient map $\widehat{f}_m : \widehat{B_{\alpha_1}}^m/\widehat{B_{\alpha_1}}^{m+1} \times \cdots \times \widehat{B_{\alpha_n}}^m/\widehat{B_{\alpha_n}}^{m+1} \to \widehat{B_{\beta_1}}^m/\widehat{B_{\beta_1}}^{m+1} \times \cdots \times \widehat{B_{\beta_p}}^m/\widehat{B_{\beta_p}}^{m+1}$. These maps form the components of a graded map

$$\widehat{\mathrm{gr}}f : \left(\prod_m \widehat{B_{\alpha_1}}^m/\widehat{B_{\alpha_1}}^{m+1}\right) \times\cdots\times \left(\prod_m \widehat{B_{\alpha_n}}^m/\widehat{B_{\alpha_n}}^{m+1}\right) \to \left(\prod_m \widehat{B_{\beta_1}}^m/\widehat{B_{\beta_1}}^{m+1}\right) \times\cdots\times \left(\prod_m \widehat{B_{\beta_p}}^m/\widehat{B_{\beta_p}}^{m+1}\right)$$

in $\mathcal{O}_{\widehat{\mathrm{gr}}\widehat{\mathcal{S}}}$, which is defined to be the collection of all such induced maps.

The functorial nature of the associated graded construction extends to completed algebraic structures in the same way as in the non-completed case. The associated graded morphism is defined as in Definition 2.1.11 with finite sums replaced by infinite sums.

The tensor product and coproduct defined on $\widehat{\mathcal{S}}$ induce a tensor product and coproduct on the associated graded. This structure is defined just as in Section 2.1.2, again with finite sums replaced by infinite sums.

For the majority of what follows only prounipotent completions are used however the m-th unipotent quotient will be required as well on occasion. $\mathcal{S}^{(m)}$ is itself a filtered algebraic structure, with filtration given by $\mathcal{F}_k\mathcal{S}^{(m)} = \mathcal{I}^k/\mathcal{I}^{m+1}$. The construction of tensor products, coproducts and associated graded can be repeated in this case as above. The only subtlety is in dealing with the filtration on the tensor product $\mathcal{S}^{(m)} \boxtimes \mathcal{S}^{(m)}$. The tensor product used here is

$$\mathcal{S}^{(m)} \boxtimes^{(m)} \mathcal{S}^{(m)} = \mathcal{S}^{(m)} \otimes \mathcal{S}^{(m)} \Big/ \sum_{i+j>m} \mathcal{F}_i\mathcal{S}^{(m)} \otimes \mathcal{F}_j\mathcal{S}^{(m)}$$

which is well-defined since $\mathcal{F}_{m+1}\mathcal{S}^{(m)} = 0$. This definition must of course be interpreted as occuring for each B_α individually as in other similar definitions throughout this section.

2.1.5 Expansions

Expansions, the fundamental objects studied in this work, are isomorphisms of the completion of an algebraic structure to its completed associated graded structure.

Definition 2.1.15. An *expansion* of the algebraic structure $\mathcal{S}$ is an isomorphism of algebraic structures $Z : \widehat{S} \to \widehat{\mathrm{gr}}\widehat{S}$ which preserves the unipotent filtration and such that $\widehat{\mathrm{gr}}Z = id_{\widehat{\mathrm{gr}}\widehat{S}}$[4].

A useful fact which will make it easier to work with expansions later is that the notion of expansion given above is equivalent to a filtration preserving map $\tilde{Z} : A[\mathcal{S}] \to \widehat{\mathrm{gr}}\widehat{\mathcal{S}}$ such that $\widehat{\mathrm{gr}}\tilde{Z}$ is the identity. When $A[\mathcal{S}]$ is not completed, $\tilde{Z}$ is no longer an isomorphism however.

In practice, it is often the case that a candidate algebraic structure $\widehat{\mathrm{gr}}\widehat{\mathcal{S}}$ can be guessed, however proving the guess is correct, which generally amounts to showing that a given list of relations is complete, is much more difficult. The following notion of an $\mathcal{A}$-expansion is one method for accomplishing this.

Definition 2.1.16. An $\mathcal{A}$-expansion of the filtered algebraic structure $\mathcal{S}$ is a filtration preserving morphism $Z_A : \widehat{\mathcal{S}} \to \mathcal{A}$ to a complete[5] graded algebraic structure $\mathcal{A}$ with the same underlying multigraph and set of types, together with a surjective, grading preserving morphism $\pi : \mathcal{A} \to \widehat{\mathrm{gr}}\widehat{\mathcal{S}}$ such that

[4]Note that $\widehat{\mathrm{gr}}Z : \widehat{\mathrm{gr}}(\widehat{S}) \to \widehat{\mathrm{gr}}(\widehat{\mathrm{gr}}(\widehat{S}))$ so in stating this condition $\widehat{\mathrm{gr}}(\widehat{\mathrm{gr}}(\widehat{S}))$ is implicitly identified with $\widehat{\mathrm{gr}}(\widehat{S})$.

[5]Complete means $\widehat{\mathcal{A}}$ is isomorphic to $\mathcal{A}$.

$\widehat{\mathrm{gr}} Z_A \circ \pi = id_{\mathcal{A}}$ as in the diagram

$$\begin{tikzcd}
& \mathcal{A} \arrow[d, bend right, "\pi"'] \\
\widehat{\mathcal{S}} \arrow[ur, "Z_A"] \arrow[r, dashed, "\pi\circ Z_A"'] & \widehat{\mathrm{gr}}\widehat{\mathcal{S}} \arrow[u, dashed, bend right, "\widehat{\mathrm{gr}}\widehat{Z_\alpha}"']
\end{tikzcd}.$$

Lemma 2.1.1. *If an $\mathcal{A}$-expansion of $\mathcal{S}$ exists then $\mathcal{A} \simeq \widehat{\mathrm{gr}}\widehat{\mathcal{S}}$ and $Z = \pi \circ Z_A$ is an expansion.*

Proof. Let $Z_A = (\{\xi_\alpha\}_{\alpha\in\Theta}, \chi)$ and let $\pi = (\{\zeta_\alpha\}_{\alpha\in\Theta}, \psi)$. Then for every set of objects $B_\alpha \in \mathcal{D}_{\widehat{\mathcal{S}}}$ of type α there is a commutative diagram

$$\begin{tikzcd}
& \xi_\alpha(B_\alpha) \arrow[d, bend right, "\zeta_{\xi_\alpha(B_\alpha)}"'] \\
B_\alpha \arrow[ur, "\xi_\alpha"] \arrow[r, dashed, "\zeta_{\xi_\alpha(B_\alpha)}\circ\xi_\alpha"'] & \widehat{\mathrm{gr}}B_\alpha \arrow[u, dashed, bend right, "\widehat{\mathrm{gr}}\widehat{\xi_\alpha}"']
\end{tikzcd}.$$

$\zeta_{\xi_\alpha(B_\alpha)}$ is surjective by assumption. Since $\widehat{\mathrm{gr}} Z_A \circ \pi$ is the identity, $\widehat{\mathrm{gr}}\widehat{\xi_\alpha} \circ \zeta_{\xi_\alpha(B_\alpha)}$ is the identity on $\widehat{\mathrm{gr}}B_\alpha$ and hence $\zeta_{\xi_\alpha(B_\alpha)}$ is bijective.

□

2.1.6 Torsors Associated to Algebraic Structures

Given any algebraic structure $\mathcal{S}$, it is always possible to construct a (possibly empty) bitorsor, that is a set with commuting left and right group actions, out of it as indicated in the diagram

$$\begin{tikzcd}
\mathrm{Aut}(\widehat{\mathcal{S}}) \arrow[r, bend left] & \widehat{\mathcal{S}} \arrow[r, Rrightarrow, "\mathbf{Iso}\left(\widehat{\mathcal{S}},\widehat{\mathrm{gr}}\widehat{\mathcal{S}}\right)"] & \widehat{\mathrm{gr}}\widehat{\mathcal{S}} & \mathrm{Aut}\left(\widehat{\mathrm{gr}}\widehat{\mathcal{S}}\right) \arrow[l, bend left]
\end{tikzcd}.$$

If nonempty, the set of isomorphisms $\mathbf{Iso}\left(\widehat{\mathcal{S}}, \widehat{\mathrm{gr}}\widehat{\mathcal{S}}\right)$ is an $\mathrm{Aut}(\widehat{\mathcal{S}})$ -$\mathrm{Aut}\left(\widehat{\mathrm{gr}}\widehat{\mathcal{S}}\right)$ bitorsor where the group multiplication is given by composition of automorphisms and the action of these groups on the set of isomorphisms is given by pre- and post-composition.

The existence of such an isomorphism is often related to a formality result in some context. The relation between expansions of groups and formality results is studied in detail in [Suciu & Wang (2019)]. In more general contexts, formality of algebraic structures is related to other notions of formality, for example to formality of the little disks operad [Tamarkin (2003),Fresse (2017b)] for the algebraic structure discussed in Section 2.5.

It is a guiding principal of Bar-Natan [Bar-Natan & Dancso (2017)] that many interesting and highly nontrivial questions regarding graded spaces fit into this framework and have topological interpretations. Two examples are included below. The example of associators will be discussed in detail later, however the second example will not be discussed in any further detail than the very brief description below and is included only to give another example of an algebraic problem that turns out to have a surprising interpretation in terms of low-dimensional topology.

Example 2.1.6. The example of Drinfel'd associators is explained in greater detail in Section 2.5. Let $\widehat{\mathcal{O}} = \widehat{\mathbf{PaB}}$ be the completed operad of parenthesized braids and $\widehat{\mathrm{gr}}\widehat{\mathcal{O}} = \widehat{\mathbf{PaCD}}$ the completed operad of parenthesized chord diagrams. Then $\mathrm{Iso}(\widehat{\mathbf{PaB}}, \widehat{\mathbf{PaCD}}) = \mathbf{ASSOC}$ is the set of Drinfel'd associators and $\mathrm{Aut}(\widehat{\mathbf{PaB}}) = \widehat{\mathbf{GT}}$ and $\mathrm{Aut}(\widehat{\mathbf{PaCD}}) = \widehat{\mathbf{GRT}}$ are the prounipotent versions of the Grothendieck-Teichmüller groups as defined by Drinfel'd [Drinfel'd (1991); Bar-Natan (1998)]. Stated this way, it is clear that the set of associators is a $\widehat{\mathbf{GT}}$-$\widehat{\mathbf{GRT}}$ bitorsor.

Example 2.1.7. Let wTF be the algebraic structure arising from w-tangled foams, constructed in [Bar-Natan & Dancso (2017)]. These are certain knotted embeddings of 2-dimensional objects in $\mathbb{R}^4$. Let $\mathcal{A}^{sw}$ be the associated graded structure. Then there is a bijection between isomorphisms $wTF \to \mathcal{A}^{sw}$ and solutions of the Kashiwara-Vergne problem in Lie theory. The details of this example can be found in [Bar-Natan & Dancso (2017)]. Thus we see that there are two groups which act freely and transitively on solutions of the Kashiwara-Vergne problem, $\mathrm{Aut}(wTF)$ and $\mathrm{Aut}(\mathcal{A}^{sw})$.

Motivated by the example of Drinfel'd associators, the notation $\widehat{\mathbf{GT}_{\mathcal{S}}} = \mathrm{Aut}(\widehat{\mathcal{S}})$ and the terminology "prounipotent Grothendieck-Teichmüller group associated to $\mathcal{S}$" as well as the notation $\widehat{\mathbf{GRT}_{\mathcal{S}}} = \mathrm{Aut}(\widehat{\mathrm{gr}}\widehat{\mathcal{S}})$ and the terminology "graded prounipotent Grothendieck-Teichmüller group associated to $\mathcal{S}$" will be used.

By replacing prounipotent completions in the above diagram with the mth unipotent quotient, pro-m versions of the Grothendieck Teichmüller group $\mathbf{GT}_{\mathcal{S}}^{(m)} = \mathrm{Aut}(\mathcal{S}^{(m)})$ and the graded Grothendieck Teichmüller group $\mathbf{GRT}_{\mathcal{S}}^{(m)} = \mathrm{Aut}(\mathrm{gr}\,\mathcal{S}^{(m)})$ are obtained as well.

2.2 Free Abelian Groups

As a first, warmup example we consider finitely generated free abelian groups. The following is an expanded version of material found in [Bar-Natan (n.d.)].

Let $G = \langle x_1, \cdots, x_n : x_i x_j = x_j x_i \rangle$ be the free abelian group on n generators. Regarding G as an algebraic structure as in Example 2.1.1, let $\mathcal{G}$ be the algebraic structure with one type of object, $\mathcal{T}_{\mathcal{G}} = \{G\}$ where G is regarded as a set equipped with a binary multiplication operation, a unary inversion operation and a nullary constant operation which is the unit 1 in G. For this example, the procedure outlined in the previous section amounts to taking the prounipotent completion and taking the associated graded algebra of the group G.

Taking formal linear combinations of elements of the same type, in this case formal linear combinations of elements in G, results in $A[\mathcal{G}] = A[x_i, x_i^{-1}]$, the algebra of Laurent polynomials in the variables $\{x_i\}_i$. The multiplication operation in $\mathcal{G}$ becomes the algebra product once extended multilinearly. The algebraic structure $A[G]$ also includes an inversion operation which sends a variable x_i to x_i^{-1} and vice versa, as well as the constant 1.

The augmentation ideal has generators $I = \langle \tilde{x_1}, \cdots, \tilde{x_n} \rangle$ where $\tilde{x_i} = x_i - 1$. Clearly every $\tilde{x_i} \in I$ by definition. Note that no generators of the form $x_i^{-1} - 1$ are required as shown by the computation

$$x_i^{-1} - 1 = -(x_i - 1)x_i^{-1}.$$

Similar straightforward computations show any term of the form $x_1^{p_1} \cdots x_n^{p_n} - 1 \in I$ where $p_j \in \mathbb{Z}$ and any $g \in G$ is of the form $g = x_1^{p_1} \cdots x_n^{p_n}$ implying $I \subset \langle \tilde{x_1}, \cdots, \tilde{x_n} \rangle$.

Turning now to the unipotent completion, I^m is generated by all m-fold products $(x_{i_1} - 1) \cdots (x_{i_m} - 1)$. To compute $A[x_i, x_i^{-1}]/I^{m+1}$, note that

$$A[x_i, x_i^{-1}]/I^{m+1} \cong A[\tilde{x}_i]/\tilde{I}^{m+1}$$

via the map $x_i \mapsto \tilde{x}_i = x_i - 1$ where $\tilde{I}$ is the ideal generated by $\tilde{x}_i$. Inverses are not required as working modulo I^{m+1},

$$x_i^{-1} = (\tilde{x}_i + 1)^{-1} = 1 - x_i + x_i^2 - \cdots \pm x_i^m$$

is already an element of $A[\tilde{x}_i]/\tilde{I}^{m+1}$.

The quotient $A[\tilde{x}_i]/\tilde{I}^{m+1}$ consists of multinomials whose terms have degree less than or equal to m, so the unipotent completion is

$$\varprojlim_m A[x_i, x_i^{-1}]/I^{m+1} = A[[\tilde{x}_1, \cdots, \tilde{x}_n]],$$

the algebra of formal infinite series in the variables $\tilde{x}_i$.

The next step is to compute the associated graded algebra

$$\operatorname{gr} \mathcal{G} = \bigoplus_m \frac{A[x_i, x_i^{-1}]/I^{m+1}}{A[x_i, x_i^{-1}]/I^m} \cong \bigoplus_m I^m/I^{m+1}$$

As observed earlier, I^m is generated by m-fold products of the $\tilde{x}_i$, and hence I^m/I^{m+1} is given by polynomials in $\overline{x}_i$ of total degree m where $\overline{x}_i$ is the equivalence class of $\tilde{x}_i$ in I/I^2. Thus

$$\operatorname{gr}\mathcal{G} = A[\overline{x}_1, \cdots, \overline{x}_n]$$

is the algebra of polynomials in the variables $\overline{x}_i$ and

$$\widehat{\operatorname{gr}}\mathcal{G} = A[[\overline{x}_1 \cdots, \overline{x}_n]]$$

is the algebra of formal power series in these variables.

To determine the coproduct in the associated graded structure, let us consider how $\Box$ acts on a generator $\overline{x}_i \in I/I^2$. Working modulo I^2,

$$\begin{aligned}
\Box(\overline{x}_i) &= \Box(x_i - 1) \\
&= x_i \otimes x_i - 1 \otimes 1 \\
&= (x_i - 1) \otimes x_i + x_i \otimes (x_i - 1) \\
&= \overline{x}_i \otimes 1 + 1 \otimes \overline{x}_i + \overline{x}_i \otimes \overline{x}_i \\
&= \overline{x}_i \otimes 1 + 1 \otimes \overline{x}_i
\end{aligned}$$

so the coproduct on the associated graded makes the generators $\overline{x}_i$ primitive.

An expansion of $\mathcal{G}$ is a map

$$Z : A[[\tilde{x}_1, \cdots \tilde{x}_n]] \to A[[\overline{x}_1, \cdots, \overline{x}_n]]$$

which must be homomorphic with respect to all the operations in our algebraic structure. This example is unusual in the sense that the unipotent completion and the completed associated graded are explicitly the same whereas in general showing these are isomorphic is a difficult problem. Even in this example it is not the case however that the obvious map $\tilde{x}_i \mapsto \overline{x}_i$ is a homomorphic expansion.

To see that this map does not commute with coproducts, note that the coproduct in $A[[\tilde{x}_i]] = \widehat{\mathcal{G}}$ is given by

$$\begin{aligned}\square(\tilde{x}_i) &= \square(x_i - 1)\\ &= x_i \otimes x_i - 1 \otimes 1\\ &= (\tilde{x}_i + 1) \otimes (\tilde{x}_i + 1) - 1 \otimes 1\\ &= \tilde{x}_i \otimes \tilde{x}_i + \tilde{x}_i \otimes 1 + 1 \otimes \tilde{x}_i\end{aligned}$$

(the term $\tilde{x}_i \otimes \tilde{x}_i$ does not cancel here since there is no quotient by I^2) while the coproduct in $A[[\overline{x}_i]] = \widehat{\mathrm{gr}}\widehat{\mathcal{G}}$ is simply $\square \overline{x}_i = \overline{x}_i \otimes 1 + 1 \otimes \overline{x}_i$ as shown above so the coproducts in these two algebras are different.

The fact that the initial algebraic structure $\mathcal{G}$ has a multiplication which must be preserved and the fact that operations are extended by multilinearity imply Z is an algebra morphism, while the fact that the algebraic structure includes an inversion implies that $Z(f^{-1}) = Z(f)^{-1}$. The last bit of structure, the nullary operation choosing the group identity, implies that $Z(1) = 1$, and so $Z(a) = a$ for any $a \in A$.

As a result, the expansion is determined by $Z(\tilde{x}_i)$, its value in the generators $\tilde{x}_i$. The condition $\widehat{\mathrm{gr}}Z = \mathrm{id}_{\widehat{\mathrm{gr}}G}$ holds if and only if

$$Z(\tilde{x}_i) = 1 + \overline{x}_i + O(\overline{x}_i^2)$$

for each x_i.

Using the fact that the coproduct is grouplike on generators in $A[[\tilde{x}_1, \cdots, \tilde{x}_n]]$ and primitive on generators in $A[[\overline{x}_1, \cdots, \overline{x}_n]]$, requiring Z to preserve coproduct amounts to the condition $Z(x_i)(y_i + z_i) = Z(x_i)(y_i)Z(x_i)(z_i)$ where

$$A[[\tilde{x}_1 \cdots, \tilde{x}_n]] \otimes A[[\tilde{x}_1, \cdots, \tilde{x}_n]] = A[[\tilde{y}_1, \tilde{z}_1, \cdots, \cdots, \tilde{y}_n, \tilde{z}_n]]$$

and similarly for the tensor product of the associated graded algebra. That is, if $Z(\tilde{x}_i) = f_i$, then $f_i(\overline{y}_i + \overline{z}_i) = f_i(\overline{y}_i)f_i(\overline{z}_i)$. The unique solution to this equation subject to the initial condition $f_i = 1 + \overline{x}_i + O(\overline{x}_i^2)$ is $f_i = \exp(\overline{x}_i)$. Hence there is a unique expansion given by $Z(\tilde{x}_i) = \exp(\overline{x}_i)$.

Since the expansion is unique the automorphism groups are trivial in this example, completing the description of the bitorsor for any finitely generated free abelian group.

2.3 Braid Groups

Let B_n be the braid group on n strands. This group can be topologically described in terms of the configuration space $\mathrm{Conf}_n(\mathcal{D}) = \mathbb{D}^n \backslash \Delta$, the configuration space of n points in a disk $\mathbb{D} = \{(x, y) :$

$x^2 + y^2 \leq 1\}$ where Δ is the thick diagonal $\{(z_1, \cdots, z_n) \in \mathbb{D}^n : z_i \neq z_j \text{ for any } i \neq j\}$. The braid group is the fundamental group of the configuration space of n non-distinct points in a disk, that is $B_n = \pi_1 \operatorname{Conf}_n(\mathbb{D})/S_n$ where the symmetric group S_n acts on $\mathbb{D}^n$ by permuting coordinates.

More intuitively, an element of the braid group can be thought of in the following way. Pick any starting position of n points in the disk and place two copies of the disk with these distinguished points directly above each other. An element of the braid group is then a way of attaching the points on the bottom copy of the disk to the top copy with strands of string which may not intersect, which must always travel upwards and such that each point on the bottom is attached to one unique point on the top.

The braid group has a presentation [Artin (1947)]

$$B_n = \langle \sigma_1, \cdots, \sigma_{n-1} : \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i \rangle$$

where $|i - j| \geq 2$.

To compute the associated graded of $A[B_n]$, note that the augmentation ideal is generated by $\sigma_i - 1$. Let $t_i = \overline{\sigma_i - 1} \in I/I^2$. The relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ implies that

$$(\sigma_i - 1)\sigma_{i+1}\sigma_i + (\sigma_{i+1} - 1)\sigma_i + \sigma_i - 1 = (\sigma_{i+1} - 1)\sigma_i\sigma_{i+1} + (\sigma_i - 1)\sigma_{i+1} + \sigma_{i+1} - 1$$

and working modulo I^2, this means $t_i = t_{i+1}$ for each i. It follows that $\widehat{\operatorname{gr}} B_n = \mathbb{Q}[[t]]$.

