VOLUME FORMULA AND INTERSECTION PAIRINGS OF N-FOLD REDUCED PRODUCTS

by

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A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy
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Abstract

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2019

Let G be a semisimple compact connected Lie group. An N-fold reduced product of G is the symplectic quotient of the Hamiltonian system of the Cartesian product of N coadjoint orbits of G under diagonal coadjoint action of G. Under appropriate assumptions, it is a symplectic orbifold. Using the technique of nonabelian localization and the residue formula of Jeffrey and Kirwan, we investigate the symplectic volume and the intersection pairings of an N-fold reduced product of G. In 2008, Suzuki and Takakura gave a volume formula of N-fold reduced products of $\mathbf{SU}(3)$ via Riemann-Roch. We compare our volume formula with theirs and prove that up to normalization constant, our volume formula completely matches theirs in the case of triple reduced products of $\mathbf{SU}(3)$.

Acknowledgements

First and foremost, I would like to express my deepest gratitude to my advisor, Lisa C. Jeffrey, for her guidance, encouragement, insight, patience and generosity. I could not have come this far without her generous support.

I would also like to thank Yael Karshon for her encouragement and inspiring conversations.

My gratitude also goes to Almut Burchard for her support and advices.

I also want to thank Alex Nabutovsky for his advices during my early years in the graduate school.

My thanks also go to Man-Duen Choi, Bernardo Galvao-Sousa, Marco Gualtieri, Abe Igelfeld, Boris Khesin, Anthony Lam, Eckhard Meinrenken, Pierre Milman, Kumar Murty and Paul Selick for their help and advices.

I would also like to thank the staff at the math department for their help during all these years.

I also want to thank my friends at the math department for their support and many cheerful conversations.

Last but not least, I want to thank my parents for their support, patience and understanding.

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Chapter 1

Introduction

In this thesis, we study objects called N-fold reduced products. The formal definitions will be given in later chapters. Roughly speaking, these objects are reduced spaces obtained from certain Hamiltonian systems.

Given a semisimple compact connected Lie group G, we may consider its adjoint orbits in its Lie algebra $\mathfrak{g}$. If we are given N such adjoint orbits, we can form their Cartesian product denoted by M. G acts on this product diagonally through adjoint action. This is a Hamiltonian system with moment map $\mu_G : M \to \mathfrak{g}$. Notice that in this thesis we will fix a G-invariant inner product on $\mathfrak{g}$ and hence we can identify adjoint orbits with coadjoint orbits, which are naturally symplectic manifolds. An N-fold reduced product is the reduced space $\mu_G^{-1}(0)/G$ of this Hamiltonian system.

In general, the geometry of these reduced products are very complicated and thus difficult to study. For example, they are in general not smooth manifolds. Even with the assumptions that we are going to make, they are still in general only orbifolds. Roughly speaking, an orbifold can be thought as a space which is almost smooth with mild singularities (see [51] and [16]). By the Marsden-Weinstein reduction theorem, such a reduced product naturally carries a symplectic structure and we are interested in understanding its symplectic geometry. One of the most important symplectic invariants is the symplectic volume and this is the topic we will focus on the most in this thesis. We will use the technique of nonabelian localization and the residue formula, developed by Jeffrey and Kirwan ([34], [35]), to study the symplectic volume and also the intersection pairings of these reduced products. In 2008, Suzuki and Takakura studied the symplectic volume of N-fold reduced products of $G = \mathbf{SU}(3)$ in their paper [53] via Riemann-Roch. Our volume formula generalizes their volume formula in the sense that their volume formula requires more restrictive inputs. Furthermore, in the case of $N = 3$, i.e., the case of triple reduced products of $\mathbf{SU}(3)$, we have proved that up to normalization constant, our volume formula matches completely with theirs.

N-fold reduced products appear in the literature in other guises as well. For example, N-fold reduced products of $G = \mathbf{SU}(2)$ can be identified with moduli spaces of polygons in $\mathbb{R}^3$ with prescribed lengths of edges, which have been studied by, for example, Hausmann and Knutson [29], Kamiyama and Tezuka [40]. By Jeffrey [31], N-fold reduced products of G can also be identified with moduli spaces of parabolic bundles on a genus 0 surface with prescribed weights, or with moduli spaces of flat G-connections on a genus 0 surface with N boundary components such that the holonomy around each boundary component is prescribed.

The organization of this thesis is as follows.

In Chapter 2, we investigate triple reduced products of $\mathbf{SU}(3)$, namely, the case of $G = \mathbf{SU}(3)$ and $N = 3$. Along the way, we also introduce the notations that can be easily generalized in the later chapters. After briefly reviewing the machinery of equivariant cohomology, we describe the method of nonabelian localization and the residue formula of Jeffrey and Kirwan ([34], [35]). Then we apply these techniques to derive our volume formula (see [32]). At the end of this chapter, we compare our volume formula with the volume formula of Suzuki and Takakura ([53]) and conclude this chapter with a proof that up to normalization constant, our volume formula matches completely with theirs in the case of triple reduced products of $\mathbf{SU}(3)$.

In Chapter 3, we first generalize our volume formula of triple reduced products of $\mathbf{SU}(3)$ to the case of N-fold reduced products of $\mathbf{SU}(3)$. Then we further generalize it to the case of N-fold reduced products of a general semisimple compact connected Lie group G. See [32].

In Chapter 4, by applying the residue formula of Jeffrey and Kirwan ([34], [35]), we compute the intersection pairings of N-fold reduced products. See [33].

Chapter 2

Volume Formula for Triple Reduced Products of $\mathbf{SU}(3)$

In this chapter we talk about triple reduced products of $\mathbf{SU}(3)$.

2.1 Definitions and Notations

Let $G = \mathbf{SU}(3)$. Thus G is the collection of all invertible complex 3×3 matrices A such that $A^* = A^{-1}$ and $\det(A) = 1$.

Let $\mathfrak{g} = \mathfrak{su}(3)$, the Lie algebra of $\mathbf{SU}(3)$. Thus $\mathfrak{g}$ is the collection of all complex 3×3 matrices X such that $X^* = -X$ and $\mathrm{tr}(X) = 0$. Notice that $\mathfrak{g}$ is a vector space over $\mathbb{R}$ and $\dim_{\mathbb{R}}(\mathfrak{g}) = 8$.

Let T be the standard maximal torus of G. Thus T is the collection of all 3×3 diagonal matrices $\mathrm{diag}(e^{\mathbf{i}\theta_1}, e^{\mathbf{i}\theta_2}, e^{-\mathbf{i}(\theta_1+\theta_2)})$ such that θ_1, θ_2 are real.

Let $\mathfrak{t}$ be the Lie algebra of T. Thus $\mathfrak{t}$ is the collection of all 3×3 diagonal matrices $\mathrm{diag}(\mathbf{i}\theta_1, \mathbf{i}\theta_2, -\mathbf{i}(\theta_1 + \theta_2))$ such that θ_1, θ_2 are real.

Let $\mathfrak{g}^* := \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathbb{R})$ be the dual vector space of $\mathfrak{g}$. Let $\mathfrak{t}^* := \mathrm{Hom}_{\mathbb{R}}(\mathfrak{t}, \mathbb{R})$ be the dual vector space of $\mathfrak{t}$.

If V is a vector space over a field $\mathbb{F}$, let $\langle \xi, X \rangle := \xi(X) \in \mathbb{F}$ denote the natural pairing between a covector $\xi \in V^* := \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ and a vector $X \in V$. In this thesis, $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, depending on the context.

For all $g \in G$, let $\mathbf{c}(g) : G \to G$ denote the map $x \mapsto gxg^{-1}$.

For all $g \in G$, let $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ denote the differential of $\mathbf{c}(g)$ at the identity $e \in G$. Thus, $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is the adjoint representation of G on $\mathfrak{g}$. The differential of Ad at the identity $e \in G$ is denoted by ad. Thus, $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$.

For all $g \in G$, let $\mathbf{K}(g) : \mathfrak{g}^* \to \mathfrak{g}^*$ be defined by:

$$\langle \mathbf{K}(g)\xi, X \rangle = \left\langle \xi, \operatorname{Ad}(g^{-1})X \right\rangle \tag{2.1}$$

for all $\xi \in \mathfrak{g}^*$, $X \in \mathfrak{g}$. Thus, $\mathbf{K} : G \to \operatorname{Aut}(\mathfrak{g}^*)$ is the coadjoint representation of G on $\mathfrak{g}^*$. The differential of $\mathbf{K}$ at the identity $e \in G$ is denoted by $\mathbf{k}$. Thus, $\mathbf{k} : \mathfrak{g} \to \operatorname{End}(\mathfrak{g}^*)$ is the coadjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^*$. Notice that for all $X \in \mathfrak{g}$,

$$\mathbf{k}(X) = -\operatorname{ad}(X)^*. \tag{2.2}$$

For all $X, Y \in \mathfrak{g}$, we define

$$(X, Y) := -\operatorname{tr}(XY). \tag{2.3}$$

Then $(\cdot,\cdot)$ is an $\operatorname{Ad}(G)$-invariant (or briefly, G-invariant) inner product on $\mathfrak{g}$.

We will use this inner product to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ through the standard identification between $X \in \mathfrak{g}$ and $(X, \cdot) \in \mathfrak{g}^*$ for all $X \in \mathfrak{g}$. Since $\mathfrak{t}$ is naturally a subspace of $\mathfrak{g}$ through the inclusion $T \subset G$, the above identification induces an identification between $\mathfrak{t}$ and $\mathfrak{t}^*$. It is in this sense that we write $\mathfrak{t}^* \subset \mathfrak{g}^*$.

Suppose $\xi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$ satisfy

$$\langle \xi, Y \rangle = (X, Y) \tag{2.4}$$

for all $Y \in \mathfrak{g}$. Namely, ξ corresponds to X under the identification through the inner product. For all $g \in G$, $Y \in \mathfrak{g}$,

$$\langle \mathbf{K}(g)\xi, Y \rangle = \left\langle \xi, \operatorname{Ad}(g^{-1})Y \right\rangle \tag{2.5}$$
$$= (X, \operatorname{Ad}(g^{-1})Y) \tag{2.6}$$
$$= (\operatorname{Ad}(g)X, Y). \tag{2.7}$$

Thus, $\mathbf{K}(g)\xi$ corresponds to $\operatorname{Ad}(g)X$ for all $g \in G$ under the identification through the inner product. It is in this sense that we say the coadjoint action of G on $\mathfrak{g}^*$ corresponds to the adjoint action of G on $\mathfrak{g}$, and it is in this sense that we identify coadjoint orbits with adjoint orbits. In this thesis, we will try to focus on the adjoint version of the theory since it is often easier to carry out computations with elements in $\mathfrak{g}$, which are matrices, than with elements in $\mathfrak{g}^*$.

The Weyl group W is defined by $\operatorname{N}(T)/T$, where $\operatorname{N}(T)$ denotes the normalizer of T

in G. Suppose $g \in \mathrm{N}(T)$. Then g is a representative of one Weyl group element $w \in W$. The element w acts on T through

$$w \cdot t = gtg^{-1} \tag{2.8}$$

for all $t \in T$. The element w acts on $\mathfrak{t}$ through

$$w \cdot X = \mathrm{Ad}(g)X \tag{2.9}$$

for all $X \in \mathfrak{t}$.

Since $G = \mathbf{SU}(3)$, W is isomorphic to the permutation group $\mathfrak{S}_3$ of 3 letters. More precisely, let

$$W = \{\mathfrak{s}_0, \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_5\} \tag{2.10}$$

where

$$\mathfrak{s}_0 \text{ corresponds to } \mathrm{Id} \in \mathfrak{S}_3, \tag{2.11}$$

$$\mathfrak{s}_1 \text{ corresponds to } (1\ 2) \in \mathfrak{S}_3, \tag{2.12}$$

$$\mathfrak{s}_2 \text{ corresponds to } (1\ 2\ 3) \in \mathfrak{S}_3, \tag{2.13}$$

$$\mathfrak{s}_3 \text{ corresponds to } (1\ 3) \in \mathfrak{S}_3, \tag{2.14}$$

$$\mathfrak{s}_4 \text{ corresponds to } (1\ 3\ 2) \in \mathfrak{S}_3, \tag{2.15}$$

$$\mathfrak{s}_5 \text{ corresponds to } (2\ 3) \in \mathfrak{S}_3. \tag{2.16}$$

The signature $\mathrm{sgn}(w)$ of a Weyl group element $w \in W$ is defined as the signature of its corresponding element $\sigma \in \mathfrak{S}_3$, i.e.

$$\mathrm{sgn}(w) := \mathrm{sgn}(\sigma). \tag{2.17}$$

Notice that we have chosen the subscripts j in $\mathfrak{s}_j$ so that

$$\mathrm{sgn}(\mathfrak{s}_j) = (-1)^j. \tag{2.18}$$

This will be convenient for later use.

In terms of matrices, we have

$$\mathfrak{s}_0 \text{ is represented by } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{N}(T), \tag{2.19}$$

$$\mathfrak{s}_1 \text{ is represented by } \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{N}(T), \tag{2.20}$$

$$\mathfrak{s}_2 \text{ is represented by } \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in \mathrm{N}(T), \tag{2.21}$$

$$\mathfrak{s}_3 \text{ is represented by } \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{N}(T), \tag{2.22}$$

$$\mathfrak{s}_4 \text{ is represented by } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{N}(T), \tag{2.23}$$

$$\mathfrak{s}_5 \text{ is represented by } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \in \mathrm{N}(T). \tag{2.24}$$

The Weyl group elements act on diagonal matrices by permuting the diagonal entries. More precisely, if $\sigma \in \mathfrak{S}_3$, then σ corresponds to a Weyl group element $w \in W$ and

$$w \cdot \mathrm{diag}(a_1, a_2, a_3) = \mathrm{diag}(a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, a_{\sigma^{-1}(3)}). \tag{2.25}$$

For example,

$$\mathfrak{s}_2 \cdot \mathrm{diag}(a_1, a_2, a_3) = \mathrm{diag}(a_3, a_1, a_2). \tag{2.26}$$

If $\sigma \in \mathfrak{S}_3$ corresponds to a Weyl group element $w \in W$, we define

$$\sigma \cdot X := w \cdot X \tag{2.27}$$

for all $X \in \mathfrak{t}$.

We sometimes just identify W with $\mathfrak{S}_3$.

Since the inner product $(\cdot, \cdot)$ is $\mathrm{Ad}(G)$-invariant on $\mathfrak{g}$, it is W-invariant on $\mathfrak{t}$.

Let $\mathcal{L}$ denote the integral lattice in $\mathfrak{t}$, that is,

$$\mathcal{L} := \left\{ H \in \mathfrak{t} \ : \ e^{2\pi H} = I \right\}. \tag{2.28}$$

Thus,

$$\mathcal{L} = \{m_1 H_1 + m_2 H_2 \ : \ m_1, m_2 \in \mathbb{Z}\} \tag{2.29}$$

where

$$H_1 = \mathrm{diag}(\mathbf{i}, -\mathbf{i}, 0), \tag{2.30}$$

$$H_2 = \mathrm{diag}(0, \mathbf{i}, -\mathbf{i}). \tag{2.31}$$

A weight is an element of $\mathfrak{t}^*$ such that it takes integer values on $\mathcal{L}$. We will use the identification through the inner product $(\cdot, \cdot)$ to regard weights as elements in $\mathfrak{t}$.

Let λ_1, λ_2 be elements in $\mathfrak{t}$ such that

$$(\lambda_i, H_j) = \delta_{ij} \tag{2.32}$$

for all $i, j \in \{1, 2\}$. Then, we obtain

$$\lambda_1 = \mathrm{diag}(\frac{2}{3}\mathbf{i}, -\frac{1}{3}\mathbf{i}, -\frac{1}{3}\mathbf{i}), \tag{2.33}$$

$$\lambda_2 = \mathrm{diag}(\frac{1}{3}\mathbf{i}, \frac{1}{3}\mathbf{i}, -\frac{2}{3}\mathbf{i}). \tag{2.34}$$

A weight $\lambda \in \mathfrak{t}$ is called dominant if and only if $(\lambda, H_j) \geq 0$ for all j. A weight $\lambda \in \mathfrak{t}$ is called integral if and only if $(\lambda, H_j) \in \mathbb{Z}$ for all j. Thus,

$$\Lambda_{\geq 0} := \{n_1\lambda_1 + n_2\lambda_2 \ : \ n_1, n_2 \text{ are both nonnegative integers}\} \tag{2.35}$$

is the collection of dominant integral weights, and

$$\mathfrak{t}_{\geq 0} := \{c_1\lambda_1 + c_2\lambda_2 \ : \ c_1, c_2 \geq 0\}, \tag{2.36}$$

$$\mathfrak{t}_{>0} := \{c_1\lambda_1 + c_2\lambda_2 \ : \ c_1, c_2 > 0\} \tag{2.37}$$

are the closed positive Weyl chamber and the open positive Weyl chamber, respectively.

