Overdetermined systems of equations, Newton Polyhedra, and Resultants

by

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Abstract

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In the first part of this thesis we develop Newton polyhedra theory for overdetermined systems of equations. Let $A_1, \ldots, A_k$ be finite sets in $\mathbb{Z}^n$ and let $Y \subset (\mathbb{C}^*)^n$ be an algebraic variety defined by a system of equations

$$f_1 = \ldots = f_k = 0,$$

where $f_1, \ldots, f_k$ are Laurent polynomials with supports in $A_1, \ldots, A_k$. Assuming that $f_1, \ldots, f_k$ are sufficiently generic, the Newton polyhedron theory computes discrete invariants of Y in terms of the Newton polyhedra of $f_1, \ldots, f_k$. It may appear that the generic system with fixed supports $A_1, \ldots, A_k$ is inconsistent. In this case one is interested in the **generic consistent** system. We extend Newton polyhedra theory to this case and compute discrete invariants generic non-empty zero sets. Unlike the classical situation, not only the Newton polyhedra of $f_1, \ldots, f_k$, but also the supports $A_1, \ldots, A_k$ themselves appear in the answers.

We proceed then to the study of overdetermined collections of linear series on algebraic varieties other then $(\mathbb{C}^*)^n$. That is let $E_i \subset H^0(X, \mathcal{L}_i)$ be a finite dimensional subspace of the space of regular sections of line bundles $\mathcal{L}_i$, so that the generic system

$$s_1 = \ldots = s_k = 0,$$

with $s_i \in E_i$ does not have any roots on X. In this case we investigate the consistency variety $R \subset \prod_{i=1}^k E_i$ (the closure of the set of all systems which have at least one common root) and study general properties of zero sets $Z_{\mathbf{s}}$ of a generic consistent system $\mathbf{s} \in R$. Then, in the case of equivariant linear series on spherical homogeneous spaces we provide a strategy for computing discrete invariants of such generic non-empty set $Z_{\mathbf{s}}$.

The second part of this thesis is devoted to the study of Δ-resultants of $(n+1)$-tuple of Laurent polynomials with generic enough Newton polyhedra. Let $Res_\Delta(f_1, \ldots, f_{n+1})$ be the Δ-*resultant* of

$(n+1)$-tuple of Laurent polynomials. We provide an algorithm for computing Res_Δ assuming that an n-tuple $(f_2, \ldots, f_{n+1})$ is *developed*. We provide a relation between the product of f_1 over roots of $f_2 = \cdots = f_{n+1} = 0$ in $(\mathbb{C}^*)^n$ and the product of f_2 over roots of $f_1 = f_3 = \cdots = f_{n+1} = 0$ in $(\mathbb{C}^*)^n$ assuming that the n-tuple $(f_1 f_2, f_3, \ldots, f_{n+1})$ is developed. If all n-tuples contained in $(f_1, \ldots, f_{n+1})$ are developed we provide a signed version of Poisson formula for Res_Δ. Interestingly, the sign of the sparse resultant is nontrivial and is defined through Parshin symbols. Our proofs are based on a topological version of the Parshin reciprocity laws.

This thesis is based on works [29], [30], and [24].

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Chapter 1

Introduction

This thesis is dedicated to the study of *overdetermined* systems of equations. The study of overdetermined systems of equations is a particular space of studying a non-generic behaviour in the space of equations and is mainly motivated by the questions about typical behaviour in the families of complete intersections. In the study of overdetermined collections one is interested in **generic solvable** systems. There are two main directions of this study. The first direction is to understand the set of solvable systems. The second direction is to extend Newton polyhedra theory to be able to compute discrete invariants of the zero set of a **generic solvable** system. In this thesis we investigate both of these directions.

1.1 Newton polyhedra theory for overdetermined systems of equations

Newton polyhedra theory connects algebraic geometry to the geometry of convex polyhedra with integral vertices in the framework of toric geometry. In particular, it studies generic complete intersections in an algebraic torus $(\mathbb{C}^*)^n$.

Let f be a Laurent polynomial in n variables. The *support* $\mathrm{supp}(f)$ of f is the set of exponents of the non-zero monomials of f. The *Newton polyhedra* $\Delta(f) \subset \mathbb{R}^n$ is the convex hull of the support of f. Fix finite sets $A_1, \ldots, A_k \subset \mathbb{Z}^n$. Let $f_1, \ldots, f_k$ be Laurent polynomials with supports in $A_1, \ldots, A_k$. Newton polyhedra theory computes discrete invariants of an algebraic variety $Y \subset (\mathbb{C}^*)^n$ defined by a generic system of equations

$$f_1 = \cdots = f_k = 0, \ \mathrm{supp}(f_i) \subset A_i. \tag{1.1}$$

More precisely, in the space of such systems there exists a Zariski open subset on which a discrete invariant of interest is constant and can be computed in terms of the combinatorics of the sets $A_1, \ldots, A_k$.

The first example of such an invariant and the starting point for Newton polyhedra theory is the Bernstein-Khovanskii-Kushnirenko (BKK) theorem ([1]). It expresses the number of solutions of a generic polynomial system of equations in terms of the volumes of their Newton polyhedra.

Theorem 1.1 (BKK)**.** *Let $f_1, \ldots, f_n$ be generic Laurent polynomials with supports in $A_1, \ldots, A_n$. Then all solutions of the system $f_1 = \ldots = f_n = 0$ in $(\mathbb{C}^*)^n$ are non-degenerate and the number of them is equal to*

$$n!Vol(\Delta_1, \ldots, \Delta_n),$$

where Δ_i is the convex hull of A_i and Vol is the mixed volume.

Newton polyhedra theory has developed a great deal since the BKK theorem. One can compute such invariants of zero sets of generic systems as algebraic genus, Euler characteristic, number of connected components, and, when certain conditions on Newton polyhedra are satisfied, mixed Hodge numbers (see Apendix A for details).

Let $A_1, \ldots, A_k$ be a collection of finite sets for which the generic system (2.1) does not have any solutions. We will call such collections *overdetermined*. Assume now, that Y is given as a zero set of system (2.1) which is generic among *solvable* systems with supports in $A_1, \ldots, A_k$. Can we say anything about topology of Y? Classical results of Newton polyhedra theory give trivial answer to this question (the generic zero set is empty). In Chapter 2 we show that all results of Newton polyhedra theory can be extended to this case.

Theorem 1.2. *Let $A_1, \ldots, A_k$ be an overdetermined collection, then for the generic solvable system (1.1) with zero set Y, one can compute all discrete invariants of Y listed above from the combinatorics of $A_1, \ldots, A_k$ alone.*

Theorem 1.2 is a direct corollary of Theorem 2.7 and for explicit examples of calculation of discrete invariants see Apendix A. One difference from the classical situation is that discrete invariants depend on not only the Newton polyhedra of $f_1, \ldots, f_k$ but the supports $A_1, \ldots, A_k$ themselves.

The main step in the proof of Theorem 1.2 is to express a generic non-empty zero set Y as a generic zero set of a system with some other supports. It turns out that one can translate this reduction result to a much more general setting, which is the subject of Chapter 3.

Let $\mathcal{E} = (E_1, \ldots, E_k)$ be a collection of base points free linear series on any irreducible algebraic variety X over $\mathbb{C}$. Such a collection is called overdetermined if the generic system $s_1 = \ldots = s_k = 0$ with $s_i \in E_i$ does not have any roots in X.

Generalizing the notion of resultant hypersurface we define *consistency variety* $R_{\mathcal{E}} \subset \prod_{i=1}^{k} E_i$ to be the closure of the set of all systems which have at least one common root. We show that for any collection $\mathcal{E}$ there exists a unique subcollection $\mathcal{E}_J$ which is responsible for all inconsistency of the collection $\mathcal{E}$.

Theorem 1.3. *Let $\mathcal{E}$ be an overdetermined collection of linear series. Then there exists a unique minimal by inclusion subcollection $\mathcal{E}_J = (E_i)_{i\in J}$ such that the generic system $f_1 = \ldots = f_k$ is solvable if and only if the subsystem given by J is solvable. In particular, $R_{\mathcal{E}} = p^{-1}(R_{\mathcal{E}_J})$, where p is the natural projection:*

$$p : \prod_{i=1}^{k} E_i \to \prod_{i\in J} E_i.$$

I use Theorem 1.3 to reduce the study of **generic solvable** systems given by the collection $\mathcal{E}$ to the study of **generic** systems given by $\mathcal{E}_{J^c} = (E_i)_{i\notin J}$. This reduction allows us to study zero sets $Z_{\mathbf{s}}$ of generic consistent system $\mathbf{s} \in R_{\mathcal{E}}$. In particular, we prove

Theorem 1.4. *If X is smooth, then generic non-empty zero sets $Z_{\mathbf{s}}$ are also smooth.*

1.2 Resultants of developed systems of Laurent polynomials

Chapters 4 and 5 are dedicated to the study of Δ-*resultant* of $(n+1)$-tuple of Laurent polynomials, whose Newton polyhedra are generic enough. Chapter 4 is mostly expository. We present there number of classical identities involving Sylvester results and a product of values of one polynomial over the roots of other in dimension 1. Our presentation of material in Chapter 4 is parallel to the one in Chapter 5 where we generalize these identities to the multidimensional case, under the assumption that Newton polyhedra of the Laurent polynomials are in general position.

A system of n equations $f_1 = \cdots = f_n = 0$ in $(\mathbb{C}^*)^n$ is called *developed* if their Newton polyhedra $\Delta(f_i)$ are located generically enough with respect to each other. The exact definition (see more detailed discussion in Section 5.2) is as follows: a collection of n polyhedra $\Delta_1, \ldots, \Delta_n \subset \mathbb{R}^n$ is called *developed* if for any covector $v \in (\mathbb{R}^n)^*$ there is i such that on the polyhedron Δ_i the inner product with v attains its biggest value precisely at a vertex of Δ_i.

A developed system resembles an equation in one unknown. A polynomial in one variable of degree d has exactly d roots counting with multiplicity. The number of roots in $(\mathbb{C}^*)^n$ counting with multiplicities of a developed system is always determined by the Bernstein-Koushnirenko formula (if the system is not developed this formula holds only for generic systems with fixed Newton polyhedra).

As in the one-dimensional case, one can explicitly compute the sum of values of any Laurent polynomial over the roots of a developed system [11],[12] and the product of all of the roots of the system regarded as elements in the group $(\mathbb{C}^*)^n$ [18]. These results can be proved topologically [12], using the topological identity between certain homology cycles related to developed system (see Section 5.3), the Cauchy residues theorem, and a topological version of the Parshin reciprocity laws (see Section 5.4).

To an $(n+1)$-tuple $A = (\mathcal{A}_1, \ldots, \mathcal{A}_{n+1})$ of finite subsets in $\mathbb{Z}^n$ one associates the A-*resultant* R_A. It is a polynomial defined up to sign in the coefficients of Laurent polynomials $f_1, \ldots, f_{n+1}$ whose supports belong to $A_1, \ldots, A_{n+1}$ respectevely. The A-resultant is equal to ± 1 if the codimension of the variety of consistent systems in the space of all systems with supports in A is greater than 1. Otherwise, R_A is a polynomial which vanishes on the variety of consistent systems and such that the degree of R_A in the coefficients of the i-th polynomial is equal to the generic number of roots of the system $f_1 = \ldots = \hat{f}_i = \ldots = f_{n+1} = 0$ (in which the equation $f_i = 0$ is removed).

The notion of A-resultant was introduced and studied in [10] under the following assumption on A: the lattice generated by the differences $a - b$ for all couples $a, b \in \mathcal{A}_i$ and all $0 \leq i \leq n+1$ is $\mathbb{Z}^n$. Under this assumption the resultant R_A is an irreducible polynomial (which was used in a definition of R_A in [10]). Later in [8] and [4] it was shown that in the general case (i.e when the differences from $\mathcal{A}_i$'s do not generate the whole lattice) R_A is some power of an irreducible polynomial. The power is equal to the generic number of roots of a corresponding consistent system (see Section **??** for more details).

To an $(n+1)$-tuple $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$ of Newton polyhedra one associates the $(n+1)$-tuple A_Δ of finite subsets $(\Delta_1 \cap \mathbb{Z}^n, \ldots, \Delta_{n+1} \cap \mathbb{Z}^n)$ in $\mathbb{Z}^n$. We define the Δ-*resultant* as A-resultant for $A = A_\Delta$. We will deal with Δ-resultants only. If a property of Δ-resultant is a known property of A-resultants for $A = A_\Delta$ we refer to a paper where the property of A-resultants is proven (without mentioning that the paper deals with A-resultants and not with Δ-resultants). Dealing with Δ-resultants only we lose nothing: A-resultants can be reduced to Δ-resultants. One can check that $R_A(f_1, \ldots, f_{n+1})$ for $A = (A_1, \ldots, A_{n+1})$ is equal to $Res_\Delta(f_1, \ldots, f_{n+1})$ for $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$, where $\Delta_1, \ldots, \Delta_{n+1}$ are the convex hulls of the sets $A_1, \ldots, A_{n+1}$.

A collection Δ is called i-*developed* if its subcollection obtained by removing the polyhedron Δ_i is developed. Using the Poisson formula (see [33], [4] and Section 5.9.3) one can show that for i-developed Δ the identity

$$Res_\Delta = \pm \Pi_\Delta^{[i]} M_i \tag{1.2}$$

holds, where $\Pi_\Delta^{[i]}$ is the product of f_i over the common zeros in $(\mathbb{C}^*)^n$ of f_j, for $j \neq i$, and M_i is an explicit monomial in the vertex coefficients (i.e. the coefficient of f_j in front of a monomial corresponding to a vertex of Δ_j) of all the Laurent polynomials f_j with $j \neq i$.

We provide an explicit algorithm for computing the term $\Pi_\Delta^{[i]}$ using the summation formula over the roots of a developed system (Corollary 5.6). Hence we get *an explicit algorithm for computing the resultant* Res_Δ for an i-developed collection Δ. This algorithm heavily uses the Poisson formula (1).

If $(n+1)$-tuple Δ is i-developed and j-developed for some $i \neq j$ the identity

$$\Pi_\Delta^{[i]} = \Pi_\Delta^{[j]} M_{i,j} s_{i,j} \tag{1.3}$$

holds, where $M_{i,j}$ is an explicit monomial in the coefficients of Laurent polynomials $f_1, \ldots, f_{n+1}$ and $s_{i,j} = (-1)^{f_{i,j}}$ is an explicitly defined sign (Corollary 5.2).

Our proof of the identity (1.3) is topological. We use the topological identity between cycles related to a developed system (see Section 5.3) and a topological version of the Parshin reciprocity laws. The identity (1.3) generalizes the formula from [18] for the product in $(\mathbb{C}^*)^n$ of all roots of a developed system of equations.

An $(n+1)$-tuple Δ is called *completely developed* if it is i-developed for every $1 \leq we \leq n+1$. For completely developed Δ the identity

$$\Pi_\Delta^{[1]} M_1 s_1 = \ldots = \Pi_\Delta^{[n+1]} M_{n+1} s_{n+1} \tag{1.4}$$

holds, where $(M_1, \ldots, M_{n+1})$ and $\left(\Pi_\Delta^{[1]}, \ldots, \Pi_\Delta^{[n+1]}\right)$ are monomials and products appearing in (5.1) and $(s_1, \ldots, s_{n+1})$ is an $(n+1)$-tuple of signs such that $s_i s_j = s_{i,j}$ where $s_{i,j}$ are the explicit signs from identity (1.3).

Our proof of the identities (1.4) uses the identity (1.3) and does not rely on the theory of resultants. Using one general fact from this theory (Theorem 5.12) one can see that the quantities in the identities (5.3) are equal to the Δ-resultant, i.e. are equal to $\pm Res_\Delta$. Thus the identities (1.4) can be considered as a signed version of the Poisson formula for completely developed systems.

1.3 Apendices

In Appendix A we summarize results of Newton polyhedra theory and provide their analogs for the overdetermined collections of supports. Results listed in Appendix A are precise version of Theorem 1.2.

In Appendix B we study combinatorial properties of defects (see Chapters 2 and 3 for a definition). We in particular show that a collection of vector subspaces in a vector space carries a matroid structure. This provides a new geometric notion of representability of matroids. Then we show that this new notion of representability is closely related to the classical one. In particular, they are equivalent over infinite fields.

Chapter 2

Discrete Invariants of Overdetermined Systems of Laurent Polynomials

2.1 Introduction.

With a Laurent polynomial f in n variables one can associate its support $supp(f) \subset \mathbb{Z}^n$ which is the set of exponents of monomials having non-zero coefficient in f and its Newton polyhedra $\Delta(f) \subset \mathbb{R}^n$ which is the convex hull of the support of f in $\mathbb{R}^n$. Consider an algebraic variety $Y \subset (\mathbb{C}^*)^n$ defined by a system of equations

$$f_1 = \cdots = f_k = 0, \tag{2.1}$$

where $f_1, \ldots, f_k$ are Laurent polynomials with the supports in finite sets $A_1, \ldots, A_k \subset \mathbb{Z}^n$. The Newton polyhedra theory computes invariants of Y assuming that the system (2.1) is generic enough. That is, there exists a proper algebraic subset Σ in the space Ω of k-tuples of Laurent polynomials $f_1, \ldots, f_k$ such that the corresponding discrete invariant is constant in $\Omega \setminus \Sigma$ and could be computed in terms of polyhedra $\Delta_1, \ldots, \Delta_k$. One of the first examples of such result is the Bernstein-Kushnirenko-Khovanskii theorem (see [1]).

Theorem 2.1 (BKK)**.** *Let $f_1, \ldots, f_n$ be generic Laurent polynomials with supports in $A_1, \ldots, A_n$. Then all solutions of the system $f_1 = \ldots = f_n = 0$ in $(\mathbb{C}^*)^n$ are non-degenerate and the number of them is* *equal to*

$$n!Vol(\Delta_1, \dots, \Delta_n),$$

where Δ_i is the convex hull of A_i and Vol is the mixed volume.

For other examples of results in Newton polyhedra theory see Apendix A or [5], [20], [23], [7]. If $(f_1, \dots, f_k) \in \Sigma$, the invariants of Y depend not only on $\Delta_1, \dots, \Delta_k$ and, in general, are much harder to compute.

In the case that $A_1, \dots, A_k$ are such that the general system is inconsistent in $(\mathbb{C}^*)^n$ one can modify the question in the following way. What are discrete invariants of a zero set of generic consistent system with given supports? The main result of this chapter is Theorem 2.7 which reduces this question to the Newton polyhedra theory. In this situation, the discrete invariants are computed in terms of supports themselves, not the Newton polyhedra. Explicit examples of applications of Theorem 2.7 are given in Apendix A (in particular we obtain a generalization of the BKK Theorem).

2.2 Preliminary facts on the set of consistency.

The material of this section is well-known (see for example [10], [36], [4]).

2.2.1 Definition of the incidence variety and the set of consistency.

Let $A = (A_1, \dots, A_k)$ be a collection of k finite subsets of the lattice $\mathbb{Z}^n$. The space Ω_A of Laurent polynomials $f_1, \dots f_k$ with supports in $A_1, \dots, A_k$ is isomorphic to $(\mathbb{C})^{|A_1|+\dots+|A_k|}$, where $|A_i|$ is the number of points in A_i.

Definition 1. The *incidence variety* $\widetilde{X}_A \subset (\mathbb{C}^*)^n \times \Omega_A$ is defined as:

$$\widetilde{X}_A = \{(p, (f_1, \dots, f_k)) \in (\mathbb{C}^*)^n \times \Omega_A | f_1(p) = \dots = f_k(p) = 0\}.$$

Let $\pi_1 : (\mathbb{C}^*)^n \times \Omega_A \to (\mathbb{C}^*)^n$, $\pi_2 : (\mathbb{C}^*)^n \times \Omega_A \to \Omega_A$ be natural projections to the first and the second factors of the product.

Definition 2. The *set of consistency* $X_A \subset \Omega_A$ is the image of $\widetilde{X}_A$ under the projection π_2.

Theorem 2.2. *The incidence variety $\widetilde{X}_A \subset (\mathbb{C}^*)^n \times \Omega_A$ is a smooth algebraic variety.*

Proof. Indeed, the projection π_1 restricted to $\widetilde{X}_A$:

$$\pi_1 : \widetilde{X}_A \to (\mathbb{C}^*)^n$$

forms a vector bundle of rank $|A_1| + \ldots + |A_k| - k$. That is because for a point $p \in (\mathbb{C}^*)^n$ the preimage $\pi_1^{-1}(p) \subset \widetilde{X}_A$ is given by k independent linear equations on the coefficients of polynomials $f_1, \ldots, f_k$. $\square$

We will say that a constructible subset X of $\mathbb{C}^N$ is irreducible if for any two polynomials f, g such that $fg|_X = 0$ either $f|_X = 0$ or $g|_X = 0$.

Corollary 2.1. *The set of consistency X_A is an irreducible constructible subset of Ω_A.*

Proof. Since $X_A = \pi_2(\widetilde{X}_A)$ is the image of an irreducible algebraic variety $\widetilde{X}_A$ under the algebraic map π_2, it is constructible and irreducible. $\square$

2.2.2 Codimension of the set of consistency.

For a collection $B = (B_1, \ldots, B_\ell)$ of finite subsets of $\mathbb{Z}^n$ let $B = B_1 + \ldots + B_\ell$ be the Minkowski sum of all subsets in the collection and let $L(B)$ be the linear subspace parallel to the minimal affine subspace containing B.

Definition 3. The defect of a collection $B = (B_1, \ldots, B_\ell)$ of finite subsets of $\mathbb{Z}^n$ is given by

$$\mathrm{def}(B_1, \ldots, B_\ell) = \dim(L(B)) - \ell.$$

For a subset $J \subset \{1, \ldots, \ell\}$ let us define the collection $B_J = (B_i)_{i \in J}$. For the simplicity we denote the defect $\mathrm{def}(B_J)$ by $\mathrm{def}(J)$, and the linear space $L(B_J)$ by $L(J)$.

The following theorem provides a criterion for a system of Laurent polynomials with supports in $A_1, \ldots, A_k$ to be generically consistent.

Theorem 2.3 (Bernstein). *A system of generic equations $f_1 = \ldots = f_k = 0$ of Laurent polynomials with supports in $A_1, \ldots, A_k$ respectively has a common root if and only if for any $J \subset \{1, \ldots, k\}$ the defect $\mathrm{def}(J)$ is nonnegative.*

According to the Bernstein theorem, if there exist subcollection of A with negative defect, the codimension of the set of consistency is positive. We will call such collections A *overdetermined.* The following theorem of Sturmfels determines the precise codimension of X_A.

Theorem 2.4 ([36], Theorem 1.1). *Let $A_1, \ldots, A_k$ be such that the generic system with supports in $A_1, \ldots, A_k$ is inconsistent. Then the codimension of the set of consistency X_A in Ω_A is equal to the maximum of the numbers $-\mathrm{def}(J)$, where J runs over all subsets of $\{1, \ldots, k\}$.*

Definition 4. For a collection $A_1, \dots, A_k$ of finite subsets of $\mathbb{Z}^n$ we will denote by $d(A)$ the smallest defect of a subcollection of A:

$$d(A) = \min\{\mathrm{def}(J) \mid J \subset \{1, \dots, k\}\}.$$

We will say that a collection A is overdetermined if the minimal defect $d(A)$ is negative.

Definition 5. For a overdetermined collection A will call a subcollection J *essential* if $\mathrm{def}(J) = d(A)$ and $\mathrm{def}(I) > d(A)$ for any $I \subset J$. In other words, J is the minimal by inclusion subcollection with the smallest defect.

This definition is related to the definition of an essential subcollection given in [36], but is different in general. Sturmfels was interested in resultants, so his definition was adapted to the case $d(A) = -1$ in which both definitions coincide.

The essential subcollection is unique. For $d = -1$ this was shown in [36] (Corollary 1.1), in Lemma 3.3 we prove this statement for arbitrary $d < 0$. In the case $d(A) = 0$ we will call the empty subcollection to be the unique essential subcollection.

Remark 1. *In the case* $d(A) = 0$ *the subcollections* J *such that* $\mathrm{def}(J) = 0$ *and* $\mathrm{def}(I) > 0$ *for any* nonempty $I \subset J$ *are also playing important role (see [23]).*

2.3 The defect and essential subcollections.

2.3.1 Uniqueness of essential subcollection.

Let $A_1, \dots, A_k$ be finite subsets of the lattice $\mathbb{Z}^n$. As before, for any $J \subset \{1, \dots k\}$, let $L(J)$ be the vector subspace parallel to the minimal affine subspace containing the Minkowski sum $A_J = \sum A_i$ with $i \in J$.

Most of the results of this section are based on the obvious observation that the dimension of vector subspaces of $\mathbb{R}^n$ is subadditive with respect to sums. That is for two vector subspaces $V, W \subset \mathbb{R}^n$ the following holds:

$$\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W) \le \dim(V) + \dim(W).$$

The immediate corollary of the relation above is the subadditivity of defect with respect to disjoint unions. More precisely, for disjoint $I, J \subset \{1, \dots k\}$ the following is true:

$$\mathrm{def}(I \cup J) = \mathrm{def}(I) + \mathrm{def}(J) - \dim(L(I) \cap L(J)) \leq \mathrm{def}(I) + \mathrm{def}(J). \tag{2.2}$$

Lemma 2.1. *Let $K = I \cap J$, then $\mathrm{def}(I \cup J) \leq def(I) + def(J) - def(K)$.*

Proof. By the definition of the defect we have:

$$\mathrm{def}(I \cup J) = \dim(L(I \cup J)) - \#(I \cup J) = \dim(L(I \cup J)) - \#I - \#J + \#K,$$

where $\#I, \#J, \#K$ are the sizes of I, J, K respectively. But also

$$def(I) + def(J) - def(K) = \dim(L(I)) + \dim(L(J)) - \dim(L(K)) - \#I - \#J + \#K,$$

so we need to compare $\dim(L(I \cup J))$ and $\dim(L(I)) + \dim(L(J)) - \dim(L(K))$. For this notice that

$$\dim(L(I \cup J)) = \dim(L(I)) + \dim(L(J)) - \dim(L(I) \cap L(J)),$$

and since $K \subset I \cap J$, the space $L(K)$ is a subspace of $L(I) \cap L(J)$, so

$$\dim(L(I \cup J)) \leq \dim(L(I)) + \dim(L(J)) - \dim(L(K)),$$

which finishes the proof. □

Corollary 2.2. *Let J and I be two not equal minimal by inclusion subcollections with minimal defect. Then $I \cap J = \varnothing$.*

Proof. Indeed, let $I \cap J = K \neq \varnothing$. Since $K \subset J$ and $K \neq J$, the defect of K is larger than the defect of J, so $def(J) - def(K) < 0$. But by Lemma 2.1

$$\mathrm{def}(I \cup J) \leq def(I) + def(J) - def(K) < \mathrm{def}(I) = \mathrm{def}(J),$$

which contradicts $\mathrm{def}(I) = \mathrm{def}(J) = d(A)$. □

Lemma 2.2. *Let A be a collection of finite subsets of $\mathbb{Z}^n$ with $d(A) \leq 0$, then the minimal by inclusion subcollection with minimal defect exists and is unique.*

Proof. In the case $d(A) = 0$ the unique essential subcollection is the empty collection $J = \varnothing$.