An expansion is determined by its values on each σ_i, and a similar analysis to the previous example shows that $Z(\sigma_i) = \exp(t)$ for each i. Note that for any braid B,

$$Z(B) = \exp\left(t \sum_{\chi \text{ a crossing}} \operatorname{sign}(\chi)\right)$$

so this invariant computes a total linking number for the braid. While linking numbers are a useful invariant, more powerful invariants will arise by considering braid groups with extra structure.

As in the previous example, the expansion is unique so the associated automorphism groups are trivial in this case as well.

2.4 Pure Braid Groups

The first highly non-trivial example is the pure braid group. This group is the subgroup of the braid group in which all strands must begin and end at the same point they began. It is defined to be

$P_n = \pi_1 \operatorname{Conf}_n(\mathbb{D}) = \pi_1(\mathbb{D}^n \backslash \Delta)$ where, as opposed to the case of the braid group, no quotient by S_n is needed as the n-points are now distinct.

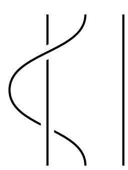
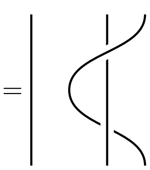
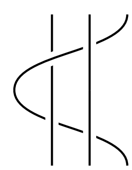
P_n has a presentation [Kassel & Turaev (2008)],

$$\langle \sigma_{ij} : 1 \leq i < j \leq n \rangle / \mathcal{R}$$

where $\mathcal{R}$ is the normal subgroup generated by the relations

$$\sigma_{ij}^{\sigma_{kl}} = \begin{cases} \sigma_{ij} & l < i \text{ or } i < k < l < j \\ \sigma_{ij}^{\sigma_{kj}^{-1}} & l = i \\ \sigma_{ij}^{\sigma_{lj}^{-1}\sigma_{ij}^{-1}} & i = k < l < j \\ \sigma_{ij}^{\sigma_{lj}\sigma_{kj}\sigma_{lj}^{-1}\sigma_{kj}^{-1}} & \end{cases} . \tag{2.1}$$

Here the exponential notation indicates conjugation: $g^h = h^{-1}gh$. Geometrically, the generator σ_{ij} with $i < j$ is the braid in which the jth strand passes underneath all strands up to and including the ith strand, then passes over the ith strand before passing underneath all the intermediate strands and ending in the jth position again. The generators of PB_3 for example are

$\sigma_{01} =$  $\sigma_{12} =$  and $\sigma_{02} =$  .

The augmentation ideal is generated by the elements $\sigma_{ij} - 1$, and the associated graded algebra is generated by t_{ij}, the images of $\sigma_{ij} - 1$ in in I/I^2. The relations in Equation 2.1 lead to the following relations betweeen the generators of the associated graded:

$$\begin{aligned} {[t_{ij}, t_{kl}]} &= 0 & |\{i,j,k,l\}| = 4 \\ {[t_{ij} + t_{ik}, t_{jk}]} &= 0 & |\{i,j,k\}| = 3 \end{aligned} \tag{2.2}$$

.

This algebra is the universal enveloping algebra $U\mathfrak{t}_n$ (in this case over the algebra A) of the Drinfeld-Kohno Lie algebra $\mathfrak{t}_n$, the Lie algebra generated by t_{ij} for $1 \leq i, j \leq n$ subject to the relations in 2.2.

The proof that this is the correct associated graded will follow from the existence of a $U\mathfrak{t}_n$-expansion via Lemma 2.1.1. Constructing an expansion for the pure braid group is highly nontrivial, and all known constructions require transcendental methods rather than purely group theoretic methods. The standard construction involves the Knizhnik-Zamolodchikov connection from conformal field theory.

Consider the $U\mathfrak{t}_n$-valued connection on the configuration space $\mathrm{Conf}_n(\mathbb{D})$ given by

$$A = \frac{1}{2\pi i} \sum_{i<j} d\log(z_i - z_j) t_{ij}$$

where z_i is a coordinate for the ith copy of $\mathbb{D}$, regarded as a subset of $\mathbb{C}$, in $\mathbb{D}^n$. The relations 2.2 imply this is a flat connection. Given any n-strand pure braid, which is a closed loop in $\mathrm{Conf}_n(\mathbb{D})$, define $Z(B) = \mathrm{Mon}_B A$ to be the monodromy of the connection A around B.

More explicitly, every braid can be represented by a smooth embedding of n line segments in $\mathbb{D}$ cross an interval, so let $\gamma : [0,1] \to \mathbb{D}^n \backslash \Delta$ be a smooth parameterization of B. Let $z_i(t)$ be the ith component of γ. Then

$$Z(B) = \sum\!\!\!\!\!\!\!\int_{\substack{n\geq 0 \\ 0<t_1<\ldots<t_n<1 \\ 1\leq i_1<j_1, i_2<j_2, \ldots, i_n<j_n\leq m}} \prod_{\alpha=1}^{n} \frac{t_{i_\alpha j_\alpha}}{2\pi i} d\log(z_{i_\alpha}(t_\alpha) - z_{j_\alpha}(t_\alpha)).$$

It follows from properties of the monodromy of a flat connection together with the relations in $U\mathfrak{t}_n$ that Z is a $U\mathfrak{t}_n$-expansion, and hence that $U\mathfrak{t}_n$ is in fact the correct associated graded algebra for P_n.

While it is expected that the automorphism groups of this structure are nontrivial since there are multiple Drinfel'd associators and each gives an invariant of pure braids (see the next section,) as far as we are aware automorphism groups of the pure braid groups organized with operations in this particular way have not been studied in detail.

2.5 Parenthesized Braids and Drinfel'd Associators

As seen in Section 2.3, considering braids as groups alone does lead to an invariant, although for many knot-theoretic applications this invariant is too weak to be of much use. To obtain a more powerful invariant extra structure needs to be given to these groups. One motivation for where this structure comes from is as follows.

Due to the fundamental relationship between braid groups and braided monoidal categories[6], it is natural to attempt to rephrase the theory of Drinfel'd associators, first developed as a way of putting not necessarily strict monoidal structures on categories of representations of algebras, purely in terms

[6]The operad formed out of all the braid groups is the operad in groups, regarded as categories with one element and invertible morphisms, whose algebras are strict monoidal categories while the operad formed out of the parenthesized braid groupoids is the operad in groupoids whose algebras are general monoidal categories.

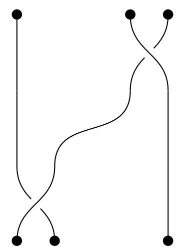


Figure 2.5.1: A parenthesized braid. This is a morphism in the groupoid $\mathbf{PaB}(3)$ from the object $(\bullet\bullet)\bullet$ to the object $\bullet(\bullet\bullet)$.

of parenthesized braid groups. Strictly speaking braid groups are related to strict monoidal categories and in order to extend to the case of a non-trivial associator, parenthesized (or non-associative) braids must be introduced. The original description of Drinfel'd associators purely in the language of braids and without reference to quasi-trangular quasi-Hopf algebras was developed in [Bar-Natan (1998)] (see also [Le & Murakami (1995)] and [Bar-Natan (1997)] for earlier related work using tangles rather than braids.) For a comprehensive treatment using the language of operads adopted here refer to the textbook account [Fresse (2017a)]. The material in this section is primarily a reformulation of the fundamental definitions and results in [Bar-Natan (1998)] which is where the proofs of results in this section not explicitly referenced elsewhere can be found.

Elements of the parenthesized braid groupoid on n-strands $\mathbf{PaB}(n)$ are usual n-strand braid diagrams but where distance between endpoints is nontrivial, as in Figure 2.5.1, and where composition is given by stacking from bottom to top. More precisely, the objects of $\mathbf{PaB}(n)$ are nonassociative words on the single letter alphabet $\{\bullet\}$. Parenthesizations of a nonassociative word in $\bullet$ will generally be indicated by horizontal distance rather than by including parentheses. For example the parenthesized word $((\bullet\bullet)\bullet)\bullet)$ can be written $\bullet\bullet\quad\bullet\qquad\qquad\bullet$.

A morphism in $\mathbf{PaB}(n)$ from a word w_1 to a word w_2, where both w_1 and w_2 have exactly n letters, is a braid diagram with n strands where each of the strands begins on a unique $\bullet$ in w_1 and ends on a unique $\bullet$ in w_2. Morphisms are composed by stacking only when the parenthesization matches which is why the $\mathbf{PaB}(n)$ form groupoids rather than groups.

The collection of all groupoids $\mathbf{PaB}(n)$ can be given the structure of an operad in groupoids by defining the partial composition operations via the gluing of one parenthesized braid into one of the strands of another parenthesized braid, an example of which is shown in Figure 2.5.2.

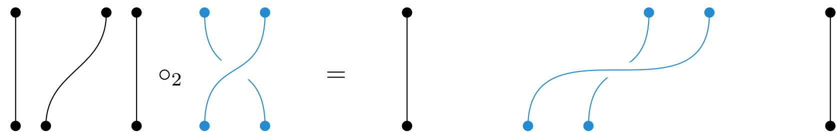
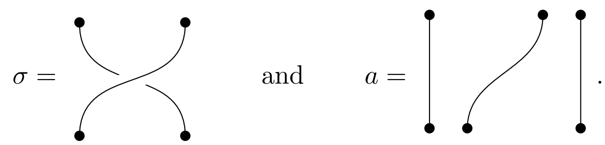


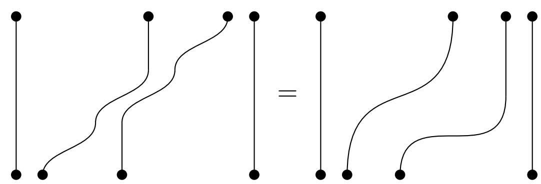
Figure 2.5.2: Partial composition operations in **PaB**. For two parenthesized braids P_1 and P_2, $P_1 \circ_i P_2$ is obtained by shrinking P_2 down so that the strands in P_2 are closer together than any two strands in P_1 and then replacing the ith strand of P_1 with P_2.

A key observation is that as an operad **PaB** is generated by only two parenthesized braids,



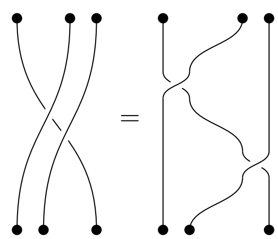
While there are several so-called "locality" relations expressing certain obvious commutativity results between these generators, there are three key relations which contain the essence of the structure, the

Pentagon Relation:

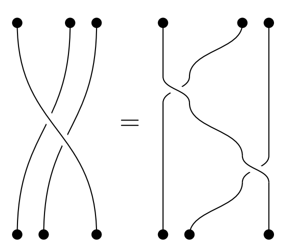


and the two hexagon relations

Hexagon Relation A:



Hexagon Relation B:

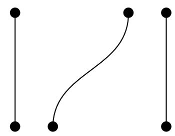
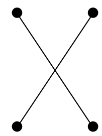
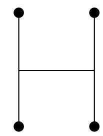


Viewing the operad **PaB** as an algebraic structure as in Example 2.1.3 [7], we can apply the procedure

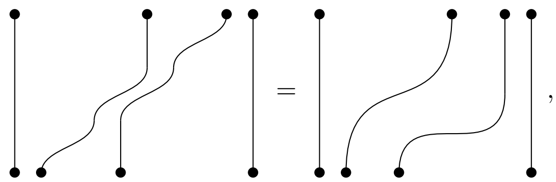
[7]There is a technicality here, since when constructing $A[\mathbf{PaB}]$ we only want to take formal linear combinations of parenthesized braids with the same underlying permutation. This can be remedied by replacing $\mathbf{PaB}(n)$ by $\mathbf{PaB}(\sigma)$ where $\mathbf{PaB}(\sigma)$ is the collection of parenthesized braid diagrams with the same underlying permutation $\sigma \in S_n$. This new

described in Chapter 2 to construct a pro-(ℓ) version $\mathbf{PaB}^{(\ell)}$, a unipotent completion $\widehat{\mathbf{PaB}}$ and an assocated graded $\widehat{\mathrm{gr}}\widehat{\mathbf{PaB}}$ of $\mathbf{PaB}$.

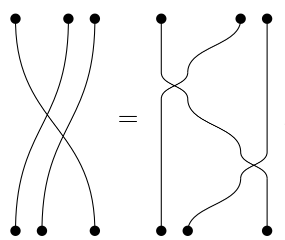
The final result of this procedure is the operad $\widehat{\mathrm{gr}}\widehat{\mathbf{PaB}} = \widehat{\mathbf{PaCD}}$ of parenthesized chord diagrams, an operad in complete Hopf algebras. This operad also has an easy to describe finite presentation. It is generated by the formal symbols

$a =$  $\mathsf{X} =$  and $\mathsf{H} =$ 

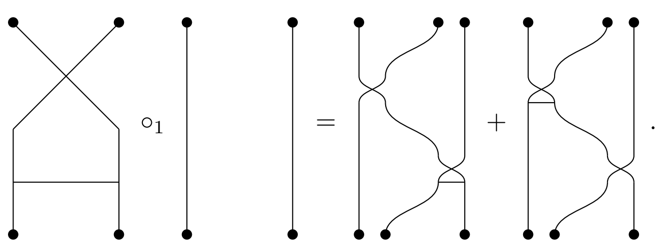
where X is its own inverse and commutes with H, subject to the usual locality relations, omitted here, and three important relations, the **pentagon relation**:



the **classical hexagon relation**[8]:



and the **semiclassical hegaxon relation**:



Applying the formalism in Section 2.1.6 we can construct a bitorsor of isomorphisms from $\widehat{\mathbf{PaB}} \to$

algebraic structure is technically no longer an operad however this example now falls exactly into the notion of an algebraic structure.

[8]Note that these diagrams are now purely combinatorial, there are no over or under crossings in **PaCD**. As a result there is only one hexagon in **PaCD**.

$\widehat{\mathbf{PaCD}}$.

Definition 2.5.1. A *Drinfel'd associator* is an operad morphism $Z : \widehat{\mathbf{PaB}} \to \widehat{\mathbf{PaCD}}$ such that $Z(\sigma) = \exp\left(\frac{1}{2}\mathsf{H}\right)$ [9]. The collection of all associators is denoted by **ASSOC**.

Definition 2.5.2. The (prounipotent) *Grothendieck-Teichmüller group* **GT** is the group of operad automorphisms $\widehat{\mathbf{PaB}} \to \widehat{\mathbf{PaB}}$ which fix σ.

Definition 2.5.3. The (prounipotent) *graded Grothendieck-Teichmüller group* **GRT** is the group of operad automorphisms $\widehat{\mathbf{PaCD}} \to \widehat{\mathbf{PaCD}}$ which fix H.

To reproduce the more usual definitions appearing in much of the literature, in particular the original definitions in [Drinfel'd (1991)], notice that an associator $Z : \widehat{\mathbf{PaB}} \to \widehat{\mathbf{PaCD}}$ is determined by the image of σ and a. $R = Z(\sigma)$ will be an invertible element of the completed algebra of chord diagrams on two strands, which is the Drinfel'd Kohno algebra $\widehat{U\mathfrak{t}_2}$. Furthermore since Z preserves coproducts, R must be grouplike and it is easy to see that the only grouplike elements of $\widehat{U\mathfrak{t}_2}$ are elements of the form $\exp(k\mathsf{H})$ for some $k \in A^\times$. We take $k = \frac{1}{2}$ in order to satisfy the expansion condition which uniquely defines $R = \exp\left(\frac{1}{2}\mathsf{H}\right)$.

Next, we need to consider $\Phi = Z(a)$. This will be an element of the algebra $\widehat{U\mathfrak{t}_3}$ of chord diagrams on 3 strands which is grouplike and invertible. In order to get a well-defined map $Z : \widehat{\mathbf{PaB}} \to \widehat{\mathbf{PaCD}}$, the pentagon and hexagon relations in the presentation of **PaB** must be satisfied. The pentagon relation leads to the equation

$$\Phi^{1,2,3}\Phi^{1,23,4}\Phi^{2,3,4} = \Phi^{12,3,4}\Phi^{1,2,34}$$

and the two hexagon relations yield the equations

$$R^{\pm 12,3} = \Phi R^{\pm 2,3}\Phi^{-1,3,2}R^{\pm 1,3}\Phi^{3,1,2}.$$

Definition 2.5.4. A *Drinfel'd associator* is an element $\Phi \in \widehat{U\mathfrak{t}_3}$[10], where Φ is grouplike, invertible, and

[9] This condition is required to ensure Z is an expansion. In many treatments of associators, it is not required that Z be an expansion and the equations given only imply $Z(\sigma) = \exp(k\mathsf{H})$ for some $k \in A^\times$ so an associator as defined in [Drinfel'd (1991)] is an element of $A^\times \times \widehat{U\mathfrak{t}_3}$.

[10] In [Drinfel'd (1991)], Φ is an element of the free Lie algebra on two generators rather than the Drinfel'd Kohno algebra. This is equivalent to our definition since $\widehat{U\mathfrak{t}_3} \simeq \widehat{FL}[x,y] \oplus \mathcal{Z}$ where $\mathcal{Z}$ is the one dimensional centre of $U\mathfrak{t}_3$. In fact $\widehat{U\mathfrak{t}_3}$ is the correct space in which associators exist since the defining equations must be expressed in it rather than the free Lie algebra.

which satisfies the equations

$$\Phi^{1,2,3}\Phi^{1,23,4}\Phi^{2,3,4} = \Phi^{12,3,4}\Phi^{1,2,34} \qquad \text{(Pentagon Equation)}$$
$$R^{\pm 12,3} = \Phi R^{\pm 2,3}\Phi^{-1,3,2}R^{\pm 1,3}\Phi^{3,1,2} \qquad \text{(Hexagon Equations)}$$

for $R = \exp\left(\frac{1}{2}t_{12}\right)$. The set of all Drinfel'd associators is denoted **ASSOC**.

A similar analysis of operad morphisms $\widehat{\mathbf{PaB}} \to \widehat{\mathbf{PaB}}$ and $\widehat{\mathbf{PaCD}} \to \widehat{\mathbf{PaCD}}$ in terms of the generators and relations of these operads as well as the composition of such morphisms leads to the following explicit descriptions of the groups **GT** and **GRT** [Bar-Natan (1998)].

Definition 2.5.5. The prounipotent group **GT** is the set of all grouplike, nondegenerate, invertible elements $\Sigma \in \widehat{\mathrm{PB}_3}$ satisfying the equations

$$d_4\Sigma \cdot d_2\Sigma \cdot d_0\Sigma = d_1\Sigma \cdot d_3\Sigma$$
$$\sigma_2\sigma_1 = \Sigma\sigma_2\Sigma^{-1}\sigma_1\Sigma$$

with group law given by

$$\Sigma_1 \times \Sigma_2 = \Sigma_1 \cdot \left(\Sigma_2|_{\sigma_1\to\Sigma^{-1}\sigma_1\Sigma,\sigma_2\to\sigma_2}\right).$$

Here, σ_1 and σ_2 are the standard generators of the braid group B_3. There is a right action of **GT** an **ASSOC** via

$$\Phi \bullet \Sigma = \Phi \cdot \Sigma|_{\sigma_1\to\Phi^{-1}\exp(t_{12}/2)X_1\cdot\Phi,\sigma_2\to\exp(t_{23}/2)X_2}$$

where this formula is defined in $\widehat{U\mathfrak{t}_3} \rtimes S_3$, $X_1 = (12)$ and $X_2 = (23)$. This action is free and transitive.

Definition 2.5.6. The prounipotent group **GRT** is the set of all grouplike, invertible elements $\Gamma \in \widehat{U\mathfrak{t}_3}$ satisfying the equations

$$\Gamma^{1,2,3}\Gamma^{1,23,4}\Gamma^{2,3,4} = \Gamma^{12,3,4}\Gamma^{1,2,34}$$
$$1 = \Gamma \cdot \Gamma^{-1,3,2}\Gamma^{3,1,2}$$
$$(t_{12})^{12,3} = \Gamma\left(t_{23}(\Gamma^{-1})^{1,3,2} + (\Gamma^{-1})^{1,3,2}t_{13}\right)\Gamma^{3,1,2}$$

with product defined by

$$\Gamma_1 \times \Gamma_2 = \Gamma_1 \cdot \Gamma_2\big|_{t_{12}\to\Gamma_1^{-1}t_{12}\Gamma_1, t_{13}\to((\Gamma_1)^{-1})^{1,3,2}t_{13}\Gamma_1^{1,3,2}, t_{23}\to t_{23}}$$

where t_{12}, t_{23} and t_{13} are the generators for the Drinfel'd-Kohno algebra $\mathfrak{t}_3$. There is a left action of **GRT** on **ASSOC** given by

$$\Gamma \bullet \Phi = \Gamma \cdot \Phi\Big|_{t_{12}\to\Gamma_1^{-1}\cdot t_{12}\cdot\Gamma_1, t_{13}\to(\Gamma_1^{-1})^{1,3,2}\cdot t_{13}\cdot\Gamma_1^{1,3,2}, t_{23}\to t_{23}}$$

which is free and transitive.

This completes the construction of the $\mathbf{GT}-\mathbf{GRT}$ bitorsor **ASSOC**. There are two key theorems in the theory of associators that are especially important for this thesis. The first theorem is due to Drinfel'd [Drinfel'd (1991)], with a knot theoretic proof given in [Bar-Natan (1998)] and says that associators can be constructed degree by degree. Since an associator $\Phi \in \widehat{U\mathfrak{t}_3}$ is an element of a graded algebra, it decomposes as $\Phi = \sum_{i=0}^{\infty} B_i$ where B_i is homogeneous of degree i.

Theorem 2.5.1. *Let $\Phi \in \widehat{U\mathfrak{t}_3}$ be grouplike and invertible and suppose that Φ satisfies the pentagon and hexagon relations to degree m, meaning that the left and right hand side of the relevant equations are equal in degrees zero through m but may differ in higher degrees. Then there is a $\varphi \in \widehat{U\mathfrak{t}_3}$ which is homogenous of degree $m+1$ and such that $\Phi + \varphi$ satisfies the pentagon and hexagon equations to degree $m+1$.*

An important corollary of this theorem is that rational associators exist. The initial construction of a Drinfel'd associator requires the use of transcendental techniques via the computation of the holonomy of a connection and is a priori only valid over $\mathbb{C}$, however it can be shown (see [Drinfel'd (1991); Bar-Natan (1998)]) via a degree by degree construction justified by this theorem that a rational associator exists, given the fact that it is known some associator exists[11].