Let

$$\alpha_1 := 2\lambda_1 - \lambda_2 = H_1, \tag{2.38}$$

$$\alpha_2 := -\lambda_1 + 2\lambda_2 = H_2 \tag{2.39}$$

be the standard simple roots for $G = \mathbf{SU}(3)$. Following the convention of [34], let

$$\mathrm{R}_+ = \{2\pi\alpha_1, 2\pi\alpha_2, 2\pi(\alpha_1 + \alpha_2)\} \tag{2.40}$$

denote the set of 2π-modified positive roots of $\mathbf{SU}(3)$. From now on, if we talk about roots, we mean the 2π-modified ones unless stated otherwise.

The Weyl group is generated by reflections across certain hyperplanes in $\mathfrak{t}$. More precisely, $W \cong \mathfrak{S}_3$ is generated by the transpositions (1 2) and (2 3), and

$$(1\ 2) \text{ acts on } \mathfrak{t} \text{ by the reflection across the hyperplane perpendicular to } \alpha_1, \tag{2.41}$$

$$(2\ 3) \text{ acts on } \mathfrak{t} \text{ by the reflection across the hyperplane perpendicular to } \alpha_2. \tag{2.42}$$

Let $\xi \in \mathfrak{g}$. Let $\mathcal{O}_\xi$ denote the adjoint orbit through ξ. Under the identification through the inner product $(\cdot,\cdot)$, it is equivalent to consider either adjoint orbits or their corresponding coadjoint counterparts. In this thesis we will mainly use the adjoint setting.

It is well known that every coadjoint orbit admits a natural symplectic form, called the Kirillov-Kostant-Souriau form, or briefly the KKS form. In the adjoint setting, it is defined as follows.

Suppose $\mathcal{O}$ is an adjoint orbit in $\mathfrak{g}$ and $\xi \in \mathcal{O}$. Then the tangent space of $\mathcal{O}$ at ξ, $T_\xi\mathcal{O}$, which is naturally a subspace of $\mathfrak{g}$, is the following collection:

$$T_\xi\mathcal{O} = \{\mathrm{ad}(X)\xi \ : \ X \in \mathfrak{g}\}\,. \tag{2.43}$$

Notice that $\mathrm{ad}(X)\xi = [X,\xi]$ for all $X \in \mathfrak{g}$.

The KKS form ω on $\mathcal{O}$ is defined by

$$\omega_\xi([X,\xi],[Y,\xi]) := (\xi,[X,Y]) \tag{2.44}$$

for all $\xi \in \mathcal{O}$, $X, Y \in \mathfrak{g}$.

ω is a 2-form such that it is closed, that is, $d\omega = 0$, and nondegenerate, that is, ω_ξ, the restriction of ω to the tangent space $T_\xi\mathcal{O}$, is a nondegenerate bilinear form, for all $\xi \in \mathcal{O}$.

Equipped with ω, $\mathcal{O}$ becomes a compact symplectic manifold. Recall that a symplectic manifold is a manifold equipped with a closed nondegenerate 2-form.

Furthermore, G acts on $\mathcal{O}$ by the adjoint action and this makes $\mathcal{O}$ a Hamiltonian G-space with the inclusion map $\mu_\mathcal{O} : \mathcal{O} \hookrightarrow \mathfrak{g}$ as the moment map. In particular, for all $X \in \mathfrak{g}$,

$$d\mu_\mathcal{O}^X = \iota_{X^\sharp}\omega \tag{2.45}$$

where $\mu_\mathcal{O}^X(\xi) := (\mu_\mathcal{O}(\xi), X) = (\xi, X)$ for all $\xi \in \mathcal{O}$ and $X^\sharp$ is the fundamental vector field on $\mathcal{O}$ generated by X and thus $X^\sharp(\xi) = [X, \xi] \in T_\xi\mathcal{O}$ for all $\xi \in \mathcal{O}$. Notice that

$$([X, Y], \xi) = (X, [Y, \xi]) \tag{2.46}$$

for all $X, Y, \xi \in \mathfrak{g}$.

Consider 3 points ξ_1, ξ_2, ξ_3 in $\mathfrak{g}$. We then have 3 adjoint orbits $\mathcal{O}_{\xi_1}, \mathcal{O}_{\xi_2}, \mathcal{O}_{\xi_3}$. Let ω_i denote the KKS form on $\mathcal{O}_{\xi_i}$.

Now consider $M := \mathcal{O}_{\xi_1} \times \mathcal{O}_{\xi_2} \times \mathcal{O}_{\xi_3}$.

Let $\mathrm{pr}_i : M \to \mathcal{O}_{\xi_i}$ be the standard projection onto the i-th factor. Then the form

$$\omega := \sum_{i=1}^{3} \mathrm{pr}_i^* \, \omega_i \tag{2.47}$$

is a symplectic form on M.

Let G act on M by

$$g \cdot (\eta_1, \eta_2, \eta_3) := (\mathrm{Ad}(g)\eta_1, \mathrm{Ad}(g)\eta_2, \mathrm{Ad}(g)\eta_3) \tag{2.48}$$

for all $g \in G$, $\eta_i \in \mathcal{O}_{\xi_i}$. Then this action makes M a Hamiltonian G-space with the moment map $\mu_G : M \to \mathfrak{g}$ such that

$$\mu_G(\eta_1, \eta_2, \eta_3) = \sum_{i=1}^{3} \mu_{\mathcal{O}_{\xi_i}}(\eta_i) = \sum_{i=1}^{3} \eta_i \tag{2.49}$$

for all $\eta_i \in \mathcal{O}_{\xi_i}$.

In this thesis, we make the following assumptions about the points ξ_1, ξ_2, ξ_3:

(A1) $M_0 := \mu_G^{-1}(0) \neq \emptyset$ and $0 \in \mathfrak{g}$ is a regular value of μ_G.

(A2) $\xi_i \in \mathfrak{t}_{\geq 0} \subset \mathfrak{g}$ for all i and each $\mathcal{O}_{\xi_i}$ is diffeomorphic to the homogeneous space G/T.

Remark. (A1) ensures that we have something to talk about and the stabilizer $\mathrm{Stab}_G(\vec{\eta})$ is finite for each $\vec{\eta} = (\eta_1, \eta_2, \eta_3) \in M_0$. Every adjoint orbit $\mathcal{O}$ will intersect $\mathfrak{t}_{\geq 0} \subset \mathfrak{g}$ at exactly one point ξ, so by only considering $\xi \in \mathfrak{t}_{\geq 0}$, we still obtain every possible orbit. This explains the first part of (A2). The second part of (A2) says that the orbits we will consider are nondegenerate, that is, they are of the highest dimension possible. In fact, assuming (A2) is equivalent as assuming that $\xi_i \in \mathfrak{t}_{>0} \subset \mathfrak{g}$ for all i.

Let M^T denote the set of fixed points in M under the action of $T \subset G$. We have the following.

Proposition 2.1.1. *M^T is the discrete set*

$$\{(w_1 \cdot \xi_1, w_2 \cdot \xi_2, w_3 \cdot \xi_3) \ : \ w_i \in W\}. \tag{2.50}$$

Thus, $\left|M^T\right| = |W|^3$.

Proof. Note that

$$\mathcal{O}_{\xi_i} \cap \mathfrak{t} = W \cdot \xi_i \tag{2.51}$$

for all i. The elements of $\mathcal{O}_{\xi_i} \cap \mathfrak{t}$ are precisely those elements in $\mathcal{O}_{\xi_i}$ that are fixed by the adjoint action of $T \subset G$. Since $\xi_i \in \mathfrak{t}_{>0}$, $\mathcal{O}_{\xi_i} \cap \mathfrak{t}$ is discrete and has the same cardinality as W. The proposition follows immediately. □

Since $G = \mathbf{SU}(3)$, we actually have $6^3 = 216$ isolated fixed points in M under the action of $T \subset G$.

Definition 2.1.2. The quotient space

$$M_{\text{red}} := M_0/G \tag{2.52}$$

is called a *triple reduced product of* G. In other words, M_{red} is the Marsden-Weinstein reduction of the Hamiltonian G-space (M, ω, G, μ_G). Sometimes we may write $M_{\text{red}}(\vec{\xi})$ to emphasize the dependence on the initial data $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$.

Remark. In general, M_{red} is not a smooth manifold. It belongs to a type of spaces called *orbifolds* or *V-manifolds* ([51]). Roughly speaking, an orbifold is almost a smooth manifold except that it has some mild singularities. At these singularities, it locally looks like U/Γ where U is an open subset of $\mathbb{R}^d$ and Γ is a finite group of linear automorphisms of U. Fortunately, [51] tells us that on such spaces the de Rham theory and Poincaré duality work basically the same way as on smooth manifolds (See also [16]).

The Marsden-Weinstein reduction theorem tells us that on M_{red}, there is a unique symplectic structure ω_{red} such that

$$i_0^*\omega = p_0^*\omega_{\text{red}} \tag{2.53}$$

where $i_0 : M_0 \hookrightarrow M$ is the inclusion map and $p_0 : M_0 \to M_0/G = M_{\text{red}}$ is the projection map.

Notice that M_{red} is compact. The symplectic volume of $(M_{\text{red}}, \omega_{\text{red}})$ is defined as

$$\text{vol}^{\mathcal{S}}(M_{\text{red}}) := \int_{M_{\text{red}}} \frac{\omega_{\text{red}}^{d/2}}{(d/2)!} \tag{2.54}$$

where d is the real dimension of M_{red}. Notice that

$$\operatorname{vol}^{\mathcal{S}}(M_{\text{red}}) = \int_{M_{\text{red}}} e^{\omega_{\text{red}}} = \frac{1}{\mathbf{i}^{d/2}} \int_{M_{\text{red}}} e^{\mathbf{i}\omega_{\text{red}}}. \tag{2.55}$$

Since M_{red} is oriented by its symplectic volume form

$$\frac{\omega_{\text{red}}^{d/2}}{(d/2)!}, \tag{2.56}$$

$\operatorname{vol}^{\mathcal{S}}(M_{\text{red}})$ is a positive real number.

We will write $\operatorname{vol}^{\mathcal{S}}(\vec{\xi})$ to denote $\operatorname{vol}^{\mathcal{S}}(M_{\text{red}}(\vec{\xi}))$ and in this way we regard $\operatorname{vol}^{\mathcal{S}}$ as a real valued function on the inputs $\vec{\xi}$.

2.2 Equivariant Cohomology

To study the symplectic volume $\operatorname{vol}^{\mathcal{S}}(M_{\text{red}})$, we are basically looking at the cohomological quantity

$$e^{\omega_{\text{red}}}[M_{\text{red}}] = \frac{1}{\mathbf{i}^{d/2}} e^{\mathbf{i}\omega_{\text{red}}}[M_{\text{red}}], \tag{2.57}$$

where $[M_{\text{red}}]$ denotes the fundamental class of M_{red}, which is picked up by the orientation induced by the symplectic volume form on M_{red}.

This quantity can be computed using the technique called the nonabelian localization due to Jeffrey and Kirwan ([34]; see also [35]) and in particular the residue formula (Theorem 8.1 in [34] and Theorem 3.1 in [35]). To state their results, we shall first review the machinery of equivariant cohomology.

In this thesis, we only consider cohomology groups over $\mathbb{C}$.

Let $\mathcal{K}$ be a compact connected Lie group. Let $\mathcal{M}$ be a $\mathcal{K}$-space.

By definition, the equivariant cohomology $H^*_{\mathcal{K}}(\mathcal{M})$ of the $\mathcal{K}$-space $\mathcal{M}$ is the ordinary cohomology $H^*(\mathcal{M}_{\mathcal{K}})$ where $\mathcal{M}_{\mathcal{K}}$ denotes the homotopy quotient

$$\mathcal{M} \times_{\mathcal{K}} E\mathcal{K} := (\mathcal{M} \times E\mathcal{K})/\mathcal{K}, \tag{2.58}$$

where $E\mathcal{K}$ denotes a contractible space on which $\mathcal{K}$ acts freely.

Let $B\mathcal{K}$ denote the quotient $E\mathcal{K}/\mathcal{K}$.

Let $H^*_{\mathcal{K}}$ denote $H^*_{\mathcal{K}}(\text{pt}) = H^*(B\mathcal{K})$.

If we consider the canonical map

$$\mathcal{M} \to \text{pt}, \tag{2.59}$$

we obtain the following natural map

$$H_{\mathcal{K}}^* \to H_{\mathcal{K}}^*(\mathcal{M}). \tag{2.60}$$

Because of this, $H_{\mathcal{K}}^*(\mathcal{M})$ is not only a ring but also an $H_{\mathcal{K}}^*$-algebra.

We will use the Cartan model ([13], [14]; see also [7]) to compute equivariant cohomology.

Let $\mathfrak{k}$ denote the Lie algebra of $\mathcal{K}$.

A $\mathcal{K}$-equivariant differential form α on $\mathcal{M}$ can be thought of as a $\mathcal{K}$-equivariant polynomial map

$$\alpha : \mathfrak{k} \to \Omega^*(\mathcal{M}). \tag{2.61}$$

Let $\Omega_{\mathcal{K}}^*(\mathcal{M})$ denote the collection of all $\mathcal{K}$-equivariant differential forms on $\mathcal{M}$. In other words,

$$\Omega_{\mathcal{K}}^*(\mathcal{M}) = (S(\mathfrak{k}^*) \otimes \Omega^*(\mathcal{M}))^{\mathcal{K}}. \tag{2.62}$$

$\Omega_{\mathcal{K}}^*(\mathcal{M})$ is graded in the following way. Suppose $\alpha \in \Omega_{\mathcal{K}}^*(\mathcal{M})$. Then the total degree of α is the differential form degree plus twice the polynomial degree. Furthermore, there is a differential $d_{\mathcal{K}}$ on $\Omega_{\mathcal{K}}^*(\mathcal{M})$ defined by

$$(d_{\mathcal{K}}\alpha)(X) := d(\alpha(X)) - \iota_{X^\sharp}(\alpha(X)) \tag{2.63}$$

for all $X \in \mathfrak{k}$, where $X^\sharp$ is the fundamental vector field induced by $X \in \mathfrak{k}$. Thus, $d_{\mathcal{K}}$ increases the total degree by 1. In this way, $(\Omega_{\mathcal{K}}^*(\mathcal{M}), d_{\mathcal{K}})$ becomes a cochain complex. In [13] and [14], H. Cartan proved that the cohomology of this cochain complex computes the equivariant cohomology $H_{\mathcal{K}}^*(\mathcal{M})$, so the above construction is called the Cartan model.

Now we return to our situation, that is, the situation involving M, M_0, M_{red}, G, T, etc.

H_G^* can be identified with the ring $S(\mathfrak{g}^*)^G$ of the G-invariant polynomial functions on $\mathfrak{g}$, and H_T^* can be identified with the ring $S(\mathfrak{t}^*)^T = S(\mathfrak{t}^*)$ since the T-action here is trivial. By restricting to $\mathfrak{t}$, $S(\mathfrak{g}^*)^G$ can be identified with $S(\mathfrak{t}^*)^W$, the ring of W-invariant polynomial functions on $\mathfrak{t}$, which is in turn a subring of $S(\mathfrak{t}^*)$. Thus, H_G^* can be identified with the subring of H_T^* that consists of all W-invariant elements:

$$H_G^* \cong S(\mathfrak{g}^*)^G \cong S(\mathfrak{t}^*)^W \subset S(\mathfrak{t}^*) \cong H_T^*. \tag{2.64}$$

The above sequence induces the canonical map

$$\tau : H_G^* \to H_T^*. \tag{2.65}$$

There is another important map called the pushforward map

$$\Pi_*^G : H_G^*(M) \to H_G^*, \tag{2.66}$$

which can be thought of as integration over M. Similarly, when we are looking at the T-action on M, the corresponding pushforward map is

$$\Pi_*^T : H_T^*(M) \to H_T^*. \tag{2.67}$$

Usually, when the context is clear, we will denote both Π_*^G and Π_*^T by the integration symbol $\int_M$ or simply Π_*.

Recall that 0 is a regular value for the moment map μ_G. By [37], the ring homomorphism

$$i_0^* : H_G^*(M) \to H_G^*(M_0) \tag{2.68}$$

is surjective. In addition, we have a canonical isomorphism

$$\pi_0^* : H^*(M_{\text{red}}) \to H_G^*(M_0) \tag{2.69}$$

induced from the map

$$\pi_0 : M_0 \times_G EG \to M_{\text{red}}. \tag{2.70}$$

Therefore, we obtain a surjective ring homomorphism

$$\kappa_0 := (\pi_0^*)^{-1} \circ i_0^* : H_G^*(M) \to H^*(M_{\text{red}}). \tag{2.71}$$

In [34], Jeffrey and Kirwan proved a formula (Theorem 8.1 in [34]) computing the following cohomological quantity

$$\kappa_0(\eta)e^{\mathbf{i}\omega_{\text{red}}}[M_{\text{red}}] \tag{2.72}$$

for any $\eta \in H_G^*(M)$. Also in [35], they rewrite the residue formula (Theorem 3.1 in [35]) computing the following cohomological quantity

$$\kappa_0(\eta)e^{\omega_{\text{red}}}[M_{\text{red}}] \tag{2.73}$$

for any $\eta \in H^*_G(M)$.

