Analytic spaces and their Tukey types

by

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Abstract

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In this Thesis we study topologies on countable sets from the perspective of Tukey reductions of their neighbourhood filters. It turns out that is closely related to the already established theory of definable (and in particular analytic) topologies on countable sets. The connection is in fact natural as the neighbourhood filters of points in such spaces are typical examples of directed sets for which Tukey theory was introduced some eighty years ago. What is interesting here is that the abstract Tukey reduction of a neighbourhood filter $\mathcal{F}_x$ of a point to standard directed sets like $\mathbb{N}^{\mathbb{N}}$ or ℓ_1 imposes that $\mathcal{F}_x$ must be analytic. We develop a theory that examines the Tukey types of analytic topologies and compare it by the theory of sequential convergence in arbitrary countable topological spaces either using forcing extensions or axioms such as, for example, the Open Graph Axiom. It turns out that in certain classes of countable analytic groups we can classify all possible Tukey types of the corresponding neighbourhood filters of identities. For example we show that if G is a countable analytic k-group then $1 = \{0\}$, $\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}}$ are the only possible Tukey types of the neighbourhood filter $\mathcal{F}_e^G$. This will give us also new metrization criteria for such groups. We also show that the study of definable topologies on countable index sets has natural analogues in the study of arbitrary topologies on countable sets in certain forcing extensions.

Dedication

To my beloved Angies.

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Chapter 1

Introduction

1.1 Convergence over directed sets

It is well known that convergence of simple sequences $(x_n)_n$ indexed by ω is not sufficient for describing topology of an arbitrary space, i.e., that we need to replace ω by an arbitrary directed set D and define convergence of nets $(x_d)_{d \in D}$. The net convergence is known in mathematics from the time it was used in the definition of the Riemann integral but it was first put in precise form in the work by Moore and Smith [28]. A particularly important work in this early stage is the work of Tukey (cf. [53]) who introduced what is today known as the *Tukey reduction* to study the Moore-Smith convergence in topology. This notion was shown to be useful in developing certain areas of general topology but its main use came later when it was started to be used as a tool for comparing two directed sets. For instance, it was used by Ginsburg and Isbell in [20, 21], and by Todorčević in [42, 44] as a way to classify directed sets and partially ordered sets in general. After this Tukey reducibility was introduced as a classification scheme also in the setting of descriptive set theory i.e., the study of analytic directed sets by Solecki and Todorčević in [40].

When we restrict our attention to the class of filters on a countable set, Tukey reducibility turns out to be quite useful to study analytic topologies. Indeed, Tukey ordering is closely related to the already established theory of definable (and in particular analytic) topologies on countable sets. The connection is in fact natural as the neighbourhood filters of points in such spaces are typical examples of directed sets.

Let us define Tukey reducibility. A pair $(D, \leq_D)$ is a partially ordered set or partial ordering if D is a set and $\leq_D$ is a relation which is reflexive, antisymmetric and transitive. A partial order $\leq_D$ on a set D is directed if any two members of D has an upper bound in D, that is, given d_0 and d_1 in D, there is d_2 in D such that $d_0 \leq_D d_2$ and $d_1 \leq_D d_2$. Given a subset $B \subseteq D$ of a partially ordered set $(D, \leq_D)$, we say that B is unbounded if for every $d \in D$ there exists $b \in B$ such that $d \leq_D b$. A map g from a partially ordered set $(D, \leq_D)$ to a partially ordered set $(E, \leq_E)$ is called a Tukey map if the image of every unbounded subset of D is an unbounded subset of E. When there is a Tukey map from a partially ordered set $(D, \leq_D)$ to a partially ordered set $(E, \leq_E)$ we say that $(D, \leq_D)$ is Tukey reducible to $(E, \leq_E)$ and we write $D \leq_T E$. When $D \leq_T E$ and $E \leq_T D$ we say that D and E are Tukey equivalent and we write $D =_T E$. The relation $=_T$ is an equivalence relation and its equivalence classes are called Tukey types.

In [34] Schmidt found an equivalent way to formulate Tukey reducibility. A subset $C \subseteq D$ of a partially ordered set $(D, \leq_D)$ is cofinal in D if for every $d \in D$ there is $c \in C$ such that $d \leq_D c$. Let $(D, \leq_D)$ and $(E, \leq_E)$ be partial orderings. We say that a function $f : E \to D$ is cofinal if the image of each cofinal subset of E is cofinal in D. When there is a cofinal map from a partial ordering $(E, \leq_E)$ to a partial ordering $(D, \leq_D)$ we say that $(D, \leq_D)$ is Tukey reducible to $(E, \leq_E)$ and we write $D \leq_T E$.

We will be interested in directed partial orders. In this case there is an interesting equivalent way of thinking about the Tukey ordering. Two partially ordered sets D and E are cofinally similar if there is a partially ordered set F so that D and E can both be embedded into F. In [53] Tukey proved that two directed ordered sets are Tukey equivalent if, and only if, they are cofinally similar.

The following are standard examples of directed partial orders

- $1 = \{0\}$

- $\mathbb{N}$
- $\mathbb{N}^{\mathbb{N}}$ ordered with the coordinate wise order
- $\ell_1 = \left\{ A \subseteq \mathbb{N} : \sum_{n \in A} \frac{1}{n+1} < \infty \right\}$ ordered with inclusion.
- An ideal over a countable set ordered with inclusion.

When working with a countable topological space we have that the Tukey reduction of a neighbourhood filter $\mathcal{F}_x$ to directed ordered sets like $\mathbb{N}^{\mathbb{N}}$ or ℓ_1 imposes that $\mathcal{F}_x$ must be analytic. If we require this to happen at every point x we have that the topology of the space is analytic. Recall that a countable topological space (X, τ) is said to be analytic if τ, viewed as a subset of 2^X, is analytic. The results of this thesis will analyze Tukey types of sequential topologies on countable sets. We shall see that in this realm the Tukey type of the directed set $\mathbb{N}^{\mathbb{N}}$ plays a rather prominent role. It might be that studying different topological properties of countable spaces different basic orders may play crucial role.

1.2 Outline of the results

Recall that a k-space is a topological space X whose topology is determined by compact sets, i.e., a set $U \subseteq X$ is open if, and only if, it has relatively open intersection with every compact subset of X. The concept of a k-space introduced long ago by N. Steenrod [41] has been rather fruitful especially in the study of topological groups and countable spaces. In the case of countable spaces it coincides with the well known concept of a sequential space, a space where the closure of a set is obtained by an iteration of sequential closures. In the realm of countable spaces it is of a particular interest to determine the sequential rank of the space and to find conditions under which this rank is either maximal, i.e., equal to ω_1 or minimal, i.e., equal to 1. Spaces of sequential rank 1 are known under the name of Fréchet spaces, a condition which in the realm of countable spaces is particularly interesting as it is related to the stronger conditions that are close to the metrizability of X. This is particularly true when we consider

Fréchet condition in the realm of countable groups. While analyzing critical examples of k-spaces in the realm of countable spaces, i.e., spaces that distinguish between various sequential ranks, one realizes that they are all *analytic*, i.e., their topologies are simple analytic subsets of the Cantor cube 2^X. Also in many potential applications of this theory one finds that one may assume that the topology of a given countable space is analytic. It turns out that this restriction indeed avoids standard pathologies in this domain and that a coherent theory of countable analytic spaces can be built ([51], [52]). In this Thesis we take another approach towards a wider class of spaces that also allows such a coherent theory, an approach based on the concept of Tukey reductions. This approach is motivated by the fact that Tukey types of all critical examples of countable k - spaces can be determined to be one of the standard directed sets like, for example the lattice $\mathbb{N}^\mathbb{N}$ of sequences of positive integers.

Recall that a $k_{\aleph_0}$-space is a topological space X whose topology is determined by countably many compact subsets of X. The problem when a k-space is $k_{\aleph_0}$-space naturally suggests itself. The following result shows that in the realm of countable analytic spaces it is closely tied to standard notions from the theory of ideals on $\mathbb{N}$.

Theorem 3.3.1 *For an analytic k-space X, the following are equivalent:*

1. *X is a $k_{\aleph_0}-$space,*
2. *$\mathcal{D}_X$ is a P-ideal*
3. *$\mathcal{D}_X$ is a weak P-ideal.*

Here, $\mathcal{D}_X$ denotes the ideal of closed discrete subsets of X.

It turns out that the condition that $\mathcal{D}_X$ is a weak P-ideal is closely related to the standard test space in this realm, the sequential fan $S(\omega)$.

Lemma 3.3.9 *If an analytic sequential group X contains a (closed) copy of* $S(\omega)$*, then the ideal of closed discrete subsets of X is a weak P-ideal.*

Adding a Tukey type restriction this leads us to the following result giving us an interesting situation where one can move from a k-space restriction to the considerably stronger $k_{\aleph_0}$-space restriction.

Theorem 4.1.5 *Every sequential non-Fréchet group Tukey reducible to* $\mathbb{N}^{\mathbb{N}}$ *is a* $k_{\aleph_0}$*-space.*

This analysis will also give us a complete characterization of the possible Tukey types of analytic sequential group topologies as the following result shows.

Theorem 4.1.18 *The only possible Tukey types realized by analytic sequential groups are* 1, $\mathbb{N}$, *and* $\mathbb{N}^{\mathbb{N}}$.

The following result shows that there is a connection between the Tukey type restrictions on topologies of countable spaces and the standard descriptive set theoretic restrictions.

Theorem 4.1.14 *Let X be a countable space such that for all* $x \in X$ *the ideal*

$$\mathcal{I}_x = \left\{ A \subseteq X : x \notin \overline{A \setminus \{x\}} \right\}$$

is Tukey reducible to $\mathbb{N}^{\mathbb{N}}$*. Then X is an analytic space, and for all* $x \in X$*, the gap* $(\mathcal{I}_x, C_x)$ *is countably separated.*

Here by C_x *we denote the ideal of sequences in X converging to x.*

The Tukey theory approach will also lead us to a new metrization criteria for countable groups with sequential topologies. The following result summarizes the metrization criteria obtained so far.

Theorem 4.2.3

Let X be a separable topological group. The following are equivalent statements

1. *X is metrizable*

2. *X is Fréchet and it has a countable dense subgroup that inherits an analytic topology (this equivalence was proven in [49, 52]).*

3. *X is Fréchet and its neighbourhood base at the identity is Tukey reducible to* $\mathbb{N}^{\mathbb{N}}$.

4. *X is Fréchet and its neighbourhood base at the identity is Tukey reducible to D for some basic order D that is analytic.*

This result places the class of countable groups Tukey equivalent to $\mathbb{N}^{\mathbb{N}}$ into a special category that is worth studying. For example, it turns out that there is only one possibility for the sequential order of sequential groups with the Tukey type $\mathbb{N}^{\mathbb{N}}$.

Theorem 4.3.1 *Let* (X, τ) *be a countable group with sequential topology that is Tukey equivalent to* $\mathbb{N}^{\mathbb{N}}$. *Then* $so(X) = \omega_1$.

In the second paper of the Thesis we use some set theoretic and axiomatic methods to study countable sequential spaces. For example, using the method of set-theoretic Forcing we obtain the following result.

Theorem 5.1.1 *If we add Cohen reals to a model of CH in the resulting model, every countable space that is sequential and selectively separable has* $\pi-$*weight at most* ω_1.

One possible application of this result comes when we consider products of sequential spaces. For example, as a corollary of this Theorem, we get the following result about finite products of countable topological spaces that are selectively separable.

Corollary 5.1.2 *If we add sufficiently many Cohen reals to a model of CH, in the resulting model, finite products of countable spaces that are sequential and selectively separable is selectively separable.*

Convergence theory in product spaces hides many pathologies as could be seen by the many open problems in this area (see [39]). Adding the restriction that the given countable space has analytic topology avoids many of these pathologies (see, [51], [52]). It turns out that there is another way towards an analogous theory of products of countable spaces that uses the principle OGA about open graphs on separable metric spaces. For example, using OGA we shall prove the following result.

Theorem 5.2.3 *Let X and Y be countable sequential spaces. If X and Y have countable fan tightness, then $X \times Y$ is selectively separable.*

Chapter 2

Preliminaries

2.1 Analytic topologies

Given a countable topological space (X, τ), we have that $\tau \subseteq \mathcal{P}(X)$. If we identify every subset of X with its characteristic function, then we can identify $\mathcal{P}(X)$ with the Cantor set 2^X. Therefore, τ, as a subset of 2^X, can be assumed to be closed, G_δ, Borel, analytic, et cetera. We say that a topological space is analytic (or closed, G_δ, ..., Borel) if it is countable and its topology is analytic (closed, G_δ, ..., Borel). All topological spaces considered in this dissertation we will be assumed to be regular and Hausdorff, and most of the times they will be countable.

As a motivation to study analytic topologies, let us recall Godefroy's characterization of separable compacta K that can be embedded in the first Baire class equipped with the pointwise convergence. We write the result as it is stated in [51].

Theorem 2.1.1 *For every separable compact space K, we have that K is embeddable into the first Baire class if, and only if, the uniformity K induces on any of its countable dense subsets is analytic.*

In addition to this we have:

Proposition 2.1.2 *[51] A (countable) regular-Hausdorff space is analytic if, and only if, it is homeomorphic to a subspace* $C_p\left(\mathbb{N}^{\mathbb{N}}\right)$.

When dealing with analytic sets that live in the Cantor set, it is often useful to have the following fact in mind. Let $[\mathbb{N}]^{<\omega}$ be the collection of all finite subsets of $\mathbb{N}$ considered as a tree under the relation of end extension. We identify the set of infinite branches of $[\mathbb{N}]^{<\omega}$ with the set $[\mathbb{N}]^{\omega}$ of all infinite subsets of $\mathbb{N}$. Consider the tree

$$[\mathbb{N}]^{<\omega} \otimes [\mathbb{N}]^{<\omega} = \left\{(s,t) \in [\mathbb{N}]^{<\omega} \times [\mathbb{N}]^{<\omega} : |s| = |t|\right\}.$$

Proposition 2.1.3 *Given an analytic set* $\mathcal{A} \subseteq 2^{\mathbb{N}}$*, there is a downwards closed subtree* T *of* $[\mathbb{N}]^{<\omega} \otimes [\mathbb{N}]^{<\omega}$ *that codes a closed subset of* $([\mathbb{N}]^{\omega})^2$ *which projects to* $\mathcal{A} \cap [\mathbb{N}]^{\omega}$.

2.2 k-spaces and $k_{\aleph_0}$-spaces

X is said to be a k-space if an arbitrary subset of X is closed when its intersection with an arbitrary compact subset of X is closed (see [41]). When we consider a countable space this reduces to the notion of sequentiality. A topological space X is sequential if every sequentially closed set is closed, where a set $F \subseteq X$ is sequentially closed if whenever a sequence of elements of F converges, then F contains the limit of the sequence. Therefore, the topology of a sequential space is determined by its convergent sequences. As a consequence of this, when X is a countable k-space we have that $A \subseteq X$ is closed in X if, and only, if $A \cap K$ is closed in K for every compact $K \subseteq X$ of the form $\{x\} \cup \{x_1, x_2, x_3, \dots\}$ where $(x_n)_n$ is a convergent sequence and x is its limit. Indeed, a compact space of cardinality $\aleph_0$ is homeomorphic to a countable successor ordinal with the order topology (see Lemma 2.3.2).

For a space X, a point $x \in X$ and a set $F \subseteq X$, we define the sequential order of x in F, denoted by $so(x, F)$ as the first ordinal α such that $x \in [F]_\alpha$, where $[F]_\alpha$ is defined as follows:

$$[F]_1 = \left\{z \in X : \exists (z_n)_n \text{ in } F \text{ with } \lim_{n\to\infty} z_n = z\right\},$$

$[F]_{\alpha+1} = [[F]_\alpha]_1$ and for α limit $[F]_\alpha = \bigcup\limits_{\beta<\alpha} [F]_\beta$. For a set $F \subseteq X$ we define $so(F)$ as the smallest ordinal α such that $\overline{F} = [F]_\alpha$. The sequential order of X, denoted by $so(X)$, is defined as the smallest ordinal α such that $\overline{F} = [F]_\alpha$, for every $F \subseteq X$. It is easily seen that the sequential order of a space is well defined for any sequential space and is at most ω_1.

A special case of sequential spaces are the ones having sequential order 1. Those spaces are called Fréchet.

A particular case of k-spaces is the $k_{\aleph_0}$-spaces. A topological space X is a $k_{\aleph_0}$-space if there is a sequence $(K_n)_n$ of compact subsets of X such that $X = \bigcup\limits_n K_n$ and a subset C of X is closed if, and only if, $C \cap K_n$ is closed in K_n, $n < \omega$.

2.3 Standard examples of sequential spaces

Amongst countable topological spaces with definable topology, there are some of special interest for us as they are usually used as test spaces to study various topological and combinatorial properties in topological spaces. We present those examples here together with some of their properties.

Example 2.3.1 1. *The Arens' space S_2. It is defined as the space on $\omega^{\leq 2}$ with the following topology. Each sequence of length 2 is declared to be isolated, a basic neighbourhood of the sequence $\langle n \rangle$ consists of the sequence itself together with all sequences of the form $\langle n, m \rangle$ for all but finitely many m's. A set $N \subseteq \omega^{\leq 2}$ is a basic neighbourhood of the empty sequence if $\emptyset \in N$, and there is a function $f : \omega \to \omega$ and an integer n with the property that $\langle m \rangle, \langle m, k \rangle \in N$, for all $m \geq n$, and for all $k \geq f(m)$. In a similar way we can define the space S_n, for $n < \omega$. S_1 is just the convergent sequence. The topology of the Arens' space is $F_{\sigma\delta}$.*

2. *The (countable) sequential fan $S(\omega)$. It is defined as the space on $\omega \times \omega \cup \{\infty\}$ with the topology described thereupon. All points that belong to $\omega \times \omega$ are declared to be isolated,*

and a set N is a neighbourhood of ∞ if $\infty \in N$, and there is a function $f : \omega \to \omega$ such that $\{(n, m) : m \geq f(n)\} \subseteq N$. The Sequential fan has a Borel topology of complexity $F_{\sigma\delta}$.

3. *The Arhangel'ski-Franklin space S_ω is the space on $\mathbb{N}^{<\mathbb{N}}$ where a set $U \subseteq \mathbb{N}^{<\mathbb{N}}$ is open if for all $s \in U$, the set $\{n : s^\frown n \in U\}$ is cofinite. S_ω has an $F_{\sigma\delta}$ topology.*

4. *Consider the Boolean group $G \subseteq 2^{2^{\mathbb{N}}}$ consisting of all clopen subsets of the Cantor set $2^{\mathbb{N}}$ with the symmetric difference as the group operation. For every dense set $K \subseteq 2^{\mathbb{N}}$, consider the topology τ_K given by the subbase having the sets of the form*

$$x^+ = \{a \in G : x \in a\} \quad \textit{and} \quad x^- = \{a \in G : x \notin a\},$$

where $x \in K$. If we take $K = 2^{\mathbb{N}}$, then τ_K is an $F_{\sigma\delta}$ topology, but if we choose K to be analytic and not Borel, then τ_K is analytic and not Borel. This example is taken from [51].

The sequential fan $S(\omega)$ has sequential order 1 but it is not metrizable. The Arens' space S_2 has sequential order 2, whereas the Arhangel'ski-Franklin space S_ω is sequential and has maximal sequential order, i.e., ω_1. If we take $A = 2^{\mathbb{N}}$ in Example 2.3.1 (4), then (G, τ_A) is not sequential. In fact, the following set is sequentially closed but is not closed: $B = \bigcup_n B_n$, where

$$B_n = \left\{ \bigcup_{i<n} [s_i] : (\forall i < n,\, s_i \in 2^{<\omega}) \text{ and } |s_i| = n \right\}$$

and for any $s \in 2^{<\omega}$, $[s] = \{z \in 2^\omega : s \sqsubseteq z\}$.

The sequential fan, the Arens' space and the Arhangel'ski-Franklin space are examples of $k_{\aleph_0}$-spaces. We can illustrate the importance of these spaces in the context of k-spaces recalling the following known facts

Lemma 2.3.2 *Let X be countable topological space with $k_{\aleph_0}$ decomposition $(K_n)_n$. Then*

1. *Each K_n is metrizable. Moreover, K_n is homeomorphic to a countable successor ordinal.*

2. *X is metrizable if, and only if it contains no copy of S_2 and no copy of the sequential fan.*

3. Assuming X is not first countable, we have that if X is Fréchet, it contains no copy of S_2.

Proof: 1. is a consequence of a Theorem of Mazurkiewicz-Sierpiński (see [27]) that asserts that every countable compact Hausdorff space is homeomorphic to some well-ordered set with the order topology. The result follows from the fact that a countable well-ordered set is metrizable as it can be embedded into $\mathbb{Q}$. On the other hand, 2. and 3. are consequence of results 23) and 24) presented in [13].

■

The following results are taken from [52]. They are of course well-known facts from the theory of topological convergence (see [39]).

Recall that a point x in a topological space X is regular-G_δ if there is a decreasing sequence $\{V_n : n < \omega\}$ of open sets such that

$$\{x\} = \bigcap_{n<\omega} V_n \left(= \bigcap_n \overline{V_n} \right).$$

Proposition 2.3.3 *Let X be a sequential space in which every point is regular-G_δ. If X is not Fréchet then it contains a closed copy of the Arens' space.*

Proof: Let us choose a set $A \subseteq X$ and a sequence $\{x_n : n < \omega\}$ in $[A]_1$ which converges to a point $x \notin [A]_1$. We may assume without loss of generality that all the x_n's are different. Now we pick a sequence $\{V_n : n < \omega\}$ of decreasing open sets such that

$$\bigcap_n \overline{V_n} = \{x\}.$$

Going to subsequences of $\{x_n : n < \omega\}$ and $\{V_n : n < \omega\}$, we may assume that $x_n \in V_n \setminus \overline{V_{n+1}}$ for all n. Since every $x_n \in [A]_1$, we can find a sequence $\{x_{nm} : m < \omega\}$ converging to x_n whose of elements are in $A \cap \left(V_n \setminus \overline{V_{n+1}}\right)$. We claim that

$$P = \{x\} \cup \{x_n : n < \omega\} \cup \bigcup_n \{x_{nm} : m < \omega\}.$$

is a closed copy of S_2.

