Annihilation of Cohomology over Gorenstein Rings

by

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Abstract

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One of the fundamental links between geometry and homological algebra is that smooth affine schemes have coordinate rings of finite global dimension. The roots of this link goes back to Hilbert's syzygy theorem and later to the work of Auslander and Buchsbaum and also of Serre.

Having finite global dimension can be characterized by Ext-modules. Namely, a ring R has finite global dimension if and only if there is a natural number n such that $\mathrm{Ext}^n_R(M,N) = 0$ for every pair M, N of R-modules. Hence, in the singular case, there are nonzero Ext-modules for arbitrarily large n. So, for a commutative Noetherian ring R, one is interested in the cohomology annihilator ideal which consists of the ring elements that annihilate all Ext-modules for arbitrarily large n.

The main theme of this thesis is to study the cohomology annihilator ideal over Gorenstein rings. Over Gorenstein rings, the cohomology annihilator ideal can be seen as the annihilator of the stable category of maximal Cohen-Macaulay modules.

The first main result concerns the cohomology annihilator ideal of a complete local coordinate ring of a reduced algebraic plane curve singularity. We show that that the cohomology annihilator ideal coincides with the conductor ideal in this case. We use this to investigate the relation between the Jacobian ideal and the cohomology annihilator ideal.

The second main result shows that if the Krull dimension of R is at most 2, then the cohomology annihilator ideal is equal to the stable annihilator ideal of a non-singular R-order. We also give several generalizations of this which brings us to the second part and the closing section of this thesis. Namely, we study the dominant dimension of orders over Cohen-Macaulay rings. We provide examples and prove results on tilting modules for orders with positive dominant dimension.

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Contents

Chapter 1

Introduction

This thesis splits roughly into two halves, the connecting theme being Cohen-Macaulay representation theory. The first part which makes the most of this thesis concerns the universal support of Tate cohomology over Gorenstein rings. A theorem due to Ragnar-Olaf Buchweitz and later rediscovered by Dmitri Orlov tells us that the Tate cohomology computations can be carried to Hom-space computations in the stable category of maximal Cohen-Macaulay modules.

Motivated by the main problem, we then study orders over Cohen-Macaulay local rings in the second part of this thesis. In particular, we study orders whose canonical module has positive projective dimension. In this setting, we introduce the notion of relative dominant dimension of an order.

Cohen-Macaulay representation theory is an active and growing research subject with connections to commutative and homological algebra, algebraic geometry, singularity theory and theoretical physics.

Similar to the classical theory of group representations where a group is studied by its action on a vector space, studying the action of rings on modules can reveal facts about rings which can not be detected otherwise. In this thesis, we adapt this point of view and we study maximal Cohen-Macaulay modules over Cohen-Macaulay rings to reveal geometric information on the scheme associated to these rings, more precisely on their singular loci.

The thesis starts with a background on maximal Cohen-Macaulay modules over Gorenstein rings, after this introduction. After defining the main ingredients of this thesis, we define the singularity category and Tate cohomology.

Chapter 3 is dedicated to preliminaries on the cohomology annihilator of a Gorenstein ring. We introduce techniques of computing cohomology annihilator ideals and discuss the geometric importance. In particular, we give a proof of the following theorem which has been to known to experts but as far as I know, a proof has not been written down.

Theorem (Proposition 3.5)**.** *For a commutative Gorenstein ring R, the cohomology annihilator ideal is the annihilator of the stable category of maximal Cohen-Macaulay modules.*

The first main theorem of this thesis is in Chapter 4. We start with this chapter by presenting the motivation for this thesis in a language which I hope to be accessible for at least an enthusiastic undergraduate student. After giving the necessary background, we prove the main theorems which are as follows.

Theorem (Theorem 4.14)**.** *Let R be a 1-dimensional Gorenstein local ring with finite normalization. Then, the cohomology annihilator ideal is contained in the conductor ideal.*

We then refer to results from Hsin-Ju Wang's PhD thesis [65] to conclude that there is indeed an equality of ideals under mild assumptions.

Theorem (Corollary 4.15)**.** *Let R be a 1-dimensional reduced complete Gorenstein local ring. Then, the cohomology annihilator ideal coincides with the conductor ideal.*

This result explains the observation which Ragnar-Olaf Buchweitz has shared with me and which is shared at the beginning of Chapter 4. We reserve a section for this explanation which uses the Milnor-Jung formula from topology.

Chapter 4 continues with applications of the main theorem. The most significant one of these applications is the following generalization of the Milnor-Jung formula.

Theorem (Theorem 4.34)**.** *Let $R = k[\![x,y]\!]/(f)$ be the coordinate ring of a reduced curve singularity where $\operatorname{char} k \neq 2$ and $R^\sharp = k[\![x,y,z_1,\ldots,z_l]\!]/(f + z_1^2 + \ldots + z_l^2)$ for some $l \geq 0$. Then,*

$$\dim_k \frac{R^\sharp}{\operatorname{jac}(R^\sharp)} = 2\dim_k \frac{R^\sharp}{\operatorname{ca}(R^\sharp)} - r + 1$$

where r is the number of branches of the curve f at its singular point.

Even though the ring elements which annihilate all Tate cohomology groups have been well studied before this thesis, there aren't many examples of precise computations of cohomology annihilator ideals in the literature. We dedicate Chapter 5 to different techniques of exact computations of cohomology annihilator ideals.

The main result of Chapter 4 relies on the fact that the normalization of a 1-dimensional Gorenstein ring has depth 1. That is, it is a maximal Cohen-Macaulay module over the base ring. In Chapter 6, we use this key point to generalize the main theorem of Chapter 4 to arbitrary finite Krull dimension. Recall that a module-finite algebra over a Cohen-Macaulay ring R is called an *R-order* if it is maximal Cohen-Macaulay as an R-module. We recall the general theory of orders in Chapter 6 and then we prove the following theorem which gives an equality up to radicals.

Theorem (Theorem 6.14)**.** *Let R be a Gorenstein ring and Λ be an R-order of finite global dimension δ. Then, for any $M \in \text{mod}R$,*

$$(\underline{\text{ann}}_R \Lambda)^{e+1} \subseteq \underline{\text{ann}}_R M_\Lambda^{\text{st}}$$

where $M_\Lambda = M \otimes_R \Lambda$, M_Λ^{st} its image in $\underline{\text{MCM}}(R)$ and e its projective dimension as a Λ-module. If, moreover, Λ has a free summand as an R-module, then

$$(\underline{\text{ann}}_R \Lambda)^{\delta+1} \subseteq \text{ca}(R) \subseteq \underline{\text{ann}}_R \Lambda.$$

However, for rings of dimension at most 2, we can get an equality by restricting ourselves to so-called non-singular orders.

Theorem (Theorem 6.16)**.** *Suppose that R is a Gorenstein ring and Λ is an R-order which contains R as a direct summand. Assume that $\dim R \leq 2$ and that Λ is non-singular. Then, we have the equality*

$$\text{ca}(R) = \underline{\text{ann}}_R(\Lambda).$$

We finish this chapter by looking at stable annihilators of cluster tilting modules.

Theorem (Theorem 6.22)**.** *Let R be a Gorenstein isolated singularity with dimension $d \geq 3$ and M be a $d-2$-cluster tilting module. Then, we have*

$$(\underline{\text{ann}}_R(M))^{2d+2} \subseteq \text{ca}(R) \subseteq \underline{\text{ann}}_R(M).$$

This finishes the first part of this thesis. We begin the second part of this thesis by noting that in dimension at most 2, we can use the ideas of Theorem 6.16 to get a better bound than the bound in Theorem 6.14 even when the order is not non-singular. With this motivation, we study the relative injective dimension of an order over a Cohen-Macaulay local ring. We define the relative dominant dimension of an order and we give basic properties of relative dominant dimension. For instance, we prove the following theorem.

Theorem (Theorem 7.19). *Let R be a maximal Cohen-Macaulay local ring with canonical module ω_R, M be a finitely generated R-module and $\Lambda = \operatorname{End}_R(M)$.*

1. *If R is of dimension 1 or 2, then Λ is an R-order and $\operatorname{rdd}\Lambda \geq 2$.*

2. *If M is maximal Cohen-Macaulay and Λ is an R-order of dimension greater than 2, we have $\operatorname{rdd}\Lambda \geq \dim R - 2$.*

The last section of Chapter 7 is dedicated to tilting theory for orders with positive relative dominant dimension. We prove theorems analogous to recent results on the finite dimensional algebra counterpart of dominant dimension.

Chapter 2

Maximal Cohen-Macaulay Modules over Gorenstein Rings

Let R be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and M be a finitely generated R-module. We say that an element $r \in R$ is a *nonzerodivisor on* M if for all $m \in M \setminus \{0\}$ we have $rm \neq 0$. A *regular sequence on* M is a sequence $(x_1, \ldots, x_m)$ in $\mathfrak{m}$ with the property that x_1 is a nonzerodivisor on M and that x_i is a nonzerodivisor on $M/\langle x_1, \ldots, x_{i-1}\rangle M$ for all $i = 2, \ldots, n$. The *depth of* M is the length of the longest regular sequence on M.

The *dimension* of a finitely generated module M is defined to be the Krull dimension of the quotient ring $R/\operatorname{ann}_R(M)$. We denote the dimension of M by $\dim M$.

For any commutative Noetherian local ring R and any nonzero finitely generated R-module M, we have

$$\operatorname{depth} M \leq \dim M \leq \dim R$$

which bounds the depth of a module by its dimension and the dimension of a module by the dimension of the ring [56].

Definition 2.1. Let R be a commutative Noetherian local ring of Krull dimension d. A nonzero finitely generated R-module M is called *maximal Cohen-Macaulay* if $\operatorname{depth} M = d$. We say that R is a *Cohen-Macaulay local ring* if it is maximal Cohen-Macaulay as a module over itself. A commutative Noetherian ring is called *Cohen-Macaulay* if for every prime ideal p, the local ring R_p is Cohen-Macaulay. We denote the category of maximal Cohen-Macaulay modules by $\mathrm{MCM}(R)$.

The theory of maximal Cohen-Macaulay modules over Cohen-Macaulay local rings has been studied intensively since the 70s (See [9, 39, 67, 50]) . In the first part of this thesis, we will be interested in a subset of Cohen-Macaulay rings, namely Gorenstein rings.

Theorem 2.2. *Let R be a commutative Noetherian local ring of Krull dimension d with residue field k. Then, the following are equivalent.*

1. *R has finite injective dimension as a module over itself.*

2. *R has injective dimension d.*

3. $\mathrm{Ext}^n_R(k, R) = 0$ *for* $n \neq d$ *and* $\mathrm{Ext}^d_R(k, R) \cong k$.

Definition 2.3. Let R be a commutative Noetherian local ring. If one of the equivalent conditions in the previous theorem holds for R, we call R a *Gorenstein local ring*. A commutative Noetherian ring is called *Gorenstein* if for every prime ideal p, the local ring R_p is Gorenstein.

Examples of Gorenstein rings are plenty. To begin with regular rings, hence coordinate rings of smooth affine varieties are Gorenstein. Moreover, every hypersurface ring and more generally every complete intersection ring is Gorenstein. However, the more traditional and motivational examples are complex valued cohomology rings of smooth orientable connected compact manifold.

Over Gorenstein rings, maximal Cohen-Macaulay modules can be defined in terms of vanishing of Ext-modules.

Lemma 2.4. *Let R be a Gorenstein ring and M be a nonzero finitely generated R-module. Then, M is maximal Cohen-Macaulay if and only if* $\mathrm{Ext}^n_R(M, R) = 0$ *for all positive integers n.*

From this description, it is immediate to see that $\mathrm{MCM}(R)$ is closed under taking direct summands and finite direct sums. In particular, every projective module over R is maximal Cohen-Macaulay. On the other hand, by Auslander-Buchsbaum formula [3], the converse holds for maximal Cohen-Macaulay modules of finite projective dimension. That is, if a maximal Cohen-Macaulay has finite projective dimension that it has to be projective. We will discuss Auslander-Buchsbaum formula with a little bit more detail in Chapter 7.

Definition 2.5. Let R be a Cohen-Macaulay local ring of Krull dimension d with residue field k. A finitely generated R-module ω_R is called a *canonical module* if

1. it has finite injective dimension,

2. it is maximal Cohen-Macaulay, and

3. $\operatorname{Ext}^d_R(k, \omega_R) \cong k$.

A Cohen-Macaulay local ring has a canonical module if and only if it is the homomorphic image of a Gorenstein local ring [28, 61, 57]. If the canonical module exists, then it is unique up to isomorphism. Note that a Cohen-Macaulay local ring is Gorenstein if and only if its canonical module is free of rank 1.

For the rest this section, we assume that R is a Gorenstein ring. We refer to [10] and [30] for more on this topic.

Let M, N be two finitely generated R-modules. With $P(M, N)$, let us denote the submodule

$$P(M, N) = \{f : M \to N \ : \ f \text{ is } R\text{-linear and it factors through a projective module.}\}$$

of the R-module $\operatorname{Hom}_R(M, N)$. We consider the quotient

$$\underline{\operatorname{Hom}}_R(M, N) := \frac{\operatorname{Hom}_R(M, N)}{P(M, N)}.$$

Definition 2.6. The *stable category of maximal Cohen-Macaulay modules*, denoted by $\underline{\operatorname{MCM}}(R)$, has the same objects as $\operatorname{MCM}(R)$ but its morphisms are given by $\underline{\operatorname{Hom}}_R(M, N)$ for all finitely generated maximal Cohen-Macaulay R-modules M and N.

Notice that if M is a projective R-module, then $\underline{\operatorname{Hom}}_R(M, N)$ and $\underline{\operatorname{Hom}}_R(N, M)$ are both zero for every maximal Cohen-Macaulay module N. We conclude that every projective R-module is isomorphic to the zero object of $\underline{\operatorname{MCM}}(R)$. In general, two maximal Cohen-Macaulay modules M and N are isomorphic in this category if there are projective modules P and Q such that $M \oplus Q \cong N \oplus P$.

If R has finite global dimension, then every maximal Cohen-Macaulay module is projective. Therefore, $\underline{\operatorname{MCM}}(R)$ is trivial. Conversely, if the global dimension of R is not finite, then $\underline{\operatorname{MCM}}(R)$ is not trivial. There are non-projective maximal Cohen-Macaulay modules in this case.

Recall that for a finitely generated module M, a *syzygy* $\Omega_R M$ is well-defined up to a projective module. Hence, we have a well-defined endofunctor $\Omega_R : \underline{\operatorname{MCM}}(R) \to \underline{\operatorname{MCM}}(R)$. In fact, this endofunctor is an autoequivalence for Gorenstein rings. We call the inverse functor Ω_R^{-1} the *cosyzygy* functor. With the syzygy functor, or the coszygy functor, $\underline{\operatorname{MCM}}(R)$ obtains the structure of a triangulated category. In this thesis, we will always consider the triangulated structure given by the cosyzygy functor.

We are now turning our attention to the singularity category. With $D^b(R)$ we denote the bounded derived category of R. We call a complex in $D^b(R)$ a *perfect complex* if it is quasi-isomorphic to a bounded complex of finitely generated projective modules. We denote the subcategory of perfect complexes by $\mathrm{Perf}(R)$.

Definition 2.7. The *stable derived category of* R, also called its singularity category, is the Verdier quotient

$$D_{sg}(R) = \frac{D^b(R)}{\mathrm{Perf}(R)} \quad .$$

Notice that just like $\underline{\mathrm{MCM}}(R)$, this category is trivial if and only if R has finite global dimension. This allows us to ask how these two categories are related.

