ON MOD p LOCAL-GLOBAL COMPABILITY FOR UNRAMIFIED GL_3

by

John Enns

A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy
Graduate Department of Mathematics
University of Toronto

Abstract

On mod p local-global compability for unramified GL_3

John Enns
Doctor of Philosophy
Graduate Department of Mathematics
University of Toronto
2018

Let K be a p-adic field. Given a continuous Galois representation $\bar{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, the mod p Langlands program hopes to associate with it a smooth admissible $\overline{\mathbb{F}}_p$-representations $\Pi_p(\bar{\rho})$ of $\mathrm{GL}_n(K)$ in a natural way. When $\bar{\rho} = \bar{r}|_{G_{F_w}}$ is the local w-part of a global automorphic Galois representation $\bar{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, for some CM field F/F^+ and place $w|p$, it is possible to construct a candidate $H^0(\bar{r})$ for $\Pi_p(\bar{r}|_{G_{F_w}})$ using spaces of mod p automorphic forms on definite unitary groups.

Assume that F_w is unramified. When $\bar{r}|_{G_{F_w}}$ is semisimple, it is possible to recover the data of $\bar{r}|_{G_{F_w}}$ from the $\mathrm{GL}_n(\mathcal{O}_{F_w})$-socle of $H^0(\bar{r})$ (also known as the set of Serre weights of $\bar{r}$). But when $\bar{r}|_{G_{F_w}}$ is wildly ramified this socle does not contain enough information. In this thesis we give an explicit recipe to find the missing data of $\bar{r}|_{G_{F_w}}$ inside the $\mathrm{GL}_3(F_w)$-action on $H^0(\bar{r})$ when $n = 3$ and $\bar{r}|_{G_{F_w}}$ is maximally nonsplit, Fontaine-Laffaille, and generic. This generalizes work of Herzig, Le and Morra [HLM17] who found analogous results when $F_w = \mathbb{Q}_p$ as well as work of Breuil and Diamond [BD14] in the case of unramified GL_2.

For Cathleen

Acknowledgements

I want to thank my advisor Florian for suggesting this problem and for his helpful advice and comments since the beginning stages of this work. I am grateful for having had the opportunity to learn about some deep and difficult areas of math under his supervision.

It would not have been possible for me to complete graduate school without the support of my family. Thank you Mom, Dad, William, and Noah. It's been hard being away from home.

Finally, to Cathleen: I can't begin to describe the positive impact you've had on my life. Thank you for your unconditional love and support. This thesis is dedicated to you.

Contents

Chapter 0

Introduction

THIS IS A DRAFT.

This thesis is about relationships between automorphic forms and Galois representations. Specifically, it is motivated by the search for a mod p local Langlands correspondence, and how such a correspondence might be realized inside the cohomology of arithmetic manifolds attached to reductive groups over number fields. The words "local-global compatibility" in the title refer to this possibility.

We begin by illustrating an example of local-global compatibility in the classical Langlands program, following [Bre12]. Let ℓ be a prime and K a finite extension of $\mathbb{Q}_\ell$ with absolute Galois group G_K and Weil group W_K. The classical local Langlands correspondence for GL_n posits the existence of a unique family of bijections

$$\begin{aligned} \mathrm{rec}_{n,p} :&\{\text{irreducible admissible representations of } \mathrm{GL}_n(K) \text{ over } \overline{\mathbb{Q}}_p\}/\cong \\ &\to \{\text{rank } n \text{ Frobenius-semisimple Weil-Deligne representations of } K \text{ over } \overline{\mathbb{Q}}_p\}/\cong \end{aligned} \tag{0.0.1}$$

for each n characterized by the list of compatibility conditions found in the introduction to [HT01], in particular compatibility with local class field theory. When $\ell \neq p$, Grothendieck's ℓ-adic monodromy theorem gives an equivalence of categories between continous Galois representations $G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ and bounded Weil-Deligne representations of K over $\overline{\mathbb{Q}}_p$. Via $\mathrm{rec}_{n,p}$ this allows us to attach an irreducible admissible $\overline{\mathbb{Q}}_p$-representation of $\mathrm{GL}_n(K)$ to any continuous p-adic representation of G_K such that Frobenius acts semisimply.