Indeed, given $y \notin P$, there must be k_0 such that $y \notin \overline{V_{k_0}}$. Consider the compact set

$$K = \{x_i : i < k_0\} \cup \{x_{im} : i < k_0 \text{ and } m < \omega\}.$$

If we let U be a neighbourhood of y such that $U \cap K = \emptyset$, then $U \cap \left(X \setminus \overline{V_{k_0}}\right)$ is a neighbourhood of y that is disjoint from P. This shows that P is a closed in X. Now, if we fix $\alpha \in \mathbb{N}^{\mathbb{N}}$ and consider a set

$$L_\alpha = \{x_{nm} : m \geq \alpha(n)\},$$

then L_α is easily seen to be (sequentially) closed in X, whence L_α is closed in P. This shows P is a closed copy of the Arens' space in X.

■

Lemma 2.3.4 *Let X be a countable topological group with sequential topology. Then*

1. *X is Fréchet if, and only if, it contains no homeomorphic copy of S_2.*
2. *If X has a copy of S_2 (or $S(\omega)$), then it has a closed copy of it.*
3. *X has a homeomorphic copy of S_2 if, and only if, it has a copy of $S(\omega)$.*

Proof:

1. It follows from Lemma 2.3.2.
2. This follows from the fact that X is countable and so every point $x \in X$ is a regular G_δ-point, ie., there is a decreasing sequence of neighbourhoods of x, say V_n $(n < \omega)$, such that $\bigcap_n \overline{V_n} = \{x\}$.
3. Let $P = \{x\} \cup \{x_n : n < \omega\} \cup \{x_{nm} : n, m < \omega\}$ be a closed copy of S_2 into X. For every $n < \omega$ let $z_{nm} = x_{nm} \cdot x_n^{-1}$, where "$\cdot$" is the group operation. Then $\lim_{m \to \infty} z_{nm} = e$, where e is the identity of the group, and

$$Q = \{e\} \cup \{z_{nm} : n, m < \omega\}$$

is a copy of $S(\omega)$ in X, where ∞ corresponds to e.

Conversely, if X has a closed copy of $S(\omega)$, since a group is homogeneous, we may assume this copy is of the form

$$Q = \{e\} \cup \{z_{nm} : n, m < \omega\}$$

where ∞ corresponds to e. Let $x_m = z_{0m}$, and take $x_{nm} = z_{(n+1)m} \cdot x_m$, for $n, m < \omega$. Then the space

$$P = \{e\} \cup \{x_n : n < \omega\} \cup \{x_{nm} : n, m < \omega\}$$

is a copy of S_2 in X. In fact, if we picked a sequence $\{x_{n\,(m_n)} : n < \omega\}$ converging to a point z, since $\lim_{n\to\infty} z_{0\,(m_n)} = e$, then

$$\lim_{n\to\infty} z_{(n+1)\,(m_n)} = \lim_{n\to\infty} x_{n\,(m_n)} \cdot z^{-1}_{0\,(m_n)} = z. \tag{2.1}$$

Since Q was chosen to be closed in X, (2.1) implies that $z \in Q$, but this contradicts that Q was a copy of $S(\omega)$ in X.

■

Definition 2.3.5 *Let X be a topological space and $x \in X$.*

1. *x has the* diagonal sequence property *if for any double-indexed sequence $\{x_{nm} : m < \omega\}$ in X such that*

$$\lim_{m\to\infty} x_{nm} = x, \quad \textit{for all } n,$$

then for each n, we can choose k_n such that

$$\lim_{n\to\infty} x_{n\,k_n}$$

If we require that some infinite subsequence of $\{x_{n\,k_n} : n < \omega\}$ converges to x instead of the sequence itself, we say that x has the weak diagonal sequence property *in X.*

2. *x is a p^+-point if for aa given decreasing sequence P_n of subsets of $X \setminus \{x\}$ $(n < \omega)$ there is a pseudo-intersection that accumulates to x, that is, there is $P \subseteq X \setminus \{x\}$ such that $x \in \overline{P}$ and $P \subseteq^* P_n$, for all $n < \omega$.*

Note that $S(\omega)$ is a space that does not have the weak diagonal sequence property. Moreover, this is a test space for the weak diagonal sequence property as the following result asserts.

Lemma 2.3.6 *(Lemma 7.68 in [49]) Given a countable topological space, if x is a Fréchet point of X, i.e. $so(x, X) = 1$, the following are equivalent statements*

1. *X has a closed copy of $S(\omega)$ where x corresponds to the point ∞.*

2. *x does not have the weak diagonal sequence property in X.*

3. *x is not a p^+-point in X.*

Proof: 1. and 2. are clearly equivalent. On the other hand, if x is not a p^+-point in X, there is a decreasing sequence P_n of subsets of $X \setminus \{x\}$ without a pseudo-intersection accumulating to x, then for every n there is a sequence $\{x_{nm} : m < \omega\} \subseteq P_n$ converging to x so that $x_{nm} \neq x_{kl}$ for $(n, m) \neq (k, l)$. Consider the subspace

$$Q = \{x\} \cup \bigcup_n \{x_{nm} : m < \omega\}.$$

Note that given $\alpha \in \mathbb{N}^{\mathbb{N}}$, since $\{x_{nm} : m \leq \alpha(n)\}$ is a pseudo-intersection we have that it does not accumulate to x. This shows that Q is homeomorphic to $S(\omega)$. On the other hand, we can choose a decreasing sequence of neighbourhoods of $x \in V_k$ such that

$$\bigcap_k \overline{V_k} = \{x\}.$$

We may assume, without loss of generality, that $\{x_{nm} : m < \omega\} \subseteq V_n$ for every $n < \omega$.

Hence, given $y \notin Q$, there must be k_0 such that $y \notin \overline{V_{k_0}}$. Consider the compact set

$$K = \{x\} \cup \{x_{im} : i < k_0 \text{ and } m < \omega\}.$$

If we let U be a neighbourhood of y such that $U \cap K = \emptyset$, then $U \cap \left(X \setminus \overline{V_{k_0}}\right)$ is a neighbourhood of y that is disjoint from Q.

This shows that Q is a closed copy of the sequential fan. Thus, 3. implies 1. is proven. The implication from 1. to 3. is clear.

■

Definition 2.3.7 *A point x in a topological space X is said to be* strongly Fréchet *if for any decreasing family $\{A_n : n < \omega\}$ of subsets of X so that $x \in \bigcap_n \overline{A_n}$, we may pick points $x_n \in A_n$ in such a way that the sequence $\{x_n : n < \omega\}$ converges to x.*

When every $x \in X$ is a strongly Fréchet point, we say that X is a strongly Fréchet space.

We also note the following immediate consequence of Lemma 2.3.6.

Lemma 2.3.8 *A Fréchet point in a topological space is strongly Fréchet if, and only if, it has the weak diagonal sequence property.*

2.4 Combinatorial properties of topological spaces

2.4.1 Ideals associated to topological spaces

Given a set X, a family $\mathcal{I} \subseteq \mathcal{P}(X)$ of subsets of X is an *ideal* (on X) if

1. whenever A and B belong to $\mathcal{I}$ we have that $A \cup B$ is in $\mathcal{I}$, and

2. if $A \in \mathcal{I}$ and $B \subseteq A$, then B belongs to $\mathcal{I}$.

   $\mathcal{I}$ is a proper ideal if

3. $X \notin \mathcal{I}$

   $\mathcal{I}$ is said to be a σ-ideal if

4. given a sequence $\{A_n : n < \omega\}$ of elements in $\mathcal{I}$, we have

$$\bigcup_n A_n \in \mathcal{I}.$$

A family $\mathcal{F} \subseteq \mathcal{P}(X)$ is a (proper) *filter* (on X) if $\mathcal{F}^* = \{X \setminus A : A \in \mathcal{F}\}$ is a (proper) ideal. $\mathcal{F}^*$ is called the *dual ideal* of $\mathcal{F}$. Analogously, if $\mathcal{I}$ is an ideal, $\mathcal{I}^* = \{X \setminus A : A \in \mathcal{F}\}$ is called the *dual filter* of $\mathcal{I}$. We will mostly consider ideals on a countable set X that are proper and contain all the finite subsets of X. Let us denote by $\mathcal{I}^+$ the family of $\mathcal{I}$*-positive* subsets of X, that is, the family of sets not in $\mathcal{I}$. Given a filter $\mathcal{F}$, we denote by $\mathcal{F}^+$ the coideal $(\mathcal{F}^*)^+$. Additionally, for a given ideal $\mathcal{I}$ on X, we define the *orthogonal* of $\mathcal{I}$, denoted by $\mathcal{I}^\perp$, as

$$\mathcal{I}^\perp = \{A \subseteq X : \forall B \in \mathcal{I}, A \cap B \text{ is finite}\}.$$

Note that $\mathcal{I} \subseteq \mathcal{I}^{\perp\perp} \left(= (I^\perp)^\perp\right)$ and $\mathcal{I}^\perp = \mathcal{I}^{\perp\perp\perp}$.

An *ultrafilter* on $\mathbb{N}$ is a maximal proper filter. Equivalently, an ultrafilter is a proper filter such that for all $A \subseteq \mathbb{N}$ either A or $\mathbb{N} \setminus A$ belongs to the ultrafilter.

Given a non-isolated point x in a topological space X, we denote by C_x^X the ideal of all convergent sequences to x, that is,

$$C_x^X = \{A \subseteq X : \forall O \in \tau_x, A \setminus O \text{ is finite}\},$$

where τ_x is the filter of neighbourhoods of x.

By $\mathcal{I}_x^X$ we will denote the dual ideal of the neighbourhood filter τ_x, that is,

$$\mathcal{I}_x^X = \left\{B \subseteq X : x \notin \overline{B \setminus \{x\}}\right\}.$$

When the topological space is determined by the context, we will omit the upper index to the ideal and we will write C_x and $\mathcal{I}_x$. Let us point out that $C_x = \mathcal{I}_x^\perp$.

By nwd (X), the ideal of nowhere dense subsets of X, and by $\mathcal{CD}$ the ideal consisting of the closed discrete subsets of X.

Given an ideal $\mathcal{I}$ on a countable set X, if we see $\mathcal{I}$ as a subset of 2^X, then we can talk of definable ideal, i.e., closed, F_σ, Borel, analytic, et cetera.

2.4.2 Points in Cantor cubes

In this section we show how the combinatorial properties of points in topological spaces correspond to some standard cardinal invariants of the continuum.

Definition 2.4.1

- *A filter $\mathcal{F}$ on a set S is a* P^+-filter *if for every decreasing sequence $X_0 {}^*\supseteq X_1 {}^*\supseteq \cdots$ of members of $\mathcal{F}^+$, there is $X \in \mathcal{F}^+$ such that $X \subseteq^* X_n$ for all $n < \omega$.*

- *$\mathcal{F}$ is a* Q^+-filter *if for every $X \in \mathcal{F}^+$ and every finite-to-one $f : \mathbb{N} \to \mathbb{N}$, there is $Y \subseteq X$, $Y \in \mathcal{F}^+$ such that $f \upharpoonright Y$ is $1-1$.*

- *$\mathcal{F}$ is* selective *if it is both P^+ and a Q^+-filter.*

Recall that *Ramsey's Theorem* asserts that, for any $k < \omega$, if the the set $[\mathbb{N}]^k$ of k-element subsets of $\mathbb{N}$ is coloured with r-different colours, then there is an infinite set $H \subseteq \mathbb{N}$ that is homogeneous, i.e., the elements of $[H]^k$ have the same colour. This is usually written using the arrow notation, that is,

$$\aleph_0 \to (\aleph_0)^k_r .$$

Given two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of $\mathbb{N}$ we write

$$\mathcal{A} \to (\mathcal{B})^k_r$$

if for every $A \in \mathcal{A}$ and every colouring $c : [A]^k \to r$, there is $B \subseteq A$ such that $B \in \mathcal{B}$ and B is c-homogeneous.

Therefore, Ramsey's theorem can be stated as

$$[\mathbb{N}]^\omega \to ([\mathbb{N}]^\omega)^k_r .$$

A set H is called *end-homogeneous* with respect to a colouring $c : [H]^2 \to 2$ if for all $x, y, z \in H$, if $x \subseteq y$ and $x \subseteq z$, then $c(x, y) = c(x, z)$.

Theorem 2.4.2 *[26] The following are equivalent for an ultrafilter $\mathcal{F}$ on $\mathbb{N}$:*

1. *$\mathcal{F}$ is selective*
2. *$\mathcal{F}^+ \to (\mathcal{F}^+)^2_2$*
3. *$\mathcal{F}^+ \to (\mathcal{F}^+)^k_r$, for any $r, k < \omega$*
4. *$\mathcal{F}^+ \to (\mathcal{F}^+)^\omega_2$ for analytic partitions.*

Definition 2.4.3 *A point x in a topological space X is a*

- Fréchet point of *X if for every countable infinite set $A \subseteq X$ which accumulates to x, there is a sequence $\{x_n : n < \omega\}$ of elements of A that converges to x. This is to say that the trace on $A_0 = \{x_n : n < \omega\}$ of the neighbourhood filter of x is equal to the Fréchet filter of A_0.*
- P^+-point *if for every countable $A \subseteq X$ which accumulates to x there is $A_0 \subseteq A$ such that $\mathcal{F} = \{U \cap A_0 : U$ is a neighbourhood of $x\}$ is a P^+-filter.*
- Q^+-point *if for every countable $A \subseteq X$ which accumulates to x there is $A_0 \subseteq A$ such that $\mathcal{F} = \{U \cap A_0 : U$ is a neighbourhood of $x\}$ is a P^+-filter.*
- Selective point of *X if for every countable infinite set $A \subseteq X$ which accumulates to x, there is an infinite set $A_0 \subseteq A$ such that the trace on $A_0 = \{x_n : n < \omega\}$ of the neighbourhood filter of x is a selective filter.*

Let us recall some known cardinal invariants.

Definition 2.4.4

- *A family $F \subseteq \mathbb{N}^\mathbb{N}$ is a* dominating *family if*

$$\forall g \in \mathbb{N}^\mathbb{N}\, \exists f \in F \text{ such that } g(n) < f(n), \text{ for all but finitely many } n's.$$

The dominating *number is defined as*

$$\mathfrak{d} = \min\{|F| : F \textit{ is a dominating family}\}.$$

- $F \subseteq \mathbb{N}^{\mathbb{N}}$ *is an* unbounded *family if*

$$\forall g \in \mathbb{N}^{\mathbb{N}}\, \exists f \in F \textit{ such that } g(n) \leq f(n), \textit{for infinitely many } n's.$$

  The bounding *number is defined as*

$$\mathfrak{b} = \min\{|F| : F \textit{ is an unbounded family}\}.$$

- *We say that a set* $P \subseteq \mathbb{N}$ *is a* pseudo-intersection *of a filter* $\mathcal{F}$ *is* $P \subseteq^* F$ *(i.e.* $P \setminus F$ *is finite) for all* $F \in \mathcal{F}$*. The* pseudo-intersection *number is defined as*

$$\mathfrak{p} = \min\{|\mathcal{B}| : \mathcal{B} \textit{ is a base of a filter without pseudo-intersection}\}.$$

- *Let* $\mathcal{M}$ *denote the* σ*-ideal of the meager subsets of* $\mathbb{R}$.

$$cov(\mathcal{M}) = \min\left\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{M} \textit{ and } \bigcup \mathcal{A} = \mathbb{R}\right\}.$$

  Recall that

$$\omega_1 \leq \mathfrak{p} \leq \mathrm{cov}(\mathcal{M}) \leq \mathfrak{d} \quad \text{and} \quad \mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}.$$

Theorem 2.4.5 *[11]*

$$\mathfrak{p} = \sup\{\alpha \in Ord : \textit{ every point of } \{0,1\}^{\alpha} \textit{ is Fréchet}\}.$$

Theorem 2.4.6 *[9] The following are equivalent*

1. $cov(\mathcal{M}) > \kappa$
2. $\forall F \subseteq \left[\mathbb{N}^{\mathbb{N}}\right]^{\kappa} \exists g \in \mathbb{N}^{\mathbb{N}}\, \forall f \in F\ (f(n) = g(n) \textit{ for infinitely many } n's).$

Theorem 2.4.6 can also be found in [10, Theorem 4.5].

Corollary 2.4.7 *The constant function* $\overline{1}$ *is not a* Q^+*-point of* $\{0,1\}^{cov(\mathcal{M})}$.

Proof: Let $\sigma = (\mathcal{M})$. From Theorem 2.4.6 we can fix a sequence $\left\{f_\xi \in \mathbb{N}^{\mathbb{N}} : \xi < \sigma\right\}$ such that

$$\forall g \in \mathbb{N}^{\mathbb{N}}\, \exists \xi < \sigma\, \exists m < \omega\, \forall n \geq m\ g(n) \neq f_\xi(n)\,.$$

Let C be the set of all finite partial functions from $\mathbb{N}$ into $\mathbb{N}$. For every $p \in C$ let $x_p \in \{0,1\}^\sigma$ be defined by $x_p(\xi) = 1$ if, and only if, there is $n \in \mathrm{dom}(p)$ such that $f_\xi(n) = p(n)$. Consider the set $A = \left\{x_p : p \in C\right\}$. Then $\overline{1} \in A'$, the set of accumulation points of A. Suppose there is a set $A_0 \subseteq A$ such that $\mathcal{F} = \left\{U \cap A_0 : U \text{ is an open set containing } \overline{1}\right\}$ is a Q^+-filter. Let us write $A_0 = \left\{x_p : p \in C_0\right\}$ for some $C_0 \subseteq C$. Since $\mathcal{F}$ is a Q^+-filter, there is $C_1 \subseteq C_0$ such that $A_1 = \left\{x_p : p \in C_1\right\} \in \mathcal{F}^+$ and $\max(p) \neq \max(q)$, for every $p \neq q$ in C_1. Moreover, for every $p \in C_1$, $\{q \upharpoonright \max(p) : q \in C_1 \text{ and } \max(p) < \max(q)\}$ forms a Δ-system. Choose $g \in \mathbb{N}^{\mathbb{N}}$ in such a way that for all $p \in C_1$, $p \upharpoonright [n,m) \leq g$ where $n = \max\{\max(q) : q \in C_1 \text{ and } \max(q) < \max(p)\}$ and $m = \max(p)$. Then there is $\xi < \sigma$ such that for infinitely many n's, $g(n) = f_\xi(n)$, a contradiction.

■

Theorem 2.4.8 *[32] The following are equivalent statements.*

1. $\kappa < cov(\mathcal{M})$
2. $\overline{1}$ *is a selective point of* $\{0,1\}^\kappa$.

Theorem 2.4.9

$$\mathfrak{d} = \sup\{\alpha \in Ord : \text{ every point of } \{0,1\}^\alpha \text{ is a } P^+\text{-point}\}\,.$$

Proof: Let $\alpha < \mathfrak{d}$ and suppose $A = \{a_m : m < \omega\} \subseteq \{0,1\}^\alpha$ is an infinite set which accumulates to $\overline{1}$ and let $\mathcal{F} = \left\{U \cap A : \overline{1} \in U \text{ and } U \text{ is open}\right\}$. Let $X_0{}^* \supseteq X_1{}^* \supseteq \cdots$ be a given sequence of elements of $\mathcal{F}^+$. For every $\xi < \alpha$ choose $f_\xi \in \mathbb{N}^{\mathbb{N}}$ so that

$$\forall n < \omega\, \exists m \leq f_\xi(n) \text{ such that } a_m \in X_n \text{ and } a_m(\xi)\,.$$

Let $g \in \mathbb{N}^{\mathbb{N}}$ be a function not dominated by any function of the form $\max\left\{f_{\xi_1}, \ldots, f_{\xi_k}\right\}$, where $\xi_1 \leq \xi_2, \ldots, \xi_k \leq \alpha$ and $k < \omega$. Consider the set $X = \bigcup_n (X_n \cap \{a_i : i \leq g(n)\})$. Then $X \in \mathcal{F}^+$ and $X \subseteq^* X_n$, for all $n < \omega$.

On the other hand, since the sequential fan $S(\omega)$ has weight $\mathfrak{d}$ then it can be embedded into $\{0, 1\}^{\mathfrak{d}}$. Moreover, the only non-isolated point of $S(\omega)$ is not a P^+-point.

■

2.5 Selectively separable spaces

Definition 2.5.1 *A space X is called* selectively separable *if for each sequence $\{D_n : n \in \omega\}$ of dense subsets of X, there is a selection sequence $\{E_n \in [D_n]^{<\omega} : n \in \omega\}$ such that $\bigcup_n E_n$ is dense.*

The notion of Selectively separable spaces was introduced by Scheepers in [33] and it has received a lot of attention since (see for instance [3, 4, 7, 8]). Let us recall some properties of selectively separable spaces.

Proposition 2.5.2 *[7] Each countable space with $\pi-$weight less than $\mathfrak{d}$ is selectively separable.*

It turns out that separable Fréchet spaces are selectively separable (see [7]). Moreover, selectively separable spaces are related to the notion of countable fan tightness.

Definition 2.5.3 *A space X has* countable fan tightness *if for any $x \in X$ and for any countable family $\{A_n : n \in \omega\}$ of subsets of X such that $x \in \bigcap_n \overline{A_n}$, there is a finite selection*

$$\{F_n \in [A_n]^{<\omega} : n \in \omega\}$$

such that $x \in \overline{\bigcup_n F_n}$.

If a space X is separable and has countable fan tightness, then X is selectively separable [3]. The converse is not true though. For instance, the sequential fan $S(\omega)$ is a selectively separable space (actually it is a countable Fréchet space with analytic topology) that has no countable fan tightness.