The bounded derived category and its shift functor is the most popular example of a triangulated category. This triangulated structure is inherited by the singularity category. Consider the functor $\mathrm{MCM}(R) \to D^b(R)$ which takes a maximal Cohen-Macaulay module and sees it as a complex concentrated in degree zero and consider its precomposition with the quotient functor $D^b(R) \to D_{sg}(R)$. Notice that under this composition, a maximal Cohen-Macaulay module is sent to the zero complex if and only if it has finite projective dimension. However, a maximal Cohen-Macaulay module having finite projective dimension has to be projective. This tells us that the functor $\mathrm{MCM}(R) \to D_{sg}(R)$ gives a functor $\underline{\mathrm{MCM}}(R) \to D_{sg}(R)$. The following theorem, which states that this functor is an equivalence of categories, is due to Buchweitz.

Theorem 2.8. *With their triangulated category structures as described above, the singularity category of R is equivalent to the stable category of maximal Cohen-Macaulay R-modules as triangulated categories.*

Remark 2.9. This equivalence has been rediscovered in geometry by Orlov in [54]. More precisely, Orlov defines the graded triangulated category of singularities of the cone over a projective variety and connects it to the bounded derived category of coherent sheaves on the base of the cone. In the Calabi-Yau case, this gives an equivalence of categories. Moreover, the singularity category is equivalent to the stable category of graded maximal Cohen-Macaulay modules over the coordinate ring of the cone. In the language of physics, this result says that the category of graded D-branes of type B in Landau-Ginzburg models with homogeneous superpotential W is equivalent to the stable category of graded maximal Cohen-Macaulay modules over the hypersurface defined by W.

Remark 2.10. In [10], Buchweitz also proves that the singularity category of a Gorenstein ring R is triangulated equivalent to the homotopy category $\underline{\mathrm{ACP}}(R)$ of acyclic complexes of projective R-modules.

Hence, given a maximal Cohen-Macaulay module M over R, there is an acyclic complex (T, d) of finitely generated projective R-modules such that $M = \text{coker}(d^0 : T^{-1} \to T^0)$. This complex is unique up to homotopy and we call it the *complete resolution of M*.

Recall that for any two finitely generated R-modules M, N, seen as objects in the bounded derived category $D^b(R)$, one has a natural isomorphism

$$\text{Ext}^i_R(M, N) = \text{Hom}_{D^b(R)}(M, N[i]).$$

Motivated from this, we have the following definition.

Definition 2.11. Let M and N be two finitely generated R-modules. Then the *i-th Tate cohomology group of M with values in N* is defined to be

$$\underline{\text{Ext}}^i_R(M, N) = \text{Hom}_{D_{sg}R}(M, N[i])$$

for any integer i.

Remark 2.12. It is worth noting that unlike Ext groups, Tate cohomology groups are possibly nonzero for negative integers.

Example 2.13. Let G be a finite group and let M be a G-group - an abelian group with a compatible G-action. Let $M^G = \{m \in M \ : \ gm = g \text{ for all } g \in G\}$ be the group of invariants. The derived functors of the association $M \mapsto M^G$ give group cohomology $H^*(G, M)$ of G with values in M. On the other hand, let M' be the subgroup of M generated by elements of the form $gm - m$ where $m \in M$ and $g \in G$. Then, the group of coinvariants of M is the quotient group $M_g = M/M'$ and the derived functors of assigning M to M_G gives group homology $H_*(M, G)$ of G with values in M. Let $D := \text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$. The group cohomology can be computed by injective resolutions of M and the group homology can be compute by free resolutions of M. Gluing together these resolutions gives a *complete* resolution of M whose cohomology $\hat{H}^*(G, M)$, defined by Tate, recovers group homology and cohomology in the following sense:

$$\hat{H}^i(G, M) = \begin{cases} H^i(G, M) & \text{for } i > 0 \\ H_i(G, M) & \text{for } i < -1 \end{cases}$$

Moreover, there is a duality

$$D\hat{H}^i(G, M) = \hat{H}^{-i-1}(G, DM)$$

for every integer i where $D = \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$. This is the origin of Tate cohomology and a special instance of our definition of Tate cohomology. Note that the group ring $\mathbb{Z}G$ is a Gorenstein ring in the noncommutative sense. That is, it has finite injective dimension both as a left and right module over itself.

Theorem 2.8 tells us that we should be able to compute this *stable* Ext groups via maximal Cohen-Macaulay modules. We first need a definition.

Definition 2.14. Let M be a finitely generated R-module. By the discussion preciding Theorem 2.8, we can see it as an object in the singularity category. By Theorem 2.8, there is a maximal Cohen-Macaulay module M^{st}, which is unique up to projective summands, whose image under the equivalence is isomorphic to M. We call M^{st} the *stabilization* or the *maximal Cohen-Macaulay approximation* of M.

Almost by definition, one has the following lemma.

Lemma 2.15. *Let M and N be finitely generated R-modules.*

1. *If M is maximal Cohen-Macaulay, then $M^{\mathrm{st}} = M$.*

2. *For any integer i, we have an isomorphism*

$$\underline{\operatorname{Ext}}^i_R(M, N) \cong \underline{\operatorname{Ext}}^i_R(M^{\mathrm{st}}, N^{\mathrm{st}}).$$

3. *For any integer $i > d$, we have an isomorphism*

$$\underline{\operatorname{Ext}}^i_R(M, N) \cong \operatorname{Ext}^i_R(M, N).$$

4. *If M and N are maximal Cohen-Macaulay modules, for any $i > 0$, we have an isomorphism*

$$\underline{\operatorname{Ext}}^i_R(M, N) \cong \operatorname{Ext}^i_R(M, N).$$

5. *If M and N are maximal Cohen-Macaulay modules, for any integer i, we have an isomorphism*

$$\underline{\mathrm{Ext}}_R^i(M,N) \cong \underline{\mathrm{Hom}}_R(\Omega_R^i M, N).$$

Remark 2.16. As mentioned throughout this section, if R is a regular local ring, then the singularity category of R is trivial and therefore the bifunctor $\underline{\mathrm{Ext}}_R^i(-,-)$ is isomorphic to the zero functor for any integer i. The main theme of the first part of this thesis is to study the support of Tate cohomology groups in the singular setting over Gorenstein rings.

Chapter 3

Preliminaries on the Annihilation of Cohomology

In this chapter, our goal is to introduce notation and preliminaries. We start with the cohomology annihilator ideal and its properties. This is followed by a discussion on the annihilation of cohomology over Gorenstein rings. We give equivalent descriptions of the cohomology annihilator in terms of the triangulated categories attached to a Gorenstein ring. The third section of this chapter is devoted to the Jacobian ideal and its relation the annihilation of cohomology. We finish this chapter with a discussion on the geometric meaning of the cohomology annihilator ideal.

3.1 The Cohomology Annihilator Ideal

Let R be a commutative Noetherian ring and let M, N be two finitely generated R-modules. Then, the Ext-group $\operatorname{Ext}^i_R(M, N)$ has the structure of an R-module for any positive integer i. Thus, we can consider its annihilator

$$\operatorname{ann}_R \operatorname{Ext}^i_R(M, N) \quad .$$

Definition 3.1. 1. The *n-th cohomology annihilator ideal* is defined to be the intersection

$$\operatorname{ca}(n,R)=\bigcap_{M,N\in \operatorname{mod}R}\operatorname{ann}_R \operatorname{Ext}^n_R(M,N) \quad .$$

2. The *cohomology annihilator ideal* is defined to be the union

$$\operatorname{ca}(R)=\bigcup_{n>0}\operatorname{ca}(n,R) \quad .$$

3. A ring element is called a *cohomology annihilator* if it belongs to the cohomology annihilator ideal.

Let M be an R-module and let

$$P: \quad \ldots\to P_2\to P_1\to P_0\to 0$$

be a projective resolution of M. Then, the n-th Ext-group $\operatorname{Ext}^n_R(M,N)$ is the n-th (co)homology of the complex

$$\operatorname{Hom}_R(P,N):0\to \operatorname{Hom}_R(P_0,N)\to \operatorname{Hom}_R(P_1,N)\to \operatorname{Hom}_R(P_2,N)\to\ldots \quad .$$

On the other hand, let $\Omega_R M$ denote the image of the map $P_1\to P_0$. It is called a *syzygy* of M and it is well-defined up to a projective summand. It is easy to see that the complex

$$\ldots\to P_2\to P_1\to 0$$

is a projective resolution of $\Omega_R M$. We can summarize this discussion as

$$\operatorname{Ext}^{n+1}_R(M,N)\cong \operatorname{Ext}^n_R(\Omega_R M,N)$$

for all $n>0$ and all finitely generated R-modules M and N. Consequently, we have

$$\operatorname{ann}_R \operatorname{Ext}^{n+1}_R(M,N)=\operatorname{ann}_R \operatorname{Ext}^n_R(\Omega_R M,N) \quad .$$

Thus, we conclude that $\operatorname{ca}(n,R)\subseteq \operatorname{ca}(n+1,R)$. Hence, we have an increasing chain of ideals inside our Noetherian ring R. We have the following lemma.

Lemma 3.2. *For a commutative Noetherian ring R, we have $\operatorname{ca}(R)=\operatorname{ca}(s,R)$ for $s\gg 0$.*

Example 3.3. Let R be the coordinate ring of a smooth algebraic variety. Then, the global dimension of R is equal to its Krull dimension by Hilbert's syzygy theorem [34, 24]. This means that for any $n > \dim R$ and for any two finitely generated modules M and N one has $\operatorname{Ext}^n_R(M, N) = 0$. Hence, $\operatorname{ca}(R) = \operatorname{ca}(n, R) = R$ for any $n > \dim R$. The cohomology annihilator is interesting only in the singular case.

3.2 Annihilation of Cohomology over Gorenstein Rings

In this section, we are focusing our attention onto the annihilation of cohomology over a Gorenstein ring and we give equivalent descriptions of the cohomology annihilator ideal. In particular, we describe the cohomology annihilator ideal in terms of the stable category of maximal Cohen-Macaulay modules, in terms of the homotopy category of acycylic complexes, in terms of the singularity category and, in the case of hypersurfaces, in terms of the homotopy category of matrix factorizations. We start with a general discussion on the annihilation of a triangulated category.

Let R be a commutative ring and $\mathcal{T}$ be an R-linear category with shift functor Σ. We say that $r \in R$ *annihilates* $X \in \mathcal{T}$ if $x \in \operatorname{ann}_R \operatorname{End}_{\mathcal{T}} X$ and we write $r \in \operatorname{ann}_{\mathcal{T}} X$ in this case. We say that r *annihilates* $\mathcal{T}$ if $r \in \operatorname{ann}_{\mathcal{T}} X$ for every $X \in \mathcal{T}$.

Note that for any $X, Y \in \mathcal{T}$, $\operatorname{Hom}_{\mathcal{T}}(X, Y)$ is naturally a module over $\operatorname{End}_{\mathcal{T}} X$ and therefore $\operatorname{ann}_{\mathcal{T}} X \subseteq \operatorname{ann} \operatorname{Hom}_{\mathcal{T}}(X, Y)$. Consequently, we have $\operatorname{ann}_{\mathcal{T}}(X \oplus Y) = \operatorname{ann}_{\mathcal{T}} X \cap \operatorname{ann}_{\mathcal{T}} Y$. On the other hand, $\operatorname{End}_{\mathcal{T}} X \cong \operatorname{End}_{\mathcal{T}} \Sigma X$ and so $\operatorname{ann}_{\mathcal{T}} X = \operatorname{ann}_{\mathcal{T}} \Sigma X$. Hence, we conclude that if $r \in \operatorname{ann}_{\mathcal{T}} X$, then r annihilates every object in $\langle X \rangle$ - the smallest subcategory containing X and closed under finite direct sums, direct summands and Σ.

We put $\langle X \rangle_1 = \langle X \rangle$. Inductively, we define $\langle X \rangle_n$ to be the full subcategory of $\mathcal{T}$ consisting of objects B such that there is a distinguished triangle $A \to B \to C \to \Sigma A$ with $A \in \langle X \rangle_{n-1}$ and $C \in \langle X \rangle$. We say that X *is a strong generator for* T if $T = \langle X \rangle_n$ for some integer n. Note that applying the cohomological functor $\operatorname{Hom}_{\mathcal{T}}(-, B)$ to the triangle $A \to B \to C \to \Sigma A$ yields an exact sequence

$$\operatorname{Hom}_{\mathcal{T}}(C, B) \to \operatorname{End}_{\mathcal{T}}(B) \to \operatorname{Hom}_{\mathcal{T}}(A, B).$$

Therefore, $\operatorname{ann} \operatorname{Hom}_{\mathcal{T}}(C, B) \operatorname{ann} \operatorname{Hom}_{\mathcal{T}}(A, B) \subseteq \operatorname{ann} \operatorname{End}_{\mathcal{T}}(B)$. The following lemma summarizes this discussion.

Lemma 3.4. *Let $\mathcal{T}$ be an R-linear triangulated category with shift functor Σ and $X \in \mathcal{T}$. For any $n \geq 1$ and $Y \in \langle X \rangle_n$, we have an inclusion $(\operatorname{ann}_{\mathcal{T}} X)^n \subseteq \operatorname{ann}_{\mathcal{T}} Y$. In particular, if X is a strong generator with $\mathcal{T} = \langle X \rangle_n$ then $(\operatorname{ann}_{\mathcal{T}} X)^n$ annihilates $\mathcal{T}$.*

We now focus on the stable category of maximal Cohen-Macaulay modules (*See Chapter 2*). Let R be a commutative Gorenstein ring with Krull dimension d. We will show that $\operatorname{ca}(R)$ is the ideal that annihilates $\underline{\mathrm{MCM}}(R)$. Even though the result has been known to the experts, the proof Is not easy to find in the literature.

Proposition 3.5. *For a commutative Gorenstein ring R, the cohomology annihilator ideal is the annihilator of the stable category of maximal Cohen-Macaulay modules.*

Proof. Recall from Lemma 3.2 that $\operatorname{ca}(R) = \operatorname{ca}(s, R)$ for a sufficiently large s. Assume that $s > d$. We have the following equalities which are almost by definition of Tate cohomology.

$$\begin{aligned}
\operatorname{ca}(R) &= \bigcap_{M,N \in \operatorname{mod} R} \operatorname{ann}_R \operatorname{Ext}_R^s(M, N) \\
&= \bigcap_{M,N \in \operatorname{mod} R} \operatorname{ann}_R \underline{\operatorname{Ext}}_R^s(M, N) \\
&= \bigcap_{M,N \in \operatorname{mod} R} \operatorname{ann}_R \underline{\operatorname{Ext}}_R^s(M^{\mathrm{st}}, N^{\mathrm{st}}) \\
&= \bigcap_{X,Y \in \mathrm{MCM}(R)} \operatorname{ann}_R \underline{\operatorname{Ext}}_R^s(X, Y) \\
&= \bigcap_{X,Y \in \mathrm{MCM}(R)} \operatorname{ann}_R \underline{\operatorname{Hom}}_R(\Omega_R^s X, Y) \\
&= \bigcap_{X,Y \in \mathrm{MCM}(R)} \operatorname{ann}_R \underline{\operatorname{Hom}}_R(X, Y)
\end{aligned}$$

where the last equality follows from the fact that Ω_R is an autoequivalence of $\underline{\mathrm{MCM}}(R)$. From our discussion on the annihilation of a triangulated category, we conclude that

$$\operatorname{ca}(R) = \bigcap_{M \in \mathrm{MCM}(R)} \operatorname{ann}_R \underline{\operatorname{End}}_R(M)$$

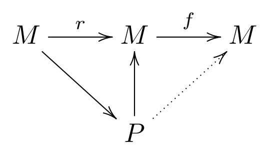
which finishes the proof. □

Remark 3.6. Note that for any commutative ring A and any A-module X, we have $\operatorname{ann}_A X = \operatorname{ann}_A \operatorname{End}_A(X)$. Therefore, for ease of notation, we make the following definition.

