The Navier-Stokes strain equation with applications to enstrophy growth and global regularity

by

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Abstract

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This thesis derives an evolution equation for the symmetric part of the gradient of the velocity (the strain tensor) for the incompressible Navier-Stokes equation. We use this equation to obtain a simplified identity for the growth of enstrophy for mild solutions that depends only on the determinant of the strain tensor, not on the nonlocal interaction of the strain tensor with the vorticity. The resulting identity allows us to prove a new family of scale-critical necessary and sufficient conditions for blow-up of the solution in finite time $T_{max} < +\infty$, which depend only on the history of the positive part of the second eigenvalue of the strain matrix. Since this matrix is trace-free, this severely restricts the geometry of any finite-time blow-up. This regularity criterion provides analytic evidence of the numerically observed tendency of the vorticity to align with the eigenvector corresponding to the middle eigenvalue of the strain matrix. We then consider a vorticity approach to the question of almost two-dimensional initial data, using this same identity for enstrophy growth and an isometry relating the third column of the strain matrix to the first two components of the vorticity. We prove a new global regularity result for initial data with two components of the vorticity sufficiently small. Finally, we prove the existence and stability of blowup for a toy model ODE of the strain equation.

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Contents

Chapter 1

Introduction

The Navier-Stokes equation, which governs viscous, incompressible flow, is one of the most fundamental equations in fluid dynamics. The incompressible Navier-Stokes equation is given by

$$\begin{aligned} \partial_t u - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f, \\ \nabla \cdot u = 0, \end{aligned} \tag{1.1}$$

where $u \in \mathbb{R}^3$ denotes the velocity, p the pressure, f the external force, and $\nu > 0$ is the viscosity. The pressure is completely determined in terms of u and f, by taking the divergence of both sides of the equation, which yields

$$-\Delta p = \sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} - \nabla \cdot f. \tag{1.2}$$

We refer here to the Navier-Stokes equation, rather than the Navier-Stokes equations, because this PDE is best viewed not as a system of equations, but as an evolution equation on the space of divergence free vector fields.

Two other objects which play a crucial role in Navier-Stokes analysis are the vorticity and the strain, which represent the anti-symmetric and symmetric parts of the $\nabla \otimes u$ respectively. The vorticity is given by taking the curl of the velocity, $\omega = \nabla \times u$, while the strain is the matrix given by $S_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$. The vorticity ω is related to the anti-symmetric part of the gradient, $A_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right)$ by

$$A = \frac{1}{2}\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}. \tag{1.3}$$

The evolution equation for vorticity is given by

$$\partial_t \omega - \nu\Delta\omega + (u \cdot \nabla)\omega - S\omega = \nabla \times f, \tag{1.4}$$

and the evolution equation for the strain is given by

$$\partial_t S + (u \cdot \nabla) S - \nu \Delta S + S^2 + \frac{1}{4}\omega \otimes \omega - \frac{1}{4}|\omega|^2 I_3 + \text{Hess}(p) = \nabla_{sym} f. \tag{1.5}$$

In addition to the curl operator, it is also useful to define a differential operator that maps a vector to the symmetric part of it's gradient tensor: $\nabla_{sym}(v)_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)$. Note that $S = \nabla_{sym} u$.

Before we proceed further we should define a number of spaces. For all $s \in \mathbb{R}, H^s\left(\mathbb{R}^3\right)$ will be the Hilbert space with norm

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}^3} \left(1 + (2\pi|\xi|)^{2s}\right) |\hat{f}(\xi)|^2 d\xi = \left\| \left(1 + (2\pi|\xi|)^{2s}\right)^{\frac{1}{2}} \hat{f} \right\|_{L^2}^2, \tag{1.6}$$

and for all $-\frac{3}{2} < s < \frac{3}{2}$, $\dot{H}^s\left(\mathbb{R}^3\right)$ will be the homogeneous Hilbert space with norm

$$\|f\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^3} (2\pi|\xi|)^{2s} |\hat{f}(\xi)|^2 d\xi = \left\| (2\pi|\xi|)^s \hat{f} \right\|_{L^2}^2. \tag{1.7}$$

Note that when referring to $H^s\left(\mathbb{R}^3\right), \dot{H}^s\left(\mathbb{R}^3\right)$, or $L^p\left(\mathbb{R}^3\right)$, the $\mathbb{R}^3$ will often be omitted for brevity's sake. All Hilbert and Lebesgue norms are taken over $\mathbb{R}^3$ unless otherwise specified. Finally we will define the subspace of divergence free vector fields inside each of these spaces.

Definition 1.1. *For all $s \in \mathbb{R}$ define $H^s_{df} \subset H^s\left(\mathbb{R}^3; \mathbb{R}^3\right)$ by*

$$H^s_{df} = \left\{u \in H^s\left(\mathbb{R}^3; \mathbb{R}^3\right) : \xi \cdot \hat{u}(\xi) = 0, \textit{almost everywhere } \xi \in \mathbb{R}^3\right\}. \tag{1.8}$$

For all $-\frac{3}{2} < s < \frac{3}{2}$, define $\dot{H}^s_{df} \subset \dot{H}^s\left(\mathbb{R}^3; \mathbb{R}^3\right)$ by

$$\dot{H}^s_{df} = \left\{u \in \dot{H}^s\left(\mathbb{R}^3; \mathbb{R}^3\right) : \xi \cdot \hat{u}(\xi) = 0, \textit{almost everywhere } \xi \in \mathbb{R}^3\right\}. \tag{1.9}$$

For all $1 \leq q \leq +\infty$, define $L^q_{df} \subset L^q\left(\mathbb{R}^3; \mathbb{R}^3\right)$ by

$$L^q_{df} = \left\{u \in L^q\left(\mathbb{R}^3; \mathbb{R}^3\right) \textit{ such that for all } f \in C_c^\infty\left(\mathbb{R}^3\right), \langle u, \nabla f\rangle = 0.\right\} \tag{1.10}$$

Note that this definition makes sense, because in $u \in H^s$ or $u \in \dot{H}^s$ implies that $\hat{u}(\xi)$ is well defined almost everywhere. We will also note that $\dot{H}^0 = L^2$, so we have two different definitions of L^2_{df}. This is not a problem as both definitions are equivalent. We will also note that throughout this thesis, we will take the magnitude of a matrix, $M \in \mathbb{R}^{3\times 3}$, to be the Euclidean norm

$$|M|^2 = \sum_{i,j=1}^{3} M_{ij}^2. \tag{1.11}$$

The standard notion of weak solutions to PDEs corresponds to integrating against test functions. Leray first proved the existence of just such weak solutions to the Navier-Stokes equation satisfying a certain energy inequality [36]. To be precise, Leray defined weak solutions, sometimes referred to as Leray-Hopf weak solutions, to the Navier-Stokes equation as follows.

Definition 1.2 (Leray weak solutions)**.** *Suppose $u^0 \in L^2_{df}$. Then $u \in L^\infty\left([0, +\infty); L^2_{df}\right) \cap$*

$L^2\left([0,+\infty);\dot{H}^1\left(\mathbb{R}^3\right)\right)$ *is a Leray weak solution to the Navier-Stokes equation if for all* $\phi \in C_c^\infty\left((0,+\infty)\times\mathbb{R}^3;\mathbb{R}^3\right), \nabla\cdot\phi = 0,$

$$\int_0^{+\infty}\int_{\mathbb{R}^3}\left(u\cdot\partial_t\phi + \nu u\cdot\Delta\phi + u\otimes u : \nabla\phi\right)dxdt = 0, \tag{1.12}$$

and for all $t > 0$

$$\frac{1}{2}\|u(\cdot,t)\|_{L^2}^2 + \nu\int_0^t \|u(\cdot,\tau)\|_{\dot{H}^1}^2 d\tau \leq \frac{1}{2}\|u^0\|_{L^2}^2. \tag{1.13}$$

We will note that this definition can also be generalized to the case with a nonzero external force, $f \neq 0$. Leray proved the existence of weak solutions in this class for all initial data $u^0 \in L^2_{df}$ by mollifying the advection term with some smooth mollifier θ, replacing $(u\cdot\nabla)u$ with $((\theta * u)\cdot\nabla)\, u$. This mollification guarantees the existence of smooth solutions globally in time to the mollified equation, and furthermore these solutions to the mollified equation satisfy an energy equality, which is (1.13) with equality. Passing to weak limits, we obtain a weak solution satisfying the energy inequality, which now does not necessarily hold with equality because the solutions to the mollified equation only converge weakly in $C_tL^2_x \cap L^2_t\dot{H}^1_x$, and do not necessarily converge in norm. The proof of existence by weak convergence of solutions to the mollified problem also means that Leray weak solutions may not be unique.

For solutions to the Navier-Stokes equation denote the energy by

$$K(t) = \frac{1}{2}\|u(\cdot,t)\|_{L^2}^2. \tag{1.14}$$

The energy inequality (1.13) holds with equality for smooth solutions to the Navier-Stokes equations, but a weak solution in $u \in L^\infty\left([0,+\infty);L^2\left(\mathbb{R}^3\right)\right)\cap L^2\left([0,+\infty);\dot{H}^1\left(\mathbb{R}^3\right)\right)$ does not have enough regularity for us to integrate by parts to conclude that $\langle (u\cdot\nabla)u, u\rangle = 0$, which is what is needed to prove that the energy equality holds.

While the global existence of Leray solutions to the Navier-Stokes equations is well established, the global existence of smooth solutions remains a major open problem. Because Leray solutions are not necessarily smooth, they are not the best adapted to studying the Navier-Stokes regularity problem. For this reason we will turn our focus to mild solutions, a notion of solution better adapted to the Navier-Stokes regularity problem that was introduced by Kato and Fujita in [18]. Before defining mild solutions, we will define the Helmholtz decomposition.

Proposition 1.3 (Helmholtz decomposition)**.** *Suppose* $1 < q < +\infty$*. For all* $v \in L^q(\mathbb{R}^3;\mathbb{R}^3)$ *there exists a unique* $u \in L^q(\mathbb{R}^3;\mathbb{R}^3)$, $\nabla\cdot u = 0$ *and* $\nabla f \in L^q(\mathbb{R}^3;\mathbb{R}^3)$ *such that* $v = u + \nabla f$*. Note because we do not have any assumptions of higher regularity, we will say that* $\nabla\cdot u = 0$*, if for all* $\phi \in C_c^\infty(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} u\cdot\nabla\phi = 0, \tag{1.15}$$

and we will say that ∇f *is a gradient if for all* $w \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3), \nabla\cdot w = 0,$ *we have*

$$\int_{\mathbb{R}^3}\nabla f\cdot w = 0. \tag{1.16}$$

Furthermore there exists $B_q \geq 1$ *depending only on* q*, such that*

$$\|u\|_{L^q} \leq B_q \|v\|_{L^q}, \tag{1.17}$$

and

$$\|\nabla f\|_{L^q} \leq B_q \|v\|_{L^q}. \tag{1.18}$$

Define $P_{df} : L^q(\mathbb{R}^3;\mathbb{R}^3) \to L^q(\mathbb{R}^3;\mathbb{R}^3)$ *and* $P_g : L^q(\mathbb{R}^3;\mathbb{R}^3) \to L^q(\mathbb{R}^3;\mathbb{R}^3)$ *by* $P_{df}(v) = u$ *and* $P_g(v) = \nabla f$*, where* $v, u,$ *and* ∇f *are taken as above.*

Furthermore, suppose $-\frac{3}{2} < s < \frac{3}{2}$*. Then for all* $v \in \dot{H}^s\left(\mathbb{R}^3;\mathbb{R}^3\right)$ *there exists a unique* $u \in \dot{H}^s_{df}, \nabla f \in \dot{H}^s\left(\mathbb{R}^3;\mathbb{R}^3\right)$ *such that* $u = v + \nabla f$ *and*

$$\|v\|^2_{\dot{H}^s} = \|u\|^2_{\dot{H}^s} + \|\nabla f\|^2_{\dot{H}^s}. \tag{1.19}$$

Likewise define $P_{df} : \dot{H}^s\left(\mathbb{R}^3;\mathbb{R}^3\right) \to \dot{H}^s\left(\mathbb{R}^3;\mathbb{R}^3\right)$ *and* $P_g : \dot{H}^s\left(\mathbb{R}^3;\mathbb{R}^3\right) \to \dot{H}^s\left(\mathbb{R}^3;\mathbb{R}^3\right)$ *by* $P_{df}(v) = u$ *and* $P_g(v) = \nabla f$*, where* $v, u,$ *and* ∇f *are taken as above.*

This is a well-known, classical result. For details, see for instance [35]. We will also note here that the L^q bounds above are equivalent to the L^q boundedness of the Riesz transform. Take the Riesz transform to be given by $R = \nabla(-\Delta)^{-\frac{1}{2}}$, then $P_{df}(v) = R \times (R \times v)$, and $P_g(v) = -R(R \cdot v)$. P_{df} is often referred to as the Leray projection because of its use by Leray in developing weak solutions to the Navier-Stokes equation.

Note that $P_{df}(\nabla p) = 0$, so the Helmholtz decomposition allows us to define solutions to the incompressible Navier-Stokes equation without making any reference to pressure at all. With this technical detail out of the way, we will now define mild solutions of the Navier-Stokes equation.

Definition 1.4 (Mild solutions)**.** *Suppose* $u \in C\left([0,T];\dot{H}^1_{df}\right) \cap L^2\left([0,T];\dot{H}^2\left(\mathbb{R}^3\right)\right)$*. Then* u *is a mild solution to the Navier-Stokes equation with external force* $f \in L^2\left([0,T];L^2\left(\mathbb{R}^3\right)\right)$ *if*

$$u(\cdot,t) = e^{\nu t\Delta}u^0 + \int_0^t e^{\nu(t-\tau)\Delta} P_{df}\left((-u\cdot\nabla)u + f\right)(\cdot,\tau)d\tau, \tag{1.20}$$

where $e^{t\Delta}$ *is the heat operator given by convolution with the heat kernel; that is to say,* $e^{t\Delta}u^0$ *is the solution of the heat equation after time* t*, with initial data* u^0*.*

Fujita and Kato proved the local existence of mild solutions for initial data in $\dot{H}^1_{df}$ in [18], a result we will state precisely below. In fact, mild solutions exist for initial data in $\dot{H}^s_{df}, s > \frac{1}{2}$. This was later extended to initial data in $L^q_{df}, q > 3$ by Kato in [26].

Theorem 1.5 (Mild solutions exist for short times)**.** *Suppose* $f = 0$*. Then there exists a constant* $C > 0$*, independent of* ν*, such that for all* $u^0 \in \dot{H}^1_{df}$*, for all* $0 < T < \frac{C\nu^3}{\|u^0\|^4_{\dot{H}^1}}$*, there exists a unique mild solution to the Navier-Stokes equation* $u \in C\left([0,T];\dot{H}^1(\mathbb{R}^3)\right)$*. Furthermore for all* $0 < \epsilon < T$*,* $u \in C\left([\epsilon,T];\dot{H}^\alpha(\mathbb{R}^3)\right)$ *for all* $\alpha > 1$*, and therefore* $u \in C^\infty\left((0,T] \times \mathbb{R}^3;\mathbb{R}^3\right)$*.*

In the case where $f \neq 0$ *for all* $u^0 \in \dot{H}^1(\mathbb{R}^3), \nabla \cdot u = 0$ *and all* $f \in L^2_{loc}\left((0,T^*);L^2(\mathbb{R}^3)\right)$ *there exists* $0 < T \leq T^*$ *and* $u \in C\left([0,T];\dot{H}^1(\mathbb{R}^3)\right) \cap L^2\left([0,T];\dot{H}^2(\mathbb{R}^3)\right)$ *such that* u *is the* *unique mild solution to the Navier-Stokes equation. Note that mild solutions with a non-smooth force are not smooth in general, because the bootstrapping argument will not work in this case.*

The proof is based on a Picard iteration scheme, as the map associated with Definition 1.4,

$$T(u) = e^{\nu t\Delta}u^0 + \int_0^t e^{\nu(t-\tau)\Delta} P_{df}\left(-(u\cdot\nabla)u + f\right)(\cdot,\tau)d\tau, \tag{1.21}$$

is a contraction mapping from L^q_{df} to L^q_{df}, for $q > 3$ and for sufficiently small times. These arguments, however, cannot guarantee the existence of a smooth solutions for arbitrarily large times. When discussing regularity for the Navier-Stokes equation it is useful to define T_{max}, the maximal time of existence for a smooth solution corresponding to some initial data.

Definition 1.6 (Maximial time of existence)**.** *For all $u^0 \in \dot{H}^1_{df}$, if there is a mild solution of the Navier-Stokes equation $u \in C\left([0,+\infty);\dot{H}^1{}_{df}\right)$, $u(\cdot,0) = u^0$, then $T_{max} = +\infty$. If there is not a mild solution globally in time with initial data u^0, then let $T_{max} < +\infty$ be the time such that $u \in C\left([0,T_{max});\dot{H}^1{}_{df}\right)$, $u(\cdot,0) = u^0$, is a mild solution to the Navier-Stokes equation that cannot be extended beyond T_{max}. That is, for all $T > T_{max}$ there is no mild solution $u \in C\left([0,T);\dot{H}^1{}_{df}\right)$, $u(\cdot,0) = u^0$.*

It remains one of the biggest open questions in nonlinear PDEs, indeed one of the Millennium Problems put forward by the Clay Mathematics Institute, whether the Navier-Stokes equation has smooth solutions globally in time for arbitrary smooth initial data [17]. Note in particular that the Clay Millennium problem can be equivalently stated in terms of Definition 1.6 as: if $f = 0$, show $T_{max} = +\infty$ for all initial data $u^0 \in H^1_{df}$, or provide a counterexample.

Theorem 1.5 states that a solution must exist locally in time for all initial data $u^0 \in \dot{H}^1$, which implies that in order for a mild solution to develop singularities in finite time it must blow up in $\dot{H}^1$. The square of the $\dot{H}^1$ norm for solutions to the Navier-Stokes equation is known as enstrophy, and can be defined equivalently as

$$E(t) = \frac{1}{2}\|u(\cdot,t)\|^2_{\dot{H}^1} = \frac{1}{2}\|\omega(\cdot,t)\|^2_{L^2} = \|S(\cdot,t)\|^2_{L^2}. \tag{1.22}$$

We will prove the equivalence of these definitions in chapter 3.

It is well known that

$$\partial_t \frac{1}{2}\|\omega(\cdot,t)\|^2_{L^2} = -\nu\|\omega\|^2_{\dot{H}^1} + \langle S, \omega\otimes\omega\rangle\,. \tag{1.23}$$

Using the Sobolev embedding of $\dot{H}^1\left(\mathbb{R}^3\right)$ into $L^6\left(\mathbb{R}^3\right)$ it follows from (1.23) that

$$\partial_t\|\omega(\cdot,t)\|^2_{L^2} \leq C\|\omega(\cdot,t)\|^6_{L^2}, \tag{1.24}$$

which is sufficient to guarantee regularity at least locally in time, but cannot prevent blowup because it is a cubic differential inequality.

In chapter 3 of this thesis, we will prove the following identity for enstrophy growth:

$$\partial_t\|S(\cdot,t)\|^2_{L^2} = -2\nu\|S\|^2_{\dot{H}^1} - \frac{4}{3}\int \operatorname{tr}(S^3). \tag{1.25}$$

Using the fact that S must be trace free, because $\mathrm{tr}(S) = \nabla \cdot u = 0$, this identity can also be expressed in terms of the determinant of S as

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 = -2\nu \|S\|_{\dot{H}^1}^2 - 4\int \det(S). \tag{1.26}$$

The nonlinearity in (1.26) is still of the same degree as in (1.23). Both nonlinearities are of degree 3, and so cannot be controlled by the dissipation in either case, however the identity (1.26) does have several advantages. First, unlike (1.23), this identity is entirely local. The identity (1.23) is nonlocal with a singular integral kernel, because S can be determined in terms of ω with a zeroth order pseudo-differential operator, $S = \nabla_{sym}(-\Delta)^{-1}\nabla \times \omega$. The identity (1.26) also reveals very significant information about the relationship between blowup and the eigenvalues of the strain tensor S. In fact, this identity leads to a new regularity criterion in terms of the middle eigenvalue of the strain tensor that encodes information about the geometric structure of potential blow-up solutions.

Theorem 1.7 (Middle eigenvalue of strain characterizes blowup time)**.** *Let* $u \in C\left([0,T]; \dot{H}^1\left(\mathbb{R}^3\right)\right)$ *for all* $T < T_{max}$ *be a mild solution to the Navier-Stokes equation with* $f = 0$*, and let* $\lambda_1(x) \leq \lambda_2(x) \leq \lambda_3(x)$ *be the eigenvalues of the strain tensor* $S(x) = \nabla_{sym}u(x)$*. Let* $\lambda_2^+(x) = \max\{\lambda_2(x), 0\}$*. If* $\frac{2}{p} + \frac{3}{q} = 2$*, with* $\frac{3}{2} < q \leq +\infty$*, then*

$$\|u(\cdot,T)\|_{\dot{H}^1}^2 \leq \|u^0\|_{\dot{H}^1}^2 \exp\left(D_q \int_0^T \|\lambda_2^+(\cdot,t)\|_{L^q(\mathbb{R}^3)}^p dt\right), \tag{1.27}$$

with the constant D_q *depending only on* q *and* ν*. In particular if* $T_{max} < +\infty$*, where* T_{max} *is the maximal existence time for a smooth solution, then*

$$\int_0^{T_{max}} \|\lambda_2^+(\cdot,t)\|_{L^q(\mathbb{R}^3)}^p dt = +\infty. \tag{1.28}$$

It goes back to the classic work of Kato [26] that smooth solutions must exist locally in time for any initial data $u^0 \in L^q_{df}$ when $q > 3$. In particular, this implies that a smooth solution of the Navier-Stokes equations developing singularities in finite time requires that the L^q norm of u must blow up for all $q > 3$. This was extended to the case $q = 3$ by Escauriaza, Seregin, and Šverák [16]. The regularity criteria implied by the local existence of smooth solutions for initial data in $L^q\left(\mathbb{R}^3\right)$ when $q > 3$ are all subcritical with respect to the scaling that preserves the solution set of the Navier-Stokes equations:

$$u^\lambda(x,t) = u(\lambda x, \lambda^2 t). \tag{1.29}$$

If u is a solution to the Navier-Stokes equations on $\mathbb{R}^3$, then so is u^λ for all $\lambda > 0$, although the time interval may have to be adjusted, depending on what notion of a solution (Leray-Hopf [36], mild, strong [18]) we are using. $L^3\left(\mathbb{R}^3\right)$ is the scale critical Lebesgue space for the Navier-Stokes equations, so the Escauriaza-Seregin-Šverák condition is scale critical.

Critical regularity criteria for solutions to the Navier-Stokes equations go back to the work of Prodi, Serrin, and Ladyzhenskaya [33, 44, 48], who proved that if a smooth solution blows up in finite time $T_{max} < +\infty$, then

$$\int_0^{T_{max}} \|u\|_{L^q}^p dt = +\infty, \tag{1.30}$$

where $\frac{2}{p} + \frac{3}{q} = 1$, and $3 < q \leq +\infty$. This result was then extended in the aforementioned Escauriaza-Seregin-Šverák paper [16] to the endpoint case $p = +\infty, q = 3$. They proved that if a smooth solution u of the Navier-Stokes equation blows up in finite time $T_{max} < +\infty$, then

$$\limsup_{t \to T_{max}} \|u(\cdot, t)\|_{L^3(\mathbb{R}^3)} = +\infty. \tag{1.31}$$

Gallagher, Koch, and Planchon [21] also proved the above statement using a different approach based on profile decomposition. The other endpoint case of this family of criteria is the Beale-Kato-Majda criterion [3], which holds for solutions of the Euler as well as for Navier-Stokes, and states that if a smooth solution to either the Euler or Navier-Stokes equations develops singularities in finite time, then

$$\int_0^{T_{max}} \|\omega(\cdot, t)\|_{L^\infty} dt = +\infty. \tag{1.32}$$

This result was also extended to the strain tensor [27].

The regularity criterion in Theorem 1.7 also offers analytical evidence of the numerically observed tendency [19] of the vorticity to align with the eigenvector corresponding to the intermediate eigenvalue λ_2. If it is true that the vorticity tends to align with the intermediate eigenvalue we would heuristically expect that

$$tr\big(S(x)\omega(x) \otimes \omega(x)\big) \sim \lambda_2(x)|\omega(x)|^2. \tag{1.33}$$

We would then heuristically expect that

$$\langle S; \omega \otimes \omega \rangle \sim \int_{\mathbb{R}^3} \lambda_2(x)|\omega(x)|^2 dx, \tag{1.34}$$

and so we would expect that there would be some inequality of the form

$$\langle S; \omega \otimes \omega \rangle \leq C \int_{\mathbb{R}^3} \lambda_2^+(x)|\omega(x)|^2 dx. \tag{1.35}$$

This is all, of course, entirely heuristic, but it is interesting that the regularity criterion we have proven is precisely of the form that would be predicted by the observed tendency of the vorticity to align with the eigenvector associated with the intermediate eigenvalue. This suggests that significant information about the geometric structure of incompressible flow is encoded in the regularity criterion in Theorem 1.7.

The family of regularity criteria in (1.30) has since been generalized to the critical Besov spaces [1, 13, 22, 29, 30, 43]. These criteria have also been generalized to criteria controlling the pressure [46, 49, 52]. In addition to strengthening regularity criteria to larger spaces, there have also been results not involving all the components of u, for instance regularity criteria on the gradient of one component ∇u_j [54], involving only the derivative in one direction, $\partial_{x_i} u$ [32], involving only one component u_j [7, 11], involving only one component of the gradient tensor $\frac{\partial u_j}{\partial x_i}$ [4], and involving only two components of the vorticity [6]. For a more thorough overview of the literature on regularity criteria for solutions to the Navier-Stokes equation see Chapter 11 in [35]. We will discuss the relationship between these results and Theorem 1.7 in chapter 5, where we will prove the following critical one direction type regularity criterion for a range of exponents for which no critical one component regularity criteria were previously known. First we must define, for any unit vector $v \in \mathbb{R}^3$, $|v| = 1$, the directional derivative in the v direction, which is given by $\partial_v = v \cdot \nabla$, and the v-th component of u, which is given by $u_v = u \cdot v$.

Theorem 1.8 (One direction regularity criterion)**.** *Let* $\{v_n(t)\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ *with* $|v_n(t)| = 1$. *Let* $\{\Omega_n(t)\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ *be Lesbesgue measurable sets such that for all* $m \neq n$, $\Omega_m(t) \cap \Omega_n(t) = \emptyset$, *and* $\mathbb{R}^3 = \bigcup_{n\in\mathbb{N}} \Omega_n(t)$. *Let* $u \in C\left([0,T]; \dot{H}^1_{df}\right)$, *for all* $T < T_{max}$ *be a mild solution to the Navier-Stokes equation with* $f = 0$. *If* $\frac{2}{p} + \frac{3}{q} = 2$, *with* $\frac{3}{2} < q \leq +\infty$, *then*

$$\|u(\cdot,T)\|_{\dot{H}^1}^2 \leq \|u^0\|_{\dot{H}^1}^2 \exp\left(D_q \int_0^{T_{max}} \left(\sum_{n=1}^{\infty} \left\|\frac{1}{2}\partial_{v_n} u(\cdot,t) + \frac{1}{2}\nabla u_{v_n}(\cdot,t)\right\|_{L^q(\Omega_n(t))}^q\right)^{\frac{p}{q}} dt\right), \tag{1.36}$$

with the constant D_q *depending only on* q *and* ν. *In particular if the maximal existence time for a smooth solution* $T_{max} < +\infty$, *then*

$$\int_0^{T_{max}} \left(\sum_{n=1}^{\infty} \|\partial_{v_n} u(\cdot,t) + \nabla u_{v_n}(\cdot,t)\|_{L^q(\Omega_n(t))}^q\right)^{\frac{p}{q}} dt = +\infty. \tag{1.37}$$

Note that if we take $v_n(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ *for each* $n \in \mathbb{N}$, *then* (1.37) *reduces to*

$$\int_0^{T_{max}} \|\partial_3 u(\cdot,t) + \nabla u_3(\cdot,t)\|_{L^q(\mathbb{R}^3)}^p dt = +\infty. \tag{1.38}$$

Theorem 1.8 is in fact a corollary of the following more general theorem, which states that for a solution of the Navier-Stokes equation to blow up, the strain must blow up in every direction.

