A groupoid approach to geometric mechanics

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1 Introduction

In 1966 V. Arnold proved that the Euler equation for an incompressible fluid describes the geodesic flow for a right-invariant metric on the group of volume-preserving diffeomorphisms of the fluid's domain [1]. This remarkable observation led to numerous advances in the study of the Hamiltonian properties, instabilities, and topological features of fluid flows. However, Arnold's approach does not apply to systems whose configuration spaces do not have a group structure. A particular example of such a system is that of a fluid with moving boundary. More generally, one can consider a system describing a rigid body moving in a fluid. Here the configurations of the fluid are identified with diffeomorphisms mapping a fixed reference domain to the exterior of the (moving) body. In general such diffeomorphisms cannot be composed, since the domain of one will not match the range of the other.

The systems we consider are numerous variations of a rigid body in an inviscid fluid. The different cases are specified by the properties of the fluid; the fluid may be compressible or incompressible, irrotational or not. By using groupoids we generalize Arnold's diffeomorphism group framework for fluid flows to show that the well-known equations governing the motion of these various systems can be viewed as geodesic equations (or more generally, Newton's equations) written on an appropriate configuration space. This extends the recent work [11], where a groupoid approach was developed to study incompressible fluid flows with vortex sheets.

We also show how constrained dynamical systems on larger algebroids are in many cases equivalent to dynamical systems on smaller algebroids, with the two systems being related by a generalized notion of Riemannian submersion. As an application, we show that incompressible fluid-body motion with the constraint that the fluid velocity is curl- and circulation-free is equivalent to solutions of Kirchhoff's equations on the finite-dimensional algebroid $\mathfrak{se}(n)$.

In order to prove these results, we further develop the theory of Lagrangian mechanics on algebroids. This approach to mechanics was initiated by Weinstein [27]. Significant advances were made by Martinez [20] et al. [5], [21]. Our approach is based on the use of vector bundle connections, which leads to new expressions for the canonical equations and structures on Lie algebroids and their duals.

There are two limiting cases of interest: the case of a fluid of zero density describes the motion of a body alone, while the case where the body is the empty set describes the motion of an ideal compressible or incompressible fluid. The case of a compressible fluid is of particular interest by itself. It turns out that for a large class of potential functions U, the gradient solutions of the compressible fluid equations can be related to solutions of Schrödinger-type equations via the *Madelung transform*, which was first introduced in 1927 [15], and more recently studied in [12] and [26]. We prove that the Madelung transform not only maps one class of equations to the other, but it also preserves the Hamiltonian properties of both equations. Namely, the non-linear Schrödinger equation is Hamiltonian with respect to the constant Poisson structure on the space of wave functions, which are complex valued (fast decaying) smooth functions on $\mathbb{R}^n$. On the other hand, the compressible Euler equation is Hamiltonian with respect to the natural Lie-Poisson structure on the space of pairs consisting of (fast decaying at infinity) fluid momenta μ and fluid densities ρ. This space is the dual of the Lie algebra of the semidirect product of the diffeomorphism group of $\mathbb{R}^n$ times the space of real-valued fast decaying functions, which is the configuration space of a compressible fluid.

The thesis is divided into three parts. The first consists of Sections 2 and 3, where the general theory of Lagrangian mechanics on algebroids is presented. The second part of the thesis comprises Sections 4 through 6, where the general theory is applied to various systems of a rigid body moving in a fluid. The last part, Section 7, studies geometric and group properties of the Madelung transform.

1.1 Main results

In this thesis we consider the following dynamical equations governing the motion of a rigid body in a fluid. In the simplest case, the fluid is incompressible and irrotational and there is no circulation around the body.[1] In this case, there are so many constraints on the fluid that its motion is completely determined by the motion of the body. The effect of the fluid is to add to the body's effective inertia. The governing equations for the body's motion are the *Kirchhoff equations*:

$$\begin{cases} \dfrac{d}{dt}\omega = [\omega, r] + \lambda \diamond l \\ \dfrac{d}{dt}\lambda = -r\lambda\,. \end{cases}$$

Here ω and λ are the effective angular and linear momenta of the fluid-body system, r and l are the angular and linear velocities, and the diamond product $\diamond : \mathbb{R}^n \times \mathbb{R}^n \to \mathfrak{so}^*(n)$ is defined by $\langle \lambda \diamond l, r\rangle := -\langle \lambda, rl\rangle$

If the fluid is no longer constrained to be irrotational, but still assumed to be incompressible, then the system is governed by the *incompressible fluid-body equations*:

$$\begin{cases} \dfrac{d}{dt}u + (u\cdot\nabla)u = -\nabla P \\ m\dfrac{d}{dt}l = \displaystyle\int_{\partial F_x} P\mathrm{n}\, i_{\mathrm{n}} d^n q \\ \dfrac{d}{dt}(r\mathbb{I}_x) = \displaystyle\int_{\partial F_x} P\mathrm{n}(q - q_x)^T\, i_{\mathrm{n}} d^n q \\ \dfrac{d}{dt}x = \xi\,. \end{cases}$$

[1]The condition that there be no circulation around the body follows from irrotationality of the fluid if the exterior of the body is simply connected. We will always assume this.

Here F_x is the domain of the fluid around the body located at position x and n is the outward pointing normal of the surface of the body ∂F_x. A superscript T denotes the transpose. The first equation is the incompressible Euler equation for the fluid with velocity u. The second and third equations are Newton's law for the body's linear momentum ml and angular momentum $r\mathbb{I}_x$ respectively. The last equation relates the body's position $x \in SE(n)$ to its velocity $\xi \in T_x SE(n)$. The function P is the pressure, which in addition to ensuring that the fluid motion remains incompressible throughout the motion, also ensures that the fluid's normal velocity at the boundary is equal to the body's normal velocity.

Finally, if there are no restrictions on the fluid, then the motion of system is governed by the *compressible fluid-body equations*:

$$
\begin{cases}
\frac{d}{dt}u + (u \cdot \nabla)u = \frac{\nabla P_1}{\tilde{\rho}} - \nabla P_2 \\
\frac{d}{dt}\tilde{\rho} + \nabla \cdot (\tilde{\rho} u) = 0 \\
m\frac{d}{dt}l = \int_{\partial F_x} (P_2 - \tilde{\rho}^{-1} P_1)\mathrm{n}\, i_{\mathrm{n}}\rho \\
\frac{d}{dt}(r\mathbb{I}_x) = \int_{\partial F_x} (P_2 - \tilde{\rho}^{-1} P_1)\mathrm{n}(q - q_x)^T\, i_{\mathrm{n}}\rho \\
\frac{d}{dt}x = \xi\,.
\end{cases}
$$

Compared to the incompressible case, the fluid density $\rho = \tilde{\rho} d^n q$ is an additional dynamical quantity. The first equation is the compressible Euler equation. The second equation is the continuity equation. The third and fourth equations, as before, are Newton's law for the body's linear and angular momenta, and the last equation relates the body's position to its velocity. The two functions P_1 and P_2 are pressure functions on F_x which play two distinct roles. The function P_1 is the same pressure that appears in the dynamics of a compressible fluid without a body. It is defined in terms of the fluid's *internal energy* $w : \mathbb{R} \to \mathbb{R}$ as $P_1 := \tilde{\rho}^2 w'(\tilde{\rho})$. The second pressure P_2 only arises when a body is present. It is responsible for maintaining the boundary condition at the fluid-body interface, as in the incompressible case.

On the other hand, rather than studying these equations, the dynamics of the fluid-body system can be approached by defining a Lagrangian L on (the tangent bundle of) the fluid-body configuration space and studying the corresponding Euler-Lagrange equations. The configuration space Q is defined in terms of a reference body, which is an open bounded subset of $\mathbb{R}^n$ defining the shape of the body, and a reference fluid density. The space Q is an infinite-dimensional manifold consisting of pairs (x, g), where $x \in SE(n)$ defines the position of the body, and g is a diffeomorphism defining the position of the fluid particles by mapping the exterior of the reference body to the exterior of the body located at position x. There is a natural L^2-type metric on Q that defines the kinetic energy $T(a) := \frac{1}{2}\langle a, a\rangle_{L^2}$ of the system. In the incompressible case, the Lagrangian $L = T$ is equal to the kinetic energy, and the corresponding Euler-Lagrange equation coincides with the geodesic equation. In the compressible case, the Lagrangian $L = T - U$ has a non-zero potential term U, and the Euler-Lagrange equation becomes Newton's equation.

Our first two main results concern the relation between these two approaches. They are reduction theorems analogous to Arnold's theorem describing incompressible fluid motion as geodesic flow on an appropriate group. A significant difference, however, is that the configuration space of the fluid-body system is *not* a group. Thus the proofs require a generalization of Arnold's approach. We have the following:

Theorem 1.1 (= Theorem 6.2 cf. [8])**.** *The incompressible fluid-body equations are equivalent to geodesic equations with respect to the natural L^2-type metric on the incompressible fluid-body configuration space.*

An analysis of the compressible system results in a similar theorem.

Theorem 1.2 (= Theorem 6.3)**.** *The compressible fluid-body equations are equivalent to Newton's equations with respect to the natural Lagrangian on the compressible fluid-body configuration space.*

The third result is a "reduction" in the sense that it relates *special* solutions of fluid-body dynamics on a large configuration space to solutions of a smaller, and in this case finite-dimensional system.

Theorem 1.3 (= Theorem 6.4)**.** *If a solution of the incompressible fluid-body equations is initially irrotational, then it remains irrotational. Moreover, such solutions of the fluid-body equations correspond to solutions of the (finite-dimensional) Kirchhoff equations.*

Rather than being compared to Arnold's theorem, this should be thought of in terms of Riemannian geometry. The incompressible fluid-body equations and the Kirchhoff equations are both (Lagrangian reductions of) geodesic equations. The incompressible fluid-body configuration space is mapped onto the Kirchhoff configuration space by (a generalization of) a Riemannian submersion. Thinking of irrotational solutions of the incompressible fluid-body system as horizontal geodesics, the theorem then is an infinite-dimensional manifestation of the fact that Riemannian submersions send horizontal geodesics to geodesics.

There are two interesting limiting cases of fluid-body dynamics. First, when the fluid density is zero, the dynamical equations reduce to the finite-dimensional equations governing the motion of a rigid body. Second, when the body is the empty set, the dynamical equations reduce to the Euler equations of an ideal compressible fluid.

We study the second case, the compressible fluid separately in detail. There is a well-known "hydrodynamical" formulation of quantum mechanics, where the fundamental dynamic quantity is not the wavefunction, but rather a compressible fluid governed by Euler fluid dynamics with a modified pressure term. The space of wavefunctions is mapped to the space of (potential) fluid momenta by means of the *Madelung transform.* In Section 7 we study the geometric properties of the Madelung transform. We prove the following main theorem.

Theorem 1.4 (= Theorem 7.10). *The Madelung transform is a Poisson map between the space of wavefunctions and the space of pairs* (ρ, μ)*, where* ρ *is a fluid density and* μ *is a potential fluid momentum, i.e. it sends one Poisson structure to the other. Moreover, the transform is a momentum map associated with a natural action of a certain semidirect product group on the space of wave functions.*

1.2 Methods and applications

Here we outline the tools from the theory of Lagrangian mechanics that we develop in order to prove the above results. Following each description of an algebroid-theoretical construction, we indicate how it is applied to the fluid-body problem.

Section 2 reviews the theory of Lie groupoids and Lie algebroids. Particular emphasis is put on vector bundle connections, as they are central to our formulation of Lagrangian mechanics.

Lie algebroids are often thought of as generalized tangent spaces. This makes them reasonable candidates to describe spaces of velocities of a physical system. In Section 5 we show that the space of velocities of the fluid-body system is naturally interpreted as an algebroid, denoted $\mathcal{FBA}$. The base of this fluid-body algebroid is the space of pairs (x, ρ), where x is an element of $SE(n)$ describing the body's position, and ρ is the density of the fluid on the exterior of the body. Elements of the fibre of $\mathcal{FBA}$ over a point (x, ρ) are pairs (ξ, u), where $\xi \in T_xSE(n)$ encodes the body's velocity, and $u \in \mathrm{vect}(F_x)$ is a vector field defined on the exterior of the body. The vector field u is the fluid velocity, and satisfies the boundary condition that the normal component of u matches the normal component of the body's velocity field. (This comes from the requirement that the fluid meets the body with no gaps or overlap between the two.) We call this the "equal normals" condition.

One of the important tools that we use is a version of the Hodge decomposition that splits $\mathcal{FBA}$ into three components.

Proposition 1.5. *We have the splitting*

$$\mathcal{FBA} = \mathcal{EFBA} \oplus \mathcal{CFBA} \oplus \mathcal{HFBA}$$

of the compressible fluid-body algebroid into exact $\mathcal{EFBA}$*, coexact* $\mathcal{CFBA}$ *and harmonic* $\mathcal{HFBA}$ *components. These subspaces are pairwise orthogonal with respect to the natural* L^2*-type metric on* $\mathcal{FBA}$*.*

Each subspace is named after the Hodge subspace that the fluid velocity resides in. For example, each element (ξ, u) of $\mathcal{EFBA}$ has a fluid velocity u that is an exact[2] vector field. The body velocity ξ is zero in each subspace except the harmonic potential component. Elements in this last component are pairs of the form $(\xi, \nabla h^\xi)$, where h^ξ is the unique harmonic field

[2]Exact vector fields are also called "potential" or "gradient" vector fields.

	Non-zero Hodge components				
Constraint	Exact	Coexact	Harmonic	Algebroid	ELA equation
compr. fluid (no constraint)	x	x	x	$\mathcal{FBA}$	Rigid body comp. Euler
incompr. fluid		x	x	$S\mathcal{FBA}$	Rigid body incomp. Euler
incompr., irrotat. fluid without circ.			x	$\mathfrak{se}(n)$	Kirchhoff

Table 1: Theorem 1.6: subsystems of a rigid body in a compressible fluid

satisfying the Neumann boundary conditions determined by the body's motion. The vector field ∇h^ξ has the lowest L^2 energy among all vector fields satisfying the "equal normals" condition. Thus the harmonic component isolates the influence of the motion of body on the fluid.

The Hodge decomposition allows us to characterize the subsystems of fluid-body dynamics. For example, the velocity of incompressible fluid motion is characterized by vanishing exact component in the Hodge splitting. Similarly, irrotational motion is characterized by vanishing coexact component.

1.2.1 Euler-Lagrange-Arnold equations and reduction

In Section 3.1 we develop, starting from a variational principle, the theory of Lagrangian mechanics on algebroids. This area of research was initiated by Weinstein [27] and developed by Martinez et al. [20], [21], [5], [22], and is very active at the moment. Our approach is novel in that it develops the theory in terms of vector bundle connections, which allows the equations and theorems to be stated geometrically without requiring much abstraction.

The central equations in Lagrangian mechanics on algebroids are what we call the Euler-Lagrange-Arnold (ELA) equations. These are equations defined on Lie algebroids that generalize both the Euler-Lagrange equations on a tangent bundle and the Euler-Arnold equations on a Lie algebra. Section 5 is devoted to bringing the fluid-body systems we consider into the unified framework of mechanics on algebroids. We prove the following:

Theorem 1.6 (= Theorems 5.15, 5.35 and 5.37)**.** *The incompressible Euler fluid-body equations, the compressible Euler fluid-body equations and the Kirchhoff equations are all examples of Euler-Arnold-Lagrange equations on certain algebroids (see Table 1).*

The key tool we use to prove Theorems 1.1 and 1.2 is the Lagrangian reduction theorem, which extends a surjective algebroid morphism relating two Lagrangians to a relation between solutions of the corresponding ELA equations.

Theorem 1.7 (= Theorem 3.23, Lagrangian reduction on algebroids, [20])**.** *Suppose $\phi : \mathcal{A}' \to \mathcal{A}$ is a surjective algebroid morphism, and suppose L and ℓ are smooth functions on $\mathcal{A}'$ and $\mathcal{A}$ respectively such that $L = \ell \circ \phi$. Then γ' is a solution of the ELA equations for L if and only if $\gamma := \phi \circ \gamma'$ is a solution of the ELA equations for ℓ on $\mathcal{A}$.*

The Lagrangian reducton theorem has the following physical interpretation. The algebroid morphism ϕ encodes the symmetries of the mechanical system, which can also be understood at the groupoid level. The groupoid is the space of arrows joining different configurations of the system (the canonical example being the pair groupoid $\mathcal{G} := Q \times Q$ formed from the configuration space Q). There is an action functional $S : \mathcal{G} \to \mathbb{R}$ on the groupoid which assigns a cost to moving the system from one configuration to another. The system has symmetries if there is a surjective groupoid morphism $\Phi : \mathcal{G} \to \mathcal{H}$ such that $S = s \circ \Phi$ for some action functional $s : \mathcal{H} \to \mathbb{R}$. The Lagrangians L and ℓ corresponding to the action functionals on groupoids are functions on the algebroids $\mathrm{Lie}(\mathcal{G})$ and $\mathrm{Lie}(\mathcal{H})$ that are related by the algebroid morphism induced by Φ.

Example 1.8. For example, if the configuration space is a group G, then the space of arrows joining different configurations is $G \times G$. If the cost of moving from state g to state h does not depend on their absolute positions, but only their relative position hg^{-1}, then the action functional $S : G \times G \to \mathbb{R}$ is of the form $S = s \circ \Phi$, where $\Phi : G \times G \to G$ is the groupoid morphism $(h, g) \mapsto hg^{-1}$. The induced algebroid morphism $\phi : TG \to \mathfrak{g}$ is the right translation to the tangent space at the identity. The relation between Lagrangians, $L = \ell \circ \phi$, encodes that L is right-invariant. In this case the reduction theorem is the classical "Euler-Poincaré" reduction of dynamics from TG to $\mathfrak{g}$.

To prove Theorem 1.3, we consider Riemannian submersions between algebroids equipped with metrics. We use Lagrangian reduction to prove the following:

Theorem 1.9 (= Theorem 3.34)**.** *Suppose $\mathcal{A}'$ and $\mathcal{A}$ are algebroids equipped with metrics, and suppose $\phi : \mathcal{A}' \to \mathcal{A}$ is a Riemannian submersion. Then a horizontal $\mathcal{A}'$-path $a' : I \to \mathcal{A}'$ is a geodesic if and only if its image $a := \phi \circ a'$ is a geodesic in $\mathcal{A}$. Moreover, if a' is a geodesic with horizontal initial vector $a'(0)$, then a' remains horizontal for all $t \in I$.*

The Lagrangian theory of mechanics on algebroids is closely related to the corresponding Hamiltonian theory. In Section 3.3.2, we give a new formula for the canonical Poisson bracket on the dual of an algebroid in terms of a vector bundle connection. We show that the Legendre transform sends solutions of the ELA equations on an algebroid to solutions of Hamilton's equations on its dual. It is shown in [4] that the Legendre transform *does not* relate the reduced Lagrangian dynamics on a semi-direct product algebra $\mathfrak{g}$ to the reduced Hamiltonian dynamics on its dual $\mathfrak{g}^*$. We show that this situation is rectified when the algebra $\mathfrak{g}$ is instead given the structure of an action algebroid (see Theorem 3.21 and Remarks 3.14, 3.22 and 3.29).

All of the above manifests that groupoid and algebroid structures are ubiquitous in Lagrangian mechanics with symmetries, and hopefully will become a natural tool in fluid dynamics.

2 Lie groupoids and Lie algebroids

In this section, we review the basic theory of Lie groupoids and Lie algebroids. It is followed by a discussion of the theory of vector bundle connections on transitivie Lie algebroids, as they are central to the formulation of mechanics that we develop in Section 3. This section concludes with a discussion of algebroid morphisms.

2.1 Definitions and examples

In this section, we recall definitions, facts and examples of Lie groupoids and Lie algebroids. More details can be found in, for example, [7], [14] and [16].

2.1.1 Lie groupoids

Definition 2.1. A *groupoid* $\mathcal{G} \rightrightarrows M$ is a pair of sets $\mathcal{G}$ and M equipped with the following structures:

1. Two maps $\operatorname{src}, \operatorname{trg} : \mathcal{G} \to M$ called the *source map* the *target map*.

2. A partial binary operation $(h, g) \mapsto hg$ on $\mathcal{G}$, defined for all pairs h, $g \in \mathcal{G}$ with $\operatorname{src}(h) = \operatorname{trg}(g)$, with the following properties

   (a) The source of the product is the source of the right factor, and the target of the product is the target of the left factor: $\operatorname{src}(hg) = \operatorname{src}(g)$ and $\operatorname{trg}(hg) = \operatorname{trg}(h)$.

   (b) Associativity: $k(hg) = (kh)g$ whenever any of these expressions is well defined.

   (c) Identities: For each $x \in M$, there is an element $\operatorname{id}_x \in \mathcal{G}$ such that $\operatorname{src}(\operatorname{id}_x) = \operatorname{trg}(\operatorname{id}_x) = x$. These identity elements satisfy $\operatorname{id}_{\operatorname{trg}(g)} g = g \operatorname{id}_{\operatorname{src}(g)} = g$ for every $g \in \mathcal{G}$.

   (d) Inverses: For each $g \in \mathcal{G}$, there is an element $g^{-1} \in \mathcal{G}$ satisfying $\operatorname{src}(g^{-1}) = \operatorname{trg}(g)$, $\operatorname{trg}(g^{-1}) = \operatorname{src}(g)$, $g^{-1}g = \operatorname{id}_{\operatorname{src}(g)}$ and $gg^{-1} = \operatorname{id}_{\operatorname{trg}(g)}$.

The set $\mathcal{G}$ is called the set of *arrows*, and M the set of *objects*. We will refer to the set of arrows as the *groupoid* and the set of objects as the *base*. A groupoid $\mathcal{G} \rightrightarrows M$ will be referred to as the *groupoid* $\mathcal{G}$ *over the base* M.

A groupoid $\mathcal{G} \rightrightarrows M$ is a *Lie groupoid* if $\mathcal{G}$ and M are manifolds and the maps $(h, g) \mapsto hg$, $x \mapsto \operatorname{id}_x$ and $g \mapsto g^{-1}$ are smooth. To make sense of smoothness of multiplication, the source and target maps are required to be submersions to guarantee that the domain of the multiplication map $\{(h, g) \in \mathcal{G} \times \mathcal{G} \mid \operatorname{src}(h) = \operatorname{trg}(g)\}$ is a submanifold of $\mathcal{G} \times \mathcal{G}$.

Example 2.2 (Standard examples of Lie groupoids)**.** We record for later reference a list of several standard ways of constructing groupoids.

1. *The pair groupoid:* One of the simplest examples of Lie groupoids is the *pair groupoid*. Given a manifold M, the pair groupoid $M \times M \rightrightarrows M$ is the Cartesian product $M \times M$ with source and target maps defined

$$\operatorname{src}(y,x) \coloneqq x \quad \text{and} \quad \operatorname{trg}(y,x) \coloneqq y\,,$$

and composition defined

$$(z,y)(y,x) \coloneqq (z,x)\,.$$

2. *Lie groups:* The other simplest example of Lie groupoids is that of a Lie group G. Any Lie group G is a Lie groupoid $G \rightrightarrows *$ over a single point $*$. The source and target maps are trivial. Groupoid composition is defined for all pairs and is given by the group multiplication on G.

3. *Action groupoids:* Let M be a manifold and let G be a group with a given left action on M. The *action groupoid* $G \ltimes M \rightrightarrows M$ is the Cartesian product $G \times M$ with source and target maps defined

$$\operatorname{src}(g,x) \coloneqq x \quad \text{and} \quad \operatorname{trg}(g,x) \coloneqq gx$$

and composition defined

$$(h,gx)(g,x) \coloneqq (hg,x)\,.$$

4. *Direct products of groupoids:* Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ be two groupoids. The direct product groupoid $\mathcal{G}\times\mathcal{H} \rightrightarrows M\times N$ is the Cartesian product $\mathcal{G}\times\mathcal{H}$ with source and target maps defined

$$\operatorname{src}(g,h) \coloneqq (\operatorname{src}(g),\operatorname{src}(h)) \quad \text{and} \quad \operatorname{trg}(g,h) \coloneqq (\operatorname{trg}(g),\operatorname{trg}(h))$$

and composition defined

$$(g_2,h_2)(g_1,h_1) \coloneqq (g_2g_1,h_2h_1)\,.$$

Example 2.3 (Groupoid for a fluid with moving boundary)**.** Here is a concrete example of a groupoid naturally arising in the description of a fluid with a moving boundary. To construct the base of the groupoid, let B_0 be some open subset of $\mathbb{R}^n$. The base of the groupoid is the orbit of B_0 under the action of the diffeomorphism group of $\mathbb{R}^n$. That is, the base $\mathcal{S}$ is the collection of sets

$$\mathcal{S} \coloneqq \bigcup_{g\in\operatorname{Diff}(\mathbb{R}^n)} g(B_0)\,.$$

This collection of sets is thought of as the collection of possible domains of the fluid. We construct a groupoid over $\mathcal{S}$ by defining

$$\mathcal{FG} := \bigcup_{B_1, B_2 \in \mathcal{S}} \mathrm{Diff}(B_1; B_2)\,.$$

The groupoid $\mathcal{FG} \rightrightarrows \mathcal{S}$ has source and target maps

$$\mathrm{src}(\phi) := \mathrm{Domain}(\phi) \quad \text{and} \quad \mathrm{trg}(\phi) := \mathrm{Range}(\phi)\,.$$

Composition of groupoid elements $\psi\phi$ is given by the usual composition of diffeomorphisms $\psi \circ \phi$, which is always well-defined when the source of the left factor equals the target of the right factor. The groupoid $\mathcal{FG}$ is interpreted as the set of all mappings between the different configurations of the fluid with moving boundary.

We finish our discussion of the basic theory of groupoids with a couple of standard definitions.

Definition 2.4. A groupoid $\mathcal{G} \rightrightarrows M$ is called *transitive* if for any $x, y \in M$ there exists a $g \in \mathcal{G}$ with $\mathrm{src}(g) = x$ and $\mathrm{trg}(g) = y$.

In this thesis we will only consider transitive groupoids.

Example 2.5. The groupoid $\mathcal{FG}$ of the fluid with moving boundary is transitive. By construction, for any points B_1 and B_2 in the base $\mathcal{S}$, there exist diffeomorphisms g_1 and g_2 of $\mathbb{R}^n$ such that $B_1 = g_1(B_0)$ and $B_2 = g_2(B_0)$. The restriction of $g_2 \circ g_1^{-1}$ to B_1 is therefore an arrow $\phi \in \mathcal{FG}$ that has $\mathrm{src}(\phi) = B_1$ and $\mathrm{trg}(\phi)B_2$.

Definition 2.6. Let $\mathcal{G} \rightrightarrows M$ be a groupoid. The *source fibre* of $\mathcal{G}$ at $x \in M$ is the set $\mathcal{G}_x := \{g \in \mathcal{G} \mid \mathrm{src}(g) = x\}$. The *isotropy group* of $\mathcal{G}$ at x is the set $\mathcal{G}_x^x := \{g \in \mathcal{G} \mid \mathrm{src}(g) = \mathrm{trg}(g) = x\}$ equipped with the group identity id_x and group multiplication induced from the groupoid multiplication.

Example 2.7. Consider again the groupoid $\mathcal{FG} \rightrightarrows \mathcal{S}$. The isotropy group at $B \in \mathcal{S}$ is the group of diffeomorphisms $\mathrm{Diff}(B)$.

2.1.2 Lie algebroids

Just as a Lie groupoid is a generalization of a Lie group, a Lie algebroid is a generalization of a Lie algebra. Its definition and properties mimic those of a Lie algebra.

Definition 2.8. A *Lie algebroid* $\mathcal{A} \to M$ is a vector bundle $\mathcal{A}$ over a base manifold M equipped with a Lie bracket $[\cdot,\cdot]$ on smooth sections and a vector bundle morphism $\# : \mathcal{A} \to TM$ such that for any sections $U, V \in \Gamma\mathcal{A}$ and any function $f \in C^\infty(M)$, the following Leibniz rule holds:

$$[U, fV] = f[U, V] + (\#U \cdot f)V\,.$$

The symbol $(\#U \cdot f)$ stands for the derivative of f along the vector field $\#U$. The given bracket on $\mathcal{A}$ is called the *algebroid bracket*, and the map $\#$ is called the *anchor map*.

Remark 2.9. It follows from the Leibniz rule and the Jacobi identity that the anchor map is a Lie algebra homomorphism from sections of $\mathcal{A}$ to vector fields on M. That is, $\#[U,V] = [\#U,\#V]$, where the bracket on the right is the Lie bracket of vector fields on M.

We will denote the fibre at x of a given algebroid $\mathcal{A}$ by $\mathcal{A}_x$.

Definition 2.10. The *Lie algebroid* $\mathrm{Lie}(\mathcal{G}) \to M$ *of a Lie groupoid* $\mathcal{G} \rightrightarrows M$ is constructed as follows. The fibre of $\mathrm{Lie}(\mathcal{G})$ at a point $x \in M$ is the tangent spaces at the identity to the source fibre G_x. The anchor map is the differential of the target map; that is, the action of $\#$ on an element $a \in \mathrm{Lie}(\mathcal{G})_x$ is computed $\#a := d/dt|_{t=0}\,\mathrm{trg}(g_t)$, where g_t is any curve in $\mathcal{G}_x$ generating a. The algebroid bracket on two sections U and V of $\mathrm{Lie}(\mathcal{G})$ is defined by extending U and V to right-invariant vector fields $\tilde{U}$ and $\tilde{V}$ on $\mathcal{G}$ tangent to source fibres. The Lie bracket of vector fields $[\tilde{U},\tilde{V}]$ is again a right-invariant vector field tangent to source fibres, so it may be identified with a section on $\mathrm{Lie}(\mathcal{G})$. This latter section is defined to be the value of the algebroid bracket of $\mathrm{Lie}(\mathcal{G})$ acting on U and V.

Throughout this thesis we always assume that any Lie algebroid comes from a Lie groupoid.

Example 2.11 (Examples of Lie algebroids)**.** Here we describe the algebroids corresponding to the groupoids listed in Example 2.2.

1. *Tangent bundles:* The algebroid of a pair groupoid $M \times M \rightrightarrows M$ is the tangent bundle TM. The algebroid bracket is the Lie bracket of vector fields, and the anchor map is the identity.

2. *Lie algebras:* The algebroid of a Lie group G is its Lie algebra $\mathfrak{g}$. The algebroid bracket is the Lie bracket on $\mathfrak{g}$, and the bundle projection and anchor map are both the zero map.

3. *Action algebroids:* The algebroid of an action groupoid $G \ltimes M \rightrightarrows M$ is the *action algebroid* $\mathfrak{g} \ltimes M \to M$. As a set, the action algebroid is the Cartesian product $\mathfrak{g} \times M$. Projection onto the second factor gives the product the structure of a vector bundle. The base point of a vector in the algebroid will be denoted with a subscript. The anchor map $\# : \mathfrak{g} \ltimes M \to TM$ is given by

$$\#\zeta_x = \zeta_x x\,. \tag{2.1}$$

The algebroid bracket acting on sections ζ and η, with values $\zeta(x) = \zeta_x$ and $\eta(x) = \eta_x$ at x, is given by

$$[\zeta,\eta](x) = [\zeta_x,\eta_x] + \#\zeta_x \cdot \eta - \#\eta_x \cdot \zeta \tag{2.2}$$

where $\#\zeta_x \cdot \eta = d/dt|_{t=0}\eta(x_t)$ for any curve x_t through x such that $d/dt|_{t=0}x_t = \#\zeta_x$.

4. *Direct product algebroids:* The algebroid of a direct product $\mathcal{G} \times \mathcal{H} \rightrightarrows M \times N$ of groupoids is the *direct product algebroid* $\mathrm{Lie}(\mathcal{G}) \times \mathrm{Lie}(\mathcal{H}) \rightrightarrows M \times N$. The projection to the base is the Cartesian product of the projections $\pi^M : \mathrm{Lie}(\mathcal{G}) \to M$ and $\pi^N : \mathrm{Lie}(\mathcal{H}) \to N$. The fibre at a point (m, n) is the direct sum of fibres $\mathrm{Lie}(\mathcal{G})_m \oplus \mathrm{Lie}(\mathcal{H})_n$. The anchor map is the Cartesian product of anchors,

$$\#(u, v) \coloneqq (\#^m u, \#^n v) \in TM \times TN \simeq T(M \times N) .$$

Let (U_i, V_i) be sections, and let $(u_i, v_i) = (U_i, V_i)(m, n)$. The algebroid bracket is given by

$$\begin{aligned} &\big[(U_1, V_1)\,,\,(U_2, V_2)\big](m, n) \\ &= \big([U_1^n, U_2^n](m) + \#^n v_1 \cdot U_2^m - \#^n v_2 \cdot U_1^m\,,\,[V_1^m, V_2^m](n) + \#^m u_1 \cdot V_2^n - \#^m u_2 \cdot V_1^n\big) . \end{aligned} \tag{2.3}$$

For each fixed n, the function $U^n \coloneqq U(\cdot, n)$ is a section of $\mathrm{Lie}(\mathcal{G})$, and for each fixed m, the function $U^m \coloneqq U(m, \cdot)$ maps N to the fibre $\mathrm{Lie}(\mathcal{G})_m$. The bracket $[U_1^n, U_2^n](m)$ is the bracket on $\mathrm{Lie}(\mathcal{G})$. The derivative $\#^n v_1 \cdot U_2^m$ is defined to be $d/dt|_{t=0}\, U_2^m(n_t)$, where n_t is a curve generating $\#^n v_1$.

Example 2.12 (Algebroid for a fluid with moving boundary)**.** The algebroid $\mathcal{FA} = \mathrm{Lie}(\mathcal{FG})$ of the groupoid $\mathcal{FG}$ has a natural interpretation. To see how the fibres look, consider a point B in the base $\mathcal{S}$ and a curve ϕ_t in the source fibre $\mathcal{FG}_B$ which passes through id_B at $t = 0$. The fibre of $\mathcal{FA}$ at the point B consists of all vectors generated by such curves, $v = d/dt\, \phi_t$. These elements v are vector fields on the set B. Furthermore, they satisfy a natural "equal normals" boundary condition, namely, the normal component of the vector field v along the boundary ∂B must be equal to the velocity of $\partial[\phi_t(B)]$ at $t = 0$. This condition is derived from the fact that the diffeomorphism ϕ_t maps the boundary ∂B to the boundary $\partial[\phi_t(B)]$ for all t.

The tangent space $T_B\mathcal{S}$ of the base manifold at the point B is identified with the set of normal vector fields along the boundary ∂B. The anchor map $\# : \mathcal{A} \to T\mathcal{S}$ is the restriction map $\#(v) \coloneqq \mathbf{n} v$, where $\mathbf{n}$ is the operator sending each vector field on B to its normal component along the boundary.

The algebroid bracket acting on sections $U, V \in \Gamma\mathcal{A}$ evaluated at a point $B \in \mathcal{S}$ can be shown to be equal to

$$[U, V](B) = [U(B), V(B)] + \#U(B) \cdot V - \#V(B) \cdot U ,$$

where the bracket on the right hand side is the usual Lie bracket of the vector fields $U(B)$ and $V(B)$, and the derivative $\#U(B) \cdot V$ is defined

$$\#U(B) \cdot V \coloneqq \frac{d}{dt} V(B_t)$$

for any curve B_t generating $\#U(B) \in T\mathcal{S}$.

The notions of transitivity and isotropy extend to algebroids.

Definition 2.13. A Lie algebroid $\mathcal{A} \to M$ is *transitive* if the anchor map is surjective.

One can prove that the Lie algebroid of a transitive Lie groupoid is transitive.

Definition 2.14. Let $\mathcal{A} \to M$ be a Lie algebroid. The *isotropy algebra* at $x \in M$ is the kernel $\ker\#_x$ of the anchor map in the fibre over x. The algebra structure is given by defining $[u,v] := [U,V](x)$ for any u, $v \in \ker\#_x$, where the bracket on the right hand side is the algebroid bracket acting on sections U and V satisfying $U(x) = u$ and $V(x) = v$.

Proposition 2.15. *If U and V are sections that extend vectors u and v in the isotropy algebra $\ker\#_x$, then the value of $[U,V](x)$ depends only on u and v and not on their extensions. The Lie algebra bracket on $\mathrm{Ker}\#_x$ is therefore well-defined. Furthermore, if $\mathcal{A}$ is the Lie algebroid of a groupoid $\mathcal{G}$, then $\mathrm{Ker}\#_x$ is the Lie algebra of the isotropy group $\mathcal{G}_x^x$.*

2.2 Connections on Lie algebroids

We develop the theory of Lagrangian mechanics on an algebroid $\mathcal{A}$ in terms of a vector bundle connection on $\mathcal{A}$. In this section we review necessary parts of the theory of vector bundle connections, and then specialize to the case when the vector bundle is an algebroid. Unlike connections on general vector bundles, a notion of torsion exists for vector bundle connections on algebroids. We also address the question of the existence of a preferred connection on an algebroid equipped with a bundle metric, analogous to the Levi-Civita connection on a Riemannian manifold.

2.2.1 Vector bundle connections

Let $E \to M$ be a vector bundle.

Definition 2.16. Let a be an element of the fibre E_x of E over $x \in M$, and consider the tangent space T_aE. The *vertical subspace* V_aE of T_aE is defined as the set of vectors tangent to curves through a which lie in the fibre E_x.

Definition 2.17. A *connection on E* is a projection $\varpi_a : T_aE \to V_aE$ of each tangent space to its vertical subspace. The kernel of a projection ϖ_a is called the *horizontal subspace* $H_aE := \ker \varpi_a$ of T_aE.

A connection therefore splits each fibre into the direct sum $T_aE = H_aE \oplus V_aE$. One can show that each H_aE is isomorphic to T_xM and each V_aE is isomorphic to E_x. Thus a connection determines a splitting

$$T_aE = T_xM \oplus E_x\,.$$

Definition 2.18. Let a_t be a curve in E over the base curve x_t. Given a connection ϖ on E, the *covariant derivative of* a_t is the new curve in E over x_t defined

$$\frac{D}{dt}a_t := \varpi_{a_t}\left(\left.\frac{d}{dt}\right|_{t=0} a_t\right) \in V_{a_t}E \simeq E_{x_t}\,.$$

The word "connection" also refers to the following directional derivative of sections of E. We will use the term "affine connection" for the derivative of sections if clarification is necessary.