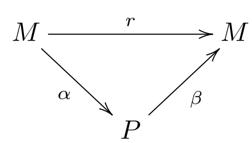
Definition 3.7. For any maximal Cohen-Macaulay R-module M over a Gorenstein ring R, the *stable* *annihilator* of M is

$$\underline{\mathrm{ann}}_R(M) := \mathrm{ann}_R\, \underline{\mathrm{End}}_R(M).$$

Remark 3.8. Let M be a maximal Cohen-Macaulay R-module and $r \in R$. By abusing the notation, we also denote the multiplication map $r \cdot \mathrm{id}_M$ on M by r. Then, $r \in \underline{\mathrm{ann}}_R(M)$ if and only if for any $f \in \mathrm{End}_R(M)$, the map $rf = fr$ factors through a projective R-module. Choosing f to be the identity map, we see that if $r \in \underline{\mathrm{ann}}_R(M)$, then multiplication by r on M factors through a projective R-module. The converse of this is also true as can be seen from the diagram below. That is, if multiplication by r on M factors through a projective module P, then $rf = fr$ factors through P for any $f \in \mathrm{End}_R(M)$.



Hence, r is a cohomology annihilator if and only if for any $M \in \mathrm{MCM}(R)$ there are maps $\alpha : M \to P$ and $\beta : P \to M$ such that P is projective and $\beta \circ \alpha = r$.



Remark 3.9. Let R be a Gorenstein ring. Let M be maximal Cohen-Macaulay R-module and P, T be projective and complete resolutions of M respectively. Then, the following are equivalent.

1. Multiplication by r on M factors through a projective module,

2. Multiplication by r on P factors through a perfect complex,

3. Multiplication by r on T is nullhomotopic.

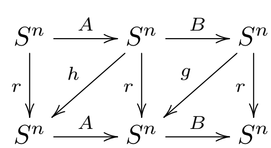
It is sometimes useful to compute the stable annihilator by making use of this equivalence. For instance, we can show that $\underline{\mathrm{ann}}_R(M) = \underline{\mathrm{ann}}_R(\Omega_R M)$ by observing that $T[-1]$ is a complete resolution of $\Omega_R M$ and multiplication by r is nullhomotopic on T if and only if it is nullhomotopic on $T[-1]$. This argument is a special case of the discussion on the annihilation of triangulated categories but it is much nicer visually.

Remark 3.10. Let R be a complete hypersurface singularity. That is, let $R = S/(f)$ for some polynomial f where $S = k[\![x_1, \ldots, x_n]\!]$. If M is a maximal Cohen-Macaulay R-module, then any minimal projective/complete resolution of M is 2-periodic [23]. Therefore, we have

$$\underline{\mathrm{ann}}_R(M) = \mathrm{ann}_R \underline{\mathrm{End}}_R(M) = \mathrm{ann}_R \underline{\mathrm{Ext}}^0_R(M,M) = \mathrm{ann}_R \underline{\mathrm{Ext}}^2_R(M,M) = \mathrm{ann}_R \mathrm{Ext}^2_R(M,M).$$

This description is useful for computation purposes. Many computer algebra systems for commutative algebra can compute this annihilator if we know a presentation for M.

Remark 3.11. Let $S = k[\![x_1, \dots, x_n]\!]$ and $f \in S$. Put $R = S/(f)$ as above. A *matrix factorization* of f is a pair of square matrices (A, B) with entries in S such that $AB = BA = f\,\mathrm{Id}$. A morphism between two matrix factorizations (A, B) and (C, D) is given by a pair of matrices (X, Y) with entries in S such that $YA = CX$ and $XB = DY$. We denote by $\mathrm{MF}(f)$ the category of matrix factorizations of f and by $\underline{\mathrm{MF}}(f)$ the corresponding homotopy category. Eisenbud proves in [23] that the homotopy category of matrix factorizations of f is equivalent to the stable category of maximal Cohen-Macaulay modules over R. Therefore, a ring element $r \in R$ is a cohomology annihilator if and only if for every matrix factorization (A, B) of f, there is a pair (h, g) of matrices such that $r = Ah + gB$.



3.3 The Jacobian Ideal

The results of this section are due to Buchweitz [10], Wang [65], Dieterich [18] and Iyengar and Takahashi [42]. Before we give the most general result due to Iyengar and Takahashi, we will look at some examples. The slogan of this section is: *the Jacobian ideal is contained in the cohomology annihilator ideal.*

Example 3.12 (Corollary 7.8.7 [10])**.** Let k be a field and let $S = k[\![x_1, \dots, x_n]\!]$ be the ring of formal power series over k. Consider a regular sequence $f_1, \dots, f_m$ in the unique maximal ideal $(x_1, \dots, x_m)$ of S and let R be the complete intersection ring $S/(f_1, \dots, f_m)$ defined by this regular sequence. The ideal generated by the maximal minors of the Jacobian matrix (whose ij-th entry is $\partial f_j / \partial x_i$) is called the Jacobian ideal of R and it annihilates all Tate cohomology groups over R. Hence, it is contained in the cohomology annihilator ideal.

The following example is a specialization of the previous example. We include it to give a more complete list of references and also because they use different techniques.

Example 3.13. Let S be the ring of formal power series as in the previous example and let $f \in S$. We consider the hypersurface ring $R = S/(f)$. The Jacobian ideal is generated by the partial derivatives of f and we know that it is contained in the cohomology annihilator ideal of R. Dieterich shows that it is actually contained in $\mathrm{ca}(n+1, R)$ [18, Proposition 18]. It is worth noting that the essential ingredient in the proof is the "product rule" for derivatives. Indeed, for any matrix factorization (A, B) of f we have

$$f = AB$$
$$\frac{\partial f}{\partial x_i} = \frac{\partial A}{\partial x_i} B + A \frac{\partial B}{\partial x_i}$$

which tells us that multiplication by $\frac{\partial f}{\partial x_i}$ is nullhomotopic (see Remark 3.11). Wang improves this result and shows that the Jacobian ideal is contained in $\mathrm{ca}(1, R)$ [65, Example 2.7].

We now give a definition for the Jacobian ideal due to Iyengar and Takahashi. This definition generalizes the classical case which we saw in previous examples.

Definition 3.14 ([42])**.** Let R be a commutative Noetherian ring.

1. A *Noether normalization* of R is a subring A with finite global dimension over which R is a finitely generated A-module.

2. For a Noether normalization A of R, the *module of Kähler differentials* $\Omega_{R/A}$ is defined to be I/I^2 where I is the kernel of the multiplication map $R \otimes_A R \to R$ which takes $r \otimes_A s$ to rs.

3. For a Noether normalization A of R, the *Kähler different* of R over A is the zeroth Fitting ideal

$$K(R/A) := \mathrm{Fitt}_0^R(\Omega_{R/A})$$

   generated by the maximal minors of a presentation matrix for $\Omega_{R/A}$.

4. The *Jacobian ideal* $\mathrm{jac}(R)$ is defined to be

$$\mathrm{jac}(R) = \sum K(R/A)$$

   where the sum is taken over all Noether normalizations of R.

Theorem 3.15 (Theorem 3.4 [42])**.** *Let R be a commutative Noetherian ring of Krull dimension d. Then, there exists a positive integer s such that*

$$(\mathrm{jac}(R))^s \subseteq \mathrm{ca}(d+1, R) \subseteq \mathrm{ca}(R) \quad .$$

If, in particular, we impose rather mild conditions on our ring, we get better results.

Theorem 3.16 (Theorem 3.8 [42]). *Let R be a commutative Noetherian ring of Krull dimension d. If R is equidimensional and each $p \in \operatorname{Spec}(R)$ satisfies $2 \operatorname{depth} R_p \geq \dim R_p$, then*

$$\operatorname{jac}(R) \subseteq \operatorname{ca}(d+1, R) \subseteq \operatorname{ca}(R) \quad .$$

In particular, if R is local, equidimensional and locally Cohen-Macaulay on the punctured spectrum and if $2 \operatorname{depth} R \geq d$, then we have the above inclusion.

3.4 The Geometric Meaning of The Cohomology Annihilator Ideal

In previous sections, we have seen that the cohomology annihilator ideal is equal to the ring if and only if the ring is of finite global dimension. We also have seen the relation between the cohomology annihilator ideal and the Jacobian ideal. Hence, we have a feeling that the cohomology annihilator ideal has some relation with the singularities of the corresponding scheme. In this section, we state a theorem due to Iyengar and Takahashi to make this idea precise.

Let R be a commutative Noetherian ring and U be a multiplicatively closed subset. Then, any finitely generated module over $U^{-1}R$ is of the form $U^{-1}M$ for some finitely generated R-module M. Moreover, we have natural isomorphisms

$$U^{-1} \operatorname{Ext}^n_R(M, N) \cong \operatorname{Ext}^n_{U^{-1}R}(U^{-1}M, U^{-1}N) \quad \text{for all } n > 0.$$

From this it follows that $U^{-1}\operatorname{ca}(n, R) \subseteq \operatorname{ca}(n, U^{-1}R)$ for all $n > 0$. Moreover, we have $U^{-1}\operatorname{ca}(R) \subseteq \operatorname{ca}(U^{-1}R)$. Now, suppose that p is a prime ideal with $p \not\supseteq \operatorname{ca}(R)$ and let U be the multiplicatively closed set $R \setminus p$. Then, the intersection $\operatorname{ca}(R) \cap U$ is nonempty. Hence, we have

$$R_p = (1) = U^{-1}\operatorname{ca}(R) \subseteq \operatorname{ca}(R_p) \subseteq R_p \quad .$$

and therefore an equality $\operatorname{ca}(R_p) = R_p$ from which we conclude $\operatorname{gldim} R_p < \infty$.

Recall that for a commutative Noetherian ring, we can define the singular locus of $\operatorname{Spec}(R)$ as

$$\operatorname{Sing}(R) = \{p \in \operatorname{Spec}(R) \ : \ \operatorname{gldim} R_p = \infty\} \quad .$$

Therefore, we have the following lemma.

Lemma 3.17. *For a commutative Noetherian ring R, we have*

$$\operatorname{Sing}(R) \subseteq V(\operatorname{ca}(R))$$

where $V(\operatorname{ca}(R)) = \{p \in \operatorname{Spec}(R) \ : \ p \supseteq \operatorname{ca}(R)\}$ *denotes the closed subset of* $\operatorname{Spec}(R)$ *defined by the cohomology annihilator ideal.*

Then a natural question to ask is "Is there equality?". In full generality, the answer is no. Indeed, note that an equality would imply that the singular locus of $\operatorname{Spec}(R)$ is closed. And "On the closedness of singular loci" is the title of a 1959 paper by Nagata in which he also studied rings with nonclosed singular loci [51, §§4.5]. Hochster [35, Example 1] and Ferrand and Reynaud [27, Proposition 3.5] produced more examples of such rings. However, under some reasonable assumptions the answer is yes.

Theorem 3.18 (Theorem 3.4, Theorem 3.5 [44])**.** *We have the equality* $\operatorname{Sing}(R) = V(\operatorname{ca}(R))$ *if one of the following conditions hold.*

1. *R is an equicharacteristic excellent local ring of finite Krull dimension, or*

2. *R is a localization of a finitely generated k-algebra of finite Krull dimension where k is a field.*

Chapter 4

The Cohomology Annihilator in Dimension One

We have seen in the previous chapter that under reasonable assumptions, the cohomology annihilator ideal of a commutative Noetherian ring contains the Jacobian ideal. However, these two ideals do not necessarily coincide. In this chapter, we are investigating how far away they are from being equal. We start with the motivating example for this chapter, in fact, for this thesis.

Example 4.1. Let $S = \mathbb{C}[\![x, y]\!]$, $f = x^3 - y^5 \in S$ and $R = S/(f)$. Then, the Jacobian ideal $\operatorname{jac}(R) = (x^2, y^4)$ is generated by the partial derivatives of f and one can compute $\operatorname{ca}(R) = (x^2, xy, y^3)$ by looking at the stable annihilators of all maximal Cohen-Macaulay R-modules (*c.f.* Example 5.2). Consider the quotient $R/\operatorname{jac}(R)$. It is an eight dimensional complex vector space whereas the quotient $R/\operatorname{ca}(R)$ has dimension four. The following picture describes bases for these vector spaces.

$$\begin{array}{ccc} & 1 & \\ x & & y \\ xy & & y^2 \\ xy^2 & & y^3 \\ & xy^3 & \end{array}$$

This picture leads to an observation due to Ragnar-Olaf Buchweitz. He observed in several other examples that the vector space dimension of the quotient $R/\operatorname{jac}(R)$ is twice that of the quotient $R/\operatorname{ca}(R)$. But there was no explanation for why this is the case. This chapter gives an explanation to this observation. The main idea comes from an ancient problem.

First, notice that 3 and 5 are relatively prime numbers. So, Euclidean algorithm tells us that given any integer n, we can find two integers a and b such that $3a+5b=n$. This is possible because we know that we can write 1 as $2\times 3+(-1)\times 5$. Thus, we would have $2n\times 3+(-n)\times 5=n$. But if we only allow non-negative integer combinations of 3 and 5, it is impossible to write 1, starting from 3 and 5. So, the question is to describe the subset $N=\{n\in\mathbb{Z}\ :\ n=3a+5b \text{ with } a,b\in\mathbb{Z}^{\geq 0}\}$ of integers. Writing the non-negative integers in three columns as below allows us to answer this question quickly.

0	1	2
3	4	5
6	7	8
9	10	11

So, any number after 7 belongs to N. Let's now investigate the numbers between 0 and 7 by writing them in the following order.

0	1	2	3
7	6	5	4

We see that there are 8 numbers here and m belongs to N if and only if $7-m$ does not belong to N. This observation tells us out of these 8 numbers, exactly 4 of them belongs to N. At least numerically, we recover Buchweitz's observation, this way!

Now, in order to make things more precise, we will consider the subring $T=\mathbb{C}[\![t^3,t^5]\!]$ of the power series ring $\mathbb{C}[\![t]\!]$ in one variable. Note that identifying t^3 with y and t^5 with x gives an isomorphism between R and T. Under this isomorphism the cohomology annihilator $\operatorname{ca}(R)$ has the image (t^8,t^9,t^{10}). Our discussion above yields that this ideal contains every power t^n of t with $n>7$ and also that it is the largest ideal of T with this property. We have

1. The cohomology annihilator ideal of T is (t^8,t^9,t^{10}).

2. An element $u\in T$ belongs to (t^8,t^9,t^{10}) if and only if $t^n u$ belongs to T for all $n\geq 0$.

This idea is our main tool to explain Buchweitz's observation.

4.1 The Conductor Ideal

The ideal (t^8, t^9, t^{10}) in the above example is called the conductor ideal of T. The aim of this section is to define the conductor ideal properly and state the properties we will use in the main theorem of the chapter. We refer to [50, 36, 37, 31, 53] and references within for this section.

Definition 4.2. Let R be a reduced ring with total ring of fractions K.

1. We say that $r \in K$ is *integral* over R if there is a monic polynomial $p(x) \in R[x]$ such that $p(r) = 0$.

2. *The integral closure* $\bar{R}$ of R is the set of all elements of K which are integral over R.