Theorem 1.9 (Blowup requires the strain to blow up in every direction)**.** *Let* $u \in C\left([0,T]; \dot{H}^1_{df}\right)$, *for all* $T < T_{max}$ *be a mild solution to the Navier-Stokes equation with* $f = 0$ *and let* $v \in L^\infty\left(\mathbb{R}^3 \times [0,T_{max}]; \mathbb{R}^3\right)$, *with* $|v(x,t)| = 1$ *almost everywhere. If* $\frac{2}{p} + \frac{3}{q} = 2$, *with* $\frac{3}{2} < q \leq +\infty$, *then*

$$\|u(\cdot,T)\|_{\dot{H}^1}^2 \leq \|u^0\|_{\dot{H}^1}^2 \exp\left(D_q \int_0^T \|S(\cdot,t)v(\cdot,t)\|_{L^q(\mathbb{R}^3)}^p dt\right), \tag{1.39}$$

with the constant D_q *depending only on* q *and* ν. *In particular if the maximal existence time for a smooth solution* $T_{max} < +\infty$, *then*

$$\int_0^{T_{max}} \|S(\cdot,t)v(\cdot,t)\|_{L^q(\mathbb{R}^3)}^p dt = +\infty. \tag{1.40}$$

Note that like the Prodi-Serrin-Ladyzhenskaya regularity criterion, the regularity criteria we prove on λ_2^+ and $\partial_3 u + \nabla u_3$ are critical with respect to scaling. The reason we require that $\frac{2}{p} + \frac{3}{q} = 2$, not $\frac{2}{p} + \frac{3}{q} = 1$ is because λ_2 is an eigenvalue of S, and therefore scales like $\nabla \otimes u$, not like u. In addition, both regularity criteria, as well as Theorem 1.7, can be generalized to the Navier-Stokes equation with an external force $f \in L_t^2 L_x^2$, which will be discussed in chapter 5, but is left out of the introduction for the sake of brevity.

Remark 1.10. *After circulating a preprint of his paper [41], the author learned of previous work by Dongho Chae on the role of the eigenvalues of the strain matrix in enstrophy growth in the context of the Euler equation [5]. In this paper, Chae proves that sufficiently smooth solutions to the Euler equation satisfy the following growth identity for enstrophy:*

$$\partial_t \|S(\cdot, t)\|_{L^2}^2 = -4 \int \det(S). \tag{1.41}$$

This is analogous to what we have proven for the growth of enstrophy for solution of the Navier-Stokes equation (1.26) *without the dissipation term, because the Euler equation has no viscosity. The methods used are somewhat different than ours; in particular the constraint space for the strain tensor and the evolution equation for the strain tensor are not used in [5]. While it is possible to establish the identity* (1.25) *without an analysis of the constraint space, we expect the results characterizing the constraint space in this paper, particularly Proposition 2.3 and Proposition 2.4, to be useful in future investigations. Chae also proves the $q = +\infty$ case of the regularity criterion in Theorem 1.7, but this criterion is new for the rest of the range of parameters. We will discuss the relationship between our method of proof and that in [5] in more detail after we have proven the identity* (1.25)*, which is Corollary 3.3 in this paper. The author would like to thank Alexander Kiselev for bringing Chae's paper to his attention.*

While global regularity for the Navier-Stokes equation with arbitrary, smooth initial data remains a major open problem, it is known that the Navier-Stokes equation must have global smooth solutions for small initial data in certain scale-critical function spaces. In particular, Fujita and Kato also proved in [18] the global existence of smooth solutions to the Navier-Stokes equation for small initial data in $\dot{H}_{df}^{\frac{1}{2}}$.

Theorem 1.11 (Global regularity for small initial data)**.** *Suppose $f = 0$. There exists $C > 0$, independent of ν, such that for all $u^0 \in \dot{H}_{df}^{\frac{1}{2}}, \|u^0\|_{\dot{H}^{\frac{1}{2}}} < C\nu$, there exists a unique global smooth solution to the Navier-Stokes equation $u \in C\left([0, +\infty); \dot{H}_{df}^{\frac{1}{2}}\right) \cap C^\infty\left((0, +\infty) \times \mathbb{R}^3; \mathbb{R}^3\right)$, $u(\cdot, 0) = u^0$.*

This result was then extended to L^3 by Kato [26] and to BMO^{-1} by Koch and Tataru [28]. We will note here that the Navier-Stokes equation is invariant under the rescaling $u^\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)$, and therefore u^0 generates a global smooth solution if and only if, $u^{0,\lambda}(x) = \lambda u^0(\lambda x)$ generates a global smooth solution for all $\lambda > 0$. It is easy to check that each of these norms are invariant with respect to this rescaling of the initial data.

In chapter 6 of this thesis, we will establish a new result guaranteeing the existence of global smooth solutions for initial data that are arbitrarily large, so long two components of the vorticity are sufficiently small in the critical Hilbert space.

Theorem 1.12 (Global regularity for two components of vorticity small)**.** *Let* $R_1 = \frac{\sqrt{3}}{2\sqrt{2}}\pi, R_2 = \frac{32\pi^4}{3(1+\sqrt{2})^4}$*. Let* $\omega_h = (\omega_1, \omega_2, 0)$ *and let* $f = 0$*. For all* $u^0 \in H^1_{df}$ *such at*

$$\|\omega_h^0\|_{\dot{H}^{-\frac{1}{2}}} \exp\left(\frac{K_0 E_0 - 6,912\pi^4\nu^4}{R_2\nu^3}\right) < R_1\nu, \tag{1.42}$$

u^0 *generates a unique, global smooth solution to the Navier-Stokes equation* $u \in C\left((0, +\infty); H^1_{df}\right)$, *that is* $T_{max} = +\infty$*. Note that the smallness condition can be equivalently stated as*

$$K_0 E_0 < 6,912\pi^4\nu^4 + R_2\nu^3 \log\left(\frac{R_1\nu}{\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}}\right). \tag{1.43}$$

Very little is known in general about the existence of smooth solutions globally in time with arbitrarily large initial data. Ladyzhenskaya proved the existence of global smooth solutions for swirl-free axisymmetric initial data [34], which gives a whole family of arbitrarily large initial data with globally smooth solutions. Mahalov, Titi, and Leibovich showed global regularity for solutions with a helical symmetry in [40]. In light of the Koch-Tataru theorem guaranteeing global regularity for small initial data in BMO^{-1}, it has been an active area of research to find examples of solutions that are large in BMO^{-1} that generate global smooth solutions, or even stronger, to find initial data large in $B^{-1}_{\infty,\infty} \supset BMO^{-1}$, which is the maximal scale invariant space. Because both swirl free, axisymetric vector fields and helically symmetric vector fields form subspaces of divergence free vector fields, clearly these are examples of initial data large in $B^{-1}_{\infty,\infty}$. Gallagher and Chemin showed the existence of initial data that generate global smooth solutions that are large in $B^{-1}_{\infty,\infty}$ on the torus by taking highly oscillatory initial data [8]. More recently Kukavica, Rusin, and Ziane exhibited a class of non-oscillatory initial data, large in $B^{-1}_{\infty,\infty}$, that generate global smooth solutions [31].

Unlike the three dimensional case, there are global smooth solutions to the Navier-Stokes equation in two dimensions. This is because in two dimensions the energy equality is scale critical, while in three dimensions the energy inequality is supercritical. This is also because vortex stretching occurs in three dimensions, but not in two dimensions, so the enstrophy is decreasing for solutions of the two dimensional Navier-Stokes equations. Given that the Navier-Stokes equation has global smooth solutions in two dimensions, one natural approach to the extending small data regularity results to arbitrarily large initial data, would be to show global regularity for the solutions that are, in some sense, approximately two dimensional.

There are also a number of previous results guaranteeing global regularity for solutions three dimensional solutions of the Navier-Stokes equations with almost two dimensional initial data. One approach to almost two dimensional initial data on the torus is to consider three dimensional initial data that is a perturbation of two dimensional initial data. Note that this approach is available on the torus, because $L^2_{df}\left(\mathbb{T}^2\right)$ forms a subspace of $L^2_{df}\left(\mathbb{T}^3\right)$, so we can consider perturbations of this subspace. It is not, however, available on the whole space, as nonzero vector fields in $L^2_{df}\left(\mathbb{R}^2\right)$, lose integrability when extended to three dimensions under the map above, and so $L^2_{df}\left(\mathbb{R}^2\right)$ does not define a subspace of $L^2_{df}(\mathbb{R}^3)$. Iftimie proved that small perturbations of two dimensional initial data must have smooth solutions to the Navier-Stokes equation globally in time. Another approach is based on re-scaling, to make the the initial data vary slowly in one direction. This approach was used by Gallagher and Chemin in [9] and extended by Gallagher, Chemin, and Paicu in [10] and by Paicu and Zhang in [42]. We will prove global regularity based on rescaling the vorticity, rather than the velocity, as this rescaling operates better with the divergence free constraint. The result we will prove is the following.

Theorem 1.13 (Global regularity for rescaled vorticity)**.** *Fix $a > 0$. For all $u^0 \in H^1_{df}, 0 < \epsilon < 1$ let*

$$\omega^{0,\epsilon}(x) = \epsilon^{\frac{2}{3}} \left(\log \left(\frac{1}{\epsilon^a} \right) \right)^{\frac{1}{4}} \left(\epsilon\omega_1^0, \epsilon\omega_2^0, \omega_3^0 \right) (x_1, x_2, \epsilon x_3), \tag{1.44}$$

and define $u^{0,\epsilon}$ using the Biot-Savart law by

$$u^{0,\epsilon} = \nabla \times (-\Delta)^{-1} \omega^{0,\epsilon}. \tag{1.45}$$

For all $u^0 \in H^1_{df}$ and for all $0 < a < \frac{4R_2\nu^3}{C_2^2 \|\omega_3^0\|^2_{L^{\frac{6}{5}}} \|\omega_3^0\|^2_{L^2}}$, there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, there is a unique, global smooth solution to the Navier-Stokes equation $u \in C\left((0, +\infty); H^1_{df}\right)$ with $u(\cdot, 0) = u^{0,\epsilon}$. Furthermore if ω_3^0 is not identically zero, then the initial vorticity is large in the critical space $L^{\frac{3}{2}}$, as $\epsilon \to 0$, that is

$$\lim_{\epsilon \to 0} \|\omega^{0,\epsilon}\|_{L^{\frac{3}{2}}} = +\infty. \tag{1.46}$$

In chapter 2, we will derive an evolution equation for the strain tensor and define mild solutions to the strain and vorticity equations. In chapter 3, we will prove an isometry showing the equivalence of defining the enstrophy in terms of the strain and in terms of the vorticity, and we will prove a new identity for enstrophy growth. In chapter 4, we will consider the question of maximal enstrophy growth locally in time. In chapter 5, we will prove Theorem 1.7, the regularity criterion on λ_2^+, as well as a number of immediate corollaries. In chapter 6, we will consider a vorticity approach to almost two dimensional initial data, proving Theorem 1.12. In chapter 7, we will discuss the relationship between this result and previous global regularity results for almost two dimensional initial data. In chapter 8, we will prove the existence and stability of blowup for toy model ODE of the strain equation. Finally, in chapter 9, we will consider the strain equation in two dimensions.

Chapter 2

Evolution equation for the strain tensor

We will begin this chapter by deriving the Navier-Stokes strain equation (1.5) in three spatial dimensions.

Proposition 2.1 (Strain reformulation of the dynamics)**.** *Suppose u is a classical solution to the Navier-Stokes equation with external force f. Then $S = \nabla_{sym}(u)$ is a classical solution to the Navier-Stokes strain equation*

$$\partial_t S + (u \cdot \nabla)S - \nu\Delta S + S^2 + \frac{1}{4}\omega \otimes \omega - \frac{1}{4}|\omega|^2 I_3 + \text{Hess}(p) = \nabla_{sym} f. \tag{2.1}$$

Proof. We begin by applying the operator ∇_{sym} to the Navier-Stokes Equation (1.1); we find immediately that

$$\partial_t S - \nu\Delta S + \text{Hess}(p) + \nabla_{sym}\left((u \cdot \nabla)u\right) = \nabla_{sym} f. \tag{2.2}$$

It remains to compute $\nabla_{sym}\left((u \cdot \nabla)u\right)$.

$$\nabla_{sym}\left((u \cdot \nabla)u\right)_{ij} = \frac{1}{2}\partial_{x_i}\sum_{k=1}^{3} u_k \frac{\partial u_j}{\partial x_k} + \frac{1}{2}\partial_{x_j}\sum_{k=1}^{3} u_k \frac{\partial u_i}{\partial x_k}. \tag{2.3}$$

$$\nabla_{sym}\left((u \cdot \nabla)u\right)_{ij} = \sum_{k=1}^{3} u_k \partial_{x_k}\left(\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\right) + \frac{1}{2}\sum_{k=1}^{3} \frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_k} + \frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_j}. \tag{2.4}$$

We can see from our definitions of S and A that

$$S_{ij}^2 = \frac{1}{4}\sum_{k=1}^{3}\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right)\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right) = \frac{1}{4}\sum_{k=1}^{3} \frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_k} + \frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_j} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j} + \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}, \tag{2.5}$$

and

$$A_{ij}^2 = \frac{1}{4}\sum_{k=1}^{3}\left(\frac{\partial u_k}{\partial x_i} - \frac{\partial u_i}{\partial x_k}\right)\left(\frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j}\right) = \frac{1}{4}\sum_{k=1}^{3} \frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_k} + \frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_j} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j} - \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}. \tag{2.6}$$

Taking the sum of these two equation, we find that

$$\left(S^2+A^2\right)_{ij} = \frac{1}{2}\sum_{k=1}^{3} \frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_k} + \frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_j}. \tag{2.7}$$

From this we can conclude that

$$\nabla_{sym}\left((u\cdot\nabla)u\right) = (u\cdot\nabla)S + S^2 + A^2. \tag{2.8}$$

Recall that

$$A = \frac{1}{2}\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}, \tag{2.9}$$

so we can express A^2 as

$$A^2 = \frac{1}{4}\omega\otimes\omega - \frac{1}{4}|\omega|^2 I_3. \tag{2.10}$$

This concludes the proof. □

We also can see that $\operatorname{tr}(S) = \nabla\cdot u = 0$, so in order to maintain the divergence free structure of the flow, we require that the strain tensor be trace free. For the vorticity the only consistency condition is that the vorticity be divergence free. Any divergence free vorticity can be inverted back to a unique velocity field, assuming suitable decay at infinity, with $u = \nabla\times(-\Delta)^{-1}\omega$. This is not true of the strain tensor, for which an additional consistency condition is required.

If we know the strain tensor S, this is enough for us to reconstruct the flow. We take

$$-2\operatorname{div}(S) = -\Delta u - \nabla(\nabla\cdot u) = -\Delta u. \tag{2.11}$$

Therefore we find that

$$u = -2\operatorname{div}(-\Delta)^{-1}S. \tag{2.12}$$

This allows us to reconstruct the flow u from the strain tensor S, but it doesn't guarantee that if we start with a general trace free symmetric matrix, the u we reconstruct will actually have this symmetric matrix as its strain tensor. We will need to define a consistency condition guaranteeing that the strain tensor is actually the symmetric part of the gradient of some divergence free vector field. This condition for the strain equation will play the same role that the divergence free condition plays in the vorticity equation. We will now define the subspace of strain matrices $L^2_{st}\subset L^2(\mathbb{R}^3;S^{3\times 3})$ as follows:

Definition 2.2 (Strain subspace)**.** *We will define the subspace of strain matrices to be*

$$L^2_{st} = \left\{\frac{1}{2}\nabla\otimes u + \frac{1}{2}(\nabla\otimes u)^* : u\in \dot{H}^1\left(\mathbb{R}^3;\mathbb{R}^3\right), \nabla\cdot u = 0\right\}. \tag{2.13}$$

This subspace of $L^2(\mathbb{R}^3;S^{3\times 3})$ can in fact be characterized by a partial differential equation, although in this case, it is significantly more complicated than the equation $\nabla\cdot u = 0$, that characterizes the space of divergence free vector fields.

Proposition 2.3 (Characterization of the strain subspace)**.** *Suppose $S\in L^2(\mathbb{R}^3;S^{3\times 3})$. Then*

$S \in L^2_{st}$ if and only if

$$\operatorname{tr}(S) = 0, \tag{2.14}$$

$$-\Delta S + 2\nabla_{sym}\left(\operatorname{div}(S)\right) = -\Delta S + (\nabla \otimes \nabla)S + ((\nabla \otimes \nabla)S)^* = 0. \tag{2.15}$$

Note that because by hypothesis we only have $S \in L^2$, we will consider S to be a solution to (2.15) *if the condition is satisfied pointwise almost everywhere in Fourier space, that is if*

$$|\xi|^2\hat{S}(\xi) - (\xi \otimes \xi)\hat{S}(\xi) - \hat{S}(\xi)(\xi \otimes \xi) = 0, \tag{2.16}$$

almost everywhere $\xi \in \mathbb{R}^3$. The partial differential equation (2.15) *can be written out in components as*

$$-\Delta S_{ij} + \sum_{k=1}^{3} \partial_{x_i}\partial_{x_k}S_{kj} + \partial_{x_j}\partial_{x_k}S_{ki} = 0. \tag{2.17}$$

Proof. First suppose $S \in L^2_{st}$, so there exists a $u \in \dot{H}^1$, $\nabla \cdot u = 0$, such that

$$S = \nabla_{sym}u. \tag{2.18}$$

As we have already shown, $\operatorname{tr}(S) = \nabla \cdot u = 0$. Next we will take the divergence of (2.18), and find that,

$$-2\operatorname{div}(S) = -2\operatorname{div}(\nabla_{sym}u) = -\Delta u - \nabla(\nabla \cdot u) = -\Delta u. \tag{2.19}$$

Applying ∇_{sym} to (2.19) we find that

$$-2\nabla_{sym}(\operatorname{div}(S)) = \nabla_{sym}(-\Delta u) = -\Delta S, \tag{2.20}$$

so the condition (2.15) is also satisfied.

Now suppose $tr(S) = 0$ and $-\Delta S + 2\nabla_{sym}(\operatorname{div}(S)) = 0$. Define u by

$$u = (-\Delta)^{-1}(-2\operatorname{div}(S)). \tag{2.21}$$

Applying ∇_{sym} to this definition we find that

$$\nabla_{sym}u = (-\Delta)^{-1}\left(-2\nabla_{sym}(\operatorname{div}(S))\right) = (-\Delta)^{-1}(-\Delta S) = S. \tag{2.22}$$

Clearly $u \in \dot{H}^1$ because $S \in L^2$ and $(-\Delta)^{-1}(-2\operatorname{div})$ is a pseudo-differential operator with order -1. It only remains to show that $\nabla \cdot u = 0$. Next we will take the trace of (2.17) and find that

$$(\operatorname{div})^2(S) = \sum_{i,j=1}^{3} \partial_{x_i}\partial_{x_j}S_{ij} = 0. \tag{2.23}$$

Using this we compute that

$$\nabla \cdot u = (-\Delta)^{-1}(-2(\operatorname{div})^2(S)) = 0. \tag{2.24}$$

This completes the proof. □

Note that the the consistency condition (2.15) is linear, so the set of matrices satisfying it form a subspace of L^2. The Navier-Stokes equation (1.1) and the vorticity equation (1.4) can best be viewed not as systems of equations, but as evolution equations on the space of divergence free vector fields. Similarly, we can view the Navier-Stokes strain equation (1.5) as an evolution equation on L^2_{st}.

When compared with the vorticity equation, the evolution equation for the strain tensor, while it requires additional terms, has a quadratic nonlinearity whose structure is far better from an algebraic point of view. This is because a vector cannot be squared, and the square of an anti-symmetric matrix (the other representation of vorticity) is a symmetric matrix, while the square of a symmetric matrix is again a symmetric matrix.

The Navier-Stokes strain equation has already been examined in [14, 19, 23], however the consistency condition (2.15) does not play a role in this analysis. The role of the strain was also considered by Chae in [5], although the evolution equation for strain does not play a role in this analysis. In [19], the authors focus on the relationship between vorticity and the strain tensor in enstrophy production, as the strain tensor and vorticity are related by a linear zero order pseudo-differential operator, $S = \nabla_{sym}(-\Delta)^{-1}\nabla \times \omega$. However, the consistency condition is actually very useful in dealing with the evolution of the strain tensor, because a number of the terms in the evolution equation (1.5) are actually in the orthogonal compliment of L^2_{st} with respect to the L^2 inner product. This will allow us to prove an identity for enstrophy growth involving only the strain, where previous identities involved the interaction of the strain and the vorticity. We will now make an observation about what matrices in $L^2(\mathbb{R}^3; S^{3\times 3})$ are in the orthogonal complement of L^2_{st} with respect to the L^2 inner product.

Proposition 2.4 (Orthogonal subspaces)**.** *For all $f \in \dot{H}^2(\mathbb{R}^3)$, for all $g \in L^2(\mathbb{R}^3)$, and for all $S \in L^2_{st}$*

$$\langle S, gI_3 \rangle = 0, \tag{2.25}$$

$$\langle S, \operatorname{Hess}(f) \rangle = 0. \tag{2.26}$$

Proof. First we'll consider the case of gI_3. Fix $S \in L^2_{st}$ and we'll take the inner product

$$\langle gI_3, S \rangle = \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} gI_{ij}S_{ij} = \int_{\mathbb{R}^3} tr(S)g = 0. \tag{2.27}$$

In order to show that $\operatorname{Hess}(f) \in \left(L^2_{st}\right)^{\perp}$, we will use the property that for $S \in L^2_{st}$

$$tr\left((\nabla \otimes \nabla)S\right) = \sum_{i,j=1}^{3} \partial x_i \partial x_j S_{ij} = 0. \tag{2.28}$$

Because $S \in L^2$ and therefore $\hat{S} \in L^2$, the above condition can be expressed as

$$\sum_{i,j=1}^{3} \xi_i \xi_j \hat{S}_{ij}(\xi) = 0, \tag{2.29}$$

almost everywhere $\xi \in \mathbb{R}^3$. Using the fact that the Fourier transform is an isometry on L^2, and

$\mathrm{Hess}(f), S \in L^2$ we compute that

$$\langle \mathrm{Hess}(f), S\rangle = \left\langle \widehat{\mathrm{Hess}(f)}, \hat{S} \right\rangle = -4\pi^2 \int_{\mathbb{R}^3} \bar{\hat{f}}(\xi) \sum_{i,j=1}^{3} \xi_i \xi_j \hat{S}_{ij}(\xi) d\xi = 0. \tag{2.30}$$

This completes the proof. □

This means that as long as u is sufficiently regular, $\mathrm{Hess}(p)$ and $-\frac{1}{4}|\omega|^2 I_3$ are in the orthogonal compliment of L^2_{st}. This fact will play a key role in the new identity for enstrophy growth that we will prove in chapter 3.

Note that u is uniquely determined in terms of both S and ω. We have already established that u can be reconstructed from S using the formula $u = -2\,\mathrm{div}(-\Delta)^{-1}S$. Likewise we know that $\nabla \times \omega = -\Delta u$, so u can be reconstructed from the vorticity using the formula $u = \nabla \times (-\Delta)^{-1}\omega$. This in particular means that S can be determined in terms of ω and vice versa with zero order pseudo-differential operators as follows: $S = \nabla_{sym} \nabla \times (-\Delta)^{-1}\omega$, and $\omega = -2\nabla \times \mathrm{div}(-\Delta)^{-1}S$. This in particular makes it possible to define mild solutions to the strain equation or the vorticity equation purely in terms of S and ω respectively.

Before we proceed further, we need to show the existence of solutions to the Navier-Stokes strain equation in a suitable space. Leray solutions are not the most well adapted to studying regularity, which is our focus, so we will work with mild solutions developed by Kato and Fujita instead [18]. Using the $\dot{H}^1$ mild solutions to the Navier-Stokes equation in Theorem 1.5, we will adapt these solutions to define for mild solutions in L^2 for the Navier-Stokes strain equation and the vorticity equation. We will define L^2 solutions to the strain evolution equation as follows.

Definition 2.5 (Mild strain solutions)**.** *Suppose $S \in C\left([0,T]; L^2_{st}\right) \cap L^2\left([0,T] : \dot{H}^1(\mathbb{R}^3)\right)$. Then we will call S a mild solution to the Navier-Stokes strain equation* (1.5) *with external force $f \in L^2\left([0,T]; L^2\left(\mathbb{R}^3\right)\right)$ if and only if for all $0 < t \leq T$,*

$$S(\cdot,t) = e^{\nu t\Delta}S^0 + \int_0^t e^{\nu(t-\tau)\Delta} \left(-(u\cdot\nabla)S - S^2 - \frac{1}{4}\omega\otimes\omega + \frac{1}{4}|\omega|^2 I_3 - \mathrm{Hess}(p) + \nabla_{sym} f\right)(\cdot,\tau)d\tau, \tag{2.31}$$

where $u = -2\,\mathrm{div}(-\Delta)^{-1}S$, $\omega = \nabla\times u$, and $p = (-\Delta)^{-1}\left(|S|^2 - \frac{1}{2}|\omega|^2 - \nabla\cdot f\right)$

We will define L^2 mild solutions to the vorticity equation likewise.

Definition 2.6 (Mild vorticity solutions)**.** *Suppose $\omega \in C\left([0,T]; L^2_{df}\right) \cap L^2\left([0,T] : \dot{H}^1\left(\mathbb{R}^3\right)\right)$. Then we will call ω a mild solution to the vorticity equation with external force $f \in L^2\left([0,T]; L^2\left(\mathbb{R}^3\right)\right)$ if and only if for all $0 < t \leq T$,*

$$\omega(\cdot,t) = e^{\nu t\Delta}\omega^0 + \int_0^t e^{\nu(t-\tau)\Delta}\left(-(u\cdot\nabla)\omega + S\omega - \nabla\times f\right)(\cdot,\tau)d\tau, \tag{2.32}$$

where $u = \nabla\times(-\Delta)^{-1}\omega$ and $S = \nabla_{sym}u$.

Proposition 2.7 (Equivalence of mild solutions)**.** *If $u \in C\left([0,T]; \dot{H}^1_{df}\right) \cap L^2\left([0,T] : \dot{H}^2\left(\mathbb{R}^3\right)\right)$ is a mild solution to the Navier-Stokes equation with external force $f \in L^2\left([0,T]; L^2\left(\mathbb{R}^3\right)\right)$ then*

$S = \nabla_{sym} u$ is an L^2 mild solution to the Navier-Stokes strain equation and $\omega = \nabla \times u$ is an L^2 mild solution to the vorticity equation

Proof. By hypothesis u satisfies

$$u(x,t) = e^{\nu t \Delta} u^0 + \int_0^t e^{\nu(t-\tau)\Delta} P_{df} \left(-(u \cdot \nabla) u + f\right) d\tau. \tag{2.33}$$

Stated in terms of the pressure, rather than the projection P_{df}, this statement becomes

$$u(x,t) = e^{\nu t \Delta} u^0 + \int_0^t e^{\nu(t-\tau)\Delta} \left(-(u \cdot \nabla) u - \nabla p + f\right) d\tau. \tag{2.34}$$

When differentiating a convolution, the derivative can be applied to either function being convolved, so taking the curl of (2.34) and applying the differential operator to $-(u \cdot \nabla) u - \nabla p + f$ rather than the heat kernel, we find that

$$\omega(\cdot,t) = e^{\nu t \Delta} \omega^0 + \int_0^t e^{\nu(t-\tau)\Delta} \left(-(u \cdot \nabla)\omega + S\omega - \nabla \times f\right)(\cdot,\tau) d\tau. \tag{2.35}$$

Likewise if we take the symmetric part of the gradient of (2.34) we find that

$$S(\cdot,t) = e^{\nu t \Delta} S^0 + \int_0^t e^{\nu(t-\tau)\Delta} \left(-(u \cdot \nabla) S - S^2 - \frac{1}{4}\omega \otimes \omega + \frac{1}{4}|\omega|^2 I_3 - \operatorname{Hess}(p) + \nabla_{sym} f\right)(\cdot,\tau) d\tau. \tag{2.36}$$

This completes the proof. □

We will note that Proposition 2.7 and Theorem 1.5 imply the existence of L^2 mild solutions to the strain and vorticity equations, simply by taking the curl or symmetric gradient of $\dot{H}^1$ mild solutions to the Navier-Stokes equation.

