Unstable Homotopy Theory Surrounding the Fibre of the p^{th} Power Map on Loop Spaces of Spheres

by

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Abstract

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In this thesis, we study the fibre of the p^{th} power map on loop spaces of spheres with a view toward obtaining homotopy decompositions. For the prime $p = 2$, we give an explicit decomposition of the fibre $\Omega^3 S^{17}\{2\}$ of the squaring map on the triple loop space of the 17-sphere, building on work of Campbell, Cohen, Peterson and Selick, who showed that such decompositions are only possible for S^5, S^9 and S^{17}. This induces a splitting of the mod 2 homotopy groups $\pi_*(S^{17}; \mathbb{Z}/2\mathbb{Z})$ in terms of the integral homotopy groups of the fibre of the double suspension $E^2\colon S^{2n-1} \to \Omega^2 S^{2n+1}$ and refines a result of Cohen and Selick, who gave similar decompositions for S^5 and S^9. For odd primes, we find that the decomposition problem for $\Omega S^{2n+1}\{p\}$ is equivalent to the strong odd primary Kervaire invariant problem. Using this unstable homotopy theoretic interpretation of the Kervaire invariant problem together with what is presently known about the 3-primary stable homotopy groups of spheres, we give a new decomposition of $\Omega S^{55}\{3\}$.

We relate the 2-primary decompositions above to various Whitehead products in the homotopy groups of mod 2 Moore spaces and Stiefel manifolds to show that the Whitehead square $[i_{2n}, i_{2n}]$ of the inclusion of the bottom cell of the Moore space $P^{2n+1}(2)$ is divisible by 2 if and only if $2n = 2, 4, 8$ or 16. As an application of our 3-primary decomposition, we prove two new cases of a longstanding conjecture which states that the fibre of the double suspension is a double loop space.

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Part I

Introduction and Mathematical Review

Chapter 1

Introduction and Summary of Results

1.1 Introduction

A basic goal of algebraic topology is to compute algebraic invariants of topological spaces and to obtain classifications of spaces up to homotopy equivalence. The method of loop space decompositions, in which the based loop space ΩX is decomposed up to homotopy into a product of simpler indecomposable factors, has played an important role in the development of both theoretical and computational aspects of the subject. Many of the most important results in algebraic topology (Adams' solution to the Hopf invariant one problem, the Kahn–Priddy theorem, Bott periodicity, etc.) have been phrased in terms of the existence (or nonexistence) of certain product decompositions in the homotopy category.

The computation of the homotopy groups of spheres remains a central problem, and while stem-by-stem calculations of individual groups have become slow and labour-intensive, one feature of loop space decompositions is that they often allow one to glean qualitative information on the global structure and properties of all homotopy groups of a space at once. An early example is Serre's classical result stating that, localized away from the prime 2, there is a homotopy equivalence

$$\Omega S^{2n} \simeq S^{2n-1} \times \Omega S^{4n-1}.$$

As nontrivial consequences, one sees immediately that (1) the odd primary components of the homotopy groups of even dimensional spheres are determined by those of odd dimensional spheres, and (2) localized at an odd prime p, S^{2n-1} has the structure of an H-space. Without localizing away from 2, Adams' solution to the Hopf invariant one problem says that ΩS^{2n} decomposes as above if and only if $n = 1, 2$ or 4.

Product decompositions of loop spaces have been particularly useful in attacking exponent problems. A space X is said to have *homotopy exponent* p^k at the prime p if p^k annihilates the p-torsion in the graded group $\pi_*(X)$. Since there is no non-contractible finite simply connected CW complex all of whose homotopy groups are known, the existence of finite simply connected complexes with homotopy exponents is not at all obvious, but a longstanding conjecture due to Moore suggests the following deep relationship with rational homotopy theory. Recall that a simply connected space X is called rationally *elliptic* if $\dim(\pi_*(X) \otimes \mathbb{Q}) < \infty$ and *hyperbolic* if $\pi_n(X) \otimes \mathbb{Q} \neq 0$ for infinitely many n.

Moore's Conjecture. *Let X be a finite simply connected CW complex.*

(a) *If X is elliptic, then X has a homotopy exponent at every prime p;*

(b) *If X is hyperbolic, then X does not have a homotopy exponent at any prime p.*

The existence of homotopy exponents for spheres was established by James [25] (for $p = 2$) and Toda [53] (for p odd) who used the James–Hopf invariant maps to prove that p^{2n} annihilates the p-torsion in $\pi_*(S^{2n+1})$, but computational evidence led Barratt to conjecture that S^{2n+1} has exponent p^n at odd primes p. The first case of this conjecture was proved by Selick [41, 42] as an immediate consequence of the following loop space decomposition. Let p be an odd prime and let $S^{2n+1}\{p\}$ denote the homotopy fibre of the degree p map on S^{2n+1}. Then there is a p-local homotopy equivalence

$$\Omega^2 S^{2p+1}\{p\} \simeq \Omega^2 S^3\langle 3\rangle \times W_p$$

where $S^3\langle 3\rangle$ is the 3-connected cover of S^3 and W_n denotes the homotopy fibre of the double suspension $E^2\colon S^{2n-1} \to \Omega^2 S^{2n+1}$. This shows that the p-primary components of the groups $\pi_k(S^3)$ for $k > 3$ are summands of the homotopy groups of $S^{2p+1}\{p\}$. Since $\Omega^2 S^{2p+1}\{p\}$ is homotopy equivalent to the mapping space $\mathrm{Map}_*(P^3(p), S^{2p+1})$ and the mod p Moore space $P^3(p)$ has suspension order p, it follows that S^3 has homotopy exponent p.

Next came the seminal work of Cohen–Moore–Neisendorfer [13, 14, 34] on the homotopy theory of odd primary Moore spaces. Let F be the homotopy fibre of the pinch map $q\colon P^{2n+1}(p) \to S^{2n+1}$ which collapses the bottom cell of the Moore space $P^{2n+1}(p) = S^{2n} \cup_p e^{2n+1}$ to a point. Cohen, Moore and

Neisendorfer investigated the homotopy fibration sequence

$$\Omega^2 S^{2n+1} \xrightarrow{\partial} \Omega F \longrightarrow \Omega P^{2n+1}(p) \xrightarrow{\Omega q} \Omega S^{2n+1}$$

and obtained explicit product decompositions of the middle two loop spaces. In particular, after localizing at the odd prime p, they found that ΩF decomposes with S^{2n-1} appearing as the factor containing the bottom cell. By composing the connecting map ∂ with the projection onto this bottom factor, they obtained a map $\varphi\colon \Omega^2 S^{2n+1} \to S^{2n-1}$ with the property that the composition with the double suspension

$$\Omega^2 S^{2n+1} \xrightarrow{\varphi} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$$

is homotopic to the p^{th} power map on $\Omega^2 S^{2n+1}$. It follows by induction on n that p^n annihilates the p-torsion in $\pi_*(S^{2n+1})$. By a result of Gray [17], if p is an odd prime, then $\pi_*(S^{2n+1})$ contains infinitely many elements of order p^n, so this is the best possible odd primary homotopy exponent for spheres.

Given that spheres have exponents, McGibbon–Wilkerson [29] proved that part (a) of Moore's conjecture above is true for almost all primes by showing that if X is a finite elliptic complex, then, for all but finitely many primes p, ΩX decomposes p-locally as a product of spheres and loop spaces of spheres.

Since spheres and single loop spaces of spheres have torsion-free homology, a wider class of spaces is required for decomposing loop spaces at primes where torsion appears. (Note that a finite simply connected complex may have p-torsion in its loop space homology for infinitely many primes p.) For example, in their study of odd primary Moore spaces $P^n(p^r)$, Cohen, Moore and Neisendorfer introduced certain p-torsion H-spaces $T_0^{2n+1}(p^r)$ with homotopy exponent p^{r+1} and proved that there are homotopy equivalences

$$\Omega P^{2n+1}(p^r) \simeq T_0^{2n+1}(p^r) \times \Omega\left(\bigvee_\alpha P^{n_\alpha}(p^r)\right)$$

$$\Omega P^{2n+2}(p^r) \simeq S^{2n+1}\{p^r\} \times \Omega\left(\bigvee_{\alpha'} P^{n_{\alpha'}}(p^r)\right)$$

where in each case the second factor is the loop space of an infinite bouquet of mod p^r Moore spaces. These loop space decompositions were used to show that $\Omega P^n(p^r)$ has the homotopy type of a weak infinite product of spaces of the form $T_0^{2m+1}(p^r)$ and $S^{2m+1}\{p^r\}$ and hence that $P^n(p^r)$ has homotopy exponent p^{r+1} for p odd and $n \geq 3$. Further examples by many other authors have shown that the spaces $S^{2n+1}\{p\}$ often occur as factors in product decompositions of loop spaces. In fact, a conjecture of Anick states that for every finite simply connected complex X, the loop space ΩX has the p-local homotopy type of a product of spaces of the form S^{2n+1}, ΩS^{2n+1}, $T_0^{2n+1}(p^r)$ and $S^{2n+1}\{p^r\}$ at almost all primes p.

Since S^{2n-1} is in general not an H-space localized at the prime 2, it cannot be a retract of the loop space of the homotopy fibre of the pinch map $P^{2n+1}(2) \to S^{2n+1}$ and an analogue of the map φ does not exist at the prime 2. The corresponding decomposition problems for mod 2 Moore spaces are much more complicated and the best possible 2-primary homotopy exponents of spheres are still unknown. Although the methods of Cohen, Moore and Neisendorfer break down in this case, the 2-primary analogue of Selick's decomposition, namely a 2-local homotopy equivalence

$$\Omega^2 S^5\{2\} \simeq \Omega^2 S^3\langle 3\rangle \times W_2,$$

was obtained by Cohen in [10]. (Here $\Omega^2 S^5\{2\}$ denotes the homotopy fibre of the squaring map on $\Omega^2 S^5$ rather than the double loop space of the fibre of the degree 2 map. For odd primes, the corresponding spaces are homotopy equivalent.) This gives a "geometric" proof of James' classical result that 4 annihilates the 2-torsion in $\pi_*(S^3)$, and this is known to be best possible. This raises the question of whether 2-primary exponent problems can be approached by recognizing other spheres or related spaces as factors in loop space decompositions of spaces which *a priori* have homotopy exponents.

Motivated by this and other applications of loop space decompositions, this thesis focuses on the decomposition problem for the homotopy fibre of the p^{th} power map on loop spaces of spheres and its relations to other classical problems in homotopy theory concerning Moore spaces, Stiefel manifolds, the double suspension map and the Kervaire invariant problem.

1.2 Summary of results

The main results of this thesis tie some loose ends in the work of Campbell, Cohen, Peterson and Selick in [9], [10], [16], [41], [42] and [43], so we briefly review their work below (see Section 5.2 for a more thorough review).

In [43], Ravenel's solution to the odd primary Arf–Kervaire invariant problem [38] was used to show that for $p \geq 5$ and $n > 1$, a nontrivial decomposition of $\Omega^2 S^{2n+1}\{p\}$ is not possible if $n \neq p$. So in this sense, Selick's decomposition of $\Omega^2 S^{2p+1}\{p\}$ and the induced splitting of mod p homotopy groups

$$\pi_k(S^{2p+1}; \mathbb{Z}/p\mathbb{Z}) \cong \pi_{k-1}(S^3) \oplus \pi_{k-3}(W_p)$$

express a special property of the $(2p+1)$-sphere which is unique among p-local spheres for these primes. The situation for $p=2$ or 3 is more interesting and an investigation of the decomposition problem for the fibre of the p^{th} power map on loop spaces of spheres for these two primes makes up the main chapters (Chapters 7 and 8) of this thesis.

We first consider the $p=2$ case. Localize all spaces and maps at the prime 2. Unlike the $p \geq 5$ case, for reasons related to the divisibility of the Whitehead square $[\iota_{2n-1}, \iota_{2n-1}] \in \pi_{4n-3}(S^{2n-1})$, the fibre of the squaring map on $\Omega^2 S^{2n+1}$ admits nontrivial product decompositions for some values of n other than $n=2$. First, in their investigation of the homology of spaces of maps from mod 2 Moore spaces to spheres, Campbell, Cohen, Peterson and Selick [9] found that if $2n+1 \neq 3, 5, 9$ or 17, then $\Omega^2 S^{2n+1}\{2\}$ is atomic and hence indecomposable up to homotopy. Following this, it was shown in [16] that after localizing at the prime 2 there is a homotopy decomposition

$$\Omega^2 S^9\{2\} \simeq BW_2 \times W_4,$$

and a by-product of the construction of this equivalence was that W_4 is a retract of $\Omega^3 S^{17}\{2\}$. Here BW_n denotes the classifying space of the fibre of the double suspension first constructed by Gray. Since BW_1 is homotopy equivalent to $\Omega^2 S^3\langle 3\rangle$ by work going back to Toda, the pattern suggested by the decompositions of $\Omega^2 S^5\{2\}$ and $\Omega^2 S^9\{2\}$ led Cohen and Selick to conjecture that $\Omega^2 S^{17}\{2\} \simeq BW_4 \times W_8$. Our first main result proves this is true after looping once more. (This weaker statement was also conjectured in [12].)

Theorem 1.1 (to be proven as Theorem 7.2)**.** *There is a 2-local homotopy equivalence*

$$\Omega^3 S^{17}\{2\} \simeq W_4 \times \Omega W_8.$$

This gives the expected splitting of the homotopy groups of S^{17} with coefficients in $\mathbb{Z}/2\mathbb{Z}$ (defined by $\pi_k(S^{17}; \mathbb{Z}/2\mathbb{Z}) := [P^k(2), S^{17}] = [\Sigma^{k-2}\mathbb{R}P^2, S^{17}]$) in terms of the integral homotopy groups of W_4 and W_8 predicted by Cohen and Selick's conjecture.

Corollary 1.2. $\pi_k(S^{17}; \mathbb{Z}/2\mathbb{Z}) \cong \pi_{k-4}(W_4) \oplus \pi_{k-3}(W_8)$ *for all* $k \geq 4$*.*

In Chapter 7 we relate the problem of decomposing $\Omega^2 S^{2n+1}\{2\}$ to several problems concerning the homotopy theory of mod 2 Moore spaces and Stiefel manifolds. Motivated by an application to the classification of knotted tori, Mukai and Skopenkov in [33] considered the problem of determining when the Whitehead square $[i_n, i_n] \in \pi_{2n-1}(P^n(2))$ of the inclusion of the bottom cell $S^n \to P^{n+1}(2)$ is divisible by 2. They conjectured that this is only possible when n or $n+1$ is a power of 2, and their main theorem proves this in case n is even. The indecomposability result for $\Omega^2 S^{2n+1}\{2\}$ in [9] mentioned above was proved by showing that for $n > 1$ the existence of a certain spherical homology class imposed by a nontrivial product decomposition implies the existence of an element $\theta \in \pi^S_{2n-2}$ of Kervaire invariant one such that $\theta\eta$ is divisible by 2, where η is the generator of the stable 1-stem π^S_1. Since such elements are well known to exist only for $2n = 4, 8$ or 16, these are the only dimensions for which $\Omega^2 S^{2n+1}\{2\}$ can decompose nontrivially. Improving on Mukai and Skopenkov's result, we show that the divisibility of the Whitehead square $[i_{2n}, i_{2n}]$ similarly implies the existence of such Kervaire invariant elements to obtain the following.

Theorem 1.3 (To be proven as Theorem 7.11)**.** *The Whitehead square* $[i_{2n}, i_{2n}] \in \pi_{4n-1}(P^{2n+1}(2))$ *is divisible by* 2 *if and only if* $2n = 2, 4, 8$ *or* 16.

The proof equates the divisibility of $[i_{2n}, i_{2n}]$ with the vanishing of a certain Whitehead product in the mod 2 homotopy of the real Stiefel manifold $V_{2n+1,2}$ (i.e., the unit tangent bundle of S^{2n}). This latter Whitehead product was considered in [44] as an obstruction to the existence of a decomposition $V_{2n,3} \simeq S^{2n-1} \times V_{2n-1,2}$ and was shown to be nontrivial for $2n > 16$. When $2n = 2, 4$ or 8, the Whitehead product vanishes for reasons related to Hopf invariant one, leaving only the boundary case $2n = 16$ unresolved. Theorem 7.11 shows that the Whitehead product is also trivial in this case. This may be relevant in settling an outstanding case of a conjecture of James on the indecomposability of Stiefel manifolds. Further calculations of Whitehead products have led to the following partial result.

Theorem 1.4 (To be proven as Theorem 7.16)**.** *The* 41*-skeletons of the* 42*-dimensional manifolds* $V_{16,3}$ *and* $S^{15} \times V_{15,2}$ *are homotopy equivalent.*

This contrasts with the Stiefel manifolds of 3-frames $V_{m,3}$ for $m > 16$ and suggests that $V_{16,3}$ is homotopically more similar to the spaces of 3-frames in $\mathbb{R}^4$ and $\mathbb{R}^8$, where there are diffeomorphisms $V_{4,3} \cong S^3 \times V_{3,2}$ and $V_{8,3} \cong S^7 \times V_{7,2}$. See Section 7.4 for more information on this problem.

A different point of contact between Stiefel manifolds and the spaces $\Omega^2 S^{2n+1}\{2\}$ leads to the following loop space decomposition which identifies $V_{5,2}$ with the 2-connected cover of $J_3(S^2)$, the third stage of the James construction on the 2-sphere.

Theorem 1.5 (To be proven as Theorem 7.19)**.** *There is a fibration*

$$V_{5,2} \longrightarrow J_3(S^2) \longrightarrow K(\mathbb{Z}, 2)$$

which is split after looping. In particular, $\pi_k(J_3(S^2)) \cong \pi_k(V_{5,2})$ *for all* $k \geq 3$.

In Chapter 8 we turn our attention to the $p = 3$ case of the decomposition problem for $\Omega^2 S^{2n+1}\{p\}$, which was left unsettled in the work of Campbell, Cohen, Peterson and Selick. Let p be an odd prime and localize all spaces and maps at p. As mentioned above, for $p \geq 5$ and $n > 1$, it was shown in [43] that $\Omega^2 S^{2n+1}\{p\}$ is indecomposable if $n \neq p$. This result was obtained by showing that a nontrivial homotopy decomposition of $\Omega^2 S^{2n+1}\{p\}$ implies the existence of a p-primary Kervaire invariant one element of order p in $\pi^S_{2n(p-1)-2}$. We will prove that the converse of this last implication is also true and that the (strong) odd primary Kervaire invariant problem is in fact equivalent to the problem of decomposing the single loop space $\Omega S^{2n+1}\{p\}$. When $p = 3$, this equivalence can be used to import results from stable homotopy theory to obtain new results concerning the unstable homotopy type of $\Omega S^{2n+1}\{3\}$ as well as some new cases of a longstanding conjecture stating that W_n is a double loop space.

Cohen, Moore and Neisendorfer conjectured in [15] that there should exist a space $T^{2n+1}(p)$ and a fibration sequence

$$\Omega^2 S^{2n+1} \xrightarrow{\varphi} S^{2n-1} \longrightarrow T^{2n+1}(p) \longrightarrow \Omega S^{2n+1}$$

in which their map φ occurs as the connecting map. The existence of such a fibration was first proved by Anick for $p \geq 5$ as the culmination of the 270 page book [4]. A much simpler construction, valid for all odd primes, was later given by Gray and Theriault in [22], in which they also show that Anick's space $T^{2n+1}(p)$ has the structure of an H-space and that all maps in the fibration above can be chosen to be H-maps.

A well-known conjecture in unstable homotopy theory states that the fibre W_n of the double suspension $E^2\colon S^{2n-1} \to \Omega^2 S^{2n+1}$ is a double loop space. Anick's space represents a potential candidate for a double classifying space of W_n, and one of Cohen, Moore and Neisendorfer's remaining open conjectures in [15] states that there should be a p-local homotopy equivalence $W_n \simeq \Omega^2 T^{2np+1}(p)$. Since Gray's construction of the classifying space BW_n shows that W_n has the homotopy type of at least a single loop space, a stronger form of the conjecture appearing in many places throughout the literature states that

$$BW_n \simeq \Omega T^{2np+1}(p).$$

Such an equivalence would provide a long-sought link between the EHP sequence and the work of Cohen, Moore and Neisendorfer on the homotopy theory of Moore spaces, and furthermore, would have implications for the differentials in the EHP spectral sequence calculating the homotopy groups of spheres. To date, equivalences $W_n \simeq \Omega^2 T^{2np+1}(p)$ are only known to exist for $n = 1$ and $n = p$. In the former case, both BW_1 and $\Omega T^{2p+1}(p)$ are well known to be homotopy equivalent to $\Omega^2 S^3\langle 3\rangle$. The latter case was proved in the strong form $BW_p \simeq \Omega T^{2p^2+1}(p)$ by Theriault [49] using in an essential way a delooping of Selick's decomposition of $\Omega^2 S^{2p+1}\{p\}$. Generalizing this argument, we prove the following in Section 8.3.

Theorem 1.6 (To be proven as Theorem 8.6)**.** *Let p be an odd prime. Then the following are equivalent:*

(a) *There exists a p-primary Kervaire invariant one element* $\theta_j \in \pi^S_{2p^j(p-1)-2}$ *of order p;*

(b) *There is a homotopy decomposition of H-spaces* $\Omega S^{2p^j+1}\{p\} \simeq T^{2p^j+1}(p) \times \Omega T^{2p^{j+1}+1}(p)$.

Moreover, if the above conditions hold, then there are homotopy equivalences

$$BW_{p^{j-1}} \simeq \Omega T^{2p^j+1}(p) \quad \text{and} \quad BW_{p^j} \simeq \Omega T^{2p^{j+1}+1}(p).$$

From this point of view, Selick's decomposition of $\Omega S^{2p+1}\{p\}$ and the previously known equivalences $BW_1 \simeq \Omega T^{2p+1}(p)$ and $BW_p \simeq \Omega T^{2p^2+1}(p)$ correspond to the existence (at all odd primes) of the Kervaire invariant one element $\theta_1 \in \pi^S_{2p^2-2p-2}$. By Ravenel's negative solution to the Kervaire invariant problem for primes $p \geq 5$, the theorem above has new content only at the prime $p = 3$. For example, in addition to the Kervaire invariant one element in $\pi^S_{2n(p-1)-2}$ for $n = p = 3$ corresponding to the decomposition of $\Omega S^7\{3\}$, it is known that there exists a 3-primary Kervaire invariant one element $\theta_3 \in \pi^S_{106}$ which we use to obtain the following decomposition of $\Omega S^{55}\{3\}$ and prove the $n = p^2$ and $n = p^3$ cases of the $BW_n \simeq \Omega T^{2np+1}(p)$ conjecture at $p = 3$.

Corollary 1.7. *There are* 3*-local homotopy equivalences*

(a) $\Omega S^{55}\{3\} \simeq T^{55}(3) \times \Omega T^{163}(3)$

(b) $BW_9 \simeq \Omega T^{55}(3)$

(c) $BW_{27} \simeq \Omega T^{163}(3)$.

Other applications in Section 8.3 include indecomposability results for $\Omega S^{2n+1}\{3\}$ for various n and results on the associativity and homotopy exponents of 3-primary Anick spaces.

Chapter 2

Homotopy Fibrations and Cofibrations

This chapter sets some notation and quickly reviews standard material on homotopy fibrations and cofibrations which will constantly be used without reference in later chapters. We work in the category of pointed topological spaces of the homotopy type of CW complexes and the corresponding homotopy category.