3. *The conductor ideal* of R is the set

$$\operatorname{co}(R) := \{r \in K \ : \ r\bar{R} \subseteq R\}.$$

Remark 4.3. Note that the integral closure of R is integrally closed. That is, the integral closure of $\bar{R}$ is $\bar{R}$ itself. In other words, $\bar{R}$ is a normal ring. Our main interest in this chapter is 1-dimensional rings and it is important to note that in this case, normality implies regular. In particular, in dimension one the integral closure $\bar{R}$ is be a Cohen-Macaulay ring.

Remark 4.4. The conductor ideal is a common ideal of R and $\bar{R}$. In fact, it is the largest ideal with this property. While $\bar{R}$ has the property that $\operatorname{End}_R(\bar{R}) \cong \bar{R}$ as R-modules, the conductor ideal can be identified with the submodule $\operatorname{Hom}_R(\bar{R}, R)$.

Remark 4.5. Every element x of the field K is of the form $x = r^{-1}s$ where $r, s \in R$ and r is a non-zerodivisor in R. So, we have $rx \in R$. Hence, if $\bar{R}$ is finitely generated as an R-module, then the product of the denominators of its generators give a non-zerodivisor inside the conductor. Conversely, if $\operatorname{co}(R)$ contains a non-zerodivisor z, then $z\bar{R} \subseteq R$ implies $\bar{R} \subseteq \frac{1}{z}R$. If R is Noetherian, this means $\bar{R}$ is finitely generated as an R-module. This is equivalent to saying that the completion $\hat{R}$ of R is reduced. In this case, we say that R is *analytically unramified.*

Example 4.6. Let k be a field and consider the subring $R = k[\![t^a, t^b]\!]$ of $k[\![t]\!]$, where a and b are relatively prime positive integers. Then, $\bar{R} = k[\![t]\!]$ and $\operatorname{co}(R) = (t^n \ : \ n \geq c)$ where c is the least positive integer such that $t^{c-1} \notin R$. In fact, it can be shown that $c = (a-1)(b-1)$. The number $c - 1$ is called the *Frobenius number* of R.

Theorem 4.7 ([59])**.** *Let k be a field and $R = k[\![x, y]\!]/(f)$ be an algebraic plane curve.*

1. *For any prime ideal* $p \in \mathrm{Spec}(R)$*, we have*

$$\dim_k \frac{\bar{R}_p}{R_p} = \dim_k \frac{R_p}{\mathrm{co}(R_p)}.$$

2. *The equality*

$$\dim_k \frac{\bar{R}}{R} = \dim_k \frac{R}{\mathrm{co}(R)}$$

holds if and only if the module of regular differentials is a free R*-module.*

Remark 4.8. In fact, the theorem is true more generally for complete intersection curves.

Theorem 4.9 (Theorem 12.2.2, [37])**.** *Let* R *be a* 1*-dimensional local analytically unramified ring with infinite residue field* k*, total ring of fractions* K*, integral closure* $\bar{R}$ *and conductor ideal* $\mathrm{co}(R)$*. Then, we have*

$$\mathrm{length}\, \frac{\bar{R}}{\mathrm{co}(R)} = 2\, \mathrm{length}\, \frac{R}{\mathrm{co}(R)}.$$

4.2 The Main Theorem

The aim of this section is to prove the following theorem.

Theorem. *Let* R *be a* 1*-dimensional reduced complete Gorenstein local ring. Then, the cohomology annihilator ideal coincides with the conductor ideal.*

One inclusion in this equality has been known over twenty years. Indeed, Hsin-Ju Wang proved in his PhD thesis under Craig Huneke the following theorem.

Theorem 4.10 (Proposition 3.1 [65])**.** *Let* R *be a* 1*-dimensional reduced complete Noetherian local ring. Then,* $\mathrm{co}(R)$ *annihilates* $\mathrm{Ext}^1_R(M, N)$ *for any* $M \in \mathrm{MCM}(R)$ *and* $N \in \mathrm{mod}R$*.*

In particular, Wang's theorem says that

$$\mathrm{co}(R) \subseteq \mathrm{ann}_R \mathrm{Ext}^1_R(\Omega^{-1}M, M) = \mathrm{ann}_R \underline{\mathrm{End}}_R(M)$$

for all $M \in \mathrm{MCM}(R)$, if we assume that R is Gorenstein. Therefore, $\mathrm{co}(R) \subseteq \mathrm{ca}(R)$. The remaining of this section is devoted to the proof of the other inclusion.

Lemma 4.11. *Suppose that R is analytically unramified. Then, we have an R-linear isomorphism*

$$\underline{\mathrm{End}}_R(\bar{R}) \cong \frac{\bar{R}}{\mathrm{co}(R)} \quad .$$

Proof. For any two finitely generated R-modules M and N we have an exact sequence

$$\mathrm{Hom}_R(M,R)\otimes_R N \xrightarrow{\nu} \mathrm{Hom}_R(M,N) \to \underline{\mathrm{Hom}}_R(M,N) \to 0$$

where $\nu(\phi\otimes n)$ is the map which sends $m\in M$ to $\phi(m)n\in N$. Letting $M=N=\bar{R}$, we get

$$\mathrm{Hom}_R(\bar{R},R)\otimes_R \bar{R} \xrightarrow{\nu} \mathrm{End}_R(\bar{R}) \to \underline{\mathrm{End}}_R(\bar{R}) \to 0$$

which gives us, by the properties we listed in the previous subsection,

$$\mathrm{co}(R)\otimes_R \bar{R} \xrightarrow{\mu} \bar{R} \to \underline{\mathrm{End}}_R(\bar{R}) \to 0$$

where μ is the multiplication map. As the conductor is an ideal of the normalization, the image of the multiplication map lies in the conductor. On the other hand, we have $\mathrm{co}(R)=\mu(\mathrm{co}(R)\otimes_R 1)$ is in the image of μ. Hence, we get the equality of the conductor ideal and the image of the multiplication map $\mathrm{co}(R)\otimes_R \bar{R} \xrightarrow{\mu} \bar{R}$. This finishes the proof. □

As a corollary of this lemma, we get the following proposition for Gorenstein rings. The proof of the proposition is checking that

$$\underline{\mathrm{ann}}_R\,\bar{R} = \mathrm{ann}_R\left(\frac{\bar{R}}{\mathrm{co}(R)}\right) = \mathrm{ann}_R\left(\frac{\bar{R}}{R}\right) = \mathrm{co}(R)$$

Proposition 4.12. *Suppose that R is an analytically unramified Gorenstein ring. Then, we have the equality*

$$\underline{\mathrm{ann}}_R\,\bar{R} = \mathrm{co}(R).$$

Remark 4.13. Suppose that R is a 1-dimensional analytically unramified Gorenstein local ring. Then, $\bar{R}$ is a 1-dimensional normal ring. Hence, $\bar{R}$ is maximal Cohen-Macaulay as an R-module. So, by the description of the cohomology annihilator in terms of the singularity category we obtain the following theorem.

Theorem 4.14. *Let R be a 1-dimensional analytically unramified Gorenstein local ring. Then, the cohomology annihilator ideal is contained in the conductor ideal.*

Hence, we have our main result.

Corollary 4.15. *Let R be a 1-dimensional reduced complete Gorenstein local ring. Then, the cohomology annihilator ideal coincides with the conductor ideal.*

Remark 4.16. Thanks are due to Ryo Takahashi who pointed out that "any complete local ring is excellent, so it is Japanese" after reviewing the first draft of [25]. This means that we do not have to assume analytically unramified in Corollary 4.15: it is automatic.

4.3 The Milnor-Jung Formula and How It Explains Buchweitz's Observation

We have shown that the cohomology annihilator ideal coincides with the conductor ideal for 1-dimensional reduced complete Gorenstein local rings. In particular, this result holds for the complete local coordinate rings of reduced algebraic plane curves. We will only consider plane curves over algebraically closed fields.

Let $R = k[\![x, y]\!]/(f)$ be the complete local coordinate ring of a reduced algebraic plane curve singularity. The *delta invariant* of R is the common number (see Theorem 4.7)

$$\dim_k \frac{\bar{R}}{R} = \dim_k \frac{R}{\mathrm{co}(R)} = \dim_k \frac{R}{\mathrm{ca}(R)}$$

which we will denote by $\delta(R)$. The delta invariant measures the number of double points concentrated at the singular point. On the other hand, R has another very important invariant called the *Milnor number*

$$\mu(R) = \dim_k \frac{R}{\mathrm{jac}(R)}$$

where we recall that $\mathrm{jac}(R)$ is the Jacobian ideal of R. The delta invariant and the Milnor number are related to each other by the Milnor-Jung formula [11]:

$$\mu(R) = 2\delta(R) - r + 1$$

where r is the number of irreducible components of the curve f. When the curve is *unibranch*, that is when $r = 1$, we have $\mu(R) = 2\delta(R)$ which explains Buchweitz's observation that was introduced at the beginning of this chapter.

Theorem 4.17. *Let $R = k[\![x, y]\!]/(f)$ be the complete local coordinate ring of a reduced algebraic plane curve singularity. Then,*

$$\mu(R) = 2\dim_k \frac{R}{\operatorname{ca}(R)} - r + 1$$

where r is the number of irreducible components of the curve f.

4.4 The Weak MCM Extending Property

For the rest of this chapter, we will investigate the *weak MCM extending property* and its applications. The first application, which we will consider in this section, is the relation between the cohomology annihilator ideal of a 1-dimensional Gorenstein local ring and its completion. The second application will be a similar relation between a ring and its Henselization. We start with the definition.

Definition 4.18. Let $f : R \to S$ be a ring homomorphism of Gorenstein rings. We say that f has the *weak* MCM *extending property* if for any $N \in \operatorname{MCM}(S)$, there is an $M \in \operatorname{MCM}(R)$ and $N' \in \operatorname{MCM}(S)$ such that

1. N' is a direct summand in $M \otimes_R S$, and
2. N can be obtained from N' in finitely many steps by taking syzygies, direct sums and direct summands.

Maps with this property carries the cohomology annihilators to cohomology annihilators.

Theorem 4.19. *Let $f : R \to S$ be a ring homomorphism of Gorenstein rings with the weak MCM extending property. Then,*

$$f(\operatorname{ca}(R)) \subseteq \operatorname{ca}(S).$$

Proof. Let $r \in \operatorname{ca}(R)$ and $N \in \operatorname{MCM}(S)$. By the weak MCM extending property there is an $M \in \operatorname{MCM}(R)$ and $N' \in \operatorname{MCM}(S)$ such that N' is a direct summand in $M \otimes_R S$ and N can be obtained from N' in finitely many steps by taking syzygies, direct sums and direct summands. We have a commutative diagram

$$\begin{tikzcd} M \arrow[rr, "r"] \arrow[dr, "\alpha"'] & & M \\ & P \arrow[ur, "\beta"'] & \end{tikzcd}$$

where P is a projective R-module. Tensoring this commutative diagram with S, we get

$$\begin{tikzcd} N' \arrow[r, hook] & M \otimes_R S \arrow[rr, "f(r)"] \arrow[dr, "\alpha \otimes_R 1"'] & & M \otimes_R S \arrow[r, two heads] & N' \\ & & P \otimes_R S \arrow[ur, "\beta \otimes_R 1"'] & & \end{tikzcd}$$

which shows that that $f(r)$ stably annihilates N'. Indeed, the composition of the three maps in the first row is multiplication by $f(r)$ and $P \otimes_R S$ is a projective S-module. Since N can be obtained from N' by taking finite direct sums, summands and syzygies, $f(r)$ also stably annihilates N. $\square$

Recall that a finitely generated R-module M *has constant rank* r if M_p is a free R_p-mod for every minimal prime p.

Lemma 4.20. *Let R be a local ring and M be an R-module. If M is locally free at the minimal primes of R, then $M \oplus \Omega_R M$ has constant rank. In particular, if R is reduced, then for any $M \in \operatorname{MCM}(R)$, the module $M \oplus \Omega_R M$ has constant rank.*

Proof. Let p be a minimal prime. Consider the short exact sequence

$$0 \to \Omega_R M \to F \to M \to 0$$

with F a free R-module. Localizing at p, we get

$$0 \to (\Omega_R M)_p \to F_p \to M_p \to 0.$$

We know that M_p is free. Hence, this sequence splits and we get

$$F_p \cong (M \oplus \Omega_R M)_p$$

which by definition tells us that $M \oplus \Omega_R M$ has a constant rank. □

Lemma 4.21 (Corollary 4.5 [45]). *Let R be a 1-dimensional local ring whose completion $\hat{R}$ is reduced and let N be a finitely generated R-module. Then, $N = M \otimes_R \hat{R}$ for some R-module M if and only if $\dim_{R_p}(N_p) = \dim_{R_q}(N_q)$ for any two prime ideals p, q of $\hat{R}$ lying over the same prime ideal of R.*

We note that a Noetherian local ring is Gorenstein if and only if its completion is Gorenstein. Moreover, the depth of a module is equal to the depth of its completion. In particular, $M \in \mathrm{MCM}(R)$ if and only if $M \otimes_R \hat{R} \in \mathrm{MCM}(\hat{R})$. Now, suppose that R has reduced completion and $N \in \mathrm{MCM}(\hat{R})$. Then, N is locally free at minimal primes. So, by Lemma 4.20 $N \oplus \Omega_{\hat{R}} N$ has constant rank. Therefore, Lemma 4.21 tells us that $N \oplus \Omega_{\hat{R}} N$ is isomorphic to $M \otimes_R \hat{R}$ as an $\hat{R}$-module for some $M \in \mathrm{MCM}(R)$. This discussion proves the following proposition.

Proposition 4.22. *Let R be a 1-dimensional Gorenstein local ring and assume that it has reduced completion $\hat{R}$. Then, the inclusion map from R to its completion has the weak MCM extending property.*

We use the weak MCM extanding property of this inclusion to prove the following theorem on the relation between the cohomology annihilators.

Theorem 4.23. *Let R be a 1-dimensional Gorenstein local ring whose completion $\hat{R}$ is reduced. Then,*

$$\mathrm{ca}(R) = \mathrm{ca}(\hat{R}) \cap R.$$

Proof. As the inclusion has the weak MCM extending property, it follows that $\mathrm{ca}(R) \subseteq \mathrm{ca}(\hat{R}) \cap R$. For the other direction, we let $M \in \mathrm{MCM}(R)$ and we look at the inclusion

$$\underline{\mathrm{End}}_R(M) \hookrightarrow \widehat{\underline{\mathrm{End}}_R(M)} = \underline{\mathrm{End}}_{\hat{R}}(\hat{M}).$$

If $r \in \mathrm{ca}(\hat{R}) \cap R$, then it annihilates $\underline{\mathrm{End}}_{\hat{R}}(\hat{M})$ and thus $\underline{\mathrm{End}}_R(M)$ for all $M \in \mathrm{MCM}(R)$. □

Remark 4.24. We note that Theorem 4.23 also follows from [6, Theorem 4.5]. Here, we gave a different proof.

Our main theorem relates the cohomology annihilator ideal to the conductor ideal and we have proved in this section a relation between the cohomology annihilator ideal of a ring with the cohomology annihilator of its completion. Naturally, we ask whether the same relation holds for the conductor ideal. There is a positive answer to this question due to Wiegand and Wiegand.

Lemma 4.25 (A.1 [66]). *Let R be a 1-dimensional Noetherian local ring. Then,*

$$\mathrm{co}(R) = \mathrm{co}(\hat{R}) \cap R.$$

As a corollary, we see that one can remove the completeness condition from Corollary 4.15.

