Semi-infinite Cohomology, Quantum Group Cohomology, and the Kazhdan-Lusztig Equivalence

by

Chia-Cheng Liu

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Abstract

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The Kazhdan-Lusztig tensor equivalence is a monoidal functor which sends modules over affine Lie algebras at a negative level to modules over quantum groups at a root of unity. A positive level Kazhdan-Lusztig functor is defined using Arkhipov-Gaitsgory's duality between affine Lie algebras of positive and negative levels. We prove that the semi-infinite cohomology functor for positive level modules factors through the positive level Kazhdan-Lusztig functor and the quantum group cohomology functor with respect to the positive part of Lusztig's quantum group. This is the main result of the thesis.

Monoidal structure of a category can be interpreted as factorization data on the associated global category. We describe a conjectural reformulation of the Kazhdan-Lusztig tensor equivalence in factorization terms. In this reformulation, the semi-infinite cohomology functor at positive level is naturally factorizable, and it is conjectured that the factorizable semi-infinite cohomology functor is essentially the positive level Kazhdan-Lusztig tensor functor modulo the Riemann-Hilbert correspondence. Our main result provides an important technical tool in a proposed approach to a proof of this conjecture.

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Chapter 1

Introduction

1.1 Basic setup

Let G be a complex reductive algebraic group, B (resp. B^-) the Borel (resp. opposite Borel) subgroup, N (resp. N^-) the unipotent radical in B (resp. B^-), and $T = B \cap B^-$ the maximal torus. Let $\mathfrak{g}, \mathfrak{b}, \mathfrak{n}, \mathfrak{b}^-, \mathfrak{n}^-, \mathfrak{t}$ be the corresponding Lie algebras. Let W be the Weyl group of G, and $h^\vee$ the dual Coxeter number of G. Denote by $\check{G}$ the Langlands dual of G.

Throughout this thesis there is an important parameter $\kappa \in \mathbb{C}^\times$, called the *level*. The level κ is called *negative* if $\kappa + h^\vee \notin \mathbb{Q}^{\geqslant 0}$, *positive* if $\kappa + h^\vee \notin \mathbb{Q}^{\leqslant 0}$, and *critical* when $\kappa = \kappa_{\text{crit}} := -h^\vee$. Given a positive level κ, the reflected level $\kappa' := -\kappa - 2h^\vee$ is negative. Clearly, if κ is both positive and negative, then κ is irrational.

All objects of algebro-geometric nature will be over the base field $\mathbb{C}$. Unless specified otherwise, by category we mean a *DG category*, i.e. an accessible stable ∞-category enriched over Vect, the unit DG category of complexes of $\mathbb{C}$-vector spaces. We denote the heart of the t-structure in a DG category C by $\mathsf{C}^\heartsuit$.

Denote by $\mathcal{O}$ the ring of functions on the formal disk at a point, and by $\mathcal{K}$ the ring of functions on the punctured formal disk. The $\mathbb{C}$-points of $\mathcal{K}$ after taking a coordinate t is the Laurent series ring $\mathbb{C}((t))$, and that of $\mathcal{O}$ becomes its ring of integers $\mathbb{C}[[t]]$. For a $\mathbb{C}$-vector space V we write $V(\mathcal{K})$ for $V((t)) \equiv V \otimes \mathbb{C}((t))$. For a scheme Y we write $Y(\mathcal{K})$ as the formal loop space of Y, whose $\mathbb{C}$-points are $\operatorname{Maps}(\operatorname{Spec} \mathbb{C}((t)), Y)$. Similarly we define $V(\mathcal{O})$ and $Y(\mathcal{O})$.

Consider the loop group $G(\mathcal{K})$ and its subgroup $G(\mathcal{O})$. Define the evaluation map $\operatorname{ev} : G(\mathcal{O}) \to G$ by $t \mapsto 0$. The Iwahori subgroup of $G(\mathcal{O})$ is defined as $I := \operatorname{ev}^{-1}(B)$. We also define $I^0 := \operatorname{ev}^{-1}(N)$.

1.2 The goal

The thesis studies the semi-infinite cohomology of modules over affine Lie algebras. Our main result is a formula (Theorem 5.3.1) which relates the semi-infinite cohomology to the quantum group cohomology, as the modules over affine Lie algebras are linked to modules over quantum groups via the Kazhdan-Lusztig tensor equivalence.

The significance of this formula is twofold:

1. Integrated with the factorization structures appearing naturally in these objects, the formula paves the way for an alternative proof of the Kazhdan-Lusztig tensor equivalence, which is widely considered overly technical. Moreover, this new approach is valid for any non-critical level κ, whereas the original Kazhdan-Lusztig equivalence was only developed for negative levels. Hence our result indicates how to generalize the Kazhdan-Lusztig equivalence to arbitrary (non-critical) level conjecturally. We will further elaborate on factorization Kazhdan-Lusztig equivalences later in the Introduction in Section 1.7.

2. The formula is instrumental in the recent progress on the quantum local geometric Langlands theory. The approach, proposed by D. Gaitsgory and J. Lurie, to the correspondence is to relate the Kac-Moody brane and the Whittaker brane by passing both to the quantum group world. The correspondence is "quantum" in the sense that we should have a family of correspondences parametrized by a non-critical level, with the critical level being the degenerate case. Our formula helps to understand computationally the bridge from the Kac-Moody brane to the quantum group world for any positive level. A brief account of the quantum local geometric Langlands theory is given in Section 1.8

1.3 Semi-infinite cohomology

The main player of the thesis, the semi-infinite cohomology, was first introduced by B. Feigin in [15], as the mathematical counterpart of the BRST quantization in theoretical physics. While the notion of semi-infinite cohomology was later generalized to broader settings [52, 1, 2, 11], our study stays within the original framework, namely, that for the affine Lie algebras. However, with the geometric interpretation of affine Lie algebras as chiral/vertex algebras, the definition of semi-infinite cohomology is naturally adapted to the geometric formulation [9, 19]. This is the version of the definition we recall in the Apendix.

Purely in terms of algebra, the setting includes a vector space M, a (finite-dimensional) Lie algebra V, and a Lie algebra action of $V(\mathcal{K})$ on M. Note that the Lie bracket on $V(\mathcal{K})$ is given by

$$[v_1 \otimes f(t), v_2 \otimes g(t)] := [v_1, v_2] \otimes (f(t) \cdot g(t)).$$

We can roughly describe the semi-infinite cohomology with respect to $V(\mathcal{K})$ as follows: we first take the Lie algebra cohomology along $V(\mathcal{O})$, and then take the Lie algebra homology along $V(\mathcal{K})/V(\mathcal{O})$. The resulting complex in Vect is denoted by $\mathfrak{C}^{\frac{\infty}{2}}(V(\mathcal{K}), M)$.

We will consider a finite-dimensional reductive Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, and the semi-infinite cohomology with respect to certain Lie subalgebras of $\mathfrak{g}(\mathcal{K})$. The modules we apply the semi-infinite cohomology to, however, will be modules over the *affine Lie algebra* associated to the loop algebra $\mathfrak{g}(\mathcal{K})$. Recall that an affine Lie algebra is a central extension of $\mathfrak{g}(\mathcal{K})$ by the central part $\mathbb{C}\mathbf{1}$, with the extension determined by specifying a complex parameter κ. We obtain the notion of modules over $\hat{\mathfrak{g}}_\kappa$, the affine Lie algebra at level κ, by requiring that $\mathbf{1}$ always acts by the number 1. The semi-infinite cohomology with respect to $\mathfrak{n}(\mathcal{K})$ of such modules comes naturally with an action of the Heisenberg algebra $\hat{\mathfrak{t}}$, which is a central extension of $\mathfrak{t}(\mathcal{K})$ by $\mathbb{C}\mathbf{1}$.

A key feature here is that the semi-infinite cohomology introduces a canonical level shift, called the *Tate shift*. More precisely, the semi-infinite cohomology $\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), M)$ of a $\hat{\mathfrak{g}}_\kappa$-module M turns out to be a module over $\hat{\mathfrak{t}}_{\kappa+\text{shift}}$, the central extension whose 2-cocycle is determined by $\kappa - \kappa_{\text{crit}}$. This is the true reason that we regard κ_{crit} as the point of origin when introducing the terminology of positive and negative level.

The abelian category of smooth representations of the Heisenberg algebra $\hat{\mathfrak{t}}$ is semi-simple, whose simple objects are called the Fock modules π_λ parametrized by the weights $\text{Hom}(T, \mathbb{G}_m)$ [34, Section 9.13]. Therefore we often take the multiplicity of π_μ in $\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), M)$ for each weight μ, and call the resulting functor the μ-component of the semi-infinite cohomology functor. This will appear on one side of our formula.

1.4 Toy example: the finite type case

Our formula can be seen as a semi-infinite analog of Kostant's Theorem on Lie algebra cohomology for finite-dimensional simple Lie algebras. We illustrate this in this section.

Recall that the complete set of finite-dimensional irreducible representations of a finite-dimensional simple Lie algebra $\mathfrak{g}$ is given by $\{V_\lambda : \lambda \text{ dominant integral weights}\}$. Kostant's Theorem [40] says that there is an isomorphism of $\mathfrak{t}$-modules

$$H^i(\mathfrak{n}, V_\lambda) \cong \bigoplus_{\ell(w)=i} \mathbb{C}_{w\cdot\lambda},$$

where the direct sum runs over Weyl group elements w with length $\ell(w) = i$. By interpreting Lie algebra cohomology as the right derived functor of $\text{Hom}_{\mathfrak{n}}(\mathbb{C}, -)$, this identity becomes a simple consequence of the BGG resolution.

On the other hand, the quantum group $U_q(\mathfrak{g})$ is known to have almost identical representation theory as $\mathfrak{g}$ when the quantum parameter is not a root of unity. In particular, we consider the quantum group cohomology $H^\bullet(U_q(\mathfrak{n}), \mathcal{V}_\lambda)$; namely, we take the right derived functor of $\text{Hom}_{U_q(\mathfrak{n})}(\mathbb{C}, -)$ applied to the finite-dimensional irreducible $U_q(\mathfrak{g})$-representation $\mathcal{V}_\lambda$. A quantum analog of the BGG resolution implies

$$H^i(U_q(\mathfrak{n}), \mathcal{V}_\lambda) \cong \bigoplus_{\ell(w)=i} \mathbb{C}_{w\cdot\lambda}.$$

Hence obviously

$$H^i(\mathfrak{n}, V_\lambda) \cong H^i(U_q(\mathfrak{n}), \mathcal{V}_\lambda).$$

The formula we are after will be the one above with Lie algebra cohomology replaced by semi-infinite cohomology on the left hand side. This will be made precise in the next section.

1.5 Irrational level

At the level of abelian categories, we can model the semi-simple category $\mathfrak{g}\text{-mod}^{\text{f.d.}}$ of finite-dimensional $\mathfrak{g}$-modules by the category $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$ of $G(\mathcal{O})$-integrable representations of $\hat{\mathfrak{g}}_\kappa$ with κ an irrational number. The category $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$ is semi-simple with simple objects given by the *Weyl modules* $\mathbb{V}^\kappa_\lambda$, and the canonical equivalence $\mathfrak{g}\text{-mod}^{\text{f.d.}} \simeq \hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$ is induced by $V_\lambda \mapsto \mathbb{V}^\kappa_\lambda$.

The $\hat{\mathfrak{g}}_\kappa$-modules when κ is irrational and the $U_q(\mathfrak{g})$-modules when q is not a root of unity are related by the *Kazhdan-Lusztig equivalence* at irrational level [37]. This is an equivalence of abelian categories

$$\mathrm{KL}^{irr}_G : \hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})} \xrightarrow{\sim} U_q(\mathfrak{g})\text{-mod}$$

which sends the simple object $\mathbb{V}^\kappa_\lambda$ to the simple object $\mathcal{V}_\lambda$.

Now, a computation (Corollary 7.1.2) using the BGG-type resolution of $\mathbb{V}^\kappa_\lambda$ shows that

$$H^{\frac{\infty}{2}+i}(\mathfrak{n}(\mathcal{K}), \mathbb{V}^\kappa_\lambda) \cong \bigoplus_{\ell(w)=i} \pi_{w\cdot\lambda}.$$

Taking the μ-components we get an isomorphism

$$H^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \mathbb{V}^\kappa_\lambda)^\mu \cong H^\bullet(U_q(\mathfrak{n}), \mathrm{KL}^{irr}_G(\mathbb{V}^\kappa_\lambda))^\mu.$$

The Kazhdan-Lusztig equivalence allows to formulate the identity for arbitrary module M in $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$. We would like to further generalize the isomorphism to the setting of DG category, which means that we should upgrade it to an isomorphism of complexes. We arrive at the desired formula at irrational level:

$$\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), M)^\mu \cong \mathrm{C}^\bullet(U_q(\mathfrak{n}), \mathrm{KL}^{irr}_G(M))^\mu. \tag{1.1}$$

An algebraic proof of (1.1) is given in Section 7.2.

1.6 Rational level

Situations are drastically more complicated at rational levels. First, the abelian category $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$ is no longer semi-simple when κ is rational. Second, the theory starts to bifurcate into the positive level case and the negative level case, and the Kazhdan-Lusztig equivalence only covers the negative one.

We deal with the negative rational level case first. We have the Kazhdan-Lusztig equivalence at negative level, which relates the negative level κ' and the quantum parameter q by

$$q = \exp(\frac{\pi\sqrt{-1}}{\kappa' - \kappa_{\text{crit}}}).$$

Clearly q is now a root of unity. To add further complication, there are more than one variants of quantum groups at a root of unity: the Lusztig form U_q^{Lus}, the Kac-De Concini form U_q^{KD}, and the small quantum group $\mathfrak{u}_q$. They fit into the following sequence

$$U_q^{\text{KD}} \twoheadrightarrow \mathfrak{u}_q \hookrightarrow U_q^{\text{Lus}}. \tag{1.2}$$

The Kazhdan-Lusztig equivalence in this case is $\text{KL}_G^{\kappa'} : \hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} \xrightarrow{\sim} U_q^{\text{Lus}}(\mathfrak{g})\text{-mod}$, which sends Weyl modules to the so-called *quantum Weyl modules*. The formula at negative rational level (Conjecture 6.2.2) is

$$\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), M)^{\mu} \cong \text{C}^{\bullet}(U_q^{\text{KD}}(\mathfrak{n}), \text{KL}_G^{\kappa'}(M))^{\mu} \tag{1.3}$$

for M in $\hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})}$.

Now we consider the positive rational level case. There is a duality between negative and positive level modules, due to Gaitsgory and Arkhipov [7]. We denote the duality functor by

$$\mathbb{D}_{G(\mathcal{O})} : \hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} \xrightarrow{\sim} \hat{\mathfrak{g}}_{\kappa}\text{-mod}^{G(\mathcal{O})},$$

where κ is positive (and so κ' is negative). A Kazhdan-Lusztig type functor at positive level can be defined using the duality as

$$\text{KL}_G^{\kappa} := \mathbb{D}^q \circ \text{KL}_G^{\kappa'} \circ \mathbb{D}_{G(\mathcal{O})}^{-1},$$

where $\mathbb{D}^q : U_q^{\text{Lus}}(\mathfrak{g})\text{-mod} \xrightarrow{\sim} U_q^{\text{Lus}}(\mathfrak{g})\text{-mod}$ is the contragredient duality for modules over quantum groups.

The crucial difference here is that the functor KL_G^{κ} at positive level only makes sense in the derived world. This is due to the fact that the duality functor $\mathbb{D}_{G(\mathcal{O})}$ is only defined on derived categories and does not preserve the heart of the t-structures. However, the functor KL_G^{κ} does send Weyl modules to quantum Weyl modules.

The formula at positive rational level (Theorem 5.3.1) is

$$\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), M)^{\mu} \cong \text{C}^{\bullet}(U_q^{\text{Lus}}(\mathfrak{n}), \text{KL}_G^{\kappa}(M))^{\mu} \tag{1.4}$$

for M in $\hat{\mathfrak{g}}_{\kappa}\text{-mod}^{G(\mathcal{O})}$. Note that the duality involved in the definition of KL_G^{κ} has the effect of swapping the quantum groups in the sequence (1.2). This is just another incarnation of the fact that the Verdier duality swaps the standard (!-) and costandard (*-) objects, while preserving the intermediate (!*-) objects. Indeed, we can realize the positive and negative level categories geometrically as D-modules on the affine flag variety by the Kashiwara-Tanisaki localization, and the functor $\mathbb{D}_{G(\mathcal{O})}$ corresponds to the Verdier dual.

The main result of this thesis is a proof of the positive level formula (1.4), whereas the negative level formula (1.3) is the subject of [25], and is still a conjecture with partial results obtained. We now explain the idea of proof at positive level, which follows the same pattern as in the work *loc. cit.* by Gaitsgory.

The quantum Frobenius gives rise to a short exact sequence of categories

$$0 \to \mathrm{Rep}(\check{B}) \to U_q^{\mathrm{Lus}}(\mathfrak{b})\text{-mod} \to \mathfrak{u}_q(\mathfrak{b})\text{-mod} \to 0. \tag{1.5}$$

The strategy is to first characterize the cohomology functor on the Kac-Moody side that corresponds to $\mathrm{C}^\bullet(\mathfrak{u}_q(\mathfrak{n}), -)^\mu$, and then pass to $\mathrm{C}^\bullet(U_q^{\mathrm{Lus}}(\mathfrak{n}), -)^\mu$ using the sequence (1.5). For this purpose we construct the !*-generalized semi-infinite cohomology functor $\mathfrak{C}^{\frac{\infty}{2}}_{!*}(\mathfrak{n}(\mathcal{K}), -)$, which is made possible by the recent discovery [24] of a non-standard t-structure on the category $\mathrm{D\text{-}mod}(\mathrm{Gr}_G)^{N(\mathcal{K})}$ of $N(\mathcal{K})$-equivariant D-modules on the affine Grassmannian Gr_G, and along with the discovery the construction of a semi-infinite intersection cohomology (IC) object $\mathrm{IC}^{\frac{\infty}{2}}$ in $\mathrm{D\text{-}mod}(\mathrm{Gr}_G)^{N(\mathcal{K})}$.

We prove (Theorem 5.3.2):

$$\mathfrak{C}^{\frac{\infty}{2}}_{!*}(\mathfrak{n}(\mathcal{K}), M)^\mu \cong \mathrm{C}^\bullet(\mathfrak{u}_q(\mathfrak{n}), \mathrm{KL}^\kappa_G(M))^\mu. \tag{1.6}$$

Identifying the coweight lattice as a sublattice of the weight lattice (depending on the parameter κ), we consider all $\check{\nu}$-components $\mathfrak{C}^{\frac{\infty}{2}}_{!*}(\mathfrak{n}(\mathcal{K}), M)^{\check{\nu}}$ at the same time; i.e. we take the direct sum over all coweights $\check{\nu}$. The resulting object acquires a $\check{B}$-action, and its $\check{B}$-invariants is precisely $\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), M)^0$ by the theory of Arkhipov-Bezrukavnikov-Ginzburg [4]. On the quantum group side this procedure produces $\mathrm{C}^\bullet(U_q^{\mathrm{Lus}}(\mathfrak{n}), \mathrm{KL}^\kappa_G(M))^0$ by the sequence (1.5). Thus we have established

$$\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), M)^0 \cong \mathrm{C}^\bullet(U_q^{\mathrm{Lus}}(\mathfrak{n}), \mathrm{KL}^\kappa_G(M))^0,$$

and for general μ the same procedure applies to the identity with a μ-shift.

1.7 Factorization

The category $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$ has a non-trivial braided monoidal structure constructed by Kazhdan and Lusztig via the Knizhnik-Zamolodchikov equations [37]. The most remarkable part of the Kazhdan-Lusztig equivalence is that it is an equivalence respecting the braided monoidal structures, where the braided monoidal structure on the quantum group side is given by the R-matrix.

As early as in the works of Felder-Wieczerkowski [17], Schechtman [49] and Schechtman-

Varchenko [50, 51], mathematicians realized that the R-matrix of a quantum group is related to topological factorizable objects on certain inductive limit of configuration spaces. The works in this direction culminated in [10] where Bezrukavnikov-Finkelberg-Schechtman established a topological realization of the category of modules over the small quantum group in terms of factorizable sheaves.

On the other hand, Khoroshkin-Schechtman [38, 39] constructed the algebro-geometric factorizable objects which they call the factorizable D-modules. Their construction gives an algebro-geometric realization of the category $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$ for κ irrational (more precisely, Drinfeld's tensor category of $\mathfrak{g}$-modules), and via the Riemann-Hilbert correspondence it corresponds to the BFS factorizable sheaves.

Therefore, after the respective realization as factorizable objects, the Kazhdan-Lusztig equivalence at irrational level is deduced from the Riemann-Hilbert correspondence, which clearly preserves the factorization structures retaining the braided monoidal structures in the original categories.

The general philosophy [44, Section 1.8 and 1.9] is that there should be a correspondence between *factorization categories* and braided monoidal categories. It is hence expected that a factorization form of the Kazhdan-Lusztig equivalence (for not just the irrational levels) exists.

We digress temporarily to discuss the notion of strong group actions on categories. We say a category C is acted on strongly by a group H if C is a module category of D-mod(H). We can twist the category D-mod(H) by a multiplicative $\mathbb{G}_m$-gerbe on H, which is equivalent to the data of a central extension $\hat{\mathfrak{h}}$ of the Lie algebra $\mathfrak{h} = \mathrm{Lie}(H)$, with a lift of the adjoint action of H on $\mathfrak{h}$ to $\hat{\mathfrak{h}}$, c.f. [32].

In the case of the loop group $G(\mathcal{K})$ of a reductive group G, corresponding to the affine Kac-Moody extension $\hat{\mathfrak{g}}_\kappa$ we have the κ-twisted category $\text{D-mod}_\kappa(G(\mathcal{K}))$. A category is acted on strongly by $G(\mathcal{K})$ at level κ if it is a module category of $\text{D-mod}_\kappa(G(\mathcal{K}))$. We will return to twisted loop group actions in the next section.

From the algebro-geometric perspective, factorization structures arise naturally from (strong) actions of a loop group [30]. The category $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$ is acted on strongly by $G(\mathcal{K})$ at level κ, which essentially comes from the action of the loop algebra $\mathfrak{g}(\mathcal{K})$ on $\hat{\mathfrak{g}}$. Then we obtain the factorization category $(\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})})_{\mathrm{Ran}(X)}$.

The main difficulty to achieve a factorization Kazhdan-Lusztig equivalence lies in the quantum group side. Since the braided monoidal structure for quantum group modules is of topological nature, the most natural factorization structure in this case should be of topological flavor. The theory of *topological factorization categories* was developed by J. Lurie in terms of algebras over the little disks operad in the $(\infty, 2)$-category of DG categories [42]. As an example, tautologically the topological factorization category associated to the braided monoidal category $\mathrm{Rep}_q(T)$ of representations of the quantum torus is $\mathrm{Shv}_q(\mathrm{Gr}_{\check{T},\mathrm{Ran}(X)})$, the constructible sheaves on the Beilinson-Drinfeld Grassmannian of $\check{T}$, twisted by a factorizable gerbe specified by q. Now the question is, how to explicitly build the topological factorization category associated to

$U_q^{\mathrm{Lus}}(\mathfrak{g})$-mod?

Owing to Lurie's theory, the answer is positive, if we replace the quantum group $U_q^{\mathrm{Lus}}(\mathfrak{g})$ by the small quantum group $\mathfrak{u}_q(\mathfrak{g})$, or by a mixed quantum group $U_q^{+\mathrm{Lus},-\mathrm{KD}}(\mathfrak{g})$ which has positive part in Lusztig form and negative part in Kac De Concini form. Nevertheless, we are still unable to construct explicitly the topological factorization category $\mathrm{Fact}(U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod})$ associated to $U_q^{\mathrm{Lus}}(\mathfrak{g})$-mod.

We can, however, modify the factorization category associated to $U_q^{+\mathrm{Lus},-\mathrm{KD}}(\mathfrak{g})$-mod in algebraic terms to get a factorization category $\mathrm{Fact}(U_q^{\frac{1}{2}}(\mathfrak{g})\text{-mod})$ that contains $\mathrm{Fact}(U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod})$ as a full subcategory. Even better, under the Riemann-Hilbert correspondence, the category $\mathrm{RH}(\mathrm{Fact}(U_q^{\frac{1}{2}}(\mathfrak{g})\text{-mod}))$ is the natural recipient of a certain factorizable functor, called the *Jacquet functor*, from $(\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})})_{\mathrm{Ran}(X)}$.

The factorization Kazhdan-Lusztig equivalence can now be formulated as

Conjecture 1.7.1. *The Jacquet functor is fully faithful, with its essential image identified with*

$$\mathrm{Fact}(U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod})$$

under the Riemann-Hilbert correspondence.

The upshot is that, in the positive level case, the Jacquet functor is precisely the semi-infinite cohomology functor. Our formula (1.4) therefore plays an instrumental role in tackling Conjecture 1.7.1. Moreover, as the definition of the semi-infinite cohomology is purely algebraic, one sees in this characterization that the transcendental nature of the Kazhdan-Lusztig equivalence exactly comes from that of the Riemann-Hilbert correspondence.

We remark that, at negative level, the definition of the Jacquet functor involves the !-generalized semi-infinite cohomology functor, which is briefly discussed in Chapter 6 and Section 8.4.

1.8 Application to quantum local geometric Langlands

Let κ be a non-critical level. Let $\check{\kappa}$ be the Langlands dual parameter such that $(\cdot,\cdot)_{\kappa-\kappa_{\mathrm{crit}}}$ and $(\cdot,\cdot)_{\check{\kappa}-\kappa_{\mathrm{crit}}}$ induce mutual inverse maps between $\mathfrak{t}$ and $\check{\mathfrak{t}} \equiv \mathfrak{t}^*$.

We denote by $G(\mathcal{K})-\mathsf{ModCat}_\kappa$ the $(\infty,2)$-category of DG categories acted on by $G(\mathcal{K})$ strongly at level κ. In its latest form ([26], circa January, 2018), the quantum local geometric Langlands conjecture is stated as

Conjecture 1.8.1. *Assume that κ is positive. There is a canonical equivalence of $(\infty,2)$-categories*

$$\mathbb{L}_G^\kappa : G(\mathcal{K})-\mathsf{ModCat}_\kappa \xrightarrow{\sim} \check{G}(\mathcal{K})-\mathsf{ModCat}_{(\check{\kappa})'}.$$

An expected feature of the above ambitious conjecture is that the *Kac-Moody brane* goes over to the *Whittaker brane*, and vice versa.

To be more precise, if $\mathsf{C} \in G(\mathcal{K})-\mathsf{ModCat}_\kappa$ and $\check{\mathsf{C}} := \mathbb{L}^\kappa_G(\mathsf{C})$, then we should have $\mathrm{KM}(\mathsf{C}) \simeq \mathrm{Whit}(\check{\mathsf{C}})$ and $\mathrm{Whit}(\mathsf{C}) \simeq \mathrm{KM}(\check{\mathsf{C}})$, where the Kac-Moody category attached to C is

$$\mathrm{KM}(\mathsf{C}) := \mathsf{Funct}_{G(\mathcal{K})}(\hat{\mathfrak{g}}_\kappa\text{-mod}, \mathsf{C})$$

and the Whittaker category attached to C is

$$\mathrm{Whit}(\mathsf{C}) := \mathsf{C}^{N(\mathcal{K}),\chi},$$

i.e. the $N(\mathcal{K})$-invariants in the category C with respect to a non-degenrate character $\chi : N(\mathcal{K}) \to \mathbb{G}_m$.

Recall from Section 1.7 that $G(\mathcal{K})$-actions give rise to factorization structures. By the above definitions, the Kac-Moody category $\mathrm{KM}(\mathsf{C})$ acquires a strong $G(\mathcal{K})$-action at level κ' (notice that the level is changed to the reflected one as the $G(\mathcal{K})$-action changes side), and the Whittaker category $\mathrm{Whit}(\mathsf{C})$ is also acted on strongly by $G(\mathcal{K})$ at level $(\check{\kappa})'$. We therefore expect that the resulting equivalence $\mathrm{KM}(\mathsf{C}) \simeq \mathrm{Whit}(\check{\mathsf{C}})$ is factorizable, and same for $\mathrm{Whit}(\mathsf{C}) \simeq \mathrm{KM}(\check{\mathsf{C}})$.

A more down-to-earch conjecture, arising as a consequence of the 2-categorical conjecture above, addresses the fundamental case when $\mathsf{C} = \text{D-mod}_\kappa(\mathrm{Gr}_G)$. The expectation of what the category $\check{\mathsf{C}} \equiv \mathbb{L}^\kappa_G(\mathsf{C})$ would be is the natural one:

$$\check{\mathsf{C}} \simeq \text{D-mod}_{(\check{\kappa})'}(\mathrm{Gr}_{\check{G}}).$$

In this case, one evaluates the Kac-Moody category of C as

$$\mathrm{KM}(\text{D-mod}_\kappa(\mathrm{Gr}_G)) \simeq \hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})},$$

and we denote the Whittaker category for $\check{\mathsf{C}}$ by

$$\mathrm{Whit}(\mathrm{Gr}_{\check{G}})_{(\check{\kappa})'} := \text{D-mod}_{(\check{\kappa})'}(\mathrm{Gr}_{\check{G}})^{N(\mathcal{K}),\chi}.$$

What the 2-categorical conjecture predicts in this case is called the *fundamental local equivalence* (FLE) at negative level:

Conjecture 1.8.2 ([27]). *There is a canonical factorizable equivalence*

$$\mathrm{FLE}_{\kappa'} : \hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} \xrightarrow{\sim} \mathrm{Whit}(\mathrm{Gr}_{\check{G}})_{(\check{\kappa})'}.$$

Switching the roles of G and $\check{G}$ in Conjecture 1.8.1 and plugging in $\mathsf{C} := \text{D-mod}_{\check{\kappa}}(\mathrm{Gr}_{\check{G}})$, we obtain the FLE at positive level:

Conjecture 1.8.3 ([27]). *There is a canonical factorizable equivalence*

$$\widetilde{\mathrm{FLE}}_\kappa : \hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})} \xrightarrow{\sim} \mathrm{Whit}(\mathrm{Gr}_{\check{G}})_{\check{\kappa}}.$$

Note that by duality, $\widetilde{\mathrm{FLE}}_\kappa$ is equivalent to the inverse of the dual functor of $\mathrm{FLE}_{\kappa'}$.

We now explain how formula (1.4) (resp. formula (1.3)) can help in the outline of proof of the FLE at positive (resp. negative) level, proposed again by D. Gaitsgory. We will discuss the positive level case, and the negative level case is similar.

Our formula only concerns the Kac-Moody side of the FLE. The treatment on the Whittaker side follows a parallel construction that is beyond the scope of this thesis. We refer the interested reader to [26, Section 5.1].

First, we note that the FLE for the group being a torus is known. Over a point (ignoring the word "factorizable"), the FLE is tautological. The factorization version follows from the Contou-Carrère's duality [46].

Recall the Jacquet functor from Section 1.7. At positive level, it is a factorization functor

$$(\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})})_{\mathrm{Ran}(X)} \to \mathrm{RH}(\mathrm{Fact}(U^{\frac{1}{2}}_{q^{-1}}(\mathfrak{g})\text{-mod}))$$

and is conjectured to be fully faithful. The category $\mathrm{RH}(\mathrm{Fact}(U^{\frac{1}{2}}_{q^{-1}}(\mathfrak{g})\text{-mod})$ by construction can be described as certain enlargement of the category of factorization modules of a factorization algebra denoted by $\Omega_\kappa^{-,\mathrm{KM},\mathrm{Lus}} \in (\hat{\mathfrak{t}}_\kappa\text{-mod}^{T(\mathcal{O})})_{\mathrm{Ran}(X)}$.

On the Whittaker side, we also construct the Jacquet functor for the Whittaker category, and the recipient is described similarly by a factorization algebra denoted by $\Omega_{\check{\kappa}}^{\mathrm{Whit},\mathrm{Lus}} \in \mathrm{D\text{-}mod}_{\check{\kappa}}(\mathrm{Gr}_{\check{T},\mathrm{Ran}(X)})$.

Conjecture 1.8.4 ([28])**.** *Under the FLE for the torus T, the factorization algebras $\Omega_\kappa^{-,\mathrm{KM},\mathrm{Lus}}$ and $\Omega_{\check{\kappa}}^{\mathrm{Whit},\mathrm{Lus}}$ are identified.*

The upshot is, if Conjecture 1.8.4 is proven, the proof of the FLE at positive level is reduced to showing that the essential images of the two Jacquet functors (for the Kac-Moody side and the Whittaker side) match each other.