2.1 Homotopy (co)fibrations and the Barratt–Puppe sequence

We assume all maps are pointed and denote the path space of X (equipped with the compact-open topology) by $PX = \{\gamma\colon I \to X \mid \gamma(0) = x_0\}$ where x_0 is the basepoint of X.

Definition 2.1. A sequence of maps $Z \to X \to Y$ is a *homotopy fibration* if there is a homotopy commutative diagram

$$\begin{array}{ccccc} Z & \longrightarrow & X & \longrightarrow & Y \\ \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} \\ F & \longrightarrow & E & \longrightarrow & B \end{array}$$

in which the vertical maps are homotopy equivalences and $F \to E \to B$ is a fibration. A sequence of maps

$$\cdots \longrightarrow X_4 \longrightarrow X_3 \longrightarrow X_2 \longrightarrow X_1$$

is a *homotopy fibration sequence* if each pair of consecutive maps is a homotopy fibration.

Definition 2.2. The *homotopy fibre* of a (pointed) map $f\colon X \to Y$ is defined by

$$F_f = \{(x, \gamma) \in X \times PY \mid \gamma(1) = f(x)\},$$

or equivalently, the pullback

$$\begin{array}{ccc} F_f & \longrightarrow & PY \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where the right vertical map is the path-loop fibration defined by $\gamma \mapsto \gamma(1)$.

Lemma 2.3 ([3])**.** *Let $f\colon X \to Y$ be a map between connected spaces. Then the homotopy type of F_f depends only on the homotopy class of f and makes $F_f \longrightarrow X \xrightarrow{f} Y$ a homotopy fibration.*

Let $q_1\colon F_f \to X$ denote the canonical projection of F_f onto X. By iterating the homotopy fibre construction, we therefore obtain a homotopy fibration sequence

$$\cdots \to F_{q_2} \xrightarrow{q_3} F_{q_1} \xrightarrow{q_2} F_f \xrightarrow{q_1} X \xrightarrow{f} Y. \tag{2.1}$$

The following lemma identifies each homotopy fibre in the sequence above in terms of X, Y and the homotopy fibre of f.

Lemma 2.4 ([3])**.** *With the notation above, the following hold:*

(a) *There are homotopy equivalences $j_1\colon \Omega Y \to F_{q_1}$ and $j_2\colon \Omega X \to F_{q_2}$ making the diagram*

$$\begin{array}{ccc} \Omega X & \xrightarrow{\Omega f} & \Omega Y \\ \simeq \downarrow j_2 & & \simeq \downarrow j_1 \\ F_{q_2} & \xrightarrow{q_3} & F_{q_1} \end{array}$$

commute up to homotopy;

(b) *There are homeomorphisms $F_{\Omega^k f} \cong \Omega^k F_f$ for each $k \geq 1$.*

Using Lemma 2.4, we may replace (2.1) by

$$\cdots \longrightarrow \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \longrightarrow \Omega F_f \longrightarrow \Omega X \xrightarrow{\Omega f} \Omega Y \longrightarrow F_f \longrightarrow X \xrightarrow{f} Y.$$

This homotopy fibration sequence is called the *Barratt–Puppe sequence* of the map $f\colon X \to Y$. It induces for any (pointed) space W a long exact sequence

$$\cdots \longrightarrow [W, \Omega^k F_f] \longrightarrow [W, \Omega^k X] \xrightarrow{(\Omega^k f)_*} [W, \Omega^k Y] \longrightarrow [W, \Omega^{k-1} F_f] \longrightarrow \cdots$$
$$\cdots \longrightarrow [W, F_f] \longrightarrow [W, X] \xrightarrow{f_*} [W, Y]$$

of groups and pointed sets.

There are Eckmann–Hilton dual versions of each of the definitions and lemmas above. For example, a *homotopy cofibration* is defined by the obvious analogue of Definition 2.1 and the *homotopy cofibre* of a map $f\colon X \to Y$ is given by the mapping cone

$$C_f = Y \cup_f CX$$

obtained by gluing the bottom of the (reduced) cone over X, $CX = X \times I / X \times \{1\} \cup \{x_0\} \times I$, to Y according to the equivalence relation $(x, 0) \sim f(x)$. Dual to the based loop space ΩX is the (reduced) suspension ΣX, and the *dual Barratt–Puppe sequence* of a map $f\colon X \to Y$ is a homotopy cofibration sequence of the form

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X \xrightarrow{\Sigma f} \Sigma Y \longrightarrow \Sigma C_f \longrightarrow \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \longrightarrow \cdots$$

which induces a long exact sequence of groups and pointed sets upon applying the functor $[-, W]$ for any (pointed) space W.

2.2 Fibre extensions of squares

In this section we describe a basic technique in homotopy theory which relates the Barratt–Puppe sequences of each of the maps in a homotopy commutative square by embedding the square in a larger homotopy commutative diagram of homotopy fibrations. These homotopy fibration diagrams were used to great effect in the work of Cohen, Moore and Neisendorfer and are referred to by Anick as *CMN diagrams* in [4].

Lemma 2.5. *Any homotopy commutative square*

$$\begin{array}{ccc} A_{22} & \xrightarrow{b_{21}} & A_{21} \\ \big\downarrow{\scriptstyle a_{12}} & & \big\downarrow{\scriptstyle a_{11}} \\ A_{12} & \xrightarrow{b_{11}} & A_{11} \end{array}$$

can be embedded in a homotopy commutative diagram

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & A_{33} & \xrightarrow{b_{32}} & A_{32} & \xrightarrow{b_{31}} & A_{31} \\
 & & \downarrow{\scriptstyle a_{23}} & & \downarrow{\scriptstyle a_{22}} & & \downarrow{\scriptstyle a_{21}} \\
\cdots & \longrightarrow & A_{23} & \xrightarrow{b_{22}} & A_{22} & \xrightarrow{b_{21}} & A_{21} \\
 & & \downarrow{\scriptstyle a_{13}} & & \downarrow{\scriptstyle a_{12}} & & \downarrow{\scriptstyle a_{11}} \\
\cdots & \longrightarrow & A_{13} & \xrightarrow{b_{12}} & A_{12} & \xrightarrow{b_{11}} & A_{11}
\end{array}$$

in which each row and each column is a homotopy fibration and there are homotopy equivalences $A_{i,j+3} \simeq \Omega A_{ij} \simeq A_{i+3,j}$ *and homotopies* $a_{i,j+3} \simeq \Omega a_{ij} \simeq a_{i+3,j}$ *and* $b_{i,j+3} \simeq \Omega b_{ij} \simeq b_{i+3,j}$*.*

Proof. See [45, Theorem 7.6.2] and [45, Theorem 7.6.3]. □

Remark 2.6. The homotopy commutative diagram in Lemma 2.5 is not uniquely determined up to homotopy by the initial homotopy commutative square unless the initial square strictly commutes or we specify a choice of homotopy between $a_{11} \circ b_{21}$ and $b_{11} \circ a_{12}$.

A special case of the above construction occurs when the initial homotopy commutative square is a homotopy pullback. Given a pair of maps $f\colon X \to Z$ and $g\colon Y \to Z$, the *homotopy pullback* of f and g is a homotopy limit which can be defined by the specific construction

$$X \times^h_Z Y = \{(x, \gamma, y) \in X \times Z^I \times Y \mid \gamma(0) = f(x),\ \gamma(1) = g(y)\}$$

but is more conveniently characterized by the equivalent conditions in the following lemma.

Lemma 2.7. *Let*

$$\begin{array}{ccc}
W & \xrightarrow{g'} & X \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
Y & \xrightarrow{g} & Z
\end{array}$$

be a homotopy commutative square where X*,* Y *and* Z *are connected. Then the following are equivalent:*

(a) *There is an induced map of homotopy fibres* $F_{f'} \to F_f$ *which is a homotopy equivalence;*

(b) *There is an induced map of homotopy fibres* $F_{g'} \to F_g$ *which is a homotopy equivalence.*

Proof. See [45, Proposition 7.6.1]. □

Note that for a homotopy commutative square satisfying the equivalent conditions above, a homotopy fibration diagram produced by Lemma 2.5 can be chosen to be of the form

$$
\begin{array}{ccccc}
 & & F_f & = & F_f \\
 & & \downarrow & & \downarrow \\
F_g & \longrightarrow & W & \xrightarrow{g'} & X \\
\| & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
F_g & \longrightarrow & Y & \xrightarrow{g} & Z
\end{array}
$$

where we have identified the homotopy fibres of f' and g' with those of f and g, respectively, and '=' indicates a homotopy equivalence. Diagrams of this form will be useful for identifying the homotopy fibres of various maps and will be referred to as *homotopy pullback diagrams*.

2.3 Homotopy decompositions

Most of the homotopy decompositions in later chapters are obtained by showing that certain homotopy fibrations $F \to E \to B$ are trivial in the sense of being fibre homotopy equivalent to the product fibration $F \times B \to B$. In this section we list some conditions under which the total space of a homotopy fibration decomposes as a product of the base and fibre.

Definition 2.8. A *right homotopy inverse* (or *homotopy section*) of a map $f\colon X \to Y$ is a map $s\colon Y \to X$ such that $f \circ s \simeq 1_Y$. A *left homotopy inverse* (or *homotopy retraction*) of $f\colon X \to Y$ is a map $r\colon Y \to X$ such that $r \circ f \simeq 1_X$.

Consider a homotopy fibration $Z \xrightarrow{i} X \xrightarrow{f} Y$. As with a short exact sequence of groups, the existence of a right homotopy inverse of f is necessary but not sufficient for there to be a homotopy equivalence $X \simeq Y \times Z$. There are two important cases, however, when a right homotopy inverse of f does imply a decomposition.

The first occurs when the homotopy fibration is the pullback along some map $Y \to B$ of the path-loop fibration over B, that is, when $X \xrightarrow{f} Y$ is the homotopy fibre of some map $Y \xrightarrow{g} B$. In this case, $Z \xrightarrow{i} X \xrightarrow{f} Y$ is a *principal fibration*, and there is a homotopy action $a\colon Z \times X \to X$ of the loop space $Z \simeq \Omega B$ on $X \simeq F_g$ given by $\alpha \cdot (y, \gamma) = (y, \alpha\gamma)$, where $\alpha\gamma$ denotes concatenation of the loop $\alpha \in \Omega B$ with the path $\gamma \in PB$. Given a right homotopy inverse s of f, this action can be used to construct a map $a \circ (1_Z \times s)\colon Z \times Y \to X$. It is easy to check that this map defines a morphism of homotopy fibrations

$$
\begin{array}{ccccc}
Z & = & = & = & Z \\
\downarrow & & & & \downarrow{\scriptstyle i} \\
Z \times Y & \xrightarrow{1_Z \times s} & Z \times X & \xrightarrow{a} & X \\
\downarrow & & & & \downarrow{\scriptstyle f} \\
Y & = & = & = & Y
\end{array}
$$

and is therefore a homotopy equivalence by the five lemma.

The second case occurs when the total space of the homotopy fibration $Z \xrightarrow{i} X \xrightarrow{f} Y$ is an H-space. Suppose that X is an H-space with multiplication $\mu\colon X \times X \to X$ and that f admits a right homotopy inverse s. Then we can multiply i and s to obtain a map

$$Z \times Y \xrightarrow{i \times s} X \times X \xrightarrow{\mu} X. \tag{2.2}$$

To see that this is a weak homotopy equivalence (and hence a homotopy equivalence, given our standing assumption that X, Y and Z have the homotopy types of CW complexes), note that after looping, the composite

$$a\colon \Omega Z \times \Omega X \xrightarrow{\Omega i \times 1_{\Omega X}} \Omega X \times \Omega X \xrightarrow{\Omega \mu} \Omega X$$

defines a homotopy action of ΩZ on ΩX and the previous argument applied to the principal fibration $\Omega Z \xrightarrow{\Omega i} \Omega X \xrightarrow{\Omega f} \Omega Y$ shows that the composite $a \circ (1_{\Omega Z} \times \Omega s)$ is a homotopy equivalence. But this last composite is precisely the loops on (2.2), so it follows that $\mu \circ (i \times s)$ is also a homotopy equivalence.

We summarize the above discussion in the following lemma.

Lemma 2.9. *Let $Z \xrightarrow{i} X \xrightarrow{f} Y$ be a homotopy fibration. Then the following hold:*

(a) *If i admits a left homotopy inverse, then $X \simeq Y \times Z$;*

(b) *If $Z \xrightarrow{i} X \xrightarrow{f} Y$ is a principal fibration and f admits a right homotopy inverse, then $X \simeq Y \times Z$;*

(c) *If X is an H-space and f admits a right homotopy inverse, then $X \simeq Y \times Z$.*

Chapter 3

James–Hopf Invariants and the EHP Sequence

In this chapter we first review the James construction, which gives a combinatorial model JX of the loop suspension $\Omega\Sigma X$ of any space X, and then describe how it leads to the James–Hopf invariant maps and the EHP sequence.

3.1 The James construction and the Bott–Samelson Theorem

For a space X with basepoint $*$, the James construction on X (defined precisely below) is the free (associative) topological monoid generated by the points of X with the baseboint $*$ as the unit. The relation between JX and the loop suspension $\Omega\Sigma X$ is hinted at by the classical Bott–Samelson Theorem, which, under mild assumptions, describes the Pontrjagin ring structure on the homology of $\Omega\Sigma X$ as the free associative algebra generated by the reduced homology of X.

For any space X, define the *suspension map* $E\colon X \to \Omega\Sigma X$ to be the adjoint of the identity map $1_{\Sigma X}\colon \Sigma X \to \Sigma X$. Given a commutative ring R and a graded R-module V, let $T(V)$ denote the tensor algebra on V.

Theorem 3.1 (Bott–Samelson [7]). *Let R be a principal ideal domain and let X be a connected space such that $H_*(X;R)$ is torsion-free. Then there is an isomorphism of algebras*

$$H_*(\Omega\Sigma X;R) \cong T(\widetilde{H}_*(X;R))$$

and $E\colon X \to \Omega\Sigma X$ *induces the canonical inclusion of generators* $\widetilde{H}_*(X;R) \to T(\widetilde{H}_*(X;R))$.

The *James construction* on X is defined as the direct limit

$$JX = \varinjlim_k J_k X$$

where $J_k X = X^k/\sim$ with $(x_1, \ldots, x_{i-1}, *, x_{i+1}, \ldots, x_k) \sim (x_1, \ldots, x_{i-1}, x_{i+1}, *, \ldots, x_k)$ for each $k \geq 0$. The *James filtration* $\{J_k X\}_{k\geq 0}$ is the filtration of JX given by word length. The fundamental properties of JX are as follows.

Theorem 3.2 ([24])**.** *Let X be a connected space with the homotopy type of a CW complex.*

(a) *There is a functorial homotopy equivalence*

$$JX \simeq \Omega\Sigma X$$

where the inclusion $X = J_1 X \to JX$ of words of length one corresponds to the suspension map $E\colon X \to \Omega\Sigma X$.

(b) *There is a functorial splitting*

$$\Sigma JX \simeq \Sigma\Omega\Sigma X \simeq \bigvee_{i=1}^{\infty} \Sigma X^{(i)}$$

where $X^{(i)}$ denotes the i-fold smash product $X \wedge \cdots \wedge X$.

Example 3.3. For $X = S^n$, there is an algebra isomorphism $H_*(\Omega S^{n+1}) \cong T(x_n)$, where $|x_n| = n$, and a homotopy equivalence $\Sigma\Omega S^{n+1} \simeq \bigvee_{i=1}^{\infty} S^{ni+1}$.

3.2 James–Hopf invariant maps and the EHP fibrations

For each $k \geq 1$, projection onto the k^{th} wedge summand in the James splitting (Theorem 3.2(b)) defines a map

$$\Sigma\Omega\Sigma X \simeq \bigvee_{i=1}^{\infty} \Sigma X^{(i)} \longrightarrow \Sigma X^{(k)}$$

which induces an epimorphism in homology. The adjoint of this map,

$$H_k\colon \Omega\Sigma X \longrightarrow \Omega\Sigma X^{(k)},$$

is called the k^{th} *James–Hopf invariant map.*

For $k = 2$ and $X = S^n$, $H_2\colon \Omega S^{n+1} \to \Omega S^{2n+1}$ induces a map $T(x_n) \to T(y_{2n})$ which is an epimorphism in degree $2n$. James identified the homotopy fibre of any such map in [25].

Theorem 3.4 (James [25])**.** *There is a 2-local homotopy fibration*

$$S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H_2} \Omega S^{2n+1}. \tag{3.1}$$

The resulting long exact homotopy sequence for the above fibration is called the (2-local) *EHP sequence*. For odd primes p, replacing the even dimensional sphere S^{2n} by the $(p-1)^{\text{st}}$ stage of the James construction $J_{p-1}(S^{2n})$, Toda gave the following odd primary analogues of James' fibration.

Theorem 3.5 (Toda [53])**.** *Let p be an odd prime. Then there are p-local homotopy fibrations*

$$\begin{gathered} J_{p-1}(S^{2n}) \xrightarrow{E} \Omega S^{2n+1} \xrightarrow{H_p} \Omega S^{2np+1} \\ S^{2n-1} \xrightarrow{E} \Omega J_{p-1}(S^{2n}) \xrightarrow{T} \Omega S^{2np-1}. \end{gathered} \tag{3.2}$$

In the first homotopy fibration, E is the inclusion $J_{p-1}(S^{2n}) \to J(S^{2n}) \simeq \Omega S^{2n+1}$. In the second, E is the inclusion of the bottom cell and T is an analogue of the p^{th} James–Hopf invariant called the Toda–Hopf invariant.

The composite $X \xrightarrow{E} \Omega\Sigma X \xrightarrow{H_2} \Omega\Sigma(X \wedge X)$ is in general not a homotopy fibration, but the following result shows that all spaces admit an EHP sequence in the metastable range.

Theorem 3.6 ([56, Theorem 12.2.2])**.** *Let W be an $(n-1)$-connected space. Then there is a long exact sequence*

$$\begin{aligned} &\pi_{3n-2}(W) \longrightarrow \pi_{3n-1}(\Sigma W) \longrightarrow \pi_{3n-1}(\Sigma W \wedge W) \longrightarrow \cdots \\ &\qquad \cdots \longrightarrow \pi_k(W) \xrightarrow{E} \pi_{k+1}(\Sigma W) \xrightarrow{H} \pi_{k+1}(\Sigma W \wedge W) \xrightarrow{P} \pi_{k-1}(W) \longrightarrow \cdots. \end{aligned}$$

Chapter 4

Primary Homotopy Theory

This chapter provides a very brief introduction to primary (or mod p) homotopy theory. In particular, we define and review the main properties of homotopy groups with coefficients in $\mathbb{Z}/p\mathbb{Z}$. A good reference for the material in this chapter is [37], where much more detail can be found.

4.1 Homotopy groups with coefficients

The homotopy groups of a space X with coefficients in an abelian group G, denoted $\pi_*(X;G)$, provide a generalization of the ordinary homotopy groups $\pi_*(X) = \pi_*(X;\mathbb{Z})$ where many of the relations between $\pi_*(X)$ and $H_*(X;\mathbb{Z})$ are easily extended to relations between $\pi_*(X;G)$ and $H_*(X;G)$. In this section we focus on the case $G = \mathbb{Z}/k\mathbb{Z}$. See [37] for a treatment of the general theory of homotopy groups with coefficients.

For $n \geq 2$, the n dimensional mod k *Moore space* $P^n(k)$ is defined as the homotopy cofibre of the degree k map on S^{n-1}. Thus $P^n(k)$ is the two-cell complex $S^{n-1} \cup_k e^n$ with exactly one nonzero reduced integral cohomology group $\widetilde{H}^n(P^n(k);\mathbb{Z}) = \mathbb{Z}/k\mathbb{Z}$.

Definition 4.1. For a pointed space X, the n^{th} homotopy group of X with coefficients in $\mathbb{Z}/k\mathbb{Z}$ is

$$\pi_n(X;\mathbb{Z}/k\mathbb{Z}) = [P^n(k), X].$$

By convention, we set $\pi_1(X;\mathbb{Z}/k\mathbb{Z}) = \pi_1(X) \otimes \mathbb{Z}/k\mathbb{Z}$ when $\pi_1(X)$ is abelian. Note that since $P^n(k)$ is a suspension for $k \geq 3$ and a double suspension for $k \geq 4$, $\pi_n(X;\mathbb{Z}/k\mathbb{Z})$ has the structure of a group for $n \geq 3$ and an abelian group for $n \geq 4$.

We will usually denote the generators of the mod k homology groups

$$\widetilde{H}_i(P^n(k);\mathbb{Z}/k\mathbb{Z}) = \begin{cases} \mathbb{Z}/k\mathbb{Z} & \text{for } i = n-1, n \\ 0 & \text{otherwise} \end{cases}$$

by u_{n-1} and v_n where subscripts denote degree. There is a natural *mod k Hurewicz homomorphism*

$$h\colon \pi_n(X;\mathbb{Z}/k\mathbb{Z}) \longrightarrow H_n(X;\mathbb{Z}/k\mathbb{Z}),$$

defined by $[f] \mapsto f_*(v_n)$, and a corresponding mod k Hurewicz Theorem (see [37, Theorem 1.8.1]).

Applying the functor $[-, X]$ to the homotopy cofibration sequence

$$\cdots \longrightarrow S^{n-1} \xrightarrow{k} S^{n-1} \longrightarrow P^n(k) \longrightarrow S^n \xrightarrow{k} S^n \longrightarrow \cdots$$

yields the long exact sequence

$$\cdots \longrightarrow \pi_n(X) \xrightarrow{k} \pi_n(X) \longrightarrow \pi_n(X;\mathbb{Z}/k\mathbb{Z}) \longrightarrow \pi_{n-1}(X) \xrightarrow{k} \pi_{n-1}(X) \longrightarrow \cdots .$$

This immediately implies the following result which shows that mod k homotopy groups are related to integral homotopy groups in the same way that mod k homology groups are related to integral homology groups.

Theorem 4.2 (Universal coefficient exact sequence)**.** *For a pointed space X and $n \geq 2$, there is a natural exact sequence*

$$0 \longrightarrow \pi_n(X) \otimes \mathbb{Z}/k\mathbb{Z} \longrightarrow \pi_n(X;\mathbb{Z}/k\mathbb{Z}) \longrightarrow \mathrm{Tor}(\pi_{n-1}(X), \mathbb{Z}/k\mathbb{Z}) \longrightarrow 0.$$

We will mainly be interested in mod p Moore spaces and mod p homotopy groups where p is a prime. The following results will be used in Chapter 5 to establish homotopy exponents for the homotopy fibre of the p^{th} power map on loop spaces of spheres.

Lemma 4.3. *If p is an odd prime, then the degree p map on $P^n(p)$ is nullhomotopic for $n \geq 4$ and hence p annihilates $\pi_n(X;\mathbb{Z}/p\mathbb{Z})$ for all spaces X.*

Proof. See [37, 1.4]. □

By contrast, the degree 2 map on $P^n(2)$ can be shown to be essential for all $n \geq 3$, but the following well-known factorization of the degree 2 map implies that the degree 4 map on $P^n(2)$ is nullhomotopic.

Lemma 4.4. *If $n \geq 3$, then the degree 2 map $\underline{2}\colon P^n(2) \to P^n(2)$ factors as the composite*

$$P^n(2) \xrightarrow{q} S^n \xrightarrow{\eta} S^{n-1} \xrightarrow{i} P^n(2)$$

where q is the pinch map onto the top cell, η is the Hopf invariant one map and i is the inclusion of the bottom cell.