Corollary 4.26. *Let R be a 1-dimensional Gorenstein local ring with reduced completion. Then, the cohomology annihilator ideal of R coincides with its conductor ideal.*

We finish this section with another application of the weak MCM extending property. We show that the cohomology annihilator ideal of a Gorenstein local ring R is contained in the cohomology annihilator ideal of its Henselization R^h. Note that R^h is also a one dimensional Cohen-Macaulay local ring with $R \subseteq R^h \subseteq \hat{R}$. We state the following lemma whose proof follows from [50, Proposition 10.5 and 10.7].

Lemma 4.27. *Let R be a Gorenstein local ring and $N \in \mathrm{MCM}(R^h)$. Then, there is an $M \in \mathrm{MCM}(R)$ such that N is a direct summand of $M \otimes_R R^h$.*

From this lemma, the following proposition follows.

Proposition 4.28. *Let R be a Gorenstein local ring. Then, the inclusion map from R to its Henselization carries the weak MCM extending property.*

As a corollary, we get

Corollary 4.29. *For a Gorenstein local ring R, we have*

$$\mathrm{ca}(R) \subseteq \mathrm{ca}(R^h).$$

4.5 A Milnor-Jung Type Formula For Branched Covers

In this section, we cover an application of the main theorem using the weak MCM extending property. More precisely, we show that a Milnor-Jung type formula holds for branched covers of algebraic plane curves. As before, let k be an algebraically closed field. Put $S = k[\![x_1, \ldots, x_n]\!]$ and $S^\sharp = S[\![y]\!]$. For a nonzero f in the maximal ideal of S and a natural number m, define $R = S/(f)$ and $R^\sharp = S^\sharp/(f + y^m)$. Then, $R^\sharp$ is called the *m-branched cover* of R.

Note that $R \cong R^\sharp/(y)$ so that every R-module has an $R^\sharp$-module structure. Let $\pi : R^\sharp \to R$ be the natural surjection so that for an $R^\sharp$-module N, $\pi N = N/yN$. We put $\Omega^\sharp = \Omega_{R^\sharp}$. We have

- If $M \in \mathrm{MCM}(R)$, then $\Omega^\sharp M \in \mathrm{MCM}(R^\sharp)$, and
- If $N \in \mathrm{MCM}(R^\sharp)$, then $\pi N \in \mathrm{MCM}(R)$

by keeping track of the depth of each module. The main ingredient of this section is the following due to Knörrer.

Lemma 4.30 ([47]). *Suppose that the characteristic of k is not 2 and let $m = 2$ so that $R^\sharp$ is the double-branched cover of R. Let $M \in \mathrm{MCM}(R)$ and $N \in \mathrm{MCM}(R^\sharp)$. Then,*

1. *$\pi\Omega^\sharp M \cong M \oplus \Omega_R M$ as R-modules, and*
2. *$\Omega^\sharp \pi N \cong N \oplus \Omega^\sharp N$ as $R^\sharp$-modules.*

Remark 4.31. For $m > 2$, the second isomorphism is no longer true. See [22, Proposition 2.3] for when exactly there is an isomorphism. However, the first isomorphism holds true by following the steps in the proof of Knörrer's lemma. In other words, π has the weak MCM extending property.

We are now ready to prove the main theorem of this section.

Theorem 4.32. *Let $R^\sharp$ be the m-branched cover of R. Then,*

$$\pi \mathrm{ca}(R^\sharp) \subseteq \mathrm{ca}(R).$$

When $m = 2$, the equality holds.

Proof. The first part follows from the fact that π has the weak MCM lifting property. We will show that when $m = 2$, the equality holds. Let $\pi(a) \in R = R^\sharp/yR^\sharp$ with $a \in R^\sharp$. Suppose that $\pi(a) \in \mathrm{ca}(R^\sharp)$. We show that $a \in \mathrm{ca}(R^\sharp)$. Since $\pi(a)\,\mathrm{Ext}^1_R(\pi M, \pi N) = 0$ for all $M, N \in \mathrm{MCM}(R^\sharp)$, we have $a\,\mathrm{Ext}^1_{R^\sharp/yR^\sharp}(M/yM, N/yN) = 0$. There are isomorphisms

$$\mathrm{Ext}^1_{R^\sharp/yR^\sharp}(M/yM, N/yN) \cong \mathrm{Ext}^2_{R^\sharp}(M/yM, N) \cong \mathrm{Ext}^1_{R^\sharp}(\Omega^\sharp(M/yM), N).$$

Since $\Omega^\sharp(M/yM) \cong \Omega^\sharp \pi M \cong M \oplus \Omega^\sharp M$, we get $a\,\mathrm{Ext}^1_{R^\sharp}(M, N) = 0$ for all $M, N \in \mathrm{MCM}(R^\sharp)$. Hence, $a \in \mathrm{ca}(1, R) \subseteq \mathrm{ca}(R)$. □

Example 4.33. Let $R^\sharp = k[\![x, y, z]\!]/(x^3 + y^4 + z^2)$, $R_1 = k[\![x, y]\!]/(x^3 + y^4)$ and $R_2 = k[\![x, z]\!]/(x^3 + z^2)$. Then, $R^\sharp$ is a double branched cover of R_1 and 4-branched cover of R_2. As we will see in Chapter 5, it is easy to compute $\mathrm{ca}(R_1) = (x^2, xy, y^2)$ and $\mathrm{ca}(R_2) = (x, z)$ using Corollary 4.15. Hence, we have $\mathrm{ca}(R^\sharp) = (x^2, xy, y^2, z)$ by Theorem 4.32. This example also shows that the theorem fails for $m > 2$. Indeed, $\mathrm{ca}(R^\sharp) \neq \pi_2^{-1}\mathrm{ca}(R_2) = (x, y, z)$ where $\pi_2 : R^\sharp \to R_2$ is the projection map.

The following theorem is a generalization of Milnor-Jung formula.

Theorem 4.34. *Let $R = k[\![x, y]\!]/(f)$ be the coordinate ring of a reduced curve singularity where* $\mathrm{char} k \neq 2$ *and $R^\sharp = k[\![x, y, z_1, \dots, z_l]\!]/(f + z_1^2 + \dots + z_l^2)$ for some $l \geq 0$. Then,*

$$\dim_k \frac{R^\sharp}{\mathrm{jac}(R^\sharp)} = 2 \dim_k \frac{R^\sharp}{\mathrm{ca}(R^\sharp)} - r + 1$$

where r is the number of branches of the curve f at its singular point.

Proof. When $l = 0$, the assertion follows from the discussion in Section 4.3. So, let $l \geq 1$. Then, we have $z_1, \dots, z_l \in \mathrm{jac}(R^\sharp) \subseteq \mathrm{ca}(R^\sharp)$. Hence, there are isomorphisms

$$R^\sharp / \mathrm{jac}(R^\sharp) \cong R/\mathrm{jac}(R), \quad R^\sharp/\mathrm{ca}(R^\sharp) \cong R/\mathrm{ca}(R)$$

as k-vector spaces, the latter of which is shown by Theorem 4.32. The theorem in the case $l = 0$ shows the assertion. □

Chapter 5

Examples

Even though the theory of cohomology annihilators has been well studied before this thesis, not many examples have been computed precisely in the literature. The focus has been on subsets of the cohomology annihilator ideal rather than exact computations. In this section, we list examples of cohomology annihilator ideal computations.

Example 5.1. Let $S = k[\![x]\!]$ and $f = x^{n+1} \in S$. Set $R = S/(f)$. Then, R has finitely many indecomposable maximal Cohen-Macaulay modules up to isomorphism: $M_0, \dots, M_n$ where $M_i = S/(x^{i+1})$. Then, the cohomology annihilator ideal is the intersection of $\underline{\operatorname{ann}}_R(M_i)$ where i ranges from 0 to n. On the other hand, since R is a hypersurface singularity, we have

$$\underline{\operatorname{End}}_R(M) \cong \operatorname{Ext}^2_R(M, M)$$

as R-modules. Therefore,

$$\operatorname{ca}(R) = \bigcap_{i=0}^{n} \operatorname{ann}_R \operatorname{Ext}^2_R(M_i, M_i).$$

Note that the complex

$$\dots \to R \xrightarrow{x^{i+1}} R \xrightarrow{x^{n-i}} R \xrightarrow{x^{i+1}} R \to 0$$

is a projective resolution of M_i. Applying $\operatorname{Hom}_R(-, M)$ to this complex, we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_i & \xrightarrow{x^{i+1}=0} & M_i & \xrightarrow{x^{n-i}} & M_i & \longrightarrow & \dots \\ & & & \nearrow & & & & \searrow & \\ & & (x^{i+1-n+i}) & & & & & & k[\![x]\!]/(x^{i+1}, x^{n-i}) \end{array}$$

and therefore,

$$\operatorname{ann}_R \operatorname{Ext}^2_R(M_i, M_i) = \operatorname{ann}_R k[\![x]\!]/(x^{i+1}, x^{n-i}) = (x^{i+1}, x^{n-i}) = (x^{\min(i+1,n-i)}).$$

Hence,

$$\operatorname{ca}(R) = \bigcap_{i=0}^{n} (x^{\min(i+1,n-i)}) = (x^{\lceil n/2 \rceil})$$

Our next example illustrates how to use matrix factorizations to compute stable annihilators. We provide a Singular code for computations.

Example 5.2. Let $S = k[\![x, y]\!]$ and f be one of the following polynomials:

- $x^2 + y^{n+1}$ where $n \geq 1$ $\quad (A_n)$,
- $x^2y + y^{n-1}$ where $n \geq 4$ $\quad (D_n)$,
- $x^3 + y^4$ $\quad (E_6)$,
- $x^3 + xy^3$ $\quad (E_7)$,
- $x^3 + y^5$ $\quad (E_8)$.

Then, $R = S/(f)$ is a simple singularity and a Gorenstein local ring of finite representation type [67]. Let us consider the E_6 case. The following list of irreducible matrix factorizations up to isomorphism is given in Yoshino's book.

$$\phi_1 = \begin{bmatrix} x & y \\ y^3 & -x^2 \end{bmatrix}, \qquad \psi_1 = \begin{bmatrix} x^2 & y \\ y^3 & -x \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} x & y^2 \\ y^2 & -x^2 \end{bmatrix}, \qquad \psi_2 = \begin{bmatrix} x^2 & y^2 \\ y^2 & -x \end{bmatrix}$$

$$\alpha = \begin{bmatrix} y^3 & x^2 & xy^2 \\ xy & -y^2 & x^2 \\ x^2 & -xy & -y^3 \end{bmatrix}, \qquad \beta = \begin{bmatrix} y & 0 & x \\ x & -y^2 & 0 \\ 0 & x & -y \end{bmatrix}$$

Thus, the irreducible maximal Cohen-Macaulay modules over $R = S/(x^3 + y^4)$ are

$$\begin{aligned} &\operatorname{coker}(\phi_1), \quad \operatorname{coker}(\psi_1) = \Omega \operatorname{coker}(\phi_1), \\ &\operatorname{coker}(\phi_2), \quad \operatorname{coker}(\psi_2) = \Omega \operatorname{coker}(\phi_2), \\ &\operatorname{coker}(\alpha), \quad \operatorname{coker}(\beta) = \Omega \operatorname{coker}(\alpha) \end{aligned}$$

and the cohomology annihilator of R is

$$\begin{aligned} \operatorname{ca}(R) &= \underline{\operatorname{ann}}_R \operatorname{coker}(\phi_1) \cap \underline{\operatorname{ann}}_R \operatorname{coker}(\phi_2) \cap \underline{\operatorname{ann}}_R \operatorname{coker}(\alpha) \\ &= \operatorname{ann}_R \operatorname{Ext}^2_R(\operatorname{coker}(\phi_1), \operatorname{coker}(\phi_1)) \cap \operatorname{ann}_R \operatorname{Ext}^2_R(\operatorname{coker}(\phi_2), \operatorname{coker}(\phi_2)) \cap \operatorname{ann}_R \operatorname{Ext}^2_R(\operatorname{coker}(\alpha), \operatorname{coker}(\alpha)) \end{aligned}$$

In order to compute these annihilators and their intersection, we use the following Singular code

```
LIB ‘‘homolog.lib’’;
LIB ‘‘primdec.lib’’;
ring S = 0, (x,y), ds;
ideal I = x3 + y4;
qring R = std(I);
 matrix phi1[2][2] = x, y , y3, -x;
 ideal a1 = std(Ann(Ext(2,phi_1,phi_1)));
print(a_1);
 matrix phi2[2][2] = x, y2, y2, -x;
 ideal a2 = std(Ann(Ext(2,phi_2,phi_2)))
```

```
print(a2);
 matrix alpha[3][3] = y3, x2, xy2, xy, -y2, x2, x2, -xy, -y3;
ideal a3 = std(Ann(Ext(2,alpha,alpha)))
print(a3);
ideal ca = std(intersect(a1,a2,a3));
print(ca);
```

to see that the cohomology annihilator ideal $\text{ca}(R)$ is generated by the three elements x^2, xy and y^2.

The next example illustrates the use of Corollary 4.15.

Example 5.3. Let $R = k[\![x, y]\!]/(x^3 - y^7)$. Then, R is a Gorenstein local ring of infinite representation type. Instead of checking the stable annihilators of maximal Cohen-Macaulay modules, we compute the conductor ideal. We have an isomorphism from R to $k[\![t^3, t^7]\!]$ which maps x to t^7 and y to x^3. So, it is enough to compute the conductor ideal of $k[\![t^3, t^7]\!]$. We look at the semigroup generated by 3 and 7.

0	1	2
3	4	5
6	7	8
9	10	11
12	13	14

This table tells us that every number after 11 belongs to the semigroup generated by 3 and 7. So, the conductor ideal of $k[\![t^3, t^7]\!]$ is generated by the three elements t^{12}, t^{13} and t^{14}. Hence,

$$\text{ca}(R) = (y^4, xy^2, x^2).$$

Example 5.4. Let $R = k[\![x, y, z]\!]/(x^3 - y^7 + z^2)$. Then, R is the double branched cover of the curve in Example 5.3. So, we get

$$\text{ca}(R) = (y^4, xy^2, x^2, z).$$

Example 5.5. Let k be an algebraically closed field of characteristic zero and G be a finite subgroup of $\text{SL}(2, k)$. The group G acts on $S = [\![x, y]\!]$ linearly. Consider the invariant ring

$$S^G = \{f \in S \ : \ g(f) = f \text{ for all } g \in G\}.$$

It is a Gorenstein local ring and it is isomorphic to a 2-dimensional ADE singularity. We also know that as an S^G-module S is maximal Cohen-Macaulay. In fact, we have an equality $\operatorname{add}_{S^G} S = \mathrm{MCM}(S^G)$ [33]. Therefore, we deduce that

$$\mathrm{ca}(S^G) = \underline{\mathrm{ann}}_{S^G}(S).$$

Example 5.6. Let k be a field and $X = (x_{ij})$ be the generic $n \times n$ square matrix. By $k[X]$, we denote the polynomial ring in variables x_{ij} where $i, j = 1, \ldots, n$. The determinant of X, $\det X$, is a polynomial in these variables. So, we can consider the $n^2 - 1$ dimensional hypersurface ring $R = k[X]/\det(X)$. The module $M = \operatorname{coker} X$ is a maximal Cohen-Macaulay module and we have an isomorphism

$$\underline{\mathrm{End}}_R(M) \cong R/J(R).$$

Hence, we conclude that $\underline{\mathrm{ann}}_R(M) = J(R)$. Thus, $\mathrm{ca}(R) \subseteq J(R)$. We already know that the Jacobian ideal is contained in $\mathrm{ca}(R)$ so we have the equality

$$\mathrm{ca}(R) = J(R).$$

Here, $J(R)$ is equal to the ideal generated by the submaximal minors of X [12, Lemma 2.6.].