We say that a local-global compatibility occurs in any situation where a purely local correspondence, such as (0.0.1), arises in connection with a correspondence between global objects, typically the cohomology of a Shimura variety or more a generally an arithmetic quotient of a locally symmetric space attached to a reductive group over a number field. For example, the cohomology of modular curves, which are Shimura varieties attached to the reductive group $\mathrm{GL}_{2/\mathbb{Q}}$, exhibits local-global compatibility. We use this example to explain the idea.

Let $\mathbb{A}$ denote the ring of adèles of $\mathbb{Q}$. Associated with any sufficiently small open compact subgroup $U \leq \mathrm{GL}_2(\mathbb{A}^\infty)$ there is an algebraic curve $Y(U)$ (the classical modular curve of level U) defined over $\mathbb{Q}$ whose space of complex points is isomorphic to $\mathrm{GL}_2(\mathbb{Q})\backslash(\mathbb{C}-\mathbb{R}) \times \mathrm{GL}_2(\mathbb{A}^\infty)/U$. We may consider the p-adic étale cohomology[1] $H^1(Y(U)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)$, which is a finite-dimensional $\overline{\mathbb{Q}}_p$-vector space with a continuous action of $G_\mathbb{Q}$. As U varies, there are corresponding maps between the curves $Y(U)$ and it makes sense to consider all levels U at once by defining $H^1 = \varinjlim_U H^1(Y(U)_{\bar{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)$. This space has a natural smooth action of $\mathrm{GL}_2(\mathbb{A}^\infty)$ commuting with its $G_\mathbb{Q}$-action. One possible strong statement of classical local-global compatibility is the following. Suppose that $r : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ is a continuous global Galois representation such that $\mathrm{Hom}_{G_\mathbb{Q}}(r, H^1) \neq 0$ (this assumption means that r is the p-adic representation associated with a classical weight 2 modular newform by the construction of Deligne). Then as a $\mathrm{GL}_2(\mathbb{A}^\infty)$-representation, we have

$$\mathrm{Hom}_{G_\mathbb{Q}}(r, H^1) \cong \pi_p \otimes {\bigotimes_{\ell \neq p}}' \mathrm{rec}_{2,p}(r|_{G_\ell}) \tag{0.0.2}$$

Here G_ℓ is a decomposition group for the prime ℓ inside $G_\mathbb{Q}$ and π_p denotes a certain smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\overline{\mathbb{Q}}_p$ that we will return to in a moment. The key point here is that we observe each local Langlands correspondence occuring inside the cohomology of the tower of Shimura varieties $(Y(U))_U$ for each prime not equal to p.

But as one sees in (0.0.2), something different happens when $\ell = p$, that is, when the field of coefficients and the local Galois representation are both p-adic. In fact there is a recipe for constructing π_p from $r|_{G_p}$, but the procedure loses information. In other words, it is not possible to reconstruct $r|_{G_p}$ from π_p. This is because there can exist no analogue to (0.0.1) that attaches smooth representations of $\mathrm{GL}_n(K)$ to *continuous* p-adic representations of G_K when $\ell = p$ in any way that doesn't lose information. The issue is that since the topologies of G_K and $\overline{\mathbb{Q}}_p$ are compatible, there is a much richer supply of local Galois representations, but on the other hand smooth representations ignore the topology of the coefficient field by definition. The problem of how to recover the local Galois representation at p in (0.0.2) is one possible starting point for the p-adic (and mod p) Langlands correspondences.