Chapter 3

Isometries and the enstrophy growth identity

We have already shown that S and ω are related to each other by zeroth order pseudo-differential operators. Because these zeroth order operators are related to the Riesz transform, which is bounded from L^q to L^q for $1 < q < +\infty$, the L^q norms of strain and vorticity are equivalent, but we will only have Calderon-Zygmund type estimates, so our control will be very bad. More precisely, for all $1 < q < +\infty$, there exists $B_q > 0$, such that $\frac{1}{B_q}\|\omega\|_{L^q} \leq \|S\|_{L^q} \leq B_q\|\omega\|_{L^q}$. We can say something much stronger in the case of L^2, and in fact for every Hilbert space $\dot{H}^\alpha, -\frac{3}{2} < \alpha < \frac{3}{2}$.

Proposition 3.1 (Hilbert space isometries for strain and vorticity)**.** *For all $-\frac{3}{2} < \alpha < \frac{3}{2}$, and for all u divergence free in the sense that $\xi \cdot \hat{u}(\xi) = 0$ almost everywhere,*

$$\|S\|_{\dot{H}^\alpha}^2 = \|A\|_{\dot{H}^\alpha}^2 = \frac{1}{2}\|\omega\|_{\dot{H}^\alpha}^2 = \frac{1}{2}\|u\|_{\dot{H}^{\alpha+1}}^2. \tag{3.1}$$

Proof. First fix s, $-\frac{3}{2} < s < \frac{3}{2}$. We will begin relating the H^s norms of the anti-symmetric part and the vorticity. Recall that

$$A = \frac{1}{2}\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}, \tag{3.2}$$

Therefore, for all $x \in \mathbb{R}^3$,

$$|(-\Delta)^{\frac{s}{2}}A(x)|^2 = \frac{1}{2}|(-\Delta)^{\frac{s}{2}}\omega(x)|^2. \tag{3.3}$$

Because in general we have that $\|f\|_{\dot{H}^s} = \|(-\Delta)^{\frac{s}{2}}f\|_{L^2}$, it immediately follows that

$$\|A\|_{\dot{H}^s}^2 = \frac{1}{2}\|\omega\|_{\dot{H}^s}^2. \tag{3.4}$$

Because u is divergence free, in Fourier space

$$|\hat{\omega}(\xi)| = |2\pi i\xi \times \hat{u}(\xi)| = 2\pi|\xi||\hat{u}(\xi)| = |\widehat{\nabla \otimes u}(\xi)|. \tag{3.5}$$

From this we can conclude that

$$\|\omega\|_{\dot{H}^s}^2 = \|\nabla \otimes u\|_{\dot{H}^s}^2 = \|u\|_{\dot{H}^{s+1}}^2. \tag{3.6}$$

Finally we will compute

$$\left|(-\Delta)^{\frac{s}{2}} (\nabla \otimes u)\right|^2 = \operatorname{tr}\left(\left((-\Delta)^{\frac{s}{2}} S + (-\Delta)^{\frac{s}{2}} A\right)\left((-\Delta)^{\frac{s}{2}} S^* + (-\Delta)^{\frac{s}{2}} A^*\right)\right). \tag{3.7}$$

However, we know that the trace of the product of a symmetric matrix and an antisymmetric matrix is always zero, so we can immediately see that

$$\left|(-\Delta)^{\frac{s}{2}} (\nabla \otimes u)\right|^2 = \left|(-\Delta)^{\frac{s}{2}} S\right|^2 + \left|(-\Delta)^{\frac{s}{2}} A\right|^2. \tag{3.8}$$

From this it follows that

$$\|\nabla \otimes u\|_{\dot{H}^s}^2 = \|S\|_{\dot{H}^s}^2 + \|A\|_{\dot{H}^s}^2, \tag{3.9}$$

but we have already established that

$$\|A\|_{\dot{H}^s}^2 = \frac{1}{2}\|\nabla \otimes u\|_{\dot{H}^s}^2, \tag{3.10}$$

so we can conclude that

$$\|A\|_{\dot{H}^s}^2 = \|S\|_{\dot{H}^s}^2 = \frac{1}{2}\|\nabla \otimes u\|_{\dot{H}^s}^2. \tag{3.11}$$

This concludes the proof. □

Now that we have established this isometry between vorticity and strain, we will proceed to proving an identity for enstrophy growth involving only S, not the interaction of S and ω.

Theorem 3.2 (Enstrophy growth identity)**.** *Suppose* $S \in C\left([0,T]; L^2_{st}\right) \cap L^2\left([0,T] : \dot{H}^1(\mathbb{R}^3)\right)$ *is a mild solution to the Navier-Stokes strain equation. Then*

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 = -2\nu\|S\|_{\dot{H}^1}^2 - \frac{4}{3}\int_{\mathbb{R}^3} \operatorname{tr}(S^3) + \langle -\Delta u, f\rangle, \tag{3.12}$$

almost everywhere $t \in (0,T]$.

Proof. Using (1.4), we can compute the rate of change of enstrophy

$$\partial_t \frac{1}{2}\|\omega(\cdot,t)\|_{L^2}^2 = -\nu\langle -\Delta\omega, \omega\rangle - \langle (u\cdot\nabla)\omega, \omega\rangle + \langle S\omega, \omega\rangle - \langle \nabla \times f, \omega\rangle. \tag{3.13}$$

Next we can integrate by parts to show that $\langle \nabla \times f, \omega\rangle = \langle f, -\Delta u\rangle$ and $\langle \omega, (u\cdot\nabla)\omega\rangle = 0$, using the divergence free condition in the latter case. Therefore we find that

$$\partial_t \frac{1}{2}\|\omega(\cdot,t)\|_{L^2}^2 = -\nu\|\omega\|_{\dot{H}^1}^2 + \langle S; \omega\otimes\omega\rangle + \langle -\Delta u, f\rangle. \tag{3.14}$$

This is the standard identity for enstrophy growth, based on the interaction of the Strain matrix and the vorticity. See chapter 7 in [35] for more details. We can use the isometry in

Proposition 2.4 to restate (3.14) in terms of strain:

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 = -2\nu \|S\|_{\dot{H}^1}^2 + \langle S; \omega \otimes \omega \rangle + \langle -\Delta u, f \rangle . \tag{3.15}$$

However we can also calculate the L^2 growth of the strain tensor directly from our evolution equation for the strain tensor (1.5),

$$\begin{aligned} \partial_t \|S(\cdot,t)\|_{L^2}^2 = & -2\nu \langle -\Delta S, S \rangle - 2 \langle (u\cdot\nabla) S, S \rangle - 2 \left\langle S^2, S \right\rangle \\ & - \frac{1}{2} \langle \omega \otimes \omega; S \rangle - 2 \langle \mathrm{Hess}(p), S \rangle + \frac{1}{2} \left\langle |\omega|^2 I_3, S \right\rangle + 2 \langle \nabla_{sym} f, S \rangle . \end{aligned} \tag{3.16}$$

Integrating by parts we know that $\langle (u\cdot\nabla)S, S\rangle = 0$. Note that $S \in C\left([0,T], L^2\right) \cap L^2\left((0,T], \dot{H}^1\right)$. In particular this implies that $S(\cdot,t), \omega(\cdot,t) \in L^2 \cap L^6$ almost everywhere $0 < t \leq T$. This means that $S(\cdot,t), \omega(\cdot,t) \in L^3$, so $\langle S; \omega \otimes \omega \rangle$ and $\int \mathrm{tr}(S^3)$ are both well defined. This also means that $|\omega(\cdot,t)|^2, \mathrm{Hess}(p)(\cdot,t) \in L^2$ almost everywhere $0 < t \leq T$. Therefore we can apply Proposition 2.4 and find that $|\omega|^2 I_3, \mathrm{Hess}(p) \in \left(L^2_{st}\right)^\perp$, so

$$\left\langle S, \frac{1}{2}|\omega|^2 I_3 \right\rangle = 0, \tag{3.17}$$

$$\langle \mathrm{Hess}(p), S \rangle = 0. \tag{3.18}$$

Now we can use the fact that S is symmetric to compute that

$$\left\langle S^2, S \right\rangle = \int_{\mathbb{R}^3} \mathrm{tr}(S^3). \tag{3.19}$$

We also compute that

$$\begin{aligned} 2 \langle \nabla_{sym} f, S \rangle &= 2 \langle \nabla \otimes f, S \rangle && (3.20) \\ &= \langle f, -2\,\mathrm{div}(S) \rangle && (3.21) \\ &= \langle f, -\Delta u \rangle . && (3.22) \end{aligned}$$

Putting all of these together we find that

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 = -2\nu \|S\|_{\dot{H}^1}^2 - \frac{1}{2} \langle S; \omega \otimes \omega \rangle - 2 \int_{\mathbb{R}^3} \mathrm{tr}(S^3) + \langle -\Delta u, f \rangle . \tag{3.23}$$

Note that the vortex stretching term $\langle S; \omega \otimes \omega \rangle$ has the opposite sign as in the well known identity for enstrophy growth (3.14). Taking advantage of this fact, we will add $\frac{1}{3}$ (3.15) to $\frac{2}{3}$ (3.23) to cancel the term $\langle S, \omega \otimes \omega \rangle$, and we find

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 = -2\nu \|S\|_{\dot{H}^1}^2 - \frac{4}{3} \int_{\mathbb{R}^3} \mathrm{tr}(S^3) + \langle -\Delta u, f \rangle . \tag{3.24}$$

Finally we will note that because the subcritical quantity $\|S(\cdot,t)\|_{L^2}$ is controlled uniformly on $[0,T]$, the smoothing due to the heat kernel guarantees that S is smooth when $f = 0$, so the identity (3.12) can be understood as a derivative of a smooth quantity in the classical sense. When $f \neq 0$, the expression for $\partial_t \|S(\cdot,t)\|_{L^2}^2$ is integrable in time because

$S \in L^2\left([0,T]; \dot{H}^1\left(\mathbb{R}^3\right)\right)$, and so must be the derivative of the continuous function $\|S(\cdot,t)\|_{L^2}^2$ almost everywhere in time. □

Now that we have improved the estimate for enstrophy growth from one that involved the interaction of the vorticity and the strain tensor to an estimate that only involves the strain tensor. We can still extract more geometric information about the flow, however. The identity for enstrophy growth in Theorem 3.2 can also be expressed in terms of $\det(S)$.

Corollary 3.3 (Alternative enstrophy growth identity). *Suppose $S \in C\left([0,T]; L^2_{st}\right) \cap L^2\left([0,T] : \dot{H}^1(\mathbb{R}^3)\right)$ is a mild solution to the Navier-Stokes strain equation. Then*

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 = -2\nu \|S\|_{\dot{H}^1}^2 - 4\int_{\mathbb{R}^3} \det(S) + \langle -\Delta u, f\rangle\,, \tag{3.25}$$

almost everywhere $0 < t \leq T$.

Proof. Because S is symmetric it will be diagonalizable with three real eigenvalues, and because S is trace free, we have $\operatorname{tr}(S) = \lambda_1 + \lambda_2 + \lambda_3 = 0$. This allows us to relate $\operatorname{tr}(S^3)$ to $\det(S)$ by

$$\begin{aligned}
\operatorname{tr}(S^3) &= \lambda_1^3 + \lambda_2^3 + \lambda_3^3 && (3.26)\\
&= \lambda_1^3 + \lambda_2^3 + (-\lambda_1 - \lambda_2)^3 && (3.27)\\
&= -3\lambda_1^2\lambda_2 - 3\lambda_1\lambda_2^2 && (3.28)\\
&= -3(\lambda_1 + \lambda_2)\lambda_1\lambda_2 && (3.29)\\
&= 3\lambda_1\lambda_2\lambda_3 && (3.30)\\
&= 3\det(S). && (3.31)
\end{aligned}$$

So we can write our growth estimate as:

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 = -2\nu \|S\|_{\dot{H}^1}^2 - 4\int_{\mathbb{R}^3} \det(S) + \langle -\Delta u, f\rangle\,. \tag{3.32}$$

This completes the proof. □

Remark 3.4. *As mentioned in the introduction, Dongho Chae proved the analogous result,*

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 = -4\int_{\mathbb{R}^3} \det(S), \tag{3.33}$$

in the context of smooth solutions to the Euler equation with no external force [5]. In this paper he shows directly that

$$\partial_t \frac{1}{2}\|\nabla \otimes u(\cdot,t)\|_{L^2}^2 = \langle (u\cdot\nabla)\, u, \Delta u\rangle = -\int_{\mathbb{R}^3} \operatorname{tr}(S^3) + \frac{1}{4}\langle S; \omega\otimes\omega\rangle\,. \tag{3.34}$$

In the context of the Euler equation, the familiar estimate for enstrophy growth following from the vorticity equation is

$$\partial_t \frac{1}{2}\|\nabla \otimes u(\cdot,t)\|_{L^2}^2 = \partial_t \frac{1}{2}\|\omega(\cdot,t)\|_{L^2}^2 = \langle S; \omega\otimes\omega\rangle\,. \tag{3.35}$$

Adding $\frac{4}{3}$ (3.34) *and* $-\frac{1}{3}$ (3.35), *it follows that*

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 = \partial_t \frac{1}{2}\|\nabla\otimes u(\cdot,t)\|_{L^2}^2 = -\frac{4}{3}\int_{\mathbb{R}^3} \operatorname{tr}(S^3) = -4\int_{\mathbb{R}^3} \det(S). \tag{3.36}$$

The identity for enstrophy growth in Corollary 3.3 gives us a significantly better understanding of enstrophy production than the classical enstrophy growth identity (3.14), because we now have the growth controlled solely in terms of the strain tensor, rather than both the strain tensor and the vorticity. This estimate also provides analytical confirmation of the well known result that the vorticity tends to align with the eigenvector corresponding to the intermediate eigenvalue of the strain matrix [19, 53]. Comparing the identities in (3.12), (3.14), and (3.25) we see that

$$\langle S, \omega\otimes\omega\rangle = -4\int_{\mathbb{R}^3} \det(S) = -\frac{4}{3}\int \operatorname{tr}(S^3). \tag{3.37}$$

When $\det(S)$ tends to be positive, it means there are two negative eigenvalues and one positive eigenvalue, so $\langle S, \omega\otimes\omega\rangle$ being negative means the vorticity tends to align, on average when integrating over the whole space, with the negative eigenspaces. Likewise, when $\det(S)$ tends to be negative, it means there are two positive eigenvalues and one negative eigenvalue, so $\langle S, \omega\otimes\omega\rangle$ being positive means the vorticity tends to align, on average when integrating over the whole space, with the positive eigenspaces. When $\det(S)$ tends to be zero when integrated over the whole space, the vorticity tends clearly to be aligned with the intermediate eigenvalue, as well. Growth in all cases geometrically corresponds to the strain matrix S stretching in two directions, while strongly contracting in the third direction.

Chapter 4

Maximal enstrophy growth

In this chapter, we will consider the maximal rate of enstrophy growth. We will prove an upper bound on the rate of enstrophy growth, which will also allow us to improve the constants in some small initial data results for Navier-Stokes. Throughout this chapter we will consider the Navier-Stokes equation with no external force, setting $f = 0$. We will begin by bounding $-4\det(S)$ in terms of $|S|^3$, and see what this matrix looks like in the sharp case of this bound.

Proposition 4.1 (Determinant bound)**.** *Let M be a three by three, symmetric, trace free matrix, then*

$$-4\det(M) \leq \frac{2}{9}\sqrt{6}|M|^3, \tag{4.1}$$

with equality if and only if $-\frac{1}{2}\lambda_1 = \lambda_2 = \lambda_3$, where $\lambda_1 \leq \lambda_2 \leq \lambda_3$ are the eigenvalues of M.

Proof. In the case where $M = 0$, it holds trivially. In the case where $M \neq 0$, then we have $\lambda_1 < 0, \lambda_3 > 0$. This allows us to define a parameter $r = -\frac{\lambda_1}{\lambda_3}$. The two parameters λ_3 and r completely define the system because $\lambda_1 = -r\lambda_3$ and $\lambda_2 = -\lambda_1 - \lambda_3 = (r-1)\lambda_3$. We must now say something about the range of values the parameter r can take on. $\lambda_1 \leq \lambda_2 \leq \lambda_3$ implies that $-r \leq r - 1 \leq 1$, so therefore $\frac{1}{2} \leq r \leq 2$. Now we can observe that

$$-4\det(M) = -4\lambda_1\lambda_2\lambda_3 = 4r(r-1)\lambda_3^3, \tag{4.2}$$

and that

$$|M|^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = (r^2 + (r-1)^2 + 1)\lambda_3^2 = (2r^2 - 2r + 2)\lambda_3^2. \tag{4.3}$$

We can combine the two equations above to find that

$$-4\det(M) = \sqrt{2}\frac{r^2 - r}{(r^2 - r + 1)^{\frac{3}{2}}}|M|^3. \tag{4.4}$$

Next we will observe that

$$\sqrt{2}\frac{r^2 - r}{(r^2 - r + 1)^{\frac{3}{2}}}\Big|_{r=2} = \sqrt{2}\frac{2}{3\sqrt{3}} = \frac{2}{9}\sqrt{6}. \tag{4.5}$$

This is exactly as we want, as $r = 2$ is the case that we want to correspond to equality. Finally we observe that for all $\frac{1}{2} \leq r < 2$, we have that

$$\sqrt{2}\frac{r^2 - r}{(r^2 - r + 1)^{\frac{3}{2}}} < \frac{2}{9}\sqrt{6}. \tag{4.6}$$

This completes the proof. □

The structure of the quadratic term in relation to $r = -\frac{\lambda_1}{\lambda_3} = 2$, the extremal case, will be investigated further in chapter 8 when we consider blow up for a toy model ODE for the Navier-Stokes strain equation. It is an interesting open question whether or not there is a strain matrix which saturates this inequality globally in space. More precisely, does there exist an $S \in L^2_{st}$, not identically zero, such that $\lambda_2(x) = \lambda_3(x)$ almost everywhere $x \in \mathbb{R}^3$?

Corollary 4.2 (Bound on enstrophy growth)**.** *Suppose* $S \in C\left([0,T]; L^2_{st}\right) \cap L^2\left([0,T] : \dot{H}^1(\mathbb{R}^3)\right)$ *is a mild solution to the Navier-Stokes strain equation. Then for all* $0 < t \leq T$,

$$\partial_t \|S(\cdot,t)\|^2_{L^2} \leq -2\nu \|S\|^2_{\dot{H}^1} + \frac{2}{9}\sqrt{6}\int_{\mathbb{R}^3} |S|^3. \tag{4.7}$$

Proof. This corollary follows immediately from Proposition 4.1 and Corollary 3.3. □

Using Corollary 4.2 and the fractional Sobolev inequality we will be able to prove a cubic differential inequality for the growth of enstrophy. The sharp fractional Sobolev inequality was first proven by Lieb [37].

Lemma 4.3 (Fractional Sobolev inequality)**.** *Let* $C_1 = \frac{1}{2^{\frac{1}{6}}\pi^{\frac{1}{3}}}$. *Then for all* $f \in \dot{H}^{-\frac{1}{2}}\left(\mathbb{R}^3\right)$,

$$\|f\|_{\dot{H}^{-\frac{1}{2}}} \leq C_1 \|f\|_{L^{\frac{3}{2}}}, \tag{4.8}$$

and for all $f \in L^3\left(\mathbb{R}^3\right)$

$$\|f\|_{L^3} \leq C_1 \|f\|_{\dot{H}^{\frac{1}{2}}}. \tag{4.9}$$

We will note in particular that the two inequalities in Lemma 4.3 are dual to each other because L^3 and $L^{\frac{3}{2}}$ are dual spaces, and $\dot{H}^{\frac{1}{2}}$ and $\dot{H}^{-\frac{1}{2}}$ are dual spaces, which is why the two ineqaulities have the same sharp constant. For more references on this inequality see also chapter 4 in [38] and the summary of these results in [15]. We can now prove a cubic differential inequality for the growth of enstrophy.

Proposition 4.4 (Cubic bound on enstrophy growth)**.** *Let* $u \in C\left([0,T_{max}); \dot{H}^1_{df}\right)$ *be a mild solution to the Navier-Stokes equation. Then for all* $0 < t < T_{max}$, *we have* $E'(t) \leq \frac{1}{3{,}456\pi^4\nu^3}E(t)^3$. *Furthermore, if* $u \in C\left([0,T_{max}); H^1_{df}\right)$, *then for all* $0 < t < T_{max}$, *we have* $K'(t) = -2\nu E(t)$.

Proof. The equality $K'(t) = -2\nu E(t)$ is the classic energy equality for smooth solutions of the Navier-Stokes equations first proven by Leray [36]. We will now prove the first inequality. We begin with the estimate for enstrophy growth in Corollary 4.2:

$$\partial_t \|S(\cdot,t)\|^2_{L^2} \leq -2\nu \|S\|^2_{\dot{H}^1} + \frac{2}{9}\sqrt{6}\|S\|^3_{L^3}. \tag{4.10}$$

Next we apply the fractional Sobolev inequality in Lemma 4.3 and interpolate between L^2 and $\dot{H}^1$ and find

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 \leq -2\nu \|S\|_{\dot{H}^1}^2 + \frac{2}{9}\sqrt{6}\frac{1}{\sqrt{2}\pi}\|S\|_{\dot{H}^{\frac{1}{2}}}^3 \tag{4.11}$$

$$\leq -2\nu \|S\|_{\dot{H}^1}^2 + \frac{2}{3^{\frac{3}{2}}\pi}\|S\|_{L^2}^{\frac{3}{2}}\|S\|_{\dot{H}^1}^{\frac{3}{2}}. \tag{4.12}$$

Substituting $r = \|S\|_{\dot{H}^1}$, we find

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 \leq \sup_{r\geq 0} -2\nu r^2 + \frac{2}{3^{\frac{3}{2}}\pi}\|S\|_{L^2}^{\frac{3}{2}} r^{\frac{3}{2}}. \tag{4.13}$$

Let $B = \frac{1}{3^{\frac{3}{2}}\pi}\|S\|_{L^2}^{\frac{3}{2}}$, and let

$$f(r) = -2\nu r^2 + 2Br^{\frac{3}{2}}. \tag{4.14}$$

Computing the derivative we find that

$$f'(r) = -4\nu r + 3Br^{\frac{1}{2}}. \tag{4.15}$$

This means f has a global maximum at $r_0 = \left(\frac{3B}{4\nu}\right)^2$. Plugging in we find that

$$f(r_0) = -2\nu\left(\frac{3B}{4\nu}\right)^4 + 2B\left(\frac{3B}{4\nu}\right)^3 = \frac{3^3 B^4}{2^7\nu^3}. \tag{4.16}$$

Recalling that $B = \frac{1}{3^{\frac{3}{2}}\pi}\|S\|_{L^2}^{\frac{3}{2}}$ and that f attains its global maximum at r_0, we conclude that

$$\sup_{r\geq 0} -2\nu r^2 + \frac{2}{3^{\frac{3}{2}}\pi}\|S\|_{L^2}^{\frac{3}{2}} r^{\frac{3}{2}} = f(r_0) = \frac{1}{3,456\pi^4\nu^3}\|S\|_{L^2}^6. \tag{4.17}$$

Therefore

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 \leq \frac{1}{3,456\pi^4\nu^3}\|S\|_{L^2}^6. \tag{4.18}$$

This completes the proof. □

The cubic bound on the growth of enstrophy is not new, however a closer analysis of the strain allows a major improvement in the constant. The best known estimate [2, 39, 45] for enstrophy growth that does not make use of the identity for enstrophy growth in terms of the determinant of strain in Proposition 3.12 is

$$E'(t) \leq \frac{27}{8\pi^4\nu^3}E(t)^3. \tag{4.19}$$

The author then improved the constant in this inequality significantly; using Proposition 3.12, the author proved in [41] a cubic differential inequality controlling the growth of enstrophy,

$$E'(t) \leq \frac{1}{1,458\pi^4\nu^3}E(t)^3, \tag{4.20}$$

in the case where $\nu = 1$, although there is no loss of generality in the proof: the proof in the case of $\nu > 0$ is entirely analogous. The proof in [41] relied on the sharp Sobolev inequality proven by Talenti [50], which we will state below.

Lemma 4.5 (Sobolev inequality). *Let* $C_2 = \frac{1}{\sqrt{3}}\left(\frac{2}{\pi}\right)^{\frac{2}{3}}$. *Then for all* $f \in L^6\left(\mathbb{R}^3\right)$

$$\|f\|_{L^6} \leq C_2\|\nabla f\|_{L^2} = C_2\|f\|_{\dot{H}^1}, \tag{4.21}$$

and for all $f \in L^{\frac{6}{5}}\left(\mathbb{R}^3\right)$

$$\|f\|_{\dot{H}^{-1}} \leq C_2\|f\|_{L^{\frac{6}{5}}}. \tag{4.22}$$

As in the fractional Sobolev inequality, we will note in particular that the two inequalities in Lemma 4.5 are dual to each other because L^6 and $L^{\frac{6}{5}}$ are dual spaces, and $\dot{H}^1$ and $\dot{H}^{-1}$ are dual spaces, which is why the constant in both inequalities is the same.

In [41], the author first interpolated between L^2 and L^6 and then applied Lemma 4.5, showing

$$\|S\|_{L^3}^3 \leq \|S\|_{L^2}^{\frac{3}{2}}\|S\|_{L^6}^{\frac{3}{2}} \leq C_2^{\frac{3}{2}}\|S\|_{L^2}^{\frac{3}{2}}\|S\|_{\dot{H}^1}^{\frac{3}{2}}. \tag{4.23}$$

It is possible to obtain a sharper constant by first applying the fractional Sobolev inequality and then interpolating between L^2 and $\dot{H}^1$. Proceeding this way, we conclude

$$\|S\|_{L^3}^3 \leq C_1^3\|S\|_{\dot{H}^{\frac{1}{2}}}^3 \leq C_1^3\|S\|_{L^2}^{\frac{3}{2}}\|S\|_{\dot{H}^1}^{\frac{3}{2}}. \tag{4.24}$$

Because $C_1^3 < C_2^{\frac{3}{2}}$, using the fractional Sobolev inequality results in a sharper bound on enstrophy growth.

Using the bounds in Proposition 4.4, we will be able to prove a small data global existence result in terms of the product of energy and enstrophy.