Example 5.7. In this example, we compute the cohomology annihilator of the one dimensional torus invariant ring with weights $[2, 1, -2, -1]$. See [67, Chapter 16] and [64, Section 8] for reference. Let k be an algebraically closed field of characteristic zero. Suppose that the torus $T = k^*$ acts on a four dimensional vector space spanned by x_1, x_2, x_3, x_4 with weights $a_1 = 2, a_2 = 1, a_3 = -2, a_4 = -1$. That is, for any $t \in T$, $t \cdot x_i = t^{a_i} x_i$ and this action extends linearly. This action extends to an action on the polynomial ring $S = k[x_1, \ldots, x_n]$. Let $R = S^T$ be the invariant ring. As the weights add up to zero, R is Gorenstein.

Assigning $\deg x_i = a_i$, we can make S into a $\mathbb{Z}$-graded ring. Let S_c be the degree c part. Clearly, $S_0 = R$ and every S_c has an R-module structure. Moreover, S_1 and S_2 are maximal Cohen-Macaulay modules over R. Hence, $\mathrm{ca}(R) \subseteq \underline{\mathrm{ann}}_R S_1 \cap \underline{\mathrm{ann}}_R S_2$.

Note that for any $-2 \leq a, b \leq 2$, we have

$$\mathrm{Hom}_R(S_a, S_b) \cong S_{b-a}.$$

In particular, $S_a^* = \operatorname{Hom}(S_a, S_0) \cong S_{-a}$ and $\operatorname{End}_R(S_a) = R$. Hence, via the exact sequence

$$S_a^* \otimes_R S_a \to \operatorname{End}_R(S_a) \to \underline{\operatorname{End}}_R(S_a) \to 0$$

one can see the stable annihilator of S_a is equal to the image of the multiplication map $S_{-a} \otimes_R S_a \to R$. It is easy to show that

$$\begin{aligned}
S_0 &= R = k[x_1x_3, x_1x_4^2, x_2^2x_3, x_3x_4] \\
S_{-1} &= Rx_4 + Rx_2x_3 \\
S_1 &= Rx_2 + Rx_1x_4 \\
S_{-2} &= Rx_3 + Rx_4^2 \\
S_2 &= Rx_1 + Rx_2^2
\end{aligned}$$

and so

$$\begin{aligned}
\underline{\operatorname{ann}}_R S_1 &= S_{-1}S_1 = (x_2x_4, x_2^2x_3, x_1x_4^2, x_1x_2x_3x_4) \\
\underline{\operatorname{ann}}_R S_2 &= S_{-2}S_2 = (x_1x_3, x_1x_4^2, x_2^2x_3, x_2^2x_4^2).
\end{aligned}$$

Next, notice that we have a k-algebra isomorphism

$$R = k[x_1x_3, x_1x_4^2, x_2^2x_3, x_3x_4] \cong \frac{k[x, y, z, w]}{(xw^2 - yz)} = P.$$

Under this isomorphism, we have the identifications

$$\begin{aligned}
\underline{\operatorname{ann}}_R S_1 &\leftrightarrow (w, z, y) \\
\underline{\operatorname{ann}}_R S_2 &\leftrightarrow (x, y, z, w^2)
\end{aligned}$$

and hence

$$\underline{\operatorname{ann}}_R S_1 \cap \underline{\operatorname{ann}}_R S_2 \leftrightarrow (w, z, y) \cap (x, y, z, w^2) = (xw, y, z, w^2) = J(P).$$

We conclude that $\operatorname{ca}(R)$ is contained in the Jacobian ideal $J(R)$ of R. We also know that the Jacobian ideal is contained in the cohomology annihilator. So, we conclude that

$$\operatorname{ca}(R) = \underline{\operatorname{ann}}_R S_1 \cap \underline{\operatorname{ann}}_R S_2 = J(R).$$

Chapter 6

Orders of Finite Global Dimension and Their Stable Annihilators

We have seen previously in Chapter 4 that for a 1-dimensional complete reduced Gorenstein local ring R, the cohomology annihilator ideal coincides with the conductor ideal (Corollary 4.15). Our key observation was the fact that the conductor ideal is equal to the stable annihilator of the normalization $\bar{R}$ (Lemma 4.11). Hence, the main result of Chapter 4 can be stated as an equality between the cohomology annihilator ideal of R and the stable annihilator of its normalization. In this chapter, we generalize this result.

Let us recall which properties of $\bar{R}$ we have used in our theory so far. First and foremost, we required that it is finitely generated as an R-module. So, in this chapter we will focus our attention on module finite algebras. Secondly, we have used that it is a normal ring of dimension one. In particular, this implied that it is a regular and therefore a Cohen-Macaulay ring. This meant that it is maximal Cohen-Macaulay as an R-module. Therefore, we will consider module finite R-algebras which are maximal Cohen-Macaulay as R-modules and we will recover the main theorem of Chapter 4 in this generality.

We will start with definition of an order over a Cohen-Macaulay ring and we will give a brief overview of the theory. We will then present the main theorem of this chapter and state its geometric importance.

6.1 Orders over Cohen-Macaulay Rings

Let R be a Cohen-Macaulay local ring with canonical module ω_R and let Λ be a module-finite R-algebra.

Definition 6.1. We say that $M \in \text{mod}\Lambda$ is *maximal Cohen-Macaulay*, if it is maximal Cohen-Macaulay as an R-module. We denote the category of maximal Cohen-Macaulay modules by $\text{MCM}(\Lambda)$.

Remark 6.2. The category $\text{MCM}(\Lambda)$ is independent of the central subring R over which Λ is a finitely generated module.

Definition 6.3. We say that Λ is an *R-order* if $\Lambda \in \text{MCM}(\Lambda)$.

Definition 6.4. The Λ-bimodule $\omega_\Lambda = \text{Hom}_R(\Lambda, \omega_R)$ is called the *canonical module of* Λ.

Remark 6.5. It is clear that Λ is an R-order if and only if Λ^{op} is an R-order. Moreover, we have an exact duality between $\text{MCM}(\Lambda)$ and $\text{MCM}(\Lambda^{\text{op}})$:

$$D := \text{Hom}_R(-, \omega_R) : \text{MCM}(\Lambda) \to \text{MCM}(\Lambda^{\text{op}}).$$

Note that by the classical Hom-tensor adjunction, we have $\text{Hom}_R(-, \omega_R) \cong \text{Hom}_\Lambda(-, \omega_\Lambda)$ on $\text{MCM}(\Lambda)$.

The reason we care about orders over Cohen-Macaulay rings is geometric. One wishes to have a non-commutative version of resolution of singularities of a commutative ring. Recall that a resolution of singularities replaces a singular algebraic variety by a non-singular one staying isomorphic on a dense open set. Note that a resolution of an affine scheme is almost never another affine scheme. Therefore, it is quite complicated from the point of view of commutative algebra. One solution to overcome this difficulty is to start considering noncommutative rings and categories associated to these rings.

Unfortunately, there are several technical issues arise immediately if one considers arbitrary noncommutative rings. We refer to Section F of [49] for examples of such difficulties and a beautiful survey which aims to answer questions like "What on earth is non-commutative geometry?" Restricting to a smaller class of rings solves some of these problems. For instance, rings which are module-finite over their centers have much better homological properties which mimic the commutative case. Restricting further, one arrives at orders over Cohen-Macaulay local rings. The following theorem is well-known to experts and we refer to [41] for more on this.

Theorem 6.6. *Let Λ be an R-order. Then the following are equivalent.*

1. $\text{gldim}\,\Lambda_p = \dim R$ *for all* $p \in \text{Spec}(R)$.

2. $\operatorname{gldim}\Lambda_{\mathfrak{m}} = \dim R$ *for the maximal ideal* $\mathfrak{m}$ *of* R.

3. *Every maximal Cohen-Macaulay* Λ*-module is projective.*

4. Λ *has finite global dimension and* ω_Λ *is a projective* Λ*-module.*

Definition 6.7. We say that an R-order Λ is *non-singular* if one of the equivalent conditions of this theorem holds.

We will continue with more motivation from geometry. Let X be a smooth complex variety over $\mathbb{C}$ and $\Omega_{X/\mathbb{C}}$ be the cotangent bundle. Then, the sheaf $\omega_X = \bigwedge^{\dim X}\Omega_{X/\mathbb{C}}$ of top differential forms on X is called the canonical sheaf of X. If X is a Cohen-Macaulay normal variety, then one considers the smooth locus X_{reg} of X and defines the canonical sheaf ω_X to be $j_*\omega_{X_{\text{reg}}}$ where j is the open immersion $X_{\text{reg}} \hookrightarrow X$. In this case, the stalks of ω_X are canonical modules as defined in Section 2.

Now, let $\pi : Y \to X$ be a resolution of singularities of a normal variety X. We say that π is a *crepant resolution* if $\pi^*\omega_X = \omega_Y$. One should note that the natural framework of this setting is Gorenstein singularities. But one of the main motivations for considering crepant resolutions comes from the study of Calabi-Yau varieties. There, one wants to have a resolution of singularities of a singular Calabi-Yau variety in which the resolution is also Calabi-Yau.

Hence, combining our discussions above, one considers noncommutative versions of crepant resolutions of Gorenstein rings. We refer to Section K of [49] for the story behind the following definition.

Definition 6.8. Let R be a Gorenstein local normal domain and Λ be an R-order. We say that Λ is a *noncommutative crepant resolution* of R if $\Lambda \cong \operatorname{End}_R(M)$ for some reflexive R-module of M and $\operatorname{gldim}\Lambda < \infty$.

Remark 6.9. If Λ is a noncommutative crepant resolution of a Gorenstein ring, then $\omega_\Lambda = \Lambda^*$ is isomorphic to Λ. In particular, Λ a is non-singular order.

We are now ready to look at stable annihilators of orders over Gorenstein rings.

6.2 On the Stable Annihilators of Orders

Let R be a Gorenstein ring. In this section, we present the main results of this chapter. Namely, we prove that under mild assumptions, the stable annihilator of an R-order of finite global dimension agrees with the cohomology annihilator ideal of R up to radicals. We also show that in low dimensions, if we further assume the order is non-singular, there is actually an equality. We start with a simple lemma and we revisit several examples from Chapter 5.

Lemma 6.10. *Let R be a Gorenstein ring and $M, N \in \mathrm{MCM}(R)$. If N has a free summand and $\Lambda = \mathrm{End}_R(M \oplus N) \in \mathrm{MCM}(R)$, then*

$$\underline{\mathrm{ann}}_R \Lambda \subseteq \underline{\mathrm{ann}}_R M.$$

Proof. If N has a free summand, then M is a direct summand in Λ. So, the stable annihilator of Λ is contained in the stable annihilator of M. □

Example 6.11. Let R be a two dimensional invariant ring defined in Example 5.5. Let $S * G$ be the skew group algebra generated as an R-module by elements of the form (s, g) with $s \in S$ and $g \in G$ where the multiplication is defined by the rule

$$(s_1, g_1)(s_2, g_2) = (s_1 g_1(s_2), g_1 g_2).$$

It is well-known that R has only finitely many nonisomorphic indecomposable maximal Cohen-Macaulay modules [2]. Furthermore, $S * G \cong \mathrm{End}_R(S)$ is a noncommutative crepant resolution of R. We have

$$\underline{\mathrm{ann}}_R(S * G) = \underline{\mathrm{ann}}_R(S^{|G|}) = \underline{\mathrm{ann}}_R(S) = \mathrm{ca}(R).$$

So in this case, the stable annihilator of a noncommutative resolution is equal to the cohomology annihilator.

Example 6.12. Let R be as in Example 5.6. The matrix X defines a $k[X]$-linear map on the free $k[X]$-module G of rank n. We also have exterior powers $\bigwedge^j X : \bigwedge^j G \to \bigwedge^j G$ for $j = 1, \ldots, n$. Let $M_j = \mathrm{coker} \bigwedge^j X$. Then, each M_j is annihilated by $\det X$ and hence has an R-module structure. Morever, M_j is maximal Cohen-Macaulay for $j = 1, \ldots, n$. Note that M_n is free of rank 1.

Let $\Lambda = \mathrm{End}_R(\bigoplus_{j=1}^n M_j)$. Then, Λ is a noncommutative crepant resolution of R [13, Theorem 6.5]. In particular, $\Lambda \in \mathrm{MCM}(R)$ and therefore $\mathrm{ca}(R) \subseteq \underline{\mathrm{ann}}_R(\Lambda)$. On the other hand, as $\bigoplus_{j=1}^n M_j$ has a free summand, Lemma 6.10 yields that

$$\underline{\mathrm{ann}}_R \Lambda \subseteq \underline{\mathrm{ann}}_R M = J(R) \subseteq \mathrm{ca}(R) \subseteq \underline{\mathrm{ann}}_R \Lambda.$$

Hence, we have a noncommutative resolution Λ whose stable annihilator is equal to the cohomology annihilator of R.

Example 6.13. Let R be as in Example 5.7. Then, $\Lambda = \mathrm{End}(R \oplus S_1 \oplus S_2)$ is a noncommutative crepant resolution of R. Hence, we have $\mathrm{ca}(R) \subseteq \underline{\mathrm{ann}}\,\Lambda$. On the other hand, by Lemma 6.10, we have $\underline{\mathrm{ann}}\,\Lambda \subseteq \underline{\mathrm{ann}}\,S_1 \cap \underline{\mathrm{ann}}\,S_2 = \mathrm{ca}(R)$. Therefore, $\underline{\mathrm{ann}}\,\Lambda = \mathrm{ca}(R)$ and we have a noncommutative resolution whose stable annihilator annihilates the entire singularity category.

Theorem 6.14. *Let R be a Gorenstein ring and Λ be an R-order of finite global dimension δ. Then, for any $M \in \mathrm{mod}R$,*

$$(\underline{\mathrm{ann}}_R\,\Lambda)^{e+1} \subseteq \underline{\mathrm{ann}}_R\,M_\Lambda^{\mathrm{st}}$$

where $M_\Lambda = M \otimes_R \Lambda$, M_Λ^{st} its image in $\underline{\mathrm{MCM}}(R)$ and e its projective dimension as a Λ-module. In particular, if Λ has a free summand as an R-module, then

$$(\underline{\mathrm{ann}}_R\,\Lambda)^{\delta+1} \subseteq \mathrm{ca}(R) \subseteq \underline{\mathrm{ann}}_R\,\Lambda.$$

Proof. If M_Λ has projective dimension $e = 0$, then it is a projective Λ-module. Hence, it is a direct summand of a finite direct sum of copies of Λ. Note that this is true both as an R-module and as a Λ-module.

Now, suppose that M_Λ has projective dimension $e \geq 1$ and consider a short exact sequence

$$0 \to X \to G \to M_\Lambda \to 0$$

with G projective so that X has projective dimension $e - 1$. This short exact sequence produces a distinguished triangle

$$X^{\mathrm{st}} \to G \to (M_\Lambda)^{\mathrm{st}} \to \Omega^{-1} X^{\mathrm{st}}$$

in $\underline{\mathrm{MCM}}(R)$. Using the discussion before Lemma 3.4 we get the desired result by induction.