In the realm of analytic topologies, Camargo and Uzcátegui [4] studied selectively separable spaces and their relation to some other combinatorial properties of a topological space.

2.6 Open graphs on separable metric spaces

Recall that a *graph* is a structure of the form $G = (X, E)$ where E is a symmetric irreflexive binary relation, i.e., $E \subseteq X^2 \setminus \Delta$, where $\Delta = \{(x, x) : x \in X\}$, and $(x, y) \in E$ implies $(y, x) \in E$. A *graph homomorphism* between two graphs $G = (X, E)$ and $H = (Y, F)$ is a mapping $f : X \to Y$ such that $(x, y) \in E$ implies $(f(x), f(y)) \in F$. We write $G \leq H$ when there is a graph homomorphism from G to H. A *discrete graph* is a graph $G = (X, E)$ for which the edge set E is empty, that is, $G = (X, \emptyset)$, a *complete graph* is a graph $G = (X, E)$ such that $E = X^2 \setminus \Delta$. The extremal cases are related to the notion of *chromatic number* $\chi(G)$ of a graph $G = (X, E)$, defined to be the minimal cardinality of a discrete graph $H = (Y, \emptyset)$ such that $G \leq H$, i.e., the minimal cardinality of a family of discrete subgraph of G that covers its vertex set X. Thus

$$\chi(G) = \min\{|Y| : G \leq (Y, \emptyset)\}.$$

Note that

$$G \leq H \text{ implies } \chi(G) \leq \chi(H).$$

A complete graph on a vertex-set Y is usually denoted by $K_Y = \left(Y, Y^2 \setminus \Delta\right)$.

It is not difficult to see that for any graph $G = (X, E)$, $K_Y \leq G$ implies $\chi(G) \geq |Y|$. The converse is actually false as one can find graphs G of large chromatic number such that $K_3 \nleq G$. The *Open Graph Axiom*, introduced in [43], addresses this implication in the case of graphs $G = (X, E)$ where X is a separable metric space and E is an open subset of X^2, that is, the case of *open graphs*.

Open Graph Axiom: An open graph $G = (X, E)$ on a separable metric space X is countably chromatic if, and only if, $K_{\aleph_1} \nleq G$.

While it is easily seen that the Continuum Hypothesis contradicts OGA, Todorčević has shown that this principle is consistent.

Theorem 2.6.1 *[43] [17, pp. 309-311] There is a proper poset (and therefore an ω_1-preserving) $\mathbb{P}$ whose forcing extension satisfies OGA together with MA.*

One can view OGA as a regularly property of a class of sets in Polish spaces, i.e., for such a set X we can consider the property OGA_X asserting the OGA-equivalence for open graphs on the vertex-set X. The first thing that should be mentioned about this is the following result.

Lemma 2.6.2 *If Y is a continuous image of X then OGA_X implies OGA_Y.*

Proof: Fix a continuous surjection $f : X \to Y$ and consider an open symmetric $F \subseteq Y^2 \setminus Y$. Define an open symmetric $E \subseteq X^2 \setminus \Delta$ by

$$(x, y) \in E \Longleftrightarrow f(x) \neq f(y) \text{ and } (f(x), f(y)) \in F.$$

Thus f is a continuous graph homomorphism, so if (Y, F) is countably chromatic so is (X, E). From this we also get that $K_{\aleph_1} \leq (X, E)$ implies $K_{\aleph_1} \leq (Y, F)$.

■

The following simple fact should also be noted.

Proposition 2.6.3 *OGA_X holds for all complete separable metric spaces X.*

Proof: Consider an open graph $G = (X, E)$. Assume $\chi(G) > \aleph_0$. Let $\mathcal{U}$ be the collection of all open subsets U of X such that $\chi\left(U, E \cap U^2\right) \leq \aleph_0$. Let $Y = X \setminus \bigcup \mathcal{U}$. Then Y is nonempty and $\chi\left(U, E \cap U^2\right) > \aleph_0$ for all nonempty relatively open subsets U of Y. So without loos of generality we may assume that X itself has this property. Note that this means that for every nonempty open $U \subseteq X$ we can find nonempty open sets $V_0, V_1 \subseteq U$ such that $V_0 \times V_1 \subseteq E$. So we can build a Cantor scheme $U_\sigma \left(\sigma \in 2^{<\mathbb{N}}\right)$ such that for all $\sigma \in 2^{<\mathbb{N}}$, $U_{\sigma^\frown 0} \times U_{\sigma^\frown 1} \subseteq E$. Using completeness of X the limit of the scheme is a perfect set $P \subseteq X$ such that $P^2 \setminus \Delta \subseteq E$.

■

Note that Lemma 2.6.2 and the proof of Proposition 2.6.3 give us the following fact from [43].

Theorem 2.6.4 *OGA_X holds for all analytic subsets X of Polish spaces and in fact we have that for open graphs G on X we have that $\chi(G) > \aleph_0$ if, and only if, $K_{2^{\mathbb{N}}} \leq G$.*

However, it should be noted that Proposition 2.6.3 and Theorem 2.6.4 are also immediate consequences of Theorem 2.6.1 via Schoenfield absoluteness. Since this observation is relevant to the principle OGA_X for other classes of sets let us make this explicit.

Lemma 2.6.5 *Assume that $\mathfrak{p} > \aleph_1$. Let $G = (X, E)$ be an open graph on a separable metric space X such that $K_{\aleph_1} \leq G$. Then there is a relatively G_δ−subset Y of X such that $Y^2 \setminus \Delta \subseteq E$.*

Proof: Fix an uncountable set $Z \subseteq X$ such that $Z^2 \setminus \Delta \subseteq E$ and assume that $|U \cap Z| = \aleph_1$ for all open $U \subseteq X$ such that $U \cap Z \neq \emptyset$. Consider a countable basis $\mathcal{B}$ of X. Let us define $\mathcal{P}$ as the collection of all pairs $p = \left\langle Y_p, \mathcal{U}_p \right\rangle$ where Y_p is a finite subset of Z, $\mathcal{U}_p$ is a finite subset of $\mathcal{B}$ which covers Y_p but no two points of Y_p lie in the same set of $\mathcal{U}_p$. Moreover, we will assume that for every $U \in \mathcal{U}_p$, $diam\,(U) \leq \mathrm{d}\,(x, y)$ for all $x \neq y$ in Y_p. We define the order by $p \leq q$ if, and only if, $Y_p \supseteq Y_q$ and if $\mathcal{U}_p$ is a refinement of $\mathcal{U}_q$. By our assumption that $\mathfrak{p} > \aleph_1$ we can find a filter $\mathcal{G} \subseteq \mathcal{P}$ such that

$$Y_{\mathcal{G}} = \bigcup_{p \in \mathcal{G}} Y_p$$

is uncountable. For every $n < \omega$, let U_n be the set of all $x \in X$ for which there is $p \in \mathcal{G}$ such that $x \in V$, for some $V \in \mathcal{U}_p$ and $\mathrm{diam}\,(V) \leq 2^{-n}$. Clearly U_n is an open set and

$$Y_{\mathcal{G}} \subseteq \bigcap_{n<\omega} U_n.$$

Now, if we let $Y = \bigcap_n U_n$, then we have that Y is uncountable and $Y^2 \setminus \Delta \subseteq E$.

■

Let OGA^*_X be the statement that for open graphs $G = (X, E)$ on X, $\chi(G) > \aleph_0$ is equivalent to $K_{2^{\mathbb{N}}} \leq G$. For a class Γ of subsets of Polish spaces, let OGA_Γ (respectively OGA^*_Γ) be the statement that OGA_X holds for all $X \in \Gamma$ (respectively OGA^*_Γ holds for all $X \in \Gamma$). When the class Γ is the class of all separable metrizable space we use the notation OGA and OGA*

instead of OGA_Γ and OGA^*_Γ. Recall that the Perfect Set Property for Γ, abbreviated PSP_Γ, is the statement that an uncountable element of Γ must contain a perfect subset.

Lemma 2.6.6 *For every class Γ of subsets of Polish spaces, OGA^*_Γ implies PSP_Γ.*

Proof: Given a set X in Γ, apply OGA^*_Γ to the open graph $\left(X, X^2 \setminus \Delta\right)$.

■

We would also like to bring to mind the following classical results.

Theorem 2.6.7 (Hausdorff) *Every uncountable analytic subset of a Polish space contains a perfect subset.*

Theorem 2.6.8 (Solovay) *Every uncountable co-analytic subset of a Polish space contains a perfect subset if and only if $\omega_1^{L[a]}$ is countable for every set $a \subseteq \omega$.*

This shows that OGA^*_Γ even for the class of co-analytic subsets of Polish spaces implies that ω_1 is an inaccessible cardinal in the constructible universe ([15]). In fact, it has been shown in [5] (see also [15]) that if $L(\mathbb{R})$ is a Solovay model then OGA_Γ, and equivalently OGA^*_Γ, holds both in $L(\mathbb{R})$ and $L(\mathbb{R})[U]$.

The following is an immediate consequence of Lemma 2.6.5.

Theorem 2.6.9 *Assume $\mathfrak{p} > \aleph_1$. Then OGA_Γ implies OGA^*_Γ for every class of sets Γ with the perfect set property that contains the class of Polish spaces and is closed under intersections.*

Thus, for any reasonable class Γ of subsets of Polish spaces, OGA^*_Γ is simply a combination of OGA_Γ and PSP_Γ.

Our interest in OGA is that many results about countable analytic spaces can be turned into results about arbitrary countable spaces using OGA. We present samples of these results hereunder.

Theorem 2.6.10 *[48] (OGA) The product $X \times Y$ of two countable strongly Fréchet spaces X and Y is strongly Fréchet provided it is Fréchet. So, in particular, for a countable space X and $n \geq 2$, the Fréchet property of X^n is equivalent to the strong Fréchet property of X^n.*

Theorem 2.6.11 *[30] (OGA) If for some $n \geq 2$ the power X^n of a countable space X is Fréchet then so is X^{n+1} provided that it is sequential.*

In this Thesis we shall also give a result a result of this sort (see Theorem 5.2.3).

Theorem 2.6.12 (OGA) *Let X and Y be countable sequential spaces. If X and Y have countable fan tightness, then $X \times Y$ is selectively separable.*

The proof of Theorem 5.2.3 will make use of the following consequence of OGA.

Proposition 2.6.13 *OGA implies that $\mathfrak{b} > \omega_1$.*

Actually, Todorčević proved in [43] that

Theorem 2.6.14 *OGA implies that $\mathfrak{b} = \aleph_2$.*

2.7 Bases and π-bases

Definition 2.7.1 *Given a topological space X and a collection $\mathcal{B}$ of nonempty open sets we say that:*

- *$\mathcal{B}$ is a* base for the topology of *X if every nonempty open set can be written as a union of members of $\mathcal{B}$. The smallest cardinality of a base is called the* weight of *X and it is denoted by $w(X)$.*
- *Given $x \in X$, $\mathcal{B}$ is a* local base at *x if every open set containing x can be written as a union of sets in $\mathcal{B}$. The* character of *x, denoted by $\chi(x, X)$, is the minimal cardinality of a local base at x. The* character of *X is defined as* $\sup\{\chi(x, X) : x \in X\}$.
- *A collection of nonempty open sets $\mathcal{B}$ is called a* π-base *for the topology of X if every nonempty open set contains a member of $\mathcal{B}$. The* π-weight *of X, denoted by $\pi w(X)$ is the smallest cardinality of a π-base of X.*

- *Given $x \in X$, $\mathcal{B}$ is a* local π-base at *x is any open set containing x contains a member of $\mathcal{B}$.*

Theorem 2.7.2 (Urysohn) *Every regular Hausdorff space with a countable basis is metrizable*

Recall that a Hausdorff group is actually regular Hausdorff. So

Theorem 2.7.3 (Birkhoff-Kakutani) *Every Hausdorff topological group with a countable neighbourhood basis at the identity is metrizable.*

Chapter 3

Analytic k-spaces

3.1 Analytic Fréchet spaces

Let X be an analytic Fréchet space with no isolated points. We assume X lives on ω and fix a downwards closed subtree T of $[\mathbb{N}]^{<\omega} \otimes [\mathbb{N}]^{<\omega}$ which projects onto the topology of X (see Proposition 2.1.3). Given a subtree W of T consider the following derivative operation:

$$\mathrm{d}\, W = W \setminus \left\{(s,t) \in W : \mathrm{int}\left(\overline{\bigcap \mathrm{proj}_1 \left[W(s,t)\right]}\right) \neq \emptyset\right\},$$

where $\mathrm{proj}_1 : [\mathbb{N}]^{<\omega} \otimes [\mathbb{N}]^{<\omega} \to [\mathbb{N}]^{<\omega}$ is the projection onto the first coordinate, $W(s,t)$ is the subtree of W consisting of all nodes compatible with (s,t), and $[W(s,t)]$ is the collection of all infinite branches of $W(s,t)$. Note that $\mathrm{proj}_1 [W(s,t)]$ is a collection of open sets and $\bigcap \mathrm{proj}_1 [W(s,t)]$ is the intersection of those open sets. Let us define $T^0 = T$ and let T^α, for $\alpha < \omega_1$, be the sequence of all subtrees of T determined by $T^\lambda = \bigcap_{\alpha<\lambda} T^\alpha$ for λ limit and $T^{\alpha+1} = \mathrm{d}\, T^\alpha$. Note that this derivative operation must be constant from some point on, i.e., there is $\beta < \omega_1$ such that $T^{\beta+1} = T^\beta$.

Lemma 3.1.1 $\left\{int\left(\overline{\bigcap proj_1 \left[T^\alpha(s,t)\right]}\right) : \alpha < \beta, (s,t) \in T^\alpha \setminus T^{\alpha+1}\right\}$ *is a* π*−base of* X

Proof:

Let U be a nonempty regular open set. There is branch b of T such that $U = \mathrm{proj}_1(b)$. Therefore, there exists $(s_0, t_0) \sqsubseteq b$ in T such that $x \in s_0$.

We may assume that for all $\lambda < \omega_1$ and for all $n > |(s_0, t_0)|$ we have that

$$\mathrm{int}\left(\overline{\bigcap \mathrm{proj}_1 \left[T^\lambda (b \restriction n)\right]}\right) = \emptyset.$$

Then $b \restriction n \in T^\beta$ for all $n < \omega$.

Notice that if we assume the opposite, the λ we obtain would have to be less than β and so we would be done.

Suppose, towards a contradiction, this is not the case and pick a nonempty open subset U of X such that non of these sets are included in closure of U.

For $(s,t) \in T^\beta$, let $N(s,t) = \bigcap \mathrm{proj}_1 \left[T^\beta (s,t)\right]$. Those sets are nowhere dense by the choice of β. Therefore, we can find a decreasing sequence $(U_n)_n$ of dense-open sets such that for all $(s,t) \in T_\beta$, there is n such that $U_n \cap N(s,t) = \emptyset$. Choose $x \in U$ and let $\{x_n : n \in \omega\} \subseteq U \setminus \{x\}$ be a sequence converging to x. For each $n < \omega$, pick $S_n \subseteq U_n$ a sequence converging to x_n. Then $x \in \overline{\bigcup_n S_n}$. Fréchetness implies again that there is a sequence $S \subseteq \bigcup_n S_n$ converging to x. Thus, for each $n < \omega$, $S \subseteq^* S_n \subseteq U_n$. Therefore, for each $(s,t) \in T^\beta$, $S \cap N(s,t)$ is finite. However, there must be $(s,t) \in T^\beta$ such that $S \subseteq^* N(s,t)$ or else we would get a branch of T^β whose first coordinate is an open set containing x which does not almost contain S.

■

It turns out that Lemma 3.1.1 is giving us another way to prove that following theorem of [52].

Theorem 3.1.2 *[52] A countable Fréchet topological group is metrizable if, and only if, its topology is analytic.*

Proof: Assume X is an analytic Fréchet group. By Lemma 3.1.1, X has a countable π-base V_n $(n < \omega)$. Then $V_n V_n^{-1}$ $(n < \omega)$ is a countable neighbourhood base of the identity of X. So G is metrizable by the Birkhoff-Kakutani theorem.

■

Let us also point out that Lemma 3.1.1 was originally proved in [37], however, the proof we offer here is different.

3.2 The orthogonal of every analytic weak P-ideal is countably generated

We say that a family $\mathcal{F} \subseteq [\omega]^\omega$ hits the family $\mathcal{G} \subseteq [\omega]^\omega$ if for every $B \in \mathcal{G}$, there is $A \in \mathcal{F}$ such that $A \cap B$ is infinite. An ideal $\mathcal{I}$ on $\mathbb{N}$ is a weak P-ideal if for every family $\{A_n : n < \omega\} \subseteq \mathcal{I} \cap [\omega]^\omega$ there is $B \in \mathcal{I}$ such that $\{B\}$ hits $\{A_n : n < \omega\}$, that is, $B \cap A_n$ is infinite for every n. Note that this is equivalent to saying that for every $\{A_n : n < \omega\} \subseteq \mathcal{I} \cap [\omega]^\omega$, there is $B \in \mathcal{I}$ such that $B \cap A_n \neq \emptyset$ for all n.

The following result has been proved, although not stated, in [45] (however, it is stated in print on several occasions including, for example, [46, 47]).

Theorem 3.2.1 *[45] The orthogonal of every analytic weak P-ideal $\mathcal{I}$ on $\mathbb{N}$ is countably generated.*

Proof: Fix a tree $T \subseteq [\mathbb{N}]^{<\omega} \otimes [\mathbb{N}]^{<\omega}$ such that $\mathcal{I} = \mathrm{proj}_1 [T]$. For a given subtree W of T (always assumed to be downwards closed) we define the derivative operation $\mathrm{d}\, W$ by

$$\mathrm{d}\, W = W \setminus \{(s,t) \in W : \mathrm{proj}_1 [W(s,t)] \text{ does not hit } \mathcal{I}\}.$$

Let $T_0 = T$, $T^\lambda = \bigcap_{\alpha<\lambda} T^\alpha$ for λ limit and $T^{\alpha+1} = \mathrm{d}\, T^\alpha$. Fix $\beta < \omega_1$ such that $T^{\beta+1} = T^\beta$. For each $\alpha < \beta$ and each $(s,t) \in T^\alpha \setminus T^{\alpha+1}$ fix a set $B_\alpha(s,t) \in \mathcal{I} \cap [\omega]^\omega$ which is not hit by any member of the family $\mathrm{proj}_1 [T^\alpha(s,t)]$. Since $\mathcal{I}$ is a weak P–ideal we can pick $B \in \mathcal{I} \cap [\omega]^\omega$such that $B \cap B_\alpha(s,t)$ is infinite, for all $(s,t) \in T^\alpha \setminus T^{\alpha+1}$. Let $f \in [T]$ be a branch such that $\mathrm{proj}_1(f) = B$. Then $f \in \left[T^\beta\right]$, so in particular $\left[T^\beta\right] \neq \emptyset$. For $(s,t) \in T^\beta$, let

$$N(s,t) = \mathbb{N} \setminus \bigcup \left\{\mathrm{proj}_1 T^\beta(s,t)\right\}.$$

Then $N(s,t) \in \mathcal{I}^{\perp}$ for all $(s,t) \in T^{\beta}$ and $\left\{N(s,t) : (s,t) \in T^{\beta}\right\}$ generates $\mathcal{I}^{\perp}$.

■

3.3 From k to $k_{\aleph_0}$ via weak P-ideals

For a space X (with no isolated points) by $\mathcal{D}_X$ we denote the collection of all closed discrete subsets of X. Note that if X is an analytic space, then $\mathcal{D}_X$ is a proper analytic ideal on X.

Theorem 3.3.1 *For an analytic k-space X, the following are equivalent:*

1. *X is a $k_{\aleph_0}$−space,*
2. *$\mathcal{D}_X$ is a P-ideal*
3. *$\mathcal{D}_X$ is a weak P-ideal.*

Proof:

The implications 1) ⇒ 2) ⇒ 3) are straightforward. As for 3) ⇒ 1), we use Theorem 3.2.1 to pick $\{A_n : n < \omega\}$ a sequence in $\mathcal{D}_X^{\perp}$ that generates $\mathcal{D}_X^{\perp}$. Note that these sets A_n's are compact, and they will witness that X is a $k_{\aleph_0}$ space. Let $Y \subseteq X$ be a non-closed set. Pick a sequence $S \subseteq Y$ such that $S \to x$ and $x \notin Y$. Since $S \in \mathcal{D}_X^{\perp}$, there is $n < \omega$ such that $S \subseteq^* A_n$ and so $x \in A_n$ and $S \subseteq^* A_n$. Therefore, S witnesses that $Y \cap A_n$ is not closed in A_n.

■

Notice that the proof of Theorem 3.3.1 gives us:

Lemma 3.3.2 *Let X be a countable k-space. Suppose that for every $x \in X$, C_x is countably generated in the ideal of precompact subsets of X, $\mathcal{K} = \left\{K \subseteq X : \overline{K} \text{ is compact}\right\}$. Then X is a $k_{\aleph_0}$-space.*

Definition 3.3.3 *A point $x \in X$ is a vD-*point *if there is a sequence $\{D_n : n < \omega\}$ of infinite closed discrete subsets of X such that $x \notin \bigcup_{n<\omega} D_n$ and such that for every open set $U \subseteq X$ containing x, $D_n \subseteq^* U$ for infinitely many n's.*

Lemma 3.3.4 *Let X be a countable sequential space with no isolated points. The following are equivalent:*

1. *$\mathcal{D}_X$ is a weak P-ideal*

2. *X has no vD-points*

Proof:

1) $\Rightarrow$ 2): Consider a point $x \in X$ and a sequence $\{D_n : n < \omega\}$ of infinite closed discrete subsets of X such that $x \notin \bigcup_{n<\omega} D_n$. Chose $D \in \mathcal{D}_X$ such that $D \cap D_n$ is infinite for all $n < \omega$. Then $U = X \setminus (D \setminus \{x\})$ is an open neighbourhood of X such that $D_n \not\subseteq^* U$ for all $n < \omega$.