Now, the formula (1.4) comes in to provide a possible way to prove Conjecture 1.8.4. The idea is to pass both the Kac-Moody side and the Whittaker side to the quantum group world, and try to verify that $\Omega_\kappa^{-,\mathrm{KM},\mathrm{Lus}}$ and $\Omega_{\check{\kappa}}^{\mathrm{Whit},\mathrm{Lus}}$ give rise to the same topological factorization algebra, the one induced from the quantum group $U^{\mathrm{Lus}}_{q^{-1}}(\mathfrak{n})$. Since at positive level the Kac-Moody Jacquet functor is given by the semi-infinite cohomology functor, a formula comparing the semi-infinite cohomology with the quantum group cohomology with respect to $U^{\mathrm{Lus}}_{q^{-1}}(\mathfrak{n})$ should make the verification a manageable task.

1.9 Structure

The thesis is organized as follows.

In Chapter 2 and Chapter 3 we recall standard constructions and results from Lie theory and geometric representation theory. In particular, in Section 2.2 we define the duality functor between negative level and positive level modules, and calculate the image of affine Verma modules and Weyl modules. In Section 2.5 we define the positive level Kazhdan-Lusztig functor by means of the duality functor.

Chapter 4 introduces the Wakimoto modules. We give two constructions, one in terms of the free field realization in the language of chiral algebra (Section 4.1), the other in terms of convolution actions on affine Verma modules (Section 4.4). To relate the two constructions, we prove Theorem 4.3.3, which identifies the type w_0 Wakimoto module with the dual affine Verma module of the same highest weight under certain conditions on the highest weight and the level. In Section 4.5 we present two formulas which compute semi-infinite cohomology using Wakimoto modules.

Chapter 5 is the main thrust of the thesis. In Section 5.2 we introduce the generalized semi-infinite cohomology functor at positive level, and define the semi-infinite IC object $\mathrm{IC}^{\frac{\infty}{2},-}$ used in defining the !*-generalized functor. Section 5.3 states our main results, the formula for the !*-functor (Theorem 5.3.2) and the formula for the original semi-infinite functor (Theorem 5.3.1).

Chapter 6 summarizes part of the results in [25], and contains a discussion on the duality pattern among the formulas at positive and negative level.

Chapter 7 gives an algebraic proof of the main formula when the level is assumed irrational.

Finally, in Chapter 8 we discuss the factorization aspect of the theory. In Section 8.1 we construct the factorization categories associated to Kac-Moody representations. In Section 8.2 we review the correspondence between Hopf algebras and topological factorization algebras, and describe the quantum group categories in factorization terms. Section 8.3 introduces the metaplectic Langlands dual group as the cokernel of Lusztig's quantum Frobenius morphism. This ultimately enables us to state the conjecture on factorization Kazhdan-Lusztig equivalence at arbitrary non-critical level in Section 8.4.

The Appendix contains definitions on chiral algebras, factorization algebras and categories, chiral differential operators, and repeats a technical construction of the semi-infinite cohomology complex in the chiral language, from [9].

1.10 Conventions on D-modules and sheaves

For a scheme Z, let $\mathcal{O}_Z$ (resp., T_Z, ω_Z, D_Z) denote its structure sheaf (resp., tangent sheaf, sheaf of top forms, sheaf of differential operators).

Let X be a scheme of finite type. The DG category of right (resp. left) D_X-modules is denoted by $\text{D-mod}(X)$ (resp. $\text{D-mod}(X)^l$). For M in $\text{D-mod}(X)^\heartsuit$, denote by $M^l := M \otimes_{\mathcal{O}_X} \omega_X^{-1}$ the corresponding left D_X-module in $(\text{D-mod}(X)^l)^\heartsuit$. This induces the side-change functor

$$-^l : \text{D-mod}(X) \to \text{D-mod}(X)^l.$$

The inverse functor is denoted by $-^r : \text{D-mod}(X)^l \to \text{D-mod}(X)$.

With the aid of higher category theory [41], we extend the notion of D-modules to arbitrary prestacks: for a prestack $\mathcal{Y}$, D-mod($\mathcal{Y}$) is defined as the limit of D-mod(S) over the category of schemes of finite type S over $\mathcal{Y}$, with structure functors given by !-pullbacks. For details see [45].

Similarly, for a scheme X of finite type, we let Shv(X) denote ind-completion of the DG category of constructible sheaves in the analytic topology on $\mathbb{C}$-points $X(\mathbb{C})$. Then we extend the definition to arbitrary prestack $\mathcal{Y}$ by taking the limit of Shv(S) over all $S \to \mathcal{Y}$.

The Riemann-Hilbert correspondence is a fully-faithful functor

$$\mathrm{RH} : \mathrm{Shv}(\mathcal{Y}) \to \text{D-mod}(\mathcal{Y})$$

whose essential image is the full subcategory of holonomic D-modules with regular singularities. The perverse t-structure on Shv($\mathcal{Y}$) matches with the usual t-structure on D-mod($\mathcal{Y}$) via RH.

On only one occasion in this thesis (Section 5.5), we mention the *ind-coherent sheaves* IndCoh($\mathcal{Y}$) of a prestack $\mathcal{Y}$. The only feature we use there is the pushforward functor f_* : IndCoh($\mathcal{Y}$) $\to$ IndCoh($\mathcal{Y}'$) of a morphism $f : \mathcal{Y} \to \mathcal{Y}'$. Note that under the *induction functor* from IndCoh($\mathcal{Y}$) to D-mod($\mathcal{Y}$), the IndCoh pushforward corresponds to the usual de Rham (*-) pushforward of right D-modules, whenever the functors are defined. We refer the reader to [33] for a full treatment of the theory of ind-coherent sheaves.

Chapter 2

Preparation: algebraic constructions

2.1 Root datum

Recall notations from Section 1.1 for algebraic groups and their Lie algebras. Let Λ (resp. $\check{\Lambda}$) be the weight (resp. coweight) lattice of G. Then by definition $\check{\Lambda}$ (resp. Λ) is the weight (resp. coweight) lattice of $\check{G}$. Write Λ^+ (resp. $\check{\Lambda}^+$) for the set of dominant weights (resp. coweights). Let R, R^+, and Π denote the set of roots, positive roots, and simple roots of G, respectively. Let ρ denote the half sum of all positive roots.

We have the standard invariant bilinear form on $\mathfrak{g}$

$$(\cdot,\cdot)_{\mathrm{st}} : \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{C},$$

which restricts to a form on $\mathfrak{t}$ and thus on the lattice $\check{\Lambda}\subset\mathfrak{t}$. We also have the natural pairing $\langle\cdot,\cdot\rangle$ between $\mathfrak{t}^*$ and $\mathfrak{t}$, which restricts to

$$\langle\cdot,\cdot\rangle : \Lambda\otimes\check{\Lambda}\to\mathbb{Z}$$

on the lattices. For each simple root α_i and coroot $\check{\alpha}_i$, let $d_i\in\{1,2,3\}$ be the integer such that $(\check{\alpha}_i,\check{\mu})_{\mathrm{st}} = d_i\langle\alpha_i,\check{\mu}\rangle$. Then there is an induced form on Λ, also denoted by $(\cdot,\cdot)_{\mathrm{st}}$ when no confusion can arise, characterized by the relations $(\mu,\alpha_i)_{\mathrm{st}} = d_i^{-1}\langle\mu,\check{\alpha}_i\rangle$ for all i.

For a number $\kappa\in\mathbb{C}^\times$, we set

$$(\cdot,\cdot)_\kappa := \kappa(\cdot,\cdot)_{\mathrm{st}} : \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{C}.$$

We then have the corresponding form $(\cdot,\cdot)_\kappa$ on Λ which satisfies

$$\frac{(\check{\alpha}_i,\check{\alpha}_j)_\kappa}{(\check{\alpha}_i,\check{\alpha}_i)_\kappa}\cdot d_i = d_j\cdot\frac{(\alpha_i,\alpha_j)_\kappa}{(\alpha_i,\alpha_i)_\kappa}.$$

Define the isomorphism $\phi_\kappa : \mathfrak{t}\to\mathfrak{t}^*$ by the relation $(\lambda,\phi_\kappa(\check{\mu}))_{\kappa-\kappa_{\mathrm{crit}}} = \langle\lambda,\check{\mu}\rangle$. We will abuse the notation by writing $\check{\mu}$ in place of $\phi_\kappa(\check{\mu})$ when both weights and coweights are present in an expression. For example, we will write $\lambda+\check{\mu}$ instead of $\lambda+\phi_\kappa(\check{\mu})$. When κ is a rational number such that $\phi_\kappa : \check{\Lambda} \to \Lambda$, let $\phi_T : T \to \check{T}$ be the map induced from ϕ_κ.

2.2 Affine Lie algebras

The affine Lie algebra $\hat{\mathfrak{g}}_\kappa$ at level κ is a central extension of $\mathfrak{g}(\mathcal{K})$ by the 1-dimensional trivial module $\mathbb{C}\mathbf{1}$, with Lie bracket defined by the 2-cocycle

$$f, g \mapsto \mathrm{Res}_{t=0}(f\,, \frac{dg}{dt})_\kappa, \quad f, g \in \mathfrak{g}(\mathcal{K}).$$

We have the Cartan decomposition of the affine Lie algebra $\hat{\mathfrak{g}}_\kappa$:

$$\hat{\mathfrak{g}}_\kappa = \mathfrak{n}(\mathcal{K}) \oplus \hat{\mathfrak{t}}_\kappa \oplus \mathfrak{n}^-(\mathcal{K}),$$

where the subalgebra $\hat{\mathfrak{t}}_\kappa := \mathfrak{t}(\mathcal{K}) \oplus \mathbb{C}\mathbf{1}$ is the Heisenberg algebra at level κ.

We define $\hat{\mathfrak{g}}_\kappa$-mod$^\heartsuit$ as the abelian category whose objects are $\hat{\mathfrak{g}}_\kappa$-modules M where (1) the central element $\mathbf{1}$ acts as the identity, and (2) each vector $m \in M$ is annhilated by $\mathfrak{g}(t^n\mathbb{C}[[t]])$ for some $n \geq 0$. The morphisms are ordinary $\hat{\mathfrak{g}}_\kappa$-equivaraint maps. The corresponding DG category is denoted by $\hat{\mathfrak{g}}_\kappa$-mod.

The full subcategory $(\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})})^\heartsuit \subset \hat{\mathfrak{g}}_\kappa\text{-mod}^\heartsuit$ consists of those modules whose $\mathfrak{g}(\mathcal{O})$-action comes from a $G(\mathcal{O})$-action. As explained in [23, Section 1.2], the $G(\mathcal{O})$-action integrating a given $\mathfrak{g}(\mathcal{O})$-action is unique at the abelian level, but not so at the derived level since higher cohomologies of $G(\mathcal{O})$ are non-trivial. Consequently, the DG category $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$ of $G(\mathcal{O})$-equivariant (equivalently, $G(\mathcal{O})$-integrable) $\hat{\mathfrak{g}}_\kappa$-modules is no longer a full subcategory of the DG category $\hat{\mathfrak{g}}_\kappa$-mod. Nevertheless, one can construct the DG category $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$ by "bootstrapping" from the abelian category $(\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})})^\heartsuit$. The details of this construction appear in Section 2 of *loc. cit.* A different approach to construct $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$, using derived algebraic geometry and higher category theory, is given in Section 4 of *loc. cit.*, which might be of interest to the reader.

For a given finite-dimensional representation V of $\mathfrak{g}$, we extend it to a module over $\mathfrak{g}(\mathcal{O}) \oplus \mathbb{C}\mathbf{1}$ by setting the action of t as 0 and the action of the central element $\mathbf{1}$ as 1. Then we perform induction to $\hat{\mathfrak{g}}_\kappa$:

$$V^\kappa := \mathrm{Ind}_{\mathfrak{g}(\mathcal{O})\oplus\mathbb{C}\mathbf{1}}^{\hat{\mathfrak{g}}_\kappa} V.$$

The resulting $\hat{\mathfrak{g}}_\kappa$-module V^κ is obviously $G(\mathcal{O})$-integrable.

Let V_λ be the finite-dimensional irreducible representation of $\mathfrak{g}$ with dominant integral highest weight λ. The corresponding $\hat{\mathfrak{g}}_\kappa$-module $\mathbb{V}_\lambda^\kappa := (V_\lambda)^\kappa$ is called the Weyl module of highest weight λ. The category $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$ is compactly generated by the subcategory of Weyl modules.

Recall the Verma module $M_\lambda := \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda$ of highest weight λ over $\mathfrak{g}$. Then we define the affine Verma module as $\mathbb{M}^\kappa_\lambda := (M_\lambda)^\kappa$. Equivalently, $\mathbb{M}^\kappa_\lambda = \mathrm{Ind}^{\hat{\mathfrak{g}}_\kappa}_{\mathrm{Lie}(I)} \mathbb{C}_\lambda$, where we regard $\mathbb{C}_\lambda$ as an $\mathrm{Lie}(I)$-module via the evalution map. By definition affine Weyl modules are objects in (the heart of) the DG category $\hat{\mathfrak{g}}_\kappa\text{-mod}^I$ of I-equivariant $\hat{\mathfrak{g}}_\kappa$-modules.

Let $\mathbb{C}_\mu$ be the 1-dimensional $\mathfrak{t}$-module of weight μ. Then similar to the definition of V^κ above we define the Heisenberg module

$$\pi^\kappa_\mu := \mathrm{Ind}^{\hat{\mathfrak{t}}_\kappa}_{\mathfrak{t}(\mathcal{O})\oplus\mathbb{C}\mathbf{1}} \mathbb{C}_\mu,$$

called the Fock module of highest weight μ.

We recall from [7] a perfect pairing between negative level and positive level $\hat{\mathfrak{g}}$-modules:

$$\langle -, - \rangle : \hat{\mathfrak{g}}_{\kappa'}\text{-mod} \times \hat{\mathfrak{g}}_\kappa\text{-mod} \to \mathrm{Vect}$$

given by $\langle N, M\rangle = \mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{g}(\mathcal{K}), N \otimes_{\mathbb{C}} M)$. Here the semi-infinite complex is taken with respect to the lattice $\mathfrak{g}(\mathcal{O}) \subset \mathfrak{g}(\mathcal{K})$. Suppose that a group K acts on $\hat{\mathfrak{g}}$-mod. Then

$$\langle -, - \rangle_K : \hat{\mathfrak{g}}_{\kappa'}\text{-mod}^K \times \hat{\mathfrak{g}}_\kappa\text{-mod}^K \xrightarrow{\langle -,-\rangle} \mathrm{Vect}^K \xrightarrow{H^\bullet_K} \mathrm{Vect} \tag{2.1}$$

defines a pairing between negative and positive level K-equivariant categories. Let $\mathbb{D}_K : \hat{\mathfrak{g}}_{\kappa'}\text{-mod}^K \to \hat{\mathfrak{g}}_\kappa\text{-mod}^K$ be the (contravariant) duality functor which satisfies

$$\langle A, B\rangle_K = \mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa}(\mathbb{D}_K A, B). \tag{2.2}$$

Let H be a subgroup of K. The pairing for equivariant categories naturally commutes with the forgetful functor $\mathrm{obliv} : \hat{\mathfrak{g}}\text{-mod}^K \to \hat{\mathfrak{g}}\text{-mod}^H$; i.e. the following diagram commutes:

$$\begin{array}{ccc} \hat{\mathfrak{g}}_{\kappa'}\text{-mod}^K & \xrightarrow{\mathbb{D}_K} & \hat{\mathfrak{g}}_\kappa\text{-mod}^K \\ {\scriptstyle \mathrm{obliv}}\downarrow & & \downarrow{\scriptstyle \mathrm{obliv}} \\ \hat{\mathfrak{g}}_{\kappa'}\text{-mod}^H & \xrightarrow{\mathbb{D}_H} & \hat{\mathfrak{g}}_\kappa\text{-mod}^H \end{array}$$

Recall that the *-averaging functor $\mathrm{Av}^{K/H}_*$ (resp. !-averaging functor $\mathrm{Av}^{K/H}_!$) is defined as the right (resp. left) adjoint to the forgetful functor $\mathrm{obliv} : \hat{\mathfrak{g}}\text{-mod}^K \to \hat{\mathfrak{g}}\text{-mod}^H$. Then it follows from the definitions that

$$\mathbb{D}_K \circ \mathrm{Av}^{K/H}_* \simeq \mathrm{Av}^{K/H}_! \circ \mathbb{D}_H. \tag{2.3}$$

We end this section by two simple but important calculations of the duality functor.

Lemma 2.2.1. *For any weight μ, $\mathbb{D}_I(\mathbb{M}^{\kappa'}_\mu) \cong \mathbb{M}^\kappa_{-\mu+2\rho}[\dim(G/B)]$.*

Proof 1. From [7, Section 2.2.8], we have

$$\mathbb{D}_I(\mathrm{Ind}^{\hat{\mathfrak{g}}_{\kappa'}}_I \mathbb{C}_\nu) \cong \mathrm{Ind}^{\hat{\mathfrak{g}}_\kappa}_I \mathbb{C}_{-\nu} \otimes \det.\ \mathrm{rel.}(\mathrm{Lie}(I), \mathfrak{g}(\mathcal{O})).$$

Note that

$$\text{det. rel.}(\mathrm{Lie}(I), \mathfrak{g}(\mathcal{O})) \cong \det(\mathfrak{g}/\mathfrak{b})^*$$

is the graded 1-dimensional B-module of weight 2ρ concentrated at degree $-\dim(G/B)$. □

Proof 2. For $L \in \hat{\mathfrak{g}}_\kappa\text{-mod}^I$, we have

$$\begin{aligned}\langle \mathbb{M}_\mu^{\kappa'}, L\rangle_I &\cong H_I^\bullet(\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{g}(\mathcal{K}), (\mathrm{Ind}_I^{\hat{\mathfrak{g}}_{\kappa'}}\mathbb{C}_\mu)\otimes L)) \cong \mathrm{Hom}_{I\text{-mod}}(\mathbb{C}_{-\mu}\otimes \det(\mathfrak{g}(\mathcal{O})/\mathrm{Lie}(I))^*, L)\\ &\cong \mathrm{Hom}_{I\text{-mod}}(\mathbb{C}_{-\mu+2\rho}[\dim(G/B)], L) \cong \mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa\text{-mod}^I}(\mathbb{M}^\kappa_{-\mu+2\rho}[\dim(G/B)], L).\end{aligned}$$

Then the assertion follows from (2.2). □

As this lemma reveals, the duality functor $\mathbb{D}$ (and its equivariant versions) does not preserve the usual t-structures. However, it does preserve the compact objects:

Lemma 2.2.2. *For $\lambda \in \Lambda^+$, $\mathbb{D}_{G(\mathcal{O})}(\mathbb{V}_\lambda^{\kappa'}) \cong \mathbb{V}^\kappa_{-w_0(\lambda)}$.*

Proof. Similar to the proofs of Lemma 2.2.1. Note that the lowest weight of V_λ is $w_0(\lambda)$. □

2.3 Quantum groups

Let ℓ be a sufficiently large positive integer divisible by all d_i's, and put $\ell_i := \ell/d_i$. Recall the following variants of quantum groups associated to $\mathfrak{g}$ and a primitive ℓ-th root of unity q (c.f. [6] and [12]):

- The Drinfeld-Jimbo quantum group $\mathbf{U}_v$, generated by Chavelley generators E_i, F_i and K_t for $i = 1, ..., \mathrm{rank}\mathfrak{g}$ and $t \in T$ over $\mathbb{C}(v)$, the rational functions in v, subject to a list of relations.
- The (Lusztig's) big quantum group $U_q^{\mathrm{Lus}}(\mathfrak{g}) \equiv U_q(\mathfrak{g})$: Take $R := \mathbb{C}[v, v^-1]_{(v-q)} \subset \mathbb{C}(v)$. The algebra $U_q(\mathfrak{g})$ is the specialization to $v = q$ of the R-subalgebra of $\mathbf{U}_v$ generated by E_i, F_i, K_t, and divided powers $E_i^{(\ell_i)}$, $F_i^{(\ell_i)}$.
- The Kac-De Concini quantum group $U_q^{\mathrm{KD}}(\mathfrak{g})$: the specialization to $v = q$ of the R-subalgebra of $\mathbf{U}_v$ generated by E_i, F_i, K_t, and $\dfrac{K_i - K_i^{-1}}{v - v^{-1}}$.
- The small quantum group $\mathfrak{u}_q(\mathfrak{g})$, defined as the $\mathbb{C}$-subalgebra of $U_q(\mathfrak{g})$ generated by (i) K_iE_i, F_i, and K_t for $t \in \mathrm{Ker}(\phi_T)$ when ℓ is even, or (ii) E_i, F_i, and K_i when ℓ is odd.

Let A be one of the above quantum groups. A-mod denotes the category of finite-dimensional modules over the Hopf algebra A.

We define the quantum Frobenius functor

$$\mathrm{Fr}_q : \mathrm{Rep}(\check{G}) \to U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod}$$

as follows: given $V \in \mathrm{Rep}(\check{G})$, let K_t act on V according to $\phi_T : T \to \check{T}$, and let $E_i^{(\ell_i)}$, $F_i^{(\ell_i)}$ act as Chevalley generators e_i, f_i of $U(\check{\mathfrak{g}})$, whereas E_i and F_i act by 0. Then the small quantum group $\mathfrak{u}_q(\mathfrak{g})$ is the Hopf subalgebra of $U_q^{\mathrm{Lus}}(\mathfrak{g})$ universal with respect to the property that $\mathfrak{u}_q(\mathfrak{g})$ acts trivially on $\mathrm{Fr}_q(V)$ for $V \in \mathrm{Rep}(\check{G})$. Namely, we have the following exact sequence of categories

$$0 \to \mathrm{Rep}(\check{G}) \to U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod} \to \mathfrak{u}_q(\mathfrak{g})\text{-mod} \to 0. \tag{2.4}$$

We can also define quantum groups for $\mathfrak{b}$, $\mathfrak{n}$, $\mathfrak{t}$ ...etc. In particular, we will consider the positive part of the various versions of quantum group defined above. In the remainder of this section we give an alternative construction of these quantum groups, where the quantum parameter q and our level κ are related in a more transparent way. Our main reference for this construction is [47].

Recall the root lattice $\mathbb{Z}\Pi$ of G. We start with a bilinear form $b : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \to \mathbb{C}^\times$. Consider the braided monoidal category $\mathrm{Rep}_q(T_{\mathrm{ad}})$ of representations of the quantum adjoint torus: the objects are $\mathbb{Z}\Pi$-graded vector spaces, bifunctor the usual tensor product $\otimes$, and braiding operator induced by

$$x \otimes y \mapsto b(\lambda, \mu)\, y \otimes x \tag{2.5}$$

for $x \in \mathbb{C}_\lambda$ and $y \in \mathbb{C}_\mu$. Consider the object $\oplus_{i\in\Pi}\mathbb{C}E_i$ in $\mathrm{Rep}_q(T_{\mathrm{ad}})$, where each E_i is in degree $\alpha_i \in \Pi$. Let

$$U_q^{\mathrm{free}}(\mathfrak{n}) := \text{the graded free associative algebra generated by } \oplus_{i\in\Pi}\mathbb{C}E_i \text{ in } \mathrm{Rep}_q(T_{\mathrm{ad}}).$$

Similarly, we define $U_q^{\mathrm{free}}(\mathfrak{n}^-)$ generated by $\oplus_{i\in\Pi}\mathbb{C}F_i$ with each F_i in degree $-\alpha_i$. Then set

$$U_q^{\mathrm{cofree}}(\mathfrak{n}) := \bigoplus_{\lambda\in\mathbb{Z}_{\geq 0}\Pi} ((U_q^{\mathrm{free}}(\mathfrak{n}^-))_{-\lambda})^*.$$

We have the canonical bialgebra structure on $U_q^{\mathrm{free}}(\mathfrak{n})$, which induces the canonical bialgebra structure on $U_q^{\mathrm{cofree}}(\mathfrak{n})$. We have a canonical bialgebra map $\varphi : U_q^{\mathrm{free}}(\mathfrak{n}) \to U_q^{\mathrm{cofree}}(\mathfrak{n})$ which sends E_i to δ_{F_i}.

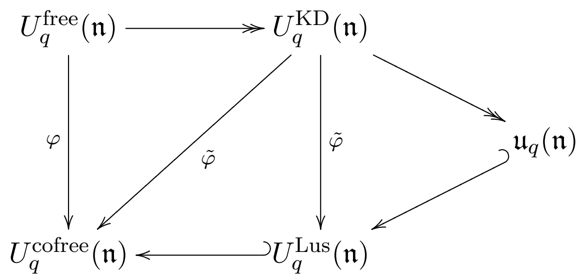
For each $i \in \Pi$, choose $v_i := b(\alpha_i, \alpha_i)^{1/2}$. For arbitrary $i, j \in \Pi$, assume that $b(\alpha_i, \alpha_j) = b(\alpha_j, \alpha_i) = v_i^{\langle\alpha_i, \check{\alpha}_j\rangle}$. For integers n, m, define the quantum binomial coefficient

$$\begin{bmatrix} n \\ m \end{bmatrix}_i = \frac{[n]_i^!}{[m]_i^![n-m]_i^!}$$

where $[n]_i = \frac{v_i^n - v_i^{-n}}{v_i - v_i^{-1}}$ and $[n]_i^! = \Pi_{s=1}^n [s]_i$. The *quantum Serre relation* corresponding to $i, j \in \Pi$ is the element

$$\sum_{p+p'=1-\langle\alpha_i,\check{\alpha}_j\rangle} (-1)^{p'} \begin{bmatrix} 1-\langle\alpha_i, \check{\alpha}_j\rangle \\ p \end{bmatrix}_i E_i^p E_j E_i^{p'}.$$

We define $U_q^{\mathrm{KD}}(\mathfrak{n})$ as the quotient of $U_q^{\mathrm{free}}(\mathfrak{n})$ by the quantum Serre relations for all $i, j \in \Pi$. It is verified in [47] that the quantum Serre relations are sent to zero under φ. Hence φ induces a map $\tilde{\varphi} : U_q^{\mathrm{KD}}(\mathfrak{n}) \to U_q^{\mathrm{cofree}}(\mathfrak{n})$. Define $U_q^{\mathrm{Lus}}(\mathfrak{n}) \subset U_q^{\mathrm{cofree}}(\mathfrak{n})$ as the sub-bialgebra linearly dual to $U_q^{\mathrm{KD}}(\mathfrak{n}^-)$. It is shown in *loc. cit.* that $\tilde{\varphi}$ factors through $U_q^{\mathrm{Lus}}(\mathfrak{n})$. Finally, the small quantum group $\mathfrak{u}_q(\mathfrak{n})$ is defined as the image $\tilde{\varphi}(U_q^{\mathrm{KD}}(\mathfrak{n}))$, regarded as a sub-bialgebra of $U_q^{\mathrm{Lus}}(\mathfrak{n})$. Therefore we have the following diagram



Various versions of representation category of the full quantum group can now be obtained by applying the notion of relative Drinfeld center to the category of modules over $U_q^{\mathrm{Lus}}(\mathfrak{n})$. Suppose that b is extended to a bilinear form on Λ, and recall $\mathrm{Rep}_q(T)$ the braided monoidal category of Λ-graded vector spaces, with the braiding operator given by the same formula as (2.5). Let $U_q^{\mathrm{Lus}}(\mathfrak{n})$-mod be the category consisting of objects in $\mathrm{Rep}_q(T)$ together with action by $U_q^{\mathrm{Lus}}(\mathfrak{n})$ compatible with the grading.

Consider the category $U_q^{\mathrm{Lus}}(\mathfrak{b})$-mod whose objects are unions of finite-dimensional $U_q^{\mathrm{Lus}}(\mathfrak{n})$-submodules. We let $U_q^{\mathrm{Lus}}(\mathfrak{n})$ act on the usual tensor product of modules by

$$x \cdot (v \otimes v') := \sum b(\deg(v), \deg(x_{(2)}))\, (x_{(1)} \cdot v) \otimes (x_{(2)} \cdot v').$$

Then both $U_q^{\mathrm{Lus}}(\mathfrak{n})$-mod and $U_q^{\mathrm{Lus}}(\mathfrak{b})$-mod are now braided monoidal categories.

We first define the category $U_q^{+\mathrm{Lus},-\mathrm{KD}}(\mathfrak{g})$-mod of representations over a "mixed" quantum group, whose positive part is of Lusztig form and negative part is of Kac-De Concini form. Concretely, the category $U_q^{+\mathrm{Lus},-\mathrm{KD}}(\mathfrak{g})$-mod is the relative Drinfeld center $\mathrm{Dr}_{\mathrm{Rep}_q(T)}(U_q^{\mathrm{Lus}}(\mathfrak{b})\text{-mod})$, whose objects are pairs (M, c), where M is in $U_q^{\mathrm{Lus}}(\mathfrak{b})$-mod and

$$c_{M,X} : M \otimes X \xrightarrow{\sim} X \otimes M$$

are isomorphisms functorial in X, such that $c_{M,X\otimes Y} = (\mathrm{Id}_X \otimes c_{M,Y}) \cdot (c_{M,X} \otimes \mathrm{Id}_Y)$ and $c_{M,\mathbb{C}_\nu}(m \otimes x) = b(\deg(m), \nu) \cdot x \otimes m$ for all $\nu \in \mathbb{Z}\Pi$.

The collection c of functorial isomorphisms of an object (M, c) in $U_q^{+\mathrm{Lus},-\mathrm{KD}}(\mathfrak{g})$-mod gives rise to a right coaction $\iota_M : M \to M \otimes U_q^{\mathrm{Lus}}(\mathfrak{n})$ of $U_q^{\mathrm{Lus}}(\mathfrak{n})$ on M, defined by

$$\iota_M(m) := (c_{M,U_q^{\mathrm{Lus}}(\mathfrak{n})})^{-1}(1 \otimes m).$$

This induces a left $(U_q^{\mathrm{Lus}}(\mathfrak{n}))^*$-action and hence a $U_q^{\mathrm{KD}}(\mathfrak{n}^-)$-action, as we have seen that $U_q^{\mathrm{Lus}}(\mathfrak{n})$

and $U_q^{\mathrm{KD}}(\mathfrak{n}^-)$ are dual to each other.

Now we define $U_q^{\frac{1}{2}}(\mathfrak{g})$-mod as the full subcategory of $U_q^{+\mathrm{Lus},-\mathrm{KD}}(\mathfrak{g})$-mod consisting of objects whose induced $U_q^{\mathrm{KD}}(\mathfrak{n}^-)$-action factors through $U_q^{\mathrm{KD}}(\mathfrak{n}^-) \twoheadrightarrow \mathfrak{u}_q(\mathfrak{n}^-)$. Finally, we can recover the category $U_q^{\mathrm{Lus}}(\mathfrak{g})$-mod of representations over the full (Lusztig's) quantum group as consisting of objects (V, ϑ) where V is in $U_q^{\frac{1}{2}}(\mathfrak{g})$-mod and $\vartheta : U_q^{\mathrm{Lus}}(\mathfrak{n}^-) \otimes V \to V$ an action extends the $\mathfrak{u}_q(\mathfrak{n}^-)$-action on V along $\mathfrak{u}_q(\mathfrak{n}^-) \hookrightarrow U_q^{\mathrm{Lus}}(\mathfrak{n}^-)$.

One can show that the forgetful functor $U_q^{\mathrm{Lus}}(\mathfrak{g})$-mod $\to U_q^{\frac{1}{2}}(\mathfrak{g})$-mod is fully faithful. Therefore we have the following embeddings of categories:

$$U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod} \hookrightarrow U_q^{\frac{1}{2}}(\mathfrak{g})\text{-mod} \hookrightarrow U_q^{+\mathrm{Lus},-\mathrm{KD}}(\mathfrak{g})\text{-mod} \tag{2.6}$$

Although we define the above representation categories in terms of abelian categories, in practice we will consider the corresponding DG categories and the notations we use above will always mean DG categories. The abelian categories will be denoted by the heart of the respective DG categories. It is important to note that the embeddings in (2.6) still hold for DG categories.

2.4 Kazhdan-Lusztig equivalence at negative level

Let κ' be a negative level and $q = \exp(\frac{\pi\sqrt{-1}}{\kappa' - \kappa_{\mathrm{crit}}})$. Given bilinear pairing $(\cdot,\cdot)_{\kappa'}$ on $\check{\Lambda}$, we define $b_{\kappa'} : \Lambda \otimes \Lambda \to \mathbb{C}^\times$ to be

$$b_{\kappa'}(\cdot,\cdot) := \exp\Big(\pi\sqrt{-1}\big((\cdot,\cdot)_{\kappa'-\kappa_{\mathrm{crit}}}|_{\mathfrak{t}}\big)^{-1}\Big) \equiv q^{(\cdot,\cdot)_{\mathrm{st}}}.$$

Then $v_i = q^{\frac{(\alpha_i,\alpha_i)_{\mathrm{st}}}{2}}$ and indeed $b_{\kappa'}(\alpha_i, \alpha_j) = b_{\kappa'}(\alpha_j, \alpha_i) = v_i^{\langle \alpha_i, \check{\alpha}_j\rangle}$. Then the constructions from Section 2.3 apply here.