Proposition 4.6 (Small data in terms of energy and enstrophy). *Suppose* $u^0 \in H^1_{df}$. *If* $K_0E_0 < 6,912\pi^4\nu^4$, *or equivalently, if* $\|u^0\|_{L^2}^2\|\omega^0\|_{L^2}^2 < 27,648\pi^4\nu^4$, *then* $T_{max} = +\infty$. *That is, there exists a unique, smooth solution to the Navier-Stokes equation* $u \in C\left([0,+\infty); H^1_{df}\right)$ *with* $u(\cdot,0) = u^0$. *Furthermore, for all* $t > 0$,

$$E(t) \leq \frac{E_0}{1 - \frac{1}{6{,}912\pi^4\nu^4}E_0K_0}. \tag{4.25}$$

Proof. Let $f(t) = K(t)E(t)$. Then we can use the product rule and Proposition 4.4 to compute that

$$f'(t) \leq -2\nu E(t)^2 + K(t)\frac{E(t)^3}{3,456\pi^4\nu^3} \tag{4.26}$$

$$\leq -2\nu E(t)^2\left(1 - \frac{f(t)}{6,912\pi^4\nu^4}\right). \tag{4.27}$$

Therefore, if $f(t) < 6,912\pi^4\nu^4$, then $f'(t) < 0$. This implies that if $f(0) < 6,912\pi^4\nu^4$, then for all $0 < t < T_{max}$, we have $f(t) < 2,916\pi^4\nu^4$. Interpolating between L^2 and $\dot{H}^1$, we can see that

$$\|u\|_{L^3}^4 \leq C_1^4\|u(\cdot,t)\|_{\dot{H}^{\frac{1}{2}}}^4 \leq C_1^4\|u(\cdot,t)\|_{L^2}^2\|u(\cdot,t)\|_{\dot{H}^1}^2 = 4C_1^4K(t)E(t) = 4C_1^4f(t). \tag{4.28}$$

Šverák, Seregin, and Escauriaza showed in [16] that if $T_{max} < +\infty$, then

$$\limsup_{t \to T_{max}} \|u(\cdot, t)\|_{L^3} = +\infty. \tag{4.29}$$

Therefore, $f(0) < 6,912\pi^4\nu^4$ implies that $T_{max} = +\infty$.

Now we will consider the bound on enstrophy globally in time. We know that

$$E'(t) \leq \frac{1}{3,456\pi^4\nu^4} E(t)^3 = \frac{1}{3,456\pi^4\nu^4} E(t)E(t)^2 \tag{4.30}$$

Fix $t > 0$. Integrating this differential inequality and making use of the energy inequality, we find

$$\frac{1}{E_0} - \frac{1}{E(t)} \leq \frac{1}{3,456\pi^4\nu^4} \int_0^t E(\tau) d\tau, \tag{4.31}$$

$$\leq \frac{1}{6,912\pi^4\nu^4} K_0. \tag{4.32}$$

Rearranging terms we find that

$$E(t) \leq \frac{E_0}{1 - \frac{1}{6,912\pi^4\nu^4} E_0 K_0}. \tag{4.33}$$

We took $t > 0$ arbitrary, so this completes the proof. □

Similar estimates were considered by Protas and Ayala in [2]. In particular, they proved that if $E_0 K_0 < \frac{16\pi^4\nu^4}{27}$, then there must be a smooth solution globally in time, and enstrophy is bounded uniformly in time, with $E(t) < \frac{E_0}{1 - \frac{27}{16\pi^4\nu^4} E_0 K_0}$, for all $t > 0$. By improving the constant for enstrophy growth instantaneously in time, we significantly expand the set of initial data for which we are guaranteed to have global smooth solutions. The initial data must be in H^1 for the product of initial energy and initial enstrophy to be bounded, so the condition in Proposition 4.6 is more restrictive than the condition in the small initial data results for $\dot{H}^{\frac{1}{2}}$ [18], L^3 [26], or BMO^{-1} [28]. However, the product of energy and enstrophy is the most physically relevant of the scale invariant quantities, and so we are able to sharpen the bound on the size initial data for which solutions are guaranteed to be smooth globally in time more effectively in this case by taking advantage of the structure of the nonlinear term. The proofs of the bounds for small initial data in $\dot{H}^{\frac{1}{2}}$, L^3, and BMO^{-1} would all work just as well for the Navier-Stokes model equation introduced by Tao [51], as would the estimates used by Protas and Ayala. The estimates used to prove Proposition 4.6, on the other hand, take advantage of the structure of the evolution equations for vorticity and strain, and the constraint spaces, and so would not hold with the same constants in Tao's model equation.

We will now prove an immediate corollary of Proposition 4.6, that any solution that blows up in finite time must be bounded away from zero that will be useful later on.

Corollary 4.7. *Suppose $u \in C\left([0, T_{max}); H^1_{df}\right)$ is a mild solution to the Navier-Stokes equation and $T_{max} < +\infty$, then for all $0 \leq t < T_{max}$,*

$$K(t)E(t) \geq 6,912\pi^4\nu^4. \tag{4.34}$$

Proof. We will prove the contrapositive. Suppose that there exists $0 \leq t < T_{max}$ such that $K(t)E(t) < 6,912\pi^4\nu^4$. Then by Proposition 4.6, $u(\cdot,t)$ generates a global smooth solution to the Navier-Stokes equations. Smooth solutions of the Navier-Stokes equations are unique, so if $u(\cdot,t)$ generates a global smooth solution to the Navier-Stokes equations, then so does u^0, and so we conclude that $T_{max} = +\infty$. □

Using Proposition 4.4, we can also prove an upper bound on blowup time, assuming there is finite time blowup, in terms of the initial energy, and a lower bound on blowup time in terms of the initial enstrophy. We will prove these results below.

Proposition 4.8 (Upper bound on T_{max})**.** *For all initial data $u^0 \in H^1_{df}$, either $T_{max} \leq \frac{K_0^2}{13{,}824\pi^4\nu^5}$ or $T_{max} = +\infty$.*

Proof. Suppose toward contradiction that $\frac{K_0^2}{13{,}824\pi^4\nu^5} < T_{max} < +\infty$. We know from the energy equality that

$$\int_0^{T_{max}} E(\tau)d\tau \leq \frac{1}{2\nu}K_0. \tag{4.35}$$

This implies that there exists $t \in (0, T_{max})$ such that $T_{max}E(t) \leq \frac{1}{2\nu}K_0$. We also know from the energy equality that $K(t) < K_0$. Combining these two inequalities as well as our hypothesis on T_{max}, we find that

$$E(t)K(t) < \frac{K_0^2}{2\nu T_{max}} < 6,912\pi^4\nu^4. \tag{4.36}$$

Using Proposition 4.6, this implies that if we take $u(\cdot,t)$ to be initial data, it generates a global smooth solution, which contradicts the assumption that $T_{max} < +\infty$. The uniqueness of strong solutions means that if $u(\cdot,t)$ generates a global smooth solution for some $0 < t < T_{max}$, then so does u^0. This contradicts the assumption that $T_{max} < +\infty$, and completes the proof. □

Proposition 4.9 (Lower bound on T_{max})**.** *For all initial data $u^0 \in \dot{H}^1_{df}$, and for all $0 < t < \frac{1{,}728\pi^4\nu^3}{E_0^2}$,*

$$E(t) \leq \frac{E_0}{\sqrt{1 - \frac{E_0^2}{1{,}728\pi^4\nu^3}t}}. \tag{4.37}$$

In particular, for all $u^0 \in \dot{H}^1_{df}$, $T_{max} \geq \frac{1{,}728\pi^4\nu^3}{E_0^2}$

Proof. Integrating the differential inequality

$$\partial_t E(t) \leq \frac{1}{3,456\pi^4\nu^3}E(t)^3, \tag{4.38}$$

we find that for all $0 < t < \frac{1{,}728\pi^4\nu^3}{E_0^2}$

$$\frac{1}{E_0^2} - \frac{1}{E(t)^2} \leq \frac{1}{1,728\pi^4\nu^3}t. \tag{4.39}$$

Rearranging terms we find that for all $0 < t < \frac{1{,}728\pi^4\nu^3}{E_0^2}$,

$$E(t) \leq \frac{E_0}{\sqrt{1 - \frac{E_0^2}{1{,}728\pi^4\nu^3}t}}. \tag{4.40}$$

The mild solution can be continued further in time as long as enstrophy is bounded, so this completes the proof. □

Chapter 5

Regularity criteria

In this chapter we will prove Theorem 1.7, as well as some immediate corollaries that were also stated in the introduction. Before we can prove these regularity criteria, we will need to prove a lemma bounding the growth of enstrophy in terms of λ_2^+.

Lemma 5.1 (Middle eigenvalue determinant bound)**.** *Suppose* $S \in C\left([0,T]; L^2_{st}\right) \cap L^2\left([0,T] : \dot{H}^1(\mathbb{R}^3)\right)$ *is a mild solution to the Navier-Stokes strain equation with external force* $f \in L^2_{loc}\left([0,T]; L^2\left(\mathbb{R}^3\right)\right)$, *and* $S(x)$ *has eigenvalues* $\lambda_1(x) \leq \lambda_2(x) \leq \lambda_3(x)$. *Define*

$$\lambda_2^+(x) = \max\{\lambda_2(x), 0\}. \tag{5.1}$$

Then

$$-\det(S) \leq \frac{1}{2}|S|^2 \lambda_2^+. \tag{5.2}$$

and for all $0 < t \leq T$,

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 \leq -\nu \|S\|_{\dot{H}^1}^2 + 2\int_{\mathbb{R}^3} \lambda_2^+ |S|^2 + \frac{2}{\nu}\|f\|_{L^2}^2. \tag{5.3}$$

Proof. We will begin by noting that $\lambda_1 \leq 0$ and $\lambda_3 \geq 0$, so clearly, $-\lambda_1\lambda_3 \geq 0$. This implies that

$$-\det(S) = (-\lambda_1\lambda_3)\lambda_2 \leq (-\lambda_1\lambda_3)\lambda_2^+. \tag{5.4}$$

Next we can apply Young's Inequality to show that

$$-\lambda_1\lambda_3 \leq \frac{1}{2}(\lambda_1^2 + \lambda_3^2) \leq \frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) = \frac{1}{2}|S|^2. \tag{5.5}$$

We can combine these inequalities and conclude that

$$-\det(S) \leq \frac{1}{2}|S|^2 \lambda_2^+. \tag{5.6}$$

Next apply Hölder's inequality, Proposition 3.1, and Young's inequality to find

$$\langle -\Delta u, f\rangle \leq \|-\Delta u\|_{L^2}\|f\|_{L^2} \tag{5.7}$$

$$= \sqrt{2}\|S\|_{\dot{H}^1}\|f\|_{L^2} \tag{5.8}$$

$$\leq \nu\|S\|_{\dot{H}^1}^2 + \frac{2}{\nu}\|f\|_{L^2}^2. \tag{5.9}$$

Recall from Corollary 3.3, that

$$\partial_t\|S\|_{L^2}^2 = -2\nu\|S\|_{\dot{H}^1}^2 - 4\int \det(S) + \langle -\Delta u, f\rangle, \tag{5.10}$$

and this completes the proof. □

With this bound, we are now ready to prove the main result of this chapter. This is Theorem 1.7 from the introduction, which is restated here for the reader's convenience.

Theorem 5.2 (Middle eigenvalue of strain characterizes the blow-up time)**.** *Let $u \in C\left([0,T];\dot{H}^1\left(\mathbb{R}^3\right)\right)\cap L^2\left([0,T];\dot{H}^2\left(\mathbb{R}^3\right)\right)$, for all $T < T_{max}$ be a mild solution to the Navier-Stokes equation with force $f \in L^2_{loc}\left((0,T^*);L^2\left(\mathbb{R}^3\right)\right)$. If $\frac{2}{p}+\frac{3}{q}=2$, with $\frac{3}{2}<q\leq+\infty$, then*

$$\|u(\cdot,T)\|_{\dot{H}^1}^2 \leq \left(\|u^0\|_{\dot{H}^1}^2 + \frac{4}{\nu}\int_0^T \|f(\cdot,t)\|_{L^2}^2\right)\exp\left(D_q\int_0^T \|\lambda_2^+(\cdot,t)\|_{L^q(\mathbb{R}^3)}^p dt\right), \tag{5.11}$$

with the constant D_q depending only on q and ν. In particular if the maximal existence time for a mild solution $T_{max} < T^*$, then

$$\int_0^{T_{max}} \|\lambda_2^+(\cdot,t)\|_{L^q(\mathbb{R}^3)}^p dt = +\infty. \tag{5.12}$$

Proof. First we will note that $\|u(\cdot,t)\|_{\dot{H}^1}^2$ must become unbounded as $t \to T_{max}$ if the mild solution cannot be extended beyond some time $T_{max} < T^*$, so it suffices to prove the bound (5.11). Applying Proposition 3.1, it is equivalent to show that

$$\|S(\cdot,T)\|_{L^2}^2 \leq \left(\|S^0\|_{L^2}^2 + \frac{2}{\nu}\int_0^T \|f(\cdot,t)\|_{L^2}^2 dt\right)\exp\left(D_q\int_0^T \|\lambda_2^+(\cdot,t)\|_{L^q(\mathbb{R}^3)}^p dt\right). \tag{5.13}$$

To begin we recall the conclusion in Lemma 5.1 (5.3)

$$\partial_t\|S(\cdot,t)\|_{L^2}^2 \leq -\nu\|S\|_{\dot{H}^1}^2 + 2\int_{\mathbb{R}^3}\lambda_2^+|S|^2 + \frac{2}{\nu}\|f\|_{L^2}^2. \tag{5.14}$$

First we will consider the case $q=+\infty$. Applying Holder's inequality with exponents 1 and $+\infty$ we see that,

$$\partial_t\|S(\cdot,t)\|_{L^2}^2 \leq 2\|\lambda_2^+\|_{L^\infty}\|S\|_{L^2}^2 + \frac{2}{\nu}\|f\|_{L^2}^2. \tag{5.15}$$

Now we can apply Gronwall's inequality and find that

$$\|S(\cdot,T)\|_{L^2}^2 \leq \left(\|S^0\|_{L^2}^2 + \frac{2}{\nu}\int_0^T \|f(\cdot,t)\|_{L^2}^2 dt\right)\exp\left(2\int_0^T \|\lambda_2^+\|_{L^\infty} dt\right). \tag{5.16}$$

Now we will consider the case $\frac{3}{2} < q < +\infty$. We will begin by applying Holder's inequality to (5.3), so take $\frac{1}{q} + \frac{1}{a} = 1$, and so

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 \leq -\nu \|S\|_{\dot{H}^1}^2 + 2\|\lambda_2^+\|_{L^q} \|S\|_{L^{2a}}^2 + \frac{2}{\nu}\|f\|_{L^2}^2. \tag{5.17}$$

Applying the Sobolev inequality we find

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 \leq -C_2\nu \|S\|_{L^6}^2 + 2\|\lambda_2^+\|_{L^q} \|S\|_{L^{2a}}^2 + \frac{2}{\nu}\|f\|_{L^2}^2. \tag{5.18}$$

Noting that $q > \frac{3}{2}$, it follows that $a < 3$, so $2a < 6$. Take $\sigma \in (0,1)$, such that $\frac{1}{2a} = \sigma\frac{1}{2} + (1-\sigma)\frac{1}{6}$. Then interpolating between L^2 and L^6 we find that

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 \leq -C_2\nu \|S\|_{L^6}^2 + 2\|\lambda_2^+\|_{L^q} \|S\|_{L^2}^{2\sigma} \|S\|_{L^6}^{2(1-\sigma)} + \frac{2}{\nu}\|f\|_{L^2}^2. \tag{5.19}$$

We know that $\frac{\sigma}{3} + \frac{1}{6} = \frac{1}{2a}$, so $\sigma = \frac{3}{2a} - \frac{1}{2}$. By definition we have that $\frac{1}{a} = 1 - \frac{1}{q}$, so $\sigma = 1 - \frac{3}{2q}$. Therefore we conclude that

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 \leq -C_2\nu \|S\|_{L^6}^2 + 2\|\lambda_2^+\|_{L^q} \|S\|_{L^2}^{2-\frac{3}{q}} \|S\|_{L^6}^{\frac{3}{q}} + \frac{2}{\nu}\|f\|_{L^2}^2. \tag{5.20}$$

Now take $b = \frac{2q}{3}$. That means $1 < b < +\infty$. Define p by $\frac{1}{p} + \frac{1}{b} = 1$, and apply Young's inequality with exponents p and b, and we find that

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 \leq -C_2\nu \|S\|_{L^6}^2 + D_q \left(\|\lambda_2^+\|_{L^q} \|S\|_{L^2}^{2-\frac{3}{q}} \right)^p + C_2\nu \|S\|_{L^6}^{b\frac{3}{q}} + \frac{2}{\nu}\|f\|_{L^2}^2. \tag{5.21}$$

Note that $\frac{1}{p} = 1 - \frac{1}{b} = 1 - \frac{3}{2q}$. This means that $p(2 - \frac{3}{q}) = 2$ and that $\frac{2}{p} + \frac{3}{q} = 2$, and we know by definition that $b\frac{3}{q} = 2$, so

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 \leq D_q \|\lambda_2^+\|_{L^q}^p \|S\|_{L^2}^2 + \frac{2}{\nu}\|f\|_{L^2}^2. \tag{5.22}$$

Applying Gronwall's inequality we find that

$$\|S(\cdot,T)\|_{L^2}^2 \leq \left(\|S^0\|_{L^2}^2 + \frac{2}{\nu}\int_0^T \|f\|_{L^2}^2 dt \right) \exp\left(D_q \int_0^T \|\lambda_2^+\|_{L^q(\mathbb{R}^3)}^p dt \right). \tag{5.23}$$

This completes the proof. □

We will note here that the case $p = 1, q = +\infty$ corresponds to the Beale-Kato-Majda criterion, so it may be possible to show that in this case the regularity criterion holds for the Euler equations as well as the Navier-Stokes equations. Note in particular that we did not use the dissipation to control the enstrophy, so there is a natural path to extend the result to solutions of the Euler equation as well. There is more work to do however, as bounded enstrophy is not sufficient to guarantee regularity for solutions to the Euler equations.

There is also an open question at the other boundary case, $p = +\infty$ $q = \frac{3}{2}$. This would likely be quite difficult as the methods used in [16,22] to extend the Prodi-Serrin-Ladyzhenskaya regularity criterion to the boundary case $p = +\infty$, $q = 3$ were much more technical than the methods in [33, 44, 48]. In particular, when $p = +\infty$ it is no longer adequate to rely on the relevant Sobolev embeddings, because we cannot apply Gronwall's inequality. Nonetheless, it is natural to suspect based on Theorem 5.2 that if u is a smooth solution to the Navier-Stokes equation with a maximal time of existence, $T_{max} < +\infty$, then

$$\limsup_{t \to T_{max}} \|\lambda_2^+(\cdot, t)\|_{L^{\frac{3}{2}}} = +\infty. \tag{5.24}$$

While we cannot prove this result, we can prove the following weaker statement.

Theorem 5.3 (Regularity criterion in the borderline case). *Let* $u \in C\left([0,T]; \dot{H}^1\left(\mathbb{R}^3\right)\right) \cap L^2\left([0,T]; \dot{H}^2\left(\mathbb{R}^3\right)\right)$, *for all* $T < T_{max}$ *be a mild solution to the Navier-Stokes equation with force* $f \in L^2_{loc}\left((0,T^*); L^2\left(\mathbb{R}^3\right)\right)$. *If* $T_{max} < T^*$, *then*

$$\limsup_{t \to T_{max}} \|\lambda_2^+(\cdot, t)\|_{L^{\frac{3}{2}}} \geq \frac{\nu}{C_2^2}, \tag{5.25}$$

where C_2 *is the constant in the sharp Sobolev inequality, Lemma 4.5.*

Proof. Suppose toward contradiction that $T_{max} < T^*$ and

$$\limsup_{t \to T_{max}} \|\lambda_2^+(\cdot, t)\|_{L^{\frac{3}{2}}} < \frac{\nu}{C_2^2}. \tag{5.26}$$

Then there must exist $\epsilon, \delta > 0$, such that for all $T_{max} - \delta < t < T_{max}$,

$$C_2^2 \|\lambda_2^+(\cdot, t)\|_{L^{\frac{3}{2}}} < \nu - \epsilon. \tag{5.27}$$

Recall from the proof of Lemma 5.1 that

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 \leq -2\nu \|S\|_{\dot{H}^1}^2 + 2\int_{\mathbb{R}^3} \lambda_2^+ |S|^2 + \sqrt{2}\|S\|_{\dot{H}^1}\|f\|_{L^2} \tag{5.28}$$

$$\leq -2\nu \|S\|_{\dot{H}^1}^2 + 2\|\lambda_2^+\|_{L^{\frac{3}{2}}}\|S\|_{L^6}^2 + \sqrt{2}\|S\|_{\dot{H}^1}\|f\|_{L^2} \tag{5.29}$$

$$\leq -2\nu \|S\|_{\dot{H}^1}^2 + 2C_2^2\|\lambda_2^+\|_{L^{\frac{3}{2}}}\|S\|_{\dot{H}^1}^2 + \sqrt{2}\|S\|_{\dot{H}^1}\|f\|_{L^2}, \tag{5.30}$$

where we have applied Hölder's inequality and the sharp Sobolev inequality.

Next we recall that by hypothesis, for all $T_{max} - \delta < t < T_{max}$,

$$C_2^2 \|\lambda_2^+\|_{L^{\frac{3}{2}}} - \nu < -\epsilon. \tag{5.31}$$

Using this fact and applying Young's inequality, we find

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 \leq -2\epsilon \|S\|_{\dot{H}^1}^2 + \sqrt{2}\|S\|_{\dot{H}^1}\|f\|_{L^2} \tag{5.32}$$

$$\leq \frac{1}{\epsilon}\|f\|_{L^2}^2. \tag{5.33}$$

Integrating this differential inequality we find that

$$\limsup_{t \to T_{max}} \|S(\cdot,t)\|_{L^2}^2 \leq \|S(\cdot, T_{max} - \delta)\|_{L^2}^2 + \frac{1}{\epsilon}\int_{T_{max}-\delta}^{T_{max}} \|f(\cdot,t)\|_{L^2}^2 dt < +\infty, \tag{5.34}$$

which is a contradiction because $T_{max} < T^*$ implies that

$$\limsup_{t \to T_{max}} \|S(\cdot,t)\|_{L^2}^2 = +\infty. \tag{5.35}$$

This completes the proof. □

Note that the boundary case in our paper is $q = \frac{3}{2}$, not $q = 3$. This is because the regularity criterion in [16, 22] is on u, whereas our regularity criterion is on an eigenvalue of the strain matrix, which scales like $\nabla \otimes u$. This is directly related to the Sobolev embedding $W^{1,\frac{3}{2}}\left(\mathbb{R}^3\right) \subset L^3\left(\mathbb{R}^3\right)$.

Theorem 5.2 is one of few regularity criteria for the Navier-Stokes equations involving a signed quantity, which is not too surprising, given that the Navier-Stokes equation is a vector valued equation. Even the scalar regularity criteria based on only one component of u do not involve signed quantities [7]. The only other regularity criterion for the Navier-Stokes equation involving a signed quantity—at least to the knowledge of the author—is the regularity criterion proved by Seregin and Šverák [46] that for a smooth solution to the Navier-Stokes equation to blowup in finite time, p must become unbounded below and $p + \frac{1}{2}|u|^2$ must become unbounded above.

We will also make a remark about the relationship between this result and the regularity criterion on one component of the gradient tensor $\frac{\partial u_j}{\partial x_i}$ in [4]. A natural question to ask in light of this regularity criterion is whether it is possible to prove a regularity criterion on just one entry of the strain tensor S_{ij}. This paper does not answer this question, however we do prove a regularity criterion on just one diagonal entry of the diagonalization of the strain tensor.

Corollary 5.4 (Any eigenvalue of strain characterizes the blow-up time)**.** *Let $u \in C\left([0,T]; \dot{H}^1\left(\mathbb{R}^3\right)\right) \cap L^2\left([0,T]; \dot{H}^2\left(\mathbb{R}^3\right)\right)$, for all $T < T_{max}$ be a mild solution to the Navier-Stokes equation with force $f \in L^2_{loc}\left((0,T^*); L^2\left(\mathbb{R}^3\right)\right)$. If $\frac{2}{p} + \frac{3}{q} = 2$, with $\frac{3}{2} < q \leq +\infty$, then*

$$\|u(\cdot,T)\|_{\dot{H}^1}^2 \leq \left(\|u^0\|_{\dot{H}^1}^2 + \frac{4}{\nu}\int_0^T \|f(\cdot,t)\|_{L^2}^2 dt\right) \exp\left(D_q \int_0^T \|\lambda_i(\cdot,t)\|_{L^q(\mathbb{R}^3)}^p dt\right), \tag{5.36}$$

with the constant D_q depending only on q and ν. In particular if $T_{max} < T^*$, then

$$\int_0^{T_{max}} \|\lambda_i(\cdot,t)\|_{L^q(\mathbb{R}^3)}^p dt = +\infty. \tag{5.37}$$

Proof. $\lambda_1 \leq \lambda_2 \leq \lambda_3$ and $\lambda_1 + \lambda_2 + \lambda_3 = 0$ implies that $|\lambda_1|, |\lambda_3| \geq |\lambda_2| \geq |\lambda_2^+|$. Therefore

$$\int_0^T \|\lambda_2^+(\cdot,t)\|_{L^q}^p dt \leq \int_0^T \|\lambda_i(\cdot,t)\|_{L^q}^p dt. \tag{5.38}$$

Applying this inequality to both conclusions in Theorem 5.2, this completes the proof. □

We will also note that there is a gap to be closed in the regularity criterion on $\frac{\partial u_j}{\partial x_i}$, because it is not the optimal result with respect to scaling and requires subcritical control on $\frac{\partial u_j}{\partial x_i}$. That is, the result only holds for $\frac{2}{p} + \frac{3}{q} = \frac{q+3}{2q} < 2$, for $i \neq j$ and $\frac{2}{p} + \frac{3}{q} = \frac{3q+6}{4q} < 2$, for $i = j$, whereas the regularity criterion on one of the eigenvalues in Corollary 5.4 is critical with respect to the scaling. It is natural, however, to ask whether Theorem 5.2 can be extended to the critical Besov spaces, so in that sense the result may be pushed further.

Corollary 5.4 is only really a new result, however, for λ_2. This is because $|\lambda_1|$ and $|\lambda_3|$ both control $|S|$. As we will see from the following proposition, the regularity criteria in terms of λ_1 or λ_3 follow immediately from the Prodi-Serrin-Ladyzhenskaya regularity criterion without needing to use strain evolution equation at all, so in this case Corollary 5.3 is just an unstated corollary of previous results.

Proposition 5.5 (Lower bounds on the magnitude of the extermal eigenvalues)**.** *Suppose $M \in S^{3\times 3}$ is a symmetric trace free matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$. Then*

$$\lambda_3 \geq \frac{1}{\sqrt{6}}|S|, \tag{5.39}$$

with equality if and only if $-\frac{1}{2}\lambda_1 = \lambda_2 = \lambda_3$, and

$$\lambda_1 \leq -\frac{1}{\sqrt{6}}|S|, \tag{5.40}$$

with equality if and only if $\lambda_1 = \lambda_2 = -\frac{1}{2}\lambda_3$.