If Λ has a free summand as an R-module, then M is a summand of M_Λ. Hence,

$$(\underline{\mathrm{ann}}_R\,\Lambda)^{\delta+1} \subseteq (\underline{\mathrm{ann}}_R\,\Lambda)^{e+1} \subseteq \underline{\mathrm{ann}}_R\,M_\Lambda \subseteq \underline{\mathrm{ann}}_R\,M.$$

As M was arbitrary, this proves the desired result. □

Theorem 6.14 shows us that up to radicals, the stable annihilator of an R-order with a free summand coincides with the cohomology annihilator of R up to radicals for any Gorenstein ring. Therefore, we get the following geometric interpretation by Theorem 3.18.

Corollary 6.15. *Let R be a Gorenstein ring and Λ be an R-order of finite global dimension. Suppose either that R is an equicharacteristic excellent local ring of finite Krull dimension or that it is a localization of a finitely generated k-algebra of finite Krull dimension where k is a field. Then, the support of $\underline{\mathrm{End}}_R(\Lambda)$ in $\mathrm{Spec}(R)$ is equal to the singular locus of R.*

Theorem 6.14 is quite general. It does not depend on the Krull dimension of R nor the global dimension of Λ. Our next result shows that if we impose conditions on these invariants, we can actually get an equality.

Theorem 6.16. *Suppose that R is a Gorenstein ring and Λ is an R-order which contains R as a direct summand. Assume that $\dim R \leq 2$ and that Λ is non-singular. Then, we have the equality*

$$\mathrm{ca}(R) = \underline{\mathrm{ann}}_R(\Lambda).$$

Proof. Let M be a maximal Cohen-Macaulay R-module and consider $M_\Lambda = M \otimes_R \Lambda$. Since Λ has a free summand, the R-module $(M_\Lambda)^*$ contains M^* as a direct summand.
Let $0 \to X \to G \to M_\Lambda \to 0$ be an exact sequence with G a projective Λ-module. Dualizing this short exact sequence over R gives a long exact sequence

$$0 \to (M \otimes_R \Lambda)^* \to G^* \to X^* \to \mathrm{Ext}^1_R(M_\Lambda, R) \to 0$$

as $\mathrm{Ext}^1_R(G, R) = 0$. Since $\dim R \leq 2$ we conclude that $(M_\Lambda)^*$ is maximal Cohen-Macaulay by the depth lemma. As Λ and therefore Λ^{op} are non-singular orders, this implies that $(M_\Lambda)^*$ is a projective Λ^{op}-module. Hence, any $r \in R$ which stably annihilates Λ also stably annihilates $(M_\Lambda)^*$ and its summand M^*. Since $\underline{\mathrm{ann}}_R M = \underline{\mathrm{ann}}_R M^*$ for every $M \in \mathrm{MCM}(R)$, we conclude

$$\underline{\mathrm{ann}}_R \Lambda \subseteq \underline{\mathrm{ann}}_R M$$

which finishes the proof. □

We will now examine the cohomology annihilator ideal with respect to strong generators of the singularity category of R. Recall from Lemma 3.4 that if M is a strong generator with $\underline{\mathrm{MCM}}(R) = \langle M \rangle_n$ then $(\underline{\mathrm{ann}}_R M)^n$ is contained in $\mathrm{ca}(R)$. The next proposition follows from this fact combined with Lemma 6.10.

Proposition 6.17. *Let R be a Gorenstein ring and $M, N \in \mathrm{MCM}(R)$. Suppose that*

1. *N contains a free summand,*

2. *M is a strong generator of $\underline{\mathrm{MCM}}(R)$ with $\underline{\mathrm{MCM}}(R) = \langle M \rangle_n$, and*

3. *$\Lambda = \mathrm{End}_R(M \oplus N)$ is an R-order.*

Then, we have

$$(\underline{\mathrm{ann}}_R(\Lambda))^n \subseteq \mathrm{ca}(R) \subseteq \underline{\mathrm{ann}}_R(\Lambda).$$

Remark 6.18. Note that Proposition 6.17 does not require Λ to be of finite global dimension.

6.3 On the Stable Annihilators of Cluster Tilting Modules

In this section, we look at the support of the stable endomorphism ring of a cluster tilting module by comparing its stable annihilator with the cohomology annihilator ideal. We start with a lemma, followed by the definition of a cluster tilting module.

Lemma 6.19. *Let R be a Gorenstein ring and $M \in \mathrm{MCM}(R)$ so that $\Lambda = \mathrm{End}_R(M)$ is also an R-order. Then, $(\underline{\mathrm{ann}}_R\, M)^2 \subseteq \underline{\mathrm{ann}}_R\, \Lambda$.*

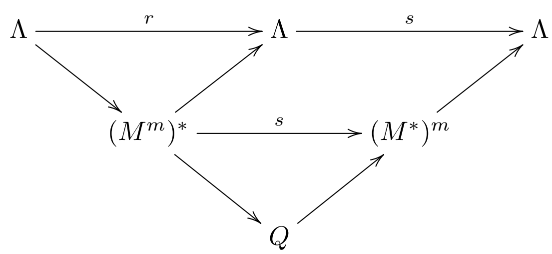
Proof. Let $r, s \in \underline{\mathrm{ann}}_R\, M$ so that there is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\ r\ } & M \\ & \searrow \quad \nearrow & \\ & F & \end{array}$$

with F being free over R of rank m. Applying $\mathrm{Hom}_R(M, -)$, we get

$$\begin{array}{ccc} \Lambda & \xrightarrow{\ r\ } & \Lambda \\ & \searrow \quad \nearrow & \\ & (M^*)^m & \end{array}$$

and extending this diagram we get



where Q is again a free R-module. This shows that $rs \in \underline{\mathrm{ann}}_R\, \Lambda$. □

Definition 6.20. Let R be a Gorenstein ring. A maximal Cohen-Macaulay R- module M is called *n-cluster tilting* if the three subcategories below coincide.

1. $\mathrm{add}_R M$,

2. $\{X \in \mathrm{MCM}(R) \ : \ \mathrm{Ext}^i_R(M, X) = 0 \text{ for } i = 1, \ldots, n\}$,

3. $\{Y \in \mathrm{MCM}(R) \ : \ \mathrm{Ext}^i_R(Y, M) = 0 \text{ for } i = 1, \ldots, n\}$,

Cluster tilting modules are not only important objects in representation theory but they also have important geometric significance. Indeed, they give noncommutative crepant resolutions. The following theorem is due to Iyama.

Theorem 6.21 (Theorem 5.2.1 [38])**.** *Let R be a Cohen-Macaulay local ring of dimension $d \geq 3$ and with canonical module ω_R and assume that R has an isolated singularity. Let M be a maximal Cohen-Macaulay R-module and set $\Lambda = \mathrm{End}_R(M)$. Then, M is $d-2$-cluster tilting if and only if M contains R and ω_R as direct summands and Λ is an R-order of global dimension d.*

In particular, Iyama's theorem says that if R is a 3-dimensional Gorenstein isolated singularity, then a maximal Cohen-Macaulay module gives a noncommutative crepant resolution if and only if it is a 1-cluster tilting module. The following theorem follows from combining Iyama's theorem with Theorem 6.14 and Lemma 6.19.

Theorem 6.22. *Let R be a Gorenstein isolated singularity with dimension $d \geq 3$ and M be a $d-2$-cluster tilting module. Then, we have*

$$(\underline{\mathrm{ann}}_R(M))^{2d+2} \subseteq \mathrm{ca}(R) \subseteq \underline{\mathrm{ann}}_R(M).$$

In particular, Theorem 6.22 says that the stable annihilator of a $d-2$-cluster tilting module and the cohomology annihilator ideal are the same up to radicals. That is, geometrically they define the same locus.

Corollary 6.23. *Let R be a Gorenstein ring of dimension d and M be a $d-2$-cluster tilting module. Suppose either that R is an equicharacteristic excellent local ring or that it is a localization of a finitely generated k-algebra where k is a field. Then, the support of $\underline{\mathrm{End}}_R(M)$ in $\mathrm{Spec}(R)$ is equal to the singular locus of R.*

Remark 6.24. Let $\mathbb{C}[\![x,y,z]\!]/(f)$ be a surface du Val singularity (*i.e.* double-branched cover of a ring from Example 5.2) and let $g \in \mathbb{C}[\![x,y,z,w]\!]$. Then a three dimensional complete local *compound du Val singularity* is a ring which is isomorphic to $\mathbb{C}[\![x,y,z,w]\!]/(f+wg)$. For a cluster tilting module M over a three dimensional compound du Val singularity, one associates the *contraction algebra* $\underline{\mathrm{End}}_R(M)$ [21, 20]. This is a finite dimensional algebra and it governs threefold flopping contractions on the schemes associated to these singularities. Corollary 6.23 shows that the annihilator of a contraction algebra is equal to the cohomology annihilator ideal. That is, any ring element which annihilates a contraction algebra annihilates the singularity category.

Chapter 7

The Relative Dominant Dimension of an Order

Let R be a Cohen-Macaulay local ring with canonical module ω_R and Λ be an R-order. In Chapter 6, we have seen two theorems relating the stable annihilator of Λ to the cohomology annihilator of R if R is a direct summand in Λ. Theorem 6.14 showed that if $\operatorname{gldim}\Lambda = \delta$, then

$$(\underline{\operatorname{ann}}_R \Lambda)^{\delta+1} \subseteq \operatorname{ca}(R) \subseteq \underline{\operatorname{ann}}_R \Lambda.$$

If we further assume that $\dim R \leq 2$ and Λ is a non-singular order, Theorem 6.16 showed us that there is indeed an equality

$$\underline{\operatorname{ann}}_R \Lambda = \operatorname{ca}(R).$$

We start this chapter by showing that if $\dim R \leq 2$, we can improve Theorem 6.14 slightly better even when Λ is not non-singular. We note that the proof of Theorem 6.16 uses the fact that if Λ is non-singular, then every maximal Cohen-Macaulay Λ-module is projective. The reason behind this fact is Auslander-Buchsbaum formula [3]. Recall that if A is a commutative Noetherian local ring and M is a nonzero finitely generated A-module of finite projective dimension, then the Auslander-Buchsbaum formula states that

$$\operatorname{pd}_A M + \operatorname{depth} M = \operatorname{depth} A.$$

In this section, we are dealing with a Cohen-Macaulay ring R. So, for us, the Auslander-Buchsbaum formula reads

$$\operatorname{pd}_R M + \operatorname{depth} M = \dim R.$$

Here is a version of this formula for non-singular orders.

Theorem 7.1 ([40], Proposition 2.3)**.** *Let Λ be an R-order with $\operatorname{gldim}\Lambda = \dim R$. For any $M \in \operatorname{mod}\Lambda$, we have*

$$\operatorname{pd}_\Lambda M + \operatorname{depth} M = \dim R.$$

The following theorem is a generalization of this and is due to Josh Stangle. It says that if a maximal Cohen-Macaulay Λ-module has finite projective dimension, then the projective dimension is bounded by that of the canonical module.

Theorem 7.2 ([62], Theorem 4.1.2)**.** *Let Λ be an R-order with $\operatorname{pd}_{\Lambda^{\mathrm{op}}} \omega_\Lambda = n$. Then, for all $M \in \operatorname{mod}\Lambda$ with $\operatorname{pd}_\Lambda M < \infty$ we have*

$$\dim R \leq \operatorname{pd}_\Lambda M + \operatorname{depth} M \leq \dim R + n.$$

In particular, if M is a maximal Cohen-Macaulay Λ-module with finite projective dimension, we have $\operatorname{pd}_\Lambda M \leq n$.

It is worth noting that the projective dimension of the bimodule ω_Λ is the same on both sides. We give a proof for this.

Lemma 7.3. *Let Λ be an R-order and assume ω_Λ has finite projective dimension. Then, the projective dimension of ω_Λ as a Λ-module equals its projective dimension as a Λ^{op}-module.*

Proof. According to Stangle's thesis Lemma 4.1.4 we have $\operatorname{Ext}^i_\Lambda(\omega_\Lambda, \Lambda) = 0$ for all $i > \operatorname{pd}_{\Lambda^{op}} \omega_\Lambda$. Since $\operatorname{Ext}^i_\Lambda(\omega_\Lambda, \Lambda) \neq 0$ for $i = \operatorname{pd}_\Lambda \omega_\Lambda$, we conclude that $\operatorname{pd}_\Lambda \omega_\Lambda \leq \operatorname{pd}_{\Lambda^{op}} \omega_\Lambda$. By symmetry, we conclude the proof. □

We are now ready to state the following theorem. The proof is a combination of the techniques from the proofs of Theorem 6.16 and Theorem 6.14. So, we do not include it in here.

Theorem 7.4. *Suppose that R is a Gorenstein ring and Λ is an R-order of finite global dimension which contains R as a direct summand. Assume that $\dim R \leq 2$ and that $\operatorname{pd}_\Lambda \omega_\Lambda \leq n$. Then, we have*

$$(\underline{\operatorname{ann}}_R \Lambda)^{n+1} \subseteq \operatorname{ca}(R) \subseteq \underline{\operatorname{ann}}_R(\Lambda).$$

In this chapter, we are going to be interested in orders with canonical modules of positive projective dimension.

7.1 The Relative Injective Dimension and The Relative Dominant Dimension

Let R be a Cohen-Macaulay local ring with canonical module ω_R and Λ be an R-order. Recall that the functor $D : \operatorname{MCM}(\Lambda) \to \operatorname{MCM}(\Lambda^{\mathrm{op}})$ defined by $D(X) = \operatorname{Hom}_R(X, \omega_R)$ is a duality.

Definition 7.5. We say that a Λ-module M is *relatively injective* if it is an injective object in $\operatorname{MCM}(\Lambda)$. That is, if

1. $M \in \operatorname{MCM}(\Lambda)$ and,
2. $\operatorname{Ext}^n_\Lambda(-, M) = 0$ on $\operatorname{MCM}(\Lambda)$ for every $n > 0$.

We denote the category of relatively injective Λ-modules by $\mathcal{RI}(\Lambda)$.

It is easy to see that $\mathcal{RI}(\Lambda)$ is closed under direct summands and finite direct sums. Hence, for any $M \in \mathcal{RI}(\Lambda)$, we have that $\operatorname{add}_\Lambda(M) \subseteq \mathcal{RI}(\Lambda)$ where $\operatorname{add}_\Lambda(M)$ denotes the smallest subcategory of $\mathcal{RI}(\Lambda)$ containing M and closed under finite direct sums, direct summands and isomorphisms.

Lemma 7.6. *1. ω_Λ is a relatively injective module. Hence, $\operatorname{add}_\Lambda \omega_\Lambda \subseteq \mathcal{RI}(\Lambda)$.*

2. If M is a relatively injective module, then $M \in \operatorname{add}_\Lambda \omega_\Lambda$. Thus, $\operatorname{add}_\Lambda \omega_\Lambda = \mathcal{RI}(\Lambda)$.

Proof. The canonical module ω_Λ is a maximal Cohen-Macaulay module by [9, Proposition 3.3.3]. Now, let $Y \in \operatorname{MCM}(\Lambda)$ and P be a projective resolution of Y as a Λ-module. Then, we have

$$\operatorname{Hom}_\Lambda(P, \omega_\Lambda) \cong \operatorname{Hom}_R(P \otimes_\Lambda \Lambda, \omega_R) \cong \operatorname{Hom}_R(P, \omega_R).$$

Hence, $\mathrm{Ext}^i_\Lambda(Y, \omega_\Lambda) \cong \mathrm{Ext}^i_R(Y, \omega_R) \cong 0$ as Y is maximal Cohen-Macaulay as an R-module. Therefore, ω_Λ is a relatively injective Λ-module.