The Kazhdan and Lusztig equivalence is a tensor equivalence of braided monoidal categories:

$$\mathrm{KL}_G : (\hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})})^{\heartsuit} \xrightarrow{\sim} (U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod})^{\heartsuit}.$$

Note that while the braided monoidal structure on $(U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod})^{\heartsuit}$ is given explicitly by the R-matrix and the Hopf algebra structure of $U_q^{\mathrm{Lus}}(\mathfrak{g})$, that on $(\hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})})^{\heartsuit}$ is a nontrivial construction by Kazhdan and Lusztig, inspired by Drinfeld's construction of the Drinfeld associator via the Knizhnik-Zamolodchikov equations, c.f. [37].

We define the quantum Weyl module $\mathcal{V}_\lambda$ of highest weight λ as the image of the Weyl module $\mathbb{V}_\lambda^{\kappa'}$ under the (negative level) Kazhdan-Lusztig functor; i.e.

$$\mathcal{V}_\lambda := \mathrm{KL}_G(\mathbb{V}_\lambda^{\kappa'}) \in (U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod})^{\heartsuit}.$$

From [37, Lemma 38.2], our quantum Weyl module coincides with what is commonly called the

Weyl module of a quantum group in the literature. Explicitly, for $\lambda \in \Lambda^+$

$$\mathcal{V}_\lambda \cong \operatorname{Ind}_{U_q^{\mathrm{Lus}}(\mathfrak{b})}^{U_q^{\mathrm{Lus}}(\mathfrak{g})} \mathbb{C}_\lambda.$$

2.5 Kazhdan-Lusztig functor at positive level

The original Kazhdan-Lusztig functor KL_G is for negative level only. In order to define a Kazhdan-Lusztig type functor for positive level modules, we invoke the duality functor $\mathbb{D}_{G(\mathcal{O})}$ defined in Section 2.2. However, as we have seen in Lemma 2.2.1, the functor $\mathbb{D}_{G(\mathcal{O})}$ does not preserve t-structures. Consequently our definition of the Kazhdan-Lusztig functor at positive level must involve DG categories.

Let κ be positive (which implies that κ' is negative) and $q = \exp(\frac{\pi\sqrt{-1}}{\kappa'-\kappa_{\mathrm{crit}}})$. We first derive the original Kazhdan-Lusztig equivalence to an equivalence of DG categories $\mathrm{KL}_G : \hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} \xrightarrow{\sim} U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod}$ (since the DG category $\hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})}$ can be recovered from its heart by "bootstrapping"; see Section 2.2). Consider the contragredient duality functor for quantum group modules

$$\mathbb{D}^q : U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod} \to U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod},$$

induced by the usual Hopf module dual at the level of abelian category. I.e. for $M \in (U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod})^\heartsuit$, we take the linear dual $\mathbb{D}^q(M) := \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ with $U_q^{\mathrm{Lus}}(\mathfrak{g})$-action twisted by the antipode. Then we can define the Kazhdan-Lusztig functor at positive level κ

$$\mathrm{KL}_G^\kappa : \hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})} \to U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod},$$

such that the following diagram commutes:

$$\begin{array}{ccc} \hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} & \xrightarrow{\mathbb{D}_{G(\mathcal{O})}} & \hat{\mathfrak{g}}_{\kappa}\text{-mod}^{G(\mathcal{O})} \\ \downarrow{\scriptstyle \mathrm{KL}_G} & & \downarrow{\scriptstyle \mathrm{KL}_G^\kappa} \\ U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod} & \xrightarrow[\mathbb{D}^q]{} & U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod} \end{array} \tag{2.7}$$

The functor KL_G^κ is not an equivalence (for it is not t-exact). Nonetheless it becomes an equivalence when restricted to the full subcategory of compact objects. In fact, all arrows in (2.7) become equivalences when restricted to compact objects.

Chapter 3

Preparation: geometric constructions

3.1 Affine flag variety and Kashiwara-Tanisaki Localization

Recall $I := \mathrm{ev}^{-1}(B)$ the Iwahori subgroup of $G(\mathcal{O})$. The affine flag variety is defined as $\mathrm{Fl} := G(\mathcal{K})/I$, and the set of I-orbits of Fl is known to be indexed by the affine Weyl group $W^{\mathrm{aff}} := \check{\Lambda} \rtimes W$. For $\tilde{w} \in W^{\mathrm{aff}}$, corresponding to the I-orbit $I\tilde{w}I \subset \mathrm{Fl}$ we denote by $j_{\tilde{w},*}$ (resp. $j_{\tilde{w},!}$) the costandard (resp. standard) object in the heart of the category $\text{D-mod}(\mathrm{Fl})^I \cong \text{D-mod}(I\backslash G(\mathcal{K})/I)$.

Let μ be a weight. In [36], Kashiwara and Tanisaki constructed the category $\text{D-mod}(\mathrm{Fl})^{I,\mu}$ of μ-twisted I-equivariant right D-modules on the affine flag variety, and proved a correspondence between $\text{D-mod}(\mathrm{Fl})^{I,\mu}$ and the category of I-equivariant modules over the affine Lie algebra at the negative level. (See [36, Section 2] for the detailed construction of $\text{D-mod}(\mathrm{Fl})^{I,\mu}$.)

The μ-twisted category admits the twisted standard and costandard objects, which will still be denoted by $j_{\tilde{w},!}$ and $j_{\tilde{w},*}$. Recall the irreducible object $j_{\tilde{w},!*}$ in the category $\text{D-mod}(\mathrm{Fl})^{I,\mu}$, defined as the image of the canonical morphism $j_{\tilde{w},!} \to j_{\tilde{w},*}$. Let $\text{D-mod}(\mathrm{Fl})^{I,\mu}_0$ be the full subcategory of $\text{D-mod}(\mathrm{Fl})^{I,\mu}$ consisting of objects whose composition factors are isomorphic to $j_{\tilde{w},!*}$ for some $\tilde{w} \in W^{\mathrm{aff}}$. We state the result of Kashiwara-Tanisaki precisely as follows:

Theorem 3.1.1. *Let κ' be a negative level. Suppose that $\mu \in \Lambda$ satisfies $\langle \mu + \rho, \check{\alpha}_i \rangle \leq 0$ for all simple coroots $\check{\alpha}_i$. There is a functor between derived categories*

$$\mathbb{H}^\bullet : \text{D-mod}(\mathrm{Fl})^{I,\mu} \to \hat{\mathfrak{g}}_{\kappa'}\text{-mod}^I$$

with the following properties:

1. *When restricted to $\text{D-mod}(\mathrm{Fl})^{I,\mu}_0$, $\mathbb{H}^n$ is trivial for all $n \neq 0$ and $\Gamma \equiv \mathbb{H}^0$ is exact.*

2. $\Gamma(j_{\tilde{w},!}) \cong \mathbb{M}^{\kappa'}_{\tilde{w}\cdot\mu}$.

Recall the dot action of $\tilde{w} \equiv \check{\lambda} w$ on a weight μ given by

$$\check{\lambda} w \cdot \mu = \check{\lambda} + w(\mu + \rho) - \rho.$$

In order to be compatible with the ordinary (non-twisted) left D-modules on $G/B \hookrightarrow \mathrm{Fl}$ and the Beilinson-Bernstein Localization Theorem for G/B, we choose the twisting to be $\mu - 2\rho$. Therefore, $\Gamma(j_{\check{\lambda} w,!}) \cong \mathbb{M}^{\kappa'}_{\check{\lambda}+w(\mu-\rho)-\rho}$.

For a weight $\nu = \check{\lambda} + w(\mu - \rho) - \rho$, we define the *dual affine Verma module* of highest weight ν as $\mathbb{M}^{\kappa',\vee}_{\nu} := \Gamma(j_{\check{\lambda} w,*})$. It is shown in [36] that this agrees with the contragredient dual of the affine Verma module $\mathbb{M}^{\kappa'}_{\nu}$. We recall the contragredient duality in Section 4.2.

More generally, $\mathbb{H}^\bullet$ intertwines the Verdier duality on D-mod$(\mathrm{Fl})^{I,\mu}$ with the contragredient duality on $\hat{\mathfrak{g}}_{\kappa'}$-mod^I. We can similarly define the *dual Weyl module* $\mathbb{V}^{\kappa',\vee}_{\nu}$, either algebraically via the contragredient duality or geometrically through Verdier duality by virtue of Kashiwara-Tanisaki's theorem (c.f. the discussion following the proof of Lemma 5.1.1.)

3.2 Convolution action on categories

Let $K \subset G(\mathcal{O})$ be a compact open subgroup, and C be a DG category acted on strongly by K on the left. This means that C is a left module category of D-mod(K).

Following [20, Section 22.5], the category D-mod$(G(\mathcal{K})/K)$ of right K-equivariant D-modules on the loop group acts on C^K by convolution, denoted by

$$- \star_K - : \text{D-mod}(G(\mathcal{K})/K) \otimes \mathsf{C}^K \to \mathsf{C}.$$

The convolution is associative in the sense that, for K, K' two open compact subgroups of $G(\mathcal{O})$, we have

$$(M_1 \star_K M_2) \star_{K'} X \simeq M_1 \star_K (M_2 \star_{K'} X)$$

for M_1, M_2, X objects in D-mod$(G(\mathcal{K})/K)$, D-mod$(K\backslash G(\mathcal{K})/K')$ and $\mathsf{C}^{K'}$ respectively. The identity object for the convolution action is $\delta_{K,G(\mathcal{K})/K}$, the delta function D-module supported on the identity coset.

Recall the DG category D-mod(Gr_G) of (right) D-modules on the affine Grassmannian $\mathrm{Gr}_G := G(\mathcal{K})/G(\mathcal{O})$. For a subgroup $H \subset G(\mathcal{K})$, we have the H-equivariant category

$$\text{D-mod}(\mathrm{Gr}_G)^H \simeq \text{D-mod}(H\backslash \mathrm{Gr}_G)$$

by considering the left H-action on D-mod(Gr_G). In particular, we will consider the $N(\mathcal{K})$ or $N^-(\mathcal{K})$-equivariant categories as defined in [24], where a special t-structure is constructed. The convolution action of D-mod$(\mathrm{Gr}_G)^{N^-(\mathcal{K})T(\mathcal{O})}$ on $\hat{\mathfrak{g}}_\kappa$-mod$^{G(\mathcal{O})}$ is:

$$- \star_{G(\mathcal{O})} - : \text{D-mod}(\mathrm{Gr}_G)^{N^-(\mathcal{K})T(\mathcal{O})} \otimes \hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})} \to (\hat{\mathfrak{g}}_\kappa\text{-mod})^{N^-(\mathcal{K})T(\mathcal{O})}.$$

We also often consider the convolution action $-\star_I - : \text{D-mod}(\text{Fl})^I \otimes \hat{\mathfrak{g}}_\kappa\text{-mod}^I \to \hat{\mathfrak{g}}_\kappa\text{-mod}^I$. We prove below some identities involving convolutions that is important for our calculations of semi-infinite cohomology. Unless otherwise specified, all convolution products in the remainder of this section are with respect to I and will be denoted $- \star -$.

Lemma 3.2.1. *Let $A \in \hat{\mathfrak{g}}_{\kappa'}\text{-mod}^I$ and $B \in \hat{\mathfrak{g}}_\kappa\text{-mod}^I$. Then we have*

$$\langle A, j_{\check{\lambda},*} \star B\rangle_I = \langle j_{-\check{\lambda},*} \star A, B\rangle_I.$$

Proof 1. Recall from (2.1) that the pairing $\langle -, -\rangle_I$ is characterized by

$$\langle M, N\rangle_I \cong H_I^\bullet(\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{g}(\mathcal{K}), M \otimes N)).$$

By [20, Proposition 22.7.3], we have

$$H_I^\bullet(\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{g}(\mathcal{K}), A \otimes (j_{\check{\lambda},*} \star B)) \cong H_I^\bullet(\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{g}(\mathcal{K}), (A \star j_{\check{\lambda},*}) \otimes B),$$

where the 'right convolution action' $A \star j_{\check{\lambda},*}$ is through the left I-equivariance structure of $j_{\check{\lambda},*}$ in $\text{D-mod}(I\backslash G(\mathcal{K})/I)$, opposite to the right I-equivariance structure we use for the left convolution action. As a consequence of this side change we have $A \star j_{\check{\lambda},*} \cong j_{-\check{\lambda},*} \star A$, hence we get $\langle A, j_{\check{\lambda},*} \star B\rangle_I = \langle j_{-\check{\lambda},*} \star A, B\rangle_I$. □

Proof 2 (Sketch). The lemma will follow from the identity

$$j_{-\check{\lambda},!} \star \mathbb{D}_I A = \mathbb{D}_I(j_{-\check{\lambda},*} \star A), \tag{3.1}$$

for we will have

$$\begin{aligned}\langle A, j_{\check{\lambda},*} \star B\rangle_I &= \text{Hom}_{\hat{\mathfrak{g}}_\kappa}(\mathbb{D}_I A, j_{\check{\lambda},*} \star B) = \text{Hom}_{\hat{\mathfrak{g}}_\kappa}(j_{-\check{\lambda},!} \star \mathbb{D}_I A, B)\\ &= \text{Hom}_{\hat{\mathfrak{g}}_\kappa}(\mathbb{D}_I(j_{-\check{\lambda},*} \star A), B) = \langle j_{-\check{\lambda},*} \star A, B\rangle_I\end{aligned}$$

by combining the identity with (2.2). But then, since $\mathbb{D}_I$ commutes with convolution actions (where $\mathbb{D}_I$ operates on D-modules to the same effect as taking the Verdier dual, c.f.[3, Theorem 1.3.4]), (3.1) follows from that $j_{-\check{\lambda},*}$ is Verdier dual to $j_{-\check{\lambda},!}$. □

Lemma 3.2.2. *Let $\tilde{w} \in W^{\text{aff}}$ and $\overline{I\tilde{w}I}$ be the closure of the orbit $I\tilde{w}I$ in* Fl. *For $F \in \text{D-mod}(\text{Gr}_G)^I$, we have the following identities:*

1. $\text{Av}_!^I(\tilde{w} \cdot F) \cong j_{\tilde{w},!} \star F[\dim \overline{I\tilde{w}I}]$,
2. $\text{Av}_*^I(\tilde{w} \cdot F) \cong j_{\tilde{w},*} \star F[-\dim \overline{I\tilde{w}I}]$.

Proof. Unravelling the definition, we see that $\tilde{w} \cdot F \cong \delta_{\tilde{w}} \star F$, where $\delta_{\tilde{w}}$ is the delta-function D-module at the coset $\tilde{w}I \in \text{Fl}$. Since F is I-equivariant, the shift $\tilde{w} \cdot F$ is $\text{Ad}_{\tilde{w}}(I)$-equivariant.

Let $I^{\tilde{w}} := I \cap \mathrm{Ad}_{\tilde{w}}(I)$ and $\mathrm{act}, \mathrm{pr} : I \times_{I^{\tilde{w}}} \mathrm{Gr}_G \to \mathrm{Gr}_G$ be the action and projection map, respectively. Then we have

$$\mathrm{Av}^I_!(\tilde{w} \cdot F) = \mathrm{act}_! \circ \mathrm{pr}^!(\delta_{\tilde{w}} \star F) \cong \mathrm{act}_!(\mathcal{O}_I \tilde{\boxtimes}(\delta_{\tilde{w}} \star F))[\dim \overline{I\tilde{w}I}] \cong j_{\tilde{w},!} \star F[\dim \overline{I\tilde{w}I}],$$

proving (1).

For (2), we apply the Verdier duality on D-modules to (1). Since the Verdier duality commutes with convolution product, we get

$$\mathrm{Av}^I_*(\tilde{w} \cdot \mathbb{D}F) \cong j_{\tilde{w},*} \star \mathbb{D}F[-\dim \overline{I\tilde{w}I}].$$

Since $\mathbb{D}$ is an equivalence, (2) is proven. □

3.3 Spherical category

We define the spherical category as

$$\mathrm{Sph}_G := \text{D-mod}(\mathrm{Gr}_G)^{G(\mathcal{O})},$$

the DG category of (left) $G(\mathcal{O})$-equivariant D-modules on Gr_G, which is a categorical analogue of the spherical Hecke algebra in number theory. The $G(\mathcal{O})$-orbits of Gr_G are parametrized by the dominant coweights of G. For each dominant coweight $\check{\lambda} \in \check{\Lambda}^+$, the corresponding $G(\mathcal{O})$-orbit is $G(\mathcal{O})t^{\check{\lambda}}G(\mathcal{O}) =: \mathrm{Gr}_G^{\check{\lambda}}$. Let $\mathrm{IC}_{\check{\lambda}} \in (\mathrm{Sph}_G)^\heartsuit$ be the IC D-module supported on the closure $\overline{\mathrm{Gr}_G^{\check{\lambda}}}$. It is known that $(\mathrm{Sph}_G)^\heartsuit$ is semisimple with simple objects given by these IC D-modules.

Recall from [43] that the spherical category is equipped with a convolution product $- \star_{G(\mathcal{O})} -$, which makes $(\mathrm{Sph}_G)^\heartsuit$ a symmetric monoidal category. In fact, this coincides with the convolution defined in Section 3.2 by taking $\mathsf{C} = \text{D-mod}(\mathrm{Gr}_G)$ and $K = G(\mathcal{O})$.

Denote by $\mathrm{Sat} : \mathrm{Rep}(\check{G})^\heartsuit \to (\mathrm{Sph}_G)^\heartsuit$ the geometric Satake equivalence, a tensor equivalence of symmetric monoidal categories. For $\check{\lambda} \in \check{\Lambda}^+$, the functor Sat sends the irreducible representation $V_{\check{\lambda}}$ of $\check{G}$ to the IC D-module $\mathrm{IC}_{\check{\lambda}}$.

We define the semi-infinite orbits $S_{\check{\mu}}$ as the $N(\mathcal{K})$-orbit $N(\mathcal{K})t^{\check{\mu}}G(\mathcal{O})$ inside Gr_G. The opposite semi-infinite orbit $T_{\check{\nu}}$ is defined to be $N^-(\mathcal{K})t^{\check{\nu}}G(\mathcal{O})$. It is known that the semi-infinite orbits and the opposite semi-infinite orbits are both parametrized by the coweight lattice $\check{\Lambda}$ of G.

The theory of weight functors developed in [43] enables us to compute cohomology of objects in the spherical category by representation theory. In particular, we will compute the !-stalk of

$\mathrm{IC}_{\check\lambda}$ at the point $t^{w_0(\check\lambda)}G(\mathcal{O}) \in \mathrm{Gr}_G$. Denote the inclusions

$$\begin{array}{ccc} t^{w_0(\check\lambda)}G(\mathcal{O}) & \xhookrightarrow{\iota} & \mathrm{Gr}_G \\ & {\scriptstyle k}\searrow \quad \nearrow{\scriptstyle s} & \\ & S_{w_0(\check\lambda)} & \end{array}$$

Then the !-stalk of $\mathrm{IC}_{\check\lambda}$ at $t^{w_0(\check\lambda)}G(\mathcal{O})$ is

$$\iota^! \, \mathrm{IC}_{\check\lambda} \cong k^! s^* \, \mathrm{IC}_{\check\lambda}[2\langle \rho, w_0(\check\lambda) - \check\lambda\rangle] \cong H_c^\bullet(S_{w_0(\check\lambda)}, \mathrm{IC}_{\check\lambda})[2\langle \rho, w_0(\check\lambda) - \check\lambda\rangle].$$

But then the cohomology $H_c^\bullet(S_{w_0(\check\lambda)}, \mathrm{IC}_{\check\lambda})$ vanishes except at degree $\langle 2\rho, w_0(\check\lambda)\rangle$, and the non-vanishing part is precisely the weight functor that computes the weight multiplicity of $V_{\check\lambda}$ at weight $w_0(\check\lambda)$. The representation theory tells us that it is one-dimensional. We conclude

$$\iota^! \, \mathrm{IC}_{\check\lambda} \cong \mathbb{C}[-\langle 2\rho, \check\lambda\rangle] \cong \mathbb{C}[\langle 2\rho, w_0(\check\lambda)\rangle]. \tag{3.2}$$

For arbitrary level κ, we define an action of $\mathrm{Rep}(\check G)$ on $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$ by

$$V, M \mapsto \mathrm{Sat}(V) \star_{G(\mathcal{O})} M.$$

According to [3, Theorem 1.3.4], we have

$$\mathrm{KL}_G(\mathrm{Sat}(V) \star_{G(\mathcal{O})} M) \cong \mathrm{Fr}_q(V) \otimes \mathrm{KL}_G(M) \tag{3.3}$$

for $V \in \mathrm{Rep}(\check G)$ and $M \in \hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})}$ where κ' is negative.

Chapter 4

Wakimoto modules

The Wakimoto modules are a class of representations of affine Kac-Moody algebras, originally introduced by M. Wakimoto [53] for $\hat{\mathfrak{sl}}_2$ and generalized to arbitrary types by B. Feigin and E. Frenkel [16]. In this section, we will give two geometric constructions of Wakimoto modules, following [20] and [25]. A more algebraic construction of Wakimoto modules can be found in [18].

4.1 First construction, via chiral differential operators

In the first construction, we follow [20]. This approach is inspired by the localization theorem of Beilinson and Bernstein [8] for finite-dimensional Lie algebras. Namely, the construction can be seen as an infinite-dimensional analog of taking sections of twisted D-modules on the big Schubert cell in G/B.

Naively one would try to make sense of D-modules on $G((t))/B((t))$. But as explained in [19, Section 11.3.3] and [20], the semi-infinite flag manifold $G((t))/B((t))$ is an ill-behaved infinite-dimensional object, and it is still not known whether a good theory of D-modules on $G((t))/B((t))$ exists. Nevertheless, the theory of chiral differential operators are created to address this issue (see [5, Section 6]). To model D-modules on $G((t))/B((t))$, we consider chiral modules over the chiral algebra $\mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G/B})_\kappa$ defined below.

Fix an arbitrary level $\kappa \in \mathbb{C}^\times$, we recall from the Appendix the chiral algebra of differential operators $\mathfrak{D}^{\mathrm{ch}}(G)_\kappa$ (by setting the pairing Q as $\kappa(\cdot,\cdot)_{\mathrm{st}}$). Denote by $\overset{\circ}{G}$ the open cell $Bw_0B \subset G$. Then we have the induced chiral differential operators $\mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G})_\kappa$ on $\overset{\circ}{G}$. We have the left- and right-invariant vector fields maps from $L_{\mathfrak{g},\kappa}$ and $L'_{\mathfrak{g},\kappa'}$ into $\mathfrak{D}^{\mathrm{ch}}(G)_\kappa$, respectively. In particular, we have a morhpism $L_{\mathfrak{n}} \to \mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G})_\kappa$ given by the composition

$$\mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G})_\kappa \leftarrow \mathfrak{D}^{\mathrm{ch}}(G)_\kappa \overset{\mathfrak{r}}{\leftarrow} L'_{\mathfrak{g},\kappa'} \leftarrow L_{\mathfrak{n}}.$$

This enables us to define the semi-infinite complex $\mathfrak{C}^{\frac{\infty}{2}}(L_{\mathfrak{n}}, \mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G})_\kappa)$. It is known [20, Lemma

10.3.1] that this complex is acyclic away from degree zero. We therefore define the chiral differential operators on $\overset{\circ}{G/N}$ as

$$\mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G/N})_\kappa := H^{\frac{\infty}{2}}(L_{\mathfrak{n}}, \mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G})_\kappa).$$

Now, consider the Lie-* subalgebra $L'_{\mathfrak{b},\kappa'} \subset L'_{\mathfrak{g},\kappa'}$ whose structure as a central extension of $L_{\mathfrak{b}}$ by ω_X comes from that of $L'_{\mathfrak{g},\kappa'}$ (defined in Section 9.3). We define $\hat{L}'_{\mathfrak{t},\kappa}$ as the central extension of $L_{\mathfrak{t}}$ induced from $\hat{L}'_{\mathfrak{b},\kappa}$, which is the Baer sum of the Tate extension $L^{\flat}_{\mathfrak{b}}$ and $L'_{\mathfrak{b},\kappa'}$. We also define the Lie-* algebra $\hat{L}_{\mathfrak{t},\kappa}$ as the Baer negative of $\hat{L}'_{\mathfrak{t},\kappa}$. The map $\mathfrak{r} : L'_{\mathfrak{g},\kappa'} \to \mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G})_\kappa$ induces a Lie-* morphism

$$\hat{L}'_{\mathfrak{t},\kappa} \to \mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G/N})_\kappa,$$

which gives rise to the chiral algebra morphism

$$U^{\mathrm{ch}}(\hat{L}'_{\mathfrak{t},\kappa}) \to \mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G/N})_\kappa.$$

Note that this involves the Tate shift owing to the construction of semi-infinite cohomology with respect to $L_{\mathfrak{n}}$, as in (9.1).

The chiral algebra $\mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G/B})_\kappa$ is defined as the Lie-* centralizer of the image of $U^{\mathrm{ch}}(\hat{L}'_{\mathfrak{t},\kappa})$ in $\mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G/N})_\kappa$. Since the left-invariant vector fields map $\mathfrak{l}$ commutes with $\mathfrak{r}$, we obtain a morphism of chiral algebras

$$\mathfrak{l} : A_{\mathfrak{g},\kappa} \to \mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G/B})_\kappa. \tag{4.1}$$

In fact, it is shown in [20, Section 10.4] that if we identify $\overset{\circ}{G} \simeq Nw_0B$ with the product $N \times B$, then we have isomorphisms

$$\mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G})_\kappa \simeq \mathfrak{D}^{\mathrm{ch}}(N) \otimes \mathfrak{D}^{\mathrm{ch}}(B)_{\kappa'}$$

$$\mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G/N})_\kappa \simeq \mathfrak{D}^{\mathrm{ch}}(N) \otimes \hat{\mathfrak{D}}^{\mathrm{ch}}(H)_\kappa$$

where $\hat{\mathfrak{D}}^{\mathrm{ch}}(H)_\kappa$ admits chiral left- and right-invariant fields morphisms

$$\mathfrak{l}_{\mathfrak{t}} : U^{\mathrm{ch}}(\hat{L}_{\mathfrak{t},\kappa}) \longrightarrow \hat{\mathfrak{D}}^{\mathrm{ch}}(H)_\kappa \longleftarrow U^{\mathrm{ch}}(\hat{L}'_{\mathfrak{t},\kappa}) : \mathfrak{r}_{\mathfrak{t}}.$$

The centralizer of $\mathfrak{r}_{\mathfrak{t}}(U^{\mathrm{ch}}(\hat{L}'_{\mathfrak{t},\kappa}))$ is precisely $U^{\mathrm{ch}}(\hat{L}_{\mathfrak{t},\kappa})$. Hence we derive from (4.1) the *free field realization*

$$\mathfrak{l} : A_{\mathfrak{g},\kappa} \to \mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G/B})_\kappa \cong \mathfrak{D}^{\mathrm{ch}}(N) \otimes U^{\mathrm{ch}}(\hat{L}_{\mathfrak{t},\kappa}). \tag{4.2}$$

In addition, by $\mathfrak{C}^{\frac{\infty}{2}}(L_{\mathfrak{t}}, \hat{\mathfrak{D}}^{\mathrm{ch}}(H)_\kappa \otimes U^{\mathrm{ch}}(\hat{L}_{\mathfrak{t},\kappa})) \cong U^{\mathrm{ch}}(\hat{L}_{\mathfrak{t},\kappa})$ [20, Section 22.6] we have

$$\mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G/B})_\kappa \cong \mathfrak{C}^{\frac{\infty}{2}}(L_{\mathfrak{t}}, \mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G/N})_\kappa \otimes U^{\mathrm{ch}}(\hat{L}_{\mathfrak{t},\kappa})), \tag{4.3}$$

where the semi-infinite complexes are taken with respect to the left-invariant vector field map of $L_{\mathfrak{t}}$.

Let us now consider the chiral $\mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G})_\kappa$-module

$$\mathrm{Dist}^{\mathrm{ch}}_G(I^0 wI)_\kappa$$

supported at a fixed point $x \in X$, corresponding to the *-extension of the D-module $\mathrm{Fun}(I^0wI)$ on $\overset{\circ}{G}[[t]]$. Then $\mathfrak{C}^{\frac{\infty}{2}}(L_{\mathfrak{n}}, \mathrm{Dist}^{\mathrm{ch}}_G(I^0wI)_\kappa)$ is naturally a $\mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G}/N)_\kappa$-module. Let $\pi_\mu^{-\kappa'-\mathrm{shift}}$ be the chiral Fock module over $U^{\mathrm{ch}}(\hat{L}_{\mathfrak{t},\kappa})$ of highest weight μ. Then for a Weyl group element w, we define the type w chiral Wakimoto module $\mathbb{W}^{\kappa,w}_\lambda$ of highest weight λ at level κ as

$$\mathbb{W}^{\kappa,w}_\lambda := \mathfrak{C}^{\frac{\infty}{2}}(L_{\mathfrak{t}}, \mathfrak{C}^{\frac{\infty}{2}}(L_{\mathfrak{n}}, \mathrm{Dist}^{\mathrm{ch}}_G(I^0wI)_\kappa) \otimes \pi^{-\kappa'-\mathrm{shift}}_{w^{-1}(\lambda+\rho)+\rho}),$$

which is acted on by $\mathfrak{D}^{\mathrm{ch}}(\overset{\circ}{G}/B)_\kappa$ due to (4.3), and becomes a chiral $A_{\mathfrak{g},\kappa}$-module via the free field realization (4.2). As in Example 9.1.1, a chiral $A_{\mathfrak{g},\kappa}$-module supported at a point $x \in X$ amounts to a module over the affine Lie algebra $\hat{\mathfrak{g}}_\kappa$. The $\hat{\mathfrak{g}}_\kappa$-module induced by $\mathbb{W}^{\kappa,w}_\lambda$ is called the type w Wakimoto module of highest weight λ at level κ, and will still be denoted by $\mathbb{W}^{\kappa,w}_\lambda$ by abuse of notation.

A crucial property of Wakimoto modules following this line of construction is:

Proposition 4.1.1 ([20] Proposition 12.5.1)**.** *For a dominant coweight $\check{\lambda} \in \check{\Lambda}^+$, we have*

$$j_{\check{\lambda},!} \star_I \mathbb{W}^{\kappa,1}_\mu \cong \mathbb{W}^{\kappa,1}_{\mu+\check{\lambda}} \quad \text{and} \quad j_{\check{\lambda},*} \star_I \mathbb{W}^{\kappa,w_0}_\mu \cong \mathbb{W}^{\kappa,w_0}_{\mu+\check{\lambda}}.$$

4.2 Digression: Modules over affine Kac-Moody algebras and contragredient duality

In order to compare Verma modules and Wakimoto modules algebraically, we need to introduce the notion of character of a module over affine Lie algebras. However, characters are well-defined only when weight spaces are finite-dimensional, which prompts us to introduce the action of the degree operator $t\partial_t$ and affine Kac-Moody algebras $\hat{\mathfrak{g}} \rtimes \mathbb{C}t\partial_t$.

Let $\hat{\mathfrak{g}} := \hat{\mathfrak{g}}_1$. Consider the affine Kac-Moody algebra $\hat{\mathfrak{g}} \rtimes \mathbb{C}t\partial_t$, where the degree operator $t\partial_t$ acts on $\hat{\mathfrak{g}}$ by

$$t\partial_t(g) := t\frac{dg}{dt} \quad \text{for}\, g \in \mathfrak{g}(K) \quad \text{and} \quad t\partial_t(\mathbf{1}) := 0.$$

Given any $\mathfrak{g}$-module V, if we let $t\partial_t$ act on V by 0 and $\mathbf{1}$ act by κ, then the induction makes

$$V^\kappa := \mathrm{Ind}^{\hat{\mathfrak{g}} \rtimes \mathbb{C}t\partial_t}_{\mathfrak{g}(\mathcal{O}) \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}t\partial_t} V$$

a module over the affine Kac-Moody algebra. We will consider the category of affine Kac-Moody modules where the central element $\mathbf{1}$ acts by κ, and we will call its objects $\hat{\mathfrak{g}}_\kappa \rtimes \mathbb{C}t\partial_t$-modules or modules over $\hat{\mathfrak{g}}_\kappa \rtimes \mathbb{C}t\partial_t$. Clearly, a $\hat{\mathfrak{g}}_\kappa \rtimes \mathbb{C}t\partial_t$-module is equivalent to a $\hat{\mathfrak{g}}_\kappa$-module with the same $t\partial_t$-action.