The second major theorem is due to Furusho [Furusho (2010)], also proved in a different way by Bar-Natan and Dansco [Bar-Natan & Dancso (2012)], and simplifies the definition of an associator by eliminating two of the equations.

Theorem 2.5.2. *Let $\Phi \in \widehat{U\mathfrak{t}_3}$ be grouplike and invertible and suppose that to degree two, Φ is given by $\frac{1}{24}t_{13}t_{23}$ plus higher order terms. If Φ satisfies the pentagon equations then Φ satisfies the two hexagon equations.*

[11] It is not however possible to prove rational associators exists without using transcendental techniques since the degree by degree proof does not work if it is not known that some associator exists first.

Chapter 3

The Algebraic Structure of Annular Braids

3.1 Annular Braid Groups

The primary topological objects we will study are braids in an annulus (for space) cross an interval (for time.)

3.1.1 Annular Braid Groups

Let $\mathbb{A} = \left\{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\right\}$ be an annulus in $\mathbb{R}^2$ and let $\mathrm{Conf}_n(\mathbb{A}) = \mathbb{A}^n \backslash \Delta$ be the configuration space of n distinct points in the annulus, where $\Delta = \{(x_1, \cdots, x_n) \in \mathbb{A}^n : x_i = x_j \text{ for some } i \neq j\}$ is the large diagonal. The symmetric group S_n acts freely on $\mathrm{Conf}_n(\mathbb{A})$ by permuting coordinates. All fundamental groups will use the basepoint in $\mathrm{Conf}_n(\mathbb{A})$ in which n-points $(x_1, \cdots, x_n) \in \mathbb{A}^n$ are evenly spaced along the portion of the x-axis to the right of the core of the annulus in numeric order from left to right as depicted in Figure 3.1.1

Definition 3.1.1. The *annular braid group on n strands* is the fundamental group

$$B_{1,n} = \pi_1 \left(\mathrm{Conf}_n(\mathbb{A})/S_n\right)$$

and the *pure annular braid group on n strands* is the fundamental group

$$\mathrm{PB}_{1,n} = \pi_1 \left(\mathrm{Conf}_n(\mathbb{A})\right).$$

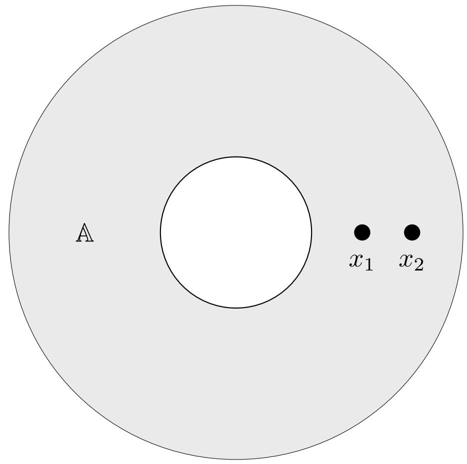


Figure 3.1.1: The basepoint for $\mathrm{Conf}_n(\mathbb{A})$ is n evenly spaced points lying on the portion of the x-axis to the right of the core of the annulus as indicated for $\mathrm{Conf}_2(\mathbb{A})$.

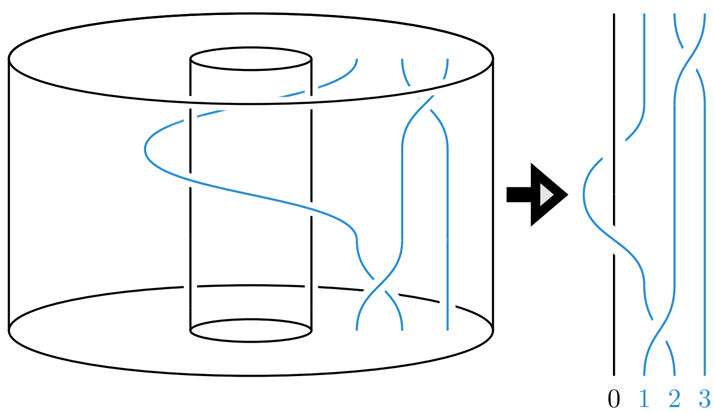


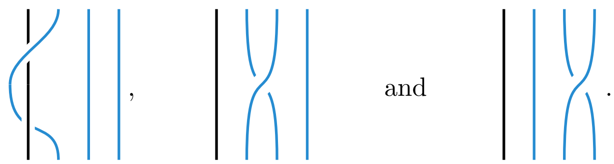
Figure 3.1.2: An annular braid with 3 strands and underlying permutation (132) as well as its representation as a usual 4 strand braid with distinguished 0th strand remaining fixed.

Geometrically, an annular braid is an embedding of n strands into the solid torus $\mathbb{A} \times [0, 1]$ in which all strands begin at the basepoint in the lower annulus $\mathbb{A} \times \{0\}$, end at the basepoint in the upper annulus $\mathbb{A} \times \{1\}$, and such that, viewing a strand as starting at the bottom and ending at the top, the vertical component of motion is always upwards. As shown in Figure 3.1.1, annular braids can be represented as usual braid diagrams in which the zeroth strand plays the role of the core of the annulus and so remains fixed throughout. The group multiplication corresponds to stacking diagrams, where we choose the convention in which B_1B_2 is the annular braid with B_2 stacked on top of B_1, so diagrams are read from bottom to top.

The group $B_{1,n}$ has a presentation [Lambropoulou (2000); Bellingeri (2003)]

$$B_{1,n} = \left\langle \tau, \sigma_1, \cdots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \\ [\sigma_i,\sigma_j]=1 \quad |i-j|>0, \\ [\tau,\sigma_1\tau\sigma_1]=1, \\ [\tau,\sigma_i]=1 \quad i>1 \end{array} \right\rangle \tag{3.1}$$

where τ represents the braid in which the first strand wraps around the core of the annulus once and σ_i represents the braid in which strand i crosses above strand $i+1$. For example, the generators of $B_{1,3}$ are



τ σ_1 σ_2

Since $\mathrm{Conf}_n(\mathbb{A})$ is homotopy equivalent to $\mathrm{Conf}_{n+1}(\mathbb{D})$[1], $\mathrm{PB}_{1,n}$ is the usual pure braid group on $n+1$ strands PB_{n+1}. In particular there is a split short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & F_n \arrow[r] & \mathrm{PB}_{1,n} \arrow[r, "\sigma"] & \mathrm{PB}_{1,n-1} \arrow[l, bend left] \arrow[r] & 1, \end{tikzcd}$$

where F_{n-1} is the free group on $n-1$ generators and the splitting embeds an n strand braid into an $n+1$ strand braid by adding a constant strand on the right. Iterating this result implies that $\mathrm{PB}_{1,n}$ is the iterated semidirect product

$$\mathrm{PB}_{1,n} = ((F_1 \rtimes F_2) \rtimes \cdots) \rtimes F_n. \tag{3.2}$$

Let $S_{1,n}$ be the subgroup of permutations of the set $\{0, \cdots, n\}$ which send 0 to itself. There is another short exact sequence

$$1 \longrightarrow \mathrm{PB}_{1,n} \longrightarrow B_{1,n} \xrightarrow{\varsigma} S_{1,n} \longrightarrow 1 \tag{3.3}$$

where the map ς, called the skeleton map, sends a braid to the permutation obtained by mapping the integer i to the end position of the i^{th} strand in the braid. For example, if B is the annular braid drawn in Figure 3.1.1 then $\varsigma(B) = (132)$.

[1]This configuration space is defined in the same way as $\mathrm{Conf}_n(\mathbb{A})$ with $\mathbb{A}$ replaced by $\mathbb{D}$

3.1.2 The Strict Monoidal Category of Annular Braids

The motivation for our construction of braidors comes from the construction of a Drinfel'd associator viewed from a braid-theoretic perspective as in [Bar-Natan (1998)], which was reviewed briefly in Section 2.5. Braidors are obtained by replacing braids in the disc by braids in an annulus, but importantly fewer operations are included in the algebraic structure of annular braids. A key point is that while each individual parenthesized braid groupoid $\mathbf{PaB}(n)$ is finitely presented, the entire structure of all braid groups is no longer finitely generated if the only permissible operation is the usual group multiplication. To remedy this, more operations are added to the structure in such a way that it is presented by only two generators and a small number of relations. The required operations to achieve this are the operadic composition operations and the resulting collection of all such groupoids together with these extra operations can be organized into the somewhat complicated structure of an operad in groupoids.

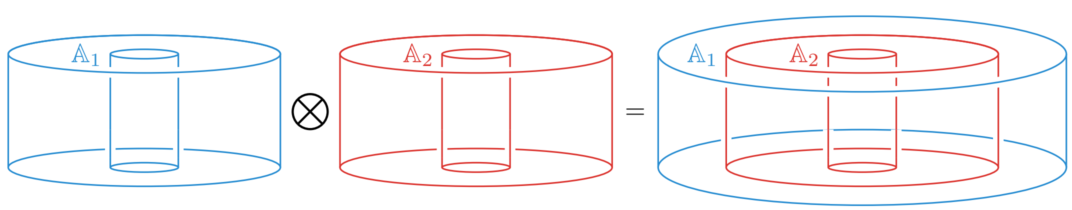
In the annular case a similar problem arises however it is possible to generate the entire structure with just a single operation, a tensor product which corresponds to doubling the core, in addition to the group multiplication rather than requiring n-doubling operations, one for each strand in an n-strand braid. This section is devoted to defining this structure in the annular case.

Definition 3.1.2. The category $\boldsymbol{B_a}$ of *braids in the annulus* has objects $\mathrm{Obj}_{\boldsymbol{B_a}} = \bigsqcup_n S_{1,n}$ comprising all permutations which fix 0 and given permutations P and Q in $S_{1,n}$ has morphism set

$$\mathrm{Mor}_{\boldsymbol{B_a}}(Q, QP) = B_{1,n}^P$$

where $B_{1,n}^P$ is the set of braids with underlying permutation P. Composition of morphisms is given by stacking braid diagrams on top of one another [2].

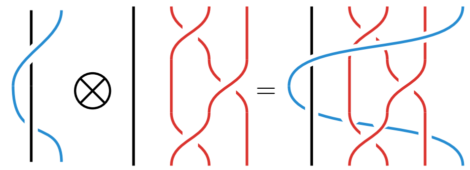
The monoidal structure on $\boldsymbol{B_a}$ is obtained by gluing one annulus into the core of another one. More precisely, $(\mathbb{A}_1 \times [0,1]) \otimes (\mathbb{A}_2 \times [0,1])$ is the annulus obtained by scaling the outer radius of A_2 to be equal to the inner radius of A_1 and then gluing A_2 into the core of A_1 as shown in the figure



This operation on annuli induces an operation on permutations and on braids in $\mathbb{A} \otimes [0,1]$ which

[2] In other words by group multiplication in the group of annular braids.

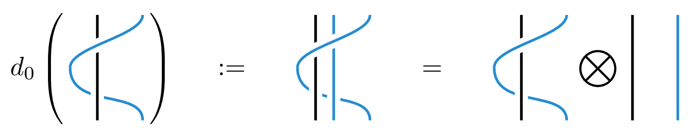
defines the monoidal structure on $\boldsymbol{B_a}$[3]. An example of this operation using representatives of annular braids as usual braid diagrams is



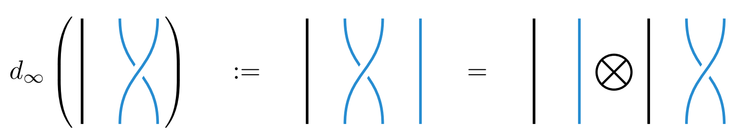
Given permutations P and Q, the monoidal product $P \otimes Q$ can be determined by taking any braid diagram with underlying permutations P and Q, computing their monoidal product and then $P \otimes Q$ is the underlying permutation $\varsigma(P \otimes Q)$ of this braid. Explicitly, if $P : \sigma : \{0, \cdots m\} \to \{0, \cdots, m\}$ and $Q : \sigma : \{0, \cdots n\} \to \{0, \cdots, n\}$, then $P \otimes Q : \{0, \cdots, m+n\} \to \{0, \cdots, m+n\}$ is defined by the formula

$$P \otimes Q(i) = \begin{cases} P(i-n)+n & i > n \\ Q(i) & i \leq n \end{cases}.$$

It will be useful to introduce notations for some auxilliary operations which, although special cases of the monoidal product, will later be convenient. The zeroth doubling operation d_0 doubles the core and regards the second copy as a strand:



The strand addition operation d_∞ adds a new strand to the right of all other strands:

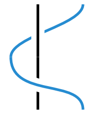
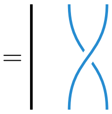
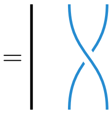


In general, if A is any annular braid diagram, then $d_0(A) = A \otimes | \; |$ and $d_\infty(A) = | \; | \otimes A$.

The category $\boldsymbol{B_a}$ together with this monoidal product forms a strict monoidal category, with unit object being the empty permutation in $S_{1,0}$. Braidors will be defined as monoidal functors out of $\boldsymbol{B_a}$ so a succinct description of $\boldsymbol{B_a}$, given in the next claim, allows simple descriptions of such functors which will be needed later.

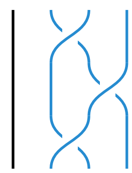
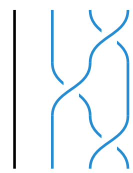
[3]Technically speaking the glued annulus must be shrunk down to the standard annulus $\mathbb{A}$ and the ends of the strands will need to be moved without any crossings to the choice of basepoint after gluing.

Lemma 3.1.1. *$\boldsymbol{B_a}$ is generated* [4] *as a strict monoidal category by $\tau^{\pm 1}$ and $\sigma^{\pm 1}$, where*

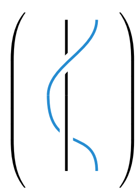
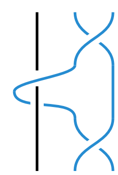
$\tau =$  , $\tau^{-1} =$  , $\sigma =$  and $\sigma^{-1} =$  ,

subject to the relations generated by

- *Braid relation: $d_\infty(\sigma)d_0(\sigma)d_\infty(\sigma) = d_0(\sigma)d_\infty(\sigma)d_0(\sigma)$ or pictorially*

 = 

- *Mixed relation: $d_0(\tau) = \sigma d_\infty(\tau)\sigma$ or pictorially*

d_0  = 

where composition of generators corresponds to stacking the corresponding diagrams.

Proof. Let $\mathcal{C}$ be the category generated as in the claim. Let $\Phi : \mathcal{C} \to \boldsymbol{B_a}$ be the monoidal functor which is the identity on objects and sends the formal generators $\tau^{\pm}$ and $\sigma^{\pm}$ to the braids indicated by the diagrams in the statement of the lemma in $\boldsymbol{B_a}$. To prove the claim requires showing that Φ is full and faithful, ie. that it induces bijections when restricted to any morphism set in $\mathcal{C}$, that it is essentially surjective, ie. that every object in $\boldsymbol{B_a}$ is isomorphic in $\boldsymbol{B_a}$ to an object in the image of Φ, and that the functor is (strong) monoidal, ie. that it preserves the tensor product structure.

Φ is well-defined since all the relations in the claim hold in the annular braid groups and it is full since the generators of any annular braid group can be constructed out of τ and σ by applying the tensor product iteratively. More precisely, the generator τ regarded as an element of the braid group $B_{1,n}$ is $\Phi(d_\infty^n(\tau))$ and the generator σ_i in $B_{1,n}$ is $\Phi(d_\infty^{n-i-1} \circ d_0^i(\sigma))$. Φ is essentially surjective since it is the identity on objects. It is also monoidal as it is defined on generators with respect to the monoidal structure and extended to tensor products in a monoidal way by construction.

[4] By the category generated in this context is meant the category with objects $\mathrm{Obj}_{\boldsymbol{B_a}} = \bigsqcup_n S_{1,n}$ and morphisms obtained by starting with the generators given together with all identity morphisms and applying all possible compositions and monoidal products iteratively

In order for Φ to be a monoidal equivalence of categories, it remains to show that Φ is faithful, which will follow from showing any relation in the presentation of $B_{1,n}$ given in Equation 3.1 is implied by the relations in the claim.

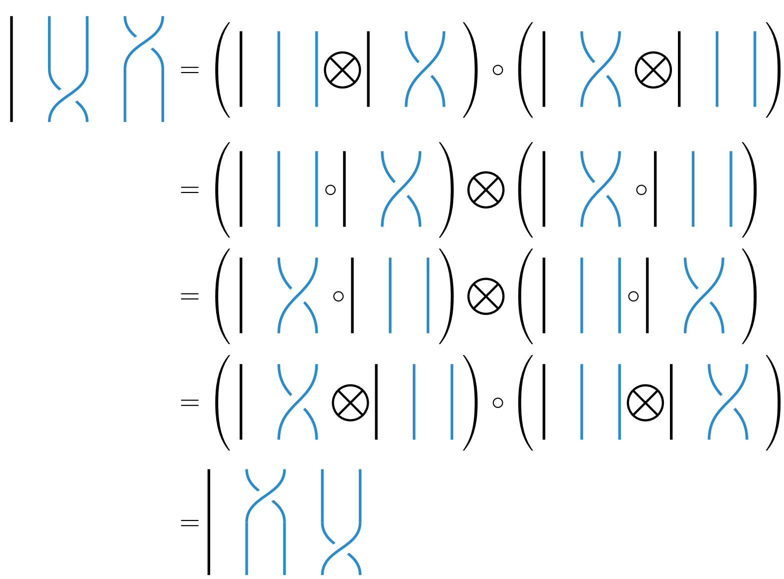
The relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ in $B_{1,3}$ is exactly the braid relation in $\mathcal{C}$, and the general case $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ in $B_{1,m}$ follows by first applying d_0^{i-1} and then d_∞^{m-i-2} to the braid relation.

There are two classes of relations in the annular braid groups, $[\sigma_i, \sigma_j] = 1$ for $|i-j| > 0$ and $[\tau, \sigma_i] = 0$ for $i > 1$ which are analogues of the so called "locality" relations appearing in [Bar-Natan (1998)] for the case of parenthesized braids. These relations are automatically accounted for in the structure of a monoidal category.

For example, since the tensor product must be a functor $\otimes : \boldsymbol{B_a} \times \boldsymbol{B_a} \to \boldsymbol{B_a}$,

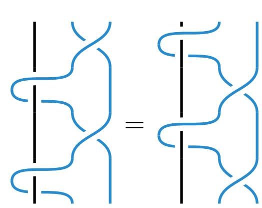
$$(B_1 \circ B_2) \otimes (C_1 \circ C_2) = (B_1 \otimes C_1) \circ (B_2 \otimes C_2).$$

In the diagrammatic representation,



which is the relation $[\sigma_1, \sigma_3] = 0$. All other locality relations can be demonstrated similarly.

Finally, the relation $[\tau, \sigma_1\tau\sigma_1^{-1}] = 1$ in $B_{1,2}$, drawn diagramatically, is



which, using the mixed relation, becomes $[d_\infty \tau, d_0 \tau] = 1$. Using the definition of d_0 and d_∞ in terms of the monoidal product and a similar argument as in the case of locality relations above, this relation is obtained as a monoidal relation as well and so holds automatically. To generalize to a relation in any $B_{1,m}$, apply d_∞^{m-2}.

We have shown Φ is a monoidal equivalence of categories, and in fact it is the identity on objects and is a set theoretic isomorphism on all morphism sets. $\square$

A monoidal category is an example of an algebraic structure. Example 2.1.2 in Chapter 2 shows how to view a category as an algebraic structure and to extend this to monoidal categories merely requires the addition of the monoidal products to the operations in the algebraic structure and the addition of the associators and unitors as constants in the algebraic structure.

The next step in the construction of braidors is to apply the procedure in Chapter 2 to $\boldsymbol{B_a}$, regarded as an algebraic structure $\mathcal{B}_a$, to obtain a prounipotent completion, and an associated graded. We will not carefully distinguish between the monoidal category $\boldsymbol{B_a}$ and the algebraic structure $\mathcal{B}_a$. The results of this procedure are described in the next few sections.

3.1.3 $\boldsymbol{B_a}$-algebras and Augmentation Ideals

The types of the algebraic structure $\boldsymbol{B_a}$ are morphism sets $\mathrm{Mor}_{\boldsymbol{B_a}}(Q, QP) = B^P_{1,n}$, so to form $A[\boldsymbol{B_a}]$ formal linear combinations of annular braids with the same underlying permutation are taken.

Definition 3.1.3. $A[\boldsymbol{B_a}]$ is the category with $\mathrm{Obj}_{A[\boldsymbol{B_a}]} = \bigsqcup_n S_{1,n}$ and for permutations P and Q in $S_{1,n}$

$$\mathrm{Mor}_{A[\boldsymbol{B_a}]}(Q, QP) = A[B^P_{1,n}].$$

Definition 3.1.4. The *augmentation ideal* I is the subcategory of $A[\boldsymbol{B_a}]$ with morphisms given by formal linear combinations $\sum_{i=1}^m a_i B_i \in A[B^P_{1,n}]$, for some fixed $P \in S_{1,n}$, which satisfy $\sum_{i=1}^m a_i = 0$. Denote by I_P those morphisms in $A[B^P_{1,n}]$ which are in I.

Definition 3.1.5. The *unipotent filtration* $\boldsymbol{B_a} \supset \mathcal{F}_1\boldsymbol{B_a} \supset \mathcal{F}_2\boldsymbol{B_a} \supset \cdots$ of the category $\boldsymbol{B_a}$ is the sequence of subcategories $\mathcal{F}_\ell\boldsymbol{B_a}$ of $\boldsymbol{B_a}$ with permutations fixing 0 as objects and with $\mathrm{Mor}_{\mathcal{F}_\ell\boldsymbol{B_a}}(Q, QP) = I^\ell_P$ where I^ℓ_P consists of all compositions with ℓ morphisms in I_R for any permutation R and an arbitrary number of other composable morphisms such that the total composition has underlying permutation P.

3.1.4 Coproduct

The tensor product of the category $\boldsymbol{B_a}$ with itself is denoted by $\boxtimes$ to avoid confusion with the monoidal structure $\otimes$.