Furthermore, for all $S \in L^2_{st}$ and for all $1 \leq q \leq +\infty$

$$\|S\|_{L^q} \leq \sqrt{6}\|\lambda_1\|_{L^q} \tag{5.41}$$

and

$$\|S\|_{L^q} \leq \sqrt{6}\|\lambda_3\|_{L^q}. \tag{5.42}$$

Proof. We will prove the statement for λ_3. The proof of the statement for λ_1 is entirely analogous and is left to the reader. First observe that if $-\frac{1}{2}\lambda_1 = \lambda_2 = \lambda_3$, then

$$|S|^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 6\lambda_3^2, \tag{5.43}$$

So we have proven that if $\lambda_2 = \lambda_3$, then $\lambda_3 = \frac{1}{\sqrt{6}}|S|$. Now suppose $\lambda_2 < \lambda_3$. Recall that

$$\operatorname{tr}(M) = \lambda_1 + \lambda_2 + \lambda_3 = 0, \tag{5.44}$$

so

$$\lambda_1 = -\lambda_2 - \lambda_3. \tag{5.45}$$

Therefore we find that

$$|S|^2 = (-\lambda_2 - \lambda_3)^2 + \lambda_2^2 + \lambda_3^2 = 2\lambda_2^2 + 2\lambda_3^2 + 3\lambda_2\lambda_3. \tag{5.46}$$

Applying Young's Inequality we can bound

$$2\lambda_2\lambda_3 \leq \lambda_2^2 + \lambda_3^2, \tag{5.47}$$

so

$$|S|^2 \leq 3\lambda_2^2 + 3\lambda_3^2 < 6\lambda_3^2. \tag{5.48}$$

$\lambda_3 \geq 0$, so this completes the proof. We leave the analogous proof for λ_1 to the reader. The L^q bounds follow immediately from integrating these bounds pointwise when one recalls that $\text{tr}(S) = 0$. We will note here that the L^q norms may be infinite, as by hypothesis we only have $S \in L^2$, but by convention the inequality is satisfied if both norms are infinite. □

In particular this implies that regularity criteria involving λ_1 or λ_3 follow immediately from regularity criteria involving S, so while the regularity criteria on λ_1 and λ_3 in Corollary 5.4 do not appear in the literature to the knowledge of the author, these criteria do not offer a real advance over the Prodi-Serrin-Ladyzhenskaya criterion [33, 44, 48], as the critical norm on u can be controlled by the critical norm on S using Sobolev embedding, which can in turn be bounded by the critical norm on λ_1 or λ_3 using Proposition 5. That is

$$\|u\|_{L^{q^*}} \leq C\|S\|_{L^q} \leq \sqrt{6}C\|\lambda_3\|_{L^q}. \tag{5.49}$$

It is the regularity criterion in terms of λ_2^+ that is really significant, because it encodes geometric information about the strain beyond just its size.

We will also note that none of the regularity criteria involving ∇u_j [54], $\partial_{x_i} u$ [32], or $\partial_{x_i} u_j$ [4], have been proven for the Navier-Stokes equation with an external force. However, the regularity criterion in Theorem 5.2 is also valid for Navier-Stokes equation with an external force. It may only be an exercise to extend the results cited above to the case with an external force, but because these papaers do not establish their regularity criteria by applying Grönwall type estimates to the enstrophy, it is not immediately clear that this is is the case.

Lemma 5.6 (The middle eigenvector is minimal). *Suppose* $S \in L^2_{st}$ *and* $v \in L^\infty(\mathbb{R}^3; \mathbb{R}^3)$ *with* $|v(x)| = 1$ *almost everywhere* $x \in \mathbb{R}^3$. *Then*

$$|\lambda_2(x)| \leq |S(x)v(x)| \tag{5.50}$$

almost everywhere $x \in \mathbb{R}^3$.

Proof. By the spectral theorem, we know that there is an orthonormal eigenbasis for $\mathbb{R}^n$. In particular, take $v_1(x), v_2(x), v_3(x)$ to be eigenvectors of $S(x)$ corresponding to eigenvalues $\lambda_1(x), \lambda_2(x), \lambda_3(x)$ such that $|v_1(x)|, |v_2(x)|, |v_3(x)| = 1$ almost everywhere $x \in \mathbb{R}^3$. Then from the spectral theorem we know that $\{v_1(x), v_2(x), v_3(x)\}$ is an orthonormal basis for $\mathbb{R}^3$ almost everywhere $x \in \mathbb{R}^3$. Therefore

$$Sv = \lambda_1(v \cdot v_1)v_1 + \lambda_2(v \cdot v_2)v_2 + \lambda_3(v \cdot v_3)v_3. \tag{5.51}$$

$\text{tr}(S) = 0$ implies that $|\lambda_2| \leq |\lambda_1|, |\lambda_3|$, so we can see that

$$|Sv|^2 = \lambda_1^2(v \cdot v_1)^2 + \lambda_2^2(v \cdot v_2)^2 + \lambda_3^2(v \cdot v_3)^2 \tag{5.52}$$

$$\geq \lambda_2^2\left((v \cdot v_1)^2 + (v \cdot v_2)^2 + (v \cdot v_3)^2\right). \tag{5.53}$$

Because $\{v_1(x), v_2(x), v_3(x)\}$ is an orthonormal basis for $\mathbb{R}^3$ almost everywhere $x \in \mathbb{R}^3$, we conclude that

$$(v \cdot v_1)^2 + (v \cdot v_2)^2 + (v \cdot v_3)^2 = |v|^2 = 1. \tag{5.54}$$

Therefore

$$|Sv|^2 \geq \lambda_2^2. \tag{5.55}$$

This concludes the proof. □

Now that we have proven Lemma 5.6, we will prove a new regularity criterion for the strain tensor. This regularity criterion is Theorem 1.9 in the introduction, and is restated here for the reader's convenience.

Theorem 5.7 (Blowup requires the strain to blow up in every direction)**.** *Let $u \in C\left([0,T];\dot{H}^1\left(\mathbb{R}^3\right)\right) \cap L^2\left([0,T];\dot{H}^2\left(\mathbb{R}^3\right)\right)$, for all $T < T_{max}$ be a mild solution to the Navier-Stokes equation with force $f \in L^2_{loc}\left((0,T^*);L^2\left(\mathbb{R}^3\right)\right)$, and let $v \in L^\infty\left(\mathbb{R}^3 \times [0,T_{max}];\mathbb{R}^3\right)$, with $|v(x,t)| = 1$ almost everywhere. If $\frac{2}{p} + \frac{3}{q} = 2$, with $\frac{3}{2} < q \leq +\infty$, then*

$$\|u(\cdot,T)\|^2_{\dot{H}^1} \leq \left(\|u^0\|^2_{\dot{H}^1} + \frac{4}{\nu}\int_0^T \|f(\cdot,t)\|^2_{L^2}\right)\exp\left(D_q\int_0^T \|S(\cdot,t)v(\cdot,t)\|^p_{L^q(\mathbb{R}^3)}dt\right), \tag{5.56}$$

with the constant D_q depending only on q and ν. In particular if the maximal existence time for a mild solution $T_{max} < T^*$, then

$$\int_0^{T_{max}} \|S(\cdot,t)v(\cdot,t)\|^p_{L^q(\mathbb{R}^3)}dt = +\infty. \tag{5.57}$$

Proof. This follows immediately from Lemma 5.6 and Theorem 5.2. □

We can use Theorem 5.7 to prove a new one-direction-type regularity criterion involving the sum of the derivative of the whole velocity in one direction, and the gradient of the component in the same direction. In fact, Theorem 5.7 allows us to prove a one direction regularity criterion that involves different directions in different regions of $\mathbb{R}^3$. First off, for any unit vector $v \in \mathbb{R}^3, |v| = 1$ we define $\partial_v = v \cdot \nabla$ and $u_v = u \cdot v$. We will now prove Theorem 1.8, which is restated here for the reader's convenience.

Corollary 5.8 (Local one direction regularity criterion)**.** *Let $\{v_n(t)\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ with $|v_n(t)| = 1$. Let $\{\Omega_n(t)\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ be Lesbesgue measurable sets such that for all $m \neq n$, $\Omega_m(t) \cap \Omega_n(t) = \emptyset$, and $\mathbb{R}^3 = \bigcup_{n\in\mathbb{N}} \Omega_n(t)$. Let $u \in C\left([0,T];\dot{H}^1\left(\mathbb{R}^3\right)\right) \cap L^2\left([0,T];\dot{H}^2\left(\mathbb{R}^3\right)\right)$, for all $T < T_{max}$ be a mild solution to the Navier-Stokes equation with force $f \in L^2_{loc}\left((0,T^*);L^2\left(\mathbb{R}^3\right)\right)$. If $\frac{2}{p} + \frac{3}{q} = 2$, with $\frac{3}{2} < q \leq +\infty$, then*

$$\|u(\cdot,T)\|^2_{\dot{H}^1} \leq \left(\|u^0\|^2_{\dot{H}^1} + \frac{4}{\nu}\int_0^T \|f(\cdot,t)\|^2_{L^2}\right)\exp\left(D_q\int_0^T \left(\sum_{n=1}^\infty \left\|\partial_{v_n}u(\cdot,t) + \nabla u_{v_n}(\cdot,t)\right\|^q_{L^q(\Omega_n(t))}\right)^{\frac{p}{q}} dt\right), \tag{5.58}$$

with the constant D_q depending only on q and ν. If the maximal existence time for a mild solution $T_{max} < T^*$, then

$$\int_0^{T_{max}} \left(\sum_{n=1}^\infty \left\|\partial_{v_n}u(\cdot,t) + \nabla u_{v_n}(\cdot,t)\right\|^q_{L^q(\Omega_n(t))}\right)^{\frac{p}{q}} dt = +\infty. \tag{5.59}$$

In particular if we take $v_n(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ *for all* $n \in \mathbb{N}$, *then* (5.59) *reduces to*

$$\int_0^{T_{max}} \|\partial_3 u(\cdot,t) + \nabla u_3(\cdot,t)\|_{L^q}^p dt = +\infty. \tag{5.60}$$

Proof. Let $v(x,t) = \sum_{n=1}^{\infty} v_n(t) I_{\Omega_n(t)}(x)$, where I_Ω is the indicator function $I_\Omega(x) = 1$ for all $x \in \Omega$ and $I_\Omega(x) = 0$ otherwise. Note that in this case we clearly have

$$S(x,t)v(x,t) = \sum_{n=1}^{\infty} I_{\Omega_n(t)}(x) S(x,t) v_n(t). \tag{5.61}$$

Because $\{\Omega_n\}_{n\in\mathbb{N}}$ are disjoint, we have

$$\|S(\cdot,t)v(\cdot,t)\|_{L^q(\mathbb{R}^3)}^q = \sum_{n=1}^{\infty} \|S(\cdot,t)v_n(t)\|_{L^q(\Omega_n(t))}^q. \tag{5.62}$$

Therefore we find that

$$\|S(\cdot,t)v(\cdot,t)\|_{L^q(\mathbb{R}^3)}^p = \left(\sum_{n=1}^{\infty} \|S(\cdot,t)v_n(t)\|_{L^q(\Omega_n(t))}^q \right)^{\frac{p}{q}}. \tag{5.63}$$

Finally observe that

$$S(x,t)v_n(t) = \frac{1}{2}\partial_{v_n} u(x,t) + \frac{1}{2}\nabla u_{v_n}(x,t), \tag{5.64}$$

so

$$\|S(\cdot,t)v(\cdot,t)\|_{L^q(\mathbb{R}^3)}^p = \left(\sum_{n=1}^{\infty} \left\| \frac{1}{2}\partial_{v_n} u(\cdot,t) + \frac{1}{2}\nabla u_{v_n}(\cdot,t) \right\|_{L^q(\Omega_n(t))}^q \right)^{\frac{p}{q}}. \tag{5.65}$$

Applying Theorem 5.7, this completes the proof. □

There are previous regularity criteria involving only one direction. For instance, Kukavica and Ziane [32] showed that if $T_{max} < +\infty$, and if $\frac{2}{p} + \frac{3}{q} = 2$, with $\frac{9}{4} \leq q \leq 3$, then

$$\int_0^{T_{max}} \|\partial_3 u(\cdot,t)\|_{L^q(\mathbb{R}^3)}^p dt = +\infty. \tag{5.66}$$

More recently, it was shown by Chemin, Zhang, and Zhang [11, 12] that if $T_{max} < +\infty$ and $4 < p < +\infty$, then

$$\int_0^{T_{max}} \|u_3(\cdot,t)\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p = +\infty. \tag{5.67}$$

Corollary 5.8 extends regularity criteria involving one fixed direction to regularity criteria in which the direction may vary in time and space. In the case where there is no external force, $f = 0$, these results both imply the special case of Corollary 5.8, that if $T_{max} < +\infty$ then

$$\int_0^{T_{max}} \|\partial_3 u(\cdot,t) + \nabla u_3(\cdot,t)\|_{L^q}^p = +\infty, \tag{5.68}$$

in the range of exponents $\frac{9}{4} \leq q \leq 3$ and $\frac{3}{2} < q < 6$ respectively. This follows from the Helmholtz decomposition in Proposition 1.3, as we will now show.

Observe that the projections associated with the Helmholtz decomposition allow us to control $\|\partial_3 u\|_{L^q}$ and $\|u_3\|_{\dot{H}^{\frac{1}{2}+\frac{p}{2}}}$ by $\|\partial_3 u + \nabla u_3\|_{L^q}$. In particular, we find

$$\|\partial_3 u\|_{L^q} = \|P_{df}\left(\partial_3 u + \nabla u_3\right)\|_{L^q} \leq B_q \|\partial_3 u + \nabla u_3\|_{L^q}. \tag{5.69}$$

Applying the the Sobolev embedding $\dot{H}^{\frac{1}{2}+\frac{p}{2}}\left(\mathbb{R}^3\right) \subset W^{1,q}\left(\mathbb{R}^3\right)$ when $\frac{2}{p} + \frac{3}{q} = 2$, and the L^q boundedness of P_g, we can also see that

$$\|u_3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \leq D\|\nabla u_3\|_{L^q} = D\|P_g\left(\partial_3 u + \nabla u_3\right)\|_{L^q} \leq DB_q \|\partial_3 u + \nabla u_3\|_{L^q}. \tag{5.70}$$

This means that the regularity criterion requiring $\partial_3 u + \nabla u_3 \in L_t^p L_x^q$ is not new in the range $\frac{3}{2} < q \leq 6$. In fact, for $\frac{3}{2} < q < +\infty$ this special case of Corollary 5.8, is equivalent to a regularity criterion on two components of the vorticity, which we will discuss in chapter 6, once we have developed the necessary isometry between $\partial_3 u + \nabla u_3$ and $(\omega_1, \omega_2, 0)$.

While the special case of Corollary 5.8 involving the regularity criterion on $\partial_3 u + \nabla u_3$ is not new, Corollary 5.7 and Corollary 5.8 are stronger than previous results in that they do not require regularity in a fixed direction, but allow this direction to vary. One interpretation of component reduction results for Navier-Stokes regularity criteria, is that if the solution is approximately two dimensional, then it must be smooth. The only reason that we have component reduction regularity criteria for the 3D Navier-Stokes equation, is because the 2D Navier-Stokes equation has smooth solutions globally in time. All of the previous component reduction regularity criteria involve some fixed direction, and so can be interpreted as saying if a solution is globally approximately two dimensional, then it must be smooth. Corollary 5.7 and Corollary 5.8 strengthen these statements to the requirement that the solution must be regular even if it is only locally two dimensional, and furthermore requires the solution to have a specific three dimensional structure with unbounded planar stretching by the strain matrix. This shows the deep geometric significance of the Theorem 5.2, that λ_2^+ controls the growth of enstrophy.

Chapter 6

A vorticity approach to almost two dimensional initial data

In order to prove the Theorem 1.12, we will need to prove some bounds on the growth of $\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}$, as well as bound the growth of enstrophy in terms of $\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}$. In order to do this we will need to consider the evolution equation for the horizontal components of vorticity, ω_h, which is given in the following proposition. Throughout this chapter and the next, we will consider the Navier-Stokes equation with no external force, setting $f = 0$.

Proposition 6.1 (Two component vorticity equation)**.** *Suppose $u \in C\left([0, T_{max}); H^1_{df}\right)$ is a mild solution, and therefore a classical solution to the Navier-Stokes equation. Then ω_h is a classical solution of*

$$\partial_t \omega_h + (u \cdot \nabla)\omega_h - \Delta\omega_h - S\omega_h - S_h\omega = 0, \tag{6.1}$$

where $\omega_h = \begin{pmatrix} \omega_1 \\ \omega_2 \\ 0 \end{pmatrix}$ *and* $S_h = \begin{pmatrix} 0 & 0 & S_{13} \\ 0 & 0 & S_{23} \\ -S_{13} & -S_{23} & 0 \end{pmatrix}$.

Proof. Kato and Fujita proved that mild solutions must be smooth [18], so clearly u is a classical solution to the Navier-Stokes equation. Therefore $\omega = \nabla \times u$ is also smooth and is a classical solution to the vorticity equation:

$$\partial_t \omega + (u \cdot \nabla)\omega - \Delta\omega - S\omega = 0. \tag{6.2}$$

Let $I_h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then we clearly have $\omega_h = I_h\omega$. Multiply the vorticity equation through by I_h and find that

$$\partial_t \omega_h + (u \cdot \nabla)\omega_h - \Delta\omega_h - I_h S\omega = 0. \tag{6.3}$$

Next we add and subtract $SI_h\omega$. Therefore,

$$\partial_t \omega_h + (u \cdot \nabla)\omega_h - \Delta\omega_h - I_h S\omega + SI_h\omega - SI_h\omega = 0. \tag{6.4}$$

Regrouping terms we find that

$$\partial_t \omega_h + (u \cdot \nabla)\omega_h - \Delta\omega_h - (I_h S - SI_h)\,\omega - S\,(I_h\omega) = 0. \tag{6.5}$$

Recall that $I_h\omega = \omega_h$ and compute that $S_h = I_h S - S I_h$, and this completes the proof. $\square$

One of the key aspects in our proof is a generalization of the isometry in Proposition 3.1 that tells us $\|S\|_{L^2}^2 = \frac{1}{2}\|\omega\|_{L^2}^2$, to an isometry that involves just one column of S and just two components of ω. In order to state this isometry, we will define the vectors v^1, v^2, v^3 as follows.

Definition 6.2. *For $i \in \{1,2,3\}$ define $v^i = \partial_i u + \nabla u_i$. Note in particular that $v^i_j = 2S_{ij}$, for all $i,j \in \{1,2,3\}$. Equivalently, note that v^1, v^2, v^3 are the columns of $2S$.*

With these vectors defined, we can restate our identity for enstrophy growth in Corollary 3.3 in terms of v^1, v^2, v^3.

Proposition 6.3 (Triple product enstrophy identity)**.** *Let $u \in C\left([0,T_{max}); H^1_{df}\right)$ be a mild solution to the Navier-Stokes equation. Then for all $0 \leq t < T_{max}$, we have*

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 = -2\nu \|S\|_{\dot{H}^1}^2 - \frac{1}{2}\int_{\mathbb{R}^3} \left(v^1 \times v^2\right) \cdot v^3. \tag{6.6}$$

Proof. We know that v^1, v^2, v^3 are the columns of $2S$, so by the triple product representation of the determinant of a three by three matrix

$$\det(2S) = \left(v^1 \times v^2\right) \cdot v^3. \tag{6.7}$$

The three by three determinant is homogeneous of order three, so

$$\det(2S) = 8\det(S). \tag{6.8}$$

Therefore we conclude that

$$-4\det(S) = \left(v^1 \times v^2\right) \cdot v^3. \tag{6.9}$$

Recalling from Proposition 3.12 that

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 = -2\nu \|S\|_{\dot{H}^1}^2 - 4\int_{\mathbb{R}^3} \det(S), \tag{6.10}$$

this completes the proof. $\square$

We will now prove an isometry that relates Hilbert norms v^3 and ω_h to each other and to $\partial_3 u$ and ∇u_3, as well as bounding Hilbert norms of S_h by ω_h.

Lemma 6.4 (Two component isometry)**.** *Suppose $u \in H^1_{df}$. Then for all $-1 \leq \alpha \leq 0$,*

$$\|v^3\|_{\dot{H}^\alpha}^2 = \|\omega_h\|_{\dot{H}^\alpha}^2 = \|\partial_3 u\|_{\dot{H}^\alpha}^2 + \|\nabla u_3\|_{\dot{H}^\alpha}^2 \tag{6.11}$$

and

$$\|S_h\|_{\dot{H}^\alpha} \leq \frac{1}{\sqrt{2}}\|\omega_h\|_{\dot{H}^\alpha}. \tag{6.12}$$

Proof. First we observe that

$$\partial_3 u - \nabla u_3 = \begin{pmatrix} \partial_3 u_1 - \partial_1 u_3 \\ \partial_3 u_2 - \partial_2 u_3 \\ 0 \end{pmatrix} = \begin{pmatrix} \omega_2 \\ -\omega_1 \\ 0 \end{pmatrix}. \tag{6.13}$$

Therefore clearly

$$\|\omega_h\|_{\dot{H}^\alpha} = \|\partial_3 u - \nabla u_3\|_{\dot{H}^\alpha}. \tag{6.14}$$

This means we can compute that

$$\|\omega_h\|_{\dot{H}^\alpha}^2 = \|\partial_3 u - \nabla u_3\|_{\dot{H}^\alpha}^2 = \|\partial_3 u\|_{\dot{H}^\alpha}^2 + \|\nabla u_3\|_{\dot{H}^\alpha}^2 - 2\left\langle \partial_3 u, \nabla u_3\right\rangle_{\dot{H}^\alpha}, \tag{6.15}$$

$$\|v^3\|_{\dot{H}^\alpha}^2 = \|\partial_3 u + \nabla u_3\|_{\dot{H}^\alpha}^2 = \|\partial_3 u\|_{\dot{H}^\alpha}^2 + \|\nabla u_3\|_{\dot{H}^\alpha}^2 + 2\left\langle \partial_3 u, \nabla u_3\right\rangle_{\dot{H}^\alpha}. \tag{6.16}$$

Next we observe that because $\nabla \cdot u = 0$, then clearly $\nabla \cdot \partial_3 u = 0$. Therefore $\partial_3 u$ and ∇u_3 are orthogonal in $\dot{H}^\alpha$, so

$$\left\langle \partial_3 u, \nabla u_3\right\rangle_{\dot{H}^\alpha} = 0. \tag{6.17}$$

Therefore we conclude that

$$\|\omega_h\|_{\dot{H}^\alpha}^2 = \|v^3\|_{\dot{H}^\alpha}^2 = \|\partial_3 u\|_{\dot{H}^\alpha}^2 + \|\nabla u_3\|_{\dot{H}^\alpha}^2. \tag{6.18}$$

Finally we see that

$$|S_h|^2 = 2S_{13}^2 + 2S_{23}^2 \leq 2S_{13}^2 + 2S_{23}^2 + 2S_{33}^2 = \frac{1}{2}|v^3|^2. \tag{6.19}$$

Therefore we can conclude that

$$\|S_h\|_{\dot{H}^\alpha}^2 \leq \frac{1}{2}\|v^3\|_{\dot{H}^\alpha}^2 = \frac{1}{2}\|\omega_h\|_{\dot{H}^\alpha}^2. \tag{6.20}$$

This completes the proof. □

Remark 6.5. *Another way to see this isometry, is that*

$$\|Se_3\|_{\dot{H}^\alpha}^2 = \frac{1}{4}\|e_3 \times \omega\|_{\dot{H}^\alpha}^2. \tag{6.21}$$

In fact, for any fixed vector $v \in \mathbb{R}^3$ we will have

$$\|Sv\|_{\dot{H}^\alpha}^2 = \frac{1}{4}\|v \times \omega\|_{\dot{H}^\alpha}^2. \tag{6.22}$$

This is directly related to Proposition 3.1, because

$$\|S\|_{\dot{H}^\alpha}^2 = \|Se_1\|_{\dot{H}^\alpha}^2 + \|Se_2\|_{\dot{H}^\alpha}^2 + \|Se_3\|_{\dot{H}^\alpha}^2 \tag{6.23}$$

$$= \frac{1}{4}\left(\|e_1 \times \omega\|_{\dot{H}^\alpha}^2 + \|e_2 \times \omega\|_{\dot{H}^\alpha}^2 + \|e_3 \times \omega\|_{\dot{H}^\alpha}^2\right) \tag{6.24}$$

$$= \frac{1}{2}\|\omega\|_{\dot{H}^\alpha}^2. \tag{6.25}$$

This shows that the isometry between the symmetric and anti-symmetric part of the gradient, between strain and vorticity, not only holds overall, but also in any fixed direction.

This isometry, together with the identity for enstrophy growth in Proposition 6.3, will allow us to prove a new bound on the growth of enstrophy in terms of the critical Hilbert norm of ω_h. Before we proceed with this estimate, we will note that there is also a generalization of this result in L^q. The L^q norms of v^3 and ω_h are also equivalent, although not necessarily equal.

Proposition 6.6 (Two component equivalence)**.** *Fix $1 < q < +\infty$ and let $B_q \geq 1$ be the constant from the Helmholtz decomposition, Proposition 1.3. Then for all $u \in \dot{W}^{1,q}_{df}\left(\mathbb{R}^3\right)$,*

$$\frac{1}{2B_q}\|\omega_h\|_{L^q} \leq \|v^3\|_{L^q} \leq 2B_q\|\omega_h\|_{L^q}. \tag{6.26}$$

Proof. As we have already seen,

$$\partial_3 u - \nabla u_3 = \begin{pmatrix} \omega_2 \\ -\omega_1 \\ 0 \end{pmatrix}, \tag{6.27}$$

so clearly

$$\|\omega_h\|_{L^q} = \|\partial_3 u - \nabla u_3\|_{L^q}. \tag{6.28}$$

Observing that $\partial_3 u = P_{df}\left(\partial_3 u - \nabla u_3\right)$, and $\nabla u_3 = P_g\left(\partial_3 u - \nabla u_3\right)$, we can apply Proposition 1.3 and find that

$$\|\partial_3 u\|_{L^q} \leq B_q\|\omega_h\|_{L^q}, \tag{6.29}$$

$$\|\nabla u_3\|_{L^q} \leq B_q\|\omega_h\|_{L^q}. \tag{6.30}$$

Recalling that $v^3 = \partial_3 u + \nabla u_3$, we apply the triangle inequality and find that

$$\|v^3\|_{L^q} \leq \|\partial_3 u\|_{L^q} + \|\nabla u_3\|_{L^q} \leq 2B_q\|\omega_h\|_{L^q}. \tag{6.31}$$

We have proven the second inequality. Now we need to show that $\|\omega_h\|_{L^q} \leq 2B_q\|v^3\|_{L^q}$. The argument is essentially the same. Observe that $\partial_3 u = P_{df}\left(v^3\right)$ and $\nabla u_3 = P_g\left(v^3\right)$. Therefore from Proposition 1.3, we find that

$$\|\partial_3 u\|_{L^q} \leq B_q\|v^3\|_{L^q}, \tag{6.32}$$

$$\|\nabla u_3\|_{L^q} \leq B_q\|v^3\|_{L^q}. \tag{6.33}$$

Applying the triangle inequality, we find that

$$\|\omega_h\|_{L^q} \leq \|\partial_3 u\|_{L^q} + \|\nabla u_3\|_{L^q} \leq 2B_q\|v^3\|_{L^q}. \tag{6.34}$$

This completes the proof. □

Proposition 6.7 (Two vorticity components control enstrophy growth)**.** *Taking C_1 and C_2 as in Lemmas 4.3 and 4.5, let $R_1 = \frac{1}{2C_1C_2}$. Then for all mild solutions to the Navier-Stokes equation $u \in C\left([0, T_{max}); H^1_{df}\right)$, we have*

$$\partial_t\|\omega(\cdot,t)\|_{L^2}^2 \leq -\frac{2}{R_1}\|\omega\|_{\dot{H}^1}^2\left(R_1\nu - \|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}\right). \tag{6.35}$$

In particular, if $T_{max} < +\infty$, then

$$\limsup_{t\to T_{max}}\|\omega_h(\cdot,t)\|_{\dot{H}^{-\frac{1}{2}}} \geq R_1\nu. \tag{6.36}$$

Proof. We begin by applying Proposition 6.3, Lemma 6.4, and the duality of $\dot{H}^{-\frac{1}{2}}$ and $\dot{H}^{\frac{1}{2}}$. We find that:

$$\begin{aligned} \partial_t \|S(\cdot,t)\|_{L^2}^2 &= -2\nu\|S\|_{\dot{H}^1}^2 - \frac{1}{2}\int_{\mathbb{R}^3} \left(v^1 \times v^2\right)\cdot v^3 && (6.37)\\ &\leq -2\nu\|S\|_{\dot{H}^1}^2 + \frac{1}{2}\|v^3\|_{\dot{H}^{-\frac{1}{2}}}\|v^1\times v^2\|_{\dot{H}^{\frac{1}{2}}} && (6.38)\\ &= -2\nu\|S\|_{\dot{H}^1}^2 + \frac{1}{2}\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}\|v^1\times v^2\|_{\dot{H}^{\frac{1}{2}}} && (6.39)\\ &= -2\nu\|S\|_{\dot{H}^1}^2 + \frac{1}{2}\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}\|\nabla\left(v^1\times v^2\right)\|_{\dot{H}^{-\frac{1}{2}}}. && (6.40)\end{aligned}$$

Next we apply the fractional Sobolev inequality, the chain rule for gradients, the generalized Hölder inequality, and the Sobolev inequality to find that

$$\begin{aligned} \partial_t \|S(\cdot,t)\|_{L^2}^2 &\leq -2\nu\|S\|_{\dot{H}^1}^2 + \frac{1}{2}C_1\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}\|\nabla\left(v^1\times v^2\right)\|_{L^{\frac{3}{2}}} && (6.41)\\ &\leq -2\nu\|S\|_{\dot{H}^1}^2 + \frac{1}{2}C_1\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}\left(\|\left(|\nabla v^1||v^2|\right)\|_{L^{\frac{3}{2}}} + \|\left(|v^1||\nabla v^2|\right)\|_{L^{\frac{3}{2}}}\right) && (6.42)\\ &\leq -2\nu\|S\|_{\dot{H}^1}^2 + \frac{1}{2}C_1\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}\left(\|\nabla v^1\|_{L^2}\|v^2\|_{L^6} + \|v^1\|_{L^6}\|\nabla v^2\|_{L^2}\right) && (6.43)\\ &\leq -2\nu\|S\|_{\dot{H}^1}^2 + C_1C_2\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}\|\nabla v^1\|_{L^2}\|\nabla v^2\|_{L^2}. && (6.44)\end{aligned}$$

Finally observe that the vectors v^i are the columns of $2S$, so

$$\|\nabla v^i\|_{L^2} = \|v^i\|_{\dot{H}^1} \leq 2\|S\|_{\dot{H}^1}. \tag{6.45}$$

Therefore we find that

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 \leq -2\nu\|S\|_{\dot{H}^1}^2 + 4C_1C_2\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}\|S\|_{\dot{H}^1}^2. \tag{6.46}$$

Applying Proposition 3.1 and recalling that $\frac{1}{R_1} = 2C_1C_2$, we find that

$$\begin{aligned} \partial_t \|\omega(\cdot,t)\|_{L^2}^2 &\leq -2\nu\|\omega\|_{\dot{H}^1}^2 + 4C_1C_2\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}\|\omega\|_{\dot{H}^1}^2 && (6.47)\\ &= -\frac{2}{R_1}\|\omega\|_{\dot{H}^2}^2\left(R_1\nu - \|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}\right). && (6.48)\end{aligned}$$

This completes the proof of the bound.