Proof. See [37, 4.8]. □

Corollary 4.5. *The degree 4 map on $P^n(2)$ is nullhomotopic for $n \geq 3$ and hence 4 annihilates $\pi_n(X;\mathbb{Z}/2\mathbb{Z})$ for all spaces X.*

4.2 Whitehead products

Recall that an *H-group* is a homotopy associative H-space with a homotopy inverse, the prototypical example being a loop space. Let G be an H-group and let $[\ ,\ ]\colon G \times G \to G$ be the commutator map $[x, y] = xyx^{-1}y^{-1}$. Given maps $f\colon X \to G$ and $g\colon Y \to G$, observe that the composite

$$X \times Y \xrightarrow{f\times g} G \times G \xrightarrow{[\ ,\ ]} G$$

is nullhomotopic on the wedge $X \vee Y$ and therefore factors up to homotopy through an extension $\langle f, g\rangle\colon X \wedge Y \to G$ called the *Samelson product* of f and g. Since the last map in the cofibration sequence

$$X \vee Y \longrightarrow X \times Y \longrightarrow X \wedge Y \longrightarrow \Sigma X \vee \Sigma Y \longrightarrow \Sigma(X \times Y)$$

has a left homotopy inverse, it follows that this extension is unique up to homotopy.

Now let X, Y and Z be any spaces. Given maps $f\colon \Sigma X \to Z$ and $g\colon \Sigma Y \to Z$, the *Whitehead product* $[f, g]\colon \Sigma(X \wedge Y) \to Z$ of f and g is defined to be the adjoint of the Samelson product $\langle \tilde{f}, \tilde{g}\rangle\colon X \wedge Y \to \Omega Z$ of the adjoints of f and g. The Whitehead product $[\ ,\ ]\colon [\Sigma X, Z] \times [\Sigma Y, Z] \to [\Sigma(X \wedge Y), Z]$ is natural with respect to maps $f'\colon X' \to X$, $g'\colon Y' \to Y$ and $h\colon Z \to Z'$ in the sense that

$$[f \circ \Sigma f', g \circ \Sigma g'] = [f, g] \circ \Sigma(f' \wedge g') \qquad \text{and} \qquad [h \circ f, h \circ g] = h \circ [f, g].$$

Just as the Whitehead product gives $\pi_*(X)$ the structure of a graded Lie algebra, the Whitehead product together with certain smash product splittings $P^m(p) \wedge P^n(p) \simeq P^{m+n}(p) \vee P^{m+n-1}(p)$ can be used to construct maps $\pi_m(X;\mathbb{Z}/p\mathbb{Z})\otimes\pi_n(X;\mathbb{Z}/p\mathbb{Z}) \to \pi_{m+n-1}(X;\mathbb{Z}/p\mathbb{Z})$ which satisfy the Lie identities for $p \geq 5$, but we won't need this as the only Whitehead products involving Moore spaces that we use in later chapters are of the form $\pi_m(X) \otimes \pi_n(X;\mathbb{Z}/2\mathbb{Z}) \to \pi_{m+n-1}(X;\mathbb{Z}/2\mathbb{Z})$.

We will make heavy use of the following basic fact, which we record for future reference.

Lemma 4.6. *The suspension of a Whitehead product is trivial.*

Part II

The Fibre of the p^{th} Power Map on Loop Spaces of Spheres

Chapter 5

Properties of $\Omega S^n\{p\}$

5.1 Basic properties

For a based loop space ΩX, let $\Omega X\{k\}$ denote the homotopy fibre of the k^{th} power map $k\colon \Omega X \to \Omega X$. Since the mod k Moore space $P^{m+1}(k)$ is the homotopy cofibre of the degree k self-map $\underline{k}\colon S^m \to S^m$, by applying the contravariant functor $\mathrm{Map}_*(-, X)$ to the homotopy cofibration

$$S^m \xrightarrow{k} S^m \longrightarrow P^{m+1}(k),$$

we obtain a homotopy fibration

$$\mathrm{Map}_*(P^{m+1}(k), X) \longrightarrow \Omega^m X \xrightarrow{k} \Omega^m X$$

which identifies the mapping space $\mathrm{Map}_*(P^{m+1}(k), X)$ with the homotopy fibre $\Omega^m X\{k\}$ of the k^{th} power map on the loop space $\Omega^m X$. This will be a useful point of view for studying the homotopy type of $\Omega S^n\{k\}$ and its loop spaces. For example, there is a homotopy equivalence

$$\Omega^2 S^n\{k\} \simeq \mathrm{Map}_*(P^3(k), S^n)$$

and since the mod k Moore space splits as

$$P^3(k) \simeq \bigvee_{i=1}^{s} P^3(p_i^{j_i})$$

where $k = p_1^{j_1} \cdots p_s^{j_s}$ is the prime factorization of k, we have

$$\begin{aligned}\Omega^2 S^n\{k\} &\simeq \prod_{i=1}^{s} \mathrm{Map}_*(P^3(p_i^{j_i}), S^n) \\ &\simeq \prod_{i=1}^{s} \Omega^2 S^n\{p_i^{j_i}\}.\end{aligned}$$

Hence the decomposition problem for $\Omega^2 S^n\{k\}$ is reduced to the case where k is a prime power. Moreover, for p odd (or for $p = 2$ and $n \in \{1, 2, 4\}$), the p-local homotopy equivalences $\Omega S^{2n} \simeq S^{2n-1} \times \Omega S^{4n-1}$ can be used to further refine the product decomposition above since

$$\begin{aligned}\Omega^2 S^{2n}\{p^r\} &\simeq \mathrm{Map}_*(P^3(p^r), S^{2n}) \\ &\simeq \mathrm{Map}_*(P^2(p^r), \Omega S^{2n}) \\ &\simeq \mathrm{Map}_*(P^2(p^r), S^{2n-1}) \times \mathrm{Map}_*(P^2(p^r), \Omega S^{4n-1}) \\ &\simeq \Omega S^{2n-1}\{p^r\} \times \Omega^2 S^{4n-1}\{p^r\}.\end{aligned}$$

Our primary interest will be in studying the homotopy fibre of the p^{th} power map on loop spaces of odd dimensional spheres, and an overarching theme of this thesis is the question of whether the spaces $\Omega S^{2n+1}\{p\}$ admit any further nontrivial product decompositions in the homotopy category (perhaps after looping).

Viewing these spaces as spaces of maps from Moore spaces to spheres also makes apparent their homotopy exponents. Note that the integral homotopy groups of $\Omega S^{2n+1}\{p\}$ are precisely the mod p homotopy groups of S^{2n+1}.

Lemma 5.1. *Let $n > 1$. The following hold:*

(a) *For $p = 2$, $\Omega S^{2n+1}\{2\}$ has homotopy exponent 4;*

(b) *For p odd, $\Omega S^{2n+1}\{p\}$ has homotopy exponent p.*

Proof. This follows immediately from Lemma 4.3 and Lemma 4.4. In fact, since the degree k map $\underline{k}\colon P^4(p) \to P^4(p)$ induces the k^{th} power map on $\mathrm{Map}_*(P^4(p), S^{2n+1}) \simeq \Omega^3 S^{2n+1}\{p\}$ and the identity map on $P^4(p)$ has order p (respectively, 4) for p odd (respectively, $p = 2$), it follows that $\Omega^3 S^{2n+1}\{p\}$ has nullhomotopic p^{th} (respectively, 4^{th}) power map. $\square$

After localizing at an odd prime p, every odd dimensional sphere S^{2n+1} is an H-space and it is easy to see that the p^{th} power map $p\colon S^{2n+1} \to S^{2n+1}$ (obtained by multiplying the identity map with itself

p times using the H-space structure on S^{2n+1}) coincides with the degree p map $\underline{p}\colon S^{2n+1} \to S^{2n+1}$ (obtained by adding the identity map to itself p times using the co-H-space structure on S^{2n+1}). We therefore denote the homotopy fibres of both by $S^{2n+1}\{p\}$. Moreover, since the loop map Ωp is homotopic to the p^{th} power map on ΩS^{2n+1}, there is no ambiguity in the notation $\Omega S^{2n+1}\{p\}$ as $\Omega(S^{2n+1}\{p\}) \simeq (\Omega S^{2n+1})\{p\}$.

For $p = 2$, the map $\Omega\underline{2}$ is in general *not* homotopic to the H-space squaring map on ΩS^{2n+1}. In fact, it is an open question whether the degree 2 map $\underline{2}\colon S^{2n+1} \to S^{2n+1}$ induces multiplication by 2 on $\pi_k(S^{2n+1})$ for all k. The homotopy fibre of $\underline{2}$ will be denoted by $S^{2n+1}\{\underline{2}\}$, and $\Omega S^{2n+1}\{2\}$ will always denote the homotopy fibre $(\Omega S^{2n+1})\{2\}$ of the squaring map.

For any prime p, consider the mod p homology Serre spectral sequence associated to the principal fibration

$$\Omega S^{2n+1} \longrightarrow S^{2n+1}\{\underline{p}\} \longrightarrow S^{2n+1}.$$

Since there is a morphism of homotopy fibrations

$$\begin{array}{ccc}
\Omega S^{2n+1} & = & \Omega S^{2n+1} \\
\downarrow & & \downarrow \\
S^{2n+1}\{\underline{p}\} & \longrightarrow & PS^{2n+1} \\
\downarrow & & \downarrow \\
S^{2n+1} & \xrightarrow{\underline{p}} & S^{2n+1}
\end{array}$$

and $\underline{p}$ induces the trivial homomorphism in mod p homology, it follows by naturality that the spectral sequence collapses at E_2 giving an isomorphism of coalgebras (and of modules over the Pontrjagin ring $H_*(\Omega S^{2n+1})$)

$$\begin{aligned}
H_*(S^{2n+1}\{\underline{p}\}) &\cong H_*(\Omega S^{2n+1}) \otimes H_*(S^{2n+1}) \\
&\cong \mathbb{Z}/p\mathbb{Z}[u_{2n}] \otimes \Lambda(v_{2n+1})
\end{aligned}$$

where the inclusion of the bottom two cells $P^{2n+1}(p) \hookrightarrow S^{2n+1}\{\underline{p}\}$ implies that the Bockstein differential is given by $\beta v_{2n+1} = u_{2n}$.

More generally, the Serre spectral sequence associated to the homotopy fibration

$$\Omega^{m+1} S^{2n+1} \longrightarrow \Omega^m S^{2n+1}\{p\} \longrightarrow \Omega^m S^{2n+1}$$

can be shown to collapse to obtain the following result.

Lemma 5.2. *If $1 \leq m \leq 2n-1$, then there is an isomorphism of coalgebras*

$$H_*(\Omega^m S^{2n+1}\{p\}) \cong H_*(\Omega^{m+1} S^{2n+1}) \otimes H_*(\Omega^m S^{2n+1}).$$

Moreover, if $2 \leq m \leq 2n-2$, then this is an isomorphism of Hopf algebras.

Proof. See [11, Lemma 15.1]. □

5.2 Review of the work of Campbell, Cohen, Peterson and Selick

This section provides a brief review of the various homotopy decompositions and indecomposability results for $\Omega^2 S^{2n+1}\{p\}$ from [9], [10], [16], [41], [42] and [43].

The first exponent results for spheres were obtained by James (for $p = 2$) and Toda (for p odd) who used the James–Hopf invariants and EHP fibrations (3.1) and (3.2) to show that p^{2n} annihilates the p-torsion in $\pi_*(S^{2n+1})$. In the case of the 3-sphere, James' upper bound on the 2-primary exponent is in fact the best possible: $\pi_*(S^3)$ contains elements of order 4 ($\pi_6(S^3) = \mathbb{Z}/12\mathbb{Z}$, for instance) but no elements of order 8. The best possible odd primary exponent for the 3-sphere was obtained as a consequence of the following decomposition due to Selick, which was also proved using the James–Hopf invariant $H_p\colon \Omega S^3 \to \Omega S^{2p+1}$.

Theorem 5.3 ([41], [42])**.** *Let p be an odd prime. Then there is a p-local homotopy equivalence*

$$\Omega^2 S^{2p+1}\{p\} \simeq \Omega^2 S^3\langle 3\rangle \times W_p$$

where $S^3\langle 3\rangle$ is the 3-connected cover of S^3 and W_p is the homotopy fibre of the double suspension $E^2\colon S^{2p-1} \to \Omega^2 S^{2p+1}$.

This shows that, for every odd prime p, the p-primary components of the groups $\pi_k(S^3)$ for $k > 3$ are summands of the homotopy groups of $S^{2p+1}\{p\}$, so as a corollary Selick obtained that S^3 has homotopy exponent p, proving a conjecture of Barratt.

This result is generalized in the celebrated exponent theorem of Cohen, Moore and Neisendorfer which states that S^{2n+1} has homotopy exponent p^n at odd primes p. The method of proof here is very different, but again loop space decompositions play a fundamental role. By decomposing into indecomposable factors the loop spaces of the mod p Moore space $P^{2n+1}(p)$ and the fibre of the pinch map $P^{2n+1}(p) \xrightarrow{q} S^{2n+1}$, they constructed a map $\varphi\colon \Omega^2 S^{2n+1} \to S^{2n-1}$ with the property that the composition with the double suspension

$$\Omega^2 S^{2n+1} \xrightarrow{\varphi} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$$

is homotopic to the p^{th} power map on $\Omega^2 S^{2n+1}$. By induction on n, this factorization implies that the $(p^n)^{\text{th}}$ power map on $\Omega^{2n} S^{2n+1}$ factors through S^1 and hence induces the trivial map on $\pi_k(\Omega^{2n} S^{2n+1})$ for all $k > 1$.

The corresponding decomposition problems for mod 2 Moore spaces are much more complicated and the best possible 2-primary exponents of spheres are still unknown. A conjecture due to Barratt and Mahowald states that S^{2n+1} has exponent $2^{n+\epsilon}$ where $\epsilon = 0$ or 1 depending on the residue of n modulo 4. The methods of Cohen, Moore and Neisendorfer break down at the prime $p = 2$, but interestingly the following 2-primary analogue of Selick's decomposition was obtained by Cohen.

Theorem 5.4 ([10])**.** *There is a 2-local homotopy equivalence*

$$\Omega^2 S^5\{2\} \simeq \Omega^2 S^3\langle 3\rangle \times W_2.$$

Since $4\cdot\pi_*(\Omega^2 S^5\{2\}) = 0$ by Lemma 5.1, this recovers James' result that S^3 has 2-primary exponent 4 and raises the question of whether the Barratt–Mahowald conjecture can be approached by recognizing other spheres or related spaces as factors in loop space decompositions of spaces which *a priori* have homotopy exponents. We remark that a different proof of Theorem 5.4 is given in [11] which uses the Lie groups $SU(4)$ and $Sp(2)$ and recovers Waggoner's [55] result that W_2 is also a retract of $\Omega^2 SU(4)\{2\}$.

In addition to their applications to exponents, loop space decompositions of $\Omega S^{2n+1}\{p\}$ also induce splittings of homotopy groups of spheres with $\mathbb{Z}/p\mathbb{Z}$ coefficients. For example, Theorem 5.3 and Theorem 5.4 imply the following.

Corollary 5.5. *For any prime p, there are p-local isomorphisms*

$$\pi_k(S^{2p+1}; \mathbb{Z}/p\mathbb{Z}) \cong \pi_{k-1}(S^3) \oplus \pi_{k-3}(W_p)$$

for all $k \geq 5$.

For primes $p \geq 5$, it was shown in [43] that similar decompositions of $\Omega^2 S^{2n+1}\{p\}$ are not possible if $n \neq 1$ or p. This was proved by first showing that, for all primes p, a nontrivial product decomposition of $\Omega^2 S^{2n+1}\{p\}$ forces a certain homology class to be spherical, which in turn implies the existence of an element of p-primary Arf–Kervaire invariant one in the stable homotopy groups of spheres. Ravenel's negative solution to the odd primary Kervaire invariant problem for primes $p \geq 5$ [38] was then used to conclude that the nonexistence of such stable homotopy classes obstructs the possibility of a nontrivial homotopy decomposition of $\Omega^2 S^{2n+1}\{p\}$ for these primes. (The various forms of the Kervaire invariant problem and its relation to loop space decompositions is discussed in more detail in Chapter 8.)

In the $p = 2$ case, a nontrivial decomposition of $\Omega^2 S^{2n+1}\{2\}$ implies more than the existence of an element of Arf–Kervaire invariant one. In their investigation of the homology of spaces of maps from mod 2 Moore spaces to spheres, Campbell, Cohen, Peterson and Selick [9] worked out the Steenrod operations in $H_*(\Omega^2 S^{2n+1}\{2\})$ and found that if $2n+1 \neq 3, 5, 9$ or 17, then $\Omega^2 S^{2n+1}\{2\}$ is atomic and hence indecomposable. These indecomposability results are summarized below.

Theorem 5.6 ([9], [43])**.** *Let $n > 1$.*

(a) *If $2n+1 \neq 5, 9$ or 17, then $\Omega^2 S^{2n+1}\{2\}$ is indecomposable.*

(b) *If $p \geq 5$ and $n \neq p$, then $\Omega^2 S^{2n+1}\{p\}$ is indecomposable.*

Following this, Cohen and Selick obtained an explicit decomposition of $\Omega^2 S^9\{2\}$ and their proof showed that $\Omega^3 S^{17}\{2\}$ also decomposes.

Theorem 5.7 ([16])**.** *After localizing at 2, the following hold:*

(a) $\Omega^2 S^9\{2\} \simeq BW_2 \times W_4$*;*

(b) W_4 *is a retract of* $\Omega^3 S^{17}\{2\}$*.*

Here BW_n denotes the classifying space of W_n first constructed by Gray [18]. Since BW_1 is known to be homotopy equivalent to $\Omega^2 S^3\langle 3\rangle$, the pattern suggested by the decompositions of $\Omega^2 S^5\{2\}$ and $\Omega^2 S^9\{2\}$ led Cohen and Selick to conjecture that $\Omega^2 S^{17}\{2\} \simeq BW_4 \times W_8$. In Section 7.1 we prove this is true with one more loop.

Missing from the above results is the $p = 3$ case. Although it follows from [43] that (as with all other primes) a nontrivial decomposition of $\Omega^2 S^{2n+1}\{3\}$ is not possible unless $n = 3^k$ for some k, the question of whether $\Omega^2 S^{2n+1}\{3\}$ decomposes for $n = 3^k$ with $k > 1$ was left unsettled in the work of Campbell, Cohen, Peterson and Selick. This question is investigated in Chapter 8 where we give a new homotopy decomposition and indicate how further progress on the 3-primary Kervaire invariant problem will lead to further decompositions or indecomposability results.

Chapter 6

Gray's Conjecture

The next two chapters will make use of a conjecture of Gray which was proved by Richter in [40], so in this short chapter we review this conjecture and spell out some of its consequences for future reference.

Let p be any prime and localize all spaces and maps at p. In his construction of a classifying space BW_n of the fibre of the double suspension, Gray [18] introduced two p-local homotopy fibrations

$$\begin{gathered} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n \\ BW_n \xrightarrow{j} \Omega^2 S^{2np+1} \xrightarrow{\phi} S^{2np-1} \end{gathered} \tag{6.1}$$

with the property that $j \circ \nu \simeq \Omega H_p$ where $H_p \colon \Omega S^{2n+1} \to \Omega S^{2np+1}$ is the p^{th} James–Hopf invariant map. In addition, Gray showed that the composite $BW_n \xrightarrow{j} \Omega^2 S^{2np+1} \xrightarrow{p} \Omega^2 S^{2np+1}$ is nullhomotopic and conjectured that the map ϕ satisfies the stronger property that the composite $\Omega^2 S^{2np+1} \xrightarrow{\phi} S^{2np-1} \xrightarrow{E^2} \Omega^2 S^{2np+1}$ is homotopic to the p^{th} power map on $\Omega^2 S^{2np+1}$. Looped versions of Gray's conjecture, giving factorizations of the p^{th} power map on the triple loop space $\Omega^3 S^{2np+1}$, were first proved by Harper and Richter for $p \geq 3$ and $p = 2$, respectively, and a proof of the full conjecture, valid at all primes, was recently given by Richter in [40].

Theorem 6.1 ([40])**.** *For any prime p, there is a homotopy fibration*

$$BW_n \xrightarrow{j} \Omega^2 S^{2np+1} \xrightarrow{\phi_n} S^{2np-1}$$

with the property that $E^2 \circ \phi_n \simeq p$.

Since $p \circ j$ is nullhomotopic, j can be lifted to $\Omega^2 S^{2np+1}\{p\}$. For p odd, it was shown in [52] that there is a homotopy fibration of the form

$$\Omega W_{np} \longrightarrow BW_n \longrightarrow \Omega^2 S^{2np+1}\{p\}$$

based on the fact that a lift $BW_n \to \Omega^2 S^{2np+1}\{p\}$ of j can be chosen to be an H-map when p is odd. One consequence of Theorem 6.1 is that this homotopy fibration exists for all primes and can be extended one step to the right by a map $\Omega^2 S^{2np+1}\{p\} \to W_{np}$.

Lemma 6.2. *For any prime p, there is a homotopy fibration*

$$BW_n \longrightarrow \Omega^2 S^{2np+1}\{p\} \longrightarrow W_{np}.$$

Proof. The homotopy pullback of ϕ_n and the fibre inclusion $W_{np} \to S^{2np-1}$ of the double suspension defines a map $\Omega^2 S^{2np+1}\{p\} \to W_{np}$ with homotopy fibre BW_n, which can be seen by comparing fibres in the homotopy pullback diagram

$$\begin{array}{ccccc}
BW_n & \longrightarrow & \Omega^2 S^{2np+1}\{p\} & \longrightarrow & W_{np} \\
\| & & \downarrow & & \downarrow \\
BW_n & \xrightarrow{j} & \Omega^2 S^{2np+1} & \xrightarrow{\phi_n} & S^{2np-1} \\
 & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle E^2} \\
 & & \Omega^2 S^{2np+1} & = & \Omega^2 S^{2np+1}.
\end{array} \qquad (6.2)$$

□

In the next two chapters, the fibration above will be shown to be split (possibly after looping) in special cases and will be used to deloop BW_n for certain values of n when $p = 3$.

The next lemma describes a more precise factorization of the looped p^{th} James–Hopf invariant, an odd primary version of which appears in [52]. By a well-known result due to Barratt, ΩH_p has order p in the group $[\Omega^2 S^{2n+1}, \Omega^2 S^{2np+1}]$ and hence lifts (non-uniquely) to a map

$$S\colon \Omega^2 S^{2n+1} \longrightarrow \Omega^2 S^{2np+1}\{p\}.$$

Improving on this, a feature of Richter's construction of the map ϕ_n is that the composite

$$\Omega^2 S^{2n+1} \xrightarrow{\Omega H_p} \Omega^2 S^{2np+1} \xrightarrow{\phi_n} S^{2np-1}$$

is already nullhomotopic [40, Lemma 4.2] before further composing with $S^{2np-1} \xrightarrow{E^2} \Omega^2 S^{2np+1}$. Note that this recovers Gray's fibration $S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n$ and the relation $j \circ \nu \simeq \Omega H_p$ since there then exists a lift $\nu\colon \Omega^2 S^{2n+1} \to BW_n$ making the diagram

$$\begin{array}{ccc} & & BW_n \\ & \overset{\nu}{\nearrow} & \big\downarrow j \\ \Omega^2 S^{2n+1} & \xrightarrow{\Omega H_p} & \Omega^2 S^{2np+1} \end{array}$$

commute up to homotopy and a Serre spectral sequence argument shows that ν has homotopy fibre S^{2n-1} which includes into $\Omega^2 S^{2n+1}$ by a degree one map. Since j factors through $\Omega^2 S^{2np+1}\{p\}$, by composing the lift ν with the map $BW_n \to \Omega^2 S^{2np+1}\{p\}$ from Lemma 6.2 we obtain a choice of lift $S\colon \Omega^2 S^{2n+1} \to \Omega^2 S^{2np+1}\{p\}$ of the looped James–Hopf invariant. Thus we have the following consequence of Richter's theorem.