2) $\Rightarrow$ 1): Let $\{D_n : n < \omega\} \subseteq \mathcal{D}_X \cap [X]^{\aleph_0}$. Let $\{x_n : n < \omega\}$ be an enumeration of X. By 2), there is an open set $U_0 \subseteq X$ containing x_0 such that $D_n^1 = D_n \setminus U_0$ is infinite for all $n < \omega$. Indeed, 2) gives us an open set $\tilde{U}_0$ containing x for which there is $l_0 < \omega$ such that for all $n > l_0$, $D_n \setminus \tilde{U}_0$ is infinite. So we can define

$$U_0 = \tilde{U}_0 \setminus \left[\left(\bigcup_{i \leq l_0} D_i \right) \setminus \{x_0\} \right].$$

We apply 2) again to $\left\{D_n^1 : n < \omega\right\}$ and $x = x_1$ to obtain an open set $U_1 \subseteq X$ containing x_1 such that $D_n^2 = D_n^1 \setminus U_1$ is infinite for all $n < \omega$. In this way we obtain a decreasing sequence $\left\{D_n^k : k < \omega\right\}$ of infinite subsets of D_n such that D_n^{k+1} is disjoint from a neighbourhood U_k of x_k. Then we can pick $D \subseteq \bigcup_{n<\omega} D_n$ such that $D \cap D_n$ is infinite for all $n < \omega$ and such that $D \cap U_k$ is finite for all $k < \omega$. Therefore D is (sequentially) closed and discrete.

■

Theorem 3.3.5 *[37] An analytic sequential space X is a $k_{\aleph_0}$-space if, and only if, X contains no vD-points.*

An example of an analytic sequential space with no vD-points is the free group over the convergent sequence (see [52]).

Remark 3.3.6 *Let us say that a space X is a (weak) PD-space if the ideal $\mathcal{D}$ of countable closed discrete sets is a (weak) P-ideal. Then we have the following reformulation of Theorem 3.3.1*

Theorem 3.3.7 *An analytic k-space is (weak) PD-space if, and only if, it is $k_{\aleph_0}$ space.*

Lemma 3.3.8 *If the product of a countable sequential space X with the sequential fan $S(\omega)$ is sequential, then $\mathcal{D}_X$ is a P-ideal.*

Proof: Suppose $\mathcal{D}_x$ is not a P-ideal and let us fix a sequence $\{D_n : n < \omega\} \subseteq \mathcal{D}$ such that for every $D \in \mathcal{D}_X$, $D_n \not\subseteq^* D$ for infinitely many n's. Then there is a point $x \in X$ such that for every open set $U \subseteq X$ containing x, $U \cap D_n$ is infinite for infinitely many n's. For $n < \omega$, let $\{d_{nm} : m < \omega\}$ enumerate $D_n \setminus \{x\}$. Consider the set $Y = \{(d_{nm}, (n, m)) : n, m < \omega\}$. Then Y is a sequentially closed subset of $X \times S(\omega)$ which is not closed since $(x, \infty) \in \overline{Y} \setminus Y$.

■

Lemma 3.3.9 *If an analytic sequential group X contains a (closed) copy of $S(\omega)$, then $\mathcal{D}_X$ is a (weak) P-ideal.*

Proof:

We will assume X lives on $\mathbb{N}$. Also, we will denote by e the identity of the group. Let $\bigcup_{n<\omega} A_n \cup \{e\}$ be a closed copy of the sequential fan $S(\omega)$ inside X.

Fix a downwards closed subtree T of $[\mathbb{N}]^{<\omega} \otimes [\mathbb{N}]^{<\omega}$ which projects onto the topology of X.

Given a subtree W of T and $(s, t) \in W$, let $A(s, t) = \bigcap \mathrm{proj}_1 [W(s, t)]$. Now let us define the following derivative operation

$$\mathrm{d}\, W = W \setminus \left\{(s, t) \in W : \left|\left\{n < \omega : \left(\overline{A(s, t)}^{-1} \cdot \overline{A(s, t)}\right) \cap A_n \neq \emptyset\right\}\right| = \aleph_0\right\}$$

Let $T^0 = T$, $T^\lambda = \bigcap_{\alpha<\lambda} T^\alpha$ for λ limit, and $T^{\alpha+1} = \mathrm{d}\, T^\alpha$. Let $\beta < \omega_1$ be such that $T^{\beta+1} = T^\beta$. Consider the family

$$\mathcal{F} = \left\{\overline{\bigcap \mathrm{proj}_1 [T^\beta(s, t)]} : (s, t) \in T^\beta\right\}$$

$\mathcal{F}$ is a countable family of closed subsets of X. Let $\{N_k : k < \omega\}$ be an enumeration of $\mathcal{F}$. Since $\mathrm{d}T^\beta = T^\beta$, then for every $k < \omega$, $N_k^{-1} \cdot N_k \cap A_n$ is nonempty for finitely many n's. Let m_k $(k < \omega)$ be a strictly increasing sequence such that for all k, $N_k^{-1} \cdot N_k \cap A_n \neq \emptyset$ implies $n < m_k$.

Claim: given $k, n < \omega$ and $x \in X$, if $|N_k \cap (x \cdot A_n)| = \aleph_0$, then $n < m_k$.

Indeed, if $|N_k \cap (x \cdot A_n)| = \aleph_0$, then $x \in N_k$. Therefore,

$$\left(N_k^{-1} \cdot N_k\right) \cap A_n \supseteq \left(x^{-1} \cdot N_k\right) \cap A_n = x^{-1} \left(N_k \cap (x \cdot A_n)\right) \neq \emptyset$$

and so by the definition of n_k, we have that $n < n_k$.

Suppose, towards a contradiction, that $\mathcal{D}_X$ is not a weak P-ideal. By Lemma 3.3.4, there is a sequence $\{B_n : n < \omega\}$ of closed discrete subsets of X such that $e \notin \bigcup_{n<\omega} B_n$ but for every open $U \subseteq X$ containing e, $B_n \subseteq^* U$ for infinitely many n's.

Since $\bigcup_{n<\omega} A_n \cup \{e\}$ and $\bigcup_{n<\omega} A_{m_n} \cup \{e\}$ are homeomorphic, we may assume that $m_n = n$ $(n < \omega)$, that is, $|N_k \cap A_n| = \aleph_0$ implies $k > n$. For every $n < \omega$, let us fix injective enumerations $A_n = \{a(n,i) : i < \omega\}$ and $B_n = \{b(n,i) : i < \omega\}$. Since $N_k \cap (b(n,i) \cdot A_n)$ is finite for all $k \leq n$ and all $i < \omega$, we can fix a strictly increasing sequence $l_n : \omega \to \omega$ such that $b(n,i) \cdot a(n, l_n(i)) \notin N_k$ for all $k \leq n$ and all $i < \omega$. Set

$$Y = \{b(n,i) \cdot a(n, l_n(i)) : n < \omega, i < \omega\}.$$

Consider an open set U containing the identity e. Let V be an open set such that $V \cdot V \subseteq U$. Find $n < \omega$ such that $B_n \subseteq^* V$. Then there exists $i < \omega$ such that $a(n, l_n(i)), b(n,i) \in V$. Therefore, $b(n,i) \cdot a(n, l_n(i)) \in U$. It follows that $e \in \overline{Y} \setminus Y$. Since X is sequential, there is a sequence S of elements of Y that converges to a point $x \notin Y$.

By definition of the derivative operation, we can choose a set $D \subseteq \bigcup_{n<\omega} A_n$ such that $|D \cap A_n| \leq 1$ for all $n < \omega$, and such that

$$D \cap \left(\bigcap \mathrm{proj}_1 \left[T^\alpha(s,t)\right]\right) \neq \emptyset \text{ for all } \alpha < \beta, \text{ and for all } (s,t) \in T^\alpha \setminus T^{\alpha+1}. \tag{3.1}$$

Since $\bigcup_{n<\omega} A_n \cup \{e\}$ is a closed copy of $S(\omega)$ in X, we have that D is a closed discrete subset of X, that we may assume it does not contain the point x. By regularity of the topology of

X, there is a symmetric open set U such that $(x \cdot U) \cap D = \emptyset$. Then (3.1) implies that $x \cdot U$ is the projection of an infinite branch of T^{β}. Therefore, there must be $(s, t) \in T^{\beta}$ such that $S \subseteq^* \bigcap \mathrm{proj}_1 \left[T^{\beta}(s, t)\right]$. On the other hand, there is $k < \omega$ such that $N_k = \overline{\bigcap \mathrm{proj}_1 \left[T^{\beta}(s, t)\right]}$ and all but finitely many elements of S have the form $b(n, i) \cdot a(n, l_n(i))$ for $n > k$. This contradiction finishes the proof.

$\blacksquare$

Chapter 4

Tukey order and analytic topologies

4.1 Tukey types of analytic topologies

A directed order $(D, \leq)$ is a partial order such that for each $x, y \in D$, there is $z \in D$ with $x, y \leq z$. A set $A \subseteq D$ is cofinal if for each $x \in D$, there is $y \in A$ with $x \leq y$. A set $A \subseteq D$ is bounded if there is $x \in D$ such that for all $y \in A$, $y \leq x$. Of particular interest here are directed sets of the following form, where x is a point in a topological space X :

$$\mathcal{I}_x = \left\{A \subseteq X \setminus \{x\} : x \notin \overline{A}\right\}$$

and

$$C_x = \{C \subseteq X \setminus \{x\} : \forall A \in \mathcal{I}_x \ \ C \cap A \text{ is finite}\}.$$

It turns out that if X is a countable analytic space, then the corresponding ideals $\mathcal{I}_x$ and C_x tend to be related to the directed sets from the following category.

Definition 4.1.1 *[40]*

A directed order D is called basic if

- D is a separable metric space,

- for every $x, y \in D$ the least upper bound $x \vee y$ exists and $\vee : D \times D \to D$ is a continuous function,

- each bounded sequence has a convergent subsequence, and

- each convergent sequence has a bounded subsequence.

Example 4.1.2 *[40] Some examples of basic directed orders:*

- $1 = \{0\}$.

- $\mathbb{N}$.

- $\mathbb{N}^{\mathbb{N}}$ *with the partial order $\alpha \leq \beta$ if, and only if, $\alpha(i) \leq \beta(i)$, for all $i < \omega$.*

- *Analytic P-Ideals (over a countable set) with inclusion as a partial order.*

- $\ell_1 = \left\{A \subseteq \omega : \sum_{n \in A} \frac{1}{1+n} < \infty\right\}$ *ordered with inclusion, with the metric given by*

$$d(A, B) = \sum_{n \in A \triangle B} \frac{1}{n+1}$$

Definition 4.1.3 *Given two directed partial ordered sets D and E, we say that a mapping $f : D \to E$ is* cofinal *if the image of every cofinal subset of D is cofinal in E. We say that a map $f : D \to E$ is a* Tukey map *if it maps unbounded subsets of D into unbounded subsets of E. It is easily seen that for two directed sets D and E there is a Tukey map from D into E if and only if there is a cofinal map from E into D. When this happens, we say that D is* Tukey reducible to *E and we write $D \leq_T E$.*

If $D \leq_T E$ and $E \leq_T D$, then we say that D and E are Tukey equivalent *or that they are* cofinally equivalent*, or that hey have the same* Tukey type *and write $D =_T E$.*

Note that if D is a cofinal subset of a directed set E then D and E are Tukey equivalent. This and the following fact explains why *Tukey types* are also called *cofinal types.*

Theorem 4.1.4 *[53] Two directed sets D and E are Tukey equivalent if, and only if, there is a directed set F such that D and E are isomorphic to cofinal subsets of F.*

In this Chapter, we intend to apply this definition to directed sets of the form [1]

$$\mathcal{I}_x^X = \left\{A \subseteq X : x \notin \overline{A \setminus \{x\}}\right\}.$$

It turns out that this kind of directed sets and their Tukey types are of special interest. In particular they lead naturally to the following definition.

Definition 4.1.5 *Given a topological space X, a directed partial order D, and a point $x \in X$, we say that x has a* Tukey-D-base *at x if $\mathcal{I}_x \leq_T D$.*

We say that X has a Tukey-*D base if it has a Tukey-D base at every point.*

If $\mathcal{I}_x$ is Tukey equivalent to D, then we say that X has Tukey type D at x. *If X has the same Tukey type D at every x, then we say that X has* Tukey type D.

Note that a point x in a topological space X has a countable neighbourhood base if, and only if, the ideal $\mathcal{I}_x$ is countably generated, so it has a cofinal subset isomorphic to 1 or $\mathbb{N}$ depending on whether x is isolated in X or not. So we have the following

Theorem 4.1.6 *1. A point x in a topological space is isolated if, and only if, $\mathcal{I}_x =_T 1$.*

2. A point x in a topological space has countable neighbourhood base if, and only if, $\mathcal{I}_x =_T 1$ or $\mathcal{I}_x =_T \mathbb{N}$.

3. A topological group G is metrizable if, and only if, it is Tukey equivalent to 1 *or* $\mathbb{N}$.

Note also the following immediate fact.

Lemma 4.1.7 *If Y is subspace of a topological space X and $x \in Y$, then*

$$\mathcal{I}_x^Y \leq_T \mathcal{I}_x^X.$$

[1] When the space X is determined by the context, we don't write it in the upper index to the ideal $\mathcal{I}$.

Proof: Define $g : \mathcal{I}_x^X \to \mathcal{I}_x^Y$ by letting $g(A) = A \cap Y$. Then g is a monotone cofinal map from $\mathcal{I}_x^X$ into $\mathcal{I}_x^Y$.

■

It turns out that one determines cofinal types of points in some of the standard examples.

Theorem 4.1.8 *The point $\emptyset$ in the Arens space S_2 has Tukey type $\mathbb{N}^{\mathbb{N}}$.*

Proof: For a strictly increasing $\alpha \in \mathbb{N}^{\mathbb{N}}$, let $f(\alpha)$ be the set of all pairs $\{n, m\}$ in $[\omega]^2$ such that $m \le \alpha(n)$. Then f is a Tukey map from a cofinal subset of $\mathbb{N}^{\mathbb{N}}$ into $\mathcal{I}_\emptyset^{S_2}$. For $A \in \mathcal{I}_\emptyset^{S_2}$, let $g(A)$ by any pair $(k, \alpha) \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ such that if for some n the singleton $\{n\}$ belongs to A or there is infinitely many m's such that $\{n, m\} \in A$ then $n \le k$ and for other n, $\{n, m\} \in A$ implies $m \le \alpha(n)$. Then g is a Tukey map from $\mathcal{I}_\emptyset^{S_2}$ into $\mathbb{N} \times \mathbb{N}^{\mathbb{N}}$. Since $\mathbb{N} \times \mathbb{N}^{\mathbb{N}} =_T \mathbb{N}^{\mathbb{N}}$, we are done.

■

Corollary 4.1.9 *Every countable sequential non Fréchet space X has a point x such that $\mathcal{I}_x \ge_T \mathbb{N}^{\mathbb{N}}$.*

Proof: This follows from Lemma 4.1.7 and Theorem 4.1.8 and the fact that every countable sequential non Fréchet space contains a closed copy of S_2 (see Proposition 2.3.3).

■

Theorem 4.1.10 *The point ∞ in the sequential fan $S(\omega)$ has Tukey type $\mathbb{N}^{\mathbb{N}}$.*

Proof: Recall that a typical neighbourhood of ∞ in $S(\omega)$ is determined by an $\alpha \in \mathbb{N}^{\mathbb{N}}$ by

$$U_\alpha = \{(n, m) : m > \alpha(n)\}.$$

This gives us an isomorphism between a cofinal subset of $\mathcal{I}_\infty^{S(\omega)}$ and $\mathbb{N}^{\mathbb{N}}$.

■

Theorem 4.1.11 *The free topological group over the convergent sequence is a sequential non-metrizable group Tukey equivalent to $\mathbb{N}^{\mathbb{N}}$.*

Proof: The free topological group over the converging sequence is a $k_{\aleph_0}$-space, so by Theorem 4.1.16 below, it is Tukey reducible to $\mathbb{N}^{\mathbb{N}}$. The other Tukey inequality follows from Corollary 4.1.9.

■

The following easy fact is also worth pointing out.

Lemma 4.1.12 *If $\mathcal{I}$ is an ideal over $\mathbb{N}$, and $\mathcal{I} \leq_T D$, for some basic order D, there is a monotone map $g : D \to \mathcal{I}$ whose range is cofinal in $\mathcal{I}$ and therefore a monotone cofinal map witnessing that $\mathcal{I}$ is Tukey below D.*

Proof: Let $f : \mathcal{I} \to D$ be a Tukey map. Then for $d \in D$, the set $\{A \in \mathcal{I} : f(A) \leq_D d\}$ is bounded in $\mathcal{I}$ so its union

$$g(d) = \bigcup \{A \in \mathcal{I} : f(A) \leq_D d\}$$

belongs to $\mathcal{I}$. It is clear that $g : D \to \mathcal{I}$ is a monotone map with cofinal range in $\mathcal{I}$.

■

Theorem 4.1.13 *[40] Let D be a separable metric space with a partial order such that the set of predecessors of each element in D is compact. Let E be a basic order. If $D \leq_T E$ and E is analytic, then D is analytic.*

Theorem 4.1.14 *Let X be a countable space such that for all $x \in X$ the ideal $\mathcal{I}_x$ is Tukey reducible to $\mathbb{N}^{\mathbb{N}}$. Then X is an analytic space, and for all $x \in X$, the gap $(\mathcal{I}_x, C_x)$ is countably separated.*

Proof:

Let $x \in X$. Note that $\mathcal{I}_x \subseteq 2^X$ is a separable metric space and if we order $\mathcal{I}_x$ with inclusion, given a set $A \in \mathcal{I}_x$, then the set of predecessors of A is contained in $\mathcal{I}_x$ and it is closed in 2^X, so it is compact. By Theorem 4.1.13 we have that $\mathcal{I}_x$ is analytic. Hence, the family

$$\mathcal{B}_x = \{O \subseteq X : \exists A \in \mathcal{I}_x \text{ such that } O = (X \setminus A) \cup \{x\}\}$$

is a neighbourhood base at x that is analytic. Since X is countable, the topology of X is analytic.

On the other hand, given $x \in X$, let $f : \mathbb{N}^{\mathbb{N}} \to \mathcal{I}_x$ be a monotone cofinal map witnessing $\mathcal{I}_x \leq_T \mathbb{N}^{\mathbb{N}}$. For every $s \in \mathbb{N}^{<\mathbb{N}}$, let $[s] = \left\{\alpha \in \mathbb{N}^{\mathbb{N}} : s \sqsubseteq \alpha\right\}$. Then the following countable family separates $(\mathcal{I}_x, C_x)$

$$\left\{\bigcup_{\alpha\in[s]} f(\alpha) : s \in \mathbb{N}^{<\mathbb{N}}\right\}.$$

To see this, take $A \in \mathcal{I}_x$ and $S \in C_x$ and let $\alpha' \in \mathbb{N}^{\mathbb{N}}$ be such that $A \subseteq f(\alpha')$. Then there must be a finite sequence $s \sqsubseteq \alpha'$ satisfying that

$$\left(\bigcup_{\alpha\in[s]} f(\alpha)\right) \cap S$$

is finite. Otherwise, we can find a sequence $\{\alpha_n : n < \omega\}$ converging (in the metric topology of $\mathbb{N}^{\mathbb{N}}$) to α' such that $S \cap f(\alpha_n) \neq \emptyset$ for infinitely many n's. Since $\mathbb{N}^{\mathbb{N}}$ is a basic order, there is a subsequence $\{\alpha_{n_k} : k < \omega\}$ bounded above by some $\alpha'' \in \mathbb{N}^{\mathbb{N}}$. Therefore, $S \cap f(\alpha'')$ is infinite. This contradiction finishes the proof.

■

Recall that an analytic sequential group that is not Fréchet has a closed copy of $S(\omega)$. Therefore, using the previous result and the results of Chapter 3, we get the following fact.

Theorem 4.1.15 *Every sequential non-Fréchet group Tukey reducible to $\mathbb{N}^{\mathbb{N}}$ is a $k_{\aleph_0}$-space.*

We should note the analogous result to Theorem 4.1.15 for the class of analytic groups was first proved by Shibakov [37] using different methods.

Let us now examine Tukey types of $k_{\aleph_0}$-spaces

Theorem 4.1.16 *Let X be a countable $k_{\aleph_0}$-space. Then for every $x \in X$, $\mathcal{I}_x \leq_T \mathbb{N}^{\mathbb{N}}$.*

Proof:

Let $X = \bigcup_n X_n$ be a $k_{\aleph_0}$ decomposition of X and let τ_n denote the subspace topology of X_n. Fix $x \in X$ and assume, without loss of generality that $x \in X_0$. For every $n < \omega$, let $\left\{O_k^n \subseteq X : k < \omega\right\}$ be a decreasing sequence of open sets in X such that $\left\{O_k^n \cap X_n : k < \omega\right\}$ is a local τ_n-base at x. Given $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$, let $U_n^\alpha = \bigcap_{i \leq n+1} O_{\alpha(i)}^i$. For $\alpha \in \mathbb{N}^{\mathbb{N}}$, let

$$U_\alpha = \bigcup_n \left(U_n^\alpha \cap X_n\right).$$

Note that U_α is a set open in X that contains x. Therefore, if we define $f : \mathbb{N}^{\mathbb{N}} \to \mathcal{I}_x$ by $f(\alpha) = X \setminus U_\alpha$, then f is a cofinal map, i.e. it maps cofinal sets to cofinal sets. Indeed, given $C \subseteq \mathbb{N}^{\mathbb{N}}$ cofinal and $B \in \mathcal{I}_x$, there is $O \subseteq X$ open in X such that $x \in O$ and $O \cap B = \emptyset$. For every $n < \omega$, let $k_n < \omega$ be such that $O_{k_n}^n \cap X_n \subseteq O \cap X_n$. If we define $\alpha(n) = k_n$, then $B \subseteq f(\alpha)$. Take $\beta \in C$ such that $\alpha \leq \beta$. Then $B \subseteq f(\beta)$.