Before we state the formula, we first fix some notations which will be convenient for later use.

Let s denote the real dimension of G.

Let l denote the real dimension of T.

Let ϖ denote the product of the positive roots of G, that is,

$$\varpi(X) = \prod_{\gamma \in \mathrm{R}_+} \gamma(X) \tag{2.74}$$

for all X in $\mathfrak{t}$ or $\mathfrak{t}_\mathbb{C}$, where $\mathfrak{t}_\mathbb{C}$ denotes the complexification of $\mathfrak{t}$. Therefore we can regard ϖ as a polynomial function on $\mathfrak{t}$ or $\mathfrak{t}_\mathbb{C}$. Notice that

$$\varpi(w \cdot X) = \operatorname{sgn}(w)\varpi(X) \tag{2.75}$$

for all $w \in W$ and all X in $\mathfrak{t}$ or $\mathfrak{t}_\mathbb{C}$.

The fixed G-invariant inner product $(\cdot,\cdot)$ induces measures on $\mathfrak{g}$ and $\mathfrak{t}$ and their corresponding complexifications, $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{t}_\mathbb{C}$. Following [34], we will reserve the Greek letter ϕ for a variable in $\mathfrak{g}$ or $\mathfrak{g}_\mathbb{C}$ and the Greek letter ψ for a variable in $\mathfrak{t}$ or $\mathfrak{t}_\mathbb{C}$.

Let $[d\phi]$ denote the induced measure on $\mathfrak{g}$ or $\mathfrak{g}_\mathbb{C}$.

Let $[d\psi]$ denote the induced measure on $\mathfrak{t}$ or $\mathfrak{t}_\mathbb{C}$.

The inner product also induces Riemannian volume forms on G and T.

Let $\operatorname{vol}^\mathcal{R}(G)$ denote the Riemannian volume of G under this induced Riemannian volume form on G.

Let $\operatorname{vol}^\mathcal{R}(T)$ denote the Riemannian volume of T under this induced Riemannian volume form on T.

Therefore, the measures $[d\phi]/\operatorname{vol}^\mathcal{R}(G)$ and $[d\psi]/\operatorname{vol}^\mathcal{R}(T)$ are independent of the choice of the inner product on $\mathfrak{g}$.

Let $\mu_T : M \to \mathfrak{t}$ denote the composition of the moment map μ_G with the orthogonal projection from $\mathfrak{g}$ to $\mathfrak{t}$. Notice that the orthogonal projection from $\mathfrak{g}$ to $\mathfrak{t}$ corresponds to the restriction map from $\mathfrak{g}^*$ to $\mathfrak{t}^*$ under the identification through the inner product. Therefore, μ_T is a moment map for the action of T on M.

Now we can state the residue formula of Jeffrey and Kirwan, adapted to our situation.

Theorem 2.2.1 (Jeffrey and Kirwan; [34], [35]). *For all $\eta \in H^*_G(M)$, we have*

$$\kappa_0(\eta)e^{\mathbf{i}\omega_{\mathrm{red}}}[M_{\mathrm{red}}] = n_0 C_G \operatorname{res}\left(\varpi^2(\psi) \sum_{F \in M^T} r^\eta_F(\psi)[d\psi]\right) \tag{2.76}$$

where n_0 is the cardinality of the stabilizer subgroup in G of a generic point in M_0, and the constant C_G is defined by

$$C_G := \frac{(-1)^{n_+}}{(2\pi)^{s-l}\,|W|\,\mathrm{vol}^{\mathcal{R}}(T)}. \tag{2.77}$$

Here, n_+ denotes the cardinality of R_+, that is, $n_+ = (s-l)/2$. ψ is a variable in $\mathfrak{t}_{\mathbb{C}}$. Notice that in our situation, all fixed points in M^T are isolated. If $F \in M^T$, the meromorphic function r_F^η on $\mathfrak{t}_{\mathbb{C}}$ is defined by

$$r_F^\eta(\psi) := e^{\mathbf{i}(\mu_T(F),\psi)} \int_F \frac{i_F^*(\eta(\psi)e^{\mathbf{i}\omega})}{\mathrm{e}_F(\psi)} \tag{2.78}$$

where $i_F : F \to M$ is the inclusion and e_F is the T-equivariant Euler class of the normal bundle to F in M.

Since F is an isolated fixed point, we have

$$\int_F \frac{i_F^*(\eta(\psi)e^{\mathbf{i}\omega})}{\mathrm{e}_F(\psi)} = \frac{i_F^*(\eta(\psi)e^{\mathbf{i}\omega})}{\mathrm{e}_F(\psi)} \tag{2.79}$$

and the quantity

$$i_F^*(\eta(\psi)e^{\mathbf{i}\omega}) \tag{2.80}$$

will be a $\mathbb{C}$-valued function on F, that is, a complex number. We denote this complex number by

$$\eta_F(\psi) := i_F^*(\eta(\psi)e^{\mathbf{i}\omega}) \in \mathbb{C}. \tag{2.81}$$

Furthermore, since $M = \mathcal{O}_{\xi_1} \times \mathcal{O}_{\xi_2} \times \mathcal{O}_{\xi_3}$, any $F \in M^T$ can be written (see Proposition **??**) by

$$F = (w_1 \cdot \xi_1, w_2 \cdot \xi_2, w_3 \cdot \xi_3) \tag{2.82}$$

for some $\vec{w} = (w_1, w_2, w_3) \in W \times W \times W$. Then we can write

$$F = \vec{w} \cdot \vec{\xi} := (w_1 \cdot \xi_1, w_2 \cdot \xi_2, w_3 \cdot \xi_3). \tag{2.83}$$

In this case, $\mathrm{e}_F(\psi)$ is the complex number

$$\mathrm{sgn}(\vec{w})\varpi^3(\psi) \tag{2.84}$$

where

$$\operatorname{sgn}(\vec{w}) := \prod_{i=1}^{3} \operatorname{sgn}(w_i). \tag{2.85}$$

In this case, we can define

$$\operatorname{sgn}(F) := \operatorname{sgn}(\vec{w}). \tag{2.86}$$

In addition, we write $\vec{w}(F)$ to mean the $\vec{w}$ such that $F = \vec{w} \cdot \vec{\xi}$.

Now, we can rewrite Equation (2.76) as

$$\kappa_0(\eta) e^{\mathbf{i}\omega_{\text{red}}}[M_{\text{red}}] = n_0 C_G \operatorname{res}\left(\varpi^2(\psi) \sum_{F \in M^T} e^{\mathbf{i}(\mu_T(F),\psi)} \frac{\eta_F(\psi)}{\operatorname{sgn}(F)\varpi^3(\psi)} [d\psi] \right). \tag{2.87}$$

This can be further simplified to

$$\kappa_0(\eta) e^{\mathbf{i}\omega_{\text{red}}}[M_{\text{red}}] = n_0 C_G \operatorname{res}\left(\sum_{F \in M^T} \operatorname{sgn}(F) \frac{\eta_F(\psi) e^{\mathbf{i}(\mu_T(F),\psi)}}{\varpi(\psi)} [d\psi] \right). \tag{2.88}$$

To obtain the symplectic volume $\operatorname{vol}^{\mathcal{S}}(M_{\text{red}})$, we let $\eta = 1 \in H_G^*(M)$ and compute:

$$\begin{aligned} \operatorname{vol}^{\mathcal{S}}(M_{\text{red}}) &= \frac{1}{\mathbf{i}^{d/2}} e^{\mathbf{i}\omega_{\text{red}}}[M_{\text{red}}] \\ &= \frac{1}{\mathbf{i}^{d/2}} \kappa_0(1) e^{\mathbf{i}\omega_{\text{red}}}[M_{\text{red}}] \\ &= \frac{1}{\mathbf{i}^{d/2}} n_0 C_G \operatorname{res}\left(\sum_{F \in M^T} \operatorname{sgn}(F) \frac{e^{\mathbf{i}(\mu_T(F),\psi)}}{\varpi(\psi)} [d\psi] \right). \end{aligned} \tag{2.89}$$

Since the fixed point set M^T can be parametrized by

$$\vec{w} = (w_1, w_2, w_3) \in W^3, \tag{2.90}$$

the sum

$$\sum_{F \in M^T} \operatorname{sgn}(F) \frac{e^{\mathbf{i}(\mu_T(F),\psi)}}{\varpi(\psi)} \tag{2.91}$$

can be rewritten as

$$\sum_{\vec{w} \in W^3} \operatorname{sgn}(\vec{w}) \frac{e^{\mathbf{i}(w_1 \cdot \xi_1 + w_2 \cdot \xi_2 + w_3 \cdot \xi_3, \psi)}}{\varpi(\psi)}. \tag{2.92}$$

We record this formula as a corollary of Theorem 2.2.1.

Corollary 2.2.2. *The symplectic volume of M_{red} can be expressed as*

$$\text{vol}^{\mathcal{S}}(M_{\text{red}}) = \frac{1}{\mathbf{i}^{d/2}} n_0 C_G \operatorname{res}\left(\sum_{\vec{w}\in W^3} \operatorname{sgn}(\vec{w}) \frac{e^{\mathbf{i}(w_1\cdot\xi_1+w_2\cdot\xi_2+w_3\cdot\xi_3,\psi)}}{\varpi(\psi)}[d\psi]\right). \tag{2.93}$$

Here, d is the real dimension of M_{red}. Suppose M is a product of N nondegenerate adjoint orbits. Then,

$$d = N(s-l) - 2s = (N-2)s - Nl. \tag{2.94}$$

To fully understand the above formulas, we need to understand the residue operator res. That is what we will study in the next section.

2.3 The Residue Operator

Now we shall study the residue operator res in detail. The main references for this section are [34] and [35].

Fourier transform is important to the development of the nonabelian localization in [34]. Following [34], we reserve the letter z for a variable in $\mathfrak{g}^*$ and the letter y for a variable in $\mathfrak{t}^*$.

If $f : \mathfrak{g} \to \mathbb{C}$ is a tempered distribution, we define its Fourier transform $F_G f$ on $\mathfrak{g}^*$ to be

$$(F_G f)(z) := \frac{1}{(2\pi)^{s/2}} \int_{\phi\in\mathfrak{g}} f(\phi) e^{-\mathbf{i}\langle z,\phi\rangle}[d\phi]. \tag{2.95}$$

Since $f : \mathfrak{g} \to \mathbb{C}$ can be regarded as a tempered distribution on $\mathfrak{t}$ by restriction to $\mathfrak{t}$, we can also define its Fourier transform $F_T f$ on $\mathfrak{t}^*$ to be

$$(F_T f)(y) := \frac{1}{(2\pi)^{l/2}} \int_{\psi\in\mathfrak{t}} f(\psi) e^{-\mathbf{i}\langle y,\psi\rangle}[d\psi]. \tag{2.96}$$

In [34], the definition of the residue operator (Definition 8.5 in [34]) is based on the following result from [30]:

Proposition 2.3.1 (Proposition 8.4 in [34]; see also [30])**.**

(i) *Suppose u is a distribution on $\mathfrak{t}^*$. Then the set*

$$\Gamma_u = \left\{\xi \in \mathfrak{t} \ :\ e^{\langle\cdot,\xi\rangle}u \textit{ is a tempered distribution}\right\} \tag{2.97}$$

is convex.

(ii) *If the interior Γ_u^0 of Γ_u is nonempty, then there is an analytic function $\hat{u}$ on $\mathfrak{t}+\mathbf{i}\Gamma_u^0$ such that the Fourier transform of $e^{\langle\cdot,\xi\rangle}u$ is $\hat{u}(\cdot+\mathbf{i}\xi)$ for all $\xi\in\Gamma_u^0$.*

(iii) *For every compact subset K of Γ_u^0, there exist a constant $C_K>0$ and an integer $N_K\geq 0$ such that*

$$|\hat{u}(\zeta)|\leq C_K(1+|\zeta|)^{N_K} \tag{2.98}$$

for all $\zeta\in\mathfrak{t}+\mathbf{i}\Gamma_u^0$ such that $\mathrm{Im}(\zeta)\in K$.

(iv) *Conversely if Γ is an open convex set in $\mathfrak{t}$ and h is a holomorphic function on $\mathfrak{t}+\mathbf{i}\Gamma$ which satisfies the condition in (iii) above, then there is a distribution u on $\mathfrak{t}^*$ such that $e^{\langle\cdot,\xi\rangle}u$ is a tempered distribution and has Fourier transform $h(\cdot+\mathbf{i}\xi)$ for all $\xi\in\Gamma$.*

(v) *If u itself is a tempered distribution on $\mathfrak{t}^*$, then its Fourier transform is the limit (in the space of tempered distributions) of the distributions*

$$\psi\mapsto(F_T(e^{\langle\cdot,t\theta\rangle}))(\psi) \tag{2.99}$$

as $t\to 0^+$, for any $\theta\in\Gamma_u^0$.

Based on the above proposition, we have the following definition:

Definition 2.3.2 (Definitions 8.5, the paragraph before Definition 8.8, and Definition 8.8 in [34])**.** Let Λ be a proper cone in $\mathfrak{t}$. Let Λ^0 denote the interior of Λ. Let h be a holomorphic function on $\mathfrak{t}-\mathbf{i}\Lambda^0\subset\mathfrak{t}_{\mathbb{C}}$ such that for every compact subset K of $\mathfrak{t}-\mathbf{i}\Lambda^0$ there exist a constant C_K and an integer $N_K\geq 0$ such that

$$|h(\zeta)|\leq C_K(1+|\zeta|)^{N_K} \tag{2.100}$$

for all $\zeta\in K$. Let $\chi:\mathfrak{g}^*\to\mathbb{R}$ be a smooth invariant function with compact support and strictly positive in some neighbourhood of 0. Let $\hat{\chi}=F_G\chi:\mathfrak{g}\to\mathbb{C}$ be its Fourier transform. Let $\hat{\chi}_\epsilon(\phi)=\hat{\chi}(\epsilon\phi)$ so that

$$(F_G\hat{\chi}_\epsilon)(z)=\chi_\epsilon(z):=\frac{1}{\epsilon^s}\chi(\frac{z}{\epsilon}). \tag{2.101}$$

Also we assume $\hat{\chi}(0)=1$. Then we define

$$\mathrm{res}^{\Lambda,\chi}(h(\psi)[d\psi]):=\lim_{\epsilon\to 0^+}\frac{1}{(2\pi\mathbf{i})^l}\int_{\psi\in\mathfrak{t}-\mathbf{i}\xi}\hat{\chi}(\epsilon\psi)h(\psi)[d\psi], \tag{2.102}$$

where ξ is any element of Λ^0. Furthermore, in the case where h is a sum of other functions h_i and $F_T h$ is smooth at 0 but the Fourier transforms of h_i may not be smooth at 0, we need to introduce a small generic parameter $\rho \in \mathfrak{t}^*$ so that all the functions in this sum have Fourier transforms that are smooth at 0. More precisely, Let Λ, χ and h be as in the above. Let $\rho \in \mathfrak{t}^*$ be such that the distribution $F_T h$ is smooth on the ray $t\rho$ for $t \in (0, \delta)$ for some $\delta > 0$, and suppose $(F_T h)(t\rho)$ tends to a well defined limit as $t \to 0^+$. Then we define

$$\mathrm{res}^{\rho,\Lambda,\chi}(h(\psi)[d\psi]) := \lim_{t\to 0^+} \mathrm{res}^{\Lambda,\chi}(h(\psi)e^{\mathbf{i}\langle t\rho,\psi\rangle}[d\psi]). \tag{2.103}$$

Remark. By the paragraph after Definition 8.5 in [34], the integral in (2.102) converges and is independent of $\xi \in \Lambda^0$. Furthermore, by Propositions 8.6, 8.7 and 8.9 in [34], the residue of a meromorphic form Ω is independent of the choices of Λ, χ and ρ if Ω is sufficiently well behaved, as is the case for Theorem 2.2.1.

2.4 Nonabelian Localization

Before we go into the computational aspects of the residue operator, we shall briefly sketch the proof of Theorem 2.2.1 by summarizing key points in [34], so that we will have a better idea about why the nonabelian localization technique in [34] works.