Clearly we can define the Verma module $\mathbb{M}^\kappa_\lambda$ and Weyl module $\mathbb{V}^\kappa_\lambda$ over $\hat{\mathfrak{g}}_\kappa \rtimes \mathbb{C}t\partial_t$ by the same induction procedure. To make the Wakimoto modules $\mathbb{W}^{\kappa,w}_\lambda$ an affine Kac-Moody module, we let the operator $t\partial_t$ act by loop rotation. Namely, the action is the unique compatible action induced from the requirement that the vacuum vector of weight λ is annihilated by $t\partial_t$.

A weight of an affine Kac-Moody algebra is a tuple $(n, \mu, v) \in (\mathbb{C}t\partial_t \oplus \mathfrak{t} \oplus \mathbb{C}\mathbf{1})^*$, with the natural pairing given by $(n, \mu, v) \cdot (pt\partial_t, h, a\mathbf{1}) = np + \mu(h) + va$. From the structure theory of affine Kac-Moody algebras, the set of roots of $\hat{\mathfrak{g}} \rtimes \mathbb{C}t\partial_t$ is

$$\hat{R} = \{(n, \alpha, 0) : n \in \mathbb{Z}, \alpha \in R \sqcup 0\} - \{(0, 0, 0)\},$$

where R is the root system of $\mathfrak{g}$. The roots of the form $(n, 0, 0)$ are called imaginary roots, each of which has multiplicity equal to $\dim \mathfrak{t}$. All the other roots are called real, with multiplicity one. The set of positive roots is

$$\hat{R}^+ = \{(0, \alpha, 0) : \alpha \in R^+\} \sqcup \{(n, \alpha, 0) : n > 0, \alpha \in R \sqcup 0\}.$$

We denote by $M(\hat{\mu})$ the $\hat{\mu}$-weight space of an affine Kac-Moody module M. Note that, due to the grading given by the action of $t\partial_t$, all weight spaces of V^κ for any $V \in \mathfrak{g}$-mod are finite-dimensional. For an affine Kac-Moody module M with finite-dimensional weight spaces, we define the character of M to be the formal sum

$$ch\, M := \sum_{\hat{\mu} \in (\mathbb{C}t\partial_t \oplus \mathfrak{t} \oplus \mathbb{C}\mathbf{1})^*} \dim M(\hat{\mu})\, e^{\hat{\mu}}.$$

The Cartan involution τ on $\mathfrak{g}$ is a linear involution which sends the Chevalley generators as follows:

$$\tau(e_i) = -f_i, \quad \tau(f_i) = -e_i, \quad \tau(h_i) = -h_i.$$

For a $\mathfrak{g}$-module V, let $V^\vee := \oplus_{\mu \in \mathfrak{t}^*} V(\mu)^*$ be the usual contragredient dual, with its $\mathfrak{g}$-action defined by $x \cdot f(v) := f(-\tau(x) \cdot v)$ for $f \in V^*$ and $x \in \mathfrak{g}$.

We extend τ to an involution $\hat{\tau}$ on $\hat{\mathfrak{g}} \rtimes \mathbb{C}t\partial_t$ by setting $\hat{\tau}(f_\theta \otimes t) := -e_\theta \otimes t^{-1}$, $\hat{\tau}(\mathbf{1}) := -\mathbf{1}$ and $\hat{\tau}(t\partial_t) = -t\partial_t$. Here θ denotes the longest root of $\mathfrak{g}$.

Given any affine Kac-Moody modules M with finite-dimensional weight spaces, we similarly define its contragredient dual $M^\vee$ as the restricted dual space

$$\bigoplus_{\hat{\mu} \in (\mathbb{C}t\partial_t \oplus \mathfrak{t} \oplus \mathbb{C}1)^*} M(\hat{\mu})^*$$

with the $\hat{\mathfrak{g}} \rtimes \mathbb{C}t\partial_t$-action given by the same formula with τ replaced by $\hat{\tau}$.

The following basic properties are evident by definition:

Proposition 4.2.1.

1. *For an affine Kac-Moody module M with finite-dimensional weight spaces, $(M^\vee)^\vee = M$.*

2. *Taking contragredient dual of a representation preserves its character.*

4.3 Camparing Wakimoto and Verma modules at negative level

When the level is negative, we show that the Wakimoto module of type w_0 is isomorphic to the dual Verma module of the same highest weight λ if λ is sufficiently dominant (Theorem 4.3.3).

Let us fix a negative level κ' from now on. The character of $\mathbb{M}_\lambda^{\kappa'}$ is by its definition

$$ch\,\mathbb{M}_\lambda^{\kappa'} = e^{(0,\lambda,\kappa')}\left(\prod_{\hat\alpha\,\in\,\hat R^+}\left(1-e^{-\hat\alpha}\right)^{mult(\hat\alpha)}\right)^{-1}.$$

The following lemma describes the character of the Wakimoto module $\mathbb{W}_\lambda^{\kappa',w_0}$. In particular, we see that the characters of Verma module $\mathbb{M}_\lambda^{\kappa'}$ and Wakimoto module $\mathbb{W}_\lambda^{\kappa',w_0}$ are identical.

Lemma 4.3.1.

$$ch\,\mathbb{W}_\lambda^{\kappa',w_0} = e^{(0,\lambda,\kappa')}\left(\prod_{\hat\alpha\,\in\,\hat R^+}\left(1-e^{-\hat\alpha}\right)^{mult(\hat\alpha)}\right)^{-1}.$$

Proof. Consider the chiral module $\mathrm{Dist}_N^{\mathrm{ch}}(\mathrm{ev}^{-1}(N))$ over the chiral algebra $\mathfrak{D}^{\mathrm{ch}}(N)$ corresponding to the D-module $\mathrm{Fun}(N[[t]])$ on $N[[t]]$. Then we can rewrite $\mathbb{W}_\lambda^{\kappa',w_0} \cong \mathrm{Dist}_N^{\mathrm{ch}}(\mathrm{ev}^{-1}(N)) \otimes \pi_{w_0(\lambda)}^{-\kappa-\mathrm{shift}}$ [20, formula (11.6)]. As a $\mathbb{C}t\partial_t \oplus \mathfrak{t} \oplus \mathbb{C}\mathbf{1}$-module, $\mathbb{W}_\lambda^{\kappa',w_0}$ is equal to

$$\mathbb{C}[x^*_{\alpha,n}]_{\alpha\in R^+,n\le 0} \otimes \mathbb{C}[x_{\alpha,m}]_{\alpha\in R^+,m<0} \otimes \mathbb{C}[y_{i,l}]_{l<0,i=1,\dots,\dim\mathfrak{t}}, \tag{4.4}$$

where the weights of $x^*_{\alpha,n}$, $x_{\alpha,m}$ and $y_{i,l}$ are $(n,-\alpha,0)$, $(m,\alpha,0)$ and $(l,0,0)$, respectively, and the vector $1\otimes 1\otimes 1$ has weight $(0,\lambda,\kappa')$. Note that $\mathbb{C}[y_{i,l}]_{l<0,i=1,\dots,\dim\mathfrak{t}}$ corresponds to the Fock module $\pi_{w_0(\lambda)}^{-\kappa-\mathrm{shift}}$ over the Heisenberg algebra, the tensor factor $\mathbb{C}[x^*_{\alpha,n}]_{\alpha\in R^+,n\le 0}$ corresponds to $\mathrm{Fun}(N[[t]])$, and $\mathbb{C}[x_{\alpha,m}]_{\alpha\in R^+,m<0}$ arises from the induction of $\mathrm{Fun}(N[[t]])$ to a chiral module over $\mathfrak{D}^{\mathrm{ch}}(N)$. The character formula follows immediately from (4.4). □

We recall the Kac-Kazhdan Theorem [35] on possible singular weights appearing in a Verma module over $\hat{\mathfrak{g}}_\kappa \rtimes \mathbb{C}t\partial_t$ (where κ is arbitrary):

Theorem 4.3.2 (Kac-Kazhdan)**.** *Let $\hat\mu = (n,\mu,\kappa)$ be a singular weight of $\mathbb{M}_\lambda^\kappa$, namely, $\hat\mu$ is a highest weight of some subquotient of M_λ^κ. Then the following condition holds: There exist a sequence of weights $(0,\lambda,\kappa) = \hat\lambda \equiv \hat\mu_1, \hat\mu_2, \dots, \hat\mu_n \equiv \hat\mu$ and a sequence of positive roots $\hat\alpha_k \in \hat R^+$, $k = 1,2,\dots,n$, such that for each k, there is $b_k \in \mathbb{Z}^{>0}$ satisfying*

$$\hat\mu_{k+1} = \hat\mu_k - b_k\cdot\hat\alpha_k$$

and

$$b_k \cdot (\hat{\alpha}_k, \hat{\alpha}_k) = 2 \cdot (\hat{\alpha}_k, \hat{\mu}_k + (0, \rho, h^\vee)).$$

Here $(\cdot,\cdot)$ *is the standard invariant bilinear form on the weights of* $\hat{\mathfrak{g}} \rtimes \mathbb{C}t\partial_t$.

The main result of this section is the following:

Theorem 4.3.3.

1. *Let* κ' *be negative and rational. Suppose that* $\lambda \in \Lambda^+$ *is sufficiently dominant. Then* $\mathbb{W}_\lambda^{\kappa',w_0}$ *is isomorphic to* $(\mathbb{M}_\lambda^{\kappa'})^\vee$ *as* $\hat{\mathfrak{g}}_{\kappa'}$*-modules.*

2. *Let* κ' *be irrational and* λ *be integral. Then* $\mathbb{W}_\lambda^{\kappa',w_0}$ *is isomorphic to* $(\mathbb{M}_\lambda^{\kappa'})^\vee$ *as* $\hat{\mathfrak{g}}_{\kappa'}$*-modules.*

Proof of 2. We will prove $\mathbb{M}_\lambda^{\kappa'} \cong (\mathbb{W}_\lambda^{\kappa',w_0})^\vee$.

First of all, since $(\mathbb{W}_\lambda^{\kappa',w_0})^\vee$ has highest weight the same as that of $\mathbb{M}_\lambda^{\kappa'}$, the universal property of Verma module implies the existence of a canonical morphism

$$\Phi : \mathbb{M}_\lambda^{\kappa'} \to (\mathbb{W}_\lambda^{\kappa',w_0})^\vee.$$

Using Lemma 4.3.1 and Proposition 4.2.1, we see that $\mathbb{M}_\lambda^{\kappa'}$ and $(\mathbb{W}_\lambda^{\kappa',w_0})^\vee$ have the same character. Hence Φ is injective if and only if it is surjective.

Suppose that Φ is not injective. Then we can pick a highest weight vector $u \in \mathrm{Ker}(\Phi)$ of weight $\hat{\mu}$. By the equality of characters, there exists $v \in (\mathbb{W}_\lambda^{\kappa',w_0})^\vee/\mathrm{Im}(\Phi)$ of the same weight $\hat{\mu}$. Now we claim that v cannot lie in $(\mathfrak{n}^-[t^{-1}] \oplus t^{-1}\mathfrak{b}[t^{-1}])(\mathbb{W}_\lambda^{\kappa',w_0})^\vee$. Indeed, if $v \in (\mathfrak{n}^-[t^{-1}] \oplus t^{-1}\mathfrak{b}[t^{-1}])(\mathbb{W}_\lambda^{\kappa',w_0})^\vee$, then we can find a vector $v' \in (\mathbb{W}_\lambda^{\kappa',w_0})^\vee$ of weight higher than $\hat{\mu}$ such that $x \cdot v' = v$ for some $x \in \hat{\mathfrak{g}}$. By the assumption on $\hat{\mu}$, we obtain a vector $u' \in M_\lambda^{\kappa'}$ with $\Phi(u') = v'$, but then $v = x \cdot \Phi(u') = \Phi(x \cdot u') \in \mathrm{Im}(\Phi)$ is a contradiction.

Therefore, the vector v projects nontrivially onto the coinvariants

$$(\mathbb{W}_\lambda^{\kappa',w_0,\vee})_{\mathfrak{n}^-[t^{-1}]\oplus t^{-1}\mathfrak{b}[t^{-1}]},$$

and in particular, as $\mathbb{C}t\partial_t \oplus \mathfrak{t} \oplus \mathbb{C}1$-modules,

$$(\mathbb{W}_\lambda^{\kappa',w_0,\vee})_{\mathfrak{n}^-[t^{-1}]\oplus t^{-1}\mathfrak{b}[t^{-1}]} \twoheadleftarrow (\mathbb{W}_\lambda^{\kappa',w_0,\vee})_{\mathfrak{n}^-[t^{-1}]\oplus t^{-1}\mathfrak{t}[t^{-1}]} \cong \mathbb{C}[x_{\alpha,m}]_{\alpha\in R^+, m<0}$$

(for the notation $x_{\alpha,m}$, see the proof of Lemma 4.3.1). We conclude that the weight $\hat{\mu}$ must be of the form

$$\hat{\mu} = (-n, \lambda + \beta, \kappa') \tag{4.5}$$

for $n \in \mathbb{Z}^{>0}, \beta \in \mathrm{Span}^+(R^+)$.

On the other hand, since $\hat{\mu}$ is a highest weight of a submodule of $\mathbb{M}_\lambda^{\kappa'}$, there exist a sequence of weights $(0, \lambda, \kappa') = \hat{\lambda} \equiv \hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_n \equiv \hat{\mu}$ and a sequence of positive roots $\hat{\alpha}_k, k = 1, 2, \ldots, n$, satisfying the conditions in the Kac-Kazhdan Theorem. For each $k = 1, 2, \ldots, n$, either $\hat{\alpha}_k$ is real or it is imaginary. We write $\hat{\mu}_k = (n_k, \mu_k, \kappa')$, $\hat{\alpha}_k = (m_k, \alpha_k, 0), m_k \geq 0, \alpha_k \in R$ if $\hat{\alpha}_k$ is real, and $\hat{\alpha}_k = (m_k, 0, 0), m_k > 0$ if $\hat{\alpha}_k$ is imaginary.

Suppose that $\hat{\alpha}_k$ is real with m_k nonzero. Then

$$\begin{aligned} b_k &= p \cdot (\hat{\alpha}_k, \hat{\mu}_k + (0, \rho, h^\vee)) \\ &= p \cdot ((m_k, \alpha_k, 0), (n_k, \mu_k + \rho, \kappa' + h^\vee)) \\ &= p \cdot (\kappa' + h^\vee) m_k + p \cdot (\mu_k + \rho, \alpha_k), \end{aligned} \tag{4.6}$$

where p is some nonzero positive rational number. From our assumption that κ' is irrational and λ is integral, we get an irrational number b_k, which contradicts the condition in the Kac-Kazhdan Theorem.

Therefore, $\hat{\alpha}_k$ has to be either real with $m_k = 0$ or imaginary for each k. This implies $\mu_{k+1} = \mu_k - \alpha_k$ for $\alpha_k \in R^+ \sqcup 0$, and so

$$\hat{\mu} = (-n, \lambda - \beta, \kappa'), \tag{4.7}$$

where $\beta \in \mathrm{Span}^+(R^+) \sqcup 0$. But then this is a contradiction to the form of $\hat{\mu}$ we obtained in (4.5). □

Proof of 1. The same argument as in the previous proof leads to formula (4.6) for b_k when $\hat{\alpha}_k$ is real. Now since κ' is assumed rationally negative and λ is sufficiently dominant, b_k can possibly be positive only when α_k is a positive root of $\mathfrak{g}$. Therefore either $\hat{\alpha}_k$ is imaginary, or $\hat{\alpha}_k$ is real and $\mu_{k+1} = \mu_k - b_k \alpha_k$ for $\alpha_k \in R^+$. Again we arrive at the expression

$$\hat{\mu} = (-n, \lambda - \beta, \kappa')$$

where $\beta \in \mathrm{Span}^+(R^+) \sqcup 0$, contradicting (4.5). □

4.4 Second construction, via convolution

Following [25], the second approach to construct the Wakimoto modules incorporates into its definition the feature that Wakimoto modules are stable under convolution with $j_{\check{\lambda},!}$ or $j_{\check{\lambda},*}$ (Proposition 4.1.1). It is sufficient for our purpose to define two types of Wakimoto modules, $\mathbb{W}_\lambda^{\kappa',*}$ and $\mathbb{W}_\lambda^{\kappa',w_0}$, at a negative level κ'.

When λ is sufficiently dominant, set $\mathbb{W}_\lambda^{\kappa',*} := \mathbb{M}_\lambda^{\kappa'}$. For general λ, write $\lambda = \lambda_1 - \check{\mu}$, where λ_1 is sufficiently dominant and $\check{\mu} \in \check{\Lambda}^+$, and define

$$\mathbb{W}_\lambda^{\kappa',*} := j_{-\check{\mu},*} \star_I \mathbb{W}_{\lambda_1}^{\kappa',*}.$$

Note that this is well-defined as $\mathbb{M}_{\check{\mu}+\lambda}^{\kappa'} \cong j_{\check{\mu},!} \star_I \mathbb{M}_\lambda^{\kappa'}$ for λ sufficiently dominant and any $\check{\mu} \in \check{\Lambda}^+$

by Kashiwara-Tanisaki localization, and $j_{-\check{\mu},*} \star j_{\check{\mu},!} \star - \simeq \mathrm{Id}$ for $\check{\mu} \in \check{\Lambda}^+$. Then by definition

$$\mathbb{W}^{\kappa',*}_{\lambda-\check{\mu}} \cong j_{-\check{\mu},*} \star_I \mathbb{W}^{\kappa',*}_{\lambda} \tag{4.8}$$

holds for all $\lambda \in \Lambda$ and $\check{\mu} \in \check{\Lambda}^+$. If λ is sufficiently anti-dominant, it can be shown that $\mathbb{W}^{\kappa',*}_{\lambda} \cong \mathbb{M}^{\kappa',\vee}_{\lambda}$.

We define $\mathbb{W}^{\kappa',w_0}_{\lambda}$ analogously. Put $\mathbb{W}^{\kappa',w_0}_{\lambda} := \mathbb{M}^{\kappa',\vee}_{\lambda}$ when λ is sufficiently dominant. For general λ, again write $\lambda = \lambda_1 - \check{\mu}$, where λ_1 is sufficiently dominant and $\check{\mu} \in \check{\Lambda}^+$, and define

$$\mathbb{W}^{\kappa',w_0}_{\lambda} := j_{-\check{\mu},!} \star_I \mathbb{W}^{\kappa',w_0}_{\lambda_1}.$$

We have seen that the type w_0 Wakimoto modules defined in Section 4.1 satisfies $\mathbb{W}^{\kappa',w_0}_{\lambda} \cong j_{-\check{\mu},!} \star_I \mathbb{W}^{\kappa',w_0}_{\lambda+\check{\mu}}$ for dominant $\check{\mu}$ (Proposition 4.1.1) and $\mathbb{W}^{\kappa',w_0}_{\lambda} \cong \mathbb{M}^{\kappa',\vee}_{\lambda}$ for sufficiently dominant λ (Theorem 4.3.3). Consequently $\mathbb{W}^{\kappa',w_0}_{\lambda}$ defined here using convolution agrees with the type w_0 Wakimoto module defined in Section 4.1. One can similarly show that $\mathbb{W}^{\kappa',*}_{\lambda}$ is identified with $\mathbb{W}^{\kappa',1}_{\lambda}$.

From this construction, it is clear that both $\mathbb{W}^{\kappa',*}_{\lambda}$ and $\mathbb{W}^{\kappa',w_0}_{\lambda}$ lie in the category $\hat{\mathfrak{g}}_{\kappa'}\text{-mod}^I$.

4.5 Relations to semi-infinite cohomology

As its construction involves semi-infinite cohomology, it is not surprising that Wakimoto modules are closely related to semi-infinite calculus. Below we present two formulas for computing semi-infinite cohomology, one at negative level and the other at positive level.

The first formula is a result from [20]:

Proposition 4.5.1 ([20] Proposition 12.4.1). *$\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \mathbb{W}^{\kappa',w_0}_{\lambda})$ is isomorphic to the Fock module $\pi^{\kappa'+\mathrm{shift}}_{\lambda}$ (placed at homological degree 0) as complexes of modules over the Heisenberg algbera $\hat{\mathfrak{t}}_{\kappa'+\mathrm{shift}}$.*

The second formula concerns the semi-infinite cohomology of modules at a positive level κ. Let $M \in \hat{\mathfrak{g}}_{\kappa}\text{-mod}^I$. We claim

Proposition 4.5.2. $\langle \mathbb{W}^{\kappa',*}_{-\mu-2\rho}[\dim(G/B)]\,,\, M\rangle_I \cong \mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), M)^{\mu}$.

Proof. The μ-component of the semi-infinite complex can be computed by

$$\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), M)^{\mu} \cong \operatorname*{colim}_{\check{\lambda}\in\check{\Lambda}} \mathrm{C}^{\bullet}(\mathrm{Lie}(I^0), \mathrm{Av}^I_*(t^{\check{\lambda}} \cdot M))^{\check{\lambda}+\mu}[\langle\check{\lambda}, 2\rho\rangle],$$

as in [25, Proposition 1.2.3]. On the other hand, by Lemma 3.2.1 and Lemma 2.2.1 we have

$$\begin{aligned}\langle \mathbb{W}^{\kappa',*}_{-\mu-2\rho}[\dim(G/B)]\,,\, M\rangle_I &\cong \operatorname*{colim}_{\check{\lambda}\in\check{\Lambda}} \langle j_{-\check{\lambda},*} \star_I \mathbb{M}^{\kappa'}_{-\mu-2\rho+\check{\lambda}}[\dim(G/B)]\,,\, M\rangle_I \\ \cong \operatorname*{colim}_{\check{\lambda}\in\check{\Lambda}} \langle \mathbb{M}^{\kappa'}_{-\mu-2\rho+\check{\lambda}}[\dim(G/B)]\,,\, j_{\check{\lambda},*} \star_I M\rangle_I &\cong \operatorname*{colim}_{\check{\lambda}\in\check{\Lambda}} \mathrm{Hom}_{\hat{\mathfrak{g}}_{\kappa}\text{-mod}^I}(\mathbb{M}^{\kappa}_{\mu+\check{\lambda}}, j_{\check{\lambda},*} \star_I M).\end{aligned}$$

Now, Lemma 3.2.2 gives

$$\begin{aligned}\mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa\text{-mod}^I}(\mathbb{M}^\kappa_{\mu+\check{\lambda}}, j_{\check{\lambda},*} \star_I M) &\cong \mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa\text{-mod}^I}(\mathbb{M}^\kappa_{\mu+\check{\lambda}}, \mathrm{Av}^I_*(t^{\check{\lambda}} \cdot M))[\langle \check{\lambda}, 2\rho\rangle] \\ &\cong \mathrm{C}^\bullet(\mathrm{Lie}(I^0), \mathrm{Av}^I_*(t^{\check{\lambda}} \cdot M))^{\check{\lambda}+\mu}[\langle \check{\lambda}, 2\rho\rangle].\end{aligned}$$

The proposition follows. □

Chapter 5

Semi-infinite cohomology vs quantum group cohomology: positive level

This chapter presents the main result of this thesis, Theorem 5.3.1 which compares the semi-infinite cohomology at positive level and the quantum group cohomology. For the proof of the theorem, we introduce the generalized semi-infinite cohomology functors, with the usual semi-infinite cohomology being a special case. It turns out that some generalized semi-infinite cohomology functors are interesting in its own right, as these functors fit into a duality pattern explained in Section 6.2.

5.1 Weyl modules revisited

In this section we prove two technical lemmas concerning Weyl modules.

Lemma 5.1.1. *Let λ be a dominant integral weight. We have isomorphic $\hat{\mathfrak{g}}_{\kappa'}$-modules*

$$\mathrm{Av}_!^{G(\mathcal{O})/I}\mathbb{M}_\lambda^{\kappa'} \cong \mathbb{V}_\lambda^{\kappa'}.$$

Proof. By viewing the induction $\mathrm{Ind}_{\mathfrak{g}(\mathcal{O})\oplus\mathbb{C}\mathbf{1}}^{\hat{\mathfrak{g}}_{\kappa'}}$ as the left adjoint functor to the restriction functor, it is easy to verify that

$$\mathrm{Av}_!^{G(\mathcal{O})/I} \circ \mathrm{Ind}_{\mathfrak{g}(\mathcal{O})\oplus\mathbb{C}\mathbf{1}}^{\hat{\mathfrak{g}}_{\kappa'}} \circ \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \simeq \mathrm{Ind}_{\mathfrak{g}(\mathcal{O})\oplus\mathbb{C}\mathbf{1}}^{\hat{\mathfrak{g}}_{\kappa'}} \circ \mathrm{Av}_!^{G/B} \circ \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}$$

as functors from $\mathfrak{b}$-mod to $\hat{\mathfrak{g}}_{\kappa'}$-mod^I.

We have $\mathrm{Av}_!^{G(\mathcal{O})/I}\mathbb{M}_\lambda^{\kappa'} \cong \mathrm{Ind}_{\mathfrak{g}(\mathcal{O})\oplus\mathbb{C}\mathbf{1}}^{\hat{\mathfrak{g}}_{\kappa'}} \circ \mathrm{Av}_!^{G/B} M_\lambda$ as $M_\lambda \cong \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}\mathbb{C}_\lambda$. Thus it remains to prove the isomorphism as $\mathfrak{g}$-modules

$$\mathrm{Av}_!^{G/B} M_\lambda \cong V_\lambda.$$

By the Beilinson-Bernstein localization combined with the translation functor (c.f. [21]), we have

$$M_\lambda^\vee \cong \Gamma(G/B, \mathrm{Dist}(Bw_0B) \otimes_{\mathcal{O}_{G/B}} \mathcal{L}_{-w_0(\lambda)}),$$

where $\mathrm{Dist}(Bw_0B)$ is the *-pushforward of the sheaf of regular functions on $Bw_0B \subset G/B$, considered as a left D-module on G/B, and $\mathcal{L}_{-w_0(\lambda)}$ is the line bundle induced from $\mathcal{O}_G \otimes_B \mathbb{C}_{w_0(\lambda)}$. It is easy to see that

$$\mathrm{Av}_*^{G/B}(\mathrm{Dist}(Bw_0B) \otimes_{\mathcal{O}_{G/B}} \mathcal{L}_{-w_0(\lambda)}) \cong \mathcal{O}_{G/B} \otimes_{\mathcal{O}_{G/B}} \mathcal{L}_{-w_0(\lambda)} \cong \mathcal{L}_{-w_0(\lambda)}. \tag{5.1}$$

The global section of the line bundle $\mathcal{L}_{-w_0(\lambda)}$ (regarded as a twisted D-module) matches the irreducible $\mathfrak{g}$-module V_λ by the Borel-Weil-Bott Theorem. Finally, we apply the Verdier dual on both sides of (5.1) and take the global sections to conclude $\mathrm{Av}_!^{G/B} M_\lambda \cong V_\lambda$, as the Verdier duality on D-mod(G/B) corresponds to the contragredient duality on $\mathfrak{g}$-modules. □

Interpreting Lemma 5.1.1 by Kashiwara-Tanisaki's theorem (Theorem 3.1.1) , we see that the Weyl module $\mathbb{V}_\lambda^{\kappa'}$ can be constructed geometrically as the global section of the μ-twisted D-module $\mathrm{Av}_!^{G(\mathcal{O})/I} j_{\tilde{w},!}$ for some $\tilde{w} \in W^{\mathrm{aff}}$ and $\mu \in \Lambda$. Note that, $\mathrm{Av}_!^{G(\mathcal{O})/I} j_{\tilde{w},!}$ is precisely the !-pushforward of $\mathcal{O}_{G(\mathcal{O})\tilde{w}I}$, the sheaf of regular functions on the $G(\mathcal{O})$-orbit $G(\mathcal{O})\tilde{w}I \subset \mathrm{Fl}$. This enables us to define the dual Weyl module as

$$\mathbb{V}_\lambda^{\kappa',\vee} := \Gamma(\mathbb{D}\mathrm{Av}_!^{G(\mathcal{O})/I} j_{\tilde{w},!}) = \Gamma(\mathrm{Av}_*^{G(\mathcal{O})/I} j_{\tilde{w},*}).$$

Lemma 5.1.2. $\mathrm{Av}_*^{G(\mathcal{O})/I} \mathbb{M}_{-\lambda}^{\kappa',\vee}[\dim G/B] \cong \mathbb{V}^{\kappa',\vee}_{-w_0(\lambda)-2\rho}$ *for any dominant integral weight* λ.

Proof. By the discussion preceeding this lemma, it suffices to prove the dual version of this isomorphism, namely

$$\mathrm{Av}_!^{G(\mathcal{O})/I} \mathbb{M}_{-\lambda}^{\kappa'}[-\dim G/B] \cong \mathbb{V}^{\kappa'}_{-w_0(\lambda)-2\rho}.$$

The same argument as in the proof of Lemma 5.1.1 reduces this to proving the $\mathfrak{g}$-module isomorphism

$$\mathrm{Av}_!^{G/B} M_{-\lambda}[-\dim G/B] \cong V_{-w_0(\lambda)-2\rho}.$$

Since λ is assumed dominant and integral, there is a unique dominant integral weight μ such that $w_0(\mu+\rho) = -\lambda+\rho$. Let $B1B$ be the identity B-orbit on G/B. We consider the $-w_0(\mu)$-twisted D-module $\mathrm{Dist}(B1B) \otimes_{\mathcal{O}_{G/B}} \mathcal{L}_{-w_0(\mu)}$, which corresponds to $M_{-\lambda}^\vee$ by the (twisted) Beilinson-Berstein localization. Then we have $\mathrm{Av}_*^{G/B}(\mathrm{Dist}(B1B) \otimes_{\mathcal{O}_{G/B}} \mathcal{L}_{-w_0(\mu)}) \cong \mathcal{L}_{-w_0(\mu)}[-\dim G/B]$. Taking global sections we obtain

$$\mathrm{Av}_*^{G/B} M_{-\lambda}^\vee[\dim G/B] \cong V_\mu \equiv V_{-w_0(\lambda)-2\rho}.$$

The assertion follows. □

5.2 Generalized semi-infinite cohomology functor

Let C be a category acted on by $G(\mathcal{K})$. We define a functor $\mathsf{p}^- : \mathsf{C}^{N^-(\mathcal{K})T(\mathcal{O})} \to (\mathsf{C}^{T(\mathcal{O})})_{N(\mathcal{K})}$ as the composition $\mathrm{Av}_*^I : \mathsf{C}^{N^-(\mathcal{K})T(\mathcal{O})} \to \mathsf{C}^I$ followed by the equivalence $\mathsf{q} : \mathsf{C}^I \simeq (\mathsf{C}^{T(\mathcal{O})})_{N(\mathcal{K})}$.

Lemma 5.2.1. *The functor* p^- *is equivalent to* $\mathrm{obliv} : \mathsf{C}^{N^-(\mathcal{K})T(\mathcal{O})} \to \mathsf{C}^{T(\mathcal{O})}$ *post-composed by the projection functor* $\mathrm{proj} : \mathsf{C}^{T(\mathcal{O})} \to (\mathsf{C}^{T(\mathcal{O})})_{N(\mathcal{K})}$.

Proof. This is essentially Section 2.1.3 in [25]. □

Assume that C is dualizable. Let $\langle\langle -, - \rangle\rangle : (\mathsf{C}^\vee)^{N(\mathcal{K})T(\mathcal{O})} \times (\mathsf{C}^{T(\mathcal{O})})_{N(\mathcal{K})} \to \mathrm{Vect}$ be the natural pairing. Note that the natural pairing is characterized by

$$\langle\langle \mathrm{Av}_!^{N(\mathcal{K})} c^\vee, \mathsf{q}(c) \rangle\rangle \cong \langle c^\vee, c \rangle_I, \tag{5.2}$$

for $c^\vee \in (\mathsf{C}^\vee)^I$ and $c \in \mathsf{C}^I$. In the followings, we take $\mathsf{C} = \hat{\mathfrak{g}}_\kappa\text{-mod}$ and its dual category $\mathsf{C}^\vee = \hat{\mathfrak{g}}_{\kappa'}\text{-mod}$ as in Section 2.2.