Lemma 6.3. *For any prime p, there is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & \xrightarrow{S} & \Omega^2 S^{2np+1}\{p\} \\ \big\downarrow \nu & & \Big\| \\ BW_n & \longrightarrow & \Omega^2 S^{2np+1}\{p\} \end{array}$$

where S is a lift of the looped James–Hopf invariant $\Omega H_p\colon \Omega^2 S^{2n+1} \to \Omega^2 S^{2np+1}$ and the map $BW_n \to \Omega^2 S^{2np+1}\{p\}$ has homotopy fibre ΩW_{np}.

Chapter 7

The $p = 2$ Case and Related Problems

Recall that for $n > 1$ and $p \geq 5$ the nonexistence of elements of p-primary Kervaire invariant one was shown to obstruct the possibility of a nontrivial homotopy decomposition of $\Omega^2 S^{2n+1}\{p\}$ unless $n = p$. So in this sense, Selick's decomposition (Theorem 5.3) expresses a special property of S^{2p+1} and its mod p homotopy groups which is unique among p-local spheres for these "large" primes. The situation for $p = 2$ or 3 is more interesting. In this chapter we turn our attention to the $p = 2$ case of the decomposition problem for $\Omega^2 S^{2n+1}\{p\}$ and some closely related phenomena in the homotopy theory of mod 2 Moore spaces and Stiefel manifolds.

Throughout this chapter all spaces and maps are assumed to be localized at the prime $p = 2$. Homology will be taken with mod 2 coefficients unless otherwise stated.

7.1 A homotopy decomposition of $\Omega^3 S^{17}\{2\}$

The following homological result was proved in [9] and used to obtain the homotopy decompositions

$$\Omega^2 S^5\{2\} \simeq BW_1 \times W_2$$

$$\Omega^2 S^9\{2\} \simeq BW_2 \times W_4$$

of [10] and [16]. (Recall that $BW_1 \simeq \Omega^2 S^3\langle 3\rangle$.)

Lemma 7.1 ([9])**.** *Let $n > 1$ and let $f\colon X \to \Omega^2 S^{2n+1}\{2\}$ be a map which induces an isomorphism on* *the module of primitives in degrees $2n-2$ and $4n-3$. If the mod 2 homology of X is isomorphic to that of $\Omega^2 S^{2n+1}\{2\}$ as a coalgebra over the Steenrod algebra, then f is a homology isomorphism.*

For $n = 2$ and $n = 4$, the primitives of $H_*(\Omega^2 S^{2n+1}\{2\})$ in degrees $2n-2$ and $4n-3$ are given by the spherical homology classes carried by the bottom cell of each factor in the product decompositions above. In both cases, the decompositions were obtained by constructing maps $BW_n \to \Omega^2 S^{4n+1}\{2\}$ and $W_{2n} \to \Omega^2 S^{4n+1}\{2\}$ (for $n = 1, 2$) which induce isomorphisms on these homology groups and then multiplying them together using the loop multiplication on $\Omega^2 S^{4n+1}\{2\}$ to obtain a map satisfying the hypotheses of Lemma 7.1.

Recall from Chapter 6 that the map $j\colon BW_n \to \Omega^2 S^{4n+1}$ from (6.1) lifts to the fibre of the squaring map $2\colon \Omega^2 S^{4n+1} \to \Omega^2 S^{4n+1}$. Since j is the homotopy fibre of a map $\phi\colon \Omega^2 S^{4n+1} \to S^{4n-1}$ which is degree two on the bottom cell, it follows that j_* is nonzero on $H_{4n-1}(\ )$. The action of the homology Bockstein on $H_{4n-1}(\Omega^2 S^{4n+1}\{2\})$ therefore implies by naturality that any lift of j must induce isomorphisms in homology in degrees $4n-1$ and $4n-2$. In particular, for all n there exist maps

$$\sigma_n\colon BW_n \longrightarrow \Omega^2 S^{4n+1}\{2\}$$

inducing isomorphisms in homology in degree $4n-2$. (For $n = 1$, $\sigma_1\colon \Omega^2 S^3\langle 3\rangle \to \Omega^2 S^5\{2\}$ is a lift of ΩH_2 restricted to the universal cover of $\Omega^2 S^3$.)

Maps of the form $W_{2n} \to \Omega^2 S^{4n+1}\{2\}$ are harder to find. For $n = 1$ and $n = 2$, Cohen and Selick first constructed maps $\lambda\colon \Omega^2 S^{8n+1} \to \Omega S^{4n+1}$ whose restrictions to S^{8n-1} are the adjoints of the Hopf invariant one maps $\nu\colon S^8 \to S^5$ and $\sigma\colon S^{16} \to S^9$, respectively. Then, observing that the H-map $\Omega\lambda$ commutes with the squaring map, they obtained an induced map of fibres from the diagram

$$\begin{array}{ccc}
\Omega^3 S^{8n+1}\{2\} & \overset{\gamma}{\dashrightarrow} & \Omega^2 S^{4n+1}\{2\} \\
\downarrow & & \downarrow \\
\Omega^3 S^{8n+1} & \xrightarrow{\Omega\lambda} & \Omega^2 S^{4n+1} \\
\downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} \\
\Omega^3 S^{8n+1} & \xrightarrow{\Omega\lambda} & \Omega^2 S^{4n+1}
\end{array}$$

where γ_* is nontrivial on $H_{8n-3}(\ )$ since $\Omega\lambda$ restricts to the double adjoint of a Hopf invariant one map on the bottom cell and hence is nontrivial in homology in degree $8n-2$. Finally, the required maps $W_{2n} \to \Omega^2 S^{4n+1}\{2\}$ were defined as the composites

$$W_{2n} \xrightarrow{\Omega\sigma_{2n}} \Omega^3 S^{8n+1}\{2\} \xrightarrow{\gamma} \Omega^2 S^{4n+1}\{2\}$$

for $n = 1$ and $n = 2$. Since $(\sigma_{2n})_*$ is nontrivial on $H_{8n-2}(\ )$, it follows that $\Omega\sigma_{2n}$ and the composite above induce nontrivial maps in homology in degree $8n-3$, so by invoking Lemma 7.1, Cohen and Selick concluded that the composites

$$BW_1 \times W_2 \xrightarrow{\sigma_1 \times (\gamma \circ \Omega\sigma_2)} \Omega^2 S^5\{2\} \times \Omega^2 S^5\{2\} \xrightarrow{\quad m \quad} \Omega^2 S^5\{2\} \tag{7.1}$$

and

$$BW_2 \times W_4 \xrightarrow{\sigma_2 \times (\gamma \circ \Omega\sigma_4)} \Omega^2 S^9\{2\} \times \Omega^2 S^9\{2\} \xrightarrow{\quad m \quad} \Omega^2 S^9\{2\} \tag{7.2}$$

are homology isomorphisms and hence homotopy equivalences.

An interesting aspect of the procedure above is that, in both cases $n = 1$ and $n = 2$, the composite $\gamma \circ \Omega\sigma_{2n}$ mapping the second factor into $\Omega^2 S^{4n+1}\{2\}$ factors through $\Omega^3 S^{8n+1}\{2\}$. Thus an immediate corollary of the first decomposition (7.1) is that W_2 is a retract of $\Omega^3 S^9\{2\}$. The second decomposition (7.2) then deloops this result and also shows that W_4 is a retract of $\Omega^3 S^{17}\{2\}$. Next, the case of $\Omega^2 S^{17}\{2\}$ is curious since, unlike $\Omega^2 S^{2n+1}\{2\}$ for all dimensions $2n+1 > 17$, there is a spherical homology class in $H_{29}(\Omega^2 S^{17}\{2\})$ making an analogous decomposition possible, but, since we are out of elements of Hopf invariant one, a suitable map $W_8 \to \Omega^2 S^{17}\{2\}$ cannot be constructed as in the procedure above. In fact, no composite $W_8 \xrightarrow{\Omega\sigma_8} \Omega^3 S^{33}\{2\} \to \Omega^2 S^{17}\{2\}$ with the homological properties of the composites $\gamma \circ \Omega\sigma_{2n}$ used above for $n = 1, 2$ can exist since this would imply that W_8 is a retract of $\Omega^3 S^{33}\{2\}$ and that a homology class in $H_{61}(\Omega^2 S^{33}\{2\})$ is spherical, a contradiction.

The next theorem shows that a map $W_8 \to \Omega^2 S^{17}\{2\}$ leading to the expected decomposition does exist after looping once.

Theorem 7.2. *There is a homotopy equivalence*

$$\Omega^3 S^{17}\{2\} \simeq W_4 \times \Omega W_8.$$

Proof. Let τ_n denote the map $BW_n \to \Omega^2 S^{4n+1}\{2\}$ appearing in Lemma 6.2. By the homotopy commutativity of the diagram (6.2), τ_n is a lift of j, implying that $(\tau_n)_*$ is nonzero on $H_{4n-2}(\ )$ by naturality of the Bockstein since j_* is nonzero on $H_{4n-1}(\ )$. We can therefore use the maps τ_n in place of the (potentially different) maps σ_n used in [16] to obtain product decompositions of $\Omega^2 S^{4n+1}\{2\}$ for $n = 1$ and 2, the advantage being that τ_n has homotopy fibre ΩW_{2n}. Explicitly, for $n = 2$ this is done as follows. By [16, Corollary 2.1], there exists a map $\gamma\colon \Omega^3 S^{17}\{2\} \to \Omega^2 S^9\{2\}$ which is nonzero in $H_{13}(\ )$.

Letting m denote the loop multiplication on $\Omega^2 S^9\{2\}$, it follows that the composite

$$\psi\colon BW_2 \times W_4 \xrightarrow{\tau_2 \times (\gamma \circ \Omega\tau_4)} \Omega^2 S^9\{2\} \times \Omega^2 S^9\{2\} \xrightarrow{\quad m \quad} \Omega^2 S^9\{2\}$$

induces an isomorphism on the module of primitives in degrees 6 and 13. Since $H_*(BW_2 \times W_4)$ and $H_*(\Omega^2 S^9\{2\})$ are isomorphic as coalgebras over the Steenrod algebra, ψ is a homology isomorphism by Lemma 7.1 and hence a homotopy equivalence.

Now the map $\Omega\tau_4$ fits in the homotopy fibration

$$W_4 \xrightarrow{\Omega\tau_4} \Omega^3 S^{17}\{2\} \longrightarrow \Omega W_8$$

and has a left homotopy inverse given by $\pi_2 \circ \psi^{-1} \circ \gamma$ where ψ^{-1} is a homotopy inverse of ψ and $\pi_2\colon BW_2 \times W_4 \to W_4$ is the projection onto the second factor. (Alternatively, composing $\gamma\colon \Omega^3 S^{17}\{2\} \to \Omega^2 S^9\{2\}$ with the map $\Omega^2 S^9\{2\} \to W_4$ of Lemma 6.2 yields a left homotopy inverse of $\Omega\tau_4$.) It follows that the homotopy fibration above is fibre homotopy equivalent to the trivial fibration $W_4 \times \Omega W_8 \to \Omega W_8$. □

Corollary 7.3. *There are isomorphisms*

$$\pi_k(S^{17}; \mathbb{Z}/2\mathbb{Z}) \cong \pi_{k-4}(W_4) \oplus \pi_{k-3}(W_8)$$

for all $k \geq 4$.

Proof. This follows immediately from Theorem 7.2 and the fact that $\Omega^3 S^{17}\{2\}$ is homotopy equivalent to the mapping space $\mathrm{Map}_*(P^4(2), S^{17})$ since

$$\begin{aligned} \pi_k(S^{17}; \mathbb{Z}/2\mathbb{Z}) &= [P^k(2),\, S^{17}] \\ &\cong [S^{k-4} \wedge P^4(2),\, S^{17}] \\ &\cong [S^{k-4},\, \mathrm{Map}_*(P^4(2), S^{17})]. \end{aligned}$$

□

We conclude this section with a discussion of H-space structures on BW_n and H-space exponents of its iterated loop spaces as they relate to the homotopy decompositions above. A question due to Gray asks for which n the classifying space BW_n has the structure of an H-space (after localizing at 2). Clearly BW_1 is an H-space since it is homotopy equivalent to the loop space $\Omega^2 S^3\langle 3\rangle$ and, as pointed out in [16], a corollary of the homotopy decomposition of $\Omega^2 S^9\{2\}$ is that BW_2 is an H-space, but beyond these two examples nothing is known. It would therefore be interesting to know whether Theorem 7.2 can be delooped to give BW_4 the structure of an H-space.

That W_n has homotopy exponent 4 (i.e., that $4 \cdot \pi_*(W_n) = 0$) for all n follows from [6] where it is shown that the 4^{th} power map on $\Omega^2 W_n$ is nullhomotopic. This result was delooped by Theriault in [50] where a factorization of the 4^{th} power map on $\Omega^2 S^{2n+1}$ through the double suspension E^2 was used to show that the 4^{th} power map on the single loop space ΩW_n is already nullhomotopic. Noting that $W_n \simeq \Omega BW_n$ is an H-space, he raised the question of whether the 4^{th} power map on W_n is also nullhomotopic. The homotopy decompositions of $\Omega^3 S^{4n+1}\{2\}$ show that this is the case when $n = 1, 2$ or 4.

Corollary 7.4. *For $n = 1, 2$ or 4, the 4^{th} power map on W_n is nullhomotopic.*

Proof. For $n = 1$ or 2, by looping the decomposition of $\Omega^2 S^{4n+1}\{2\}$ we obtain an equivalence of H-spaces

$$\Omega^3 S^{4n+1}\{2\} \simeq W_n \times \Omega W_{2n},$$

so the result follows immediately from the fact that $\Omega^3 S^{4n+1}\{2\}$ has H-space exponent 4. For $n = 4$, the homotopy equivalence of Theorem 7.2 is not necessarily an equivalence of H-spaces, but since W_4 is mapped into $\Omega^3 S^{17}\{2\}$ by an H-map $\Omega\tau_4$ with a left homotopy inverse r, the 4^{th} power map on W_4 factors as

$$\begin{aligned} 4 &\simeq r \circ \Omega\tau_4 \circ 4 \\ &\simeq r \circ 4 \circ \Omega\tau_4 \\ &\simeq * \end{aligned}$$

since the 4^{th} power map on $\Omega^3 S^{17}\{2\}$ is nullhomotopic by the proof of Lemma 5.1. □

Of course, it is clear that W_n has *some* H-space exponent for all n since by Lemma 6.2 there is a homotopy fibration of H-spaces and H-maps

$$\Omega^2 W_{2n} \longrightarrow W_n \longrightarrow \Omega^3 S^{4n+1}\{2\}$$

which gives an upper bound of 16 for the H-space exponent of W_n since $\Omega^3 S^{4n+1}\{2\}$ has H-space exponent 4.

7.2 The lift of the looped James–Hopf invariant

One consequence of the splitting of the homotopy fibration

$$W_n \longrightarrow \Omega^3 S^{4n+1}\{p\} \longrightarrow \Omega W_{2n}$$

when $n \in \{1, 2, 4\}$ is a corresponding homotopy decomposition of the fibre of the lift S of the looped James–Hopf invariant appearing in Lemma 6.3. As in [50], we define the space Y and the map t by the homotopy fibration

$$Y \xrightarrow{t} \Omega^2 S^{2n+1} \xrightarrow{S} \Omega^2 S^{4n+1}\{2\}.$$

This space and its odd primary analogue play a central role in the construction of Anick's fibration in [50, 52] and the alternative proof given in [51] of Cohen, Moore and Neisendorfer's determination of the odd primary homotopy exponent of spheres. Unlike at odd primes, Theriault [50] showed that the lift S of ΩH_2 cannot be chosen to be an H-map. This stems from a difference in the Pontrjagin ring structure on the homology of $\Omega^2 S^{2n+1}$ with mod 2 coefficients versus mod p coefficients for $p \geq 3$. We recall that there are algebra isomorphisms

$$H_*(\Omega^2 S^{2n+1}; \mathbb{Z}/p\mathbb{Z}) \cong \begin{cases} \bigotimes_{i=0}^{\infty} \mathbb{Z}/p\mathbb{Z}[a_{2np^i-1}] & \text{for } p = 2 \\ \left(\bigotimes_{i=0}^{\infty} \Lambda(a_{2np^i-1})\right) \otimes \left(\bigotimes_{i=1}^{\infty} \mathbb{Z}/p\mathbb{Z}[b_{2np^i-2}]\right) & \text{for } p \geq 3 \end{cases}$$

where subscripts denote degrees and

$$\beta(a_{2np^{i+1}-1}) = \begin{cases} a_{2np^i-1}^2 & \text{for } p = 2 \\ b_{2np^{i+1}-2} & \text{for } p \geq 3. \end{cases}$$

Proposition 7.5. *The lift $S\colon \Omega^2 S^{2n+1} \longrightarrow \Omega^2 S^{4n+1}\{2\}$ of ΩH_2 cannot be chosen to be an H-map.*

Proof. Let S denote any lift of $\Omega H_2\colon \Omega^2 S^{2n+1} \to \Omega^2 S^{4n+1}$ and suppose S is an H-map. It follows that there is a homotopy commutative diagram

$$\begin{array}{ccccc} S^{2n-1} \times \Omega^2 S^{2n+1} & \xrightarrow{E^2 \times 1} & \Omega^2 S^{2n+1} \times \Omega^2 S^{2n+1} & \xrightarrow{m} & \Omega^2 S^{2n+1} \\ \downarrow{\scriptstyle \pi_2} & & & & \downarrow{\scriptstyle S} \\ \Omega^2 S^{2n+1} & & \xrightarrow{S} & & \Omega^2 S^{4n+1}\{2\}. \end{array}$$

Clockwise around the diagram, we have

$$(m \circ (E^2 \times 1))_*(\iota_{2n-1} \otimes a_{2n-1}) = m_*(a_{2n-1} \otimes a_{2n-1}) = a_{2n-1}^2 = \beta(a_{4n-1}).$$

Since a_{4n-1} is sent under $(\Omega H_2)_*$ to the generator of $H_{4n-1}(\Omega^2 S^{4n+1})$, and since S is a lift of ΩH_2, it follows that $S_*(a_{4n-1})$ is nonzero in $H_{4n-1}(\Omega^2 S^{4n+1}\{2\})$. But since the $(4n-1)$-skeleton of $\Omega^2 S^{4n+1}\{2\}$ is $P^{4n-1}(2)$, it follows by naturality of the Bockstein that

$$S_*(a_{2n-1}^2) = S_*(\beta(a_{4n-1})) = \beta(S_*(a_{4n-1})) \neq 0,$$

a contradiction since, counter-clockwise around the diagram, we clearly have $(S \circ \pi_2)_*(\iota_{2n-1} \otimes a_{2n-1}) = 0$. □

Nevertheless, the following corollary of Theorem 7.2 shows that the fibre of S has the structure of an H-space in cases of Hopf invariant one.

Corollary 7.6. *There is a homotopy fibration $S^{2n-1} \xrightarrow{f} Y \xrightarrow{g} \Omega W_{2n}$ with the property that the composite $S^{2n-1} \xrightarrow{f} Y \xrightarrow{t} \Omega^2 S^{2n+1}$ is homotopic to the double suspension E^2. Moreover, if $n = 1, 2$ or 4 then the fibration splits, giving a homotopy equivalence*

$$Y \simeq S^{2n-1} \times \Omega W_{2n}.$$

Proof. By Lemma 6.3, the homotopy fibration defining Y fits in a homotopy pullback diagram

$$\begin{array}{ccccc}
S^{2n-1} & = & S^{2n-1} & & \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle E^2} & & \\
Y & \xrightarrow{t} & \Omega^2 S^{2n+1} & \xrightarrow{S} & \Omega^2 S^{4n+1}\{2\} \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \nu} & & \| \\
\Omega W_{2n} & \longrightarrow & BW_n & \longrightarrow & \Omega^2 S^{4n+1}\{2\},
\end{array}$$

which proves the first statement. Note that when $n = 1, 2$ or 4, the map $\Omega W_{2n} \to BW_n$ is nullhomotopic by the proof of Theorem 7.2, hence $\nu \circ t$ is nullhomotopic and t lifts through the double suspension. Since any choice of a lift $Y \to S^{2n-1}$ is degree one in $H_{2n-1}(\ )$, it also serves as a left homotopy inverse of f, which implies the asserted splitting by Lemma 2.9. □

Remark 7.7. The first part of Corollary 7.6 and an odd primary version are proved by different means in [50] and [51], respectively (see Remark 6.2 in [50]). At odd primes, there is an analogous splitting for $n = 1$:

$$Y \simeq S^1 \times \Omega W_p \simeq S^1 \times \Omega^3 T^{2p^2+1}(p)$$

where $T^{2p^2+1}(p)$ is Anick's space (see Section 8.1).

7.3 Relations to Whitehead products in Moore spaces and Stiefel manifolds

The special homotopy decompositions of $\Omega^3 S^{2n+1}\{2\}$ discussed in Section 7.1 are made possible by the existence of special elements in the stable homotopy groups of spheres, namely elements of Arf–Kervaire invariant one $\theta \in \pi^S_{2n-2}$ such that $\theta\eta$ is divisible by 2. In this section, we give several reformulations of the existence of such elements in terms of mod 2 Moore spaces and Stiefel manifolds.

We first aim to relate the problem of decomposing $\Omega^2 S^{2n+1}\{2\}$ to a problem considered by Mukai and Skopenkov in [33] of computing a certain summand in a homotopy group of the Moore space $P^{2n+1}(2)$. More specifically, we answer a question raised in [33] by determining for which n the Whitehead square of the inclusion $S^{2n} \to P^{2n+1}(2)$ is divisible by 2. This will follow from a preliminary result (Proposition 7.9 below) equating the divisibility of this Whitehead square with the vanishing of a Whitehead product in the mod 2 homotopy of the Stiefel manifold $V_{2n+1,2}$.

Let $V_{2n+1,2}$ denote the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{2n+1}$ or, equivalently, the unit tangent bundle over S^{2n}. Its homology is given by

$$H_*(V_{2n+1,2}) \cong \Lambda(u_{2n-1}, v_{2n})$$

with Bockstein $\beta v_{2n} = u_{2n-1}$. Let $i_{n-1}\colon S^{n-1} \to P^n(2)$ denote the inclusion of the bottom cell and let $j_n\colon P^n(2) \to P^n(2)$ denote the identity map. Similarly, let $i'_{2n-1}\colon S^{2n-1} \to V_{2n+1,2}$ and $j'_{2n}\colon P^{2n}(2) \to V_{2n+1,2}$ denote the inclusions of the bottom cell and bottom Moore space, respectively.