The first key point is the abelian localization formula:

Theorem 2.4.1 (Berline and Vergne, [8]; Theorem 2.1 in [34]). *If $\sigma \in H_T^*(M)$, then*

$$(\Pi_*\sigma)(\psi) = \sum_{F\in M^T} \int_F \frac{i_F^*(\sigma(\psi))}{\mathrm{e}_F(\psi)}. \tag{2.104}$$

We are interested in the case when $\sigma = \eta e^{\mathbf{i}\bar{\omega}}$ where $\eta \in H_G^*(M)$ and

$$\bar{\omega}(\phi) = \omega + \mu_G(\phi) \tag{2.105}$$

is the standard equivariant extension of ω. Notice that here $\sigma = \eta e^{\mathbf{i}\bar{\omega}}$ is regarded as a T-equivariant cohomology class through the restriction map $\mathfrak{g}^* \to \mathfrak{t}^*$.

Let $r^\eta := \Pi_*(\eta e^{\mathbf{i}\bar{\omega}}) \in H_T^*$. Now, by Theorem 2.4.1, we have

$$r^\eta(\psi) = \sum_{F\in M^T} r_F^\eta(\psi), \tag{2.106}$$

where

$$r_F^\eta(\psi) = e^{\mathbf{i}(\mu_T(F),\psi)} \int_F \frac{i_F^*(\eta(\psi)e^{\mathbf{i}\omega})}{\mathrm{e}_F(\psi)}. \tag{2.107}$$

It turns out that the Fourier transform of $\Pi_*(\eta e^{\mathbf{i}\bar{\omega}})$ will be closely related to the cohomological quantity $\kappa_0(\eta)e^{\mathbf{i}\omega_{\text{red}}}[M_{\text{red}}]$. More precisely, we have the following key point:

Proposition 2.4.2 (Proposition 8.10 in [34])**.**

(a) *The distribution $F_G(\Pi_*(\eta e^{\mathbf{i}\bar{\omega}}))(z')$ defined for $z' \in \mathfrak{g}^*$ is represented by a smooth function for z' in a sufficiently small neighbourhood of* 0*.*

(b) *We have*

$$\kappa_0(\eta)e^{\mathbf{i}\omega_{\text{red}}}[M_{\text{red}}] = \frac{1}{(2\pi)^{s/2}\mathbf{i}^s \operatorname{vol}^{\mathcal{R}}(G)} F_G(\Pi_*(\eta e^{\mathbf{i}\bar{\omega}}))(0) \tag{2.108}$$

$$= \frac{(2\pi)^{l/2}}{(2\pi)^s\,|W| \operatorname{vol}^{\mathcal{R}}(T)\mathbf{i}^s} F_T(\varpi^2\Pi_*(\eta e^{\mathbf{i}\bar{\omega}}))(0). \tag{2.109}$$

Remark. The formulas in the above proposition hold if we assume G acts freely on M_0. In general, the G-action is not free on M_0 but has finite stabilizers. Let n_0 be the cardinality of the stabilizer in G of a generic point of M_0, where being generic here means having stabilizer with minimum cardinality, e.g. for $G = \mathbf{SU}(n)$, $n_0 = |\mathrm{Z}(\mathbf{SU}(n))| = n$. Then, if we multiply the right hand sides of the formulas above by n_0, the formulas still hold in this more general situation. Roughly, this is because in this more general situation, $M_0 \to M_{\text{red}}$ is an orbifold principal bundle instead of a smooth manifold principal bundle and we need to work on finite local covers over points in M_{red} instead of working on M_{red} directly.

The link between (2.108) and (2.109) is provided by the following key point:

Lemma 2.4.3 (Weyl integration formula; Lemma 3.1 in [34])**.** *Suppose $f : \mathfrak{g} \to \mathbb{R}$ is G-invariant. Then*

$$\int_{\phi\in\mathfrak{g}} f(\phi)[d\phi] = \frac{\operatorname{vol}^{\mathcal{R}}(G)}{|W| \operatorname{vol}^{\mathcal{R}}(T)} \int_{\psi\in\mathfrak{t}} f(\psi)\varpi^2(\psi)[d\psi]. \tag{2.110}$$

The rest of the proof for Proposition 2.4.2 depends on the following important result (see Gotay [21], Guillemin and Sternberg [28] and Marle [47]):

Proposition 2.4.4 (Proposition 5.2 in [34])**.** *Assume* 0 *is a regular value of μ_G. Then there is a neighbourhood*

$$U \cong M_0 \times \{z \in \mathfrak{g}^* \;:\; |z| < h\} \subset M_0 \times \mathfrak{g}^*, \tag{2.111}$$

where $h > 0$ is some sufficiently small number, of M_0 on which the symplectic form ω can be given as follows. Recall that $p_0 : M_0 \to M_{\text{red}}$ is the orbifold principal G-bundle. Let $\theta \in \Omega^1(M_0) \otimes \mathfrak{g}$ be a connection on this principal bundle. Recall that on M_{red} there is a symplectic structure ω_{red} such that $p_0^*\omega_{\text{red}} = i_0^*\omega$. Let α be a 1-form on $U \subset M_0 \times \mathfrak{g}^*$ defined by

$$\alpha_{(p,z)}(v, \xi) := \langle z, \theta_p(v) \rangle \tag{2.112}$$

for all $p \in M_0$, $z \in \mathfrak{g}^*$ with $|z| < h$, $v \in T_pM_0$ and $\xi \in T_z\mathfrak{g}^* = \mathfrak{g}^*$. Then, the symplectic form ω on U can be given by

$$\omega = \mathrm{pr}_1^*\, p_0^*\omega_{\text{red}} + d\alpha. \tag{2.113}$$

Moreover, the moment map μ_G on U is given by $\mu_G(p, z) = z$.

Roughly speaking, by considering a sequence of carefully chosen test functions $\chi_\epsilon : \mathfrak{g}^* \to \mathbb{R}_{\geq 0}$ such that as $\epsilon \to 0$, the functions χ_ϵ tend to the Dirac delta distribution on $\mathfrak{g}^*$, Jeffrey and Kirwan integrated $F_G(\Pi_*(\eta e^{\mathbf{i}\bar{\omega}}))$ against this sequence of test functions χ_ϵ. By invoking Proposition 2.4.4, they were able to concentrate on arbitrarily small neighbourhoods of M_0 and obtain the estimate which eventually established the link between $F_G(\Pi_*(\eta e^{\mathbf{i}\bar{\omega}}))(0)$ and the cohomological quantity $\kappa_0(\eta)e^{\mathbf{i}\omega_{\text{red}}}[M_{\text{red}}]$, finishing the proof for Proposition 2.4.2.

To understand the residue formula, i.e. Theorem 2.2.1, we need to see one more link, that is, the link between the expression in (2.109) and the residue in (2.76).

This link is established basically by the following two points.

The first point is a consequence of Proposition 2.3.1:

Proposition 2.4.5 (Proposition 8.7 in [34]). *Let $u : \mathfrak{t}^* \to \mathbb{C}$ be a distribution, and assume the set Γ_u defined in Proposition 2.3.1 contains $-\Lambda^0$. Then, $h = F_T u$ is a holomorphic function on $\mathfrak{t} - \mathbf{i}\Lambda^0$ and h satisfies the hypotheses in Definition 2.3.2. Assume in addition that u is smooth at 0. Then $\mathrm{res}^{\Lambda,\chi}(h(\psi)[d\psi])$ is independent of the test function χ, and moreover,*

$$\mathrm{res}^{\Lambda,\chi}(h(\psi)[d\psi]) = \frac{1}{\mathbf{i}^l(2\pi)^{l/2}}u(0). \tag{2.114}$$

The second point is related to the terms r_F^η as in Equations (2.106) and (2.107). To compute $\mathrm{res}(\varpi^2(\psi)r^\eta(\psi)[d\psi])$, we need to make sense and compute $\mathrm{res}(\varpi^2(\psi)r_F^\eta(\psi)[d\psi])$ for each F. To do so, Jeffrey and Kirwan chose some generic parameter $\rho \in \mathfrak{t}^*$ so that there exists $\delta > 0$ such that for all F, the distribution $F_T r_F^\eta$ is smooth along the ray $t\rho$ for $t \in (0, \delta)$ and tends to a well-defined limit as $t \to 0^+$. They showed that this is possible by first realizing that the functions $r_F^\eta(\psi)$ are sums of terms of the form

$$\frac{e^{\mathbf{i}\langle \mu_T(F),\psi\rangle}}{\prod_j \beta_{F,j}(\psi)^{n_j}} \tag{2.115}$$

with $\beta_{F,j} \in \mathfrak{t}^*$ satisfying certain conditions and thus realizing the Fourier transforms $F_T r_F^\eta$ as piecewise polynomial functions of the form $H_{\bar\beta}(y)$ whose definition will be given shortly. This last important realization is provided by the following point due to Guillemin, Lerman, Prato and Sternberg ([23], [24] and [25]):

Proposition 2.4.6 (Proposition 3.6 in [34])**.**

(a) *(Part (a) of Proposition 3.6 in [34]; see [23], Section 3.2 in [24], and [25]) Define*

$$H_{\bar\beta}(y) = \mathrm{vol}\left\{ (s_1, \ldots, s_\nu) \ : \ s_i \geq 0,\ y = \sum_j s_j \beta_j \right\} \tag{2.116}$$

for some ν-tuple $\bar\beta = (\beta_1, \ldots, \beta_\nu)$ with $\beta_j \in \mathfrak{t}^*$ such that all β_j lie in the interior of some half-space of $\mathfrak{t}^*$. Here vol *denotes the standard Euclidean volume multiplied by a normalization constant. Thus, $H_{\bar\beta}$ is a piecewise polynomial function supported on the cone*

$$C_{\bar\beta} := \left\{ \sum_j s_j \beta_j \ : \ s_j \geq 0 \right\}. \tag{2.117}$$

Let $h(y) := H_{\bar\beta}(y + \tau)$ for some $\tau \in \mathfrak{t}^*$. Then the Fourier transform of h is given for ψ in the complement of the union of the hyperplanes $\{\psi \in \mathfrak{t} \ : \ \beta_j(\psi) = 0\}$ by the formula

$$F_T h(\psi) = \frac{e^{\mathbf{i}\langle \tau,\psi\rangle}}{\mathbf{i}^\nu \prod_{j=1}^\nu \beta_j(\psi)}. \tag{2.118}$$

(b) *(Part (c) of Proposition 3.6 in [34]; see Section 2 in [23]) The function $H_{\bar\beta}$ satisfies the differential equation*

$$\prod_{j=1}^{\nu} \beta_j\left(\frac{\partial}{\partial y} H_{\bar\beta}(y)\right) = \delta_0(y), \tag{2.119}$$

where δ_0 is the Dirac delta distribution.

(c) *(Part (d) of Proposition 3.6 in [34]; see Proposition 2.6 in [23]) The function $H_{\bar\beta}$ is smooth at any $y \in U_{\bar\beta}$, where $U_{\bar\beta}$ consists of the points in $\mathfrak{t}^*$ that are not in any cone spanned by a subset of $\{\beta_1, \ldots, \beta_\nu\}$ containing fewer than l elements.*

By these considerations, $\mathrm{res}(r_F^\eta(\psi)[d\psi])$ can be computed as

$$\mathrm{res}^{\rho,\Lambda,\chi}(r_F^\eta(\psi)[d\psi]) = \lim_{t\to 0^+} \frac{1}{(2\pi)^{l/2}\mathbf{i}^l} F_T r_F^\eta(t\rho) \tag{2.120}$$

by Definition 2.3.2 and Proposition 2.4.5. Thus,

$$\sum_{F\in M^T} \mathrm{res}^{\rho,\Lambda,\chi}(\varpi^2(\psi) r_F^\eta(\psi)[d\psi]) = \mathrm{res}^{\rho,\Lambda,\chi}(\varpi^2(\psi) r^\eta(\psi)[d\psi]) \tag{2.121}$$

$$= \frac{1}{(2\pi)^{l/2}\mathbf{i}^l} F_T(\varpi^2 r^\eta)(0), \tag{2.122}$$

providing the last link for the proof of Theorem 2.2.1.

2.5 Computing the Residue

In this section, we will focus on computing the residue $\mathrm{res}(\Omega_\lambda)$ of a special class of meromorphic forms Ω_λ such that

$$\Omega_\lambda(\psi) = \frac{e^{\mathbf{i}\lambda(\psi)}[d\psi]}{\prod_{j=1}^{\nu} \beta_j(\psi)}, \tag{2.123}$$

where λ is some point in $\mathfrak{t}^*$ and β_j all lie in the dual cone of a proper cone Λ in $\mathfrak{t}$. A proper cone is an open cone such that its apex is the origin and it is properly contained in some half space. Given a proper cone $\Lambda \subset \mathfrak{t}$, its dual cone Λ^* is defined to be the following collection of elements in $\mathfrak{t}^*$:

$$\Lambda^* := \{\beta \in \mathfrak{t}^* \ : \ \beta(\psi) > 0 \text{ for all } \psi \in \Lambda\}. \tag{2.124}$$

Jeffrey and Kirwan gave a list of properties satisfied by the residue operator in Proposition 8.11 in [34] (see also Proposition 3.2 in [35]) and this list of properties will be the basis for the computations in this thesis. We record these properties here:

Proposition 2.5.1 (Proposition 8.11 in [34]; see also Proposition 3.2 in [35])**.** *Let Λ be a proper cone in $\mathfrak{t}$. Suppose that $\beta_1, \dots, \beta_\nu \in \mathfrak{t}^*$ all lie in Λ^*. Let λ be an element in $\mathfrak{t}^*$ such that it does not lie in any cone spanned by a subset of $\{\beta_1, \dots, \beta_\nu\}$ containing fewer than l elements. Let $\psi = (\psi_1, \dots, \psi_l)$ be any system of coordinates on $\mathfrak{t}$. For any multi-index $J = (j_1, \dots, j_l)$, where $j_1, \dots, j_l$ are nonnegative integers, let ψ^J denote $\psi_1^{j_1} \dots \psi_l^{j_l}$. Let $|J| = j_1 + \cdots + j_l$. Then res^Λ satisfies the following properties:*

(i)

$$(2\pi\mathbf{i})^l \operatorname{res}^\Lambda \left(\frac{e^{\mathbf{i}\lambda(\psi)}[d\psi]}{\prod_{j=1}^{\nu} \beta_j(\psi)} \right) = \mathbf{i}^\nu H_{\bar\beta}(\lambda), \tag{2.125}$$

where $H_{\bar\beta}$ *is as in Proposition 2.4.6.*

(ii)

$$\operatorname{res}^\Lambda \left(\frac{e^{\mathbf{i}\lambda(\psi)}[d\psi]}{\prod_{j=1}^{\nu} \beta_j(\psi)} \right) = 0 \tag{2.126}$$

unless $\lambda \in C_{\bar\beta}$ *where* $C_{\bar\beta}$ *is the cone spanned by* $\{\beta_1, \ldots, \beta_\nu\}$*.*

(iii)

$$\operatorname{res}^\Lambda \left(\frac{\psi^J e^{\mathbf{i}\lambda(\psi)}[d\psi]}{\prod_{j=1}^{\nu} \beta_j(\psi)} \right) = \sum_{m=0}^{\infty} \lim_{s\to 0^+} \operatorname{res}^\Lambda \left(\frac{\psi^J (\mathbf{i}\lambda(\psi))^m e^{\mathbf{i}s\lambda(\psi)}[d\psi]}{m! \prod_{j=1}^{\nu} \beta_j(\psi)} \right). \tag{2.127}$$

(iv) *If* $\{\beta_1, \ldots, \beta_\nu\}$ *does not span* $\mathfrak{t}^*$ *as a vector space then*

$$\lim_{s\to 0^+} \operatorname{res}^\Lambda \left(\frac{\psi^J e^{\mathbf{i}s\lambda(\psi)}[d\psi]}{\prod_{j=1}^{\nu} \beta_j(\psi)} \right) = 0. \tag{2.128}$$

(v)

$$\lim_{s\to 0^+} \operatorname{res}^\Lambda \left(\frac{\psi^J e^{\mathbf{i}s\lambda(\psi)}[d\psi]}{\prod_{j=1}^{\nu} \beta_j(\psi)} \right) = 0 \tag{2.129}$$

unless $\nu - |J| = l$*.*

(vi) *If* $\nu = l$ *and* $\{\beta_1, \ldots, \beta_l\}$ *spans* $\mathfrak{t}^*$ *as a vector space then*

$$\lim_{s\to 0^+} \operatorname{res}^\Lambda \left(\frac{e^{\mathbf{i}s\lambda(\psi)}[d\psi]}{\prod_{j=1}^{l} \beta_j(\psi)} \right) = 0 \tag{2.130}$$

unless $\lambda = \sum_{j=1}^{l} \lambda_j \beta_j$ *such that* $\lambda_j > 0$ *for all* j*, i.e., unless* $\lambda \in C_{\bar\beta}^0$ *where* $C_{\bar\beta}^0$ *denotes the interior of* $C_{\bar\beta}$*. If* $\lambda \in C_{\bar\beta}^0$*, then*

$$\lim_{s\to 0^+} \operatorname{res}^\Lambda \left(\frac{e^{\mathbf{i}s\lambda(\psi)}[d\psi]}{\prod_{j=1}^{l} \beta_j(\psi)} \right) = \frac{1}{\det(\bar\beta)}, \tag{2.131}$$

where here $\bar\beta$ *is regarded as an* $l \times l$ *matrix whose columns are the coordinates of* $\beta_1, \ldots, \beta_l$ *with respect to any orthonormal basis of* $\mathfrak{t}^*$ *defining the same orientation as* $\beta_1, \ldots, \beta_l$*.*