For $d(A) < 0$, existence is clear. For uniqueness, assume that I and J are two different minimal by inclusion subcollections with minimal defect, then by Lemma 1 $I \cap J = \varnothing$. But for disjoint subcollections I, J by relation (2.2) we have:

$$\mathrm{def}(I \cup J) \leq \mathrm{def}(I) + \mathrm{def}(J) < \mathrm{def}(I) = \mathrm{def}(J),$$

since $\mathrm{def}(I) = \mathrm{def}(J) = d(A) < 0$. But this contradicts the minimality of I and J. □

2.3.2 Some properties of the essential subcollection.

Let $A = (A_1, \dots, A_k)$ be a collection of finite subsets of the lattice $\mathbb{Z}^n$. For the subcollection J denote by $J^c = \{1, \dots, k\} \setminus J$ the compliment subcollection and by $\pi_J : \mathbb{R}^n \to \mathbb{R}^n / L(J)$ the natural projection.

Lemma 2.3. *In the notations above let $\pi_J(J^c)$ be the collection $(\pi_J(A_i))_{i \in J^c}$. Then the following relations hold:*

1. $\mathrm{def}(A) = \mathrm{def}(J) + \mathrm{def}(\pi_J(J^c))$,

2. $d(A) \geq d(J) + d(\pi_J(J^c))$,

3. if J furthermore is the unique essential subcollection of A, then $d(\pi_J(J^c)) = 0$.

Proof. The proof of the part 1. is a direct calculation:

$$\mathrm{def}(J \cup J^c) = \dim(L(J \cup J^c)) - \#(J \cup J^c) =$$

$$\dim(L(J)) + \dim L(\pi_J(J^c)) - \#(J) - \#(J^c) = \mathrm{def}(J) + \mathrm{def}(\pi_J(J^c)).$$

For the part 2. note, that for any $B \subset J$, $C \subset J^c$ one has $L(B) \subset L(J)$ and hence the following is true:

$$\mathrm{def}(\pi_B(C)) \geq \mathrm{def}(\pi_J(C)).$$

This implies that:

$$\mathrm{def}(B \cup C) = \mathrm{def}(B) + \mathrm{def}(\pi_B(C)) \geq \mathrm{def}(B) + \mathrm{def}(\pi_J(C)) \geq d(J) + d(\pi_J(J^c)).$$

For the part 3. assume that $\mathrm{def}(\pi_J(I)) < 0$ for some $I \subset J^c$. Then by part 1. we have

$$\mathrm{def}(J \cup I) = \mathrm{def}(J) + \mathrm{def}(\pi_J(I)) < \mathrm{def}(J).$$

Since the defect of empty collection is 0, the minimal defect $d(\pi_J(I))$ is also 0. $\square$

Proposition 2.1. *Let $A = (A_1, \dots, A_k)$ be a collection of finite subsets of $\mathbb{Z}^n$ such that* $\mathrm{def}(A) = d(A) < 0$. *Let J be the unique essential subcollection of the collection A. Then for any $i \in J$, the following is true:*

$$\mathrm{def}(A \setminus \{i\}) = d(A \setminus \{i\}) = d(A) + 1.$$

Proof. For a collection B and an element $b \in B$ the defect can not increase by more then 1 after removing b:

$$\mathrm{def}(B \setminus \{b\}) \leq \mathrm{def}(B) + 1,$$

where the equality holds if and only if $L(B \setminus \{b\}) = L(B)$. For the essential subcollection J and any $i \in J$, the defect $\mathrm{def}(J \setminus \{i\})$ is strictly greater then $\mathrm{def}(J)$, so it is equal to $\mathrm{def}(J) + 1 = \mathrm{def}(A) + 1$. Hence, $L(J) = L(J \setminus \{i\})$, and in particular $\pi_J = \pi_{J \setminus \{i\}}$.

Since $\mathrm{def}(A) = d(A) = \mathrm{def}(J)$, the defect $\mathrm{def}(\pi_J J^c)$ is equal to zero by part 1. of Lemma 2.3. Moreover, one has:

$$\mathrm{def}(A \setminus \{i\}) = \mathrm{def}(J \setminus \{i\}) + \mathrm{def}(\pi_{J \setminus \{i\}} J^c) = \mathrm{def}(J \setminus \{i\}) + \mathrm{def}(\pi_J J^c) = \mathrm{def}(A) + 1.$$

By part 2. and part 3. of Lemma 2.3 one has:

$$d(A \setminus \{i\}) \geq d(J \setminus \{i\}) + d(\pi_{J \setminus \{i\}} J^c) = d(J \setminus \{i\}) + d(\pi_J J^c) = d(J) + 1.$$

But since $\mathrm{def}(A \setminus \{i\}) = \mathrm{def}(A) + 1$, the minimal defect $d(A \setminus \{i\})$ is also equal to $\mathrm{def}(A) + 1$. $\square$

Corollary 2.3. *Let $A = (A_1, \dots, A_{n+d})$ be a collection of finite subsets of $\mathbb{Z}^n$ such that A is an essential collection of defect $-d$, i.e.*

$$-d = d(A) = \mathrm{def}(A) < \mathrm{def}(J),$$

for any proper $J \subset \{1, \dots, n+d\}$. Then there exists a subcollection I of size $\dim(L(A)) = n$ with $d(I) = 0$.

Proof. Apply Proposition 2.1 successively. $\square$

2.4 The main theorem.

In this section we will prove the main theorem. For a collection $A = (A_1, \dots, A_k)$ of finite subsets of $\mathbb{Z}^n$ and subcollection J let $A_J, L(J)$, and π_J be as before. For the subgroup G of $\mathbb{Z}^n$ we will denote by $\ker(G)$ the set of points $p \in (\mathbb{C}^*)^n$ such that $g(p) = 1$ for any $g \in G$. Furthermore, denote by

- $\Lambda(J) = L(J) \cap \mathbb{Z}^n$ the lattice of integral points in $L(J)$;
- $G(J)$ the group generated by all the differences of the form $(a - b)$ with $a, b \in A_i$ for any $i \in J$;
- $ind(J)$ the index of $G(J)$ in $\Lambda(J)$;
- $\ker(G)$ the set of points $p \in (\mathbb{C}^*)^n$ such that $g(p) = 1$ for any $p \in G(J)$.

2.4.1 Independence properties of systems

In this subsection we will prove independence theorems for the roots of generically consistent systems.

Lemma 2.4. *Let $A \subset \mathbb{Z}^n$ be a finite subset of size at least 2 and let $p, q \in (\mathbb{C})^*$ be such that $p/q \notin \ker(G(A))$. Then the set of Laurent polynomials f with support in A, such that $f(p) = f(q) = 0$ has codimension 2 in Ω_A.*

Proof. Vanishing of f at points p and q gives two linear conditions on the coefficients of f:

$$\sum_{k \in A} a_k p^k = 0, \quad \sum_{k \in A} a_k q^k = 0.$$

The relations above are independent unless $(p/q)^k = \lambda$ for some λ, and any $k \in A$. The later implies that $(p/q)^{k_1 - k_2} = 1$ for any $k_1, k_2 \in A$, i.e. $p/q \in \ker(G(A))$. □

Definition 6. Let T be an algebraic subgroup of $(\mathbb{C}^*)^n$ and $A = (A_1, \dots A_n)$ be a collection of finite subsets of $\mathbb{Z}^n$ such that $d(A) = 0$. We would say that A is T-independent if the generic system of Laurent polynomials $f_1, \dots, f_n$ with supports in A does not have two different roots $p, q \in (\mathbb{C}^*)^n$ with $p/q \in T$.

Corollary 2.4. *Let $A = (A_1, \dots A_n)$ be a collection of finite subsets of $\mathbb{Z}^n$ and $G \subset (\mathbb{C}^*)^n$ be a finite subgroup such that $G \cap \ker(A_1, \dots A_n) = 1$, then the collection is G-independent.*

Proof. Indeed, since $G cap \ker(G(A)) = 1$, for each $g \in G$ there exist i such that $g \notin \ker(A_i)$. So the space of systems which vanish at a pair of different points p and q with $p/q \in G$ is a finite union of codimension at least 1 subspaces, which finishes the proof. □

For an algebraic subgroup T of $(\mathbb{C}^*)^n$ let $Lie(T)$ be its Lie algebra and $L_T \subset \mathbb{R}^n$ be its annihilator in the space of characters. In other words, L_T is a linear span of the set of monomials which have value 1 on the identity component of a group T.

Theorem 2.5. *Let $A = (A_1, \dots A_n)$ be a collection of finite subsets of $\mathbb{Z}^n$ such that $d(A) = 0$. Let T be an algebraic subgroup of $(\mathbb{C}^*)^n$ such that $\ker G(A) \cap T = 1$ and for any subcollection J such that $L_T \subset L_J$ the defect of J is positive. Then the collection A is T-independent.*

Proof. Let k be the dimension of T, then $\dim L_T = n - k$. Since for any J with $L_T \subset L_J$ the defect of J is positive and $d(A) = 0$, there are at most $n - k - 1$ supports A_i such that $L_i \subset L_T$. Indeed, assume there is a subcollection J of size $n - k$ with $L_J \subset L_T$, then $\mathrm{def}(J) = 0$ and $L_J = L_T$, which contradicts the assumptions. Therefore, there are at least $k + 1$ supports A_i, say for $i = 1, \dots, k + 1$, with $\dim(T \cap \ker A_i) < k$.

Define T_1 to be the union $\bigcup_{i=1}^{k+1}(T \cap \ker A_i)$ and $T' = T \setminus T_1$ to be its compliment. By Lemma 2.4, the codimension of the set of systems $\mathbf{f}$ with supports in A having roots x and px for the fixed $x \in (\mathbb{C}^*)^n$ and $p \in T'$ is at least $n + k + 1$. Hence, the space of systems with supports in A having two different roots p, q with $p/q \in T'$ has codimension at least 1.

If the dimension of T_1 is positive, notice that $L_T \subset L_{T_1}$, and, therefore, for any J such that $L_{T_1} \subset L_J$ the defect of J is positive. Hence, we can apply the above argument to T_1, and continue inductively until we obtain T_l of dimension 0 (with $T'_{l-1} = T_{l-1} \setminus T_l$).

Since $\dim T_l = 0$, i.e. T_l is a finite subgroup of $(\mathbb{C}^*)^n$, by Corollary 2.4 the space of systems with two different roots p, q with $p/q \in T_1$ has codimension at least 1.

In this manner we obtained the decomposition of T in the finite disjoint union of subsets $\amalg_{i=0}^{l} T'_i$ (where $T'_0 = T'$ and $T'_l = T_l$) such that for any i the space of systems with a pair of different roots p, q with $p/q \in T_i$ has codimension at least 1. Therefore, the space of system with a pair of different roots p and q with $p/q \in T$ is a finite union of codimension at least 1 subspaces, and the theorem is proved. □

Corollary 2.5. *Let $\chi : (\mathbb{C}^*)^n \to \mathbb{C}^*$ be any character and $A = (A_1, \dots A_n)$ be a collection of finite subsets of $\mathbb{Z}^n$ such that $G(A) = \mathbb{Z}^n$ and $\mathrm{def}(J) > 0$ for any proper nonempty subcollection J. Then the generic system of Laurent polynomials with supports in A does not have a pair of different roots $p, q \in (\mathbb{C}^*)^n$ with $\chi(p) = \chi(q)$.*

Proof. Indeed, $\chi(p) = \chi(q)$ if and only if $p/q \in \ker(\chi)$, but the collection A is $\ker(\chi)$-independent since it satisfies the assumptions of Theorem 2.5 for any algebraic subgroup of $(\mathbb{C}^*)^n$. □

2.4.2 Zero set of the generic essential system.

In this subsection we will work with the systems of Laurent polynomials $f_1 = \ldots = f_k = 0$ with supports in $A = (A_1, \ldots, A_k)$ such that the essential subcollection is A itself. We call such systems essential.

Theorem 2.6. *Let $A = (A_1, \ldots, A_{n+d})$ be a collection of finite subsets of $\mathbb{Z}^n$ such that $ind(A) = 1$. Let also A be an essential collection, i.e.*

$$-d = d(A) = \operatorname{def}(A) < \operatorname{def}(J),$$

for any proper $J \subset \{1, \ldots, n+d\}$. Then for a generic consistent system $\mathbf{f} = (f_1, \ldots, f_k) \in X_A \subset \Omega_A$, the corresponding zero set $Y_{\mathbf{f}}$ is a single point.

Here, and everywhere in this chapter, by a generic point in algebraic variety X parametrizing systems of Laurent polynomials we mean a point in $X \setminus \Sigma$ for a fixed Zariski closed subvariety Σ of smaller dimension.

Proof. By Proposition 2.3 there exists a subcollection I of A of size n with $d(I) = 0$. Without loss of generality let us assume that $I = \{1, \ldots, n\}$. The space Ω_A of polynomials with supports in A could be thought as a product

$$\Omega_A = \Omega_I \times \Omega_{I^c},$$

where Ω_I and Ω_{I^c} are the spaces of systems of Laurent polynomials with supports in I and I^c respectively. Let $p : \Omega_A \to \Omega_I$ be the natural projection on the first factor.

By the Bernstein criterion the subsystem $f_1 = \ldots = f_n = 0$ is generically consistent. Moreover, the BKK Theorem asserts that the generic number of solutions in $(\mathbb{C}^*)^n$ is $n!Vol(\Delta_1, \ldots, \Delta_n)$, where Δ_i is the convex hull of A_i, and in particular is finite. Let us denote by $\Omega_I^{gen} \subset \Omega_I$ the Zariski open subset of systems $f_1 = \ldots = f_n = 0$ with exactly $n!Vol(\Delta_1, \ldots, \Delta_n)$ roots.

For each point $\mathbf{f}_I \in \Omega_I^{gen}$ the preimage $p^{-1}(\mathbf{f}_I)$ of the projection p restricted to the set of consistency X_A is a union of $n!Vol(\Delta_1, \ldots, \Delta_n)$ vector spaces $V_j(\mathbf{f}_I)$'s of dimension $|A_{n+1}| + \ldots + |A_{n+d}| - d$ each. The intersection of any two of these vector spaces has smaller dimension for generic $\mathbf{f}_I \in \Omega_I^{gen}$. Indeed, since $G(A) = \mathbb{Z}^n$ and A is essential, the ussumptions of Theorem 2.5 are satisfied for the collection I and subgroup $\ker(I^c)$ of $(\mathbb{C}^*)^n$. Hence I is $\ker(I^c)$-independent by Theorem 2.5.

Denote by $X'_A \subset X_A$ the set of points which belongs to exactly one of the $V_j(\mathbf{f}_I)$'s. By construction, the dimension of X'_A is equal to $|A_1| + \ldots + |A_{n+d}| - d = \dim(X_A)$. Since X_A is irreducible, the complement $\Sigma = X_A \setminus X'_A$ is an algebraic subvariety of smaller dimension. But for any $\mathbf{f} \in X'_A$ the zero set $Y_{\mathbf{f}}$ is a single point, so the theorem is proved. □

Corollary 2.6. *Let $A = (A_1, \ldots, A_k)$ be an essential collection of finite subsets of $\mathbb{Z}^n$ of defect $d(A) = \operatorname{def}(A) = -d$. Then for the generic $\mathbf{f} \in X_A \subset \Omega_A$ the zero set $Y_{\mathbf{f}}$ is a finite disjoint union of $ind(A)$ subtori of dimension $n-k+d$ which are different by a multiplication by elements of $(\mathbb{C}^*)^n$.*

Proof. The lattice $G(A)$ generated by all of the differences in A_i's defines a torus $T \simeq (\mathbb{C}^*)^{k-d}$ for which $G(A)$ is the lattice of characters. The inclusion $G(A) \hookrightarrow \mathbb{Z}^n$ defines the homomorphism:

$$p : (\mathbb{C}^*)^n \to T.$$

The kernel of the homomorphism p is the subgroup of $(\mathbb{C}^*)^n$ consisting of finite disjoint union of $ind(A)$ subtori of dimension $n-k+d$ which are different by a multiplication by elements of $(\mathbb{C}^*)^n$.

The multiplication of Laurent polynomials by monomials does not change the zero set of a system. For any i let $\widetilde{A}_i$ be any translation of A_i belonging to $G(J)$. We can think of $\widetilde{A}_i$ as support of a Laurent polynomial on T. We will denote by $\widetilde{A}$ the collection $(\widetilde{A}_1, \ldots, \widetilde{A}_k)$ understood as a collection of supports of Laurent polynomials on the torus T. The collection $\widetilde{A}$ satisfies the assumptions of Theorem 2.6.

With a system $\mathbf{f} \in \Omega_A$ one can associate a system of Laurent polynomials $\widetilde{\mathbf{f}}$ on T in a way described above. The zero set of $Y_{\mathbf{f}}$ of a system $\mathbf{f}$ is given by

$$Y_{\mathbf{f}} = p^{-1}(Y_{\widetilde{\mathbf{f}}}) \,(\text{in particular}\, Y_{\mathbf{f}} \simeq Y_{\widetilde{\mathbf{f}}} \times ker(p)),$$

where $Y_{\widetilde{\mathbf{f}}}$ is the zero set of the system $\widetilde{\mathbf{f}}$ on T. By Theorem 2.6 for the generic system $\widetilde{\mathbf{f}} \in X_{\widetilde{A}} \subset \Omega_{\widetilde{A}}$ the zero set $Y_{\widetilde{\mathbf{f}}}$ which finishes the proof. □

2.4.3 General systems

Theorem 2.7. *Let $A = (A_1, \ldots, A_k)$ be a collection of finite subsets of $\mathbb{Z}^n$ with the essential subcollection J. Then for the generic system $\mathbf{f} \in X_A \subset \Omega_A$ the zero set $Y_{\mathbf{f}}$ is a disjoint union of $ind(J)$ varieties $Y_1, \ldots, Y_{ind(J)}$ each of which is given by a Δ-nondegenerate system with the same Newton polyhedra.*

Theorem 2.7 provides a solution for the problem of computing discrete invariants of the zero set of generic consistent system with overdetermined supports by reducing it to the classical Newton polyhedra theory. The concrete examples of applications of Theorem 2.7 are given in the next section.

Proof. Without loss of generality let us assume that $J = \{1, \ldots, l\}$. By Corollary 2.6 there exists a Zariski open subset $X'_A \subset X_A$, such that for any $\mathbf{f} = (f_1, \ldots, f_k) \in X'_A$ the zero set of the system

$f_1 = \ldots = f_l = 0$ is a finite disjoint union of $ind(J)$ subtori $V_1, \ldots, V_{ind(J)}$ which are different by a multiplication by an element of $(\mathbb{C}^*)^n$.

For the generic point $\mathbf{f} = (f_1, \ldots, f_k) \in X'_A$ the restrictions of Laurent polynomials $f_{l+1}, \ldots, f_k$ to each V_i are non-degenerate Laurent polynomials with Newton polyhedra $\pi_J(\Delta_{l+1}), \ldots, \pi_J(\Delta_k)$, respectively. □

Corollary 2.7. *For the generic system* $\mathbf{f} \in X_A \subset \Omega_A$ *the zero set* $Y_{\mathbf{f}}$ *is a non-degenerate complete intersection, and in particular is smooth. That is* $Y_{\mathbf{f}}$ *is defined by* $codim(Y_{\mathbf{f}})$ *equations with independent differentials.*

Proof. Indeed, each of the components $Y_i \subset V_i$ of $Y_{\mathbf{f}}$ is defined by the restrictions of Laurent polynomials $f_{l+1}, \ldots, f_k$ to V_i, and hence is a non-degenerate complete intersection in V_i for generic consistent system $\mathbf{f}$.

But the union of shifted subtori $V_1, \ldots, V_{ind(J)}$ could be defined by the $codim(V_i)$ more independent equations in $(\mathbb{C}^*)^n$, which finishes the proof. □

2.5 Necessary and sufficient condition for T-independence

For an algebraic subgroup $T \subset (\mathbb{C}^*)^n$, Theorem 2.5 gives sufficient condition for the system of Laurent polynomials to be T-independent. In this section we will prove more precise version of Theorem 2.5, which will provide a necessary and sufficient condition for T-independence. This section is not directly related to the main topic of the chapter, but it uses some results proved earlier.

Theorem 2.8. *Let* $A = (A_1, \ldots A_n)$ *be a collection of finite subsets of* $\mathbb{Z}^n$ *such that* $d(A) = 0$*. Let* T *be an algebraic subgroup of* $(\mathbb{C}^*)^n$*. Then the collection* A *is* T*-independent if and only if:*

(i) for any subcollection J *such that* $L_T \subset L_J$ *and* $\operatorname{def}(J) = 0$*, the system* $\pi_J(J^c)$ *generically has one root;*

(ii) $\ker G(A) \cap T = 1$.

Proof. ($\rightarrow$)For necessity Let $\operatorname{def}(J) = 0$, $L_T \subset L_J$ and $Vol(\pi_J(J^c)) > 1$. Then for the generic system the solution of the subsystem J is the finite set of shifted subtori $V_1, \ldots, V_l$ such that for each i the shifted subgroup T either do not intersect V_i or contains V_i.

The restriction of the subsystem J^c to each of V_i has has supports $\pi_J(J^c)$ and generically have $Vol(\pi_J(J^c))$ roots. Therefore, for $p \in (\mathbb{C}^*)^n$ such that $V_i \subset pT$, the shifted subgroup pT generically contains at least $Vol(\pi_J(J^c)) > 1$ roots of the system.

($\leftarrow$)Let J be the minimal subcollection so that $L_T \subset L_J$ and $\mathrm{def}(J) = 0$ (if such collection does not exist we are done by Theorem 2.5). The generic solutions of subsystem J is the finitely many moved subtori V_i's of $(\mathbb{C}^*)^n$, and the restriction of the compliment system J^c to each of them has one root. Hence, to prove sufficiency it is enough to show that for the generic subsystem J, the translates of T contains at most of the V_i's.

The lattice $G(J)$ could be viewed as a character lattice of a torus W and the inclusion of the lattice $G(J)$ to $\mathbb{Z}^n$ defines a projection of $(\mathbb{C}^*)^n$ to W. Let $\widetilde{T}$ be the image of T under this projection. Notice that the system J (viewed as a system on W) and $\widetilde{T}$ satisfies the assumptions of Theorem 2.5, so the system J is $\widetilde{T}$-independent which finishes the proof. $\square$

Condition (i) is not very explicit, however Theorem 2.9 proven in [9] gives explicit description of all of the systems which have generically unique solution. For simplicity, here by the lattice volume on $\mathbb{R}^n$ we will mean a integer volume normalized such that the volume of a parallelepiped generated by a lattice basis is equal to $n!$.

Theorem 2.9. *A collection A of n integer polytopes in V has the mixed lattice volume 1 if and only if*
1) the mixed volume is not zero, and
2) there exists $k > 0$ such that, up to translations, k of the polytopes are faces of the same k-dimensional lattice volume 1 integer simplex in a k-dimensional rational subspace $U \subset V$, and the images of the other $n - k$ polytopes under the projection $V \to V/U$ have the mixed lattice volume 1.

Chapter 3

Overdetermined Systems on Spherical, and Other Algebraic Varieties

Newton polyhedra theory has generalizations to other classes of algebraic varieties such as spherical homogeneous spaces G/H with a collection of G-invariant linear systems. The first result in this direction was a generalization of the BKK Theorem and was obtained by Brion and Kazarnovskii in [2, 17]. For more results see for example [26, 27, 16]. The role of the Newton polytope in these results is played by the Newton-Okounkov polytope, which is a polytope fibered over the moment polytope with string polytopes as fibers.

Even more generally, in [15] and [28] Newton polyhedra theory was generalized to the theory of Newton-Okounkov bodies. For a linear series E on an irreducible algebraic variety X one can associate a convex body $\Delta(E)$ called the Newton-Okounkov body in such a way that the number of roots of a generic system

$$s_1 = \ldots = s_n = 0$$

with $s_i \in E_i$ can be expressed in terms of volumes of Newton-Okounkov bodies $\Delta(E_i)$.

As in classical Newton polyhedra theory all these results work for a generic system. That is, as before, in the space of systems $\mathbf{E} = E_1 \times \ldots \times E_k$ there exists a Zariski closed subset Z such that for any system $\mathbf{s} \in \mathbf{E} \backslash \mathbf{Z}$ discrete invariants of the zero set $Z_{\mathbf{s}}$ are the same and can be computed combinatorially. In particular, for overdetermined systems all the answers provided by these results are trivial.

In this chapter we generalize some of the results of Chapter 2 to the case of overdetermined linear systems on algebraic varieties other then algebraic torus. Our main goal is to extend the results mentioned above to the case of overdetermined linear series.