In order to get a p-adic local Langlands correspondence the idea is to replace smooth representations of $\mathrm{GL}_n(K)$ with something that sees the p-adic topology, namely Banach pace representations of $\mathrm{GL}_n(K)$ over (a finite subextension of) $\overline{\mathbb{Q}}_p$. The attempt to prove results in this direction has been a point of heavy research activity for some time, and the only cases that have been settled are $n = 1$ (which is as usual a consequence of local class field theory) and $n = 2, K = \mathbb{Q}_p$. See [Bre10] for an overview. Indeed, it is now known that there is a functorial link between categories of unitary Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and continuous p-adic representations of $G_{\mathbb{Q}_p}$ inducing a bijection between irreducible objects on both sides (see [CDPs14] for a precise statement). Analogues of these statements for $n > 2$ or $K \neq \mathbb{Q}_p$ are

[1] Technically one should have parabolic cohomology. See [Eme06] for more precise statements.

highly sought after.

In [Eme11], Emerton proved the p-adic analogue of (0.0.2). Of course, one cannot use simply the p-adic étale cohomology exactly as above since we just saw this doesn't work. Emerton has instead introduced a theory of p-adically completed cohomology where H^1 is replaced by a kind of completion $\widehat{H}^1$ that is not a smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, but a Banach representation. Then the analogue of (0.0.2) holds. See the introduction to [Eme11] for the precise statement, or below for the mod p version.

Since Banach representations admit a canonical notion of reduction mod p (by reducing their unit ball), it is hoped that in addition to a p-adic correspondence there will also be a mod p local Langlands correspondence compatible with reducing lattices mod p on the Galois side. We state the hope precisely: there should be a map

$$\begin{aligned}&\{\text{continuous representations } G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)\}/\cong\\ &\qquad\qquad\hookrightarrow \{\text{admissible representations of } \mathrm{GL}_n(K) \text{ over } \overline{\mathbb{F}}_p\}/\cong .\end{aligned} \tag{0.0.3}$$

which we denote $\bar{\rho} \mapsto \Pi_p(\bar{\rho})$ compatible with the (conjectural) p-adic local Langlands correspondence and hopefully with the mod p cohomology of Shimura varieties.

As with the p-adic case, the existence of this correspondence is known in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$ (it is actually easier than the p-adic result and was known before), and the analogue of (0.0.2) holds as a corollary to Emerton's p-adic result. We will actually state a weaker version, since it is analogous to the situation considered in this thesis. Fix an open compact subgroup $U^p \leq \mathrm{GL}_2(\mathbb{A}^{\infty,p})$ (a tame level). Consider the mod p étale cohomology of the modular curve with infinite level at p: $\bar{H}^1(U^p) := \varinjlim_{U_p} H^1(Y(U_pU^p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)$ where U_p runs over compact open subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$. By its construction this object has a smooth action of $\mathrm{GL}_2(\mathbb{Q}_p)$ commuting with its $G_{\mathbb{Q}}$-action. Suppose that $\bar{r} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is a continuous irreducible modular representation satisfying some technical assumptions. Then as a smooth $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ we have

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(\bar{r}, \bar{H}^1(U^p)) \cong \Pi_p(\bar{r}|_{G_p})^{d(U^p)} \tag{0.0.4}$$

where $d(U^p) \in \mathbb{N}$ is some multiplicity and Π_p is the previously established mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ [Eme11].

The reason that no mod p (or p-adic) local Langlands correspondence has yet been established for groups other than $\mathrm{GL}_2(\mathbb{Q}_p)$ is that the mod p representation theory of any other group appears to be infinitely more difficult than for $\mathrm{GL}_2(\mathbb{Q}_p)$. For example, there are infinitely many irreducible admissible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(K)$ for any field $K \neq \mathbb{Q}_p$ with a fixed central character (whereas for $\mathrm{GL}_2(\mathbb{Q}_p)$ there are only finitely many). Breuil and Paškūnas in [BP12] made a detailed study of smooth $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(\mathbb{Q}_{p^f})$. Their results made it clear that one could not expect a 1-1 correspondence in (0.0.3) for $\mathrm{GL}_2(\mathbb{Q}_{p^f})$ if $f > 1$, and in light of this the same is expected for any group other than $\mathrm{GL}_2(\mathbb{Q}_p)$. The main problem in the classification of irreducible smooth $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_n(K)$ is in understanding the supersingular representations (which are analogous to the supercuspidals in the characteristic 0 case). Despite recent advances in classifying the irreducible admissible representations of $\mathrm{GL}_n(K)$ (and even more general reductive groups) *in terms of* supersingular representations ([AHHV17]), the classification of the supersingular representations themselves does not appear to be forthcoming.