Now, we are finally ready for the computation of the symplectic volume of the triple ($N = 3$) reduced product M_{red}, i.e., the cohomological quantity

$$\frac{1}{\mathbf{i}^{d/2}} e^{\mathbf{i}\omega_{\text{red}}}[M_{\text{red}}]. \tag{2.132}$$

By Corollary 2.2.2, we have

$$\frac{1}{\mathbf{i}^{d/2}} e^{\mathbf{i}\omega_{\text{red}}}[M_{\text{red}}] = \frac{1}{\mathbf{i}^{d/2}} n_0 C_G \operatorname{res}\left(\sum_{\vec{w} \in W^3} \operatorname{sgn}(\vec{w}) \frac{e^{\mathbf{i}(w_1 \cdot \xi_1 + w_2 \cdot \xi_2 + w_3 \cdot \xi_3, \psi)}}{\varpi(\psi)} [d\psi] \right). \tag{2.133}$$

To compute the residue on the right hand side of the above equation, we need to first choose an open cone Λ in $\mathfrak{t}$. We choose $\Lambda = \mathfrak{t}_{>0}$, the open positive Weyl chamber. One reason that we make this choice is that we observe that (here recall that we are considering $G = \mathbf{SU}(3)$)

$$\varpi(\psi) = \prod_{j=1}^{3} \beta_j(\psi), \tag{2.134}$$

where

$$\beta_1 = (\operatorname{diag}(2\pi\mathbf{i}, -2\pi\mathbf{i}, 0), \cdot), \tag{2.135}$$
$$\beta_2 = (\operatorname{diag}(0, 2\pi\mathbf{i}, -2\pi\mathbf{i}), \cdot), \tag{2.136}$$
$$\beta_3 = (\operatorname{diag}(2\pi\mathbf{i}, 0, -2\pi\mathbf{i}), \cdot). \tag{2.137}$$

Thus $\beta_3 = \beta_1 + \beta_2$ and the collection $\{\beta_1, \beta_2, \beta_3\}$ is just the set of positive roots of $G = \mathbf{SU}(3)$. Notice that all of $\beta_1, \beta_2, \beta_3$ lie in Λ^*, the dual cone of Λ.

Let $\bar{\beta} = (\beta_1, \beta_2, \beta_3)$.

Let

$$\vec{w} \odot \vec{\xi} := \sum_{i=1}^{N} w_i \cdot \xi_i. \tag{2.138}$$

Here we are considering $N = 3$.

Let us rewrite the residue part in Equation (2.133):

$$\operatorname{res}\left(\sum_{\vec{w}\in W^3}\operatorname{sgn}(\vec{w})\frac{e^{\mathbf{i}(w_1\cdot\xi_1+w_2\cdot\xi_2+w_3\cdot\xi_3,\psi)}}{\varpi(\psi)}[d\psi]\right) \tag{2.139}$$

$$=\operatorname{res}^{\Lambda}\left(\sum_{\vec{w}\in W^3}\operatorname{sgn}(\vec{w})\frac{e^{\mathbf{i}(\vec{w}\odot\vec{\xi})(\psi)}[d\psi]}{\prod_{j=1}^{3}\beta_j(\psi)}\right) \tag{2.140}$$

$$=\sum_{\vec{w}\in W^3}\operatorname{sgn}(\vec{w})\operatorname{res}^{\Lambda}\left(\frac{e^{\mathbf{i}(\vec{w}\odot\vec{\xi})(\psi)}[d\psi]}{\prod_{j=1}^{3}\beta_j(\psi)}\right) \tag{2.141}$$

$$=\sum_{\vec{w}\in W^3}\operatorname{sgn}(\vec{w})\frac{\mathbf{i}^3}{(2\pi\mathbf{i})^2}H_{\vec{\beta}}(\vec{w}\odot\vec{\xi}). \tag{2.142}$$

Notice that the last step above is derived by Equation (2.125).

Combining the above formula with the constant part in Equation (2.133), we have

$$\operatorname{vol}^{\mathcal{S}}(M_{\text{red}}(\vec{\xi}))=\frac{1}{\mathbf{i}^{d/2}}n_0C_G\frac{\mathbf{i}^3}{(2\pi\mathbf{i})^2}\sum_{\vec{w}\in W^3}\operatorname{sgn}(\vec{w})H_{\vec{\beta}}(\vec{w}\odot\vec{\xi}) \tag{2.143}$$

$$=\frac{1}{\mathbf{i}}\cdot 3\cdot\frac{(-1)^3}{(2\pi)^6\cdot 6\cdot\operatorname{vol}^{\mathcal{R}}(T)}\cdot\frac{\mathbf{i}^3}{(2\pi\mathbf{i})^2}\cdot\sum_{\vec{w}\in W^3}\operatorname{sgn}(\vec{w})H_{\vec{\beta}}(\vec{w}\odot\vec{\xi}) \tag{2.144}$$

$$=\frac{-1}{2\cdot(2\pi)^8\cdot\operatorname{vol}^{\mathcal{R}}(T)}\sum_{\vec{w}\in W^3}\operatorname{sgn}(\vec{w})H_{\vec{\beta}}(\vec{w}\odot\vec{\xi}). \tag{2.145}$$

Notice that since we are considering triple reduced products of $G=\mathbf{SU}(3)$ here, we have

$$d=N(s-l)-2s=3\cdot(8-2)-2\cdot 8=2, \tag{2.146}$$

$$n_+=(s-l)/2=(8-2)/2=3, \tag{2.147}$$

$$n_0=|\mathrm{Z}(\mathbf{SU}(3))|=3. \tag{2.148}$$

Therefore, we need to compute $H_{\vec{\beta}}(\vec{w}\odot\vec{\xi})$ to obtain an explicit formula for the symplectic volume of a triple reduced product.

Remark. Before we move on, we want to comment about the normalization constant appearing before the sum in the above formula. Since this normalization constant depends on the convention we choose, the explicit value of the normalization constant usually does not have huge significance. Different authors may use different conventions. Thus their normalization constant may be different than ours. Therefore the significance of an explicit formula usually comes from its qualitative behaviours, e.g., the order of growth, the places where the function fails to be smooth, etc.

Since later we will compare our volume formula with the volume formula obtained by Suzuki and Takakura [53], we will parametrize things in a way similar to theirs.

Since we have identified $\mathfrak{t}^*$ with $\mathfrak{t}$ using the inner product, we will regard the β_j as matrices. Explicitly,

$$\beta_1 = \mathrm{diag}(2\pi\mathbf{i}, -2\pi\mathbf{i}, 0), \tag{2.149}$$

$$\beta_2 = \mathrm{diag}(0, 2\pi\mathbf{i}, -2\pi\mathbf{i}), \tag{2.150}$$

$$\beta_3 = \mathrm{diag}(2\pi\mathbf{i}, 0, -2\pi\mathbf{i}). \tag{2.151}$$

Any element $\xi \in \mathfrak{t}_{>0}$ can be written as

$$\xi = (\ell - m) \cdot \frac{2\beta_1 + \beta_2}{3} + m \cdot \frac{\beta_1 + 2\beta_2}{3} \tag{2.152}$$

for some $\ell > m > 0$. If ℓ and m can be any real number, then the above formula parametrize all $\xi \in \mathfrak{t}$.

Let

$$\Omega_1 := \frac{2\beta_1 + \beta_2}{3}, \tag{2.153}$$

$$\Omega_2 := \frac{\beta_1 + 2\beta_2}{3}. \tag{2.154}$$

Then,

$$\xi = (\ell - m) \cdot \Omega_1 + m \cdot \Omega_2 = \ell \cdot \Omega_1 + m \cdot (\Omega_2 - \Omega_1). \tag{2.155}$$

We fix $\{\Omega_1, \Omega_2 - \Omega_1\}$ as the basis for $\mathfrak{t}$ for this computation.

Our goal here is to express $H_{\bar{\beta}}(\xi)$ as a function of ℓ and m.

It is helpful to express $H_{\bar{\beta}}(\lambda_1\beta_1 + \lambda_2\beta_2)$ in terms of λ_1 and λ_2 first.

By the definition of $H_{\bar{\beta}}$, we have

$$H_{\bar{\beta}}(\lambda_1\beta_1 + \lambda_2\beta_2) = \mathrm{vol}\left\{(s_1, s_2, s_3) \in \mathbb{R}_+^3 \ : \ \sum_{j=1}^{3} s_j\beta_j = \lambda_1\beta_1 + \lambda_2\beta_2\right\}, \tag{2.156}$$

where $\mathbb{R}_+$ denotes the set of all nonnegative real numbers. Therefore we want to solve the following equation:

$$s_1\beta_1 + s_2\beta_2 + s_3(\beta_1 + \beta_2) = \lambda_1\beta_1 + \lambda_2\beta_2. \tag{2.157}$$

Notice that $\beta_3 = \beta_1 + \beta_2$. Collecting terms, we then have:

$$(s_1 + s_3)\beta_1 + (s_2 + s_3)\beta_2 = \lambda_1\beta_1 + \lambda_2\beta_2. \tag{2.158}$$

Thus we have the following linear system:

$$s_1 + s_3 = \lambda_1, \tag{2.159}$$

$$s_2 + s_3 = \lambda_2. \tag{2.160}$$

The solution set S is:

$$\begin{aligned} S &= \{(\lambda_1 - s_3, \lambda_2 - s_3, s_3) \ : \ \lambda_1 - s_3 \geq 0,\ \lambda_2 - s_3 \geq 0,\ s_3 \geq 0\} & (2.161)\\ &= \{(\lambda_1 - s_3, \lambda_2 - s_3, s_3) \ : \ s_3 \leq \lambda_1,\ s_3 \leq \lambda_2,\ s_3 \geq 0\} & (2.162)\\ &= \{(\lambda_1 - s_3, \lambda_2 - s_3, s_3) \ : \ 0 \leq s_3 \leq \min(\lambda_1, \lambda_2)\}. & (2.163)\end{aligned}$$

Notice that $S = \emptyset$ if $\lambda_1 < 0$ or $\lambda_2 < 0$. Therefore,

$$H_{\bar{\beta}}(\lambda_1\beta_1 + \lambda_2\beta_2) = \mathrm{vol}(S) = C \cdot \max(\min(\lambda_1, \lambda_2), 0), \tag{2.164}$$

where C is some constant.

Now, we compute $H_{\bar{\beta}}(\xi)$ for $\xi = \ell \cdot \Omega_1 + m \cdot (\Omega_2 - \Omega_1)$.

First, we rewrite ξ in the form of $\lambda_1\beta_1 + \lambda_2\beta_2$:

$$\begin{aligned} \xi &= \ell \cdot \Omega_1 + m \cdot (\Omega_2 - \Omega_1) & (2.165)\\ &= (\ell - m) \cdot \Omega_1 + m \cdot \Omega_2 & (2.166)\\ &= (\ell - m) \cdot \frac{2\beta_1 + \beta_2}{3} + m \cdot \frac{\beta_1 + 2\beta_2}{3} & (2.167)\\ &= \frac{2\ell - m}{3} \cdot \beta_1 + \frac{\ell + m}{3} \cdot \beta_2. & (2.168)\end{aligned}$$

Therefore,

$$H_{\bar{\beta}}(\xi) = C \cdot \max(\min(\frac{2\ell - m}{3}, \frac{\ell + m}{3}), 0) \tag{2.169}$$

for some constant C.

Now, consider the triple reduced product $M_{\mathrm{red}}(\vec{\xi})$ of $G = \mathbf{SU}(3)$ with the input $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$ where

$$\xi_i = (\ell_i - m_i) \cdot \Omega_1 + m_i \cdot \Omega_2 = \ell_i \cdot \Omega_1 + m_i \cdot (\Omega_2 - \Omega_1),\ \ell_i > m_i > 0. \tag{2.170}$$

Notice that each ξ_i lies in the open positive Weyl chamber $\mathfrak{t}_{>0}$. Our goal is to express the symplectic volume of $M_{\text{red}}(\vec{\xi})$ in terms of $\ell_1, \ell_2, \ell_3, m_1, m_2, m_3$.

First, we need to write each Weyl group element as a 2×2 matrix with respect to the basis $\{\Omega_1, \Omega_2 - \Omega_1\}$.

Recall that we have enumerated the Weyl group as $\{\mathfrak{s}_0, \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_5\}$ and associated the $\mathfrak{s}_j$ with elements in $\mathfrak{S}_3$ bijectively through Equations (2.11–2.16). Each $\mathfrak{s}_j$ can be regarded as a linear transformation from $\mathfrak{t}$ to itself. With respect to the basis $\{\Omega_1, \Omega_2 - \Omega_1\}$, they can be written as the following matrices:

$$\mathfrak{s}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.171}$$

$$\mathfrak{s}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.172}$$

$$\mathfrak{s}_2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \tag{2.173}$$

$$\mathfrak{s}_3 = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \tag{2.174}$$

$$\mathfrak{s}_4 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.175}$$

$$\mathfrak{s}_5 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}. \tag{2.176}$$

Now, combining Equations (2.145) and (2.169), we obtain the following explicit formula for the symplectic volume of triple reduced products of $G = \mathbf{SU}(3)$:

Theorem 2.5.2. *The symplectic volume of the triple reduced product $M_{\text{red}}(\vec{\xi})$ of $G = \mathbf{SU}(3)$ with the input $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$, where*

$$\xi_i = (\ell_i - m_i) \cdot \Omega_1 + m_i \cdot \Omega_2 = \ell_i \cdot \Omega_1 + m_i \cdot (\Omega_2 - \Omega_1), \ \ \ell_i > m_i > 0, \tag{2.177}$$

can be computed by the following explicit formula:

$$\operatorname{vol}^{\mathcal{S}}(M_{\text{red}}(\vec{\xi})) = K \cdot \sum_{i=0}^{5} \sum_{j=0}^{5} \sum_{k=0}^{5} (-1)^{i+j+k} \cdot \max\left(\min\left(P_{ijk}^{(1)}(\vec{\xi}), P_{ijk}^{(2)}(\vec{\xi})\right), 0\right), \tag{2.178}$$

where K is a constant and

$$P^{(1)}_{ijk}(\vec{\xi}) = (\frac{2}{3}\,\mathrm{pr}_1 - \frac{1}{3}\,\mathrm{pr}_2)(P_{ijk}(\vec{\xi})), \tag{2.179}$$

$$P^{(2)}_{ijk}(\vec{\xi}) = (\frac{1}{3}\,\mathrm{pr}_1 + \frac{1}{3}\,\mathrm{pr}_2)(P_{ijk}(\vec{\xi})), \tag{2.180}$$

with

$$P_{ijk}(\vec{\xi}) = \mathfrak{s}_i \cdot \begin{pmatrix} \ell_1 \\ m_1 \end{pmatrix} + \mathfrak{s}_j \cdot \begin{pmatrix} \ell_2 \\ m_2 \end{pmatrix} + \mathfrak{s}_k \cdot \begin{pmatrix} \ell_3 \\ m_3 \end{pmatrix} \in \mathbb{R}^2. \tag{2.181}$$

Here pr_i *denotes the standard projection onto the i-th coordinate.*

Remark. It is clear that the above formula is a piecewise linear function. While this function is continuous, there are places, called "walls", where this function is not differentiable. These walls are introduced by the max and min operators. If we cross a wall, we will see that the gradient vector "jumps".

2.6 A Recent Result of Suzuki and Takakura

It will be interesting to compare our result with a recent result on symplectic volume of N-fold reduced products of $G = \mathbf{SU}(3)$ by Suzuki and Takakura [53] in 2008.

In this section, we will describe their result in the case when $N = 3$.

The settings in their paper [53] are almost the same as ours except that their choice of inner product on $\mathfrak{g}$ is $(\cdot,\cdot)/(4\pi^2)$ where $(\cdot,\cdot)$ is our choice of inner product.

Their initial input is more restrictive than ours in the following sense:

Let $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$ be the input such that

$$\xi_i = (\ell_i - m_i) \cdot \Omega_1 + m_i \cdot \Omega_2, \tag{2.182}$$

where $\ell_i > m_i > 0$ and all ℓ_i and m_i are integers that are divisible by 3 and also

$$(\vec{w} \odot \vec{\xi}, \Omega_1) \neq 0 \tag{2.183}$$

for all $\vec{w} \in W^3$.