3.1 Introduction

Let X be an irreducible algebraic variety over $\mathbb{C}$ and let $\mathcal{E} = (E_1, \ldots, E_k)$ be a collection of base-point free linear series on X. That is, $E_i \subset H^0(X, \mathcal{L}_i)$ is a finite dimensional subspace of the space of regular sections of globally generated line bundles $\mathcal{L}_i$, such that there are no points $x \in X$ with $s(x) = 0$ for any $s \in E_i$.

A collection of linear series $\mathcal{E}$ defines systems of equations on X of the form

$$s_1 = \cdots = s_k = 0, \tag{3.1}$$

where $s_i \in E_i$. A collection $\mathcal{E}$ is called *overdetermined* if system (3.1) does not have any roots on X for the generic choice of $\mathbf{s} = (s_1, \ldots, s_k) \in \mathbf{E} = E_1 \times \ldots \times E_k$. If generic system (3.1) has a solution we will say that $\mathcal{E}$ is generically solvable. Here, and everywhere in this thesis, by saying that some property is satisfied by a generic point of an irreducible algebraic variety Y we mean that there exists a Zariski closed subset $Z \subset Y$ such that for any $y \in Y \setminus Z$ this property is satisfied.

Structure of the chapter and formulation of the results. In Section 3.2, for an overdetermined collection of linear series, we define the consistency variety $R_{\mathcal{E}} \subset \mathbf{E}$ which is the closure of the set of all solvable systems and prove that it is irreducible. Basic geometric properties of $R_{\mathcal{E}}$ are studied in Theorems 3.4 and 3.5. In Subsection 3.2.4 we consider the case when $codim(R_{\mathcal{E}}) = 1$ and define the resultant polynomial of a collection $\mathcal{E}$. The resultant of a collection of linear series is a generalization of the $\mathcal{L}$-resultant defined in [10]. We prove that all the basic properties of $\mathcal{L}$-resultant are also satisfied by the resultant of a collection of linear series.

Section 3.3 is devoted to studying the generic non-empty zero set of an overdetermined collection of linear systems $\mathcal{E}$ on an irreducible variety X. One of the main results of this section is Theorem 3.8 which expresses a generic non-empty zero set of system (3.1), defined by $\mathcal{E}$, as the generic zero set of another linear series which is generically solvable. This allows one to use classical results described in the previous subsection to find the topology of the generic non-empty zero set in number of examples.

In Section 3.4 we study G-equivariant linear series on homogeneous G-spaces. Spherical homogeneous spaces are of special interest for us. We apply Theorem 3.8 to obtain Theorem 3.12 which provides a strategy for computing discrete invariants of a generic non-empty zero set of an overdetermined linear series on a spherical homogeneous space. An example of an application of Theorem 3.12 is given in Subsection 3.4.3.

3.2 Consistency variety and resultant of a collection of linear series on a variety

In this section we define the consistency variety of a collection of linear series and describe its main properties.

3.2.1 Background

Let X be an irreducible complex algebraic variety. Let $\mathcal{L}_1, \dots \mathcal{L}_k$ be globally generated line bundles on X. For $i = 1, \dots, k$, let $E_k \subset H^0(X, \mathcal{L}_i)$ be finite-dimensional, base points free linear series. Let $\mathbf{E}$ denote the k-fold product $E_1 \times \cdots \times E_k$.

Definition 7. The *incidence variety* $\widetilde{R}_{\mathbf{E}} \subset X \times \mathbf{E}$ is defined as:

$$\widetilde{R}_{\mathbf{E}} = \{(p, (s_1, \dots, s_k)) \in X \times \mathbf{E} \mid f_1(p) = \dots = f_k(p) = 0\}.$$

Let $\pi_1 : X \times \mathbf{E} \to X$, $\pi_2 : X \times \mathbf{E} \to \mathbf{E}$ be natural projections to the first and the second factors of the product.

Definition 8. The *consistency variety* $R_{\mathbf{E}} \subset \Omega_A$ is the closure of the image of $\widetilde{R}_L$ under the projection π_2.

Theorem 3.1. *The incidence variety $\widetilde{R}_{\mathbf{E}} \subset X \times L$ and the consistency variety $R_{\mathbf{E}}$ are irreducible algebraic varieties.*

Proof. Since $E_1, \dots E_k$ are base points free, the preimage $\pi_1^{-1}(p) \subset \widetilde{R}_{\mathbf{E}}$ of any point $p \in X$ is defined by k independent linear equations on elements of $\mathbf{E}$. Therefore, the projection π_1 restricted to $\widetilde{R}_{\mathbf{E}}$:

$$\pi_1 : \widetilde{R}_{\mathbf{E}} \to X$$

forms a vector bundle of rank $\dim(L) - k$ and in particular is irreducible.

The set of concistency systems $R_{\mathbf{E}} = \pi_2(\widetilde{R}_{\mathbf{E}})$ is the image of an irreducible algebraic variety $\widetilde{R}_{\mathbf{E}}$ under the algebraic map π_2, so it is irreducible constructible set. Hence it's closure is an irreducible algebraic variety. □

For two linear systems E_i, E_j, let their product E_iE_j be a vector subspace of $H^0(X, \mathcal{L}_i \otimes \mathcal{L}_j)$ generated by all the elements of the form $f \otimes g$ with $f \in E_i, g \in E_j$. For any $J \subset \{1, \ldots, k\}$, by E_J we will denote the product $\prod_{j \in J} E_j$.

To a base points free linear system E, one can associate a morphism $\Phi_E : X \to \mathbb{P}(E^*)$ called a *Kodaira map*. It is defined as follows: for a point $x \in X$ its image $\Phi_E(x) \in \mathbb{P}(E^*)$ is the hyperplane $E_x \in E$ consisting of all the sections $g \in E$ which vanish at x. We will denote by Y_E the image of the Kodaira map Φ_E and by τ_E the dimension of Y_E. For a collection of linear series $E_1, \ldots, E_k$ and $J \subset \{1, \ldots, k\}$ we will write Φ_J, Y_J and τ_J for Φ_{E_J}, Y_{E_J} and τ_{E_J} respectively.

Definition 9. For a collection of linear series $E_1 \ldots, E_k$, the defect of a subcollection $J \subset \{1, \ldots, k\}$ is defined as

$$\operatorname{def}(E_J) = \tau_J - |J|.$$

The above definition is motivated by the fact that if $E_1 = E_{A_1}, \ldots, E_k = E_{A_k}$ are linear systems on $(\mathbb{C}^*)^n$ given by the spaces of Laurent polynomials with supports in $A_1, \ldots A_k \subset \mathbb{Z}^n$, the defect $\operatorname{def}(E_J) = \operatorname{def}(J)$ (where the later defect is as in Chapter 2). The relation of Definition 9 to the one given in Chapter 2 in more general setting is given in Section 3.2.2. The following theorem of Kaveh and Khovanskii generalizes Bernstein's criterion (Theorem 2.3) and gives a condition on a collection of linear series to be generically solvable in terms of defects.

Theorem 3.2 ([16] Theorems 2.14 and 2.19)**.** *The generic system of equations $s_1 = \ldots = s_k = 0$ with $f_i \in E_i$ is solvable if and only if $\operatorname{def}(E_J) \geq 0$ for any $J \subset \{1, \ldots, k\}$.*

In other words, Theorem 3.2 states that the codimension of the consistency variety is equal to 0 in $\mathbf{E}$ if and only if $\operatorname{def}(E_J) \geq 0$ for any $J \subset \{1, \ldots, k\}$. In Subsection 3.2.3 we will generalize this result by finding the codimension of R_E in terms of defects of subcollections.

3.2.2 The defect of vector subspaces and essential subcollections

In this subsection we will remind a definition of a combinatorial version of the defect (as in Chapter **??**) and relate it to the one given in Definition 9. Let k be any field, $\mathcal{V} = (V_1, \ldots, V_k)$ be a collection of vector subspaces of a vector space $W \cong k^n$ (in this paper we will always work with $k = \mathbb{C}$ or $\mathbb{R}$). For $J \subset \{1, \ldots, k\}$ let V_J be the Minkowski sum $\sum_{j \in J} V_j$ and $\pi_J : W \to W/V_J$ be the natural projection.

Definition 10. For a collection of vector subspaces $\mathcal{V} = (V_1, \ldots, V_k)$ of W we define

i) the defect of a subcollection $J \subset \{1, \ldots, k\}$ by $\operatorname{def}(J) = \dim(V_J) - |J|$;

ii) the minimal defect $\mathrm{d}(\mathcal{V})$ of a collection V to be the minimal defect of $J \subset \{1, \dots, k\}$.

iii) an essential subcollection to be a subcollection J so that $\mathrm{def}(J) = d(\mathcal{V})$ and $\mathrm{def}(I) > \mathrm{def}(J)$ for any proper subset I of J.

The essential subcollections has proved to be useful in studying systems of equations which are generically inconsistent and their resultants. The above definition is related to the definition of an essential subcollection given in [36]: two definitions coincide if $\mathrm{d}(\mathcal{V}) = -1$, but are different in general. The results of Section 2.3 which we are going to use later in this chapter can be collected in the following theorem.

Theorem 3.3 ([29] Section 3)**.** *Let $\mathcal{V} = (V_1, \dots, V_k)$ be a collection of vector subspaces of W with $\mathrm{d}(\mathcal{V}) \leq 0$, then:*

i) an essential subcollection exists and is unique.

ii) if J is the unique essential subcollection of $\mathcal{V}$, then $\mathrm{d}(\pi_J(J^c)) = 0$.

iii) if J is the essential subcollection there exists a subcollection $I \subset J$ of size $dim(V_J)$ with $\mathrm{d}(I) = 0$.

Parts *i*) and *ii*) of Theorem 3.3 are still true in the case $\mathrm{d}(\mathcal{V}) > 0$, the unique essential subcollection in this case is the empty subcollection. A subcollection I from part *iii*) of Theorem 3.3 is almost never unique.

To relate the combinatorial version of defect to the geometric version defined in Definition 9 we introduce a collection of distributions (and codistributions) on X related to a collection of linear systems $E_1, \dots, E_k$. Let X and $E_1, \dots, E_k$ be as before, denote further by X^{sing} the singular locus of X, by Y_J^{sing} the singular locus of $Y_J = \Phi_J(X)$.

Let also $\Sigma_J^c \subset X \setminus \Big(X^{sing} \cup \Phi_J^{-1}(Y_J^{sing})\Big)$ be the set of all critical points of Φ_J. Finally, let $B_J = X^{sing} \cup \Phi_J^{-1}(Y_J^{sing}) \cup \Sigma_J^c$ and let $U \subset X$ be a Zariski open subset defined by:

$$U = X \setminus \bigcup_J B_J.$$

So we get that U is a smooth algebraic variety and the restriction of Φ_J to U is a regular map for any $J \subset \{1, \dots, k\}$.

Definition 11. Let $a \in U$ and $\widetilde{F}_J(a)$ be the subspace of the tangent space T_aU defined by the linear equations $dg_a = 0$ for all $g \in E_J$. Let also $\widetilde{F}_J^\vee(a) \subset T_a^*U$ be the annihilator of $\widetilde{F}_J(a)$. Then

(1) $\widetilde{F}_J$ is an $(n - \tau_J)$-dimensional distribution on the Zariski open set $U \subset X$ defined by the collection of subspaces $\widetilde{F}_J(a)$.

(2) $\widetilde{F}_J^\vee$ is a τ_J-dimensional codistribution on the Zariski open set $U \subset X$ defined by the collection of

subspaces $\widetilde{F}_J^{\vee}(a)$.

The next lemma is a corollary of the Implicit Function Theorem.

Lemma 3.1. *The foliation $\widetilde{F}_J$ on U is completely integrable. Its leaves are connected components of the fibers the Kodaira map $\Phi_J : U \to Y_J$.*

Fibers of the Kodaira maps Φ_J can be described in terms of systems of equations defined by collection of lynear series $E_1, \dots, E_k$.

Lemma 3.2 ([16] Lemma 2.11)**.** *For $a, b \in X$ we have $\Phi_J(a) = \Phi_J(b)$ if and only if for every $i \in J$ the sets $\{g_i \in E_i | g_i(a) = 0\}$ and $\{g_i \in E_i | g_i(b) = 0\}$ coincide.*

Corollary 3.1. *Let $U \subset X$ be as before, then for any $a \in U$ one has:*

$$F_{E_J}(a) = \bigcap_{i \in J} F_i(a) \quad F_{E_J}^{\vee}(a) = \sum_{i \in J} F_i^{\vee}(a)$$

Proof. Indeed, by Lemma 3.1 $F_{E_J}(a)$ is a tangent space to the fiber of Φ_J at a point $a \in U \setminus \Sigma_c$. But by Lemma 3.2 the fiber of Φ_J through the point a is equal to the intersection of fibres of Φ_{E_i}, with $i \in J$ passing through the same point. □

The following Proposition 3.1 relates two definitions of the defect and is the main result of this subsection. Proposition 3.1 will allow us to apply combinatorial results of Theorem 3.3 to the geometric version of defect.

Proposition 3.1. *Let $U \subset X$ be as before, then the defect* $\operatorname{def}(E_J)$ *of linear system E_J (as in Definition 9) is equal to the defect of a collection of vector subspaces $(F_{E_i}^{\vee}(a))_{i \in J}$ of T^*U_a (as in Definition 10) for any $a \in U$.*

Proof. By construction, $F_{E_J}^{\vee}$ is τ_J-dimensional codistribution on U, so $\dim(F_{E_J}^{\vee}(a)) = \tau_J$ for any $a \in U$. Therefore, by Corollary 3.1 we have

$$\begin{aligned} Def(E_J) = \tau_J - |J| = \dim(F_{E_J}^{\vee}(a)) - |J| = \dim(F_{E_J}^{\vee}(a)) - |J| \\ = \dim\left(\sum_{i \in J} F_i^{\vee}(a)\right) - |J| = \operatorname{def}(F_{E_i}^{\vee}(a))_{i \in J}. \end{aligned} \qquad \square \quad (3.2)$$

3.2.3 Properties of the consistency variety

In this subsection we will investigate basic properties of the consistency variety. One of the main results of this section is the following theorem which computes the codimension of the consistency variety in terms of defects.

Theorem 3.4. *Let $\mathcal{E} = (E_1, \dots E_k)$ be a collection of base points free linear systems on a quasi projective irreducible variety X. Then the codimension of the consistency variety $R_{\mathbf{E}}$ is equal to $-\mathrm{d}(\mathcal{E})$ where $\mathrm{d}(\mathcal{E})$ is the minimal possible defect $\mathrm{def}(E_J)$ for $J \subset \{1, \dots, k\}$.*

We say that a collection of linear series $\mathcal{E} = (E_1, \dots, E_k)$ on X is *injective* if linear series from $\mathcal{E}$ separate points of X. In other words, $\mathcal{E}$ is injective if the product of Kodaira maps $\prod_{i=1}^{k} \Phi_{E_i}$ is injective on X. Note that by Lemma 3.2 the product of Kodaira maps $\prod_{i=1}^{k} \Phi_{E_i}$ is injective if and only if the Kodaira map Φ_E for $E = \prod_{i=1}^{k} E_i$ is such. Therefore, equivalently $\mathcal{E}$ is injective if the Kodaira map Φ_E is injective.

Any collection of linear series $\mathcal{E}$ on X could be reduced to an injective collection $\widetilde{\mathcal{E}}$ such that zero sets of $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are related in an easy way. In order to do so, let us describe the zero set $Z_{\mathbf{s}}$ of a system of equations $s_1 = \ldots = s_k = 0$ with $s_i \in E_i$ in terms of Kodaira maps Φ_{E_i}.

For the product of projective spaces $\mathbb{P}_{\mathbf{E}} = \mathbb{P}(E_1^*) \times \ldots \times \mathbb{P}(E_k^*)$ let $p_i : \mathbb{P}_{\mathbf{E}} \to \mathbb{P}(E_i^*)$ be the natural projection on the i-th factor. Each function $s_i \in E_i$ defines a hyperplane H_{s_i} on $\mathbb{P}(E_i^*)$, with slight abuse of notation let us denote its preimage under p_i by the same letter. Let $\Phi_{\mathbf{E}} : X \to \mathbb{P}_{\mathbf{E}}$ be the product of Kodaira maps and $Y_{\mathbf{E}} = \Phi_{\mathbf{E}}(X)$ be its image. In this notation the zero set $Z_{\mathbf{s}}$ is given by

$$Z_{\mathbf{s}} = \Phi_{\mathbf{E}}^{-1} \left(Y_{\mathbf{E}} \cap \bigcap_{i=1}^{k} H_{s_i} \right).$$

Therefore for any collection of linear series $\mathcal{E} = (E_1, \dots, E_k)$ on X one can associate an injective collection $\widetilde{\mathcal{E}}$ on $Y_{\mathbf{E}}$, where $\widetilde{\mathcal{E}}$ is the restriction of $E_1, \dots, E_k$ to $Y_{\mathbf{E}}$ such that

$$Z_{\mathbf{s}} = \Phi_{\mathbf{E}}^{-1}(Z_{\widetilde{\mathbf{s}}}).$$

Proposition 3.2. *Let X be an irreducible algebraic variety over $\mathbb{C}$ and $\mathcal{E} = (E_1, \dots, E_k)$ be an overdetermined collection of linear systems on X, let J be the essential subcollection of $\mathcal{E}$. Let also X_a be a fiber of Φ_J passing through a point $a \in X$. Then for generic $a \in X$, the restriction of the collection $\mathcal{E}_{J^c} = (E_i)_{i \notin J}$ on X_a is generically solvable.*

Proof. Here one can take any $a \in U$, with $U = X \setminus \bigcup_J B_J$ as before. For $J = \{i_1, \dots, i_s\}$ and any point $a \in U$ the defect $\mathrm{def}(J)$ can be computed as

$$\mathrm{def}(J) = \mathrm{def}(\widetilde{F}_{i_1}^{\vee}, \dots, \widetilde{F}_{i_s}^{\vee})_a.$$

So it is enough to show that the minimal defect of the restriction of $\mathbf{E}_{J^c}$ on X_a is nonnegative. But the codistribution $\widetilde{F}^\vee$ of the restriction of $\mathbf{E}_{J^c}$ on X_a is given by $\pi_J(F^\vee_{J^c})$, where

$$\pi_J : T^*U \to T^*(X_a \cap U) \cong T^*U/F^\vee_J,$$

is the natural projection. By the second part of Theorem 3.3 we know that $d(\pi_J(J^c)) = 0$. Therefore by Theorem 3.2 the collection $\mathbf{E}_{J^c}$ on X_a is generically solvable. □

Proof of Theorem 3.4. Without loss of generality we can assume that $\mathcal{E}$ is injective. Indeed, $\widetilde{\mathcal{E}}$ is solvable in codimension r if and only if $\mathcal{E}$ is solvable in codimension r.

By Proposition 3.2 we can assume that the essential subcollection of $E_1, \dots, E_k$ is the collection itself. We will call such collections of linear series *essential*. For an essential collection, by Theorem 3.3 there exists a solvable subcollection of size $\tau_{\mathbf{E}}$, assume it is equal to $J = \{1, \dots, r\}$. Then the dimension $\tau_{\mathbf{E}}$ is equal to the dimension τ_J. Therefore the generic solution linear conditions from J on $\Phi_J(X)$ is a union of finitely many points.

Since linear systems E_i's are base points free, the condition on the sections from $E_{r+1}, \dots, E_k$ to vanish at any of these points is union of $k - r = -\operatorname{def}(\mathbf{E})$ clearly independent linear conditions, which finishes the proof of the theorem in this case. □

We finish this section with another corollary of Proposition 3.2 which reduces the study of resultant subvarieties to the study of resultant subvarieties of essential collections of linear systems.

Theorem 3.5. *Let X be a complex irreducible quasi-projective algebraic variety and $\mathcal{E} = (E_1, \dots, E_k)$ be overdetermined collection of linear systems on X with the essential subcollection J. Then the consistency variety $R_{\mathcal{E}}$ does not depend on E_i with $i \notin J$. In other words:*

$$R_{\mathcal{E}} = p^{-1}(R_{\mathcal{E}_J}), \ \textit{where } p : \mathbf{E} \to \mathbf{E}_J = \prod_{i \in J} E_i$$

is the natural projection.

Proof. By Proposition 3.2 there exists a Zariski open subset W of $R_{\mathcal{E}_J}$ such that for any $\mathbf{s} \in W$ the collection of linear systems J^c restricted to a zero set of $\mathbf{s}$ is generically solvable. Therefore if $V \subset R_{\mathcal{E}}$ is a set of solvable systems $\mathbf{s}$ such that $p(\mathbf{s}) \in U$ one has $\operatorname{codim}(V) = \operatorname{codim}(R_{\mathcal{E}}) = -d(\mathcal{E})$. Since $R_{\mathbf{E}}$ is an irreducible variety it coincides with the closure of V, which finishes the proof. □

3.2.4 Resultant of a collection of linear series on a variety

In this subsection we will define resultant of a collection of linear series and translate results of previous subsection to the language of resultants. The notion of resultant defined here is a generalization of $\mathcal{L}$-resultant defined in [10], and most of the results are analogous to the results on $\mathcal{L}$-resultants in [10].

Let $\mathcal{E} = (E_1, \dots, E_{n+1})$ be a collection of linear systems on an irreducible variety X of dimension n. Assume also, that the codimension of the consistency variety $R_{\mathcal{E}}$ is equal 1.

Lemma 3.3. *Let $X, \mathcal{E}$ be as before, then there exists an Zariski open subset $U \subset R_{\mathcal{E}}$ so that for any $\mathbf{s} = (s_1, \dots, s_{n+1}) \in U$, the zero set $Z_{\mathbf{s}}$ of the system $s_1 = \dots = s_{n+1} = 0$ on X is finite. Moreover, the cordiality of $Z_{\mathbf{s}}$ is the same for any $\mathbf{s} \in U$.*

Proof. Let π_1, π_2 be restrictions of two natural projections from $X \times \mathbf{E}$ to $X, \mathbf{E}$ respectively to a incidence variety $\widetilde{R}_{\mathcal{E}}$. For a system $\mathbf{s} \in R_{\mathcal{E}}$ the zero set $Z_{\mathbf{s}}$ is given by $\pi_1(\pi_2^{-1}(\mathbf{s}))$, in particular if $\pi_2^{-1}(\mathbf{s})$ is finite of cordiality k such is $Z_{\mathbf{s}}$. Easy dimension counting shows that $\dim \widetilde{R}_{\mathcal{E}} = \dim R_{\mathcal{E}}$, so for the generic $\mathbf{s} \in R_{\mathcal{E}}$ the preimage $\pi_2^{-1}(\mathbf{s})$ is finite, with fixed cordiality. □

Note that the conditions that $\operatorname{codim} R_{\mathcal{E}} = 1$ and generic non-empty zero set is finite forces the number of linear series to be $n+1$.

Definition 12. Let $X, \mathcal{E}$ be as before, then the resultant $Res_{\mathcal{E}}$ is a polynomial which defines the hypersurface $R_{\mathcal{E}}$ with multiplicity equal to the cordiality of the generic non-empty zero set $Z_{\mathbf{s}}$ [1]. Since R_E is irreducible such polynomial is well defined up to multiplicative constant. For $(s_1, \dots, s_{n+1}) \in \mathbf{E}$ by $Res_{\mathcal{E}}(s_1, \dots, s_{n+1})$ we will denote the value of resultant on the tuple $s_1, \dots, s_{n+1}$.

The next theorem is an immediate corollary of Theorems 3.4 and 3.5. This theorem was proved by Sturmfels in [36] for equivariant linear systems on an algebraic torus. In that setting resultant of a collection of linear series on a variety is usually called *sparse resultant*.

Theorem 3.6. *The consistency variety $R_{\mathcal{E}}$ of a collection of linear systems $\mathcal{E} = (E_1, \dots, E_{n+1})$ has codimension 1 if and only if $\mathrm{d}(E_1, \dots, E_{n+1}) = -1$. Moreover, if J is essential subcollection of $\mathcal{E}$, then the resultant $Res_{\mathcal{E}}$ depends only on equations from E_i with $i \in J$.*

It was shown by Kaveh and Khovanskii (see for example [15]) that for a collection $E_1, \dots, E_n$ of linear series on an irreducible variety X the number of roots of a system $s_1 = \dots = s_n = 0$ is constant for the generic $s_i \in E_i$. The generic number of roots of a system $s_1 = \dots = s_n = 0$ is called *the intersection index* of $E_1, \dots, E_n$ and is denoted by $[E_1, \dots, E_n]$.

[1] In some places the resultant is defined as unique up to constant irreducible polynomial defining $R_{\mathcal{E}}$, but the definition provided here seems more natural. See [4] for details.

Theorem 3.7. *The resultant $Res_{\mathcal{E}}$ is a quasihomogeneous polynomial with degree in the i-th entry equal to the intersection index $[E_1, \ldots, \hat{E}_i, \ldots, E_{n+1}]$. In particular, if J is essential subcollection of E and $i \notin J$ the degree in the i-th entry is 0.*

Proof. The resultant $Res_{\mathcal{E}}$ is a quasihomogeneous polynomial since system $s_1 = \ldots = s_n = 0$ has a root on X, if and only if system $\lambda_1 s_1 = \ldots = \lambda_n s_n = 0$ for any $\lambda_i \in \mathbb{C}^*$. To find the degree of $Res_{\mathcal{E}}$ in the i-th entry consider

$$Res_{\mathcal{E}}(s_1, \ldots, s_i + \lambda s_i', \ldots, s_{n+1})$$

as a polynomial of λ for the fixed generic choice of $s_1, \ldots, s_i, s_i', \ldots, s_{n+1}$. It is easy to see that the number of roots of $Res_{\mathcal{E}}(s_1, \ldots, s_i + \lambda s_i', \ldots, s_{n+1})$ counting with multiplicities is equal to the number of common roots of $s_1 = \ldots = \hat{s}_i = \ldots = s_{n+1} = 0$, so the degree of $Res_{\mathcal{E}}$ in the i-th entry is $[E_1, \ldots, \hat{E}_i, \ldots, E_{n+1}]$. □

3.3 Generic non-empty zero set and reduction theorem

In this section we first study generic non-empty zero sets. In particular, we show in Theorem 3.8 that a generic non-empty zero set given by an overdetermined collection of linear series can be also defined as a generic zero set of generically solvable collection. Then we define a notion of equivalence of two collections of linear systems. Informally speaking, two collections of linear systems are equivalent if they have the same generic nonempty zero sets. We show that every generically inconsistent collection of linear series is equivalent to a collection of minimal defect -1.