Since none of these difficulties appear on the Galois side of (0.0.3) (it is easy to classify representations $G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ for any p-adic field K), these discoveries have caused some uncertainty about the nature of the conjectural mod p local Langlands correspondence. Nevertheless, we expect some kind of local-global compatibility, and it has been the viewpoint of a number of papers (and is the viewpoint of this thesis) that the shape of the hypothetical $\Pi(\bar{\rho})$ for groups other than $\mathrm{GL}_2(\mathbb{Q}_p)$ might be studied via analogues of the globally defined space $\mathrm{Hom}_{G_{\mathbb{Q}}}(\bar{r}, \bar{H}^1(U^p))$. In particular, one would at least like to show that (an analogue of) this space depends only on the *local* Galois representation $\bar{r}|_{G_p}$. Unfortunately, even this is too much to ask right now, given how poorly smooth mod p representations are understood.

This work

This thesis studies local-global compatibility for the group $\mathrm{GL}_3(\mathbb{Q}_{p^f})$. It is customary in the burgeoning literature of the area to use the global set-up of a definite unitary group G defined over a totally imaginary CM field F/F^+. This is because the associated arithmetic manifolds are 0-dimensional and hence admit cohomology only in degree 0, which allows for numerous simplifications. On the other hand, one knows how to attach Galois representations to automorphic representations of G satisfying (classical) local-global compatibility and the Taylor-Wiles patching machinery is well-developed in this context. So these groups form the simplest possible global set-up in which to test local conjectures.

We now describe our results. Let F be a number field which is a quadratic extension of a totally real field F^+ such that p is unramified in F^+ and each place of F^+ above p splits in F. Fix a definite unitary group G defined over F^+ that becomes isomorphic to GL_3 over F and choose places $w|v$ of F/F^+ unramified of degree f over p. We begin with an irreducible automorphic global Galois representation $\bar{r} : G_F \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$. Here automorphic means that $\bar{r}$ is the reduction mod p of a p-adic Galois representation attached to an automorphic representation of G. We also assume that $\bar{r}$ satisfies some technical hypotheses needed for the Taylor-Wiles method. Let $U^v \leq G(\mathbb{A}_{F^+}^{\infty,v})$ be a fixed well-chosen compact open subgroup which is in particular unramified at all places of F^+ inert in F, and consider

$$H^0(U^v, \overline{\mathbb{F}}_p) := \varinjlim_{U_v} H^0(U_v U^v, \overline{\mathbb{F}}_p),$$

where U_v runs over compact open subgroups of $G(F_v^+)$ and $H^0(U, \overline{\mathbb{F}}_p)$ denotes the 0th cohomology[2] of the (finite) arithmetic quotient $G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U$. By its construction $H^0(U^v,\overline{\mathbb{F}}_p)$ has a smooth admissible action of $G(F_v^+)\cong \mathrm{GL}_3(F_w)=\mathrm{GL}_3(\mathbb{Q}_{p^f})$. There is also a commuting action of a Hecke algebra $\mathbb{T}$ consisting of Hecke operators at all places of F^+ not dividing p that split in F. Let $\mathfrak{m}_{\bar r}$ denote the maximal ideal of $\mathbb{T}$ corresponding to $\bar r$ via the Satake isomorphism. Since there is no Galois action on the cohomology group we have chosen, the natural analoge of the $\bar r$-isotypic subspace that was considered in (0.0.4) above is the $\mathfrak{m}_{\bar r}$-torsion subspace $H^0(U^v,\overline{\mathbb{F}}_p)[\mathfrak{m}_{\bar r}]$. This is a smooth admissible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_3(F_w)$, nonzero by the assumption of automorphy. By analogy with the local-global compatibility for $\mathrm{GL}_2(\mathbb{Q}_p)$ in (0.0.4) we expect this representation to be a number of copies of $\Pi_p(\bar r|_{G_{F_w}})$, where the number of copies depends only on U^v.