Definition 2.19. An *affine connection* ∇ on a vector bundle E is a map $\nabla : \Gamma TM \times \Gamma E \to \Gamma E$ denoted by $(V, A) \mapsto \nabla_V A$ satisfying the following properties:

1. $\nabla_{fV+gW}A = f\nabla_V A + g\nabla_W A \quad (C^\infty(M)$-linearity in the first argument)
2. $\nabla_V(A+B) = \nabla_V A + \nabla_V B \quad$ (linearity in the second argument)
3. $\nabla_V(fA) = f\nabla_V A + V(f)A \quad$ (Leibniz rule),

where A, B are sections of E, the sections V, W are sections of TM, and f, g are smooth functions on M.

Remark 2.20. It can be shown that $C^\infty(M)$-linearity in the first argument implies that $\nabla_V A(x)$ only depends on the value of V at the point x. It therefore makes sense to write $\nabla_v A(x)$, where $v = V(x)$.

Remark 2.21. Affine connections and covariant differentiation are related as follows. If $E \to M$ is a vector bundle with a covariant differentiation D/dt, the corresponding affine connection ∇ acting on V and A is the section defined at each point $x \in M$ by

$$\nabla_V A(x) := \left.\frac{D}{dt}\right|_{t=0} A(x_t) \in E_x\,,$$

where x_t is a curve in M through $x_0 = x$ such that $d/dt|_{t=0}x_t = V(x)$.

Conversely, given an affine connection, the corresponding covariant differentiation of a curve a_t is defined

$$\left.\frac{D}{dt}\right|_{t=0} a_t := \left.\frac{d}{dt}\right|_{t=0} A_t(x) + \nabla_{\frac{dx_t}{dt}} A_0(x)\,,$$

where $x_t := \pi(a_t)$ is the base curve of a_t, and A_t is a time-dependent section of E such that $a_t = A_t(x_t)$.

Induced connection on the dual

In addition to splitting the tangent space of a vector bundle, any connection induces a splitting of the dual bundle as well. For a bundle E with a connection, we have

$$T_a^* E = T_x^* M \oplus E_x^* \,.$$

We briefly recall how this identification works. First, for each $\beta \in T_a^* E$, we define an associated pair (β_H, β_V). The the *horizontal part* β_H is the unique covector in $T_x^* M$ such that $\langle \beta_H, v \rangle = \langle \beta, \bar{v} \rangle$ for every $v \in T_x M$, where $\bar{v}$ is the element of $H_a E$ identified with v. Similarly, the *vertical part* β_V is the unique covector in E_x^* such that $\langle \beta_V, b \rangle = \langle \beta, \bar{b} \rangle$ for every $b \in E_x$, where $\bar{b}$ is the element of $V_a E$ canonically identified with b.

Conversely, every pair (β_H, β_V) has an associated β in $T_a^* E$ defined, for all $\dot{a} \in T_a E$, by

$$\langle \beta, \dot{a} \rangle \coloneqq \langle \beta_V, b \rangle + \langle \beta_H, v \rangle \,,$$

where $b \in \mathcal{A}_x$ is identified with $\varpi(\dot{a}) \in V_a \mathcal{A}$ and $v \in T_x M$ is identified with $\varpi^\perp(\dot{a}) \in H_a \mathcal{A}$. It is easy to check that these associations are inverses of each other.

Consider a function $L : E \to \mathbb{R}$ and its differential $dL \in T^* E$. The components of $dL(a)$ with respect to the splitting just described are called the *vertical* and *horizontal* differentials of L at a, and are denoted

$$dL(a) \simeq (d_H L(a), d_V L(a)) \in T_x^* M \oplus E_x^* \,.$$

Remark 2.22. We wish to write the derivative of L along a curve in E in terms of the vertical and horizontal differentials. Let a_t be a curve in E generating a vector $\dot{a} \in T_a E$. The derivative $D/dt|_0\, a_t \in E_x$ is identified with the vertical projection $\varpi(\dot{a}) \in V_a E$, and $d/dt|_0\, \pi(a_t) \in T_x M$ is identified with the horizontal projection $\varpi^\perp(\dot{a}) \in H_a E$. Therefore

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} L(a_t) = \langle dL(a), \dot{a} \rangle &= \langle dL(a), \varpi^\perp(\dot{a}) \rangle + \langle dL(a), \varpi(\dot{a}) \rangle \\ &= \left\langle d_H L(a), \left.\frac{d}{dt}\right|_{t=0} \pi(a_t) \right\rangle + \left\langle d_V L(a), \left.\frac{D}{dt}\right|_{t=0} a_t \right\rangle . \end{aligned}$$

In particular, the vertical and horizontal differentials may be found by differentiating L along vertical and horizontal curves respectively.

A covariant derivative D/dt on a vector bundle E induces a covariant derivative on the dual E^*, again denoted by D/dt, in the following way.

Definition 2.23. Given a curve α in E^*, we define $D/dt\, \alpha_t \in E^*$ by requiring

$$\frac{d}{dt} \langle \alpha_t, a_t \rangle = \left\langle \frac{D}{dt} \alpha_t, a_t \right\rangle + \left\langle \alpha_t, \frac{D}{dt} a_t \right\rangle$$

for all curves a_t in E that lie over the same base curve as α_t.

Metrics on vector bundles

We recall some definitions associated with metrics on vector bundles.

Definition 2.24. A *metric* on a vector bundle E is a smoothly varying choice of non-degenerate inner product on each fibre E_x. We write the value of the metric on $a, b \in E_x$ as $\langle a, b\rangle_E$. The *inertia operator* $\mathcal{I} : E \to E^*$ associated with a metric is defined as

$$\langle \mathcal{I}(a), b\rangle := \langle a, b\rangle_E \,,$$

where the pairing on the left is the dual pairing.

Definition 2.25. A connection on E is *metric compatible* if for all curves a_t, b_t in E over a common base curve x_t,

$$\frac{d}{dt}\langle a_t, b_t\rangle_E = \left\langle \frac{D}{dt}a_t, b_t\right\rangle_E + \left\langle a_t, \frac{D}{dt}b_t\right\rangle_E \,.$$

It is not hard to prove that the covariant differentiation of a metric compatible connection commutes with the inertia operator.

Lemma 2.26. *Suppose E is equipped with a metric and a compatible connection. Then the covariant differentiation of curves through E and the covariant differentiation of curves through E^* are related by the inertia operator:*

$$\frac{D}{dt}\mathcal{I}(a_t) = \mathcal{I}\left(\frac{D}{dt}a_t\right) \,.$$

2.2.2 $\mathcal{A}$-connections, Levi-Civita connections

The structure of algebroids allows for a number of constructions that are unavailable on more general vector bundles. The first we discuss is that of an $\mathcal{A}$-connection.

Definition 2.27. An $\mathcal{A}$-connection is a map

$$\begin{aligned} \nabla^{\mathcal{A}} : \Gamma\mathcal{A} \times \Gamma\mathcal{A} &\to \Gamma\mathcal{A} \\ (A, B) &\mapsto \nabla^{\mathcal{A}}_A B \end{aligned}$$

that is $C^\infty(M)$-linear in A, linear in B and satisfies a Leibniz rule in B.

Note that the usual formula for torsion extends directly to $\mathcal{A}$-connections:

$$T^{\mathcal{A}}_{\nabla}(A, B) := \nabla^{\mathcal{A}}_A B - \nabla^{\mathcal{A}}_B A - [A, B] \,.$$

We sometimes call the vector bundle connection defined in the previous section a TM-*connection* to distinguish it from the $\mathcal{A}$-connection we have just defined. In the former connection, the "direction of differentiation" input is a section of TM, while in the latter connection, the direction input is a section of $\mathcal{A}$. Later on, TM-connections will be more useful for us than $\mathcal{A}$-connections.

For general vector bundle connections, there is no analog of the torsion of a connection. On algebroids, however, the anchor map may be used to define the torsion of a TM-connection.

Definition 2.28. The *torsion* $T_\nabla : \mathcal{A}_x \times \mathcal{A}_x \to \mathcal{A}_x$ of a connection ∇ is defined

$$T_\nabla(a,b) := \nabla_{\#A}B(x) - \nabla_{\#B}A(x) - [A,B](x)\,, \tag{2.4}$$

where A, $B \in \Gamma\mathcal{A}$ are any sections such that $A(x) = a$ and $B(x) = b$. The notation $T_\nabla(A,B)$ is used to mean the section of $\mathcal{A}$ defined by $T_\nabla(A,B)(x) := T_\nabla(A(x),B(x))$.

Remark 2.29. If the base of $\mathcal{A}$ is finite dimensional, the torsion T_∇ can be shown to be independent of the extending sections W_i by the usual argument. If the base is infinite dimensional, this independence needs to be checked explicitly on a case by case basis.

Suppose now that $\mathcal{A}$ is equipped with a metric. A natural question is if this metric specifies a preferred TM-connection on $\mathcal{A}$ analogous to the Levi-Civita connection on a Riemannian manifold. First, note that the Levi-Civita theorem holds for $\mathcal{A}$-connections [7].

Theorem 2.30. *Let $\mathcal{A}$ be a Lie algebroid with a metric $\langle\cdot,\cdot\rangle$. There exists a unique $\mathcal{A}$-connection $\nabla^{\mathcal{A}}$, termed the* Levi-Civita $\mathcal{A}$-connection, *that is*

1. *metric compatible: $\#A\langle B,C\rangle = \langle\nabla^{\mathcal{A}}_A B,C\rangle + \langle B,\nabla^{\mathcal{A}}_A C\rangle$ for all A, B, $C \in \Gamma\mathcal{A}$ and*
2. *torsion free: $T^{\mathcal{A}}_\nabla(A,B) = 0$ for all A, $B \in \Gamma\mathcal{A}$.*

A preferred "Levi-Civita TM-connection" may be defined for transitive algebroids as follows. The metric splits each fibre of $\mathcal{A}$ into the orthogonal components $\mathcal{A}_x = \ker\#_x \oplus \ker\#_x^\perp$. For transitive algebroids, the anchor map is an isomorphism between $\ker\#_x^\perp$ and T_xM. We may define a lifting map $\#^{-1} : TM \to \mathcal{A}$ which sends $v \in T_xM$ to the unique a in $\ker\#_x^\perp$ satisfying $\#a = v$. This lifting map is then used to relate TM-connections and $\mathcal{A}$-connections.

Definition 2.31. Let a transitive algebroid $\mathcal{A}$ be equipped with a metric. The *Levi-Civita TM-connection* (or simply *Levi-Civita connection* when there is no ambiguity) is the unique TM-connection defined by

$$\nabla_v A := \nabla^{\mathcal{A}}_{\#^{-1}(v)} A$$

for all $v \in TM$ and all $A \in \Gamma\mathcal{A}$, where $\nabla^{\mathcal{A}}$ is the Levi-Civita $\mathcal{A}$-connection.

The Levi-Civita TM-connection is metric compatible but not torsion-free.

Proposition 2.32. *Let $\mathcal{A}$ be an algebroid with a metric. Given a section $A \in \Gamma\mathcal{A}$, let $A^{\parallel}(x)$ and $A^{\perp}(x)$ be the components of $A(x)$ lying in $\ker\#_x$ and $\ker\#_x^{\perp}$ respectively. Let $[\cdot,\cdot]_{\ker\#_x}$ be the bracket on the isotropy algebra $\ker\#_x$. Then the Levi-Civita TM-connection ∇ on $\mathcal{A}$ has the properties*

1. *metric compatibility: $v\langle A, B\rangle = \langle\nabla_v A, B\rangle + \langle A, \nabla_v B\rangle$ for all $v \in TM$ and A, $B \in \Gamma\mathcal{A}$;*
2. *the torsion is given by the formula*

$$T_\nabla(A,B)(x) = -[A^{\parallel}(x), B^{\parallel}(x)]_{\ker\#_x} - \nabla^{\mathcal{A}}_{A^{\parallel}} B^{\perp}(x) + \nabla^{\mathcal{A}}_{B^{\parallel}} A^{\perp}(x)\,, \tag{2.5}$$

where $\nabla^{\mathcal{A}}$ is the Levi-Civita $\mathcal{A}$-connection.

Proof. Metric compatibility follows from a short computation. If v is a vector in TM and A, B are sections of $\mathcal{A}$, we have

$$\langle\nabla_v A, B\rangle + \langle A, \nabla_v B\rangle = \langle\nabla^{\mathcal{A}}_{\#^{-1}(v)} A, B\rangle + \langle A, \nabla^{\mathcal{A}}_{\#^{-1}(v)} B\rangle = v\langle A, B\rangle$$

by the metric compatibility of $\nabla^{\mathcal{A}}$.

To derive the formula for the torsion, recall that $\nabla^{\mathcal{A}}_A B - \nabla^{\mathcal{A}}_B - [A,B] = 0$. By definition of torsion (2.4),

$$\begin{aligned} T_\nabla(A,B)(x) &= \nabla_{\#A} B(x) - \nabla_{\#B} A(x) - [A,B](x) = \nabla^{\mathcal{A}}_{A^{\perp}} B(x) - \nabla^{\mathcal{A}}_{B^{\perp}} A(x) - [A,B](x) \\ &= -\nabla^{\mathcal{A}}_{A^{\parallel}} B(x) + \nabla^{\mathcal{A}}_{B^{\parallel}} A(x) = -\nabla^{\mathcal{A}}_{A^{\parallel}} B^{\parallel}(x) + \nabla^{\mathcal{A}}_{B^{\parallel}} A^{\parallel}(x) - \nabla^{\mathcal{A}}_{A^{\parallel}} B^{\perp}(x) + \nabla^{\mathcal{A}}_{B^{\parallel}} A^{\perp}(x) \\ &= -[A^{\parallel}(x), B^{\parallel}(x)]_{\ker\#_x} - \nabla^{\mathcal{A}}_{A^{\parallel}} B^{\perp}(x) + \nabla^{\mathcal{A}}_{B^{\parallel}} A^{\perp}(x)\,. \end{aligned}$$

□

Remark 2.33. In the case that $\mathcal{A}$ is the tangent bundle of a Riemannian manifold, the Levi-Civita TM-connection defined above coincides with the usual one.

In the case that $\mathcal{A} = \mathfrak{g}$ is a Lie algebra, the trivial connection (the only vector bundle connection available) is the Levi-Civita TM-connection. Indeed, this connection is trivially metric compatible. It is also torsion-minimal in the sense of (2.5), since the isotropy algebra is equal to the algebra itself, and the torsion of the trivial connection is equal to the (negative of the) Lie algebra bracket.

2.3 Algebroid morphisms

The definition of an algebroid morphism is subtle. To see why this is so, consider first two algebroids $\mathcal{A}$ and $\mathcal{A}'$ that are over the same base manifold M. If $\phi : \mathcal{A}' \to \mathcal{A}$ is a base-preserving vector bundle morphism of algebroids (i.e. the base map $\underline{\phi}$ is the identity on M), then ϕ maps sections of $\mathcal{A}'$ to sections of $\mathcal{A}$. In this case we say that ϕ is an algebroid morphism if

$$\phi([A,B]') = [\phi(A), \phi(B)] \quad \text{for all } A,\, B \in \Gamma\mathcal{A}'. \tag{2.6}$$

If ϕ does not preserve the base manifold, then in general the image of a section under ϕ is not a section itself and the above bracket-preserving condition does not make sense.

Defining a morphism ϕ between two general algebroids can be done by transfering structures on $\mathcal{A}'$ and $\mathcal{A}$ to a third vector bundle, the pullback bundle of $\mathcal{A}$ by $\underline{\phi}$, where the left- and right-hand sides of condition (2.6) can be given meaning. It turns out that vector bundle connections are essential to this characterization of algebroid morphisms.

Definition 2.34. Given a vector bundle $E \to M$ and a map $f : M' \to M$, the *pullback bundle* $f^*E \to M'$ of E by f is defined

$$f^*E \coloneqq \{(x, e) \in M' \times E \mid f(x) = \pi(e)\}\,.$$

The bundle projection $\mathrm{pr} : f^*E \to M'$, $\mathrm{pr}(x, e) \coloneqq x$ is projection onto the first factor. Fibres of f^*E are denoted f^*E_x. For each $x \in M'$, the fibre f^*E_x is isomorphic to $E_{f(x)}$.

Remark 2.35. Sections W of f^*E may be written as $W = (\mathrm{Id}, A)$, where Id is the identity function on M' and $A : M' \to E$ is a smooth function such that $\pi(A(x)) = f(x)$ for all $x \in M'$.

Definition 2.36. Let $\phi : E' \to E$ be a vector bundle morphism from $E' \to M'$ to $E \to M$ over the base map $\underline{\phi} : M' \to M$. Define maps $\phi^! : E' \to \underline{\phi}^*E$ and $\theta : \underline{\phi}^*E \to E$ by

$$\phi^! e' \coloneqq (\pi(e'), \phi(e')) \quad \text{and} \quad \theta(x, e) \coloneqq e\,.$$

Roughly, $\phi^!$ changes the fibres and preserves the base, while θ changes the base and preserves the fibres. Clearly we have $\phi = \theta \circ \phi^!$.

Consider now a bundle morphism $\phi : \mathcal{A}' \to \mathcal{A}$ between algebroids. The map $\phi^!$ is base-preserving and therefore maps sections of $\mathcal{A}'$ to sections of $\underline{\phi}^*\mathcal{A}$. Thus we may attempt to replace the morphism condition (2.6) with $\phi^!([a,b]') = [\phi^!(a), \phi^!(b)]^*$. This approach requires making sense of the bracket on the right-hand side, which is now a bracket $[\cdot,\cdot]^*$ on $\underline{\phi}^*\mathcal{A}$. Morally speaking, $[\cdot,\cdot]^*$ should be the pullback of the bracket $[\cdot,\cdot]$ on $\mathcal{A}$ by the map θ, but this is not possible in general since θ does not map sections to sections. Instead, using a connection on $\mathcal{A}$, we may write $[A, B] = \nabla_{\#A}B - \nabla_{\#B}A - T_\nabla(A, B)$ and pull back the torsion and covariant derivatives.

Definition 2.37. Let $\phi : \mathcal{A}' \to \mathcal{A}$ be a vector bundle morphism, and let $\mathcal{A}$ have a connection ∇. The *pullback covariant derivative* $\theta^*\nabla_v W$ of a section $W = (\mathrm{Id}, A) \in \Gamma\underline{\phi}^*\mathcal{A}$ in the direction $v \in \Gamma TM'$ is the section of $\underline{\phi}^*\mathcal{A}$ defined at each point $x \in M'$ by

$$\left(\theta^*\nabla_v W\right)(x) \coloneqq \left(x, \left.\frac{D}{dt}\right|_{t=0} \theta(x_t, A(x_t))\right) = \left(x, \left.\frac{D}{dt}\right|_{t=0} A(x_t)\right)\,,$$

where x_t is a curve through $x_0 = x$ in M' such that $d/dt|_{t=0} x_t = v(x)$.

Let T_∇ be the torsion of the connection ∇. The *pullback torsion* $\theta^* T_\nabla$ applied to sections $W_i \in \Gamma \underline{\phi}^* \mathcal{A}$ is the section of $\underline{\phi}^* \mathcal{A}$ defined by

$$\theta^* T_\nabla(W_1, W_2)(x) := \big(x, T_\nabla(\theta(x, A_1(x)), \theta(x, A_2(x)))\big) = \big(x, T_\nabla(A_1(x), A_2(x))\big)$$

for all x in M'.

Note that these pullbacks are well defined; we never need to consider the image of sections under θ, only the image of curves (to define $\theta^* \nabla_v W$) and the image of points (to define $\theta^* T_\nabla$).

We can now formulate the appropriate replacement of condition (2.6) to define general algebroid morphisms.

Definition 2.38. Let $\mathcal{A}' \to M'$ and $\mathcal{A} \to M$ be Lie algebroids. A vector bundle morphism $\phi : \mathcal{A}' \to \mathcal{A}$ over a map $\underline{\phi} : M' \to M$ is an *algebroid morphism* if ϕ

1. preserves the anchor: $d\underline{\phi} \circ \#' = \# \circ \phi$
2. preserves the bracket:
$$\phi^![A, B]' = \theta^*_\nabla[\phi^! A, \phi^! B] \tag{2.7}$$
for all A, $B \in \Gamma \mathcal{A}$, with
$$\theta^*_\nabla[\phi^! A, \phi^! B] := \theta^* \nabla_{\#' A} \phi^! B - \theta^* \nabla_{\#' B} \phi^! A - \theta^* T_\nabla(\phi^! A, \phi^! B)\,.$$

The next lemma shows that the definition of algebroid morphism is independent of the choice of this connection. The proof is a straightforward but tedious computation.

Lemma 2.39. *Let $\phi : \mathcal{A}' \to \mathcal{A}$ be an anchor-preserving vector bundle morphism between algebroids. The section $\theta^*_\nabla[\phi^! A, \phi^! B] \in \Gamma \underline{\phi}^* \mathcal{A}$ is independent of the connection ∇.*

Remark 2.40. In the case that ϕ is base preserving, Definition 2.38 coincides with condition (2.6).

Remark 2.41. The bracket condition (2.7) can be phrased as a generalized Maurer-Cartan equation [7]. Let ϕ, $\psi : \mathcal{A}' \to \mathcal{A}$ be anchor-preserving bundle morphisms. Given a connection ∇ on $\mathcal{A}$, define the differential operator $d_\nabla \phi$ and bracket $[\phi, \psi]_\nabla$ as follows. For each A, $B \in \Gamma \mathcal{A}'$, the section $d_\nabla \phi(A, B) \in \Gamma \underline{\phi}^* \mathcal{A}$ is given by

$$d_\nabla \phi(A, B) := \theta^* \nabla_{\#' A} \phi^! B - \theta^* \nabla_{\#' B} \phi^! A - \phi^! [A, B]'$$

and the section $[\phi, \psi]_\nabla(A, B) \in \Gamma \underline{\phi}^* \mathcal{A}$ is given by

$$[\phi, \psi]_\nabla(A, B) := -\big(\theta^* T_\nabla(\phi^! A, \psi^! B) + \theta^* T_\nabla(\psi^! A, \phi^! B)\big)\,.$$

The bracket condition (2.7) is easily seen to be equivalent to the *generalized Maurer-Cartan equation*

$$d_\nabla \phi + \frac{1}{2}[\phi, \phi]_\nabla = 0\,. \tag{2.8}$$

Remark 2.42. Since ϕ is anchor-preserving, ϕ sends $\mathcal{A}'$-paths to $\mathcal{A}$-paths. Indeed, if $\gamma' : I \to \mathcal{A}'$ is an $\mathcal{A}'$-path, then

$$\frac{d}{dt}\underline{\phi}(\underline{\gamma}'(t)) = d\underline{\phi}(\#'\gamma'(t)) = \#\phi(\gamma'(t)) \,.$$

We conclude this section with two standard results on morphisms (see, eg. [14]). They will be useful for building algebroid morphisms.

Theorem 2.43. *If $\Phi : \mathcal{G}' \to \mathcal{G}$ is a Lie groupoid morphism over the base map $\underline{\phi} : M' \to M$, then the induced map $\phi : \mathcal{A}' \to \mathcal{A}$ defined by $\phi(a) := \frac{d}{dt}\big|_{t=0} \Phi(g_t)$, where g_t is a curve in $\mathcal{G}$ generating a, is an algebroid morphism over $\underline{\phi}$.*

Theorem 2.44. *The composition of two groupoid morphisms is again a groupoid morphism. The composition of two algebroid morphisms is again an algebroid morphism.*

2.3.1 Algebroid morphisms from $T(I^2)$ to $\mathcal{A}$

The algebroid morphisms from the algebroid $T(I^2)$, the tangent bundle of the unit square, to an arbitrary algebroid $\mathcal{A}$ are of special importance. They are the correct notion of "homotopy of $\mathcal{A}$-paths", which is required to do variational calculus on functionals of $\mathcal{A}$-paths. It also turns out that such algebroid morphisms are easier to characterize than general algebroid morphisms.

Note that any vector bundle morphism $\alpha : T(I^2) \to \mathcal{A}$ may be written in the form

$$\alpha = a dt + b d\epsilon \quad \text{for } a,\, b : I^2 \to \mathcal{A} \text{ smooth functions.}$$

We will use the notation $a_t^\epsilon = a^\epsilon(t) = a_t(\epsilon) = a(t,\epsilon)$ to express the dependence of a map $a : I^2 \to \mathcal{A}$ on its arguments.

Definition 2.45. Let $\mathcal{A}$ be an algebroid and let $a : I^2 \to \mathcal{A}$ be a smooth function such that $a^\epsilon := a(\cdot,\epsilon)$ is an $\mathcal{A}$-path for all ϵ. A function $b : I^2 \to \mathcal{A}$ such that $b_t := b(t,\cdot)$ is an $\mathcal{A}$-path for all t is called *conjugate* to a if

$$\alpha := a dt + b d\epsilon : T(I^2) \to \mathcal{A}$$

is an algebroid morphism.

The following characterization of algebroid morphisms from $T(I^2)$ to $\mathcal{A}$ simplifies our proofs in Section 3.

Lemma 2.46 (See, eg. [9])**.** *Given an algebroid $\mathcal{A}$ with a connection ∇, any anchor-preserving vector bundle morphism $\alpha = a dt + b d\epsilon : T(I^2) \to \mathcal{A}$ is an algebroid morphism if and only if*

$$\frac{D}{dt} b - \frac{D}{d\epsilon} a - T_\nabla(a,b) = 0 \,. \tag{2.9}$$

Proof. Suppose α is an algebroid morphism. Then the bracket condition (2.7) holds for all sections $A, B \in \Gamma T(I^2)$, in particular the coordinate vector fields ∂_t and ∂_ϵ;

$$\begin{aligned} 0 &= \theta^* \nabla_{\#'\partial_t} \alpha^! \partial_\epsilon - \theta^* \nabla_{\#'\partial_\epsilon} \alpha^! \partial_t - \theta^* T_\nabla(\alpha^! \partial_t, \alpha^! \partial_\epsilon) \\ &= \Big((t,\epsilon), \frac{D}{dt} \alpha(\partial_\epsilon(t,\epsilon)) \Big) - \Big((t,\epsilon), \frac{D}{d\epsilon} \alpha(\partial_t(t,\epsilon)) \Big) - \big((t,\epsilon), T_\nabla(\alpha(\partial_t(t,e)), \alpha(\partial_\epsilon(t,\epsilon))) \big) . \end{aligned}$$

Applying the map $\theta : \underline{\alpha}^* \mathcal{A} \to \mathcal{A}$ to this equation and noting that $\alpha(\partial_t(t,\epsilon)) = a(t,\epsilon)$ and $\alpha(\partial_\epsilon(t,\epsilon)) = b(t,\epsilon)$ shows that equation (2.9) holds.

Conversely, suppose that equation (2.9) holds. The above reasoning shows that this condition is equivalent to

$$d_\nabla \alpha(\partial_t, \partial_\epsilon) + \frac{1}{2} [\alpha, \alpha]_\nabla (\partial_t, \partial_\epsilon) = 0 ,$$

which is the Maurer-Cartan equation (2.8) evaluated on the standard basis sections of $T(I^2)$. To show that α is an algebroid morphism it is enough to show that the Maurer-Cartan equation for α holds when evaluated on *any* pair of sections. But this follows from the fact that any section A of $T(I^2)$ may be written $A = f\partial_t + g\partial_\epsilon$ for some smooth functions $f, g \in C^\infty(I^2)$, and from the fact that $d_\nabla \alpha$ and $[\alpha,\alpha]_\nabla$ are $C^\infty(I^2)$-linear. □

3 Mechanics on algebroids

In this section we present the Lagrangian theory of mechanics on algebroids in terms of vector bundle connections. We define a generalization to algebroids of Hamilton's principle, a generalization of the Euler-Lagrange equation, and derive the relation between them. We call the analog of the Euler-Lagrange equation the *Euler-Lagrange-Arnold* (ELA) equation (Definition 3.6). We show that many equations of geometric mechanics are special cases of the ELA equation.

Next, the relationship with Hamiltonian mechanics is established. We review Hamilton's equation on the dual of an algebroid, and show that it is related to our ELA equation by means of the Legendre transform. In doing so, we give a new and useful formula for the Poisson bracket on the dual of an algebroid in terms of a vector bundle connection.

Finally we turn to Lagrangian reduction (Theorem 3.24), which in the language of algebroids is the following statement: *If two Lagrangians $L' : \mathcal{A}' \to \mathbb{R}$ and $L : \mathcal{A} \to \mathbb{R}$ are related by a surjective algebroid morphism ϕ, then a' is a solution of the ELA equation for L' if and only if $\phi \circ a'$ is a solution of the ELA equation for L.* We give examples of Lagrangian reduction and recover some known results, including the theorem of Lagrangian semidirect product reduction.

We also generalize to algebroids the notion of a Riemannian submersion and use Lagrangian reduction to prove that a Riemannian submersion of algebroids projects horizontal geodesics to geodesics.

3.1 Hamilton's principle and the Euler-Lagrange-Arnold equation

Our development of Lagrangian mechanics on algebroids is meant to mimic the classical theory of Lagrangian mechanics on a Riemannian manifold M (or more accurately, on the tangent bundle TM). In the classical theory, we have a Lagrangian $L : TM \to \mathbb{R}$ and an action functional $S(\gamma_t) := \int L(\gamma_t)\, dt$ defined on curves γ_t in TM that are prolongations of curves in M. The dynamics of the system is given by Hamilton's principle, which is a variational principle saying that the system evolves along a critical path of S. These critical paths of S are shown to satisfy the Euler-Lagrange equations. In the generalization to algebroids, the Lagrangian is taken to be a function $L : \mathcal{A} \to \mathbb{R}$, and the action functional S is now defined on $\mathcal{A}$-paths. Again the dynamics is determined by Hamilton's principle. Critical paths of S are shown to satisfy the Euler-Lagrange-Arnold equations.

To begin, we need to define a suitable notion of variation of paths so that we may define a critical path for a functional.

Definition 3.1. An *admissible variation of an $\mathcal{A}$-path* $\gamma : I \to \mathcal{A}$ is defined to be a smooth function $a : I^2 \to \mathcal{A}$ such that

1. $a_t^0 = a(t, 0) = \gamma_t$
2. $a^\epsilon = a(\cdot, \epsilon)$ is an $\mathcal{A}$-path for all $\epsilon \in I$
3. there exists a $b : I^2 \to \mathcal{A}$ conjugate to a such that $b(0, \epsilon) = b(1, \epsilon) = 0$ for all $\epsilon \in I$.

Remark 3.2. An admissible variation of an $\mathcal{A}$-path is the natural notion of a *homotopy of an $\mathcal{A}$-path* in an algebroid. If $g : I^2 \to \mathcal{G}$ is a homotopy of $\mathcal{G}$-paths, then the derivatives $a = D_t^R g$ and $b = D_\epsilon^R g$ are conjugate pairs in an admissible variation of $\gamma_t = a_t^0$. Thus b_t^0 is interpreted as the vector field along $\pi(\gamma_t)$ describing an infinitesimal variation of γ, (see [6] for details).

The next lemma guarantees that one is able to find admissible variations with prescribed infinitesimal variations.

Lemma 3.3. *Given an $\mathcal{A}$-path $\gamma : I \to \mathcal{A}$ and a smooth function $\delta\gamma_t : I \to \mathcal{A}$ such that $\pi \circ \delta\gamma_t = \pi \circ \gamma_t$, there exists a conjugate pair $a, b : I^2 \to \mathcal{A}$ such that $a_t^0 = \gamma_t$ and $b_t^0) = \delta\gamma_t$.*

Proof. We construct a homotopy of $\mathcal{G}$-paths $g : I^2 \to \mathcal{G}$ with the properties $D_t^R g(t, 0) = a_t^0 = \gamma_t$ and $D_\epsilon^R g(t, 0) = b_t^0 = \delta\gamma_t$. By Remark 3.2, the functions $a = D_t^R g$ and $b = D_\epsilon^R g$ are conjugate pairs in an admissible variation of γ.

Let Δ be a (possibly time-dependent) section such that $\Delta(\pi \circ \delta\gamma_t) = \delta\gamma_t$, and let $\phi_\Delta^\epsilon : \mathcal{G} \to \mathcal{G}$ be the flow induced by Δ (see [6]). Let g_t^0 be the $\mathcal{G}$-path integrating $\gamma_t = a_t^0$. The homotopy defined by $g : I^2 \to \mathcal{G}$ by $g(t, \epsilon) := \phi_\Delta^\epsilon(g_t^0)$ has the desired properties. □

Definition 3.4. A *critical path* of a functional $S : C^\infty(I, \mathcal{A}) \to \mathbb{R}$ is an $\mathcal{A}$-path γ such that

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} S(a^\epsilon) = 0$$

for all admissible variations a of γ.

The generalization of Hamilton's principle to algebroids is thus:

Definition 3.5 (Hamilton's principle)**.** *A physical system defined by a Lagrangian L on an algebroid $\mathcal{A}$ evolves along critical $\mathcal{A}$-paths of the action functional $S(\gamma) := \int_I L(\gamma)\, dt$.*

Such critical paths are solutions of a generalization of the Euler-Lagrange equation.

Definition 3.6. An $\mathcal{A}$-path $\gamma_t : I \to \mathcal{A}$ is said to satisfy the *Euler-Lagrange-Arnold (ELA) equation* for a function $L : \mathcal{A} \to \mathbb{R}$ if, for some connection ∇ on $\mathcal{A}$,

$$\frac{D}{dt}\big(d_V L(\gamma_t)\big) + T^*_\nabla\big(d_V L(\gamma_t), \gamma_t\big) - \#^* d_H L(\gamma_t) = 0\,. \tag{3.1}$$

Here the derivatives D/dt, d_V and d_H are defined with respect to the connection on $\mathcal{A}$, and $T^*_\nabla : \mathcal{A}^*_x \times \mathcal{A}_x \to \mathcal{A}^*_x$ is the *cotorsion*, a bilinear function defined by $\langle T^*_\nabla(\mu, a), b\rangle := \langle \mu, T_\nabla(a, b)\rangle$.

Remark 3.7. One could check directly that solutions of the ELA equation do not depend on the choice of connection. Instead we set this issue aside for now, and revisit it in Section 3.3.2, where we show that the ELA equation is equivalent to Hamilton's equation on the dual of the algebroid. Hamilton's equation may be defined without reference to a connection, so it follows that the ELA equation must be independent of the choice of connection.

The relation between Hamilton's principle and the ELA equations is a direct analog of the classical case.

Theorem 3.8. *An $\mathcal{A}$-path γ_t is a critical point of $S(\gamma) := \int_I L(\gamma)\, dt$ if and only if γ_t solves the ELA equation for L.*

We postpone the proof of this theorem until Section 6.

3.2 Examples of Euler-Lagrange-Arnold equations

In this section we present a number of well-known dynamical equations in geometric mechanics as special cases of the ELA equation.

3.2.1 Euler-Lagrange equations, Newton's equations, and geodesic equations

The Euler-Lagrange-Arnold equations coincide with the Euler-Lagrange equations when the algebroid is the tangent bundle of a Riemannian manifold M. Suppose TM is equipped with the Levi-Civita connection. Since the Levi-Civita connection is torsion-free, and since the anchor map $\# : TM \to TM$ is the identity, The Euler-Lagrange-Arnold equations become

$$\frac{D}{dt} d_V L(v) - d_H L(v) = 0 \tag{3.2}$$

for a TM-path $v = v(t)$. Being a TM-path means that $v = d/dt\, \pi(v) = \dot{x}$, so that v is the prolongation of a curve x in M. These are the Euler-Lagrange equations. For future reference, we derive Newton's equation as a special case.

Proposition 3.9. *Suppose M is a Riemannian manifold with metric $\langle \cdot, \cdot \rangle_M$. Let the Lagrangian $L : TM \to \mathbb{R}$ be defined as*

$$L(v) := \frac{1}{2} \langle v, v \rangle_M - U(\pi(v))$$

for some potential function $U : M \to \mathbb{R}$. Then the Euler-Lagrange-Arnold equation on TM for L is Newton's equation for a curve x in M

$$\mathcal{I}\left(\frac{D}{dt}\dot{x}\right) = -\nabla U(x)\,,$$

where ∇ is the gradient defined with respect to the metric on M, $\mathcal{I}$ is the inertia operator for the metric, and the covariant derivative D/dt is given by the Levi-Civita connection.

If the potential U is the zero function, then Newton's equation reduces to the geodesic equation

$$\frac{D}{dt}\dot{x} = 0\,.$$

Proof. As argued above, the Euler-Lagrange-Arnold equation on TM is the Euler-Lagrange equation (3.2). To compute the fibre differential of L at v, let $w \in T_{\pi(v)}M$ be arbitrary. Let γ be a vertical curve passing through v at $t = 0$ such that $D/dt|_{t=0}\, \gamma(t) = w$. Because γ is vertical, $U(\pi \circ \gamma)$ is constant. We have

$$\langle d_V L(v), w \rangle = \frac{d}{dt}\bigg|_{t=0} L(\gamma(t)) = \frac{d}{dt}\bigg|_{t=0} \frac{1}{2} \langle \gamma(t), \gamma(t) \rangle_M = \left\langle \gamma(0), \frac{D}{dt}\bigg|_{t=0} \gamma(t) \right\rangle_M = \langle v, w \rangle_M$$

where the second to last equality uses the metric compatibility of D/dt. This shows that $d_V L(v) = \mathcal{I}(v)$.

To compute the base differential of L at v, let $w \in T_{\pi(v)}M$ be arbitrary, and let h be the horizontal lift of w to $H_v TM$. Let γ be a horizontal curve passing through v at $t = 0$ such that $d/dt|_{t=0}\, \gamma(t) = h$. Since γ is horizontal, $\langle \gamma, \gamma \rangle_M$ is constant. We compute that

$$\langle d_H L(v), w \rangle = \frac{d}{dt}\bigg|_{t=0} L(\gamma(t)) = -\frac{d}{dt}\bigg|_{t=0} U\big(\pi(\gamma(t))\big) = -\left\langle dU(\pi(v)), \frac{d}{dt}\bigg|_{t=0} \pi(\gamma(t)) \right\rangle.$$

Note that $d/dt|_{t=0}\,\pi(\gamma(t)) = \pi_*(h) = w$, so we conclude $d_H L(v) = -dU(\pi(v))$.

Inserting these expressions for the fibre and base differentials of L into the Euler-Lagrange-Arnold equation and applying the metric raising operator (which commutes with D/dt by metric compatibility), we obtain

$$\frac{D}{dt}\mathcal{I}(v) = \mathcal{I}\frac{D}{dt}v = -\nabla U(\pi(v))$$

Finally, letting $x \coloneqq \pi(v)$, we have $\dot{x} = \#v = v$, since v is a TM-path.

If U is the zero function, then Newton's equation becomes $\mathcal{I}(D/dt\,v) = 0$, which reduces to the geodesic equation after an application of $\mathcal{I}^{-1}$. □

3.2.2 Euler-Poincaré equations

When the algebroid is a Lie algebra, the ELA equations coincide with the well-known Euler-Arnold equations.