We define generalized semi-infinite cohomology functors (at positive level κ) by the following procedure: An object $\mathcal{F}$ in $\mathrm{D\text{-}mod}(\mathrm{Gr}_G)^{N^-(\mathcal{K})T(\mathcal{O})}$ defines a functor

$$\mathsf{p}^-(\mathcal{F} \star_{G(\mathcal{O})} -) : \hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})} \to (\hat{\mathfrak{g}}_\kappa\text{-mod}^{T(\mathcal{O})})_{N(\mathcal{K})}.$$

We pair the resulting object with $\mathrm{Av}_!^{N(\mathcal{K})} \mathbb{W}^{\kappa',*}_{-\mu-2\rho}[\dim(G/B)]$ to get the generalized semi-infinite cohomology functor corresponding to $\mathcal{F}$ and weight μ:

$$\mathfrak{C}^{\frac{\infty}{2}}_{\mathcal{F}}(\mathfrak{n}(\mathcal{K}), -)^\mu := \langle\langle \mathrm{Av}_!^{N(\mathcal{K})} \mathbb{W}^{\kappa',*}_{-\mu-2\rho}[\dim(G/B)]\, ,\, \mathsf{p}^-(\mathcal{F} \star_{G(\mathcal{O})} -) \rangle\rangle : \hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})} \to \mathrm{Vect}. \tag{5.3}$$

To justify that it really is a generalization of the usual semi-infinite cohomology, we let $\mathcal{F} = \delta_{G(\mathcal{O})}$ be the identity with respect to $\star_{G(\mathcal{O})}$. Indeed, by (5.2) and Proposition 4.5.2

$$\begin{aligned}\mathfrak{C}^{\frac{\infty}{2}}_{\delta_{G(\mathcal{O})}}(\mathfrak{n}(\mathcal{K}), M)^\mu &\cong \langle\langle \mathrm{Av}_!^{N(\mathcal{K})} \mathbb{W}^{\kappa',*}_{-\mu-2\rho}[\dim(G/B)]\, ,\, \mathsf{p}^-(M) \rangle\rangle \cong \\ &\cong \langle \mathbb{W}^{\kappa',*}_{-\mu-2\rho}[\dim(G/B)]\, ,\, \mathrm{Av}_*^I(M) \rangle_I \cong \mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), M)^\mu.\end{aligned}$$

The most important generalized semi-infinite cohomology functor for us will be the !*-semi-infinite cohomology functor, given by the semi-infinite IC object $\mathrm{IC}^{\frac{\infty}{2},-}$ in $\mathrm{D\text{-}mod}(\mathrm{Gr}_G)^{N^-(\mathcal{K})T(\mathcal{O})}$ which we construct below.

Consider $\check{\Lambda}^+$ as a poset with the (non-standard) partial order

$$\check{\lambda}_1 \le \check{\lambda}_2 \Leftrightarrow \check{\lambda}_2 - \check{\lambda}_1 \in \check{\Lambda}^+.$$

Recall the semi-infinite IC object

$$\mathrm{IC}^{\frac{\infty}{2},-} := \underset{\check{\lambda} \in \check{\Lambda}^+}{\mathrm{colim}}\, t^{\check{\lambda}} \cdot \mathrm{Sat}((V_{\check{\lambda}})^*)[\langle 2\rho, \check{\lambda} \rangle] \in \mathrm{D\text{-}mod}(\mathrm{Gr}_G)^{N^-(\mathcal{K})T(\mathcal{O})},$$

analogous to $\mathrm{IC}^{\frac{\infty}{2}} \in \text{D-mod}(\mathrm{Gr}_G)^{N(\mathcal{K})T(\mathcal{O})}$ constructed and studied in [24]. For $\check{\mu} \in \check{\Lambda}^+$, the transition morphsim

$$t^{\check{\lambda}} \cdot \mathrm{Sat}((V_{\check{\lambda}})^*)[\langle 2\rho, \check{\lambda}\rangle] \to t^{\check{\lambda}+\check{\mu}} \cdot \mathrm{Sat}((V_{\check{\lambda}+\check{\mu}})^*)[\langle 2\rho, \check{\lambda}+\check{\mu}\rangle]$$

in the colimit is given by

$$\begin{aligned} &t^{\check{\lambda}} \cdot \mathrm{Sat}((V_{\check{\lambda}})^*)[\langle 2\rho, \check{\lambda}\rangle] \to t^{\check{\lambda}} \cdot (t^{\check{\mu}} \cdot \mathrm{Sat}((V_{\check{\mu}})^*)[\langle 2\rho, \check{\mu}\rangle]) \star \mathrm{Sat}((V_{\check{\lambda}})^*)[\langle 2\rho, \check{\lambda}\rangle] \\ &\to t^{\check{\lambda}+\check{\mu}} \cdot \mathrm{Sat}((V_{\check{\lambda}})^* \otimes (V_{\check{\mu}})^*)[\langle 2\rho, \check{\lambda}+\check{\mu}\rangle] \to t^{\check{\lambda}+\check{\mu}} \cdot \mathrm{Sat}((V_{\check{\lambda}+\check{\mu}})^*)[\langle 2\rho, \check{\lambda}+\check{\mu}\rangle]. \end{aligned} \tag{5.4}$$

Here the first arrow is given by the natural map $\delta_{t^{-\check{\mu}}G(\mathcal{O})} \to \mathrm{Sat}((V_{\check{\mu}})^*)[\langle 2\rho, \check{\mu}\rangle]$, which in turn is induced from the identification of the !-fiber of $\mathrm{Sat}((V_{\check{\mu}})^*)$ at the coset $t^{-\check{\mu}}G(\mathcal{O})$ with $\mathbb{C}$ (see (3.2)). The second arrow follows from the geometric Satake equivalence, and the third arrow arises from the dual of the embedding $V_{\check{\lambda}+\check{\mu}} \to V_{\check{\lambda}} \otimes V_{\check{\mu}}$.

If $\check{\lambda}$ is dominant, it is well-known that $\dim \overline{It^{\check{\lambda}}I} = \langle 2\rho, \check{\lambda}\rangle$. Then by Lemma 3.2.2 we have $\mathrm{Av}^I_*(t^{\check{\lambda}} \cdot F) \cong j_{\check{\lambda},*} \star F[-\langle 2\rho, \check{\lambda}\rangle]$ for any $G(\mathcal{O})$-equivariant F. Applying the functor Av^I_* to $\mathrm{IC}^{\frac{\infty}{2},-}$, we see that the transition morphism (5.4) becomes

$$\begin{aligned} &j_{\check{\lambda},*} \star \mathrm{Sat}((V_{\check{\lambda}})^*) \to j_{\check{\lambda},*} \star (j_{\check{\mu},*} \star \mathrm{Sat}((V_{\check{\mu}})^*)) \star \mathrm{Sat}((V_{\check{\lambda}})^*) \\ &\to j_{\check{\lambda}+\check{\mu},*} \star \mathrm{Sat}((V_{\check{\lambda}})^* \otimes (V_{\check{\mu}})^*) \to j_{\check{\lambda}+\check{\mu},*} \star \mathrm{Sat}((V_{\check{\lambda}+\check{\mu}})^*), \end{aligned}$$

where the first arrow comes from $\delta_{G(\mathcal{O})} \to j_{\check{\mu},*} \star j_{-\check{\mu},!} \star \delta_{G(\mathcal{O})} \to j_{\check{\mu},*} \star \mathrm{Sat}((V_{\check{\mu}})^*)$, induced by the natural morphism $\mathrm{Sat}(V_{\check{\mu}}) \cong \mathrm{IC}_{\mathrm{Gr}^{\check{\mu}}_G} \to j_{\check{\mu},*} \star \delta_{G(\mathcal{O})}$. We conclude that

$$\mathrm{Av}^I_*(\mathrm{IC}^{\frac{\infty}{2},-} \star_{G(\mathcal{O})} M) \cong \underset{\check{\lambda}\in\check{\Lambda}^+}{\mathrm{colim}}\, j_{\check{\lambda},*} \star_I \mathrm{Sat}((V_{\check{\lambda}})^*) \star_{G(\mathcal{O})} M \tag{5.5}$$

in the category $\text{D-mod}(\mathrm{Gr})^I$.

Now we consider the generalized semi-infinite cohomology functor $\mathfrak{C}^{\frac{\infty}{2}}_{\mathrm{IC}^{\frac{\infty}{2},-}}(\mathfrak{n}(\mathcal{K}), -)^{\mu}$. By definition

$$\begin{aligned} \mathfrak{C}^{\frac{\infty}{2}}_{\mathrm{IC}^{\frac{\infty}{2},-}}(\mathfrak{n}(\mathcal{K}), M)^{\mu} =& \langle\langle \mathrm{Av}^{N(\mathcal{K})}_! \mathbb{W}^{\kappa',*}_{-\mu-2\rho}[\dim(G/B)]\, ,\, \mathsf{p}^-(\mathrm{IC}^{\frac{\infty}{2},-} \star_{G(\mathcal{O})} M)\rangle\rangle \\ =& \langle\langle \mathrm{Av}^{N(\mathcal{K})}_! \mathbb{W}^{\kappa',*}_{-\mu-2\rho}[\dim(G/B)]\, ,\, \mathsf{q} \circ \mathrm{Av}^I_*(\mathrm{IC}^{\frac{\infty}{2},-} \star_{G(\mathcal{O})} M)\rangle\rangle, \end{aligned}$$

which again by (5.2), (5.5) and Proposition 4.5.2 is isomorphic to

$$\begin{aligned} &\langle \mathbb{W}^{\kappa',*}_{-\mu-2\rho}[\dim(G/B)]\, ,\, \mathrm{Av}^I_*(\mathrm{IC}^{\frac{\infty}{2},-} \star_{G(\mathcal{O})} M)\rangle_I \\ \cong\ & \mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), \mathrm{Av}^I_*(\mathrm{IC}^{\frac{\infty}{2},-}) \star_{G(\mathcal{O})} M)^{\mu} \cong \underset{\check{\lambda}\in\check{\Lambda}^+}{\mathrm{colim}}\, \mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}),\, j_{\check{\lambda},*} \star_I \mathrm{Sat}((V_{\check{\lambda}})^*) \star_{G(\mathcal{O})} M)^{\mu}. \end{aligned} \tag{5.6}$$

Definition 5.2.1. The functor $\mathfrak{C}^{\frac{\infty}{2}}_{!*}(\mathfrak{n}(\mathcal{K}), -)^\mu : \hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})} \to \mathrm{Vect}$ defined by

$$M \mapsto \operatorname*{colim}_{\check{\lambda}\in\check{\Lambda}^+} \mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}),\, j_{\check{\lambda},*} \star_I \mathrm{Sat}((V_{\check{\lambda}})^*) \star_{G(\mathcal{O})} M)^\mu$$

is called the μ-component of the !*-generalized semi-infinite cohomology functor.

We similarly define the μ-component of the !-generalized semi-infinite cohomology functor as

$$\mathfrak{C}^{\frac{\infty}{2}}_{!}(\mathfrak{n}(\mathcal{K}), -)^\mu := \operatorname*{colim}_{\check{\lambda}\in\check{\Lambda}^+} \mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}),\, j_{\check{\lambda},*} \star_I j_{-\check{\lambda},*} \star_{G(\mathcal{O})} -)^\mu$$

The original semi-infinite cohomology functor should then be regarded as the *-version of the construction.

5.3 The formulas

Recall that we have fixed a positive level κ and the quantum parameter is set to $q = \exp(\frac{\pi\sqrt{-1}}{\kappa'-\kappa_{\mathrm{crit}}})$. Let A be one of the algebras $U_q^{\mathrm{KD}}(\mathfrak{n})$, $\mathfrak{u}_q(\mathfrak{n})$, or $U_q^{\mathrm{Lus}}(\mathfrak{n})$ defined in Section 2.3. Set $\mathrm{C}^\bullet(A, -)$ to be the derived functor of A-invariants, and $\mathrm{C}^\bullet(A, -)^\mu$ the μ-component of the resulting (complex of) Λ-graded vector spaces. Here, we take as input a $U_q^{\mathrm{Lus}}(\mathfrak{g})$-module, regarded as an A-module via restriction.

The goal of this chapter is to prove the following formula:

Theorem 5.3.1. *For each weight μ we have an isomorphism in* Vect*:*

$$\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), M)^\mu \cong \mathrm{C}^\bullet(U_q^{\mathrm{Lus}}(\mathfrak{n}), \mathrm{KL}_G^\kappa(M))^\mu.$$

Our proof of Theorem 5.3.1 relies on the following analogous formula for the !*-generalized semi-infinite cohomology at positive level:

Theorem 5.3.2. *The isomorphism*

$$\mathfrak{C}^{\frac{\infty}{2}}_{!*}(\mathfrak{n}(\mathcal{K}), M)^\mu \cong \mathrm{C}^\bullet(\mathfrak{u}_q(\mathfrak{n}), \mathrm{KL}_G^\kappa(M))^\mu$$

holds for all weights μ.

The proofs of Theorem 5.3.1 and Theorem 5.3.2 will occupy the remaining sections of this chapter.

We state the corresponding conjectural formula for the !-generalized semi-infinite cohomology functor:

Conjecture 5.3.3. *For all $\mu \in \Lambda$,*

$$\mathfrak{C}^{\frac{\infty}{2}}_{!}(\mathfrak{n}(\mathcal{K}), M)^\mu \cong \mathrm{C}^\bullet(U_q^{\mathrm{KD}}(\mathfrak{n}), \mathrm{KL}_G^\kappa(M))^\mu.$$

5.4 Proof of Theorem 5.3.2

By the definition of the !*-functor

$$\mathfrak{C}^{\frac{\infty}{2}}_{!*}(\mathfrak{n}(\mathcal{K}), M)^{\mu} \cong \underset{\check{\lambda}\in\check{\Lambda}^+}{\operatorname{colim}} \langle \mathbb{W}^{\kappa',*}_{-\mu-2\rho}[\dim(G/B)]\,,\ j_{\check{\lambda},*} \star_I \operatorname{Sat}((V_{\check{\lambda}})^*) \star_{G(\mathcal{O})} M\rangle_I.$$

We have by Lemma 3.2.1

$$\begin{aligned}\langle \mathbb{W}^{\kappa',*}_{-\mu-2\rho}[\dim(G/B)]\,, j_{\check{\lambda},*} \star_I \operatorname{Sat}((V_{\check{\lambda}})^*) \star_{G(\mathcal{O})} M\rangle_I \\ \cong \langle j_{-\check{\lambda},*} \star_I \mathbb{W}^{\kappa',*}_{-\mu-2\rho}[\dim(G/B)]\,, \operatorname{Sat}((V_{\check{\lambda}})^*) \star_{G(\mathcal{O})} M\rangle_I,\end{aligned}$$

and by (4.8) the latter is isomorphic to

$$\langle \mathbb{W}^{\kappa',*}_{-\check{\lambda}-\mu-2\rho}[\dim(G/B)]\,, \operatorname{Sat}((V_{\check{\lambda}})^*) \star_{G(\mathcal{O})} M\rangle_I.$$

When $\check{\lambda}$ is sufficiently dominant, the above pairing becomes

$$\langle \mathbb{M}^{\kappa',\vee}_{-\check{\lambda}-\mu-2\rho}[\dim(G/B)]\,, \operatorname{Sat}((V_{\check{\lambda}})^*) \star_{G(\mathcal{O})} M\rangle_I,$$

which is then isomorphic to

$$\operatorname{Hom}_{\hat{\mathfrak{g}}_\kappa\text{-mod}^I}\left(\mathbb{D}_I\left(\mathbb{M}^{\kappa',\vee}_{-\check{\lambda}-\mu-2\rho}[\dim(G/B)]\right), \operatorname{Sat}((V_{\check{\lambda}})^*) \star_{G(\mathcal{O})} M\right)$$

by (2.2).

Since $\operatorname{Sat}((V_{\check{\lambda}})^*) \star_{G(\mathcal{O})} M$ is $G(\mathcal{O})$-equivariant, by the left adjointness of $\operatorname{Av}_!^{G(\mathcal{O})/I}$ and (2.3), the above is isomorphic to

$$\operatorname{Hom}_{\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}}\left(\mathbb{D}_{G(\mathcal{O})} \operatorname{Av}_*^{G(\mathcal{O})/I}(\mathbb{M}^{\kappa',\vee}_{-\check{\lambda}-\mu-2\rho}[\dim(G/B)])\,, \operatorname{Sat}((V_{\check{\lambda}})^*) \star_{G(\mathcal{O})} M\right),$$

and from Lemma 5.1.2 this becomes

$$\operatorname{Hom}_{\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}}\left(\mathbb{D}_{G(\mathcal{O})} \mathbb{V}^{\kappa',\vee}_{-w_0(\check{\lambda}+\mu)}\,, \operatorname{Sat}((V_{\check{\lambda}})^*) \star_{G(\mathcal{O})} M\right). \tag{5.7}$$

Recall the dual quantum Weyl module $\mathcal{V}^{\vee}_{\nu}$, defined as the image of $\mathbb{V}^{\kappa',\vee}_{\nu}$ under the negative level Kazhdan-Lusztig functor; i.e. $\mathcal{V}^{\vee}_{\nu} := \operatorname{KL}_G(\mathbb{V}^{\kappa',\vee}_{\nu})$. Then with $\mathbb{D}^q(\mathcal{V}^{\vee}_{\nu}) \cong \mathcal{V}_{-w_0(\nu)}$ and the definition of $\operatorname{KL}^{\kappa}_G$, we deduce that (5.7) is isomorphic to

$$\operatorname{Hom}_{U_q^{\mathrm{Lus}}(\mathfrak{g})}\left(\mathcal{V}_{\mu+\check{\lambda}}, \operatorname{KL}^{\kappa}_G(\operatorname{Sat}((V_{\check{\lambda}})^*) \star_{G(\mathcal{O})} M)\right).$$

Since $\operatorname{Sat}(V_{\check{\nu}}) \cong \operatorname{IC}_{\check{\nu}}$ is Verdier self-dual, $\mathbb{D}_{G(\mathcal{O})}(\operatorname{Sat}(V_{\check{\nu}}) \star M) \cong \operatorname{Sat}(V_{\check{\nu}}) \star \mathbb{D}_{G(\mathcal{O})}(M)$ by the same argument as in the proof of (3.1). Combined with (3.3) we have

$$\begin{aligned}&\mathrm{KL}^{\kappa}_{G}(\,\mathrm{Sat}((V_{\check\lambda})^*)\star_{G(\mathcal{O})} M\,)\cong \mathbb{D}^q\circ \mathrm{KL}_{G}(\,\mathrm{Sat}((V_{\check\lambda})^*)\star_{G(\mathcal{O})} \mathbb{D}_{G(\mathcal{O})}M\,)\cong\\ &\cong \mathbb{D}^q(\mathrm{Fr}_q((V_{\check\lambda})^*))\otimes \mathrm{KL}^{\kappa}_{G}(M)\cong \mathrm{Fr}_q(V_{\check\lambda})\otimes \mathrm{KL}^{\kappa}_{G}(M).\end{aligned}$$

So far we have shown that

$$\mathfrak{C}^{\frac{\infty}{2}}_{!*}(\mathfrak{n}(\mathcal{K}),M)^{\mu}\cong \underset{\check\lambda\in\check\Lambda^+}{\mathrm{colim}}\ \mathrm{Hom}_{U_q^{\mathrm{Lus}}(\mathfrak{g})}(\mathcal{V}_{\mu+\check\lambda},\mathrm{Fr}_q(V_{\check\lambda})\otimes \mathrm{KL}^{\kappa}_{G}(M)). \tag{5.8}$$

On the quantum group side, we follow the same derivation as in the proof of [25, Theorem 3.2.2]. Recall $\overset{\bullet}{\mathfrak{u}}_q(\mathfrak{b})$-mod the category of representations of the small quantum Borel with full Lusztig's torus. The coinduction functor $\mathrm{CoInd}^{U_q^{\mathrm{Lus}}(\mathfrak{b})}_{\overset{\bullet}{\mathfrak{u}}_q(\mathfrak{b})}$ is the right adjoint to the restriction functor from $U_q^{\mathrm{Lus}}(\mathfrak{b})$-mod to $\overset{\bullet}{\mathfrak{u}}_q(\mathfrak{b})$-mod. The functor $\mathrm{CoInd}^{U_q^{\mathrm{Lus}}(\mathfrak{b})}_{\overset{\bullet}{\mathfrak{u}}_q(\mathfrak{b})}$ sends the trivial representation to

$$\mathrm{CoInd}^{U_q^{\mathrm{Lus}}(\mathfrak{b})}_{\overset{\bullet}{\mathfrak{u}}_q(\mathfrak{b})}(\mathbb{C})\cong \mathrm{Fr}_q(\mathcal{O}_{\check B/\check T})$$

by [25, Section 3.1.4]. Moreover by [3, Proposition 3.1.2] we have $\mathcal{O}_{\check B/\check T}\simeq \underset{\check\lambda\in\check\Lambda^+}{\mathrm{colim}}\,\mathbb{C}_{-\check\lambda}\otimes V_{\check\lambda}$ as $\check B$-modules. Then

$$\begin{aligned}\mathrm{C}^{\bullet}(\mathfrak{u}_q(\mathfrak{n}),\mathrm{KL}^{\kappa}_{G}(M))^{\mu}&:=\mathrm{Hom}_{\overset{\bullet}{\mathfrak{u}}_q(\mathfrak{b})}(\mathbb{C}_{\mu},\ \mathrm{Res}^{U_q^{\mathrm{Lus}}(\mathfrak{g})}_{\overset{\bullet}{\mathfrak{u}}_q(\mathfrak{b})}\mathrm{KL}^{\kappa}_{G}(M))\\ &\cong \mathrm{Hom}_{U_q^{\mathrm{Lus}}(\mathfrak{b})}(\mathbb{C}_{\mu},\ \mathrm{CoInd}^{U_q^{\mathrm{Lus}}(\mathfrak{b})}_{\overset{\bullet}{\mathfrak{u}}_q(\mathfrak{b})}\circ \mathrm{Res}^{U_q^{\mathrm{Lus}}(\mathfrak{g})}_{\overset{\bullet}{\mathfrak{u}}_q(\mathfrak{b})}\mathrm{KL}^{\kappa}_{G}(M))\\ &\cong \mathrm{Hom}_{U_q^{\mathrm{Lus}}(\mathfrak{b})}(\mathbb{C}_{\mu},\ \mathrm{Fr}_q(\mathcal{O}_{\check B/\check T})\otimes \mathrm{Res}^{U_q^{\mathrm{Lus}}(\mathfrak{g})}_{\overset{\bullet}{\mathfrak{u}}_q(\mathfrak{b})}\mathrm{KL}^{\kappa}_{G}(M))\\ &\cong \underset{\check\lambda\in\check\Lambda^+}{\mathrm{colim}}\ \mathrm{Hom}_{U_q^{\mathrm{Lus}}(\mathfrak{b})}(\mathbb{C}_{\mu},\ \mathrm{Fr}_q(\mathbb{C}_{-\check\lambda}\otimes V_{\check\lambda})\otimes \mathrm{Res}^{U_q^{\mathrm{Lus}}(\mathfrak{g})}_{\overset{\bullet}{\mathfrak{u}}_q(\mathfrak{b})}\mathrm{KL}^{\kappa}_{G}(M))\\ &\cong \underset{\check\lambda\in\check\Lambda^+}{\mathrm{colim}}\ \mathrm{Hom}_{U_q^{\mathrm{Lus}}(\mathfrak{b})}(\mathbb{C}_{\mu+\check\lambda},\ \mathrm{Fr}_q(V_{\check\lambda})\otimes \mathrm{Res}^{U_q^{\mathrm{Lus}}(\mathfrak{g})}_{\overset{\bullet}{\mathfrak{u}}_q(\mathfrak{b})}\mathrm{KL}^{\kappa}_{G}(M))\\ &\cong \underset{\check\lambda\in\check\Lambda^+}{\mathrm{colim}}\ \mathrm{Hom}_{U_q^{\mathrm{Lus}}(\mathfrak{g})}(\mathcal{V}_{\mu+\check\lambda},\mathrm{Fr}_q(V_{\check\lambda})\otimes \mathrm{KL}^{\kappa}_{G}(M)),\end{aligned}$$

which agrees with (5.8).

5.5 Proof of Theorem 5.3.1

The proof follows the same idea as in [25, Section 3.3] for the negative level case.

Let $\overset{\bullet}{\mathcal{F}}_{!*}$ denote the object

$$\bigoplus_{\check\nu\in\check\Lambda}\left(\underset{\check\lambda\in\check\Lambda^+}{\mathrm{colim}}\ j_{\check\nu+\check\lambda,*}\star \mathrm{Sat}((V_{\check\lambda})^*)\right)$$

in the category $\mathrm{D\text{-}mod}(\mathrm{Gr}_G)^I$.

By the theory of Arkhipov-Bezrukavnikov-Ginzburg [4], we have an equivalence

$$\text{D-mod}(\mathrm{Gr}_G)^I \simeq \mathrm{IndCoh}((\mathrm{pt} \times_{\check{\mathfrak{g}}} \tilde{\check{\mathcal{N}}})/\check{G}).$$

Under this equivalence, $\overset{\bullet}{\mathcal{F}}_{!*}$ corresponds to $s_*\mathcal{O}(\check{B})$, where

$$s : \mathrm{pt}/\check{B} \simeq (\check{G}/\check{B})/\check{G} \hookrightarrow (\mathrm{pt} \times_{\check{\mathfrak{g}}} \tilde{\check{\mathcal{N}}})/\check{G}.$$

It follows that $\overset{\bullet}{\mathcal{F}}_{!*}$ is equipped with a $\check{B}$-action, such that the $\check{B}$-invariant of $\overset{\bullet}{\mathcal{F}}_{!*}$ corresponds to $s_*\mathcal{O}(0)$, where $\mathcal{O}(0)$ is the trivial line bundle on $\mathrm{pt}/\check{B}$.

Again by the equivalence in [4], the delta function $\delta_{G(\mathcal{O})}$ on the identity coset in Gr_G corresponds to $s_*\mathcal{O}(0)$. Consequently, the $\check{B}$-invariant of $\overset{\bullet}{\mathcal{F}}_{!*}$ is identified with $\delta_{G(\mathcal{O})}$.

Now, we consider the object

$$\langle \mathbb{W}^{\kappa',*}_{-\mu-2\rho}[\dim G/B]\,,\, \overset{\bullet}{\mathcal{F}}_{!*} \star_{G(\mathcal{O})} M\rangle_I$$

in Vect. From the above discussion, this object inherits a $\check{B}$-action, such that the $\check{B}$-invariant is equal to $\langle \mathbb{W}^{\kappa',*}_{-\mu-2\rho}[\dim G/B]\,,\, M\rangle_I \cong \mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), M)^\mu$. By Theorem 5.3.2, we have

$$\begin{aligned}\langle \mathbb{W}^{\kappa',*}_{-\mu-2\rho}[\dim G/B]\,,\, \overset{\bullet}{\mathcal{F}}_{!*} \star_{G(\mathcal{O})} M\rangle_I &\cong \bigoplus_{\check{\nu}\in\check{\Lambda}} \mathfrak{C}^{\frac{\infty}{2}}_{!*}(\mathfrak{n}(\mathcal{K}), M)^{\mu+\check{\nu}} \\ &\cong \bigoplus_{\check{\nu}\in\check{\Lambda}} \mathrm{C}^\bullet(\mathfrak{u}_q(\mathfrak{n}), \mathrm{KL}^\kappa_G(M))^{\mu+\check{\nu}}.\end{aligned}$$

It follows that $\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), M)^\mu$ is isomorphic to the $\check{B}$-invariant of the space

$$\bigoplus_{\check{\nu}\in\check{\Lambda}} \mathrm{C}^\bullet(\mathfrak{u}_q(\mathfrak{n}), \mathrm{KL}^\kappa_G(M))^{\mu+\check{\nu}} \cong \mathrm{Hom}_{\mathfrak{u}_q(\mathfrak{b})}(\mathbb{C}_\mu, \mathrm{KL}^\kappa_G(M)),$$

which is identified with

$$\mathrm{Hom}_{U_q^{\mathrm{Lus}}(\mathfrak{b})}(\mathbb{C}_\mu, \mathrm{KL}^\kappa_G(M)) \cong \mathrm{C}^\bullet(U_q^{\mathrm{Lus}}(\mathfrak{n}), \mathrm{KL}^\kappa_G(M))^\mu.$$

This proves the theorem.

Chapter 6

The parallel story at negative level

In this chapter we briefly summarize the negative level counterpart of the theory, carried out in [25]. Throughout this chapter we fix a negative level κ'.

6.1 Generalized semi-infinite cohomology functors at negative level

Gaitsgory defined the !-Wakimoto modules at the positive level as

$$\mathbb{W}^{\kappa,!}_{\mu} := \mathbb{D}_I(\mathbb{W}^{\kappa',*}_{-\mu}).$$

Let N be an object in $\hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})}$. Analogous to the positive level case, define the μ-component of the !- and !*-generalized semi-infinite cohomology functors as

$$\mathfrak{C}^{\frac{\infty}{2}}_{!}(\mathfrak{n}(\mathcal{K}), N)^{\mu} := \langle N, \mathbb{W}^{\kappa,!}_{-\mu}\rangle_I,$$

$$\mathfrak{C}^{\frac{\infty}{2}}_{!*}(\mathfrak{n}(\mathcal{K}), N)^{\mu} := \underset{\check{\lambda}\in\check{\Lambda}^+}{\operatorname{colim}} \langle j_{-\check{\lambda},*} \star_I \operatorname{Sat}(V_{\check{\lambda}}) \star_{G(\mathcal{O})} N, \mathbb{W}^{\kappa,!}_{-\mu}\rangle_I,$$

whereas the *-functor is the original semi-infinite cohomology functor at negative level. As in Section 5.2, one can also write the definitions in terms of the pairing $\langle\langle -,-\rangle\rangle : (\mathsf{C}^{\vee})^{N(\mathcal{K})T(\mathcal{O})} \times (\mathsf{C}^{T(\mathcal{O})})_{N(\mathcal{K})} \to \text{Vect}$, where we now take $\mathsf{C} = \hat{\mathfrak{g}}_{\kappa'}\text{-mod}$ and $\mathsf{C}^{\vee} = \hat{\mathfrak{g}}_{\kappa}\text{-mod}$.

6.2 Formulas at negative level and duality pattern

The formulas which compare generalized semi-infinite cohomology at negative level with quantum group cohomology are stated below:

Theorem 6.2.1 ([24] Theorem 3.2.2 and Theorem 3.2.4)**.** *Let $N \in \hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})}$. The isomorphisms*

$$\mathfrak{C}^{\frac{\infty}{2}}_{!}(\mathfrak{n}(\mathcal{K}), N)^{\mu} \cong \mathrm{C}^{\bullet}(U_q^{\mathrm{Lus}}(\mathfrak{n}), \mathrm{KL}_G(N))^{\mu} \tag{6.1}$$

and

$$\mathfrak{C}^{\frac{\infty}{2}}_{!*}(\mathfrak{n}(\mathcal{K}), N)^{\mu} \cong \mathrm{C}^{\bullet}(\mathfrak{u}_q(\mathfrak{n}), \mathrm{KL}_G(N))^{\mu} \tag{6.2}$$

hold for all weights μ.

Conjecture 6.2.2 ([24] Conjecture 4.1.4)**.** *Let $N \in \hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})}$. The isomorphism*

$$\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), N)^{\mu} \cong \mathrm{C}^{\bullet}(U_q^{\mathrm{KD}}(\mathfrak{n}), \mathrm{KL}_G(N))^{\mu} \tag{6.3}$$

hold for all weights μ.

Remark 6.2.1. Conjecture 6.2.2 is verified when $\mu = \check{\nu} - 2\rho$ for all $\check{\nu} \in \check{\Lambda}$ in [24].

Comparing the formulas here with those in Section 5.3, we see a consistent duality picture: On the one hand, the positive level category $\hat{\mathfrak{g}}_{\kappa}\text{-mod}^{G(\mathcal{O})}$ is dual to the negative level one via the duality functor $\mathbb{D}_{G(\mathcal{O})}$, which corresponds to the Verdier dual when rendered into geometry using the Kashiwara-Tanisaki localization. Therefore $\mathbb{D}_{G(\mathcal{O})}$ should swap standard (i.e. !-) objects and costandard (i.e. *-) objects, and the intermediate (i.e. !*-) objects are preserved. On the other hand, at the quantum group side we have seen the pattern of standard, costandard and intermediate objects in the sequence

$$U_q^{\mathrm{KD}}(\mathfrak{n}) \twoheadrightarrow \mathfrak{u}_q(\mathfrak{n}) \hookrightarrow U_q^{\mathrm{Lus}}(\mathfrak{n})$$

and that $U_q^{\mathrm{KD}}(\mathfrak{n}^-) \cong (U_q^{\mathrm{Lus}}(\mathfrak{n}))^*$.