■

Lemma 4.1.17 *If X is a countable $k_{\aleph_0}$-space, then for every $x \in X$ the ideal $\mathcal{C}_x$ is an F_σ-ideal.*

From Theorem 3.3.1 and Lemma 3.3.9 we obtain

Theorem 4.1.18 *Let X be a countable k-group with identity e. If $\mathcal{I}_e \leq_T \mathbb{N}^{\mathbb{N}}$, then either X is metrizable or X is a $k_{\aleph_0}$-space.*

Theorem 4.1.19 *Let X be an analytic sequential group. Then X has Tukey type* 1, $\mathbb{N}$, *or* $\mathbb{N}^{\mathbb{N}}$.

Proof: Let e be the identity of the group and note that the ideal $\mathcal{I}_e$ is countably generated if, and only if, the the group is first countable, i.e., metrizable, whence $\mathcal{I}_e$ has Tukey type 1 or $\mathbb{N}$, according to Theorem 4.1.6. Therefore, if $\mathcal{I}_x$ is not Tukey equivalent to either 1 or $\mathbb{N}$, from Theorem 3.1.2, X is not Fréchet so by Corollary 4.1.9, we have $\mathbb{N}^{\mathbb{N}} \leq_T \mathcal{I}_x$. Moreover, from Corollary **??** and Theorem 4.1.16 we have that $\mathcal{I}_x \leq_T \mathbb{N}^{\mathbb{N}}$.

■

This theorem and Theorem 4.1.13 give us the following corollary.

Corollary 4.1.20 *A countable group with sequential topology is analytic if, and only if, its topology is Tukey equivalent to* 1, $\mathbb{N}$, *or* $\mathbb{N}^{\mathbb{N}}$.

Let us point out that in [31, 56] it is shown that there are $\aleph_1$-many non homeomorphic $k_{\aleph_0}$ groups that are not metrizable. Therefore there are $\aleph_1$-many groups with Tukey type $\mathbb{N}^{\mathbb{N}}$.

Question 4.1.21 *Is there an example of a countable topological group so that $\mathcal{I}_e$ is Tukey equivalent to ℓ_1? Since ℓ_1 is an analytic basic order, this group would have an analytic topology. Moreover, since ℓ_1 not Tukey equivalent to $\mathbb{N}^{\mathbb{N}}$, such group could not be sequential by Corollary 4.1.20.*

We would like to finish this section calculating the Tukey type of a non-sequential topological group.

Example 4.1.22 *Let $G \subseteq 2^{2^{\mathbb{N}}}$ denote the the Boolean group consisting of all clopen subsets of the Cantor set $2^{\mathbb{N}}$ with the symmetric difference as the group operation. For every dense set $K \subseteq 2^{\mathbb{N}}$, consider the topology τ_K given by the subbase having the sets of the form*

$$x^+ = \{a \in G : x \in a\} \quad \text{and} \quad x^- = \{a \in G : x \notin a\},$$

where $x \in K$.

Fix $K \subseteq 2^N$ and order $[K]^{<\omega}$, the set of finite subsets of K, with inclusion. Given $a \in G$, we will prove that $[K]^{<\omega} \leq_T \mathcal{I}_a^{(G,\tau_k)}$. In order to do so, let us define a Tukey map

$$\phi : \left[2^{\mathbb{N}}\right]^{<\omega} \longrightarrow \mathcal{I}_a^{(G,\tau_k)}$$

by

$$\phi(k) = G \setminus \left(\bigcap \{x^+ : x \in a \text{ and } x \in k\} \cap \bigcap \{x^- : x \notin a \text{ and } x \in k\}\right).$$

We will prove that preimage of bounded sets is bounded. Let $\mathcal{B} \subseteq \mathcal{I}_a$ be a bounded set. Then there is a τ_K-basic open set $U \subseteq G$ containing a such that for all $A \in \mathcal{B}$, $A \subseteq G \setminus U$. Let $y_0, \ldots, y_{n_0}, z_0, \ldots, z_{n_1} \in K$ be such that

$$U = \bigcap_{i \leq n_0} y_i^+ \cap \bigcap_{j \leq n_1} z_j^-.$$

Consider the set

$$k_0 = \{y_0, \ldots, y_{n_0}\} \cup \{z_0, \ldots, z_{n_1}\} \in [K]^{<\omega}.$$

Clearly $\phi(k_0) = U$. *We claim that* k_0 *bounds* $\phi^{-1}[\mathcal{B}]$. *Otherwise, there would be* $k \in \phi^{-1}[\mathcal{B}]$ *such that* $k \nsubseteq k_0$. *Take* $x \in k \setminus k_0$. *Assume* $x \in a$ *and note that*

$$U \subseteq G \setminus \phi(k) \subseteq x^+.$$

Recall that for a given sequence $s \in 2^{<\mathbb{N}}$, *the set* $[s] = \left\{y \in 2^{\mathbb{N}} : s \sqsubseteq y\right\}$ *is clopen in* $2^{\mathbb{N}}$. *Furthermore, since* $k \subseteq K$ *is finite, there is* $m < \omega$ *(*m *can be thought as the* m*-th level of the tree* $2^{<\mathbb{N}}$*) such that for all* $y, z \in k \cup \{x\}$, *with* $y \neq z$, *we have that*

$$[y \restriction m] \cap [z \restriction m] = \emptyset.$$

Consider the set

$$a_0 = \bigcup_{i \leq n_0} [y_i \restriction m].$$

a_0 *is a clopen set in* $2^{\mathbb{N}}$. *Moreover,* $a_0 \in U$ *but* $a_0 \notin x^+$, *a contradiction.*

If we assume that $x \notin a$, *then*

$$U \subseteq G \setminus \phi(k) \subseteq x^-$$

and we can use a similar idea to find $a_0 \in U$ *such that* $a_0 \notin x^-$.

Thus, for any $K \subseteq 2^{\mathbb{N}}$ *we have that*

$$[K]^{<\omega} \leq_T \mathcal{I}_a^{(G,\tau_K)}.$$

If $K \subseteq 2^{\mathbb{N}}$ *is an analytic uncountable dense set, then* K *has cardinality* $\mathfrak{c}$. *Therefore,*

$$[\mathfrak{c}]^{<\omega} \leq_T \mathcal{I}_a^{(G,\tau_K)}.$$

Since $\mathcal{I}_a^{(G,\tau_K)}$ *has cardinality at most* $\mathfrak{c}$, *then it is always the case that*

$$\mathcal{I}_a^{(G,\tau_K)} \leq_T [\mathfrak{c}]^{<\omega}.$$

In consequence, $\mathcal{I}_a^{(G,\tau_K)} =_T [\mathfrak{c}]^{<\omega}$.

As shown in example 5.6 in [51] there is an analytic non-Borel dense set $K_0 \subseteq 2^{\mathbb{N}}$ *such that* τ_{K_0} *is analytic non-Borel. Therefore, the space* (G, τ_{K_0}) *has maximal Tukey type and its topology is analytic non-Borel. On the other hand,* $\left(G, \tau_{(2^{\mathbb{N}})}\right)$ *also has maximal Tukey type and its topology is* $F_{\sigma\delta}$. *Finally, note that Corollary 4.1.20 shows that these examples, that have maximal Tukey type, cannot be sequential, a fact can be proven in a direct way.*

4.2 Tukey order and metrizability of groups

We will now turn our attention to a special class of topological groups that will allow us to obtain a characterization of separable metrizable groups in terms of the Tukey ordering. Recall that for topological groups, being Hausdorff is equivalent to be regular Hausdorff. So we may just assume our topological groups are Hausdorff.

Definition 4.2.1 *[16] Given a topological space X, and* $x \in X$, *a family*

$$\mathcal{U} = \left\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$$

is called a $\mathfrak{G}-$base at x *if* $\mathcal{U}$ *is a base of neighbourhoods at x and for all* $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$, $\alpha \leq \beta$ *implies that* $U_\beta \subseteq U_\alpha$. *If the space has a* $\mathfrak{G}-$*base at each point, we say that X has a* $\mathfrak{G}-$base.

Lemma 4.2.2 *A point x of a topological space X has a* $\mathfrak{G}-$*base if, and only if,* $\mathcal{I}_x \leq_T \mathbb{N}^{\mathbb{N}}$.

Proof:

Suppose $\mathcal{U} = \left\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$ is a $\mathfrak{G}-$base at x. Then $\alpha \mapsto U_\alpha$ is cofinal map from $\mathbb{N}^{\mathbb{N}}$ into $\mathcal{I}_x$, so $\mathcal{I}_x \leq_T \mathbb{N}^{\mathbb{N}}$. Conversely, suppose $f : \mathcal{I}_x \to \mathbb{N}^{\mathbb{N}}$ is a Tukey reduction. For $\alpha \in \mathbb{N}^{\mathbb{N}}$, set

$$U_\alpha = X \setminus \bigcup \{A \in \mathcal{I}_x : f(A) \leq \alpha\}.$$

Then $\mathcal{U} = \left\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$ is a $\mathfrak{G}-$base at x.

■

It is not difficult to prove that having a Tukey-D-base is inherited by subspaces. In addition to that, when we consider topological groups, having a Tukey-D-base reduces to having one at the identity.

The main purpose of this section is to prove the following theorem, in which we do not require the group to be countable but separable.

Theorem 4.2.3 *Let X be a separable topological group. The following are equivalent statements*

1. *X is metrizable*
2. *[52, 49] X is Fréchet and it has a countable dense subgroup that inherits an analytic topology.*
3. *X is Fréchet and its neighbourhood base at the identity is Tukey reducible to $\mathbb{N}^{\mathbb{N}}$.*
4. *X is Fréchet and its neighbourhood base at the identity is Tukey reducible to D for some basic order D that is analytic.*

Proof:

1) $\Rightarrow$ 2) is clear and 2) $\Rightarrow$ 3) can be obtained from Theorem 4.1.19. 3) $\Rightarrow$ 4) is also direct.

For the implication 4) $\Rightarrow$ 1), let D be a basic order that is analytic and such that there is a Tukey-D-base at the identity. Let G be a countable dense subgroup of X. Then $\mathcal{I}_e \leq_T D$, where $\mathcal{I}_e = \left\{A \subseteq G : e \notin \overline{A \setminus \{e\}}\right\}$. By Theorem 4.1.14, the topology of G is analytic. Thus, X is metrizable by Lemma 3.1.1.

■

4.3 Sequential order of groups Tukey below $\mathbb{N}^{\mathbb{N}}$

Recall that an analytic space is homeomorphic to a subspace of $C_p\left(\mathbb{N}^{\mathbb{N}}\right)$ (see Theorem 2.1.1). Additionally, it was proven in [14] that given a topological space X, the space $C_p(X)$ has sequential order 1 or ω_1. Moreover, in [35] the author showed that a sequential topological group with a point-countable k-network is either metrizable of it has sequential order ω_1. In this section we use their methodology to study the sequential order of groups Tukey below $\mathbb{N}^{\mathbb{N}}$.

Given two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of ω (or some countable set), we say that $\mathcal{A}$ and $\mathcal{B}$ are orthogonal if for each $A \in \mathcal{A}$ and each $B \in \mathcal{B}$, $A \cap B$ is finite. We say that $\mathcal{A}$ and $\mathcal{B}$ are countably separated (see [45]) if there is a countable family $\{M_n \subseteq \omega : n < \omega\}$ such that for all $(A, B) \in \mathcal{A} \times \mathcal{B}$ there is $n < \omega$ such that $A \subseteq^* M_n$ (i.e. $A \setminus M_n$ is finite) and $M_n \cap B$ is finite. Given a countable topological space (X, τ) and $x \in X$, C_x and $\mathcal{I}_x$ are orthogonal families. In section we study the sequential order of a sequential topological group Tukey below $\mathbb{N}^{\mathbb{N}}$.

Theorem 4.1.13 gives us that a topological group Tukey below the analytic directed order $\mathbb{N}^{\mathbb{N}}$ has analytic topology. If in addition to that we assume the group is sequential, then the topology has only two possible Tukey types: $\mathbb{N}$ or $\mathbb{N}^{\mathbb{N}}$ (see Theorem 4.1.19). The sequential order of countable group with topology Tukey equivalent to $\mathbb{N}$ is 1 as in this case the group is metrizable. For countable groups having a topology Tukey equivalent to $\mathbb{N}^{\mathbb{N}}$ we have the following theorem.

Theorem 4.3.1 *Let (X, τ) be a countable group with sequential topology that is Tukey equivalent to $\mathbb{N}^{\mathbb{N}}$. Then $so(X) = \omega_1$.*

Proof:

By Theorem 4.1.14 there is a family $\mathcal{M} = \{M_n : n < \omega\}$ such that for any $x \in X$, any sequence $C \in C_x$, and any open set $O \subseteq X$ containing x, there exists $n < \omega$ such that $C \subseteq^* M_n \subseteq O$.

We will prove that there are subsets of X of sequential order as high as we want. More precisely, we will prove the following:

$$\forall \alpha < \omega_1 \, \forall x \in X \, \forall \, U \in \tau \, \forall C \in C_x \cap \mathcal{P}(U), \exists A \subseteq U \text{ such that} \tag{4.1}$$

$\flat$. $\overline{A} \setminus \{x\} \subseteq U$, bb. $x \in [A]_{\alpha+1} \setminus \bigcup\limits_{\xi \leq \alpha} [A]_{\xi}$, and

bbb. $\sup\limits_{c \in C} so(c, A) = \alpha$.

Case $\alpha = 1$: fix $x \in X$, an open set $U \in \tau$, and a sequence $(x_n)_n$ contained in U and converging to x. We will prove there is a closed copy of S_2

$$P = \{x_{\langle m,n \rangle} : m, n < \omega\} \cup \{x_{\langle m \rangle} : m < \omega\} \cup \{x_{\emptyset}\}$$

such that:

- Every $t \in S_2$ corresponds to x_t, and $x_{\emptyset} = x$,
- for all $m < \omega$, $x_m = x_{\langle m \rangle}$, and
- $P \setminus \{x\} \subseteq U$

Theorems 4.1.13 and 3.1.2 give us that a countable sequential group with topology Tukey equivalent to $\mathbb{N}^{\mathbb{N}}$ is not Fréchet. From this and Lemma 2.3.4, there is a closed copy $F \subseteq X$ of the sequential fan $S(\omega)$. Given that a topological space is homogeneous, we may assume that $\emptyset$ corresponds to e, the identity element of X. Let $F = \{w(m, n) : m, n < \omega\} \cup \{e\}$, where $w(m, n) \neq w(l, p)$ for all $(m, n) \neq (l, p)$, and $\lim_{n \to \infty} w(m, n) = e$, for all $m < \omega$. Let $x_{\langle m,n \rangle} = w(m, n) \cdot x_m$, where " $\cdot$ " is the group operation. It is not difficult to see that

$$P = \{x_{\langle m,n \rangle} : m, n < \omega\} \cup \{x_{\langle m \rangle} : m < \omega\} \cup \{x_{\emptyset}\}$$

is a copy of S_2. Since X is countable, we may assume P is closed in X. Now, shrink P if necessary to get that $\bigcup\limits_{m<\omega} \{x_{\langle m,n \rangle} : n < \omega\} \subseteq U$. This proves the case $\alpha = 1$.

Suppose 4.1 holds for all $\beta < \alpha < \omega_1$. We will analyze the successor case $\alpha = \mu + 1$ first. Let us fix $x \in X$ and $U \in \tau$, and $C = \{x_m : m < \omega\} \in C_x$ such that $C \subseteq U$. We use the case $\alpha = 2$ to get a closed copy of S_2 inside U satisfying 4.1. We will denote this copy by

$$P = \{x_{\langle m,n \rangle} : m, n < \omega\} \cup \{x_{\langle m \rangle} : m < \omega\} \cup \{x_{\emptyset}\},$$

where the sequence $t \in S_2$ corresponds to x_t, for all $m < \omega$, $x_{\langle m \rangle} = x_m$, and $x = x_\emptyset$.

Since P is a closed copy of S_2 in X, we may assume there is a sequence $(U_m)_m$ of open subsets of X such that, for all $m \neq k$, we have that $\overline{U_m} \cap \overline{U_k} = \emptyset$, and for all $m < \omega$, $\overline{U_m} \subseteq U$ and $\{x_{\langle m,n \rangle} : n < \omega\} \cup \{x_{\langle m \rangle}\} \subseteq U_m$ and $x \notin U_m$.

For every $D \subseteq X$, let us denote by $\{M_l(D) : l < \omega\}$ the family consisting of all sets $M \in \mathcal{M}$ such that $M \cap D \neq \emptyset$ and for some $m_D < \omega$, $\overline{M} \cap \{x_{\langle m,n \rangle} : m \geq m_D \text{ and } n < \omega\} = \emptyset$.

Claim 1: there is a strictly increasing sequence $(n_i)_i$ and a sequence of sets $(A_i)_i$ such that for all $i < \omega$ we have:

1. $x_{\langle n_i \rangle} \in [A_i]_{\mu+1} \setminus \bigcup_{\xi \leq \mu} [A_i]_\xi$, and
2. $\overline{A_i} \setminus \{x_{\langle n_i \rangle}\} \subseteq U_{n_i} \setminus \bigcup_{k,l<i} \overline{M_k\left(\overline{A_l}\right)}$.

Proof of claim 1: Define $n_0 = 0$ and A_0 as any set given by the induction hypothesis applied to U_0 and the sequence $\{x_{\langle 0,n \rangle} : n < \omega\}$. Assume that for all $j < i$, n_j and A_j have been defined satisfying the desired properties. For every $k, l < i$ let $m(k,l) < \omega$ be such that

$$\overline{M_k\left(\overline{A_l}\right)} \cap \{x_{\langle m,n \rangle} : m \geq m(k,l) \text{ and } n < \omega\} = \emptyset.$$

If we let $m_0 = \max\left(\{m(k,l) : k,l < i\} \cup \left\{n_j : j < i\right\}\right)$, then

$$\left(\bigcup_{k,l<i} \overline{M_k\left(\overline{A_l}\right)}\right) \cap \{x_{\langle m,n \rangle} : m \geq m_0 \text{ and } n < \omega\} = \emptyset.$$

We will let $n_i = m_0 + 1$. Using the induction hypothesis there is a set A_i satisfying that

- $A_i \subseteq U_{n_i} \setminus \bigcup_{k,l<i} \overline{M_k\left(\overline{A_l}\right)}$,
- $\overline{A_i} \setminus \{x_{\langle n_i \rangle}\} \subseteq U_{n_i} \setminus \bigcup_{k,l<i} \overline{M_k\left(\overline{A_l}\right)}$ and
- $x_{\langle n_i \rangle} \in [A_i]_{\mu+1} \setminus \bigcup_{\xi \leq \mu} [A_i]_\xi$.

- $\sup_{n<\omega} so\left(x_{\langle m_0+1,n\rangle}, A_i\right) = \mu.$

□

Notice that the definition of the sequence $(U_m)_m$ implies that, for all $i < \omega$,

$$\overline{A_i} \cap \left(\left\{x_{\langle j\rangle} : j \neq n_i\right\} \cup \{x\}\right) = \emptyset.$$

and for all $j \neq i$, $\overline{A_j} \cap \overline{A_i} = \emptyset$.

Now, given that any countable subset of $\mathbb{N}^{\mathbb{N}}$ is almost dominated, there is a function $f : \mathbb{N} \to \mathbb{N}$ such that if $E = \{x_{\langle k,l\rangle} : k < \omega, \text{ and } l \leq f(k)\}$, then for all $n < \omega$ either

- $\overline{M_n} \cap E = \emptyset$, or

(4.2)

- there is $m < \omega$ such that $\overline{M_n} \cap \{x_{\langle k,l\rangle} : k \geq m \text{ and } l < \omega\} = \emptyset$

Let $K = \bigcup_{i<\omega} \overline{A_i} \cup \{x\}$.

Claim 2: For every $i < \omega$, let $z_i \in \overline{A_i} \setminus \{x_{n_i}\}$. If $z_i \to z$, then $z \in E$.

Proof of claim 2: Since $E = \{x_{\langle k,l\rangle} : k \in \omega, \text{ and } l \leq f(k)\} \subseteq S_2$, then E is closed in X. Therefore, if $z \in X \setminus E$, then there is an open set $O \subseteq X$ such that $z \in O \subseteq \overline{O} \subseteq X \setminus E$. By construction of $\mathcal{M}$, there is there is $n^* < \omega$ such that $M_{n^*} \subseteq O$ and $\{z_i : i < \omega\} \subseteq^* M_{n^*}$. Hence, $\overline{M_{n^*}} \cap E = \emptyset$. By 4.2, there is $m < \omega$ such that

$$\overline{M_{n^*}} \cap \{x_{\langle k,l\rangle} : k \geq m \text{ and } l < \omega\} = \emptyset$$

Notice that this implies that for all $i < \omega$ such that $z_i \in M_{n^*}$, there is $n_i < \omega$ such that $M_{n^*} = M_{n_i}\left(\overline{A_i}\right)$. Fix one such $i < \omega$. If $j > \max\{i, n_i\}$ is such that $z_j \in M_{n^*}$, then $z_j \notin \overline{M_{n_i}\left(\overline{A_i}\right)} = \overline{M_{n^*}}$, a contradiction.

□

Note that **Claim 2** implies that $K \cup E$ is a closed subset of X. If we let A denote the set $\bigcup_i A_i$, then we must have that $\overline{A} \setminus \{x\} \subseteq U$.

Claim 3: Let $\gamma < \omega_1$, and $z \in [A]_{\gamma+1} \setminus \bigcup_{\xi \le \gamma} [A]_\xi$. If $z \notin E \cup \{x\}$, then there is $p_z < \omega$ such that $z \in \left[A_{p_z}\right]_{\gamma+1}$.