Now we will prove the second piece. Suppose $T_{max} < +\infty$. Then

$$\limsup_{t\to T_{max}} \|\omega(\cdot,t)\|_{L^2}^2 = +\infty. \tag{6.49}$$

Therefore, for all $\epsilon > 0$, $\|\omega(\cdot,t)\|_{L^2}$ is not nonincreasing on the interval $(T_{max}-\epsilon, T_{max})$. Therefore, for all $\epsilon > 0$, there exists $t \in (T_{max}-\epsilon, T_{max})$, such that $\partial_t\|\omega(\cdot,t)\|_{L^2} > 0$. Applying the bound we have just proven, this implies that for all $\epsilon > 0$, there exists $t \in (T_{max}-\epsilon, T_{max})$ such that

$$\|\omega_h(\cdot,t)\|_{\dot{H}^{-\frac{1}{2}}} > R_1\nu. \tag{6.50}$$

Therefore,

$$\limsup_{t \to T_{max}} \|\omega_h(\cdot, t)\|_{\dot{H}^{-\frac{1}{2}}} \geq R_1 \nu. \tag{6.51}$$

This completes the proof. □

We will note that this is the $\dot{H}^{-\frac{1}{2}}$ version of a theorem proved in $L^{\frac{3}{2}}$ by Chae and Choe in [6]. Their result is the following.

Theorem 6.8 (Two component regularity criterion)**.** *Let $u \in C\left([0, T_{max}); \dot{H}^1_{df}\right)$, be a mild solution to the Navier-Stokes equation. There exists $C > 0$ independent of ν such that if $T_{max} < +\infty$, then*

$$\limsup_{t \to T_{max}} \|\omega_h\|_{L^{\frac{3}{2}}} \geq C\nu. \tag{6.52}$$

Furthermore, for all $\frac{3}{2} < q < +\infty$, let $\frac{3}{q} + \frac{2}{p} = 2$. There exists $D_q > 0$ defending on only p, q and ν such that

$$E(t) \leq E_0 \exp\left(D_q \int_0^t \|\omega_h(\cdot, t)\|_{L^q}^p dt\right). \tag{6.53}$$

Proposition 6.7 extends the result of Chae and Choe from a lower bound on ω_h in $L^{\frac{3}{2}}$ near a possible singularity to a lower bound in $\dot{H}^{-\frac{1}{2}}$ near a possible singularity. The analysis of the relationship between ω_h and v^3 also sheds some light on a relationship between Theorem 6.8 and Corollary 5.8.

We will note here that Proposition 6.6 implies that the regularity criterion on $\partial_3 u + \nabla u_3$, the special case of Corollary 5.8 when the direction is taken to be fixed, is equivalent to Chae and Choe's result in Theorem 6.8 for $\frac{3}{2} < q < +\infty$, because we have shown that for $1 < q < +\infty$, $\|\omega_h\|_{L^q}$ and $\|\partial_3 u + \nabla u_3\|_{L^q}$ are equivalent norms.

We previously found a bound for enstrophy growth in terms of $\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}$. The next step will be to prove a bound on the growth of $\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}$ using the evolution equation for ω_h in Proposition 6.1 and the bounds in Proposition 6.4.

Proposition 6.9 (Grönwall type bound for two vorticity components)**.** *Taking C_1 and C_2 as in Lemmas 4.3 and 4.5, let $\frac{1}{R_2} = \frac{27}{128}\left(1+\sqrt{2}\right)^4 C_1^4 C_2^4$. Then for all mild solutions to the Navier-Stokes equation $u \in C\left([0, T_{max}); H^1_{df}\right)$ and for all $0 \leq t < T_{max}$,*

$$\partial_t \|\omega_h(\cdot, t)\|_{\dot{H}^{-\frac{1}{2}}}^2 \leq \frac{1}{R_2 \nu^3} \|\omega\|_{L^2}^4 \|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}^2. \tag{6.54}$$

Furthermore, for all $0 \leq t < T_{max}$

$$\|\omega_h(\cdot, t)\|_{\dot{H}^{-\frac{1}{2}}}^2 \leq \|\omega_h^0\|_{\dot{H}^{-\frac{1}{2}}}^2 \exp\left(\frac{1}{R_2 \nu^3} \int_0^t \|\omega(\cdot, \tau)\|_{L^2}^4 d\tau\right). \tag{6.55}$$

Proof. We begin by using the evolution equation for ω_h in Proposition 6.1 to compute that

$$\partial_t \frac{1}{2} \|\omega_h(\cdot, t)\|_{\dot{H}^{-\frac{1}{2}}}^2 = -\nu \|\omega_h\|_{\dot{H}^{\frac{1}{2}}}^2 - \left\langle (-\Delta)^{-\frac{1}{2}} \omega_h, (u \cdot \nabla)\omega_h \right\rangle + \left\langle (-\Delta)^{-\frac{1}{2}} \omega_h, S\omega_h + S_h \omega \right\rangle. \tag{6.56}$$

Next we bound the last term using the duality of $\dot{H}^1$ and $\dot{H}^{-1}$:

$$\left\langle (-\Delta)^{-\frac{1}{2}}\omega_h, S\omega_h + S_h\omega \right\rangle \leq \|(-\Delta)^{-\frac{1}{2}}\omega_h\|_{\dot{H}^1}\|S\omega_h + \omega_h S\|_{\dot{H}^{-1}} \tag{6.57}$$

$$= \|\omega_h\|_{L^2}\|S\omega_h + \omega_h S\|_{\dot{H}^{-1}} \tag{6.58}$$

$$\leq C_2\|\omega_h\|_{L^2}\|S_h\omega + S\omega_h\|_{L^{\frac{6}{5}}}, \tag{6.59}$$

where we have applied the definition of the $\dot{H}^1$ to show that $\|(-\Delta)^{-\frac{1}{2}}\omega_h\|_{\dot{H}^1} = \|\omega_h\|_{L^2}$, and then applied the Sobolev inequality in Lemma 4.5. Applying the triangle inequality, the generalized Hölder inequality, and the fractional Sobolev inequality we can see that

$$\left\langle (-\Delta)^{-\frac{1}{2}}\omega_h, S\omega_h + S_h\omega \right\rangle \leq C_2\|\omega_h\|_{L^2}\left(\|S_h\omega\|_{L^{\frac{6}{5}}} + \|S\omega_h\|_{L^{\frac{6}{5}}}\right) \tag{6.60}$$

$$\leq C_2\|\omega_h\|_{L^2}\left(\|S_h\|_{L^3}\|\omega\|_{L^2} + \|S\|_{L^2}\|\omega_h\|_{L^3}\right) \tag{6.61}$$

$$\leq C_1C_2\|\omega_h\|_{L^2}\left(\|S_h\|_{\dot{H}^{\frac{1}{2}}}\|\omega\|_{L^2} + \|S\|_{L^2}\|\omega_h\|_{\dot{H}^{\frac{1}{2}}}\right). \tag{6.62}$$

Applying Lemma 6.4 we observe that $\|S_h\|_{\dot{H}^{\frac{1}{2}}} \leq \frac{1}{\sqrt{2}}\|\omega_h\|_{\dot{H}^{\frac{1}{2}}}$, and applying Proposition 3.1 we observe that $\|S\|_{L^2} = \frac{1}{\sqrt{2}}\|\omega\|_{L^2}$. Finally we can conclude that

$$\left\langle (-\Delta)^{-\frac{1}{2}}\omega_h, S\omega_h + S_h\omega \right\rangle \leq \sqrt{2}C_1C_2\|\omega_h\|_{L^2}\|\omega\|_{L^2}\|\omega_h\|_{\dot{H}^{\frac{1}{2}}} \tag{6.63}$$

$$\leq \sqrt{2}C_1C_2\|\omega\|_{L^2}\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{2}}\|\omega_h\|_{\dot{H}^{\frac{1}{2}}}^{\frac{3}{2}}, \tag{6.64}$$

where we have interpolated between $\dot{H}^{-1}$ and $\dot{H}^1$, observing that $\|\omega_h\|_{L^2} \leq \|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{2}}\|\omega_h\|_{\dot{H}^{\frac{1}{2}}}^{\frac{1}{2}}$.

We now turn our attention to the term $-\left\langle (-\Delta)^{-\frac{1}{2}}\omega_h, (u\cdot\nabla)\omega_h \right\rangle$. First we note that $u \in C\left((0,T_{max}); H^\infty\right)$, due to the higher regularity of mild solutions, so we have sufficient regularity to integrate by parts. Using the fact that $\nabla\cdot u = 0$, conclude that

$$-\left\langle (-\Delta)^{-\frac{1}{2}}\omega_h, (u\cdot\nabla)\omega_h \right\rangle = \left\langle (u\cdot\nabla)(-\Delta)^{-\frac{1}{2}}\omega_h, \omega_h \right\rangle. \tag{6.65}$$

Applying the generalized Hölder inequality, the Sobolev inequality, and the isometry in Proposition 3.1, and interpolating between $\dot{H}^{-1}$ and $\dot{H}^1$ as above, we find that

$$\left\langle (u\cdot\nabla)(-\Delta)^{-\frac{1}{2}}\omega_h, \omega_h \right\rangle \leq \|u\|_{L^6}\|\nabla(-\Delta)^{-\frac{1}{2}}\omega_h\|_{L^2}\|\omega_h\|_{L^3} \tag{6.66}$$

$$= \|u\|_{L^6}\|\omega_h\|_{L^2}\|\omega_h\|_{L^3} \tag{6.67}$$

$$\leq C_1C_2\|u\|_{\dot{H}^1}\|\omega_h\|_{L^2}\|\omega_h\|_{\dot{H}^{\frac{1}{2}}} \tag{6.68}$$

$$= C_1C_2\|\omega\|_{L^2}\|\omega_h\|_{L^2}\|\omega_h\|_{\dot{H}^{\frac{1}{2}}} \tag{6.69}$$

$$\leq C_1C_2\|\omega\|_{L^2}\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{2}}\|\omega_h\|_{\dot{H}^{\frac{1}{2}}}^{\frac{3}{2}}. \tag{6.70}$$

Combining the bounds in (6.64) and (6.70), we find that

$$\partial_t\frac{1}{2}\|\omega_h(\cdot,t)\|_{\dot{H}^{-\frac{1}{2}}}^2 \leq -\nu\|\omega_h\|_{\dot{H}^{\frac{1}{2}}}^2 + \left(1+\sqrt{2}\right)C_1C_2\|\omega\|_{L^2}\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{2}}\|\omega_h\|_{\dot{H}^{\frac{1}{2}}}^{\frac{3}{2}}. \tag{6.71}$$

Setting $r = \|\omega_h\|_{\dot{H}^{\frac{1}{2}}}$, we can see that

$$\partial_t \frac{1}{2}\|\omega_h(\cdot,t)\|^2_{\dot{H}^{-\frac{1}{2}}} \leq \sup_{r>0}\left(-\nu r^2 + \left(1+\sqrt{2}\right) C_1 C_2 \|\omega\|_{L^2} \|\omega_h\|^{\frac{1}{2}}_{\dot{H}^{-\frac{1}{2}}} r^{\frac{3}{2}}\right). \tag{6.72}$$

Let $f(r) = -\nu r^2 + M r^{\frac{3}{2}}$, where $M = \left(1+\sqrt{2}\right) C_1 C_2 \|\omega\|_{L^2} \|\omega_h\|^{\frac{1}{2}}_{\dot{H}^{-\frac{1}{2}}}$. Observe that

$$f'(r) = -2\nu r + \frac{3}{2} M r^{\frac{1}{2}}. \tag{6.73}$$

Therefore f has a global max at $r_0 = \sqrt{\frac{3M}{4\nu}}$. This implies that

$$\sup_{r>0}\left(-\nu r^2 + \left(1+\sqrt{2}\right) C_1 C_2 \|\omega\|_{L^2} \|\omega_h\|^{\frac{1}{2}}_{\dot{H}^{-\frac{1}{2}}} r^{\frac{3}{2}}\right) = f(r_0) = \frac{27}{256\nu^3} M^4. \tag{6.74}$$

Substituting in for M, we find that

$$\partial_t \frac{1}{2}\|\omega_h(\cdot,t)\|^2_{\dot{H}^{-\frac{1}{2}}} \leq \frac{27\left(1+\sqrt{2}\right)^4 C_1^4 C_2^4}{256\nu^3} \|\omega\|^4_{L^2} \|\omega_h\|^2_{\dot{H}^{-\frac{1}{2}}}. \tag{6.75}$$

Multiplying both sides by 2, and substituting in $\frac{1}{R_2} = \frac{27\left(1+\sqrt{2}\right)^4 C_1^4 C_2^4}{128}$, observe that

$$\partial_t \|\omega_h(\cdot,t)\|^2_{\dot{H}^{-\frac{1}{2}}} \leq \frac{1}{R_2 \nu^3} \|\omega\|^4_{L^2} \|\omega_h\|^2_{\dot{H}^{-\frac{1}{2}}}. \tag{6.76}$$

Applying Grönwall's inequality, this completes the proof. □

With this bound, we now have developed all the machinery we need to prove the main result of this chapter, Theorem 1.12, which is restated here for the reader's convenience.

Theorem 6.10 (Global regularity for two components of vorticity small)**.** *For each initial condition $u^0 \in \dot{H}^1_{df}$ such that*

$$\|\omega_h^0\|_{\dot{H}^{-\frac{1}{2}}} \exp\left(\frac{K_0 E_0 - 6,912\pi^4\nu^4}{R_2 \nu^3}\right) < R_1 \nu, \tag{6.77}$$

u^0 generates a unique, global smooth solution to the Navier-Stokes equation $u \in C\left((0,+\infty); H^1_{df}\right)$, that is $T_{max} = +\infty$. Note that the smallness condition can be equivalently stated as

$$K_0 E_0 < 6,912\pi^4\nu^4 + R_2\nu^3 \log\left(\frac{R_1\nu}{\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}}\right), \tag{6.78}$$

and that the constants R_1 and R_2 are taken as in Propositions 6.7 and 6.9.

Proof. We will prove the contrapositive. That is we will show that $T_{max} < +\infty$ implies that

$$\|\omega_h^0\|_{\dot{H}^{-\frac{1}{2}}} \exp\left(\frac{K_0 E_0 - 6,912\pi^4\nu^4}{R_2 \nu^3}\right) \geq R_1 \nu. \tag{6.79}$$

Using Proposition 4.6, $T_{max} < +\infty$ implies that $K_0 E_0 \geq 2,916\pi^4\nu^4$. This means that

$$\exp\left(\frac{K_0 E_0 - 6,912\pi^4\nu^4}{R_2\nu^3}\right) \geq 1. \tag{6.80}$$

If $\|\omega_h^0\| \geq R_1\nu$, this completes the proof.

Now Suppose $\|\omega_h^0\| < R_1\nu$. We know that

$$\limsup_{t\to T_{max}} \|\omega(\cdot,t)\|_{L^2} = +\infty. \tag{6.81}$$

Therefore $\|\omega(\cdot,t)\|_{L^2}$ cannot be non-decreasing on $(0,T_{max})$. There exists $0 < \tilde{t} < T_{max}$ such that $\partial_t\|\omega(\cdot,t)\|_{L^2}^2 > 0$. By Proposition 6.7, we can conclude that there exists $0 < \tilde{t} < T_{max}$ such that $\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}} > R_1\nu$. $\omega_h \in C\left([0,T_{max});\dot{H}^{-\frac{1}{2}}\right)$, so by the intermediate value theorem, there exists $0 < t < \tilde{t}$, such that $\|\omega_h(\cdot,t)\|_{\dot{H}^{-\frac{1}{2}}} = R_1\nu$. Let T be the first such time. That is, define $T < T_{max}$ by

$$T = \inf\left\{t < T_{max} : \|\omega_h(\cdot,t)\|_{\dot{H}^{-\frac{1}{2}}} = R_1\nu\right\}. \tag{6.82}$$

It is clear from the intermediate value theorem and the fact that $\|\omega_h^0\|_{\dot{H}^{-\frac{1}{2}}} < R_1\nu$, that for all $t < T$, $\|\omega_h(\cdot,t)\|_{\dot{H}^{-\frac{1}{2}}} < R_1\nu$.

Applying Proposition 6.7, this implies that for all $t < T$, $\partial_t\|\omega(\cdot,t)\|_{L^2}^2 < 0$. Using Proposition 6.9, observe that

$$R_1\nu = \|\omega_h(\cdot,T)\|_{\dot{H}^{-\frac{1}{2}}} \leq \|\omega_h^0\|_{\dot{H}^{-\frac{1}{2}}} \exp\left(\frac{1}{2R_2\nu^3}\int_0^T \|\omega(\cdot,t)\|_{L^2}^4 dt\right). \tag{6.83}$$

Using the fact that $\|\omega(\cdot,t)\|_{L^2}$ is decreasing on the interval $[0,T]$, we can pull out a factor of $\|\omega^0\|_{L^2}^2$, and conclude

$$R_1\nu \leq \|\omega_h^0\|_{\dot{H}^{-\frac{1}{2}}} \exp\left(\frac{1}{2R_2\nu^3}\|\omega^0\|_{L^2}^2\int_0^T \|\omega(\cdot,t)\|_{L^2}^2 dt\right). \tag{6.84}$$

We know from the energy equality that

$$\int_0^T \|\omega(\cdot,t)\|_{L^2}^2 dt = \frac{1}{2}\|u^0\|_{L^2} - \frac{1}{2}\|u(\cdot,T)\|_{L^2}^2. \tag{6.85}$$

Therefore

$$R_1\nu \leq \|\omega_h^0\|_{\dot{H}^{-\frac{1}{2}}} \exp\left(\frac{1}{2R_2\nu^3}\|\omega^0\|_{L^2}^2\left(\frac{1}{2}\|u^0\|_{L^2}^2 - \frac{1}{2}\|u(\cdot,T)\|_{L^2}^2\right)\right). \tag{6.86}$$

Again using the fact that $\|\omega(\cdot,t)\|_{L^2}$ is decreasing on the interval $[0,T]$, and therefore that $\|\omega(\cdot,T)\|_{L^2} < \|\omega^0\|_{L^2}$, we may conclude that

$$R_1\nu \leq \|\omega_h^0\|_{\dot{H}^{-\frac{1}{2}}} \exp\left(\frac{1}{R_2\nu^3}\left(\frac{1}{2}\|u^0\|_{L^2}^2\frac{1}{2}\|\omega^0\|_{L^2}^2 - \frac{1}{2}\|\omega(\cdot,T)\|_{L^2}^2\frac{1}{2}\|u(\cdot,T)\|_{L^2}^2\right)\right). \tag{6.87}$$

This means that

$$R_1\nu \leq \|\omega_h^0\|_{\dot{H}^{-\frac{1}{2}}} \exp\left(\frac{1}{R_2\nu^3}\left(K_0E_0 - K(T)E(T)\right)\right). \tag{6.88}$$

Applying Corollary 4.7, $K(T)E(T) \geq 6,912\pi^4\nu^4$, so

$$\|\omega_h^0\|_{\dot{H}^{-\frac{1}{2}}} \exp\left(\frac{K_0E_0 - 6,912\pi^4\nu^4}{R_2\nu^3}\right) \leq \|\omega_h^0\|_{\dot{H}^{-\frac{1}{2}}} \exp\left(\frac{1}{R_2\nu^3}\left(K_0E_0 - 6,912\pi^4\nu^4\right)\right). \tag{6.89}$$

Therefore $T_{max} < +\infty$ implies that

$$\|\omega_h^0\|_{\dot{H}^{-\frac{1}{2}}} \exp\left(\frac{1}{R_2\nu^3}\left(K_0E_0 - 6,912\pi^4\nu^4\right)\right) \geq R_1\nu. \tag{6.90}$$

This completes the proof. □

Chapter 7

Relationship of the vorticity approach to previous results

In this chapter we will consider the relationship between the vorticity approach to almost two dimensional initial data developed in chapter 6 and previous global regularity results for almost two dimensional initial data. Gallagher and Chemin proved in [9] that initial data re-scaled so it varies slowly in one direction must generate global smooth solutions.

Theorem 7.1 (Global regularity in the well prepared case)**.** *Let $v_h^0 = (v_1, v_2)$ be a smooth divergence free vector field on $\mathbb{R}^3$ that belongs, along with all of its derivatives, to $L^2\left(\mathbb{R}_{x_3}; \dot{H}^{-1}\left(\mathbb{R}^2\right)\right)$, and let w^0 be any smooth divergence free vector field from $\mathbb{R}^3$ to $\mathbb{R}^3$. For each $\epsilon > 0$ define the re-scaled initial data by*

$$u^{0,\epsilon}(x) = (v_h^0 + \epsilon w_h^0, w_3^0)(x_h, \epsilon x_3). \tag{7.1}$$

Then there exists $\epsilon_0 > 0$, such that for all $0 < \epsilon < \epsilon_0$, the initial data $u^{0,\epsilon}$ generates a global smooth solution to the Navier-Stokes equations.

This is often referred to as the well-prepared case, because $v_3^0 = 0$, and so $v^{0,\epsilon}$ converges to a two dimensional vector field in the sense that for all $x \in \mathbb{R}^3$.

$$\lim_{\epsilon \to 0} u^{0,\epsilon}(x) = (v_h^0, w_3^0)(x_h, 0). \tag{7.2}$$

We will also note that global regularity in Theorem 7.1 is not a consequence of Koch and Tataru's theorem on global regularity for small initial data in BMO^{-1}, because, subject to certain conditions, $v^{0,\epsilon}$ is large in $B^{-1}_{\infty,\infty}$, the largest scale-critical space.

Gallagher, Chemin, and Paicu generalized this result to the ill-prepared case in [10].

Theorem 7.2 (Global regularity in the ill prepared case)**.** *Let u^0 be a divergence free vector field on $\mathbb{T}^2 \times \mathbb{R}$, and for each $\epsilon > 0$ let our rescaling be given by*

$$u^{0,\epsilon}(x) = (u_h^0, \frac{1}{\epsilon} u_3^0)(x_h, \epsilon x_3). \tag{7.3}$$

For all $a > 0$ there exists $\epsilon_0, \mu > 0$ such that if

$$\| \exp(a|D_3|)u^0\|_{H^4(\mathbb{T}^2 \times \mathbb{R})} \leq \mu, \tag{7.4}$$

then for all $0<\epsilon<\epsilon_0$, the initial data $u^{0,\epsilon}$ generates a global smooth solution to the Navier-Stokes equation.

This is referred to as the ill-prepared case because whenever u_3^0 is not identically zero, this clearly does not converge to any almost two dimensional vector field. The proof of this result is quite technical, in particular because all control over $u_3^{0,\epsilon}$ is lost as $\epsilon \to 0$. This means that the proofs do not rely on L^p or Sobolev space estimates, but are based on controlling regularity via a Banach space, B^s that is introduced. The theorem in the paper is actually proved in terms of $B^{\frac{7}{2}}$ and the result in terms of H^4 follows as a corollary.

The underlying reason for these technical difficulties is that, in order to maintain the divergence free structure needed for the Navier-Stokes equation, making the solution vary slowly in x_3 requires us to make $u_3^{0,\epsilon}$ large, so that applying the chain rule,

$$\nabla \cdot u^{0,\epsilon}(x) = (\partial_1 u_1^0 + \partial_2 u_2^0 + \epsilon \frac{1}{\epsilon} \partial_3 u_3^0)(x_h, \epsilon x_3) = (\nabla \cdot u^0)(x_h, \epsilon x_3) = 0. \tag{7.5}$$

One way to get around this technical difficulty without the restriction that $v_3^0 = 0$, is to perform the rescaling in terms of the vorticity, rather than the velocity. For a solution to be almost two dimensional, we want both and u_3 to be small and for the solution to vary slowly with respect to x_3, but the divergence free condition doesn't let us scale both out simultaneously.

On the vorticity side however, a two dimensional flow has its vorticity in the vertical direction, so an almost two dimensional flow corresponds to one in which ω_1 and ω_2 are small, and which varies slowly with respect to x_3. Take

$$\omega^{0,\epsilon} = (\epsilon \omega_h, \omega_3)(x_h, \epsilon x_3). \tag{7.6}$$

This re-scaling preserves the divergence free condition, because applying the chain rule

$$\nabla \cdot \omega^{0,\epsilon}(x) = \epsilon (\nabla \cdot \omega^0)(x_h, \epsilon x_3) = 0. \tag{7.7}$$

Furthermore, this is a re-scaling which allows us to to converge to an almost two dimensional initial data without any restrictions such as $v_3^0 = 0$. Theorem 1.12, is not strong enough to prove there is global regularity for sufficiently small ϵ with this re-scaling, because it is only a logarithmic correction. We will, however prove an analogous result that is slightly weaker in terms of scaling, because it grows more slowly in the critical norms as $\epsilon \to 0$, but still becomes large in the critical space $L^{\frac{3}{2}}$; this result in Theorem 1.13 in the introduction, which is restated here for the reader's convenience.