Remark 7.8. Note that we index these maps by the dimension of their source rather than their target, so the element of $\pi_{4n-1}(P^{2n+1}(2))$ we call $[i_{2n}, i_{2n}]$ is called $[i_{2n+1}, i_{2n+1}]$ in [33].

Proposition 7.9. *The Whitehead square $[i_{2n}, i_{2n}] \in \pi_{4n-1}(P^{2n+1}(2))$ is divisible by 2 if and only if the Whitehead product $[i'_{2n-1}, j'_{2n}] \in \pi_{4n-2}(V_{2n+1,2}; \mathbb{Z}/2\mathbb{Z})$ is trivial.*

Proof. Let $\lambda\colon S^{4n-2} \to P^{2n}(2)$ denote the attaching map of the top cell in

$$V_{2n+1,2} \simeq P^{2n}(2) \cup_\lambda e^{4n-1}$$

and observe that $[i'_{2n-1}, j'_{2n}] = j'_{2n} \circ [i_{2n-1}, j_{2n}]$ by naturality of the Whitehead product. The map $[i_{2n-1}, j_{2n}]\colon P^{4n-2}(2) \to P^{2n}(2)$ is essential since its adjoint is the Samelson product $\langle \tilde{i}_{2n-1}, \tilde{j}_{2n} \rangle$ of the adjoints of i_{2n-1} and j_{2n} which has nontrivial mod 2 Hurewicz image

$$[u, v] = u \otimes v - v \otimes u \in H_{4n-3}(\Omega P^{2n}(2)),$$

where $H_*(\Omega P^{2n}(2))$ is isomorphic as an algebra to the tensor algebra $T(u, v)$ with $|u| = 2n - 2$ and $|v| = 2n - 1$ by the Bott–Samelson Theorem. This can be seen by considering that $\langle \tilde{i}_{2n-1}, \tilde{j}_{2n} \rangle$ factors as the composite

$$\begin{array}{ccccccc} P^{4n-3}(2) & \xrightarrow{\sim} & S^{2n-2} \wedge P^{2n-1}(2) & \xrightarrow{\tilde{i} \wedge \tilde{j}} & \Omega P^{2n}(2) \wedge \Omega P^{2n}(2) & \xrightarrow{[\ ,\ ]} & \Omega P^{2n}(2) \\ v_{4n-3} & \longmapsto & \iota_{2n-2} \otimes v_{2n-1} & \longmapsto & u_{2n-2} \otimes v_{2n-1} & \longmapsto & [u_{2n-2}, v_{2n-1}] \end{array}$$

where $[\ ,\ ]$ is the commutator map and subscripts denote degrees of homology generators. Since the homotopy fibre of the inclusion $j'_{2n}\colon P^{2n}(2) \to V_{2n+1,2}$ has $(4n-2)$-skeleton S^{4n-2} which maps into $P^{2n}(2)$ by the attaching map λ, it follows that $[i'_{2n-1}, j'_{2n}] = j'_{2n} \circ [i_{2n-1}, j_{2n}]$ is trivial if and only if $[i_{2n-1}, j_{2n}]$ is homotopic to the composite

$$P^{4n-2}(2) \xrightarrow{q} S^{4n-2} \xrightarrow{\lambda} P^{2n}(2)$$

where q is the pinch map onto the top cell. It therefore suffices to show that $[i_{2n-1}, j_{2n}] \simeq \lambda \circ q$ if and only if $[i_{2n}, i_{2n}]$ is divisible by 2.

To ease notation, let P^n denote the mod 2 Moore space $P^n(2)$. Consider the morphism of EHP sequences (see Theorem 3.6)

$$\begin{array}{ccccccc} [S^{4n}, P^{2n+1}] & \xrightarrow{H} & [S^{4n}, \Sigma P^{2n} \wedge P^{2n}] & \xrightarrow{P} & [S^{4n-2}, P^{2n}] & \xrightarrow{E} & [S^{4n-1}, P^{2n+1}] \\ \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} \\ [P^{4n}, P^{2n+1}] & \xrightarrow{H} & [P^{4n}, \Sigma P^{2n} \wedge P^{2n}] & \xrightarrow{P} & [P^{4n-2}, P^{2n}] & \xrightarrow{E} & [P^{4n-1}, P^{2n+1}] \end{array}$$

induced by the pinch map. A homology calculation shows that the $(4n)$-skeleton of $\Sigma P^{2n} \wedge P^{2n}$ is homotopy equivalent to $P^{4n} \vee S^{4n}$. Let $k_1 \colon P^{4n} \to \Sigma P^{2n} \wedge P^{2n}$ and $k_2 \colon S^{4n} \to \Sigma P^{2n} \wedge P^{2n}$ denote the composites

$$P^{4n} \hookrightarrow P^{4n} \vee S^{4n} \simeq \mathrm{sk}_{4n}(\Sigma P^{2n} \wedge P^{2n}) \hookrightarrow \Sigma P^{2n} \wedge P^{2n}$$

and

$$S^{4n} \hookrightarrow P^{4n} \vee S^{4n} \simeq \mathrm{sk}_{4n}(\Sigma P^{2n} \wedge P^{2n}) \hookrightarrow \Sigma P^{2n} \wedge P^{2n}$$

defined by the left and right wedge summand inclusions, respectively. Then we have that $\pi_{4n}(\Sigma P^{2n} \wedge P^{2n}) = \mathbb{Z}/4\mathbb{Z}\{k_2\}$ and $P(k_2) = \pm 2\lambda$ by [31, Lemma 12]. It follows from the universal coefficient exact sequence (Theorem 4.2)

$$0 \to \pi_{4n}(\Sigma P^{2n} \wedge P^{2n}) \otimes \mathbb{Z}/2\mathbb{Z} \to \pi_{4n}(\Sigma P^{2n} \wedge P^{2n}; \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Tor}(\pi_{4n-1}(\Sigma P^{2n} \wedge P^{2n}), \mathbb{Z}/2\mathbb{Z}) \to 0$$

that

$$\begin{aligned} \pi_{4n}(\Sigma P^{2n} \wedge P^{2n}; \mathbb{Z}/2\mathbb{Z}) &= [P^{4n}, \Sigma P^{2n} \wedge P^{2n}] \\ &= \mathbb{Z}/2\mathbb{Z}\{k_1\} \oplus \mathbb{Z}/2\mathbb{Z}\{k_2 \circ q\} \end{aligned}$$

and that the generator $k_2 \circ q$ is in the kernel of P since $P(k_2) = \pm 2\lambda$ implies

$$P(k_2 \circ q) = P(q^*(k_2)) = q^*(P(k_2)) = \pm\lambda \circ 2 \circ q = 0$$

by the commutativity of the above diagram and the fact that $P^{4n-2} \xrightarrow{q} S^{4n-2}$ and $S^{4n-2} \xrightarrow{2} S^{4n-2}$ are consecutive maps in a cofibration sequence. Therefore, by exactness of the second row, we must have that $[i_{2n-1}, j_{2n}] = P(k_1)$ since the suspension of a Whitehead product is trivial. On the other hand, $\Sigma\lambda$ is homotopic to the composite $S^{4n-1} \xrightarrow{[\iota_{2n}, \iota_{2n}]} S^{2n} \xrightarrow{i_{2n}} P^{2n+1}$ by [31], which implies

$$E(\lambda \circ q) = i_{2n} \circ [\iota_{2n}, \iota_{2n}] \circ q = [i_{2n}, i_{2n}] \circ q,$$

so $E(\lambda \circ q)$ is trivial in $[P^{4n-1}, P^{2n+1}]$ precisely when $[i_{2n}, i_{2n}]$ is divisible by 2. Hence $[i_{2n}, i_{2n}]$ is divisible by 2 if and only if $\lambda \circ q = P(k_1) = [i_{2n-1}, j_{2n}] \in [P^{4n-2}, P^{2n}]$, and the proposition follows. $\square$

We use Proposition 7.9 in two ways. First, since the calculation of $\pi_{31}(P^{17}(2))$ in [32] shows that $[i_{16}, i_{16}] = 2\widetilde{\sigma}^2_{16}$ for a suitable choice of representative $\widetilde{\sigma}^2_{16}$ of the Toda bracket $\{\sigma^2_{16}, 2\iota_{16}, i_{16}\}$, it follows from Proposition 7.9 that the Whitehead product

$$[i'_{15}, j'_{16}] \colon P^{30}(2) \longrightarrow V_{17,2}$$

is nullhomotopic and hence there exists a map $S^{15} \times P^{16}(2) \to V_{17,2}$ extending the wedge of skeletal inclusions $S^{15} \vee P^{16}(2) \to V_{17,2}$. This resolves the only case left unsettled by Theorem 3.2 of [44].

In the other direction, note that such maps $S^{2n-1} \times P^{2n}(2) \to V_{2n+1,2}$ restrict to maps $S^{2n-1} \times S^{2n-1} \to V_{2n+1,2}$ which exist only in cases of Kervaire invariant one by [57, Proposition 2.27], so Proposition 7.9 shows that when $2n \neq 2^k$ for some $k \geq 1$ the Whitehead square $[i_{2n}, i_{2n}] \in \pi_{4n-1}(P^{2n+1}(2))$ cannot be divisible by 2 for the same reasons that the Whitehead square $[\iota_{2n-1}, \iota_{2n-1}] \in \pi_{4n-3}(S^{2n-1})$ cannot be divisible by 2. Moreover, since maps of the form $S^{2n-1} \times P^{2n}(2) \to V_{2n+1,2}$ extending the inclusions of S^{2n-1} and $P^{2n}(2)$ are shown not to exist for $2n > 16$ in [44], Proposition 7.9 implies that the Whitehead square $[i_{2n}, i_{2n}]$ is divisible by 2 if and only if $2n = 2, 4, 8$ or 16. In all other cases it generates a $\mathbb{Z}/2\mathbb{Z}$ summand in $\pi_{4n-1}(P^{2n+1}(2))$. This improves on the main theorem of [33] which shows by other means that $[i_{2n}, i_{2n}]$ is not divisible by 2 when $2n$ is not a power of 2.

These results are summarized in Theorem 7.11 below. First we recall the following well-known equivalent formulations of the Kervaire invariant problem.

Theorem 7.10 ([11], [57]). *The following are equivalent:*

(a) *The Whitehead square $[\iota_{2n-1}, \iota_{2n-1}] \in \pi_{4n-3}(S^{2n-1})$ is divisible by 2;*

(b) *There exists a map $P^{4n-2}(2) \to \Omega S^{2n}$ which is nonzero in homology;*

(c) *There exists a space X with mod 2 cohomology $\widetilde{H}^i(X) \cong \mathbb{Z}/2\mathbb{Z}$ for $i = 2n$, $4n-1$, $4n$ and zero otherwise with $Sq^{2n} \colon H^{2n}(X) \to H^{4n}(X)$ and $Sq^1 \colon H^{4n-1}(X) \to H^{4n}(X)$ isomorphisms;*

(d) *There exists a map $f \colon S^{2n-1} \times S^{2n-1} \to V_{2n+1,2}$ such that $f|_{S^{2n-1} \times *} = f|_{* \times S^{2n-1}}$ is the inclusion of the bottom cell;*

(e) *$n = 1$ or there exists an element $\theta \in \pi^S_{2n-2}$ of Kervaire invariant one.*

The above conditions hold for $2n = 2, 4, 8, 16, 32$ and 64, and the recent solution to the Kervaire invariant problem by Hill, Hopkins and Ravenel [23] implies that, with the possible exception of $2n = 128$, these are the only values for which the conditions hold. Using Proposition 7.9 and mimicking the reformulations above we obtain the following.

Theorem 7.11. *The following are equivalent:*

(a) *The Whitehead square* $[i_{2n}, i_{2n}] \in \pi_{4n-1}(P^{2n+1}(2))$ *is divisible by* 2*;*

(b) *There exists a map* $P^{4n}(2) \to \Omega P^{2n+2}(2)$ *which is nonzero in homology;*

(c) *There exists a space* X *with mod* 2 *cohomology* $\widetilde{H}^i(X) \cong \mathbb{Z}/2\mathbb{Z}$ *for* $i = 2n+1,\ 2n+2,\ 4n+1,$ $4n+2$ *and zero otherwise with* $Sq^{2n} \colon H^{2n+1}(X) \to H^{4n+1}(X)$*,* $Sq^1 \colon H^{2n+1}(X) \to H^{2n+2}(X)$ *and* $Sq^1 \colon H^{4n+1}(X) \to H^{4n+2}(X)$ *isomorphisms;*

(d) *There exists a map* $f \colon S^{2n-1} \times P^{2n}(2) \to V_{2n+1,2}$ *such that* $f|_{S^{2n-1}\times *}$ *and* $f|_{*\times P^{2n}(2)}$ *are the skeletal inclusions of* S^{2n-1} *and* $P^{2n}(2)$*, respectively;*

(e) $n = 1$ *or there exists an element* $\theta \in \pi^S_{2n-2}$ *of Kervaire invariant one with* $\theta\eta$ *divisible by* 2*;*

(f) $2n = 2, 4, 8$ *or* 16*.*

Proof. (a) is equivalent to (b): In the $n = 1$ case, $[\iota_2, \iota_2] = 2\eta_2$ implies $[i_2, i_2] = 0$, and since $\eta_3 \in \pi_4(S^3)$ has order 2 its adjoint $\widetilde{\eta}_3 \colon S^3 \to \Omega S^3$ extends to a map $P^4(2) \to \Omega S^3$. If this map desuspended, then $\widetilde{\eta}_3$ would be homotopic to a composite $S^3 \to P^4(2) \to S^2 \xrightarrow{E} \Omega S^3$, a contradiction since $\pi_3(S^2) \cong \mathbb{Z}$ implies that any map $S^3 \to S^2$ that factors through $P^4(2)$ is nullhomotopic. Hence the map $P^4(2) \to \Omega S^3$ has nontrivial Hopf invariant in $[P^4(2), \Omega S^5]$ from which it follows that $P^4(2) \to \Omega S^3$ is nonzero in $H_4(\ )$. Composing with the inclusion $\Omega S^3 \to \Omega P^4(2)$ gives a map $P^4(2) \to \Omega P^4(2)$ which is nonzero in $H_4(\ )$.

Now suppose $n > 1$ and $[i_{2n}, i_{2n}] = 2\alpha$ for some $\alpha \in \pi_{4n-1}(P^{2n+1}(2))$. Then $\Sigma\alpha$ has order 2 so there is an extension $P^{4n+1}(2) \to P^{2n+2}(2)$ whose adjoint $f \colon P^{4n}(2) \to \Omega P^{2n+2}(2)$ satisfies $f|_{S^{4n-1}} = E \circ \alpha$. Since $\Omega\Sigma(P^{2n+1}(2) \wedge P^{2n+1}(2))$ has $4n$-skeleton S^{4n}, to show that f_* is nonzero on $H_{4n}(P^{4n}(2))$ it suffices to show that $H_2 \circ f$ is nontrivial in $[P^{4n}(2), \Omega\Sigma(P^{2n+1}(2) \wedge P^{2n+1}(2))]$ where $H_2 \colon \Omega P^{2n+2}(2) \to \Omega\Sigma(P^{2n+1}(2) \wedge P^{2n+1}(2))$ is the second James–Hopf invariant. If $H_2 \circ f$ is nullhomotopic, then there is a map $g \colon P^{4n}(2) \to P^{2n+1}(2)$ making the diagram

$$\begin{array}{ccccc} P^{2n+1}(2) & \xrightarrow{E} & \Omega P^{2n+2}(2) & \xrightarrow{H_2} & \Omega\Sigma(P^{2n+1}(2) \wedge P^{2n+1}(2)) \\ \uparrow{\scriptstyle g} & \nearrow{\scriptstyle f} & & & \\ P^{4n}(2) & & & & \end{array}$$

commute. But then $\alpha - g|_{S^{4n-1}}$ is in the kernel of $E_* \colon \pi_{4n-1}(P^{2n+1}(2)) \to \pi_{4n}(P^{2n+2}(2))$ which is generated by $[i_{2n}, i_{2n}]$, so $\alpha - g|_{S^{4n-1}}$ is a multiple of $[i_{2n}, i_{2n}]$. Since $[i_{2n}, i_{2n}]$ has order 2 and clearly $2g|_{S^{4n-1}} = 0$, it follows that $[i_{2n}, i_{2n}] = 2\alpha = 0$, a contradiction. Therefore f_* is nonzero on $H_{4n}(P^{4n}(2))$.

Conversely, assume $n > 1$ and $f\colon P^{4n}(2) \to \Omega P^{2n+2}(2)$ is nonzero in $H_{4n}(\ )$. Since the restriction $f|_{S^{4n-1}}$ lifts through the $(4n-1)$-skeleton of $\Omega P^{2n+2}(2)$, there is a homotopy commutative diagram

$$\begin{array}{ccc} S^{4n-1} & \longrightarrow & P^{4n}(2) \\ \downarrow g & & \downarrow f \\ P^{2n+1}(2) & \xrightarrow{E} & \Omega P^{2n+2}(2) \end{array}$$

for some map $g\colon S^{4n-1} \to P^{2n+1}(2)$. Since $E \circ 2g$ is nullhomotopic, $2g$ is a multiple of $[i_{2n}, i_{2n}]$. But if $2g = 0$, then g admits an extension $e\colon P^{4n}(2) \to P^{2n+1}(2)$ and it follows that $f - E \circ e$ restricts trivially to S^{4n-1} and hence factors through the pinch map $q\colon P^{4n}(2) \to S^{4n}$. This makes the Pontrjagin square $u^2 \in H_{4n}(\Omega P^{2n+2}(2))$ a spherical homology class, and this is a contradiction which can be seen as follows. If u^2 is spherical, then the $4n$-skeleton of $\Omega P^{2n+2}(2)$ is homotopy equivalent to $P^{2n+1}(2) \vee S^{4n}$. On the other hand, it is easy to see that the attaching map of the $4n$-cell in $\Omega P^{2n+2}(2)$ is given by the Whitehead square $[i_{2n}, i_{2n}]$ which is nontrivial as $n > 1$, whence $P^{2n+1} \cup_{[i_{2n}, i_{2n}]} e^{4n} \not\simeq P^{2n+1}(2) \vee S^{4n}$.

(a) is equivalent to (d): Since the Whitehead product $[i'_{2n-1}, j'_{2n}] \in \pi_{4n-2}(V_{2n+1,2}; \mathbb{Z}/2\mathbb{Z})$ is the obstruction to extending the map

$$i'_{2n-1} \vee j'_{2n}\colon S^{2n-1} \vee P^{2n}(2) \longrightarrow V_{2n+1,2}$$

to the product $S^{2n-1} \times P^{2n}(2)$, this follows immediately from Proposition 7.9.

As described in [44], applying the Hopf construction to a map $f\colon S^{2n-1} \times P^{2n}(2) \to V_{2n+1,2}$ as in (d) yields a map

$$H(f)\colon \Sigma(S^{2n-1} \wedge P^{2n}(2)) = P^{4n}(2) \longrightarrow \Sigma V_{2n+1,2}$$

with Sq^{2n} acting nontrivially on $H^{2n}(C_{H(f)})$. Since $\Sigma^2 V_{2n+1,2} \simeq P^{2n+2}(2) \vee S^{4n+1}$, composing the suspension of the Hopf construction $H(f)$ with a retract $\Sigma^2 V_{2n+1,2} \to P^{2n+2}(2)$ defines a map $g\colon P^{4n+1}(2) \to P^{2n+2}(2)$ with Sq^{2n} acting nontrivially on $H^{2n+1}(C_g)$, so (d) implies (c).

By the proof of [44, Theorem 3.1], (c) implies (e), and (e) implies (f). The triviality of the Whitehead product $[i'_{2n-1}, j'_{2n}] \in \pi_{4n-2}(V_{2n+1,2}; \mathbb{Z}/2\mathbb{Z})$ when $n = 1, 2$ or 4 is implied by [44, Theorem 2.1], for example, and Proposition 7.9 implies $[i'_{15}, j'_{16}] \in \pi_{30}(V_{17,2}; \mathbb{Z}/2\mathbb{Z})$ is trivial as well since $[i_{16}, i_{16}] \in \pi_{31}(P^{17}(2))$ is divisible by 2 by [32, Lemma 3.10]. Thus (f) implies (d). $\square$

7.4 A Whitehead product in $V_{n,3}$

We next consider another Whitehead product whose vanishing could be added to the list of equivalent conditions in Theorem 7.11. Let $V_{n,k}$ denote the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$. A Whitehead product obstructing a splitting of the fibre bundle

$$V_{n-1,2} \longrightarrow V_{n,3} \longrightarrow S^{n-1}$$

was shown to be nontrivial for all $n > 16$ by Selick [44] in his proof of the indecomposability of these Stiefel manifolds. This obstruction vanishes for $n = 4$ or 8 in which cases quaternion or octonion multiplication can be used to give explicit diffeomorphisms $V_{n,3} \cong S^{n-1} \times V_{n-1,2}$. The case of $V_{16,3}$ occupies a grey area similar to $\Omega^2 S^{17}\{2\}$. The goal of this section is to show that the Whitehead product obstructing a splitting also vanishes in this case and that the 42-dimensional manifolds $V_{16,3}$ and $S^{15} \times V_{15,2}$ are homotopy equivalent up through their 41-skeletons.

In his celebrated solution to the vector fields on spheres problem, Adams [2] determined for which values of n and k the fibre bundle

$$V_{n-1,k-1} \longrightarrow V_{n,k} \longrightarrow S^{n-1}$$

admits a section. A natural next question is: when does the total space split up to homotopy as a product of the base and fibre in those cases where a section exists? The existence of such splittings is well known for $n = 2, 4$ and 8, and an early result of James showed that such a splitting cannot exist if n is not a power of 2. In his 1976 book on the topology of Stiefel manifolds [26], James noted that $V_{16,3}$ was the first unsettled case and conjectured that all Stiefel manifolds $V_{n,k}$ are indecomposable except for the special cases occuring when $n = 2, 4$ or 8. Forty years later, independent results of various authors have verified this conjecture for all but a finite list of Stiefel manifolds, and $V_{16,3}$ is among the last unsettled cases [27, 44].

Note that for all $n = 2^k$, $k \geq 2$, the presence of a section of the map $V_{n,3} \to S^{n-1}$ implies by Lemma 2.9 that there is a loop space decomposition

$$\Omega V_{n,3} \simeq \Omega S^{n-1} \times \Omega V_{n-1,2},$$

so homotopy groups alone cannot distinguish the homotopy types of $V_{n,3}$ and $S^{n-1} \times V_{n-1,2}$. Instead, the Lie algebraic structure given by Whitehead products was used in [44] to prove the indecomposability of $V_{n,3}$ for $n > 16$ as follows. First, the question of whether there exists a homotopy equivalence $S^{n-1} \times V_{n-1,2} \to V_{n,3}$ can be filtered by restricting to a skeleton of $V_{n-1,2}$ and asking whether there exists a $(2n-3)$-equivalence $S^{n-1} \times P^{n-2}(2) \to V_{n,3}$, that is, whether the Whitehead product of the inclusions of the two factors is trivial. Selick showed that if this Whitehead product is trivial, then so is the Whitehead product $[i'_{n-1}, j'_n] \in \pi_{2n-2}(V_{n+1,2}; \mathbb{Z}/2\mathbb{Z})$ and hence there exists a map $S^{n-1} \times P^n(2) \to V_{n+1,2}$ extending the wedge of inclusions $i'_{n-1} \vee j'_n \colon S^{n-1} \vee P^n(2) \to V_{n+1,2}$. Next, the existence of such a map was used to construct an element $\theta \in \pi^S_{n-2}$ of Kervaire invariant one with $\theta\eta$ divisible by 2 as described in the proof of Theorem 7.11. For $n > 16$, the nonexistence of such elements therefore shows that $[i'_{n-1}, j'_n] \in \pi_{2n-2}(V_{n+1,2}; \mathbb{Z}/2\mathbb{Z})$ is nontrivial and rules out the possibility of a nontrivial decomposition of $V_{n,3}$.