Definition 3.10. The *Euler-Arnold* equations on a Lie algebra $\mathfrak{g}$ for a Lagrangian $L : \mathfrak{g} \to \mathbb{R}$ are defined to be

$$\frac{d}{dt}dL(\xi) - \mathrm{ad}^*_\xi dL(\xi) = 0\,.$$

Proposition 3.11. *The Euler-Lagrange-Arnold equations on a Lie algebra $\mathfrak{g}$, viewed as a Lie algebroid over a single point, coincide with the Euler-Arnold equations.*

Proof. View a given Lie algebra $\mathfrak{g}$ as an algebroid over a single point. The anchor map is therefore the zero map, and only the trivial connection exists.[3] The torsion of the trivial connection is easily computed to be the negative of the Lie algebra bracket. It follows that, for $\xi, \zeta \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$, we have

$$\langle T^*_\nabla(\mu,\xi),\zeta\rangle = \langle \mu, T_\nabla(\xi,\zeta)\rangle = -\langle \mu, [\xi,\zeta]\rangle\,,$$

so that $T^*_\nabla(\mu,\xi) = -\mathrm{ad}^*_\xi \mu$.

Since the algebroid consists of a single fibre, the derivative D/dt is equal to d/dt, the base differential d_b is zero and the fibre differential d_f coincides with the usual full differential d. With all these considerations taken into account, the Euler-Lagrange-Arnold equations on $\mathfrak{g}$ for a Lagrangian L become

$$\frac{d}{dt}dL(\xi) - \mathrm{ad}^*_\xi dL(\xi) = 0$$

for a curve $\xi = \xi(t)$ in $\mathfrak{g}$. These are the Euler-Poincaré equations. □

[3] Let P be the single base point, so that TP is the trivial vector space. The only map $\nabla : TP \times \mathfrak{g} \to \mathfrak{g}$ that is linear in the first argument is the zero map.

3.2.3 Dynamics on action algebroids

In this section we compute the Euler-Lagrange-Arnold equations on an action algebroid $\mathfrak{g} \ltimes V$, where V is a vector space. The resulting equations are important since they are the result of "Lagrangian semidirect product reduction" (see [4] and our Section 3.5.2). We refer to Examples 2.2 and 2.11 for definitions and notation associated with action groupoids and action algebroids.

Definition 3.12. Suppose $\mathfrak{g}$ is a Lie algebra acting via a left action on a vector space V. Let $\zeta_x = (\zeta, x)$ be an element of the trivial vector bundle $\mathfrak{g} \times V$ with bundle structure defined by projection onto the second factor. Let $L : \mathfrak{g} \times V \to \mathbb{R}$ be a Lagrangian. The *Euler-Poincaré equations with an advected quantity* x are defined by

$$\frac{d}{dt} d_V L(\zeta_x) = \mathrm{ad}^*_{\zeta_x} d_V L(\zeta_x) - d_H L(\zeta_x) \diamond x \qquad \frac{d}{dt} x_t = \zeta_x x \, . \tag{3.3}$$

Here the diamond operator $\diamond : V^* \times V \to \mathfrak{g}^*$ is defined by the condition that $\langle \lambda \diamond x, \zeta \rangle = -\langle \lambda, \zeta x \rangle$ for all $\lambda \in V^*$, $x \in V$ and $\zeta \in \mathfrak{g}$. The vertical and horizontal differentials are defined with respect to the trivial bundle connection.

Such equations arise in the context of systems with advected quantities. For example, a compressible fluid has its density as an advected quantity.

Proposition 3.13. *The Euler-Lagrange-Arnold equations on an action algebroid* $\mathfrak{g} \ltimes V$ *coincide with the Euler-Poincaré equations with an advected quantity* x*.*

Proof. To compute the ELA equations on $\mathfrak{g} \ltimes V$, we need to compute the cotorsion operator T^*_∇ and the adjoint $\#^*$ of the anchor map. Let ∇ be the trivial bundle connection on $\mathfrak{g} \ltimes V$. Notice that for this connection,

$$\nabla_{\#\zeta_x} \eta = \left. \frac{d}{dt} \right|_{t=0} \eta(x_t) = \#\zeta_x \cdot \eta$$

for any curve x_t generating $\#\zeta_x$. Using this identity and formula (2.2) for the action algebroid bracket, the torsion becomes $T_\nabla(\zeta_x, \eta_x) = -[\zeta_x, \eta_x]$. Thus the cotorsion T^*_∇ is the negative of the coadjoint operator:

$$T^*_\nabla(\mu_x, \zeta_x) = -\mathrm{ad}^*_{\zeta_x} \mu_x \, .$$

Next, the adjoint of the anchor map is computed. For $\lambda \in T^*_x V \simeq V^*$ and for η_x in the fibre of $V \rtimes \mathfrak{g}$ over x, we have

$$\langle \#^* \lambda, \eta_x \rangle = \langle \lambda, \#\eta_x \rangle = \langle \lambda, \eta_x x \rangle = -\langle \lambda \diamond x, \eta_x \rangle \, .$$

Substituting these terms into the ELA equation (3.1) results in the Euler-Poincaré equations with an advected quantity. □

Remark 3.14. Suppose that a group G acts from the left on a vector space V. The set $\mathfrak{g} \times V$ can be given either the structure of a semidirect product algebra or the structure of an action algebroid. It is noted in [4] that the Euler-Poincaré equations on the semidirect product do not coincide with the modified Euler-Poincaré equations on the action algebroid. Furthermore, the latter equations on the action algebroid are the ones that give physically correct dynamics. It is therefore preferable to treat $\mathfrak{g} \ltimes V$ as an action algebroid.

3.3 Relation to Hamiltonian mechanics

In this section we relate the Lagrangian theory we have developed to the corresponding Hamiltonian theory. We will define the canonical Poisson bracket (and the corresponding Poisson tensor) on the dual of an algebroid, as well as Hamilton's equation. There is a straightforward generalization of the Legendre transform which allows us to associate a Hamiltonian function $H : \mathcal{A}^* \to \mathbb{R}$ to any regular Lagrangian $L : \mathcal{A} \to \mathbb{R}$. We show that the Legendre transform relates solutions of the ELA equation for L to solutions of Hamilton's equation for H. We give an expression for the Poisson tensor in terms of a given connection, which will be useful in proving this result.

3.3.1 Natural Poisson bracket on the dual of an algebroid

The Lie bracket on a Lie algebroid $\mathcal{A}$ induces a Poisson bracket on its dual $\mathcal{A}^*$. Since any Poisson bracket is a derivation, it only depends on the derivatives of its arguments $F, G \in C^\infty(\mathcal{A}^*)$. Thus any Poisson bracket is characterized by its action on a class functions whose derivatives span the cotangent space of A^*. Lemma 3.16 below shows that such a class is given by the fibre-wise affine functions. Such functions are sums of fibre-wise linear functions $\Psi_W : \mathcal{A}^* \to \mathbb{R}$, $\Psi_W(\alpha) := \langle \alpha, W(\pi(\alpha)) \rangle$, where $W \in \Gamma\mathcal{A}$ is a section of $\mathcal{A}$, and fibre-wise constant functions $f \circ \pi : A^* \to \mathbb{R}$. We have the following definitions.

Definition 3.15. 1. Let $W \in \Gamma\mathcal{A}$ be a section of $\mathcal{A}$, and let $f \in C^\infty(B)$. In terms of fibre-wise affine functions, the *canonical Poisson bracket* $\{\cdot,\cdot\}$ on the dual $\mathcal{A}^*$ of a Lie algebroid $\mathcal{A}$ with Lie bracket $[\cdot,\cdot]$ is defined by the properties

$$\{\Psi_{W_1}, \Psi_{W_2}\} = \Psi_{[W_1,W_2]} \qquad \{\Psi_W, f \circ \pi\} = \#W(f)\,, \qquad \{f_1 \circ \pi, f_2 \circ \pi\} = 0\,. \tag{3.4}$$

2. The corresponding *Poisson tensor* $\mathcal{P}_\alpha : T^*_\alpha\mathcal{A}^* \times T^*_\alpha\mathcal{A}^* \to \mathbb{R}$ is defined as

$$\mathcal{P}_\alpha(\sigma_1, \sigma_2) := \{F_1, F_2\}(\alpha)\,, \tag{3.5}$$

where $F_i \in C^\infty(\mathcal{A}^*)$ are any functions such that $dF_i(\alpha) = \sigma_i$.

If $\mathcal{A}$ is equipped with a connection, $\mathcal{A}^*$ is given the dual connection, and $T^*\mathcal{A}^*$ is given the dual splitting, then an explicit formula for the Poisson tensor may be found in terms of the torsion of that connection.

Lemma 3.16. *Suppose $\alpha \in \mathcal{A}^*$ is given, $x := \pi(\alpha) \in M$ is the base point of α, and suppose $\sigma \in T^*_\alpha \mathcal{A}^*$ is decomposed with respect to a connection on $T\mathcal{A}^*$ as $\sigma = (\gamma, a) \in T^*_x M \oplus \mathcal{A}_x$. Then there exists a section $W \in \Gamma\mathcal{A}$ and a function $f \in C^\infty(M)$ such that*

$$(\gamma, a) = d(\Psi_W + f \circ \pi)(\alpha)\,.$$

Proof. Let $\sigma = (\gamma, a) \in T^*_x M \oplus \mathcal{A}_x$ be given. Consider $F \in C^\infty(\mathcal{A}^*)$ given by

$$F := \Psi_W - f \circ \pi + g \circ \pi, ,$$

where the section $W \in \Gamma\mathcal{A}$ is such that $W(x) = a$, and the functions f, $g \in C^\infty(M)$ satisfy, for all $\dot{x} \in T_x M$,

$$\langle df(x), \dot{x}\rangle = \langle \alpha, \nabla_{\dot{x}} W(x)\rangle \quad \text{and} \quad \langle dg(x), \dot{x}\rangle = \langle \gamma, \dot{x}\rangle\,.$$

To compute the differential of F, let α_t be a curve in $\mathcal{A}^*$ through $\alpha_0 = \alpha$ with base curve $x_t := \pi(\alpha_t) \in B$. We have

$$\begin{aligned}
\left\langle dF(\alpha), \frac{\delta}{\delta t}\bigg|_{t=0} \alpha_t \right\rangle = \frac{d}{dt}\bigg|_{t=0} F(\alpha_t) &= \frac{d}{dt}\bigg|_{t=0} \left[\langle \alpha_t, W(x_t)\rangle - f(x_t) + g(x_t)\right] \\
&= \left\langle \frac{D}{dt}\bigg|_{t=0} \alpha_t, a \right\rangle + \langle \alpha, \nabla_{\dot{x}} W(x)\rangle - \langle df(x), \dot{x}\rangle + \langle dg(x), \dot{x}\rangle \\
&= \left\langle \frac{D}{dt}\bigg|_{t=0} \alpha_t, a \right\rangle + \langle \gamma, \dot{x}\rangle\,,
\end{aligned}$$

so that $dF(\alpha) = (\gamma, a)$ as desired. □

Proposition 3.17. *Let σ_i be in $T^*_\alpha \mathcal{A}^*$ be decomposed via a connection as $\sigma_i = (\gamma_i, a_i) \in T^*_x M \oplus \mathcal{A}_x$. The Lie-Poisson tensor $\mathcal{P}$ on $\mathcal{A}^*$ at the point α is given in terms of this decomposition by the formula*

$$\mathcal{P}_\alpha(\sigma_1, \sigma_2) = -\langle \alpha, T_\nabla(a_1, a_2)\rangle + \langle \# a_1, \gamma_2\rangle - \langle \# a_2, \gamma_1\rangle\,, \tag{3.6}$$

where T_∇ is the torsion of the connection.

Proof. Let $F_i = \Psi_{W_i} - f_i \circ \pi + g_i \circ \pi \in C^\infty(\mathcal{A}^*)$ be defined as in Lemma 3.16. By the property $dF_i(\alpha) = \sigma_i$ and the definition of the Poisson tensor, we have

$$\begin{aligned}
\mathcal{P}_\alpha(\sigma_1, \sigma_2) &= \{F_1, F_2\}(\alpha) \\
&= \{\Psi_{W_1}, \Psi_{W_2}\}(\alpha) + \{\Psi_{W_1}, -f_2 \circ \pi + g_2 \circ \pi\}(\alpha) - \{\Psi_{W_2}, -f_1 \circ \pi + g_1 \circ \pi\}(\alpha) \\
&= \langle \alpha, [W_1, W_2](x)\rangle + \langle d(-f_2 + g_2)(x), \# a_1\rangle - \langle d(-f_1 + g_1)(x), \# a_2(x)\rangle \\
&= \langle \alpha, [W_1, W_2](x)\rangle - \langle \alpha, \nabla_{\# a_1} W_2(x)\rangle + \langle \alpha, \nabla_{\# a_2} W_1(x)\rangle + \langle \# a_1, \gamma_2\rangle - \langle \# a_2, \gamma_1\rangle\,.
\end{aligned}$$

Applying the definition of $T_\nabla(a_1, a_2)$ gives the result. □

3.3.2 Hamilton's equations and the Legendre transform

We have seen how to construct canonical equations (the ELA equation) on a Lie algebroid given a function $L : \mathcal{A} \to \mathbb{R}$. Now we outline the dual construction; given a function $H : \mathcal{A}^* \to \mathbb{R}$, we define dynamical equations on $\mathcal{A}^*$ that are canonically associated with H.

Definition 3.18. The *Hamiltonian vector field* X_H *associated with* $H : \mathcal{A}^* \to \mathbb{R}$ is the unique vector field on $\mathcal{A}^*$ satisfying

$$\langle X_H(a), \sigma \rangle = \mathcal{P}_\alpha(dH(\alpha), \sigma) \quad \forall \sigma \in T^*_\alpha \mathcal{A}^* \,.$$

A curve $\alpha_t : I \to \mathcal{A}^*$ satifies *Hamilton's equation for* H if

$$\left.\frac{d}{dt}\right|_{t=0} \alpha_t = X_H(\alpha_t) \,.$$

The fundamental link between dynamics on $\mathcal{A}$ and dynamics on $\mathcal{A}^*$ is the Legendre transform.

Definition 3.19. The *Legendre transform with respect to* $L : \mathcal{A} \to \mathbb{R}$ is the function $\mathbb{F}L : \mathcal{A} \to \mathcal{A}^*$ defined by

$$\mathbb{F}L(a) := d_V L(a) \,.$$

In other words, $\langle \mathbb{F}L(a), b \rangle = \langle d_V L(a), b \rangle = d/dt|_{t=0} L(a + tb)$ for all $b \in \mathcal{A}_{\pi(a)}$.

If a Lagrangian is such that $\mathbb{F}L$ is invertible, we say that the Lagrangian is *regular*. For regular Lagrangians $L : \mathcal{A} \to \mathbb{R}$, we can define associated Hamiltonians $H : \mathcal{A}^* \to \mathbb{R}$ by the formula $H(\alpha) := \langle \alpha, a \rangle - L(a)$, where $a = \mathbb{F}L^{-1}(\alpha)$. We prove below that the Legendre transform puts solutions of the ELA equation for L in one-to-one correspondence with solutions of Hamilton's equation for the associated H. We first prove the following lemma.

Lemma 3.20. *Suppose a Lagrangian* L *is such that the Legendre transform* $\mathbb{F}L$ *is invertible, so that the Hamiltonian* H *corresponding to* L *may be defined. Then the inverse* $\mathbb{F}L$ *is given by*

$$\mathbb{F}L^{-1} = \mathbb{F}H \,,$$

where the Legendre transform $\mathbb{F}H : \mathcal{A}^* \to \mathcal{A}$ *is defined* $\mathbb{F}H(\alpha) := d_V H(\alpha)$. *Furthermore, for any connection, the horizontal differentials* d_H *of* L *and* H *are related by*

$$d_H L(a) = -d_H H(\alpha) \,,$$

where $\alpha = \mathbb{F}L(a)$.

Proof. Let a connection on $\mathcal{A}$ be given. Let $a \in \mathcal{A}$ and $\alpha \in \mathcal{A}^*$ be related by $\mathbb{F}L(a) = \alpha$. Note that to prove the first assertion we must show $d_V H(\alpha) = a$. Thus, computing the differential $dH(\alpha)$ and reading off its vertical and horizontal parts will prove both claims simultaneously. Let $\delta\beta \in T_\alpha \mathcal{A}^*$ be arbitrary, and denote its splitting with respect to the given connection by $(v, \beta) \in T_{\pi(\alpha)}B \oplus \mathcal{A}^*_{\pi(\alpha)}$. Consider a curve α_t in $\mathcal{A}^*$ that generates $\delta\beta$, so that

$$\frac{d}{dt}\bigg|_{t=0} \alpha_t = \left(\frac{d}{dt}\bigg|_{t=0} \pi(\alpha_t), \frac{D}{dt}\bigg|_{t=0} \alpha_t \right) = (v, \beta) \,.$$

Let a_t be the curve in $\mathcal{A}$ that is related to α_t by $\mathbb{F}L$. We compute

$$\begin{aligned} \langle dH(\alpha), \delta\beta \rangle &= \frac{d}{dt}\bigg|_{t=0} H(\alpha_t) = \frac{d}{dt}\bigg|_{t=0} [\langle \alpha_t, a_t \rangle - L(a_t)] \\ &= \left\langle \frac{D}{dt}\bigg|_{t=0} \alpha_t, a \right\rangle + \left\langle \alpha, \frac{D}{dt}\bigg|_{t=0} a_t \right\rangle - \left\langle d_V L(a), \frac{D}{dt}\bigg|_{t=0} a_t \right\rangle - \langle d_H L(a), v \rangle \\ &= \langle \beta, a \rangle - \langle d_H L(a), v \rangle \,. \end{aligned}$$

Thus we have shown $d_V H(\alpha) = a$ and $d_H H(\alpha) = -d_H L(a)$. ☐

Theorem 3.21. *Suppose $L : \mathcal{A} \to \mathbb{R}$ is such that $\mathbb{F}L : \mathcal{A} \to \mathcal{A}^*$ is invertible. Define $H : \mathcal{A}^* \to \mathbb{R}$ by $H(\alpha) := \langle \alpha, a \rangle - L(a)$, where $a = \mathbb{F}L^{-1}(\alpha)$. A curve a_t in $\mathcal{A}$ solves the Euler-Lagrange-Arnold equation for L if and only if its Legendre transform $\alpha_t := \mathbb{F}L(a_t)$ satifies Hamilton's equation for H.*

Proof. A curve $\alpha_t : I \to \mathcal{A}^*$ solves Hamilton's equation for H, ie. $d/dt\, \alpha_t = X_H(\alpha_t)$, if and only if

$$\left\langle \frac{d}{dt}\bigg|_{t=t_0} \alpha_t, \sigma \right\rangle = \mathcal{P}_\alpha\big(dH(\alpha), \sigma\big) \quad \forall \sigma \in T^*_\alpha \mathcal{A}^*, \quad \forall t_0 \in I \,,$$

Let $\sigma = (\gamma, b) \in T^*_{\pi(\alpha)}B \oplus \mathcal{A}_{\pi(\alpha)}$ be the splitting of σ with respect to some connection ∇ on $\mathcal{A}$. We write out the above in terms of the splitting and use the formula for the Poisson tensor to obtain the equivalent expression

$$\left\langle \frac{D}{dt}\bigg|_{t=t_0} \alpha_t, b \right\rangle + \left\langle \frac{d}{dt}\bigg|_{t=t_0} \pi(\alpha_t), \gamma \right\rangle = -\big\langle T^*_\nabla(\alpha, d_V H(\alpha)), b \big\rangle - \big\langle \#^* d_H H(\alpha), b \big\rangle + \big\langle \# d_V H(\alpha), \gamma \big\rangle \,.$$

This can be re-written in terms of a. Using Lemma 3.20 to make the substitutions $d_V H(\alpha) = a$, $\alpha = d_V L(a)$, $d_H H(\alpha) = -d_H L(a)$, and $\pi(\alpha) = \pi(a)$, we have

$$\left\langle \frac{D}{dt}\bigg|_{t=t_0} d_V L(a_t), b \right\rangle + \left\langle \frac{d}{dt}\bigg|_{t=t_0} \pi(a_t), \gamma \right\rangle = -\big\langle T^*_\nabla(d_V L(a), a), b \big\rangle + \big\langle \#^* d_H L(a), b \big\rangle + \big\langle \# a, \gamma \big\rangle \,.$$

Since $\sigma = (\gamma, b)$ is arbitrary, the above equation holds if and only if

$$\frac{D}{dt}\bigg|_{t=t_0} d_V L(a_t) = -T^*_\nabla(d_V L(a), a) + \#^* d_H L(a) \quad \text{and} \quad \frac{d}{dt}\bigg|_{t=t_0} \pi(a_t) = \# a \quad \forall t_0 \in I \,,$$

that is, if and only if a_t is an $\mathcal{A}$-path that solves the ELA equation for L. ☐

Remark 3.22. Consider a group G acting on a vector space V. The set $\mathfrak{g} \times V$ can be given either the structure of a semidirect product algebra or the structure of an action algebroid. The action algebroid has base manifold V and typical fibre $\mathfrak{g}$, while the semidirect product algebra has trivial base and a single fibre $\mathfrak{g} \times V$. The difference in base and fibre structure leads to different ELA equations for a given function L on the set $\mathfrak{g} \times V$, depending on which structure we use. (The ELA equations for each structure are computed in Sections 3.2.2 and 3.2.3.)

On the other hand, one can check that the Poisson bracket on the dual of an action algebroid coincides with the Poisson bracket on the dual of a semidirect product algebra, so Hamilton's equation is the same regardless of which structure we use. This situation does not contradict Theorem 3.21, since the Legendre transform relating an action algebroid to its dual is *different* from the Legendre transform relating a semidirect product algebra to its dual. Indeed, the definition of the Legendre transform depends on the base and fibre structure given to the set $\mathfrak{g} \times V$.

As noted in [4], the ELA equations on the semidirect product algebra are physically incorrect. The correct Lagrangian equations are instead ELA equations on the action algebroid. This suggests that even though Hamilton's equations on the dual $\mathfrak{g} \times V$ are the same regardless of which structure the set is given, it is preferable to consider the set as an action algebroid. This way the correspondence via the Legendre transform between Hamiltonian dynamics and the correct Lagrangian dynamics is recovered.

3.4 Lagrangian reduction

By *Lagrangian reduction* we mean the following theorem:

Theorem 3.23 (Lagrangian reduction on algebroids, [20], cf. [27])**.** *Suppose $\phi : \mathcal{A}' \to \mathcal{A}$ is a surjective algebroid morphism, and suppose $L' : \mathcal{A}' \to \mathbb{R}$ and $L : \mathcal{A} \to \mathbb{R}$ are Lagrangians such that $L' = L \circ \phi$. Then γ' is a solution of the ELA equations for L if and only if $\gamma := \phi \circ \gamma'$ is a solution of the ELA equations for L on $\mathcal{A}$.*

The condition $L' = L \circ \phi$ means that L' is a function of fewer variables than the variables of $\mathcal{A}'$ (since the target of ϕ is typically a smaller algebroid than $\mathcal{A}'$). We sometimes call $(\mathcal{A}, L)$ the *reduced system* and $(\mathcal{A}', L')$ the *unreduced system.* The Lagrangian reduction theorem says that dynamics on the reduced system are equivalent to dynamics on the unreduced system.

Both the reduction theorem and the statement that Hamilton's principle is equivalent to ELA dynamics are contained in the following:

Theorem 3.24. *Suppose $\phi : \mathcal{A}' \to \mathcal{A}$ is a surjective algebroid morphism. Let $L' : \mathcal{A}' \to \mathbb{R}$ and $L : \mathcal{A} \to \mathbb{R}$ be Lagrangians such that $L' = L \circ \phi$. The following are equivalent:*

1. *$\gamma' : I \to \mathcal{A}'$ is a critical point of*

$$S'(\gamma') := \int_0^1 L'\big(\gamma'_t\big)\, dt\,.$$

2. γ' satisfies the Euler-Lagrange-Arnold equation for L'.

3. $\gamma := \phi \circ \gamma' : I \to \mathcal{A}$ is a critical point of

$$S(\gamma) := \int_0^1 L\big(\gamma_t\big)\, dt\,.$$

4. $\gamma := \phi \circ \gamma'$ satisfies the Euler-Lagrange-Arnold equation for L.

Our proof of the above will mirror the proof of the classical Euler-Poincaré reduction theorem, which is itself modelled on the standard calculus of variations derivation of the Euler-Lagrange equation (see, eg. [19]). To that end, we need two lemmas. The first says that algebroid morphisms send admissible variations to admissible variations. The second is a generalization of the fundamental lemma of calculus of variations.

Lemma 3.25. *Let γ' be an $\mathcal{A}'$-path and $\phi : \mathcal{A}' \to \mathcal{A}$ be an algebroid morphism. If a' is an admissible variation of γ', then $\phi(a')$ is an admissible variation of $\gamma := \phi \circ \gamma'$. Moreover, if b' is conjugate to a' and equal to zero at the endpoints, then $\phi(b')$ is conjugate to $\phi(a')$ and is zero at the endpoints.*

Proof. We check the three properties that $\phi(a')$ must satisfy in order to be an admissible variation of $\phi(a')$. Clearly $\phi(a'^{\epsilon}_t) = \phi(\gamma'_t)$. By Remark 2.42, $\phi(a'^{\epsilon})$ is an $\mathcal{A}$-path for all $\epsilon \in I$. Lastly, let $b' : I^2 \to \mathcal{A}'$ be a function conjugate to a' such that $b'^{\epsilon}_0 = b'^{\epsilon}_1 = 0$. By the composition property of algebroid morphisms,

$$\phi(\alpha') = \phi(a')dt + \phi(b')d\epsilon : T(I^2) \to \mathcal{A}$$

is again an algebroid morphism, showing that $\phi(b')$ is conjugate to $\phi(a')$. At the endpoints, $\phi(b'^{\epsilon}_0)) = \phi(b'^{\epsilon}_1)) = 0$ since ϕ is linear on fibres. □

Lemma 3.26. *Let $\phi : \mathcal{A}' \to \mathcal{A}$ be a surjective algebroid morphism over the base map $\underline{\phi} : M' \to M$. Suppose $\gamma' : I \to \mathcal{A}'$ is an $\mathcal{A}'$-path, and define an $\mathcal{A}$-path $\gamma := \phi \circ \gamma' : I \to \mathcal{A}$. Let $\mu : I \to \mathcal{A}^*$ be a smooth curve over the same base curve $x := \pi \circ \gamma$ as γ. If μ is not identically zero, then there exists an admissible variation a' of γ' such that*

$$\int_0^1 \big\langle \mu_t\,,\, \phi \circ b'^{0}_t \big\rangle\, dt \neq 0\,,$$

where b' is the conjugate of a'.

Proof. Suppose μ is not identically zero, so that there is a $t_0 \in I$ such that $\mu_0 := \mu(t_0) \neq 0$. Then, letting $x_0 := x(t_0)$, there exists $y \in \mathcal{A}_{x_0}$ such that $\langle \mu_0, y\rangle \neq 0$. Since ϕ is surjective, there is an $x'_0 \in M'$ and a $y' \in \mathcal{A}'_{x'_0}$ such that $\phi(y') = y$.

We construct a smooth curve c' over $x' := \pi \circ \gamma'$ such that $c'(t_0) = y'$. Extend y' to a section $Y' \in \Gamma\mathcal{A}'$, say by choosing Y' to be the constant section in a local chart around y'

multiplied by a smooth cutoff function whose support is in the coordinate neighbourhood. (The cutoff is used to extend the section beyond the local neighbourhood of y'.) Then we define $c'_t := Y'(x'_t)$.

Note that $f(t) := \langle \mu_t, \phi \circ c'_t \rangle$ is a smooth function on I, so there is a neighbourhood $J \subset I$ of t_0 such that f is either positive or negative on J. Letting $\sigma : I \to \mathbb{R}$ denote a smooth cutoff with $\sigma(t_0) = 1$ and support contained in J, define $\delta\gamma'_t := \sigma(t) c'_t$. Then the function $g : I \to \mathbb{R}$ defined by

$$g(t) := \langle \mu_t, \phi \circ \delta\gamma'_t \rangle = \sigma(t) f(t)$$

is smooth, non-negative or non-positive, and non-zero on an open neighbourhood of t_0. Hence

$$\int_0^1 \langle \mu_t, \phi \circ \delta\gamma'_t \rangle \, dt \neq 0 \, .$$

Finally, note that by Lemma 3.3, there exists an admissible variation a' of γ' whose conjugate b' satisfies $b'^0_t = \delta\gamma'_t$ for all $t \in I$. The result now follows. □

We are now ready to prove the main theorem.

Proof of Theorem 3.24. Suppose the equivalence $1 \Leftrightarrow 4$ is established. Then the equivalences $2 \Leftrightarrow 1$ and $4 \Leftrightarrow 3$ follow as special cases when the morphism ϕ is chosen to be the identity. So we are left with proving the following:

Suppose $\phi : \mathcal{A}' \to \mathcal{A}$ is a surjective algebroid morphism. Let $L' : \mathcal{A}' \to \mathbb{R}$ and $L : \mathcal{A} \to \mathbb{R}$ be Lagrangians such that $L' = L \circ \phi$. Then $\gamma' : I \to \mathcal{A}'$ is a critical point of $S'(\gamma') := \int_0^1 L'(\gamma'_t)\, dt$ if and only if $\gamma := \phi \circ \gamma'$ is a solution to the Euler-Lagrange equation for L.

Let $\mu_\gamma : I \to \mathcal{A}^*$ be defined

$$\mu_\gamma(t) := \frac{D}{dt}\big(d_V L(\gamma_t)\big) + T^*_{\bar{\nabla}}\big(d_V L(\gamma_t), \gamma_t\big) - \#^* d_H L(\gamma_t) \, .$$

Assume first that γ' is a critical point of $S'(\gamma') := \int_0^1 L'(\gamma'_t)\, dt$. We show that μ_γ is identically zero.

Suppose μ_γ is not identically zero. Then by Lemma 3.26 there is a variation a' of γ' such that

$$\int_0^1 \big\langle \mu_\gamma(t) \, , \, \phi \circ b'^0_t) \big\rangle \, dt \neq 0 \qquad (3.7)$$

for the conjugate b' of a'. On the other hand, we will show that the integral on the left hand side of (3.7) must be zero, showing that the assumption that μ is somewhere non-zero is inconsistent. Since ϕ is an algebroid morphism, $\phi(a')$ is a variation of $\phi(\gamma') = \gamma$ with conjugate $\phi(b')$ (Lemma 3.25). This implies four things:

1. $\phi(a'_0) = \phi(\gamma') = \gamma$.

2. The endpoints $\phi(b'^\epsilon_0)$ and $\phi(b'^\epsilon_1)$ are zero.

3. $\phi(a')dt + \phi(b')d\epsilon : T(I^2) \to \mathcal{A}$ is a morphism, implying

$$\frac{D}{dt}\phi(b') - \frac{D}{d\epsilon}\phi(a') - T_\nabla(\phi(a'), \phi(b')) = 0$$

for any connection ∇ on $\mathcal{A}$.

4. The equation $d/d\epsilon|_{\epsilon=0}\underline{\phi}(\underline{a}'^{\epsilon}_t) = \#\phi(b'^0_t)$ holds, where $\underline{\phi}$ is the base map of ϕ and $\underline{a}'^{\epsilon}_t$ is the base map of a'^{ϵ}_t. This follows from the facts that b'_t is an $\mathcal{A}'$-path in ϵ for all t and that ϕ is anchor-preserving, so that

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\underline{\phi}(\underline{a}'^{\epsilon}_t) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0}\underline{\phi}(\underline{b}'^{\epsilon}_t) = d\underline{\phi}(\#' b'^0_t) = \#\phi(b'^0_t).$$

Now, since γ' is a critical point,

$$\begin{aligned}
0 &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} S'(a'^{\epsilon}) = \int_0^1 \frac{d}{d\epsilon}\bigg|_{\epsilon=0} L'(a'^{\epsilon}_t)\, dt = \int_0^1 \frac{d}{d\epsilon}\bigg|_{\epsilon=0} L\big(\phi \circ a'^{\epsilon}_t\big)\, dt \\
&= \int_0^1 \left\langle d_V L\big(\phi \circ a'^0_t\big), \frac{D}{d\epsilon}\bigg|_{\epsilon=0} \phi \circ a'^{\epsilon}_t \right\rangle + \left\langle d_H L\big(\phi \circ a'^0_t\big), \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \underline{\phi}\big(\underline{a}'^{\epsilon}_t\big)\right\rangle dt \\
&= -\int_0^1 \left\langle d_V L\big(\gamma_t\big), \frac{D}{dt}\phi \circ b'^0_t - T_\nabla\big(\gamma_t, \phi \circ b'^0_t\big)\right\rangle + \big\langle d_H L\big(\gamma_t\big), \#\phi \circ b'^0_t\big\rangle\, dt \\
&= \int_0^1 \left\langle -\frac{D}{dt}\big(d_V L(\gamma_t)\big) - T^*_\nabla\big(d_V L(\gamma_t), \gamma_t\big) + \#^* d_H L\big(\gamma_t\big), \phi \circ b'^0_t \right\rangle dt \\
&= -\int_0^1 \langle \mu_\gamma(t), \phi \circ b'^0_t \rangle\, dt\,,
\end{aligned}$$

contradicting (3.7), showing that μ must be identically zero.

Next assume μ_γ is the zero function. The above computation of $d/d\epsilon|_{\epsilon=0}\, S(a'_\epsilon)$ shows that

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} S'(a'_\epsilon) = -\int_0^1 \langle \mu_\gamma(t), \phi \circ b'^0_t \rangle\, dt\,,$$

which is zero for any admissible variation a'^{ϵ} of γ'. Thus γ' is a critical path of S'. □

3.5 Recovery of some known results on reduction

In this section we show how classical Euler-Poincaré reduction on groups may be recovered from the more general algebroid reduction. Then we turn to the Lagrangian theory of semidirect product reduction and show that it naturally fits into the theory of algebroid reduction as well. We finish with Remark 3.29, which along with Remarks 3.14 and 3.22, argues that *in geometric mechanics, semidirect product groups should instead be given the structure of action algebroids.*

3.5.1 Euler-Poincaré reduction

The classical Euler-Poincaré reduction theorem applies in situations where the configuration space of the physical system may be identified with a Lie group G, and the Lagrangian $L : TG \to \mathbb{R}$ is invariant under the left (or right) action of G on TG. The symmetry in the Lagrangian allows the Euler-Lagrange dynamics on TG to be reduced to dynamics on the algebra $\mathfrak{g}$ governed by the *Euler-Arnold* (or *Euler-Poincaré*) equation

$$\frac{d}{dt} d\ell(\xi) = -\mathrm{ad}^*_\xi d\ell(\xi)\,,$$

where ξ is a curve through $\mathfrak{g}$ and ℓ is the restriction of L to $\mathfrak{g}$. Examples of systems governed by Euler-Arnold equations are numerous, including the motion of rigid bodies and the motion of incompressible fluids in a fixed domain.

Theorem 3.27 (Euler-Poincaré reduction)**.** *Suppose $L : TG \to \mathbb{R}$ is a Lagrangian on a Lie group G that is invariant under the right action of G on TG. Then a cuve g through G solves the Euler-Lagrange equation for L if and only if the curve $\xi := \dot{g}g^{-1}$ through the algebra $\mathfrak{g}$ solves the Euler-Arnold equation for the Lagrangian ℓ defined to be the restriction of L to $\mathfrak{g}$.*

Proof. Let $\Phi : G \times G \to G$ be the groupoid morphism defined by $\Phi(h, g) := hg^{-1}$ over the trivial base map sending G to the point set $*$. We write elements ξ_g of a tangent space T_gG as $\xi_g = \xi g$, where ξ is an element of the algebra $\mathfrak{g}$. The induced algebroid morphism $\phi : G \to \mathfrak{g}$ is given by right translation to the identity, $\phi(\xi_g) = \xi_g g^{-1} = \xi$. Since L is right invariant, we have

$$L(\xi_g) = L(\xi_g g^{-1}) = L(\xi) = \ell(\xi) = \ell \circ \phi(\xi_g)\,.$$

In Section 3.2 it is shown that the Euler-Lagrange equation on G is the ELA equation on TG, and the Euler-Arnold equation on $\mathfrak{g}$ is the ELA equation on $\mathfrak{g}$. The result therefore follows by Lagrangian reduction on algebroids, Theorem 3.24. □

3.5.2 Semidirect product reduction

The motivating example for this section is the compressible fluid, which has the Lagrangian $L_{\rho_0} : T\mathrm{Diff}(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$L(u_g) := \frac{1}{2}\int_{\mathbb{R}^n} u_g \cdot u_g\, \rho_0 - \int_{\mathbb{R}^n} w\Big(\mathrm{Det}(Dg^{-1})\tilde{\rho}_0 \circ g^{-1}\Big)\, g_*\rho_0\,,$$

which depends on a reference density $\rho_0 = \tilde{\rho}_0 dq$. This Lagrangian has some features that we describe in the abstract for purposes of developing the general theory.

Suppose we have a group G and a dynamical system on TG defined by a Lagrangian $L_{\rho_0} : TG \to \mathbb{R}$. The Lagrangian depends on a parameter ρ_0, which is an element of a vector space V on which G acts transitively from the left. We may denote the whole family of such Lagrangians by $L : TG \times V \to \mathbb{R}$, where L is defined such that $L_\rho(u_g) = L(u_g, \rho)$.

We assume that L is invariant under the left action of G on $TG \times V$ given by $h \cdot (v_g, \rho) := (v_g h^{-1}, h\rho)$. This means, in particular, that L_{ρ_0} is invariant under the corresponding left action of G_{ρ_0}, the isotropy group of ρ_0. The reduction theorem in this setting is known as "Lagrangian semidirect product reduction". It says that since L_{ρ_0} has G_{ρ_0}-invariance, the dynamics may be reduced to $\mathfrak{g} \ltimes V$. This should be compared to the usual Lagrangian reduction theorem; since L_{ρ_0} has a weaker invariance property than full G-invariance, we may only conclude a weaker reduction to a space larger than $\mathfrak{g}$.

Theorem 3.28 ([4]). *Let L and L_{ρ_0} be as above. Define $l : \mathfrak{g} \times V \to \mathbb{R}$ by restricting the second argument of L to the tangent space of the identity. Let g be a curve in G and let $(\rho, \xi) := (\rho_0 g^{-1}, \dot{g} g^{-1})$ be a curve in $\mathfrak{g} \ltimes V$. The following are equivalent.*

1. *With ρ_0 held fixed, the variational principle*

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_0^1 L_{\rho_0}\big(g_\epsilon(t), \dot{g}_\epsilon(t)\big)\, dt = 0 \tag{3.8}$$

holds for variations g_ϵ of g with fixed endpoints.