Chapter 7

An algebraic approach at irrational level

The theory we developed so far is greatly simplified when we are in the case of irrational levels. As the quantum parameter q is no longer a root of unity, Lusztig's, Kac-De Concini's and the small quantum groups all coincide, with the category of representations being semi-simple. The category of $G(\mathcal{O})$-equivariant representations of the affine Lie algebra is semi-simple at irrational level as well, and in this case the (negative level) Kazhdan-Lusztig functor is an equivalence tautologically. That Kazhdan-Lusztig functor is a monoidal functor can be seen as a reformulation of Drinfeld's theorem on Knizhnik-Zamolodchikov associators [37, Part III and Part IV].

Fix an irrational level κ throughout this chapter. We will give an algebraic proof of the formula that appears in Theorem 5.3.2 at an irrational level.

7.1 BGG-type resolutions

Let ℓ be the usual length function on the Weyl group W of $\mathfrak{g}$, and recall the dot action of the Weyl group on weights by $w \cdot \mu := w(\mu+\rho)-\rho$. Then the celebrated *Bernstein-Gelfand-Gelfand* (BGG) *resolution* is stated as follows:

Proposition 7.1.1. *Let λ be a dominant integral weight of $\mathfrak{g}$. Then we have a resolution of V_λ given by*

$$0 \to M_{w_0\cdot\lambda} \to \cdots \to \bigoplus_{\ell(w)=i} M_{w\cdot\lambda} \to \cdots \to M_\lambda \twoheadrightarrow V_\lambda.$$

Note that $\mathbb{V}^\kappa_\lambda \cong (\mathbb{V}^\kappa_\lambda)^\vee$ since it is irreducible (when κ is irrational). We apply the induction functor $(\cdot)^\kappa$ to the BGG resolution and then take the contragredient dual to get

$$\mathbb{V}^\kappa_\lambda \hookrightarrow (\mathbb{M}^\kappa_\lambda)^\vee \to \cdots \to \bigoplus_{\ell(w)=i} (\mathbb{M}^\kappa_{w\cdot\lambda})^\vee \to \cdots \to (\mathbb{M}^\kappa_{w_0\cdot\lambda})^\vee \to 0. \tag{7.1}$$

Combine Theorem 4.3.3 and (7.1) we get

$$\mathbb{V}_\lambda^\kappa \hookrightarrow \mathbb{W}_\lambda^{\kappa,w_0} \to \cdots \to \bigoplus_{\ell(w)=i} \mathbb{W}_{w\cdot\lambda}^{\kappa,w_0} \to \cdots \to \mathbb{W}_{w_0\cdot\lambda}^{\kappa,w_0} \to 0.$$

Now we can compute the semi-infinite cohomology of Weyl modules at irrational level by applying Proposition 4.5.1:

Corollary 7.1.2. *For $\lambda \in \Lambda^+$, we have an isomorphism of $\hat{\mathfrak{t}}_{\kappa+\mathrm{shift}}$-modules*

$$H^{\frac{\infty}{2}+i}(\mathfrak{n}(\mathcal{K}), \mathbb{V}_\lambda^\kappa) \cong \bigoplus_{\ell(w)=i} \pi_{w\cdot\lambda}^{\kappa+\mathrm{shift}}.$$

Now we turn to the quantum group side. When q is not a root of unity, the quantum Weyl module $\mathcal{V}_\lambda$ coincides with the irreducible module $\mathcal{L}_\lambda$, constructed by the usual procedure of taking irreducible quotient of the quantum Verma module $\mathcal{M}_\lambda$. Analogous to the non-quantum case, we have the BGG resolution for representations of $U_q(\mathfrak{g})$:

$$0 \to \mathcal{M}_{w_0\cdot\lambda} \to \cdots \to \bigoplus_{\ell(w)=i} \mathcal{M}_{w\cdot\lambda} \to \cdots \to \mathcal{M}_\lambda \twoheadrightarrow \mathcal{V}_\lambda.$$

As a consequence, we deduce

$$H^i(U_q(\mathfrak{n}), \mathcal{V}_\lambda) = \bigoplus_{\ell(w)=i} \mathrm{Hom}_{U_q(\mathfrak{n})}(\mathbb{C}, \mathcal{M}_{w\cdot\lambda}) = \bigoplus_{\ell(w)=i} \mathbb{C}_{w\cdot\lambda}. \tag{7.2}$$

7.2 Commutativity of the diagram

Recall the tautological equivalence $\mathrm{KL}_T : \hat{\mathfrak{t}}_{\kappa+\mathrm{shift}}\text{-mod}^{T(\mathcal{O})} \to \mathrm{Rep}_q(T)$ which is induced by the assignment

$$\pi_\lambda^{\kappa+\mathrm{shift}} \mapsto \mathbb{C}_\lambda.$$

We will verify the commutativity of the following diagram

$$\begin{array}{ccc} \hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})} & \xrightarrow{\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}),-)} & \hat{\mathfrak{t}}_{\kappa+\mathrm{shift}}\text{-mod}^{T(\mathcal{O})} \\ \downarrow{\scriptstyle \mathrm{KL}_G} & & \downarrow{\scriptstyle \mathrm{KL}_T} \\ U_q(\mathfrak{g})\text{-mod} & \xrightarrow{\mathrm{C}^\bullet(U_q(\mathfrak{n}),-)} & \mathrm{Rep}_q(T) \end{array} \tag{7.3}$$

which clearly implies

$$\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), M)^\mu \cong \mathrm{C}^\bullet(U_q(\mathfrak{n}), \mathrm{KL}_G(M))^\mu$$

for all μ.

We will need the theory of compactly generated categories. A detailed treatise of the theory is given in [41]. For a brief review of definitions and facts, see [14]. We recall the following proposition from [41].

Proposition 7.2.1 ([41] Proposition 5.3.5.10). *Let C be a cocomplete category, D be a small category, and $F : \mathsf{D} \to \mathsf{C}$ be a functor. Then F uniquely induces a continuous functor $\bar{F} : \mathsf{Ind}(\mathsf{D}) \to \mathsf{C}$ with $\bar{F}|_{\mathsf{D}} = F$. Here, $\mathsf{Ind}(\mathsf{D})$ denotes the ind-completion of the category D.*

We shall take D to be the full subcategory of compact generators of $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$, i.e., the subcategory whose objects consist of Weyl modules $\mathbb{V}_\lambda^\kappa$ for dominant integral weights λ. Then from the definition of compactly generated categories, we have $\mathsf{Ind}(\mathsf{D}) = \hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$. Let C be the category $\mathrm{Rep}_q(T)$.

To prove the commutativity of (7.3), by Proposition 7.2.1 it suffices to show that the two functors $\mathrm{C}^\bullet(U_q(\mathfrak{n}), -) \circ \mathrm{KL}_G$ and $\mathrm{KL}_T \circ \mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -)$ restricted to D are the same.

Since $\mathrm{KL}_G(\mathbb{V}_\lambda^\kappa) = \mathcal{V}_\lambda$, from (7.2) and Corollary 7.1.2 we get

$$H^i(\mathrm{C}^\bullet(U_q(\mathfrak{n}), -) \circ \mathrm{KL}_G(\mathbb{V}_\lambda^\kappa)) \cong H^i(\mathrm{KL}_T \circ \mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -)(\mathbb{V}_\lambda^\kappa)) \quad \forall i.$$

As objects in the category $\mathrm{Rep}_q(T)^\heartsuit$ have no nontrivial extensions, this shows that the two functors $\mathrm{C}^\bullet(U_q(\mathfrak{n}), -) \circ \mathrm{KL}_G$ and $\mathrm{KL}_T \circ \mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -)$ have the same image for objects in D.

Let λ and μ be arbitrary dominant integral weights of $\mathfrak{g}$, and let f be a morphism in $\mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa}(\mathbb{V}_\lambda^\kappa, \mathbb{V}_\mu^\kappa)$. It remains to show that the following two morphisms

$$\mathrm{C}^\bullet(U_q(\mathfrak{n}), -) \circ \mathrm{KL}_G(f)$$

and

$$\mathrm{KL}_T \circ \mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -)(f)$$

are identical.

Since κ is irrational, by [37, Proposition 27.4] all Weyl modules $\mathbb{V}_\lambda^\kappa$ are irreducible. Then $\mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa}(\mathbb{V}_\lambda^\kappa, \mathbb{V}_\mu^\kappa)$ vanishes when $\lambda \neq \mu$. Now, $\mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa}(\mathbb{V}_\lambda^\kappa, \mathbb{V}_\lambda^\kappa)$ is one-dimensional and generated by the identity morphism $\mathrm{Id}_{\mathbb{V}_\lambda^\kappa}$. Clearly we have

$$\mathrm{C}^\bullet(U_q(\mathfrak{n}), -) \circ \mathrm{KL}_G\,(\mathrm{Id}_{\mathbb{V}_\lambda^\kappa}) = \mathrm{Id}_{C^\bullet} = \mathrm{KL}_T \circ \mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -)(\mathrm{Id}_{\mathbb{V}_\lambda^\kappa}),$$

where $C^\bullet$ is the complex in $\mathrm{Rep}_q(T)$ with $C^i = \bigoplus_{\ell(w)=i} \mathbb{C}_{w\cdot\lambda}$. It follows that $\mathrm{C}^\bullet(U_q(\mathfrak{n}), -) \circ \mathrm{KL}_G$ and $\mathrm{KL}_T \circ \mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -)$ agree on morphisms in D. Therefore the two functors are isomorphic when restricted to D and so the diagram (7.3) commutes.

Chapter 8

Bringing in the factorization

The goal of the chapter is to outline an approach to the factorization Kazhdan-Lusztig equivalence at arbitrary non-critical level, proposed by D. Gaitsgory. In the negative level case, this approach can be seen as giving a new proof of the original Kazhdan-Lusztig equivalence. We will adopt the modern theory of factorization algebras and categories, systematically developed in [9, 44]. A brief review of the theory is given in Section 9.4.

The contents in this chapter are entirely borrowed from the talk notes of Winter School on Local Geometric Langlands Theory [26], in which most of the results are due to D. Gaitsgory and the speakers. The exposition here is somewhat informal, with technical details omitted.

8.1 The Kac-Moody factorization categories

In Example 9.2.1 we defined the affine Kac-Moody chiral algebra $A_{\mathfrak{g},\kappa}$ associated to the bilinear form $(\cdot,\cdot)_\kappa$, and by Proposition 9.1.1 the category $A_{\mathfrak{g},\kappa}\text{-mod}_x^{\mathrm{ch}}$ of chiral $A_{\mathfrak{g},\kappa}$-modules supported at a point $x \in X$ is equivalent to the category $\hat{\mathfrak{g}}_\kappa$-mod.

By Theorem 9.4.1 and Theorem 9.4.2, we have the corresponding affine Kac-Moody factorization algebra $\Upsilon_{\mathfrak{g},\kappa}$ and the global category of chiral $A_{\mathfrak{g},\kappa}$-modules on X is now equivalent to $\Upsilon_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}}(X)$.

Through the procedure of external fusion, we organize the factorization modules on X^I for any I into a factorization category

$$\Upsilon_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}}(\mathrm{Ran}(X)),$$

whose fibre over the finite subset $\{x_1, \ldots, x_n\} \subset X$ is

$$\hat{\mathfrak{g}}_\kappa\text{-mod}_{x_1} \otimes \ldots \otimes \hat{\mathfrak{g}}_\kappa\text{-mod}_{x_n} \hookrightarrow \Upsilon_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}}(\mathrm{Ran}(X))$$

via the above identification of module cateogries. We call $\Upsilon_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}}(\mathrm{Ran}(X))$ the Kac-Moody factorization category at level κ.

We are most interested in the factorization category associated to $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$. Let $\mathcal{O}_x$ be the ring of functions on the formal disc centered at $x \in X$, and $D_x := \operatorname{Spec} \mathcal{O}_x$ be the formal disc. For subset $\{x_1, \ldots, x_n\} \subset X$, define

$$D_{\{x_1,\ldots,x_n\}} := \operatorname{Spec}(\mathcal{O}_{x_1} \otimes \ldots \otimes \mathcal{O}_{x_n}) \simeq \bigsqcup_i D_{x_i}.$$

Consider the multi-jets space defined as the moduli

$$\text{Jets}_{X^I}(G) := \{(x_i) \in X^I, \phi : D_{\{x_i : i \in I\}} \to G\}.$$

From the trivial factorization property

$$D_{\{x_1,\ldots,x_n\}} = D_{\{x_1,\ldots,x_k\}} \bigsqcup D_{\{x_{k+1},\ldots,x_n\}}$$

for $\{x_1, \ldots, x_k\} \cap \{x_{k+1}, \ldots, x_n\} = \varnothing$, it is clear that the fibre of $\text{Jets}_{X^I}(G)$ at $\{x_1, \ldots, x_n\}$ with all x_i distinct is equal to $G(\mathcal{O}_{x_1}) \times \ldots \times G(\mathcal{O}_{x_n})$. Therefore we get a factorization category

$$I \rightsquigarrow \text{D-mod}(\text{Jets}_{X^I}(G))$$

whose fibre at $\{x_1, \ldots, x_n\} \subset X$ is equivalent to $\text{D-mod}(G(\mathcal{O}_{x_1})) \otimes \ldots \otimes \text{D-mod}(G(\mathcal{O}_{x_n}))$. Denote this factorization category by $\text{D-mod}(\text{Jets}(G))_{\text{Ran}(X)}$.

The group $G(\mathcal{O}_x)$ acts on the category $\hat{\mathfrak{g}}_\kappa\text{-mod}_x$; i.e. $\hat{\mathfrak{g}}_\kappa\text{-mod}_x \simeq \Upsilon_{\mathfrak{g},\kappa}\text{-mod}^{\text{fact}}(X)_x$ is a comodule category of $\text{D-mod}(G(\mathcal{O}_x))$. Varying the point x in X, we see that $\Upsilon_{\mathfrak{g},\kappa}\text{-mod}^{\text{fact}}(X)$ becomes a comodule of $\text{D-mod}(\text{Jets}_X(G))$. Then the $\text{D-mod}(\text{Jets}_X(G))$-invariants

$$\left(\Upsilon_{\mathfrak{g},\kappa}\text{-mod}^{\text{fact}}(X)\right)^{G(\mathcal{O})}$$

is given by the standard Milnor construction

$$\lim \left(\Upsilon_{\mathfrak{g},\kappa}\text{-mod}^{\text{fact}}(X) \rightrightarrows \Upsilon_{\mathfrak{g},\kappa}\text{-mod}^{\text{fact}}(X) \otimes \text{D-mod}(\text{Jets}_X(G)) \substack{\rightarrow\\\rightarrow\\\rightarrow} \ldots \right).$$

The resulting sheaf of categories on X has fibre at a point equivalent to $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$.

Now, again through external fusion, we define the $G(\mathcal{O})$-invariant factorization module categories on X^I for any I and organize them into a factorization category which we denote by

$$(\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})})_{\text{Ran}(X)}.$$

This is the sought after factorization category associated to $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$.

Remark 8.1.1. Alternatively, one expects to obtain the same factorization category of $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$ by taking the Milnor construction of $\text{D-mod}(\text{Jets}(G))_{\text{Ran}(X)}$-invariants of $\Upsilon_{\mathfrak{g},\kappa}\text{-mod}^{\text{fact}}(\text{Ran}(X))$.

8.2 Lurie's functor

Our goal in this section is to construct the topological factorization categories associated to the braided monoidal categories of quantum group modules. In the case of the small quantum group, the theory in [10] is recovered.

In the topological world, the analogy between braided monoidal categories and factorization categories actually becomes a correspondence, constructed by Jacob Lurie using the notion of E_2-algebras [42]. We will treat the theory as a black box and only give a summary of its consequences. The exposition here follows [29] and [22, Section 5], where some details of the theory are given.

We start with a Hopf algebra A in a braided monoidal category C. Assume that the augmentation comodule k of A is compact in A-comod, the category of A-comodules in C. The Koszul dualtiy

$$A \mapsto (\mathrm{Hom}_{A\text{-comod}}(k,k))^{op} =: \mathrm{Kosz}(A)$$

defines an equivalence A-comod $\to$ Kosz(A)-mod. Since A is a Hopf algebra, which by definition is an associative algebra in the category of coassociative coalgebras, Kosz(A) becomes an E_2-algebra, i.e. an associative algebra in the category of associative algebras.

We consider the relative Drinfeld center $\mathrm{Dr}_{\mathsf{C}}(A\text{-comod})$.[1] On the Kosz$(A)$-mod side, the same procedure produces the category of E_2-modules of the E_2-algebra Kosz(A). Hence we have the equivalence

$$\mathrm{Dr}_{\mathsf{C}}(A\text{-comod}) \simeq \mathrm{Kosz}(A)\text{-mod}^{E_2}.$$

Lurie's construction gives a functor Fact which sends an E_2-algebra to a topological factorization algebra on $\mathrm{Ran}(\mathbb{A}^1)$, c.f. [48]. The same construction is categorified to a functor from E_2-categories to topological factorization categories. Note that, if unwinding the definitions, one sees that E_2-categories precisely correspond to braided monoidal categories; see Section 2.1 in *loc. cit.* We thus obtain a factorization algebra $\Omega(A) := \mathrm{Fact}(\mathrm{Kosz}(A))$ in the factorization category $\mathrm{Fact}(\mathsf{C})$, and the category $\mathrm{Kosz}(A)\text{-mod}^{E_2}$ is then equivalent to the category $\Omega(A)\text{-mod}_0^{\mathrm{fact}}$ of topological factorization modules at $0 \in \mathbb{A}^1$ in $\mathrm{Fact}(\mathsf{C})$. In summary, there is a canonical equivalence

$$\mathrm{Dr}_{\mathsf{C}}(A\text{-comod}) \simeq \Omega(A)\text{-mod}_0^{\mathrm{fact}}.$$

We now describe the topological factoriztion category $\mathrm{Fact}(\mathrm{Rep}_q(T))$ corresponding to the braided monoidal category $\mathrm{Rep}_q(T)$ introduced in Section 2.3. Recall the Beilinson-Drinfeld Grassmannian for the dual torus $\mathrm{Gr}_{\check{T},\mathrm{Ran}(X)}$, the space over $\mathrm{Ran}(X)$ whose $\mathbb{C}$-points are Λ-colored subsets of $X(\mathbb{C})$. The topological factorization category is

$$\mathrm{Fact}(\mathrm{Rep}_q(T)) \simeq \mathrm{Shv}_q(\mathrm{Gr}_{\check{T},\mathrm{Ran}(\mathbb{A}^1)}), \tag{8.1}$$

[1] The definition of the relative Drinfeld center is briefly mentioned in Section 2.3

where the subscript q in $\mathrm{Shv}_q(\mathrm{Gr}_{\check{T},\mathrm{Ran}(\mathbb{A}^1)})$ means that the factorization category is twisted by a *factorization gerbe* specified by the parameter q (or equivalently, by the form b). For details on this twisting, see [48, Section 2.3] and [22, Section 4.1].

Take the Hopf algebra $U_q^{\mathrm{KD}}(\mathfrak{n}^-)$ in $\mathrm{Rep}_q(T)$. Denote the corresponding topological factorization algebra by $\Omega_q^{\mathrm{Lus}} := \Omega(U_q^{\mathrm{KD}}(\mathfrak{n}^-))$. The above procedure gives the equivalence

$$\mathrm{Dr}_{\mathrm{Rep}_q(T)}(U_q^{\mathrm{KD}}(\mathfrak{n}^-)\text{-comod}) \simeq \Omega_q^{\mathrm{Lus}}\text{-mod}_0^{\mathrm{fact}}.$$

Recall from Section 2.3, the category $U_q^{+\mathrm{Lus},-\mathrm{KD}}(\mathfrak{g})$-mod is defined as the relative Drinfeld center $\mathrm{Dr}_{\mathrm{Rep}_q(T)}(U_q^{\mathrm{Lus}}(\mathfrak{b})\text{-mod})$. As $U_q^{\mathrm{KD}}(\mathfrak{n}^-)$ is linearly dual to $U_q^{\mathrm{Lus}}(\mathfrak{n})$, we see that

$$U_q^{+\mathrm{Lus},-\mathrm{KD}}(\mathfrak{g})\text{-mod} \simeq \Omega_q^{\mathrm{Lus}}\text{-mod}_0^{\mathrm{fact}}. \tag{8.2}$$

Similarly, if we take $U_q^{\mathrm{Lus}}(\mathfrak{n}^-)$ and denote $\Omega_q^{\mathrm{KD}} := \Omega(U_q^{\mathrm{Lus}}(\mathfrak{n}^-))$, then the relative Drinfeld center gives the category $U_q^{+\mathrm{KD},-\mathrm{Lus}}(\mathfrak{g})$-mod of representations of the quantum group which has Kac-De Concini positive part and Lusztig negative part, and we have the equivalence

$$U_q^{+\mathrm{KD},-\mathrm{Lus}}(\mathfrak{g})\text{-mod} \simeq \Omega_q^{\mathrm{KD}}\text{-mod}_0^{\mathrm{fact}}. \tag{8.3}$$

Consider the Hopf algebra $\mathfrak{u}_q(\mathfrak{n}^-)$ in $\mathrm{Rep}_q(T)$ instead. Let $\Omega_q^{\mathrm{small}} := \Omega(\mathfrak{u}_q(\mathfrak{n}^-))$. The same construction gives the equivalence

$$\overset{\bullet}{\mathfrak{u}}_q(\mathfrak{g})\text{-mod} \simeq \Omega_q^{\mathrm{small}}\text{-mod}_0^{\mathrm{fact}}, \tag{8.4}$$

as $\mathfrak{u}_q(\mathfrak{n}^-)$ is linearly dual to $\mathfrak{u}_q(\mathfrak{n})$.

The factorization description of the categories $U_q^{\mathrm{Lus}}(\mathfrak{g})$-mod and $U_q^{\frac{1}{2}}(\mathfrak{g})$-mod will be postponed to Section 8.4.

Back to the abstract setting, assume further that our braided monoidal category C is *ribbon*; i.e. for each object M there is a ribbon automorphism $\theta_M : M \to M$ compatible with the braiding. Suppose that the Hopf algebra A is equivariant with respect to the ribbon structure of C. Then for any curve X we can generalize the functor Fact to produce topological factorization categories (and algebras inside them) on $\mathrm{Ran}(X)$ [48, Section 2.2]. Given $x \in X$, we can twist the braided monoidal category $\mathrm{Dr}_{\mathsf{C}}(A\text{-comod})$ by the tangent line $T_x(X)$ at x using the ribbon structure of C. This amounts to attaching the factorization modules of $\Omega(A)$ to $x \in X$. Namely, we have a canonical equivalence

$$\mathrm{Dr}_{\mathsf{C}}(A\text{-comod})_{T_x(X)} \simeq \Omega(A)\text{-mod}_x^{\mathrm{fact}}.$$

Define a ribbon structure θ on $\mathrm{Rep}_q(T)$ by $\theta_{C_\lambda}(m) = b(\lambda, \lambda + 2\rho) \cdot m$. One can check that $U_q^{\mathrm{KD}}(\mathfrak{n}^-)$, $U_q^{\mathrm{Lus}}(\mathfrak{n}^-)$ and $\mathfrak{u}_q(\mathfrak{n}^-)$ are all equivariant with respect to the ribbon structure. Then we update all three equivalences (8.2), (8.3) and (8.4) to the corresponding versions over an arbitrary point $x \in X$.

Similar to the algebro-geometric situation in Section 8.1, we organize topological factorization modules over multiple points into a factorization category

$$\Omega(A)\text{-mod}^{\text{fact}}(\text{Ran}(\text{X})),$$

whose fibre over the finite subset $\{x_1, \dots, x_n\} \subset X$ is

$$\Omega(A)\text{-mod}^{\text{fact}}_{x_1} \otimes \dots \otimes \Omega(A)\text{-mod}^{\text{fact}}_{x_n} \hookrightarrow \Omega(A)\text{-mod}^{\text{fact}}(\text{Ran}(X)).$$

8.3 Quantum Frobenius revisited

Our approach to obtain a factorization description of $U_q^{\text{Lus}}(\mathfrak{g})$-mod is to modify the factorization category for $U_q^{+\text{Lus},-\text{KD}}(\mathfrak{g})$-mod by using the quantum Frobenius map. This section serves to give a reformulation of the quantum Frobenius for this purpose.

Recall from Section 2.3 that the quantum Frobenius is a functor $\text{Fr}_q : \text{Rep}(\check{G}) \to U_q^{\text{Lus}}(\mathfrak{g})$-mod, which fits into the following exact sequence of categories

$$0 \to \text{Rep}(\check{G}) \to U_q^{\text{Lus}}(\mathfrak{g})\text{-mod} \to \mathfrak{u}_q(\mathfrak{g})\text{-mod} \to 0. \tag{8.5}$$

However, in order to understand the relations between $U_q^{\text{Lus}}(\mathfrak{g})$-mod and $\mathfrak{u}_q(\mathfrak{g})$-mod when q varies, we would like all categories in the above sequence to be "quantum"; namely, a version of $\text{Rep}(\check{G})$ that is sensitive to the parameter q. This is given by what is called the *metaplectic Langlands dual* of G, a reductive group that is determined combinatorially by G and q.

Recall from Section 2.4 that the parameter q is determined by the form b, which in turn is defined by the bilinear pairing $(\cdot,\cdot)_{\kappa'}$ for a negative level κ'. We are in the setting that κ' is rational and each $v_i := b(\alpha_i, \alpha_i)^{1/2}$ is a root of unity (but not equal to 1). Let l_i be the order of v_i^2.

Following [31], we define the metaplectic Langlands dual group as follows:

Let $\check{\Lambda}^\sharp \subset \check{\Lambda}$ be the sublattice

$$\check{\Lambda}^\sharp := \{\check{\lambda} \in \check{\Lambda} : (\check{\lambda}, \check{\mu})_{\kappa'} \in \mathbb{Z}, \forall \check{\mu} \in \check{\Lambda}\},$$

and $\Lambda^\sharp$ be the dual lattice of $\check{\Lambda}^\sharp$ in $\Lambda \otimes \mathbb{Q}$. Consider the embedding $\check{R} \hookrightarrow \check{\Lambda}$ defined by $\check{\alpha}_i \mapsto \check{\alpha}_i^\sharp := l_i \cdot \check{\alpha}_i$ for each i. Denote the image as $\check{R}^\sharp$. Dually we define $R^\sharp$ as the image of the map $R \to \Lambda \otimes \mathbb{Q}$, defined by $\alpha_i \mapsto \alpha_i^\sharp := (1/l_i) \cdot \alpha_i$. One checks that $R^\sharp \subset \Lambda^\sharp$, $\check{R}^\sharp \subset \check{\Lambda}^\sharp$ and $(R^\sharp \subset \Lambda^\sharp, \check{R}^\sharp \subset \check{\Lambda}^\sharp)$ defines a finite type root datum. Then we define the metaplectic Langland's dual group H to be the reductive group associated to the root datum $(\check{R}^\sharp \subset \check{\Lambda}^\sharp, R^\sharp \subset \Lambda^\sharp)$.

Write B_H, N_H and T_H for the Borel, the unipotent radical (inside the Borel) and the torus of H, respectively. Let $\mathfrak{n}_H := \text{Lie}(N_H)$. We can now reformulate the quantum Frobenius as a map $U_q^{\text{Lus}}(\mathfrak{n}) \to U(\mathfrak{n}_H)$ of Hopf algebras in $\text{Rep}_q(T)$, where $U(\mathfrak{n}_H)$ is regarded as an algebra in

$\mathrm{Rep}_q(T)$ via $\mathrm{Rep}(T_H) \to \mathrm{Rep}_q(T)$. The Frobenius map fits into the short exact sequence

$$1 \to \mathfrak{u}_q(\mathfrak{n}) \to U_q^{\mathrm{Lus}}(\mathfrak{n}) \to U(\mathfrak{n}_H) \to 1.$$

Dualizing the sequence, we get

$$1 \to \mathcal{O}_{N_H} \to U_q^{\mathrm{KD}}(\mathfrak{n}^-) \to \mathfrak{u}_q(\mathfrak{n}^-) \to 1.$$

A key property that will be used in Section 8.4 is that $\mathcal{O}_{N_H}$ is mapped centrally (as associative algebras) and cocentrally (as coassociative coalgebras) into $U_q^{\mathrm{KD}}(\mathfrak{n}^-)$.

For the full Lusztig's quantum group, we have a short exact sequence of categories

$$0 \to \mathrm{Rep}(H) \to U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod} \to \mathfrak{u}_q(\mathfrak{g})\text{-mod} \to 0. \tag{8.6}$$

Remark 8.3.1. In (8.5), the category $\mathrm{Rep}(\check{G})$ is fixed and the functor Fr_q deforms according to q, while in (8.6) the category $\mathrm{Rep}(H)$ depends on q and the definition of the functor $\mathrm{Rep}(H) \to U_q^{\mathrm{Lus}}(\mathfrak{g})$-mod is fixed for different q.

The sequence (8.6) gives an action of the monoidal category $\mathrm{Rep}(H)$ on $U_q^{\mathrm{Lus}}(\mathfrak{g})$-mod. We have the following equivalences as consequences of the metaplectic Langlands dual construction:

$$U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod} \underset{\mathrm{Rep}(H)}{\otimes} \mathrm{Rep}(B_H) \simeq U_q^{\frac{1}{2}}(\mathfrak{g})\text{-mod}; \tag{8.7}$$

$$U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod} \underset{\mathrm{Rep}(H)}{\otimes} \mathrm{Rep}(T_H) \simeq \overset{\bullet}{\mathfrak{u}}_q(\mathfrak{g})\text{-mod}. \tag{8.8}$$

By (8.7), the fact that the forgetful functor $U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod} \to U_q^{\frac{1}{2}}(\mathfrak{g})$-mod is fully faithful now follows from the fully-faithfulness of the restriction functor $\mathrm{Rep}(H) \to \mathrm{Rep}(B_H)$.

8.4 Towards a factorization Kazhdan-Lusztig equivalence

In this section we summarize the conjectural theory of factorization Kazhdan-Lusztig equivalence. All constructions, results and conjectures are due to D. Gaitsgory [22, 26].

Let $\mathrm{C}^{\cdot}(\mathfrak{n}_H)$ be the Chevalley complex of $\mathfrak{n}_H$ in the symmetric monoidal category $\mathrm{Rep}(T_H)$, regarded as a commutative DG algebra. Note that an E_2-algebra in a symmetric monoidal category is nothing but a commutative algebra. Then by Lurie's functor Fact, we produce a factorization algebra $\Omega^{\mathrm{cl}} := \mathrm{Fact}(\mathrm{C}^{\cdot}(\mathfrak{n}_H))$ in the factorization category $\mathrm{Fact}(\mathrm{Rep}(T_H))$. Note also that $\Omega^{\mathrm{cl}} \cong \Omega(\mathcal{O}_{N_H})$.

The functor Fact actually gives more in the above situation: with the input of a commutative algebra in a symmetric monoidal category, Fact outputs a *commutative factorization algebra* in a *commutative factorization category*. We define these notions below.

Recall from Section 9.4 that a factorization category $\mathsf{C}_{\mathrm{Ran}(X)}$ is an assignment $I \rightsquigarrow \mathsf{C}_I$ of sheaf of categories on X^I for each finite set I, with the equivalences

$$(\mathsf{C}_I \boxtimes \mathsf{C}_J)|_{[X^I \times X^J]_{\text{disj}}} \xrightarrow{\sim} \mathsf{C}_{I \sqcup J}|_{[X^I \times X^J]_{\text{disj}}}. \tag{8.9}$$

We say a factorization category is commutative if the equivalence (8.9) extend to a functor

$$\mathsf{C}_I \boxtimes \mathsf{C}_J \longrightarrow \mathsf{C}_{I \sqcup J}$$

for all I, J. We similarly define a commutative factorization algebra to be a factorization algebra whose factorization isomorphisms (over the disjoint loci) extend to morphisms on the whole space for any partition of products of the curve X.

We can further enhance the definition of a factorization module to a *commutative factorization module* by similarly requiring extra morphisms extending isomorphisms (9.6). Now, if a factorization algebra Υ is acted on by a commutative factorization algebra Υ', then we define the category

$$\Upsilon\text{-mod}^{\text{fact}}_{\Upsilon'\text{-com}}$$

as the category of factorization modules of Υ with a compatible commutative factorization Υ'-module structure.