In the remaining case $n = 16$, the Whitehead product $[i'_{15}, j'_{16}] \in \pi_{30}(V_{17,2}; \mathbb{Z}/2\mathbb{Z})$ is in fact trivial by Theorem 7.11, so a decomposition of $V_{16,3}$ cannot be ruled out as easily. We will show by a direct computation that the Whitehead product in $V_{16,3}$ is also trivial, which suggests that this space is homotopically more similar to the Stiefel manifolds of 3-frames in $\mathbb{R}^4$ and $\mathbb{R}^8$ than in $\mathbb{R}^n$ for $n > 16$.

It will be useful to recall that there are coalgebra isomorphisms

$$H_*(V_{16,3}) \cong H_*(S^{15} \times V_{15,2}) \cong \Lambda(x_{13}, x_{14}, x_{15})$$

where $\beta x_{14} = x_{13}$. Let $\hat{\imath}_{13} \colon S^{13} \to V_{16,3}$ and $\hat{\jmath}_{14} \colon P^{14}(2) \to V_{16,3}$ denote the inclusions of the bottom cell and bottom Moore space, respectively. Note that these maps factor through the inclusions $i'_{13} \colon S^{13} \to V_{15,2}$ and $j'_{14} \colon P^{14}(2) \to V_{15,2}$ and that there is a homotopy fibration diagram

$$\begin{array}{ccccc}
S^{13} & \xrightarrow{i'_{13}} & V_{15,2} & \longrightarrow & S^{14} \\
\| & & \downarrow & & \downarrow \\
S^{13} & \xrightarrow{\hat{\imath}_{13}} & V_{16,3} & \longrightarrow & V_{16,2} \\
 & & \downarrow \uparrow s & & \downarrow \\
 & & S^{15} & = & S^{15}
\end{array} \tag{7.3}$$

where s is a section of the projection $V_{16,3} \to S^{15}$ onto the last vector of a 3-frame. To show that $[\hat{\jmath}_{14}, s] \colon P^{28}(2) \to V_{16,3}$ is nullhomotopic, we first consider the restriction

$$[\hat{\jmath}_{14}, s] \circ i_{27} = [\hat{\imath}_{13}, s] \colon S^{27} \longrightarrow P^{28}(2) \longrightarrow V_{16,3}.$$

Lemma 7.12. *The Whitehead product $[\hat{\imath}_{13}, s] \in \pi_{27}(V_{16,3})$ is trivial.*

Proof. By naturality of the Whitehead product, $[\hat{\imath}_{13}, s]$ composes trivially into $V_{16,2}$ since $\hat{\imath}_{13}$ does. Thus $[\hat{\imath}_{13}, s]$ factors as

$$S^{27} \xrightarrow{f} S^{13} \xrightarrow{i'_{13}} V_{15,2} \longrightarrow V_{16,3}$$

for some map f. Since the fibre inclusion i'_{13} factors as a composite of inclusions $S^{13} \to P^{14}(2) \to V_{15,2}$ and since every element of

$$\pi_{27}(S^{13}) = \mathbb{Z}/16\mathbb{Z}\{\sigma_{13}^2\} \oplus \mathbb{Z}/2\mathbb{Z}\{\kappa_{13}\}$$

is a suspension and hence commutes with the degree 2 map, we may assume f is not divisible by 2, otherwise $[\hat{\imath}_{13}, s]$ factors through the cofibration $S^{13} \xrightarrow{2} S^{13} \longrightarrow P^{14}(2)$ and we are done. But then, since the homotopy group $\pi_{27}(S^{13})$ stabilizes to $\pi_{14}^S = \mathbb{Z}/2\mathbb{Z}\{\sigma^2\} \oplus \mathbb{Z}/2\mathbb{Z}\{\kappa\}$ and $P^{14}(2)$ is a stable retract of $V_{15,2}$ and $V_{16,3}$ by Miller's stable splitting of Stiefel manifolds [30], it follows that the composite $S^{27} \xrightarrow{f} S^{13} \to P^{14}(2)$ is stably nontrivial and hence that

$$S^{27} \xrightarrow{f} S^{13} \longrightarrow P^{14}(2) \longrightarrow V_{15,2} \longrightarrow V_{16,3}$$

is stably nontrivial. This is a contradiction since the suspension of a Whitehead product is trivial by Lemma 4.6, so the result follows. □

We will also use the following computation from [32] which was used in the previous section.

Lemma 7.13 ([32]). *For suitable choices of representatives of Toda brackets $\widetilde{\kappa}_{16} \in \{\kappa_{16}, 2\iota_{16}, i_{16}\}$ and $\widetilde{\sigma}_{16}^2 \in \{\sigma_{16}^2, 2\iota_{16}, i_{16}\}$, we have*

$$\pi_{31}(P^{17}(2)) = \mathbb{Z}/4\mathbb{Z}\{\widetilde{\kappa}_{16}\} \oplus \mathbb{Z}/4\mathbb{Z}\{\widetilde{\sigma}_{16}^2\} \oplus \mathbb{Z}/2\mathbb{Z}\{i_{16}\rho_{16}\}$$

where $2\widetilde{\kappa}_{16} = i_{16}\eta_{16}\kappa_{17}$ and $2\widetilde{\sigma}_{16}^2 = i_{16}[\iota_{16}, \iota_{16}]$.

Lemma 7.14. *The Whitehead product $[\hat{\jmath}_{14}, s] \in \pi_{28}(V_{16,3}; \mathbb{Z}/2\mathbb{Z})$ is trivial.*

Proof. Since $[\hat{\jmath}_{14}, s]$ composes trivially into the base space S^{15} and since the projection $V_{16,3} \to S^{15}$ admits a section, $[\hat{\jmath}_{14}, s]$ lifts uniquely to the fibre $V_{15,2}$. Denote this lift by $\overline{[\hat{\jmath}_{14}, s]} \colon P^{28}(2) \to V_{15,2}$. The restriction of $\overline{[\hat{\jmath}_{14}, s]}$ to S^{27} is clearly a lift of $[\hat{\imath}_{13}, s]$. But by Lemma 7.12 the constant map $S^{27} \xrightarrow{*} V_{15,2}$ is also a lift of $[\hat{\imath}_{13}, s]$, and since this lift is also unique, the restriction of $\overline{[\hat{\jmath}_{14}, s]}$ to S^{27} must be nullhomotopic. We therefore have an extension through the pinch map

$$\begin{array}{ccccc} S^{27} & \longrightarrow & P^{28}(2) & \xrightarrow{q} & S^{28} \\ & & \big\downarrow{\scriptstyle \overline{[\hat{\jmath}_{14}, s]}} & \swarrow{\scriptstyle e} & \\ & & V_{15,2}. & & \end{array}$$

Let $\partial\colon \Omega S^{14} \to S^{13}$ denote the connecting map of the fibration $S^{13} \to V_{15,2} \to S^{14}$. It is easy to see that the composite $S^{13} \xrightarrow{E} \Omega S^{14} \xrightarrow{\partial} S^{13}$ is homotopic to the degree 2 map from which it follows that the connecting homomorphism $\partial_*\colon \pi_{28}(S^{14}) \to \pi_{27}(S^{14})$ is given by

$$\mathbb{Z}/8\mathbb{Z}\{\sigma_{14}^2\} \oplus \mathbb{Z}/2\mathbb{Z}\{\kappa_{14}\} \longrightarrow \mathbb{Z}/16\mathbb{Z}\{\sigma_{13}^2\} \oplus \mathbb{Z}/2\mathbb{Z}\{\kappa_{13}\}$$

$$\sigma_{14}^2 \longmapsto 2\sigma_{13}^2$$

$$\kappa_{14} \longmapsto 0.$$

Since the composite $S^{28} \xrightarrow{e} V_{15,2} \to S^{14}$ represents an element of the kernel of ∂_*, it must be either κ_{14} or 0. The former is impossible since this would imply that there is a homotopy commutative diagram

$$\begin{array}{ccc} P^{28}(2) & \xrightarrow{q} & S^{28} \\ \big\downarrow{\scriptstyle \overline{[\hat{\jmath}_{14}, s]}} & & \big\downarrow{\scriptstyle \kappa_{14}} \\ V_{15,2} & \longrightarrow & S^{14} \\ \big\downarrow & & \big\downarrow \\ V_{16,3} & \longrightarrow & V_{16,2} \end{array}$$

and hence that the Whitehead product $[\hat{\jmath}_{14}, s]\colon P^{28}(2) \to V_{16,3}$ is stably nontrivial since $\kappa \in \pi_{14}^S$ is not divisible by 2 and there is a stable splitting $\Sigma^\infty V_{16,2} \simeq \Sigma^\infty(S^{14} \vee S^{15} \vee S^{29})$. Thus $S^{28} \xrightarrow{e} V_{15,2} \to S^{14}$ is nullhomotopic and e lifts to the fibre S^{13}.

To summarize, we have now factored the lift $\overline{[\hat{\jmath}_{14}, s]}$ of $[\hat{\jmath}_{14}, s]$ as a composite

$$P^{28}(2) \xrightarrow{q} S^{28} \xrightarrow{f} S^{13} \xrightarrow{i'_{13}} V_{15,2} \tag{7.4}$$

and it remains to analyze the middle map f. By [54], we have $\pi_{28}(S^{13}) = \mathbb{Z}/32\mathbb{Z}\{\rho_{13}\} \oplus \mathbb{Z}/2\mathbb{Z}\{\bar{\epsilon}_{13}\}$ and $\bar{\epsilon}_{13} = \eta_{13}\kappa_{14} = \kappa_{13}\eta_{27}$. Write $f = a\rho_{13} + b\bar{\epsilon}_{13}$ for some $a \in \mathbb{Z}/32\mathbb{Z}$, $b \in \mathbb{Z}/2\mathbb{Z}$. Suppose a is a unit. Then note that $S^{31} \xrightarrow{\Sigma^3 f} S^{16} \xrightarrow{i_{16}} P^{17}(2)$ is not divisible by 2 by Lemma 7.13. Suspending once more, $i_{17} \circ \Sigma^4 f$ is nontrivial since ρ_{13} is stably not divisible by 2. We wish to show that $i_{17} \circ \Sigma^4 f$ is also not divisible by 2. A Serre spectral sequence argument shows that the homotopy fibre of the pinch map $P^{18}(2) \xrightarrow{q} S^{18}$ has 33-skeleton S^{17}, so there is a commutative diagram

$$\begin{array}{ccccccccc} \pi_{32}(S^{17}) & \xrightarrow{2} & \pi_{32}(S^{17}) & \xrightarrow{i_{17*}} & \pi_{32}(P^{18}(2)) & \xrightarrow{q_*} & \pi_{32}(S^{18}) & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \Sigma^\infty & & \downarrow \cong & & \\ \pi^S_{15} & \xrightarrow{2} & \pi^S_{15} & \longrightarrow & [\Sigma^\infty S^{32}, \Sigma^\infty P^{18}(2)] & \longrightarrow & \pi^S_{14} & \longrightarrow & 0 \end{array}$$

where both rows are exact sequences and suspensions of the elements labelled $\widetilde{\sigma}^2_{16}$ and $\widetilde{\kappa}_{16}$ in Lemma 7.13 are mapped under q_* to the generators of $\pi_{32}(S^{18}) \cong \pi^S_{14} = \mathbb{Z}/2\mathbb{Z}\{\sigma^2\} \oplus \mathbb{Z}/2\mathbb{Z}\{\kappa\}$. Since the images under i_{17*} of the generators of $\pi_{32}(S^{17}) \cong \pi^S_{15} = \mathbb{Z}/32\mathbb{Z}\{\rho\} \oplus \mathbb{Z}/2\mathbb{Z}\{\bar{\epsilon}\}$ both have order 2, clearly if $S^{32} \xrightarrow{\Sigma^4 f} S^{17} \xrightarrow{i_{17}} P^{18}(2)$ is divisible by 2, then $i_{17} \circ \Sigma^4 f = 2\alpha$ where $\alpha = \Sigma\widetilde{\sigma}^2_{16}$, $\Sigma\widetilde{\kappa}_{16}$ or their sum. But since $2\Sigma\widetilde{\sigma}^2_{16} = 0$ and $2\Sigma\widetilde{\kappa}_{16} = i_{17}\eta_{17}\kappa_{18} = i_{17}\bar{\epsilon}_{17}$ by Lemma 7.13, this is impossible so $i_{17} \circ \Sigma^4 f$ is not divisible by 2. By the five lemma, the stabilization map $\Sigma^\infty \colon \pi_{32}(P^{18}(2)) \to [\Sigma^\infty S^{32}, \Sigma^\infty P^{18}(2)]$ is an isomorphism, so this shows that $i_{13} \circ f$ is stably not divisible by 2 and hence the composite $P^{28}(2) \xrightarrow{q} S^{28} \xrightarrow{f} S^{13} \xrightarrow{i_{13}} P^{14}(2)$ is stably nontrivial. Since $P^{14}(2)$ is a stable retract of $V_{15,2}$ and $V_{16,3}$, this implies that the composite (7.4) and $[\hat{\jmath}_{14}, s]$ are also stably nontrivial, contradicting the fact that the suspension of a Whitehead product is trivial by Lemma 4.6. Therefore a is not a unit and we can now assume that $f = a\rho_{13} + b\bar{\epsilon}_{13} \in \pi_{28}(S^{13})$ is either $\bar{\epsilon}_{13}$ or 0.

If f is $\bar{\epsilon}_{13} = \kappa_{13}\eta_{27}$, then since the degree 2 map on $P^n(2)$ factors as the composite $P^n(2) \xrightarrow{q} S^n \xrightarrow{\eta} S^{n-1} \to P^n(2)$ for $n \geq 3$, there is a homotopy commutative diagram

$$\begin{array}{ccccc} P^{28}(2) & \xrightarrow{\underline{2}} & P^{28}(2) & \xrightarrow{\kappa'} & S^{13} \\ \downarrow q & & \uparrow i_{27} & \nearrow \kappa_{13} & \\ S^{28} & \xrightarrow{\eta_{27}} & S^{27} & & \end{array}$$

where κ' denotes an extension of κ_{13}. Since this extension may be chosen to be a suspension, κ' commutes with the degree 2 map, so the factorization (7.4) reads

$$\begin{aligned} \overline{[\hat{\jmath}_{14}, s]} &\simeq i'_{13} \circ f \circ q \\ &\simeq i'_{13} \circ \kappa_{13} \circ \eta_{27} \circ q \\ &\simeq i'_{13} \circ \kappa' \circ \underline{2} \\ &\simeq i'_{13} \circ \underline{2} \circ \kappa' \\ &\simeq * \end{aligned}$$

since the composite $S^{13} \xrightarrow{2} S^{13} \xrightarrow{i'_{13}} V_{15,2}$ is nullhomotopic. $\square$

Combined with the results of [44], Lemma 7.14 implies the following.

Corollary 7.15. *There exists a map* $S^{n-1} \times P^{n-2}(2) \to V_{n,3}$ *inducing a monomorphism in homology if and only if* $n = 4, 8$ *or* 16.

When $n = 4$ or 8, the map above extends to a homotopy equivalence $S^{n-1} \times V_{n-1,2} \xrightarrow{\sim} V_{n,3}$. We do not know if this is true for $n = 16$ but Lemma 7.14 implies the following partial result.

Theorem 7.16. *The* 41*-skeletons of the* 42*-dimensional manifolds* $V_{16,3}$ *and* $S^{15} \times V_{15,2}$ *are homotopy equivalent.*

Proof. Recall that $H_*(V_{16,3}) \cong H_*(S^{15} \times V_{15,2}) \cong \Lambda(x_{13}, x_{14}, x_{15})$. Define P by the pushout of inclusions

$$\begin{array}{ccc} S^{15} \vee P^{14}(2) & \longrightarrow & S^{15} \vee V_{15,2} \\ \downarrow & & \downarrow \\ S^{15} \times P^{14}(2) & \longrightarrow & P \end{array}$$

and observe that the mod 2 homology of P is given by

$$H_k(P) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } k \in \{13, 14, 15, 27, 28, 29\} \\ 0 & \text{otherwise.} \end{cases}$$

The induced map $P \to S^{15} \times V_{15,2}$ induces a monomorphism in homology, hence $P \simeq \mathrm{sk}_{29}(S^{15} \times V_{15,2}) \simeq \mathrm{sk}_{41}(S^{15} \times V_{15,2})$. By Lemma 7.14, there exists a map $S^{15} \times P^{14}(2) \to V_{16,3}$ extending $s \vee \hat{\jmath}_{14} \colon S^{15} \vee P^{14}(2) \to V_{16,3}$ where the inclusion $\hat{\jmath}_{14}$ is a restriction of the inclusion $V_{15,2} \to V_{16,3}$, so there is similarly an induced map $P \to V_{16,3}$ which is a 41-equivalence. $\square$

7.5 The 2-connected cover of $J_3(S^2)$

In this section we consider some relations between the fibre bundle

$$S^{4n-1} \longrightarrow V_{4n+1,2} \longrightarrow S^{4n}$$

defined by projection onto the first vector of an orthonormal 2-frame in $\mathbb{R}^{4n+1}$ (equivalently, the unit tangent bundle over S^{4n}) and the homotopy fibration

$$BW_n \longrightarrow \Omega^2 S^{4n+1}\{2\} \longrightarrow W_{2n}$$

of Lemma 6.2. Letting $\partial\colon \Omega S^{4n} \to S^{4n-1}$ denote the connecting map of the first fibration, we will show that there is a morphism of homotopy fibrations

$$\begin{array}{ccccc} \Omega^2 S^{4n} & \xrightarrow{\Omega\partial} & \Omega S^{4n-1} & \longrightarrow & \Omega V_{4n+1,2} \\ \downarrow & & \downarrow & & \downarrow \\ \Omega W_{2n} & \longrightarrow & BW_n & \longrightarrow & \Omega^2 S^{4n+1}\{2\} \end{array} \tag{7.5}$$

from which it will follow that for $n = 1, 2$ or 4, $\Omega\partial$ lifts through $\Omega\phi_n\colon \Omega^3 S^{4n+1} \to \Omega S^{4n-1}$. If this lift can be chosen to be $\Omega^2 E$, then it follows that there is a homotopy fibration diagram

$$\begin{array}{ccccc} \Omega^2 V_{4n+1,2} & \longrightarrow & \Omega^2 S^{4n} & \xrightarrow{\Omega\partial} & \Omega S^{4n-1} \\ \downarrow & & \downarrow{\scriptstyle \Omega^2 E} & & \Vert \\ W_n & \xrightarrow{\Omega j} & \Omega^3 S^{4n+1} & \xrightarrow{\Omega\phi_n} & \Omega S^{4n-1} \\ \downarrow & & \downarrow{\scriptstyle \Omega^2 H} & & \\ \Omega^3 S^{8n+1} & = & \Omega^3 S^{8n+1} & & \end{array} \tag{7.6}$$

which identifies $\Omega^2 V_{4n+1,2}$ with the homotopy pullback of Ωj and $\Omega^2 E$. We verify this and deloop it for $n = 1$ since it leads to an interesting identification of $V_{5,2}$ with the 2-connected cover of $J_3(S^2)$ which gives isomorphisms $\pi_k(V_{5,2}) \cong \pi_k(J_3(S^2))$ for all $k \geq 3$.

In his factorization of the 4^{th} power map on $\Omega^2 S^{2n+1}$ through the double suspension, Theriault constructs in [50] a space A and a map $\overline{E}\colon A \to \Omega S^{2n+1}\{2\}$ with the following properties:

(a) $H_*(A) \cong \Lambda(x_{2n-1}, x_{2n})$ with Bockstein $\beta x_{2n} = x_{2n-1}$;

(b) $\overline{E}$ induces a monomorphism in homology;

(c) There is a homotopy fibration $S^{2n-1} \to A \to S^{2n}$ and a homotopy fibration diagram

$$\begin{array}{ccccc} S^{2n-1} & \longrightarrow & A & \longrightarrow & S^{2n} \\ \downarrow{\scriptstyle E^2} & & \downarrow{\scriptstyle \overline{E}} & & \downarrow{\scriptstyle E} \\ \Omega^2 S^{2n+1} & \longrightarrow & \Omega S^{2n+1}\{2\} & \longrightarrow & \Omega S^{2n+1}. \end{array}$$

Noting that the homology of A is isomorphic to the homology of the unit tangent bundle $\tau(S^{2n})$ as a coalgebra over the Steenrod algebra, Theriault raises the question of whether A is homotopy equivalent to $\tau(S^{2n}) = V_{2n+1,2}$. Our next proposition shows this is true for any space A with the properties above.

Proposition 7.17. *There is a homotopy equivalence $A \simeq V_{2n+1,2}$.*

Proof. First we show that A splits stably as $P^{2n} \vee S^{4n-1}$. As in [50], let Y denote the $(4n-1)$-skeleton of $\Omega S^{2n+1}\{2\}$. Consider the homotopy fibration

$$\Omega S^{2n+1}\{2\} \longrightarrow \Omega S^{2n+1} \xrightarrow{\ 2\ } \Omega S^{2n+1}$$

and recall that $H_*(\Omega S^{2n+1}\{2\}) \cong H_*(\Omega S^{2n+1}) \otimes H_*(\Omega^2 S^{2n+1})$. Restricting the fibre inclusion to Y and suspending once we obtain a homotopy commutative diagram

$$\begin{array}{ccccccc}
 & & & & S^{2n+1} & \xrightarrow{\underline{2}} & S^{2n+1} \\
 & & & \nearrow^{\ell} & \downarrow & & \downarrow \\
\Sigma Y & \longrightarrow & \Sigma\Omega S^{2n+1}\{2\} & \longrightarrow & \Sigma\Omega S^{2n+1} & \xrightarrow{\Sigma 2} & \Sigma\Omega S^{2n+1}
\end{array}$$

where $\underline{2}$ is the degree 2 map, the vertical maps are inclusions of the bottom cell of $\Sigma\Omega S^{2n+1}$ and a lift ℓ inducing an isomorphism in $H_{2n+1}(\ )$ exists since ΣY is a $4n$-dimensional complex and $\mathrm{sk}_{4n}(\Sigma\Omega S^{2n+1}) = S^{2n+1}$. It follows from the James splitting $\Sigma\Omega S^{2n+1} \simeq \bigvee_{i=1}^{\infty} S^{2ni+1}$ and the commutativity of the diagram that $\underline{2} \circ \ell$ is nullhomotopic, so in particular $\Sigma\ell$ lifts to the fibre $S^{2n+2}\{\underline{2}\}$ of the degree 2 map on S^{2n+2}. Since $H_*(S^{2n+2}\{\underline{2}\}) \cong \mathbb{Z}/2\mathbb{Z}[u_{2n+1}] \otimes \Lambda(v_{2n+2})$ with $\beta v_{2n+2} = u_{2n+1}$, this implies $\Sigma\ell$ factors through a map $r\colon \Sigma^2 Y \to P^{2n+2}(2)$ which induces an epimorphism in homology by naturality of the Bockstein. Precomposing r with the inclusion $P^{2n+2}(2) \to \Sigma^2 Y$ therefore shows that $P^{2n+2}(2)$ is a retract of $\Sigma^2 Y$. (Alternatively, the retraction r can be obtained by suspending a lift $\Sigma Y \to S^{2n+1}\{\underline{2}\}$ of ℓ and using the well-known fact that $\Sigma S^{2n+1}\{\underline{2}\}$ splits as a wedge of Moore spaces.) Now since $\overline{E}\colon A \to \Omega S^{2n+1}\{2\}$ factors through Y and induces a monomorphism in homology, composing $\Sigma^2 A \to \Sigma^2 Y$ with the retraction r shows that $\Sigma^2 A \simeq P^{2n+2}(2) \vee S^{4n+1}$.