Now, let M be any relatively injective module and consider $DM \in \mathrm{MCM}(\Lambda^{\mathrm{op}})$. Consider a short exact sequence

$$0 \to L \to G \to DM \to 0$$

with a projective Λ^{op}-module G. Then, by depth lemma L is also maximal Cohen-Macaulay, so is DL. Applying D to the above short exact sequence we get a short exact sequence

$$0 \to M \to DG \to DL \to \mathrm{Ext}^1_R(DM, \omega_R) \cong 0$$

as DM is maximal Cohen-Macaulay. Since M is relatively injective, we have $\mathrm{Ext}^1_\Lambda(DL, M) = 0$. So, this short exact sequence splits and we get that M is a direct summand of DG which is in $\mathrm{add}_\Lambda \omega_\Lambda$. □

As an immediate consequence of Lemma 7.6, we conclude

Lemma 7.7. *The duality D takes relatively injective Λ-modules to projective Λ^{op}-modules and projective Λ-modules to relatively injective Λ^{op}-modules.*

Remark 7.8. Note that if we consider ω_Λ as a Λ-module, then its image under D in $\mathrm{MCM}(\Lambda^{\mathrm{op}})$ is Λ^{op}. Every Λ^{op}-module has a projective resolution and we can dualize that resolution to conclude that every maximal Cohen-Macaulay Λ-module has a relatively injective coresolution.

Definition 7.9. *The relative injective dimension* of a maximal Cohen-Macaulay Λ-module M, denoted $\mathrm{rid}_\Lambda M$, is the projective dimension of DM as a Λ^{op}-module.

Remark 7.10. Let A be a ring that is both left and right Noetherian. Suppose that A has finite injective dimension as a right and left module over itself. Then, Zaks proves that these injective dimensions coincide [68]. Lemma 7.3 is a version of Zaks's theorem for relative injective dimensions for orders.

Remark 7.11. If Λ is a non-singular order, then by [**?**] we have

$$\mathcal{RI}(\Lambda) = \mathrm{MCM}(\Lambda) = \mathrm{proj}(\Lambda).$$

That is, non-singular orders are self-relatively-injective. The converse also holds if we assume that Λ has finite global dimension. Indeed, if Λ has finite global dimension with a projective canonical module, then it is non-singular.

We now recall the definition of *dominant dimension* for finite dimensional algebras. Let A be a finite dimensional (or more generally an artinian) algebra. Consider A as a right module over itself and let

$$0 \to A \to I^0 \to I^1 \to \ldots \to I^m \to \ldots$$

be a minimal injective resolution of A. Then, the dominant dimension of A is defined to be the largest integer k or ∞ such that $I^0, I^1, \ldots, I^{k-1}$ are projective.

Dominant dimension was introduced by Nakayama in his study of complete homology theory and has been studied intensively over the decades. For instance, it has been used to classify finite dimensional algebras of finite representation type [5]. Unlike other notions of dimension, it is desired that the dominant dimension is large. This is related to self-orthogonality which plays an important role in Iyama's higher Auslander-Reiten theory and Rouquier's cover theory (See [26] and references within). We are interested in the notion of dominant dimension inside the category of maximal Cohen-Macaulay modules over an order.

Definition 7.12. Let $0 \to \Lambda \to I^0 \to I^1 \ldots \to I^m \to 0$ be a minimal relatively injective coresolution of Λ. Then, the *relative dominant dimension of* Λ, denoted $\operatorname{rdd}(\Lambda)$, is defined to be the largest integer k or ∞ such that $I^0, I^1, \ldots, I^{k-1}$ are projective.

Remark 7.13. If Λ is non-singular, then $\operatorname{rdd}\Lambda = \infty$.

The proof of the following proposition is immediate by definition and by Theorem **??**.

Proposition 7.14. *If Λ has finite relative dominant dimension, then*

$$\operatorname{rdd}\Lambda \leq \operatorname{rid}\Lambda \leq \operatorname{gldim}\Lambda \leq \dim R + \operatorname{rid}\Lambda.$$

7.2 The Relative Dominant Dimension of an Endomorphism Ring

In this section, we discuss the relative dominant dimension of an endomorphism ring. In particular, we prove the following theorem.

Theorem. *Let R be a maximal Cohen-Macaulay local ring with canonical module ω_R, M be maximal Cohen-Macaulay R-module and $\Lambda = \operatorname{End}_R(M)$.*

1. *If R is of dimension 1 or 2, then Λ is an R-order and* $\operatorname{rdd}\Lambda \geq 2$.

2. *If M is maximal Cohen-Macaulay and Λ is an R-order of dimension greater than 2, we have* $\operatorname{rdd}\Lambda \geq \dim R - 2$.

Let R be a maximal Cohen-Macaulay local ring with canonical module ω_R, M be a finitely generated R-module and put $\Lambda = \operatorname{End}_R(M)$. Consider $DM = \operatorname{Hom}_R(M, \omega_R)$ and let $P \to DM$ be a projective resolution of DM as an R-module:

$$P: \quad \ldots \to R^{n_2} \to R^{n_1} \to R^{n_0} \to DM \to 0.$$

Applying D once again, we obtain a relatively injective coresolution $M \to DP$:

$$DP: \quad 0 \to \omega_R^{n_0} \to \omega_R^{n_1} \to \omega_R^{n_2} \to \ldots \quad .$$

Next, we apply $\operatorname{Hom}_R(M, -)$ to this complex to get $\operatorname{End}_R(M) \to \operatorname{Hom}_R(M, DP)$:

$$\operatorname{Hom}_R(M, DP): \quad 0 \to \operatorname{Hom}_R(M, \omega_R^{n_0}) \to \operatorname{Hom}_R(M, \omega_R^{n_1}) \to \operatorname{Hom}_R(M, \omega_R^{n_2}) \to \ldots$$

which may have positive cohomology.

Remark 7.15. If we assume that M contains ω_R as a direct summand, then $\operatorname{Hom}_R(M, \omega_R)$ is a direct summand in Λ. Therefore, $\operatorname{Hom}_R(M, \omega_R^{n_j})$ are projective as Λ-modules.

Remark 7.16. Assume Λ is an R-order. If we assume that M contains R as a direct summand, then we have $M \cong \operatorname{Hom}_R(R, M)$ is a direct summand in Λ. So, $DM = \operatorname{Hom}_R(M, \omega_R)$ is a direct summand in $\omega_\Lambda = \operatorname{Hom}_R(\Lambda, \omega_R)$. Thus, it is relatively injective. So are $\operatorname{Hom}_R(M, \omega_R^{n_j})$.

Therefore, if M contains R and ω_R as a direct summand, then the complex $\operatorname{Hom}_R(M, DP)$ above consists of projective and relatively injective modules. As mentioned above, $\operatorname{Hom}_R(M, DP)$ may have positive cohomology. Let us look at this cohomology closely.

The functor $\operatorname{Hom}_R(M, -)$ is left exact and D is an exact duality. We know that the zeroth cohomology of P is isomorphic to DM by definition. Hence, the zeroth homology of $\operatorname{Hom}_R(M, DP)$ is isomorphic to $\operatorname{Hom}_R(M, M)$. So, we have an exact sequence

$$0 \to \Lambda \to \operatorname{Hom}_R(M, \omega_R^{n_0}) \to \operatorname{Hom}_R(M, \omega_R^{n_1})$$

This means that if $\dim R = 1$ or 2, then Λ is maximal Cohen-Macaulay as an R-module for being a second syzygy of the cokernel. That is, Λ is an R-order. Hence, we can talk about its relative dominant dimension. We proved

Proposition 7.17. *Let R be a maximal Cohen-Macaulay local ring with canonical module ω_R, M be a finitely generated R-module and $\Lambda = \mathrm{End}_R(M)$. If $\dim_R = 1$ or 2, then Λ is an R-order and $\operatorname{rdd} \Lambda \geq 2$.*

We continue looking at the cohomology of $\mathrm{Hom}_R(M, DP)$. By the classical Hom-tensor adjunction and by the commutativity of R, we have

$$\begin{aligned}\mathrm{Hom}_R(M, DP) &= \mathrm{Hom}_R(M, \mathrm{Hom}_R(P, \omega_R)) \\ &\cong \mathrm{Hom}_R(M \otimes_R P, \omega_R) \\ &\cong \mathrm{Hom}_R(P \otimes_R M, \omega_R) \\ &\cong \mathrm{Hom}_R(P, \mathrm{Hom}_R(M, \omega_R)) = \mathrm{Hom}_R(P, DM) \quad .\end{aligned}$$

Hence, the cohomology of $\mathrm{Hom}_R(M, DP)$ is isomorphic to $\mathrm{Ext}^*_R(DM, DM) \cong \mathrm{Ext}^*_R(M, M)$. The following theorem is due to Dao and Huneke.

Theorem 7.18 ([16], Theorem 3.2)**.** *Let R be a Cohen-Macaulay ring of dimension $d \geq 3$ and M be a maximal Cohen-Macualay R-module. Then, $\mathrm{End}_R(M)$ is an R-order if and only if $\mathrm{Ext}^i_R(M, M) = 0$ for $i = 1, \ldots, d-2$.*

We summarize the discussion of this subsection in a theorem.

Theorem 7.19. *Let R be a maximal Cohen-Macaulay local ring with canonical module ω_R, M be a maximal Cohen-Macaulay R-module and $\Lambda = \mathrm{End}_R(M)$.*

1. *If R is of dimension 1 or 2, then Λ is an R-order and $\operatorname{rdd} \Lambda \geq 2$.*
2. *If M is maximal Cohen-Macaulay and Λ is an R-order of dimension greater than 2, we have $\operatorname{rdd} \Lambda \geq \max\{2, \dim R - 2\}$.*

7.3 Special Tilting Modules for Orders with Finite Relative Injective Dimension

In this section, we are studying special tilting modules. In the case of Artinian algebras, these modules have been studied in [52, 55, 14]. We start with a brief overview of tilting theory.

Tilting theory, originated in the 1970s by the works of Bernstein, Gelfand, Ponomarev [8, 29] and Auslander, Platzeck and Reiten [4] is a tool to construct equivalences between categories. The word tilting comes from the reflection functors in the [8] and their effect on a root system by a change of basis that tilts the axes relative to the positive roots. Originally studied in the context of Morita theory in the representation theory of finite dimensional algebras, tilting theory is now an essential tool in a Morita theory of derived categories by the work of Rickard [58] and of triangulated categories by the work of Keller [46]. We refer to the *Handbook of Tilting Theory* for more on tilting theory [1].

Let R be a Cohen-Macaulay local ring and let Λ be an R-order.

Definition 7.20. Let $k \geq 0$.

1. We say that $T \in \mathrm{MCM}(\Lambda)$ is a *k-tilting module* if
   (a) $\mathrm{pd}_\Lambda T \leq k$,
   (b) $\mathrm{Ext}^i_\Lambda(T, T) = 0$ for $i \geq 1$, and
   (c) There is an exact sequence $0 \to \Lambda \to X_0 \to X_1 \to \ldots \to X_k \to 0$ where $X_j \in \mathrm{add}_\Lambda T$ for $j = 0, 1, \ldots, k$.
2. We say that $C \in \mathrm{MCM}(\Lambda)$ is a *k-cotilting module* if
   (a) $\mathrm{rid}_\Lambda C \leq k$,
   (b) $\mathrm{Ext}^i_\Lambda(T, T) = 0$ for $i \geq 1$, and
   (c) There is an exact sequence $0 \to Y^k \to \ldots \to Y_1 \to Y_0 \to \omega_\Lambda \to 0$ where $Y_j \in \mathrm{add}_\Lambda C$ for $j = 0, 1, \ldots, k$.

If T is a tilting module, then there is a derived equivalence between $\Gamma = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$ and Λ [32].

The following lemma will give us our main ingredient in constructing our tilting and cotilting modules.

Lemma 7.21. *Let Λ be an R-order and*

$$0 \to \Lambda \to I^0 \to I^1 \to \ldots \to I^m$$

be the first m terms in a relatively injective coresolution of Λ. Then $K_j := \mathrm{Im}(I^{j-1} \to I^j)$ is maximal Cohen-Macaulay for $j = 1, \ldots, m$.

Proof. If the relative injective coresolution is infinite, then each K_j is maximal Cohen-Macaulay by depth lemma. If the relative injective coresolution is finite, then one can use descending induction starting from the last term in the coresolution in order to conclude again by depth lemma. □

Theorem 7.22. *Let $0 \to \Lambda \to I^0 \to I^1 \to \ldots \to I^{k-1}$ be an exact sequence where I^j are projective and relatively injective. Then*

$$T_j := I^0 \oplus I^1 \oplus \ldots \oplus I^{j-1} \oplus K_j$$

is a j-tilting module where $K_j := \operatorname{Im}(I^{j-1} \to I^j)$ for $j = 1, \ldots, k-1$.

Proof. Suppose K_j is as in the proposition and consider the exact sequence

$$0 \to \Lambda \to I^0 \to I^1 \to \ldots \to I^{j-1} \to K_j \to 0$$

Firstly, it shows that $\operatorname{pd}_\Lambda K_j = \operatorname{pd}_\Lambda T_j \leq j$ and secondly it gives an $\operatorname{add}_\Lambda T_j$-coresolution of Λ. So, in order to show T_j is tilting, it is enough to show $\operatorname{Ext}^i_\Lambda(T_j, T_j) = 0$ for all $i \geq 1$. As $I^0, \ldots, I^{j-1}$ are projective and relative injective, we can reduce the problem to the vanishing of $\operatorname{Ext}^i_\Lambda(K_j, K_j)$ for $i \geq 1$. The proof is by induction on j.

Let $j = 1$ so that we have a short exact sequence $0 \to \Lambda \to I^0 \to K_1 \to 0$. Applying $\operatorname{Hom}_\Lambda(K_1, -)$ to this exact sequence yields an exact sequence

$$\operatorname{Ext}^i_\Lambda(K_1, I^0) \to \operatorname{Ext}^i_\Lambda(K_1, K_1) \to \operatorname{Ext}^{i+1}_\Lambda(K_1, \Lambda) \quad i \geq 1.$$

Note that the term on the left is zero as I^0 is relatively injective and the term on the right is zero as $\operatorname{pd}_\Lambda K_1 \leq 1$. Now, suppose that our desired result holds for $j-1$. We have a short exact sequence

$$0 \to K_{j-1} \to I^{j-1} \to K_j \to 0 \quad .$$

Applying $\operatorname{Hom}_\Lambda(-, K_{j-1})$ to this short exact sequence gives an exact sequence

$$\operatorname{Ext}^i_\Lambda(K_{j-1}, K_{j-1}) \to \operatorname{Ext}^{i+1}_\Lambda(K_j, K_{j-1}) \to \operatorname{Ext}^{i+1}_\Lambda(I^{j-1}, K_{j-1}) \quad i \geq 1.$$

The term on the left is zero by induction and the term on the right is zero because I^{j-1} is projective. So, we conclude $\operatorname{Ext}^{i+1}_\Lambda(K_j, K_{j-1}) = 0$ for $i \geq 1$. Now, to our short exact sequence we apply $\operatorname{Hom}_\Lambda(K_j, -)$

to get

$$\mathrm{Ext}^i_\Lambda(K_j, I^{j-1}) \to \mathrm{Ext}^i_\Lambda(K_j, K_j) \to \mathrm{Ext}^{i+1}_\Lambda(K_j, K_{j-1}) \quad i \geq 1.$$

We found that the term on the right is zero and the term on the left is zero because I^{j-1} is relatively injective. This finishes the proof. □

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