Now, let $L = \ell_1 + \ell_2 + \ell_3$ and $M = m_1 + m_2 + m_3$.

If I is a subset of $\{1, 2, 3\}$,

$$\ell_I := \sum_{i \in I} \ell_i, \tag{2.184}$$

$$m_I := \sum_{i \in I} m_i. \tag{2.185}$$

If I and J are two disjoint subsets of $\{1, 2, 3\}$,

$$\ell_{I,J} := \ell_I + \ell_J = \sum_{i \in I \sqcup J} \ell_i, \tag{2.186}$$

$$m_{I,J} := m_I + m_J = \sum_{i \in I \sqcup J} m_i. \tag{2.187}$$

If $(I_1, \ldots, I_6)$ is a 6-tuple of subsets of $\{1, 2, 3\}$, $(I_1, \ldots, I_6)$ is called a 6-partition of $\{1, 2, 3\}$ if and only if:

$$I_1 \cup \cdots \cup I_6 = \{1, 2, 3\} \text{ and } I_j \cap I_k = \emptyset \text{ whenever } j \neq k. \tag{2.188}$$

For $N = 3$, the result of Suzuki and Takakura is the following:

Theorem 2.6.1 (Theorem 4.5 in [53], in the case $N = 3$)**.** *Let $\mathcal{I}_{\vec{\xi}}$ denote the set of those 6-partitions $(I_1, \ldots, I_6)$ of $\{1, 2, 3\}$ such that*

$$\ell_{I_1,I_2} + m_{I_4,I_5} < \frac{L+M}{3}, \text{ and} \tag{2.189}$$

$$\ell_{I_3,I_4} + m_{I_6,I_1} < \frac{L+M}{3}. \tag{2.190}$$

Let $\mathcal{J}_{\vec{\xi}}$ denote the set of those 6-partitions $(I_1, \ldots, I_6)$ of $\{1, 2, 3\}$ such that

$$\ell_{I_3,I_4} + m_{I_6,I_1} > \frac{L+M}{3}, \text{ and} \tag{2.191}$$

$$\ell_{I_5,I_6} + m_{I_2,I_3} > \frac{L+M}{3}. \tag{2.192}$$

Let $A_{\vec{\xi}} : \mathcal{I}_{\vec{\xi}} \to \mathbb{R}$ be defined by:

$$A_{\vec{\xi}}(I_1, \ldots, I_6) := \frac{-(-1)^{|I_1|+|I_3|+|I_5|}}{6} \left(\frac{L+M}{3} - \ell_{I_1,I_2} - m_{I_4,I_5} \right). \tag{2.193}$$

Let $B_{\vec{\xi}} : \mathcal{J}_{\vec{\xi}} \to \mathbb{R}$ *be defined by:*

$$B_{\vec{\xi}}(I_1, \ldots, I_6) := \frac{-(-1)^{|I_1|+|I_3|+|I_5|}}{6} \left(\ell_{I_5,I_6} + m_{I_2,I_3} - \frac{L+M}{3} \right). \tag{2.194}$$

Then, the symplectic volume of $M_{\text{red}}(\vec{\xi})$ *is given by:*

$$\mathcal{V}(\vec{\xi}) = \sum_{(I_1,\ldots,I_6)\in\mathcal{I}_{\vec{\xi}}} A_{\vec{\xi}}(I_1, \ldots, I_6) + \sum_{(I_1,\ldots,I_6)\in\mathcal{J}_{\vec{\xi}}} B_{\vec{\xi}}(I_1, \ldots, I_6). \tag{2.195}$$

We shall briefly explain this result.

First, 6-partitions $(I_1, \ldots, I_6)$ of $\{1, 2, 3\}$ correspond bijectively to fixed points of T acting on M, i.e., to M^T, in the following way.

Suzuki and Takakura have used a different enumeration $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$ of the Weyl group W of $\mathbf{SU}(3)$ in their paper [53]. Relating their enumeration with ours, we have:

$$\sigma_1 = \text{Id} = \mathfrak{s}_0, \tag{2.196}$$
$$\sigma_2 = (2\ 3) = \mathfrak{s}_5, \tag{2.197}$$
$$\sigma_3 = (1\ 2\ 3) = \mathfrak{s}_2, \tag{2.198}$$
$$\sigma_4 = (1\ 2) = \mathfrak{s}_1, \tag{2.199}$$
$$\sigma_5 = (1\ 3\ 2) = \mathfrak{s}_4, \tag{2.200}$$
$$\sigma_6 = (1\ 3) = \mathfrak{s}_3. \tag{2.201}$$

For each $\vec{w} = (w_1, w_2, w_3) \in W^3$, let I_j be defined as

$$I_j := \{i \in \{1, 2, 3\}\ :\ w_i = \sigma_j\} \tag{2.202}$$

for $j = 1, \ldots, 6$. Then $(I_1, \ldots, I_6)$ is a 6-partition of $\{1, 2, 3\}$. For example, given $(\sigma_2, \sigma_5, \sigma_2) \in W^3$, the corresponding 6-partition of $\{1, 2, 3\}$ is

$$(\emptyset, \{1, 3\}, \emptyset, \emptyset, \{2\}, \emptyset). \tag{2.203}$$

In other words, I_j tells us in which coordinates (in this case, the first, the second, or the third coordinate of $\vec{w}$) σ_j appears in $\vec{w}$.

Suzuki and Takakura have observed that $\vec{w} \odot \vec{\xi}$ is the matrix

$$2\pi\mathbf{i}\cdot\mathrm{diag}(\ell_{I_1,I_2}+m_{I_4,I_5}-\frac{L+M}{3},\ell_{I_3,I_4}+m_{I_6,I_1}-\frac{L+M}{3},\ell_{I_5,I_6}+m_{I_2,I_3}-\frac{L+M}{3}). \quad (2.204)$$

It will be interesting to see when the above matrix is in the cone spanned by $\bar{\beta} = (\beta_1, \beta_2, \beta_3)$.

$\vec{w} \odot \vec{\xi}$ is in the cone spanned by $\bar{\beta}$ if and only if

$$(\vec{w} \odot \vec{\xi}, \Omega_1) > 0, \text{ and} \quad (2.205)$$

$$(\vec{w} \odot \vec{\xi}, \Omega_2) > 0. \quad (2.206)$$

Recall that

$$\Omega_1 = \frac{2\pi\mathbf{i}}{3} \cdot \mathrm{diag}(2, -1, -1), \quad (2.207)$$

$$\Omega_2 = \frac{2\pi\mathbf{i}}{3} \cdot \mathrm{diag}(1, 1, -2). \quad (2.208)$$

Thus, $\vec{w} \odot \vec{\xi}$ is in the cone spanned by $\bar{\beta}$ if and only if

$$2\ell_{I_1,I_2} + 2m_{I_4,I_5} - \ell_{I_3,I_4} - m_{I_6,I_1} - \ell_{I_5,I_6} - m_{I_2,I_3} > 0, \text{ and} \quad (2.209)$$

$$\ell_{I_1,I_2} + m_{I_4,I_5} + \ell_{I_3,I_4} + m_{I_6,I_1} - 2\ell_{I_5,I_6} - 2m_{I_2,I_3} > 0. \quad (2.210)$$

Notice that

$$(\vec{w} \odot \vec{\xi}, \Omega_1) = \frac{4\pi^2}{3}(2\ell_{I_1,I_2} + 2m_{I_4,I_5} - \ell_{I_3,I_4} - m_{I_6,I_1} - \ell_{I_5,I_6} - m_{I_2,I_3}) \quad (2.211)$$

$$= 4\pi^2\left(\ell_{I_1,I_2} + m_{I_4,I_5} - \frac{L+M}{3}\right), \text{ and} \quad (2.212)$$

$$(\vec{w} \odot \vec{\xi}, \Omega_2) = \frac{4\pi^2}{3}(\ell_{I_1,I_2} + m_{I_4,I_5} + \ell_{I_3,I_4} + m_{I_6,I_1} - 2\ell_{I_5,I_6} - 2m_{I_2,I_3}) \quad (2.213)$$

$$= 4\pi^2\left(\frac{L+M}{3} - \ell_{I_5,I_6} - m_{I_2,I_3}\right). \quad (2.214)$$

It will be interesting to compare the above condition with the conditions for $\mathcal{I}_{\vec{\xi}}$ and $\mathcal{J}_{\vec{\xi}}$ used in Theorem 2.6.1.

If $\vec{w} \odot \vec{\xi}$ satisfies the condition for $\mathcal{I}_{\vec{\xi}}$, then

$$
\begin{aligned}
& 2\ell_{I_1,I_2} + 2m_{I_4,I_5} - \ell_{I_3,I_4} - m_{I_6,I_1} - \ell_{I_5,I_6} - m_{I_2,I_3} && (2.215)\\
&= 3\ell_{I_1,I_2} + 3m_{I_4,I_5} - L - M && (2.216)\\
&= 3(\ell_{I_1,I_2} + m_{I_4,I_5} - \frac{L+M}{3}) < 0. && (2.217)
\end{aligned}
$$

Thus, $\vec{w} \odot \vec{\xi}$ is not in the cone spanned by $\bar{\beta}$.

If we look closely at the first inequality of the condition for $\mathcal{I}_{\vec{\xi}}$, namely,

$$\ell_{I_1,I_2} + m_{I_4,I_5} < \frac{L+M}{3}, \tag{2.218}$$

this means exactly

$$(\vec{w} \odot \vec{\xi}, \Omega_1) < 0. \tag{2.219}$$

Also, the second inequality of the condition for $\mathcal{I}_{\vec{\xi}}$, namely,

$$\ell_{I_3,I_4} + m_{I_6,I_1} < \frac{L+M}{3}, \tag{2.220}$$

means exactly

$$(\vec{w} \odot \vec{\xi}, \Omega_2 - \Omega_1) < 0. \tag{2.221}$$

Thus, we have translated the condition for $\mathcal{I}_{\vec{\xi}}$ into the following two inequalities:

$$
\begin{aligned}
(\vec{w} \odot \vec{\xi}, \Omega_1) < 0, \text{ and} && (2.222)\\
(\vec{w} \odot \vec{\xi}, \Omega_2 - \Omega_1) < 0. && (2.223)
\end{aligned}
$$

Similarly, we can translate the condition for $\mathcal{J}_{\vec{\xi}}$ into the following two inequalities:

$$
\begin{aligned}
(\vec{w} \odot \vec{\xi}, \Omega_2 - \Omega_1) > 0, \text{ and} && (2.224)\\
(\vec{w} \odot \vec{\xi}, \Omega_2) < 0. && (2.225)
\end{aligned}
$$

Notice that if $\vec{w} \odot \vec{\xi}$ satisfies either the condition for $\mathcal{I}_{\vec{\xi}}$ or the condition for $\mathcal{J}_{\vec{\xi}}$, we always have that $\vec{w} \odot \vec{\xi}$ is in the cone spanned by $-\bar{\beta} = (-\beta_1, -\beta_2, -\beta_1 - \beta_2)$. In other words, only those $\vec{w} \odot \vec{\xi}$ contained in the cone spanned by $-\bar{\beta}$ will contribute to the sum in the volume formula of Suzuki and Takakura. Also notice that the sign

$$-(-1)^{|I_1|+|I_3|+|I_5|} \tag{2.226}$$

is exactly the signature of $\vec{w}$.

The above observations lead us to guess that our volume formula (Theorem 2.5.2) and the volume formula of Suzuki and Takakura (Theorem 2.6.1) are very closely related.

The first key observation supporting that our formula should indeed agree with theirs is that we derived our formula by using the residue formula (Theorem 2.2.1) with a choice of cone, namely the cone spanned by $\bar{\beta}$ and as a result, only those $\vec{w} \odot \vec{\xi}$ in the cone spanned by $\bar{\beta}$ will contribute to the sum in our volume formula. However, the total sum in the residue formula does not depend on the choice of cone. Therefore, we could equally well choose the cone spanned by $-\bar{\beta}$ to carry out the computations of the individual terms in the sum. Let's carry this out.

More precisely, we start from the choice of $\Lambda_- = -\mathfrak{t}_{>0}$. In this way, all of $-\beta_1, -\beta_2, -\beta_3 = -\beta_1 - \beta_2$ lie in the dual cone Λ_-^*.

With this new choice of cone, we carry out the computation, starting from Equation (2.139):

$$\operatorname{res}\left(\sum_{\vec{w}\in W^3} \operatorname{sgn}(\vec{w}) \frac{e^{\mathbf{i}(\vec{w}\odot\vec{\xi},\psi)}[d\psi]}{\varpi(\psi)}\right) \tag{2.227}$$

$$= \operatorname{res}^{\Lambda_-}\left(\sum_{\vec{w}\in W^3} \operatorname{sgn}(\vec{w}) \frac{e^{\mathbf{i}(\vec{w}\odot\vec{\xi},\psi)}[d\psi]}{\prod_{j=1}^3 \beta_j(\psi)}\right) \tag{2.228}$$

$$= \sum_{\vec{w}\in W^3} \operatorname{sgn}(\vec{w}) \operatorname{res}^{\Lambda_-}\left(\frac{e^{\mathbf{i}(\vec{w}\odot\vec{\xi},\psi)}[d\psi]}{\prod_{j=1}^3 \beta_j(\psi)}\right) \tag{2.229}$$

$$= -\sum_{\vec{w}\in W^3} \operatorname{sgn}(\vec{w}) \operatorname{res}^{\Lambda_-}\left(\frac{e^{\mathbf{i}(\vec{w}\odot\vec{\xi},\psi)}[d\psi]}{\prod_{j=1}^3 (-\beta_j)(\psi)}\right) \tag{2.230}$$

$$= -\sum_{\vec{w}\in W^3} \operatorname{sgn}(\vec{w}) \frac{\mathbf{i}^3}{(2\pi\mathbf{i})^2} H_{(-\bar{\beta})}(\vec{w}\odot\vec{\xi}). \tag{2.231}$$

Now, our volume formula corresponding to the cone Λ_- can be written as

$$\operatorname{vol}^{\mathcal{S}}(M_{\mathrm{red}}(\vec{\xi})) = K' \sum_{\vec{w}\in W^3} \operatorname{sgn}(\vec{w}) H_{(-\bar{\beta})}(\vec{w}\odot\vec{\xi}), \tag{2.232}$$

where K' is a positive constant. Notice that the constant K in our volume formula corresponding to the cone Λ, i.e., Equation (2.178), is negative. In fact, $K' = -K$.

$H_{(-\bar{\beta})}$ is supported in the cone spanned by $-\bar{\beta}$. Therefore, only those $\vec{w} \odot \vec{\xi}$ inside this cone will contribute to the sum. This gives us the common ground to compare our volume formula (using the cone Λ_-) and the volume formula of Suzuki and Takakura.

Let's compute $H_{(-\bar{\beta})}(\lambda_1 \cdot (-\beta_1) + \lambda_2 \cdot (-\beta_2))$ where $\lambda_1, \lambda_2 \in \mathbb{R}$.

We have:

$$H_{(-\bar{\beta})}(\lambda_1 \cdot (-\beta_1) + \lambda_2 \cdot (-\beta_2)) \tag{2.233}$$

$$= \mathrm{vol}\left\{ (s_1, s_2, s_3) \in \mathbb{R}^3_+ \;:\; \sum_{j=1}^{3} s_j \cdot (-\beta_j) = \lambda_1 \cdot (-\beta_1) + \lambda_2 \cdot (-\beta_2) \right\}. \tag{2.234}$$

Thus, we need to solve the following equation:

$$s_1 \cdot (-\beta_1) + s_2 \cdot (-\beta_2) + s_3 \cdot (-\beta_1 - \beta_2) = \lambda_1 \cdot (-\beta_1) + \lambda_2 \cdot (-\beta_2). \tag{2.235}$$

Therefore, we need to solve the following linear system:

$$s_1 + s_3 = \lambda_1, \tag{2.236}$$

$$s_2 + s_3 = \lambda_2. \tag{2.237}$$

The solution set S_- is:

$$S_- = \{(\lambda_1 - s_3, \lambda_2 - s_3, s_3) \;:\; \lambda_1 - s_3 \geq 0,\; \lambda_2 - s_3 \geq 0,\; s_3 \geq 0\} \tag{2.238}$$

$$= \{(\lambda_1 - s_3, \lambda_2 - s_3, s_3) \;:\; s_3 \leq \lambda_1,\; s_3 \leq \lambda_2, s_3 \geq 0\} \tag{2.239}$$

$$= \{(\lambda_1 - s_3, \lambda_2 - s_3, s_3) \;:\; 0 \leq s_3 \leq \min(\lambda_1, \lambda_2)\}. \tag{2.240}$$

Therefore,

$$H_{-\bar{\beta}}(\lambda_1 \cdot (-\beta_1) + \lambda_2 \cdot (-\beta_2)) = \mathrm{vol}(S_-) = C \cdot \max(\min(\lambda_1, \lambda_2), 0), \tag{2.241}$$

where C is the same constant as in Equation (2.164).