3.3.1 Zero sets of essential collection of linear systems

First, we will study *essential* collections of linear series i.e. collections $\mathcal{E} = (E_1, \ldots, E_k)$ such that $\mathrm{def}(J) > \mathrm{d}(\mathcal{E})$ for any $J \subset \{1, \ldots, k\}$.

Let $Y \subset \mathbb{P}_{\mathbf{E}} = \mathbb{P}(E_1^*) \times \ldots \times \mathbb{P}(E_k^*)$ be an irreducible variety of dimension d. For a subset $J \subset \{1, \ldots, k\}$ denote by $\mathbb{P}_J$ the product $\prod_{i \in J} \mathbb{P}(E_i)$ and by π_J the natural projection $\mathbb{P}(\mathbf{E}^*) \to \mathbb{P}_J$. With slight abuse of notation let us denote the restriction of this projection on Y also by π_J and by Y_J its image. Assume also, that $\mathcal{E} = (E_1, \ldots, E_k)$ is an essential collection of linear systems on Y with a solvable subcollection J of size d which exists by Theorem 3.3 and Proposition 3.1.

Lemma 3.4. *In situation as above for the generic pair of points $x_1, x_2 \in Y_J$ the sets $F_{x_i} = \pi_{J^c}(\pi_J^{-1}(x_i))$, for $i = 1, 2$ are disjoint.*

Proof. The condition on sets F_{x_1} and F_{x_2} to be disjoint is open in the space of pairs, so since Y is irreducible it is enough to show that there exists at least one pair x_1, x_2 with $F_{x_1} \cap F_{x_2} = \varnothing$.

Assume otherwise, then for a given point $x_0 \in Y_J$ there exists an preimage $y_0 \in \pi_J^{-1}(x_0)$ and an open set $U \in Y_J$ such that for any $x \in U$ there exists $y \in \pi_J^{-1}(x)$ with $\pi_{J^c}(y) = \pi_{J^c}(y_0)$. So there exists a section $s : U \to Y$ of π_J defined by $s(x) = y$ with a property that $\pi_{J^c} \circ s$ is a constant map on U. But since π_J is a finite morphism, the image $s(U)$ is an Zariski open in Y, and, therefore π_{J^c} is constant on Y, which contradicts the essentiallity of $E_1, \ldots, E_k$. □

The main result of this subsection is the following proposition.

Proposition 3.3. *Let $E_1, \ldots, E_k$ be an essential collection of linear systems on a quasi-projective irreducible variety X. Then for the generic point $\mathbf{s} \in R_{\mathbf{E}}$, the zero set $Z_{\mathbf{s}}$ of a system given by $\mathbf{s}$ is a unique fiber of the Kodaira map Φ_E.*

Proof. As in subsection 3.2.3 we can assume that $\mathcal{E}$ is injective by replacing X with $Y_{\mathbf{E}} = \Phi_{\mathbf{E}}(X) \subset \mathbb{P}_{\mathbf{E}} = \prod_i E_i^*$. Note that the collection $\mathcal{E}$ restricted to $Y_{\mathbf{E}}$ is still essential.

Therefore, it is enough to show that for an irreducible variety $Y_{\mathbf{E}} \subset \mathbb{P}_{\mathbf{E}}$ so that $\mathcal{E}$ is an essential collection on $Y_{\mathbf{E}}$, and for the generic choice of $\mathbf{s} \in R_{\mathcal{E}}$ the intersection $Y \cap H_{s_1} \cap \ldots \cap H_{s_k}$ is a point.

Let $J \subset \{1, \ldots, k\}$ be solvable subcollection of size $\tau_{\mathbf{E}}$, J^c be its complement and let π_J, π_{J^c} be two natural projections restricted to $Y_{\mathbf{E}}$:

$$\begin{array}{ccc} Y_{\mathbf{E}} & \xrightarrow{p_{J^c}} & \mathbb{P}_{J^c} = \prod_{i \notin J} \mathbb{P}(E_i^*) \\ \big\downarrow p_J & & \\ \mathbb{P}_J = \prod_{i \in J} \mathbb{P}(E_i^*) & & \end{array}$$

The generic intersection $Y_J \cap \left(\bigcap_{i\in J} H_{s_i}\right)$ is nonempty and finite, and hence of the same cardinality. It is enough to show that for for generic choice of s_i's with $i \in J$ there are no two points x, y in $Y_J \cap \left(\bigcap_{i\in J} H_{s_i}\right)$ with $p_{J^c}(p_J^{-1}(x)) \cap p_{J^c}(p_J^{-1}(y)) \neq \varnothing$. Indeed, in such a case any two points in the finite intersection

$$Y_{\mathbf{E}} \cap \bigcap_{i\in J} H_{s_i} = \pi_J^{-1}\left(Y_J \cap \bigcap_{i\in J} H_{s_i}\right)$$

would be separated by generic hyperplanes H_{s_i}'s with $i \notin J$.

Let the cardinality of the generic intersection $Y_J \cap \left(\bigcap_{i\in J} H_{s_i}\right)$ be equal to r, we will show that for the generic r-tuple of points $x_1, \ldots, x_r$ the sets $F_{x_i} = \pi_{J^c}(\pi_J^{-1}(x_i))$, for $i = 1, \ldots, r$ are mutually disjoint. Since this condition is open in the space of tuples $x_1, \ldots, x_r$ and Y_J is irreducible it is enough to show that there exist at least one tuple with such property.

Assume otherwise, that for any tuple $x_1, \dots, x_r$ the sets F_{x_i}, for $i = 1, \dots, r$ are not mutually disjoint. This is only possible if for any pair of point x_1, x_2 the sets F_{x_1}, F_{x_2} are not disjoint, but this contradicts Lemma 3.4 since $Y_{\mathbf{E}}$ satisfy its conditions. □

3.3.2 Generic non-empty zero set

In this subsection we study the generic non-empty zero set of a system of equations $s_1 = \ldots = s_k = 0$ with $s_i \in E_i$. First let us summarize results of the last two sections on the generic non-empty zero set $Z_{\mathbf{s}}$.

Theorem 3.8. *Let $\mathcal{E}$ be an overdetermined collection of linear series on an irreducible variety X, with the essential subcollection J. Then for the generic solvable system $\mathbf{s} \in R_{\mathcal{E}}$, the zero set $Z_{\mathbf{s}}$ is the generic zero set of the collection $\mathcal{E}_{J^c} = (E_i)_{i \notin J}$ restricted to a fiber of a Kodaira map Φ_J.*

Proof. By Theorem 3.5 and Proposition 3.3 the zero set of a generic solvable system $\mathbf{s} \in R_{\mathcal{E}}$ is the zero set of a generic system $\mathcal{E}_{J^c} = (E_i)_{i \notin J}$ restricted to a unique fiber of the Kodaira map Φ_J. Moreover, such a restriction is generically solvable by Proposition 3.2. □

Theorem 3.8 expresses the zero set of a system generic in the space of solvable systems defined by the collection $\mathcal{E}$ as the zero set of the system which is generic in the space of *all systems* defined by collection $\mathcal{E}_{J^c}$ restricted to a fiber of Φ_J. We will use this result in coming sections to reduce questions about topology of generic non-empty zero set to questions about topology of generic zero set.

Proposition 3.4. *Let $f : X \to Y$ be a morphism between two algebraic varieties with X - smooth. Then, the generic fiber is smooth.*

Proof. Since the condition is local in Y we can assume Y to be affine. By Noether normalization lemma there exists a finite morphism $g : Y \to \mathbb{C}^k$, with $k = \dim Y$. By Bertini theorem the generic fiber of the composition $g \circ f : X \to \mathbb{C}^k$ is smooth. But since the generic fiber of $g \circ f$ is a finite union of disjoint fibers of f, the generic fiber of f is also smooth. □

Theorem 3.9. *Let X be smooth algebraic variety and let $E_1, \dots, E_k$ be base points free linear systems on X. Then for generic solvable k-tuple $\mathbf{s} = (s_1, \dots, s_k) \in R_E$, the zero set $Z_{\mathbf{s}}$ is smooth. Moreover, the arithmetic genus of $Z_{\mathbf{s}}$ is constant for a generic choice of $\mathbf{s}$.*

Proof. Let π_1, π_2 be two natural projections from $X \times \mathbf{E}$ to $X, \mathbf{E}$ respectively. Denote by π_1, π_2 also their restrictions to a incidence variety $\widetilde{R}_E$. For a system $\mathbf{s} \in R_E$ the zero set $Z_{\mathbf{s}}$ is given by $\pi_1(\pi_2^{-1}(\mathbf{s}))$, in particular is isomorphic to $\pi_2^{-1}(\mathbf{s})$. But since $\widetilde{R}_E$ is smooth ($\widetilde{R}_E$ is a vector bundle over X), the fiber of π_2 over the generic point $\mathbf{s} \in R_E$ is smooth by Proposition 3.4, and therefore such is the generic zero set $Z_{\mathbf{s}}$.

For the second part notice that any algebraic morphism to an irreducible variety is flat over Zariski open subset. So for some Zariski open $U \subset R_E$ the projection $\pi_2^{-1}(U) \to U$ is a flat family. The statement then follows from a fact that arithmetic genus is constant in flat families. □

3.3.3 Reduction theorem

In this subsection we will formulate and prove Reduction theorem. First we define what does it mean for two collections of linear systems to be equivalent.

Definition 13. Two collections $E_1, \dots, E_k$ and $W_1, \dots, W_l$ of linear systems on a quasi-projective irreducible variety X are called equivalent if there exist Zariski open subsets $U \subset R_{\mathbf{E}}$ and $V \subset R_{\mathbf{W}}$ such that for any $u \in U$ $(v \in V)$ there exists $v \in V$ $(u \in U)$ such that the zero sets X_u and X_v coincide.

Theorem 3.10 (Reduction theorem)**.** *Any collection $\mathcal{E} = (E_1, \dots, E_k)$ of generically nonsolvable linear series is equivalent to some collection $\mathcal{W}$ of minimal defect -1. Moreover, if $\mathrm{d}(\mathcal{E}) = -d$ and $\mathcal{E}_J = (E_1, \dots, E_r)$ is an essential subcollection of $\mathcal{E}$, then $\mathcal{W}$ can be defined as*

$$W_1 = \ldots = W_{r-d+1} = E_J, \quad W_{r-d+2} = E_{r+1}, \dots, W_{k-d+1} = E_k.$$

Proof. First note that collections $\mathcal{E}_J = (E_1, \dots, E_r)$ and $\mathcal{W}_K = (W_1, \dots, W_{r-d+1})$ are equivalent collections which are the essential subcollections of $\mathcal{E}$ and $\mathcal{W}$ respectively. Indeed, since both collections are essential, by Proposition 3.3 they are equivalent if and only if generic fibres of their Kodaira maps coincide. But this follows directly from the fact that $W_1 = \ldots = W_{r-d+1} = E_J = E_1 \cdot \ldots \cdot E_r$.

Therefore, we have two collections $\mathcal{E}$ and $\mathcal{W}$ with equivalent essential subcollections $J = (E_1, \dots, E_r)$ and $K = (W_1, \dots, W_{r-d+1})$ and coinciding complements:

$$\mathcal{E}_{J^c} = (E_{r+1}, \dots, E_k) = (W_{r-d+2}, \dots, W_{k-d+1}) = \mathcal{W}_{K^c}.$$

But any two such collections are equivalent since the zero set $Z_{\mathbf{u}}$ of a generic system is a solution of the system compliment to the essential subsystem restricted to the zero set of the essential subsystem. □

3.4 Equivariant linear systems on homogeneous varieties

This section is devoted to the study of G-invariant linear systems on a complex variety X with a transitive G-action. First, we work with general homogeneous space and prove Theorem 3.11 which reduces the study of generic non-empty zero sets of overdetermined systems to the study of generic complete intersections.

We apply then this result to obtain Theorem 3.12 which together with results of [16] provides a strategy for computation of discrete invariants of generic non-empty zero set of a system of equations associated to a collection of overdetermined linear series on a spherical homogeneous space.

3.4.1 Linear systems on homogeneous varieties

In this subsection we will study some general results on linear systems on homogeneous varieties. The main result of this subsection is a reduction of an overdetermined collection of linear series to several isomorphic generically solvable collections.

Let G be a connected algebraic group, and $X = G/H$ be a G-homogeneous space. Let us denote by $x_0 \in X$ the class of identity element $e \cdot H \in G/H$. Let $\mathcal{L}_1, \dots, \mathcal{L}_k$ be globally generated G-linearized line bundles on G/H. For each $i = 1, \dots, k$ let E_i be a nonzero G-invariant linear system for $\mathcal{L}_i$ i.e. E_i is a finite dimensional G-invariant subspace of $H^0(X, \mathcal{L}_i)$.

Each E_J is G-invariant and hence it is base points free on G/H. Thus the Kodaira map Φ_J is defined on the whole G/H. Since $E_1, \dots, E_k$ are G-invariant, E_J^* is a linear representation of G for any $J \subset \{1, \dots, k\}$ and, therefore, there is a natural action of G on $\mathbb{P}(E_J^*)$. It is easy to see that the Kodaira map Φ_J is equivariant for this action. Therefore, the image $\Phi_J(X)$ is a quasi-projective homogeneous G-variety isomorphic to $G/(G_{\Phi_J(x_0)})$, where $G_{\Phi_J(x_0)}$ is a stabilizer of $\Phi_J(x_0) \in \mathbb{P}(E_J^*)$. For $J \subset \{1, \dots, k\}$ we will denote the stabilizer $G_{\Phi_J(x_0)}$ by $\Gamma_J \subset G$.

Definition 14. Two collections of G-invariant linear systems $\mathcal{E} = (E_1, \dots, E_k)$, $\mathcal{E}' = (E_1', \dots E_k')$ on homogeneous spaces X, X' respectively are isomorphic if there exists an G equivariant isomorphism $f : X \to X'$ and an isomorphism of G-linearized line bundles $\phi_i : \mathcal{L}_i \to f^*\mathcal{L}_i'$ for any $i = 1, \dots, k$ such that $\phi_i^* \circ f^*(E_i') = E_i$.

Proposition 3.5. *Let $\Phi_J : X \to \mathbb{P}(E_J^*)$ and Γ_J be as before then:*

(i) for any $y \in \Phi_J(X)$ the fiber $F_y := \Phi_J^{-1}(y)$ has a structure of Γ_J-variety;

(ii) any fiber F_y is isomorphic to $F_{y_0} \cong \Gamma_J/H$ as Γ_J-variety;

(iii) for any G equivariant linear system V on X and any point $y \in \Phi_J(X)$ the restriction $E|_{F_y}$ is

Γ_J-invariant. Moreover, a pair $F_y, V\big|_{F_y}$ is isomorphic to the pair $F_{y_0}, V\big|_{F_{y_0}}$.

Proof. For the parts (i) and (ii) let G_y be the stabilizer of a point $y \in \Phi_J(X)$, then the fiber F_y is an homogeneous G_y variety. But also G_y is conjugated to Γ_J, i.e. $G_y = g\Gamma_J g^{-1}$ for some $g \in G$. Then the equivariant isomorphism between F_{y_0} and F_y (which also will define Γ_J on F_y) can be defined by

$$\Gamma_J \times F_{y_0} \to G_y \times F_y, \quad (\gamma, p) \mapsto (g\gamma g^{-1}, gp).$$

The part (iii) follows directly from the construction above and the fact that linear system V is G-invariant. □

By Proposition 3.5, any fiber of the Kodaira map associated to a G-invariant linear series is a homogeneous variety and in particular is smooth. Therefore, irreducible components of fibers coincide with connected components. Next proposition is a more precise version of Proposition 3.5, which deals with connected components of fibers of the Kodaira map.

Proposition 3.6. *Let $\Phi_J : X \to \mathbb{P}(E_J^*)$ and Γ_J be as before then:*
(i) connected components of fibres of Φ_J have a structure of Γ_J^0-variety;
(ii) any two connected components of any two fibres are isomorphic as Γ_J^0-varieties and, in particular are isomorphic to $\Gamma_J^0/(\Gamma_J^0 \cap H)$, where Γ_J^0 is the connected component of identity in Γ_J;
(iii) for any G equivariant linear system V on X and two connected components C_1, C_2 of any two fibres, the restrictions $E\big|_{C_1}, E\big|_{C_2}$ are Γ_J^0-invariant. Moreover, pairs $C_1, V\big|_{C_1}$ and $C_2, V\big|_{C_2}$ are isomorphic.

Proof. Most of the proof is absolutely analogous to the proof of Proposition 3.5. The only statement which needs clarification is that any connected component of a fiber is isomorphic to $\Gamma_J^0/(\Gamma_J^0 \cap H)$. By Proposition 3.5 it is enough to check this for a connected component of a given fiber, say F_{y_0}. The rest easily follows from the fact that $F_{y_0} = \Gamma_J/H$. □

For a subcollection $J \subset \{1, \dots, k\}$ we will call the number of connected components of a fiber of the Kodaira map $\Phi_J : G/H \to \mathbb{P}(E_J^*)$ *the index* of J and denote it by $ind(J)$. One can describe $ind(J)$ in a group theoretic way, this description clarifies the term "index".

Connected components of identity Γ_J^0, H^0 are normal subgroups of groups Γ_J, H respectively. There exists a natural homomorphism between $i : H/H^0 \to \Gamma_J/\Gamma_J^0$ induced by the inclusion of $H \subset \Gamma_J$. The homomorphism i is well-defined since H^0 is a subgroup of Γ_J^0. It is easy to see that connected components of Γ_J/H and hence of any other fiber of Φ_J are in one to one correspondence with elements of the coset set

$$(\Gamma_J/\Gamma_J^0)\Big/i(H/H^0),$$

i.e. $ind(J)$ is equal to the index of $i(H/H^0)$ in Γ_J/Γ_J^0.

Theorem 3.11. *Let $X = G/H$ be a homogeneous space and let $E_1,\dots,E_k$ be G - invariant linear systems on X, with the essential subcollection $J \subset \{1,\dots,k\}$. Then the zero set $X_{\mathbf{s}}$ for generic $\mathbf{s} \in R_{\mathcal{E}}$ is a disjoint union of $ind(J)$ subvarieties $Y_1,\dots,Y_{ind(J)}$. Moreover, for any $i = 1,\dots,ind(J)$ Y_i is a generic zero set of a collection of linear series isomorphic to $\mathcal{E}_{J^c} = (E_i)_{i\notin J}$ restricted to $\Gamma_J^0/(\Gamma_J^0 \cap H)$.*

Proof. By Theorem 3.8 the zero set $Z_{\mathbf{s}}$ of a generic solvable system given by $\mathbf{s} \in R_{\mathcal{E}}$ is the zero set of a generic system $\mathcal{E}_{J^c}$ restricted to the fiber of the Kodaira map Φ_J. But by Proposition 3.6 connected components with the restrictions of the collection $\mathcal{E}_{J^c}$ to them are isomorphic. □

3.4.2 Linear series in spherical varieties

In this subsection we will study G-invariant linear series on spherical varieties. A homogeneous space G/H of a reductive group G is called *spherical* if some (and hence any) Borel subgroup $B \subset G$ has an open dense orbit in G/H. Starting from this point a group G will assumed to be reductive and a homogeneous space G/H will assumed to be spherical.

Any G-invariant linear series V on G/H is a representation of a reductive group G, therefore it is a direct sum of irreducible representations:

$$V = \bigoplus_{\lambda} V_\lambda.$$

It is well known that the decomposition above is multiplicity free, that is each irreducible representation appears at most once in it. Indeed, let V_λ and V_λ' be two different irreducible representations with the same highest weight appearing in decomposition of V, and let s and s' be highest weight vectors in V_λ and V_λ' respectively. In particular, both s, s' are B-eigensections of weight λ of some G-linearised line bundle $\mathcal{L}$. Therefore the ratio s/s' is a B-invariant rational function on G/H. Since X has an open B-orbit we conclude that s/s' is constant, so $V_\lambda = V_\lambda'$. The set of weights appearing in the decomposition of V is called *G-spectrum* of V and is denoted by $\mathrm{Spec}_G(V)$. The pair of a G-linearized line bundle $\mathcal{L}$ and a finite subset A of $\mathrm{Spec}_G(H^0(X,\mathcal{L}))$ determines uniquely a G-invariant linear series on X.

The main result of this subsection is Theorem 3.12 which realizes a generic non-empty zero set defined by overdetermined collection of linear series, as a zero set defined by generically solvable collection. In [16], for a collection of linear series $\mathcal{E} = (E_1,\dots,E_k)$ and for a generic choice of $\mathbf{s} \in \mathbf{E}$ some discrete invariants of the zero set $Z_{\mathbf{s}}$ were computed in terms of combinatorics of the Newton-Okounkov polytope.

The Newton-Okounkov polytope is constructed as a polytope fibered over moment polytope with string polytopes as fibers. The construction of Newton-Okounkov polytope depends only on G-spectra of linear series E^k (for more details see [16]). These results together with Theorem 3.12 provide a strategy for computing discrete invariants of generic non-empty zero set defined by overdetermined collection of linear series.

Lemma 3.5. *Let H be a spherical subgroup of a reductive group G, let K be a connected reductive subgroup of G which contains H (i.e $H \subset K \subset G$). Then H^0 is a spherical subgroup of K.*

Proof. First notice that if H is spherical subgroup of G then such is H^0 so without loss of generality we can assume that $H = H^0$.

Now consider a point $x \in K/H \subset G/H$, there exists a Borel subgroup B of G such that $B \cdot x$ is a dense in G/H. Let U be the intersection of the dense orbit $B \cdot x$ with K/H, in other words $U = (B \cap K) \cdot x$. Note that $(B \cap K)$ is solvable and therefore $(B \cap K)^0$ is contained in some Borel subgroup B_K of K. From the other hand, since $B \cdot x$ is open in G/H, U is open and dense in K/H, and since K/H is irreducible the orbit $(B \cap K)^0 \cdot x$ is dense in K/H. We conclude that the orbit $B_K \cdot x$ is dense in K/H as B_K contain $(B \cap K)^0$. □

Proposition 3.7. *Let $E_1, \ldots, E_k$ be G-invariant linear systems on a spherical homogeneous space G/H and Γ_J be a reductive group for some $J \subset \{1, \ldots, k\}$. Then the connected components of fibers of the Kodaira map Φ_J are isomorphic spherical homogeneous spaces. Moreover, for any G-invariant linear series V, all restrictions of V to connected components of fibers are isomorphic.*

Proof. By part (ii) of Proposition 3.6 connected components of the fibers of the Kodaira map Φ_J are homogeneous spaces which are isomorphic to $\Gamma_J^0/(\Gamma_J^0 \cap H)$, and since Γ_J^0 is reductive by Lemma 3.5 they are also spherical. The last statement of the above proposition is identical to part (iii) of Proposition 3.6. □

Theorem 3.12. *Let $\mathcal{E} = (E_1, \ldots, E_k)$ and G/H be as before. Let also J be the essential subcollection of $\mathcal{E}$ such that Γ_J is a reductive group. Then for the generic system $\mathbf{s} \in R_{\mathbf{E}}$ the zero set $Z_{\mathbf{s}}$ is a disjoint union of $ind(J)$ varieties $Y_1, \ldots, Y_{ind(J)}$. Moreover, subvarieties Y_j's are defined by isomorphic Γ_J^0-invariant collections of linear series on the spherical variety $\Gamma_J^0/(\Gamma_J^0 \cap H)$.*

Proof. By Theorem 3.11 the zero set $Z_{\mathbf{s}}$ of a generic solvable system $\mathbf{s} \in R_{\mathcal{E}}$ is a union of $ind(J)$ subvarieties $Y_1, \ldots, Y_{ind(J)}$, with Y_j's defined by isomorphic Γ_J^0-invariant collections of linear series on connected components of Kodaira map Φ_J. But connected components of Kodaira map Φ_J are spherical and isomorphic to $\Gamma_J^0/(\Gamma_J^0 \cap H)$ by Proposition 3.7. □

Corollary 3.2. *In situation as above, the arithmetic genus $g(Z_{\mathbf{s}})$ of the generic non-empty zero set $Z_{\mathbf{s}}$ could be computed as*

$$g(Z_{\mathbf{s}}) = ind(J)g(Y_i),$$

for any $i = 1, \dots, ind(J)$, and can be computed in terms combinatorics of Newton-Okounkov polytopes. If in addition all linear series from collection $\mathcal{E}_{J^c}$ restricted to $\Gamma_J^0/(\Gamma_J^0 \cap H)$ are injective, mixed Hodge numbers $h^{p,0}(Z_{\mathbf{s}})$ of the generic non-empty zero set $Z_{\mathbf{s}}$ can be computed as

$$h^{p,0}(Z_{\mathbf{s}}) = ind(J)h^{p,0}(Y_i),$$

for any $i = 1, \dots, ind(J)$, and can be computed in terms combinatorics of Newton-Okounkov polytopes.

Proof. Corollary follows immediately from Theorem 3.12 and Theorems 1, 2 of [16]. □

Theorem 3.12 involves condition on Γ_J to be reductive, the following theorem provides a geometric criterion for a subgroup of a reductive group to be reductive.