Definition 3.1.6. $A[\boldsymbol{B_a}]\boxtimes A[\boldsymbol{B_a}]$ is the monoidal category with objects $P\boxtimes P$ for permutations $P \in S_{1,n}$ and morphisms

$$\mathrm{Mor}_{\boldsymbol{B_a}\boxtimes\boldsymbol{B_a}}(Q \boxtimes Q, QP \boxtimes QP) = A[B_{1,n}^P] \otimes_A A[B_{1,n}^P].$$

Definition 3.1.7. The coproduct $\square : \boldsymbol{B_a} \to \boldsymbol{B_a}\boxtimes\boldsymbol{B_a}$ is the monoidal functor which sends a permutation P to $P \boxtimes P$ and makes any braid $B \in B_{1,n}$ grouplike, that is $\square(B) = B \otimes B$.

3.1.5 Unipotent Completion

Definition 3.1.8. The *ℓth unipotent quotient* ${\boldsymbol{B_a}}^{(\ell)}$ of the category $\boldsymbol{B_a}$ has permutations fixing 0 as objects and has morphisms

$$\mathrm{Mor}_{{\boldsymbol{B_a}}^{(\ell)}}(Q, QP) = A\left[B_{1,n}^P\right]/I_P^{(\ell+1)}.$$

Definition 3.1.9. The *prounipotent completion* $\widehat{\boldsymbol{B_a}}$ of the category $\boldsymbol{B_a}$ has permutations fixing 0 as objects and has morphisms

$$\mathrm{Mor}_{\widehat{\boldsymbol{B_a}}}(Q, QP) = \varprojlim \left(A\left[B_{1,n}^P\right]/I_P^{(\ell+1)}\right).$$

All of the structure which was added to $\boldsymbol{B_a}$ is inherited by $\widehat{\boldsymbol{B_a}}$ and ${\boldsymbol{B_a}}^{(\ell)}$ as shown in general in Chapter 2 so these are both strict monoidal categories with a coproduct functor and unipotent filtrations $\mathcal{F}_\bullet\widehat{\boldsymbol{B_a}}$ and $\mathcal{F}_\bullet{\boldsymbol{B_a}}^{(\ell)}$.

3.2 The Category of Annular Chord Diagrams

Next the associated graded construction can be applied to the algebraic structure $\boldsymbol{B_a}$ to obtain the second important category, the category of chord diagrams for annular braids.

Definition 3.2.1. $\mathrm{gr}\,{\boldsymbol{B_a}}^{(\ell)}$ is the monoidal category with $\mathrm{Obj}_{\mathrm{gr}\,{\boldsymbol{B_a}}^{(\ell)}} = \bigsqcup_{n\geq 0} S_{1,n}$ and with

$$\mathrm{Mor}_{\mathrm{gr}\,{\boldsymbol{B_a}}^{(\ell)}}(Q, QP) = \prod_{m=0}^{\ell} I_P^m/I_P^{m+1}.$$

Definition 3.2.2. The completed associated graded cateogry $\widehat{\mathrm{gr}}\widehat{\boldsymbol{B_a}}$ of $\widehat{\boldsymbol{B_a}}$ has $\mathrm{Obj}_{\widehat{\mathrm{gr}}\widehat{\boldsymbol{B_a}}} = \bigsqcup_{n\geq 0} S_{1,n}$

and has
$$\mathrm{Mor}_{\widehat{\mathrm{gr}}\widehat{\boldsymbol{B}_{\mathbf{a}}}}(Q,QP)=\prod_{m=0}^{\infty} I_P^m/I_P^{m+1}.$$

Following the procedure detailed in Section 2.1.5, to determine $\widehat{\mathrm{gr}}\widehat{\boldsymbol{B}_{\mathbf{a}}}$ we will guess a candidate graded category $\widehat{\mathbf{CD}_{\mathbf{a}}}$ and prove it correct by showing a $\widehat{\mathbf{CD}_{\mathbf{a}}}$-expansion $Z:\widehat{\boldsymbol{B}_{\mathbf{a}}}\to\widehat{\mathbf{CD}_{\mathbf{a}}}$ exists.

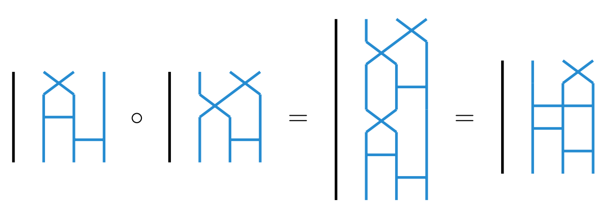
Definition 3.2.3. $\widehat{\mathbf{CD}_{\mathbf{a}}}$, the category of *chord diagrams for annular braids*, has $\mathrm{Obj}_{\widehat{\mathbf{CD}_{\mathbf{a}}}}=\bigsqcup_{n\geq 0} S_{1,n}$ and
$$\mathrm{Mor}_{\widehat{\mathbf{CD}_{\mathbf{a}}}}(Q,QP)=\widehat{U\mathfrak{t}_{1,n}}\cdot P$$

where the notation $\widehat{U\mathfrak{t}_{1,n}}\cdot P$ denotes all formal products $D\cdot P$ with $P\in S_{1,n}$ and $D\in\widehat{U\mathfrak{t}_{1,n}}$ an (infinite) A-linear combination of elements of the Drinfel'd-Kohno algebra. The composition law in this category is given by $(X\cdot P)\circ(Y\cdot Q)=XY^P\cdot PQ$ for composible permutations P and Q and elements $X,Y\in\widehat{U\mathfrak{t}_{n+1}}$ where the action of $S_{1,n}$ on $\widehat{U\mathfrak{t}_{1,n}}$ is obtained by sending t_{ij} to $t_{\sigma^{-1}(i)\sigma^{-1}(j)}$.

The morphisms in $\widehat{\mathbf{CD}_{\mathbf{a}}}$ can be graphically represented using the identification of $\mathfrak{t}_{1,n}$ with horizontal chord diagrams for which the diagrammatic representation of the generator t_{ij} has a chord connecting strand i to j and by indicating permutations at the top of diagrams as a permutation of the strands. For example, $t_{12}t_{23}\cdot(12)$ is represented by the diagram

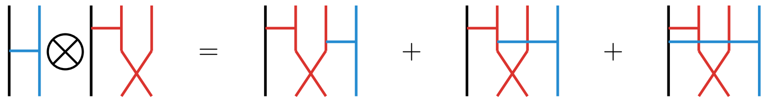
$t_{23}t_{12}\cdot(12) \quad =$  .

The composition law is diagramatically represented by stacking chord diagrams and sliding chords downward along strands until all chords are beneath all permutations. For example, the composition $\Big(t_{23}t_{12}\cdot(12)\Big)\circ\Big(t_{23}\cdot(12)(23)\Big)=t_{23}t_{12}t_{13}\cdot(23)$ is obtained by stacking and sliding as in the diagram

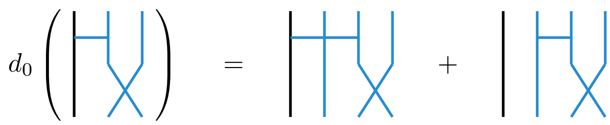


We will often draw diagrams which are not in the standard form and have chords above permutations of strands for convenience.

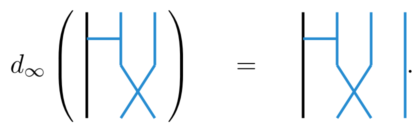
$\widehat{\mathbf{CD}_{\mathbf{a}}}$ has the structure of a strict monoidal category with a coproduct functor which will now be described. The monoidal structure is defined by gluing into the core and summing all ways of connecting chords that were connected to the core to the new strands. For example,



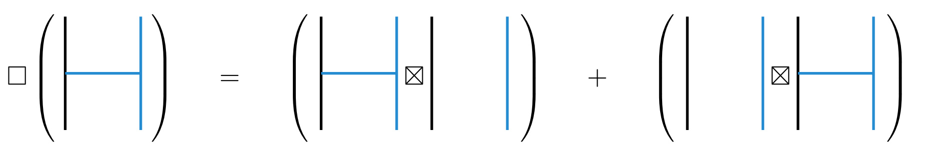
As before we introduce the notation d_0 and d_∞ via $d_0(C) = C \otimes |\,|$ and $d_\infty(C) = |\,| \otimes C$. In the diagrammatic notation we have for example



and



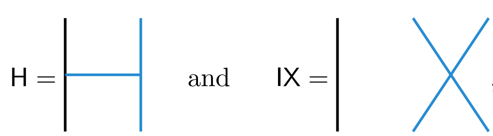
The coproduct $\square : \mathbf{CD_a} \to \mathbf{CD_a} \boxtimes \mathbf{CD_a}$ is defined by making individual chords primitive [5], so for example



Finally, the graded component of degree m of $\mathbf{CD_a}$ comprises linear combinations of diagrams each of which has exactly m chords.

There is a structural result which will allow us to describe functors into and out of $\mathbf{CD_a}$ as was the case for the the category $\boldsymbol{B_a}$.

Definition 3.2.4. Let C be the category with objects $\mathrm{Obj}_C = \bigsqcup_n S_{1,n}$ and with morphisms generated as a strict monoidal category enriched over graded A-algebras by generators

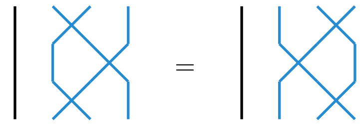


where H has degree 1 and IX has degree 0, subject to the relations generated by

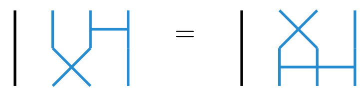
- Idempotency: $(\mathsf{IX})^2 = \mathrm{id}_3$

[5]The reason chords, ie. the generators t_{ij}, are primitive is the same as in the Example of free Abelian groups in Section 2.2.

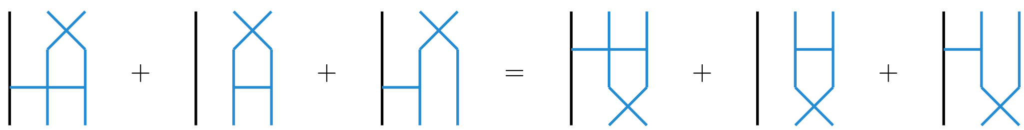
- Braid : $d_\infty(\mathsf{IX})d_0(\mathsf{IX})d_\infty(\mathsf{IX}) = d_0(\mathsf{IX})d_\infty(\mathsf{IX})d_0(\mathsf{IX})$



- Chord slide: $\Big[d_\infty(\mathsf{IX}), d_0(\mathsf{IX}) \cdot \big(d_0\mathsf{H} - \mathsf{IX}\cdot\mathsf{HI}\cdot\mathsf{IX}\big) \cdot d_0(\mathsf{IX})\Big] = 0$



- Chord Flip : $[d_0\mathsf{H} + \mathsf{HI}, \mathsf{IX}] = 0$



Lemma 3.2.1. *C is isomorphic to $\mathbf{CD_a}$.*

Proof. The proof will consist of defining a functor $\Phi : C \to \mathbf{CD_a}$ and proving it is a monoidal isomorphism. On objects, Φ is simply the identity. On morphisms, Φ sends IX and H to the horizontal chords as indicated in the diagrams in the statement of the Lemma. Φ can be homomorphically extended to compositions and monoidal products of generators in order to get a monoidal functor.

The diagrams in the staement of the lemma illustrate the image under Φ of each relation in C as a chord diagram. These equalities hold in the algebra of horizontal chord diagrams, so that defining a functor on the generators as in the previous paragraph is well-defined.

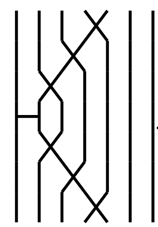
It remains to show that the monoidal functor Φ is an isomorphism. Φ is the identity on objects in the category C, and we show below that Φ induces an isomorphism on each morphism set. Beginning with surjectivity, every transposition of neighbouring strands in any morphism set in $\mathbf{CD_a}$ is of the form $\sigma_{i,i+1} = \Phi(d_\infty^n d_0^m \mathsf{IX})$, and since transpositions generate all permutations, any permutation is in the image of Φ.

To see that all chords are in the image of Φ, note first that

$$t_{01} = \Phi(\mathsf{H})$$

$$t_{02} = \Phi(\mathsf{IX} \cdot \mathsf{HI} \cdot \mathsf{IX})$$

$$t_{05} = \Phi\left(d_0^2\mathsf{IX} \cdot d_0\mathsf{IX} \cdot \mathsf{IX} \cdot \mathsf{H} \cdot \mathsf{IX} \cdot d_0\mathsf{IX} \cdot d_0^2\mathsf{IX}\right) =$$

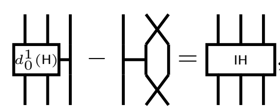


The pattern in the above examples leads to the general formula

$$t_{0m} = \left[\prod_{i=0}^{m-3} d_\infty^i d_0^{m-i-3}(\mathsf{IX})\right] \cdot d_\infty^{m-2}(\mathsf{H}) \cdot \left[\prod_{i=0}^{m-3} d_\infty^{m-i-3} d_0^i(\mathsf{IX})\right].$$

To shift a chord away from the zeroth strand, the fundamental relation needed is

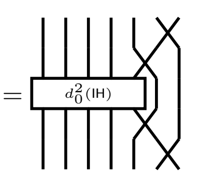
$$t_{12} = d_0(\mathsf{H}) - \mathsf{IX} \cdot \mathsf{HI} \cdot \mathsf{IX} =$$



and the left end of the chord can be shifted further to the right by applying d_0 to t_{12}.

This can be combined with the above method of shifting the right end of the strand further to the right as in the example

$$t_{36} =$$



to obtain the formula

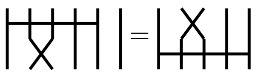
$$t_{mn} = \left[\prod_{i=0}^{n-m-1} d_\infty^i d_0^{n-i-2}(\mathsf{IX})\right] \cdot \left[d_\infty^{n-m-1} d_0^{n-1}\Big(d_0(\mathsf{H}) - \mathsf{IX} \cdot \mathsf{HI} \cdot \mathsf{IX}\Big)\right] \cdot \left[\prod_{i=0}^{n-m-1} d_\infty^{n-i-2} d_0^i(\mathsf{IX})\right]$$

when $m > 1$. Finally, as many strands to the right of the chord as is required can be added using d_∞ to obtain t_{mn} as an element of a morphism set given by chord diagrams of more than n strands.

It remains to show Φ is injective on morphism sets. This will be achieved by showing all relations which hold in the algebra of chord diagrams are images of relations which hold in C. In each of the below cases, the description of allowable diagrams refers to the diagram on the left of the equality and the drawn diagram is an example of one of the relations covered by the case.

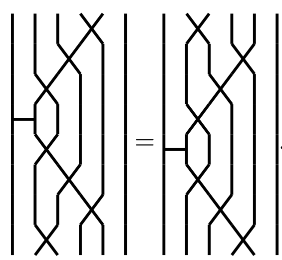
Step 0: |X's behave like permutations. The image of the idempotency relation and the braid relation under Φ are the relations $\sigma_i^2 = e$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ which generate all relations in the symmetric group $S_{1,p}$.

Step 1: Moving noninteracting |X's past chords.

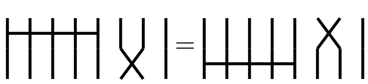
Case 1.a: 

The chord must start at the core but can span arbitrarily many strands. The crossing can be between any two adjacent strands between the endpoints of the chord (not including the endpoints themselves.)

Using the above expansions of chords in terms of elements of the category C,

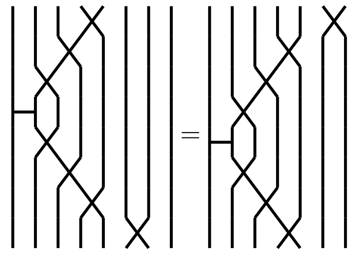


To get from the left hand diagram to the right hand one, slide the bottom-most crossing through all other crossings preceeding the chord using the fact that the |X's behave like permutations. Interchanging the crossing with the chord is a monoidal relation since the crossing involves only strands to the right of the chord. Finally, slide the crossing above all the remaining crossings to obtain the right hand diagram.

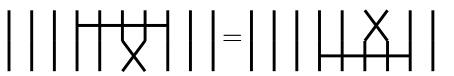
Case 1.b: 

The chord must start at the core but can span arbitrarily many strands. The crossing can be between any two adjacent strands to the right of the chord.

Using the above expansions of chords in terms of elements of the category C,

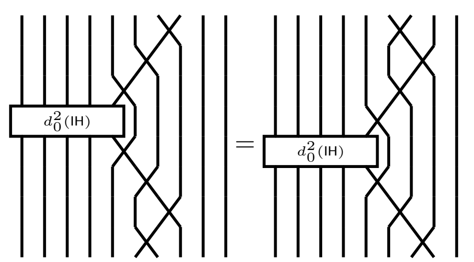


The bottom-most crossing in the left hand diagram slides past all other crossings using the permutation relations and slides past the chord using monoidal relations to result in the right hand diagram.

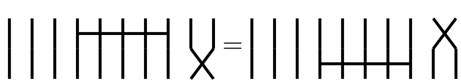
Case 1.c: 

The chord can start on any strand other than the zeroth and can end on any strand at least 3 strands to the right of the starting strand. The crossing can be between any two adjacent strands which lie between the start and end of the chord, but not involving the endpoints.

Using the above expansions of chords in terms of the elements of the category C,

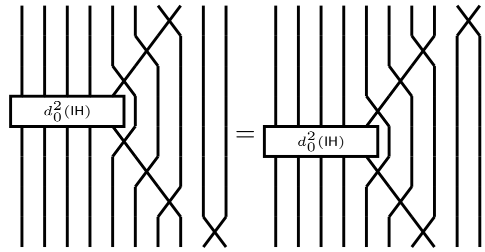


The bottom-most crossing in the left hand diagram slides past all other crossings using the permutation relations. Since all nonidentity strands in $d_0^2(\mathsf{IH})$ are to the right of the strands which cross in this crossing, it also slides past $d_0^2(\mathsf{IH})$ using monoidality to obtain the right diagram.

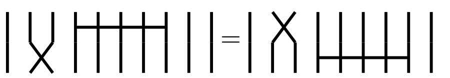
Case 1.d: 

The chord can start anywhere other than the zeroth strand. The crossing must involve adjacent strands to the right of the chord.

Using the above expansions of chords in terms of the elements of the category C,

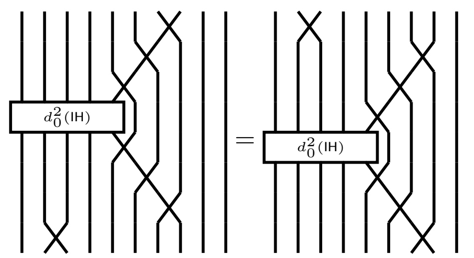


The bottom crossing in the left diagram slides past all other crossings using permutation relations and slides past $d_0^2(\mathsf{IH})$ by using a monoidal relation.

Case 1.e: 

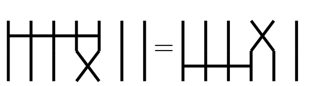
The crossing can involve any two adjacent strands to the left of the chord.

Using the above expansions of chords in terms of the elements of the category C,

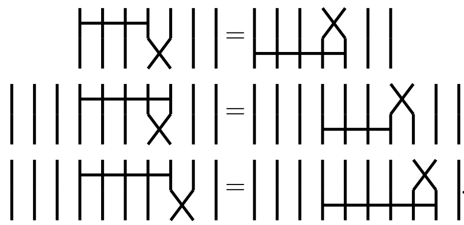


The bottom permutation in the right diagram can slide up just beneaath the chord using permutation relations. To commute the permutation with the chord we need to show $[d_0^m(\mathsf{IX}), d_0^n(\mathsf{IH})] = 0$ where $n \geq m+2$. Using the definition $\mathsf{IH} = d_0\mathsf{H} - \mathsf{IX} \cdot \mathsf{HI} \cdot \mathsf{IX}$, this reduces to the condition that $[d_0^m(\mathsf{IX}), d_0^n(\mathsf{H})] = 0$ for $n \geq m+2$ by setting each term of the commutator to zero individually and sliding $d_0^m(\mathsf{IX})$ past $d_0^n(\mathsf{IX})$.

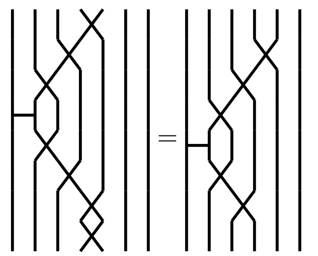
Step 2: Sliding chords along crossings:

Case 2.a: 

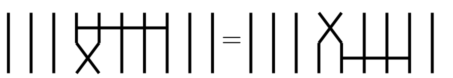
The chord must begin to the right of the crossing, and can end on either of the two strands making up a crossing. There are three other types of diagrams than the one given above,



Using the above expansions of chords in terms of the elements of the category C, the first equality becomes

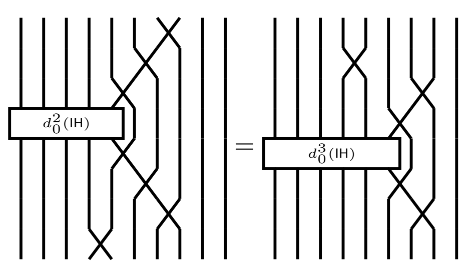


The left diagram becomes the right diagram by simply removing the two bottommost crossings using the relation $\mathsf{IX}^2 = \mathrm{id}_3$. All of the other cases reduce to cancelling a IX^2 factor somewhere in the diagram in a similar way.

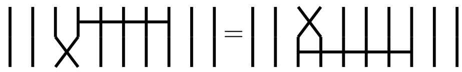
Case 2.b: 

The chord can begin on any strand other than the zeroth. The crossing must be between the strand the chord ends on and the next strand to the right, which can't be the strand the chord ends on.

Using the above expansions of chords in terms of the elements of the category C,

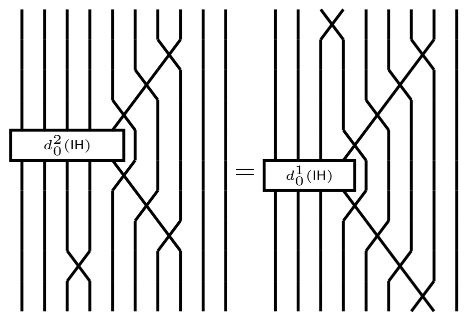


Starting with the bottommost crossing in the left hand diagram, slide it past the next two crossings. To continue sliding the chord up, we need to slide this crossing, say $d_0^m(\mathsf{IX})$ past $d_0^{m+1}(\mathsf{IX})d_0^m(\mathsf{IH})d_0^{m+1}(\mathsf{IX})$. In other words, we need to show $d_0^m(\mathsf{IX})d_0^{m+1}(\mathsf{IX})d_0^m(\mathsf{IH})d_0^{m+1}(\mathsf{IX}) = d_0^{m+1}(\mathsf{IH}) \cdot d_0^m(\mathsf{IX})$ which can be obtained from the chord slide relation by applying d_0 repeatedly.