In particular, one would hope that the smooth $\mathrm{GL}_3(F_w)$-representation $H^0(U^v,\overline{\mathbb{F}}_p)[\mathfrak{m}_{\bar r}]$ depends only on the *local* Galois representation $\bar r|_{G_{F_w}}$. Unfortunately, this is out of reach at the moment. We are instead concerned with whether the passage from $\bar r|_{G_{F_w}}$ to $H^0(U^v,\overline{\mathbb{F}}_p)[\mathfrak{m}_{\bar r}]$ loses information. Our main result (Theorem 3.5.6 + Theorem 3.6.1) is that it does not. A somewhat vague version of this result is Theorem 0.0.5 below; see Chapter 3 for the precise version. Unfamiliar terminology will be described in the following paragraphs.

Theorem 0.0.5. *In the above setting, let $\bar r|_{G_{F_w}}$ be upper-triangular and maximally nonsplit. Assume that it is Fontaine-Laffaille and sufficiently generic, and has a single modular Serre weight. Then it is possible to recover the data of the local Galois representation $\bar r|_{G_{F_w}}: G_{\mathbb{Q}_{p^f}}\to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ from the $\mathrm{GL}_3(\mathbb{Q}_{p^f})$-action on $H^0(U^v,\overline{\mathbb{F}}_p)[\mathfrak{m}_{\bar r}]$ via an* explicit *recipe.*

In order to explain the meaning of the hypotheses in the theorem we need some additional background. Let us begin with the concept of a Serre weight of a representation $\bar r$. If K is any p-adic field, a Serre weight of K is defined to be an (isomorphism class of an) irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_n(\mathcal{O}_K)$. All Serre weights factor through $\mathrm{GL}_n(k)$ where k is the residue field of K, so there are finitely many of them. Going back to the setting of (0.0.3), we see that if the correspondence Π exists, then associated with $\bar\rho$ there should be the set of Serre weights given by the set of irreducible constituents of the $\mathrm{GL}_n(\mathcal{O}_K)$-socle of $\Pi(\bar\rho)$.

In the global setting described above, the Serre weights of $\bar r|_{G_{F_w}}$ should therefore arise as the irreducible constituents of the $\mathrm{GL}_3(\mathcal{O}_{F_w})$-socle of $H^0(U^v,\overline{\mathbb{F}}_p)[\mathfrak{m}_{\bar r}]$, so we define the set of modular Serre weights of $\bar r$ to be this set and call it $W(\bar r|_{G_{F_w}})$ even though *a priori* it depends on all of $\bar r$ (technically, one should take into account all places above p when defining this set but we ignore this point in the Introduction; see Definition 3.4.2). Conjectures about the modular Serre weights of an automorphic Galois representation have played an important role in the development of the mod p Langlands program, since the $\mathrm{GL}_n(\mathcal{O}_K)$-socle is the most accessible piece of $\Pi(\bar\rho)$. In particular, when $\bar r|_{G_{F_w}}$ is semisimple, [Her09] gave a conjectural set $W^?(\bar r|_{G_{F_w}})$ of regular weights, this time truly depending only on $\bar r|_{G_{F_w}}$, and in fact only on the restriction to inertia, that should coincide with the set of regular weights in $W(\bar r|_{G_{F_w}})$.

[2]Technically we take coefficients in a certain local system V', but this is irrelevant for the Introduction.

Here regularity is a genericity condition. If $\bar{r}|_{G_{F_w}}$ is sufficiently generic then it is known that all weights occurring in $W(\bar{r}|_{G_{F_w}})$ are regular [Enn17] so this point is not so important for us. We also refer to [GHS] for an overview of various sets of Serre weights that can be attached to a local mod p Galois representation.