Theorem 3.13 ([37])**.** *Let G be a reductive algebraic group then a closed subgroup H of G is reductive if and only if the coset space G/H is an affine algebraic variety.*

Remark 3.1. The condition on Γ_J to be reductive is quite restrictive. However, for any triple $H \subset K \subset G$ where H is a spherical subgroup of G and K is reductive one can realize K as Γ_J for some collection of G-invariant linear series on G/H. Indeed, let $\pi : G/H \to G/K$ be natural projection and let $\mathcal{E} = (E_1, \dots, E_k)$ be an injective (i.e. such that E_J is very ample) essential collection of G-invariant linear system on G/K. Then the pullback collection $\pi^*\mathcal{E} = (\pi^*E_1, \dots, \pi^*E_k)$ is an essential collection of linear series on G/H with $\Gamma_J = K$. It will be interesting to classify all such triples $H \subset K \subset G$. Some of the results in this direction could be found in [13].

3.4.3 Example

In this subsection we will give a concrete example of an application of Theorem 3.12. We will work with homogeneous space GL_n/U. Let $\mathcal{E} = (E_1, E_2, E_3)$ be a collection of linear series defined as

$$E_1 = E_2 = Span(c, \det^k), \quad E_3,$$

where c is a constant function, $\det^k(g \cdot U) = \det(g)^k$, and E_3 is a very ample linear series.

The minimal defect of $\mathcal{E}$ is -1, the essential subcollection is $J = \{1, 2\}$, and $\Gamma_J = \mathrm{SL}_n[k]$, where

$$\mathrm{SL}_n[k] = \left\{g \in \mathrm{GL}_n \,|\, det(g)^k = 1\right\}.$$

Therefore, a connected component of a fiber of the Kodaira map Φ_J is isomorphic to SL_n/U and the number of connected components of a fiber is $ind(J) = k$.

It follows by Theorem 3.12 that the generic non-empty zero set $Z_{\mathbf{s}}$ of a system $s_1 = s_2 = s_3 = 0$ is a union of k subvarieties $Y_1, \ldots, Y_k$ such that each of Y_i is a hypersurface in SL_n/U cut out by a generic section $s \in E_3|_{\mathrm{SL}_n/U}$. In particular geometric genus and mixed Hodge numbers $h^{p,0}$ of $Z_{\mathbf{s}}$ could be computed using results of [16].

Chapter 4

Resultants in dimension one

In Chapter 5, in the assumption that Newton polyhedra of $n+1$-tuple of Laurent polynomials $(f_1, \dots, f_{n+1})$ are generic enough, we will give a number of identities involving the value of Δ-*resultant* $Res_\Delta(f_1, \dots, f_{n+1})$ and the product of a polynomial f_i over the common roots of remaining. To prepare the reader to Chapter 5, in this chapter we present analogous results in one dimensional case. Most of results of this chapter are classical, but we present them in a way which is parallel to the presentation in Chapter 5.

4.1 Sylvester's formula for resultant

Let $P_1 = a_0 + \cdots + a_k z^k$, $P_2 = b_0 + \cdots + b_n z^n$ be polynomials in one complex variable of degrees $\leq k$ and $\leq n$. Sylvester defined resultants $Res^{[1]}$ and $Res^{[2]}$ which are equal up to a sign. $Res^{[1]}$ and $Res^{[2]}$ are defined as the determinant of the $(n+k) \times (n+k)$ matrices M_1, M_2 respectively, where:

$$M_1(P_1, P_2) = \begin{pmatrix} a_k & a_{k-1} & \dots & a_0 & 0 & 0 \\ 0 & \ddots & \ddots & \dots & \ddots & 0 \\ 0 & 0 & a_k & a_{k-1} & \dots & a_0 \\ b_n & b_{n-1} & \dots & b_0 & 0 & 0 \\ 0 & \ddots & \ddots & \dots & \ddots & 0 \\ 0 & 0 & b_n & b_{n-1} & \dots & b_0 \end{pmatrix}$$

and $M_2(P_1, P_2) = M_1(P_2, P_1)$.

One can see that:

- $Res^{[1]} = (-1)^{kn} Res^{[2]}$;

- resultants are polynomials in the coefficients of P_1 and P_2 with degrees n and k in coefficients of P_1 and P_2 respectively;

- the polynomials $Res^{[1]}$ and $Res^{[2]}$ have integer coefficients;

- the coefficient of the monomial $a_k^n b_0^k$ in $Res^{[1]}$ is $+1$ and, hence, the coefficient of the same monomial in $Res^{[2]}$ is $(-1)^{kn}$,

- the coefficients of $Res^{[1]}$ and $Res^{[2]}$ are coprime integers (this follows from two previous points).

Under the assumption $a_n \neq 0$, $b_k \neq 0$ the resultant $Res^{[1]}$ (the resultant $Res^{[2]}$) is equal to zero if and only if the polynomials P_1 and P_2 have common root. One can show that the variety of pairs of polynomials P_1 and P_2 having a common root is an irreducible quasi projective variety X (a simple proof of a multidimensional version of this statement can be found in [10]). Let D be an irreducible polynomial equal to zero on X. According to the previous statement $Res^{[1]}$ and $Res^{[2]}$ are equal to the same power of D, multiplied by some coefficients.

The Sylvester resultants can be generalized for Laurent polynomials. Consider two Laurent polynomials

$$f_1 = a_k z^k + \cdots + a_n z^n, \quad f_2 = b_l z^l + \cdots + b_m z^m, \tag{4.1}$$

on $\mathbb{C}^*$ whose Newton polyhedra belong to the segments Δ_1, Δ_2 defined by inequalities $k \leq x \leq n$, $l \leq x \leq m$. With f_1, f_2 let us associate the pair of polynomials P_1, P_2, where

$$P_1 = z^{-k} f_1 = a_k + \cdots + a_n z^{n-k}, \quad P_2 = z^{-l} f_2 = b_l + \cdots + b_m z^{m-l}. \tag{4.2}$$

For $\Delta = (\Delta_1, \Delta_2)$ we define resultants $Res_\Delta^{[1]}$, $Res_\Delta^{[2]}$ as follows:

$$Res_\Delta^{[1]}(f_1, f_2) = (-1)^{n(m-l)} Res^{[1]}(P_1, P_2), \quad Res_\Delta^{[2]}(f_1, f_2) = (-1)^{l(n-k)} Res^{[2]}(P_1, P_2).$$

The definitions of Δ-resultants $Res_\Delta^{[1]}$ and $Res_\Delta^{[2]}$ are made in such a way that they coincide with the product resultants which are defined in the next section (see Theorem 3).

4.2 Product formula in dimension one

For Laurent polynomials f_1, f_2 as in (4.1), let $\Pi_\Delta^{[1]} = \prod_{y_j \in Y} f_1^{m_{y_j}}(y_j)$ and $\Pi_\Delta^{[2]} = \prod_{x_i \in X} f_2^{m_{x_i}}(x_i)$ where $X = \{x_i\}$ and $Y = \{y_j\}$ are the sets of non zero roots of f_1 and f_2 and m_{y_j}, m_{x_i} are their multiplicities.

Theorem 4.1. *If $a_k a_n b_l b_m \neq 0$ then the following identity holds:*

$$b_l^{-k} b_m^n \Pi_\Delta^{[1]} = (-1)^{kl+nm} a_k^{-l} a_n^m \Pi_\Delta^{[2]}. \tag{4.3}$$

Let us recall an elementary proof of this classical theorem.

Proof. We have $f_1(z) = a_n(z-x_1)^{m_{x_1}} \cdot \dots \cdot (z-x_s)^{m_{x_s}} z^k$, so

$$\Pi_\Delta^{[1]} = \prod_{y_j \in Y} \left[a_n y_j^k \prod_{x_i \in X} (y_j - x_i)^{m_{x_i}} \right]^{m_{y_j}} = a_n^{m-l} \left[(-1)^{m-l} (b_l/b_m) \right]^k \Pi_\Delta^{[1,2]},$$

where $\Pi_\Delta^{[1,2]} = \prod_{x_i \in X, y_j \in Y} (y_j - x_i)^{m_{x_i} m_{y_j}}$. Here we used the Vieta relation $\prod_{y_j \in Y} y_j^{m_{y_j}} = (-1)^{m-l} b_l/b_m$. Thus we proved the identity

$$b_l^{-k} b_m^n \Pi_\Delta^{[1]} = a_n^{m-l} b_m^{n-k} \Pi_\Delta^{[1,2]} (-1)^{mk-lk}.$$

In a similar way

$$a_k^{-l} a_n^m \Pi_\Delta^{[2]} = a_n^{m-l} b_m^{n-k} \Pi_\Delta^{[2,1]} (-1)^{nl-kl},$$

where $\Pi_\Delta^{[2,1]} = \prod_{x_i \in X, y_j \in Y} (x_i - y_j)^{m_{x_i} m_{y_j}}$. But $\Pi_\Delta^{[1,2]} = \Pi_\Delta^{[2,1]} (-1)^{(m-l)(n-k)}$. Theorem 4.1 is proved. □

For Laurent polynomials f_1, f_2 as in (4.1) let us define their product resultants $Res_{\Pi,\Delta}^{[1]}(f_1, f_2)$ and $Res_{\Pi,\Delta}^{[2]}(f_1, f_2)$ by the formulas

$$Res_{\Pi,\Delta}^{[1]} = b_l^{-k} b_m^n \Pi_\Delta^{[1]}, \qquad Res_{\Pi,\Delta}^{[2]} = a_k^{-l} a_n^m \Pi_\Delta^{[2]}. \tag{4.4}$$

Theorem 4.2. *(1) The product resultants are polynomials in the coefficients of f_1 and f_2 (the expressions in (4.4) themselves could have removable singularities at the hyperplanes where the extreme coefficients vanish);*

(2) If the extreme coefficients are nonzero, i.e. $a_k a_n b_l b_m \neq 0$, the product resultants equal to zero exactly on the pairs of Laurent polynomials having common root in $\mathbb{C}^*$;

(3) The product resultants have degrees $(m-l)$ and $(n-k)$ in the coefficients of f_1 and f_2 correspondingly;

(4) The coefficient of the monomial $a_k^{m-l} b_m^{n-k}$ in $Res_{\Pi,\Delta}^{[1]}$ and $Res_{\Pi,\Delta}^{[2]}$ is equal to $(-1)^{k(m-l)}$ and $(-1)^{l(n-k)}$ respectively.

Proof. The expression $b_l^{-k} b_m^n \Pi_\Delta^{[1]}$ obviously is a polynomial of degree $m-l$ in coefficients of f_1 and the expression $a_k^{-l}a_n^m\Pi_\Delta^{[1]}$ is obviously a polynomial in coefficients of f_2 of degree $n-k$. Since two expressions are equal up to sign we have proved (1) and (3).

It is clear that $Res_{\Pi,\Delta}^{[1]}$ vanishes if and only if $f_1(z)=0$ for some root z of f_2, so z is a common root. The same is true for $Res_{\Pi,\Delta}^{[2]}$.

For part (4) let us note that the values of the monomial $a_k^{m-l}b_m^{n-k}$ in $b_l^{-k}b_m^n\Pi_\Delta^{[1]}$ come from multiplying the term a_kz^k over roots of f_2. Using the Vieta formula we have: $b_l^{-k}b_m^n\prod a_kz^k = (-1)^{k(m-l)}a_k^{m-l}b_m^{n-k}$. □

Theorem 4.3. *Assume that $k=l=0$. Then the product resultants coincide with the Sylvester resultants:*

$$Res_\Delta^{[1]} = Res_{\Pi,\Delta}^{[1]} = (-1)^{kl+nm}Res_{\Pi,\Delta}^{[2]} = (-1)^{kl+nm}Res_\Delta^{[2]}.$$

Proof. Both functions $Res_\Delta^{[1]}$ and $Res_{\Pi,\Delta}^{[1]}$ are polynomials in the coefficients of f_1, f_2 of the same degree. They both vanish on the set of pairs of polynomials having a common root in $\mathbb{C}^*$. Since the set of pairs of polynomials having a common root is irreducible the polynomials $Res_\Delta^{[1]}$ and $Res_{\Pi,\Delta}^{[1]}$ are proportional. They have the same coefficient in front of the monomial $a_k^{m-l}b_m^{n-k}$, so they are equal. A similar argument works for $Res_\Delta^{[2]}$ and $Res_{\Pi,\Delta}^{[2]}$. □

Keeping in mind multidimensional generalizations in the Section 4.3 we will present another topological proof of Theorem 4.1 and in the Section 4.4 we will present an algorithm for computing the product resultant which does not rely on the Sylvester determinant.

4.3 Weil reciprocity law

4.3.1 Weil symbol and Weil law

Let f and g be two meromorphic functions on a compact Riemann surface S. About each point $p\in S$ one can choose a local parameter u such that $u(p)=0$ and consider the Laurent expansions $f=c_1u^{k_1}+\ldots$, $g=c_2u^{k_2}+\ldots$ of f and g, with dots are standing for higher order terms. One can check that the expression

$$\{f,g\}_p = (-1)^{k_1k_2}c_1^{-k_2}c_2^{k_1}$$

is independent of the choice of u. The number $\{f,g\}_p$ is called the *Weil symbol* of f and g at the point p.

Example 1. *If p is a zero of g of multiplicity m_p and $f(p) \neq 0, \infty$, then $\{f, g\}_p = f^{-m_p}(p)$. If p is a zero of f of multiplicity m_p and $g(p) \neq 0, \infty$, then $\{f, g\}_p = g(p)^{m_p}$.*

Example 2. *Consider f_1, f_2 from (4.1) as the functions on $\mathbb{C}P^1$. The main terms of Laurent expansions of f_1, f_2 at 0 are $a_k z^k, b_l z^l$ respectively, so $\{f_1, f_2\}_0 = (-1)^{kl} a_k^{-l} b_l^k$.*

Let $w = 1/z$ be the local parameter on $\mathbb{C}P^1$ at ∞. Then the main terms of Laurent expansions of f_1, f_2 at ∞ are $a_n w^{-n}, b_m w^{-m}$ respectively, so $\{f_1, f_2\}_\infty = (-1)^{nm} a_n^m b_m^{-n}$.

Let $D \subset S$ be a finite set containing all points where f or g is equal to 0 or to ∞ (we assume that each function f,g is not identically equal to zero at each connected component of S).

Theorem 4.4 (Weil reciprocity law)**.** *For any couple of meromorphic functions f and g the following relation holds:*

$$\prod_{p \in D} \{f, g\}_p = 1. \tag{4.5}$$

A compact Riemann surface S equipped with its field of meromorphic functions can be considered as an algebraic curve equipped with its field of rational functions. Under such consideration Theorem 4.4 becomes purely algebraic.

Corollary 4.1. *Consider f_1, f_2 from (4.1) as the functions on $\mathbb{C}P^1$. Let X, Y be sets of non zero roots of f_1, f_2. Assume that $X \cap Y = \varnothing$ and roots $x_i \in X, y_j \in Y$ have multiplicities m_{x_i}, m_{y_j}. Then according to the Weil reciprocity law and examples 1 and 2*

$$b_l^{-k} b_m^n \prod_{y_j \in Y} f_1^{m_{y_j}}(y_j) = (-1)^{kl+nm} a_k^{-l} a_n^m \prod_{x_i \in X} f_2^{m_{x_i}}(x_i).$$

Thus Theorem 4.1 could be considered as a corollary of the Weil reciprocity law. On the other hand Theorem 4.1 provides an elementary proof of the Weil reciprocity law in the case under consideration. In the general case the Weil reciprocity law also can be reduced using Newton polygons to similar elementary arguments [18].

4.3.2 Topological extension of the Weil reciprocity law.

Let S be a Riemann surface (not necessary compact) and let $D \subset S$ be a discrete subset. The *Leray coboundary operator* δ associates to every point $p \in D$ an element $\delta(p) \in H_1(S \setminus D, \mathbb{Z})$ represented by a small circle centered at p with the counterclockwise orientation. Let M be the multiplicative group of meromorphic functions on S which are regular and nonzero on $S \setminus D$.

Theorem 4.5. *To each couple $f, g \in M$ one can associate a map $\{f, g\} : H^1(S \setminus D, \mathbb{Z}) \to \mathbb{C}^*$ such that the following properties hold:*

(1) for each $p \in D$ the image $\{f, g\}(\delta(p))$ of the cycle $\delta(p)$ under the map $\{f, g\}$ is equal to the Weil symbol $\{f, g\}_p$;

(2) $\{f, g\} = \{g, f\}^{-1}$;

(3) for any triple $f, g, \phi \in M$ the identity $\{f, g\phi\} = \{f, g\}\{f, \phi\}$ holds.

A simple proof of Theorem 4.5 can be found in [19]. If the surface S is compact then the following relation between the cycles $\delta(p)$ holds:

Lemma 4.1. *The element $\sum_{p\in D} \delta(p) \in H_1(S \setminus D, \mathbb{Z})$ is equal to zero.*

Proof. Indeed the cycle $-\sum_{p\in D} \delta(p)$ is the boundary of $S \setminus \bigcup_{p\in D} B_p$ where B_p is the open ball centered in p with the boundary $\delta(p)$. □

The Weil reciprocity law follows from Theorem 4.5 and Lemma 4.1: $\prod_{p\in D}\{f, g\}_p = 1$ because in $H_1(S \setminus D)$ the identity $\sum_{p\in D} \delta(p) = 0$ holds.

Let us reformulate Lemma 4.1 in the case related to the torus $\mathbb{C}^* = \mathbb{C}P^1 \setminus \{0, \infty\}$. We will work with $S = \mathbb{C}P^1$ and $D = \{0, \infty\} \cup D'$ where D' is a finite set containing the sets X, Y of non zero roots of the functions f_1, f_2 from (4.1), i.e. containing non zero roots of $P = f_1f_2$. Let Δ be the Newton polyhedron of P. Then Δ is the segment with vertices $A_0 = k + l$ and $A_\infty = n + m$. Let $T^1 \subset \mathbb{C}^*$ be the circle $|z| = 1$ orientated by the form $d(\arg z)$. Let $T^1_{A_0}, T^1_{A_\infty}$ be the cycles in $\mathbb{C}^* \setminus D'$ given by $\frac{1}{\lambda}T^1, \lambda T^1$ where $|\lambda|$ is big enough. Let $k_{A_0} = 1$ and $k_{A_\infty} = -1$.

Theorem 4.6 (one dimensional topological theorem)**.** *In the notations above the identity $\sum_{p\in D'} \delta(p) = -(k_{A_0}T^1_{A_0} + k_{A_\infty}T^1_{A_\infty})$ holds.*

Proof. Theorem 4.6 immediately follows from Lemma 4.1 because $T^1_{A_0} = \delta(0)$ and $T^1_{A_\infty} = -\delta(\infty)$. □

The identity $\prod_{p\in D}\{f_1, f_2\}^{-1}_p = \{f_1, f_2\}_0\{f_1, f_2\}_\infty$ follows from Theorems 4.5 and 4.6. It can be rewritten as $\Pi^{[1]}_\Delta/\Pi^{[2]}_\Delta = (-1)^{-kl+nm}a_k^{-l}b_l^k a_n^m b_m^{-n}$. Thus we obtained a topological proof of Theorem 4.1.

4.4 Sums over roots of Laurent polynomial and elimination theory

4.4.1 Sums over roots of Laurent polynomial

Let $z \in \mathbb{C}^*$ be a root of multiplicity $\mu(z)$ of the Laurent polynomial P. Then for any Laurent polynomial f the following theorem holds.

Theorem 4.7. *The sum $\sum f(z)\mu(z)$ over all roots $z \in \mathbb{C}^*$ of P is equal to $-(res_0\omega + res_\infty\omega)$, where $\omega = f\frac{dP}{P} = \frac{\partial P}{\partial z} \cdot \frac{fz}{P} \cdot \frac{dz}{z}$.*

Proof. Theorem 4.7 follows from the Cauchy residue formula since the residue of the form ω at a root z is equal to $f(z)\mu(z)$. □

Theorem 4.7 provides an explicit formula for the sum $\sum f(z)\mu(z)$: in contrast to the roots of P, the points $0, \infty$ are independent of the coefficients of P and therefore the residues of ω at 0 and ∞ could be explicitly computed.

4.4.2 Elimination theory related to the one-dimensional case

Let P and f be Laurent polynomials as above. Here we explain how to find any symmetric function of the set of values $\{f(z)\}$ over all roots $z \in \mathbb{C}^*$ (each root z is taken with multiplicity $\mu(z)$).

Denote by $f^{(k)}$ the number $f^{(k)} = \sum_z f^k(z)\mu(z)$. By Theorem 4.7 one can calculate $f^{(k)}$ for any k explicitly. The power sum symmetric polynomials form a generating set for the ring of symmetric polynomials.

Corollary 4.2. *One can find explicitly all symmetric functions of $\{f(z)\}$, construct a monic polynomial whose roots are $\{f(z)\}$, and eliminate z from the definition of the set $\{f(z)\}$. In particular, one can compute the products $\Pi_\Delta^{[1]}, \Pi_\Delta^{[2]}$ defined in the Section 4.3.*

Chapter 5

The Resultant of Developed Systems of Laurent Polynomials

5.1 Introduction

As before, to a Laurent polynomial f in n variables one associates its Newton polyhedron $\Delta(f)$ which is a convex lattice polyhedron in $\mathbb{R}^n$. A system of n equations $f_1 = \cdots = f_n = 0$ in $(\mathbb{C}^*)^n$ is called *developed* if (roughly speaking) their Newton polyhedra $\Delta(f_i)$ are located generically enough with respect to each other. The exact definition (see also Section 5.2) is as follows: a collection of n polyhedra $\Delta_1, \ldots, \Delta_n \subset \mathbb{R}^n$ is called *developed* if for any covector $v \in (\mathbb{R}^n)^*$ there is i such that on the polyhedron Δ_i the inner product with v attains its biggest value precisely at a vertex of Δ_i.

A developed system resembles an equation in one unknown. A polynomial in one variable of degree d has exactly d roots counting with multiplicity. The number of roots in $(\mathbb{C}^*)^n$ counting with multiplicities of a developed system is always determined by the Bernstein-Kushnirenko formula (see [1]) (if the system is not developed this formula holds only for generic systems with fixed Newton polyhedra).

As in the one-dimensional case, one can explicitly compute the sum of values of any Laurent polynomial over the roots of a developed system (see [11], [12]) and the product of all of the roots of the system regarded as elements in the group $(\mathbb{C}^*)^n$ (see [18]). These results can be proved topologically (as in [12]), using the topological identity between certain homology cycles related to developed system (see Section 5.3), the Cauchy residues theorem, and a topological version of the Parshin reciprocity laws (see Section 5.4).

To an $(n+1)$-tuple $A = (\mathcal{A}_1, \ldots, \mathcal{A}_{n+1})$ of finite subsets in $\mathbb{Z}^n$ one associates the *A-resultant* R_A. It is a polynomial defined up to sign in the coefficients of Laurent polynomials $f_1, \dots, f_{n+1}$ whose supports belong to $A_1, \dots, A_{n+1}$ respectively. The A-resultant is equal to ± 1 if the codimension of the variety of consistent systems in the space of all systems with supports in A is greater than 1. Otherwise, R_A is a polynomial which vanishes on the variety of consistent systems and such that the degree of R_A in the coefficients of the i-th polynomial is equal to the generic number of roots of the system $f_1 = \dots = \hat{f}_i = \dots = f_{n+1} = 0$ (in which the equation $f_i = 0$ is removed).

The notion of A-resultant was introduced and studied in [10] under the following assumption on A: the lattice generated by the differences $a - b$ for all couples $a, b \in \mathcal{A}_i$ and all $0 \leq i \leq n+1$ is $\mathbb{Z}^n$. Under this assumption the resultant R_A is an irreducible polynomial (which was used in a definition of R_A in [10]). Later in [8] and [4] it was shown that in the general case (i.e when the differences from $\mathcal{A}_i$'s do not generate the whole lattice) R_A is some power of an irreducible polynomial. The power is equal to the generic number of roots of a corresponding consistent system (see Section **??** for more details).

To an $(n+1)$-tuple $\Delta = (\Delta_1, \dots, \Delta_{n+1})$ of Newton polyhedra one associates the $(n+1)$-tuple A_Δ of finite subsets $(\Delta_1 \cap \mathbb{Z}^n, \dots, \Delta_{n+1} \cap \mathbb{Z}^n)$ in $\mathbb{Z}^n$. We define the Δ-*resultant* as A-resultant for $A = A_\Delta$. In the paper we deal with Δ-resultants only. If a property of Δ-resultant is a known property of A-resultants for $A = A_\Delta$ we refer to a paper where the property of A-resultants is proven (without mentioning that the paper deals with A-resultants and not with Δ-resultants). Dealing with Δ-resultants only we lose nothing: A-resultants can be reduced to Δ-resultants. One can check that $R_A(f_1, \dots, f_{n+1})$ for $A = (A_1, \dots, A_{n+1})$ is equal to $Res_\Delta(f_1, \dots, f_{n+1})$ for $\Delta = (\Delta_1, \dots, \Delta_{n+1})$, where $\Delta_1, \dots, \Delta_{n+1}$ are the convex hulls of the sets $A_1, \dots, A_{n+1}$.

A collection Δ is called i-*developed* if its subcollection obtained by removing the polyhedron Δ_i is developed. Using the Poisson formula (see [33], [4] and Section 5.9.3) one can show that for i-developed Δ the identity

$$Res_\Delta = \pm \Pi_\Delta^{[i]} M_i \tag{5.1}$$

holds, where $\Pi_\Delta^{[i]}$ is the product of f_i over the common zeros in $(\mathbb{C}^*)^n$ of f_j, for $j \neq i$, and M_i is an explicit monomial in the vertex coefficients (i.e. the coefficient of f_j in front of a monomial corresponding to a vertex of Δ_j) of all the Laurent polynomials f_j with $j \neq i$.

We provide an explicit algorithm for computing the term $\Pi_\Delta^{[i]}$ using the summation formula over the roots of a developed system (Corollary 5.6). Hence we get *an explicit algorithm for computing the resultant* Res_Δ for an i-developed collection Δ. This algorithm heavily uses the Poisson formula (1).

If $(n+1)$-tuple Δ is i-developed and j-developed for some $i \neq j$ the identity

$$\Pi_{\Delta}^{[i]} = \Pi_{\Delta}^{[j]} M_{i,j} s_{i,j} \tag{5.2}$$

holds, where $M_{i,j}$ is an explicit monomial in the coefficients of Laurent polynomials $f_1, \ldots, f_{n+1}$ and $s_{i,j} = (-1)^{f_{i,j}}$ is an explicitly defined sign (Corollary 5.2).