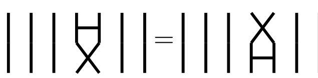
Case 2.c: 

The chord can begin on any strand other than the zeroth. The crossing must be between the strand the chord ends on and the previous strand. which can't be the zeroth strand.

Using the above expansions of chords in terms of the elements of the category C,

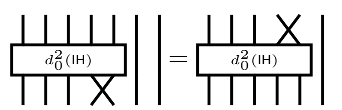


These relations are a consequence of the previous case since they can be obtained by conjugating relations in the previous case by $d_\infty^m d_0(\mathsf{IX})$.

Case 2.d: 

The chord can be between any two adjacent strands not including the zeroth. The crossing must be between these same two adjacent strands.

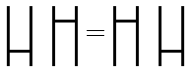
Using the above expansions of chords in terms of the elements of the category C,



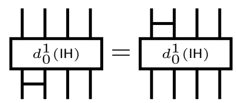
Algebraically, this relation is $[d_0^m(\mathsf{IH}), d_0^m(\mathsf{IX})]$ which follows from the relation $[\mathsf{IH}, \mathsf{IX}]$ by doubling. Expanding using the definition $\mathsf{IH} = d_0\mathsf{H} + \mathsf{IX} \cdot \mathsf{HI} \cdot \mathsf{IX}$, $[\mathsf{IH}, \mathsf{IX}]$ is equivalent to the chord flip relation.

So far, we have shown that we can slide chords along any crossings as is standard in the algebra of chord diagrams. Given any further relation, we can always conjugate the relation by appropriate permutations and slide the relation through the permutations to obtain a relation in which the left most strand involved in the relation is either the zeroth or the first strand. We will use this throughout the remainder of the proof so that only one relation needs to be checked in each case now.

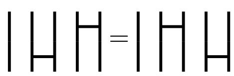
Step 3: Passing disjoint chords by each other

Case 3.a: 

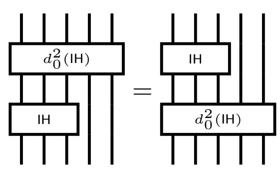
Using the above expansions of chords in terms of the elements of the category C,



The relation $[d_\infty \mathsf{HI}, d_0 \mathsf{IH}]$ is a monoidal relation.

Case 3.b: 

Using the above expansions of chords in terms of the elements of the category C,



The relation $[d_\infty^2 \mathsf{IH}, d_0^2 \mathsf{IH}]$ is a monoidal relation.

Step 4: 4T Relations

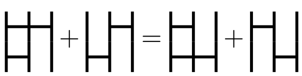
In order to reduce checking the 4T relations

$$[t_{ij}, t_{ik} + t_{jk}] = 0$$

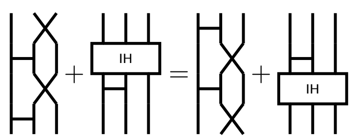
in the Drinfel'd Kohno algebra are in the image of the relations we will use our previous work to slide any such relation to involve either the core and the first 3 strands or else the first 4 strands. For example, consider the 4T relation $[t_{45}, t_{48} + t_{58}]$. Let P be the permutation $(14)(25)(38)$ and let P_{X} be this permutation written in terms of the generators X and IX. Then assuming the 4T relation holds for the first three strands (and regarding t_{ij} as an element of C as described above,)

$$\begin{aligned}
t_{45}(t_{48}+t_{58}) &= t_{45}(t_{48}+t_{58})\cdot P_{\mathsf{X}}P_{\mathsf{X}}^{-1} && \text{Slide the chords over the crossings}\\
&= P_X t_{12}(t_{13}+t_{23})P_{\mathsf{X}}^{-1}\\
&= P_X(t_{13}+t_{23}t_{12})P_{\mathsf{X}}^{-1}\\
&= (t_{48}+t_{58})t_{45}\cdot P_{\mathsf{X}}P_{\mathsf{X}}^{-1}\\
&= (t_{48}+t_{58})t_{45}.
\end{aligned}$$

Using this idea of sliding chords as far left as possible, any 4T relation can be reduced to one of the following two cases.

Case 4.a: 

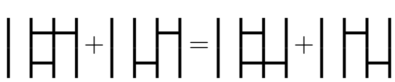
Using the above expansions of chords in terms of the elements of the category C,



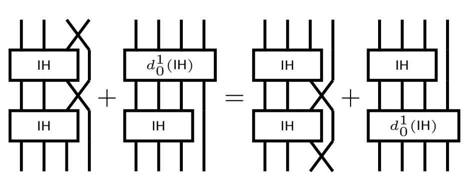
Recalling the definition, $\mathsf{IH} = d_0H - \mathsf{IX}\cdot\mathsf{HI}\cdot\mathsf{IX}$, algebraically this relation is

$$0 = [\mathsf{HI}, \mathsf{IX}\cdot\mathsf{HI}\cdot\mathsf{IX} + d_0H - \mathsf{IX}\cdot\mathsf{HI}\cdot\mathsf{IX}] = [d_\infty\mathsf{H}, d_0\mathsf{H}]$$

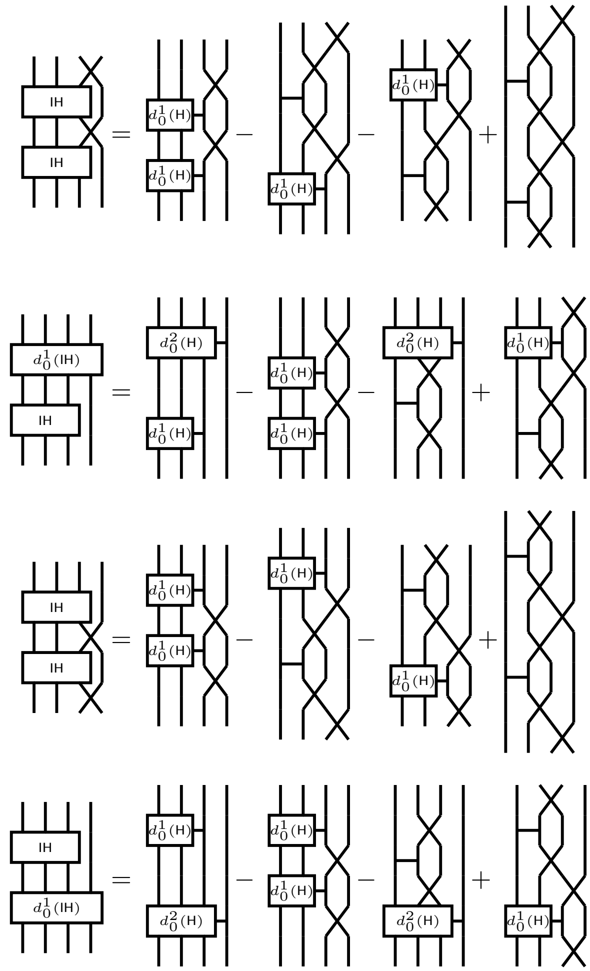
which is a monoidal relation.

Case 4.b: 

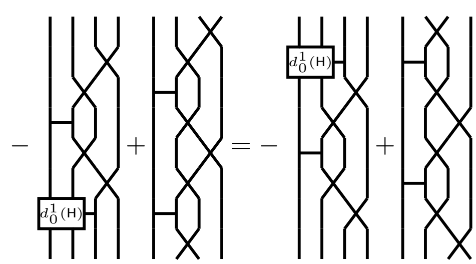
Using the above expansions of chords in terms of elements of the category C,



Recalling the definition of IH,



After cancellation, the relation now involves only the four terms



Algebraically, this relation is

$$0 = [d_\infty\left(d_0\mathsf{H} - \mathsf{IX}\cdot\mathsf{HI}\cdot\mathsf{IX}\right), d_0\mathsf{IX}\cdot d_\infty\mathsf{IX}\cdot d_\infty\mathsf{HI}\cdot d_\infty\mathsf{IX}\cdot d_0\mathsf{IX}]$$

The left entry in the commutator is the expansion in the category C for the chord t_{12} while the second entry is t_{03}. We have already shown disjoint chords commute in Step 3 above so the relation holds

$\square$

3.3 Braidors and Grothendieck-Teichmüller Groups

3.3.1 The General Setup

We now have two algebraic structures, $\widehat{\boldsymbol{\mathcal{B}_a}}$ and $\widehat{\boldsymbol{\mathcal{CD}_a}}$ out of which, following Section 2.1.6, two groups and a bitorsor (a set with compatible left and right free transitive group actions) can be constructed as in the diagram

$$\mathrm{Aut}(\widehat{\boldsymbol{\mathcal{B}_a}}) \circlearrowright \widehat{\boldsymbol{\mathcal{B}_a}} \overset{\mathbf{Iso}(\widehat{\boldsymbol{\mathcal{B}_a}},\widehat{\boldsymbol{\mathcal{CD}_a}})}{\Rrightarrow} \widehat{\boldsymbol{\mathcal{CD}_a}} \circlearrowleft \mathrm{Aut}(\widehat{\boldsymbol{\mathcal{CD}_a}}) .$$

Using the terminology taken from the case of Drinfel'd associators, we use the notation $\mathrm{Aut}(\widehat{\boldsymbol{\mathcal{B}_a}}) = \mathbf{GT}_a$, called the annular Grothendieck-Teichmüller group and $\mathrm{Aut}(\widehat{\boldsymbol{\mathcal{CD}_a}}) = \mathbf{GRT}_a$, called the graded annular Grothendieck-Teichmüller group. The set of ismorphisms $\mathbf{Iso}(\widehat{\boldsymbol{\mathcal{B}_a}}, \widehat{\boldsymbol{\mathcal{CD}_a}}) = \mathbf{BRAID}$ is the set of braidors and is a $\mathbf{GT}_a - \mathbf{GRT}_a$ bitorsor.

Pro-ℓ version of the space of braidors and the two annular Grothendieck-Teichmüller groups can be obtained by replacing $\widehat{\boldsymbol{B_a}}$ and $\widehat{\mathbf{CD_a}}$ by the pro-ℓ version as in the diagram

$$\mathbf{GT}_a^{(\ell)} = \mathrm{Aut}\left(\boldsymbol{\mathcal{B}_a}^{(\ell)}\right) \circlearrowright \boldsymbol{\mathcal{B}_a}^{(\ell)} \overset{\mathbf{Iso}(\widehat{\boldsymbol{B_a}},\mathbf{CD_a}^{(\ell)})}{\Rrightarrow} \boldsymbol{\mathcal{CD}_a}^{(\ell)} \circlearrowleft \mathrm{Aut}\left(\boldsymbol{\mathcal{CD}_a}^{(\ell)}\right) = \mathbf{GRT}_a^{(\ell)}$$

and $\mathbf{Iso}(\widehat{\boldsymbol{B_a}}, \mathbf{CD_a}^{(\ell)}) = \mathbf{BRAID}^{(\ell)}$ is the pro-ℓ version of the set of braidors, consisting of braidors up to degree ℓ.

3.3.2 Braidors

To provide an explicit description of a braidor we need to study isomorphisms $Z : \widehat{\boldsymbol{\mathcal{B}_a}} \to \widehat{\boldsymbol{\mathcal{CD}_a}}$. Such an isomorphism of algebraic structures will consist of a strict monoidal functor $Z : \widehat{\boldsymbol{B_a}} \to \widehat{\mathbf{CD_a}}$ which preserves the coproduct.

Using Claim 3.1.1, to construct Z it suffices to specify the images $Z(\sigma) = \mathsf{IX} \cdot B$ and $Z(\tau) = R$ of the generators, where $B \in \widehat{U\mathfrak{t}_{1,2}}$ and $R \in \widehat{U\mathfrak{t}_{1,1}}$, and check that the relations in Claim 3.1.1 and $\square$ are preserved. Since σ and τ are both invertible, B and R must be also. In order for $\square$ to be preserved, B and R must be grouplike. Since a grouplike element in $\widehat{U\mathfrak{t}_{1,1}}$ is simply an exponential, set

$R = \exp(t_{01})$[6]. Using the notation $B^{f^{-1}(0),f^{-1}(1),f^{-1}(2)}$ for a map $f : \{0,1,2\} \to \{0,1,2,3\}$ as defined in the introduction, the two relations in Claim 3.1.1 yield two equations that must be satisfied by B and R, summarised in the

Definition 3.3.1. A braidor is an invertible, grouplike element $B \in \widehat{U\mathfrak{t}_3}$ which satisfies the equations

$$B^{0,1,2}B^{02,1,3}B^{0,2,3} = B^{01,2,3}B^{0,1,3}B^{03,1,2} \qquad \text{(Braid Equation)}$$

$$R^{01,2} = BR^{0,2}B^{0,2,1} \qquad \text{(Mixed Equation)}$$

where $R = \exp(t_{01})$.

A similar argument for $\mathbf{BRAID}^{(\ell)}$ leads to the notion of braidors up to degree ℓ in the pro-ℓ case.

Definition 3.3.2. A braidor of degree ℓ is an invertible, grouplike element $B \in \bigoplus_{n=0}^{\ell}(U\mathfrak{t}_3)_n$, where $(U\mathfrak{t}_3)_n$ is the degree n component of $\widehat{U\mathfrak{t}_3}$, which, modulo degree $(\ell+1)$ and higher terms, satisfies the braid and mixed relations in Definition 3.3.1. The set of all braidors of degree ℓ will be denoted $\mathbf{BRAID}^{(\ell)}$.

It follows immediately from the definitions that $\mathbf{BRAID} = \varprojlim_{\ell} \mathbf{BRAID}^{(\ell)}$.

3.3.3 The Annular Grothendieck-Teichmüller Group $\mathbf{GT}_a$

While we make no use of $\mathbf{GT}_a$, for completeness a brief summary of its definition and properties is given. There is a free, transitive action of the group of strict monoidal functors $S : \widehat{\boldsymbol{B_a}} \to \widehat{\boldsymbol{B_a}}$ which preserve $\square$ and fix τ[7] on $\mathbf{BRAID}$. The action is given by precomposition and these functors form a group via composition.

Such a functor is determined by the image $S(\sigma) \in \widehat{B_{1,2}}$ of σ. Using the fact that S preserves the coproduct and the fact that σ is invertible, we can write $S(\sigma) = \Sigma \cdot \sigma$ for some grouplike, invertible $\Sigma \in \widehat{\mathrm{PB}_{1,2}} \cong \widehat{\mathrm{PB}_3}$. In order to be well defined the two relations in Claim 3.1.1 must be preserved. The two equations resulting from these relations are recorded in Definition 3.3.3.

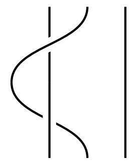
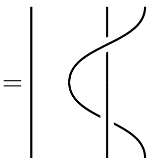
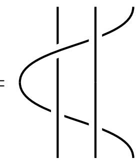
To express the group law explicitly in terms of braid groups, let $O : \mathbf{CD_a} \to \mathbf{CD_a}$ be another such functor with corresponding $\Omega \in \widehat{\mathrm{PB}_3}$. Then

$$S \circ O(\sigma) = S(\Omega \cdot \sigma) = S(\Omega) \cdot S(\sigma) = S(\Omega) \cdot \Sigma \cdot \sigma.$$

[6]In fact $R = \exp(k \cdot t_{01})$ for any $k \in A^{\times}$ is the most general solution however only in the case $k = 1$ will the resulting invariant of annular braids be an expansion (ie. gr $Z = Z$) so we enforce this condition in the definition of a braidor.

[7]This condition arises as expansions had to send τ to the fixed element R and hence $\mathbf{GT}_a$ elements must preserve τ.

In order to determine $S(\Omega)$ the following generators of PB_3 will be used

$\sigma_{01} =$  $\sigma_{12} =$  and $\sigma_{02} =$  .

Rewriting in terms of the generators of $\boldsymbol{B_a}$, $\sigma_{01} = d_\infty(\tau)$, $\sigma_{12} = \sigma^2$ and $\sigma_{02} = \sigma \cdot d_\infty(\tau) \cdot \sigma^{-1}$. Applying S and rewriting the result using these generators as well as the generators σ_i of the braid groups where σ_i is the braid in which strand i crosses over strand $i+1$[8],

$$S(\sigma_{01}) = S(d_\infty \tau) = d_\infty \tau = \sigma_{01}$$
$$S(\sigma_{12}) = S(\sigma^2) = (\Sigma \cdot \sigma)^2 = (\Sigma \sigma_1)^2$$
$$S(\sigma_{02}) = S(\sigma \cdot d_\infty(\tau)\sigma^{-1}) = \Sigma\sigma_1\sigma_{01}(\sigma_1\Sigma)^{-1}.$$

The action of S on Ω is thus obtained by replacing all occurences of σ_{ij} by these expressions:

$$S(\Omega) = \Omega|_{\sigma_{01}\to\sigma_{01},\ \sigma_{12}\to(\Sigma\sigma_1)^2,\ \sigma_{02}\to\Sigma\sigma_1\sigma_{01}(\sigma_1\Sigma)^{-1}}.$$

Definition 3.3.3. $\widehat{\mathbf{GT}}_a$ is the collection of all grouplike, invertible elements $\Sigma \in \widehat{\mathrm{PB}_{1,2}}$ which satisfy the equations

$$d_\infty(\Sigma) \cdot \sigma_1 \cdot d_0(\Sigma) \cdot \sigma_2 \cdot d_\infty(\Sigma) \cdot \sigma_1 = d_0(\Sigma) \cdot \sigma_2 \cdot d_\infty(\Sigma) \cdot \sigma_1 \cdot d_0(\Sigma) \cdot \sigma_2$$
$$d_0(\sigma_{01}) = \Sigma\sigma_1 \cdot \sigma_{12} \cdot \Sigma \cdot \sigma_1$$

in $\widehat{B_{1,2}}$ where $\sigma_1 = d_\infty(\sigma)$ and $\sigma_2 = d_0(\sigma)$. The group law in $\widehat{\mathbf{GT}_a}$ is

$$\Omega \times \Sigma = \Omega\Big|_{\sigma_{01}\to\sigma_{01},\ \sigma_{12}\to(\Sigma\sigma_1)^2,\ \sigma_{02}\to\Sigma\sigma_1\sigma_{01}(\sigma_1\Sigma)^{-1}} \cdot \Sigma.$$

Proposition 3.3.1. *There is a free, transitive right action of the group* $\widehat{\mathbf{GT}_a}$ *on* **BRAID** *via*

$$B \bullet \Sigma = \Sigma\Big|_{\sigma_{01}\to\exp(t_{01}),\sigma_{12}\to BB^{0,2,1},\sigma_{02}\to B\exp(t_{01})(B^{-1})^{0,2,1}} \cdot B$$

[8] While σ_i is not itself in the pure braid group, all three of the final expressions for $S(\sigma_{ij})$ are pure braids

for $\Sigma \in \widehat{\mathbf{GT}_a}$ *and* $B \in \mathbf{BRAID}$.

Proof. The action corresponds to the action of $\mathrm{Aut}(\widehat{\boldsymbol{\mathcal{B}}_{\boldsymbol{a}}})$ on $\mathrm{Iso}(\widehat{\boldsymbol{\mathcal{B}}_{\boldsymbol{a}}}, \widehat{\boldsymbol{\mathcal{CD}}_{\boldsymbol{a}}})$ by precomposition which is free and transitive. The explicit formula for the action comes from a similar computation as for the explicit formula for the group law. If Z is the functor corresponding to the braidor B and S is the functor corresponding to Σ, then

$$Z(S(\sigma)) = Z(\Sigma \cdot \sigma) = Z(\Sigma) \cdot B \cdot \sigma.$$

Applying Z to the generators of the pure braid group yields

$$\begin{aligned} Z(\sigma_{01}) &= \exp(t_{01}) \\ Z(\sigma_{12}) &= BB^{0,2,1} \\ Z(\sigma_{02}) &= B\exp(t_{02})B^{0,2,1} \end{aligned}$$

so that

$$Z(\Sigma) = \Sigma\Big|_{\sigma_{01}\to\exp(t_{01}),\sigma_{12}\to BB^{0,2,1},\sigma_{02}\to B\exp(t_{01})(B^{-1})^{0,2,1}}$$

$\square$

3.3.4 The Graded Annular Grothendieck-Teichmüller Group $\mathbf{GRT}_a$

The group of strict monoidal functors $\gamma : \widehat{\mathbf{CD_a}} \to \widehat{\mathbf{CD_a}}$ which preserve the coproduct and which fix t_{01} also acts freely and transitively on **BRAID**, by post-composition now. Doing a similar analysis as was done for $\mathbf{GT}_a$, $\gamma(\mathsf{H}) = \mathsf{H}$ as any braidor sends τ to the same thing and we can write $\gamma(\mathsf{IX}) = \Gamma \cdot \mathsf{IX}$ for some grouplike, invertible $\Gamma \in \widehat{U\mathfrak{t}_{1,2}}$ which must satisfy the two equations in Definition 3.3.4 in order to be well-defined with respect to the relations in Claim 3.2.1. Although the explicit group law is not used anywhere, it can be determined as follows. Let γ_1 and γ_2 be elements of $\mathbf{GRT}_a$ with $\gamma_1(\mathsf{IX}) = \Gamma_1 \cdot \mathsf{IX}$ and $\gamma_2(\mathsf{IX}) = \cdot\Gamma_2 \cdot \mathsf{IX}$.