There has been a great deal of progress on comparing the sets $W(\bar{r}|_{G_{F_w}})$ and $W^?(\bar{r}|_{G_{F_w}})$ in the setting of definite unitary groups ([EGH13, LLHL17]) when $\bar{r}|_{G_{F_w}}$ is semisimple. On the other hand, when $\bar{r}|_{G_{F_w}}$ is *not* semisimple, the conjecture of [Her09] does not apply and there isn't yet a concrete way to predict the set of modular weights. Nevertheless, one expects $W(\bar{r}|_{G_{F_w}}) \subseteq W^?(\bar{r}|^{\mathrm{ss}}_{G_{F_w}})$ in all cases and that there will be a geometric interpretation of how the weight set behaves ([GHS], Section 6). Moreover, forthcoming work of Le, Le Hung, Levin and Morra [LLHLM] should give the complete picture on the modular Serre weights in the setting considered in this paper.

Going back to the statement of Theorem 0.0.5, when $\bar{r}|_{G_{F_w}}$ is semisimple the data of $\bar{r}|_{G_{F_w}}$ may be read off from its set of modular Serre weights, that is, off from the $\mathrm{GL}_3(\mathcal{O}_{F_w})$-socle of $H^0(U^v, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\bar{r}}]$. But when $\bar{r}|_{G_{F_w}}$ is not semisimple, the $\mathrm{GL}_3(\mathcal{O}_{F_w})$-socle cannot contain all of the data of $\bar{r}|_{G_{F_w}}$. For example, we assume that $\bar{r}|_{G_{F_w}}$ is upper-triangular and maximally nonsplit, which is to say

$$\bar{r}|_{G_{F_w}} \sim \begin{pmatrix} \psi_2 & * & * \\ & \psi_1 & * \\ & & \psi_0 \end{pmatrix} \tag{0.0.6}$$

for some characters $\psi_i : G_{F_w} \to \overline{\mathbb{F}}_p^\times$ where the two off-diagonal Ext^1 classes are nonzero. Then one "sees" the data of the diagonal characters ψ_2, ψ_1, ψ_0 in the $\mathrm{GL}_3(\mathcal{O}_{F_w})$-socle of $H^0(U^v, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\bar{r}}]$, but this socle cannot contain the data of the extension classes. So the question to answer in Theorem 0.0.5 is how to recover these Ext^1 classes from the $\mathrm{GL}_3(F_w)$-action on $H^0(U^v, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\bar{r}}]$. To do so is then to investigate the structure of the cohomology representation $H^0(U^v, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\bar{r}}]$ (hence $\Pi(\bar{r}|_{G_{F_w}})$) beyond its $\mathrm{GL}_3(\mathcal{O}_{F_w})$-socle.

The remaining hypotheses in Theorem 0.0.5 are there to simplify the question. The strongest of them is the assumption that $\bar{r}|_{G_{F_w}}$ is Fontaine-Laffaille. This condition is crucial for our method since we use Fontaine-Laffaille theory to normalize the extension classes; see the strategy of proof below. The hypothesis of genericity is a condition on the inertial weights of $\bar{r}|_{G_{F_w}}$ (Definition 1.1.2) and is critical in the sense that were it absent nearly everything in the proof would go wrong, but not so important in the sense that for a fixed degree of genericity the "proportion" of local mod p Galois representations that are generic approaches 1 as $p \to \infty$. But also note that the assumption of genericity does imply a lower bound on p.

Finally, it was first observed (almost, up to a shadow weight, and for $f = 1$) in [HLM17] that "most" upper-triangular maximally nonsplit Galois representations had a unique modular Serre weight (see Remark 3.5.5). This is confirmed for all f, based on discussions with the authors of the forthcoming [LLHLM], so the final assumption in Theorem 0.0.5 again represents the most generic case.

Strategy of proof

We now briefly explain the explicit recipe in Theorem 0.0.5. The case $f = 1$ was done by Herzig, Le and Morra in [HLM17], and our strategy follows theirs. The first step is local - using Fontaine-Laffaille theory we define parameters $\mathrm{FL}_\tau \in \overline{\mathbb{F}}_p^\times$ which, along with the diagonal characters in (0.0.6) determine the isomorphism class of $\bar{r}|_{G_{F_w}}$ (Definition 1.2.14). These Fontaine-Laffaille parameters are indexed by a collection of inertial types τ (principal series types).