Our proof of the identity (5.2) is topological. We use the topological identity between cycles related to a developed system (see Section 5.3) and a topological version of the Parshin reciprocity laws. The identity (5.2) generalizes the formula from [18] for the product in $(\mathbb{C}^*)^n$ of all roots of a developed system of equations.

An $(n+1)$-tuple Δ is called *completely developed* if it is i-developed for every $1 \leq i \leq n+1$. For completely developed Δ the identity

$$\Pi_{\Delta}^{[1]} M_1 s_1 = \ldots = \Pi_{\Delta}^{[n+1]} M_{n+1} s_{n+1} \tag{5.3}$$

holds, where $(M_1, \ldots, M_{n+1})$ and $\left(\Pi_{\Delta}^{[1]}, \ldots, \Pi_{\Delta}^{[n+1]}\right)$ are monomials and products appearing in (5.1) and $(s_1, \ldots, s_{n+1})$ is an $(n+1)$-tuple of signs such that $s_i s_j = s_{i,j}$ where $s_{i,j}$ are the explicit signs from identity (5.2).

Our proof of the identities (5.3) uses the identity (5.2) and does not rely on the theory of resultants. Using one general fact from this theory (Theorem 5.12) one can see that the quantities in the identities (5.3) are equal to the Δ-resultant, i.e. are equal to $\pm Res_{\Delta}$. Thus the identities (5.3) can be considered as a signed version of the Poisson formula for completely developed systems.

5.2 Developed systems and combinatorial coefficients

Let $\Delta = (\Delta_1, \ldots, \Delta_n)$ be an n-tuple of convex polyhedra in $\mathbb{R}^n$, and let $\sum \Delta_i = \Delta_1 + \cdots + \Delta_n$ be their Minkowski sum. Each face Γ of the polyhedron $\sum \Delta_i$ can be uniquely represented as a sum

$$\Gamma = \Gamma_1 + \cdots + \Gamma_n,$$

where Γ_i is a face of Δ_i.

An n-tuple Δ is called *developed* if for each face Γ of the polyhedron $\sum \Delta_i$, at least one of the terms Γ_i in its decomposition is a vertex.

The system of equations $f_1 = \ldots = f_n = 0$ on $(\mathbb{C}^*)^n$, where $f_1, \ldots, f_n$ are Laurent polynomials, is called developed if the n-tuple $(\Delta_1, \dots, \Delta_n)$ of their Newton polyhedra is developed.

For a developed n-tuple of polyhedra Δ, a map $h : \partial \sum \Delta_i \to \partial \mathbb{R}^n_+$ of the boundary $\partial \sum \Delta_i$ of $\sum \Delta_i$ into the boundary of the positive octant is called *characteristic* if the component h_i of the map $h = (h_1, \dots, h_n)$ vanishes precisely on the faces Γ, for which the i-th term Γ_i in the decomposition is a point (a vertex of the polyhedron Δ_i). One can show that the space of characteristic maps is nonempty and connected. The preimage of the origin under a characteristic map is precisely the set of all vertices of the polyhedron $\sum \Delta_i$.

The *combinatorial coefficient* k_A of a vertex A of $\sum \Delta_i$ is the local degree of the germ

$$h : (\partial \sum \Delta_i, A) \to (\partial \mathbb{R}^n_+, 0)$$

of a characteristic map restricted to the boundary $\partial \sum \Delta_i$ of $\sum \Delta_i$. The combinatorial coefficient is independent of a choice of a characteristic map, but it depends on the choice of the orientation of $\sum \Delta_i$ and $\mathbb{R}^n_+$. The first one is given by the orientation of the space of characters on $(\mathbb{C}^*)^n$. The second is defined by an ordering of the polyhedra $\Delta_1, \dots, \Delta_n$ in n-tuple Δ. Both orientations are an arbitrary choice, and after changing each of them the combinatorial coefficient will change sign. For more detailed discussion of combinatorial coefficients see [12],[35],[18].

Let us discuss combinatorial coefficients in the two-dimensional case. Two polygons $\Delta_1, \Delta_2 \subset \mathbb{R}^2$ are developed if and only if they do not have parallel sides with the same direction of the outer normals (see figure 5.1). If Δ_1, Δ_2 are developed each side of $\Delta_1 + \Delta_2$ comes either from Δ_1 or from Δ_2. That is each side of $\Delta_1 + \Delta_2$ is either the sum of a side of Δ_1 and a vertex of Δ_2, or the sum of a vertex of Δ_1 and a side of Δ_2.

The two types of sides are labeled by 2 (dashed in the picture) and 1 (solid in the picture) respectively. Giving $\mathbb{R}^2$ and the positive octant the standard orientations we can find the local degree of a characteristic map at a vertex. It is equal to 0 if neighbouring edges have the same label, to $+1$ if the label at A is changing from 2 to 1 in the counter clockwise direction, and to -1 if it is changing from 1 to 2.

So the combinatorial coefficient k_A of a vertex $A \in \Delta$ is equal to 0 if neighbouring edges have the same label, and $+1$ or -1 (depending on the orientation) if the labeling changes at A. The only possible value of combinatorial coefficient in dimension 2 is $-1, 0$ or $+1$ because the local mapping degree of one dimensional manifolds could take only these values. The combinatorial coefficient in dimension ≥ 3 could be any integer number.

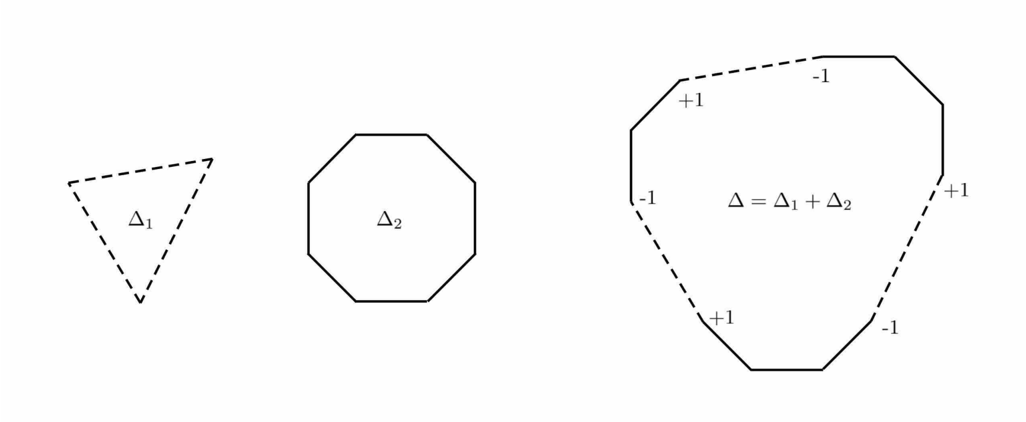

Figure 5.1: Combinatorial coefficient in dimension 2.

5.3 Topological theorem.

5.3.1 Grothendieck cycle.

Let z be an isolated root of a system $f_1 = \ldots = f_n = 0$ on $(\mathbb{C}^*)^n$ where $f_1, \ldots, f_n$ are Laurent polynomials. The Grothendieck cycle γ_z is a class in the group of n-dimensional homologies of the complement $(U \setminus \Gamma)$ of a small neighborhood U of the point z, of the hyperplane Γ defined by the equation $P = f_1 \ldots f_n = 0$. For almost all small enough $\varepsilon = (\varepsilon_1, \ldots \varepsilon_n) \in \mathbb{R}^n_+$, the subset $\gamma_{z,\varepsilon}$ defined by $|f_i| = \varepsilon_i$ is a smooth compact real submanifold of $(U \setminus \Gamma)$. The Grothendieck cycle γ_z is the cycle of the submanifold $\gamma_{z,\varepsilon}$ for small enough ε oriented by the form $d(arg f_1) \wedge \ldots \wedge d(arg f_n)$. The orientation of the Grothendieck cycle depends on the order of the equations $f_1 = 0, \ldots, f_n = 0$.

5.3.2 The cycle related to a vertex of the Newton polyhedron.

Let $\Gamma \subset (\mathbb{C}^*)^n$ be a hypersurface $P = 0$, where P is a Laurent polynomial with Newton polyhedron $\Delta(P)$. Let $T^n \subset (\mathbb{C}^*)^n$ be the torus $|z_1| = \ldots = |z_n| = 1$ orientated by $\omega = d(\arg z) \wedge \ldots \wedge d(\arg(z_n))$. The sign of ω depends on the order of variables, so the sign of a cycle T^n depends on the orientation of the space $\mathbb{R}^n$ of characters on $(\mathbb{C}^*)^n$.

For every vertex A of $\Delta(P)$ we will assign an n-dimensional cycle T^n_A in $(\mathbb{C}^*)^n \setminus \Gamma$ defined up to homological equivalence. For this denote by $\xi_A = (\xi_1, \ldots, \xi_n)$ an integer covector such that the inner product of $x \in \sum \Delta_1$ with ξ_A attains its maximum value at A. Consider the 1-parameter subgroup $\lambda(t) = (t^{\xi_1}, \ldots, t^{\xi_n})$ of $(\mathbb{C}^*)^n$. For t with large enough absolute value $|t|$ the translation $\lambda(t)T^n$ of T^n by the subgroup $\lambda(t)$ does not intersect the hypersurface Γ.

Let us define T^n_A as a cycle $\lambda(t)T^n$ with $|t|$ large enough. The definition makes sense because the homology class of $\lambda(t)T^n$ in $(\mathbb{C}^*)^n \setminus \Gamma$ does not depend on the choice of ξ_A and t, provided $|t|$ is large enough.

5.3.3 Topological theorem for n Laurent polynomials.

Let $f_1, \dots, f_n$ be Laurent polynomials with developed Newton polyhedra $\Delta_1, \dots, \Delta_n$. Let Γ be a hypersurface in $(\mathbb{C}^*)^n$ defined by the equation $P = f_1 \dots f_n = 0$. In [11] and [12] the following theorem is proved.

Theorem 5.1. *In $(\mathbb{C}^*)^n \setminus \Gamma$ the sum of the Grothendieck cycles γ_z over all roots z of the system $f_1 = \ldots = f_n = 0$ is homologous to the cycle $(-1)^n \sum k_A T^n_A$, where the sum is taken over all vertices A of $\sum \Delta_i$ and k_A is the combinatorial coefficient at the vertex A.*

The signs in the topological theorem depend on the choice of order of variables $z_1, \dots, z_n$ and on the choice of order of functions $f_1, \dots, f_n$. In the statement these orders are fixed in an arbitrary way. Changing the order of the variables changes the sign of cycle at vertices T^n_A and all of the combinatorial coefficients. Choosing a different order for the equations will change the signs of all the Grothendieck cycles γ_z, and all of the combinatorial coefficients as well.

5.3.4 Topological theorem for $(n + 1)$ Laurent polynomials.

We will say that the collection of polyhedra $\Delta_1, \dots, \Delta_{n+1}$ is *i-developed* if the collection with Δ_i removed is developed. In this section we present a version of the topological theorem applicable for a collection of $n + 1$ Laurent polynomials which is i-developed and j-developed for some $1 \le i < j \le n$

Let i, j be indexes such that $1 \le i < j \le n + 1$. We will associate with i, j the permutation $\{k_1, \dots, k_{n+1}\}$ of $\{1, \dots, n + 1\}$ defined by the following relations $k_1 = i$, $k_2 = j$, and $k_3 < \cdots < k_{n+1}$.

The following lemma is obvious.

Lemma 5.1. *A collection of $(n + 1)$ polyhedra $\Delta_1, \dots, \Delta_{n+1}$ in $\mathbb{R}^n$ is i-developed and j-developed for some $i < j$ if and only if the collection $(\Delta_i + \Delta_j), \Delta_{k_3}, \dots, \Delta_{k_{n+1}}$ is developed.*

Let $\Delta_1, \dots, \Delta_{n+1}$ be an i-developed and j-developed collection of Newton polyhedra. Let us denote by $k^{i,j}_A$ the combinatorial coefficient of a vertex $A \in \sum \Delta_i$ associated with the the collection $\Delta_i + \Delta_j, \Delta_{k_3}, \dots, \Delta_{k_{n+1}}$. Let $f_1, \dots, f_{n+1}$ be Laurent polynomials with Newton polyhedra $\Delta_1, \dots, \Delta_{n+1}$ such that the system

$$f_1 = \cdots = f_{n+1} = 0 \tag{5.4}$$

is not consistent in $(\mathbb{C}^*)^n$. Denote by X_i the set of all roots x of the system

$$f_1 = \cdots = \hat{f}_i = \cdots = f_{n+1} = 0, \tag{5.5}$$

where the equation $f_i = 0$ is removed. Denote by X_j the set of all roots y of the system

$$f_1 = \cdots = \hat{f}_j = \cdots = f_{n+1} = 0, \tag{5.6}$$

where the equation $f_j = 0$ is removed.

Theorem 5.2. *Assume that the Newton polyhedra $\Delta_1, \ldots, \Delta_{n+1}$ of the Laurent polynomials $f_1, \ldots, f_{n+1}$ are i-developed and j-developed and that the system (5.4) is not consistent in $(\mathbb{C}^*)^n$. Then in the group $H_n((\mathbb{C}^*)^n \setminus \Gamma, \mathbb{Z})$ the identity*

$$(-1)^{j-2} \sum_{x \in X_i} \gamma_x + (-1)^{i-1} \sum_{y \in X_j} \gamma_y = (-1)^n \sum k_A^{i,j} T_A^n$$

holds, where γ_x and γ_y are the Grothendieck cycles of the roots x and y of the systems (5.5), (5.6) and the summation on the right is taken over all vertices A of $\Delta = \Delta_1 + \cdots + \Delta_n$.

Proof. According to the topological theorem the sum of the Grothendieck cycles γ_z over the set $X_{i,j}$ of all roots z of the system

$$f_i f_j = f_{k_1} = \cdots = f_{k_{n+1}} = 0$$

is equal to $(-1)^n k_A^{i,j} T_A$. The set $X_{i,j}$ is equal to $X_i \cup X_j$ where X_i is the set of roots x of the system $f_j = f_{k_1} = \cdots = f_{k+1} = 0$ and X_j is the set of roots y of the system $f_i = f_{k_1} = \cdots = f_{k+1} = 0$. If $z = x \in X_i$ then the cycle γ_z is equal to the cycle $(-1)^{j-2}\gamma_x$ for the system (5.5). The sign $(-1)^{j-2}$ in the identity appears because of the change of the order equations from $f_j = f_{k_1} = \cdots = f_{k_{n+1}} = 0$ to $f_1 = \cdots = \hat{f}_i = \cdots = f_{n+1} = 0$. In a similar way if $z = y \in X_j$ then the cycle γ_z is equal to the cycle $(-1)^{i-1}\gamma_y$ for the system (5.6). □

5.4 Parshin reciprocity laws

5.4.1 Analog of the determinant of $n+1$ vectors in n-dimesional space over $\mathbb{F}_2$.

A determinant of n vectors in n-dimensional space L_n over the field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ is unique non-zero multilinear function on n-tuples of vectors in L_n which is invariant under the $GL(n, \mathbb{F}_2)$ action and which has value 0 if the n-tuple is dependent.

It turns out that there exists unique function on $(n+1)$-tuples of vectors in L_n having exactly the same properties (see [18], [25]).

Theorem 5.3. *There exists a unique non zero function D on $(n+1)$-tuples of vectors in L_n satisfying the following properties:*

(i) D is $GL(n, \mathbb{F}_2)$ invariant, i.e. for any $A \in GL(n, \mathbb{F}_2)$ the equality $D(k_1, \dots, k_{n+1}) = D(A(k_1), \dots, A(k_{n+1}))$ holds;

(ii) if the rank of $k_1, \dots, k_{n+1}$ is $< n$ then $D(k_1, \dots, k_{n+1}) = 0$;

(iii) D is multilinear.

Let us present two explicit formulas for D:

I) If the rank of $k_1, \dots, k_{n+1}$ is $< n$ then $D(k_1, \dots, k_{n+1}) = 0$ (see (ii)). If the rank is n then there is a unique non-zero collection $\lambda_1, \dots, \lambda_{n+1} \in \mathbb{F}_2$ such that $\lambda_1 k_1 + \dots + \lambda_{n+1} k_{n+1} = 0$ and . In this case $D(k_1, \dots, k_{n+1}) = 1 + \lambda_1 + \dots + \lambda_{n+1}$.

II) If on the space L_n the coordinates are fixed, then $D(k_1, \dots, k_{n+1}) = \sum_{j>i} \Delta_{ij}$, where Δ_{ij} is the determinant of $(n \times n)$ matrix whose first $n-1$ columns are coordinates of vectors $k_1, \dots, k_{n+1}$ with vectors k_i, k_j removed and the last column is the coordinatewise product of vectors k_i and k_j.

For a vector $v \in \mathbb{Z}^n$ let $\widetilde{v} \in \mathbb{F}_2^n$ be its mod 2 reduction. For an $(n+1)$-tuple of vectors $v_1 \dots v_{n+1} \in \mathbb{Z}^n$ we define $D(v_1 \dots v_{n+1})$ as $D(\widetilde{v}_1 \dots \widetilde{v}_{n+1})$.

The determinant of a matrix A over $\mathbb{R}$ is the volume of the oriented parallelepiped spanned by the columns of A. It turns out that the function D also computes the volume of some figure (see [22]). This property of D allows to fit it into the topological version of Parshin reciprocity laws (see [25] and the Section 5.4.3).

5.4.2 Parshin symbols of monomials

Consider $n+1$ monomials $c_1 \cdot \mathbf{z}^{\mathbf{k}_1}, \dots, c_{n+1} \cdot \mathbf{z}^{\mathbf{k}_{n+1}}$ with nonzero coefficients $c_i \in \mathbb{C}^*$ in n complex variables, where

$$\mathbf{z} = (z_1, \dots, z_n),$$

$$\mathbf{k}_i = (k_{i,1}, \dots, k_{i,n}) \text{ with } \mathbf{k}_i \in (\mathbb{Z})^n,$$

$$c_i \cdot \mathbf{z}^{\mathbf{k}_i} = c_i z_1^{k_{i,1}} \cdot \dots \cdot z_n^{k_{i,n}}.$$

The *Parshin Symbol* $[c_1 \cdot \mathbf{z}^{\mathbf{k}_1}, \dots, c_{n+1} \cdot \mathbf{z}^{\mathbf{k}_{n+1}}]$ of the sequence $c_1 \cdot \mathbf{z}^{\mathbf{k}_1}, \dots, c_{n+1} \cdot \mathbf{z}^{\mathbf{k}_{n+1}}$ is equal by definition to

$$(-1)^{D(\mathbf{k}_1, \dots, \mathbf{k}_{n+1})} \cdot c_1^{-\det(\mathbf{k}_2, \dots, \mathbf{k}_{n+1})} \cdot \dots \cdot c_{n+1}^{(-1)^{n+1} \det(\mathbf{k}_1, \dots, \mathbf{k}_n)} = (-1)^{D(\mathbf{k}_1, \dots, \mathbf{k}_{n+1})} \cdot \exp\left[-\det(A)\right], \text{ where}$$

$$A = \begin{pmatrix} \ln c_1 & k_{1,1} & \dots & k_{1,n+1} \\ \vdots & \vdots & & \vdots \\ \ln c_{n+1} & k_{n+1,1} & \dots & k_{n+1,n+1} \end{pmatrix},$$

and $D : (\mathbb{Z}^n)^{n+1} \to \mathbb{Z}/2\mathbb{Z}$ is the function defined in the previous section.

Example 3. *The Parshin symbol* $[c_1 z^{k_1}, c_2 z^{k_2}]$ *of functions* $c_1 z^{k_1}, c_2 z^{k_2}$ *in one variable* z *is equal to* $(-1)^{k_1 k_2} c_1^{-k_2} c_1^{k_1}$*, thus it is equal to the Weil symbol* $\{c_1 z^{k_1}, c_2 z^{k_2}\}_0$ *of these functions at the origin* $z = 0$.

By definition, the Parshin symbol is skew-symmetric, so for example,

$$[c_1 \cdot \mathbf{z}^{\mathbf{k}_1}, c_2 \cdot \mathbf{z}^{\mathbf{k}_2}, \dots, c_{n+1} \cdot \mathbf{z}^{\mathbf{k}_{n+1}}] = [c_2 \cdot \mathbf{z}^{\mathbf{k}_2}, c_1 \cdot \mathbf{z}^{\mathbf{k}_1}, \dots, c_{n+1} \cdot \mathbf{z}^{\mathbf{k}_{n+1}}]^{-1},$$

and multiplicative, so for example, if $c_1 \cdot \mathbf{z}^{\mathbf{k}_1} = a_1 b_1 \cdot \mathbf{z}^{\mathbf{l}_1 + \mathbf{m}_1}$, then

$$[c_1 \cdot \mathbf{z}^{\mathbf{k}_1}, \dots, c_{n+1} \cdot \mathbf{z}^{\mathbf{k}_{n+1}}] = [a_1 \cdot \mathbf{z}^{\mathbf{l}_1}, \dots, c_{n+1} \cdot \mathbf{z}^{\mathbf{l}_{n+1}}][b_1 \cdot \mathbf{z}^{\mathbf{m}_1}, \dots, c_{n+1} \cdot \mathbf{z}^{\mathbf{m}_{n+1}}].$$

5.4.3 Topological version of Parshin laws.

The Parshin reciprocity laws (see [32]) are applicable to $(n+1)$ rational functions on an n-dimensional algebraic variety over an algebraically closed field of any characteristic. They contain several general relations between the Parshin symbols of these $(n+1)$ functions analogous to the relation between the Weil symbols of two functions given in the Weil reciprocity law for an algebraic curve. We will need a topological version of Parshin's laws over $\mathbb{C}$ in the special situation that the algebraic variety is $(\mathbb{C}^*)^n$

and the $(n+1)$ functions are Laurent polynomials $f_1, \dots, f_{n+1}$. Let us state needed facts for that special situation (for general case see [25]).

Let Γ be a hypersurface in $(\mathbb{C}^*)^n$ defined by the equation $P = f_1 \dots f_n = 0$. According to the topological version of the Parshin reciprocity laws there is a map $[f_1, \dots, f_{n+1}] : H^n((\mathbb{C}^*)^n \setminus \Gamma, \mathbb{Z}) \to \mathbb{C}^*$ having the following properties:

1) The map $[f_1, \dots, f_n]$ depends skew symmetrically on the components f_i, so for example $[f_1, f_2, \dots, f_{n+1}] = [f_2, f_1, \dots, f_{n+1}]^{-1}$.

2) Let $A = A_1 + \dots + A_{n+1}$ be a vertex of $\Delta_1 + \dots + \Delta_{n+1}$ where A_i is a vertex in Δ_i. Let $c_i \cdot \mathbf{z}^{\mathbf{k}_i}$ be the monomial with the coefficient c_i in f_i corresponding to $A_i \in \Delta_i$. Then $[f_1, \dots, f_{n+1}](T^n_A) = [c_1 \cdot \mathbf{z}^{\mathbf{k}_1}, \dots, c_{n+1} \cdot \mathbf{z}^{\mathbf{k}_{n+1}}]$ where T^n_A is the cycle corresponding to the vertex A.

3) Let z be a root of multiplicity $\mu(z)$ of the system $f_1 = \dots = \hat{f_i} = \dots = f_{n+1} = 0$, where the equation $f_i = 0$ has been removed. Let γ_z be the corresponding Grothendieck cycle. Assume that $f_i(z) \neq 0$. Then $[f_1, f_2, \dots, f_{n+1}](\gamma_z) = f_i(z)^{(-1)^i \mu(z)}$.

5.5 Product over roots of a system of equations

Let $\Delta = (\Delta_1, \dots, \Delta_{n+1})$ be $(n+1)$ Newton polyhedra in the lattice $\mathbb{Z}^n$ and let Ω_Δ be the space of $(n+1)$-tuples of Laurent polynomials $(f_1, \dots f_{n+1})$ such that the Newton polyhedron of f_i is contained in Δ_i. We will define a rational function $\Pi^{[i]}_\Delta$ on the space Ω_Δ which we will call *the product of f_i over the common zeros of f_j for $j \neq i$*. Let $U^i_\Delta \subset \Omega_\Delta$ be the Zariski open set defined by the following condition: $(f_1, \dots, f_{n+1}) \in U^i_\Delta$ if and only if the set $Y^i_\Delta \subset (\mathbb{C}^*)^n$ of common zeros of f_j for $j \neq i$ is finite and the number of points in Y^i_Δ (counting with multiplicities) is equal to $n! Vol(\Delta_1, \dots, \hat{\Delta}_i, \dots, \Delta_{n+1})$ (the polyhedron Δ_i is omitted in this mixed volume).

Definition 15. We define the function $\Pi^{[i]}_\Delta$ on U^i_Δ as follows. If the set $Y^i_\Delta \subset (\mathbb{C}^*)^n$ is empty (i.e. if $n! Vol(\Delta_1, \dots, \hat{\Delta}_i, \dots, \Delta_{n+1}) = 0$) then $\Pi^{[i]}_\Delta \equiv 1$. Otherwise

$$\Pi^{[i]}_\Delta(f_1, \dots, f_{n+1}) = \prod_{x \in Y^i_\Delta} f_i^{m_x}(x),$$

where m_x is the multiplicity of the common zero $x \in Y^i_\Delta$ of the Laurent polynomials f_j for $j \neq i$.

Lemma 5.2. *The function $\Pi^{[i]}_\Delta$ is regular on U^i_Δ. It can be extended to a rational function on Ω_Δ.*

Proof. Let $\widetilde{U}^i_\Delta \subset U^i_\Delta$ be the Zariski open set in which common zeros of f_j with $j \neq i$ have multiplicity one. The function $\Pi^{[i]}_\Delta$ is obviously regular on $\widetilde{U}^i_\Delta$. By the removable singularity theorem it is regular in U^i_Δ. By definition $\Pi^{[i]}_\Delta$ is an algebraic single-valued function on U^i_Δ. Thus, it is rational function on Ω_Δ. □

Note that even if $(f_1, \dots, f_{n+1}) \notin U^i_\Delta$ then the product of f_i over the common roots of f_j for $j \neq i$ is well defined if the set of common roots is finite. But this product is not necessarily equal to $\Pi^{[i]}_\Delta(f_1, \dots, f_{n+1})$.