Writing $\gamma_1 \circ \gamma_2(\mathsf{IX}) = \gamma_1(\Gamma_2 \cdot \mathsf{IX}) = \gamma_1(\Gamma_2)\Gamma_1 \cdot \mathsf{IX}$ and using the generators $t_{01} = \mathsf{H}$, $t_{02} = \mathsf{IX} \cdot \mathsf{H} \cdot \mathsf{IX}$ and $t_{12} = d_0\mathsf{H} - \mathsf{IX} \cdot \mathsf{H} \cdot \mathsf{IX}$ of $\mathfrak{t}_3$ we see that

$$\begin{aligned} \gamma_1(t_{01}) &= t_{01} \\ \gamma_1(t_{02}) &= \Gamma_1 \cdot t_{02} \cdot \Gamma_1^{0,2,1} \\ \gamma_1(t_{23}) &= t_{02} + t_{12} - \Gamma_1 \cdot t_{02} \cdot \Gamma_1^{0,2,1} \end{aligned}$$

and so

$$\gamma_1(\Gamma_2) = \Gamma_2|_{t_{01}\to t_{01},t_{02}\to\Gamma_1\cdot t_{02}\cdot\Gamma_1^{0,2,1},t_{12}\to t_{02}+t_{12}-\Gamma_1 t_{01}\Gamma_1^{0,2,1}}$$

leading to the

Definition 3.3.4. $\widehat{\mathbf{GRT}}_a$ is the collection of all grouplike, invertible elements $\Gamma \in \widehat{U\mathfrak{t}_3}$ such that $\Gamma^{-1} = \Gamma^{0,2,1}$ and which satisfy the equations

$$\Gamma^{0,1,2}\Gamma^{02,1,3}\Gamma^{0,2,3} = \Gamma^{01,2,3}\Gamma^{0,1,3}\Gamma^{03,1,2} \qquad \text{(Braid Equation)}$$

$$t_{01}^{012,3} - \Gamma^{01,2,3}t_{01}^{01,3}\Gamma^{01,3,2} = \Gamma\Gamma^{02,1,3}\left(t_{02}^{02,3} - \Gamma^{0,2,3}t_{03}\Gamma^{0,3,2}\right)\Gamma^{02,3,1}\Gamma^{0,2,1} \qquad \text{(Slide Equation)}$$

The group law in $\widehat{\mathbf{GRT}}_a$ is

$$\Gamma_1 \times \Gamma_2 = \Gamma_2|_{t_{01}\to t_{01},t_{02}\to\Gamma_1\cdot t_{02}\cdot\Gamma_1^{0,2,1},t_{12}\to t_{02}+t_{12}-\Gamma_1 t_{02}\Gamma_1^{0,2,1}} \cdot \Gamma_1.$$

There is also a pro-ℓ version of $\mathbf{GRT}_a$ obtained by replacing $\widehat{\boldsymbol{\mathcal{CD}_a}}$ by $\boldsymbol{\mathcal{CD}_a}^{(\ell)}$ in the above definition.

Definition 3.3.5. $\mathbf{GRT}_a^{(\ell)}$ is the set of grouplike, invertible $\Gamma \in \bigoplus_{n=0}^{(\ell)}(U\mathfrak{t}_3)_n$ which satisfy the braid and slide equations modulo terms of degreee $(\ell+1)$ and higher. The group law is as in Definition 3.3.4 again working modulo terms of degree $(\ell+1)$ and higher.

An immediate consequence of the definitions is that $\mathbf{GRT}_a = \varprojlim_\ell \mathbf{GRT}_a^{(\ell)}$.

Proposition 3.3.2. *There is a free and transitive action of the group* $\widehat{\mathbf{GRT}}_a$ *on* **BRAID** *via*

$$\Gamma \bullet B = B|_{t_{01}\to t_{01},t_{02}\to\Gamma\cdot t_{01}\cdot\Gamma^{0,2,1},t_{12}\to t_{02}+t_{12}-\Gamma t_{01}\Gamma^{0,2,1}} \cdot \Gamma.$$

Working modulo degree $\ell+1$, the above formula also defines a free and transitive action of $\mathbf{GRT}_a^{(\ell)}$ *on* $\mathbf{BRAID}^{(\ell)}$.

Proof. The action corresponds to the action of $\mathrm{Aut}(\widehat{\mathbf{CD_a}})$ on $\mathrm{Iso}(\widehat{\boldsymbol{B_a}}, \widehat{\mathbf{CD_a}})$ by post-composition which is free and transitive.

To determine the explicit formula given for the action, Let Z be the functor corresponding to B and let γ be the functor corresponding to Γ. Then $\gamma(Z(\sigma)) = \gamma(B)\cdot\Gamma\cdot\mathsf{IX}$. Since

$$\gamma(t_{01}) = t_{01}$$
$$\gamma(t_{02}) = \Gamma \cdot t_{02}\Gamma^{0,2,1}$$
$$\gamma(t_{23}) = t_{02} + t_{12} - \Gamma \cdot t_{02}\Gamma^{0,2,1}$$

so

$$\gamma(B) = B|_{t_{01}\to t_{01}, t_{02}\to \Gamma\cdot t_{02}\cdot\Gamma^{0,2,1}, t_{12}\to t_{02}+t_{12}-\Gamma t_{01}\Gamma^{0,2,1}}.$$

□

Proposition 3.3.3. *Each* $\mathbf{GRT}_a^{(\ell)}$ *is a unipotent affine algebraic group scheme over* $\mathbb{Q}$*. Hence it is reduced and connected and* $\mathbf{GRT}_a$ *is prounipotent.*

Proof. We have constructed a group $\mathbf{GRT}_a^{(\ell)}$ for any $\mathbb{Q}$-algebra A in a functorial way. That is, we have constructed a functor $\mathbf{GRT}_a^{(\ell)}$ from the category of commutative $\mathbb{Q}$-algebras to the category of groups and so $\mathbf{GRT}_a^{(\ell)}$ is an affine group scheme for every ℓ.

The action of $\mathbf{GRT}_a^{(\ell)}$ on $\mathbf{CD_a}^{(\ell)}$ defines a faithful representation of $\mathbf{GRT}_a^{(\ell)}$ on the vector space of chord diagrams whose underlying permutation is the identity. Thus $\mathbf{GRT}_a^{(\ell)}$ may be regarded as an algebraic matrix group. Given any $\Gamma \in \mathbf{GRT}_a$, $\Gamma(\mathsf{H}) = \mathsf{H}$ by definition and note that $\Gamma(\mathsf{IX}) = \mathsf{IX}+$(higher order terms) since Γ must preserve the underlying permutation. Hence $\mathbf{GRT}_a^{(\ell)}$ is a unipotent affine group scheme over a field of characteristic zero. Standard results from the theory of algebraic groups (see for example [Waterhouse (1979)]) imply that $\mathbf{GRT}_a$ is reduced and connected. Since $\mathbf{GRT}_a = \varprojlim_\ell \mathbf{GRT}_a^{(\ell)}$ is an inverse limit of unipotent affine group schemes, $\mathbf{GRT}_a$ is prounipotent. □

Proposition 3.3.4. *The annular graded Grothendieck-Teichmüller Lie algebra* $\mathfrak{grt}_a = \mathrm{Lie}(\mathbf{GRT}_a)$ *is the set of all primitive* $\gamma \in \widehat{U\mathfrak{t}_3}$ *which satisfy the equations*

$$\gamma^{0,1,2} + \gamma^{02,1,3} + \gamma^{0,2,3} = \gamma^{01,2,3} + \gamma^{0,1,3} + \gamma^{03,1,2}$$
$$t_{01}^{012,3} - \left[\Gamma^{01,2,3}, t_{01}^{01,3}\right] = \left[\Gamma, \left[\Gamma^{02,1,3}, \left[t_{02}^{02,3} - \left[\Gamma^{0,2,3}, t_{03}\right]\right]\right]\right].$$

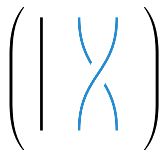
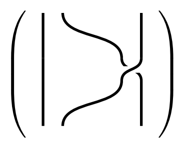
$\mathfrak{grt}_a^{(\ell)} = \mathrm{Lie}(\mathbf{GRT}_a^{(\ell)})$ *is obtained by working modulo terms of degree* $(\ell+1)$ *and higher in the above formulas.*

Proof. From the results in Proposition 3.3.3 it follows (see [Waterhouse (1979)] for example) that the Lie algebra is defined by the linearizations of the equations defining the group, as indicated. □

3.3.5 Existence of Braidors

The first question to ask about braidors is of course whether any exist. While existence is a very difficult question for associators, it is easy to see that a braidor can be constructed out of any associator so the existence of braidors reduces easily to the existence of associators.[9]

There is a map $C : \mathbf{ASSOC} \to \mathbf{BRAID}$ which, given an associator Φ regarded as a functor $\widehat{\boldsymbol{B_a}} \to \widehat{\mathbf{CD_a}}$, is defined on the generator σ of $\widehat{\boldsymbol{B_a}}$ by

$C(\Phi)$  $= \Phi$  .

Regarding Φ as an element of $\widehat{U\mathfrak{t}_{1,2}}$ rather than $\widehat{U\mathfrak{t}_3}$, which amounts to shifting the labelling of all strands up by one, C is defined by $C(\Phi) = \Phi \exp\left(\frac{1}{2}t_{12}\right)\Phi^{-0,2,1}$. That $C(\Phi)$ satisfies the required three equations in the definition of a braidor follows from the fact that Φ satisfies the pentagon and the hexagon equations.

Corollary 3.3.1. **BRAID** *is nonempty for any* $\mathbb{Q}$*-algebra* A*.*

Proof. **ASSOC** is nonempty [Drinfel'd (1991)]. Furthermore rational braidors exist [Drinfel'd (1991); Bar-Natan (1998)] and it is clear from the formula $C(\Phi) = \Phi \exp\left(\frac{1}{2}t_{12}\right)\Phi^{-0,2,1}$ that if all the coefficients of Φ are rational then so are the coefficients of the braidor $C(\Phi)$. Thus rational braidors, and a forteriori braidors with coefficients in any $\mathbb{Q}$-algebra A, exist. $\square$

Corollary 3.3.2. $\widehat{\mathbf{CD_a}} = \widehat{\mathrm{gr}}\,\widehat{\boldsymbol{B_a}}$.

Proof. The existence of a braidor implies there is a $\widehat{\boldsymbol{\mathcal{CD}_a}}$-expansion $Z : \widehat{\boldsymbol{\mathcal{B}_a}} \to \widehat{\boldsymbol{\mathcal{CD}_a}}$ so apply Lemma 2.1.1. $\square$

Lemma 3.3.1. *The map* $C : \mathbf{ASSOC} \to \mathbf{BRAID}$ *is injective.*

Proof. Let Φ, Ψ be Drinfel'd associators and suppose $C(\Phi) = C(\Psi)$. Since Φ and Ψ both satisfy the positive hexagon equation

$$R^{01,2} = \Phi R^{1,2}\Phi^{-0,2,1}R^{0,2}\Phi^{2,0,1} = C(\Phi)R^{0,2}\Phi^{2,0,1}$$

[9] Since our proof of the existence of braidors depends on the existence of associators, we are unfortunately still able to construct a braidor only by using transcendental techniques.

where $R = \exp\left(\frac{1}{2}t_{12}\right)$, it follows that $C(\Phi) = R^{01,2}\Phi^{-2,0,1}R^{-0,2}$ and $C(\Psi) = R^{01,2}\Psi^{-2,0,1}R^{-0,2}$. Hence

$$R^{01,2}\Phi^{-2,0,1}R^{-0,2} = R^{01,2}\Psi^{-2,0,1}R^{-0,2}$$

which implies $\Phi = \Psi$ since R is invertible. □

3.4 Braidors in Degree One

A direct computation using the equations in low degree shows that an associator is of the form

$$\Phi = 1 + \frac{1}{24}t_{13}t_{23} + \cdots$$

and hence, using the explicit formula for C, that

$$B = 1 + \frac{1}{2}t_{12} + \cdots .$$

On the other hand, using the equations defining a braidor, we get

$$B = 1 + \frac{1}{2}t_{12} + k(t_{01} - t_{02}) + \cdots .$$

In both of these equations the ellipsis indicate terms of degree higher than 2[10].

The origin of this discrepency is the existence of degree one automorphisms $\Gamma_k \in \mathbf{GT}_a$ of annular braids for which there is no corresponding automorphism of braids in a disc.

To determine Γ_k explicitly, let β be an n strand annular braid. Let γ_n be the n strand annular braid defined in the following way, using the interpretation of a braid is an element of the fundamental group of the configuration of n points in an annulus. First let the outermost point circle the core counterclockwise one time. After completing this, the outermost two points circle the core of the annulus one time. Continue adding one more point until finally all n points circle the annulus one time counterclockwise. The resulting annular braid is γ_n.

Γ_k is determined via conjugation by γ_n^k. That is, on an n-strand annular braid, Γ_k is defined by $\Gamma_k(\beta) = \gamma_n^{-k}\beta\gamma^k$.[11] Using the action of $\mathbf{GT}_a$ on **BRAID**, given an expansion $Z \in$ **BRAID**, a new expansion $Z_k \in$ **BRAID** is obtained where $Z_k(\beta) = Z \circ \Gamma$.

[10] A priori it is not clear that this equation extends to all degrees to give an actual braidor, however our argument below shows how to appropriately modify a braidor coming from an associator to obtain braidors of this form for any k.

[11] Powers by real numbers are well-defined since we are working in a prounipotent completion.

To give an explicit formula using only the Drinfel'd-Kohno algebra, let

$$C_n = \sum_{0 \leq i < j \leq n} j \; t_{ij}$$

and let $C \in \widehat{U}\mathfrak{t}_n$ be the element with graded components C_n. Then $Z_k(\beta) = e^{-kC} Z(\beta) e^{kC}$. If Z is an expansion with corresponding braidor of the form $B = 1 + \frac{1}{2}t_{12} + \cdots$ with $k = 0$, then Z_k defines a corresponding braidor B_k which is identical except that it now includes a $k(t_{01} - t_{02})$ term in degree one.

The net result is that there is a decomposition $\mathbf{BRAID} = \mathbf{BRAID}_0 \times A$ where $\mathbf{BRAID}_0$ is the collection of braidors with $k = 0$. So, while there are braidors not coming from associators, they are all easily obtained from associators as described above. Note however that looking at just this new degree one term in braidors not coming from associators does yield an invariant of braids which may not be contained within an associator.

3.5 Reduction of Equations for Braidors

One of the classic results in the theory of associators due to Furusho [Furusho (2010)] is Theorem 2.5.2 stating that the pentagon equation implies the two hexagon equations. This theorem has an annular counterpart which is proven in this section.

Theorem 3.5.1. *If $B \in \widehat{U\mathfrak{t}_{1,2}}$ satisfies the braid equation and has degree one term equal to $\frac{1}{2}t_{12}$ then it satisfies the mixed relation.*

Proof. Assume B satisfies the braid equation. Assume also that B satisfies the mixed relation up to degree n and let E be the error in the mixed relation in degree $n+1$. E is the degree $n+1$ homogeneous element of $\widehat{U\mathfrak{t}_{1,2}}$ which measures the difference $R^{01,2} - BR^{0,2}B^{0,2,1}$ in degree $n+1$. In order to show the mixed equation is in fact valid it suffices to show E must be zero and the strategy to do this is to find equations which are satisfied by E in the hopes of showing no element of $\widehat{U\mathfrak{t}_{1,2}}$ satisfies these equations except the zero element.

In order to find such an equation satisfied by E, we need to find a syzygy, that is a relation between mixed relations which is accomplished in Figure 3.5.1.

Each move in the diagram in which the mixed relation is applied to move from one diagram to the next will pick up an error E installed on the relevant strands. Comparing the errors along the two paths in the diagram (see the caption for more detail) from the leftmost node to the rightmost node results in

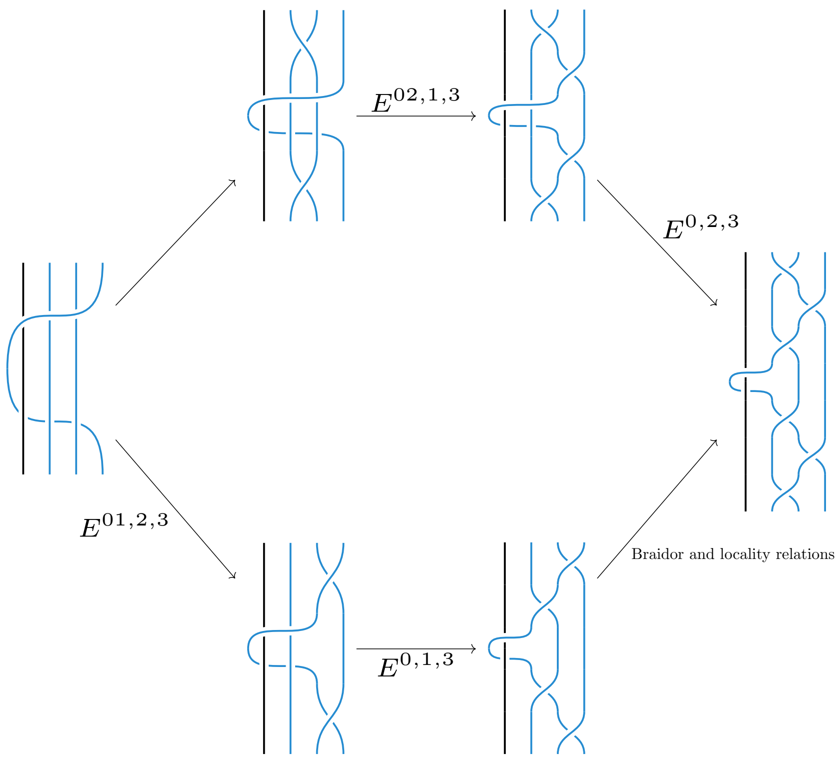


Figure 3.5.1: The derivation of an equation satisfied by the error E in the mixed relation in degree $n+1$. Start at the left-most chord diagram in the figure, then follow the two paths to the right-most diagram. Each time the slide relation must be applied, to get from one diagram to the next, the corresponding error term accrued is written above the error. Summing over the error terms along each path and setting the result to be equal since we end on the same diagram yields the equation $E^{01,2,3} - E^{0,2,3} - E^{02,1,3} + E^{0,1,3} = 0$

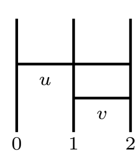
the equation

$$E^{01,2,3} - E^{0,2,3} - E^{02,1,3} + E^{0,1,3} = 0 \tag{3.4}$$

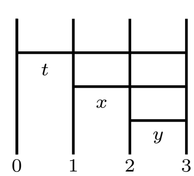
which must be satisfied by E.

Both B and R are grouplike, and by definition E is the lowest degree term in the difference $R^{01,2} - BR^{0,2}B^{0,2,1}$. The lowest degree term in the difference between two grouplike objects must be primitive. E is thus a homogeneous primitive element of $\widehat{U\mathfrak{t}_{1,2}}$, and so must in fact be a Lie polynomial of homogeneous degree $n+1$.

Consider the free Lie algebras $FL[u,v]$ and $FL[x,y,z]$ contained in $\mathfrak{t}_{1,2}$ and $\mathfrak{t}_{1,3}$ as indicated in the diagrams



and



.

Modulo a term in the centre of $\widehat{U\mathfrak{t}_3}$[12], we may assume $E \in FA[u,v]$ which implies that every term in Equation 3.4 lies in $FA[t,x,y]$. In fact, by primitivity, E becomes a Lie polynomial $F \in \mathrm{Lie}[u,v]$.

To determine what Equation 3.4 becomes as an equation in $FL[t,x,y]$, note that $d_0(u) = t + x$ and $d_0(v) = y$. Combining this operation with an appropriate permutation of strands,

$$\begin{aligned} E^{01,2,3} &\mapsto F(t+x,y) \\ E^{0,2,3} &\mapsto F(t,y) \\ E^{02,1,3} &\mapsto F(t+y,x) \\ E^{0,1,3} &\mapsto F(t,x) \end{aligned}$$

and hence Equation 3.4 becomes

$$F(t+x,y) - F(t,y) - F(t+y,x) + F(t,x) = 0. \tag{3.5}$$

in $FL[t,x,y]$.

We will now view the Lie series as lying within the corresponding universal enveloping algebra and work with the full algebra rather than only the primitive part of it. As a primitive part of the free associative algebra, F can be written in the form

$$F(u,v) = F_1(u,v)u + F_2(u,v)v.$$

Equation 3.5 can be encoded via the map $d : FA[u,v] \to FA[t,x,y]$ defined by $dF = F(t+x,y) - F(t,y) - F(t+y,x) + F(t,x)$. To determine more information, we will consider compositions of d with certain projections. First let us consider the composition $ev_{t=0} \circ d$ where $ev_{t=0} : FA[t,x,y] \to FA[x,y]$ evaluates the noncommutative polynomial at $t = 0$. Using the decomposition $F(u,v) = F_1(u,v)u + F_2(u,v)v$ and then applying the mapping $ev_{t=0}$ yields

$$\begin{aligned} F(t+x,y) &= F_1(t+x,y)(t+x) + F_2(t+x,y)y \mapsto F_1(x,y)x + F_2(x,y)y \\ F(t,y) &= F_1(t,y)t + F_2(t,y)y \mapsto F_2(0,y)y \\ F(t+y,x) &= F_1(t+y,x)(t+y) + F_2(t+y,x)x \mapsto F_1(y,x)y + F_2(y,x)x \\ F(t,x) &= F_1(t,x)t + F_2(t,x)x \mapsto F_2(0,x)x. \end{aligned}$$

[12]Recall that $FL[u,v] = \mathfrak{t}_3/\mathcal{Z}(\mathfrak{t}_3)$

The equation $ev_{t=0} \circ d(F) = 0$ implies

$$\begin{aligned}
&F_1(x,y)x + F_2(x,y)y - F_2(0,y)y - F_1(y,x)y - F_2(y,x)x + F_2(0,x)x = 0 \\
\implies &\Big(F_1(x,y) - F_2(y,x) + F_2(0,x)\Big)x + \Big(F_2(x,y) - F_1(y,x) - F_2(0,y)\Big)y = 0 \\
\implies &F_1(x,y) - F_2(y,x) + F_2(0,x) = 0.
\end{aligned}$$

Next consider the composition $\pi_y \circ d$ where $\pi_y : FA[t,x,y] \to FA[t,x]$ projects onto words having only a single y in them which must appear at the right of the word and then discards this trailing y. In other words, π_y sends a noncommutative polynomial of the form $h(t,x)y$ to $h(t,x)$ and sends all other polynomials to zero.