2. *The curve $g(t)$ satisfies the Euler-Lagrange equations for L_{ρ_0} on G.*

3. *The variational principle*

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_0^1 l\big(\xi_\epsilon(t), \rho_\epsilon(t)\big)\, dt = 0 \tag{3.9}$$

holds on $\mathfrak{g} \ltimes V$, where the variations of ξ and ρ satisfy

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \xi_\epsilon = \dot{\eta} + [\xi, \eta] \quad \text{and} \quad \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \rho_\epsilon = \eta\rho\,. \tag{3.10}$$

for some curve η in $\mathfrak{g}$ which vanishes at the endpoints.

4. *The curve (ξ, ρ) satisfies the Euler-Poincaré equations on $\mathfrak{g} \ltimes V$,*

$$\frac{d}{dt} d_V l(\xi, \rho) = \mathrm{ad}^*_\xi d_V l(\xi, \rho) - d_H l(\xi, \rho) \diamond \rho \quad \text{and} \quad \dot{\rho} = \xi\rho\,. \tag{3.11}$$

We provide a proof of this theorem using the reduction techniques on algebroids developed in Section 6. See Examples 2.2 and 2.11 for definitions and notation associated with action groupoids and action algebroids.

Proof of Theorem 3.28. We apply the Reduction Theorem 3.24. First we demonstrate that the Lagrangians $L_{\rho_0} : TG \to \mathbb{R}$ and $l : \mathfrak{g} \ltimes V$ are related by an algebroid morphism $\phi_{\rho_0} : TG \to \mathfrak{g} \ltimes V$.

Define the map $\Phi_{\rho_0} : G \times G \to G \ltimes V$ over the map $\underline{\Phi}_{\rho_0} : G \to V$ by

$$\Phi_{\rho_0}(h, g) := (hg^{-1}, g\rho_0)\,, \qquad \underline{\Phi}_{\rho_0}(g) := g\rho_0\,.$$

It is easy to check that this is a groupoid morphism. The induced algebroid morphism ϕ_{ρ_0} acting on $v_g = \xi g \in T_g G$ is computed [4]

$$\phi_{\rho_0}(v_g) = \xi_{g\rho_0} = (\xi, g\rho_0)\,.$$

This morphism is surjective since G acts transitively on V. By definition of L_{ρ_0} and the left-invariance of L, we have for any $v_g \in T_g G$,

$$L_{\rho_0}(v_g) = L(v_g, \rho_0) = L(v_g g^{-1}, g\rho_0) = L(\xi, g\rho_0) = l \circ \phi_{\rho_0}(v_g)\,.$$

Next we observe that each of the four statements in Theorem 3.28 is equivalent to the corresponding statement in the Reduction Theorem 3.24 in the special case $\mathcal{A}' = TG$ and $\mathcal{A} = \mathfrak{g} \ltimes V$.

First, let $\gamma' : I \to TG$. The curve γ' is admissible if and only if it is the prolongation of a curve g in G. Admissible variations of γ' are exactly those coming from variations of the underlying curve g. It follows that γ' is a critical path (in the sense of Definition 3.4) of $S'(\gamma') = \int_I L_{\rho_0}(\gamma'(t))\, dt$ if and only if g is a critical path of the variational problem (3.8).

Second, by Example 3.2.1, the ELA equations for γ' are the same as the Euler-Lagrange equations for the underlying curve g.

Third, let $\gamma := \phi_{\rho_0} \circ \gamma' : I \to \mathfrak{g} \ltimes V$, and write $\gamma = (\xi, \rho)$. One may check that a variation of γ is admissible if and only if there exists an η satisfying (3.10). Thus γ is a critical path of $S(\gamma) = \int_I l(\gamma(t))\, dt$ if and only if (ρ, ξ) is a critical path of the variational problem (3.9).

Fourth, by Example 3.2.3, the ELA equations for γ are the same as the Euler-Poincaré equations for (ξ, ρ). This completes the proof. □

Remark 3.29. We have shown that the results of the Lagrangian theory of semidirect product reduction follow naturally from algebroid reduction when the reduced space is given the structure of an action algebroid rather than that of a semidirect product group. In particular, our proof explains the origin of the special form of the variations (3.10) as the condition that $(\xi, \rho)_\epsilon$ is admissible in the sense of Definition 3.1.

3.6 Riemannian submersions of algebroids

We present a generalization of the notion of a Riemannian submersion between Riemannian manifolds to the case where the tangent bundles are replaces by more general algebroids. The definitions and theorems are closely analogous to their classical counterparts, but we note that our notion of geodesic makes essential use of the algebroid structure.

[4] Recall the notation for action algebroids from Example 2.11: Each fibre of $\mathfrak{g} \ltimes V \to V$ is a copy of $\mathfrak{g}$. An element $\xi \in \mathfrak{g}$ in the fibre over $\rho \in V$ is denoted by ξ_ρ or by (ρ, ξ).

Suppose $E' \to M'$ and $E \to M$ are vector bundles equipped with metrics $\langle\cdot,\cdot\rangle_{M'}$ and $\langle\cdot,\cdot\rangle_M$, and suppose $\phi : E' \to E$ is a surjective vector bundle morphism over the map $\underline{\phi} : M' \to M$. The kernel of ϕ defines a vector bundle $\ker\phi \to M'$, called the *vertical bundle*, and its orthogonal compliment $\ker\phi^\perp \to M'$, called the *horizontal bundle*.

Definition 3.30. Suppose $\phi : E' \to E$ is a surjective vector bundle morphism. A curve $a' : I \to E'$ is *horizontal* if $a'(t)$ is an element of the horizontal bundle $\ker\phi^\perp$ for all $t \in I$.

Definition 3.31. Define the *lift operator* $\text{lift}_{x'} : E_{\underline{\phi}(x')} \to \ker\phi^\perp_{x'}$, which maps the fibre of E at $\underline{\phi}(x') \in M$ to the fibre of $\ker\phi^\perp$ at $x' \in M'$, by assigning to each $a \in E_{\underline{\phi}(x')}$ the unique vector a' in $\ker\phi^\perp_{x'}$ such that $\phi(a') = a$.

Definition 3.32. A vector bundle morphism $\phi : E' \to E$ between two bundles equipped with metrics is a *Riemannian submersion* if its restriction to the horizontal bundle is an isometry. That is, for any $a', b' \in \ker\phi^\perp_x$, we have $\langle a', b'\rangle_{M'} = \langle\phi(a'), \phi(b')\rangle_M$

The geodesics of a Riemannian manifold M can be characterized as curves through TM that satisfy two conditions: they must be prolongations of curves in M and they must satisfy the Euler-Lagrange equation for the kinetic energy Lagrangian $L(\gamma, \dot\gamma) = \frac{1}{2}|\dot\gamma|^2$. These conditions have natural generalizations from TM to more general algebroids.

Definition 3.33. Suppose $\mathcal{A} \to M$ is an algebroid equipped with a metric $\langle\cdot,\cdot\rangle_M$. An $\mathcal{A}$-path $a : I \to \mathcal{A}$ is a *geodesic* if it solves the Euler-Lagrange-Arnold equation for $L : \mathcal{A} \to \mathbb{R}$ defined by $L(a) := \frac{1}{2}\langle a, a\rangle_M$.

By the reduction theorem, an $\mathcal{A}$-path is a geodesic if and only if it is a critical path of the action functional S associated to L,

$$S(a) := \frac{1}{2}\int_I \langle a, \rangle_M \, dt \,.$$

As in classical Riemannian geometry, Riemannian submersions of algebroids send horizontal geodesics to geodesics.

Theorem 3.34. *Suppose $\mathcal{A}'$ and $\mathcal{A}$ are algebroids equipped with metrics, and suppose $\phi : \mathcal{A}' \to \mathcal{A}$ is a Riemannian submersion. Then a horizontal $\mathcal{A}'$-path a' is a geodesic in $\mathcal{A}'$ if and only if its image $a := \phi \circ a'$ is a geodesic in $\mathcal{A}$. Moreover, if a' is a geodesic with horizontal initial vector $a'(0)$, then a' remains horizontal for all $t \in I$.*

Proof. Let Π and $\Pi^\perp$ be the orthogonal projections of $\mathcal{A}'$ onto $\ker\phi$ and $\ker\phi^\perp$ respectively. Then the metric may be written

$$\langle a', b'\rangle_{\mathcal{A}'} = \big\langle \Pi(a'), \Pi(b')\big\rangle_{\mathcal{A}'} + \big\langle \Pi^\perp(a'), \Pi^\perp(b')\big\rangle_{\mathcal{A}'} = \big\langle \Pi(a'), \Pi(b')\big\rangle_{\mathcal{A}'} + \big\langle \phi(a'), \phi(b')\big\rangle_{\mathcal{A}} \,,$$

where we have used the fact that ϕ is an isometry between $\ker\phi^\perp$ and $\mathcal{A}$. Defining $S^{\ker}(a') := \frac{1}{2}\int_I \langle\Pi(a'), \Pi(a')\rangle_{\mathcal{A}'} \, dt$, we may write $S'(a') = S^{\ker}(a') + S \circ \phi(a')$

Let a' be a horizontal $\mathcal{A}'$-path. Horizontality of a' implies that it is a minimum of the functional $S^{\ker}(a')$. Thus, for any admissible variation a'_ϵ of a',

$$\frac{d}{dt}\bigg|_{t=0} S'(a'_\epsilon) = \frac{d}{dt}\bigg|_{t=0} S^{\ker}(a'_\epsilon) + S \circ \phi(a'_\epsilon) = \frac{d}{dt}\bigg|_{t=0} S \circ \phi(a'_\epsilon)\,,$$

showing that a' is a critical path of S' if and only if it is a critical path of $S \circ \phi$. It follows that a' solves the ELA equation for L' if and only if it solves the ELA equation for $L \circ \phi$. By the reduction theorem 3.24, this latter condition is equivalent to the statement that $\phi(a')$ solves the ELA equation for L on $\mathcal{A}$. This completes the proof that a horizontal $\mathcal{A}'$-path is a geodesic if and only if its image under ϕ is a geodesic.

Next, suppose a' is a geodesic with a horizontal initial vector $a'(0)$. Consider the geodesic a in $\mathcal{A}$ with initial vector $\phi(a'(0))$. Lift a to $\mathcal{A}'$ by first solving

$$\frac{d}{dt}x'(t) = \#\mathrm{lift}_{x'(t)}a(t)\,, \qquad x'(0) = \pi(a'(0))$$

to obtain a base path x' in M', then defining $a'(t) \coloneqq \mathrm{lift}_{x'(t)}a(t)$. The $\mathcal{A}'$-path a' has initial vector $a'(0)$ and it is a geodesic since a is a geodesic. By uniqueness, a' must be the geodesic we were considering in the first place, and it is horizontal by construction. □

Example 3.35 (Otto's Riemannian submersion)**.** Consider a Riemannian manifold M. Let $\mathrm{Diff}(M) \ltimes \mathrm{Dens}(M) \rightrightarrows \mathrm{Dens}(M)$ be the action groupoid corresponding to the action of diffeomorphisms of on densities by pushforward, and let $\mathrm{vect}(M) \ltimes \mathrm{Dens}(M) \to \mathrm{Dens}(M)$ be the corresponding action algebroid. Equip each fibre $(\mathrm{vect}(M) \ltimes \mathrm{Dens}(M))_\rho$ with the (metric-induced) L^2 inner product of vector fields weighted by ρ. Equip $\mathrm{Dens}(M)$ with the Wasserstein metric corresponding to the metric on M. Then the anchor map $\#$ from the fibre $(\mathrm{vect}(M) \ltimes \mathrm{Dens}(M))_{\rho_0}$ to the tangent space $T_{\rho_0}\mathrm{Dens}(M)$ is exactly the Riemannian submersion introduced by Otto in [23], with respect to the reference density ρ_0.

4 Fluid-body kinematics

4.1 Rigid body kinematics

Here we set up the definitions and notations related to the rigid body. We start with a description of the body's configuration space.

The shape of the body is defined by an open bounded subset $B_0 \subset \mathbb{R}^n$. The set B_0 is called the *reference body*, and is thought of as the set of labels for the particles of the body. By a *configuration* of the body we mean, for each label $Q \in B_0$, a specification of the position of the body particle so labeled.

Rigidity of the body means that for any two points Q_1 and Q_2 in the label space, the Euclidean distance between Q_1 and Q_2 must be equal to the Euclidean distance between their physical locations q_1 and q_2 in any configuration. Clearly, given an element x in the

Euclidean group $SE(n)$, assigning the location $q := x \cdot Q$ to each label Q specifies a valid configuration of the rigid body. Conversely, every valid configuration is associated with exactly one element $x \in SE(n)$. In this way we identify $SE(n)$ with the configuration space of the rigid body. We define the subset $B_x := x \cdot B_0 \subset \mathbb{R}^n$. This is the set of points in physical space that the body occupies when it is at position x.

Notation

Before continuing, we remark on the notation used for vectors, covectors and dual pairings.

As is standard, velocities are represented by vectors and momenta are represented by covectors. Vectors in $\mathbb{R}^n$ are thought of as $n \times 1$ matrices (columns) and covectors are thought of as $1 \times n$ matrices (rows). Thus the transpose operator, denoted by superscript T, sends vectors to covectors and vice versa. Writing a vector u adjacent to a covector v denotes matrix muliplication. Note that vu is a 1×1 matrix, while uv is an $n \times n$ matrix.

If u is a vector and v is a covector, we define the pairing

$$\langle u, v \rangle := \mathrm{Tr}(uv) .$$

Note that the product uv is a matrix whose trace is equal to the number vu, so the ordering does not matter; we have $\langle u, v \rangle = \langle v, u \rangle$.

Angular quantities are represented as matrices; the angular position of a rigid body is an element of $SO(n)$, the angular velocity is an element of $\mathfrak{so}(n)$, and so on. The dual pairing of an element R of a matrix space and an element Ω of the dual matrix space is defined similarly to the pairing of vectors and covectors.

$$\langle R, \Omega \rangle := \mathrm{Tr}(R\Omega) .$$

For future reference we record the group composition law and form of inverses for the semidirect product group $SE(n)$. Given $x_i \in SE(n)$, composition is defined

$$x_1 x_2 = (O_1, L_1)(O_2, L_2) := (O_1 O_2, O_1 L_2 + L_1) ,$$

and the inverse of an element x is given by

$$x^{-1} = (O, L)^{-1} = (O^{-1}, -O^{-1} L) .$$

4.1.1 Velocities of the body

The velocity of the body at position x is described by an element ξ in the tangent space $T_x SE(n)$. As a set, $T_x SE(n) = T_O SO(n) \times \mathbb{R}^n$, and we write $\xi = (R, l)$. The matrix $R \in T_O SO(n)$ may be written as the product $R = rO$, where $r \in \mathfrak{so}(n)$ is an element of the Lie algebra. The vector l is the linear velocity and the matrix r is the angular velocity.

The pairing between $TSE(n)$ and its dual is defined component-wise. Given $\xi = (R, l) \in T_xSE(n)$ and $\eta = (\Omega, \lambda) \in T_x^*SE(n)$, we define

$$\langle \xi, \eta \rangle = \langle (R, l), (\Omega, \lambda) \rangle := \langle R, \Omega \rangle + \langle l, \lambda \rangle . \tag{4.1}$$

The velocity of the body at position x may also be described as a vector field on the set B_x. We show how such a vector field is constructed from an element $\xi \in T_xSE(n)$. Let x_t be a curve in $SE(n)$ that generates ξ. As the body's position advances along the curve x_t, we trace the motion of a particle in the body. A particle located at $q \in B_x$ at $t = 0$ is labelled by the point $Q = x^{-1}q$ in the reference body B_0. To see where q ends up at later times t, we apply x_t to its label point Q. Therefore the particle located at $q \in B_x$ at time $t = 0$ is located at $x_t x^{-1} \cdot q \in B_{x_t}$ at time t. Thus the curve x_t generates a family of diffeomorphisms $\tau_t : B_x \to B_{x_t}$ describing the motion of the body:

$$\tau_t(q_b) := x_t x^{-1} \cdot q_b .$$

Since $\tau_0 = \mathrm{Id}$, the derivative at $t = 0$ of τ_t is a vector field which clearly encodes the instantaneous velocity of the body moving according to x_t. Vectors in $T_xSE(n)$ can then be associated with body velocity fields in a natural way.

Definition 4.1. Given a vector $v \in T_xSE(n)$, we define the associated *body velocity field* $\tilde{v}$ on B_x by

$$\tilde{v}(q) := \frac{d}{dt}\bigg|_{t=0} \tau_t(q) = \frac{d}{dt}\bigg|_{t=0} x_t x^{-1} \cdot q = v x^{-1} \cdot q ,$$

where $\tau_t(q) = x_t x^{-1} \cdot q$ motion associated to any curve x_t in $SE(n)$ satisfying $x_0 = x$ and $d/dt|_{t=0} x_t = v$.

Proposition 4.2. *Given a body position $x = (O, L) \in SE(n)$ and $v = (rO, l) \in T_xSE(n)$, the body vector field $\tilde{v}$ is given by*

$$\tilde{v}(q) = l + r(q - q_x) . \tag{4.2}$$

Proof. Recall that the reference body is chosen so that its centre of mass is at the origin, ie. $\int_{B_0} Q \, dQ = 0$. It follows that

$$\begin{aligned} q_x &= \frac{1}{m} \int_{B_x} q \, dq = \frac{1}{m} \int_{B_0} x \cdot Q \, dQ = \frac{1}{m} \int_{B_0} OQ + L \, dQ \\ &= \frac{1}{m} O \left[\int_{B_0} Q \, dQ \right] + L = L . \end{aligned}$$

By definition of the body velocity field, we have

$$\tilde{v}(q) = v x^{-1} \cdot q = (rO, l)(O^{-1}, -O^{-1}L) \cdot q = (r, l - rL) \cdot q = l + r(q - L) ,$$

and the result follows. ∎

4.1.2 Metric on the body configuration space

We will need to define the kinetic energy of the body, which is half the norm squared of a metric on the configuration space. This section is devoted to defining that metric in terms of the body's mass and moment of inertia. The mass of the body is defined

$$m := \int_{B_0} 1\, dQ = \int_{B_x} 1\, dq\,,$$

where the second equality holds for all $x \in SE(n)$ and follows from the change of variables $Q = x^{-1} \cdot q$. Here we are choosing the mass density of the body to be uniform and equal to 1, though the result can be extended to bodies with non-uniform density as well. The centre of mass of the body depends on its position x and is given by

$$q_x := \frac{1}{m} \int_{B_x} q\, dq\,.$$

This vector-valued integral is computed component-wise over $q \in \mathbb{R}^n$. We assume that the centre of mass of the reference body is at the origin, so that $q_0 = 0$. The moment of inertia is the symmetric invertible matrix

$$\mathbb{I} := \int_{B_0} QQ^T\, dQ\,.$$

Recall that we are multiplying a column vector Q on the left by a row vector Q^T on the right, so that QQ^T is an $n \times n$ matrix. Again, the integral is computed component-wise.

Definition 4.3. Let $\xi_i = (R_i, l_i)$ be vectors in $T_x SE(n)$. The *body L^2 metric* is defined

$$\langle \xi_1, \xi_2 \rangle_B := \langle \xi_1, \mathcal{I}_B \xi_2 \rangle\,,$$

where the operator $\mathcal{I}_B : T_x SE(n) \to T_x^* SE(n)$ given by

$$\mathcal{I}_B(\xi) = \mathcal{I}_B(R, l) := (\mathbb{I} R^T, m l^T)$$

is the *body inertia operator.*

Proposition 4.4. *The body metric is invariant under the left action of $SE(n)$.*

Proof. Let $\xi = (R, l) \in T_y SE(n)$ be given and let $y_t = (A_t, B_t)$ be a curve generating ξ. The lifted left action of $x = (O, L) \in SE(n)$ on ξ is computed

$$x \cdot \xi := \left.\frac{d}{dt}\right|_{t=0} x y_t = \left.\frac{d}{dt}\right|_{t=0} (OA_t, OB_t + L) = (OR, Ol) \in T_{xy} SE(n)\,.$$

It follows that

$$\langle x \cdot \xi_1, x \cdot \xi_2 \rangle_B = \langle OR_1, \mathbb{I} R_2^T O^T \rangle + \langle Ol_1, m l_2^T O^T \rangle = \langle R_1, \mathbb{I} R_2^T \rangle + \langle l_1, m l_2^T \rangle = \langle \xi_1, \xi_2 \rangle_B\,.$$

□

The body metric is natural in the sense that it is a special case of the standard kinetic energy of a continunous medium.

Proposition 4.5. *The value of the body metric on two elements ξ_1 and ξ_2 in $T_xSE(n)$ is equal to the value of the L^2 metric of vector fields applied to the corresponding body velocity fields $\tilde{\xi}_1$ and $\tilde{\xi}_2$:*

$$\langle \xi_1, \xi_2 \rangle_B = \int_{B_x} \tilde{\xi}_1 \cdot \tilde{\xi}_2 \, dq \,. \tag{4.3}$$

Proof. Let $\xi_i = (R_i, l_i)$. Starting with the definition of the body velocity fields,

$$\begin{aligned} \int_{B_x} \langle \tilde{\xi}_1, \tilde{\xi}_2^T \rangle \, dq &= \int_{B_x} \langle \xi_1 x^{-1} \cdot q, (\xi_2^{-1} \cdot q)^T \rangle \, dq = \int_{B_0} \langle \xi_1 \cdot Q, (\xi_2 \cdot Q)^T \rangle \, dQ \\ &= \int_{B_0} \langle R_1 Q + l_1, (R_2 Q)^T + l_2^T \rangle \, dQ \\ &= \int_{B_0} \langle R_1 Q, (R_2 Q)^T \rangle + \langle R_1 Q, l_2^T \rangle + \langle l_1, (R_2 Q)^T \rangle + \langle l_1, l_2^T \rangle \, dQ \,. \end{aligned} \tag{4.4}$$

Now we use the following identities. The first term in the above becomes

$$\int_{B_0} \langle R_1 Q, (R_2 Q)^T \rangle \, dQ = \left\langle R_1, \left[\int_{B_0} Q Q^T \, dQ \right] R_2^T \right\rangle = \langle R_1, \mathbb{I} R_2^T \rangle \,.$$

The second and third terms are zero, since

$$\int_{B_0} \langle R_i Q, l_j^T \rangle \, dQ = \left\langle R_i, \left[\int_{B_0} Q \, dQ \right] l_j^T \right\rangle = \langle R_i, m q_0 l_j^T \rangle$$

and the centre of mass q_0 of the reference body is at the origin. The last term becomes

$$\int_{B_0} \langle l_1, l_2^T \rangle \, dQ = \left\langle l_1, \left[\int_{B_0} 1 \, dQ \right] l_2^T \right\rangle = \langle l_1, m l_2^T \rangle \,.$$

Substituting these identities into formula (4.4) and using the definition (4.3) of the body metric proves equation (4.3). □

Levi-Civita connection on $SE(n)$

In this section we discuss the covariant differentiation with respect to the Levi-Civita connection of the body metric. It will be viewed as coming from a flat Levi-Civita connection on a larger space, the space of $(n+1)$-square matrices $\mathrm{Mat}(n+1)$, into which $SE(n)$ is embedded.

Equip $\mathrm{Mat}(n+1)$ with the metric $\langle A, B \rangle_{\mathrm{Mat}} := \langle \mathrm{diag}(\mathbb{I}, m) A^T, B \cdot \rangle$. Embed $SE(n)$ into $\mathrm{Mat}(n+1)$ by the map

$$(O, L) \mapsto \begin{pmatrix} O & L \\ 0 & 1 \end{pmatrix} .$$

The group $SE(n)$ with the body metric is a Riemannian submanifold of $\mathrm{Mat}(n+1)$ via this embedding. It is easily checked that the usual derivative d/dt is the Levi-Civita covariant derivative with respect to $\langle \cdot, \cdot \rangle_{\mathrm{Mat}}$. The Levi-Civita covariant derivative on $SE(n)$ is therefore given by

$$\frac{D}{dt}\zeta = \Pi\left(\frac{d}{dt}\zeta\right),$$

where Π is the orthogonal projection from $T\mathrm{Mat}(n+1)$ to $TSE(n)$ with respect to $\langle \cdot, \cdot \rangle_{\mathrm{Mat}}$. The next lemma is useful because it allows us to avoid computing D/dt explicitly if we are only interested in $\mathcal{I}_B(D/dt)$.

Lemma 4.6. *If D/dt is the Levi-Civita covariant derivative on $SE(n)$ with respect to the body metric, then*

$$\left\langle \frac{D}{dt}\bigg|_{t=0} \zeta_t, \theta \right\rangle_B = \left\langle \frac{d}{dt}\bigg|_{t=0} \zeta_t, \theta \right\rangle_{\mathrm{Mat}} = \left\langle \mathbb{I}\, \frac{d}{dt}\bigg|_{t=0} (r_t O_t)^T, sO \right\rangle + \left\langle m\, \frac{d}{dt}\bigg|_{t=0} (l_t)^T, k \right\rangle$$

for all curves $\zeta_t = (r_t O_t, l_t)$ through ζ in $T_x SE(n)$ and all $\theta = (sO, k) \in T_x SE(n)$.

Proof. By the above characterization of the Levi-Civita covariant derivative on $SE(n)$, and since Π is an orthogonal projection with respect to the metric $\langle \cdot, \cdot \rangle_{\mathrm{Mat}}$, we have

$$\left\langle \frac{D}{dt}\bigg|_{t=0} \zeta_t, \theta \right\rangle_B = \left\langle \Pi\left(\frac{d}{dt}\bigg|_{t=0} \zeta_t \right), \theta \right\rangle_{\mathrm{Mat}} = \left\langle \frac{d}{dt}\bigg|_{t=0} \zeta_t, \Pi(\theta) \right\rangle_{\mathrm{Mat}} = \left\langle \frac{d}{dt}\bigg|_{t=0} \zeta_t, \theta \right\rangle_{\mathrm{Mat}}.$$

The result now follows from the definition of $\langle \cdot, \cdot \rangle_{\mathrm{Mat}}$. □

4.2 The fluid-body configuration space

We suppose the fluid-body system resides in $\mathbb{R}^n$ with $n \geq 3$. The configuration space of the system is defined in terms of a reference configuration, which is a bounded open subset B_0 describing the shape of the body as in Section 4.1. The set B_0 is thought of as the set of labels for points in the body, and its compliment, $F_0 := \mathbb{R}^n \setminus B_0$ is thought of as the set of labels for the points in the fluid. If the fluid is compressible, we assume that there is a reference density of the fluid $\rho_0 \in \mathrm{Dens}(F_0)$. We assume that the compliment of the body is simply connected.

By a *configuration* of the fluid-body system we mean, for each label $Q_f \in F_0$, a specification of the position of the fluid particle so labeled, and for each label $Q_b \in B_0$, a specification of the position of the body particle so labeled. The configurations must satisfy some natural constraints. First, the motion of the body must be rigid. As discussed in Section 4.1, this means that the valid configurations of the body are of the form $B_x := x \cdot B_0$, where x is an element of the Euclidean group $SE(n)$. Second, the fluid must meet the body with no gaps or overlap between the two. This means that in any valid configuration with body position B_x, there is a diffeomorphism $\phi : F_0 \to F_x := \mathbb{R}^n \setminus B_x$ such that for each label $Q_f \in F_0$ the position of the corresponding fluid particle is given by $q_f := \phi(Q_f) \in F_x$. We therefore make the following definition of the fluid-body configuration space.

Definition 4.7. Given a reference body B_0 and a reference fluid density $\rho_0 \in \mathrm{Dens}(F_0)$, we define the *configuration space of the fluid-body system* to be the set

$$Q := \bigcup_{x \in SE(n)} \{x\} \times \mathrm{Diff}(F_0, F_x)\,.$$

The configuration space SQ of the *incompressible* fluid-body system is defined in the same way, except the reference density is no longer needed and the diffeomorphism group Diff is replaced with the group of volume-preserving diffeomorphisms SDiff.

Remark 4.8. It is also possible to consider fluid-body systems in bounded domains $D \subset \mathbb{R}^n$. In this case the restriction on the dimension may be relaxed, and 2-dimensional systems can be considered in addition to those in 3 dimensions and higher. This discrepancy in the dimension is traced back to our use of the Hodge decomposition in Section 4.4.1, which requires stronger hypotheses for unbounded domains than for bounded ones.

Remark 4.9. The case where the compliment of the body is not simply connected allows for non-trivial circulation of the fluid around the body. It is an interesting problem to study the motion of a rigid body in an incompressible, irrotational fluid with circulation. In 2 dimensions this system is governed by the so-called Chaplygin-Lamb equations [25].

4.3 Fluid densities on exterior domains

In order to consider compressible fluids, we will need to keep track of the fluid density. Here we define the space of fluid densities on the exterior of a rigid body. A description of its tangent space is then given.

Definition 4.10. Let $\mathrm{Dens}(*) := \bigcup_{g \in \mathrm{Diff}(\mathbb{R}^n)} \mathrm{Dens}(g(F_0))$ be the set of all densities over all fluid domains exterior to any (non-rigid) body. The set of densities exterior to a given *rigid* body is defined to be

$$M := \big\{(x, \rho) \in SE(n) \times \mathrm{Dens}(*) \mid \rho \in \mathrm{Dens}(F_x)\big\}\,.$$

Proposition 4.11. *The tangent space $T_{(x,\rho)}M$ may be identified with pairs $(\xi, -di_{\nabla p}\rho) \in T_xSE(n) \times \mathrm{Dens}(F_x)$ such that $(\xi, \nabla p)$ is in $\mathcal{EFBA}_{(x,\rho)} \oplus \mathcal{HFBA}_{(x,\rho)}$.*

Proof. Given a vector in $T_{(x,\rho)}M$, we construct a pair $(\xi, -di_{\nabla p}\rho)$ with the required properties. Suppose (x_t, ρ_t) is a curve in M generating the given vector. Let $\xi := d/dt|_{t=0}\, x_t$. Let $\rho_t = \phi_{t*}\rho$ for some family of diffeomorphisms $\phi_t : F_x \to F_{x_t}$. Then we have that $d/dt|_{t=0}\, \rho_t = -di_u\rho$ for $u = d/dt|_{t=0}\, \phi_t$. We claim that there are functions f and h^ξ on F_x such that $f = \text{const.}$ on ∂F_x and h^ξ is harmonic satisfying the Neumann boundary condition $\mathbf{n}\tilde{\xi} = \mathbf{n}\nabla h^\xi$, and such that $p := f + h^\xi$ solves

$$di_{\nabla p}\rho = di_u\rho\,. \tag{4.5}$$

Indeed, equation (4.5) is equivalent to the elliptic equation $\nabla \cdot (\tilde{\rho}\nabla p) = \nabla \cdot (\tilde{\rho} u)$. There exists a unique f solving

$$\nabla \cdot (\tilde{\rho}\nabla f) = \nabla \cdot (\rho u)\,, \quad f|_{\partial F_x} = 0$$

and a unique h^ξ solving

$$\nabla \cdot (\tilde{\rho}\nabla h^\xi) = 0\,, \quad \mathbf{n}\nabla h^\xi = \mathbf{n}\tilde{\xi}\,.$$

Then by construction, $(\xi, \nabla p) = (0, \nabla f) + (\xi, \nabla h^\xi)$ is in $\mathcal{EFBA}_{(x,\rho)} \oplus \mathcal{HFBA}_{(x,\rho)}$, and

$$\left.\frac{d}{dt}\right|_{t=0} (x_t, \rho_t) = (\xi, -id_u\rho) = (\xi, -id_{\nabla p}\rho)\,.$$

Conversely, given a pair $(\xi, -id_{\nabla p}\rho)$, we can construct a curve x_t in $SE(n)$ generating ξ, and a family of diffeomorphisms $\phi_t : F_x \to F_{x_t}$ generating ∇p such that $(x_t, \phi_{t*}\rho)$ is a curve in M. It is not hard to show that $(\xi, \nabla p)$ is recovered from $(x_t, \phi_{t*}\rho)$ by the algorithm presented at the beginning of the proof. Thus we have established an isomorphism between the tangent space $T_{(x,\rho)}M$ and the space of pairs $(\xi, -di_{\nabla p}\rho)$. □

4.4 Velocities of the fluid-body system and the Hodge decomposition

It is intuitively reasonable that the instantaneous velocity of the fluid-body system in a configuration $(x, g) \in SE(n) \times \mathrm{Diff}(F_0, F_x)$ should be a pair (ξ, u) in $T_xSE(n) \times \mathrm{vect}(F_x)$ satisfying the "equal normals" boundary condition $\mathrm{n} \cdot \tilde{\xi} = \mathrm{n} \cdot u$. We wish to develop some general notions for the space of fluid-body velocity pairs.

Consider the vector bundle $TSE(n) \oplus_M \mathrm{vect} \to M$, defined over the base manifold M of Section 4.3, that has fibre over $(x, \rho) \in M$ given by

$$(TSE(n) \oplus_M \mathrm{vect})_{(x,\rho)} := T_xSE(n) \oplus \mathrm{vect}(F_x)\,.$$

Elements of this bundle are pairs (ξ, u) describing the velocity of a body and a fluid, but since these pairs may not satisfy the equal normals condition, not all elements are admissible velocity pairs of the fluid-body system. This bundle will be called the *extended fluid-body velocity space.*

The goal of this section is to define an L^2-type metric on $TSE(n) \oplus_M \mathrm{vect} \to M$ and use it obtain an extraordinarily useful splitting of this bundle, modelled on the Hodge decomposition. The splitting will give an othogonal projection from the extended velocity space to the space of admissible fluid-body velocities. It will also allow us to easily identify the isotropy algebras of the fluid-body algebroids that we are interested in.

Remark 4.12. The reason why we keep track of the density ρ in the base manifold M is that, even though each fibre $(TSE(n) \oplus_M \mathrm{vect})_{(x,\rho)}$ with a fixed x is identical, the splitting of the fibre will depend on ρ.

We begin by defining the natural L^2-type metric on the extended space of fluid-body velocity pairs.

Definition 4.13. Let $\langle\cdot,\cdot\rangle_B$ be the body metric. The *L^2-metric on $TSE(n)\oplus_M$ vect* is defined on the fibre over $(x,\rho)\in M$ as

$$\big\langle(\xi,u)\,,\,(\omega,v)\big\rangle_{L^2} := \langle\xi,\omega\rangle_B + \int_{F_x} u\cdot v\,\rho = \langle\xi,\omega\rangle_B + \langle u,v\rangle_F\,.$$

The *inertia operator* $\mathcal{I} : TSE(n)\oplus_M \text{vect} \to (TSE(n)\oplus_M \text{vect})^*$ is defined by the condition

$$\big\langle\mathcal{I}(\xi,u)\,,\,(\omega,v)\big\rangle(x,\rho) = \big\langle(\xi,u)\,,\,(\omega,v)\big\rangle_{L^2}(x,\rho)$$

for all (ξ,u) and (ω,v) in $TSE(n)\oplus_M$ vect. Explicitly, $\mathcal{I}$ is given by the formula $\mathcal{I}(\xi,u) = (\mathcal{I}_B(\xi),\tilde{\rho}u)$.

4.4.1 The Hodge decomposition

The aforementioned splitting of the extended fluid-body velocity bundle is based on the Hodge decomposition of vector fields on exterior domains in $\mathbb{R}^n$. We direct the reader to [24] for further details on the Hodge decomposition in exterior domains.

Weighted L^2 metrics

Throughout this section, we will often omit the distinction between k-vector fields and k-forms whenever it does not bring ambiguity. Whenever necessary, the components of a k-vector field written with respect to the standard basis of $\mathbb{R}^n$ will be considered as components of a k-form. This allows us to interpret the differential d and the codifferential δ as operators on k-vector fields. This is the same identification of vector fields and forms that is given by the Euclidean metric. We caution that even when considering metrics other than Euclidean, as we are about to do, we still identify vectors and forms via the Euclidean metric.

Recall that the Hodge decomposition splits vector fields into their exact, coexact, and harmonic parts. The definitions of "coexact" and "harmonic" depend on a choice of Riemannian metric through the definition of the codifferential operator δ. We review the basic definitions and point out some relevant special cases and results.

Remark 4.14. We require a Hodge decomposition on an unbounded domain $F\subset\mathbb{R}^n$ that is orthogonal with respect to a *weighted* L^2 metric. Given a volume form ρ on F (that will represent the fluid density), we define the $L^2_\rho(F)$ metric by

$$\langle u,v\rangle_{L^2_\rho(F)} := \int_F u\cdot v\,\rho\,.$$

(Note that this is exactly the term appearing in Definition 4.13). We will obtain the $L^2_\rho(F)$ Hodge decomposition from the more general Hodge decomposition on Riemannian manifolds.

To do this, we must find for each ρ a Riemannian metric on F whose metric pairing of vector fields is equal to the $L^2_\rho(F)$ pairing. Such a Riemannian metric on F is defined by

$$u \cdot_\rho v := \tilde{\rho}^a u \cdot v\,,$$

where $\tilde{\rho}$ is the smooth, bounded, positive function such that $\rho = \tilde{\rho} dq$, and $a := (n/2+1)^{-1}$, and the dot $\cdot$ on the right denotes the Euclidean metric. Indeed, the volume form of the metric $\cdot_\rho$ satisfies $d\mathrm{Vol}_\rho = \sqrt{|\tilde{\rho}|^{na}}dq$, so the $\cdot_\rho$ pairing of vector fields is equal to

$$\int_F u \cdot_\rho v \, d\mathrm{Vol}_\rho = \int_F \tilde{\rho}^a u \cdot v \sqrt{|\tilde{\rho}|^{na}} dq = \int_F u \cdot v \, \rho\,,$$

which is the $L^2_\rho(F)$ pairing.

The Hodge star operator $* : \Omega^k(F) \to \Omega^{n-k}(F)$ depends on the metric $\cdot_\rho$ and is defined by the condition

$$u \wedge *v = u \cdot_\rho v \, d\mathrm{Vol}_\rho = u \cdot v \, \tilde{\rho} dq = u \cdot v \, \rho\,.$$

The codifferential operator $\delta : \Omega^k(F) \to \Omega^{k-1}(F)$ is defined

$$\delta u := (-1)^{nk+n+1} * d(*u)\,.$$

We sometimes write δ_ρ to emphasize that the codifferential depends on the metric (and in turn, on the density ρ). The most important property of the codifferential that we use is highlighted in the next Proposition.