Now, let's look at those $\vec{w} \odot \vec{\xi}$ inside the cone spanned by $-\bar{\beta}$. Without loss of generality we can assume that $\vec{\xi}$ is generic so that for all $\vec{w} \in W^3$,

$$(\vec{w} \odot \vec{\xi}, \Omega_1) \neq 0, \tag{2.242}$$

$$(\vec{w} \odot \vec{\xi}, \Omega_2) \neq 0, \tag{2.243}$$

$$(\vec{w} \odot \vec{\xi}, \Omega_2 - \Omega_1) \neq 0. \tag{2.244}$$

Therefore, the collection of those $\vec{w} \odot \vec{\xi}$ inside the cone spanned by $-\bar{\beta}$ is the disjoint union of the two sets $\mathcal{A}_{\vec{\xi}}$ and $\mathcal{B}_{\vec{\xi}}$, where $\mathcal{A}_{\vec{\xi}}$ denotes the set of those $\vec{w} \odot \vec{\xi}$ such that

$$(\vec{w} \odot \vec{\xi}, \Omega_1) < 0, \text{ and} \tag{2.245}$$

$$(\vec{w} \odot \vec{\xi}, \Omega_2 - \Omega_1) < 0, \tag{2.246}$$

and $\mathcal{B}_{\vec{\xi}}$ denotes the set of those $\vec{w} \odot \vec{\xi}$ such that

$$(\vec{w} \odot \vec{\xi}, \Omega_2 - \Omega_1) > 0, \text{ and} \tag{2.247}$$

$$(\vec{w} \odot \vec{\xi}, \Omega_2) < 0. \tag{2.248}$$

Notice that the above grouping is in complete match with the grouping by $\mathcal{I}_{\vec{\xi}}$ and $\mathcal{J}_{\vec{\xi}}$ in the formula of Suzuki and Takakura.

For each $\vec{w} \odot \vec{\xi} \in \mathcal{A}_{\vec{\xi}}$, it is easy to see that $\vec{w} \odot \vec{\xi}$ can be written as $\lambda_1 \cdot (-\beta_1) + \lambda_2 \cdot (-\beta_2)$ with some λ_1, λ_2 satisfying $0 < \lambda_1 < \lambda_2$. Therefore, the contribution of this $\vec{w} \odot \vec{\xi}$ to our volume formula is

$$K' \cdot \operatorname{sgn}(\vec{w}) \cdot H_{(-\vec{\beta})}(\vec{w} \odot \vec{\xi}) = K' \cdot \operatorname{sgn}(\vec{w}) \cdot \lambda_1. \tag{2.249}$$

Notice that this $\vec{w}$ will correspond to a 6-partition $(I_1, \ldots, I_6)$ and we have

$$\operatorname{sgn}(\vec{w}) = -(-1)^{|I_1|+|I_3|+|I_5|}. \tag{2.250}$$

Now we only need to figure out how we can express this λ_1 in terms of the ℓ_i's and m_i's.

First, given $a, b \in \mathbb{R}$ and

$$2\pi\mathbf{i} \cdot \operatorname{diag}(a, b, -a-b) = \lambda_1 \cdot (-\beta_1) + \lambda_2 \cdot (-\beta_2), \tag{2.251}$$

we want to express λ_1 and λ_2 in terms of a, b. This is equivalent as solving the following linear system:

$$-\lambda_1 = a, \tag{2.252}$$

$$\lambda_1 - \lambda_2 = b. \tag{2.253}$$

Thus, we have

$$\lambda_1 = -a, \tag{2.254}$$

$$\lambda_2 = -a - b. \tag{2.255}$$

Recall that $\vec{w} \odot \vec{\xi}$ is the matrix

$$2\pi\mathbf{i}\cdot\operatorname{diag}(\ell_{I_1,I_2}+m_{I_4,I_5}-\frac{L+M}{3}, \ell_{I_3,I_4}+m_{I_6,I_1}-\frac{L+M}{3}, \ell_{I_5,I_6}+m_{I_2,I_3}-\frac{L+M}{3}). \tag{2.256}$$

Hence, if $\vec{w} \odot \vec{\xi} \in \mathcal{A}_{\vec{\xi}}$, the contribution of this $\vec{w} \odot \vec{\xi}$ to our volume formula is

$$K' \cdot \operatorname{sgn}(\vec{w}) \cdot \lambda_1 = K' \cdot \left(-(-1)^{|I_1|+|I_3|+|I_5|}\right) \cdot \left(\frac{L+M}{3} - \ell_{I_1,I_2} - m_{I_4,I_5}\right), \tag{2.257}$$

which totally matches the term

$$A_{\vec{\xi}}(I_1, \ldots, I_6) = \frac{-(-1)^{|I_1|+|I_3|+|I_5|}}{6}\left(\frac{L+M}{3} - \ell_{I_1,I_2} - m_{I_4,I_5}\right) \tag{2.258}$$

for the contribution of this $\vec{w} \odot \vec{\xi}$ to the volume formula of Suzuki and Takakura, provided that $K' = 1/6$ and this is only a matter of convention of normalization.

For each $\vec{w} \odot \vec{\xi} \in \mathcal{B}_{\vec{\xi}}$, it is easy to see that $\vec{w} \odot \vec{\xi}$ can be written as $\lambda_1 \cdot (-\beta_1) + \lambda_2 \cdot (-\beta_2)$ with some λ_1, λ_2 satisfying $0 < \lambda_2 < \lambda_1$. Therefore, the contribution of this $\vec{w} \odot \vec{\xi}$ to our volume formula is

$$K' \cdot \operatorname{sgn}(\vec{w}) \cdot H_{(-\vec{\beta})}(\vec{w} \odot \vec{\xi}) \tag{2.259}$$

$$=K' \cdot \operatorname{sgn}(\vec{w}) \cdot \lambda_2 \tag{2.260}$$

$$=K' \cdot \left(-(-1)^{|I_1|+|I_3|+|I_5|}\right) \cdot \left(\ell_{I_5,I_6} + m_{I_2,I_3} - \frac{L+M}{3}\right), \tag{2.261}$$

which totally matches the term

$$B_{\vec{\xi}}(I_1, \ldots, I_6) = \frac{-(-1)^{|I_1|+|I_3|+|I_5|}}{6}\left(\ell_{I_5,I_6} + m_{I_2,I_3} - \frac{L+M}{3}\right) \tag{2.262}$$

for the contribution of this $\vec{w} \odot \vec{\xi}$ to the volume formula of Suzuki and Takakura, provided that an appropriate normalization convention is chosen so that $K' = 1/6$.

Combining the observation that the total sum in the residue formula does not depend on the choice of cone, we have proved the following result:

Theorem 2.6.2. *Our volume formula in Theorem 2.5.2 completely matches and hence extends the volume formula of Suzuki and Takakura (Theorem 2.6.1) for triple reduced products of* $\mathbf{SU}(3)$*, provided that an appropriate normalization convention is chosen so that* $K = -K' = -1/6$.

Chapter 3

Generalizations of Volume Formula

In this chapter, we generalize some of the results in Chapter 2.

3.1 Volume Formula for N-fold Reduced Products of $\mathbf{SU}(3)$

In this section, our group G is still $\mathbf{SU}(3)$. Let T be the standard maximal torus in G as in Chapter 2. Let W denote the Weyl group.

Let $N \geq 3$ be a positive integer. Notice that in Chapter 2 our N is equal to 3.

Let $\vec{\xi} = (\xi_1, \ldots, \xi_N)$ be the initial input, where the ξ_i's satisfy the conditions specified in Chapter 2.

Let $M = \mathcal{O}_{\xi_1} \times \cdots \times \mathcal{O}_{\xi_N}$. This is a compact symplectic manifold with G acting diagonally on it in a Hamiltonian fashion.

Let μ_G and μ_T be the moment maps of the G-action and the T-action respectively.

Let M^T, M_0, M_{red} be defined similarly as in Chapter 2.

The following proposition is an easy generalization to Proposition 2.1.1.

Proposition 3.1.1. *M^T is the discrete set*

$$\{(w_1 \cdot \xi_1, \ldots, w_N \cdot \xi_N) \ : \ w_i \in W\}\,. \tag{3.1}$$

Thus, $\left|M^T\right| = |W|^N$.

As a result, M^T is parametrized by $\vec{w} \in W^N$. Moreover, 6-partitions of $\{1, \ldots, N\}$ can be defined similarly and each 6-partition $(I_1, \ldots, I_6)$ of $\{1, \ldots, N\}$ corresponds to a unique $\vec{w} \in W^N$ in the same way as before:

$$I_j = \{i \in \{1, \ldots, N\} \ : \ w_i = \sigma_j\}, \tag{3.2}$$

for $j = 1, \ldots, 6$. Thus, M^T can also be parametrized by all the 6-partitions of $\{1, \ldots, N\}$.

In addition, $\vec{w} \cdot \vec{\xi}$ and $\vec{w} \odot \vec{\xi}$ can be similarly defined.

Before we move on, we briefly discuss why we do not consider the cases $N = 1$ or $N = 2$.

For the case $N = 1$, M is just a single adjoint orbit while μ_G is the inclusion map from M into $\mathfrak{g}$. Since we assume that $\mu_G^{-1}(0) \neq \emptyset$, $0 \in M$ and hence M is just the degenerate adjoint orbit $\{0\}$. However, we do not allow degenerate adjoint orbits and hence the case $N = 1$ simply does not exist under our assumptions.

For the case $N = 2$, M is a Cartesian product of two adjoint orbits, i.e., $M = \mathcal{O}_{\xi_1} \times \mathcal{O}_{\xi_2}$. Since we assume that $\mu_G^{-1}(0) \neq \emptyset$, there exists a point $(\eta_1, \eta_2) \in M$ such that $\eta_1 + \eta_2 = 0$ and this implies that $\mathcal{O}_{\xi_2} = -\mathcal{O}_{\xi_1}$ as subsets in the vector space $\mathfrak{g}$. Hence, for each $(\eta, -\eta) \in \mu_G^{-1}(0)$, the stabilizer is a maximal torus of G and thus is infinite, contradicting the assumption that 0 is a regular value for μ_G. Hence, the case $N = 2$ does not exist either under our assumptions.

When computing the symplectic volume of $M_{\mathrm{red}}(\vec{\xi})$, the only essential difference between the $N = 3$ case and the general $N \geq 3$ case lies in the equivariant Euler class e_F of the normal bundle of the fixed points of T. In the general $N \geq 3$ case, for each fixed point $F = \vec{w} \cdot \vec{\xi} \in M^T$,

$$\mathrm{e}_F(\psi) = \mathrm{sgn}(\vec{w}) \cdot \varpi^N(\psi). \tag{3.3}$$

Hence, we have the following result.

Theorem 3.1.2. *In the general $N \geq 3$ case, the symplectic volume of $M_{\mathrm{red}}(\vec{\xi})$ is:*

$$\mathrm{vol}^{\mathcal{S}}(M_{\mathrm{red}}(\vec{\xi})) = \frac{1}{\mathbf{i}^{d/2}} n_0 C_G \,\mathrm{res}\left(\sum_{\vec{w} \in W^N} \mathrm{sgn}(\vec{w}) \frac{e^{\mathbf{i}(\vec{w} \odot \vec{\xi}, \psi)}[d\psi]}{\varpi^{N-2}(\psi)} \right), \tag{3.4}$$

where n_0 and C_G are as the same as in Theorem 2.2.1 and

$$d = N(s - l) - 2s = (N - 2)s - Nl, \tag{3.5}$$

where s is the real dimension of G and l is the real dimension of T.

To obtain a more explicit volume formula, we want to compute the residue in the above formula.

As in Section 2.6, we will later compare our formula with the general formula of Suzuki and Takakura [53]. Therefore, by the comparison argument in Section 2.6, when computing the residue, we will use the cone $\Lambda_- = -\mathfrak{t}_{>0}$ instead of $\Lambda = \mathfrak{t}_{>0}$ so that the comparison can be made more directly.

Before we compute the residue, we first write down here the general formula of Suzuki and Takakura [53]:

Theorem 3.1.3 (Theorem 4.5 in [53]). *Let $N \geq 3$ be an integer. Let*

$$\xi_i = (\ell_i - m_i) \cdot \Omega_1 + m_i \cdot \Omega_2, \tag{3.6}$$

where $\ell_i > m_i > 0$ are all integers divisible by 3 and

$$(\vec{w} \odot \vec{\xi}, \Omega_1) \neq 0 \tag{3.7}$$

for all $\vec{w} \in W^N$. Let

$$L = \sum_{i=1}^{N} \ell_i, \quad M = \sum_{i=1}^{N} m_i. \tag{3.8}$$

Let $\mathcal{I}_{\vec{\xi}}$ denote the set of 6-partitions $(I_1, \ldots, I_6)$ of $\{1, \ldots, N\}$ such that

$$\ell_{I_1,I_2} + m_{I_4,I_5} < \frac{L+M}{3}, \tag{3.9}$$

$$\ell_{I_3,I_4} + m_{I_6,I_1} < \frac{L+M}{3}. \tag{3.10}$$

Let $\mathcal{J}_{\vec{\xi}}$ denote the set of 6-partitions $(I_1, \ldots, I_6)$ of $\{1, \ldots, N\}$ such that

$$\ell_{I_3,I_4} + m_{I_6,I_1} > \frac{L+M}{3}, \tag{3.11}$$

$$\ell_{I_5,I_6} + m_{I_2,I_3} > \frac{L+M}{3}. \tag{3.12}$$

Let $A_{\vec{\xi}} : \mathcal{I}_{\vec{\xi}} \to \mathbb{R}$ be defined by

$$A_{\vec{\xi}}(I_1, \ldots, I_6) := \frac{-(-1)^{|I_1|+|I_3|+|I_5|}}{6(3N-8)!} \sum_{j=0}^{N-3} \binom{3N-8}{j} \binom{2N-6-j}{N-3} \left(\frac{L+M}{3} - \ell_{I_3,I_4} - m_{I_6,I_1}\right)^j \left(\frac{L+M}{3} - \ell_{I_1,I_2} - m_{I_4,I_5}\right)^{3N-8-j}. \tag{3.13}$$

Let $B_{\vec{\xi}} : \mathcal{J}_{\vec{\xi}} \to \mathbb{R}$ *be defined by*

$$B_{\vec{\xi}}(I_1, \ldots, I_6) := \frac{-(-1)^{|I_1|+|I_3|+|I_5|}}{6(3N-8)!} \sum_{j=0}^{N-3} \binom{3N-8}{j} \binom{2N-6-j}{N-3} \left(\ell_{I_3,I_4} + m_{I_6,I_1} - \frac{L+M}{3}\right)^j \left(\ell_{I_5,I_6} + m_{I_2,I_3} - \frac{L+M}{3}\right)^{3N-8-j}. \quad (3.14)$$

Then, the symplectic volume of $M_{\text{red}}(\vec{\xi})$ *is given by*

$$\mathcal{V}(\vec{\xi}) = \sum_{(I_1,\ldots,I_6)\in\mathcal{I}_{\vec{\xi}}} A_{\vec{\xi}}(I_1, \ldots, I_6) + \sum_{(I_1,\ldots,I_6)\in\mathcal{J}_{\vec{\xi}}} B_{\vec{\xi}}(I_1, \ldots, I_6). \quad (3.15)$$

Now we return to continue computing the residues in our formula.

Recall that all of $-\beta_1, -\beta_2, -\beta_3 = -\beta_1 - \beta_2$ lie in the dual cone Λ_-^*.

We have:

$$\operatorname{res}\left(\sum_{\vec{w}\in W^N} \operatorname{sgn}(\vec{w}) \frac{e^{\mathbf{i}(\vec{w}\odot\vec{\xi},\psi)}[d\psi]}{\varpi^{N-2}(\psi)}\right) \quad (3.16)$$

$$= \operatorname{res}^{\Lambda_-}\left(\sum_{\vec{w}\in W^N} \operatorname{sgn}(\vec{w}) \frac{e^{\mathbf{i}(\vec{w}\odot\vec{\xi},\psi)}[d\psi]}{\varpi^{N-2}(\psi)}\right) \quad (3.17)$$

$$= \sum_{\vec{w}\in W^N} \operatorname{sgn}(\vec{w}) \operatorname{res}^{\Lambda_-}\left(\frac{e^{\mathbf{i}(\vec{w}\odot\vec{\xi},\psi)}[d\psi]}{\varpi^{N-2}(\psi)}\right) \quad (3.18)$$

$$= \sum_{\vec{w}\in W^N} \operatorname{sgn}(\vec{w}) \operatorname{res}^{\Lambda_-}\left(\frac{e^{\mathbf{i}(\vec{w}\odot\vec{\xi},\psi)}[d\psi]}{\left((-1)\cdot\prod_{j=1}^3(-\beta_j)(\psi)\right)^{N-2}}\right) \quad (3.19)$$

$$= (-1)^{N-2} \cdot \sum_{\vec{w}\in W^N} \operatorname{sgn}(\vec{w}) \operatorname{res}^{\Lambda_-}\left(\frac{e^{\mathbf{i}(\vec{w}\odot\vec{\xi},\psi)}[d\psi]}{\left(\prod_{j=1}^3(-\beta_j)(\psi)\right)^{N-2}}\right). \quad (3.20)$$

To compute

$$\operatorname{res}^{\Lambda_-}\left(\frac{e^{\mathbf{i}(\vec{w}\odot\vec{\xi},\psi)}[d\psi]}{\left(\prod_{j=1}^3(-\beta_j)(\psi)\right)^{N-2}}\right), \quad (3.21)$$

we introduce the notation $(-\bar{\beta})^{N-2}$ to denote the following:

$$(-\bar{\beta})^{N-2} := (-\beta_1, -\beta_2, -\beta_3, \ldots, -\beta_1, -\beta_2, -\beta_3), \quad (3.22)$$

where the sequence $-\beta_1, -\beta_2, -\beta_3$ repeats itself for $N-2$ times.