Next, let $E^\infty\colon A \to QA$ denote the stabilization map (where $Q = \Omega^\infty\Sigma^\infty$) and let F denote the homotopy fibre of a map $g\colon QP^{2n}(2) \to K(\mathbb{Z}/2\mathbb{Z}, 4n-2)$ representing the mod 2 cohomology class $u_{2n-1}^2 \in H^{4n-2}(QP^{2n}(2))$. A homology calculation shows that the $(4n-1)$-skeleton of F is a three-cell complex with homology isomorphic to $\Lambda(x_{2n-1}, x_{2n})$ as a coalgebra. The splitting $\Sigma^2 A \simeq \Sigma^2(P^{2n}(2) \vee S^{4n-1})$ gives rise to a map $\pi_1\colon QA \simeq QP^{2n}(2) \times QS^{4n-1} \to QP^{2n}(2)$ inducing isomorphisms on $H_{2n-1}(\ )$

and $H_{2n}(\ )$, and since the composite

$$A \xrightarrow{E^\infty} QA \xrightarrow{\pi_1} QP^{2n}(2) \xrightarrow{g} K(\mathbb{Z}/2\mathbb{Z}, 4n-2)$$

is nullhomotopic for degree reasons, there is a lift $A \to F$ of $\pi_1 \circ E^\infty$ inducing isomorphisms on $H_{2n-1}(\ )$ and $H_{2n}(\ )$. The coalgebra structure of $H_*(A)$ then implies this lift is a $(4n-1)$-equivalence and the result follows as $V_{2n+1,2}$ can similarly be seen to be homotopy equivalent to the $(4n-1)$-skeleton of F. □

The morphism of homotopy fibrations (7.5) is now obtained by noting that the composite $\Omega S^{4n-1} \to \Omega V_{4n+1,2} \xrightarrow{\Omega\overline{E}} \Omega^2 S^{4n+1}\{2\}$ is homotopic to $\Omega S^{4n-1} \xrightarrow{\Omega E^2} \Omega^3 S^{4n+1} \to \Omega^2 S^{4n+1}\{2\}$, which in turn is homotopic to a composite $\Omega S^{4n-1} \to BW_n \to \Omega^2 S^{4n+1}\{2\}$ since by Theorem 6.1 there is a homotopy fibration diagram

$$\begin{array}{ccccccc}
\Omega S^{4n-1} & \longrightarrow & BW_n & \xrightarrow{j} & \Omega^2 S^{4n+1} & \xrightarrow{\phi_n} & S^{4n-1} \\
\downarrow{\scriptstyle \Omega E^2} & & \downarrow & & \| & & \downarrow{\scriptstyle E^2} \\
\Omega^3 S^{4n+1} & \longrightarrow & \Omega^2 S^{4n+1}\{2\} & \longrightarrow & \Omega^2 S^{4n+1} & \xrightarrow{2} & \Omega^2 S^{4n+1}.
\end{array}$$

Specializing to the $n = 1$ case, the proof of Theorem 7.19 will show that $\Omega V_{5,2}$ fits in a delooping of diagram (7.6). We will need the following cohomological characterization of $V_{5,2}$.

Lemma 7.18. *Let E be the total space of a fibration $S^3 \to E \to S^4$. If E has integral cohomology group $H^4(E;\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and mod 2 cohomology ring $H^*(E)$ an exterior algebra $\Lambda(u,v)$ with $|u| = 3$ and $|v| = 4$, then E is homotopy equivalent to the Stiefel manifold $V_{5,2}$.*

Proof. As shown in [57, Theorem 5.8], the top row of the homotopy pullback diagram

$$\begin{array}{ccccc}
X^4 & \longrightarrow & P^4(2) & \longrightarrow & BS^3 \\
\downarrow & & \downarrow{\scriptstyle q} & & \| \\
S^7 & \xrightarrow{\nu} & S^4 & \longrightarrow & BS^3
\end{array}$$

induces a split short exact sequence

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \pi_6(P^4(2)) \longrightarrow \pi_5(S^3) \longrightarrow 0$$

from which it follows that $\pi_6(P^4(2)) = \mathbb{Z}/4\mathbb{Z}\{\lambda\} \oplus \mathbb{Z}/2\mathbb{Z}\{\widetilde{\eta}_3^2\}$ where λ is the attaching map of the top cell of $V_{5,2}$ and $\widetilde{\eta}_3^2$ maps to the generator η_3^2 of $\pi_5(S^3)$. It follows from the cohomological assumptions that

$E \simeq P^4(2) \cup_f e^7$, where $f = a\lambda + b\widetilde{\eta}_3^2$ for some $a \in \mathbb{Z}/4\mathbb{Z}$, $b \in \mathbb{Z}/2\mathbb{Z}$, and that $H_*(\Omega E)$ is isomorphic to a polynomial algebra $\mathbb{Z}/2\mathbb{Z}[u_2, v_3]$. Since the looped inclusion $\Omega P^4(2) \to \Omega E$ induces the abelianization map $T(u_2, v_3) \to \mathbb{Z}/2\mathbb{Z}[u_2, v_3]$ in homology, it is easy to see that the adjoint $f' \colon S^5 \to \Omega P^4(2)$ of f has Hurewicz image $[u_2, v_3] = u_2 \otimes v_3 - v_3 \otimes u_2$ and hence f is not divisible by 2. Moreover, since E is an S^3-fibration over S^4, the pinch map $q \colon P^4(2) \to S^4$ must extend over E. This implies the composite $S^6 \xrightarrow{f} P^4(2) \xrightarrow{q} S^4$ is nullhomotopic and therefore $b = 0$ by the commutativity of the diagram above. It now follows that $f = \pm\lambda$ which implies $E \simeq V_{5,2}$. □

Theorem 7.19. *There is a homotopy fibration*

$$V_{5,2} \longrightarrow J_3(S^2) \longrightarrow K(\mathbb{Z}, 2)$$

which is split after looping.

Proof. Let h denote the composite $\Omega S^3\langle 3\rangle \to \Omega S^3 \xrightarrow{H} \Omega S^5$ and consider the pullback

$$\begin{array}{ccc} P & \longrightarrow & S^4 \\ \downarrow & & \downarrow E \\ \Omega S^3\langle 3\rangle & \xrightarrow{h} & \Omega S^5. \end{array}$$

Since h has homotopy fibre S^3, so does the map $P \to S^4$. Next, observe that P is the homotopy fibre of the composite $\Omega S^3\langle 3\rangle \xrightarrow{h} \Omega S^5 \xrightarrow{H} \Omega S^9$ and since ΩS^9 is 7-connected, the inclusion of the 7-skeleton of $\Omega S^3\langle 3\rangle$ lifts to a map $\mathrm{sk}_7(\Omega S^3\langle 3\rangle) \to P$. Recalling that $H^4(\Omega S^3\langle 3\rangle; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_*(\Omega S^3\langle 3\rangle) \cong \Lambda(u_3) \otimes \mathbb{Z}/2\mathbb{Z}[v_4]$ with generators in degrees $|u_3| = 3$ and $|v_4| = 4$, it follows that this lift must be a homology isomorphism and hence a homotopy equivalence. So P is homotopy equivalent to the total space of a fibration satisfying the hypotheses of Lemma 7.18 and there is a homotopy equivalence $P \simeq V_{5,2}$.

It is well known that the iterated composite of the p^{th} James–Hopf invariant $H^{\circ k} \colon \Omega S^{2n+1} \to \Omega S^{2np^k+1}$ has homotopy fibre $J_{p^k-1}(S^{2n})$, the $(p^k-1)^{\text{st}}$ stage of the James construction on S^{2n}. The argument above identifies $V_{5,2}$ with the homotopy fibre of the composite

$$\Omega S^3\langle 3\rangle \longrightarrow \Omega S^3 \xrightarrow{H} \Omega S^5 \xrightarrow{H} \Omega S^9,$$

so there is a homotopy pullback diagram

$$\begin{CD} V_{5,2} @>>> J_3(S^2) @>>> K(\mathbb{Z},2)\\ @VVV @VVV @|\\ \Omega S^3\langle 3\rangle @>>> \Omega S^3 @>>> K(\mathbb{Z},2)\\ @VV{H\circ h}V @VV{H^{\circ 2}}V \\ \Omega S^9 @= \Omega S^9 \end{CD}$$

where the maps into $K(\mathbb{Z}, 2)$ represent generators of $H^2(J_3(S^2); \mathbb{Z}) \cong \mathbb{Z}$ and $H^2(\Omega S^3; \mathbb{Z}) \cong \mathbb{Z}$. To see that the homotopy fibration along the top row splits after looping, note that the connecting map $\Omega K(\mathbb{Z}, 2) = S^1 \to V_{5,2}$ is nullhomotopic since $V_{5,2}$ is simply connected. Therefore the looped projection map $\Omega J_3(S^2) \to S^1$ has a right homotopy inverse producing a splitting $\Omega J_3(S^2) \simeq S^1 \times \Omega V_{5,2}$ (by Lemma 2.9). □

Corollary 7.20. *$\pi_k(J_3(S^2)) \cong \pi_k(V_{5,2})$ for all $k \geq 3$.*

Chapter 8

The $p = 3$ Case and Related Problems

8.1 Anick's spaces

Let p be an odd prime and localize all spaces and maps at p. In their seminal work on the homotopy theory of odd primary Moore spaces [13, 14, 34], Cohen, Moore and Neisendorfer determined a decomposition of the loop space of $P^n(p^r)$, for $n \geq 3$ and $r \geq 1$, into a product of indecomposable spaces up to homotopy. Using this and similar loop space decompositions of some related spaces, they constructed a map $\varphi\colon \Omega^2 S^{2n+1} \to S^{2n-1}$ with the property that the composite

$$\Omega^2 S^{2n+1} \xrightarrow{\varphi} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$$

is homotopic to the $(p^r)^{\text{th}}$ power map on $\Omega^2 S^{2n+1}$. The existence of such a map for $r = 1$ immediately implies by induction on n that the $(p^n)^{\text{th}}$ power map on $\Omega^{2n} S^{2n+1}$ factors through S^1, and hence that p^n annihilates all p-torsion in $\pi_*(S^{2n+1})$ for odd primes p.

Cohen, Moore and Neisendorfer conjectured in [15] that there should exist a space $T^{2n+1}(p^r)$ and a fibration sequence

$$\Omega^2 S^{2n+1} \xrightarrow{\varphi} S^{2n-1} \longrightarrow T^{2n+1}(p^r) \longrightarrow \Omega S^{2n+1}$$

in which their map occurs as the connecting map. The existence of such a fibration was first proved by Anick for $p \geq 5$ as the culmination of the 270 page book [4]. A much simpler construction, valid for all odd primes, was later given by Gray and Theriault in [22], in which they also show that Anick's space

$T^{2n+1}(p^r)$ has the structure of an H-space and that all maps in the fibration above can be chosen to be H-maps.

Theorem 8.1 ([22])**.** *Let $p \geq 3$, $r \geq 1$ and $n \geq 1$. Then there is an H-space $T^{2n+1}(p^r)$ and a homotopy fibration sequence of H-spaces and H-maps*

$$\Omega^2 S^{2n+1} \xrightarrow{\varphi} S^{2n-1} \longrightarrow T^{2n+1}(p^r) \longrightarrow \Omega S^{2n+1}$$

with the property that $E^2 \circ \varphi \simeq p^r$.

The existence of Anick's spaces was also anticipated in work of Gray [19] in which the author considered the possibility of constructing certain unstable filtrations of Smith–Toda spectra $V(m)$ suitable for generalizing the classical EHP sequences and "composition methods" pioneered by Toda in studying the homotopy groups of spheres. In particular, Gray conjectured the existence of secondary EHP sequences given by homotopy fibrations

$$T_{2n-1} \xrightarrow{\quad E \quad} \Omega T_{2n} \xrightarrow{\quad H \quad} BW_n \tag{8.1}$$

$$T_{2n} \xrightarrow{\quad E \quad} \Omega T_{2n+1} \xrightarrow{\quad H \quad} BW_{n+1}$$

where the sequence of spaces $\{T_i\}$ together with structure maps given by (adjoints of) the maps $E\colon T_i \to \Omega T_{i+1}$ would represent a spectrum equivalent to the Moore spectrum $S^0 \cup_p e^1 = V(0)$, just as the spaces in the classical EHP fibrations (3.1) and (3.2) form an unstable filtration of the sphere spectrum $S^0 = V(-1)$. Gray found that the fibrations above could be constructed if T_{2n} was taken to be $S^{2n+1}\{p\}$ and T_{2n-1} was the total space of a fibration as in Theorem 8.1. The existence of these secondary EHP fibrations was proved in [22] as a consequence of the construction of Anick's space given there. We give a brief description of this construction next since the main result of this chapter states that the first of the secondary EHP fibrations above splits in certain cases having to do with the Kervaire invariant, similar to how the 2-primary EHP fibration (3.1) splits in Hopf invariant one cases.

The construction of Anick's fibration in [22] was accomplished by proving that there is an extension

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & \xrightarrow{\partial} & \Omega S^{2n+1}\{p^r\} \\ \Big\downarrow{\scriptstyle \nu} & \swarrow{\scriptstyle H} & \\ BW_n & & \end{array}$$

of the map ν from (6.1) through the connecting map ∂ from the homotopy fibration sequence

$$\Omega^2 S^{2n+1} \xrightarrow{\partial} \Omega S^{2n+1}\{p^r\} \longrightarrow \Omega S^{2n+1} \xrightarrow{p^r} \Omega S^{2n+1}.$$

Given such an extension, Anick's space $T^{2n+1}(p^r)$ can be defined as the homotopy fibre of H, and the homotopy fibration projection map of Theorem 8.1 is then given by the composite $T^{2n+1}(p^r) \to \Omega S^{2n+1}\{p^r\} \to \Omega S^{2n+1}$ as in the homotopy fibration diagram

$$\begin{array}{ccccccc}
\Omega^2 S^{2n+1} & \xrightarrow{\varphi} & S^{2n-1} & \longrightarrow & T^{2n+1}(p^r) & \longrightarrow & \Omega S^{2n+1} \\
\| & & \downarrow{\scriptstyle E^2} & & \downarrow{\scriptstyle E} & & \| \\
\Omega^2 S^{2n+1} & \xrightarrow{p^r} & \Omega^2 S^{2n+1} & \xrightarrow{\partial} & \Omega S^{2n+1}\{p^r\} & \longrightarrow & \Omega S^{2n+1} \\
& & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle H} & & \\
& & BW_n & = & BW_n. & &
\end{array} \qquad (8.2)$$

Note that this simultaneously constructs Anick's fibration (of Theorem 8.1) and the secondary EHP fibration (8.1). (The other secondary EHP fibration is easily obtained by defining the secondary Hopf invariant map $H\colon \Omega T^{2n+3}(p^r) \to BW_{n+1}$ to be the composite $\Omega T^{2n+3}(p^r) \to \Omega^2 S^{2n+3} \xrightarrow{\nu} BW_{n+1}$ and checking that its homotopy fibre is $S^{2n+1}\{p^r\}$.)

In addition to its relevance to Cohen, Moore and Neisendorfer's exponent theorem and Gray's secondary EHP sequences, Anick's space also represents a potential candidate for a double classifying space of the fibre W_n of the double suspension $E^2\colon S^{2n-1} \to \Omega^2 S^{2n+1}$. A longstanding conjecture in homotopy theory states that, after localizing at an odd prime p, W_n is a double loop space, and one of Cohen, Moore and Neisendorfer's remaining open conjectures in [15] states that there should be a p-local homotopy equivalence $W_n \simeq \Omega^2 T^{2np+1}(p)$. Since Gray's construction of the classifying space BW_n shows that W_n has the homotopy type of at least a single loop space, a stronger form of the conjecture appearing in many places throughout the literature states that

$$BW_n \simeq \Omega T^{2np+1}(p).$$

Such an equivalence would provide a long-sought link between the EHP sequence and the work of Cohen, Moore and Neisendorfer on the homotopy theory of Moore spaces, and furthermore, would have implications for the differentials in the EHP spectral sequence calculating the homotopy groups of spheres.

Note that both spaces BW_n and $\Omega T^{2np+1}(p)$ appear as fibres of maps of the form $\Omega^2 S^{2np+1} \to S^{2np-1}$

and the conjecture would follow if it were known that the np (and $r = 1$) case of the connecting map φ in Anick's fibration (Theorem 8.1) was homotopic to the map ϕ_n in Theorem 6.1. In view of Lemma 6.2, one could also ask whether the projection map in the homotopy fibration

$$BW_n \longrightarrow \Omega^2 S^{2np+1}\{p\} \longrightarrow W_{np}$$

is homotopic to the loops on the map H in the homotopy fibration

$$T^{2np+1}(p) \xrightarrow{E} \Omega S^{2np+1}\{p\} \xrightarrow{H} BW_{np}$$

appearing in (8.2) above.

To date, equivalences $W_n \simeq \Omega^2 T^{2np+1}(p)$ are only known to exist for $n = 1$ and $n = p$.

Theorem 8.2 ([46], [49]). *Let $p \geq 3$. Then there are homotopy equivalences of H-spaces*

(a) $BW_1 \simeq \Omega T^{2p+1}(p)$

(b) $BW_p \simeq \Omega T^{2p^2+1}(p)$.

In the former case, both BW_1 and $\Omega T^{2p+1}(p)$ are well known to be homotopy equivalent to $\Omega^2 S^3\langle 3\rangle$. Using Anick's fibration, Selick showed in [46] that $T^{2p+1}\{p\} \simeq \Omega S^3\langle 3\rangle$ and that his decomposition of $\Omega^2 S^{2p+1}\{p\}$ can be delooped to a homotopy equivalence

$$\Omega S^{2p+1}\{p\} \simeq \Omega S^3\langle 3\rangle \times BW_p.$$

Part (b) of the theorem above was proved by Theriault [49] using in an essential way this decomposition of $\Omega S^{2p+1}\{p\}$. Generalizing this argument, we prove in Section 8.3 the $n = p^2$ and $n = p^3$ cases of the strong form of the conjecture for $p = 3$.

8.2 The Kervaire invariant problem

The purpose of this section is to briefly review the Kervaire invariant problem and prove a lemma comparing various forms of the problem which we will need in the next section.

Originating in differential topology, the Kervaire invariant refers to a $\mathbb{Z}/2\mathbb{Z}$-valued invariant of a framed $(4m+2)$-dimensional manifold defined as the Arf invariant of a certain quadratic form

$$q\colon H^{2m+1}(M; \mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

determined by the framing on M and the intersection form on the middle dimensional cohomology group. Viewed as a framed cobordism invariant, the Kervaire invariant determines a homomorphism

$$K\colon \Omega^{\mathrm{fr}}_{4m+2} \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

where Ω^{fr}_* denotes the framed cobordism ring.

Tracing this construction through the Pontrjagin–Thom isomorphism $\Omega^{\mathrm{fr}}_* \cong \pi^S_*$, Browder showed in [8] that the Kervaire invariant can only be nontrivial for manifolds of dimension $2^{j+1}-2$ and that there exists a framed manifold of this dimension with Kervaire invariant one if and only if a certain element

$$h_j^2 \in E_2^{2,2^{j+1}} = \mathrm{Ext}^{2,2^{j+1}}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$$

is a permanent cycle in the mod 2 Adams spectral sequence $\{E_r^{*,*}, d_r\}$, which converges to the 2-component of π^S_*. Here, $h_j \in \mathrm{Ext}^{1,2^j}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$ is the Hopf invariant element which corresponds to the indecomposable element Sq^{2^j} of the mod 2 Steenrod algebra $\mathcal{A}_2$ in the sense that h_j is a permanent cycle representing a stable homotopy class $\alpha \in \pi^S_{2^j-1}$ if and only if Sq^{2^j} detects α, i.e., acts nontrivially on $H^*(C_\alpha)$. Similarly, h_j^2 corresponds to the secondary cohomology operation $\Phi_{j,j}$ defined by the Adem relation

$$Sq^{2^j} Sq^{2^j} = \sum_{i=0}^{j-1} Sq^{2^{j+1}-2^i} Sq^{2^i},$$

so Browder's result says there exists a framed $(2^{j+1}-2)$-manifold with nontrivial Kervaire invariant if and only if there exists an element $\theta_j \in \pi^S_{2^{j+1}-2}$ which is detected by $\Phi_{j,j}$.

For odd primes $p \geq 3$, there are also Hopf invariant elements $h_j \in \mathrm{Ext}^{1,2p^j(p-1)}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$ corresponding to the indecomposable elements $\mathcal{P}^{p^j}$ of the mod p Steenrod algebra $\mathcal{A}_p$ and, analogous to the h_j^2 for $p = 2$, there are elements denoted

$$b_{j-1} \in E_2^{2,2p^j(p-1)} = \mathrm{Ext}^{2,2p^j(p-1)}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$$

which correspond to a secondary cohomology operation Γ_{j-1} defined by the Adem relation expanding

$$\mathcal{P}^{(p-1)p^{j-1}} \mathcal{P}^{p^{j-1}}.$$

Definition 8.3. Let p be odd (respectively, $p = 2$). We say that $\pi^S_{2p^j(p-1)-2}$ contains an element of *p-primary Kervaire invariant one* if b_{j-1} (respectively, h_j^2) is a permanent cycle in the mod p Adams spectral sequence. In this case, we denote any stable homotopy class detected by b_{j-1} (respectively, h_j^2) by $\theta_j \in \pi^S_{2p^j(p-1)-2}$ and say that θ_j exists.

For any prime p, one can distinguish between various weak and strong forms of the p-primary Kervaire invariant problem. In its present stable homotopy theoretic form, the weak Kervaire invariant problem asks when the hypothetical element $\theta_j \in \pi^S_{2p^j(p-1)-2}$ exists, that is, for which j the element

$$\begin{Bmatrix} h_j^2 & \text{if } p = 2 \\ b_{j-1} & \text{if } p \geq 3 \end{Bmatrix} \in \mathrm{Ext}^{2,2p^j(p-1)}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$$

is a permanent cycle in the Adams spectral sequence. The strong form asks for which j there exists a p-primary Kervaire invariant one element $\theta_j \in \pi^S_{2p^j(p-1)-2}$ with $p\theta_j = 0$. To date, in all cases where an element of Kervaire invariant one is known to exist, θ_j can be chosen to have order p, and it is conjectured that the strong and weak forms are equivalent.

Another form of the problem, which we could call the unstable strong Kervaire invariant problem, asks for which n there exists a three-cell complex with nontrivial Bockstein β and Steenrod operation $\mathcal{P}^n$ (or Sq^n if $p = 2$). This form of the problem is used in [43] where several equivalent reformulations are given. In the next section, we will need to known that this form of the problem is equivalent to the strong Kervaire invariant problem above. The next lemma proves this. As usual, assume all spaces and maps have been localized at p.