Proposition 4.15. *Vector fields u are co-closed if and only if they leave the density ρ invariant. That is,*

$$\delta_\rho u = 0 \qquad \Longleftrightarrow \qquad \mathcal{L}_u \rho = d i_u \rho = 0\,.$$

Proof. Note that for any vector fields u and v, we have $v \wedge *u = v \cdot_\rho u \, d\mathrm{Vol}\rho = v \cdot u \, \rho = v \wedge i_u \rho$, so that $*u = i_u \rho$. So, using the definition of δ_ρ, we find that co-closed vector fields u satisfy $0 = \delta_\rho u = (-1)^{nk+n+1} * d i_u \rho$. This equality holds if and only if $d i_u \rho = 0$. □

Hodge components

The components of the Hodge decomposition are now defined. Let

$$\begin{aligned} \mathcal{E}x(F) &:= \{df \in \mathrm{vect}(F) \mid \mathrm{t} \cdot df = 0\} \\ \mathcal{C}o_\rho(F) &:= \{\delta_\rho w \in \mathrm{vect}(F) \mid \mathrm{n} \cdot \delta_\rho w = 0\} \quad \text{and} \\ \mathcal{H}_\rho(F) &:= \{y \in \mathrm{vect}(F) \mid dy = \delta_\rho y = 0\}\,. \end{aligned}$$

denote the subspaces of exact, co-exact and harmonic vector fields.

Remark 4.16. In this thesis we are considering the case where the fluid domain F is simply connected with a compact boundary ∂F. Simple-connectedness of F implies that all harmonic fields are also exact. Thus $\mathcal{H}(F)$ is also equal to

$$\{y \in \mathrm{vect}(F) \mid y = dh, \quad h : F \to \mathbb{R} \text{ satisfies } \delta_\rho dh = 0\} .$$

However, we emphasize that even when F is simply connected, $\mathcal{H}(F)$ is *not* a subspace of $\mathcal{E}x(F)$, because the only field in $\mathcal{H}(F)$ that satisfies the boundary condition $\mathbf{t}df = 0$ is the zero field.

Weighted Sobolev spaces and the Hodge theorem

To get Hodge decomposition results on non-compact domains, we need to control the behaviour at infinity of the vector fields. It suffices to assume the fields lie in certain weighted Sobolev spaces which we now define.

Let $C_c^\infty(F)$ be the space of compactly supported smooth real valued functions on a subset F of $\mathbb{R}^n$. The *weighted Sobolev norm* $\|\cdot\|_{H_1^s}$ on $C_c^\infty(F)$ is defined

$$\|g\|_{H_1^s} := \sum_{0 \le |\gamma| \le s} \|\partial^\gamma g \sigma^{\frac{1+|\gamma|}{2}}\|_{L^2} ,$$

where γ is a multi-index for the partial derivatives ∂ and $\sigma(x) := (1 + x^2)$.

Definition 4.17. The *weighted Sobolev space* $H_1^s(F)$ is defined to be the H_1^s-completion of $C_c^\infty(F)$. This definition is extended to vector-valued functions in the natural way.

$$H_1^s(F;\mathbb{R}^n) := \{\mathbf{g} = (g_1, \dots, g_n) \in C^\infty(\mathbb{R}^n;\mathbb{R}^n) \mid g_i \in H_1^\infty(\mathbb{R}^n) \text{ for all } 1 \le i \le n\} .$$

Definition 4.18. The *weighted Sobolev space of vector fields* is defined to be those vector fields whose component functions lie in $H_1^s(F)$:

$$H_1^s\mathrm{vect}(F) := \mathrm{vect}(F) \bigcap H_1^s(F;\mathbb{R}^n) .$$

Similarly, if $X(F)$ is any subspace of vect(F), we define $H_1^s X(F) := X(F) \cap H_1^s(F;\mathbb{R}^n)$.

We quote the Hodge decomposition theorem on exterior domains. A proof may be found in [24].

Theorem 4.19 (Hodge decomposition). *Let F be the closure of the compliment of a bounded open set in $\mathbb{R}^n$, with $n \ge 3$. The weighted Sobolev space $H_1^\infty\mathrm{vect}(F)$ of vector fields on F splits into the $L_\rho^2(F)$-orthogonal direct sum*

$$H_1^s\mathrm{vect}(F) = H_1^s\mathcal{E}x(F) \oplus H_1^s\mathcal{C}o_\rho(F) \oplus H_1^s\mathcal{H}_\rho(F) . \tag{4.6}$$

Remark 4.20. For the remainder of the text, we will assume that all diffeomorphisms from one domain F_1 to another F_2 lie in the *weighted Sobolev space* defined

$$H_1^s\mathrm{Diff}(F_1;F_2) := \{\phi = \mathrm{Id} + \mathbf{g} \mid \mathbf{g} \in H_1^s(F_1;\mathbb{R}^n) , \det(\mathrm{Id} + Dg) > 0\} .$$

We also assume all vector fields over a domain F are in $H_1^s\mathrm{vect}(F)$. From now on, we drop the prefix H_1^s and write simply Diff$(F_1;F_2)$, vect(F), $\mathcal{E}x(F)$, and so on.

4.4.2 Splitting the fluid-body velocity bundle

The goal of this section is to identify important subspaces of $TSE(n) \oplus_M \mathrm{vect}$ that are orthogonal with respect to this L^2 metric. These spaces mirror the Hodge components. Their suggestive names will be justified in the discussions of the fluid-body algebroids in Sections 5.2 and 5.3.

First we embed $\mathcal{E}x(F_x)$ and $\mathcal{C}o_\rho(F_x)$ into the extended space of velocity pairs simply by appending the zero element of $T_xSE(n)$:

$$\begin{aligned}
\mathcal{EFBA}_{(x,\rho)} &:= \left\{(0,u) \in T_xSE(n) \oplus \mathrm{vect}(F_x) \mid u \in \mathcal{E}x(F_x)\right\} \\
\mathcal{CFBA}_{(x,\rho)} &:= \left\{(0,u) \in T_xSE(n) \oplus \mathrm{vect}(F_x) \mid u \in \mathcal{C}o_\rho(F_x)\right\},
\end{aligned}$$

Next, consider the subspace of velocity pairs consisting of those pairs where the fluid velocity is a ρ-harmonic vector field that satisfies the "equal normals" boundary condition.

$$\mathcal{HFBA}_{(x,\rho)} := \left\{(\xi, \nabla h^\xi) \in T_xSE(n) \times \mathcal{H}_\rho(F_x) \mid \mathrm{n} \cdot \tilde{\xi} = \mathrm{n} \cdot \nabla h^\xi\right\}.$$

Notice that h^ξ is completely determined by ξ, since it is harmonic and required to satisfy fixed Neumann boundary conditions once ξ is chosen. Thus $\mathcal{HFBA}_{(x,\rho)}$ is finite-dimensional. Basic linear algebra then proves the L^2-orthogonal splitting

$$T_xSE(n) \times \mathcal{H}_\rho(F_x) = \mathcal{HFBA}_{(x,\rho)} \oplus \mathcal{HFBA}^\perp_{(x,\rho)}.$$

Proposition 4.21. *Each fibre of the extended fluid-body velocity space admits the following L^2-orthogonal splitting:*

$$T_xSE(n) \oplus \mathrm{vect}(F_x) = \mathcal{EFBA}_{(x,\rho)} \oplus \mathcal{CFBA}_{(x,\rho)} \oplus \mathcal{HFBA}_{(x,\rho)} \oplus \mathcal{HFBA}^\perp_{(x,\rho)}. \tag{4.7}$$

Proof. The Hodge theorem 4.19 allows us to write

$$\begin{aligned}
T_xSE(n) \oplus \mathrm{vect}(F_x) &= T_xSE(n) \oplus \mathcal{E}x(F_x) \oplus \mathcal{C}o_\rho(F_x) \oplus \mathcal{H}_\rho(F_x) \\
&= \mathcal{E}x(F_x) \oplus \mathcal{C}o_\rho(F_x) \oplus \mathcal{H}_\rho(F_x) \oplus \mathcal{HFBA}_{(x,\rho)} \oplus \mathcal{HFBA}^\perp_{(x,\rho)} \\
&= \mathcal{EFBA}_{(x,\rho)} \oplus \mathcal{CFBA}_{(x,\rho)} \oplus \mathcal{HFBA}_{(x,\rho)} \oplus \mathcal{HFBA}^\perp_{(x,\rho)}. \qquad (4.8)
\end{aligned}$$

It is not difficult to show that this splitting is L^2-orthogonal, proving (4.7). □

For an incompressible fluid, we assume ρ is the uniform density $\rho = dq$. The above theory carries through in the exact same manner.

Definition 4.22. The *extended incompressible fluid-body velocity space* is the vector bundle $TSE(n) \oplus_{SE(n)} \mathrm{vect} \to M$ defined over the base manifold $SE(n)$ that has fibre over $x \in SE(n)$ given by

$$(TSE(n) \oplus_{SE(n)} \mathrm{vect})_x := T_xSE(n) \oplus \mathrm{vect}(F_x)$$

The Hodge components $\mathcal{E}x(F_x)$, $\mathcal{C}o(F_x)$ and $\mathcal{H}(F_x)$ are defined with respect to the Euclidean metric on $\mathbb{R}^n$ and the uniform density $\rho = dq$. As before, these spaces are embedded into the extended velocity space:

$$\begin{aligned}
\mathcal{EFBA}_x &:= \left\{(0,u) \in T_xSE(n) \oplus \mathrm{vect}(F_x) \mid u \in \mathcal{E}x(F_x)\right\} \\
\mathcal{CFBA}_x &:= \left\{(0,u) \in T_xSE(n) \oplus \mathrm{vect}(F_x) \mid u \in \mathcal{C}o(F_x)\right\} \\
\mathcal{HFBA}_x &:= \left\{(\xi, \nabla h^\xi) \in T_xSE(n) \times \mathcal{H}(F_x) \mid \mathrm{n} \cdot \tilde{\xi} = \mathrm{n} \cdot \nabla h^\xi\right\},
\end{aligned}$$

with $\mathcal{HFBA}_x^\perp$ defined as the L^2-orthogonal compliment of $\mathcal{HFBA}_x$ in $T_xSE(n) \times \mathcal{H}(F_x)$

The extended incompressible fluid-body velocity space admits an orthogonal splitting.

Proposition 4.23. *Each fibre of the extended incompressible fluid-body velocity space admits the following L^2-orthogonal splitting:*

$$T_xSE(n) \oplus \mathrm{vect}(F_x) = \mathcal{EFBA}_x \oplus \mathcal{CFBA}_x \oplus \mathcal{HFBA}_x \oplus \mathcal{HFBA}_x^\perp . \tag{4.9}$$

5 Fluid-body dynamics

In this section we derive various equations in fluid-body mechanics as Euler-Lagrange-Arnold equations on certain natural algebroids equipped with certain natural Lagrangians. The process of deriving the ELA equations is always the same. First the underlying groupoid appropriate to the system is presented and its algebroid is computed. Then the algebroid is given a metric, and the Levi-Civita TM connection is found. Formulas for the covariant differentiation of curves and for the torsion are derived; this is necessary since the ELA equations are defined in terms of these object. Finally, the appropriate Lagrangian is defined, and the ELA equations are computed.

In the "limiting" cases where the algebroid of the system is a tangent space or a Lie algebra, these steps are shortened, since the ELA equations in these cases have already been computed in general in Section 3.2.

5.1 Unreduced dynamics

We describe now fluid-body dynamics in its "unreduced" form. That is, we define the appropriate Lagrangian $L : TSQ \to \mathbb{R}$ on the configuration space SQ of the incompressible fluid-body system, and show that the dynamics are governed by the geodesic equation. Then we treat the compressible system in a similar manner, and show that the dynmaics there are governed by Newton's equation. These descriptions are "unreduced" in the sense that they does not make use of any symmetry in the configuration spaces or the Lagrangians. The Lagrangian reduction theorem will be used in Section 6 to relate the unreduced equations to their symmetry-reduced counterparts.

5.1.1 Incompressible dynamics as geodesic flow

The "unreduced" incompressible fluid-body system is defined analogously to the compressible case. The underlying groupoid is the pair groupoid $SQ \times SQ \rightrightarrows SQ$ formed by the incompressible configuration space SQ. As before, the associated algebroid is the tangent bundle TSQ, and each tangent space $T_{(x,g)}SQ$ is the subspace of $T_x SE(n) \times T_g S\mathrm{Diff}(F_0, F_x)$ consisting of those pairs (ξ, u_g) which satisfy the "equal normals" boundary condition

$$\mathrm{n} \cdot u_g \circ g^{-1} = \mathrm{n} \cdot \tilde{\xi} \, .$$

The natural L^2-type metric on TSQ is given by the sum of the body metric and the L^2 metric of vector fields acting on the fluid velocities:

$$\big\langle (\xi, u_g), (\omega, v_g) \big\rangle_{L^2} \coloneqq \langle \xi, \omega \rangle_B + \int_{F_0} (u_g \cdot v_g) \circ g \, dq \, . \tag{5.1}$$

Note that in the incompressible case it is common to assume the reference density is uniform, though in principle one could consider some non-uniform reference density ρ_0 that is advected by the incompressible flow.

The Lagrangian $L : TSQ \to \mathbb{R}$ for the unreduced system is given by the kinetic energy $L = T$, where

$$T(\xi, u_g) \coloneqq \frac{1}{2} \big\langle (\xi, u_g), (\xi, u_g) \big\rangle_{L^2} \, .$$

Thus the ELA equation on TSQ is the geodesic equation

$$\frac{D^2}{dt^2}(x, g) = 0 \, , \tag{5.2}$$

where D/dt is the covariant derivative associated with the Levi-Civita connection on TQ with respect to the L^2 metric on TQ.

5.1.2 Compressible dynamics as Newton's equation

The "unreduced" groupoid of the compressible fluid-body system is the pair groupoid $Q \times Q \rightrightarrows Q$.

The associated algebroid is the tangent bundle TQ. Each tangent space $T_{(x,g)}Q$ is the subspace of $T_x SE(n) \times T_g \mathrm{Diff}(F_0, F_x)$ consisting of those pairs (ξ, u_g) which satisfy the boundary condition $\mathbf{n} u_g \circ g^{-1} = \mathbf{n} \tilde{\xi}$. That is, on the boundary ∂B_x, the normal component of the fluid velocity field $u \circ g^{-1}$ matches that of the body velocity field $\tilde{\xi}$.

The natural L^2-type metric on TQ is given by

$$\big\langle (\xi, u_g), (\omega, v_g) \big\rangle_{L^2} \coloneqq \langle \xi, \omega \rangle_B + \int_{F_0} (u_g \cdot v_g) \circ g \, \rho_0 \, . \tag{5.3}$$

The Lagrangian $L : TQ \to \mathbb{R}$ for the unreduced system is of the standard form $L = T - U$. The kinetic energy T is defined with respect to the above metric. We have

$$T(\xi, u_g) = \frac{1}{2}\big\langle(\xi, u_g), (\xi, u_g)\big\rangle_{L^2}.$$

The potential energy U is defined in terms of a given (smooth) *internal energy* (or *constitutive relation*) $w : \mathbb{R}^+ \to \mathbb{R}^+$

$$U(\xi, u_g) = \int_{F_x} w\big(\mathrm{Det}(Dg^{-1})\tilde{\rho}_0 \circ g^{-1}\big)\, g_* \rho_0.$$

Note that U only depends on the base point (x, g), ie. it is of the form $U(\xi, v) = U(\pi(\xi, v))$. Thus, as in Example 3.2.1, the Euler-Lagrange-Arnold equation for L on TQ is Newton's equation

$$\mathcal{I}\frac{D^2}{dt^2}(x, g) = -\nabla U(x, g)\,, \tag{5.4}$$

where $\mathcal{I} : TQ \to T^*Q$ is the inertia operator for the L^2 metric and D/dt is the covariant derivative associated with the Levi-Civita connection on TQ with respect to the L^2 metric on TQ.

5.2 Reduced incompressible dynamics

In the previous section, we saw that the incompressible fluid-body system is governed by the geodesic equation on the configuration space SQ equipped with a natural L^2-type metric. Now we give another set of equations describing the motion of the fluid, and show that these too are the ELA equation on a certain algebroid equipped with a natural Lagrangian.

Remark 5.1. Our present description is a "reduced" description, since it takes advantage of the so-called "particle relabelling symmetry" of the fluid. It may also be called the "Eulerian" description of the fluid, since the fluid velocity is the dynamical quantity that is being solved for, rather than the fluid particle positions.

A rigid body with position $x = (O, L) \in SE(n)$, velocity $\xi = (rO, l) \in T_x SE(n)$ and inertia $\mathbb{I}_x := O\mathbb{I}O^T$ moves through an incompressible fluid with velocity $u \in \mathfrak{svect}(F_x)$ and pressure P according to the *incompressible fluid-body equations:*

$$\begin{cases} \dfrac{d}{dt}u + (u \cdot \nabla)u = -\nabla P \\ m\dfrac{d}{dt}l = \displaystyle\int_{\partial F_x} P\mathrm{n}\, i_{\mathrm{n}} dq \\ \dfrac{d}{dt}(r\mathbb{I}_x) = \displaystyle\int_{\partial F_x} P\mathrm{n}(q - q_x)^T\, i_{\mathrm{n}} dq \\ \dfrac{d}{dt}x = \xi \end{cases} \tag{5.5}$$

The goal of this section is to show that these equations are the Euler-Lagrange-Arnold equations for the natural fluid-body Lagrangian defined on a natural algebroid. Recall the definition of the ELA equation:

$$\frac{D}{dt}d_V L(a_t) = -T^*_\nabla\big(d_V L(a_t)\,,\,a_t\big) + \#^* d_H L(a_t)\,.$$

To show that equations (5.5) are an example of an ELA equation, we must plug the following information into the above, and show that the substitutions result in the incompressible fluid-body equations. We require

1. A Lagrangian L on an algebroid.
2. A connection on that algebroid, preferably the Levi-Civita connection for a metric that is natural to the problem. This is necessary to compute the vertical and horizontal differentials d_V and d_H, as well as the covariant differentiation of curves D/dt and the cotorsion T^*_∇.

We describe the algebroid of the incompressible fluid-body system by first defining its underlying groupoid. The *incompressible fluid-body groupoid* $\mathcal{SFBG} \rightrightarrows SE(n)$ is the set of triples

$$(y,x,g) \in SE(n)\times SE(n)\times \mathrm{SDiff}(F_x,F_y)\,.$$

The source and target of (y,x,g) are x and y respectively. Groupoid multiplication is defined

$$(z,y,h)(y,x,g) \coloneqq (z,x,h\circ g)\,.$$

An element (y,x,g) of $\mathcal{SFBG}$ is naturally interpreted as a map from the space of fluid-body configurations with body position x to those configurations with body position y.

Remark 5.2. The constructions in this thesis can be carried out in the framework of Sobolev spaces. We refer the reader to [10] and [2] for further details.

We now derive the Lie algebroid $\mathcal{SFBA} \to SE(n)$ of the incompressible fluid-body groupoid $\mathcal{SFBG} \rightrightarrows SE(n)$. We will see that elements of the fibre $\mathcal{SFBA}_x$ over $x \in SE(n)$ are naturally interpreted as a fluid-body velocity pairs (ξ,u), with $\xi \in T_xSE(n)$ the body's velocity, and $u \in \mathfrak{svect}(F_x)$ the fluid's velocity. Each fluid-body velocity pair satisfies the natural boundary condition that the normal components of the fluid's velocity and the body's velocity agree on the interface ∂F_x.

Proposition 5.3. *The Lie algebroid $\mathcal{SFBA} \to SE(n)$ of the incompressible fluid-body groupoid $\mathcal{SFBG}$ has the following structure.*

1. *The fibre $\mathcal{SFBA}_x$ over $x \in SE(n)$ is the vector space*

$$\mathcal{SFBA}_x = \left\{(\xi,u)\in T_xSE(n)\times\mathfrak{svect}(F_x)\mid \mathrm{n}\cdot u = \mathrm{n}\cdot\tilde{\xi}\right\}\,.$$

   Here n *is the outward pointing normal vector field along ∂F_x. We refer to the condition $\mathrm{n}\cdot u = \mathrm{n}\cdot\tilde{\xi}$ as the "equal normals" boundary condition.*

2. *The anchor map* $\# : \mathcal{SFBA} \to TSE(n)$ *is given by*

$$\#(\xi, u) = \xi \, .$$

3. *The Lie algebroid bracket on sections* $(\Xi_i, U_i) \in \Gamma\mathcal{SFBA}$ *is the section defined at each* $x \in SE(n)$ *by*

$$\big[(\Xi_1, U_1), (\Xi_2, U_2)\big](x) = \big([\Xi_1, \Xi_2](x), [U_1(x), U_2(x)] + \#U_1(x)\cdot U_2 - \#U_2(x)\cdot U_1\big) \, , \quad (5.6)$$

where the operations appearing on the right are defined in Remark 5.19 below.

Remark 5.4. In the above, Ξ is a section of $TSE(n)$ and U is a section of $\bigcup_{x \in SE(n)} \mathfrak{svect}(F_x)$. The operations appearing in expression of the algebroid bracket are defined:

1. $[\Xi_i, \Xi_j]$ is the usual Lie bracket of a vector fields on $SE(n)$.
2. For each x, $U(x)$ is a vector field on F_x, so $[U_i(x), U_j(x)]$ is the usual Lie bracket of vector fields on F_x.
3. The derivative of U in the direction $\xi \in T_x SE(n)$ is the vector field on F_x defined at q as

$$(\xi \cdot U)\big|_q \coloneqq \frac{d}{dt}\bigg|_{t=0} \left(U(x_t)|_q\right) \, ,$$

where x_t is any curve in $SE(n)$ such that $d/dt|_{t=0} x_t = \xi$.

Proof of 5.3. Part 1: By definition of $\mathcal{SFBA} = \mathrm{Lie}(\mathcal{SFBG})$, vectors in $\mathcal{SFBA}_x$ are generated by curves in $\mathcal{SFBG}$ of the form (x_t, x, g_t), where $x_0 = x$ and $g_0 = \mathrm{id}_{F_x}$. Thus every element $a \in \mathcal{FBA}_x$ has the form

$$a = \frac{d}{dt}\bigg|_{t=0} (x_t, x, g_t) = (\xi, 0, u) \simeq (\xi, u) \in T_x SE(n) \times \mathrm{vect}(F_x) \, .$$

The boundary condition is derived from the fact that both the diffeomorphisms[5] g_t and τ_t map the initial boundary ∂F_x to the time-advanced boundary F_{x_t}. Specifically, let $\Gamma_t : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $\partial F_t = \{\Gamma_t = 0\}$. Differentiating the relation $\Gamma_t \circ g_t(q) = 0$ for all $q \in \partial F_0$, we find $\partial_t \Gamma_0(q) + d\Gamma_{0,q}(u) = 0$. Similarly, $\partial_t \Gamma_0(q) + d\Gamma_{0,q}(\tilde{\xi}) = 0$. Subtracting these two equations, we find

$$d\Gamma_{0,q}(u - \tilde{\xi}) = 0 \, ,$$

which means that the vector field $u - \tilde{\xi}$ is tangent to the boundary ∂F_0. Equivalently, $\mathrm{n} \cdot u = \mathrm{n} \cdot \tilde{\xi}$.

Conversely, given a pair (ξ, u) satisfying the above boundary condition, we can construct a curve in the source fibre $\mathcal{SFBG}_x$ that generates (ξ, u).

[5] Recall from Section 4.1.1 that the body motion is defined $\tau_t(q) \coloneqq x_t x^{-1} \cdot q$.

Part 2: Let $(\xi, u) \in \mathcal{SFBA}_x$ be generated by (x_t, x, g_t). The anchor map is computed

$$\#(\xi, u) := \frac{d}{dt}\bigg|_{t=0} \operatorname{trg}(x_t, x, g_t) = \frac{d}{dt}\bigg|_{t=0} x_t = \xi\,.$$

Part 3: We derive a formula for the bracket on $\mathcal{SFBA}$ in two steps. We show there exists an "extended" algebroid $\mathcal{SFBA}^{ext}$ over the same base $SE(n)$ which maps surjectively onto $\mathcal{SFBA}$ by an algebroid morphism. This extended algebroid is itself a subalgebroid of the direct product algebroid $TSE(n) \times \mathfrak{svect}(\mathbb{R}^n)$. The formula for the bracket on $\mathcal{SFBA}^{ext}$ is given by restricting the bracket on $TSE(n) \times \mathfrak{svect}(\mathbb{R}^n)$. Then, the relation between $\mathcal{FBA}^{ext}$ and $\mathcal{FBA}$ turns out to be simple enough to allow us to write the bracket on $\mathcal{SFBA}$ in terms of the one on $\mathcal{SFBA}^{ext}$.

Consider the pair groupoid $SE(n) \times SE(n) \rightrightarrows SE(n)$ and the group of volume preserving diffeomorphisms $\mathrm{SDiff}(\mathbb{R}^n)$. Form the direct product groupoid

$$\mathcal{PG} := (SE(n) \times SE(n)) \times \mathrm{SDiff}(\mathbb{R}^n) \rightrightarrows SE(n)\,.$$

Define the extended fluid-body groupoid $\mathcal{FBG}^{ext}$ by

$$\mathcal{FBG}^{ext} := \left\{(y, x, g)\mathcal{PG} \mid g(F_x) = F_y\right\} \rightrightarrows SE(n)\,,$$

As in Example 2.11, the direct product algebroid $\mathcal{PA} := \mathrm{Lie}(\mathcal{PG}) = TSE(n) \times \mathfrak{svect}(\mathbb{R}^n) \to SE(n)$ has the bracket structure

$$\big[(\Xi_1, U_1), (\Xi_2, U_2)\big](x) = \big([\Xi_1, \Xi_2](x), [U_1(x), U_2(x)] + \#U_1(x) \cdot U_2 - \#U_2(x) \cdot U_1\big)\,. \quad (5.7)$$

The inclusion $I : \mathcal{SFBG}^{ext} \to \mathcal{PG}$ over the identity base map $\underline{I} : SE(n) \to SE(n)$ is easily seen to be a groupoid morphism. Thus the algebroid $\mathcal{SFBA}^{ext} := \mathrm{Lie}(\mathcal{SFBG}^{ext})$ is a subalgebroid of $\mathcal{PA}$, and the bracket on $\mathcal{SFBA}^{ext}$ is given by restricting the bracket on $\mathcal{PA}$.

Define the surjective morphism $\Phi : \mathcal{SFBG}^{ext} \to \mathcal{SFBG}$ over the identity base map by $\Phi(y, x, g) := (y, x, g|_{F_x})$. The induced algebroid morphism $\phi : \mathcal{SFBA}^{ext} \to \mathcal{SFBA}$ is given by a similar restriction, $\phi(\xi, u) = (\xi, u|_{F_x})$.

We compute the bracket on $\mathcal{SFBA}$. Let (Ξ_i, U_i) be sections of $\mathcal{SFBA}$. Define sections $(\Xi_i, \tilde{U}_i)$ of $\mathcal{SFBA}^{ext}$ by extending, for each $x \in SE(n)$, the vector field $U_i(x) \in \mathfrak{svect}(F_x)$ to a vector field $\tilde{U}_i(x)$ defined on all of $\mathbb{R}^n$. Thus $\phi(\Xi_i, \tilde{U}_i) = (\Xi_i, U_i)$. We have

$$\big[(\Xi_1, U_1)\,,\, (\Xi_2, U_2)\big]^{\mathcal{SFBA}} = \big[\phi(\Xi_1, \tilde{U}_1)\,,\, \phi(\Xi_2, \tilde{U}_2)\big]^{\mathcal{SFBA}} = \phi\big[(\Xi_1, \tilde{U}_1),\, (\Xi_2, \tilde{U}_2)\big]^{\mathcal{SFBA}^{ext}}\,.$$

The formula (5.6) follows from (5.7) and the fact that each vector field $\tilde{U}_i(x)$ equals $U_i(x)$ on the open set F_x, so that

$$[\tilde{u}_1, \tilde{u}_2]|_{F_x} = [u_1, u_2] \quad \text{and} \quad \big(\#(\xi_i, \tilde{u}_i) \cdot \tilde{U}_j\big)\big|_{F_x} = \#(\xi_i, u_i) \cdot U_j\,.$$

□

Remark 5.5. The technique we use to compute the bracket on the fluid-body algebroid $\mathcal{FBA}$ may also be used to compute the bracket on a related algebroid appearing in [11], where a fluid with a vortex sheet is studied.

5.2.1 Metric and connection for the incompressible system

In this section we give the incompressible fluid-body algebroid $\mathcal{SFBA}$ a metric structure, and then compute the corresponding Levi-Civita connection and its torsion. Here is our approach. We define the metric on $\mathcal{SFBA}$ as the restriction of the natural L^2-type metric on the extended space $TSE(n) \oplus_M \text{vect}$ (Definition 4.13). This in particular allows us to define an orthogonal projection $\Pi : TSE(n) \oplus_M \text{vect} \to \mathcal{SFBA}$ (Corollary 5.8). We may then identify the Levi-Civita connection on the incompressible fluid-body algebroid by first computing the Levi-Civita connection on the extended space, and then applying the orthogonal projection Π. This is analogous to the classical characterization of the Levi-Civita connection on a Riemannian submanifold.

The orthogonal projection, as well as the orthogonal decomposition $\mathcal{SFBA} = \ker \# \oplus \ker \#^{\perp}$ necessary to compute the Levi-Civita connection, are computed using the Hodge decomposition of Theorem 4.19.

We begin by equipping $\mathcal{SFBA}$ with the natural metric.

Definition 5.6. The L^2 metric on $\mathcal{SFBA}$ is the one induced by the L^2 metric on the extended incompressible fluid-body velocity space $TSE(n) \oplus_{SE(n)} \text{vect}$ via restriction. That is, for (ξ, u), $(\omega, v) \in \mathcal{SFBA}_x$, we define

$$\big\langle (\xi, u)\,,\, (\omega, v) \big\rangle_L^2 := \langle \xi, \omega \rangle_B + \int_{F_x} u \cdot v \, dq\,.$$

To determine the associated Levi-Civita connection, it is necessary to compute the isotropy subbundle ker# and its orthogonal compliment. Proposition 4.23 allows for a convenient description of these bundles.

Proposition 5.7. *The bundles $\mathcal{SFBA}$,* ker# *and* ker#$^{\perp}$ *are given in terms of the Hodge components of the fluid-body velocity bundle as*

$$\begin{aligned} \mathcal{SFBA}_x &= \mathcal{CFBA}_x \oplus \mathcal{HFBA}_x\,, \\ \ker\#_x &= \mathcal{CFBA}_x\,, \qquad\qquad \textit{and} \\ \ker\#_x^{\perp} &= \mathcal{HFBA}_x\,. \end{aligned}$$

Proof. Step 1: We begin by showing the first equality. If (ξ, u) is a pair in the direct sum $\mathcal{CFBA}_x \oplus \mathcal{HFBA}_x$, then $\mathrm{n} \cdot u = \mathrm{n} \cdot \tilde{\xi}$, and $\delta u = \nabla \cdot u = 0$. Thus (ξ, u) is in $\mathcal{SFBA}_x$. To prove the converse, we will show that any (ξ, u) in $\mathcal{SFBA}_x$ is orthogonal to $\mathcal{EFBA}_x$ and $\mathcal{HFBA}_x^{\perp}$. It follows from Proposition 4.23 that (ξ, vu) is in the direct sum $\mathcal{CFBA}_x \oplus \mathcal{HFBA}_x$.

If (ξ, u) is in $\mathcal{SFBA}_x$, then for any $(0, \nabla f)$ in $\mathcal{EFBA}_x$, we have

$$\langle (\xi, u), (0, \nabla f) \rangle_{L^2} = \int_{F_x} u \cdot \nabla f \, dq = \int_{F_x} \nabla \cdot (f u) \, dq = \int_{\partial F_x} f \, i_u dq$$

since u is divergence-free. Note that $\mathrm{t} \cdot \nabla f = 0$ implies that f is constant (say, c) on the boundary ∂F_x, so

$$\int_{\partial F_x} f\, i_u dq = c \int_{\partial F_x} i_u dq = 0\,,$$

where the last equality follows from another application of Stokes' theorem and the fact that v is divergence-free. Thus (ξ, u) is orthogonal to $\mathcal{EFBA}_x$.

Next, let $(\eta, \nabla f)$ be in $\mathcal{HFBA}_x^{\perp}$. Let h^ξ be the unique harmonic function on F_x satisfying the Neumann boundary conditions $\mathrm{n} \cdot \nabla h^\xi = \mathrm{n} \cdot \tilde{\xi}$. Then $i_{\nabla h^\xi} dq = i_{\tilde{\xi}} dq$, which in turn equals $i_u dq$ since (ξ, u) is in $S\mathcal{FBA}_x$ and thus satisfies the "equal normals" boundary condition. This implies

$$\int_{F_x} \nabla f \cdot u\, dq = \int_{\partial F_x} f\, i_u dq = \int_{\partial F_x} f\, i_{\nabla h^\xi} dq = \int_{F_x} \nabla f \cdot \nabla h^\xi\, dq\,,$$

from which it follows that

$$\langle (\xi, u), (\eta, \nabla f) \rangle_{L^2} = \langle (\xi, \nabla h^\xi), (\eta, \nabla f) \rangle_{L^2} = 0\,,$$

since $(\xi, \nabla h^\xi)$ is in $\mathcal{HFBA}_x$. This proves the first equality, which writes $S\mathcal{FBA}_x$ in terms of its Hodge components.

Step 2: By definition, every element of the isotropy algebra $\ker \#_x$ is of the form $(0, u)$ for some divergence-free vector field u satisfying the Neumann boundary condition $\mathrm{n} \cdot u = 0$. Thus the isotropy algebra is contained in $\mathcal{CFBA}_x$. The reverse containment is also obvious.

Lastly, the description given above of $S\mathcal{FBA}$ and $\ker \#_x$ in terms of their Hodge components makes it clear that $\ker \#_x^{\perp} = \mathcal{HFBA}_x$. □

An immediate corollary of Propositions 5.7 and 4.23 is the following.

Corollary 5.8. *There exists an orthogonal projection operator* $\Pi : TSE(n) \oplus_{SE(n)} \mathrm{vect} \to S\mathcal{FBA}$*.*

5.2.2 Levi-Civita connection

The Levi-Civita connection on $S\mathcal{FBA}$ that we are interested in is the vector bundle connection (or TM-connection) described in general in Section 2.2.2. Before computing it in Proposition 5.10, we highlight the definition and notation of the lifting isomorphism between the tangent bundle of the base and the compliment of the isotropy algebra.

Definition 5.9. Let $\#^{-1} : T_x SE(n) \to \ker \#_x^{\perp}$ be the "lift" isomorphism defined

$$\#^{-1}(\xi) \coloneqq (\xi, \nabla h^\xi)\,,$$

where h^ξ is the harmonic function uniquely determined by ξ satisfying the Neumann boundary condition $\mathrm{n} \cdot \nabla h^\xi = \mathrm{n} \cdot \tilde{\xi}$.

Proposition 5.10. *Let ∇^E be the Levi-Civita connection on $TSE(n)$ with respect to the body metric. Let ξ be a vector in $T_xSE(n)$, let (Ω, V) be a section of $\mathcal{SFBA}$, and let $(\omega, v) := (\Omega, V)(x)$. The Levi-Civita connection on $\mathcal{SFBA}$ with respect to the L^2 metric is given by*

$$\nabla_\xi(\Omega, V)(x) = \Pi\left(\nabla^E_\xi \Omega(x)\,,\, (\nabla h^\xi \cdot \nabla)v + \xi \cdot V(x)\right).$$

Proof. If the torsion-free, metric compatible $\mathcal{A}$-connection $\nabla^{\mathcal{A}}$ on $\mathcal{SFBA}$ is known, then the formula for the Levi-Civita connection follows directly from Definition 2.31,

$$\nabla_\xi(\Omega, V)(x) := \nabla^{\mathcal{A}}_{\#^{-1}(\xi)}(\Omega, V)(x)$$

and the definition of the lift isomorphism given above. We give a formula for the connection $\nabla^{\mathcal{A}}$ and prove that it is torsion-free and metric compatible. Define $\nabla^{\mathcal{A}}$ by

$$\nabla^{\mathcal{A}}_{(\xi,u)}(\Omega, V)(x) := \Pi\left(\nabla^E_\xi Om\,,\, (u \cdot \nabla)v + \xi \cdot V(x)\right). \tag{5.8}$$

Note that the projection Π is included to ensure that the resulting fluid-body velocity pair has compatible boundary conditions.

Consider two sections (Ξ, U) and (Ω, V) with values (ξ, u) and (ω, v) at x. It is clear that $\nabla^{\mathcal{A}}_{(\xi,u)}(\Omega, V)(x) - \nabla^{\mathcal{A}}_{(\omega,v)}(\Xi, U)(x) = [(\Xi, U), (\Omega, V)](x)$ for all $x \in SE(n)$, so the connection is torsion-free.