Now we can characterize the category $U_q^{\frac{1}{2}}(\mathfrak{g})$-mod in factorization terms. The fact that $\mathcal{O}_{N_H}$ is mapped centrally into $U_q^{\text{KD}}(\mathfrak{n}^-)$ implies the existence of a canonical action of Ω^{cl} on Ω_q^{Lus}. The following proposition essentially follows from the identifications

$$U_q^{\frac{1}{2}}(\mathfrak{g})\text{-mod} \simeq U_q^{+\text{Lus},-\text{KD}}(\mathfrak{g})\text{-mod} \underset{\text{Dr}_{\text{Rep}(T_H)}(\text{Rep}(B_H))}{\otimes} \text{Rep}(B_H),$$

$$\text{Dr}_{\text{Rep}(T_H)}(\text{Rep}(B_H)) \simeq \Omega^{\text{cl}}\text{-mod}^{\text{fact}},$$

and

$$\text{Rep}(B_H) \simeq \Omega^{\text{cl}}\text{-mod}^{\text{fact}}_{\Omega^{\text{cl}}\text{-com}}.$$

Proposition 8.4.1 ([28] Proposition 2.2.4)**.** *There is a canonical equivalence*

$$U_q^{\frac{1}{2}}(\mathfrak{g})\text{-mod} \simeq \Omega_q^{\text{Lus}}\text{-mod}^{\text{fact}}_{\Omega^{\text{cl}}\text{-com}}.$$

Consequently, there is an equivalence of factorization categories

$$\text{Fact}(U_q^{\frac{1}{2}}(\mathfrak{g})\text{-mod}) \xrightarrow{\sim} \Omega_q^{\text{Lus}}\text{-mod}^{\text{fact}}_{\Omega^{\text{cl}}\text{-com}}(\text{Ran}(X)).$$

As the forgetful functor $U_q^{\text{Lus}}(\mathfrak{g})\text{-mod} \to U_q^{\frac{1}{2}}(\mathfrak{g})$-mod is fully faithful, we realize the factorization category $\text{Fact}(U_q^{\text{Lus}}(\mathfrak{g})\text{-mod})$ as a full factorization subcategory of $\text{Fact}(U_q^{\frac{1}{2}}(\mathfrak{g})\text{-mod})$.

We turn to the Kac-Moody side of the Kazhdan-Lusztig functor. We will first describe the (conjectural) factorization version of the Kazhdan-Lusztig equivalence at positive level. As usual, let κ be a positive (rational) level.

Recall the generalized semi-infinite cohomology functor $\mathfrak{C}^{\frac{\infty}{2}}_{\mathcal{F}}(\mathfrak{n}(\mathcal{K}),-) : \hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})} \to \operatorname{Rep}(T)$ from Section 5.2. (We organize all the μ-components to a functor with output as Λ-graded vector spaces.) The group $N(\mathcal{K})$ and $T(\mathcal{O})$ naturally factorizes, and the functor p^- factorizes due to Lemma 5.2.1. Therefore the generalized functors defined in (5.3) via natural pairing of factorizable objects can be naturally upgraded to factorization functors. In particular, we get a factorization functor

$$\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}),-)_{\operatorname{Ran}(X)} : (\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})})_{\operatorname{Ran}(X)} \longrightarrow \text{D-mod}_{\kappa+\text{shift}}(\operatorname{Gr}_{\check{T},\operatorname{Ran}(X)}),$$

where the subscript κ+shift in $\text{D-mod}_{\kappa+\text{shift}}(\operatorname{Gr}_{\check{T},\operatorname{Ran}(X)})$ denotes the twisting given by a factorization gerbe corresponding to the level κ and a Tate shift, which arises naturally from the semi-infinite cohomology; c.f. (9.1). The twisting on $\operatorname{Shv}_{q^{-1}}(\operatorname{Gr}_{\check{T},\operatorname{Ran}(X)})$ introduced in (8.1) matches with the twisting here, in the sense that we have the twisted Riemann-Hilbert functor

$$\operatorname{RH} : \operatorname{Shv}_{q^{-1}}(\operatorname{Gr}_{\check{T},\operatorname{Ran}(X)}) \hookrightarrow \text{D-mod}_{\kappa+\text{shift}}(\operatorname{Gr}_{\check{T},\operatorname{Ran}(X)}). \tag{8.10}$$

The vacuum module $\mathbb{V}_0^\kappa$ is the unit object in the braided monoidal category $\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})}$. The unit object is naturally upgraded to a factorization algebra in the corresponding factorization category. Let

$$\Omega_\kappa^{-,\text{Lus}} \in \text{D-mod}_{\kappa+\text{shift}}(\operatorname{Gr}_{\check{T},\operatorname{Ran}(X)})$$

be the factorization algebra associated to $\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}),\mathbb{V}_0^\kappa)$. Moreover, $\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}),-)_{\operatorname{Ran}(X)}$ canonically induces a functor

$$(\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})})_{\operatorname{Ran}(X)} \longrightarrow \Omega_\kappa^{-,\text{Lus}}\text{-mod}^{\text{fact}}(\operatorname{Ran}(X)). \tag{8.11}$$

Conjecture 8.4.2. *Under the twisted Riemann-Hilbert functor* (8.10)*,* $\Omega_\kappa^{-,\text{Lus}}$ *is identified with* $\Omega_{q^{-1}}^{\text{Lus}}$*.*

Assume Conjecture 8.4.2. Denote by

$$\Omega_\kappa^{-,\text{Lus}}\text{-mod}^{\text{fact}}_{\Omega^{\text{cl}}\text{-com}}$$

the category obtained by the same construction as $\Omega_{q^{-1}}^{\text{Lus}}\text{-mod}^{\text{fact}}_{\Omega^{\text{cl}}\text{-com}}$ on the D-module side of the Riemann-Hilbert correspondence. Then by Proposition 8.4.1 the twisted Riemann-Hilbert identifies

$$\Omega_\kappa^{-,\text{Lus}}\text{-mod}^{\text{fact}}_{\Omega^{\text{cl}}\text{-com}}(\operatorname{Ran}(X)) \simeq \operatorname{Fact}(U_{q^{-1}}^{\frac{1}{2}}(\mathfrak{g})\text{-mod}). \tag{8.12}$$

The following conjectural statement is the factorization Kazhdan-Lusztig equivalence at positive level κ that we are after:

Conjecture 8.4.3. *The functor* (8.11) *induces a canonical fully faithful embedding of factorization categories*

$$(\hat{\mathfrak{g}}_\kappa\text{-mod}^{G(\mathcal{O})})_{\mathrm{Ran}(X)} \hookrightarrow \Omega_\kappa^{-,\mathrm{Lus}}\text{-mod}^{\mathrm{fact}}_{\Omega^{\mathrm{cl}}\text{-com}}(\mathrm{Ran}(X)).$$

Moreover, the essential image of the above functor is equivalent to $\mathrm{Fact}(U^{\mathrm{Lus}}_{q^{-1}}(\mathfrak{g})\text{-mod})$ *under the identification* (8.12).

We now briefly discuss the negative level case. As the duality pattern in Section 6.2 suggests, we should start with the !-generalized functor $\mathfrak{C}_!^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -) : \hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} \to \mathrm{Rep}(T)$. However, what makes things more complicated is that, $\mathfrak{C}_!^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -)$ is known to be non-factorizable. To remedy this, we define the Jacquet functor $\mathsf{J}_!^{\kappa'}$ as the composition

$$\hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})} \xrightarrow{\Delta_0 \star_{G(\mathcal{O})} -} \hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{N(\mathcal{K})T(\mathcal{O})} \xrightarrow{\mathfrak{C}_!^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}),-)} \mathrm{Rep}(T)$$

where $\Delta_0 \in \text{D-mod}(\mathrm{Gr}_G)^{N(\mathcal{K})T(\mathcal{O})}$ is the !-extension of the D-module ω_{S_0} on the semi-infinite orbit $S_0 \subset \mathrm{Gr}_G$.

Proposition 8.4.4. *The Jacquet functor* $\mathsf{J}_!^{\kappa'}$ *is factorizable.*

We can now proceed with the construction similar to the positive level case. Namely, we construct the factorization algebra

$$\Omega^{\mathrm{Lus}}_{\kappa'} \in \text{D-mod}_{\kappa'+\mathrm{shift}}(\mathrm{Gr}_{\check{T},\mathrm{Ran}(X)})$$

associated to $\mathsf{J}_!^{\kappa'}(\mathbb{V}_0^{\kappa'})$, which should conjecturally be identified with Ω_q^{Lus} via the twisted Riemann-Hilbert functor

$$\mathrm{RH} : \mathrm{Shv}_q(\mathrm{Gr}_{\check{T},\mathrm{Ran}(X)}) \hookrightarrow \text{D-mod}_{\kappa'+\mathrm{shift}}(\mathrm{Gr}_{\check{T},\mathrm{Ran}(X)}). \tag{8.13}$$

The Jacquet functor induces a factorization functor

$$(\hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})})_{\mathrm{Ran}(X)} \longrightarrow \Omega^{\mathrm{Lus}}_{\kappa'}\text{-mod}^{\mathrm{fact}}(\mathrm{Ran}(X)), \tag{8.14}$$

with the target identified with $\mathrm{Fact}(U_q^{\frac{1}{2}}(\mathfrak{g})\text{-mod})$ via (8.13). The main conjecture for the negative level case is

Conjecture 8.4.5. *The functor* (8.14) *induces a canonical fully faithful embedding of factorization categories*

$$(\hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{G(\mathcal{O})})_{\mathrm{Ran}(X)} \hookrightarrow \Omega^{\mathrm{Lus}}_{\kappa'}\text{-mod}^{\mathrm{fact}}_{\Omega^{\mathrm{cl}}\text{-com}}(\mathrm{Ran}(X)).$$

Moreover, the essential image of the above functor is sent to $\mathrm{Fact}(U_q^{\mathrm{Lus}}(\mathfrak{g})\text{-mod})$ *under the twisted Riemann-Hilbert functor* (8.13).

We expect to recover the original (derived) Kazhdan-Lusztig equivalence at negative level by taking the fibre of the functor in Conjecture 8.4.5 over any point $x \in X$.

Finally, for completeness we record what is conjectured to be the category on the Kac-Moody side corresponding to $U_{q^{-1}}^{\frac{1}{2}}(\mathfrak{g})$-mod and $U_q^{\frac{1}{2}}(\mathfrak{g})$-mod:

Conjecture 8.4.6.

- *The functor* $\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -)$ *defines an equivalence*

$$\hat{\mathfrak{g}}_\kappa\text{-mod}_{N(\mathcal{K})T(\mathcal{O})} \xrightarrow{\sim} \Omega_\kappa^{\mathrm{Lus}}\text{-mod}^{\mathrm{fact}} \left(\simeq U_{q^{-1}}^{\frac{1}{2}}(\mathfrak{g})\text{-mod} \right).$$

- *The functor* $\mathfrak{C}_!^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), -)$ *defines an equivalence*

$$\hat{\mathfrak{g}}_{\kappa'}\text{-mod}^{N(\mathcal{K})T(\mathcal{O})} \xrightarrow{\sim} \Omega_{\kappa'}^{\mathrm{Lus}}\text{-mod}^{\mathrm{fact}} \left(\simeq U_q^{\frac{1}{2}}(\mathfrak{g})\text{-mod} \right).$$

Chapter 9

Appendix

In section 9.1 and 9.3, we recall the formalism of Lie-* algebras, chiral algebras and the construction of chiral differential operators as developed in [9] and reviewed in [5].

9.1 Lie-* algebras and chiral algebras

Let X be a smooth algebraic curve over $\mathbb{C}$. Denote by $\Delta : X \hookrightarrow X^2$ the diagonal map and $j : X^2 \setminus \Delta \hookrightarrow X^2$ the open embedding of the complement of diagonal.

We define a Lie-* algebra on X to be a right D-module L , equipped with a Lie-* bracket

$$\{\cdot,\cdot\} : L \boxtimes L \to \Delta_! L,$$

which is a morphism between right D-modules on X^2 satisfying the following conditions:

- (Anti-symmetry) Let $\sigma = (1,2) \in \mathcal{S}_2$ be the transposition acting on $X \times X$. Then

$$\sigma(\{\cdot,\cdot\}(s)) = -\{\cdot,\cdot\}(\sigma(s)),$$

  where s is a section of $L \boxtimes L$.

- (Jacobi identity) Let $\tau = (1,2,3) \in \mathcal{S}_3$ be the permutation acting on $X \times X \times X$. Then

$$\{\{\cdot,\cdot\},\cdot\}(s) + \tau^{-1}(\{\{\cdot,\cdot\},\cdot\}(\tau(s))) + \tau^{-2}(\{\{\cdot,\cdot\},\cdot\}(\tau^2(s)))$$

  equals the zero section of $\Delta_! L$, where s is a section of $L \boxtimes L \boxtimes L$.

Let A be a right D-module on X. We similarly define a chiral bracket on A as a morphism

$$\{\cdot,\cdot\}^{ch} : j_* j^* A \boxtimes A \to \Delta_! A$$

between D-modules on X^2, which is anti-symmetric and satisfies the Jacobi identity. A (unital) chiral algebra $(A, \{\cdot,\cdot\}^{ch}_A, u_A)$ is the data of a right D-module A with a chiral bracket $\{\cdot,\cdot\}^{ch}_A$, and a unit morphism $u_A : \omega_X \to A$ such that $\{\cdot,\cdot\}^{ch}_A \circ j_*j^*(u_A \boxtimes \mathrm{Id}_A) : j_*j^*\omega_X \boxtimes A \to \Delta_! A$ is the canonical map arising from the standard exact triangle

$$\omega_X \boxtimes A \to j_*j^*(\omega_X \boxtimes A) \to \Delta_!\Delta^!(\omega_X \boxtimes A) \cong \Delta_! A \to$$

associated to $\omega_X \boxtimes A$.

For a chiral algebra A, we define a chiral module M over A as a right D-module with an action morphism act : $j_*j^*(A \boxtimes M) \to \Delta_! M$, such that chiral analogs of the usual axioms for modules over a (universal enveloping algebra of) Lie algebra hold (c.f. [5, Section 1.1]).

We have the category $\mathsf{Alg}^{\mathrm{chiral}}(X)$ (resp. $\mathsf{Alg}^{\mathrm{Lie}^*}(X)$) of chiral (resp. Lie-*) algebras on X, where morphisms are morphisms of D-modules respecting the chiral (resp. Lie-*) brackets. A canonical tensor product structure can be defined for chiral brackets, which turns $\mathsf{Alg}^{\mathrm{chiral}}(X)$ into a symmetric monoidal category; c.f. [9, Section 3.4.15]. For a chiral algebra A and $x \in X$, we will consider the category of chiral A-modules supported at x, denoted by $A\text{-mod}^{\mathrm{ch}}_x$. By letting the point x vary in X, we obtain a global category (i.e. sheaf of categories) $A\text{-mod}^{\mathrm{ch}}(X)$ on X.

Given a chiral algebra A, by pre-composing the chiral bracket with the natural map $A \boxtimes A \to j_*j^*A \boxtimes A$, we get a Lie-* bracket on A. This defines a forgetful functor from the category of chiral algebras to the category of Lie-* algebras. We introduce the functor of universal enveloping chiral algebra $L \mapsto U^{\mathrm{ch}}(L)$ as the left adjoint to the forgetful functor. Namely, for a Lie-* algebra L and a chiral algebra A, we have

$$\mathrm{Hom}_{\text{Lie-}*}(L, A) = \mathrm{Hom}_{\mathrm{chiral}}(U^{\mathrm{ch}}(L), A).$$

Proposition 9.1.1 (c.f. [5] Section 1.4)**.** *Let $x \in X$ be any point, and L a Lie-* algebra.*

1. $U^{\mathrm{ch}}(L)_x \cong \mathrm{Ind}^{H_{\mathrm{dR}}(\mathcal{D}^*_x, L)}_{H_{\mathrm{dR}}(\mathcal{D}_x, L)}(\mathbb{C})$, *where $\mathbb{C}$ is the trivial representation over the topological Lie algebra $H_{\mathrm{dR}}(\mathcal{D}_x, L)$.*

2. *There is an equivalence between the chiral modules over $U^{\mathrm{ch}}(L)$ supported at x and the continuous modules over $H_{\mathrm{dR}}(\mathcal{D}^*_x, L)$.*

3. $U^{\mathrm{ch}}(L)$ *has a unique filtration* $U^{\mathrm{ch}}(L) = \bigcup_{i \geq 0} U^{\mathrm{ch}}(L)_i$ *with the following properties:*

   (a) $U^{\mathrm{ch}}(L)_0 = \omega_X$;

   (b) $U^{\mathrm{ch}}(L)_1 / U^{\mathrm{ch}}(L)_0 \cong L$;

   (c) $\{\cdot,\cdot\}^{\mathrm{ch}} : j_*j^*(U^{\mathrm{ch}}(L)_i \boxtimes U^{\mathrm{ch}}(L)_j) \to \Delta_!(U^{\mathrm{ch}}(L)_{i+j})$
   $\{\cdot,\cdot\}^{\mathrm{ch}} : U^{\mathrm{ch}}(L)_i \boxtimes U^{\mathrm{ch}}(L)_j \to \Delta_!(U^{\mathrm{ch}}(L)_{i+j-1})$;

   (d) *The natural embedding $L \to U^{\mathrm{ch}}(L)$ induces an isomorphism* $\mathrm{Sym}(L) \cong \mathrm{gr}(U^{\mathrm{ch}}(L))$.

   (e) *At the level of fibers, the filtration of*

$$U^{\mathrm{ch}}(L)_x \cong \mathrm{Ind}^{H_{\mathrm{dR}}(\mathcal{D}^*_x, L)}_{H_{\mathrm{dR}}(\mathcal{D}_x, L)}(\mathbb{C}) \cong U(H_{\mathrm{dR}}(\mathcal{D}^*_x, L)) \otimes_{U(H_{\mathrm{dR}}(\mathcal{D}_x, L))} \mathbb{C}$$

comes from the natural filtration of the universal enveloping algebra $U(H_{\mathrm{dR}}(\mathcal{D}_x^*, L))$.

Example 9.1.1. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. Then $L_{\mathfrak{g}} := \mathfrak{g}\otimes_{\mathbb{C}}\mathcal{D}_X$ is naturally a Lie-* algebra. Let $Q : \mathfrak{g}\otimes\mathfrak{g} \to \mathbb{C}$ be a G-invariant symmetric bilinear form, which defines an $\mathcal{O}_X$-pairing

$$\phi_Q : \mathfrak{g}\otimes\mathcal{O}_X \times \mathfrak{g}\otimes\mathcal{O}_X \to \omega_X, \quad \phi_Q(a\otimes f, b\otimes g) := Q(a,b) f\, dg.$$

Then ϕ_Q induces a $\mathcal{D}_X$-pairing $L_{\mathfrak{g}}\boxtimes L_{\mathfrak{g}} \to \Delta_! \omega_X$ which satisfies the condition of a 2-cocycle (c.f. [9, Section 2.5.9]). We define the Lie-* algebra extension $L_{\mathfrak{g},Q}$ of $L_{\mathfrak{g}}$ by ω_X using this 2-cocycle, called the affine Kac-Moody extension of $L_{\mathfrak{g}}$.

Taking the de Rham cohomology $H_{\mathrm{dR}}(\mathcal{D}_x^*, L_{\mathfrak{g},Q})$, we recover the usual affine Lie algebra $\hat{\mathfrak{g}}_Q = \mathfrak{g}(\mathcal{K}_x)\oplus\mathbb{C}\mathbf{1}$ associated to the form Q. By Proposition 9.1.1 we have an equivalence between $\hat{\mathfrak{g}}_Q$-modules and chiral $U^{\mathrm{ch}}(L_{\mathfrak{g},Q})$-modules supported at a point $x\in X$. On the other hand,

$$U^{\mathrm{ch}}(L_{\mathfrak{g},Q})_x \cong \mathrm{Ind}_{H_{\mathrm{dR}}(\mathcal{D}_x, L_{\mathfrak{g},Q})}^{H_{\mathrm{dR}}(\mathcal{D}_x^*, L_{\mathfrak{g},Q})}(\mathbb{C}) \cong \mathrm{Ind}_{\mathfrak{g}(\mathcal{O}_x)\oplus\mathbb{C}\mathbf{1}}^{\hat{\mathfrak{g}}_Q}(\mathbb{C}) \equiv V_Q(\mathfrak{g})$$

is the space underlying the affine Kac-Moody vertex algebra associated to $\mathfrak{g}$ and Q. It is proven in [9] that chiral $U^{\mathrm{ch}}(L_{\mathfrak{g},Q})$-modules supported at a point are equivalent to modules over the vertex algebra $V_Q(\mathfrak{g})$.

9.2 Semi-infinite cohomology

Let L be a Lie-* algebra. Denote by L° its dual Lie-* algebra, and by

$$\langle\, ,\rangle : L[1]\boxtimes L^\circ[-1] \to \Delta_!\omega_X$$

the natural pairing in the DG super convention.

Now we consider $L[1]\oplus L^\circ[-1]$, and further extend the pairing $\langle\, ,\rangle$ to a skew-symmetric (in the DG super sense) pairing on $L[1]\oplus L^\circ[-1]$:

$$\langle\, ,\rangle_{L,L^\circ} : (L[1]\oplus L^\circ[-1])\boxtimes(L[1]\oplus L^\circ[-1]) \to \Delta_!\omega_X$$

by setting the kernel of $\langle\, ,\rangle_{L,L^\circ}$ as $(L[1]\boxtimes L[1])\oplus(L^\circ[-1]\boxtimes L^\circ[-1])$. This defines a Lie-* algebra $(L[1]\oplus L^\circ[-1])^\flat := L[1]\oplus L^\circ[-1]\oplus\omega_X$, which is a central extension of $L[1]\oplus L^\circ[-1]$ via the pairing $\langle\, ,\rangle_{L,L^\circ}$.

We define the Clifford algebra associated to L as the twisted enveloping chiral algebra $\mathcal{C}l$ of the ω_X-extension $(L[1]\oplus L^\circ[-1])^\flat$. By definition, this is the universal enveloping chiral algebra $U^{\mathrm{ch}}((L[1]\oplus L^\circ[-1])^\flat)$ modulo the ideal generated by $1-1^\flat$, where 1 is the section of $\omega_X = U^{\mathrm{ch}}((L[1]\oplus L^\circ[-1])^\flat)_0$ and $1^\flat$ is the section of ω_X in $(L[1]\oplus L^\circ[-1])^\flat \cong U^{\mathrm{ch}}((L[1]\oplus L^\circ[-1])^\flat)_1/U^{\mathrm{ch}}((L[1]\oplus L^\circ[-1])^\flat)_0$.

There is a canonical PBW filtration $\mathcal{C}l_\cdot$ with

$$\mathcal{C}l_1 = (L[1] \oplus L^\circ[-1])^\flat$$

and the associated graded D_X-algebra is equal to $\mathrm{Sym}^\cdot(L[1] \oplus L^\circ[-1])$. We define an additional $\mathbb{Z}$-grading $^{(\cdot)}$ on $\mathcal{C}l$ by setting

$$L[1] \subseteq \mathcal{C}l^{(-1)}, \quad L^\circ[-1] \subseteq \mathcal{C}l^{(1)} \quad \text{and} \quad \omega_X \subseteq \mathcal{C}l^{(0)}.$$

Clearly $\omega_X = \mathcal{C}l_0^{(0)}$, so by the PBW theorem $\mathcal{C}l_2^{(0)}/\omega_X \cong L \otimes L^\circ$. In other words, $\mathcal{C}l_2^{(0)}$ is a central extension of $\mathfrak{gl}(L) \cong L \otimes L^\circ$ by ω_X, called the Tate extension and will be denoted by $\mathfrak{gl}(L)^\flat$. Consider the morphism $ad : L \to \mathfrak{gl}(L)$ induced from the adjoint action of L on itself. Then we can pull-back the Tate extension along ad to get a central extension $L^\flat$ of L by ω_X. Denote the morphism $L^\flat \to \mathfrak{gl}(L)^\flat$ by β.

Suppose that we are given a chiral algebra A and a morphism of Lie-* algebras $\alpha : L^\flat \to A$ with $1^\flat \mapsto -1_A$. (We always denote the associated Lie-* algebra of a chiral algebra by the same notation when no confusion can arise.) Consider the graded chiral algebra $A \otimes \mathcal{C}l^{(\cdot)}$. For simplicity of notations, let $\mu \equiv \{\ ,\ \}^{\mathrm{ch}}_{A\otimes\mathcal{C}l}$ denote the chiral product of $A \otimes \mathcal{C}l$. The *BRST differential* is an odd derivation d on $A \otimes \mathcal{C}l^{(\cdot)}$ of degree 1 with respect to both the $^{(\cdot)}$ and the PBW (structural) gradings, as defined in the following.

Since α and β send $1^\flat \in L^\flat$ to $-1_A \in A$ and $1^\flat \in \mathfrak{gl}(L)^\flat$ respectively, we put

$$\ell^{(0)} := \alpha + \beta : L \to A \otimes \mathcal{C}l^{(0)}.$$

Also set

$$\ell^{(-1)} : L[1] \hookrightarrow \mathcal{C}l^{(-1)} \hookrightarrow A \otimes \mathcal{C}l^{(-1)}.$$

Recall that we have the DG super Chevalley complex $(\mathrm{Sym}(L^\circ[-1]), \delta)$ sitting inside $\mathcal{C}l \subset A \otimes \mathcal{C}l$. The differential δ is induced (by taking S_n-coinvariants of $L^\circ[-1]^{\otimes n}$ for each n) from the map $L^\circ[-1]^{\otimes n} \to L^\circ[-1]^{\otimes n+1}$ given by

$$\phi \mapsto -\frac{1}{2}\phi(\{\ ,\ \}_L \otimes \mathrm{Id}_{L[1]} \otimes \cdots \otimes \mathrm{Id}_{L[1]}).$$

It is easy to check that $ad_{\ell^{(0)}} = \delta ad_{\ell^{(-1)}} + ad_{\ell^{(-1)}}\delta$ as $*$-operations

$$L \boxtimes \mathrm{Sym}(L^\circ[-1]) \to \Delta_*\mathrm{Sym}(L^\circ[-1]) \hookrightarrow \Delta_* A \otimes \mathcal{C}l.$$

Here $ad_{\ell^{(0)}}$ means the adjoint action of the image of $\ell^{(0)}$ via the Lie-* bracket of $A \otimes \mathcal{C}l$, and similar for $ad_{\ell^{(-1)}}$. Since δ acts as zero on $\omega_X = \mathrm{Sym}^0(L^\circ[-1])$, the relation restricts to $L^\circ[-1] \subset \mathrm{Sym}(L^\circ[-1])$ as

$$\{\ell^{(0)}, \mathrm{Id}_{L^\circ[-1]}\}_{A\otimes\mathcal{C}l} = \{\ell^{(-1)}, \delta|_{L^\circ[-1]}\}_{A\otimes\mathcal{C}l},$$

where $\{\ \}_{A\otimes\mathcal{C}l}$ is the Lie-* bracket of $A\otimes\mathcal{C}l$.

Now we define

$$\tilde{\chi} := \mu(\ell^{(0)}, \mathrm{Id}_{L^\circ[-1]}) - \mu(\ell^{(-1)}, \delta|_{L^\circ[-1]}),$$

a chiral operation $j_*j^*(L\boxtimes L^\circ) \to \Delta_*(A\otimes\mathcal{C}l^{(1)}[1])$. Since $\tilde{\chi}$ vanishes under the pull-back $L\boxtimes L^\circ \hookrightarrow j_*j^*(L\boxtimes L^\circ)$, it induces a morphism

$$\chi : L\otimes L^\circ \to A\otimes\mathcal{C}l^{(1)}[1].$$

Plugging in $\mathrm{Id}_L \in \mathrm{End}(L) = L\otimes L^\circ$, we obtain $\mathfrak{d} := \chi(\mathrm{Id}_L) \in A\otimes\mathcal{C}l^{(1)}[1]$. Finally we define the BRST differential $d := \{\mathfrak{d}, \cdot\}_{A\otimes\mathcal{C}l}$. Indeed, it has degree 1 with respect to both the two gradings as desired. Moreover, it satisfies $d^2 = 0$. (See [9] for the proofs of this fact and the theorem below.) We call the DG chiral algebra $(A\otimes\mathcal{C}l^{(\cdot)}, d)$ the BRST or semi-infinite complex associated to L and α.

The following theorem is called the BRST property of d.

Theorem 9.2.1. *The BRST differential d is a unique odd derivation of $A\otimes\mathcal{C}l^{(\cdot)}$ of structure and $^{(\cdot)}$ degrees 1 such that $d\circ\ell^{(-1)} = \ell^{(0)}$.*

Given an A-chiral module M, we form $(M\otimes\mathcal{C}l_{\mathfrak{n}}, d_M)$, the chiral module complex over the DG chiral algebra $(A\otimes\mathcal{C}l_{\mathfrak{n}}, d)$, with differential d_M induced from d. When the morphism $L^\flat \to A$ is clear from the context, we simply write

$$\mathfrak{C}^{\frac{\infty}{2}}(L, M) := (M\otimes\mathcal{C}l_{\mathfrak{n}}, d_M)$$

and call $\mathfrak{C}^{\frac{\infty}{2}}(L, M)$ the semi-infinite complex of M with respect to L. We will denote by

$$H^{\frac{\infty}{2}+i}(L, M)$$

the i-th cohomology of the semi-infinite complex.

Example 9.2.1. Consider the $\mathcal{O}_X$-Lie algebra $\mathfrak{n}\otimes\mathcal{O}_X$. It has dual $\mathfrak{n}^*\otimes\omega_X$, along with the natural $\mathcal{O}_X$-pairing

$$(\ ,\ ) : (\mathfrak{n}\otimes\mathcal{O}_X)\otimes(\mathfrak{n}^*\otimes\omega_X) \to \omega_X.$$

We denote by $L_{\mathfrak{n}} = \mathfrak{n}\otimes D_X$ and $L_{\mathfrak{n}}^\circ = \mathfrak{n}^*\otimes\omega_X\otimes D_X$ the corresponding right D_X-modules, and upgrade $(\ ,\ )$ to a pairing in the DG super setting

$$\langle\ ,\ \rangle : L_{\mathfrak{n}}[1]\boxtimes L_{\mathfrak{n}}^\circ[-1] \to \Delta_!(\omega_X\otimes D_X) \to \Delta_!\omega_X.$$

Then we have the Clifford algebra $\mathcal{C}l_{\mathfrak{n}}$ associated to $L_{\mathfrak{n}}$. When restricted to the formal disc $\mathcal{D}_x$ at $x\in X$, by Proposition 9.1.1.3(e) the chiral algebra $\mathcal{C}l_{\mathfrak{n}}$ has fiber at x isomorphic to the semi-infinite Fermionic vertex superalgebra associated to $\mathfrak{n}$, as defined in [19, section 15.1]. The pairing $(\ ,\ )$ restricts to the usual residue pairing on $\mathfrak{n}(\mathcal{K}_x)$. The induced Tate extension

$\mathfrak{n}(\mathcal{K}_x)^\flat \subset \mathfrak{gl}(\mathfrak{n}(\mathcal{K}_x))^\flat$ is given by the 2-cocycle

$$u \otimes f, v \otimes g \mapsto \mathrm{tr}(ad_u ad_v)\mathrm{Res}_x(f dg),$$

where ad_u, ad_v are adjoint actions of u, v on $\mathfrak{n}$, respectively.

For $\kappa \in \mathbb{C}$, consider the pairing $Q = \kappa(\cdot,\cdot)_{\mathrm{st}}$, where $(\cdot,\cdot)_{\mathrm{st}}$ is the standard bilinear form on $\mathfrak{g}$. Recall the affine Kac-Moddy extension $L_{\mathfrak{g},Q} \equiv L_{\mathfrak{g},\kappa}$ constructed in Example 9.1.1. Set $A_{\mathfrak{g},\kappa}$ as the twisted enveloping chiral algebra of $L_{\mathfrak{g},\kappa}$.

By definition there is a canonical embedding $L_{\mathfrak{g},\kappa} \hookrightarrow A_{\mathfrak{g},\kappa}$ as Lie-* algebras. Since κ vanishes on $\mathfrak{n}$, the Kac-Moody extension splits on $L_{\mathfrak{n}}$, and thus $L_{\mathfrak{n}} \hookrightarrow L_{\mathfrak{g},\kappa} \hookrightarrow A_{\mathfrak{g},\kappa}$. This map actually lifts to $L_{\mathfrak{n}}^\flat$: the extension $L_{\mathfrak{n}}^\flat$ is actually trivialized due to the fact that $\mathfrak{n}$ has nilpotent adjoint action. Hence we are free to send $1^\flat \in L_{\mathfrak{n}}^\flat$ to $-1 \in L_{\mathfrak{g},\kappa}$. This defines $\alpha : L_{\mathfrak{n}}^\flat \to A_{\mathfrak{g},\kappa}$, and we form the BRST complex $(A_{\mathfrak{g},\kappa} \otimes \mathcal{C}l_{\mathfrak{n}}, d)$ by the general construction above.