Proof of claim 3: this is clear for $\gamma = 1$. Suppose this claim holds for all $\beta < \gamma$ and fix $z \in [A]_{\gamma+1} \setminus \bigcup_{\xi \le \gamma} [A]_\xi$ such that $z \notin E \cup \{x\}$. Assume $\gamma = \mu + 1$. Let $\left(w_j\right)_j$ be a sequence in $[A]_\gamma \setminus \bigcup_{\xi \le \mu} [A]_\xi$ converging to z. From the induction hypothesis, for each $j < \omega$, there is $p_j < \omega$ such that $w_j \in \left[A_{p_j}\right]_\gamma$.

Since E is closed and $z \notin E$, we may assume $w_j \notin E$, for all $j < \omega$. On the other hand, for every $j < \omega$, $w_j \in \overline{A} \subseteq K \cup E$, and so $w_j \in K$. Again, since $z \neq x$, we may assume $w_j \neq x$ for all $j < \omega$. Therefore, every w_j belongs to $\bigcup_i \overline{A_i}$. In addition to that, we may assume that $\left\{w_j : j < \omega\right\} \cap \{x_{\langle n_i \rangle} : i < \omega\} = \emptyset$. Therefore, every w_j belongs to $\bigcup_i \left(\overline{A_i} \setminus \{x_{\langle n_i \rangle}\}\right)$. Recall that $\left(\overline{A_i}\right)_i$ is a pairwise disjoint sequence. As a consequence of this, for every $j < \omega$, we may let $p'_j < \omega$ denote the only one index such that

$$w_j \in \overline{A_{p'_j}} \setminus \left\{x_{\left\langle n_{p'_j} \right\rangle}\right\}.$$

Note that if $\left\{p'_j : j < \omega\right\}$ were infinite, then **Claim 2** would imply that $z \in D$, which is impossible. Therefore, $\left\{p'_j : j < \omega\right\}$ is finite. Hence, we may assume that there is $p < \omega$ such that $w_j \in \overline{A_p} \setminus \left\{x_{\langle n_p \rangle}\right\}$, for all $j < \omega$. This implies that every p_j is equal to p. Let $p_z = p$. Since $w_j \in \left[A_{p_z}\right]_\gamma$, for all $j < \omega$, we have that $z \in \left[A_{p_z}\right]_{\gamma+1}$. The limit case is done in a similar way.

□

Claim 4: $x \in [A]_{\alpha+1} \setminus \bigcup_{\xi \le \alpha} [A]_\xi$.

Proof of claim 4: **Claim 1** gives us that $x \in [A]_{\alpha+1}$. Suppose $(z_i)_i$ is a sequence converging to x such that for all $i < \omega$, there is $\gamma_i \leq \mu$ satisfying that

$$z_i \in [A]_{\gamma_i+1} \setminus \bigcup_{\xi \leq \gamma_i} [A]_\xi .$$

Since $x \notin E$, we may assume that $\left\{z_j : j < \omega\right\} \cap E = \emptyset$. Therefore, we can use **Claim 3** to obtain $p_i < \omega$ such that $z_i \in \left[A_{p_i}\right]_{\gamma_i+1} \subseteq \overline{A_{p_i}}$. Since $x \notin \overline{A_{p_i}}$, we have that $\left\{z_j : j < \omega\right\} \cap \overline{A_{p_i}}$ is finite, for every $i < \omega$. Again, using that $x \notin E$ and **Claim 3**, we have that

$$\left\{z_j : i < \omega\right\} \subseteq^* \left\{x_{n\langle i\rangle} : i < \omega\right\}.$$

We will actually assume that $\left\{z_j : j < \omega\right\} \subseteq \left\{x_{n\langle i\rangle} : i < \omega\right\}$. But this together with the definition of $(A_i)_i$ implies that $\gamma_i \geq \mu$, whence $\gamma_i = \mu$, for all (almost) $i < \omega$.

□

The case α limit is done in a similar way and so we will omit it.

■

4.4 Tukey type of sequential topological spaces

Any ideal $\mathcal{I}$ over $\mathbb{N}$ (or any countable set) we can be identified with a topological space with base set $X = \mathbb{N} \cup \{\infty\}$. The topology, denoted by $\tau_{\mathcal{I}}$, is defined as follows. Every point in $\mathbb{N}$ is declared to be isolated, and the filter of neighbourhoods of ∞ is determined by the dual filter of $\mathcal{I}$, denoted by $\mathcal{I}^*$, that is,

$$\tau_\infty = \{A \subseteq \mathbb{N} \cup \{\infty\} : \mathbb{N} \setminus (A \cup \{\infty\}) \in \mathcal{I} \text{ and } \infty \in A\}.$$

If $\mathcal{I}^*$ is non-principal, this is a Hausdorff topology with Borel complexity at least the one of $\mathcal{I}$. It is not difficult to see that $\mathcal{I}_\infty = \mathcal{I}$. Therefore, $\tau_{\mathcal{I}}$ is a Fréchet topology if, and only if, $\mathcal{I}$ is a Fréchet ideal $\left(\text{i.e., } \mathcal{I} = (\mathcal{I}^\perp)^\perp\right)$.

This idea can be extended to construct other more complex topological spaces. Perhaps the most well known example is the Arhangel'ski-Franklin space S_ω (and its variations) which have been studied by several people (see for instance [2, 25, 38, 52]). We will follow the presentation given in [52] where they study a topology $\tau_{\mathcal{F}}$ on $\mathbb{N}^{<\omega}$ where $\mathcal{F}$ is a filter over $\mathbb{N}$, such that $(\mathbb{N}^{<\omega}, \tau_{\mathcal{F}})$ is a sequential space if, and only if, $\mathcal{F}$ is a Fréchet filter (i.e. its dual ideal is Fréchet).

Given a filter $\mathcal{F}$ on $\mathbb{N}$ containing the cofinite sets, we define the following topology on $\mathbb{N}^{<\omega}$. An nonempty set $U \subseteq \mathbb{N}^{<\omega}$ is open if

$$(\forall t \in U)\,(\{n \in \mathbb{N} : t^\frown n \in U\} \in \mathcal{F})\,.$$

We denote this topology by $\tau_{\mathcal{F}}$.

The prototypical sequential space of sequential rank ω_1 is the well known Arhangel'ski-Franklin space S_ω which turns out to be homeomorphic to $(\mathbb{N}^{<\omega}, \tau_{\mathcal{F},})$, where $\mathcal{F}$ is the filter of cofinite sets.

Proposition 4.4.1 *[52] Let $\mathcal{F}$ be a filter on $\mathbb{N}$ containing the cofinite sets. Then*

(i) $(\mathbb{N}^{<\omega}, \tau_{\mathcal{F}})$ *is T_2, zero dimensional and has no isolated points.*

(ii) $(\mathbb{N}^{<\omega}, \tau_{\mathcal{F}})$ *is sequential if, and only if, $\mathcal{F}$ is a Fréchet filter.*

(iii) There is no group structure on $\mathbb{N}^{<\omega}$ compatible with $(\mathbb{N}^{<\omega}, \tau_{\mathcal{F}})$ for $\mathcal{F}$ Fréchet.

(iv) The space $(\mathbb{N}^{<\omega}, \tau_{\mathcal{F}})$ is homogeneous.

(v) If $\mathcal{F}$ is Borel, then $\tau_{\mathcal{F}}$ is Borel (as a subset of $2^{\mathbb{N}^{<\omega}}$).

The purpose of this section is to calculate the Tukey type of $\tau_{\mathcal{F}}$ when $\mathcal{F}$ is a filter on $\mathbb{N}$ which is Freéchet and contains the cofinite sets. To that end, we will use the following known fact.

Lemma 4.4.2 *Consider the direct orders $\left(\mathbb{N}^{\mathbb{N}}\right)^{\mathbb{N}}$ and $\left[\left(\mathbb{N}^{\mathbb{N}}\right)^{\mathbb{N}}\right]^{\mathbb{N}}$ ordered coordinate wise. Then*

$$\mathbb{N}^{\mathbb{N}} =_T \left(\mathbb{N}^{\mathbb{N}}\right)^{\mathbb{N}} =_T \left[\left(\mathbb{N}^{\mathbb{N}}\right)^{\mathbb{N}}\right]^{\mathbb{N}}.$$

Theorem 4.4.3 *If $\mathcal{F}$ is a filter Tukey reducible to $\mathbb{N}^{\mathbb{N}}$, then $\mathcal{I}_{\emptyset}^{\tau_{\mathcal{F}}}$ is Tukey reducible to $\mathbb{N}^{\mathbb{N}}$.*

Proof: We will prove that the dual filter of $\mathcal{I}_{\emptyset}^{\tau_{\mathcal{F}}}$, denoted by $\mathcal{F}_{\emptyset}$, is Tukey reducible to $\left[\left(\mathbb{N}^{\mathbb{N}}\right)^{\mathbb{N}}\right]^{\mathbb{N}}$. Then the result will follow from Lemma 4.4.2.

Let us fix a cofinal monotone map $g : \mathbb{N}^{\mathbb{N}} \to \mathcal{F}$. Given $\varphi \in \left[\left(\mathbb{N}^{\mathbb{N}}\right)^{\mathbb{N}}\right]^{\mathbb{N}}$ let $f(\varphi) \in \mathcal{F}_{\emptyset}$ be defined by levels, that is,

$$f(\varphi) = \bigcup_{l<\omega} L_l$$

where $L_l \subseteq \mathbb{N}^{[l]}$, the set of sequences of length l. We put $L_0 = \{\emptyset\}$. If L_l has been defined, we can consider an enumeration $\{s_m : m < \omega\}$ of L_l and we define L_{l+1} by

$$L_{l+1} = \bigcup_{m<\omega} g\left(\varphi_{(l+1)m}\right)$$

Then f is a monotone map from a cofinal subset of $\left[\left(\mathbb{N}^{\mathbb{N}}\right)^{\mathbb{N}}\right]^{\mathbb{N}}$ into $\mathcal{F}_{\emptyset}$. Thus,

$$\mathcal{F}_{\emptyset} \leq_T \left[\left(\mathbb{N}^{\mathbb{N}}\right)^{\mathbb{N}}\right]^{\mathbb{N}}.$$

■

Corollary 4.4.4 *$\tau_{\mathcal{F}}$ has Tukey type $\mathbb{N}^{\mathbb{N}}$ whenever $\mathcal{F}$ is a Fréchet filter containing the cofinite sets that is Tukey reducible to $\mathbb{N}^{\mathbb{N}}$.*

Proof:

From Proposition 4.4.1 we have that if $\mathcal{F}$ is a Fréchet filter, then $\tau_{\mathcal{F}}$ is a sequential space that is not a Fréchet topological space. From this and Theorem 4.1.8 we obtain that the Tukey type of $\tau_{\mathcal{F}}$ is at least $\mathbb{N}^{\mathbb{N}}$. The remaining Tukey reduction follows from Theorem 4.4.3.

■

When $\mathcal{F}$ consists of the filter that contains only the cofinite subsets of $\mathbb{N}$ we have that $(\mathbb{N}^{<\omega}, \tau_{\mathcal{F}})$ is homeomorphic to S_ω, the Arhangel'ski-Franklin space. Therefore, the Tukey type of S_ω is $\mathbb{N}^{\mathbb{N}}$.

We now turn our attention to computing the Tukey type of a family of Fréchet ideals found in [18]. This will provide examples of filters for which Theorem 4.4.3 and Corollary 4.4.4 can be applied.

Definition 4.4.5 *Given a countable set X, let $\{K_n : n \in F\}$ be a partition of X, where $F \subseteq \mathbb{N}$. For $n \in F$, let I_n be an ideal on K_n. The* direct sum*, denoted by $\bigoplus_{n \in F} I_n$, is defined by*

$$A \in \bigoplus_{n \in F} I_n \Leftrightarrow (\forall n \in F)\,(A \cap K_n \in I_n)\,.$$

In general, given a sequence of ideals I_n over a countable set X_n, we define $\oplus_n I_n$ by taking a partition $\{K_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ and an isomorphic copy I'_n of I_n on K_n and let $\oplus_n I_n$ be $\oplus_n I'_n$. It should be clear that $\oplus_n I_n$ is, up to isomorphism, independent of the partition and the copy used.

Definition 4.4.6

1. *Given a family $\mathcal{A}$ subset of $\mathcal{P}(\mathbb{N})$, we define $\mathcal{A}^\perp$ as the family of all subsets B of $\mathbb{N}$ such that for all $A \in \mathcal{A}$, we have that $A \cap B$ is finite.*

2. *An ideal $\mathcal{I}$ is* Fréchet *if $\mathcal{I} = \mathcal{I}^{\perp\perp}$.*

   Notice that $\mathcal{A}^\perp$ is a Fréchet ideal whenever $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$.

Definition 4.4.7 *An ideal $\mathcal{I}$ is said to be* Borel (or F_σ, G_δ, analytic, et cetera) *if it is so when viewed as a subset of $\mathcal{P}(\mathbb{N})$, where $\mathcal{P}(\mathbb{N})$ is identified with $2^{\mathbb{N}}$ via characteristic functions.*

Example 4.4.8

- *Fin, the ideal of finite subsets of $\mathbb{N}$, is F_σ.*
- *$\emptyset \times Fin = \oplus_n \mathcal{F}_n$, where every $\mathcal{F}_n$ is isomorphic to Fin, is $F_{\sigma\delta}$*
- *Consider the family $\mathfrak{B}$ of ideals containing Fin and closed under the operations of direct sums and orthogonal. It is shown in [18] that this family has $\aleph_1$-many non-isomorphic Fréchet ideals of Borel complexity at most $F_{\sigma\delta}$.*

The Tukey type of Fin is $\mathbb{N}$ and the Tukey type of $\emptyset \times Fin$ is $\mathbb{N}^{\mathbb{N}}$. The next results will allow us to compute the Tukey type of the other ideals in $\mathfrak{B}$.

Proposition 4.4.9 *Suppose $I = \bigoplus_k I_k$. If $I_k \leq_T \mathbb{N}^{\mathbb{N}}$, for all $k < \omega$, then $I \leq_T \mathbb{N}^{\mathbb{N}}$.*

Proof: Let $(N_k)_k$ be a partition of $\mathbb{N}$ into infinite sets such that $A \in \mathcal{I}$ iff for all $k < \omega$ $A \cap N_k \in \mathcal{I}_k$.

Since $\mathbb{N}^{\mathbb{N}} \leq_T N_k^{\mathbb{N}}$ (they are actually Tukey equivalent), we can fix a cofinal map $f_k : \mathbb{N}_k^{\mathbb{N}} \to \mathcal{I}_k$, $k < \omega$. This f_k will be assumed to be monotone.

Let us define a cofinal map $f : \mathbb{N}^{\mathbb{N}} \to \mathcal{I}$ as follows:

Given $\alpha \in \mathbb{N}^{\mathbb{N}}$, consider $\widehat{\alpha}_k = \alpha \restriction N_k \in \mathbb{N}_k^{\mathbb{N}}$, $k < \omega$, and let

$$f(\alpha) = \bigcup_k f_k(\widehat{\alpha}_k)$$

In order to prove that f is a cofinal map, we fix a cofinal set $C \subseteq \mathbb{N}^{\mathbb{N}}$ and $A \in \mathcal{I}$. Since f_k is a cofinal map, there is $\widehat{\alpha}_k \in N_k^{\mathbb{N}}$ such that $A \cap N_k \subseteq f_k(\widehat{\alpha}_k)$, $k < \omega$. Now let $\alpha \in \mathbb{N}^{\mathbb{N}}$ be defined by $\alpha(r) = \widehat{\alpha}_k(r)$, for $r \in N_k$, $k < \omega$. Choose $\beta \in C$ such that $\alpha \leq \beta$ in the $\mathbb{N}^{\mathbb{N}}$ order. Given $k < \omega$, and $r \in N_k$, we have that $\alpha(r) \leq \beta(r)$, and so $\widehat{\alpha}_k(r) \leq \widehat{\beta}_k(r)$, that is, $\widehat{\alpha}_k \leq \widehat{\beta}_k$ in the $\mathbb{N}_k^{\mathbb{N}}$ order. Since f_k is monotone, this implies that $A \cap N_k \subseteq f_k\left(\widehat{\beta}_k\right)$. Thus, $A \subseteq f(\beta)$. This proves f is a cofinal map and finishes the proof.

■

Proposition 4.4.10 *Let $\mathcal{J} = \left(\bigoplus_k \mathcal{I}_k\right)^{\perp}$. If $\mathcal{I}_k^{\perp} \leq_T \mathbb{N}^{\mathbb{N}}$, for all $k < \omega$, then $\mathcal{J} \leq_T \mathbb{N}^{\mathbb{N}}$.*

Proof: Let $(N_k)_k$ be a partition of $\mathbb{N}$ into infinite sets such that $A \in \mathcal{J}^{\perp}$ if, and only if, there is $k_0 < \omega$ such that $A \subseteq \bigcup_{k \leq k_0} N_k$, and for all $k \leq k_0$, $A \cap N_k \in \mathcal{I}_k^{\perp}$ (see Lemma 2.2 in [18]).

For every $k < \omega$, let $f_k : N_k^{\mathbb{N}} \to \mathcal{I}_k^{\perp}$ be a cofinal map (recall we this map is assumed to be monotone). Let us define $f : \mathbb{N}^{\mathbb{N}} \to \mathcal{J}$ as follows: given $\alpha \in \mathbb{N}^{\mathbb{N}}$, we let $\widehat{\alpha}_k = \alpha \restriction N_k$ and

$$f(\alpha) = \bigcup_{k \leq \alpha(0)} f_k(\widehat{\alpha}_k).$$

Let $C \subseteq \mathbb{N}^{\mathbb{N}}$ be a cofinal set and $A \in \mathcal{J}$. Denote by $k_0 < \omega$ the first integer such that $A \subseteq \bigcup_{k \leq k_0} N_k$, and for all $k \leq k_0$, $A \cap N_k \in \mathcal{I}_k^{\perp}$. For every $k \leq k_0$, let $\widehat{\alpha}_k \in N_k^{\mathbb{N}}$ be such that

$$A \cap N_k \subseteq f_k(\widehat{\alpha}_k).$$

Now we will define $\alpha \in \mathbb{N}^{\mathbb{N}}$ so that $A \subseteq f(\alpha)$. Pick $k_1 < \omega$ such that $0 \in N_{k_1}$. If $k_1 \leq k_0$, then $\alpha(0) = \max\{k_0, \widehat{\alpha}_{k_1}(0)\}$, and if $k_1 > k_0$, then $\alpha(0) = k_0$. On the other hand, for $k \leq k_0$, and

$r \in N_k \setminus \{0\}$, we put $\alpha(r) = \widehat{\alpha}_k(r)$. Choose $\beta \in C$ so that $\alpha \leq \beta$. If $k \leq \alpha(0)$ and $r \in N_k$, then $\alpha(r) \leq \beta(r)$, i.e., $\widehat{\alpha}_k(r) \leq \widehat{\beta}_k(r)$, whence $\widehat{\alpha}_k \leq \widehat{\beta}_k$. Therefore,

$$A \cap N_k \subseteq f_k(\widehat{\alpha}_k) \subseteq f_k\left(\widehat{\beta}_k\right).$$

Hence, $A \subseteq f(\beta)$. This proves that f is a cofinal map and ends the proof.

■

Corollary 4.4.11 *Every ideal in the family $\mathfrak{B}$ is Tukey reducible to $\mathbb{N}^{\mathbb{N}}$.*

Proof:

Recall that $1 \leq_T \mathbb{N} \leq_T \mathbb{N}^{\mathbb{N}}$, so $\mathcal{P}(\mathbb{N})$ and Fin are Tukey both reducible to $\mathbb{N}^{\mathbb{N}}$. Additionally, the other ideals in $\mathfrak{B}$ are obtained from Fin by taking orthogonal and direct sums. This together with Propositions 4.4.9 and 4.4.10 give us the desired conclusion.

■

The following lemma will give us a lower bound to the Tukey type of most ideals in $\mathfrak{B}$. Recall that given an ideal $\mathcal{J}$ on $\mathbb{N}$ and $K \subseteq \mathbb{N}$ the restriction of $\mathcal{J}$ to K, denoted by $\mathcal{J} \upharpoonright K$, is the ideal given by

$$\mathcal{J} \upharpoonright K = \{A \cap K : A \in \mathcal{J}\} = \{B \subseteq K : B \in \mathcal{J}\}.$$

Lemma 4.4.12 *Given an ideal $\mathcal{J}$ and $K \subseteq \mathbb{N}$, we have that $\mathcal{J} \upharpoonright K \leq_T \mathcal{J}$.*

Proof: Let $f : \mathcal{J} \to \mathcal{J} \upharpoonright K$ given by $f(A) = A \cap K$. Given $C \subseteq \mathcal{J}$ cofinal and $B \in \mathcal{J} \upharpoonright K$, since $B \in \mathcal{J}$, there is $C \in C$ such that $B \subseteq C$. Therefore, $B = B \cap K \subseteq C \cap K = f(C)$. Thus, f is a cofinal Tykey map, that is, $\mathcal{J} \upharpoonright K \leq_T \mathcal{J}$.

■

Corollary 4.4.13 *$\mathbb{N}^{\mathbb{N}}$ is Tukey below every ideal in $\mathfrak{B}$ that is not countably generated.*

Proof: Given an ideal $\mathcal{I} \in \mathfrak{B}$ not countably generated, we have that $\mathcal{I}$ is isomorphic to $\mathcal{I} \oplus (\emptyset \times Fin)$ (see Lemma 3.3 in [18]). Therefore, there is a set $K \subseteq \mathbb{N}$ such that $\emptyset \times Fin \cong \mathcal{I} \upharpoonright K$. From this and Lemma 4.4.12 we have that $\mathbb{N}^{\mathbb{N}} =_T \emptyset \times Fin \leq_T \mathcal{I}$.

■

Corollaries 4.4.11 and 4.4.13 give us the following result.