All that remains is to check that $\nabla^{\mathcal{A}}$ is compatible with the L^2 metric on $\mathcal{SFBA}$. Let (θ, w) be a vector in $\mathcal{SFBA}_x$, and let x_t be a curve in $SE(n)$ generating θ. First observe that

$$\left.\frac{d}{dt}\right|_{t=0} \int_{F_{x_t}} U(x_t) \cdot V(x_t)\, dq = \int_{\partial F_x} u \cdot v\, i_{\tilde\theta} dq + \int_{F_x} \left(\theta \cdot U(x)\right) \cdot v\, dq + \int_{F_x} u \cdot \left(\theta \cdot V(x)\right) dq\,,$$

and since $\mathrm{n} \cdot w = \mathrm{n} \cdot \tilde\theta$,

$$\int_{\partial F_x} u \cdot v\, i_{\tilde\theta} dq = \int_{\partial F_x} u \cdot v\, i_w dq\,.$$

Moreover, using Stokes' theorem and identifying vector fields and 1-forms, we have

$$\int_{\partial F_x} u \cdot v\, i_w dq = \int_{F_x} i_w d\big(u \cdot v\big)\, dq = \int_{F_x} \big((w \cdot \nabla)u\big) \cdot v\, dq + \int_{F_x} u \cdot \big((w \cdot \nabla)v\big)\, dq\,.$$

It is now straightforward to check metric compatibility. Note that $\#(\theta, w) = \theta$. We have

$$\begin{aligned}
\#(\theta, w)\big\langle (\Xi, U), (\Omega, V)\big\rangle_{L^2} &= \left.\frac{d}{dt}\right|_{t=0} \left(\big\langle \Xi, \Omega \big\rangle_B (x_t) + \int_{F_{x_t}} U(x_t) \cdot V(x_t)\, dq \right) \\
&= \big\langle \nabla^E_\theta \Xi, \Omega \big\rangle_B (x) + \big\langle \Xi, \nabla^E_\theta \Omega \big\rangle_B (x) \\
&\quad + \int_{F_x} \big((w \cdot \nabla)u\big) \cdot v\, dq + \int_{F_x} u \cdot \big((w \cdot \nabla)v\big)\, dq \\
&\quad\quad + \int_{F_x} \big(\theta \cdot U(x)\big) \cdot v\, dq + \int_{F_x} v \cdot \big(\theta \cdot V(x)\big)\, dq \\
&= \big\langle \nabla^{\mathcal{A}}_{(\theta,w)}(\Xi, U), (\Omega, V)\big\rangle_{L^2}(x) + \big\langle (\Xi, U), \nabla^{\mathcal{A}}_{(\theta,w)}(\Omega, V)\big\rangle_{L^2}(x)\,.
\end{aligned}$$

□

Lemma 5.11. *Let $a = (\xi, u)$ and $b = (\omega, v)$ be two points in $\mathcal{SFBA}_x$. The torsion of the Levi-Civita connection on $\mathcal{SFBA}$ is*

$$T_\nabla(a,b) = \Pi\big(0\,,\, (\nabla h^\xi \cdot \nabla)v - (\nabla h^\omega \cdot \nabla)u - [u,v]\big)\,.$$

Proof. Let $A = (\Xi, U)$ and $B = (\Omega, V)$ be two sections of $\mathcal{SFBA}$ such that $A(x) = a = (\xi, u)$ and $B(x) = b = (\omega, v)$. A simple computation shows

$$\begin{aligned} T_\nabla(a,b) &= \nabla_{\#a}B(x) - \nabla_{\#b}A(x) - [A,B](x) \\ &= \nabla_\xi(\Omega,V)(x) - \nabla_\omega(\Xi,U)(x) - [(\Xi,U),(\Omega,V)](x) \\ &= \Pi\big(0\,,\,(\nabla h^\xi\cdot\nabla)v - (\nabla h^\omega\cdot\nabla)u - [u,v]\big)\,, \end{aligned}$$

where we have used the fact that ∇^E is torsion-free, and the terms $\xi \cdot V(x)$ and $\omega \cdot U(x)$, which appear in both the connection and the algebroid bracket, have cancelled. □

5.2.3 Incompressible fluid-body equations

In this section we show that the incompressible fluid-body equations (5.5) are the Euler-Lagrange-Arnold equations on $\mathcal{SFBA}$ for the *incompressible fluid-body Lagrangian*

$$L(a) := \frac{1}{2}\langle a, a\rangle_{L^2}\,.$$

With the connection and Lagrangian on $\mathcal{SFBA}$ now in hand, we begin computing the terms that appear in the ELA equation:

$$\frac{D}{dt}d_V L(a_t) = -T^*_\nabla\big(d_V L(a_t)\,,\, a_t\big) + \#^* d_H L(a_t)\,.$$

We start by computing the differentials $d_V L(a)$ and $d_H L(a)$ (see Remark 2.22). Let $b \in \mathcal{SFBA}_x$ be given and let a_t be a vertical curve in $\mathcal{SFBA}$ through a such that $D/dt|_{t=0}a_t = b$. Then the vertical differential of L acting on b is computed

$$\langle d_V L(a), b\rangle = \frac{d}{dt}\bigg|_{t=0} L(a_t) = \left\langle a,\, \frac{D}{dt}\bigg|_{t=0} a_t\right\rangle_{L^2} = \langle \mathcal{I}(a), b\rangle\,,$$

so that $d_V L(a) = \mathcal{I}(a)$. Similarly, let $\eta \in T_x SE(n)$ be given and let a_t now be a horizontal curve through a such that $d/dt|_{t=0}\pi(a_t) = \eta$. Then we have

$$\langle d_H L(a), \eta\rangle = \frac{d}{dt}\bigg|_{t=0} L(a_t) = \left\langle a,\, \frac{D}{dt}\bigg|_{t=0} a_t\right\rangle_{L^2} = 0\,,$$

so that $d_H L(a) = 0$.

It is also necessary to compute $T^*_\nabla(d_V L(a), a)$.

Lemma 5.12. *Consider the bilinear operator B_∇ associated with the Levi-Civita connection ∇ on $\mathcal{SFBA}$. Let L be the incompressible fluid-body Lagrangian, and let $a = (\xi, u) \in \mathcal{SFBA}_x$. The following formula holds:*

$$T^*_\nabla(d_V L(a), a) = \mathcal{I} \circ \Pi\big(0\,, (v\cdot\nabla)v - (\nabla h^\xi \cdot \nabla)v\big)\,. \tag{5.9}$$

Proof. Let $b = (\omega, v) \in \mathcal{SFBA}_x$ be arbitrary. Using the expression for $d_V L(a)$ above and formula for the torsion in Lemma 5.11 , we have

$$\begin{aligned}\big\langle T^*_\nabla(d_V L(a), a), b\big\rangle &= \big\langle d_V L(a), T_\nabla(a,b)\big\rangle = \big\langle a, T_\nabla(a,b)\big\rangle_{L^2} \\ &= \big\langle (\xi,u), \Pi\big(0\,,\, (\nabla h^\xi\cdot\nabla)v - (\nabla h^\omega \cdot\nabla)u - [u,v]\big)\big\rangle_{L^2} \\ &= \underbrace{\int_{F_x} u\cdot\big((\nabla h^\xi\cdot\nabla)v\big)\,dq}_{1} - \underbrace{\int_{F_x} u\cdot\big((\nabla h^\omega\cdot\nabla)u\big)\,dq}_{2} - \underbrace{\int_{F_x} u\cdot[u,v]\,dq}_{3}\,.\end{aligned}$$

Each of these three terms may be rewritten. Starting with the first,

$$\mathbf{1} = \int_{F_x} \nabla(u\cdot v)\cdot\nabla h^\xi - \big((\nabla h^\xi\cdot\nabla)u\big)\cdot v\,dq\,.$$

The first term on the right hand side can be written entirely in terms of u and v using the fact that ∇h^ξ and u are divergence-free and have equal normal components on the boundary. We have

$$\int_{F_x}\nabla(u\cdot v)\cdot\nabla h^\xi\,dq = \int_{\partial F_x} u\cdot v\, i_{\nabla h^\xi} dq = \int_{\partial F_x} u\cdot v\, i_u dq = \int_{F_x}\nabla(u\cdot v)\cdot u\,dq\,.$$

Making this substitution into the expression for $\mathbf{1}$ and expanding the gradient results in

$$\mathbf{1} = \int_{F_x}\big((u\cdot\nabla)u\big)\cdot v + u\cdot\big((u\cdot\nabla)v\big) - \big((\nabla h^\xi\cdot\nabla)u\big)\cdot v\,dq\,.$$

Similar steps may be used to rewrite the second term. Also, the Lie bracket in the third term may be expanded. We have

$$\mathbf{2} = \int_{F_x} u\cdot\big((v\cdot\nabla)u\big)\,dq \qquad\text{and}\qquad \mathbf{3} = \int_{F_x} u\cdot\big((u\cdot\nabla)v - (v\cdot\nabla)u\big)\,dq\,.$$

Combining these expressions results in

$$\begin{aligned}\big\langle T^*_\nabla(d_V L(a), a), b\big\rangle &= \int_{F_x}\big((u\cdot\nabla)u - (\nabla h^\xi\cdot\nabla)u\big)\cdot v\,dq \\ &= \big\langle\big(0\,,(u\cdot\nabla)u - (\nabla h^\xi\cdot\nabla)u\big)\,,\,(\omega,v)\big\rangle_{L^2}\,.\end{aligned}$$

Since this is true for all $b = (\omega, v) \in \mathcal{SFBA}_x$, we have shown that formula (5.9) holds. □

Remark 5.13. The action of the projection operator Π may be written more explicitly;

$$T^*_\nabla(d_V L(a), a) = \mathcal{I}\big(0\,,\,(u\cdot\nabla)u - (\nabla h^\xi\cdot\nabla)u\big) + \mathcal{I}(\eta, \nabla P)\,,$$

where the "correction term" $(\eta, \nabla p) = -\Pi^\perp(0\,,\,(u\cdot\nabla)u - (\nabla h^\xi\cdot\nabla)u)$ is the unique element of $\mathcal{SFBA}_x^\perp$ which ensures that the right-hand side of the above equation lies in $\mathcal{SFBA}_x$. In other words, the correction term guarantees that the fluid velocity is divergence-free and has compatible boundary conditions along the body. We will see that the function p is the pressure of the fluid.

Lemma 5.14. *The correction term $(\eta, \nabla P)$ satisfies*

$$\langle \eta, \theta\rangle_B = -\left\langle s\,,\,\int_{\partial F_x} P(q-q_x)\mathrm{n}^T\, i_{\mathrm{n}} dq\right\rangle - \left\langle k\,,\,\int_{\partial F_x} P\mathrm{n}^T\, i_{\mathrm{n}} dq\right\rangle$$

for all $\theta = (sO, k) \in T_x SE(n)$.

Proof. Note that $(\theta, \nabla h^\theta)$ is in $\mathcal{SFBA}_x$, and is therefore orthogonal to $(\eta, \nabla P)$:

$$0 = \big\langle (\eta, \nabla P)\,,\,(\theta, \nabla h^\theta)\big\rangle_{L^2} = \langle \eta, \theta\rangle_B + \int_{F_x} \nabla P\cdot\nabla h^\theta\, dq\,.$$

Since ∇h^θ is divergence-free, we can use Stokes' theorem to write

$$\langle \eta, \theta\rangle_B = -\int_{\partial F_x} P\, i_{\nabla h^\theta} dq = -\int_{\partial F_x} P\, i_{\tilde\theta} dq\,,$$

where the last equality holds since ∇h^θ and $\tilde\theta$ have equal normal components on the boundary. Moreover, when restricted to vectors tangent to the boundary, the forms $i_{\tilde\theta}dq$ and $i_{\langle\tilde\theta,\mathrm{n}^T\rangle\mathrm{n}}dq$ are equal. The normal component of the body velocity field $\tilde\theta$ is

$$\langle\tilde\theta, \mathrm{n}^T\rangle\mathrm{n} = \langle s(q-q_x) + k, \mathrm{n}^T\rangle\mathrm{n} = \langle s, (q-q_x)\mathrm{n}^T\rangle\mathrm{n} + \langle k, \mathrm{n}^T\rangle\mathrm{n}\,.$$

The result follows from straightforward substitutions. □

We are now ready to state the main theorem of this section.

Theorem 5.15. *The incompressible fluid-body equations* (5.5) *are the Euler-Lagrange-Arnold equations on the incompressible fluid-body algebroid $\mathcal{SFBA}$ with respect to the Lagrangian*

$$L(a) := \frac{1}{2}\langle a, a\rangle_{L^2}\,.$$

Proof. Recall that an $\mathcal{A}$-path a satisfies the Euler-Lagrange-Arnold equations if

$$\frac{D}{dt} d_v L(a) = -T^*_\nabla(d_V L(a), a) + \#^* d_H L(a)\,.$$

First note that the last equation in the incompressible fluid body equations, $d/dt\, x = \xi$, is the condition that a is an $\mathcal{A}$-path. Using the above-computed expressions for $d_V L(a)$, $d_H L(a)$ and $T^*_\nabla(d_V L(a), a)$,

$$\frac{D}{dt}a = \left(\frac{D}{dt}\mathcal{I}_B(\xi)\,,\,\frac{d}{dt}u + (\nabla h^\xi \cdot \nabla)u\right) = -\big(\mathcal{I}_B(\eta)\,,\,(u\cdot\nabla)u - (\nabla h^\xi\cdot\nabla)u + \nabla P\big)\,.$$

Writing the above out component-wise, we have

$$\frac{D}{dt}\mathcal{I}_B(\xi) = -\mathcal{I}_B(\eta) \tag{5.10}$$

$$\frac{d}{dt}u + (u\cdot\nabla)v = -\nabla P\,. \tag{5.11}$$

Equation (5.11) is the usual incompressible Euler equation which governs the fluid.

It remains to show that (5.10) is equivalent to the equations that govern the motion of the body. Let $\theta = (sO, k) \in T_x SE(n)$ be arbitrary, and take the L^2 pairing of it with equation (5.10). On the right hand side we have, by Lemma 5.14,

$$-\langle\mathcal{I}_B(\eta),\theta\rangle = -\langle\eta,\theta\rangle_B = \left\langle s\,,\,\int_{\partial F_x} P(q-q_x)\mathrm{n}^T\, i_{\mathrm{n}} dq\right\rangle + \left\langle k\,,\,\int_{\partial F_x} P\mathrm{n}^T\, i_{\mathrm{n}} dq\right\rangle\,.$$

On the left, by Lemma 4.6, we have

$$\left\langle\frac{D}{dt}\mathcal{I}_B(\xi),\theta\right\rangle_B = \left\langle\mathbb{I}\frac{d}{dt}(rO)^T, sO\right\rangle + \left\langle m\frac{d}{dt}l^T, k\right\rangle\,.$$

For the rotational term, we write

$$\left\langle\mathbb{I}\frac{d}{dt}(rO)^T, sO\right\rangle = \left\langle O\mathbb{I}\frac{d}{dt}(O^Tr^T), s\right\rangle = \left\langle\frac{d}{dt}(O\mathbb{I}O^Tr^T), s\right\rangle - \left\langle\frac{d}{dt}(O)\mathbb{I}O^Tr^T, s\right\rangle\,.$$

Since $d/dt\, x = \xi$, we have in particular $d/dt\, O = rO$. It follows that $\langle d/dt(O)\mathbb{I}O^Tr^T, s\rangle = \langle rO\mathbb{I}O^Tr^T, s\rangle = 0$, since it is the trace of the product of a symmetric matrix $rO\mathbb{I}O^Tr^T$ and an antisymmetric matrix s. We have now shown

$$\left\langle\frac{d}{dt}(\mathbb{I}_x r^T), s\right\rangle + \left\langle m\frac{d}{dt}l^T, k\right\rangle = \left\langle s\,,\,\int_{\partial F_x} P(q-q_x)\mathrm{n}^T\, i_{\mathrm{n}} dq\right\rangle + \left\langle k\,,\,\int_{\partial F_x} P\mathrm{n}^T\, i_{\mathrm{n}} dq\right\rangle$$

for all s and k. The equations of motion for the body follow.

□

5.3 Reduced compressible dynamics

The goal of this section is to show that the dynamical equations of a rigid body moving in a compressible fluid are ELA equations on what we call the compressible fluid-body algebroid.

The "Eulerian" or "reduced" compressible fluid-body equations are

$$\begin{cases} \frac{d}{dt}u + (u \cdot \nabla)u = \dfrac{\nabla P_1}{\tilde{\rho}} - \nabla P_2) \\ \frac{d}{dt}\tilde{\rho} + \nabla \cdot (\tilde{\rho} u) = 0 \\ m\frac{d}{dt}l = \int\limits_{\partial F_x} (P_2 - \tilde{\rho}^{-1} P_1)\mathrm{n}\, i_{\mathrm{n}}\rho \\ \frac{d}{dt}(r\mathbb{I}_x) = \int\limits_{\partial F_x} (P_2 - \tilde{\rho}^{-1} P_1)\mathrm{n}(q - q_x)^T\, i_{\mathrm{n}}\rho \\ \frac{d}{dt}x = \xi\,. \end{cases} \tag{5.12}$$

Here F_x is the domain of the fluid around the body located at position x and n is the outward pointing normal of the surface of the body ∂F_x. The first equation is the compressible Euler equation for the fluid with velocity v and density ρ. The second equation is the continuity equation. The third and fourth equations are Newton's law for the body's linear momentum ml and angular momentum $r\mathbb{I}_x$ respectively. The last equation relates the body's position $x \in SE(n)$ to its velocity $\zeta \in T_x SE(n)$. The two functions P_1 and P_2 are functions on F_x with two distinct roles that we now clarify.

Remark 5.16. The fluid is assumed to have a *constitutive relation* or *internal energy per unit mass* given by a function $w : \mathbb{R} to \mathbb{R}$. The function P_1 is defined explicitly in terms of the density function $\tilde{\rho}$ and w by $P_1 := \tilde{\rho}^2 w'(\tilde{\rho})$. This is the standard pressure term that appears in the compressible Euler equations without a rigid body. With a rigid body present, the ρ-harmonic function P_2 is included to maintain the boundary condition $\mathbf{n}u = \mathbf{n}\tilde{\xi}$.

The derivation of the compressible fluid-body equations from the ELA equation is quite involved, so we outline the main points here. First a description of the underlying fluid-body groupoid is given and its algebroid is computed (Proposition 5.20). Then the fluid-body algebroid is equipped with a metric, and the relevant objects are given in Proposition 5.25 (the orthogonal compliment of the isotropy algebra), Remark 5.29 (covariant differentiation of curves with respect to the Levi-Civita connection), and Lemma 5.30 (the torsion). The compressible fluid-body Lagrangian is defined in Section 5.3.3, and the terms of the ELA equation are computed. Finally, the dynamical equations are derived in Theorem 5.35.

5.3.1 Compressible fluid-body algebroid

We begin by defining the compressible fluid-body groupoid.

Remark 5.17. The following construction is analogous to that of an action groupoid. To see why that should be, consider the compressible Euler equations without a rigid body. These equations are an example of "Euler-Poincaré equations with an advected quantity", which Example 3.2.3 shows are just the ELA equation on an action algebroid. Specifically, the compressible Euler equations are ELA equations on the algebroid of the action groupoid $\mathrm{Diff}(F) \ltimes \mathrm{Dens}(F)$. With the inclusion of a rigid body, this groupoid is modified to keep track of the body's position.

Recall the definition of M, the space of fluid densities, from Section 4.3.

Definition 5.18. The *compressible fluid-body groupoid* $\mathcal{FBG} \rightrightarrows M$ is the set

$$\mathcal{FBG} := \left\{ (y, x, \rho, g) \in SE(n) \times SE(n) \times \mathrm{Dens}(F_x) \times \mathrm{Diff}(F_x, F_y) \right\}$$

over the base M. The source and target maps are given by

$$\mathrm{src}(y, x, \rho, g) := (x, \rho) \quad \text{and} \quad \mathrm{trg}(y, x, \rho, g) := (y, g_* \rho)\,,$$

and composition is defined

$$(z, y, g_* \rho, h)(y, x, \rho, g) := (z, x, \rho, h \circ g)\,.$$

Before describing the algebroid of the fluid-body groupoid, it will be helpful to understand a related construction. Recall from Section 4.3 the definition of $\mathrm{Dens}(*)$, the space of densities on the exterior of a *non-rigid* body. Consider the action groupoid $\mathrm{Diff}(\mathbb{R}^n) \ltimes \mathrm{Dens}(*) \rightrightarrows \mathrm{Dens}(*)$. Let

$$\mathcal{PG} := \big(SE(n) \times SE(n) \big) \times \big(\mathrm{Diff}(\mathbb{R}^n) \ltimes \mathrm{Dens}(*) \big) \rightrightarrows SE(n) \times \mathrm{Dens}(*)$$

be a direct product groupoid, and let

$$\mathcal{PA} := \mathrm{Lie}(\mathcal{PG}) = TSE(n) \times \big(\mathrm{vect}(\mathbb{R}^n) \ltimes \mathrm{Dens}(\mathbb{R}^n) \big)$$

be its algebroid.

Remark 5.19. We use the following notation to describe the bracket on $\mathcal{PA}$. The factors of a section (Ξ, U) of $\mathcal{PA}$ are the function $\Xi : SE(n) \times \mathrm{Dens}(*) \to TSE(n)$ and the function $U : SE(n) \times \mathrm{Dens}(*) \to \mathrm{vect}\mathbb{R}^n$. We define $\Xi^\rho := \Xi(\cdot, \rho)$ and $\Xi^x := \Xi(x, \cdot)$. For each fixed ρ, the function Ξ^ρ is a section of $TSE(n)$, and for each fixed x, the function Ξ^x is a map from $\mathrm{Dens}(*)$ to the tangent space $T_x SE(n)$. We make similar definitions for U^x (a section of $\mathrm{vect}\mathbb{R} \ltimes \mathrm{Dens}(*)$ for each fixed x) and U^ρ (a map from $SE(n)$ to $\mathrm{vect}(\mathbb{R}^n)$ for each fixed ρ). Lowercase letters denote the value of sections at a point, $(\xi, u) := (\Xi, U)(x, \rho)$.

The following operations appear in the expression for the algebroid bracket on $\mathcal{PA}$ and are defined:

1. $[\Xi_i^\rho, \Xi_j^\rho]$ is the usual Lie bracket of a vector fields on $SE(n)$.
2. $\#^\rho u \cdot \Xi^x := d/dt|_{t=0}\, \Xi^x(\rho_t)$, where ρ_t is a curve generating $\#^\rho u$.

3. $[u_i, u_j]$ is the usual Lie bracket of vector fields on F_x.

4. $\#(\xi, u) \cdot U := d/dt|_{t=0}\, U(x_t, \rho_t)$, where (x_t, ρ_t) is a curve generating $\#(\xi, u)$.

By Example 2.11, the direct product algebroid $\mathcal{PA}$ has the bracket

$$\begin{aligned}&\big[(\Xi_1, U_1), (\Xi_2, U_2)\big]^{\mathcal{PA}}(x, \rho)\\ &\quad = \Big([\Xi_1^\rho, \Xi_2^\rho](x) + \#^\rho u_1 \cdot \Xi_2^x - \#^\rho u_2 \cdot \Xi_1^x\,,\ [U_1^x, U_2^x](\rho) + \#^x \xi_1 \cdot U_2^\rho - \#^x \xi_2 \cdot U_1^\rho\Big)\,. \end{aligned} \tag{5.13}$$

The bracket in the left-hand factor is the Lie bracket on the vector fields $\Xi^\rho \in \Gamma TSE(n)$. The bracket in the right-hand factor is the action algebroid bracket acting on sections $U_i^x \in \Gamma\mathrm{vect}(\mathbb{R}^n) \ltimes \mathrm{Dens}(*)$. The latter is written out explicitly as

$$\begin{aligned}[U_1^x, U_2^x](\rho) &= [U_1^x(\rho), U_2^x(\rho)] + \#^\rho U_1^x(\rho) \cdot U_2^x - \#^\rho U_1^x(\rho) \cdot U_2^x\\ &= [u_1, u_2] + \#^\rho u_1 \cdot U_2^x - \#^\rho u_2 \cdot U_2^x\,.\end{aligned}$$

Also note that $\#(\xi_i, u_i) \cdot U_j = \#^x \xi_1 \cdot U_2^\rho + \#^\rho u_i \cdot U_j^x$. Substituting these identities into the expression for $[\cdot,\cdot]^{\mathcal{PA}}$,

$$\begin{aligned}&\big[(\Xi_1, U_1), (\Xi_2, U_2)\big]^{\mathcal{PA}}(x, \rho)\\ &\quad = \Big([\Xi_1^\rho, \Xi_2^\rho](x) + \#^\rho u_1 \cdot \Xi_2^x - \#^\rho u_2 \cdot \Xi_1^x\,,\ [u_1, u_2] + \#(\xi_1, u_1) \cdot U_2 - \#(\xi_2, u_2) \cdot U_1\Big)\,. \end{aligned} \tag{5.14}$$

We are now ready to describe the fluid-body algebroid.

Proposition 5.20. *The Lie algebroid $\mathcal{FBA} \to M$ of the compressible fluid-body groupoid $\mathcal{FBG}$ has the following structure.*

1. *The fibre $\mathcal{FBA}_{(x,\rho)}$ over $(x, \rho) \in M$ is the vector space*

$$\mathcal{FBA}_{(x,\rho)} = \left\{(\xi, u) \in T_x SE(n) \times \mathrm{vect}(F_x) \mid \mathbf{n}u = \mathbf{n}\tilde{\xi}\right\}\,.$$

2. *The anchor map $\# : \mathcal{FBA}_{(x,\rho)} \to T_{(x,\rho)}M$ is given by*

$$\#(\xi, u) = (\#^x \xi, \#^\rho u)(\xi, -d i_u \rho)\,.$$

3. *Let $(\Xi_i, U_i) \in \Gamma\mathcal{FBA}$ be sections and let $(\xi_i, u_i) := (\Xi_i, U_i)(x, \rho)$. The Lie algebroid bracket is the section defined at each $(x, \rho) \in M$ by*

$$\begin{aligned}&\big[(\Xi_1, U_1), (\Xi_2, U_2)\big](x, \rho)\\ &\quad = \Big([\Xi_1^\rho, \Xi_2^\rho](x) + \#^\rho u_1 \cdot \Xi_2^x - \#^\rho u_2 \cdot \Xi_1^x\,,\ [u_1, u_2] + \#(\xi_1, u_1) \cdot U_2 - \#(\xi_2, u_2) \cdot U_1\Big)\,, \end{aligned} \tag{5.15}$$

where the operations appearing on the right are defined in a manner similar to Remark 5.19.

Proof. Part 1: The proof is entirely similar to that of the corresponding statement for the incompressible case, Proposition 5.3.

Part 2: Let $(\xi, u) \in \mathcal{FBA}_{(x,\rho)}$ be generated by (x_t, x, ρ, g_t). The anchor map is computed

$$\#(\xi, u) := \frac{d}{dt}\bigg|_{t=0} \operatorname{trg}(x_t, x, \rho, g_t) = \frac{d}{dt}\bigg|_{t=0} (x_t, g_{t*}\rho) = (\xi, -\mathcal{L}_u \rho) = (\xi, -di_u\rho)\,.$$

Part 3: We derive a formula for the bracket on $\mathcal{FBA}$ in two steps. We show there exists an "extended" algebroid $\mathcal{FBA}^{ext}$ over the same base M which maps surjectively onto $\mathcal{FBA}$ by an algebroid morphism. This extended algebroid is itself a subalgebroid of the direct product algebroid $\mathcal{PA}$. The formula for the bracket on $\mathcal{FBA}^{ext}$ is given by restricting the bracket (5.14) on $\mathcal{PA}$. Then, the relation between $\mathcal{FBA}^{ext}$ and $\mathcal{FBA}$ turns out to be simple enough to allow us to write the bracket on $\mathcal{FBA}$ in terms of the one on $\mathcal{FBA}^{ext}$.

Define the extended fluid-body groupoid $\mathcal{FBG}^{ext}$ by

$$\mathcal{FBG}^{ext} := \left\{(y, x, g, \rho) \in \mathcal{PG} \mid g(F_x) = F_y\right\} \rightrightarrows M\,,$$

The inclusion $I : \mathcal{FBG}^{ext} \to \mathcal{PG}$ over $\underline{I} : M \to SE(n) \times \operatorname{Dens}(*)$ is easily seen to be a groupoid morphism. Thus the algebroid $\mathcal{FBA}^{ext} := \operatorname{Lie}(\mathcal{FBG}^{ext})$ is a subalgebroid of $\mathcal{PA} := \operatorname{Lie}(\mathcal{PG}) = TSE(n) \times (\operatorname{vect}(\mathbb{R}^n) \ltimes \operatorname{Dens}(\mathbb{R}^n))$, and the bracket on $\mathcal{FBA}^{ext}$ is given by restricting the bracket on $\mathcal{PA}$.

Define the surjective morphism $\Phi : \mathcal{FBG}^{ext} \to \mathcal{FBG}$ over the identity $\mathrm{id} : M \to M$ by $\Phi(y, x, g, \rho) := (y, x, g|_{F_x}, \rho)$. The induced algebroid morphism $\phi : \mathcal{FBA}^{ext} \to \mathcal{FBA}$ is a similar restriction, $\phi(\xi, u) = (\xi, u|_{F_x})$.

We compute the bracket on $\mathcal{FBA}$. Let (Ξ_i, U_i) be sections of $\mathcal{FBA}$. Define sections $(\Xi_i, \tilde{U}_i)$ of $\mathcal{FBA}^{ext}$ by extending, for each (x, ρ), the vector field $U_i(x, \rho) \in \operatorname{vect}(F_x)$ to a vector field $\tilde{U}_i(x, \rho)$ defined on all of $\mathbb{R}^n$. Thus $\phi(\Xi_i, \tilde{U}_i) = (\Xi_i, U_i)$. We have

$$\big[(\Xi_1, U_1)\,,\,(\Xi_2, U_2)\big]^{\mathcal{FBA}} = \big[\phi(\Xi_1, \tilde{U}_1)\,,\,\phi(\Xi_2, \tilde{U}_2)\big]^{\mathcal{FBA}} = \phi\big[(\Xi_1, \tilde{U}_1),\,(\Xi_2, \tilde{U}_2)\big]^{\mathcal{FBA}^{ext}}\,.$$

The formula (5.15) follows from (5.14) and the fact that each vector field $\tilde{U}_i(x, \rho)$ equals $U_i(x, \rho)$ on the open set F_x, so that

$$[\tilde{u}_1, \tilde{u}_2]|_{F_x} = [u_1, u_2] \quad \text{and} \quad \big(\#(\xi_i, \tilde{u}_i) \cdot \tilde{U}_j\big)\big|_{F_x} = \#(\xi_i, u_i) \cdot U_j\,.$$

□

5.3.2 Metric and connection for the compressible system

In this section we give the fluid-body algebroid $\mathcal{FBA}$ a metric structure, and then compute the corresponding Levi-Civita connection and its torsion. Here is our approach. We define the metric on $\mathcal{FBA}$ as the restriction of the natural L^2-type metric on the extended space $TSE(n) \oplus_M \text{vect}$. This in particular allows us to define an orthogonal projection $\Pi : TSE(n) \oplus_M \text{vect} \to \mathcal{FBA}$ (Corollary 5.24). We may then identify the Levi-Civita connection on the fluid-body algebroid by first computing the Levi-Civita connection on the extended space, and then applying the orthogonal projection Π. This is analogous to the classical characterization of the Levi-Civita connection on a Riemannian submanifold.

The orthogonal projection, as well as the orthogonal decomposition $\mathcal{FBA} = \ker \# \oplus \ker \#^{\perp}$ necessary to compute the Levi-Civita connection, are computed using the Hodge decomposition of Theorem 4.19.

Definition 5.21. The L^2 metric on $\mathcal{FBA}$ is the one induced by the L^2 metric on the extended fluid-body velocity space $TSE(n) \oplus_M \text{vect}$ via restriction.

The next proposition shows that the fluid-body algebroid splits nicely in terms of the exact, coexact, and harmonic subspaces of the extended velocity space that we identified in Section 4.4.2.

Proposition 5.22. *We have the following L^2-orthogonal splitting of $\mathcal{FBA}$:*

$$\mathcal{FBA}_{(x,\rho)} = \mathcal{EFBA}_{(x,\rho)} \oplus \mathcal{CFBA}_{(x,\rho)} \oplus \mathcal{HFBA}_{x,\rho} , \tag{5.16}$$

Furthermore, the L^2 orthogonal compliment of $\mathcal{FBA}$ in the extended fluid-body velocity space $TSE(n) \oplus_M \text{vect}$ is the subspace

$$\mathcal{FBA}^{\perp}_{(x,\rho)} = \mathcal{HFBA}^{\perp}_{x,\rho} . \tag{5.17}$$

Proof. First we show (5.16) holds. Note that pairs (ξ, u) in each of $\mathcal{EFBA}_{(x,\rho)}$, $\mathcal{CFBA}_{(x,\rho)}$ and $\mathcal{HFBA}_{x,\rho}$ all satisfy the boundary condition $\mathbf{n}\tilde{\xi} = \mathbf{n}u$ required to be in $\mathcal{FBA}_{(x,\rho)}$. Conversely, by the Hodge theorem, any pair in $\mathcal{FBA}$ may be decomposed as

$$(\xi, u) = (0, df) + (0, \delta_\rho w) + (\xi, dh) ,$$

which is an element of $\mathcal{EFBA}_{(x,\rho)} \oplus \mathcal{CFBA}_{(x,\rho)} \oplus \mathcal{HFBA}_{x,\rho}$. Equation (5.17) now follows immediately from Proposition 4.21. □

Remark 5.23. We have the following interpretations of the subspaces comprising $\mathcal{FBA}$. Fluid-body velocity pairs in $\mathcal{EFBA}_{(x,\rho)}$ have zero body velocity and a fluid velocity given by an exact vector field. Such pairs therefore correspond to infinitesimal motions of the system that leave the body fixed and move the fluid density. Pairs in $\mathcal{CFBA}_{(x,\rho)}$ also have zero body velocity, but have a fluid velocity that is ρ-divergence free, and therefore correspond to fluid motions that leave the fluid density fixed. Finally, pairs in $\mathcal{HFBA}_{x,\rho}$ have non-zero body velocity, and a fluid velocity given by a harmonic exact vector field. These pairs encode the influence of the motion of the body on the surrounding fluid.

The following corollary is immediate from Proposition 5.22.

Corollary 5.24. *There exists an L^2-orthogonal projection $\Pi_{(x,\rho)} : T_xSE(n) \oplus \mathrm{vect}F_x \to \mathcal{FBA}_{(x,\rho)}$.*

Proof. The desired projection is projection onto the first three factors of the decomposition of $T_xSE(n) \oplus \mathrm{vect}F_x$ given in Proposition 4.21. □

The projection onto the compressible fluid-body algebroid gives us a tool to guarantee that any fluid-body velocity pairs that we generate in our computations lie in the fluid-body algebroid. We think of Π as "fixing" the fluid and body velocities so that they satisfy the equal normals boundary condition.

Descriptions of the isotropy algebra and its orthogonal compliment are needed to derive a formula for the Levi-Civita connection on $\mathcal{FBA}$. These subspaces again coincide with the Hodge components we identified in Section 4.4.2.

Proposition 5.25. *The isotropy algebra $\ker \#(x,\rho)$ and its L^2-orthogonal compliment $\ker \#^{\perp}_{(x,\rho)}$ are equal to*

$$\ker \#_{(x,\rho)} = \mathcal{CFBA}_{(x,\rho)}\,,$$
$$\ker \#^{\perp}_{(x,\rho)} = \mathcal{EFBA}_{(x,\rho)} \oplus \mathcal{HFBA}_{x,\rho}\,.$$

Proof. Suppose (ξ, u) is in $\ker \#_{(x,\rho)}$. Then $\#(\xi, u) = (\xi, -di_u\rho) = 0$. We therefore have, for any $(\theta, df) \in \mathcal{EFBA}_{(x,\rho)} \oplus \mathcal{HFBA}_{x,\rho}$,

$$\big\langle (\xi, u)\,,\,(\theta, df)\big\rangle_{L^2} = \langle u, df\rangle_{L^2_\rho(F_x)} = \int_{F_x} u \cdot df\,\rho$$
$$= -\int_{F_x} f\,di_u\rho + \int_{\partial F_x} f\,i_u\rho = 0\,.$$

Here we have used the fact that $\mathbf{n}u = 0$ implies $i_u\rho = \tilde{\rho} i_u dq = 0$ on the boundary ∂F_x. This shows, by Proposition 5.22, that $\ker \#_{(x,\rho)}$ is contained in $\mathcal{CFBA}_{(x,\rho)}$.

Next suppose (ξ, u) is in $\mathcal{CFBA}_{(x,\rho)}$. Then $\xi = 0$ and u is ρ-co-exact. In particular, u is co-closed, and by Proposition 4.15, we have $-di_u\rho = 0$. This shows that $\#(\xi, u) = 0$, proving that $\mathcal{CFBA}_{(x,\rho)}$ is contained in $\ker \#_{(x,\rho)}$.

The expression for $\ker \#^{\perp}_{(x,\rho)}$ now follows from Proposition 5.22. □

Remark 5.26. To simplify notation, we will sometimes write $-\Delta_\rho p$ for $-di_\nabla\rho$.

Corollary 5.27. *The lift operator $\#^{-1} : TM \to \ker \#^{\perp}$ is given by*

$$\#^{-1}(\xi, -\Delta_\rho p) = (\xi, \nabla p)\,.$$

Proof. By Propositions 5.25 and 4.11, for every $(\xi, -di_{\nabla p}\rho) \in T_{(x,\rho)}M$, the pair $(\xi, \nabla p)$ is in $\ker \#^{\perp}_{(x,\rho)}$. It is also immediate from the formula for $\#$ that $\#(\xi, \nabla p) = (\xi, -di_{\nabla p}\rho)$. □

We can now compute the Levi-Civita connection on $\mathcal{FBA}$. The formula involves a lot of notation, but we are ultimately only interested in the associated covariant derivative D/dt of curves in $\mathcal{FBA}$ and the torsion T_∇, and these latter objects are much simpler.

Proposition 5.28. *Let ∇^E be the Levi-Civita connection on $TSE(n)$ with respect to the body metric. Let $(\zeta, -\Delta_\rho p)$ be a vector in T_xM, let (Ξ, U) be a section of $\mathcal{FBA}$, and let $(\xi, u) \coloneqq (\Xi, U)(x)$. The Levi-Civita connection on $\mathcal{FBA}$ with respect to the L^2 metric is given by*

$$\nabla_{(\theta,-\Delta_\rho p)}(\Xi, U)(x,\rho) = \Pi\big(\nabla^E_\theta \Xi^\rho(x) + (-\Delta_\rho p)\cdot \Xi^x(\rho)\,,\, (\nabla p\cdot\nabla)u + (\theta, -\Delta_\rho p)\cdot U(x,\rho)\big)\,. \quad (5.18)$$

Proof. We proceed by first giving a formula for the Levi-Civita $\mathcal{A}$-connection on $TSE(n)\oplus_M$ vect, then we apply the projection Π to get the Levi-Civita $\mathcal{A}$-connection on $\mathcal{FBA}$, and finally we use the lift map $\#^{-1}$ above to get the Levi-Civita TM connection we are looking for, according to the prescription 2.31.