Now, the fiber $\mathcal{C}l_{\mathfrak{n},x}$ is a chiral module over $\mathcal{C}l_{\mathfrak{n}}$ supported at $x \in X$. By Proposition 9.1.1.2, a $\hat{\mathfrak{g}}_\kappa$-module M corresponds to a chiral $A_{\mathfrak{g},\kappa}$-module at x, also denoted by M by abuse of notation. The BRST differential d induces a unique differential d_M on the chiral module $M \otimes \mathcal{C}l_{\mathfrak{n},x}$ compatible with the action of $(A_{\mathfrak{g},\kappa} \otimes \mathcal{C}l_{\mathfrak{n}}, d)$. The resulting complex $(M \otimes \mathcal{C}l_{\mathfrak{n},x}, d_M)$, called the semi-infinite complex of M with respect to $\mathfrak{n}(\mathcal{K})$, is independent of the choice of x and will be denoted by $\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), M)$. For an explicit algebraic construction of the complex $\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), M)$, see [19, Section 15.1].

In the remaining of this section, we will show that the semi-infinite complex $\mathfrak{C}^{\frac{\infty}{2}}(\mathfrak{n}(\mathcal{K}), M)$ of a $\hat{\mathfrak{g}}_\kappa$-module M admits a canonical structure of a (complex of) module over the Heisenberg algebra.

Let $L_{\mathfrak{b},\kappa}$ be the Lie-* subalgebra of $L_{\mathfrak{g},\kappa}$ which normalizes $L_{\mathfrak{n}}$. Then clearly $L_{\mathfrak{b},\kappa}$ is a central extension of $L_{\mathfrak{b}} = \mathfrak{b} \otimes D_X$ by ω_X, induced from the affine Kac-Moody extension. We have another extension of $L_{\mathfrak{b}}$. The adjoint action of $\mathfrak{b}$ on $\mathfrak{n}$ induces a map $L_{\mathfrak{b}} \to \mathfrak{gl}(L_{\mathfrak{n}})$, so the pull-back of Tate extension $\mathfrak{gl}(L_{\mathfrak{n}})^\flat$ gives an extension of $L_{\mathfrak{b}}$ by ω_X, denoted by $L_{\mathfrak{b}}^\flat$. By construction we have a map $L_{\mathfrak{n}}^\flat \hookrightarrow L_{\mathfrak{b}}^\flat$.

Let $L_{\mathfrak{b}}^\natural$ be the Baer sum of the extensions $L_{\mathfrak{b},\kappa}$ and $L_{\mathfrak{b}}^\flat$. Note that $1^\flat \in L_{\mathfrak{n}}^\flat$ is sent to $-1 \in L_{\mathfrak{b},\kappa}$ by α and to $1^\flat \in L_{\mathfrak{b}}^\flat$, so we have an embedding $s : L_{\mathfrak{n}} \hookrightarrow L_{\mathfrak{b}}^\natural$. We would like to define a map $\ell_{\mathfrak{b}}^{(0)}$ on $L_{\mathfrak{b}}^\natural$ which extends $\ell^{(0)}$; namely, it satisfies $\ell_{\mathfrak{b}}^{(0)}|_{L_{\mathfrak{n}}} \equiv \ell_{\mathfrak{b}}^{(0)} \circ s = \ell^{(0)}$. Indeed, we set $\ell_{\mathfrak{b}}^{(0)} := \alpha_{\mathfrak{b}} + \beta_{\mathfrak{b}}$, where

$$\alpha_{\mathfrak{b}} : L_{\mathfrak{b},\kappa} \hookrightarrow L_{\mathfrak{g},\kappa} \hookrightarrow A_{\mathfrak{g},\kappa} \otimes \mathcal{C}l_{\mathfrak{n}}^{(0)}$$

arises from the canonical embedding of $L_{\mathfrak{b},\kappa}$ as a subalgebra, and

$$\beta_{\mathfrak{b}} : L_{\mathfrak{b}}^\flat \to \mathfrak{gl}(L_{\mathfrak{n}})^\flat \hookrightarrow A_{\mathfrak{g},\kappa} \otimes \mathcal{C}l_{\mathfrak{n}}^{(0)}$$

is the natural map obtained from pulling back the Tate extension. Obviously this map satisfies $\ell_{\mathfrak{b}}^{(0)}|_{L_{\mathfrak{n}}} = \ell^{(0)}$. By the BRST property, $\ell_{\mathfrak{b}}^{(0)}|_{L_{\mathfrak{n}}} = \ell^{(0)} = d\ell^{(-1)}$, so the image of $L_{\mathfrak{n}}$ under $\ell_{\mathfrak{b}}^{(0)}$ lies in Im d. Moreover, we have the following lemma:

Lemma 9.2.2. *The image of* $\ell_{\mathfrak{b}}^{(0)}$ *is contained in* Ker d.

Proof. We again write $\mu \equiv \{\ ,\}^{\mathrm{ch}}_{A_{\mathfrak{g},\kappa}\otimes\mathcal{C}l_{\mathfrak{n}}}$ (resp. $\mu_{\mathcal{C}l}$) as the chiral bracket of $A_{\mathfrak{g},\kappa}\otimes\mathcal{C}l_{\mathfrak{n}}$ (resp. $\mathcal{C}l_{\mathfrak{n}}$), for simplicity of notations. Recall that $\{\ ,\}_{A_{\mathfrak{g},\kappa}\otimes\mathcal{C}l_{\mathfrak{n}}}$ is the corresponding Lie-* bracket of $A_{\mathfrak{g},\kappa}\otimes\mathcal{C}l_{\mathfrak{n}}$.

We need to show $\{\mathfrak{d},\ell_{\mathfrak{b}}^{(0)}\}_{A_{\mathfrak{g},\kappa}\otimes\mathcal{C}l_{\mathfrak{n}}}=0$. Since $d\ell_{\mathfrak{b}}^{(0)}|_L = d\ell^{(0)} = d^2\ell^{(-1)}=0$, it suffices to show $\{\mathfrak{d},\ell_{\mathfrak{b}}^{(0)}(h)\}_{A_{\mathfrak{g},\kappa}\otimes\mathcal{C}l_{\mathfrak{n}}}=0$ for $h\in L_{\mathfrak{b}}^{\natural}/(L_{\mathfrak{n}}\oplus\omega_X)$.

Fix the structure constants $c_\gamma^{\alpha,\beta}$ such that $[e_\alpha,e_\beta]=\sum_\gamma c_\gamma^{\alpha,\beta}e_\gamma$, where α,β and γ run over positive roots of $\mathfrak{g}$. We denote by $e_\alpha^*\in L_{\mathfrak{n}}^{\circ}$ the dual element to e_α. Then explicitly we have

$$\ell^{(0)}(e_\alpha)=e_\alpha\otimes 1+1\otimes\left(\sum_{\beta,\gamma}c_\gamma^{\alpha,\beta}\mu_{\mathcal{C}l}(e_\gamma,e_\beta^*)\right),$$
$$\delta(e_\alpha^*)=-\sum_{\sigma,\rho}c_\alpha^{\sigma,\rho}\mu_{\mathcal{C}l}(e_\sigma^*,e_\rho^*).$$

Now we consider $\chi(\mathrm{Id}_{L_{\mathfrak{n}}})$ where $\mathrm{Id}_{L_{\mathfrak{n}}}$ is written as $\sum_\alpha e_\alpha\otimes e_\alpha^*$ for α runs over positive roots of $\mathfrak{g}$.

$$\begin{aligned}\chi(\mathrm{Id}_{L_{\mathfrak{n}}}) &= \sum_\alpha\Big(\mu(\ell^{(0)}(e_\alpha),1\otimes e_\alpha^*)-\mu(\ell^{(-1)}(e_\alpha),1\otimes\delta(e_\alpha^*))\Big)\\
&=\sum_\alpha\Big(\mu\Big(e_\alpha\otimes 1+1\otimes\Big(\sum_{\beta,\gamma}c_\gamma^{\alpha,\beta}\mu_{\mathcal{C}l}(e_\gamma,e_\beta^*)\Big),1\otimes e_\alpha^*\Big)+\mu\Big(1\otimes e_\alpha,1\otimes\Big(\sum_{\sigma,\rho}c_\alpha^{\sigma,\rho}\mu_{\mathcal{C}l}(e_\sigma^*,e_\rho^*)\Big)\Big)\Big)\\
&=\sum_\alpha\Big(\mu(e_\alpha\otimes 1,1\otimes e_\alpha^*)+\sum_{\beta,\gamma}c_\gamma^{\alpha,\beta}1\otimes\mu_{\mathcal{C}l}(\mu_{\mathcal{C}l}(e_\gamma,e_\beta^*),e_\alpha^*)+\sum_{\sigma,\rho}c_\alpha^{\sigma,\rho}1\otimes\mu_{\mathcal{C}l}(e_\alpha,\mu_{\mathcal{C}l}(e_\sigma^*,e_\rho^*))\Big).\end{aligned}$$

The second and third terms of the last line above can be simplified using the Jacobi identity: (Here we omit the tensor factor $1\in A_{\mathfrak{g},\kappa}$ of $A_{\mathfrak{g},\kappa}\otimes\mathcal{C}l_{\mathfrak{n}}$.)

$$\begin{aligned}&\sum_{\beta,\gamma}c_\gamma^{\alpha,\beta}\mu_{\mathcal{C}l}(\mu_{\mathcal{C}l}(e_\gamma,e_\beta^*),e_\alpha^*)+\sum_{\sigma,\rho}c_\alpha^{\sigma,\rho}\mu_{\mathcal{C}l}(e_\alpha,\mu_{\mathcal{C}l}(e_\sigma^*,e_\rho^*))\\
&=\sum_{\beta,\gamma}c_\gamma^{\alpha,\beta}\Big(\mu_{\mathcal{C}l}(e_\gamma,\mu_{\mathcal{C}l}(e_\beta^*,e_\alpha^*))+\mu_{\mathcal{C}l}(e_\beta^*,\mu_{\mathcal{C}l}(e_\alpha^*,e_\gamma))\Big)-\sum_{\sigma,\rho}c_\alpha^{\rho,\sigma}\mu_{\mathcal{C}l}(e_\alpha,\mu_{\mathcal{C}l}(e_\sigma^*,e_\rho^*))\\
&=\sum_{\beta,\gamma}c_\gamma^{\alpha,\beta}\mu_{\mathcal{C}l}(e_\beta^*,\mu_{\mathcal{C}l}(e_\alpha^*,e_\gamma)).\end{aligned}$$

Finally, we write $\ell_{\mathfrak{b}}^{(0)}(h) = h \otimes 1 + 1 \otimes ad_h$, and then see that

$$\begin{aligned}
&\{\mathfrak{d}, \ell_{\mathfrak{b}}^{(0)}(h)\}_{A_{\mathfrak{g},\kappa}\otimes\mathcal{C}l_{\mathfrak{n}}} \\
=&\Big\{\sum_{\alpha} \mu(e_\alpha \otimes 1, 1 \otimes e_\alpha^*), \ell_{\mathfrak{b}}^{(0)}(h)\Big\}_{A_{\mathfrak{g},\kappa}\otimes\mathcal{C}l_{\mathfrak{n}}} + \Big\{\sum_{\alpha,\beta,\gamma} c_\gamma^{\alpha,\beta} 1 \otimes \mu_{\mathcal{C}l}(e_\beta^*, \mu_{\mathcal{C}l}(e_\alpha^*, e_\gamma)), \ell_{\mathfrak{b}}^{(0)}(h)\Big\}_{A_{\mathfrak{g},\kappa}\otimes\mathcal{C}l_{\mathfrak{n}}} \\
=&\sum_{\alpha,\beta,\gamma} c_\gamma^{\alpha,\beta}(\beta(h) + \alpha(h) - \gamma(h)) 1 \otimes \mu_{\mathcal{C}l}(e_\beta^*, \mu_{\mathcal{C}l}(e_\alpha^*, e_\gamma)) = 0,
\end{aligned}$$

as the structure coefficient $c_\gamma^{\alpha,\beta}$ is nonzero only when $\alpha + \beta = \gamma$. □

By the lemma we obtain a morphism of Lie-* algebras

$$L_{\mathfrak{t}}^{\natural} \cong L_{\mathfrak{b}}^{\natural}/L_{\mathfrak{n}} \to H^\bullet(A_{\mathfrak{g},\kappa} \otimes \mathcal{C}l_{\mathfrak{n}}, d).$$

The universal property of twisted enveloping chiral algebra then yields a morphism of chiral algebras

$$U^{\mathrm{ch}}(L_{\mathfrak{t}})^{\natural} \to H^\bullet(A_{\mathfrak{g},\kappa} \otimes \mathcal{C}l_{\mathfrak{n}}, d).$$

The Lie-* algebra $L_{\mathfrak{t}}^{\natural}$ is a central extension of $\mathfrak{t} \otimes D_X$ by ω_X, and the corresponding Lie algebra $H_{\mathrm{dR}}(\mathcal{D}_x^*, L_{\mathfrak{t}}^{\natural})$ is the Heisenberg algebra with the 2-cocycle given by the form $(\kappa - \kappa_{\mathrm{crit}})|_{\mathfrak{t}} \cdot (\ , \ )_{\mathrm{st}}$ (see [13]). We call $L_{\mathfrak{t}}^{\natural}$ the Heisenberg Lie-* algebra with a Tate shift, and denote the corresponding Lie algebra

$$H_{\mathrm{dR}}(\mathcal{D}_x^*, L_{\mathfrak{t}}^{\natural}) =: \hat{\mathfrak{t}}_{\kappa+\mathrm{shift}}. \tag{9.1}$$

9.3 Chiral differential operators

Recall the jet construction $J(\cdot)^l$ as the left adjoint functor of the forgetful functor from the category of D-algebras to the category of $\mathcal{O}$-algebras. Namely, for a D-algebra B^l and an $\mathcal{O}$-algebra C we have

$$\mathrm{Hom}_{\text{D-algebra}}(J(C)^l, B^l) = \mathrm{Hom}_{\mathcal{O}\text{-algebra}}(C, B^l).$$

Let $Z := X \times G$. Set $\Theta_Z := T_Z \otimes_{\mathcal{O}_Z} (J(\mathcal{O}_Z)^l \otimes_{\mathcal{O}_X} D_X)$, which becomes a Lie-* algebra when endowed with a canonical Lie-* bracket (c.f. [5, Section 2.4]). The Lie-* algebra Θ_Z is called the Lie-* algebra of vector fields on Z. Let $\mathfrak{g} := \mathrm{Lie}(G)$. The map $\mathfrak{g} \to T_G$ of left invariant vector fields induces a canonical Lie-* morphism $L_{\mathfrak{g}} \to \Theta_Z$.

By fixing a form Q on $\mathfrak{g}$ as in Example 9.1.1, we can construct a canonical chiral algebra of differential operators $\mathfrak{D}^{\mathrm{ch}}(G)_Q$ on $Z = X \times G$, by imitating the construction of the ring of differential operators from the structure sheaf and the tangent sheaf [5, Theorem 3.4]. Similar to the ring of differential operators case, there is a canonical filtration on $\mathfrak{D}^{\mathrm{ch}}(G)_Q = \bigcup_{i\geq 0}(\mathfrak{D}^{\mathrm{ch}}(G)_Q)_i$ with the following properties (see [5, Theorem 3.4 and Theorem 3.7]):

- $(\mathfrak{D}^{\mathrm{ch}}(G)_Q)_0 \cong J(Z)$ as chiral algebras.

- There is a morphism of Lie-* algberas $\Theta_Z \to (\mathfrak{D}^{\mathrm{ch}}(G)_Q)_1/(\mathfrak{D}^{\mathrm{ch}}(G)_Q)_0$, which induces an isomorphism $\mathrm{Sym}_{J(Z)^l}(\Theta_Z^l) \cong \mathrm{gr}(\mathfrak{D}^{\mathrm{ch}}(G)_Q^l)$ of D-algebras.

- (Left-invariant vector fields) There is an embedding of Lie-* algebras

$$\mathfrak{l} : L_{\mathfrak{g},Q} \to (\mathfrak{D}^{\mathrm{ch}}(G)_Q)_1,$$

  such that $\omega_X \subset L_{\mathfrak{g},Q}$ is sent identically to $\omega_X \subset J(Z) \cong (\mathfrak{D}^{\mathrm{ch}}(G)_Q)_0$ and the composition $L_{\mathfrak{g},Q} \to (\mathfrak{D}^{\mathrm{ch}}(G)_Q)_1/(\mathfrak{D}^{\mathrm{ch}}(G)_Q)_0 \cong \Theta_Z$ agrees with the canonical morphism $L_{\mathfrak{g},Q} \to L_{\mathfrak{g}} \to \Theta_Z$ that corresponds to left-invariant vector fields on G.

- (Right-invariant vector fields) Let Q' be the bilinear form $-(\cdot,\cdot)_{\mathrm{Killing}} - Q$. Let $L'_{\mathfrak{g},Q'}$ be the Baer sum of the two central extensions $L_{\mathfrak{g}}^{-\flat}$ (the Baer negative of the Tate extension) and $L_{\mathfrak{g},-Q}$. Then there is an embedding of Lie-* algberas

$$\mathfrak{r} : L'_{\mathfrak{g},Q'} \to (\mathfrak{D}^{\mathrm{ch}}(G)_Q)_1,$$

  such that $\omega_X \subset L'_{\mathfrak{g},Q'}$ is sent identically to $\omega_X \subset J(Z) \cong (\mathfrak{D}^{\mathrm{ch}}(G)_Q)_0$ and the composition $L'_{\mathfrak{g},Q'} \to (\mathfrak{D}^{\mathrm{ch}}(G)_Q)_1/(\mathfrak{D}^{\mathrm{ch}}(G)_Q)_0 \cong \Theta_Z$ agrees with the canonical morphism $L'_{\mathfrak{g},Q'} \to L_{\mathfrak{g}} \to \Theta_Z$ that corresponds to right-invariant vector fields on G.

- The morphisms $\mathfrak{l}$ and $\mathfrak{r}$ commute. I.e. the composition

$$L_{\mathfrak{g},Q} \boxtimes L'_{\mathfrak{g},Q'} \xrightarrow{\mathfrak{l}\boxtimes\mathfrak{r}} \mathfrak{D}^{\mathrm{ch}}(G)_Q \boxtimes \mathfrak{D}^{\mathrm{ch}}(G)_Q \xrightarrow{\{,\}} \Delta_! \mathfrak{D}^{\mathrm{ch}}(G)_Q$$

  is identically zero.

9.4 Factorization algebras and categories

The (algebro-geometric) notion of factorization algebras was introduced by Beilinson and Drinfeld in [9], for the purpose of introducing the chiral homology. As factorization algerbas and chiral algebras are equivalent objects, the presentation of factorization algebras allows one to define a categorified version of the structure, called the factorization categories. The theory of factorization categories (a.k.a. chiral categories) is developed in [44]. Here we will skip most technical details and only give an informal review of the subject.

We start with an intuitive definition of factorization algebras. Let X be a smooth algebraic curve. A factorization algebra Υ is defined as an assignment: to each subset $\{x_1, x_2, \dots, x_n\}$ of X, we associate a vector space $\Upsilon_{x_1,x_2,\dots,x_n}$ with isomorphism

$$\Upsilon_{x_1,x_2,\dots,x_n} \cong \Upsilon_{x_1} \otimes \Upsilon_{x_2} \otimes \dots \otimes \Upsilon_{x_n} \tag{9.2}$$

such that when the points move or collide the assignment and the isomorphism (9.2) change continuously. One can repeat the above word by word with vector spaces replaced by categories to get an intuitive definition of factorization categories.

In algebro-geometric terms, the intuitive definition for factorization algebras is translated to assigning a D-module Υ_I on X^I to each finite non-empty set I, together with the following isomorphisms:

- (Ran condition) For each surjection $I \twoheadrightarrow J$, an isomorphism

$$\Delta^!_{I/J}\Upsilon_I \xrightarrow{\sim} \Upsilon_J$$

  where $\Delta_{I/J} : X^J \hookrightarrow X^I$ is the induced diagonal embedding.

- (Factorization condition) For finite sets I and J, let $\iota : [X^I \times X^J]_{\text{disj}} \hookrightarrow X^{I \sqcup J}$ be the open locus where points in X^I are disjoint from points in X^J. The isomorphism is

$$\iota^!(\Upsilon_I \boxtimes \Upsilon_J) \xrightarrow{\sim} \iota^! \Upsilon_{I \sqcup J}.$$

These isomorphisms are required to satisfy a list of compatibility relations that we choose to omit. Finally, for $|I| \geq 2$ we require that Υ_I has no nonzero sections supported on diagonal divisors.

Denote the category of factorization algebras (on the curve X) by $\mathsf{Alg}^{\text{fact}}(X)$. It has a natural symmetric monoidal structure given by the usual tensor product.

Theorem 9.4.1 ([9] Theorem 3.4.9)**.** *The functor* $\mathsf{Alg}^{\text{fact}}(X) \to \mathsf{Alg}^{\text{chiral}}(X)$ *given by* $\Upsilon \mapsto (\Upsilon_{\{*\}})^r$ *is a monoidal equivalence.*

We can repackage the assignment $I \rightsquigarrow \Upsilon_I$ and the Ran condition as saying that Υ is a D-module on the *Ran space* $\text{Ran}(X)$. Explicitly, $\text{Ran}(X)$ is the prestack defined as

$$\text{Ran}(X) := \operatorname*{colim}_I X^I$$

where the colimit is taken along the embedding $X^J \hookrightarrow X^I$ for each surjection $I \twoheadrightarrow J$. Given a test scheme S, the S-point $\text{Ran}(X)(S) = \operatorname*{colim}_I X(S)^I$ is the collection of non-empty finite subsets of $X(S)$. Define the prestack of disjoint locus of $\text{Ran}(X) \times \text{Ran}(X)$ as

$$[\text{Ran}(X) \times \text{Ran}(X)]_{\text{disj}} := \operatorname*{colim}_{I,J} [X^I \times X^J]_{\text{disj}}.$$

We have the canonical inclusion map (m_1) and union map (m_2):

$$\text{Ran}(X) \times \text{Ran}(X) \xleftarrow{m_1} [\text{Ran}(X) \times \text{Ran}(X)]_{\text{disj}} \xrightarrow{m_2} \text{Ran}(X). \tag{9.3}$$

We refer to (9.3) as the chiral multiplicative structure of $\text{Ran}(X)$, which abides by associativity and commutativity constraints (in the sense of an object in a ∞-category) and can thus be seen as a commutative algebra object in a certain correspondence 2-category, c.f. [44, Section 5]. Now a factorization algebra $\mathcal{A}$ is equivalent to a D-module Υ on $\mathrm{Ran}(X)$ equipped with an isomorphism

$$m_1^!(\Upsilon \boxtimes \Upsilon) \xrightarrow{\sim} m_2^! \Upsilon \tag{9.4}$$

and the corresponding n-ary multiplication analogues. The intuitive definition given above now derscribes the behavior of stalks of Υ over finite subsets of X.

To categorify a sheaf of sets, one naturally considers a sheaf of categories. We thus define a factorization category $\mathsf{C}_{\mathrm{Ran}(X)}$ as a sheaf of D-mod(Ran(X))-module categories on Ran(X), together with an equivalence

$$m_1^!(\mathsf{C}_{\mathrm{Ran}(X)} \boxtimes \mathsf{C}_{\mathrm{Ran}(X)}) \xrightarrow{\sim} m_2^! \mathsf{C}_{\mathrm{Ran}(X)}. \tag{9.5}$$

and its n-ary multiplication analogues. Note that one should develop a theory of factorization categories where the categories are derived, as this is often the case for applications in geometric representation theory. However, in the setting of derived cateogries, to give a definition without mentioning the Ran space and its chiral multiplicative structure, one has to specify a long list of homotopically coherent compatibility conditions for Ran and factorization equivalences. Although this approach is not impossible and actually is realized in [44, Section 8], we choose to not discuss this approach here. We will nevertheless write C_I to denote the corresponding sheaf of categories on X^I and make use of the factorization equivalences

$$(\mathsf{C}_I \boxtimes \mathsf{C}_J)|_{[X^I \times X^J]_{\mathrm{disj}}} \xrightarrow{\sim} \mathsf{C}_{I \sqcup J}|_{[X^I \times X^J]_{\mathrm{disj}}}$$

for convenience.

The machinery of commutative algebras in a correspondence 2-category automatically produces the notion of functors between factorization categories; c.f. [44, Section 5.23]. Intuitively a factorization functor $F_{\mathrm{Ran}(X)} : \mathsf{C}_{\mathrm{Ran}(X)} \to \mathsf{D}_{\mathrm{Ran}(X)}$ should induce functors over finite subsets of X such that the following diagram commutes:

$$\begin{array}{ccc} \mathsf{C}_{x_1,\dots,x_n} & \xrightarrow{F_{x_1,\dots,x_n}} & \mathsf{D}_{x_1,\dots,x_n} \\ \downarrow \simeq & & \downarrow \simeq \\ \mathsf{C}_{x_1} \otimes \dots \otimes \mathsf{C}_{x_n} & \xrightarrow{F_{x_1} \otimes \dots \otimes F_{x_n}} & \mathsf{D}_{x_1} \otimes \dots \otimes \mathsf{D}_{x_n} \end{array}$$

Given a factorization category $\mathsf{C}_{\mathrm{Ran}(X)}$, one define a factorization algebra object Υ in $\mathsf{C}_{\mathrm{Ran}(X)}$ as a sheaf of objects on Ran(X) equipped with isomorphisms as in (9.4). It should not be surprising that a factorization algebra in $\mathsf{C}_{\mathrm{Ran}(X)}$ is sent to a factorization algebra in $\mathsf{D}_{\mathrm{Ran}(X)}$ under a factorization functor $F_{\mathrm{Ran}(X)} : \mathsf{C}_{\mathrm{Ran}(X)} \to \mathsf{D}_{\mathrm{Ran}(X)}$.

While Ran(X) is a commutative algebra object in the correspondence 2-category, one can define the notion of a Ran(X)-module object in the same category. Then a sheaf of categories on a module object with action equivalences similar in form to (9.5) will be called a factorization module category. Below we give a concrete account of a Ran(X)-module object of particular interest.

Fix a (non-empty) finite set I. Consider the category fSet_I whose objects are arbitrary maps $I \to J$ for finite set J, and morphisms are commutative diagrams

$$\begin{array}{ccc} & I & \\ \swarrow & & \searrow \\ J & \twoheadrightarrow & J' \end{array}$$

The I-marked Ran space $\mathrm{Ran}_I(X)$ is defined as the prestack

$$\mathrm{Ran}_I(X) := \underset{(I\to J)\in \mathsf{fSet}_I^{op}}{\mathrm{colim}} X^J,$$

which becomes a Ran(X)-module in the correspondence 2-category via the following diagram

$$\begin{array}{ccc} & [\mathrm{Ran}(X) \times \mathrm{Ran}_I(X)]_{\mathrm{disj}} & \\ {\scriptstyle act_1} \swarrow & & \searrow {\scriptstyle act_2} \\ \mathrm{Ran}(X) \times \mathrm{Ran}_I(X) & & \mathrm{Ran}_I(X) \end{array}$$

Here the map act_2 sends $(K, (I \to J))$ to $(I \to J \sqcup K)$ for I, J, K finite subsets of X, whereas the map act_1 is obvious.

With the above setup, given a factorization category $\mathsf{C}_{\mathrm{Ran}(X)}$, we define a factorization module category $\mathsf{M}_{\mathrm{Ran}_I(X)}$ on X^I (for $\mathsf{C}_{\mathrm{Ran}(X)}$) as a sheaf of D-mod($\mathrm{Ran}_I(X)$)-module categories, together with an equivalence

$$act_1^!(\mathsf{C}_{\mathrm{Ran}(X)} \boxtimes \mathsf{M}_{\mathrm{Ran}_I(X)}) \xrightarrow{\sim} act_2^! \, \mathsf{M}_{\mathrm{Ran}_I(X)}$$

and its n-ary action analogues.

Example 9.4.1. There is a canonical map $\mathrm{Ran}_I(X) \to X^I$. Given a factorization category $\mathsf{C}_{\mathrm{Ran}(X)}$, the corresponding sheaf of categories C_I on X^I can be regarded as a factorization $\mathsf{C}_{\mathrm{Ran}(X)}$-module category on $\mathrm{Ran}_I(X)$ by pulling back along the canonical map. We abuse the terminology and say that C_I is a factorization module category on X^I.

Again, the machinery of Ran(X)-module objects immediately gives the notion of a functor between factorization module categories. In particular, a functor from a $\mathsf{C}_{\mathrm{Ran}(X)}$-module $\mathsf{M}_{\mathrm{Ran}_I(X)}$ to a $\mathsf{D}_{\mathrm{Ran}(X)}$-module $\mathsf{N}_{\mathrm{Ran}_I(X)}$ specifies a factorization functor $\mathsf{C}_{\mathrm{Ran}(X)} \to \mathsf{D}_{\mathrm{Ran}(X)}$ and a functor $\mathsf{M}_{\mathrm{Ran}_I(X)} \to \mathsf{N}_{\mathrm{Ran}_I(X)}$ compatible with the actions. For convenience we write

$$F : (\mathsf{C}_{\mathrm{Ran}(X)}, \mathsf{M}_{\mathrm{Ran}_I(X)}) \to (\mathsf{D}_{\mathrm{Ran}(X)}, \mathsf{N}_{\mathrm{Ran}_I(X)})$$

with $F^{(1)}$ and $F^{(2)}$ functors for the first and second component, respectively.

One categorical level down, given a factorization algerba Υ in $\mathsf{C}_{\mathrm{Ran}(X)}$, we define a factorization Υ-module in $\mathsf{M}_{\mathrm{Ran}_I(X)}$ as an object $\Xi \in \mathsf{M}_{\mathrm{Ran}_I(X)}$ with isomorphisms

$$act_1^!(\Upsilon \boxtimes \Xi) \xrightarrow{\sim} act_2^! \, \Xi \tag{9.6}$$

and its n-ary action analogues. Let $\Upsilon\text{-mod}^{\text{fact}}(\mathsf{M}_{\mathrm{Ran}_I(X)})$ denote the category of factorization Υ-modules in $\mathsf{M}_{\mathrm{Ran}_I(X)}$. Consider a functor $F : (\mathsf{C}_{\mathrm{Ran}(X)}, \mathsf{M}_{\mathrm{Ran}_I(X)}) \to (\mathsf{D}_{\mathrm{Ran}(X)}, \mathsf{N}_{\mathrm{Ran}_I(X)})$ and a factorization algebra $\Upsilon \in \mathsf{C}_{\mathrm{Ran}(X)}$. We get a factorization algebra $F^{(1)}(\Upsilon) \in \mathsf{D}_{\mathrm{Ran}(X)}$, and F induces a canonical functor

$$\Upsilon\text{-mod}^{\text{fact}}(\mathsf{M}_{\mathrm{Ran}_I(X)}) \longrightarrow F^{(1)}(\Upsilon)\text{-mod}^{\text{fact}}(\mathsf{N}_{\mathrm{Ran}_I(X)}).$$

Let $\mathsf{C}_{\mathrm{Ran}(X)}$ be a factorization category and $\Upsilon \in \mathsf{C}_{\mathrm{Ran}(X)}$ a factorization algebra. As in Example 9.4.1 C_I is a factorization module category on X^I. In the special case of $I = \{*\}$, we denote the category of factorization Υ-modules on X by $\Upsilon\text{-mod}^{\text{fact}}(X)$. In view of Theorem 9.4.1, let $A := (\Upsilon_{\{*\}})^r$ be the chiral algebra corresponding to Υ. The following theorem identifies factorization modules and chiral modules:

Theorem 9.4.2 ([9] Proposition 3.4.19)**.** *There is a canonical equivalence of sheaves of categories*

$$\Upsilon\text{-mod}^{\text{fact}}(X) \xrightarrow{\sim} A\text{-mod}^{\text{ch}}(X).$$

Finally, we discuss the *external fusion* of factorization modules. We consider the factorization module category $\mathsf{C}_{I,J,\text{disj}}$ defined by restricting $\mathsf{C}_{I \sqcup J}$ to the locus $[X^I \times X^J]_{\text{disj}}$. Then the external fusion is a canonical functor

$$\Upsilon\text{-mod}^{\text{fact}}(\mathsf{C}_I) \otimes \Upsilon\text{-mod}^{\text{fact}}(\mathsf{C}_J) \to \Upsilon\text{-mod}^{\text{fact}}(\mathsf{C}_{I,J,\text{disj}}).$$

For detailed constructions, see [44, Section 6.22-26]

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