Assume that the collection $\Delta = (\Delta_1, \dots, \Delta_{n+1})$ is i-developed. Consider the space Ω_Δ of $(n+1)$-tuples $(f_1, \dots, f_{n+1})$ of Laurent polynomials whose Newton polyhedra are contained correspondingly in $(\Delta_1, \dots, \Delta_{n+1})$. Denote by Ω^i_Δ the open subset in Ω_Δ defined by the condition that the Newton polyhedron of f_j is Δ_j for $j \neq i$.

Theorem 5.4. *The function $\Pi^{[i]}_\Delta$ is regular on Ω^i_Δ. Moreover, there exists a monomial M_i in vertex coefficients of all the f_j for $j \neq i$ such that the product $M_i\Pi^{[i]}_\Delta$ is a polynomial.*

Proof. Since the system $f_1 = \ldots = \hat{f}_i = \cdots = f_{n+1} = 0$ is developed, the number of roots counting with multiplicities is constant on Ω^i_Δ and $\Omega^i_\Delta \subset U^i_\Delta$. Therefore the function $\Pi^{[i]}_\Delta$ is regular on Ω^i_Δ. Thus there exists a monomial M_i in the vertex coefficients of f_j for $j \neq i$ such that the product $M_i\Pi^{[i]}_\Delta$ is a polynomial on Ω_Δ. □

In Section 5.9.4 we will present an algorithm for computing the function $\Pi^{[i]}_\Delta$ for i-developed systems.

5.6 Identity for i- and j- developed system.

Theorem 5.5. *Assume that $\Delta = (\Delta_1, \dots, \Delta_{n+1})$ is i-developed and j-developed for some $i \neq j$. Assume also that $(f_1, \dots, f_{n+1}) \in \Omega_\Delta$ satisfies the assumptions of Theorem 5.2. Then*

$$\Pi^{[i]}_\Delta(f_1, \dots, f_{n+1})(\Pi^{[j]}_\Delta(f_1, \dots, f_{n+1}))^{-1} = \prod [f_1, \dots, f_{n+1}]_A^{(-1)^{n+i+j} k_A^{i,j}},$$

where the product on the right is taken over the vertices A of $\sum \Delta_i$.

Proof. Let us apply the element $[f_1, \dots, f_{n+1}] \in H^n((\mathbb{C}^*)^n \setminus \Gamma, \mathbb{C}^*)$ to the identity from Theorem 5.2. According to the Section 5.4.3 we have

$$\Pi^{[i]}_\Delta(f_1, \dots, f_{n+1})^{(-1)^i(-1)^{j-2}} \Pi^{[j]}_\Delta(f_1, \dots, f_{n+1}))^{(-1)^j(-1)^{i-1}} =$$

$$= \prod [f_1, \dots, f_{n+1}]_A^{(-1)^n k_A^{i,j}}.$$

To complete the proof it is enough to raise each side of this identity to the power $(-1)^{i+j}$. □

Theorem 5.5 contains the formula from [18] for the product in $(\mathbb{C}^*)^n$ of all the roots of a developed system of n equations. To find such a product it is enough to compute the product over all roots of any monomial $\mathbf{z}^\mathbf{m}$: taking the coordinate functions $z_1, \ldots, z_n$ as such monomials one obtains all coordinates of the product of all roots. Assume that $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$ is, say, 1-developed and that $\Delta_1 = \{m\}$ is a single point. Consider an $(n+1)$ tuple of Laurent polynomials $f_1, \ldots, f_{n+1}$ with Newton polyhedra $\Delta_1, \ldots, \Delta_{n+1}$.

Then: 1) f_1 is the monomial $\mathbf{z}^\mathbf{m}$ with a nonzero coefficient c, i.e. $f_1 = c \cdot \mathbf{z}^\mathbf{m}$;
2) Δ is j-developed for any $1 < j \leq (n+1)$ because the collection Δ with Δ_j skipped contains the point Δ_1.

Let us apply Theorem 5.5 to the case under consideration with $i = 1$, $j = 2$. We have: a) $\Pi_\Delta^{[2]} = 1$ because the system $cx^m = f_3 = \cdots = f_{n+1} = 0$ has no roots in $(\mathbb{C}^*)^n$;
b) $\Pi_\Delta^{[1]}$ is equal to the product of $c \cdot \mathbf{z}^\mathbf{m}$ over all roots of the system $f_2 = \cdots = f_{n+1} = 0$, i.e. is equal to $c^{n! Vol(\Delta_2, \ldots, \Delta_{n+1})}$ multiplied by the product of $\mathbf{z}^\mathbf{m}$ over all roots of the system.

Corollary 5.1. *With the assumptions of Theorem 5.5 for $i = 1$, $j = 2$ and $f_1 = c \cdot \mathbf{z}^\mathbf{m}$ the product of $\mathbf{z}^\mathbf{m}$ over the roots of the system $f_2 = \cdots = f_{n+1} = 0$ multiplied by $c^{n! Vol(\Delta_2, \ldots, \Delta_{n+1})}$ is equal to $\prod [f_1, \ldots, f_{n+1}]_A^{(-1)^{n+1} k_A^{1,2}}$.*

Corollary 5.2. *If $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$ is i-developed and j-developed then on Ω_Δ the relation*

$$\Pi_\Delta^{[i]} / \Pi_\Delta^{[j]} = M_{i,j} s_{i,j} \tag{5.7}$$

holds, where $M_{i,j}$ is an explicit monomial in the vertex coefficients of all f_k and $s_{i,j} = \prod_A (-1)^{D(A_1, \ldots, A_{n+1}) k_A^{i,j}}$, where $A_1, \ldots, A_{n+1}$ are vertices of $\Delta_1, \ldots, \Delta_{n+1}$ such that $A_1 + \cdots + A_{n+1} = A$.

Proof. By definition $[f_1, \ldots, f_{n+1}]_A$ is an explicit monomial in coefficients of $f_1, \ldots, f_{n+1}$ corresponding to the vertices $A_1, \ldots, A_{n+1}$ multiplied by $(-1)^{D(A_1, \ldots, A_{n+1})}$. □

5.7 Identities for a completely developed system.

Definition 16. A collection $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$ of polyhedra is called *completely developed* if it is i-developed for all $1 \leq i \leq n+1$.

Theorem 5.6. *For a completely developed Δ there is a $(n+1)$-tuple $(M_1, \ldots, M_{n+1})$ where M_k is a monomial depending on the vertex coefficients of $(f_1, \ldots, f_{n+1}) \in \Omega_\Delta$ with f_k removed and a $(n+1)$-tuple*

$(s_1, \ldots, s_{n+1})$ *where* $s_i = \pm 1$*, such that:*

$$\Pi_{\Delta}^{[1]} M_1 s_1 = \ldots = \Pi_{\Delta}^{[n+1]} M_{n+1} s_{n+1}. \tag{5.8}$$

The $(n+1)$*-tuple* $(M_1, \ldots, M_{n+1})$ *of monials is unique and the* $(n+1)$*-tuple of signs* $(s_1, \ldots, s_{n+1})$ *is unique up to simultaneous multiplication by* -1*. Moreover the relation* $s_i s_j = s_{i,j}$ *holds.*

Proof. To prove existence we will use the identities (5.7) for $j > i = 1$. Let us represent each monomial $M_{1,j}$ as a product $\prod_{1 \le k \le n+1} m_{1,j}^{(k)}$ where $m_{1,j}^{(k)}$ is a monomial depending on the vertex coefficients of f_k only. Denote $m = \prod_{j \ne 1} m_{1,j}^{(j)}$ and divide each identity (5.7) for $j > i = 1$ by m. We obtain a needed representation with

$$(M_1, \ldots, M_{n+1}) = (m^{-1}, M_{1,2} m^{-1}, \ldots, M_{1,n+1} m^{-1}),$$

and

$$(s_1, \ldots, s_{k+1}) = (1, s_{1,2}, \ldots, s_{1,n+1}).$$

To show uniqueness assume that $(M'_1, \ldots, M'_{n+1})$ and $(s'_1, \ldots, s'_{n+1})$ are another pair of $(n+1)$-tuples of monomials and signs such that:

$$\Pi_{\Delta}^{[1]} M'_1 s'_1 = \ldots = \Pi_{\Delta}^{[n+1]} M'_{n+1} s'_{n+1}.$$

For any i the ratio M_i/M'_i is a monomial which does not depend on coefficients of f_i . But since $M_1 s_1 / M'_1 s'_1 = \ldots = M_{n+1} s_{n+1} / M'_{n+1} s'_{n+1}$, the ratio M_i/M'_i is equal to 1 and collections of signs are proportional. The relation $s_i s_j = s_{i,j}$ follows from (5.7). □

Remark 2. *Let* $G = (\mathbb{Z}/2\mathbb{Z})^{n+1}/D$ *be the factor group of* $(\mathbb{Z}/2\mathbb{Z})^{n+1}$ *by the diagonal subgroup* $D = \{(1, \ldots, 1), (-1, \ldots, -1)\}$*. Assigning to a collection of completely developed polyhedra* $\Delta_1, \ldots, \Delta_{n+1}$ *the collection of signs* $(s_1, \ldots, s_{n+1})$ *defined up to simultaneous multiplication by* -1 *gives a map to* G*. This map is a coordinatewise homomorphism with respect to Minkowski sum, for example the relation*

$$\phi(\Delta_1 + \Delta'_1, , \ldots, \Delta_{n+1}) = \phi(\Delta_1, \ldots, \Delta_{n+1}) \phi(\Delta'_1, \ldots, \Delta_{n+1}),$$

holds for any completely developed collections $(\Delta_1, \Delta_2, \ldots, \Delta_{n+1})$ *and* $(\Delta'_1, \Delta_2, \ldots, \Delta_{n+1})$*.*

The multihomomorphism ϕ *is closely related to the resultants. Take any collection of Laurent polynomials* $f = (f_1, \ldots, f_{n+1})$ *with completely developed collection of Newton polyhedra* $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$ *such that all vertex coefficients of* f_i*'s are equal to 1. Then the values of the* $n+1$ *product resultants on* *the collection f would coincide up to a sign, and so $\phi(\Delta) = (R^{[1]}_{\Pi\Delta}, \ldots, R^{[n+1]}_{\Pi\Delta})$ up to multiplication by common factor.*

In a contrast to the Δ-resultants, the map ϕ in general is not translation invariant (although, it is invariant under simultaneous translation of Δ_i's) and is not symmetric.

Theorems 5.5 and 5.6 allow to describe the monomials $M_1, \ldots, M_{n+1}$ in terms of Parshin symbols. Now we will describe these monomials in terms of Newton polyhedra. Let us introduce some notation.

For an i-developed collection $\Delta_1, \ldots, \Delta_{n+1}$ denote by $\widetilde{\Delta}_i$ the sum $\Delta_1 + \cdots + \hat{\Delta}_i + \cdots + \Delta_{n+1}$ where Δ_i is removed. For each facet $\Gamma \subset \widetilde{\Delta}_i$ denote by v_Γ an irreducible integral covector such that the inner product with v_Γ attains its maximum value on $\widetilde{\Delta}_i$ at the facet Γ. With v_Γ one associates the value $H_{\Delta_i}(v_\Gamma)$ of the support function of Δ_i on v_Γ, the faces $\Delta_j^{v_\Gamma}$ of Δ_j at which the inner product with v_Γ attains the maximal value. The facet Γ is *essential* if among the faces $\Delta_j^{v_\Gamma}$ with $j \neq i$ exactly one face $\Delta_{j(v_\Gamma)}^{v_\Gamma}$ is a vertex. With an essential facet Γ one associates a coefficient $a_{j(v_\Gamma)}$ of the Laurent polynomial $f_{j(v_\Gamma)}$ at the vertex $\Delta_{j(v_\Gamma)}^{v_\Gamma}$, and the integral mixed volume $V(v_\Gamma)$ of the collection of polyhedra $\{\Delta_j^{v_\Gamma}\}$ in which the polyhedra $\Delta_i^{v_\Gamma}$ and $\Delta_{j(v_\Gamma)}^{v_\Gamma}$ are removed.

Let $L(\Gamma)$ be a linear subspace parallel to the minimal affine subspace containing Γ. We define the integral volume on $L(\Gamma)$ as the translation invariant volume normalized by the following condition: for any $v_1, \ldots, v_{n-1}$ the generators of the lattice $L(\Gamma) \cap \mathbb{Z}^n$, the volume the parallelepiped with sides $v_1, \ldots, v_{n-1}$ is equal to 1.

Any polyhedron in the collection $\{\Delta_j^{v_\Gamma}\}$ in which the polyhedra $\Delta_i^{v_\Gamma}$ and $\Delta_{j(v_\Gamma)}^{v_\Gamma}$ are removed could be translated to $L(\Gamma)$. By $V(v_\Gamma)$ we mean the integral mixed volume of these translations (note that $V(v_\Gamma)$ could vanish for some Γ).

Theorem 5.7. *For the monomial M_i the following formula holds*

$$M_i = \prod a_{j(v_\Gamma)}^{(n-1)! H_{\Delta_i}(v_\Gamma) V(v_\Gamma)}$$

where the product is taken over all essential facets Γ of $\widetilde{\Delta}_i$.

Proof. Let us sketch a proof for M_1. In Theorem 5.6 we represented M_1 in the form $M_1 = (\prod_{j\neq 1} m_{1,j}^{(j)})^{-1}$. One can deal with each factor $m_{1,j}^{(j)}$ separately. We will show that $m_{1,2}^{(2)} = \prod_\Gamma a_{2(v_\Gamma)}^{d(v_\Gamma)}$, where $a_{2(v_\Gamma)}$ is the coefficient of f_2 at the vertex $\Delta_2^{v_\Gamma}$, $d(v_\Gamma) = (n-1)! V(v_\Gamma) H_{\Delta_1}(v_\Gamma)$, and the product is taken over all facets Γ of $\Delta_{1,2} = \Delta_3 + \cdots + \Delta_{n+1}$.

Let $C \subset (\mathbb{C}^*)^n$ be the curve defined by the system $f_3 = \ldots = f_{n+1} = 0$ (we assume that this system is generic enough). The normalization $\widetilde{C}$ of C has a very explicit description: it can be obtained as the closure of C in the toric comactification X of $(\mathbb{C}^*)^n$ associated with the polyhedron Δ_{12}. In particular, each facet Γ of Δ_{12} corresponds to a codimension 1 orbit X_Γ in X. The equality $\widetilde{C} \setminus C = \bigcup_\Gamma (\widetilde{C} \cap X_\Gamma)$ holds. Moreover the number of points in $\widetilde{C} \cap X_\Gamma$ is equal to $(n-1)!V(v_\Gamma)$, (see [21] for details).

By (5.7) we have $M_{1,2} = \pm \Pi^{[1]}_\Delta / \Pi^{[2]}_\Delta$. By definition $M_{1,2} = m^{(1)}_{1,2} m^{(2)}_{1,2} F$, where $F = \prod_{k>2} m^{(k)}_{1,2}$ is independent of f_1, f_2. On the other hand $\Pi^{[1]}_\Delta / \Pi^{[2]}_\Delta$ is equal to the product of $\{f_1, f_2\}^{-1}_p$ over all zeros p of $f_1 f_2$ on the curve C. By Weil's theorem this product is equal to $\prod_{q \in (\widetilde{C} \setminus C)} \{f_1, f_2\}_q$.

Explicit calculations show that for any $g \in \widetilde{C} \cap X_\Gamma$ the following identity holds: $\{f_1, f_2\}_p = a^{H_{\Delta_2}(v_\Gamma)}_{1(v_\Gamma)} a^{-H_{\Delta_1}(v_\Gamma)}_{2(v_\Gamma)} G$, where G is independent of $f_1, f_2)$. The number of points in $\widetilde{C} \cap X_F$ is equal to $(n-1)!Vol(\Delta^v_3, \ldots, \Delta^v_{n+1})$. Putting everything together we get the needed identity $m^{(2)}_{1,2} = \prod_\Gamma a^{d(v_\Gamma)}_{2(v_\Gamma)}$. □

Definition 17. For completely developed $(n+1)$-tuple Δ and $1 \leq i \leq n+1$ we define the *i-th product resultant* $R^{[i]}_{\Pi\Delta}$ on Ω_Δ as $R^{[i]}_{\Pi\Delta} = \Pi^{[i]}_\Delta M_i$. By (5.8) all the product resultants $R^{[i]}_{\Pi\Delta}$ are equal up to sign.

Theorem 5.8. *Let $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$ be a completely developed collection. Then:*

1) each product resultant $R^{[i]}_{\Pi\Delta}$ is a polynomial on Ω_Δ. The degree of $R^{[i]}_{\Pi\Delta}$ in the coefficients of f_j is equal to the number of roots of the generic system $f_1 = \ldots = f_{n+1} = 0$ with f_j skipped (i.e is equal to $n!\,Vol(\Delta_1, \ldots, \hat{\Delta}_j, \ldots, \Delta_{n+1})$).

2) the function $R^{[i]}_{\Pi\Delta}$ is equal to zero at $(f_1, \ldots, f_{n+1}) \in \Omega^{[i]}_\Delta$ if and only if the system $f_1 = \ldots = f_{n+1} = 0$ has a root in $(\mathbb{C}^*)^n$.

Proof. The expression $M_i \Pi^{[i]}_\Delta = R^{[i]}_{\Pi\Delta}$ is obviously a polynomial of degree $n!\,Vol(\Delta_1, \ldots, \hat{\Delta}_j, \ldots, \Delta_{n+1})$ in the coefficients of f_i. Since all product resultants are equal up to sign we have proven 1).

Statement 2) is obvious from the definitions. □

Let $m \in \Delta_i \cap \mathbb{Z}^n$. Denote by c the coefficient in front of z^m in the Laurent polynomial f_i with Newton polyhedron Δ_i.

Theorem 5.9. *In the notations from Theorem 15, the degree of $R^{[j]}_{\Pi\Delta}$ in a specific coefficient c of f_i is equal to $n!\,Vol(\Delta_1, \ldots, \hat{\Delta}_i, \ldots, \Delta_{n+1})$. Each polynomial $R^{[j]}_{\Pi\Delta}$ contains exactly one monomial of the highest degree in c and the coefficient in front of this monomial is ± 1.*

Proof. Without loss of generality we can assume that $i = 1$. Since all product resultants are equal up to sign it is enough to prove the statement for $R^{[1]}_{\Pi\Delta} = M_1 \Pi^{[1]}_\Delta$. The monomial M_1 is independent of the coefficients of f_1 and monomial of the highest degree in c comes from multiplying the monomial cz^m over roots of $f_2 = \cdots = f_{n+1} = 0$. Now the theorem follows from Corollary 5.1. □

5.8 Sums of Grothendieck residues over roots of developed system.

In this section we discuss a formula from [11], [12] for the sum of Grothendieck residues over the roots of a developed system. As a corollary we provide an algorithm for computing the product of values of a Laurent polynomial over the roots of a developed system.

5.8.1 Grothendieck residue.

Consider the system $f_1 = \ldots = f_n = 0$ in $(\mathbb{C}^*)^n$ and the hypersurface Γ defined by $f_1 \cdot \ldots \cdot f_n = 0$. Let ω be a holomorphic n-form on $(\mathbb{C}^*)^n \setminus \Gamma$.

Definition 18. The Grothendieck residue of ω at the root z of the system $f_1 = \ldots = f_n = 0$ is defined as the number $\frac{1}{(2\pi i)^n} \int_{\gamma_z} \omega$, where γ_z is the Grothendieck cycle at z.

As ω is automatically closed, the Grothendieck residue at the root z is well defined.

5.8.2 The residue of the form at a vertex of a polyhedron.

For each vertex A of the Newton Polyhedron $\Delta(P)$ of a Laurent polynomial P, we will construct the Laurent series of the function f/P, for any Laurent polynomial f.

Let $q_A \neq 0$ be the coefficient of the monomial in P which corresponds to the vertex A of $\Delta(P)$. The constant term of the Laurent polynomial $\widetilde{P} = P/(q_A z^a)$ equals one. We will define the Laurent series of $1/\widetilde{P}$ by the formula:

$$1/\widetilde{P} = 1 + (1 - \widetilde{P}) + (1 - \widetilde{P})^2 + \ldots.$$

Since each monomial z^b appears only in finitely many summands $(1 - \widetilde{P})^k$, the above sum is well defined. The *Laurent series of the rational function f/P at the vertex A of* $\Delta(P)$ is the product of the series $1/\widetilde{P}$ and the Laurent polynomial $q_A z^a f$.

Consider the n-form $\omega_f = f dz_1 \wedge \ldots \wedge dz_n / P z_1 \cdot \ldots \cdot z_n$.

Definition 19. The Grothendieck residue $res_A \omega$ of $\omega = \omega_f$ at the vertex A of $\Delta(P)$ is defined as the number $\frac{1}{(2\pi i)^n} \int_{T_A} \omega_f$, where T_A is the cycle assigned to a vertex A (see sec.7.2).

Lemma 5.3. *The residue $res_A \omega_f$ is equal to the coefficient in from of the monomial $(z_1 \cdot \ldots \cdot z_n)^{-1}$ in Laurent series of f/P at the vertex A.*

We will not prove this simple lemma. See [12] for the details.

5.8.3 Summation formula.

Consider the developed system of equations $f_1 = \ldots = f_n = 0$ in $(\mathbb{C}^*)^n$ with Newton polyhedra $\Delta_1, \ldots, \Delta_n$. Denote by P the product $f_1 \cdot \ldots \cdot f_n$.

Theorem 5.10. *For any Laurent polynomial f the sum of the Grothendieck residues of the form $\omega_f = f dz_1 \wedge \ldots \wedge dz_n / P z_1 \cdot \ldots \cdot z_n$ over all the roots of the system is equal to $(-1)^n \sum k_A res_A \omega_f$, where the summation is taken over all vertices A of $\Delta_1 + \ldots + \Delta_n$.*

Proof. Theorem 5.10 follows from Theorem 5.1 (see Sections 5.8.1, 5.8.2). □

Corollary 5.3. *The sum $\sum f(z)\mu(z)$ of the values of any Laurent polynomial f over all roots z of a developed system, counted with multiplicities $\mu(z)$, is equal to $(-1)^n \sum k_A res_A \omega_\varphi$ where $\varphi = f \det M$ and M is $(n \times n)$-matrix with the entries $M_{i,j} = \partial f_i / \partial z_j$.*

Proof. The Grothendieck residue of $\omega = f df_1 \wedge \ldots \wedge df_n / f_1 \cdot \ldots \cdot f_n$ at the root z is equal to $f(z)\mu(z)$. It is easy to see that $\omega = \omega_\varphi$. So Corollary 5.3 follows from Theorem 5.10. □

5.8.4 Elimination theory.

Here we use notations from the previous section. Let f be a Laurent polynomial. We will explain how to find any symmetric function of the sequence of the numbers $\{f(z)\}$ for all roots z of the system (each root z is taken with multiplicity $\mu(z)$)

Denote by $f^{[k]}$ the number $f^{[k]} = \sum_z f^k(z)\mu(z)$. By Corollary 5.3 one can calculate $f^{[k]}$ for any k explicitly. The power sum symmetric polynomials form a generating set for the ring of symmetric polynomials.

Corollary 5.4. *One can find explicitly all symmetric functions of $\{f(z)\}$ and construct a monic polynomial whose roots are $\{f(z)\}$. In particular one can compute $\Pi_\Delta^{[i]} = \prod_z f(z)^{\mu(z)}$ for any i-developed system.*

5.9 Δ-Resultants.

5.9.1 Definition and some properties of Δ-resultant.

Following [10] we define the Δ-resultant for a collection $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$ of $(n+1)$ Newton polyhedra in $\mathbb{R}^n$. We also generalize this definition to a collection Δ in $\mathbb{R}^N$ with $N \geq n$ such that the polyhedron $\Delta_1 + \cdots + \Delta_{n+1}$ has dimension $\leq n$.

For $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$ with $\Delta_i \subset \mathbb{R}^n$ denote by $X_\Delta \subset \Omega_\Delta$ the quasi-projective set of points $(f_1, \ldots, f_{n+1}) \in \Omega_\Delta$ such the system $f_1 = \cdots = f_{n+1} = 0$ in $(\mathbb{C}^*)^n$ is consistent. The set X_Δ is irreducible, an easy proof of this fact can be found in [10].

Definition 20. The Δ-*resultant* Res_Δ is a polynomial on Ω_Δ satisfying the following conditions:

(i) The degree of Res_Δ in the coefficients of f_i is equal to the number of roots of the generic system $f_1 = \ldots = f_{n+1} = 0$ with f_i skipped (i.e is equal to $n!\,Vol(\Delta_1, \ldots, \hat{\Delta}_i, \ldots, \Delta_{n+1})$.) The coefficients of Res_Δ are coprime integers.

(ii) If the codimension of X_Δ in Ω_Δ is greater then 1 then $Res_\Delta \equiv \pm 1$.

(iii) $Res_\Delta(f_1, \ldots, f_{n+1}) = 0$ if and only if $(f_1, \ldots, f_{n+1})$ belongs to the closure $\overline{X}_\Delta$ of X_Δ in Ω_Δ.