Using the decomposition $F(u,v) = F_1(u,v)u + F_2(u,v)v$ and then applying the mapping π_y yields

$$\begin{aligned}
F(t+x,y) &= F_1(t+x,y)(t+x) + F_2(t+x,y)y \mapsto F_2(t+x,0) \\
F(t,y) &= F_1(t,y)t + F_2(t,y)y \mapsto F_2(t,0) \\
F(t+y,x) &= F_1(t+y,x)(t+y) + F_2(t+y,x)x \mapsto F_1(t,x) \\
F(t,x) &= F_1(t,x)t + F_2(t,x)x \mapsto 0.
\end{aligned}$$

so $\pi_y \circ d(F) = 0$ becomes

$$F_2(t+x,0) - F_2(t,0) - F_1(t,x) = 0.$$

Next do a similar thing with $\pi_x \circ d : FA[u,v] \to FA[x,y]$, where π_t projects onto words ending with t and having no other occurences of t in them, then drops this trailing t. Using the decomposition $F(u,v) = F_1(u,v)u + F_2(u,v)v$ and then applying the mapping π_t yields

$$\begin{aligned}
F(t+x,y) &= F_1(t+x,y)(t+x) + F_2(t+x,y)y \mapsto F_1(x,y) \\
F(t,y) &= F_1(t,y)t + F_2(t,y)y \mapsto F_1(0,y) \\
F(t+y,x) &= F_1(t+y,x)(t+y) + F_2(t+y,x)x \mapsto F_1(y,x) \\
F(t,x) &= F_1(t,x)t + F_2(t,x)x \mapsto F_1(0,x).
\end{aligned}$$

so $\pi_t \circ d(F) = 0$ becomes

$$F_1(x,y) - F_1(0,y) - F_1(y,x) + F_1(0,x) = 0.$$

If we standardize these three equations by changing variables back to u and v, we get the following three equations

$$F_1(u,v) = F_2(u+v,0) - F_2(u,0) \tag{3.6}$$

$$F_1(u,v) - F_2(v,u) = -F_2(0,u) \tag{3.7}$$

$$F_1(u,v) - F_1(v,u) = F_1(0,v) + F_1(0,u) = 0$$

Equation 3.6 implies that $F_1(u,v) = f(u+v) - f(u)$ where $F_2(t,0) = f(t)$, and this together with Equation 3.7 implies that $F_2(t,v) = f(u+v) - f(v) + g(v)$ where $F_2(0,t) = g(t)$. Thus,

$$F(u,v) = \Big(f(u+v) - f(u)\Big)u + \Big(f(t+v) - f(v) + g(v)\Big)v. \tag{3.8}$$

Finally, we claim that primitivity of $F(u,v)$ implies it must be a multiple of v. To see why, consider the left bracketing map $\mathcal{L} : FA[u,v] \to FA[u,v]$ which sends a word in the alphabet $\{u,v\}$ to the iterated commutator obtained by nesting commutators of the letters of the word so that innermost commutators are furthest to the left (constants are sent to zero.) For example $\mathcal{L}(uvuu) = [[[u,v],u],u]$. Consider also the Euler operator $D : FA[u,v] \to FA[u,v]$, the map which sends a word w of length n in the alphabet $\{u,v\}$ to nw. It is a standard fact [Reutenauer (2003)] that on primitive elements, acting via left bracketing is equivalent to acting via minus the Euler operator.

For a function of a single variable like $f(u)$, left bracketing kills any monomial term au^n other than the degree one term, so that $\mathcal{L}(f(u)) = f'(0)u$ and similarly $\mathcal{L}(g(u)) = g'(0)u$. Applying the left bracketing operation to the formula for $F(u,v)$ in Equation 3.8, we get

$$\begin{aligned}
\mathcal{L}(F) &= \Big[\mathcal{L}\Big(f(u+v) - f(u)\Big), u\Big] + \Big[\mathcal{L}\Big(f(u+v) - f(v) + g(v)\Big), v\Big] + g(0)v \\
&= [f'(0)v, u] + [f'(0)u, v] + g(0)v \\
&= g(0)v.
\end{aligned}$$

Enforcing primitivity, it must therefore be the case that $D(F) = -g(0)v$ but the only way this is possible is if $F = \alpha v$ is a multiple of v, and so is homogeneous of degree one.

Since our assumption on the degree one part of the braidor automatically implies the mixed equation holds in degree 1, F can have no degree one component and hence the error in the mixed equation vanishes. $\square$

Corollary 3.5.1. BRAID *is the collection of all grouplike, invertible elements* $B \in \widehat{U\mathfrak{t}_3}$ *such that the* *coefficient of* t_{12} *is* $\frac{1}{2}$ *and which satisfy the single equation*

$$B^{0,1,2}B^{02,1,3}B^{0,2,3} = B^{01,2,3}B^{0,1,3}B^{03,1,2}$$

Proof. By Theorem 3.5.1, the mixed relation can only fail to hold in degree 1, however a direct computation shows that when the degree one term is of the form given then the mixed equation holds in degree one automatically. □

3.6 Extension of Braidors

One of the fundamental results in the theory of associators is that they can be constructed degree by degree, which in fact is an essential part of the proof that rational associators exist [Drinfel'd (1991); Bar-Natan (1998)]. A sketch of a proof with a significant gap is presented in this section, together with some partial progress in filling this gap. Some computational evidence that this conjecture holds is presented in the next chapter.

The unproven conjecture is a reduction of the equations defining an element of $\mathbf{GRT}_a$ to a single equation.

Conjecture 3.6.1. *Let* $\Gamma \in \widehat{U\mathfrak{t}_{1,2}}$ *be a grouplike, invertible element which satisfies the braid equation. Then* Γ *satisfies the slide relation as well.*

Before giving some results dealing with this conjecture, let us assume the conjecture is true and show how it implies that braidors extend degree by degree. All results that are stated as theorems or corollaries on this section must be understand as depending on the as yet unproven Conjecture 3.6.1.

Theorem 3.6.1. *All braidors of degree* n *extend to braidors of degree* $n+1$*. In other words, the map* $\mathbf{BRAID}^{(n+1)} \to \mathbf{BRAID}^{(n)}$ *is surjective.*

The proof of this theorem is inspired by and similar to Drinfel'd's proof of the same result for associators [Drinfel'd (1991)], especially as reformulated in [Bar-Natan (1998)]. The idea of the proof is to use the fact that **BRAID** is a $\mathbf{GRT}_a$ torsor to reduce to showing that the group morphism $\mathbf{GRT}_a^{(n+1)} \to \mathbf{GRT}_a^{(n)}$ is surjective. Since these are unipotent affine algebraic group schemes, it suffices to check this for the derivative maps $\mathfrak{grt}_a^{(n+1)} \to \mathfrak{grt}_a^{(n)}$ of Lie algebras. This becomes trivial after using Conjecture 3.6.1, that one of the equations defining an element in $\mathbf{GRT}_a$ is implied by the others, since then we can simply extend by zero.

Corollary 3.6.1. *The natural homomorphism* $\mathbf{GRT}_a^{(n+1)} \to \mathbf{GRT}_a^{(n)}$ *is surjective.*

Proof. By Propositions 3.3.3 and 3.3.4, $\mathbf{GRT}_a^{(n)}$ is a unipotent affine algebraic group scheme so it is enough to check this for the derivative map $\mathfrak{grt}_a^{(n+1)} \to \mathfrak{grt}_a^{(n)}$. By Conjecture 3.6.1 the slide equation is implied by the braid equation in degrees greater than 1 so it is not needed as part of the definition of $\mathbf{GRT}_a$. Hence the Lie algebra is defined by the single equation

$$\gamma^{0,1,2} + \gamma^{02,1,3} + \gamma^{0,3,2} = \gamma^{01,2,3} + \gamma^{0,1,3} + \gamma^{03,1,2}.$$

in degrees greater than 1.

A direct computation shows that elements of $\mathbf{GRT}_a^{(1)}$ can be extended to degree 2, and if γ satisfies this equation up to degree n, it can be extended to satisfy this equation up to degree $n+1$ by simply defining it to be 0 in degree $n+1$[13]. □

Corollary 3.6.2. *The natural map* $\mathbf{BRAID}^{(n+1)} \to \mathbf{BRAID}^{(n)}$ *is surjective.*

Proof. Let B be a braidor up to degree n. We've already observed braidors exist by constructing a braidor out of an associator so there is at least one degree n braidor B_0 which extends to a degree $n+1$ braidor. Since $\mathbf{GRT}_a^{(n)}$ acts transitively on $\mathbf{BRAID}^{(n)}$ there is some $G \in \mathbf{GRT}_a^{(n)}$ with $G \bullet B = B_0$. Since $\mathbf{GRT}_a^{(n+1)} \to \mathbf{GRT}^{(n)}$ is surjective, there is a $\Gamma \in \mathbf{GRT}_a^{(n+1)}$ which agrees with G up to degree n. Then $\Gamma^{-1} \bullet B_0$ is the required extension of B to a braidor of degree $n+1$. □

A similar strategy as was used to show the braid equation implies the mixed equation for braidors can be used to begin attempting to deal with Conjecture 3.6.1. Let $\Gamma \in \widehat{U\mathfrak{t}_{1,2}}$ be a grouplike element which satisfies the braid equation. Assume in addition that Γ satisfies the slide equation up to degree n and let $E \in \widehat{U\mathfrak{t}_{1,3}}$ be the error in the slide relation in degree $n+1$.

As in the proof of Theorem 3.5.1 syzygies, that is relations betwewen the relations defining $\mathbf{GRT}_a$, are required in order to derive equations which must be satisfied by E, in the hopes of showing that no error can satisfy these equations other than the zero element.

Three such relations can be obtained from the diagrams in Figure 3.6.1 and Figure 3.6.2. Recall from the proof of Theorem 3.5.1 that indicated in this figure is a sequence of diagrams which should be equivalent in the algebra of chord diagrams, however each time a slide relations is required to move from one diagram to the next an instance of the error term installed on the correct strands is required as Γ does not satisfy the braid equation in degree $n+1$. Having gone all the way around the diagram, the

[13]It is here that the argument fails if the slide equation is required. In order for the linearization of the slide equation to hold true in degree $n+1$ puts a new condition on γ in degree n due to the presence of the t_{ij} factors in the equation so that it would not be possible to simply extend by zero anymore if this relation is part of the definition of the Lie algebra.

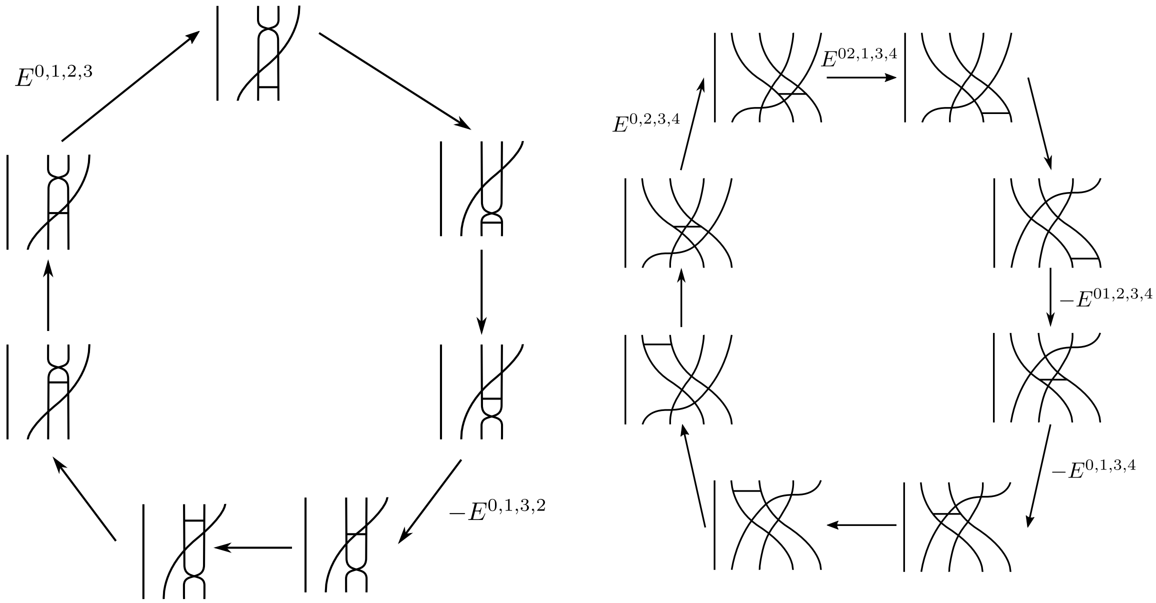


Figure 3.6.1: The derivation of the syzygies S1 and S2. Start at any chord diagram in the figure, then follow the entire loop indicated by the arrows back to the start. Each time the slide relation must be applied, to get from one diagram to the next, the corresponding error term accrued is written above the arrow. Summing over all these error terms and setting the result equal to zero in the left diagram yields the equation $E^{0,1,2,3} = E^{0,1,3,2}$ while doing this for the right diagram yields $E^{0,2,3,4} + E^{02,1,3,4} = E^{0,1,3,4} + E^{01,2,3,4}$.

sum of the accumulated errors must be equal to zero since we return to the original diagram and hence an equation which must be satsified by E is obtained.

$$E^{0,1,2,3} = E^{0,1,3,2} \qquad (S1)$$

$$E^{0,2,3,4} + E^{02,1,3,4} = E^{0,1,3,4} + E^{01,2,3,4} \qquad (S2)$$

$$[t_{12}, E^{0,1,3,4} + E^{01,2,3,4}] = [t_{34}, E^{0,3,1,4} + E^{03,4,1,2}] \qquad (S3).$$

These three syzygies do appear to put significant restrictions on the error term E, however there are nonzero solutions and so far we have been unable to either find other equations E must satisfy or else to show using other properties an error in the slide relation must satisfy that only the zero element can satisfy all three equations plus these other properties.

Direct computations in low degree appear to indicate that a Lie polynomial $E \in \widehat{U\mathfrak{t}_{1,3}}$ of degree n is constrained by these equations to lie in "small" subspaces of $\widehat{U\mathfrak{t}_{1,3}}$. More precisely, there is an isomorphism $\mathfrak{t}_{1,3} \cong FL_1 \rtimes FL_2 \rtimes FL_3$ (recall Equation 3.2) so the error in degree n is a priori an element of an algebra which is the size of the degree n component of this iterated semidirect product.

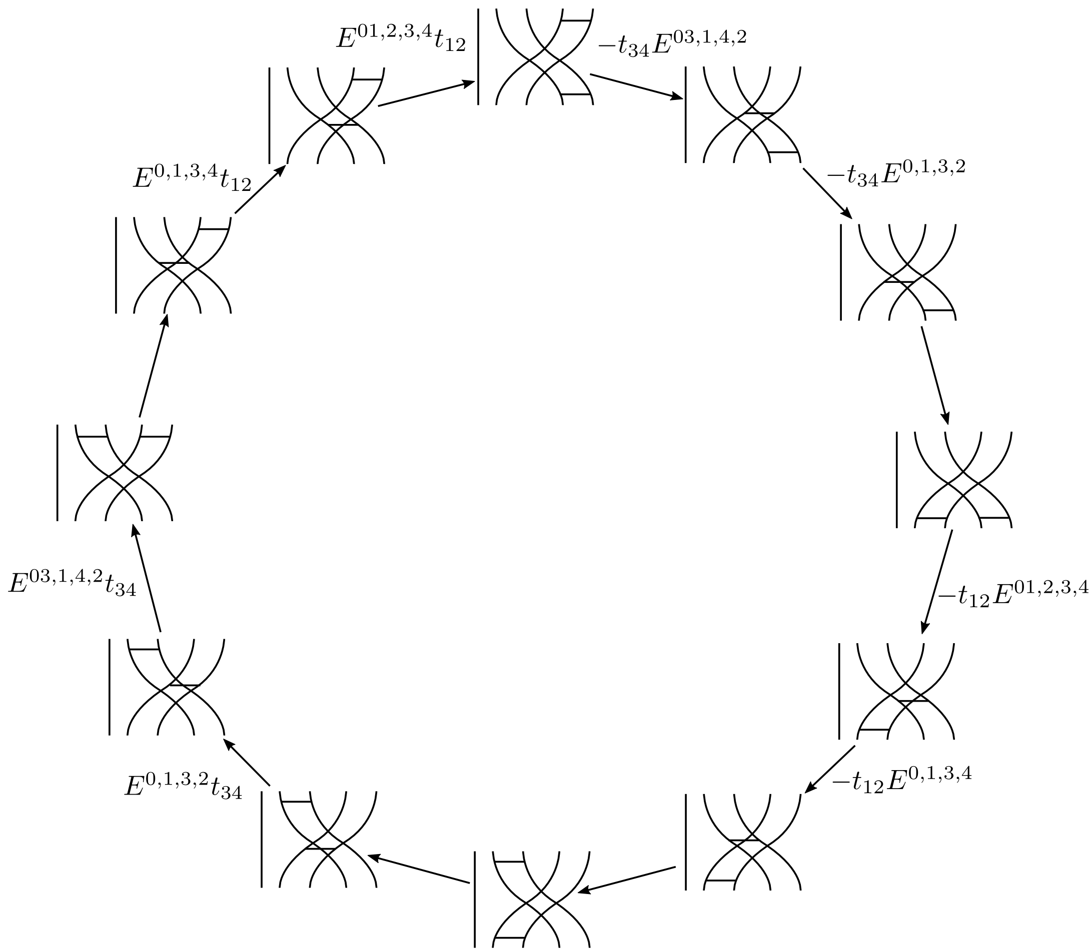


Figure 3.6.2: The derivation of the syzygy S3. Start at any chord diagram in the figure, then follow the entire loop indicated by the arrows back to the start. Each time the slide relation must be applied, to get from one diagram to the next, the corresponding error term accrued is written above the error. Summing over all these error terms and setting the result equal to zero yields the equation $[E^{0,1,3,4} + E^{01,2,3,4}, t_{12}] + [E^{0,1,3,2} + E^{03,1,4,2}, t_{34}] = 0$.

Computations in low degrees indicate that syzygy $S2$ alone already constrains the degree n error E to lie in a subspace whose dimension is twice that of the degree n component of a free Lie algebra on two generators minus one, a significant reduction of the size of the full semidirect product. Furthermore, although only computed to low degree, it appears that solutions to all three syzygy equations in degree n are constrained to lie in subspaces which grow linearly with n. It may however be the case that growth becomes superlinear in higher degrees.

While it therefore seems Conjecture 3.6.1 can be reduced to dealing with a relatively small and manageable space, we have not been able to resolve the conjecture at this point.

Chapter 4

Computations, Conjectures and Future Work

4.1 Computations Using FreeLie.m

Computations in free Lie algebras (and related algebras like the Drinfel'd-Kohno algebra appearing in this paper) have been implemented by Bar-Natan in the Mathematica package **FreeLie.m**. Some documentation as well as many examples of nontrivial computations using this and related packages can be found in Bar-Natan (2015). For this paper some additional functionality not contained in previous versions was required. The version of **FreeLie.m** used in this paper can be obtained from www.math.toronto.edu/drorbn/People/Ens/thesis/FreeLie.m. The Mathematica notebook itself is available at www.math.toronto.edu/drorbn/People/Ens/thesis/FreeLie.nb.

In particular the package allows the computation of all solutions of equations like those defining braidors and associators up to some fixed degree n. Unfortunately, in order for such computations to be truly meaningful, in the sense that information about braidors (or elements of Grothendieck-Teichmüller groups) is actually being obtained, it first must be shown that all braidors up to degree n actually extend to full braidors. As a result, all the computations in this section depend on the unproven Conjecture 3.6.1 in order to have relevance for braidors.

Even assuming Conjecture 3.6.1, computations up to a given degree can not generally be used to prove conjectures, however they do allow conjectures to be formulated and provide some evidence that a conjecture is true by verfying it to a high degree. On the other hand, these computations can be used to disprove conjectures by explicitly constructing counterexamples up to some degree n computationally and assuming the conjecture that all braidors extend concluding the existence of full braidors as counterexamples.

Throughout this chapter, we will rewrite all equations involving braidors and $\mathbf{GRT}_a$ as equations in $\widehat{U\mathfrak{t}_{n+1}}$ rather than $\widehat{U\mathfrak{t}_{1,n}}$ as this is the indexing used in **FreeLie.m**. This amounts to shifting the indices i and j in t_{ij} up by 1.

4.1.1 Braidors and Associators in the Drinfel'd-Kohno Algebra

As explained in the introduction, it seems reasonable to expect that there are more braidors than associators. In other words a priori it seems reasonable to expect the injective map C : **ASSOC** $\to$ **BRAID** not to be surjective. To test this we can solve for both an associator and a braidor up to some fixed degree and check how many arbitrary choices must be made in each degree to compute the dimension of the solution space. This is done up to degree 10 in the Mathematica notebook **BraidorsInDK10.nb** (www.math.toronto.edu/drorbn/People/Ens/thesis/BraidorsInDK10.nb), which we now summarise.

To begin, ensure the working directory contains the **FreeLie.m** package. This package is then loaded and set to display infinite series to degree 3 and to pick random values between −100 and 100 when making arbitrary choices in solving equations.

```
<< FreeLie.m;
$SeriesShowDegree = 3;
Arb = Arbitrator → (Replace[#, _ → RandomInteger[{-100, 100}], 1] &);
```

Next, Φ is defined to be an infinite formal Lie series in $\widehat{U\mathfrak{t}_3}$[1] (a Drinfel'd-Kohno Series DKS in the notation of the code) with coefficients labelled by $\Phi s[i_1, \cdots, i_n]$. Recall that in all the code samples in this chapter, the indicies labelling the Drinfeld-Kohno generators t_{ij} are shifted up by one compared to the notation used previously since in the Mathematica implementation the numbering begins at 1 rather than 0.

Since associators are fixed in the first few degrees, these coefficients are set manually. The **SeriesSolve** function from **FreeLie.m** is then called which defines each coefficient $\Phi s[i_1, \cdots, i_n]$ to be a program which will solve the equation and store some information about the solution (but not the solution itself) in the variable **assocInfo**. Note that the concept of lazy evaluation is used to deal with infinite series here. Calling **SeriesSolve** merely defines Φ to be a program which takes as input an integer n and outputs a solution to the equations in degree n rather than actually computing anything. These programs are then run by Mathematica only to as high degree as is needed in any given computation so that there is no need to explicitly truncate infinite series by hand.