We claim that the Levi-Civita $\mathcal{A}$-connection on $TSE(n)\oplus_M$ vect is given by

$$\nabla^{\mathcal{A}}_{(\theta,w)}(\Xi,U)(x,\rho) \coloneqq \big(\nabla^B_{(\theta,w)}\Xi\,,\, \nabla^F_{(\theta,w)}U\big)(x,\rho)\,,$$

where

$$\nabla^B_{(\theta,w)}\Xi(x,\rho) \coloneqq \nabla^E_\theta \Xi^\rho(x) + \#^\rho w\cdot \Xi^x(\rho)$$

and

$$\nabla^F_{(\theta,w)}U(x,\rho) \coloneqq (w\cdot\nabla)u + \#(\theta,w)\cdot U(x,\rho)\,.$$

It is easy to see that $\nabla^{\mathcal{A}}$ is torsion-free. To check metric compatibility, we wish to show

$$\#(\theta,w)\big\langle(\Xi,U)\,,\,(\Omega,V)\big\rangle_{L^2} = \big\langle\nabla^{\mathcal{A}}_{(\theta,w)}(\Xi,U)\,,\,(\Omega,V)\big\rangle_{L^2} + \big\langle(\Xi,U)\,,\,\nabla^{\mathcal{A}}_{(\theta,w)}(\Omega,V)\big\rangle_{L^2}\,.$$

It suffices to show, for a curve (x_t,ρ_t) in M generating $\#(\theta,w)$,

$$\frac{d}{dt}\Big|_{t=0}\big\langle\Xi(x_t,\rho_t)\,,\,\Omega(x_t,\rho_t)\big\rangle_B = \big\langle\nabla^B_{(\theta,w)}\Xi(x,\rho)\,,\,\Omega(x,\rho)\big\rangle_B + \big\langle\Xi(x,\rho)\,,\,\nabla^B_{(\theta,w)}\Omega(x,\rho)\big\rangle_B \quad (5.19)$$

and

$$\frac{d}{dt}\Big|_{t=0}\big\langle U(x_t,\rho_t)\,,\,V(x_t,\rho_t)\big\rangle_F = \big\langle\nabla^F_{(\theta,w)}U(x,\rho)\,,\,V(x,\rho)\big\rangle_F + \big\langle U(x,\rho)\,,\,\nabla^F_{(\theta,w)}V(x,\rho)\big\rangle_F\,. \quad (5.20)$$

To see that (5.19) is true, treat $\Xi(x_t,\rho_t)$ as a time-dependent section Ξ^{ρ_t} evaluated on a curve x_t, and treat $\Omega(x_t,\rho_t)$ the same. Then we have

$$\frac{d}{dt}\Big|_{t=0}\big\langle\Xi(x_t,\rho_t)\,,\,\Omega(x_t,\rho_t)\big\rangle_B = \left\langle\frac{D}{dt}\Big|_{t=0}\Xi^{\rho_t}(x_t)\,,\,\Omega(x,\rho)\right\rangle_B + \left\langle\Xi(x,\rho)\,,\,\frac{D}{dt}\Big|_{t=0}\Omega^{\rho_t}(x_t)\right\rangle_B\,,$$

where $D/dt|_{t=0}$ here is the covariant derivative of curves for the Levi-Civita connection ∇^E on $SE(n)$. Note that

$$\frac{D}{dt}\bigg|_{t=0} \Xi^{\rho_t}(x_t) = \nabla^E_\theta \Xi^\rho(x) + \frac{d}{dt}\bigg|_{t=0} \Xi^x(\rho_t) = \nabla^B_{\theta,w}\Xi(x,\rho)\,,$$

so (5.19) holds.

Next we show equation (5.20). Starting with the definition of the inner product $\langle\cdot,\cdot\rangle_F$,

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} \big\langle U(x_t,\rho_t)\,,\,V(x_t,\rho_t)\big\rangle_F &= \frac{d}{dt}\bigg|_{t=0} \int_{F_{x_t}} U(x_t,\rho_t)\cdot V(x_t,\rho_t)\,\rho_t \\ &= \int_{F_x}\left[\frac{d}{dt}\bigg|_{t=0} U(x_t,\rho_t)\cdot v + u\cdot\frac{d}{dt}\bigg|_{t=0} V(x_t,\rho_t)\right]\rho \qquad (5.21)\\ &\quad + \int_{\partial F_x} u\cdot v\, i_{\tilde\theta}\rho + \int_{F_x} u\cdot v\,\frac{d}{dt}\bigg|_{t=0}\rho_t\,.\end{aligned}$$

By definition, $d/dt|_{t=0}\,U(x_t,\rho_t) = \#(\theta,w)\cdot U(x,\rho)$. We also have $i_{\tilde\theta}\rho = i_w\rho$ on ∂F_x since (θ,w) is in $\mathcal{FBA}$. And since (x_t,ρ_t) generates $\#(\theta,w) = (\theta,-di_w\rho)$, we have $d/dt|_{t=0}\,\rho_t = -di_w\rho$. The last line of (5.21) can therefore be written

$$\begin{aligned}\ldots &= \int_{F_x} d(u\cdot v\, i_w\rho) - u\cdot v\, di_w\rho = \int_{F_x} i_w d(u\cdot v)\,\rho \\ &= \int_{F_x}\big[(w\cdot\nabla)u\big]\cdot v) + u\cdot\big[(w\cdot\nabla)v\big]\,\rho\,.\end{aligned}$$

Thus equation (5.21) becomes

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} \big\langle U(x_t,\rho_t)\,,\,V(x_t,\rho_t)\big\rangle_F &= \int_{F_x}\big[[\#(\theta,w)\cdot U(x,\rho)]\cdot v + u\cdot[\#(\theta,w)\cdot V(x,\rho)]\big]\,\rho \\ &\quad + \int_{F_x}\big[(w\cdot\nabla)u\big]\cdot v) + u\cdot\big[(w\cdot\nabla)v\big]\,\rho\,,\end{aligned}$$

which is the equality (5.20) that we wished to show.

Having identified the Levi-Civita $\mathcal{A}$-connection on $TSE(n)\oplus_M$ vect, the Levi-Civita TM-connection for $\mathcal{FBA}$ is given by

$$\nabla_{(\theta,-\Delta_\rho p)}(\Xi,U)(x,\rho) = \Pi\big[\nabla^{\mathcal{A}}_{\#^{-1}(\theta,-\Delta_\rho p)}(\Xi,U)\big](x,\rho)\,.$$

Using the definitions of $\nabla^{\mathcal{A}}$ and $\#^{-1}$, one recovers the expression (5.18). □

Remark 5.29. The associated covariant derivative of a curve $a_t = (\xi_t,u_t)$ through $\mathcal{FBA}$ has a much simpler formula.

$$\frac{D}{dt}a_t = \left(\frac{D}{dt}\xi_t, \frac{d}{dt}u_t + (\nabla p\cdot\nabla)u\right)$$

The derivative D/dt on the right hand side is the covariant derivative of curves in $TSE(n)$ associated with the Levi-Civita connection ∇^E on $SE(n)$ with respect to the body metric. The derivative $d/dt\, u_t$ is computed pointwise in $q \in F_x$. The function p satisfies $d/dt|_{t=0}\, \pi(\xi_t, u_t) = (\zeta, -di_{\nabla p}\rho)$.

Lemma 5.30. *Let $a = (\xi, u)$ and $b = (\omega, v)$ be two points in $\mathcal{FBA}_{(x,\rho)}$. Let p^a and p^b be functions on F_x such that $-di_u\rho = -di_{\nabla p^a}\rho$ and $-di_v\rho = -di_{\nabla p^b}\rho$. The torsion of the Levi-Civita connection on $\mathcal{FBA}$ is*

$$T_\nabla(a,b) = \Pi\big(0\,,\,(\nabla p^a \cdot \nabla)v - (\nabla p^b \cdot \nabla)u - [u,v]\big)\,.$$

Proof. Let $A = (\Xi, U)$ and $B = (\Omega, V)$ be two sections of $\mathcal{FBA}$ such that $A(x,\rho) = a = (\xi, u)$ and $B(x) = b = (\omega, v)$. A simple but notation-heavy computation shows

$$\begin{aligned} T_\nabla(a,b) &= \nabla_{\#a}B(x) - \nabla_{\#b}A(x) - [A,B](x) \\ &= \nabla_{\#(\xi,u)}(\Omega,V)(x,\rho) - \nabla_{\#(\omega,v)}(\Xi,U)(x,\rho) - [(\Xi,U),(\Omega,V)](x,\rho) \\ &= \Pi\big(0\,,\,(\nabla p^a\cdot\nabla)v - (\nabla p^b\cdot\nabla)u - [u,v]\big)\,, \end{aligned}$$

where we have used the fact that ∇^E is torsion-free. □

5.3.3 Compressible fluid-body equations

The goal of this section is to show that the compressible fluid-body equations (5.12) are the Euler-Lagrange-Arnold equations on $\mathcal{FBA}$ for the *compressible fluid-body Lagrangian*

$$L(a) \coloneqq \frac{1}{2}\langle a,a\rangle_{L^2} - U(\pi(a))\,,$$

where $(U) : M \to \mathbb{R}$ is the potential energy defined in terms of the fluid's internal energy $w : \mathbb{R} \to \mathbb{R}$ as

$$U(x,\rho) \coloneqq \int_{F_x} w(\tilde\rho)\rho\,.$$

We compute each term appearing in the ELA equation, starting with the vertical derivative $d_V L(a)$. Let $b \in \mathcal{FBA}_{(x,\rho)}$ be given and let a_t be a vertical curve in $\mathcal{FBA}$ through a such that $D/dt|_{t=0}\, a_t = b$. Then $U \circ \pi$ is constant on a_t and the vertical differential of L acting on b is computed

$$\langle d_V L(a), b\rangle = \frac{d}{dt}\bigg|_{t=0} L(a_t) = \left\langle a,\, \frac{D}{dt}\bigg|_{t=0} a_t\right\rangle_{L^2} = \langle \mathcal{I}(a), b\rangle\,,$$

so that $d_V L(a) = \mathcal{I}(a)$.

Next we compute $\#^* d_H L(a)$.

Lemma 5.31. *The action of $\#^*$ on the horizontal derivative $d_H L(a) \in T^*_{\pi(a)}M$ is given by*

$$\#^* d_H L(a) = (\beta\,,\,,dP_1) \in \mathcal{FBA}^*_{(x,\rho)}\,, \tag{5.22}$$

where $P_1 : F_x \to \mathbb{R}$ is defined in terms of the density function $\tilde{\rho}$ and the constitutive relation w as $P_1 := \tilde{\rho}^2 w'(\tilde{\rho})$, and where $\beta_1 \in T_x^* SE(n)$ is defined

$$\langle \beta_1, \xi \rangle := -\int_{\partial F_x} P_1 \, i_{\tilde{\xi}} dq \quad \forall_{\_} \xi \in T_x SE(n) \,.$$

If the rotational an translational components of ξ are $\xi = (sO, k)$, then β_1 satisfies

$$\langle \beta_1, \xi \rangle = -\left\langle s \,, \int_{\partial F_x} \frac{P_1}{\tilde{\rho}} (q - q_x) \mathrm{n}^T \, i_{\mathrm{n}} \rho \right\rangle - \left\langle k \,, \int_{\partial F_x} \frac{P_1}{\tilde{\rho}} \mathrm{n}^T \, i_{\mathrm{n}} \rho \right\rangle . \tag{5.23}$$

Proof. Note that the horizontal derivative of the kinetic energy is zero, so $\#^* d_H L(a) = -\#^* U(\pi(a))$. Now let $(\xi, u) \in \mathcal{FBA}_{(x,\rho)}$ be given and let a_t be a horizontal curve through a over the base curve (x_t, ρ_t). Suppose also that a_t is an $\mathcal{A}$-path, so that that $d/dt|_{t=0}(x_t, \rho_t) = \#(\xi, u) = (\xi, -d i_u \rho)$. Then we have

$$\begin{aligned} \langle \#^* d_H U(\pi(a)), (\xi, u) \rangle &= \langle d_H U(\pi(a)), (\xi, -d i_u \rho) = \left.\frac{d}{dt}\right|_{t=0} \int_{F_{x_t}} w(\tilde{\rho}_t) \, \rho_t \rangle \\ &= \int_{\partial F_x} w(\tilde{\rho}) \, i_{\tilde{\xi}} \rho + \int_{F_x} w'(\tilde{\rho}) \left.\frac{d}{dt}\right|_{t=0} \tilde{\rho} \, \rho + \int_{F_x} w(\tilde{\rho}) \left.\frac{d}{dt}\right|_{t=0} \rho_t \end{aligned}$$

Since (ξ, u) is in $\mathcal{FBA}_{(x,\rho)}$, we have $i_{\tilde{\xi}} \rho = i_u \rho$ on the boundary ∂F_x. Also note that, in addition to $d/dt|_{t=0} \, \rho_t = -d i_u \rho$, we have

$$\left.\frac{d}{dt}\right|_{t=0} \tilde{\rho}_t \, \rho = \tilde{\rho} \left.\frac{d}{dt}\right|_{t=0} \tilde{\rho}_t dq = \tilde{\rho} \left.\frac{d}{dt}\right|_{t=0} \rho_t = -\tilde{\rho} \, d i_u \rho \,.$$

We continue the derivation making these substitutions and using Stokes' theorem.

$$\begin{aligned} \ldots &= \int_{F_x} d\big(w(\tilde{\rho}) \, i_u \rho \big) - w'(\tilde{\rho}) \tilde{\rho} \, d i_u \rho - w(\tilde{\rho}) \, d i_u \rho \\ &= \int_{F_x} w'(\tilde{\rho}) d\tilde{\rho} \wedge i_u \rho - d\big(w'(\tilde{\rho}) \tilde{\rho} \, i_u \rho \big) + d\big(w'(\tilde{\rho}) \tilde{\rho} \big) \wedge i_u \rho \\ &= -\int_{\partial F_x} P_1 \, i_{\tilde{\xi}} dq + \int_{F_x} i_u dP_1 \, dq \,, \end{aligned}$$

where the last equality is easy to confirm using the definition of P_1. Thus we have derived the expression (5.22) for $\#^* d_H L(a)$. To get the explicit form of β, note that when restricted to vectors tangent to the boundary, the forms $i_{\tilde{\xi}} \rho$ and $i_{\langle \tilde{\xi}, \mathrm{n}^T \rangle \mathrm{n}} \rho$ are equal. The normal component of the body velocity field $\tilde{\xi}$ is

$$\langle \tilde{\xi}, \mathrm{n}^T \rangle \mathrm{n} = \langle s(q - q_x) + k, \mathrm{n}^T \rangle \mathrm{n} = \langle s, (q - q_x) \mathrm{n}^T \rangle \mathrm{n} + \langle k, \mathrm{n}^T \rangle \mathrm{n} \,.$$

Equation (5.23) now follows from straightforward substitutions. □

It is also necessary to compute $T_\nabla^*(d_V L(a), a)$.

Lemma 5.32. *Consider the cotorsion T^*_∇ associated with the Levi-Civita connection ∇ on $\mathcal{FBA}$. Let L be the compressible fluid-body Lagrangian. Let $a = (\xi, u) \in \mathcal{FBA}_{(x,\rho)}$, and let $(\xi, \nabla p^a)$ be the* $\ker \#^\perp$*-component of a. The following formula holds:*

$$T^*_\nabla(d_V L(a), a) = \mathcal{I} \circ \Pi\Big(0\,, (u\cdot\nabla)u - (\nabla p^a \cdot \nabla)u\Big)\,. \tag{5.24}$$

Proof. Let $b = (\omega, v) \in \mathcal{FBA}_{(x,\rho)}$ be arbitrary. Using the expression for $d_V L(a)$ above and formula for the torsion in Lemma 5.30 , we have

$$\begin{aligned}
\Big\langle T^*_\nabla(d_V L(a), a)\,, b\Big\rangle &= \Big\langle d_V L(a)\ ,, T_\nabla(a,b)\Big\rangle = \Big\langle a\,, T_\nabla(a,b)\Big\rangle_{L^2} \\
&= \Big\langle (\xi,u), \Pi\Big(0\,, (\nabla p^a\cdot\nabla)v - (\nabla p^b\cdot\nabla)u - [u,v]\Big)\Big\rangle_{L^2} \\
&= \underbrace{\int_{F_x} u\cdot\Big((\nabla p^a\cdot\nabla)v\Big)\,\rho}_{1} - \underbrace{\int_{F_x} u\cdot\Big((\nabla p^b a\cdot\nabla)u\Big)\,\rho}_{2} - \underbrace{\int_{F_x} u\cdot[u,v]\,\rho}_{3}\,.
\end{aligned}$$

Each of these three terms may be rewritten. Starting with the first,

$$\mathbf{1} = \int_{F_x} \nabla(u\cdot v)\cdot\nabla p^a - \Big((\nabla p^a\cdot\nabla)u\Big)\cdot v\,\rho\,.$$

The first term on the right hand side can be written entirely in terms of u and v. We have

$$\begin{aligned}
\int_{F_x} \nabla(u\cdot v)\cdot\nabla p^a\,\rho = \int_{F_x} i_{\nabla p^a} d(u\cdot v)\,\rho &= \int_{F_x} d(u\cdot v)\wedge i_{\nabla p^a}\rho \\
&= \int_{F_x} d(u\cdot v\, i_{\nabla p^a}\rho) - u\cdot v d i_{\nabla p^a}\rho \\
&= \int_{\partial F_x} u\cdot v\, i_u\rho - \int_{F_x} u\cdot v\, d i_u\rho
\end{aligned}$$

This last equality follows from the fact that $(\xi, \nabla p^a)$ is the $\ker\#^\perp$-component of (ξ, u), so $i_{\nabla p^a}\rho = i_{\tilde{\xi}}\rho = i_u\rho$ on ∂F_x, and $d i_{\nabla p^a}\rho = d i_u\rho$. We can now reverse the above steps, with u in place of ∇p^a, to conclude

$$\int_{F_x} \nabla(u\cdot v)\cdot\nabla p^a\,\rho = \int_{F_x} \nabla(u\cdot v)\cdot u\,\rho\,.$$

Making this substitution into the expression for **1** and expanding the gradient yields

$$\mathbf{1} = \int_{F_x} \Big((u\cdot\nabla)u\Big)\cdot v + u\cdot\Big((u\cdot\nabla)v\Big) - \Big((\nabla p^a\cdot\nabla)u\Big)\cdot v\,\rho\,.$$

Similar steps may be used to rewrite the second term. Also, the Lie bracket in the third term may be expanded. We have

$$\mathbf{2} = \int_{F_x} u\cdot\Big((v\cdot\nabla)u\Big)\,\rho \qquad \text{and} \qquad \mathbf{3} = \int_{F_x} u\cdot\Big((u\cdot\nabla)v - (v\cdot\nabla)u\Big)\,\rho\,.$$

Combining these expressions results in

$$\Big\langle T^*_\nabla(d_V L(a), a), b\Big\rangle = \int_{F_x} \Big((u\cdot\nabla)u - (\nabla p^a\cdot\nabla)u\Big)\cdot v\,\rho$$
$$= \Big\langle\Big(0\,,\,(u\cdot\nabla)u - (\nabla p^a\cdot\nabla)u\Big)\,,\,(\omega, v)\Big\rangle_{L^2}.$$

Since this is true for all $b = (\omega, v) \in \mathcal{FBA}_{(x,\rho)}$, we have shown that formula (5.24) holds. □

Remark 5.33. The action of the projection operator Π may be written somewhat more explicitly;

$$T^*_\nabla(d_V L(a), a) = \mathcal{I}\big(0\,,\,(u\cdot\nabla)u - (\nabla p^a\cdot\nabla)u\big) + \mathcal{I}(\beta_2, \nabla P_2)\,,$$

where the "correction term" $(\beta_2, \nabla P_2) = -\Pi^\perp(0\,,\,(u\cdot\nabla)u - (\nabla p^a\cdot\nabla)u)$ is the unique element of $\mathcal{FBA}^\perp_{(x,\rho)}$ which ensures that the right-hand side of the above equation lies in $\mathcal{FBA}_{(x,\rho)}$. In other words, the correction term guarantees that the fluid velocity has compatible boundary conditions along the body.

Lemma 5.34. *The correction term $(\beta_2, \nabla P_2)$ satisfies*

$$\langle\beta_2, \theta\rangle_B = -\Big\langle s\,,\,\int_{\partial F_x} P_2(q - q_x)\mathrm{n}^T\, i_{\mathrm{n}}\rho\Big\rangle - \Big\langle k\,,\,\int_{\partial F_x} P_2\mathrm{n}^T\, i_{\mathrm{n}}\rho\Big\rangle \tag{5.25}$$

for all $\theta = (sO, k) \in T_x SE(n)$.

Proof. Note that $(\theta, \nabla h^\theta)$ is in $\mathcal{FBA}_{(x,\rho)}$, and is therefore orthogonal to $(\beta_2, \nabla P_2)$:

$$0 = \Big\langle(\beta, \nabla P_2)\,,\,(\theta, \nabla h^\theta)\Big\rangle_{L^2} = \langle\beta_2, \theta\rangle_B + \int_{F_x} \nabla P_2\cdot\nabla h^\theta\,\rho\,.$$

Since ∇h^θ is ρ-divergence-free, we can use Stokes' theorem to write

$$\langle\beta_2, \theta\rangle_B = -\int_{\partial F_x} P_2\, i_{\nabla h^\theta}\rho = -\int_{\partial F_x} P_2\, i_{\tilde\theta}\rho\,,$$

where the last equality holds since ∇h^θ and $\tilde\theta$ have equal normal components on the boundary. Formula (5.25) now follows along similar lines as the proof of Lemma 5.31. □

We are now ready to state the main theorem of this section.

Theorem 5.35. *The incompressible fluid-body equations* (5.12) *are the Euler-Lagrange-Arnold equations on the compressible fluid-body algebroid $\mathcal{FBA}$ with respect to the Lagrangian*

$$L(a) \coloneqq \frac{1}{2}\langle a, a\rangle_{L^2} - U(\pi(a))\,,$$

where $(U): M \to \mathbb{R}$ is the potential energy defined in terms of the fluid's constitutive relation $w: \mathbb{R}\to\mathbb{R}$ as

$$U(x, \rho) \coloneqq \int_{F_x} w(\tilde\rho)\,\rho\,.$$

Proof. Recall that an $\mathcal{A}$-path a satisfies the Euler-Lagrange-Arnold equations if

$$\frac{D}{dt} d_V L(a) = -T^*_\nabla(d_V L(a), a) + \#^* d_H L(a)\,.$$

First note that the equation $d/dt\, x = \xi$ and the conservation equation $d/dt\, \rho = -\nabla \cdot (\rho u)$ together are equivalent to the condition that a is an $\mathcal{A}$-path. The other equations of (5.12) are derived from the ELA equation using the above-computed expressions for $d_V L(a)$, $d_H L(a)$ and $T^*_\nabla(d_V L(a), a)$. On the left-hand side we have

$$\frac{D}{dt} d_V L(a) = \frac{D}{dt}\mathcal{I}(a) = \mathcal{I}\left(\frac{D}{dt}a\right) = \left(\mathcal{I}_B\left[\frac{D}{dt}\xi\right],\, \tilde{\rho}\left[\frac{d}{dt}u + (\nabla p^a \cdot \nabla)u\right]\right),$$

and on the right-hand side,

$$-T^*_\nabla(d_V L(a), a) + \#^* d_H L(a) = \Big(-\mathcal{I}_B(\beta_2) + \beta_1\,,\, -\tilde{\rho}\big[(u\cdot\nabla)u - (\nabla p^a \cdot \nabla)u + dP_2\big] + dP_1\Big)\,.$$

Writing out the ELA equation component-wise, we have

$$\mathcal{I}_B\left[\frac{D}{dt}\xi\right] = \beta_1 - \mathcal{I}_B(\beta_2) \tag{5.26}$$

$$\tilde{\rho}\left[\frac{d}{dt}u + (u\cdot\nabla)u\right] = dP_1 - \tilde{\rho}\, dP_2\,. \tag{5.27}$$

Equation (5.27) is the usual incompressible Euler equation which governs the fluid.

It remains to show that (5.26) is equivalent to the equations that govern the motion of the body. Let $\theta = (sO, k) \in T_x SE(n)$ be arbitrary, and take the dual pairing of it with equation (5.26). On the right hand side we have, by Lemmas 5.31 and 5.14,

$$\langle \beta_1 - \mathcal{I}_B(\beta_2), \theta\rangle = \left\langle s\,,\, \int_{\partial F_x}(P_2 - \tilde{\rho}^{-1}P_1)(q - q_x)\mathrm{n}^T\, i_{\mathrm{n}}\rho\right\rangle + \left\langle k\,,\, \int_{\partial F_x}(P_2 - \tilde{\rho}^{-1}P_1)\mathrm{n}^T\, i_{\mathrm{n}}\rho\right\rangle\,.$$

On the left, by Lemma 4.6, we have

$$\left\langle \mathcal{I}_B\left[\frac{D}{dt}\xi\right], \theta\right\rangle = \left\langle \mathbb{I}\frac{d}{dt}(rO)^T, sO\right\rangle + \left\langle m\frac{d}{dt}l^T, k\right\rangle\,.$$

For the rotational term, we write

$$\left\langle \mathbb{I}\frac{d}{dt}(rO)^T, sO\right\rangle = \left\langle O\mathbb{I}\frac{d}{dt}(O^T r^T), s\right\rangle = \left\langle \frac{d}{dt}(O\mathbb{I}O^T r^T), s\right\rangle - \left\langle \frac{d}{dt}(O)\mathbb{I}O^T r^T, s\right\rangle\,.$$

Since $d/dt\, x = \zeta$, we have in particular $d/dt\, O = rO$. It follows that $\langle d/dt(O)\mathbb{I}O^T r^T, s\rangle = \langle rO\mathbb{I}O^T r^T, s\rangle = 0$, since it is the trace of the product of a symmetric matrix $rO\mathbb{I}O^T r^T$ and an antisymmetric matrix s. We have now shown

$$\left\langle \frac{d}{dt}(\mathbb{I}_x r^T), s\right\rangle + \left\langle m\frac{d}{dt}l^T, k\right\rangle = \left\langle s\,,\, \int_{\partial F_x}(P_2 - \tilde{\rho}^{-1}P_1)(q - q_x)\mathrm{n}^T\, i_{\mathrm{n}}\rho\right\rangle + \left\langle k\,,\, \int_{\partial F_x}(P_2 - \tilde{\rho}^{-1}P_1)\mathrm{n}^T\, i_{\mathrm{n}}\rho\right\rangle$$

for all s and k. The equations of motion for the body follow.

□

5.4 Kirchhoff dynamics

The *Kirchhoff system* is the special case of fluid-body motion in $\mathbb{R}^n$ where the fluid is incompressible, irrotational, and without circulation around the body. In this section we define the Kirchhoff equations and show that they are Euler-Lagrange-Poincaré equations. In Section 6.3 we justify their validity by showing that the solutions of these equations can be identified with incompressible, irrotational and circulation-free solutions of the full fluid-body system.

The assumptions on the fluid imply that the only non-zero Hodge component of the fluid velocity is the exact harmonic part. This part is completely determined by Neumann boundary conditions prescribed by the body's velocity. We will see that the motion of the body is described with no explicit reference to the fluid; the only effect of the fluid is to modify the body's inertia.

Define the *added mass* $\mathcal{I}_{AM} : \mathfrak{se}(n) \to \mathfrak{se}^*(n)$ by

$$\langle (r_1, l_1), \mathcal{I}_{AM}(r_2, l_2) \rangle := \int_{F_0} \nabla h^{\xi_1} \cdot \nabla h^{\xi_2} \, dQ \,,$$

where $\xi_i = (r_i, l_i) \in \mathfrak{se}(n)$, and h^{ξ_i} is the harmonic function on F_0 satisfying the Neumann boundary conditions $\mathbf{n}\nabla h^{\xi_i}(Q) = \mathbf{n}(l_i + r_i Q) = \tilde{\xi}(Q)$. Note that it depends on the shape of the body and the body's velocity, but not on the fluid's motion. The *total mass* is then defined to be the sum of the body inertia and the added mass, $\mathcal{I}_T := \mathcal{I}_B + \mathcal{I}_{AM}$.[6]

Remark 5.36. Recall that the harmonic functions h^ξ generate the minimal fluid motion required to accomodate the motion of the body. The added mass is therefore interpreted as the kinetic energy of the fluid as it is displaced by the body.

Let $(\rho, \lambda) := \mathcal{I}_T(r, l)$ denote the total momentum of the body in the fluid. The motion of the body is governed by *Kirchhoff's equations*,

$$\begin{cases} \dfrac{d}{dt}\rho = [\rho, r] + \lambda \diamond l \\ \dfrac{d}{dt}\lambda = -r\lambda \,. \end{cases} \tag{5.28}$$

Here the diamond product $\diamond : \mathbb{R}^n \times \mathbb{R}^n \to \mathfrak{so}^*(n)$ is defined by

$$\langle \lambda \diamond l, r \rangle := -\langle \lambda, rl \rangle \,.$$

We now derive the Kirchhoff equations as the Euler-Lagrange-Poincaré equations for a particular Lagrangian system on $\mathfrak{se}(n)$.

[6]Here we abuse notation slightly and consider $\mathcal{I}_B$ not to be the map $TSE(n) \to T^*SE(n)$ defined previously, but its restriction to a map $\mathfrak{se}(n) \to \mathfrak{se}^*(n)$.

Theorem 5.37. *The Kirchhoff equations are the Euler-Lagrange-Poincaré equations on the algberoid* $\mathfrak{se}(n)$ *with Lagrangian*

$$\ell(r,l) := \frac{1}{2}\langle (r,l), \mathcal{I}_T(r,l)\rangle \,. \tag{5.29}$$

Proof of Proposition 5.37. By Example [reference the example], the Euler-Lagrange-Poincaré equation on $\mathfrak{se}(n)$ coincides with the Euler-Poincaré equation,

$$\frac{d}{dt} d\ell(r,l) = \mathrm{ad}^*_{(r,l)} d\ell(r,l) \,. \tag{5.30}$$

The differential of L evaluated at (r,l) is easily found to be

$$d\ell(r,l) = \mathcal{I}_T(r,l) =: (\rho,\lambda) \,.$$

We recall that the coajoint action on $\mathfrak{se}^*(n)$ is given by the formula

$$\mathrm{ad}^*_{(r,l)}(\rho,\lambda) = ([\rho,r] + \lambda \diamond l, -r\lambda) \,.$$

The Kirchhoff equations (5.28) follow immediately. □

Remark 5.38. In $\mathbb{R}^3$, a skew-symmetric matrix r may be identified with a vector $\hat{r}$ by the relation

$$\hat{r}_i := \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk} r_{jk} \,,$$

where ϵ_{ijk} is the alternating symbol. With this identification, we find

$$r\lambda = \hat{r}\times\lambda \,, \quad \widehat{[\rho,r]} = \hat{\rho}\times\hat{r} \quad \text{and} \quad \widehat{\lambda\diamond l} = \lambda\times l \,.$$

The Kirchhoff equations in $\mathbb{R}^3$ may therefore be written

$$\frac{d}{dt}\hat{\rho} = \hat{\rho}\times\hat{r} + \lambda\times l$$
$$\frac{d}{dt}\lambda = \lambda\times\hat{r} \,.$$

6 Reduction of fluid-body systems

In this section we apply the Lagrangian reduction theorem to relate the dynamics of the fluid-body systems we have discussed. We show that solutions, possibly subject to constraints, of the ELA equation for one fluid-body system are equivalent to solutions of the ELA equation for another. In order to apply Lagrangian reduction, we construct a morphism of the underlying groupoids, then show that the induced algebroid morphism relates the Lagrangians of the systems.

Remark 6.1. In Section 5.1, we saw that the motion of the incompressible (resp. compressible) fluid-body system is governed by the geodesic equation (resp. Newton's equation). In this formulation of the problem, the dynamical quantity describing the fluid is the family of diffeomorphisms g that give the *position* of the fluid particles. Compare this to the incompressible and compressible fluid-body equations (5.5) and (5.12), where the quantity being solved for is the fluid *velocity*. The former description is called the "Lagrangian" description of the fluid, and the latter is called the "Eulerian" description.

The reduction theorems 6.2 and 6.3 can be interpretted as establishing the equivalence of the Lagrangian and Eulerian descriptions of the fluid-body system.

6.1 Incompressible fluid-body dynamics as geodesic flow

We start with the incompressible fluid-body system.

Theorem 6.2 (cf. [8])**.** *The incompressible fluid-body equations* (5.5) *are equivalent to geodesic equations* (5.2) *on the incompressible fluid-body configuration space.*

Proof. The map $\Phi : SQ \times SQ \to S\mathcal{FBG}$ defined by $\Phi((y,h)\,,(x,g)) := (y,x,h\circ g^{-1})$ is easily seen to be a groupoid morphism over the base map $\underline{\Phi} : SQ \to SE(n)$ defined by $\underline{\Phi}(x,g) := x$. The induced algebroid morphism $\phi : TSQ \to S\mathcal{FBA}$ is then, for each (ξ, u_g) in the fibre $T_{(x,g)}SQ$,

$$\phi(\xi,u_g) = (\xi,u)\,,$$

where u is a vector field on F_x and $u_g := u\circ g$. We show that ϕ relates the Lagrangian $L' : TSQ \to \mathbb{R}$ of the unreduced system to the Lagrangian $L : S\mathcal{FBA} \to \mathbb{R}$ of the reduced system. Recall that L' is defined

$$L'(\xi,u_g) := \frac{1}{2}\langle(\xi,u_g),(\xi,u_g)\rangle_{L^2} = \frac{1}{2}\langle\xi,\xi\rangle_B + \frac{1}{2}\int_{F_0} u_g\cdot u_g\,dQ\,.$$

Applying the change of variables $q = g(Q)$, the Lagrangian is written

$$L'(\xi,u_g) = \frac{1}{2}\langle\xi,\xi\rangle_B + \frac{1}{2}\int_{F_x} u\cdot u\,dq = \frac{1}{2}\big\langle\phi(\xi,u_g)\,,\,\phi(\xi,u_g)\big\rangle = L\circ\phi(\xi,u_g)\,,$$

showing that L' and L are related by the morphism ϕ. The Reduction Theorem 3.24 then implies that the geodesic equations on the incompressible fluid-body configuration space SQ, which are the ELA equations for L' on TSQ, are equivalent to the incompressible fluid-body equations, which, by Theorem 5.15, are the ELA equations for L on $S\mathcal{FBA}$. □

6.2 Compressible fluid-body dynamics as Newton's equation

Next we turn to the compressible fluid-body system. The technique we use to prove the next theorem should be compared to the proof of the semidirect product reduction theorem of Section 3.5.2.

Theorem 6.3. *The compressible fluid-body equations* (5.12) *are equivalent to Newton's equations* (5.4) *on the compressible fluid-body configuration space.*

Proof. Recall that ρ_0 denotes the reference density on the reference fluid domain F_0. Define the map $\Phi_{\rho_0} : Q \times Q \to \mathcal{FBG}$ by

$$\Phi_{\rho_0}((y,h)\,,(x,g)) := (y,x,g_*\rho_0,h\circ g^{-1})\,,$$

It is easily seen to be a groupoid morphism over the base map $\underline{\Phi}_{\rho_0} : Q \to M$ defined by $\underline{\Phi}_{\rho_0}(x,g) := (x,g_*\rho)$. The induced algebroid morphism $\phi : TQ \to \mathcal{FBA}$ is then, for each (ξ,u_g) in the fibre $T_{(x,g)}Q$,

$$\phi(\xi,u_g) = (\xi,,u) \in \mathcal{FBA}_{(x,g_*\rho_0)}\,,$$

where u is a vector field on F_x and $u_g := u\circ g$. We show that ϕ relates the Lagrangian $L' : TQ \to \mathbb{R}$ of the unreduced system to the Lagrangian $L : \mathcal{FBA} \to \mathbb{R}$ of the reduced system. Recall that L' is defined

$$\begin{aligned} L'(\xi,u_g) &:= \frac{1}{2}\langle(\xi,u_g),(\xi,u_g)\rangle_{L^2} - U'(\xi,u_g) \\ &= \frac{1}{2}\langle\xi,\xi\rangle_B + \frac{1}{2}\int_{F_0} u_g\cdot u_g\,\rho_0 - \int_{F_x} w\big(\mathrm{Det}(Dg^{-1})\tilde{\rho}_0\circ g^{-1}\big)\,g_*\rho_0\,. \end{aligned}$$

Applying the change of variables $q = g(Q)$, the Lagrangian is written

$$L'(\xi,u_g) = \frac{1}{2}\langle\xi,\xi\rangle_B + \frac{1}{2}\int_{F_x} u\cdot u\,g_*\rho_0 - \int_{F_x} w\big(\mathrm{Det}(Dg^{-1})\tilde{\rho}_0\circ g^{-1}\big)\,g_*\rho_0 = L\circ\phi(\xi,u_g)\,,$$

showing that L' and L are related by the morphism ϕ. The Reduction Theorem 3.24 then implies that Newton's equation on the compressible fluid-body configuration space Q, which are the ELA equations for L' on TQ, are equivalent to the compressible fluid-body equations, which, by Theorem 5.35, are the ELA equations for L on $\mathcal{FBA}$. □

6.3 Projection of potential solutions to the Kirchhoff system

For this section we will emphasize the dependence on x of vectors ξ in $T_xSE(n)$ by writing ξ_x.

We now establish the relation between potential solutions of the incompressible fluid-body system and the Kirchhoff system.

Theorem 6.4. *If $a : I \to S\mathcal{FBA}$ is solution of the incompressible fluid-body equations that is initially irrotational, then it remains irrotational. Moreover, if $\phi : S\mathcal{FBA} \to \mathfrak{se}(n)$ is defined by $\phi(\xi_x,u) = -x^{-1}\xi_x$, then a is an irrotational solution of the incompressible fluid-body equations if and only if $\phi(a)$ is a solution of the Kirchhoff equations.*

The proof is an application of the theorem (3.34) on the projection of horizontal geodesics by Riemannian submersions. In order to apply the theorem, we need to interpret solutions of the incompressible fluid-body equations and solutions of the Kirchhoff equations as geodesics. We then need to show that the map ϕ is a Riemannian submersion, and interpret the irrotational condition on the fluid as a horizontality condition.

Theorem 5.15 says that solutions of the incompressible fluid-body equations are geodesics in $S\mathcal{FBA}$ in the sense of Section 3.6. We would like to interpret solutions of the Kirchhoff equations as geodesics as well, and in order to do this, we need to specify the appropriate metric on $\mathfrak{se}(n)$. Define the *Kirchhoff metric* on $\mathfrak{se}(n)$ by

$$\langle \xi, \omega \rangle_{Kirch} := \langle \xi, \mathcal{I}_T \omega \rangle = \left\langle \xi, \omega \right\rangle_B + \int_{F_0} \nabla h^{\xi} \cdot \nabla h^{\omega} \, dQ \,.$$

Remark 6.5. The Kirchhoff metric is the restriction to $\mathfrak{se}(\mathfrak{n})$ of the following metric on $SE(n)$:

$$\left\langle \xi_x, \omega_x \right\rangle_B + \left\langle \xi_x, \omega_x \right\rangle_{AM} , \tag{6.1}$$

where the *added mass* is defined

$$\left\langle \xi_x, \omega_x \right\rangle_{AM} := \int_{F_x} \nabla h^{\xi_x} \cdot \nabla h^{\omega_x} \, dq \,.$$

We show that the added mass is invariant under the left action of $SE(n)$. To do this we require a lemma about the harmonic functions that generate the fluid flow caused by the moving body.