Theorem 4.4.14 *Let $\mathcal{I}$ be an ideal in $\mathfrak{B}$.*

1. *If $\mathcal{I}$ is not countably generated, then $\mathcal{I}$ is Tukey equivalent to $\mathbb{N}^{\mathbb{N}}$.*

2. *Fin and $Fin \times \emptyset$ are Tukey equivalent to $\mathbb{N}$.*

3. *$\mathcal{P}(\mathbb{N})$ is Tukey equivalent to 1.*

Proposition 4.4.15 *(Proposition 5.3 in [18]) Let $\mathcal{I}$ and $\mathcal{J}$ be ideals in $\mathfrak{B}$ such that both $\mathcal{I}$ and $\mathcal{J}$ are obtained as a countable infinite sum of ideals in $\mathfrak{B}$. If $\mathcal{I}$ is not isomorphic to $\mathcal{J}$, then $(\mathbb{N}^{<\omega}, \tau_{\mathcal{I}^*})$ is not homeomorphic to $(\mathbb{N}^{<\omega}, \tau_{\mathcal{J}^*})$.*

Example 4.4.16 *Note that Theorems 4.4.3 and 4.4.14 together with Proposition 4.4.15 give us $\aleph_1$-many different examples of sequential topologies with Tukey type $\mathbb{N}^{\mathbb{N}}$.*

The following proposition is a consequence of Theorem 4.1.13.

Proposition 4.4.17 *If an ideal over a countable set is Tukey below $\mathbb{N}^{\mathbb{N}}$, then it is analytic.*

Corollary 4.4.18 *If an ideal $\mathcal{I}$ and its orthogonal are Tukey below $\mathbb{N}^{\mathbb{N}}$, and $\mathcal{I}$ is Fréchet, then both $\mathcal{I}$ and $\mathcal{I}^{\perp}$ are Borel.*

The hypotheses of Corollary 4.4.18 is realized by any ideal in $\mathfrak{B}$ but there are ideals of low Borel complexity that do not realize it, as the following example shows.

Example 4.4.19 *Let $\mathcal{I}_d$ be the ideal on $\mathbb{N}^{<\mathbb{N}}$ generated by the dominated sets, that is,*

$$\mathcal{I}_d = \left\{A \subseteq \mathbb{N}^{<\mathbb{N}} : \exists \alpha \in \mathbb{N}^{\mathbb{N}} \text{ such that } A \subseteq \left\{s \in \mathbb{N}^{<\mathbb{N}} : s(i) \leq \alpha(i), \text{ for all } i < |s|\right\}\right\}$$

Then $\mathcal{I}_d$ is Tukey equivalent to $\mathbb{N}^{\mathbb{N}}$. To show that $\mathcal{I}_d \leq_T \mathbb{N}^{\mathbb{N}}$, let $f : \mathbb{N}^{\mathbb{N}} \to \mathcal{I}_d$ be defined by

$$f(\alpha) = \left\{s \in \mathbb{N}^{<\mathbb{N}} : s(i) \leq \alpha(i), \text{ for all } i < |s|\right\}.$$

f is a cofinal map, Indeed, given a cofinal set $C \subseteq \mathbb{N}^{\mathbb{N}}$, and $A \in \mathcal{I}_d$, we can let $\alpha \in \mathbb{N}^{\mathbb{N}}$ be the leftmost branch such that $A \subseteq \left\{s \in \mathbb{N}^{<\mathbb{N}} : s(i) \leq \alpha(i), \text{ for all } i < |s|\right\}$. Pick $\beta \in C$ such that $\alpha \leq \beta$ in the $\mathbb{N}^{\mathbb{N}}$ order. Then $A \subseteq f(\alpha) \subseteq f(\beta)$. Thus, f is a cofinal map.

To prove that $\mathbb{N}^{\mathbb{N}}$ is Tukey reducible to $\mathcal{I}_d$ we can use Lemma 4.4.12 and the fact that $\emptyset \times Fin$ is isomorphic to a restriction of I_d (see Theorem 4.5 in [18]).

Note that $\mathcal{I}_d$ is an $F_{\sigma\delta}$ ideal that is Fréchet, and its orthogonal is the ideal $\mathcal{I}_{wf}$ generated by the well-founded trees. This ideal is complete co-analytic and so it is not Tukey below $\mathbb{N}^{\mathbb{N}}$.

Question 4.4.20

1. *If an ideal $\mathcal{I}$ is Fréchet and both $\mathcal{I}$ and its orthogonal are Tukey below $\mathbb{N}^{\mathbb{N}}$, does $\mathcal{I}$ belong to the family $\mathfrak{B}$?*
2. *If an ideal $\mathcal{I}$ is Fréchet and both $\mathcal{I}$ and its orthogonal are Tukey below $\mathbb{N}^{\mathbb{N}}$, What is the Borel complexity of $\mathcal{I}$ and $\mathcal{I}^{\perp}$? Note that if the previous question has a positive answer, this question would also be answered.*
3. *If an ideal $\mathcal{I}$ is Fréchet, Borel, with Borel orthogonal, and $\mathcal{I}$ is Tukey below $\mathbb{N}^{\mathbb{N}}$, is $\mathcal{I}^{\perp}$ Tukey below $\mathbb{N}^{\mathbb{N}}$?*

4.5 Countably separated ideals

This section gives an application of Theorem 4.1.14.

Definition 4.5.1 *[22, 54] An ideal $\mathcal{I}$ is* countably separated *if there is a countable collection $\{X_n \subseteq \mathbb{N} : n < \omega\}$ such that for all $A \in \mathcal{I}$ and all $B \notin \mathcal{I}$, there is n such that $A \cap X_n = \emptyset$ and $B \cap X_n \notin \mathcal{I}$.*

Definition 4.5.2

1. *Given two families $\mathcal{A}$ and $\mathcal{B}$ subsets of $\mathcal{P}(\mathbb{N})$ we say that $\mathcal{A}$ and $\mathcal{B}$ are* orthogonal families *if for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$, we have that $A \cap B$ is finite.*
2. *A pair $(\mathcal{A}, \mathcal{B})$ of orthogonal families is* countably separated *if there is a countable family $\{X_n \subseteq \mathbb{N} : n < \omega\}$ such that for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$ there is $n < \omega$ such that the sets $A \cap X_n$ and $B \setminus X_n$ are both finite.*

When $A \setminus B$ is a finite set we say that A is almost contained in B and we write $A \subseteq^* B$.

Proposition 4.5.3 *A Fréchet ideal $\mathcal{I}$ is countably separated whenever the pair $(\mathcal{I}, \mathcal{I}^\perp)$ is countably separated.*

Proof: Let $\{X_n : n < \omega\}$ be a family that countably separates $(\mathcal{I}, \mathcal{I}^\perp)$ and let $\{a_m : m < \omega\}$ be an enumeration of Fin, the collection of finite subsets of $\mathbb{N}$. For every $n, m < \omega$, consider the set $X_{nm} = X_n \setminus a_m$. We claim that the family $\{X_{nm} : n, m < \omega\}$ countably separates $\mathcal{I}$. Indeed, given $A \in \mathcal{I}$ and $B \notin \mathcal{I}$, since $\mathcal{I}$ is Fréchet, there is an infinite set $B' \subseteq B$ in $\mathcal{I}^\perp$. Let n be such that $A \cap X_n$ is finite and $B' \subseteq^* X_n$. Take $m < \omega$ such that $a_m = A \cap X_n$. Then $A \cap X_{nm} = \emptyset$ and $B' \subseteq^* X_{mn}$. Therefore, $B' \cap X_{nm} \in \mathcal{I}^\perp$ is infinite, whence $B \cap X_{nm} \notin \mathcal{I}^{\perp\perp} = \mathcal{I}$.

■

If a pair $(\mathcal{A}, \mathcal{B})$ is countably separated by $\{X_n : n < \omega\}$, then the pair $(\mathcal{B}, \mathcal{A})$ is countably separated by $\{\mathbb{N} \setminus X_n : n < \omega\}$. This gives us the following lemma.

Lemma 4.5.4 *If $\mathcal{I}$ is a Fréchet ideal such that the pair $(\mathcal{I}, \mathcal{I}^\perp)$ is countably separated, then $\mathcal{I}^\perp$ is countably separated.*

Proof: This follows from the fact that $(\mathcal{I}^\perp, \mathcal{I})$ is countably separated, $\mathcal{I} = \mathcal{I}^{\perp\perp}$ and Proposition 4.5.3.

■

Theorem 4.5.5 *If a Fréchet ideal $\mathcal{I}$ is Tukey reducible to $\mathbb{N}^\mathbb{N}$, then $\mathcal{I}$ is countably separated and analytic. Moreover, $\mathcal{I}^\perp$ is also countably separated.*

Proof: Since $\mathbb{N}^\mathbb{N}$ is an analytic basic order we have that $\mathcal{I}$ is analytic from Proposition 4.4.17. The countably separability of $\mathcal{I}$ and $\mathcal{I}^\perp$ follows from Theorem 4.1.14, Proposition 4.5.3 and Lemma 4.5.4.

■

Example 4.5.6 *The ideal $\mathcal{I}_d$ is Tukey reducible to $\mathbb{N}^{\mathbb{N}}$ and it is Fréchet. Its orthogonal is the ideal $\mathcal{I}_{wf}$ generated by the well-founded trees on $\mathbb{N}$. I_{wf} is a complete co-analytic ideal (see [12]) and from Theorem 4.5.5 it is countably separated. Since ideals Tukey reducible to $\mathbb{N}^{\mathbb{N}}$ are analytic we have that $\mathcal{I}_{wf}$ is not Tukey reducible to $\mathbb{N}^{\mathbb{N}}$.*

Example 4.5.7 *All ideals in the family $\mathfrak{B}$ are countably separated. This follows from the fact that those ideals are Tukey reducible to $\mathbb{N}^{\mathbb{N}}$.*

Remark 4.5.8 *Theorem 4.5.5 gives an answer to Question* 9.23 *of [54].*

4.6 Analytic selectively separable spaces

We begin our study of selectively separable spaces by analyzing the minimal size of a π-base in a definable sequential space that is selectively separable. We will reproduce some known proofs for the sake of completeness.

Let X be an analytic sequential space with no isolated points. We assume X lives on ω and fix a downwards closed subtree T of $[\mathbb{N}]^{<\omega} \otimes [\mathbb{N}]^{<\omega}$ which projects onto the topology of X. Given a subtree W of T consider the following derivative operation:

$$\mathrm{d}\, W = W \setminus \left\{(s,t) \in W : \mathrm{int}\left(\bigcap \mathrm{proj}_1\left[W(s,t)\right]\right) \neq \emptyset\right\},$$

where $\mathrm{proj}_1 : [\mathbb{N}]^{<\omega} \otimes [\mathbb{N}]^{<\omega} \to [\mathbb{N}]^{<\omega}$ is the projection onto the first coordinate, $W(s,t)$ is the subtree of W consisting of all nodes compatible with (s,t), and $[W(s,t)]$ is the collection of all infinite branches of $W(s,t)$. Let us define $T^0 = T$ and let T^α, for $\alpha < \omega_1$, be the sequence of all subtrees of T determined by $T^\lambda = \bigcap_{\alpha<\lambda} T^\alpha$ for λ limit and $T^{\alpha+1} = \mathrm{d}\, T^\alpha$. Note that this derivative operation must be constant from some point on, i.e., there is $\beta < \omega_1$ such that $T^{\beta+1} = T^\beta$.

From Corollary 4.1.20 we have the following result.

Proposition 4.6.1 *The character of an analytic sequential group is either $\aleph_0$ or $\mathfrak{d}$.*

We will prove in this section (see Corollary 4.6.9) that an analytic sequential group with character less than $\mathfrak{d}$ is metrizable in a different way, using the notion of selective separability. On the other hand, from [7] we have that:

Proposition 4.6.2 *[7] Each countable space with $\pi-$weight less than $\mathfrak{d}$ has the following property:*

Given a sequence of dense sets $\{D_n : n \in \omega\}$, there is a selection sequence

$$\{E_n \in [D_n]^{<\omega} : n \in \omega\}$$

so that $\bigcup\limits_n E_n$ is dense.

Definition 4.6.3 *[33] A space X is called* selectively separable, *if given a sequence $\{D_n : n \in \omega\}$ of dense subsets of X, there is a selection sequence $\{E_n \in [D_n]^{<\omega} : n \in \omega\}$ such that $\bigcup_n E_n$ is dense.*

The following proposition is also a known fact:

Proposition 4.6.4 *[3, 7] Each space with countable $\pi-$weight is selectively separable.*

Proposition 4.6.1 implies that an analytic sequential group with character less than $\mathfrak{d}$ has a countable base, so in particular a countable π-base. The following theorem gives a more general version via Proposition 4.6.4.

Theorem 4.6.5 *Let X be an analytic sequential space. If X is selectively separable, then X has countable $\pi-$weight.*

Proof:

Let $\mathcal{B} = \left\{ \operatorname{int}\left(\overline{\bigcap \operatorname{proj}_1 \left[T^\alpha (s,t)\right]} \right) : \alpha < \beta, (s,t) \in T^\alpha \setminus T^{\alpha+1} \right\}$.

Claim 1: Given $z \in X$, if $\mathcal{B}$ is not a local $\pi-$base at z, then there is $(s_1, t_1) \in T^\beta$ such that $z \in s_1$.

PROOF: There is a nonempty regular open set U not containing any member of $\mathcal{B}$ and such that $z \in U$.

Given $(s, t) \in T$, put

$$O(s,t) = \operatorname{int}\left(\overline{\bigcap \operatorname{proj}_1 [T^\alpha (s,t)]}\right)$$

There is a branch b of $[T]$ so that $U = \operatorname{proj}_1 (b)$. Then there is (s_0, t_0) initial segment of b such that $z \in s_0$.

If there is $n_0 > |(s_0, t_0)|$ such that $b \upharpoonright n_0 \in T \setminus T'$, then $O(b \upharpoonright n_0) \subseteq U$ and $O(b \upharpoonright n_0) \in \mathcal{B}$. Therefore, we may assume

$$\forall n > |(s_0, t_0)|, b \upharpoonright n \in T'$$

If there is $n_0 > |(s_0, t_0)|$ such that $b \upharpoonright n_0 \in T' \setminus T''$, then $O(b \upharpoonright n_0) \subseteq U$ and $O(b \upharpoonright n_0) \in \mathcal{B}$. Therefore, we may assume

$$\forall n > |(s_0, t_0)|, b \upharpoonright n \in T''$$

Continuing this way we may assume that for all $\alpha < \beta$, and for all $|(s_0, t_0)| < n < \omega$, $b \upharpoonright n \in T^\alpha$. Since β is a successor ordinal, then this implies that $b \upharpoonright n \in T^\beta$, for all $n < \omega$ (recall T^β is a downwards closed tree). Thus, there is $(s_1, t_1) \in T^\beta$ such that $z \in s_1$. $\square$

Fix $x \in X$. For $(s, t) \in T^\beta$, let $N(s,t) = \bigcap \operatorname{proj}_1 \left[T^\beta (s,t)\right]$. Those sets are nowhere dense by the choice of β. Therefore, we can find a decreasing sequence $(U_n)_n$ of dense-open sets not containing x such that for all $(s, t) \in T_\beta$, there is n such that $U_n \cap N(s,t) = \emptyset$. Since our space is selectively separable, there is a selection $\{E_n \in [U_n]^{<\omega} : n < \omega\}$ such that $D = \bigcup_n E_n$ is dense.

Claim 2: $\mathcal{B}$ is a local π-base at each point $z \in \overline{D} \setminus D$.

PROOF: We proceed by induction on the sequential order of each point. Take $z \in [D]_1 \setminus D$ and suppose $\mathcal{B}$ is not a local π−base at z. By claim 1, there is $(s_1, t_1) \in T^\beta$ such that $z \in s_1$. Fix an infinite $S \subseteq D$ converging to z. By the choice of D we must have that $S \cap N(s,t)$ is finite for all $(s, t) \in T^\beta$. However, there must be $(s, t) \in T^\beta$ such that $S \subseteq^* N(s,t)$ or else we would get a branch of T^β whose first coordinate is an open set containing x which does not almost contain S. This contradiction shows that $\mathcal{B}$ is a local π-base at z.

Suppose the result for any point $y \in \overline{D} \setminus D$ having sequential order less than α ($\alpha > 1$). Take $z \in [D]_\alpha \setminus D$ and U a regular open set containing z. We may assume that

$$z \in [D]_\alpha \setminus \bigcup_{\beta<\alpha} [D]_\beta .$$

Therefore, there must be an infinite sequence $(z_n)_n$ converging to z such that for all $n \in \omega$ we have that

$$z_n \in \bigcup_{\beta<\alpha} [D]_\beta \setminus D.$$

By inductive hypothesis we have that $\mathcal{B}$ is a local π-base at z_n for each $n < \omega$. Finally, the convergence of $(z_n)_n$ to z guarantees the existence of $O \in \mathcal{B}$ such that $O \subseteq U$. $\square$

To finish the proof, use claim 2 and notice that $x \in \overline{D} \setminus D$.

$\blacksquare$

From the proof of Theorem 4.6.5 we see that if nwd (X), the ideal of nowhere dense subsets of X, were a p-ideal, then we could obtain the same conclusion of the cited theorem. However, we have the following result.

Theorem 4.6.6 *(Theorem 8.5, [51]) Let (X, τ) be a countable Hausdorff topological space without isolated points. If τ is an analytic topology, then the ideal nwd (X) is not a p-ideal.*

Example 4.6.7 *Consider the set $G \subseteq 2^{2^{\mathbb{N}}}$ consisting of all clopen subsets of the Cantor set $2^{\mathbb{N}}$ with the topology τ obtained as a subset of $C_p\left(2^{\mathbb{N}}\right)$. This space has the following properties:*

- *τ is a Borel topology (example 5.6 of [51]).*
- *(G, τ) is not sequential. In fact, the following set is sequentially closed but is not closed: $B = \bigcup_n B_n$, where*

$$B_n = \left\{ \bigcup_{i<n} [s_i] : \left(\forall i < n, s_i \in 2^{<\mathbb{N}}\right) \text{ and } |s_i| = n \right\}$$

and for any $s \in 2^{<\mathbb{N}}$, $[s] = \left\{z \in 2^{\mathbb{N}} : s \sqsubseteq z\right\}$.

- *(G, τ) is selectively separable (Corollary 2.15 of [7]).*

- *τ has uncountable $\pi-$weight.*

Therefore, we cannot remove sequentiality in Theorem 4.6.5.

Corollary 4.6.8 *Let (G, τ) be a topological group with analytic sequential topology. If (G, τ) is selectively separable, then G is metrizable.*

Proof: Assume G is an analytic sequential group which is selectively separable. By Theorem 4.6.5, G has a countable π-base V_n $(n < \omega)$. Then $V_n V_n^{-1}$ $(n < \omega)$ is a countable neighbourhood base of the identity of G. So X is metrizable by the Birkhoff-Kakutani theorem.

■

From Proposition 4.6.4 and Corollary 4.6.8 we obtain:

Corollary 4.6.9 *An analytic sequential group is metrizable if, and only if, its character is less than $\mathfrak{d}$.*

Theorem 4.6.10 *[7] If X and Y are countable selectively separable spaces, and $\pi w(Y) < \mathfrak{b}$, then $X \times Y$ is selectively separable.*

Proposition 4.6.4 together with Theorems 4.6.5 and 4.6.10 give us:

Corollary 4.6.11

1. *Finite products of analytic sequential spaces that are selectively separable is selectively separable.*
2. *Let X and Y be countable selectively separable spaces. If Y is sequential and has analytic topology, then $X \times Y$ is selectively separable.*

■

From the previous corollary we have that among such spaces we cannot find a counterexample to the question made in Problem 5.1 of [7].

Proposition 4.6.12 *[7] Every separable Fréchet space is selectively separable.*

Corollary 4.6.13 *[37] Every analytic Fréchet space has countable π-weight.*

Remark 4.6.14 *In Corollary 4.6.13 we may try to replace Fréchetness by sequentiality to see if the result is still true. If the space has sequential order ω_1 the result is not true. Indeed, the Arkhangelski-Franklin space S_ω has sequential order ω_1 and a standard diagonalization process shows that it does not have a countable π-base.*

On the other hand, if we assume that $1 < so(X) < \omega_1$ we do not know if the result is still true. If we look at the test spaces S_k (see [52]) we have that $so(S_k) = k$ and $S_k \cap \mathbb{N}^k$ is a dense set of isolated points whence $\mathcal{B} = \left\{\{t\} : t \in S_k \cap \mathbb{N}^k\right\}$ is a countable π-base for S_k. An example of a countable sequential space with sequential order less than ω_1 without isolated points could be constructed adding a copy of $\mathbb{Q}$ above every isolated point in S_2. This example however still has a countable π-base.

Chapter 5

k-spaces in forcing extensions

In this chapter we present some results about countable sequential spaces without the assumption that they are analytic.

5.1 Sequential spaces in Cohen extensions

In [8], the authors proved the product of two countable Fréchet spaces is selectively separable. Similar proof shows that the product of two countable sequential spaces that are selectively separable is again selectively separable. In [8], the authors also showed that if we add Cohen reals over a model of CH all countable Frécher spaces have π-weight at most ω_1. Using their methodology we get the following result.

Theorem 5.1.1 *If we add Cohen reals to a model of CH in the resulting model, every countable space that is sequential and selectively separable has $\pi-$weight at most ω_1.*

Proof:

Assume the ground model satisfies CH and assume, we force with the poset $P = Fn(\aleph_2, 2)$ of finite partial functions from ω_2 into 2. Let $\dot{\tau}$ be a P-name for a topology on ω so that $X = (\omega, \dot{\tau})$ is forced to be a sequential space that is selectively separable. Let $\dot{A}_n$ denote a P-name that is forced to be the collection of all sequences converging to $n \in \omega$. Since $Fn(\lambda, 2)$

is completely embedded in $Fn(\xi, 2)$, for $\lambda < \xi$, we can use Corollary 4.5 of [6]. Let $\theta = 2^{\mathfrak{c}^+}$ and $M \prec H_\theta$ be an elementary submodel such that $M^\omega \subseteq M$ and $|M| = \omega_1$. Furthermore, let's assume that X, $\dot{\tau}$, and $\left\{\dot{A}_n : n < \omega\right\}$ are in M. We claim that $\dot{\tau} \cap M$ is forced to be a $\pi-$base for $\dot{\tau}$. In order to prove this, we will use the fact that M is closed under $\omega-$sequences. From this fact, we have that if G is a $P-$generic filter, then $V[G \cap M]$ is a submodel of $V[G]$ which will satisfy that the interpretation of $\dot{\tau} \cap M$ will be a sequential topology on ω that is selectively separable. We will denote by τ' such topology, i.e. $\tau' = int_{G \cap M}(\dot{\tau} \cap M) \in V[G \cap M]$. Moreover in such topology, for each n, the interpretation of $\dot{A}_n \cap M$ will be the collection of sequences converging to n.