Theorem 7.3 (Global regularity for rescaled vorticity)**.** *Fix $a>0$. For all $u^0 \in H^1_{df}$, $0<\epsilon<1$ let*

$$\omega^{0,\epsilon}(x) = \epsilon^{\frac{2}{3}} \left(\log \left(\frac{1}{\epsilon^a} \right) \right)^{\frac{1}{4}} \left(\epsilon \omega_1^0, \epsilon \omega_2^0, \omega_3^0 \right) (x_1, x_2, \epsilon x_3), \tag{7.8}$$

and define $u^{0,\epsilon}$ using the Biot-Savart law by

$$u^{0,\epsilon} = \nabla \times (-\Delta)^{-1} \omega^{0,\epsilon}. \tag{7.9}$$

For all $u^0 \in H^1_{df}$ and for all $0 < a < \frac{4 R_2 \nu^3}{C_2^2 \|\omega_3^0\|^2_{L^{\frac{6}{5}}} \|\omega_3^0\|^2_{L^2}}$, there exists $\epsilon_0>0$ such that for all $0<\epsilon<$

ϵ_0, there is a unique, global smooth solution to the Navier-Stokes equation $u \in C\left((0,+\infty); H^1_{df}\right)$ with $u(\cdot,0) = u^{0,\epsilon}$. Furthermore if ω_3^0 is not identically zero, then the initial vorticity becomes large in the critical space $L^{\frac{3}{2}}$ as $\epsilon \to 0$, that is

$$\lim_{\epsilon \to 0} \|\omega^{0,\epsilon}\|_{L^{\frac{3}{2}}} = +\infty. \tag{7.10}$$

We note that while Theorem 7.3 is weaker in terms of scaling than Theorem 7.2 proven in [10], it is stronger in the sense that it allows us to take as initial data the re-scalings of arbitrary $u^0 \in H^1_{df}$, whereas Theorem 7.2 requires that the we re-scale $u^0 \in H^4$ that is also smooth with respect to x_3. The regularity hypotheses on u^0 in Theorem 7.3 are the weakest available in order to ensure global regularity for initial data rescaled to be almost two dimensional. Before proving Theorem 7.3, we will need to state a corollary of Theorem 1.12 that guarantees global regularity purely in terms of L^p norms of ω.

Corollary 7.4. *For all $u^0 \in \dot{H}^1_{df}$ such at*

$$C_1 \|\omega_h^0\|_{L^{\frac{3}{2}}} \exp\left(\frac{\frac{1}{4}C_2^2 \|\omega^0\|_{L^{\frac{6}{5}}} \|\omega^0\|_{L^2}^2 - 6,912\pi^4\nu^4}{R_2\nu^3}\right) < R_1\nu, \tag{7.11}$$

u^0 generates a unique, global smooth solution to the Navier-Stokes equation $u \in C\left((0,+\infty); H^1_{df}\right)$, that is $T_{max} = +\infty$, with C_2 taken as in Lemma 4.5, and R_1 and R_2 taken as in Theorem 1.12.

Proof. This is a corollary of Theorem 1.12. Suppose

$$C_1 \|\omega_h^0\|_{L^{\frac{3}{2}}} \exp\left(\frac{\frac{1}{4}C_2^2 \|\omega^0\|_{L^{\frac{6}{5}}}^2 \|\omega^0\|_{L^2}^2 - 6,912\pi^4\nu^4}{R_2\nu^3}\right) < R_1\nu. \tag{7.12}$$

We know from the fractional Sobolev inequality, Lemma 4.3, that

$$\|\omega_h^0\|_{\dot{H}^{-\frac{1}{2}}} \leq C_1 \|\omega_h^0\|_{L^{\frac{3}{2}}}, \tag{7.13}$$

and from the Sobolev inequality, Lemma 4.5, that

$$K_0 = \frac{1}{2}\|\omega^0\|_{L^2}^2 \leq \frac{1}{2} C_2^2 \|\omega^0\|_{L^{\frac{6}{5}}}^2. \tag{7.14}$$

Therefore we can conclude that

$$\|\omega_h^0\|_{\dot{H}^{-\frac{1}{2}}} \exp\left(\frac{K_0E_0 - 6,912\pi^4\nu^4}{R_2\nu^3}\right) \leq C_1 \|\omega_h^0\|_{L^{\frac{3}{2}}} \exp\left(\frac{\frac{1}{4}C_2^2 \|\omega^0\|_{L^{\frac{6}{5}}} \|\omega^0\|_{L^2}^2 - 6,912\pi^4\nu^4}{R_2\nu^3}\right). \tag{7.15}$$

This implies that

$$\|\omega_h^0\|_{\dot{H}^{-\frac{1}{2}}} \exp\left(\frac{K_0E_0 - 6,912\pi^4\nu^4}{R_2\nu^3}\right) < R_1\nu. \tag{7.16}$$

Applying Theorem 1.12, this completes the proof. □

Remark 7.5. *For all* $1 \leq q < +\infty$, *and for all* $f \in L^q\left(\mathbb{R}^3\right)$

$$\|f^\epsilon\|_{L^q} = \epsilon^{-\frac{1}{q}}\|f\|_{L^q}, \tag{7.17}$$

where $f^\epsilon(x) = f(x_1, x_2, \epsilon x_3), \epsilon > 0$. *This is an elementary computation for the rescaling of the* L^q *norm in one direction.*

We will now prove Theorem 7.3.

Proof. Fix $u^0 \in H^1_{df}$ and $0 < a < \frac{4R_2\nu^3}{C_2^2\|\omega_3^0\|^2_{L^{\frac{6}{5}}}\omega_3^0\|\|^2_{L^2}}$. We will prove the result using Corollary 7.4. Applying Remark 7.5, we find that

$$\|\omega_h^{0,\epsilon}\|_{L^{\frac{3}{2}}} = \epsilon \log\left(\epsilon^{-a}\right)^{\frac{1}{4}}\|\omega_h^0\|_{L^{\frac{3}{2}}}. \tag{7.18}$$

Similarly we apply Remark 7.5, to compute the other relevant L^q norms in Corollary 7.4:

$$\|\omega_3^{0,\epsilon}\|_{L^2} = \epsilon^{\frac{1}{6}} \log\left(\epsilon^{-a}\right)^{\frac{1}{4}}\|\omega_h^0\|_{L^2}, \tag{7.19}$$

$$\|\omega_h^{0,\epsilon}\|_{L^2} = \epsilon^{\frac{7}{6}} \log\left(\epsilon^{-a}\right)^{\frac{1}{4}}\|\omega_h^0\|_{L^2}, \tag{7.20}$$

$$\|\omega_3^{0,\epsilon}\|_{L^{\frac{6}{5}}} = \epsilon^{-\frac{1}{6}} \log\left(\epsilon^{-a}\right)^{\frac{1}{4}}\|\omega_h^0\|_{L^{\frac{6}{5}}}, \tag{7.21}$$

$$\|\omega_h^{0,\epsilon}\|_{L^{\frac{6}{5}}} = \epsilon^{\frac{5}{6}} \log\left(\epsilon^{-a}\right)^{\frac{1}{4}}\|\omega_h^0\|_{L^{\frac{6}{5}}}. \tag{7.22}$$

Using the triangle inequality for norms we can see that

$$\|\omega^{0,\epsilon}\|_{L^2} \leq \|\omega_3^{0,\epsilon}\|_{L^2} + \|\omega_h^{0,\epsilon}\|_{L^2} \tag{7.23}$$

$$= \epsilon^{\frac{1}{6}} \log\left(\epsilon^{-a}\right)^{\frac{1}{4}}\|\omega_3^0\|_{L^2} + \epsilon^{\frac{7}{6}} \log\left(\epsilon^{-a}\right)^{\frac{1}{4}}\|\omega_h^0\|_{L^2}. \tag{7.24}$$

Likewise we may compute that

$$\|\omega^{0,\epsilon}\|_{L^{\frac{6}{5}}} \leq \|\omega_3^{0,\epsilon}\|_{L^{\frac{6}{5}}} + \|\omega_h^{0,\epsilon}\|_{L^{\frac{6}{5}}} \tag{7.25}$$

$$= \epsilon^{-\frac{1}{6}} \log\left(\epsilon^{-a}\right)^{\frac{1}{4}}\|\omega_3^0\|_{L^{\frac{6}{5}}} + \epsilon^{\frac{5}{6}} \log\left(\epsilon^{-a}\right)^{\frac{1}{4}}\|\omega_h^0\|_{L^{\frac{6}{5}}}. \tag{7.26}$$

Combining these inequalities and factoring out the $\log\left(\epsilon^{-a}\right)^{\frac{1}{4}}$ terms we find that

$$\|\omega^{0,\epsilon}\|^2_{L^{\frac{6}{5}}}\|\omega^{0,\epsilon}\|^2_{L^2} \leq \log\left(\epsilon^{-a}\right)\left(\|\omega_3^0\|_{L^2} + \epsilon\|\omega_h^0\|_{L^2}\right)^2\left(\|\omega_3^0\|_{L^{\frac{6}{5}}} + \epsilon\|\omega_h^0\|_{L^{\frac{6}{5}}}\right)^2. \tag{7.27}$$

Dividing by $R_2\nu^3$ and taking the exponential of both sides of this inequality, we find that

$$\exp\left(\frac{C_2^2\|\omega^{0,\epsilon}\|^2_{L^{\frac{6}{5}}}\|\omega^{0,\epsilon}\|^2_{L^2}}{4R_2\nu^3}\right) \leq \epsilon^{-a\frac{C_2^2\left(\|\omega_3^0\|_{L^2}+\epsilon\|\omega_h^0\|_{L^2}\right)^2\left(\|\omega_3^0\|_{L^{\frac{6}{5}}}+\epsilon\|\omega_h^0\|_{L^{\frac{6}{5}}}\right)^2}{4R_2\nu^3}}. \tag{7.28}$$

Combining this with the estimate (7.18), we find that

$$\|\omega_h^{0,\epsilon}\|_{L^{\frac{3}{2}}} \exp\left(\frac{C_2^2\|\omega^{0,\epsilon}\|_{L^{\frac{6}{5}}}^2\|\omega^{0,\epsilon}\|_{L^2}^2}{4R_2\nu^3}\right) \leq \epsilon^{1-a\frac{C_2^2\left(\|\omega_3^0\|_{L^2}+\epsilon\|\omega_h^0\|_{L^2}\right)^2\left(\|\omega_3^0\|_{L^{\frac{6}{5}}}+\epsilon\|\omega_h^0\|_{L^{\frac{6}{5}}}\right)^2}{4R_2\nu^3}} \log\left(\epsilon^{-a}\right)^{\frac{1}{4}}\|\omega_h^0\|_{L^{\frac{3}{2}}}. \tag{7.29}$$

We know from the definition of a that

$$a\frac{\|w_3^0\|_{L^2}^2\|\omega_3^0\|_{L^{\frac{6}{5}}}^2}{R_2\nu^3} < 1, \tag{7.30}$$

so fix

$$0 < \delta < 1 - a\frac{\|w_3^0\|_{L^2}^2\|\omega_3^0\|_{L^{\frac{6}{5}}}^2}{R_2\nu^3}. \tag{7.31}$$

Clearly we can see that

$$\lim_{\epsilon\to 0} 1 - a\frac{\left(\|\omega_3^0\|_{L^2}+\epsilon\|\omega_h^0\|_{L^2}\right)^2\left(\|\omega_3^0\|_{L^{\frac{6}{5}}}+\epsilon\|\omega_h^0\|_{L^{\frac{6}{5}}}\right)^2}{R_2\nu^3} = 1 - a\frac{\|w_3^0\|_{L^2}^2\|\omega_3^0\|_{L^{\frac{6}{5}}}^2}{R_2\nu^3}. \tag{7.32}$$

Therefore, there exists $r > 0$, such that for all $0 < \epsilon < r$,

$$1 - a\frac{\left(\|\omega_3^0\|_{L^2}+\epsilon\|\omega_h^0\|_{L^2}\right)^2\left(\|\omega_3^0\|_{L^{\frac{6}{5}}}+\epsilon\|\omega_h^0\|_{L^{\frac{6}{5}}}\right)^2}{R_2\nu^3} > \delta. \tag{7.33}$$

Then for all $0 < \epsilon < \min(1, r)$,

$$\epsilon^{1-a\frac{\left(\|\omega_3^0\|_{L^2}+\epsilon\|\omega_h^0\|_{L^2}\right)^2\left(\|\omega_3^0\|_{L^{\frac{6}{5}}}+\epsilon\|\omega_h^0\|_{L^{\frac{6}{5}}}\right)^2}{R_2\nu^3}} < \epsilon^{\delta}. \tag{7.34}$$

Combining this estimate with the estimate (7.29), we find

$$\lim_{\epsilon\to 0}\|\omega_h^{0,\epsilon}\|_{L^{\frac{3}{2}}} \exp\left(\frac{C_2^2\|\omega^{0,\epsilon}\|_{L^{\frac{6}{5}}}^2\|\omega^{0,\epsilon}\|_{L^2}^2}{4R_2\nu^3}\right) \leq \lim_{\epsilon\to 0}\|\omega_h^0\|_{L^{\frac{3}{2}}}\epsilon^{\delta}\log\left(\epsilon^{-a}\right)^{\frac{1}{4}}. \tag{7.35}$$

Making the substitution $k = \frac{1}{\epsilon}$, we find

$$\lim_{\epsilon\to 0}\|\omega_h^0\|_{L^{\frac{3}{2}}}\epsilon^{\delta}\log\left(\epsilon^{-a}\right)^{\frac{1}{4}} = \lim_{k\to+\infty}\|\omega_h^0\|_{L^{\frac{3}{2}}}\frac{\log\left(k^a\right)^{\frac{1}{4}}}{k^{\delta}} = 0, \tag{7.36}$$

because the logarithm grows more slowly than any power. Putting these inequalities together we find that

$$\lim_{\epsilon\to 0}\|\omega_h^{0,\epsilon}\|_{L^{\frac{3}{2}}} \exp\left(\frac{C_2^2\|\omega^{0,\epsilon}\|_{L^{\frac{6}{5}}}^2\|\omega^{0,\epsilon}\|_{L^2}^2}{4R_2\nu^3}\right) \leq 0. \tag{7.37}$$

This limit is clearly non-negative, so we can conclude that

$$\lim_{\epsilon\to 0}\|\omega_h^{0,\epsilon}\|_{L^{\frac{3}{2}}}\exp\left(\frac{C_2^2\|\omega^{0,\epsilon}\|_{L^{\frac{6}{5}}}^2\|\omega^{0,\epsilon}\|_{L^2}^2}{4R_2\nu^3}\right)=0. \tag{7.38}$$

Therefore there exists $\epsilon_0>0$, such that for all $0<\epsilon<\epsilon_0$,

$$\|\omega_h^{0,\epsilon}\|_{L^{\frac{3}{2}}}\exp\left(\frac{C_2^2\|\omega^{0,\epsilon}\|_{L^{\frac{6}{5}}}^2\|\omega^{0,\epsilon}\|_{L^2}^2}{4R_2\nu^3}\right)<\exp\left(\frac{6,912\pi^4\nu^4}{R_2\nu^3}\right)R_1\nu. \tag{7.39}$$

Applying Corollary 7.4, this means for all $0<\epsilon<\epsilon_0$ there is a unique global smooth solution for initial data $u^{0,\epsilon}\in H^1_{df}$.

Next we will show that unless ω_3^0 is identically zero,

$$\lim_{\epsilon\to 0}\|\omega^{0,\epsilon}\|_{L^{\frac{3}{2}}}=+\infty. \tag{7.40}$$

We know that

$$\|\omega^{0,\epsilon}\|_{L^{\frac{3}{2}}}\geq\|\omega_3^{0,\epsilon}\|_{L^{\frac{3}{2}}}, \tag{7.41}$$

so it suffices to show that

$$\lim_{\epsilon\to 0}\|\omega_3^{0,\epsilon}\|_{L^{\frac{3}{2}}}=+\infty. \tag{7.42}$$

We can see from Remark 7.5, that

$$\|\omega_3^{0,\epsilon}\|_{L^{\frac{3}{2}}}=\log\left(\epsilon^{-a}\right)\|\omega_3^0\|_{L^{\frac{3}{2}}}. \tag{7.43}$$

Therefore we may compute that

$$\lim_{\epsilon\to 0}\|\omega_3^{0,\epsilon}\|_{L^{\frac{3}{2}}}=\|\omega_3^0\|_{L^{\frac{3}{2}}}\lim_{\epsilon\to 0}\log\left(\epsilon^{-a}\right)=+\infty. \tag{7.44}$$

This completes the proof. □

Iftimie proved the global existence of smooth solutions for the Navier-Stokes equation with three dimensional initial data that are a perturbation of two dimensional initial data. As we mentioned in the introduction, this is possible on the torus, but not on the whole space, in particular because $L^2\left(\mathbb{T}^2\right)$ defines a subspace of $L^2\left(\mathbb{T}^3\right)$, but $L^2\left(\mathbb{R}^2\right)$ does not define a subspace of $L^2\left(\mathbb{R}^3\right)$ because we lose integrability. The precise result Iftime showed is the following [24].

Theorem 7.6 (Perturbations of two dimensional initial data)**.** *There exists $C>0$, such that for all $v^0\in L^2_{df}(\mathbb{T}^2;\mathbb{R}^3)$, and for all $w^0\in H^{\frac{1}{2}}_{df}\left(\mathbb{T}^3;\mathbb{R}^3\right)$, such that*

$$\|w^0\|_{\dot{H}^{\frac{1}{2}}}\exp\left(\frac{\|v^0\|_{L^2}^2}{C\nu^2}\right)\leq C\nu, \tag{7.45}$$

there exists a unique, global smooth solution to the Navier-Stokes equation with initial data $u^0=v^0+w^0$.

In fact, Iftimie proves something slightly stronger. The result still holds if the space $H^{\frac{1}{2}}$ is replaced by the anisotrophic space $H^{\delta,\delta,\frac{1}{2}-\delta}, 0<\delta<\frac{1}{2}$ which is the space given by taking the $H^{\frac{1}{2}-\delta}$ norm with respect to x_3, leaving x_1, x_2 fixed, giving us a function of x_1 and x_2, then taking the H^δ norm with respect to x_2 and so forth. In the range $0 < \delta < \frac{1}{2}$, these spaces strictly contain $H^{\frac{1}{2}}$. This result was also extended to the case of the Navier-Stokes equation with an external force by Gallagher [20], but only where the control in w^0 is in the critical Hilbert space $\dot{H}^{\frac{1}{2}}$, not in these more complicated, anisotropic spaces. These anisotropic spaces are quite messy; in particular we will note that for $\alpha \neq 0$, $H^{\alpha,\alpha,\alpha} \neq H^\alpha\left(\mathbb{T}^3\right)$. For this reason, and because the results in this thesis deal with Hilbert spaces, we will focus our comparison of Iftimie's result with ours in the setting of $\dot{H}^{\frac{1}{2}}$. For more details on these anisotropic spaces, see [25].

We will find that Iftimie's result neither implies, nor is implied by, our result, but that they are closely related. In order to compare the results in this thesis to the result proven by Iftimie, it is first necessary to state a version of Theoerem 1.12 on the torus. The result will be essentially the same, although possibly with different constants.

Theorem 7.7 (Global regularity for two components of vorticity on the torus)**.** *There exists $\tilde{R}_1, \tilde{R}_2, \tilde{R}_3 > 0$ independent of ν, such that or all $u^0 \in H^1_{df}\left(\mathbb{T}^3\right)$ such at*

$$\|\omega_h^0\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{T}^3)} \exp\left(\frac{K_0 E_0 - \tilde{R}_3 \nu^4}{\tilde{R}_2 \nu^3}\right) < \tilde{R}_1 \nu, \tag{7.46}$$

u^0 generates a unique, global smooth solution to the Navier-Stokes equation $u \in C\left((0,+\infty); H^1_{df}\left(\mathbb{T}^3\right)\right)$, that is $T_{max} = +\infty$.

The proof of the this result on the torus is exactly the same as the proof of the result on the whole space. The only reason the constants may be different is because the sharp Sobolev constant may be worse on the torus than the whole space. We will note that when considering solutions to the Navier-Stokes equations on the torus, we include the stipulation that the flow over the whole torus integrates to zero, so

$$\hat{u}(0,0,0) = \int_{\mathbb{T}^3} u(x)dx = 0. \tag{7.47}$$

This normalization is necessary in order to mod out constant functions on the torus, so without this stipulation, we would not in fact be able to make use of Sobolev and fractional Sobolev inequalities.

In order to relate Theorem 7.6 and Theorem 7.7, we will need to define a projection from three dimensional vector fields to two dimensional vector fields, following the approach of Iftimie [24] and Gallagher [22].

Proposition 7.8 (Projection onto two dimensional velocities)**.** *Define P_{2d} by*

$$P_{2d}(u)(x_h) = \int_0^1 u(x_h, x_3)dx_3. \tag{7.48}$$

Then for all $1 \leq q \leq +\infty$, $P_{2d} : L^q_{df}\left(\mathbb{T}^3\right) \to L^q_{df}\left(\mathbb{T}^2\right)$. In particular,

$$\nabla \cdot P_{2d}(u) = 0, \tag{7.49}$$

and

$$\|P_{2d}(u)\|_{L^q(\mathbb{T}^2)} \leq \|u\|_{L^q(\mathbb{T}^3)}. \tag{7.50}$$

Proof. Notice that we are projecting onto two dimensional vector fields by taking the average in the vertical direction. First we will observe that P_{2d} is a bounded linear map from L^q to L^q. Linearity is clear. As for boundedness, applying Minkowski's inequality, we find

$$\|P_{2d}(u)\|_{L^q(\mathbb{T}^2)} \leq \int_0^1 \|u_h(\cdot, x_3)\|_{L^q(\mathbb{T}^2)} dx_3. \tag{7.51}$$

Let $f(x_3) = \|u_h(\cdot, x_3)\|_{L^q(\mathbb{T}^3)}, g(x_3) = 1$, and let $\frac{1}{p} + \frac{1}{q} = 1$, then apply Hölder's inequality to observe

$$\int_0^1 \|u(\cdot, x_3)\|_{L^q} dx_3 \leq \|f\|_{L^q} \|g\|_{L^p} = \|u\|_{L^q(\mathbb{T}^3)} 1. \tag{7.52}$$

So we may conclude that

$$\|P_{2d}(u)\|_{L^q(\mathbb{T}^2)} \leq \|u\|_{L^q(\mathbb{T}^3)}. \tag{7.53}$$

Now we need to show that for all $u \in L^q_{df}\left(\mathbb{T}^3\right), \nabla \cdot P_{2d}(u) = 0$. First we will show this by formal computation for u smooth, and then we will extend by density. Fix $u \in C^\infty\left(\mathbb{T}^3\right), \nabla \cdot u = 0$. Observe that

$$\nabla \cdot P_{2d}(u)(x_1, x_2) = \int_0^1 (\partial_1 u_1 + \partial_2 u_2)(x_1, x_2, x_3) dx_3. \tag{7.54}$$

Using the fact that $\nabla \cdot u = 0$, we can conclude that $\partial_1 u_1 + \partial_2 u_2 = -\partial_3 u_3$. Applying the fundamental theorem of calculus, and using the fact that u^3 is continuous and periodic, we find

$$\nabla \cdot P_{2d}(u)(x_1, x_2) = -\int_0^1 \partial_3 u_3(x_1, x_2, x_3) dx_3 = -u^3(x_1, x_2, 1) + u^3(x_1, x_2, 0) = 0. \tag{7.55}$$

We will proceed to proving that $\nabla \cdot P_{2d}(u)$ for all $u \in L^q_{df}\left(\mathbb{T}^3\right)$. Note, we here refer to divergence free in the sense of integrating against test functions, as u is not differentiable a priori. Fix $u \in L^q_{df}\left(\mathbb{T}^3\right)$ and $f \in C^\infty\left(\mathbb{T}^2\right)$. $C^\infty\left(\mathbb{T}^3\right)$ is dense in $L^q_{df}\left(\mathbb{T}^3\right)$, so for some arbitrary $\epsilon > 0$, fix $v \in C^\infty\left(\mathbb{T}^3\right), \nabla \cdot v = 0$, such that

$$\|u - v\|_{L^q(\mathbb{T}^3)} < \epsilon. \tag{7.56}$$

As we have shown above $\nabla \cdot P_{2d}(v) = 0$ in the classical sense, so clearly

$$\langle P_{2d}(v), \nabla f \rangle = 0. \tag{7.57}$$

Using the linearity of P_{2d} observe that

$$\langle P_{2d}(u), \nabla f \rangle = \langle P_{2d}(u - v), \nabla f \rangle. \tag{7.58}$$

Applying Hölder's inequality we find that

$$|\langle P_{2d}(u - v), \nabla f \rangle| \leq \|P_{2d}(u - v)\|_{L^q} \|\nabla f\|_{L^p}. \tag{7.59}$$

We know from the bound we have already shown that

$$\|P_{2d}(u-v)\|_{L^q(\mathbb{T}^2)} \leq \|u-v\|_{L^q(\mathbb{T}^3)} < \epsilon, \tag{7.60}$$

so therefore

$$|\langle P_{2d}(u), \nabla f\rangle| < \epsilon \|\nabla f\|_{L^p}. \tag{7.61}$$

But $\epsilon > 0$ was arbitrary, so taking $\epsilon \to 0$, we find that

$$\langle P_{2d}(u), \nabla f\rangle = 0. \tag{7.62}$$

This completes the proof. □

We will also define the projection onto the subspace orthogonal to $L^2_{df}\left(\mathbb{T}^2;\mathbb{R}^3\right)$.

Definition 7.9 (Projection onto the perpendicular space). *Let $P_{2d}^{\perp}: L^2_{df}\left(\mathbb{T}^3;\mathbb{R}^3\right) \to L^2_{df}\left(\mathbb{T}^3;\mathbb{R}^3\right)$, be given by*

$$P_{2d}^{\perp}(u) = u - P_{2d}(u). \tag{7.63}$$

Note that this is well defined, because we have already shown that $u \in L^2_{df}\left(\mathbb{T}^3\right)$ implies that $P_{2d}(u) \in L^2_{df}\left(\mathbb{T}^3\right)$, so clearly their difference, $u - P_{2d}(u)$, is also in this space, which means it is a well defined linear map.

Remark 7.10. *Note that Theorem 7.6 can be reformulated in terms of P_{2d} and $P_{2d}^{\perp}$ as saying there exists $C>0$ such that for all $u^0 \in H^{\frac{1}{2}}_{df}\left(\mathbb{T}^3\right)$, such that*

$$\|P_{2d}^{\perp}\left(u^0\right)\|_{\dot{H}^{\frac{1}{2}}} \exp\left(\frac{\|P_{2d}\left(u^0\right)\|_{L^2}^2}{C\nu^2}\right) \leq C\nu, \tag{7.64}$$

u^0 generates a global smooth solution to the Navier-Stokes equation.

Next we will note that P_{2d} decomposes the support of the Fourier transform of u into the plane where $k_3 = 0$ and the rest of $\mathbb{Z}^3$.

Proposition 7.11 (Fourier decomposition). *Fix $u \in H^{\frac{1}{2}}_{df}\left(\mathbb{T}^3\right)$. Let $v = P_{2d}(u), w = P_{2d}^{\perp}(u)$. Then*

$$\hat{v}(k) = \begin{cases} \hat{u}(k), k_3 = 0 \\ 0, k_3 \neq 0 \end{cases} \tag{7.65}$$

and

$$\hat{w}(k) = \begin{cases} \hat{u}(k), k_3 \neq 0 \\ 0, k_3 = 0 \end{cases}. \tag{7.66}$$

Proof. First we note that it is obvious that $\hat{w} = \hat{u} - \hat{v}$, so it suffices to prove (7.65). First note that $\partial_3 v = 0$, so

$$\widehat{\partial_3 v}(k) = 2\pi i k_3 \hat{v}(k) = 0. \tag{7.67}$$

Therefore we see that $k_3 \neq 0$ implies that $\hat{v}(k) = 0$. Now we will proceed to the case where $k_3 = 0$. Observe that

$$\hat{v}(k_1,k_2,0) = \int_{\mathbb{T}^2} v(x_h) \exp\left(-2\pi i(k_1x_1 + k_2x_2)\right) dx_h. \tag{7.68}$$

Recalling the definition of P_{2d}, we can see that

$$\hat{v}(k_1, k_2, 0) = \int_{\mathbb{T}^2} \int_0^1 u(x_h, z) \exp\left(-2\pi i(k_1 x_1 + k_2 x_2)\right) dx_h dz. \tag{7.69}$$

Taking $x = (x_h, z) \in \mathbb{T}^3$ we can express this integral as

$$\hat{v}(k_1, k_2, 0) = \int_{\mathbb{T}^3} u(x) \exp\left(-2\pi i(k_1 x_1 + k_2 x_2)\right) dx = \hat{u}(k_1, k_2, 0). \tag{7.70}$$

This completes the proof. □

This Fourier decomposition allows us to control $P_{2d}^{\perp}(u)$ by $\partial_3 u$, although in doing so we lose criticality.