Lemma 6.6. *The harmonic functions h^{ξ_y} and $h^{x\xi_y}$ are related by $h^{x\xi_y}(q) = h^{\xi_y}(x^{-1} \cdot q)$.*

Proof. Define the function h on F_{xy} by $h(q) := h^{\xi_y}(x^{-1} \cdot q)$. It is easily checked that h is harmonic, so to show that $h = h^{x\xi_y}$, it suffices to show that these two functions satisfy the same Neumann boundary conditions. Let $x = (O, L) \in SE(n)$, so that O is an orthogonal matrix. It is not hard to show that $\nabla h(q) = O \nabla h^{\xi_y}(x^{-1} \cdot q)$. Also, if n^y and n^{xy} are the normal vector fields on ∂F_y and $\partial F_x y$ respectively, then $n^{xy}(q) = O n^y (x^{-1} \cdot q)$. The body velocity vector fields are related similalry, as $\widetilde{x\xi_y}(q) = O\tilde{\xi}_y(x^{-1} \cdot q)$. To see this last identity, note that by definition,

$$\tilde{\xi}_y(x^{-1} \cdot q) = \left.\frac{d}{dt}\right|_{t=0} y_t y^{-1} x^{-1} \cdot q \,,$$

so that

$$\widetilde{x\xi_y}(q) = \left.\frac{d}{dt}\right|_{t=0} x y_t (xy)^{-1} \cdot q = O \left[\left.\frac{d}{dt}\right|_{t=0} y_t y^{-1} x^{-1} \cdot q \right] = O \tilde{\xi}_y (x^{-1} q) \,.$$

The Neumann boundary conditions for h therefore read, for all $q \in \partial F_{xy}$,

$$\langle n^{xy}(q), \nabla h(q) \rangle = \langle n^y(x^{-1} \cdot q), \nabla h^{\xi_y}(x^{-1} \cdot q) \rangle = \langle n^y (x^{-1} \cdot q), \tilde{\xi}_y (x^{-1} \cdot q) \rangle = \langle n^{xy}(q), \widetilde{x\xi_y}(q) \rangle \,,$$

which are the same boundary conditions for $h^{x\xi_y}$. □

Proposition 6.7. *The added mass term is invariant under the left action of $SE(n)$.*

Proof. From Lemma 6.6 it follows that $\nabla h^{x\xi_y}(q) = O\nabla h^{\xi_y}(x^{-1}\cdot q)$. So, starting with the definition of the added mass term and applying the change of variables $q = x\cdot Q$,

$$\begin{aligned}\langle \xi_y, \eta_y\rangle_{AM}(y) &= \int_{F_y} \nabla h^{\xi_y}(Q)\cdot \nabla h^{\eta_y}(Q)\, dQ = \int_{F_{xy}} \nabla h^{\xi_y}(x^{-1}\cdot q)\cdot \nabla h^{\eta_y}(x^{-1}\cdot q)\, dq \\ &= \int_{F_{xy}} \nabla h^{x\xi_y}(q)\cdot \nabla h^{x\eta_y}(q)\, dq = \langle x\xi_y, x\eta_y\rangle_{AM}(xy)\,.\end{aligned}$$

□

Having established the necessary lemmas, we now prove the main result.

Proof of Theorem 6.4. Observe that solutions of the incompressible fluid-body equations are, in the sense of Section 3.6, geodesics for the L^2 metric in $S\mathcal{FBA}$, while solutions of the Kirchhoff equations are geodesics for the Kirchhoff metric on $\mathfrak{se}(n)$. We will show that the irrotational solutions of the fluid-body equations are exactly the horizontal geodesics with respect to a Riemannian submersion $\phi : S\mathcal{FBA} \to \mathfrak{se}(n)$. The result then follows from Theorem 3.34.

Consider the groupoid morphism $\Phi : S\mathcal{FBG} \to SE(n)$ defined $\Phi(y,x,g) := y^{-1}x$ (Note that the base of the groupoid $SE(n)$ is a single point, so the base map of Φ is trivial.) The induced algebroid morphism $\phi : S\mathcal{FBA} \to \mathfrak{se}(n)$ is then

$$\phi(\xi_x, u) = -x^{-1}\xi_x\,.$$

We show that ϕ is a Riemannian submersion in the sense of Section 3.6. Recall the L^2-orthogonal splitting $S\mathcal{FBA} = \mathcal{CFBA} \oplus \mathcal{HFBA}$. It is easy to see that $\ker\phi = \mathcal{CFBA}$, so that the horizontal subbundle is given by $\ker\phi^{-1} = \mathcal{HFBA}$. Thus a curve in the incompressible fluid-body algebroid is horizontal if and only if its coexact component is zero, ie. its fluid velocity field is irrotational.

To show that ϕ is a Riemannian submersion, we need to check

$$\Big\langle (\xi_x, u)\,,\,(\omega_x, v)\Big\rangle_{L^2} = \Big\langle \phi(\xi_x, u)\,,\,\phi(\omega_x, v)\Big\rangle_{Kirch}$$

for all (ξ_x, u), $(\omega_x, v) \in \mathcal{HFBA}$. Indeed, when restricted to horizontal vectors, the L^2 metric on $S\mathcal{FBA}$ reads

$$\begin{aligned}\Big\langle (\xi_x, u)\,,\,(\omega_x, v)\Big\rangle_{L^2} &= \Big\langle \xi_x, \omega_x\Big\rangle_B + \Big\langle \xi_x, \omega_x\Big\rangle_{AM} = \Big\langle -x^{-1}\xi_x, -x^{-1}\omega_x\Big\rangle_B + \Big\langle -x^{-1}\xi_x, -x^{-1}\omega_x\Big\rangle_{AM} \\ &= \Big\langle \phi(\xi_x, u)\,,\,\phi(\omega_x, v)\Big\rangle_{Kirch}\,,\end{aligned}$$

where we have used the left invariance of the body metric and the added mass, as well as Lemma 6.5. Thus ϕ is an isometry from the horizontal bundle $\mathcal{HFBA}$ to $\mathfrak{se}(n)$. □

Remark 6.8. The morphism ϕ can be thought of as encoding a two-step process. The map $\phi : \mathcal{SFBA} \to \mathfrak{se}(n)$ is the composition $\psi \circ \#$, where $\# : \mathcal{SFBA} \to TSE(n)$ is the anchor map and $\psi : TSE(n) \to \mathfrak{se}(n)$ is the map $\psi(\xi_x) \coloneqq -x^{-1}\xi_x$. The anchor map $\#$ is a Riemannian submersion with respect to the L^2 metric on $\mathcal{SFBA}$ and the Riemannian metric on $SE(n)$ defined by (6.1). This latter metric is left invariant on $SE(n)$, and the corresponding Euler-Poincaré reduction is encoded by the map ψ.

7 The Madelung transform

The Madelung transform was introduced in 1927, soon after the birth of quantum mechanics, as a way to relate Schrödinger-type equations to hydrodynamical equations [15]. It turns out that the Madelung transform not only maps one equation to the other, but it also preserves the Hamiltonian properties of both equations. Namely, the non-linear Schrödinger equation is Hamiltonian with respect to the constant Poisson structure on the space of wave functions, which are complex valued fast decaying smooth functions on $\mathbb{R}^n$. On the other hand, the compressible Euler equation is Hamiltonian with respect to the natural Lie-Poisson structure on the space of pairs consisting of fluid momenta μ and fluid densities ρ. This space is the dual of the Lie algebra of the semidirect product group of the group of diffeomorphisms of $\mathbb{R}^n$ times the space of real-valued fast decaying functions, which is the configuration space of a compressible fluid. In this section we show that the Madelung transform sends one Poisson structure to the other. Moreover, the transform is a momentum map associated with a natural action of this semidirect product group on the space of wave functions.

Let complex valued functions $\psi \in \Psi = H^\infty(\mathbb{R}^n;\mathbb{C}) \coloneqq \cap_{k\geq 0} H^k(\mathbb{R}^n;\mathbb{C})$ evolve according to the non-linear Schrödinger equation

$$\partial_t \psi = \frac{i}{2}\left(\Delta\psi - 2f\left(|\psi|^2\right)\psi\right). \tag{NLS}$$

Here $f : \mathbb{R} \to \mathbb{R}$ is a smooth function which characterizes the type of non-linear Schrödinger equation being considered. For example, the Gross-Pitaevskii equation corresponds to $f(r) = r-1$. Depending on the type of non-linearity, one may consider ψ belonging to an appropriate function space other than $H^\infty(\mathbb{R}^n;\mathbb{C})$, but for simplicity this will not be discussed in the present section, see [3] for details.

The Madelung transform is a map between the NLS-type and hydrodynamical equations. It takes a nonvanishing complex valued function ψ to the pair of real valued functions (τ,ρ) defined by $\psi = \sqrt{\rho}e^{i\tau}$. Such a substitution for ψ sends NLS into a system of equations for the functions ρ and $v \coloneqq \nabla\tau$ known as the "quantum hydrodynamical system"

$$\begin{cases} \partial_t \rho = -\nabla\cdot(\rho v) \\ \partial_t v = -v\cdot\nabla v - \nabla\left(f(\rho) - \dfrac{\Delta\left(\sqrt{\rho}\right)}{2\sqrt{\rho}}\right) \end{cases}. \tag{QHDa}$$

The first equation of this system is the continuity equation for a density ρ moved by a flow with velocity v. The second equation would be the classical Euler equation of a barotropic fluid except for the fact that the "quantum pressure" $\mathcal{P}(\rho) := \frac{\Delta\sqrt{\rho}}{2\sqrt{\rho}}$ depends on both ρ and its derivatives rather than just on ρ itself.

This system can be written in terms of the momentum, which is the 1-form $\mu = \rho v^\flat$ defined with respect to the Euclidean metric on $\mathbb{R}^n$. (Throughout this section $v^\flat$ is identified with v, so we write $\mu = \rho v$ as well.) Assuming ρ is always positive, the system QHDa is equivalent to the following:

$$\begin{cases} \partial_t \rho = -\nabla \cdot \mu \\ \partial_t \mu = -\nabla \cdot \left(\dfrac{1}{\rho} \mu \otimes \mu \right) - \rho \nabla \left(f(\rho) - \dfrac{\Delta\left(\sqrt{\rho}\right)}{2\sqrt{\rho}} \right) \end{cases} . \tag{QHD}$$

The term $\nabla \cdot \left(\frac{1}{\rho}\mu \otimes \mu\right)$, in components, is given by

$$\left(\nabla \cdot \left(\frac{1}{\rho} \mu \otimes \mu \right) \right)_j = \sum_i \partial_i \left(\frac{1}{\rho} \mu_i \mu_j \right).$$

Note that ρ and μ are natural coordinates on the dual $\mathfrak{s}^*$ of the Lie algebra of the semidirect Lie group $S = \mathrm{Diff}(\mathbb{R}^n) \ltimes H^\infty(\mathbb{R}^n)$ which is the configuration space of the compressible fluid. Here $\mathrm{Diff}(\mathbb{R}^n)$ stands for the group of diffeomorphisms that asymptotically approach the identity map at infinity.

Remark 7.1. While we formulate and prove the main result in the setting of $\mathbb{R}^n$, the definitions and all proofs can be extended to an arbitrary manifold with volume form $d\mathrm{Vol}$ by replacing all gradients with exterior differentiation and by defining the divergence of a vector field in terms of $d\mathrm{Vol}$. Note however that one needs a metric structure to define the equations NLS and QHD on manifolds, where $v \cdot \nabla v$ stands for the covariant derivative $\nabla_v v$. A metric is also required to define the kinetic energy term in the Hamiltonians for these equations.

We start the section with a review of the relevant geometric structures associated with NLS and QHD to establish the context for the Madelung transform. The Hamiltonian structures of these systems are reviewed in Section 7.1.1, while Section 7.1.2 defines the action of the semidirect product group S on the space of wave functions Ψ. The proof of the main result, that the Madelung transform is a momentum map for this action, is given in Section 7.2.2, Theorem 7.10. For more details on applications of the Madelung transform we refer to [3].

7.1 Geometric preliminaries

7.1.1 Hamiltonian structures of non-linear Schrödinger and the quantum hydrodynamical system

Both NLS and QHD are Hamiltonian systems with respect to the following Hamiltonians and Poisson structures. Note that these Poisson structures are only defined a subclasses $\mathcal{A}_\Psi \subset C^\infty(\Psi)$ and $\mathcal{A}_{\mathfrak{s}^*} \subset C^\infty(\mathfrak{s}^*)$ of smooth functionals on Ψ and $\mathfrak{s}^*$. As always with Poisson brackets on infinite-dimensional spaces, the definition of the Poisson algebra of functionals is a subtle question. For instance, functionals on a space of functions u defined as integrals of polynomials (in u and finitely many derivatives of u) are closed under the Poisson bracket but not under multiplication. The same is true of functionals having smooth L^2 gradients: they are closed under the Poisson bracket but not under multiplication, cf. [13].

We start with NLS and consider the space $\Psi = H^\infty(\mathbb{R}^n;\mathbb{C})$ of complex valued functions ψ. The real Hermitian inner product on Ψ is defined by $\langle f, g\rangle := \operatorname{Re}\int \bar{f} g\, dx$, and the gradient ∇ is defined with respect to this inner product. The Poisson bracket on Ψ is given by

$$\{F, G\}_{\text{NLS}}(\psi) = \left\langle \nabla F, -\tfrac{i}{2}\nabla G\right\rangle.$$

The Hamiltonian associated with NLS is

$$H_{\text{NLS}} = \int_{\mathbb{R}^n} \frac{1}{2}|\nabla\psi|^2 + U(|\psi|^2)dx,$$

where $U : \mathbb{R} \to \mathbb{R}$ is a function satisfying $U' = f$. One finds that the Hamiltonian vector field $X_{H_{\text{NLS}}}$ associated with this Hamiltonian functional and Poisson bracket $\{\cdot,\cdot\}_{\text{NLS}}$ is given by

$$X_{H_{\text{NLS}}} = -\frac{i}{2}\nabla H = \frac{i}{2}(\Delta\psi - 2f(|\psi|^2)\psi),$$

which is the right-hand side of NLS.

Now consider the equation QHD, which describes the motion of a compressible isentropic-type fluid. The Poisson geometry of such fluids was studied in [18], and we outline the results below.

Consider the semidirect product group $S = \operatorname{Diff}(\mathbb{R}^n) \ltimes H^\infty(\mathbb{R}^n;\mathbb{R})$. Here and below we assume that the elements "decay sufficiently fast at infinity." The space $H^\infty(\mathbb{R}^n;\mathbb{R})$ is defined as $H^\infty(\mathbb{R}^n;\mathbb{R}) := \cap_{k\geq 0} H^k(\mathbb{R}^n;\mathbb{R})$, while $\operatorname{Diff}(\mathbb{R}^n)$ is the set of diffeomorphisms on $\mathbb{R}^n$ of the form $g = \mathrm{id} + f$ with $f \in H^\infty(\mathbb{R}^n;\mathbb{R}^n)$. The Lie algebra of the group S is $\mathfrak{s} = \operatorname{vect}(\mathbb{R}^n) \ltimes H^\infty(\mathbb{R}^n;\mathbb{R})$, and the (regular) dual of this Lie algebra is $\mathfrak{s}^* = \operatorname{vect}^*(\mathbb{R}^n) \oplus H^{\infty *}(\mathbb{R}^n;\mathbb{R})$. The space $\mathfrak{s}^*$ is the space of elements (μ, ρ), where ρ is a density and $\mu = \rho v^\flat$ is a 1-form defined with respect to the Euclidean metric on $\mathbb{R}^n$. As mentioned above, we also identify μ with ρv. There is a natural linear Poisson structure on the dual of any Lie algebra, the Lie-Poisson bracket. For this reason we work with the momentum μ rather than the velocity v. The Lie-Poisson bracket on $\mathfrak{s}^*$ is given by

$$\begin{aligned}\{F, G\}_{\mathrm{CF}}(\mu, \rho) = &\int_{\mathbb{R}^n} \mu \cdot \left[\left(\frac{\delta G}{\delta \mu} \cdot \nabla\right) \frac{\delta F}{\delta \mu} - \left(\frac{\delta F}{\delta \mu} \cdot \nabla\right) \frac{\delta G}{\delta \mu}\right] dx \\ &+ \int_{\mathbb{R}^n} \rho \left[\left(\frac{\delta G}{\delta \mu} \cdot \nabla\right) \frac{\delta F}{\delta \rho} - \left(\frac{\delta F}{\delta \mu} \cdot \nabla\right) \frac{\delta G}{\delta \rho}\right] dx.\end{aligned}$$

This bracket is also called the *compressible fluid bracket*. The Hamiltonian for QHD, written in terms of momentum μ, is

$$H_{\mathrm{CF}} = \int_{\mathbb{R}^n} \frac{1}{2} \frac{|\mu|^2}{\rho} + \frac{1}{8} \frac{|\nabla \rho|^2}{\rho} + U(\rho)\, dx.$$

We denote the associated Hamiltonian vector field on the (μ, ρ)-space by $X_{H_{\mathrm{CF}}}$. One can check that it agrees with the right-hand side of QHD: if $X^{\rho}_{H_{\mathrm{CF}}}$ and $X^{\mu}_{H_{\mathrm{CF}}}$ denote the ρ and μ components of $X_{H_{\mathrm{CF}}}$, then

$$\begin{aligned}X^{\rho}_{H_{\mathrm{CF}}} &= -\nabla \cdot \mu\,, \\ X^{\mu}_{H_{\mathrm{CF}}} &= -\nabla \cdot \left(\frac{1}{\rho} \mu \otimes \mu\right) - \rho \nabla \left(f(\rho) - \frac{\Delta\left(\sqrt{\rho}\right)}{2\sqrt{\rho}}\right).\end{aligned}$$

7.1.2 Lie group and Lie algebra actions on the space of wave functions

It turns out that it is natural to think of Ψ as being a space of complex valued half-densities on $\mathbb{R}^n$ since ψ is square-integrable and $|\psi|^2$ is often interpreted as a probability measure. Half-densities are characterized by how they are transformed under diffeomorphisms of the underlying space: if ψ is a half-density on $\mathbb{R}^n$ and g is a diffeomorphism of $\mathbb{R}^n$, the pushforward $g_*(\psi)$ of ψ is $g_*(\psi) = \sqrt{|\mathrm{Det}(Dg^{-1})|}\, \psi \circ g^{-1}$. With this in mind, the following action of S on Ψ is natural.

Definition 7.2. The semidirect product group $S = \mathrm{Diff}(\mathbb{R}^n) \ltimes H^{\infty}(\mathbb{R}^n)$ *acts on the Poisson space* $(\Psi, \{\cdot, \cdot\}_{\mathrm{NLS}})$ as follows: if $(g, a) \in S$ is a group element, then

$$(g, a) : \psi \mapsto (g, a) \cdot \psi := \sqrt{|\mathrm{Det}(Dg^{-1})|}\, e^{-ia} (\psi \circ g^{-1}). \tag{7.1}$$

In other words, ψ is pushed forward under the diffeomorphism g as a complex-valued half-density, followed by a pointwise phase adjustment e^{-ia}.

For the above Lie group action 7.1, the Lie algebra action is as follows.

Proposition 7.3. *Given an element* $\xi = (v, \alpha) \in \mathfrak{s} = \mathrm{vect}(\mathbb{R}^n) \ltimes H^\infty(\mathbb{R}^n)$*, the infinitesimal action of* ξ *on* Ψ *corresponding to the action 7.1 is the vector field* $\xi_\Psi \in \mathfrak{X}(\Psi)$ *defined at each point* ψ *by*

$$\xi_\Psi(\psi) = -\frac{1}{2}\psi\nabla \cdot v - i\alpha\psi - \nabla\psi \cdot v. \tag{7.2}$$

Proof. The proof is a direct computation. □

In the next section we define the Madelung transform and show that it is a momentum map associated with the action 7.2.

7.2 The Madelung transform and a geometric interpretation

7.2.1 The Madelung transform

The classical Madelung transform is a map $(\tau, \rho) \mapsto \psi = \sqrt{\rho}e^{i\tau}$ defined for positive ρ. We consider the inverse map as a more fundamental object and define it below.

Definition 7.4. The (inverse) *Madelung transform* is the map $\mathbf{M} : \Psi \to \mathfrak{s}^*$ defined by

$$\mathbf{M}(\psi) := \begin{pmatrix} \mathrm{Im}\,\bar{\psi}\nabla\psi \\ \mathrm{Re}\,\bar{\psi}\psi \end{pmatrix} = \begin{pmatrix} \mu \\ \rho \end{pmatrix}.$$

Remark 7.5. On a general manifold where $\mathfrak{s}^* = \mathrm{vect}^*(\mathbb{R}^n) \oplus H^{\infty *}(\mathbb{R}^n; \mathbb{R})$ is not identified with $\mathrm{vect}(\mathbb{R}^n) \oplus H^\infty(\mathbb{R}^n; \mathbb{R})$, the μ-component of the Madelung transform is defined to be the 1-form $\mu := \mathrm{Im}\,\bar{\psi}d\psi$.

Proposition 7.6. *The map* $\mathbf{M}$ *is the inverse of the classical Madelung transform in the sense that if* $\psi = \sqrt{\rho}e^{i\tau}$*, then* $\mathbf{M}(\psi) = (\rho\nabla\tau, \rho)$*. If* ρ *is positive, then* $(\rho\nabla\tau, \rho)$ *can be identified with* $([\tau], \rho)$*, where* $[\cdot]$ *is the equivalence class of functions on* $\mathbb{R}^n$ *modulo an additive constant.*

Proof. By the definition of $\mathbf{M}$,

$$\begin{aligned} \mathbf{M}(\sqrt{\rho}e^{i\tau}) &= \begin{pmatrix} \mathrm{Im}\sqrt{\rho}e^{-i\tau}\nabla\left(\sqrt{\rho}e^{i\tau}\right) \\ \mathrm{Re}\sqrt{\rho}e^{-i\tau}\sqrt{\rho}e^{i\tau} \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{Im}\sqrt{\rho}e^{-i\tau}\left(\frac{1}{2}\rho^{-\frac{1}{2}}e^{i\tau}\nabla\rho + i\sqrt{\rho}e^{i\tau}\nabla\tau\right) \\ \rho \end{pmatrix} = \begin{pmatrix} \rho\nabla\tau \\ \rho \end{pmatrix}. \end{aligned}$$

If ρ is positive, one can recover $[\tau]$ from $\rho\nabla\tau$ by dividing by ρ and integrating. □

Remark 7.7. Note that the equivalence of functions τ and τ' differing by an additive constant corresponds to the physical equivalence of two wave functions ψ and ψ' differing by a constant phase factor.

Proposition 7.8. (cf. eg. [15]) *The (inverse) Madelung transform* $\mathbf{M}$ *sends NLS to QHD:* $d\mathbf{M}(X_{H_{\rm NLS}}) = X_{H_{\rm CF}}$.

Proof. The pushforward $d\mathbf{M}_\psi(\phi)$ of a tangent vector $\phi \in T_\psi\Psi$ by the map $\mathbf{M}$ is given by

$$d\mathbf{M}_\psi(\phi) = \begin{pmatrix} \mathrm{Im}(\bar\psi\nabla\phi + \bar\phi\nabla\psi) \\ 2\mathrm{Re}(\bar\psi\phi) \end{pmatrix} =: \begin{pmatrix} (d\mathbf{M}_\psi(\phi))^\mu \\ (d\mathbf{M}_\psi(\phi))^\rho \end{pmatrix}.$$

Recall that $X_{H_{\rm NLS}} = \frac{i}{2}(\Delta\psi - 2f(|\psi|^2)\psi)$. Now substitute $\psi = \sqrt{\rho}e^{i\tau}$ into the expression for $X_{H_{\rm NLS}}$, which results in

$$\begin{aligned} X_{H_{\rm NLS}} &= \frac{i}{2}\left(\Delta\left(\sqrt{\rho}e^{i\tau}\right) - 2f(\rho)\sqrt{\rho}e^{i\tau}\right) \\ &= \frac{i}{2}\left(-\frac{|\nabla\rho|^2}{4\rho^{\frac{3}{2}}}e^{i\tau} + \frac{\Delta\rho}{2\sqrt{\rho}}e^{i\tau} - \sqrt{\rho}|\nabla\tau|^2e^{i\tau} - 2f(\rho)\sqrt{\rho}e^{i\tau}\right) \\ &\quad - \frac{1}{2}\left(\frac{\nabla\rho\cdot\nabla\tau}{\sqrt{\rho}}e^{i\tau} + \sqrt{\rho}\Delta\tau e^{i\tau}\right). \end{aligned}$$

The ρ-component of the image $d\mathbf{M}_\psi(X_{H_{\rm NLS}})$ is obtained by multiplying 7.3 by $2\bar\psi = 2\sqrt{\rho}e^{-i\tau}$ and then taking the real part. We have

$$\begin{aligned} (d\mathbf{M}_\psi(X_{H_{\rm NLS}}))^\rho &= 2\mathrm{Re}(\bar\psi X_{H_{\rm NLS}}) = -\left(\nabla\rho\cdot\nabla\tau + \rho\Delta\tau\right) \\ &= -\nabla\cdot(\rho\nabla\tau) = -\nabla\cdot\mu = X^\rho_{H_{\rm CF}}, \end{aligned}$$

which is the right-hand side of the continuity equation in QHD.

The μ-component of $d\mathbf{M}_\psi(X_{H_{\rm NLS}})$ is found in a similar fashion. Namely, after straightforward computations, one obtains

$$\begin{aligned} (d\mathbf{M}_\psi(X_{H_{\rm NLS}}))^\mu &= \mathrm{Im}\left(\bar\psi\nabla X_{H_{\rm NLS}} + \bar X_{H_{\rm NLS}}\nabla\psi\right) \\ &= \frac{|\nabla\rho|^2\nabla\rho}{4\rho^2} - \frac{\nabla|\nabla\rho|^2}{8\rho} - \frac{\Delta\rho\nabla\rho}{4\rho} + \frac{\nabla\Delta\rho}{4} \\ &\quad - \frac{\rho\nabla|\nabla\tau|^2}{2} - \nabla\tau\nabla\cdot(\rho\nabla\tau) - \rho\nabla f(\rho). \end{aligned}$$

Direct computation shows that the first line of the right-hand side of the last equality is equal to $\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{2\sqrt{\rho}}\right)$. The terms involving τ simplify to

$$\begin{aligned} -\frac{\rho\nabla|\nabla\tau|^2}{2} - \nabla\tau\nabla\cdot(\rho\nabla\tau) &= -\left[(\rho\nabla\tau\cdot\nabla)\nabla\tau + \nabla\tau\nabla\cdot(\rho\nabla\tau)\right] \\ &= -\left[(\mu\cdot\nabla)\left(\frac{\mu}{\rho}\right) + \frac{\mu}{\rho}\nabla\cdot\mu\right] = -\nabla\cdot\left(\frac{1}{\rho}\mu\otimes\mu\right), \end{aligned}$$

so that

$$(d\mathbf{M}_\psi(X_{H_{\rm NLS}}))^\mu = -\nabla\cdot\left(\frac{1}{\rho}\mu\otimes\mu\right) - \rho\nabla\left(f(\rho) - \frac{\Delta\sqrt{\rho}}{2\sqrt{\rho}}\right) = X^\mu_{H_{\rm CF}}.$$

□

7.2.2 The Madelung transform as a momentum map

We first recall the definition of a momentum map.

Suppose we are given a Poisson manifold P, a Lie algebra $\mathfrak{g}$, and an action $A : \mathfrak{g} \to \mathfrak{X}(P)$, $A(\xi) = \xi_P$. Let $\langle\!\langle , \rangle\!\rangle$ denote the pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$. The Lie algebra action A *admits a momentum map* if there exists a map $\mathbf{J} : P \to \mathfrak{g}^*$ satisfying the following definition:

Definition 7.9. A *momentum map* associated with a Lie algebra action $A(\xi) = \xi_P$ is a map $\mathbf{J} : P \to \mathfrak{g}^*$ such that for every $\xi \in \mathfrak{g}$ the function $J(\xi) : P \to \mathbb{R}$ defined by $J(\xi)(p) := \langle\!\langle \mathbf{J}(p), \xi \rangle\!\rangle$ satisfies

$$X_{J(\xi)} = \xi_P.$$

Thus Lie algebra actions that admit momentum maps are Hamiltonian actions on P, and the pairing of the momentum map at a point with an element $\xi \in \mathfrak{g}$ returns a Hamiltonian function associated with the Hamiltonian vector field ξ_P. We now show that $\mathbf{M}$ is a momentum map associated with the action 7.2. In the proofs below we will use the notation of Definition 7.9 and define $M(\xi) : \Psi \to \mathbb{R}$ by $M(\xi)(\psi) := \langle\!\langle \mathbf{M}(\psi), \xi \rangle\!\rangle$.

Theorem 7.10. *For the Lie algebra $\mathfrak{s} = \mathrm{vect}(\mathbb{R}^n) \ltimes H^\infty(\mathbb{R}^n)$, its action 7.2 on the Poisson space $\Psi = H^\infty(\mathbb{R}^n; \mathbb{C})$ equipped with the Poisson structure $\{F, G\}_{\mathrm{NLS}}(\psi) = \left\langle \nabla F, -\frac{i}{2} \nabla G \right\rangle$ admits a momentum map. The map $\mathbf{M} : \Psi \to \mathfrak{s}^*$ defined by 7.3 is a momentum map associated with this Lie algebra action.*

Proof. The Hamiltonian vector field of $M(\xi)$ is given by

$$X_{M(\xi)} = -\frac{i}{2} \nabla M(\xi),$$

where the gradient is defined with respect to the inner product $\langle f, g \rangle = \mathrm{Re} \int \bar{f} g \, dx$. Let $\xi = (v, \alpha)$ be an element of $\mathfrak{s} = \mathrm{vect}(\mathbb{R}^n) \ltimes H^\infty(\mathbb{R}^n)$ and its pairing with $(\mu, \rho) \in \mathfrak{s}^*$ is given by $\langle\!\langle (v, \alpha), (\mu, \rho) \rangle\!\rangle := \int_{\mathbb{R}^n} \rho \cdot \alpha + \mu \cdot v \, dx$. We have

$$\begin{aligned} M(\xi)(\psi) &= \int_{\mathbb{R}^n} \mathbf{M}(\psi)^\rho \cdot \alpha + \mathbf{M}(\psi)^\mu \cdot v \, dx \\ &= \mathrm{Re} \int_{\mathbb{R}^n} \bar{\psi} \psi \alpha - i \bar{\psi} \nabla \psi \cdot v \, dx \ . \end{aligned}$$

To find the gradient, or variational derivative, let $\phi \in C_c^\infty(\mathbb{R}^n; \mathbb{C})$ be a test function and consider the variation of ψ in the direction ϕ:

$$\begin{aligned} \frac{d}{d\epsilon} M(\xi)(\psi + \epsilon \phi) \Big|_{\epsilon = 0} &= \mathrm{Re} \int_{\mathbb{R}^n} \bar{\psi} \phi \alpha + \bar{\phi} \psi \alpha - i \bar{\phi} \nabla \psi \cdot v - i \bar{\psi} \nabla \phi \cdot v \, dx \\ &= \mathrm{Re} \int_{\mathbb{R}^n} 2 \bar{\phi} \psi \alpha - i \bar{\phi} \nabla \psi \cdot v + i \phi \nabla \cdot (\bar{\psi} v) \, dx \\ &= \mathrm{Re} \int_{\mathbb{R}^n} \bar{\phi} \left[2 \psi \alpha - 2 i \nabla \psi \cdot v - i \psi \nabla \cdot v \right] \, dx \,, \end{aligned}$$

so that $\nabla M(\xi)(\psi) = 2\psi\alpha - 2i\nabla\psi\cdot v - i\psi\nabla\cdot v$. Finally we conclude that

$$X_{M(\xi)}(\psi) = -i\alpha\psi - \nabla\psi\cdot v - \frac{1}{2}\psi\nabla\cdot v.$$

Comparing this with 7.2, one obtains that $X_{M(\xi)}(\psi) = \xi_\Psi(\psi)$. □

Any momentum map $\mathbf{J} : P \to \mathfrak{g}^*$ is also a Poisson map taking the bracket on P to the Lie-Poisson bracket on $\mathfrak{g}^*$ provided that it is *infinitesimally equivariant* (see, for example, [19, Thm. 12.4.1]). Recall that a momentum map $\mathbf{J}$ of a Lie algebra $\mathfrak{g}$ is infinitesimally equivariant if for all $\xi, \eta \in \mathfrak{g}$ the following holds:

$$J([\xi,\eta]) = \{J(\xi), J(\eta)\}. \tag{7.3}$$

Theorem 7.11. *The map $\mathbf{M} : \Psi \to \mathfrak{s}^*$ is infinitesimally equivariant for the action of the semidirect product Lie algebra $\mathfrak{s}$.*

Proof. We start by computing the left hand side of 7.3. In our context, the Lie bracket on a pair of elements $\xi = (u,\alpha)$, $\eta = (v,\beta) \in \mathfrak{s} = \mathrm{vect}(\mathbb{R}^n) \ltimes H^\infty(\mathbb{R}^n)$ is given by

$$[(u,\alpha),(v,\beta)] = ([u,v], v\cdot\nabla\alpha - u\cdot\nabla\beta). \tag{7.4}$$

We refer to [18] for the general formula of the Lie bracket on a semidirect product algebra. Using the bracket 7.4 and the definition of $\mathbf{M}$, we have

$$M([\xi,\eta])(\psi) = \mathrm{Re}\int_{\mathbb{R}^n} -i\bar\psi\nabla\psi\cdot[u,v] + \bar\psi\psi\,(v\cdot\nabla\alpha - u\cdot\nabla\beta)\,dx.$$

For the right-hand side of 7.3 we obtain

$$\begin{aligned}\{M(\xi),M(\eta)\}_{\mathrm{NLS}}(\psi) &= \langle\nabla M(\xi), \tfrac{-i}{2}\nabla M(\eta)\rangle(\psi)\\ &= \frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^n}\left[2\bar\psi\alpha + 2i\nabla\bar\psi\cdot u + i\bar\psi\nabla\cdot u\right]\times\left[-2i\psi\beta - 2\nabla\psi\cdot v - \psi\nabla\cdot v\right]dx\\ &= \mathrm{Re}\int_{\mathbb{R}^n}\left[-2\bar\psi\nabla\psi\cdot v\alpha - \bar\psi\psi\nabla\cdot v\alpha + 2\psi\nabla\bar\psi\cdot u\beta + \bar\psi\psi\nabla\cdot u\beta\right]\\ &+ \left[-i\nabla\bar\psi\cdot u\nabla\psi\cdot v + i\nabla\psi\cdot u\nabla\bar\psi\cdot v - i\psi\nabla\bar\psi\cdot u\nabla\cdot v + i\psi\nabla\bar\psi\cdot v\nabla\cdot u\right]dx.\end{aligned} \tag{7.5}$$

At this point we use two identities that are easily verified. The first is

$$-2\bar\psi\nabla\psi\cdot v\alpha - \bar\psi\psi\nabla\cdot v\alpha = -\nabla\cdot\left(\bar\psi\psi v\alpha\right) + \bar\psi\psi v\cdot\nabla\alpha + \psi\nabla\bar\psi\cdot v\alpha - \bar\psi\nabla\psi\cdot v\alpha. \tag{7.6}$$

Notice that $\psi\nabla\bar\psi\cdot v\alpha - \bar\psi\nabla\psi\cdot v\alpha$ is purely imaginary, so this term will not contribute to the integral in 7.5. Neither will $-\nabla\cdot\left(\bar\psi\psi v\alpha\right)$ since it is an exact derivative. There is a similar identity involving u and β in the place of v and α.

The second identity we use is

$$
\begin{aligned}
&- i\nabla\bar{\psi}\cdot u\nabla\psi\cdot v + i\nabla\psi\cdot u\nabla\bar{\psi}\cdot v - i\psi\nabla\bar{\psi}\cdot u\nabla\cdot v + i\psi\nabla\bar{\psi}\cdot v\nabla\cdot u \\
&\qquad = -i\nabla\cdot(\psi\nabla\bar{\psi}\cdot uv) + i\nabla\cdot(\psi\nabla\bar{\psi}\cdot vu) - i\psi\nabla\bar{\psi}\cdot[u,v]. \quad (7.7)
\end{aligned}
$$

The first two terms on the right-hand side of 7.7 do not contribute to 7.5.

So, using 7.6 and 7.7, we can rewrite the Poisson bracket in 7.5 as

$$\{M(\xi), M(\eta)\}_{\mathrm{NLS}}(\psi) = \operatorname{Re}\int_{\mathbb{R}^n} \bar{\psi}\psi\,[v\cdot\nabla\alpha - u\cdot\nabla\beta] - i\bar{\psi}\nabla\psi\cdot[u,v]\,dx.$$

Since the latter expression is equal to $M([\xi,\eta])(\psi)$, this completes the proof. □

Corollary 7.12. *The map* $\mathbf{M}$ *is a Poisson map sending the bracket* $\{\cdot,\cdot\}_{\mathrm{NLS}}$ *to the bracket* $\{\cdot,\cdot\}_{\mathrm{CF}}$.

Proof. This is an immediate corollary of Theorems 7.10 and 7.11. □

One can also check the Poisson property of $\mathbf{M}$ by a direct computation very similar to the proof of Theorem 7.11.

Remark 7.13. In the language of [17], the map $\mathbf{M}$ is an example of *symplectic* or *Clebsch variables* for the "gradient subspace" of the space $\mathfrak{s}^*$, where $(\mu,\rho)\in\mathfrak{s}^*$ with $\mu = \rho v^\flat$ and $v = \nabla\tau$. The term "Clebsch variables" means a Poisson map $\psi : R \to P$ from a symplectic space R to a Poisson space P. Corollary 7.12 shows that our map $\mathbf{M}$ is a Poisson map from the symplectic space Ψ (with symplectic form given by $\omega(U,V) = -2\langle U, iV\rangle$) to the gradient subspace of the Poisson space $\mathfrak{s}^*$. Such a map ψ is also called a *symplectic realization* [28].

Remark 7.14. In [12, 26] the Madelung transform is understood somewhat differently. It is a surjection σ from the space $\mathcal{C}(M)$ of smooth non-vanishing complex functions on a Riemannian manifold M to the tangent bundle $T\mathcal{P}(M)$ of the space $\mathcal{P}(M)$ of probability measures over M. Here $T\mathcal{P}(M)$ plays the role of the phase space for potential motions of the compressible fluid. It is shown in [26] that σ is a symplectic submersion of $\mathcal{C}(M)$ equipped with the constant symplectic structure on complex functions into $T\mathcal{P}(M)$ equipped with the natural symplectic structure induced by the Wasserstein metric.

Remark 7.15. One of possible applications of the momentum map nature of the Madelung transform is its use for Hamiltonian reduction to the space of singular solutions such as vortex sheets and vortex membranes.

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