Now we have:

$$\operatorname{res}^{\Lambda_-}\left(\frac{e^{\mathbf{i}(\vec{w}\odot\vec{\xi},\psi)}[d\psi]}{\left(\prod_{j=1}^{3}(-\beta_j)(\psi)\right)^{N-2}}\right) \tag{3.23}$$

$$= \frac{\mathbf{i}^{3(N-2)}}{(2\pi\mathbf{i})^2}\cdot H_{(-\bar{\beta})^{N-2}}(\vec{w}\odot\vec{\xi}). \tag{3.24}$$

Therefore, we have the following result:

Theorem 3.1.4. *In the general $N \geq 3$ case, the symplectic volume of $M_{\mathrm{red}}(\vec{\xi})$ is:*

$$\mathrm{vol}^{\mathcal{S}}(M_{\mathrm{red}}(\vec{\xi})) = C\cdot\sum_{\vec{w}\in W^N}\mathrm{sgn}(\vec{w})H_{(-\bar{\beta})^{N-2}}(\vec{w}\odot\vec{\xi}), \tag{3.25}$$

where C is a constant.

Given any $\lambda_1, \lambda_2 \in \mathbb{R}$, we want to compute $H_{(-\bar{\beta})^{N-2}}(\lambda_1\cdot(-\beta_1)+\lambda_2\cdot(-\beta_2))$. This expression is equal to the Euclidean volume of a subset S of $\mathbb{R}_+^{3(N-2)}$, up to a normalization constant. This subset S consists of all the elements of the form

$$(s_1^{(1)}, s_2^{(1)}, s_3^{(1)}, \ldots, s_1^{(N-2)}, s_2^{(N-2)}, s_3^{(N-2)}) \in \mathbb{R}_+^{3(N-2)} \tag{3.26}$$

such that

$$\sum_{k=1}^{N-2}\left(s_1^{(k)}(-\beta_1)+s_2^{(k)}(-\beta_2)+s_3^{(k)}(-\beta_3)\right) = \lambda_1\cdot(-\beta_1)+\lambda_2\cdot(-\beta_2), \tag{3.27}$$

which is equivalent to the following linear system:

$$\sum_{k=1}^{N-2}\left(s_1^{(k)}+s_3^{(k)}\right) = \lambda_1, \tag{3.28}$$

$$\sum_{k=1}^{N-2}\left(s_2^{(k)}+s_3^{(k)}\right) = \lambda_2. \tag{3.29}$$

If $\lambda_1 < 0$ or $\lambda_2 < 0$, then clearly $S = \emptyset$. Since we are interested in the generic situation, without loss of generality, we can assume that both $\lambda_1 > 0$ and $\lambda_2 > 0$.

Then, clearly, S is a bounded convex $(3N-8)$-polytope.

Let the hyperplane in $\mathbb{R}^{3(N-2)}$ determined by Equation (3.28) be denoted P_1. It is perpendicular to the vector

$$(1, 0, 1, 1, 0, 1, \ldots, 1, 0, 1) \in \mathbb{R}^{3(N-2)}. \tag{3.30}$$

Let the hyperplane in $\mathbb{R}^{3(N-2)}$ determined by Equation (3.29) be denoted P_2. It is perpendicular to the vector

$$(0, 1, 1, 0, 1, 1, \ldots, 0, 1, 1) \in \mathbb{R}^{3(N-2)}. \tag{3.31}$$

Then we have

$$S = P_1 \cap P_2 \cap \mathbb{R}_+^{3(N-2)}. \tag{3.32}$$

In the generic situation, S is a codimension 2 polytope in $\mathbb{R}^{3(N-2)}$. When N increases, the geometry of S becomes increasingly complicated. Even though in principle it is possible to compute the Euclidean volume of S, when N is large, carrying out the computations by hand becomes practically infeasible. Therefore, we will use another method to compute the residue, namely, the method of iterated one-dimensional residues due to Jeffrey and Kirwan [35], which we will describe now.

Definition 3.1.5 (Definition 3.3 in [35]). Let $f(z)$ be a meromorphic function of $z \in \mathbb{C}$ that can be expressed as a finite sum as follows:

$$f(z) = \sum_{j=1}^{m} g_j(z) e^{\mathbf{i}\lambda_j z}, \tag{3.33}$$

where the g_j's are rational functions of z and the λ_j's are nonzero real numbers. Then,

$$\mathrm{Res}_z^+ f(z)dz = \sum_{\lambda_j \geq 0} \sum_{b \in \mathbb{C}} \mathrm{Res}_z(g_j e^{\mathbf{i}\lambda_j z}; z = b), \tag{3.34}$$

where Res_z denotes the usual notion of residue with respect to $z \in \mathbb{C}$ in complex analysis.

The following result of Jeffrey and Kirwan [35] can be used to compute the residues in our formula.

Proposition 3.1.6 (Proposition 3.4 in [35]). *Let*

$$h(\psi) = \frac{q(\psi) e^{\mathbf{i}\lambda(\psi)}}{\prod_{j=1}^{\nu} \beta_j(\psi)}, \tag{3.35}$$

where $q(\psi)$ is a polynomial function of $\psi \in \mathfrak{t}$ and $\lambda, \beta_1, \ldots, \beta_\nu$ are elements of $\mathfrak{t}^*$. Suppose that λ is not in any proper subspace of $\mathfrak{t}^*$ spanned by a subset of $\{\beta_1, \ldots, \beta_\nu\}$. Let Λ be *a proper cone in* $\mathfrak{t}$ *contained in some connected component of*

$$\{\psi \in \mathfrak{t} \;:\; \beta_j(\psi) \neq 0,\ 1 \leq j \leq \nu\}\,. \tag{3.36}$$

Then, for a generic choice of coordinate system $\psi = (\psi_1, \dots, \psi_l)$ *on* $\mathfrak{t}$ *such that*

$$(0, \dots, 0, 1) \in \Lambda, \tag{3.37}$$

we have

$$\mathrm{res}^{\Lambda}(h(\psi)[d\psi]) = \Delta \cdot \mathrm{Res}^{+}_{\psi_1} \dots \mathrm{Res}^{+}_{\psi_l}\, h(\psi) d\psi_l \dots d\psi_1, \tag{3.38}$$

where the variables $\psi_1, \dots, \psi_{k-1}$ *are regarded as constants when computing* $\mathrm{Res}^{+}_{\psi_k}$. Δ *is the determinant of the* $l \times l$ *matrix whose columns are the coordinates of an orthonormal basis of* $\mathfrak{t}$ *defining the same orientation as the chosen coordinate system.*

3.2 Volume Formula for General N-fold Reduced Products

The method of nonabelian localization and the residue formula apply not only for $G = \mathbf{SU}(3)$, but also for any compact connected Lie groups. However, to apply Theorem 2.2.1 in our situation, namely the situation where the group G acts diagonally on the product of adjoint orbits by the adjoint action, we need to make sure that the stabilizer of any point in $M_0 = \mu_G^{-1}(0)$ is finite. Therefore, in addition to the Lie group G being compact and connected, we assume that G is also semisimple.

Let T denote a chosen maximal torus of G. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{t}$ be the Lie algebra of T. Let W denote the Weyl group $\mathrm{N}(T)/T$. Let $(\cdot, \cdot)$ denote a chosen G-invariant inner product on $\mathfrak{g}$. Let R_+ denote the collection of positive roots of G. Let $\mathfrak{t}_{>0}$ denote the open positive Weyl chamber.

Let s denote the real dimension of G. Let l denote the real dimension of T. Let $N \geq 3$ be a positive integer.

Let $\vec{\xi} = (\xi_1, \dots, \xi_N)$ be an N-tuple of elements in $\mathfrak{g}$. Let $M = \mathcal{O}_{\xi_1} \times \cdots \times \mathcal{O}_{\xi_N}$ be the product of the corresponding adjoint orbits. Then G acts on M through the diagonal adjoint action. This is a Hamiltonian action, so we have the moment maps μ_G and μ_T as before. Let $M_0 = \mu_G^{-1}(0)$. Let $M_{\mathrm{red}} = M_0/G$ be the reduced space, which we call an N-fold reduced product of G.

The input $\vec{\xi} = (\xi_1, \dots, \xi_N)$ satisfies the following assumptions:

(A1) $\mu_G^{-1}(0) \neq \emptyset$ and 0 is a regular value for μ_G.

(A2) All ξ_i's lie in $\mathfrak{t}_{>0}$.

Given $\vec{w} = (w_1, \ldots, w_N) \in W^N$, let

$$\vec{w} \cdot \vec{\xi} = (w_1 \cdot \xi_1, \ldots, w_N \cdot \xi_N) \tag{3.39}$$

and

$$\vec{w} \odot \vec{\xi} = \sum_{i=1}^{N} w_i \cdot \xi_i, \tag{3.40}$$

just as before. Then, we have

$$M^T = \left\{ \vec{w} \cdot \vec{\xi} \ : \ \vec{w} \in W^N \right\}, \tag{3.41}$$

where M^T denotes the fixed point set of the action of T on M. Notice that M^T is discrete and $\left|M^T\right| = |W|^N$.

Let

$$\varpi(\psi) = \prod_{\gamma \in \mathrm{R}_+} \gamma(\psi) \tag{3.42}$$

for all $\psi \in \mathfrak{t}$.

At each fixed point $F = \vec{w} \cdot \vec{\xi} \in M^T$, the T-equivariant Euler class of the normal bundle over F is

$$\mathrm{e}_F(\psi) = \varpi^N(\psi). \tag{3.43}$$

Therefore, we have the following general result:

Theorem 3.2.1. *The symplectic volume of* $M_{\mathrm{red}}(\vec{\xi})$ *is*

$$\mathrm{vol}^{\mathcal{S}}(M_{\mathrm{red}}(\vec{\xi})) = \frac{1}{\mathbf{i}^{d/2}} n_0 C_G \operatorname{res}\left(\sum_{\vec{w} \in W^N} \operatorname{sgn}(\vec{w}) \frac{e^{\mathbf{i}(\vec{w} \odot \vec{\xi}, \psi)}[d\psi]}{\varpi^{N-2}(\psi)} \right), \tag{3.44}$$

where

$$d = N(s-l) - 2s = (N-2)s - Nl, \tag{3.45}$$

and n_0 *is the cardinality of the stabilizer* $\mathrm{Stab}_G(p)$ *of a generic point* p *in* M_0 *and the constant* C_G *is defined by*

$$C_G := \frac{(-1)^{n_+}}{(2\pi)^{s-l} |W| \, \mathrm{vol}^{\mathcal{R}}(T)}. \tag{3.46}$$

Here, n_+ *is the cardinality of* R_+*, i.e.,* $n_+ = (s-l)/2$*.*

To compute the residue in the above formula, we can either use the method of iterated one-dimensional residues (Proposition 3.1.6) or compute the residue using the function $H_{\bar{\beta}}$ as in the previous sections (this requires a choice of a proper cone Λ in $\mathfrak{t}$).

Chapter 4

Intersection Pairings of N-fold Reduced Products

In this chapter, we investigate the intersection pairings of N-fold Reduced Products. We will use the same settings as in Section 3.2.

Let $\mathcal{O}$ be a nondegenerate adjoint orbit of G, i.e., $\mathcal{O} \cong G/T$. Then the cohomology of $\mathcal{O}$ is generated by the first Chern classes of line bundles $L_{\tilde{\beta}}$ over $\mathcal{O}$, where

$$L_{\tilde{\beta}} = G \times_{T,\tilde{\beta}} \mathbb{C}. \tag{4.1}$$

Here $\mathcal{O}$ is identified with G/T and the equivalence relation on $G \times \mathbb{C}$ is

$$(g, z) \sim (gt, \tilde{\beta}(t)^{-1} z) \tag{4.2}$$

for all $g \in G$, $t \in T$ and $z \in \mathbb{C}$, where $\tilde{\beta} \in \mathrm{Hom}(T, \mathbf{U}(1))$ is a weight.

The references for the above result are Fulton [20] (Proposition 3 in Section 10.2) and Tu [54] (Theorem 5 in Section 4).

For instance, if $G = \mathbf{SU}(n)$, the collection of simple roots of G gives rise to a basis for the cohomology of G/T.

For each weight $\tilde{\beta} : T \to \mathbf{U}(1)$, there is a linear map $\beta : \mathfrak{t} \to \mathbb{R}$ such that

$$\tilde{\beta}(\exp(\psi)) = e^{\mathbf{i}\beta(\psi)} \tag{4.3}$$

for all $\psi \in \mathfrak{t}$, where $\exp : \mathfrak{t} \to T$ is the exponential map. Then, when restricted to the fixed points of T on $\mathcal{O}$, the equivariant first Chern class of the line bundle $L_{\tilde{\beta}}$ is

$$c_1^{\mathrm{eq}}(L_{\tilde{\beta}})(\psi) = c_1(L_{\tilde{\beta}}) + \beta(\psi) \tag{4.4}$$

for all $\psi \in \mathfrak{t}$.

Now, we return to our situation of a product of N adjoint orbits of G, i.e.,

$$M = \mathcal{O}_{\xi_1} \times \cdots \times \mathcal{O}_{\xi_N}. \tag{4.5}$$

By Proposition 5.8 in [37], we have

$$H_G^*(M) = H^*(M) \otimes H^*(BG). \tag{4.6}$$

$H^*(M)$ can be obtained from $H^*(\mathcal{O}_{\xi_i})$'s by the Künneth theorem.

Recall that we have the map

$$\kappa_0 : H_G^*(M) \to H^*(M_{\text{red}}). \tag{4.7}$$

Therefore, $H^*(M_{\text{red}})$ can be generated by elements in the form $\kappa_0(\eta)$ such that η is a product of powers of equivariant first Chern classes:

$$\eta = \prod_{i=1}^{N} \left(c_1^{\text{eq}}(\text{pr}_i^* L_{\tilde{\beta}_i})\right)^{p_i}, \tag{4.8}$$

where p_i's are nonnegative integers and $\text{pr}_i^* L_{\tilde{\beta}_i}$ is the pullback line bundle of the line bundle $L_{\tilde{\beta}_i}$over $\mathcal{O}_{\xi_i}$ with weight $\tilde{\beta}_i$ through the standard projection

$$\text{pr}_i : \mathcal{O}_{\xi_1} \times \cdots \times \mathcal{O}_{\xi_N} \to \mathcal{O}_{\xi_i} \tag{4.9}$$

onto the i-th factor.

In addition, we observe that when restricted to a fixed point $\vec{w} \cdot \vec{\xi}$ of T on M, we have

$$\eta(\psi)\Big|_{\vec{w}\cdot\vec{\xi}} = \prod_{i=1}^{N} \left(\beta_i(\psi)\right)^{p_i}. \tag{4.10}$$

Therefore, by Theorem 2.2.1, we have the following result:

Theorem 4.0.1. *Let*

$$\eta = \prod_{i=1}^{N} \left(c_1^{\text{eq}}(\text{pr}_i^* L_{\tilde{\beta}_i})\right)^{p_i} \tag{4.11}$$

as above. Then we have

$$\kappa_0(\eta)e^{\mathbf{i}\omega_{\text{red}}}[M_{\text{red}}] = n_0 C_G \operatorname{res}\left(\sum_{\vec{w}\in W^N} \text{sgn}(\vec{w}) \frac{e^{\mathbf{i}(\vec{w}\odot\vec{\xi},\psi)} \prod_{i=1}^{N} \left(\beta_i(\psi)\right)^{p_i}}{\varpi^{N-2}(\psi)}[d\psi]\right). \tag{4.12}$$

Notice that the above theorem describes all intersection pairings between cohomology classes of M_{red}. The residue in the above formula can be computed by the method of iterated one-dimensional residues (Proposition 3.1.6).

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