Proposition 7.12. *For all $u \in H_{df}^{\frac{1}{2}}\left(\mathbb{T}^3\right)$,*

$$\|P_{2d}^{\perp}(u)\|_{\dot{H}^{\frac{1}{2}}} \leq \frac{1}{2\pi} \|\partial_3 u\|_{\dot{H}^{\frac{1}{2}}}. \tag{7.71}$$

Proof. Let $w = P_{2d}^{\perp}(u) = u - P_{2d}(u)$. Observe that

$$\|w\|_{\dot{H}^{\frac{1}{2}}}^2 = \sum_{k \in \mathbb{Z}^3} 2\pi |k| |\hat{w}(k)|^2 = \sum_{k_3 \neq 0} 2\pi |k| |\hat{u}(k)|^2. \tag{7.72}$$

Note that for all $k_3 \neq 0, k_3^2 \geq 1$, so we can see that

$$\|w\|_{\dot{H}^{\frac{1}{2}}}^2 \leq \sum_{k_3 \neq 0} 2\pi |k| k_3^2 |\hat{u}(k)|^2 = \frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}^3} 2\pi |k| |2\pi i k_3 \hat{u}(k)|^2. \tag{7.73}$$

Recalling that $\hat{\partial_3 u}(k) = 2\pi i k_3 \hat{u}(k)$, we can compute that

$$\|w\|_{\dot{H}^{\frac{1}{2}}}^2 \leq \sum_{k_3 \neq 0} 2\pi |k| k_3^2 |\hat{u}(k)|^2 = \frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}^3} 2\pi |k| |\widehat{\partial_3 u}(k)|^2 = \frac{1}{4\pi^2} \|\partial_3 u\|_{\dot{H}^{\frac{1}{2}}}^2. \tag{7.74}$$

□

This inequality allows us to prove a corollary of Iftimie's result, Theorem 7.6, that is stated as bound on in terms of the size of $\partial_3 u$ in $H^{\frac{1}{2}}$, rather than in terms of perturbations of $L_{df}^2\left(\mathbb{T}^2\right)$.

Corollary 7.13. *There exists $C > 0$ independent of ν, such that for all $u^0 \in H_{df}^{\frac{1}{2}}\left(\mathbb{T}^3\right)$,*

$$\|\partial_3 u^0\|_{\dot{H}^{\frac{1}{2}}} \exp\left(\frac{\|u^0\|_{L^2}^2}{C\nu^2}\right) \leq 2\pi C\nu, \tag{7.75}$$

implies u^0 generates a global, smooth solution to the Navier-Stokes equations.

Proof. We will take $C > 0$ as in Theorem 7.6. Suppose $u^0 \in H_{df}^{\frac{1}{2}}$ and

$$\|\partial_3 u^0\|_{\dot{H}^{\frac{1}{2}}} \exp\left(\frac{\|u^0\|_{L^2}^2}{C\nu^2}\right) \leq 2\pi C\nu. \tag{7.76}$$

Note that we do not assume that $u \in H^{\frac{3}{2}}$, but the bound on $\|\partial_3 u\|_{\dot{H}^{\frac{1}{2}}}$ clearly implies that $\partial_3 u \in H^{\frac{1}{2}}$ nonetheless. Let $v^0 = P_{2d}\left(u^0\right)$ and let $w^0 = u^0 - P_{2d}\left(u^0\right)$. From Proposition 7.8, we know that

$$\|v^0\|_{L^2(\mathbb{T}^2)} \leq \|u^0\|_{L^2(\mathbb{T}^3)}. \tag{7.77}$$

We also know from Proposition 7.12, that

$$\|w^0\|_{\dot{H}^{\frac{1}{2}}} \leq \frac{1}{2\pi}\|\partial_3\|_{\dot{H}^{\frac{1}{2}}}. \tag{7.78}$$

Putting these two inequalities together we find that

$$\|w^0\|_{\dot{H}^{\frac{1}{2}}} \exp\left(\frac{\|v^0\|_{L^2}^2}{C\nu^2}\right) \leq C\nu. \tag{7.79}$$

Applying Theorem 7.6, this completes the proof. □

We should note here that Corollary 7.13 is not equivalent to Iftimie's result Theorem 7.6; the corollary is implied by this result, but does not imply it. That is because Iftimie's result involves controlling $\|P_{2d}^{\perp}\left(u^0\right)\|_{\dot{H}^{\frac{1}{2}}}$, which is scale critical, but Corollary 7.13 involves controlling $\|\partial_3 u\|_{\dot{H}^{\frac{1}{2}}}$, which is not scale critical.

Corollary 7.13 neither implies, nor is implied by Theorem 7.7, which is the main result of this chapter translated to the setting of the torus rather than the whole space. This is because on the torus, as on the whole space,

$$\|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}^2 = \|\partial_3 u\|_{\dot{H}^{-\frac{1}{2}}}^2 + \|\nabla u_3\|_{\dot{H}^{-\frac{1}{2}}}^2. \tag{7.80}$$

This means that Theorem 7.7 is weaker than Corollary 7.13 in the sense that it requires control on both $\partial_3 u$ and ∇u_3, but it is stronger in the sense that it requires control in the critical space $\dot{H}^{-\frac{1}{2}}$, rather than the subcritical space $\dot{H}^{\frac{1}{2}}$.

In fact we will show that Theorem 7.7 is not implied by Theorem 7.6, because it is not possible to control $\|P_{2d}^{\perp}\left(u^0\right)\|_{\dot{H}^{\frac{1}{2}}}$ by $\|\omega_h^0\|_{\dot{H}^{-\frac{1}{2}}}$. The precise result will be as follows.

Proposition 7.14.

$$\sup_{\substack{u \in H_{df}^{\frac{1}{2}}(\mathbb{T}^3) \\ \|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}=1}} \|P_{2d}^{\perp}(u)\|_{\dot{H}^{\frac{1}{2}}} = +\infty. \tag{7.81}$$

Proof. For all $n \in \mathbb{N}$, define $u^n \in H_{df}^{\frac{1}{2}}$, in terms of its Fourier transform by

$$\widehat{u^n}(k) = a_n \begin{cases} (n,-1,0), k = \pm(1,n,1) \\ 0, \text{otherwise} \end{cases}, \tag{7.82}$$

where a_n is a normalization factor given by

$$a_n = \left(\frac{\sqrt{n^2+2}}{4\pi\left(n^2+1\right)}\right)^{\frac{1}{2}}. \tag{7.83}$$

It is easy to check that for all $n, k \in \mathbb{N}, k \cdot \widehat{u^n}(k) = 0$, so $\nabla \cdot u^n = 0$, and for each $n \in \mathbb{N}, u^n \in H^{\frac{1}{2}}_{df}\left(\mathbb{T}^3\right)$.

It is not essential to the proof, but we will also note for the sake of clarity that

$$u^n(x) = 2a_n(n, -1, 0) \cos\left(2\pi(x_1 + nx_2 + x_3)\right). \tag{7.84}$$

Note that for all $n \in \mathbb{N}$ $u_3^n = 0$, so we have

$$\|\omega_h^n\|_{\dot{H}^{-\frac{1}{2}}} = \|\partial_3 u^n\|_{\dot{H}^{-\frac{1}{2}}}. \tag{7.85}$$

We know that $\widehat{\partial_3 u}(k) = 2\pi i k_3 \widehat{u^n}(k)$, so we can conclude that

$$\widehat{\partial_3 u^n}(k) = 2\pi i a_n \begin{cases} (n, -1, 0), k = \pm(1, n, 1) \\ 0, \text{otherwise} \end{cases}. \tag{7.86}$$

Therefore we can compute that

$$\|\omega_h^n\|^2_{\dot{H}^{-\frac{1}{2}}} = \|\partial_3 u^n\|^2_{\dot{H}^{-\frac{1}{2}}} = 2\frac{1}{2\pi|(1, n, 1)|}|a_n 2\pi i(n, -1, 0)|^2 \tag{7.87}$$

Simplifying terms we find that

$$\|\omega_h^n\|^2_{\dot{H}^{-\frac{1}{2}}} = \frac{4\pi a_n^2\left(n^2+1\right)}{\sqrt{n^2+2}}. \tag{7.88}$$

Recalling that

$$a_n^2 = \frac{\sqrt{n^2+2}}{4\pi\left(n^2+1\right)}, \tag{7.89}$$

we conclude that for all $n \in \mathbb{N}$,

$$\|\omega_h^n\|_{\dot{H}^{-\frac{1}{2}}} = \|\partial_3 u^n\|_{\dot{H}^{-\frac{1}{2}}} = 1. \tag{7.90}$$

We know from Proposition 7.11, that the Fourier transform of $P_{2d}(u)$ is supported on the plane $k_3 = 0$ in $\mathbb{Z}^3$. For all $k_1, k_2 \in \mathbb{N}, \widehat{u^n}(k_1, k_2, 0) = 0$. This implies that for all $n \in \mathbb{N}, P_{2d}(u^n) = 0$, and therefore $P_{2d}^{\perp}(u^n) = u^n$. Observe that

$$\|u^n\|^2_{\dot{H}^{\frac{1}{2}}} = 2\left(2\pi|(1, n, 1)|\right) a_n^2 |(n, -1, 0)|^2 = 4\pi a_n^2 (n^2+1)\sqrt{n^2+2}. \tag{7.91}$$

Again recalling that

$$a_n^2 = \frac{\sqrt{n^2+2}}{4\pi\left(n^2+1\right)}, \tag{7.92}$$

we conclude that for all $n \in \mathbb{N}$,

$$\|u^n\|^2_{\dot{H}^{\frac{1}{2}}} = n^2 + 2. \tag{7.93}$$

Note that we have shown that for all $n \in \mathbb{N}$, $\|\omega_h^n\|_{\dot{H}^{-\frac{1}{2}}} = 1$, and $\|P_{2d}^{\perp}(u^n)\|_{\dot{H}^{\frac{1}{2}}} = \sqrt{n^2+2}$.

Therefore we may conclude that

$$\sup_{\substack{u \in H^{\frac{1}{2}}_{df}(\mathbb{T}^3) \\ \|\omega_h\|_{\dot{H}^{-\frac{1}{2}}}=1}} \|P_{2d}^{\perp}(u)\|_{\dot{H}^{\frac{1}{2}}} = +\infty. \tag{7.94}$$

□

By proving that $\|P_{2d}(u^0)\|_{\dot{H}^{\frac{1}{2}}}$ cannot be controlled by $\|\omega_h^n\|_{\dot{H}^{-\frac{1}{2}}}$, we have shown definitively that Theorem 7.7 is not a corollary of earlier work by Iftimie and separately by Gallagher, and so this result is new on the torus as well as on the whole space.

Chapter 8

Blowup for a toy model ODE of the strain equation

Now that we have outlined the main advances for Navier-Stokes regularity that are possible by utilizing strain equation, we will consider a toy model ODE. The main advantage of the strain equation formulation of the Navier-Stokes equation compared with the vorticity formulation is that the quadratic term $S^2+\frac{1}{4}w\otimes w$ has a much nicer structure than the quadratic term $S\omega$ in the vorticity formulation. The price we pay for this is that there are additional terms, particularly $\text{Hess}(p)$ which are not present in the vorticity formulation. There is also the related difficulty that the consistency condition in the strain formulation is significantly more complicated than in the vorticity formulation.

We will now examine a toy model ODE, prove the existence and stability of blowup, and examine asymptotic behavior near blowup. The simplest toy model equation would be to keep only the local part of the quadratic term (vorticity depends non-locally on S), and to study the ODE $\partial_t M + M^2 = 0$. As long as the initial condition $M(0)$ is an invertible matrix, this has the solution $(M(t))^{-1} = (M(0))^{-1} + tI_3$. This equation will blow up in finite time assuming that $M(0)$ has at least one negative eigenvalue. Blowup is unstable in general, because any small perturbation into the complex plane will mean there will not be blowup. However, if we restrict to symmetric matrices, then blowup is stable, because then the eigenvalues must be real valued, so a small perturbation will remain on the negative real axis. The negative real axis is an open set of $\mathbb{R}$, but not of $\mathbb{C}$, so blowup is stable only when we are restricted to matrices with real eigenvalues, which is the case we are concerned with as the strain tensor is symmetric. This equation does not preserve the family of trace free matrices however, because $tr(M^2) = |M|^2 \neq 0$, and therefore doesn't really capture any of the features of the strain equation (1.5). We will instead take our toy model ODE on the space of symmetric, trace free matrices to be

$$\partial_t M + M^2 - \frac{1}{3}|M|^2 I_3 = 0. \tag{8.1}$$

Because every symmetric matrix is diagonalizable over $\mathbb{R}$, and every diagonalizable matrix is mutually diagonalizable with the identity matrix, this equation can be treated as a system of ODEs for the evolution of the eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$ with for every $1 \leq i \leq 3$,

$$\partial_t \lambda_i = -\lambda_i^2 + \frac{1}{3}(\lambda_1^2 + \lambda_2^2 + \lambda_3^3). \tag{8.2}$$

This equation has two families of solutions with a type of scaling invariance. Let $S(0) = Cdiag(-2,1,1)$, with $C > 0$ then $S(t) = f(t)diag(-2,1,1)$, where $f_t = f^2, f(0) = C$. Therefore we have blowup in finite time, with $S(t) = \frac{1}{\frac{1}{C}-t}diag(-2,1,1)$. The reverse case, one positive eigenvalue and two equal, negative eigenvalues, also preserves scaling, but decays to zero as $t \to \infty$. Let $S(0) = Cdiag(-1,-1,2)$, with $C > 0$. Then $S(t) = \frac{1}{\frac{1}{C}+t}diag(-1,-1,2)$.

We will show that the blow up solution is stable, while the decay solution is unstable. Furthermore the blow up solution is asymptotically a global attractor except for the unstable family of solutions that decay to zero (i.e two equal negative eigenvalues and the zero solution). To prove this we will begin by rewriting our system. First of all, we will assume without loss of generality, that $S \neq 0$, because clearly if $S(0) = 0$, then $S(t) = 0$, is the solution. If $S \neq 0$, then clearly $\lambda_1 < 0$ and $\lambda_3 > 0$. Our system of equations really only has two degrees of freedom, because of the condition $tr(S) = \lambda_1 + \lambda_2 + \lambda_3 = 0$, but because we are interested in the ratios of the eigenvalues asymptotically, we will reduce the system to the two parameters λ_3 and $r = -\frac{\lambda_1}{\lambda_3}$. These two parameters completely determine our system because $\lambda_1 = -r\lambda_3$ and $\lambda_2 = -\lambda_1 - \lambda_3 = (r-1)\lambda_3$. We now will rewrite our system of ODEs:

$$\partial_t \lambda_3 = \frac{1}{3}(\lambda_1^2 + \lambda_2^2 - 2\lambda_3^2) = \frac{1}{3}\lambda_3^2\left(r^2 + (r-1)^2 + 2\right), \tag{8.3}$$

$$\partial_t \lambda_3 = \frac{1}{3}\lambda_3^2\left(2r^2 - 2r - 1\right). \tag{8.4}$$

$$\partial_t r = \frac{\lambda_1 \partial_t \lambda_3 - \lambda_3 \partial_t \lambda_1}{\lambda_3^2} = \lambda_3\left(-r(-\frac{1}{3} - \frac{2}{3}r + \frac{2}{3}r^2) + (-\frac{2}{3} + \frac{2}{3}r + \frac{1}{3}r^2)\right), \tag{8.5}$$

$$\partial_t r = \frac{1}{3}\lambda_3(-2r^3 + 3r^2 + 3r - 2). \tag{8.6}$$

At this point it will be useful to remark on the range of values our two variables can take. Clearly the largest eigenvalue $\lambda_3 \geq 0$, and $\lambda_3 = 0$ if and only if $\lambda_1, \lambda_2, \lambda_3 = 0$. Now we turn to the range of values for r. Recall that $\lambda_2 = (r-1)\lambda_3$, and that $\lambda_1 \leq \lambda_2 \leq \lambda_3$. Therefore $-r \leq r - 1 \leq 1$, so $\frac{1}{2} \leq r \leq 2$. If we take $f(r) = -2r^3 + 3r^2 + 3r - 2$, we find that $f(r)$ is positive for $\frac{1}{2} < r < 2$ with $f(\frac{1}{2}), f(2) = 0$. This is the basis for the blowup solution being the asymptotic attractor. We are now ready to state our theorem on the existence and algebraic structure of finite time blow up solutions.

Theorem 8.1 (Toy model dynamics). *Suppose $\lambda_3(0) > 0$ and $r(0) > \frac{1}{2}$, then there exists $T > 0$ such that $\lim_{t \to T} \lambda_3(t) = +\infty$, and furthermore $\lim_{t \to T} r(t) = 2$*

Proof. We'll start by showing that finite time blow up exists, and then we will show that r goes to 2 as we approach the blow up time. First we observe that $g(r) = 2r^2 - 2r - 1$, has a zero at $\frac{1+\sqrt{3}}{2}$. $g(r) < 0$, for $\frac{1}{2} \leq r < \frac{1+\sqrt{3}}{2}$, and g is both positive and increasing on $\frac{1+\sqrt{3}}{2} < r \leq 2$. We will begin with the case where $r(0) = r_0 > \frac{1+\sqrt{3}}{2}$. Clearly $\partial_t r \geq 0$, so $r(t) > r_0$, and $g\left(r(t)\right) > g(r_0)$. Let $C = \frac{1}{3}g(r_0)$, then we find that:

$$\partial_t \lambda_3 = \frac{1}{3}g\left(r(t)\right)\lambda_3^2 \geq C\lambda_3^2. \tag{8.7}$$

From this differential inequality, we find that

$$\lambda_3(t) \geq \frac{1}{\frac{1}{\lambda_3(0)} - Ct}, \tag{8.8}$$

so clearly there exists a time $T \leq \frac{1}{C\lambda_3(0)}$, such that $\lim_{t\to T} \lambda_3(t) = +\infty$.

Now we consider the case where $\frac{1}{2} < r_0 \leq \frac{1+\sqrt{3}}{2}$. It suffices to show that there exists a $T_a > 0$ such that $r(T_a) > \frac{1+\sqrt{3}}{2}$, then the proof above applies. Note that g is increasing on the interval $\left[-\frac{1}{2}, 2\right]$, so $g\left(r(t)\right) > g(r_0)$. Let $B = -\frac{1}{3}g(r_0) > 0$, and let $C = \frac{1}{3}\min\left(f(r_0), f(\frac{1+\sqrt{3}}{2})\right)$. Suppose towards contradiction that for all $t > 0$, $r(t) \leq \frac{1+\sqrt{3}}{2}$. Then we will have the differential inequalities,

$$\partial_t r \geq C\lambda_3, \tag{8.9}$$

$$\partial \lambda_3 \geq -B\lambda_3^2. \tag{8.10}$$

From (8.10) it follows that

$$\lambda_3(t) \geq \frac{1}{\frac{1}{\lambda_3(0)} + Bt}. \tag{8.11}$$

Plugging (8.11) into (8.9), we find that

$$r(t) \geq r_0 + C\int_0^t \frac{1}{\frac{1}{\lambda_3(0)} + B\tau} d\tau = r_0 + \frac{C}{B}\log\left(1 + B\lambda_3(0)t\right). \tag{8.12}$$

However, this estimate (8.12) clearly contradicts our hypothesis that $r(t) \leq \frac{1+\sqrt{3}}{2}$ for all $t > 0$. Therefore, we can conclude that there exists $T_a > 0$, such that $r(T_a) > \frac{1+\sqrt{3}}{2}$, and then we have reduced the problem to the case that we have already proven.

Now we will show that $\lim_{t\to T} r(t) = 2$. Suppose toward contradiction that $\lim_{t\to T} r(t) = r_1 < 2$. First take $a(t) = \frac{1}{3}f\left(r(t)\right)$. Observe that $a(t) > 0$ for $0 \leq t \leq T$. Our differential equation is now given by $\partial_t \lambda_3 = a(t)\lambda_3^2$, which must satisfy

$$\frac{1}{\lambda_3(t_1)} - \frac{1}{\lambda_3(t_2)} = \int_{t_1}^{t_2} a(\tau) d\tau. \tag{8.13}$$

If we take $t_2 = T$, the blow up time, then (8.13) reduces to

$$\frac{1}{\lambda_3(t)} = \int_t^T a(\tau) d\tau. \tag{8.14}$$

Let $A(t) = \int_t^T a(\tau)d\tau$. Clearly $A(T) = 0, A'(T) = -a(T) < 0$. By the fundamental theorem of calculus, for all $m > a(T)$, there exists $\delta > 0$, such that for all $t, T - \delta < t < T$,

$$A(t) \leq -m(t - T) = m(T - t). \tag{8.15}$$

Using the definition of A and plugging in to (8.14) we find that for all $T = \delta < t < T$,

$$\lambda_3(t) \geq \frac{1}{m(T - t)}. \tag{8.16}$$

Let $B = \frac{1}{3} min\left(f(r_0), f(r_1)\right)$. It then follows from our hypothesis that

$$\partial_t r \geq B\lambda_3. \tag{8.17}$$

Therefore we can apply the estimate (8.16) to the differential inequality (8.17) to find that for all $T - \delta < t < T$,

$$r(t) \geq r(T-\delta) + B\int_{T-\delta}^{t} \frac{1}{m(T-\tau)} d\tau = r(T-\delta) + \frac{B}{M}\log\left(\frac{\delta}{T-t}\right). \tag{8.18}$$

However, it is clear from (8.18) that $\lim_{t\to T} r(t) = +\infty$, contradicting our hypothesis that $\lim_{t\to T} r(t) < 2$, so we can conclude that $\lim_{t\to T} r(t) = 2$. $\square$

This toy model ODE shows that the local part of the quadratic nonlinearity tends to drive the intermediate eigenvalue λ_2 upward to λ_3, unless $\lambda_1 = \lambda_2$. Given the nature of the regularity criterion on λ_2^+, the dynamics of the eigenvalues of the strain matrix are extremely important. The fact that the toy model ODE blows up from all initial conditions where $\lambda_1 < \lambda_2$, and that $\lambda_2 = \lambda_3$ is a global attractor on all initial conditions where $\lambda_1 < \lambda_2$, provides a mechanism for blowup, but of course the very complicated nonlocal effects make it impossible to say anything definitive about blowup for the full Navier-Stokes strain equation without a much more detailed analysis.

Chapter 9

The strain equation in two dimensions

We will conclude this thesis with a brief analysis of the Navier-Stokes strain equation in two spatial dimensions. It is natural, given the difficulties that exist in three dimensions, to want to look at the simpler two dimensional case. However, none of the interesting features of the three dimensional case will turn up in two dimensions, there simply are not enough degrees of freedom for the eigenvalues of the strain matrix. We will be able to prove a statement about the change in enstrophy for two dimensions, however this will not be a new result, as the vorticity equation is already well understood in two dimensions.

First we will define the scalar vorticity as $w = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$. The evolution equation for the vorticity is given by

$$\partial_t \omega + (u \cdot \nabla)\omega - \nu \Delta \omega = 0. \tag{9.1}$$

Note in particular that there is no vortex stretching in two dimensions, there is only the advection term, and the dissipation term. This means that enstrophy will be non-decreasing, in particular that for a smooth solution

$$\partial_t \frac{1}{2} \|\omega(\cdot, t)\|_{L^2}^2 = -\nu \|\omega\|_{\dot{H}^1}^2. \tag{9.2}$$

The equation for the strain will have more terms than just advection and dissipation, but nonetheless this identity for enstrophy growth can be proven using the strain equation as well, which we will state now.

Proposition 9.1 (Strain equation in two dimensions)**.** *The Navier-Stokes strain equation can be written as an evolution equation on L^2_{st} in two dimensions as*

$$\partial_t S + (u \cdot \nabla)S - \nu \Delta S + \left(\frac{1}{2}|S|^2 - \frac{1}{4}w^2 \right) I_2 + \text{Hess}(p) = 0 \tag{9.3}$$

Proof. We will begin by recalling that the general form of the Navier-Stokes vorticity equation in n dimensions is

$$\partial_t S + (u \cdot \nabla)S - \nu \Delta S + S^2 + A^2 + \text{Hess}(p) = 0. \tag{9.4}$$

Note that in two dimensions the entries A are defined by the scalar vorticity with

$$A = \frac{1}{2}\begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix}, \tag{9.5}$$

so clearly we have

$$A = -\frac{1}{4}w^2 I_2 \tag{9.6}$$

Next observe that because S is symmetric and real valued and trace free it will have the form

$$S = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \tag{9.7}$$

for some $a, b \in \mathbb{R}$. This means that

$$S^2 = \begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix} = \frac{1}{2}|S|^2 I_2. \tag{9.8}$$

This completes the proof. □

We will note here that while the Navier-Stokes strain equation has more terms than the vorticity equation in two dimensions, beyond just dissipation and advection, $\left(\frac{1}{2}|S|^2 + \frac{1}{4}\omega^2\right) I_2 + \text{Hess}(p) \in \left(L^2_{st}\right)^\perp$, so these additional terms are only projecting back into the constraint space, and cannot drive blowup in L^2 as we will now see. This contrasts with the vorticity, which is a scalar in two dimensions, and so there is no constraint space–the vorticity is a generic scalar function.

Theorem 9.2 (Enstrophy in two dimensions)**.** *For all $S^0 \in L^2$ satisfying the consistency condition there exists a global smooth solution to the Navier-Stokes strain equation with for all $t > 0$*

$$\|S(\cdot,t)\|_{L^2}^2 + 2\nu \int_0^t \|S(\cdot,\tau)\|_{\dot{H}^1}^2 d\tau = \|S^0\|_{L^2}^2. \tag{9.9}$$

Proof. Just as in the three dimensional case, here we have $\|S^0\|_{L^2}^2 = \frac{1}{2}\|w^0\|_{L^2}^2$. It is well known that for two dimensional Navier-Stokes, enstrophy is a monotone quantity and therefore that initial vorticity in L^2 is sufficent to guaruntee global smooth solutions. For the second piece we can observe that integrating by parts $\langle (u \cdot S)S, S\rangle = 0$. We also know that

$$\left\langle \left(\frac{1}{2}|S|^2 - \frac{1}{4}w^2\right) I_2, S \right\rangle = \int_{\mathbb{R}^2} \left(\frac{1}{2}|S|^2 - \frac{1}{4}w^2\right) tr(S) = 0. \tag{9.10}$$

Finally we observe that as in the three dimensional case

$$\langle \text{Hess}(p), S\rangle = 0. \tag{9.11}$$

From this we can conclude that

$$\partial_t \|S(\cdot,t)\|_{L^2}^2 = -2\|S(\cdot,t)\|_{\dot{H}^1}^2 \tag{9.12}$$

Integrating this differential equality, this completes the proof. □

This does not provide any new identity, though; this is simply equivalent to what is already known about enstrophy for two dimensional Navier-Stokes using the scalar vorticity equation. We cannot get any insight into the three dimensional Navier-Stokes strain equation by looking at the two dimensional case, because the trace free condition in two dimensions means that the eigenvalues of the strain matrix have only one degree of freedom, so none of the difficult aspects from the three dimensional equation can play a role in two dimensions. Studying the Navier-Stokes strain equation in two dimensions, therefore, will unfortunately not be of any use in understanding the three dimensional case.

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