Theorem 5.11 ([10])**.** [1] *There exists a unique up to sign polynomial Res_Δ on Ω_Δ satisfying the conditions $(i) - (iii)$.*

The Δ-resultant obviously has the following properties: 1) it is independent of the ordering of the polyhedra from the set $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$; 2) it is invariant under translations of the polyhedra from the set Δ; 3) it is invariant under linear transformations of $\mathbb{R}^n$ inducing an automorphism of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

Example 4. *Suppose $\Delta_1 = \{m\}$ is a one point set, i.e. $f_1 = cz^m$ is a monomial z^m with some coefficient c. Then $Res_\Delta = \pm c^{n!\,Vol(\Delta_2, \ldots, \Delta_{n+1})}$.*

Assume that $\Delta = (\Delta_1, \ldots, \Delta_{n+1})$ with $\Delta_i \subset \mathbb{R}^N$ satisfying inequality $\dim(\Delta_1 + \cdots + \Delta_{n+1}) \leq n$. Choose any linear isomorphism $A : \mathbb{R}^N \to \mathbb{R}^N$ preserving the lattice $\mathbb{Z}^N$ and choose vectors $v_1, \ldots, v_{n+1} \in \mathbb{Z}^N$ such that the polyhedra $\Delta'_i = A(\Delta_i) + v_i$ belong to the n dimensional coordinate subspace $\mathbb{R}^n \subset \mathbb{R}^N$.

Definition 21. The generalized Δ-resultant for Δ as above is defined as the Δ'-resultant for $\Delta' = (\Delta'_1, \ldots, \Delta'_{n+1})$. The generalized Δ-resultant is well defined, i.e. is independent of the choice of A and $v_1, \ldots, v_{n+1}$.

5.9.2 Product resultants and Δ-resultant

In this section we show that for a completely developed collection Δ all product resultants are equal up to sign to the Δ-resultant.

[1] In [10] the A-resultant is defined, the connection between A-resultants and Δ-resultants is described in introduction.

In [10] A-resultant is defined under some assumption on $(n+1)$-tuple of supports A. The general definition of the A-resultant is given in [8].

The condition (i) on the degrees of resultant could be replaced by the condition that the resultant is a polynomial which vanishes on X_Δ with multiplicity equal to the generic number of solutions of consistent system (see [4]).

The Sylvester formula represents the Δ-resultants of two polynomials in one variable as a determinant of one of two explicitly written matrices (see Section 4.1). In [3] the Sylvester formula was beautifully generalized in the following way. For the collection of $n+1$ Newton polyhedra Δ, Canny and Emiris construct $n+1$ matrices M_i (which coincide with Sylvester's matrix up to permutation of rows in dimension 1) such that the Δ-resultant Res_Δ divides their determinants. Moreover, Res_Δ is the greatest common divisor of polynomials $det(M_i)$, thus they obtain a practical algorithm for computing Res_Δ. The construction in [3] heavily uses geometry of Newton polyhedra.

By the extreme monomials of a polynomial P we will mean the monomials corresponding to the vertices of the Newton polyhedron of P. In [36] Sturmfels generalized the construction in [3] and using this generalization proved the following theorem.

Theorem 5.12. *All extreme monomials of the Δ-resultant have coefficient -1 or $+1$.*

Now we are able to prove the following theorem.

Theorem 5.13. *For a completely developed Δ for any $1 \le i \le n+1$ the product resultant $R^{[i]}_{\Pi\Delta}$ is equal up to sign to the Δ-resultant Res_Δ.*

Proof. Without the loss of generality we can assume that $i = 1$. Both functions Res_Δ and $Res^{[1]}_{\Pi,\Delta}$ are polynomials in the coefficients of $f_1, \dots, f_{n+1}$ of the same degrees and they both vanish on the set $\overline{X}_\Delta$ (see Theorem 15). Since the set $\overline{X}_\Delta$ is irreducible polynomials Res_Δ and $Res^{[1]}_{\Pi,\Delta}$ are proportional. According to Corollary 5.1 for any chosen coefficient c of f_1, in the polynomial $Res^{[1]}_{\Pi,\Delta}$ there is a unique monomial having the highest degree in c and the coefficient in $Res^{[1]}_{\Pi,\Delta}$ in front of this monomial is ± 1. But according to Theorem 5.12 the coefficient in Res_Δ in front of this monomial is also ± 1. So $Res_\Delta = \pm Res^{[1]}_{\Pi,\Delta}$. □

5.9.3 The Poisson formula.

The inductive Poisson formula for Δ-resultant (see [33], [4]) and the summation formula (see Section 5.8.3) allow to provide an algorithm computing the Δ-resultant for 1-developed systems. To state the formula let us define all terms appearing in it. Let Δ be $(\Delta_1, \dots, \Delta_{n+1})$ and let $(f_1, \dots, f_{n+1})$ be a point in Ω_Δ. The only term in the formula depending on the coefficients of f_1 is the term $\Pi^{[1]}_\Delta$. To present the other terms we need some notation.

Denote by $\widetilde{\Delta}_1$ the sum $\Delta_2 + \dots + \Delta_{n+1}$. For each facet Γ of $\widetilde{\Delta}_1$ denote by v_Γ the irreducible integral covector such that the inner product of $x \in \widetilde{\Delta}_1$ with v_Γ attains its maximum value on Γ. With v_Γ one associates the value $H_{\Delta_1}(v_\Gamma)$ of the support function of Δ_1 on v_Γ, the faces $\Delta_j^{v_\Gamma}$ of Δ_j on which the inner product of $x \in \Delta_j$ with v_Γ attains its maximal value. By $f_j^{v_\Gamma}$ we denote the sum $\sum c_m x^m$ over $m \in \Delta_j^{v_\Gamma} \cap \mathbb{Z}^n$, where c_m is the coefficient in front of x^m in f_j. For each v_Γ the collection of n polyhedra $\widetilde{\Delta}_1^{v_\Gamma} = (\Delta_2^{v_\Gamma}, \dots, \Delta_{n+1}^{v_\Gamma})$ satisfies the inequality $\dim(\Delta_2^{v_\Gamma} + \cdots + \Delta_{n+1}^{v_\Gamma}) \leq (n-1)$. This is why the generalized $\widetilde{\Delta}_1^{v_\Gamma}$ resultant $Res_{\widetilde{\Delta}_1^{v_\Gamma}}(f_2^{v_\Gamma}, \dots, f_{n+1}^{v_\Gamma})$ is defined. Now we are ready to state the Poisson formula.

Theorem 5.14. *The following Poisson formula holds:*

$$Res_\Delta(f_1, \dots, f_{n+1}) = \pm \Pi_\Delta^{[1]}(f_1, \dots, f_{n+1}) \prod Res_{\widetilde{\Delta}_1^{v_\Gamma}}^{H_1(v_\Gamma)}(f_2^{v_\Gamma}, \dots, f_{n+1}^{v_\Gamma}),$$

where the product is taken over all facets Γ of $\widetilde{\Delta}_1$.

In a subsequent paper we are going to give an elementary proof of Theorem 21 (in fact, we will generalize the Poisson formula from the toric case to a larger class of algebraic varieties).

Remark 3. *Mixed volume and Δ-resultants have many similar properties. For example the non symmetric formula for the mixed volume*

$$Vol(\Delta_1, \dots, \Delta_n) = \frac{1}{n} \sum_v H_1(v) Vol(\Delta_2^v, \dots, \Delta_n^v)$$

is analogues to the Poisson formula for the Δ-resultants.

For a 1-developed collection $\Delta = (\Delta_1, \dots, \Delta_{n+1})$ of Newton polyhedra the Poisson formula becomes much simpler: in this case the resultants $Res_{\widetilde{\Delta}_1^{v_\Gamma}}(f_1^{v_\Gamma}, \dots, f_{n+1}^{v_\Gamma})$ can be computed explicitly. Below we present such computation.

By definition of Δ being 1-developed collection, for any facet Γ of $\widetilde{\Delta}_1$ in the set $\{\Delta_j^{v_\Gamma}\}$ with $j > 1$ at least one polyhedron $\Delta_{j(v_\Gamma)}^{v_\Gamma}$ is a point (if more then one polyhedron is a point denote by $\Delta_{j(v_\Gamma)}^{v_\Gamma}$ any of them). Denote by $Vol(v_\Gamma)$ the integral $(n-1)$ dimensional mixed volume of collection $\{\Delta_j^{v_\Gamma}\}$ with $j > 1$ in which the polyhedron $\Delta_{j(v_\Gamma)}^{v_\Gamma}$ is skipped. Denote by $a_{j(v_\Gamma)}$ the coefficient of the Laurent polynomial $f_{j(v_\Gamma)}$ at the vertex $\Delta_{j(v_\Gamma)}^{v_\Gamma}$. In the above notation Example 4 provides us the formula:

$$Res_{\widetilde{\Delta}_1}^{v_\Gamma}(f_2^{v_\Gamma}, \dots, f_{n+1}^{v_\Gamma}) = \pm a_{j(v_\Gamma)}^{(n-1)!\, Vol(v_\Gamma)}.$$

Corollary 5.5. *With the notation as above the Δ-resultant of the 1-developed collection Δ is given by:*

$$Res_\Delta(f_1, \dots, f_{n+1}) = \pm \Pi_\Delta^{[1]}(f_1, \dots, f_{n+1}) \prod_\Gamma a_{j(v_\Gamma)}^{(n-1)!\, Vol(v_\Gamma) H_1(v_\Gamma)}.$$

Corollary 5.6. *Using Corollary 5.4 one can produce an algorithm for computing the* Δ*-resultant of a 1-developed collection* Δ*.*

Indeed, the only implicit term in the formula from the Corollary 5.5 is the term $\Pi_\Delta^{[1]}$. Corollary 5.4 provides an algorithm for its computation.

5.9.4 A sign version of Poisson formula

Let Δ be a developed collection. According to Theorem 5.6 with such Δ, an $(n+1)$-tuple of monomials $M_1, \dots, M_{n+1}$ and an $(n+1)$-tuple of signs are defined. According to the formula from Corollary 5.5 the Poisson formula in that case can be written as $Res_\Delta = \pm\Pi_\Delta^{[1]} M_1$. By definition Δ is not only 1-developed, it is i-developed for any i. Thus one can write the formula from Corollary 5.5 putting instead of f_1 any f_i.

Corollary 5.7. *The following equalities hold:*

$$\pm\Pi_\Delta^{[1]} M_1 = \cdots = \pm\Pi_\Delta^{[n+1]} M_{n+1} = \pm Res_\Delta.$$

Thus from the theory of Δ-resultants one can prove the product identities from Theorem 5.6 up to sign. It is impossible to reconstruct the signs in these identities using Δ-resultants: the Δ-resultant itself is defined up to sign only. Our Theorem 5.6 provided the sign version

$$\Pi_\Delta^{[1]} M_1 s_1 = \cdots = \Pi_\Delta^{[n+1]} M_{n+1} s_{n+1}$$

of Poisson type identities and Theorem 5.13 provides identities $\Pi_\Delta^{[i]} M_i = \pm Res_\Delta$ (which unavoidable could be up to sign only).

Appendices

Appendix A

Newton polyhedra theory and computation of discrete invariants

In this appendix we will recap the state of art for Newton Polyhedra theory and will give the versions of these results for overdetermined collections.

A.1 Classical Newton polyhedra theory

Consider a system of k Laurent polynomials $f_1 = \cdots = f_k = 0$ with Newton polyhedra $\Delta_1, \dots, \Delta_k$ in $(\mathbb{C}^*)^n$. Assume that the system is non-degenerate, which is a generic condition in the space of Laurent polynomials with fixed Newton polyhedra. Many discrete invariants of the set of solutions are identical and are expressed in terms of the polyhedra.

A.1.1 Euler characteristics of complete intersections

The following theorem computes Euler characteristic of a complete intersection.

Theorem A.1. *The Euler characteristic of the nondegenerate complete intersection* $f_1 = \ldots = f_k = 0$ *in* $(\mathbb{C}^*)^n$, $(k \leq n)$, *with the Newton polyhedra* $\Delta_1, \dots, \Delta_k$ *is equal to*

$$(-1)^{n-k} n! \sum Vol(\Delta_1, \dots, \Delta_k, \Delta_{i_1}, \dots, \Delta_{i_{n-k}}),$$

where the sum is taken over all sets $1 \leq i_1 \leq \cdots \leq i_{n-k} \leq k$.

Theorem A.1 has especially nice form for a hypersurface given by a generic Laurent polynomial with Newton Polyhedron Δ.

Corollary A.1. *The Euler characteristic of a hypersurface in $(\mathbb{C}^*)^n$ defined by non degenerate equation with Newton polyhedron Δ is equal to $(-1)^{n-1}n!Vol(\Delta)$.*

A.1.2 Number of connected components of complete intersections

Let J be the a biggest with respect to inclusion subcollection of polyhedra $\Delta_1, \ldots, \Delta_k$ subcollections with zero defect. Such subset exists and is unique (see for example Corollary B.1 or [23]). Polyhedra Δ_{i_j}, for $i_j \in J$ can be shifted by parallel translation into the space L_J Thus the mixed volume $Vol(\Delta_{i_1}, \ldots, \Delta_{i_s})$ is well defined.

Theorem A.2. *The number of connected components b_0 of a generic complete intersection defined by a system $f_1 = \ldots = f_k = 0$ of Laurent polynomials is given by*

$$b_0 = s!Vol(\Delta_{i_1}, \ldots, \Delta_{i_s}),$$

here if $J = \varnothing$ we say that $Vol(\Delta_{i_1}, \ldots, \Delta_{i_s}) = 1$.

A.1.3 Genus of complete intersections

The zero set of a non-degenerate system $f_1 = \cdots = f_k = 0$, of Laurent polynomials, is a smooth quasi-projective algebraic variety, but not necessarily complete. The arithmetic genus of a smooth complete variety Y over $\mathbb{C}$ is the alternating sum $\sum(-1)^p h^{p,0}(\mathrm{Y})$ of the numbers $h^{p,0}(\mathrm{Y})$.

It turns out that the arithmetic genus is a birational invariant of an algebraic variety, so let us define the arithmetic genus of the zero Z set of a non-degenerate system $f_1 = \cdots = f_k = 0$ as the arithmetic genus of any birational comactification of Z.

Let us define the number $B(\Delta)$ as the number of integer points lying in the interior of the polyhedron Δ (in the topology of the affine span of Δ), multiplied by $(-1)^{\dim(\Delta)}$.

Theorem A.3. *The arithmetic genus of the nondegenerate complete intersection $f_1 = \ldots = f_k = 0$ in $(\mathbb{C}^*)^n$, $(k \leq n)$, with the Newton polyhedra $\Delta_1, \ldots, \Delta_k$ is equal to*

$$1 - \sum B(\Delta_i) + \sum_{j>i} B(\Delta_i + \Delta_j) - \cdots + (-1)^k B(\Delta_1 + \cdots + \Delta_k).$$

A.2 Newton polyhedra theory for overdetermined systems

Theorem 2.7 asserts that any discrete invariant which can be computed by means of the theory of Newton polyhedra could be also computed for the zero set $Y_{\mathbf{f}}$ of generic consistent system with overdetermined supports. In this section we give several explicit examples of such calculations.

Through this section let $A_1, \dots A_k \subset \mathbb{Z}^n$ be a collection of finite sets with the essential subcollection $J \subset \{1, \dots, k\}$. As before let

- Δ_i be the convex hull of A_i;
- L_J be the vector subspace of $\mathbb{R}^n$ parallel to the affine span of $\sum_{i \in J} A_i$;
- $\pi_J : \mathbb{R}^n \to \mathbb{R}^n / L_J$ be the natural projection;
- $ind(J)$ be the index of the lattice generated by the differences from A_i's with $i \in J$ in the lattice $L_J \cap \mathbb{Z}^n$.

Finally, by the volume on a vector space V with a lattice Λ inside we mean the translation invariant volume normalized by the following condition: for any $v_1, \dots, v_k$ which are generators of the lattice Λ, the volume of the parallelepiped with sides $v_1, \dots, v_k$ is equal to 1.

The proofs of all the theorems of this section are absolutely analogous, so we are omitting all of them but the proof of Theorem A.4.

A.2.1 Number of roots

The following theorem is a generalization of BKK theorem for overdetermined systems of Laurent Polynomials.

Theorem A.4. *Let $A_1, \dots, A_{n+k} \subset \mathbb{Z}^n$ be such that $d(A) = -k$ and J be the unique essential subcollection. Then the zero set $Y_{\mathbf{f}}$ of the generic consistent system has dimension 0, and the number of points in $Y_{\mathbf{f}}$ is equal to*

$$(n - \#J + k)! \cdot ind(J) \cdot Vol(\pi_J(\Delta_i)_{i \notin J}),$$

where Δ_i is the convex hull of A_i and Vol is the mixed volume on $\mathbb{R}^n / L(J)$ normalized with respect to the lattice $\mathbb{Z}^n / \Lambda(J)$.

If $k = 0$ this theorem coincides with the BKK theorem. In the case $k = 1$ the generic number of solution appears as the corresponding degree of A-resultant and was computed in [4].

Proof. First note that for generic $\mathbf{f} \in X_A$ the dimension $\dim(Y_{\mathbf{f}})$ is equal to $\dim(\widetilde{X}_A) - \dim(X_A) = 0$. By Theorem 2.7 the generic zero set $Y_{\mathbf{f}}$ is a disjoint union of $ind(J)$ varieties $Y_1, \ldots, Y_{ind(J)}$ each of which is defined by generic system with Newton polyhedra $\pi_J(\Delta_i)$ for $i \notin J$. By the BKK formula the number of points in Y_i is finite and is equal to $(n - \#J + k)! Vol(\pi_J(\Delta_i)_{i \notin J}))$. Therefore, the number of points in $Y_{\mathbf{f}}$ is

$$|Y_{\mathbf{f}}| = \sum_{i=1}^{ind(J)} |Y_i| = (n - \#J + k)! \cdot ind(J) \cdot Vol(\pi_J(\Delta_i)_{i \notin J}).$$

□

A.2.2 Euler characteristics of complete intersections

The following theorem computes Euler characteristic of a generic non empty zero set of a system of Laurent polynomials $f_1 = \ldots = f_k = 0$ with supports of f_i in A_i.

Theorem A.5. *The Euler characteristic of the generic nonempty set $f_1 = \ldots = f_k = 0$ in $(\mathbb{C}^*)^n$, with supports in $A_1, \ldots, A_k$ is equal to*

$$ind(J)(-1)^{n-k-\mathrm{def}(J)} \dim(\mathbb{R}^n/L_J)! \sum Vol\left(\pi_J(\Delta_1), \ldots, \pi_J(\Delta_k), \pi_J(\Delta_{i_1}), \ldots, \pi_J(\Delta_{i_{n-k-\mathrm{def}(J)}})\right),$$

where the sum is taken over all sets $i_1 \leq \cdots \leq i_{n-k-\mathrm{def}(J)} \notin J$.

A.2.3 Number of connected components of complete intersections

Let as before $J^c = \{1, \ldots, k\} \setminus J$ be the compliment of the essential subcollection J. Let furthermore, $K \subset J^c$ be the maximal by inclusion subcollection of zero defect in the collection $\pi_J(J^c)$.

Theorem A.6. *The number of connected components b_0 $f_1 = \ldots = f_k = 0$ in $(\mathbb{C}^*)^n$, with supports in $A_1, \ldots, A_k$ is given by*

$$b_0 = ind(J) \cdot |K|! \cdot Vol(\pi_J(\Delta_i))_{i \in K},$$

here if $K = \varnothing$ we say that $Vol(\pi_J(\Delta_i))_{i \in K} = 1$.

A.2.4 Genus of complete intersections

By Corollary 2.7 the zero set of a generic solvable system $f_1 = \cdots = f_k = 0$ is a smooth quasi-projective algebraic variety. As in Section A.1.3 The following theorem computes arithmetic genus of a generic non-empty set.

Theorem A.7. *The arithmetic genus of the generic nonempty set* $f_1 = \ldots = f_k = 0$ *in* $(\mathbb{C}^*)^n$, $(k \leq n)$, *with supports in* $A_1, \ldots, A_k$ *is equal to*

$$ind(J) \cdot \left(1 - \sum B\big(\pi_J(\Delta_i)\big) + \sum_{i,j \notin J\, j>i} B\big(\pi_J(\Delta_i + \Delta_j)\big) - \cdots + (-1)^k B\big(\pi_J(\Delta_{J^c})\big)\right).$$

Appendix B

Matroid structure coming from defects

In this apendix we study matroids coming from collection of vector subspaces of a vector space. Matroid is a combinatorial object which is created to mimic the independence relation of a collection of vectors in a vector space. For an introduction to matroid theory see [31].

Definition. A finite set M together with a collection of subsets $\mathcal{I} \subset 2^M$ is called a matroid if

i) $\varnothing \in \mathcal{I}$;

ii) for any $B \in \mathcal{I}$ and $A \subset B$ we have $A \in \mathcal{I}$;

iii) for any $A, B \in \mathcal{I}$ with $|A| > |B|$ there exists $x \in A \setminus B$ so that $B \cup \{x\} \in \mathcal{I}$.

Example 5. *A collection of vectors $v_1, \ldots, v_k \in V$ defines a matroid structure on $\{1, \ldots, k\}$ by the rule that $\{i_1, \ldots, i_s\} \in \mathcal{I}$ iff vectors $v_{i_1}, \ldots, v_{i_s}$ are independent.*

Let us call a collection of vector subspaces $L_1, \ldots, L_k \subset V$ independent if $\mathrm{d}(L_1, \ldots, L_k) = 0$. The main result of this subsection is the following theorem which generalizes Example 5.

Theorem B.1. *A collection of vector subspaces $L_1, \ldots, L_k \subset V$ defines a matroid structure on $\{1, \ldots, k\}$ by the rule that $\{i_1, \ldots, i_s\} \in \mathcal{I}$ iff subspaces $L_{i_1}, \ldots, L_{i_s}$ are independent.*

For the proof of Theorem B.1 we will need the following lemma.

Lemma B.1. *Let $L_1, \ldots, L_k$ be a collection of independent vector subspaces. Then*

$$\dim \left(\sum_{J,\, def(J)=0} L_J \right) = \left| \bigcup_{J,\, def(J)=0} J \right|,$$

where both the summation and the union are taken over all subcollections of $L_1, \ldots, L_k$*. In other words,*

$$\operatorname{def}\left(\bigcup_{J,\, def(J)=0} J\right) = 0.$$

Proof. First notice, that if J_1, J_2 are two subcollection with $\operatorname{def}(J_1) = \operatorname{def}(J_2) = 0$, then $\operatorname{def}(J_1 \cup J_2) = 0$. Indeed, since $L_1, \ldots, L_k$ are independent, $\operatorname{def}(J_1 \cup J_2) \geq 0$ and $\operatorname{def}(J_1 \cap J_2) \geq 0$, and therefore

$$0 \leq \operatorname{def}(J_1 \cup J_2) \leq \operatorname{def}(J_1) + \operatorname{def}(J_1) - \operatorname{def}(J_1 \cap J_2) = -\operatorname{def}(J_1 \cap J_2) \leq 0.$$

By applying the above successively one get

$$\operatorname{def}\left(\bigcup_{J,\, def(J)=0} J\right) = 0.$$

□

Corollary B.1. *For any collection of independent vector subspaces, there exists unique maximal subcollection of defect 0, i. e. the subcollection of defect 0 which contains any other subcollection of defect 0.*

Proof. A subcollection $\left(\bigcup_{J,\, def(J)=0} J\right)$ is the unique maximal subcollection of defect 0. □

Now we are ready to prove Theorem B.1.

Proof of Theorem B.1. Properties $i)$ and $ii)$ are clearly satisfied by $\mathcal{I}$, let us prove that property $iii)$ is also satisfied. Let A, B be two independent subcollection of vector subspaces with $r = |A| > |B| = s$. Since $\operatorname{def}(J \cup \{L_i\}) \geq \operatorname{def}(J) + 1$, the only problem we might have is with subcollections of B of defect 0. Let C be the maximal subcollection of defect 0 of an independent collection B. Since, the collection A is independent as well there exists at most $|C|$ subspaces $L_i \in A$ with $L_i \subset L_C$ ($\dim(L_C) = |C|$).

Therefore there exists at least $r - |C|$ members of a collection A which we can add to B without making it dependent. And since $|B \setminus C| = s - |C| < r - |A|$ at least one of this members is not contained in B. □

B.0.1 Representability of a matroid defined by collection of vector subspaces

Representable matroids are the ones coming from collection of vectors in the vector space. Representable matroids play important role in matroid theory.

Definition 22. A matroid M is called representable over a field k if there exists a vector space V over k with a collection of vectors $v_i \in V$ which defines matroid M.

In analogy with Definition 22, we would say that a matroid is *representable by vector subspaces* over field k if it can be realized as a matroid coming from a collection of subspaces of a vector space V over field k.

It is not so hard to give an example of a matroid which is representable by vector subspaces over a field k but not representable over k in the sense of Definition 22. Nevertheless, the following theorem shows that two notions are very closely related. This new notion of representability of matroids might be interesting in the connection with the correlation constants of fields (see [34], [14]).

Theorem B.2. *Any matroid representable by vector subspaces over infinite field k is representable over the same field k.*
Any matroid representable by vector subspaces over finite field k is representable over suitable finite extension $\widetilde{k}$ of k.

Theorem B.2, in particular, provides another proof of Theorem B.1.

Proof. It is enough to show that in a collection $L_1, \ldots, L_k$ one can replace the first space L_1 with a space of dimension 1. If one replaces L_1 with $V \subset L_1$ any collection which was dependant will remain dependent. We are going to show that one can choose $V \subset L_1$ of dimension 1 in such a way that any independent subcollection will remain independent as well.

The only subcollections which can become dependent after such a replace are collections A for which $\operatorname{def}(A \setminus \{1\}) = 0$. But for any such collection $\dim(L_1 \cap L_{A\setminus\{1\}}) < \dim L_1$. Therefore if we work over an infinite field k,

$$L_1 \setminus \left(\bigcup_A L_{A\setminus\{1\}} \right) \neq \varnothing,$$

Since the number of subcollections is finite. In case of finite field k, there exists a finite extension $\widetilde{k}$, so that

$$(L_1 \otimes \widetilde{k}) \setminus \left(\bigcup_A L_{A\setminus\{1\}} \otimes \widetilde{k} \right) \neq \varnothing.$$

By replacing L_1 with any V of dimension 1 in the above difference, one does not change the dependencies of subcollections. □

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