[1] Note that **FreeLie.m** always implements these series on the level of Lie algebras and so the grouplike series in $\widehat{U\mathfrak{t}_3}$ described in earlier chapters is the exponential of this Lie series.

```
Φ = DKS[3, Φs];
Φs[2, 1] = Φs[3, 1] = Φs[3, 2] = 0; Φs[3, 1, 2] = 1 / 24;
assocInfo =
  SeriesSolve[Φ, Φ^σ[3,2,1] ≡ -Φ && Φ ** Φ^σ[1,23,4] ** Φ^σ[2,3,4] ≡ Φ^σ[12,3,4] ** Φ^σ[1,2,34]];
```

We next ask Mathematica to evaluate Φ to degree 6 and time the result. The output displays a rational Drinfel'd associator to this degree. The Lie bracket is denoted in this code by an overbracket.

```
Φ@{6} // Timing
```

$$\Big\{1.20667,\ \mathrm{DKS}\Big[0,\ \frac{1}{24}[t_{13},t_{23}],\ 0,\ -\frac{7\,[[[t_{13},t_{23}],t_{23}],t_{23}]}{5760}+\frac{7\,[t_{13},[[t_{13},t_{23}],t_{23}]]}{5760}-\frac{[t_{13},[t_{13},[t_{13},t_{23}]]]}{1440},$$

$$0,\ \frac{31\,[[[[[t_{13},t_{23}],t_{23}],t_{23}],t_{23}],t_{23}]}{967680}-\frac{157\,[[t_{13},[[t_{13},t_{23}],t_{23}]],[t_{13},t_{23}]]}{1935360}-\frac{31\,[[t_{13},t_{23}],[[[t_{13},t_{23}],t_{23}],t_{23}]]}{387072}-$$

$$\frac{31\,[t_{13},[[[[t_{13},t_{23}],t_{23}],t_{23}],t_{23}]]}{483840}+\frac{11\,[t_{13},[[t_{13},[t_{13},t_{23}]],[t_{13},t_{23}]]]}{290304}+\frac{31\,[t_{13},[[t_{13},t_{23}],[[t_{13},t_{23}],t_{23}]]]}{725760}+$$

$$\frac{83\,[t_{13},[t_{13},[[[t_{13},t_{23}],t_{23}],t_{23}]]]}{967680}-\frac{13\,[t_{13},[t_{13},[t_{13},[[t_{13},t_{23}],t_{23}]]]]}{241920}+\frac{[t_{13},[t_{13},[t_{13},[t_{13},[t_{13},t_{23}]]]]]}{60480},\ \ldots\Big]\Big\}$$

Finally, the number of arbitrary choices **SeriesSolve** made in each degree to construct the associator is computed from the information stored in **assocInfo**

```
ArbAssoc = Length[Last[#]] & /@ Read[assocInfo]
```

{0, 0, 1, 0, 1, 0}

Next we repeat the same procedure replacing the pentagon equation with the braid equation. Recall that by Corollary 3.5.1, the single braid equation defines braidors except in degree 1, and to ensure compatibility with braidors coming from associators as explained in Section 3.4, we assume the braidor is in $\mathbf{BRAID}_0$ and set the coefficient of t_{12} and t_{13} to zero.

```
B = DKS[3, Bs];
Bs[2, 1] = Bs[3, 1] = 0; Bs[3, 2] = 1 / 2;
braidInfo =
  SeriesSolve[B, B^σ[1,2,3] ** B^σ[13,2,4] ** B^σ[1,3,4] ≡ B^σ[12,3,4] ** B^σ[1,2,4] ** B^σ[14,2,3]];
```

Finally we compute an arbitrary braidor up to degree 6 and display the number of arbitrary choices made in each degree.

```
B@{6} // Timing
```

SeriesSolve: In degree 3 arbitrarily setting {Bs[3, 1, 2, 2] → 0}.

SeriesSolve: In degree 5 arbitrarily setting {Bs[3, 1, 1, 1, 2, 2] → 0}.

$$\Big\{2.50333,\ \mathrm{DKS}\Big[\frac{t_{23}}{2},\ \frac{1}{12}\, t_{13}\, t_{23},\ 0,\ -\frac{t_{13}\, t_{23}\, t_{23}\, t_{23}}{5760} + \frac{1}{720}\, t_{13}\, t_{13}\, t_{23}\, t_{23} - \frac{1}{720}\, t_{13}\, t_{13}\, t_{13}\, t_{23},$$

$$-\frac{t_{13}\, t_{23}\, t_{23}\, t_{23}\, t_{23}}{7680} + \frac{t_{13}\, t_{23}\, t_{13}\, t_{23}\, t_{23}}{8640} + \frac{t_{13}\, t_{13}\, t_{23}\, t_{23}\, t_{23}}{3840},$$

$$\frac{t_{13}\, t_{23}\, t_{23}\, t_{23}\, t_{23}\, t_{23}}{645\,120} - \frac{t_{13}\, t_{13}\, t_{23}\, t_{23}\, t_{13}\, t_{23}}{145\,152} - \frac{71\, t_{13}\, t_{23}\, t_{13}\, t_{23}\, t_{23}\, t_{23}}{2\,903\,040} -$$

$$\frac{23\, t_{13}\, t_{13}\, t_{23}\, t_{23}\, t_{23}\, t_{23}}{483\,840} + \frac{t_{13}\, t_{13}\, t_{13}\, t_{23}\, t_{13}\, t_{23}}{20\,160} + \frac{t_{13}\, t_{13}\, t_{23}\, t_{13}\, t_{23}\, t_{23}}{22\,680} +$$

$$\frac{13\, t_{13}\, t_{13}\, t_{13}\, t_{23}\, t_{23}\, t_{23}}{161\,280} - \frac{t_{13}\, t_{13}\, t_{13}\, t_{13}\, t_{23}\, t_{23}}{15\,120} + \frac{t_{13}\, t_{13}\, t_{13}\, t_{13}\, t_{13}\, t_{23}}{30\,240},\ \ldots\Big]\Big\}$$

```
ArbBraid = Length[Last[#]] & /@ Read[braidInfo]
```

{0, 0, 1, 0, 1, 0}

For complete output up to degree 10, see the Mathematica notebook **BraidorsInDK10.nb** linked to above. The fact that the dimensions of the solution spaces turn out to be the same for both braidors and associators up to degree 10 in these computations leads to the

Conjecture 4.1.1. *The map C : **ASSOC** $\to$ **BRAID**$_0$ defined in Section 3.3.5 is surjective. Since **BRAID** and **ASSOC** are torsors for the Grothendieck-Teichmüller groups, we further conjecture that **GRT** $\cong$ **GRT**$_{a,0}$ and **GT** $\cong$ **GT**$_{a,0}$ as prounipotent affine group schemes where **GRT**$_{a,0}$ and **GT**$_{a,0}$ are the versions of these groups which are zero in degree* 1.

There is a related, but not equivalent result in [Lochak & Schneps (1993)] which ought to be mentioned. In that paper the full braid groups plus a zeroth doubling operation are considerd rather than the annular versions. The result from that paper could be translated, with some modifications to account for the fact that profinite completions are used there rather then prounipotent completions, to show that a subset of all braidors which preserve some extra structure not included in our algebraic structures come from associators but the question of whether all braidors are of this special form remains open.

4.1.2 Braidors In the Kashiwara-Vergne Algebra

The computations up until this point have assumed braidors and associators are elements of the completed Drinfel'd-Kohno algebra. It is possible however to ask about the existence and properties of these objects in any other space in which the equations make sense.

As a first exploration of braidors in other algebras and to test whether the conjectured equality of braidors and associators is a general feature of the equations or something unique to the Drinfel'd-Kohno algebra, the Kashiwara-Vergne and related algebras provide a good testing ground.

For completeness we give a brief review of the relevant definitions found in [Alekseev & Torossian (2012)] (see also [Bar-Natan & Dancso (2017)] for an interpretation of the following algebras and maps in terms of the topology of knotted objects.) Let $\mathfrak{der}_n$ be the Lie algebra of derivations of the free Lie algebra $FL_n = FL[x_1, \cdots, x_n]$ on n generators. Let $\mathfrak{tr}_n$ be the quotient of the positive degree part of the free associative algebra $FA_n = FA[x_1, \cdots, x_n]$ on n generators by commutators:

$$\mathfrak{tr}_n = \left(\prod_{k=1}^{\infty} FA_n\right) \Big/ \langle ab - ba \ : \ a, b \in FA_n \rangle.$$

The quotient map $\mathrm{tr} : FA_n \to \mathfrak{tr}_n$ is called the trace map.

A derivation $u \in \mathfrak{der}_n$ is called *tangential* if there exist $a_i \in FL_n$ such that $u(x_i) = [x_i, a_i]$ for $1 \leq i \leq n$. A tangential derivation is called *special* if

$$u\left(\sum_{i=1}^{n} x_i\right) = 0.$$

The Lie algebra of tangential derivations is denoted by $\mathfrak{tder}_n$ and the Lie algebra of special derivations is denoted by $\mathfrak{sder}_n$. That these both form Lie subalgebras of $\mathfrak{der}_n$ can be verified by a direct computation.

There is a map $\mathrm{div} : \mathfrak{tder}_n \to \mathfrak{tr}_n$ which, given a tangential derivation u represented by $(a_1, \cdots, a_n)$ as in the previous paragraph, is defined by

$$\mathrm{div}(u) = \sum_{k=1}^{n} \mathfrak{tr}(x_k(\partial_k a_k)).$$

The *Kashiwara-Vergne* Lie algebras $\mathfrak{kv}_n$ are the special derivations with vanishing divergence and also form a Lie subalgebra.

There is an action of S_n on $\mathfrak{tder}_n$ defined by sending u, represented by $(a_1, \cdots, a_n)$, to the derivation represented by

$$\big(a_{\sigma^{-1}(1)}(x_{\sigma(1)}, \cdots, x_{\sigma(n)}), \cdots, a_{\sigma^{-1}(n)}(x_{\sigma(1)}, \cdots, x_{\sigma(n)})\big)$$

for any $\sigma \in S_n$.

Using the action of S_n, the relevant operators appearing in the definitions of braidors and associators, usually called simplicial and coproduct maps in this context, can all be constructed out of the following two basic ones. The simplicial map, which when applied to a derivation $u \in \mathfrak{tder}_{n-1}$ is denoted

$u^{1,2,\cdots,n-1} \in \mathfrak{tder}_n$, maps $(a_1, \cdots, a_{n-1}) \to (a_1, \cdots, a_{n-1}, 0)$. The coproduct, which when applied to a derivation $u \in \mathfrak{tder}_{n-1}$ is denoted by $u^{12,3,\cdots,n} \in \mathfrak{tder}_n$, sends a representative $(a_1, \cdots, a_{n-1})$ to

$$\big(a_1(x_1+x_2, x_3, \cdots, x_n), a_2(x_1+x_2, x_3, \cdots, x_n), \cdots, a_n(x_1+x_2, x_3, \cdots, x_n)\big).$$

These operations, together with all the variants obtained by applying the S_n action, are well-defined on $\mathfrak{tder}_n, \mathfrak{sder}_n$ and $\mathfrak{kv}_n$ and hence can be defined on the corresponding Lie groups $\mathrm{TAut}_n, \mathrm{SAut}_n$ and KV_n of these Lie algebras. All the questions we have been asking about braidors in the Drinfel'd-Kohno algebra can thus be asked in each of these spaces as well, now that the operators required in the pentagon and braidor equations are defined.

To begin we solve for a braidor in each of the three new spaces in the Mathematica notebook **BraidorsInDer.nb** (www.math.toronto.edu/drorbn/People/Ens/thesis/BraidorsInDer.nb). The initialization is as before.

```
<< FreeLie.m;
$SeriesShowDegree = 3;
Arb = Arbitrator → (Replace[#, _ :> RandomInteger[{-100, 100}], 1] &);
```

The remaining computation is nearly identical to the one in the Drinfel'd-Kohno algebra, remembering that a braidor $B_1 \in \mathrm{TAut}$ will be represented in **FreeLie.m** by three Lie series in the variables $1, 2$ and 3. $\mu\mathbf{1}, \nu\mathbf{1}$ and $\eta\mathbf{1}$ are defined to be Lie Series in $\widehat{FL[1,2,3]}$ and $\mathbf{B_1}$ is the derivation which sends $1 \mapsto \mu 1$, $2 \mapsto \nu 1$ and $3 \mapsto \eta 1$. We then solve for $\mathbf{B_1}$ up to degree 6. Note that the proof that a braidor is defined by just a single equation is only valid in the Drinfel'd-Kohno algebra so both equations are required here.

```
μ1 = LS[{1, 2, 3}, μ1s]; ν1 = LS[{1, 2, 3}, ν1s]; η1 = LS[{1, 2, 3}, η1s];
B₁ = ⟨1 → μ1, 2 → ν1, 3 → η1⟩;
BraidorTang = SeriesSolve[{μ1, ν1, η1},
   (B₁^σ[1,2,3] ** B₁^σ[13,2,4] ** B₁^σ[1,3,4] ≡ B₁^σ[12,3,4] ** B₁^σ[1,2,4] ** B₁^σ[14,2,3]) ∧
    (RR^σ[1,2] ** B₁^σ[1,2,3] ** RR^σ[1,3] ** B₁^σ[1,3,2]) ≡
     (B₁^σ[1,2,3] ** RR^σ[1,3] ** B₁^σ[1,3,2] ** RR^σ[1,2]) ∧
    (RR^σ[12,3] ≡ B₁^σ[1,2,3] ** RR^σ[1,3] ** B₁^σ[1,3,2]), Arb];
B₁@{6}; // Timing
```

```
SeriesSolve: No solution in degree 2.
```

Since there is no solution in degree 2, we see that the analogue of Conjecture 1.1.1 does not hold in TAut, braidors need not be constructible degree by degree here. This is not surprising since the locality relations do not hold in TAut and so the theory would be expected to exhibit pathologies.

Doing the same computation for special derivations requires only adding the extra equation $B(1 + 2 + 3) = 0$ when solving for a braidor.

```
μ2 = LS[{1, 2, 3}, μ2s]; ν2 = LS[{1, 2, 3}, ν2s]; η2 = LS[{1, 2, 3}, η2s];
B2 = ⟨1 → μ2, 2 → ν2, 3 → η2⟩;
BraidorSpec = SeriesSolve[{μ2, ν2, η2},
   ℏ^-1 (b[LW@1, μ2] + b[LW@2, ν2] + b[LW@3, η2] ≡ LS[0]) ∧
    (B2^σ[1,2,3] ** B2^σ[13,2,4] ** B2^σ[1,3,4] ≡ B2^σ[12,3,4] ** B2^σ[1,2,4] ** B2^σ[14,2,3]) ∧
    (RR^σ[1,2] ** B2^σ[1,2,3] ** RR^σ[1,3] ** B2^σ[1,3,2]) ≡
     (B2^σ[1,2,3] ** RR^σ[1,3] ** B2^σ[1,3,2] ** RR^σ[1,2]) ∧
    (RR^σ[12,3] ≡ B2^σ[1,2,3] ** RR^σ[1,3] ** B2^σ[1,3,2]), Arb];
B2@{6}; // Timing
```

```
SeriesSolve: No solution in degree 6.
```

Once again, the computation fails to find a solution, in degree 6 this time. It is somewhat more suprising that Conjecture 1.1.1 fails in SAut since the locality relations do hold here.

Finally, to do the computation in KV_3 simply add the equation requiring the divergence to vanish.

```
μ3 = LS[{1, 2, 3}, μ3s]; ν3 = LS[{1, 2, 3}, ν3s]; η3 = LS[{1, 2, 3}, η3s];
B3 = ⟨1 → μ3, 2 → ν3, 3 → η3⟩;
BraidorKV = SeriesSolve[{μ3, ν3, η3},
   ℏ^-1 (b[LW@1, μ3] + b[LW@2, ν3] + b[LW@3, η3] ≡ LS[0]) ∧
    B3^σ[1,2,3] ** B3^σ[13,2,4] ** B3^σ[1,3,4] ≡ B3^σ[12,3,4] ** B3^σ[1,2,4] ** B3^σ[14,2,3] ∧
    div[B3] ≡ CWS[0] ∧ (RR^σ[1,2] ** B3^σ[1,2,3] ** RR^σ[1,3] ** B3^σ[1,3,2]) ≡
     (B3^σ[1,2,3] ** RR^σ[1,3] ** B3^σ[1,3,2] ** RR^σ[1,2]) ∧
    (RR^σ[12,3] ≡ B3^σ[1,2,3] ** RR^σ[1,3] ** B3^σ[1,3,2]), Arb];
B3@{6}; // Timing
```

```
{55.0567, Null}
```

This time the computation finishes. Looking at the number of free choices made in each degree,

```
ArbBraidor = Length[Last[#]] & /@ Read[BraidorKV]
```

```
{1, 0, 1, 0, 1, 0}
```

we see that there is agreement with what happens for associators in the Drinfel'd-Kohno algebra. This leads to the

Conjecture 4.1.2. *Braidors in* KV *can be constructed degree by degree and the map* C : **ASSOC** $\to$ $\mathbf{BRAID}_0$ *is surjective when interpreted inside* KV*. Furthermore any braidor in* KV *is the image of a braidor in the Drinfel'd-Kohno algebra via the map* $\widehat{U\mathfrak{t}_3} \to \mathrm{KV}_3$ *described in [Alekseev & Torossian (2012)].*

These conjectures are verified to degree 8 in the Mathematica notebook **BraidorsInDer.nb** linked to above.

4.2 Future Work

The work carried out in this thesis is just the beginning of a study of the algebraic structure of annulur braids and braidors. The next step is to fill in the gap in the proof of Conjecture 3.6.1, that braidors can be constructed degree by degree, in order to make the computations of this chapter truly meaningful. As observed in Section 3.6 we believe this can be reduced to a question about a relatively small and controllable subspace of $\widehat{U\mathfrak{t}_{1,3}}$ and hence expect this to be an achievable goal, despite having failed thus far.

Conjecture 4.1.1, that essentially all braidors come from associators or the corresponding isomorphism of the Grothendieck-Teichmüller groups with their annular counterparts, would be the next goal however this problem has the potential to be significantly more difficult. In fact there are a whole collection of prounipotent affine group schemes, all related via inclusions as in the diagram

$$\begin{tikzcd} & \mathrm{GRT}_a \arrow[dl, dashed, leftrightarrow] \arrow[dr, dashed, leftrightarrow] \arrow[drr, dashed, leftrightarrow] & & \\ \pi_1\big(\mathrm{MT}(\mathbb{Z})\big) \arrow[r, hook] & \mathrm{GRT} \arrow[u] \arrow[r, hook] & \mathrm{DMR}_0 \arrow[r, hook] & \mathrm{KRV}. \end{tikzcd}$$

Here $\pi_1(\mathrm{MT}(\mathbb{Z}))$ is the motivic fundamental group, DMR_0 is the regularised double shuffle group and KRV is the group of degenerate solutions of the Kashiwara-Vergne problem. A brief discussion of these group and the relations between can be found in [Furusho (2010)]. These five groups are all speculated to be isomorphic and the indicated inclusions are already nontrivial results however no isomorphisms have yet been shown to exist. The question of showing a group which is related to the Grothendieck-Teichmüller group is actually isomorphic to it thus has a history of being very difficult.

On the other hand, if the conjectured equivalence of the annular and non-annular versions of the Grothendieck-Teichmüller groups could be proven then the existence of a new group isomorphic to **GRT**, together with its interpretation in terms of braids in the annulus, may lead to new ways of obtaining information about the other groups in this diagram via the as yet unkown maps indicated by blue dashed arrows so exploring the relationship between $\mathbf{GRT}_{a,0}$ and the other four groups could provide interesting results .

Moving on to the knot-theoretic aspects of braidors, a universal finite-type invariant of braids in the annulus is constructed in this thesis. More interesting would be a universal finite-type invariant of tangles in the annulus, since in particular this would yield a universal finite type invariant of usual knots. In order to achieve this requires determining how to extend a functor $Z : \boldsymbol{B_a} \to \mathbf{CD_a}$ so that it acts on the cups and caps which appear in tangle diagrams in an invariant way, and study the resulting invariant of knots in an annulus and usual knots. The extension to tangles in an annulus is expected to be a straightforward short term goal.

Finally, while Conjectures 4.1.1 and 4.1.2 may appear to stymie the original goal of finding an easier to construct and compute universal finite type invariant of knots using braidors, as every braidor comes from an associator anyway, recall that these conjectures only apply to braidors and associators in the Drinfeld-Kohno algebra or the Kashiwara-Vergne algebra. It may be possible to work around these conjectures by replacing the Drinfel'd -Kohno algebra by other algebras. Since braidors require only the monoidal product operation, it is possible to replace the Drinfeld-Kohno algebras, in which all the operadic partial compositions are defined, by other algebras in which the monoidal product operation is well-defined but the other operadic partial compositions are not. A braidor in such an algebra can then be defined using the same formulas as in Definition 3.3.1, and such a braidor could not possible come from an associator and so may be new and simpler.

As an example of such an algebra, let $A_n^0 = \langle \{\sigma\}_{\sigma\in S_n}, x, t_1, \cdots, t_n\rangle$ be the free associative algebra with given generators where S_n is the symmetric group on $\{1, \cdots, n\}$. Quotient A_n^0 by the following relations

1. x is in the center of the algebra.
2. The relations in the symmetric group S_n are satisfied by permutations σ.
3. $t_i\sigma = \sigma t_{\sigma i}$
4. $[t_i, t_j] = x\sigma_{ij}(t_i - t_j)$ where σ_{ij} is the transposition interchanging i and j.

Let A_n be the degree completion of this quotient where $\deg x = 1, \deg t_i = 1$ and $\deg \sigma = 0$ for any permutation σ.

The operation $d_\infty : A_n \to A_{n+1}$ is the obvious inclusion obtained by regarding $\{1, \cdots, n\}$ as a subset of $\{1, \cdots, n+1\}$. The doubling operation $d_0 : A_n \to A_{n+1}$ is defined on generators by $d_0(x) = x$, $d_0(t_i) = t_{i+1} + x\sigma_{i,i+1}$ and d_0 acts on the permutation σ by identifying $\{1, \cdots, n\}$ with $\{2, \cdots, n+1\} \subset \{1, \cdots, n+1\}$.

Having defined the operations d_0 and d_∞, the equations defining a braidor make sense in $\prod_{i=1}^{\infty} A_n$. Developing the theory of braidors in this algebra, some preliminary results of which can be found at http://drorbn.net/index.php?title=The_HOMFLY_Braidor_Algebra, or other algebras with the appropriate operations allowing the braidor equations to be defined may lead to interesting and simpler invariants of braids and tangles, yet another direction to be pursued.

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