Recall that $V[G]$ can be obtained by forcing over $V[G \cap M]$ with the poset $Fn(\aleph_2 \setminus M, 2)$. Having this in mind, we now proceed by working within the model $V[G \cap M]$ (which we refer to as the ground model). Even though it is not strictly correct, we will continue to denote by $\dot{\tau}$ the name for the final topology in $V[G]$. Now suppose $\dot{U}$ is a name for a set forced to be non-empty and a member of $\dot{\tau}$. For each condition p, let $\dot{U}_{\bar{p}}$ denote the set $\left\{x \in \omega : p \Vdash x \in \dot{U}\right\}$. Notice that $\dot{U}_{\bar{p}}$ is a set in the ground model and it is forced by p to be contained in $\dot{U}$. In addition to that, by the elementary assumptions on M, it also follows that p forces the ground model closure of $\dot{U}_{\bar{p}}$ to be contained in the closure of $\dot{U}$ (i.e., in $V[G \cap M]$, the τ'-closure of $\dot{U}_{\bar{p}}$ is contained in the τ'-closure of $\dot{U}$).

Assume, towards a contradiction, that it is forced that the closure of $\dot{U}$ contains no ground model open set. Now, from the assumptions on M, there must be a condition p_0 and an integer x such that $p_0 \Vdash x \in \dot{U}$ and for all condition $p \leq p_0$, $\dot{U}_p$ is nowhere dense.

On another hand, since $\dot{U}$ is a name for a subset of ω, we may choose a countable $L \subseteq \aleph_2 \setminus M$ so that $dom(p_0) \subseteq L$ and for each $k \in \omega$ and each condition $p \in Fn(\aleph_2, 2)$, $p \Vdash k \in \dot{U}$ implies $p \restriction L \Vdash \dot{U}$. Now, since $\dot{U}$ is a $Fn(L, 2)$-name, we may let $\{p_l : l < \omega\}$ enumerate all conditions of $Fn(L, 2)$ which extend p_0. Therefore, for each $n < \omega$, if denote by D_n the complement of the closure of $\dot{U}_{\bar{p}_0} \cup \cdots \cup \dot{U}_{\bar{p}_n}$, then D_n is dense. Since our topology is selectively separable, there is a finite selection $\{F_n \in [D_n]^{<\omega} : n < \omega\}$ such that $F = \bigcup_n F_n$ is dense. Therefore, $x \in \overline{F}$.

Now we analyze the sequential order of x.

The basic idea is the following: if $so(x, F) = 1$, there is a sequence $S_x \subseteq F$ converging to x. Moreover, S_x is almost disjoint from $\dot{U}_{\bar{p}}$, for every $p \in Fn(L, 2)$ which extends p_0. On the other hand, since S_x converges to x, we have, by the elementarity, that S_x converges to x in the final model. Therefore, there must be a condition q which forces that S_x is almost contained in $\dot{U}$, a contradiction.

Claim: Suppose z is an integer for which there is a condition q_z such that $q_z \Vdash z \in \dot{U}$. Then there is an integer x_z such that: there is a condition q_{x_z} which forces x_z to be in $\dot{U}$, and there is a sequence $S_{x_z} \subseteq F$ converging to x_z.

Proof of the claim: if $so(z, F) = 1$, we proceed as above. Suppose the result if $so(z, F) < \alpha$. We may assume that α is a successor ordinal. Since F is dense, z is in the closure of F, and so there is a sequence $C_z = \{z_n : n < \omega\}$ converging to z such that $so(z_n, F) < \alpha$, for all n. Since C_z converges to z, then, by elementarity, C_z also converges to z in the final model. Therefore, there must be a condition q_{n_0} and an integer $z_{n_0} \in C_z$ such that $q_{n_0} \Vdash z_{n_0} \in \dot{U}$. Since $so(z_{n_0}, F) < \alpha$, the induction gives us the desire x_z. This completes the prof of the claim.

Choose a y that satisfies the conclusion of the claim. Then there is a condition q_y which forces y to be in $\dot{U}$, and a sequence $S_y \subseteq F$ converging to y. As before, by elementarity, S_y converges to y in the final model and so there is a condition p_y which forces that S_y is almost contained in $\dot{U}$. On the other hand, Since $S_y \subseteq F$, then S_y is almost disjoint from $\dot{U}_{\bar{p}}$, for every $p \in Fn(L, 2)$, which extends p_0. Let r be a common extension of p_0, p_y, and q_y. Then $r \upharpoonright L \leq p_0$ and $r \upharpoonright L \Vdash S_y \subseteq^* \dot{U}$ $\left(\text{whence } S_y \subseteq^* \dot{U}_{\overline{r \upharpoonright L}}\right)$. This gives us the desired contradiction.

■

Corollary 5.1.2 *If we add at least ω_2 Cohen reals to a model of CH in the resulting model, finite products of countable spaces that are sequential and selectively separable is selectively separable.*

Proof: It is well known that in the resulting model $\mathfrak{d} > \omega_1$. From Theorem 2.4 of [7] we get that if a space has π-weight less than $\mathfrak{d}$, then it is selectively separable. Also, from Theorem 5.1.1 we get that in the final model, a space satisfying the hypothesis of our corollary has π-weight at most ω_1. Therefore, finite products of such spaces will also have π-weight at most ω_1. On the other hand, in the final model, ω_1 is less that $\mathfrak{d}$ and therefore, such finite products will be selectively separable.

■

5.2 Sequential spaces and the Open Graph Axiom

In [8] (Theorem 3.3), the authors prove that PFA implies that finite products of countable Fréchet spaces is selectively separable. It would be interesting to see if the same result holds for countable sequential spaces. If we assume that the spaces have countable fan tightness, we can obtain a similar result from the Open Graph Axiom.

Definition 5.2.1 *A space X has* countable fan tightness *if for any $x \in X$ and for any countable family $\{A_n : n < \omega\}$ of subsets of X such that $x \in \bigcap_n \overline{A_n}$, there is a finite selection*

$$\{F_n \in [A_n]^{<\omega} : n < \omega\}$$

such that $x \in \overline{\bigcup_n F_n}$.

It is known ([3], Proposition 2.3) that if a space X is separable and has countable fan tightness, then X is selectively separable. The converse is not true though. For instance, the sequential fan $S(\omega)$ is a selectively separable space (actually it is a countable Fréchet space with analytic topology) that has no countable fan tightness. We also have the following result about this notion.

Proposition 5.2.2 *Let X be a countable space with sequential topology. X is a Fréchet space with the weak diagonal sequence property (at each of its points) if, and only if, X has countable fan tightness.*

Proof: From Lemma 2.3.8 we have that a countable Fréchet space has the weak diagonal sequence property if, and only if it is strongly Fréchet. Additionally, from Corollary 3 of [1] we have that for a countable space with sequential topology, being strongly Fréchet is equivalent to having countable fan tightness.

■

Theorem 5.2.3 *(OGA) Let X and Y be countable sequential spaces. If X and Y have countable fan tightness, then $X \times Y$ is selectively separable.*

Proof: By Proposition 5.2.2 we have that X and Y have the weak diagonal sequence property.

Fix $(x, y) \in X \times Y$ and let $(A_m)_m$ be a decreasing sequence of dense subsets of $X \times Y$.

Let A_m be enumerated by $\{x_{mn}, y_{mn} : n < \omega\}$, and let X and Y be enumerated by $\{q_i : i < \omega\}$ and $\left\{r_j : j < \omega\right\}$, respectively. We will assume that $q_0 = x$ and $r_0 = y$. In addition to that, we will use the following fact:

Given a sequence $(R_m)_m$ of closed subsets of $X \times Y$ such that for all $m < \omega$, $(x, y) \notin R_m$, then we may assume that $A_m \cap R_m = \emptyset$. Indeed, $(x, y) \in \overline{A_m \setminus R_m}$, for all $m < \omega$.

Using this fact and the fact that for all $m < \omega$ the sets $\{q_1, \dots, q_m\} \times Y$ and $X \times \{r_1, \dots, r_m\}$ are closed subsets of $X \times Y$, we may assume that each of these sets are disjoint from A_m.

On the other hand, we may assume that there exists $m_0 < \omega$ such that for all $m > m_0$, $(x, y) \notin \overline{(\{x\} \times Y) \cap A_m}$. Otherwise, there would be an increasing sequence of integers $(m_k)_k$ such that $(x, y) \in \overline{(\{x\} \times Y) \cap A_{n_k}}$, and since the space $\{x\} \times Y$ has countable fan tightness, then we could find a finite selection from the sequence (A_{n_k}) whose union would contain (x, y) in its closure. But this finite selection would give us a finite selection from $(A_m)_m$ whose union would also contain (x, y) in its closure.

Similarly, we may assume that there exists $l_0 < \omega$ such that for all $l > l_0$, $(x, y) \notin \overline{(X \times \{y\}) \cap A_l}$. Hence, we may assume, without loss of generality, that for all $m < \omega$, (x, y) is not in either of the sets $\overline{(\{x\} \times Y) \cap A_m}$ and $\overline{(X \times \{y\}) \cap A_m}$. This together with our assumption made above, allow us to assume that for all $m \in \omega$ we have

$$(\{q_0, \ldots, q_m\} \times Y) \cap A_m = \emptyset \text{ and } (X \times \{r_0, \ldots, r_m\}) \cap A_m = \emptyset.$$

Now, given a subset $a \subseteq A_0$, we will denote by $(a)_X$ the set of first coordinates of a and by $(a)_Y$ the set of second coordinates of a. For every $f \in \omega^\omega$, let Γ_f be the finite selection determined by f, i.e., $\Gamma_f = \{(x_{mn}, y_{mn}) : m \in \omega \text{ and } n \leq f(m)\}$. Consider the families

$$\mathcal{A} = \left\{a \subseteq A_0 : (a)_X \to x, \text{ and } \exists f \in \omega^\omega \text{ such that } a \subseteq \Gamma_f\right\},$$

and

$$\mathcal{B} = \left\{b \subseteq A_0 : (b)_Y \to y, \text{ and } \exists f \in \omega^\omega \text{ such that } b \subseteq \Gamma_f\right\}.$$

Notice that if $a \in \mathcal{A}$ and $b \in \mathcal{B}$, then a, b, $(a)_X$, and $(b)_Y$ are infinite; moreover, if $a \cap b$ is infinite, then we are done. Therefore, we may assume $a \cap b$ is finite for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Additionally, since X and Y are both strongly Fréchet spaces, then $\mathcal{A}$ and $\mathcal{B}$ are both nonempty.

Consider now the set

$$\mathfrak{X} = \{(a, b) \in \mathcal{A} \times \mathcal{B} : a \cap b = \emptyset\}.$$

We will apply OGA to the colouring determined by

$$\mathfrak{K}_0 = \{\{(a, b), (a', b')\} : (a \cap b') \cup (a' \cap b) \neq \emptyset\},$$

which is an open subset of $2^{(A_0)^2} \cong 2^\omega$.

We now study the two alternatives of OGA:

Case 1: there is an uncountable $\mathfrak{Y} \subseteq \mathfrak{X}$ so that $[\mathfrak{Y}]^2 \subseteq \mathfrak{K}_0$. Since OGA implies $\mathfrak{b} > \omega_1$ (see Proposition 2.6.13), we may consider an uncountable subfamily $\mathfrak{Y}' \subseteq \mathfrak{Y}$ and a function $f \in \omega^\omega$ such that for all $(a, b) \in \mathfrak{Y}'$, $a \cup b \subseteq \Gamma_f$. We claim that Γ_f is a selection that contains (x, y) in its closure. In order to prove that, let us fix open sets $O \subseteq X$ and $U \subseteq Y$ with $(x, y) \in O \times U$. For all $(a, b) \in \mathfrak{Y}'$, since $(a)_X \to x$, and $(b)_Y \to y$ then $(a)_X \setminus O$ and $(b)_Y \setminus U$ are finite. From this and the fact that for every $m < \omega$, A_m is disjoint from the sets $\{q_0, \cdots, q_{m-1}\} \times Y$ and $X \times \{r_0, \cdots, r_{m-1}\}$ we obtain that $a \setminus (O \times Y)$, and $b \setminus (X \times U)$ are finite sets. Therefore, we may refine further to obtain an uncountable subfamily $\mathfrak{Y}'' \subseteq \mathfrak{Y}'$ and two finite sets s_0 and t_0 such that for all $(a, b) \in \mathfrak{Y}''$, $a \setminus (O \times Y) = s_0$, and $b \setminus (X \times U) = t_0$. Let us fix two different elements $(a, b), (a', b') \in \mathfrak{Y}''$. We claim that $(a \cap b') \cup (a' \cap b) \subseteq (O \times Y) \cap (X \times U) = O \times U$. Indeed,

$$(a \cap b') \setminus (O \times Y) \subseteq s_0 \cap b' \subseteq a' \cap b' = \emptyset,$$

and

$$(a \cap b') \setminus (X \times U) \subseteq a \cap t_0 \subseteq a \cap b = \emptyset.$$

Similarly, we obtain that $(a' \cap b) \subseteq (O \times Y) \cap (X \times U)$. Therefore,

$$\emptyset \neq (a \cap b') \cup (a' \cap b) \subseteq (O \times U) \cap \Gamma_f.$$

Thus, $(x, y) \in \overline{\Gamma_f}$.

Case 2: $\mathfrak{X} = \bigcup_m \mathfrak{X}_m$, where each $\mathfrak{X}_n$ satisfies that $[\mathfrak{X}_m]^2 \cap \mathfrak{K}_0 = \emptyset$. We will show this alternative is impossible. For every $m < \omega$, consider the sets

$$C_m = \bigcup \{a : \exists b \text{ such that } (a, b) \in \mathfrak{X}_m\} \subseteq X \times Y,$$

and

$$D_m = \bigcup \{b : \exists a \text{ such that } (a, b) \in \mathfrak{X}_m\} \subseteq X \times Y.$$

Since $[\mathfrak{X}_m]^2 \cap \mathfrak{K}_0 = \emptyset$, we have that $C_m \cap D_m = \emptyset$.

Fix $a_0 \in \mathcal{A}$ and $b_0 \in \mathcal{B}$ and let $\{x_i : i < \omega\}$ and $\left\{y_j : j < \omega\right\}$ denote $(a)_X$, and $(b)_Y$, respectively. We may assume $x_p \neq x_l$, and $y_p \neq y_l$ for all $p \neq l$.

We will define a decreasing sequence of sets $\{Z_m : m < \omega\}$ such that

(a) $Z_0 = A_0$

(b) For every $m \in \omega$ we have that $Z_m \subseteq A_m$, and

$$(\forall l < \omega) \quad (\forall k < \omega) \quad (\exists i > k) \quad (\exists j > k) \quad \text{such that } \left(x_i, y_j\right) \in \overline{Z_m \cap A_l}.$$

(c) For every $m \in \omega$, $Z_{m+1} \subseteq Z_m \setminus C_m$ or $Z_{m+1} \subseteq Z_m \setminus D_m$.

Assume we have defined Z_m satisfying a), b), and c).

Subcase 1: if we have

$$(\forall l < \omega) \;\; (\forall k < \omega) \;\; (\exists i > k) \;\; (\exists j > k) \;\; \text{such that } \left(x_i, y_j\right) \in \overline{(Z_m \setminus C_m) \cap A_l},$$

then we take $Z_{m+1} = (Z_m \setminus C_m) \cap A_{m+1}$

Subcase 2: if we have

$$(\exists l_0 < \omega)\,(\exists k_0 < \omega)\,(\forall i > k_0)\,(\forall j > k_0)\;,\; \left(x_i, y_j\right) \notin \overline{(Z_m \setminus C_m) \cap A_{l_0}}, \tag{5.1}$$

we claim that

$$(\forall l < \omega) \;\; (\forall k < \omega) \;\; (\exists i > k) \;\; (\exists j > k) \;\; \text{such that } \left(x_i, y_j\right) \in \overline{(Z_m \setminus D_m) \cap A_l},$$

Once we prove the claim, we take $Z_{m+1} = (Z_m \setminus D_m) \cap A_{m+1}$.

Proof of the claim: Fix $l \geq l_0$ and $k < \omega$. From the inductive hypothesis, there exist $i, j > \max\{k_0, k\}$ such that

$$\left(x_i, y_j\right) \in \overline{Z_m \cap A_l} \tag{5.2}$$

We will prove that $\left(x_i, y_j\right) \in \overline{(Z_m \setminus D_m) \cap A_l}$: fix open sets $O \subseteq X$ and $U \subseteq Y$ such that $\left(x_i, y_j\right) \in O \times U$. From (5.1) we have that there exist open sets $O' \subseteq X$ and $U' \subseteq Y$ such that $\left(x_i, y_j\right) \in O' \times U'$ and $(O' \times U') \cap (Z_m \setminus C_m) \cap A_{l_0} = \emptyset$. Since $l \geq l_0$, we have that $A_l \subseteq A_{l_0}$. Therefore,

$$((O \cap O') \times (U \cap U')) \cap (Z_m \setminus C_m) \cap A_l = \emptyset.$$

On the other hand, from (5.2) we have that

$$((O \cap O') \times (U \cap U')) \cap Z_m \cap A_l \neq \emptyset.$$

This together with the fact that $C_m \cap D_m = \emptyset$ gives us that

$$((O \cap O') \times (U \cap U')) \cap (Z_m \setminus D_m) \cap A_l \neq \emptyset.$$

To finish the construction notice that if $l < l_0$, since $A_{l_0} \subseteq A_l$, we can use the same $i, j > \max\{k_0, k\}$ obtained before (from the case $l = l_0$), to get $\left(x_i, y_j\right) \in \overline{(Z_m \setminus D_m) \cap A_l}$. Our construction is now complete.

Notice that $(x_0, y_0) \in \overline{Z_0}$ implies that $x_0 \in \overline{(Z_0)_X}$ and $y_0 \in \overline{(Z_0)_Y}$. Since X and Y are Fréchet, there exist infinite sequences $c_0 \subseteq Z_0$ and $d_0 \subseteq Z_0$ such that $(c_0)_X \to x_0$ and $(d_0)_Y \to y_0$. Let $k_1 = 1$. There exist $i_1, j_1 > k_1$ such that $\left(x_{i_1}, y_{j_1}\right) \in \overline{Z_{k_1}}$. Again, using Fréchetness, we can find infinite sequences $c_1 \subseteq Z_{k_1}$ and $d_1 \subseteq Z_{k_1}$ such that $(c_1)_X \to x_{i_1}$ and $(d_1)_Y \to y_{j_1}$. We now let $k_2 = \max\{i_1, j_1\} + 1$. Recursively, we can find strictly increasing sequences $\{k_l : l < \omega\}$, $\{i_l : l < \omega\}$, and $\{j_l : l < \omega\}$ such that $\left(x_{i_l}, y_{j_l}\right) \in \overline{Z_{k_l}}$, and infinite sequences $c_l \subseteq Z_{k_l}$ and $c_l \subseteq Z_{k_l}$ such that

$$(c_l)_X \to x_{i_l}$$

and

$$(d_l)_Y \to y_{j_l}.$$

Since X and Y are Hausdorff, we may assume that $c_p \cap c_l = \emptyset$, and $d_p \cap d_l = \emptyset$, for all $p \neq l$. Now, since $x \in \overline{\bigcup\{(c_l)_X : l < \omega\}}$, and $y \in \overline{\bigcup\{(d_l)_Y : l < \omega\}}$, using Fréchetness, there are sequences

$$\left\{v_p : p < \omega\right\} \subseteq \bigcup\{(c_l)_Y : l < \omega\} \text{ and } \left\{w_p : p < \omega\right\} \subseteq \bigcup\{(d_l)_Y : l < \omega\}$$

converging to x and y respectively. These sequences will satisfy that for all $l < \omega$,

$$\left\{v_p : p < \omega\right\} \cap (c_l)_X \text{ and } \left\{w_p : p < \omega\right\} \cap (d_l)_X \text{ are finite.}$$

We may further assume that

$$\left|\left\{v_p : p < \omega\right\} \cap (c_l)_X\right| \leq 1 \text{ and } \left|\left\{w_p : p < \omega\right\} \cap (d_l)_X\right| \leq 1.$$

In addition to that, these sequences determine a sequence $a = \left\{c_p\,(p) \in Z_{n_p} : p < \omega\right\} \in \mathcal{A}$ and $b = \left\{d_p\,(p) \in Z_{n_p} : p < \omega\right\} \in \mathcal{B}$ such that $\left(c_p\,(p)\right)_X = v_p$ and $\left(d_p\,(p)\right)_Y = w_p$. Since $a \cap b$ is finite, by removing finitely many points, we may assume $a \cap b = \emptyset$. Therefore, $(a, b) \in \mathfrak{X}$. So there is $m < \omega$ such that $(a, b) \in \mathfrak{X}_m$. This implies that $a \subseteq C_m$ and $b \subseteq D_m$. However, $c_{m+1}\,(m+1)\,, d_{m+1}\,(m+1) \in Z_{n_{m+1}} \subseteq Z_m$. This contradicts that Z_{m+1} is a subset of either $Z_m \setminus C_m$ or $Z_m \setminus D_m$.

Let us point out that the effective version is true, i.e., if the topologies on the underlying spaces are analytic, then the assumptions in Theorem 5.2.3 make our spaces bisequential and bisequentiality is productive.

Moreover, it is important to bring to mind that the product of two spaces with countable fan tightness does not necessarily have countable fan tightness (see [1, Remark 3]).

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