Lemma 8.4. *Let $n > 1$. Then there exists a p-primary Kervaire invariant one element of order p in $\pi^S_{2n(p-1)-2}$ if and only if there exists a three-cell complex*

$$X = S^{2n+1} \cup e^{2np} \cup e^{2np+1}$$

with β and $\mathcal{P}^n$ (or Sq^{2n} if $p = 2$) acting nontrivially on $H^*(X)$.

Proof. First, let $p \geq 3$ and suppose $\pi^S_{2n(p-1)-2}$ contains an element of p-primary Kervaire invariant one with order p. Then $n = p^j$ for some $j \geq 1$ and we denote the element by θ_j. Since W_{n+1} is $(2(n+1)p-4)$-connected, the double suspension $E^2 \colon S^{2n+1} \to \Omega^2 S^{2n+3}$ is a $(2np + 2p - 3)$-equivalence and it follows that $\pi_{2np-1}(S^{2n+1})$ is stable. Therefore θ_j desuspends down to an element

$$\alpha \in \pi_{2np-1}(S^{2n+1}) \cong \pi^S_{2n(p-1)-2}.$$

Since $p\alpha = 0$, there is an extension of α to a map $\alpha' \colon P^{2np}(p) \to S^{2n+1}$. Let X be the homotopy cofibre

$C_{\alpha'}$ and consider the morphism of homotopy cofibrations

$$\begin{array}{ccccc} S^{2np-1} & \xrightarrow{\alpha} & S^{2n+1} & \longrightarrow & C_\alpha \\ \downarrow & & \| & & \downarrow \\ P^{2np}(p) & \xrightarrow{\alpha'} & S^{2n+1} & \longrightarrow & X. \end{array}$$

Since the stable secondary cohomology operation Γ_{j-1} acts nontrivially on $H^*(C_\alpha)$ and the induced map of cofibres is an isomorphism on $H^{2n+1}(\ )$ and $H^{2np}(\ )$, it follows that $\Gamma_{j-1}\colon H^{2n+1}(X) \to H^{2np}(X)$ is an isomorphism. The long exact cohomology sequence induced by the bottom row shows that the Bockstein $\beta\colon H^{2np}(X) \to H^{2np+1}(X)$ also acts by an isomorphism. Therefore the factorization of $\mathcal{P}^{p^j}$ by secondary cohomology operations

$$\mathcal{P}^{p^j} = \sum_{i=1}^{j-1} a_{j,i}\Psi_i + b_j\mathcal{R} + \sum_\gamma c_{j,\gamma}\Gamma_\gamma$$

given by Liulevicius [28] and Shimada and Yamanoshita [47] in their solution to the mod p Hopf invariant one problem shows that $\mathcal{P}^{p^j}$ must act by an isomorphism $H^{2n+1}(X) \to H^{2np+1}(X)$.

Conversely, suppose a space X has a cell structure $S^{2n+1} \cup e^{2np} \cup e^{2np+1}$ with nontrivial cohomology operations β and $\mathcal{P}^n$. Note that since $n > 1$ and $\mathcal{P}^n$ is decomposable in $\mathcal{A}_p$ unless n is a power of p, we have that $n = p^j$ for some $j \geq 1$. The formula above then implies that the Liulevicius–Shimada–Yamanoshita operation $\Gamma_{j-1} : H^{2n+1}(X) \to H^{2np}(X)$ is nontrivial and hence the attaching map $a\colon S^{2np-1} \to S^{2n+1}$ for the middle cell of X represents an element $\theta_j \in \pi^S_{2n(p-1)-2}$ of Kervaire invariant one. It remains to show that θ_j has order p. Let $i\colon S^{2n+1} \hookrightarrow X$ denote the inclusion of the bottom cell and observe that $i \circ a$ is nullhomotopic as i is the composite of skeletal inclusions $S^{2n+1} \hookrightarrow C_a \hookrightarrow X$ and $S^{2np-1} \xrightarrow{a} S^{2n+1} \hookrightarrow C_a$ is a homotopy cofibration. Thus a lifts through the $(2np)$-skeleton of the homotopy fibre of i, which by a Serre spectral sequence argument is homotopy equivalent to $P^{2np}(p)$. Since this shows that a extends to the Moore space $P^{2np}(p)$, we have that $pa = p\theta_j = 0$.

For $p = 2$, the lemma follows by a similar argument using Adams' [1] factorization of Sq^{2^j} for $j \geq 4$ by secondary cohomology operations in place of the factorization of $\mathcal{P}^{p^j}$ for $j \geq 1$ used above. In the remaining cases $j = 1, 2$ and 3, it is easy to check that the lemma holds using the first three 2-primary Kervaire invariant elements η^2, ν^2 and σ^2. $\square$

8.3 A homotopy decomposition of $\Omega S^{55}\{3\}$

In this section we consider the decomposition problem for loop spaces of the fibre $S^{2n+1}\{3\}$ of the degree 3 map on S^{2n+1}. For primes $p \geq 5$, it was shown in [43] that $\Omega^2 S^{2n+1}\{p\}$ is indecomposable if $n \neq 1$ or p. This result was obtained by showing that, for any odd prime p, a nontrivial homotopy decomposition of $\Omega^2 S^{2n+1}\{p\}$ implies the existence of a p-primary Kervaire invariant one element of order p in $\pi^S_{2n(p-1)-2}$. We will prove that the converse of this last implication is also true and that the (strong) odd primary Kervaire invariant problem is in fact equivalent to the problem of decomposing the single loop space $\Omega S^{2n+1}\{p\}$. When $p = 3$, this equivalence can be used to import results from stable homotopy theory to obtain new results concerning the unstable homotopy type of $\Omega S^{2n+1}\{3\}$ as well as some cases of the conjecture that W_n is a double loop space.

The following extension lemma, originally proved by Anick and Gray in [5] for $p \geq 5$ and later extended to include the $p = 3$ case by Gray and Theriault in [22], will be crucial.

Lemma 8.5. *Let p be an odd prime. Let X be an H-space such that $p^k \cdot \pi_{2np^k-1}(X; \mathbb{Z}/p^{k+1}\mathbb{Z}) = 0$ for $k \geq 1$. Then any map $P^{2n}(p) \to X$ extends to a map $T^{2n+1}(p) \to X$.*

Consider the homotopy fibration

$$T^{2n+1}(p) \xrightarrow{E} \Omega S^{2n+1}\{p\} \xrightarrow{H} BW_n \tag{8.3}$$

from diagram (8.2), where E induces in homology the inclusion of

$$H_*(T^{2n+1}(p)) \cong \Lambda(a_{2n-1}) \otimes \mathbb{Z}/p\mathbb{Z}[c_{2n}]$$

into

$$H_*(\Omega S^{2n+1}\{p\}) \cong \left(\bigotimes_{i=0}^{\infty} \Lambda(a_{2np^i-1})\right) \otimes \left(\bigotimes_{i=1}^{\infty} \mathbb{Z}/p\mathbb{Z}[b_{2np^i-2}]\right) \otimes \mathbb{Z}/p\mathbb{Z}[c_{2n}],$$

and H induces the projection onto

$$H_*(BW_n) \cong \left(\bigotimes_{i=1}^{\infty} \Lambda(a_{2np^i-1})\right) \otimes \left(\bigotimes_{i=1}^{\infty} \mathbb{Z}/p\mathbb{Z}[b_{2np^i-2}]\right).$$

When $n = p$, it follows from the proof in [46] of Selick's decomposition of $\Omega S^{2p+1}\{p\}$ that H admits a right homotopy inverse $s \colon BW_p \to \Omega S^{2p+1}\{p\}$ splitting the homotopy fibration (8.3) in this case.

Restricting to the bottom cell of BW_p, Theriault in [49] extended the composite

$$S^{2p^2-2} \hookrightarrow BW_p \xrightarrow{s} \Omega S^{2p+1}\{p\}$$

to a map $P^{2p^2-1}(p) \to \Omega S^{2p+1}\{p\}$ and then applied Lemma 8.5 to the adjoint map $P^{2p^2}(p) \to S^{2p+1}\{p\}$ to obtain an extension $T^{2p^2+1}(p) \to S^{2p+1}\{p\}$. Finally, looping this last map, he showed that the composite

$$\Omega T^{2p^2+1}(p) \longrightarrow \Omega S^{2p+1}\{p\} \xrightarrow{H} BW_p$$

is a homotopy equivalence, thus proving the $n = p$ case of the conjecture that $BW_n \simeq \Omega T^{2np+1}(p)$.

In our case, we will use Lemma 8.5 to first construct a right homotopy inverse of $H\colon \Omega S^{2n+1}\{p\} \to BW_n$ in dimensions $n = p^j$ for which there exists an element $\theta_j \in \pi^S_{2p^j(p-1)-2}$ of (strong) Kervaire invariant one and then follow the same strategy as above to obtain both a homotopy decomposition of $\Omega S^{2p^j+1}\{p\}$ and a homotopy equivalence $BW_{p^j} \simeq \Omega T^{2p^{j+1}+1}(p)$. These equivalences can then be used to compare the loops on (8.3) with the $n = p^{j-1}$ case of the homotopy fibration

$$BW_n \longrightarrow \Omega^2 S^{2np+1}\{p\} \longrightarrow W_{np}$$

of Lemma 6.2 to further obtain a homotopy equivalence of fibres $BW_{p^{j-1}} \simeq \Omega T^{2p^j+1}(p)$.

From this point of view, Selick's decomposition of $\Omega S^{2p+1}\{p\}$ and the previously known equivalences $BW_1 \simeq \Omega T^{2p+1}(p)$ and $BW_p \simeq \Omega T^{2p^2+1}(p)$ correspond to the existence (at all odd primes) of the Kervaire invariant one element $\theta_1 = \beta_1 \in \pi^S_{2p^2-2p-2}$ given by the first element of the periodic beta family in the stable homotopy groups of spheres. To emphasize this, we phrase the following theorem in terms of an arbitrary odd prime, although by Ravenel's negative solution to the Kervaire invariant problem for primes $p \geq 5$, the theorem has new content only at the prime $p = 3$.

Theorem 8.6. *Let p be an odd prime. Then the following are equivalent:*

(a) *There exists a p-primary Kervaire invariant one element $\theta_j \in \pi^S_{2p^j(p-1)-2}$ of order p;*

(b) *There is a homotopy decomposition of H-spaces $\Omega S^{2p^j+1}\{p\} \simeq T^{2p^j+1}(p) \times \Omega T^{2p^{j+1}+1}(p)$.*

Moreover, if the above conditions hold, then there are homotopy equivalences

$$BW_{p^{j-1}} \simeq \Omega T^{2p^j+1}(p) \quad \textit{and} \quad BW_{p^j} \simeq \Omega T^{2p^{j+1}+1}(p).$$

Proof. We first show that (b) implies (a). Given any homotopy equivalence

$$\psi\colon T^{2p^j+1}(p) \times \Omega T^{2p^{j+1}+1}(p) \xrightarrow{\simeq} \Omega S^{2p^j+1}\{p\},$$

let $n = p^j$ and let f denote the composite

$$f\colon S^{2np-2} \hookrightarrow \Omega T^{2np+1}(p) \xrightarrow{i_2} T^{2n+1}(p) \times \Omega T^{2np+1}(p) \xrightarrow{\psi} \Omega S^{2n+1}\{p\}$$

where the first map is the inclusion of the bottom cell of $\Omega T^{2np+1}(p)$ and the second map i_2 is the inclusion of the second factor. Then

$$f_*(\iota) = b_{2np-2} \in H_{2np-2}(\Omega S^{2n+1}\{p\})$$

where ι is the generator of $H_{2np-2}(S^{2np-2})$. Since by [43], the homology class b_{2np-2} is spherical if and only if there exists a stable map $g\colon P^{2n(p-1)-1}(p) \to S^0$ for which the Steenrod operation $\mathcal{P}^n$ acts nontrivially on $H^*(C_g)$, it follows from Lemma 8.4 that $\pi^S_{2n(p-1)-2}$ contains an element of p-primary Kervaire invariant one and order p.

Conversely, suppose there exists a p-primary Kervaire invariant one element $\theta_j \in \pi^S_{2p^j(p-1)-2}$ of order p. Then by Lemma 8.4 and [43], the homology class $b_{2p^{j+1}-2} \in H_{2p^{j+1}-2}(\Omega S^{2p^j+1}\{p\})$ is spherical, so there exists a map $f\colon S^{2p^{j+1}-2} \to \Omega S^{2p^j+1}\{p\}$ with Hurewicz image $b_{2p^{j+1}-2}$. Since $\Omega S^{2p^j+1}\{p\}$ has homotopy exponent p by Lemma 5.1, it follows that f has order p and hence extends to a map

$$e\colon P^{2p^{j+1}-1}(p) \longrightarrow \Omega S^{2p^j+1}\{p\}.$$

Let $\hat{e}\colon P^{2p^{j+1}}(p) \to S^{2p^j+1}\{p\}$ denote the adjoint of e. Again by Lemma 5.1,

$$p \cdot \pi_*(S^{2p^j+1}\{p\}; \mathbb{Z}/p^k\mathbb{Z}) = 0$$

for all $k \geq 1$, so since $S^{2p^j+1}\{p\}$ is an H-space [35], the map $\hat{e}$ satisfies the hypotheses of Lemma 8.5 and therefore admits an extension

$$s\colon T^{2p^{j+1}+1}(p) \longrightarrow S^{2p^j+1}\{p\}.$$

Note that this factorization of $\hat{e}$ through s implies that the adjoint map e factors through Ωs, so we have a commutative diagram

$$\begin{array}{ccccc} S^{2p^{j+1}-2} & \longrightarrow & P^{2p^{j+1}-1}(p) & \longrightarrow & \Omega T^{2p^{j+1}+1}(p) \\ & \searrow^{f} & & \searrow^{e} & \downarrow{\scriptstyle \Omega s} \\ & & & & \Omega S^{2p^j+1}\{p\} \end{array}$$

where the maps along the top row are skeletal inclusions, and hence $(\Omega s)_*$ is an isomorphism on $H_{2p^{j+1}-2}(\ )$ since f_* is. Since $H\colon \Omega S^{2p^j+1}\{p\} \to BW_{p^j}$ induces an epimorphism in homology, the composite

$$\Omega T^{2p^{j+1}+1}(p) \xrightarrow{\Omega s} \Omega S^{2p^j+1}\{p\} \xrightarrow{H} BW_{p^j}$$

induces an isomorphism of the lowest nonvanishing reduced homology group

$$H_{2p^{j+1}-2}(\Omega T^{2p^{j+1}+1}(p)) \cong H_{2p^{j+1}-2}(BW_{p^j}) \cong \mathbb{Z}/p\mathbb{Z}.$$

By [21], any map $\Omega T^{2np+1}(p) \to BW_n$ which is degree one on the bottom cell is a homotopy equivalence. Thus $H \circ \Omega s$ is a homotopy equivalence. Composing a homotopy inverse of $H \circ \Omega s$ with Ωs, we obtain a right homotopy inverse of H, which shows (by Lemma 2.9) that the homotopy fibration

$$T^{2p^j+1}(p) \xrightarrow{E} \Omega S^{2p^j+1}\{p\} \xrightarrow{H} BW_{p^j}$$

splits. Moreover, letting m denote the loop multiplication on $\Omega S^{2p^j+1}\{p\}$, the composite

$$T^{2p^j+1}(p) \times \Omega T^{2p^{j+1}+1}(p) \xrightarrow{E \times \Omega s} \Omega S^{2p^j+1}\{p\} \times \Omega S^{2p^j+1}\{p\} \xrightarrow{m} \Omega S^{2p^j+1}\{p\}$$

is a multiplicative homotopy equivalence since E and Ωs are H-maps and m is homotopic to the loops on the H-space multiplication on $S^{2p^j+1}\{p\}$.

It remains to show that there is a homotopy equivalence $BW_{p^{j-1}} \simeq \Omega T^{2p^j+1}(p)$. Let ϕ denote the homotopy equivalence $T^{2p^j+1}(p) \times \Omega T^{2p^{j+1}+1}(p) \xrightarrow{\simeq} \Omega S^{2p^j+1}\{p\}$ constructed above and consider the homotopy fibration

$$BW_{p^{j-1}} \longrightarrow \Omega^2 S^{2p^j+1}\{p\} \longrightarrow W_{p^j}$$

from Lemma 6.2. Since $\Omega\phi$ is also a homotopy equivalence, it has a homotopy inverse $(\Omega\phi)^{-1}$. We define a map $BW_{p^{j-1}} \to \Omega T^{2p^j+1}(p)$ by composing the fibre inclusion $BW_{p^{j-1}} \to \Omega^2 S^{2p^j+1}\{p\}$ above with $\pi_1 \circ (\Omega\phi)^{-1}$ as in the diagram

$$\begin{array}{ccccc} BW_{p^{j-1}} & \longrightarrow & \Omega^2 S^{2p^j+1}\{p\} & \longrightarrow & W_{p^j} \\ \downarrow & & \downarrow{\scriptstyle (\Omega\phi)^{-1}} & & \\ \Omega T^{2p^j+1}(p) & \xleftarrow{\pi_1} & \Omega T^{2p^j+1}(p) \times \Omega^2 T^{2p^{j+1}+1}(p) & & \end{array}$$

where π_1 denotes the projection onto the first factor. Since all three maps in the composition induce isomorphisms on $H_{2p^j-2}(\ )$, it again follows from the atomicity result in [21] that the composite defines a homotopy equivalence $BW_{p^{j-1}} \simeq \Omega T^{2p^j+1}(p)$. □

Remark 8.7. In the $j = 1$ case, it was shown in [46] and [49], respectively, that (for all odd primes p) the homotopy equivalences $BW_1 \simeq \Omega T^{2p+1}(p) \simeq \Omega^2 S^3\langle 3\rangle$ and $BW_p \simeq \Omega T^{2p^2+1}(p)$ can be chosen to be equivalences of H-spaces. Using the techniques of [49], the analogous homotopy equivalences in the conclusion of Theorem 8.6 can similarly be chosen to be multiplicative.

Since, by [43], $\Omega S^{2n+1}\{p\}$ is atomic for all n such that $\pi^S_{2n(p-1)-2}$ has no element of p-primary Kervaire invariant one, $\Omega S^{2n+1}\{p\}$ is indecomposable for $n \neq p^j$ and it follows from Theorem 8.6 that for odd primes the decomposition problem for $\Omega S^{2n+1}\{p\}$ is equivalent to the strong p-primary Kervaire invariant problem. Both problems are open for $p = 3$, but the elements

$$b_{j-1} \in \mathrm{Ext}^{2,2p^j(p-1)}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$$

in the E_2-term of the Adams spectral sequence which potentially detect elements of odd primary Kervaire invariant one are known to behave differently for $p = 3$ than they do for primes $p \geq 5$. Recall that, for all odd primes, b_0 is a permanent cycle in the Adams spectral sequence detecting the Kervaire invariant element $\theta_1 \in \pi^S_{2p(p-1)-2}$. In [38], Ravenel showed that for $j > 1$ and $p > 3$, the elements b_{j-1} support nontrivial differentials in the Adams spectral sequence and hence there do not exist any other Kervaire invariant elements $\theta_j \in \pi^S_{2p^j(p-1)-2}$ for $j > 1$ and $p > 3$. For $p = 3$ however, it is known (see [38, 39]) that while b_1 supports a nontrivial differential, b_2 is a permanent cycle representing a 3-primary Kervaire invariant element $\theta_3 \in \pi^S_{106}$. Since, according to [39], $\pi^S_{106} \cong \mathbb{Z}/3\mathbb{Z}$ (after localizing at $p = 3$), this stable homotopy class must have order 3, so we have the following consequences of Theorem 8.6.

Corollary 8.8. *There are 3-local homotopy equivalences*

(a) $\Omega S^{55}\{3\} \simeq T^{55}(3) \times \Omega T^{163}(3)$

(b) $BW_9 \simeq \Omega T^{55}(3)$

(c) $BW_{27} \simeq \Omega T^{163}(3)$.

Corollary 8.9. $\pi_k(S^{55}; \mathbb{Z}/3\mathbb{Z}) \cong \pi_{k-2}(T^{55}(3)) \oplus \pi_{k-1}(T^{163}(3))$

$$\cong \pi_{k-4}(W_9) \oplus \pi_{k-3}(W_{27}).$$

Corollary 8.10. $\Omega S^{19}\{3\}$ *is atomic.*

Two useful properties of Anick's space $T^{2n+1}(p^r)$ conjectured by Anick and Gray [4, 5] are that (i) $T^{2n+1}(p^r)$ is a homotopy commutative and homotopy associative H-space, and (ii) $T^{2n+1}(p^r)$ has homotopy exponent p^r. Both properties have been established for all $p \geq 5$, $r \geq 1$ and $n \geq 1$, but only partial results have been obtained in the $p = 3$ case. For example, in [48], it was shown that $T^7(3^r)$ is homotopy commutative and homotopy associative if $r = 1$ but not homotopy associative if $r > 1$, and moreover that $T^{11}(3^r)$ is not homotopy associative for any $r \geq 1$. More generally, Gray showed in [20] that for $n > 1$, $T^{2n+1}(3^r)$ is not homotopy associative if $r > 1$, and that if $T^{2n+1}(3)$ is homotopy associative, then $n = 3^k$ for some $k \geq 0$.

Concerning property (ii), [36] shows that $T^{2n+1}(3^r)$ has homotopy exponent 3^r for $r \geq 2$, so $T^{2n+1}(3)$ is the only remaining open case. The $p = 3$ cases of the homotopy equivalences $\Omega T^{2p+1}(p) \simeq BW_1$ and $\Omega T^{2p^2+1}(p) \simeq BW_p$ imply that $T^7(3)$ and $T^{19}(3)$ each have homotopy exponent 3, but in general $T^{2n+1}(3)$ is only known to have homotopy exponent bounded above by 9.

Corollary 8.11.

(a) *The H-space $T^{55}(3)$ is homotopy commutative and homotopy associative.*

(b) *$T^{55}(3)$ and $\Omega T^{163}(3)$ each have H-space exponent 3, and hence*

$$3 \cdot \pi_*(T^{55}(3)) = 3 \cdot \pi_*(T^{163}(3)) = 0.$$

Proof. Since the homotopy equivalence of Corollary 8.8(a) is an equivalence of H-spaces, part (b) follows immediately from the fact that $\Omega S^{2n+1}\{3\}$ has H-space exponent 3 [35], and part (a) follows from the fact that the loop space $\Omega S^{2n+1}\{p\}$ is homotopy associative and homotopy commutative (since, in particular, it is the loop space of an H-space). □

The proof in [20] that the 3-primary Anick space $T^{2n+1}(3)$ can only be homotopy associative when n is a power of 3 also shows that if $T^{2n+1}(3)$ is homotopy associative, then there exists a three-cell complex

$$S^{2n+1} \cup_3 e^{2n+2} \cup e^{6n+1}$$

with nontrivial mod 3 Steenrod operation $\mathcal{P}^n$, which implies (by Spanier–Whitehead duality and Lemma 8.4) the existence of an element of strong Kervaire invariant one. Note that Theorem 8.6 shows that the converse is also true, which gives the following interpretation of the 3-primary Kervaire invariant problem.

Theorem 8.12. *Let $n > 1$. Then there exists a 3-primary Kervaire invariant one element $\theta \in \pi^S_{4n-2}$ of order 3 if and only if Anick's space $T^{2n+1}(3)$ is homotopy associative.*

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