For each type τ, we define explicit group algebra operators $S'_\tau, S_\tau \in \overline{\mathbb{F}}_p[\mathrm{GL}_3(k_w)]$ and our main result (Theorem 3.5.6) shows (roughly) that there is an equality of *injective* operators

$$S'_\tau \Pi = \kappa_\tau \mathrm{FL}_\tau S_\tau \tag{0.0.7}$$

between explicit Iwahori eigenspaces of $H^0(U^v, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\bar{r}}]$ where κ_τ is an explicit constant in $\overline{\mathbb{F}}_p^\times$, and $\Pi \in \mathrm{GL}_3(F_w)$ is a normalizer of the Iwahori subgroup.

The main idea of the proof is to relate the mod p cohomology $H^0(U^v, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\bar{r}}]$ directly to spaces of classical automorphic forms on G in characteristic 0, which is easy since cohomology occurs only in degree 0. It is known how to attach Galois representations to automorphic representations of G (Theorem 3.3.1) satisfying classical local-global compability (0.0.1) linking the local Galois and automorphic representations. We study how lifts of the operators S'_τ, S_τ behave in characteristic 0, where we can relate the local Galois and automorphic sides, and then study how both sides reduce mod p.

The automorphic representations which contribute to the Iwahori eigenspaces in question are principal series at w by design, and their Galois representations at w are potentially crystalline lifts of $\bar{r}|_{G_{F_w}}$ of parallel Hodge type $\mu = (2, 1, 0)$ and inertial type τ. Accordingly, the first chapter of the thesis studies potentially crystalline lifts of $\bar{r}|_{G_{F_w}}$ using p-adic Hodge theory. The main result (Theorem 1.7.1) relates the Fontaine-Laffaille invariants of $\bar{r}|_{G_{F_w}}$ with certain Frobenius eigenvalues on the weakly admissible modules associated with the lifts. We make use of the recent work of [LLHLM18a] giving explicit descriptions of potentially crystalline deformation rings of Hodge type μ.

The subject of Chapter 2 is the local automorphic side. The results in characteristic 0 are easier compared to the Galois side (Section 2.6), but the most technical part of this thesis is to show the injectivity of the operators in (0.0.7). This boils down to a statement about nonvanishing of S_τ and S'_τ on certain quotients of principal series representations for the finite group $\mathrm{GL}_3(k_w)$. The proof of this result (Theorem 2.3.2) is inspired by [LLHLM18b] - by embedding the principal series into an algebraic representation of $\prod_{j=0}^{f-1} \mathrm{GL}_{3/\overline{\mathbb{F}}_p}$, we may use Lie theory and known results about the structure of Weyl modules to do explicit computations that show nonvanishing. We remark that one technical improvement in this thesis to previous work consists of using nonsplit algebraic groups, which makes the aforementioned computations feasible. We also spend some time in this chapter generalizing a result of Pillen about restricting algebraic representations to finite subgroups to the nonsplit case (Section 2.2), although we don't end up needing this.

The proof of (0.0.7) in Chapter 3 uses a certain assumption about freeness over a Hecke algebra. We conclude the thesis by showing that this hypothesis is satisfied when U^v is well-chosen and $\bar{r}$ satisfies some technical conditions using Taylor-Wiles patching and the Diamond-Fujiwara method (Theorem 3.6.1). The proof of this result is a very mild generalization of the analogous result in [HLM17]; in fact it is easier in our case because of the assumption about a single Serre weight.

We remark that on both the Galois and automorphic sides, the particular choice of types τ is very important and it is not clear why these types work whereas others don't.

Relation to previous work

As mentioned above, the $f = 1$ case is due to [HLM17]. They work in a similar setting to us, but without the assumption of a single Serre weight. It would be possible to study the Serre weights of $\bar{r}|_{G_{F_w}}$ when $f > 1$ using generalizations of their methods, and in fact we use our group operators to prove a mild result on the Serre weights of $\bar{r}|_{G_{F_w}}$ in our main result Theorem 3.5.6, but we avoided a more serious investigation because the Serre weights of $\bar{r}$ are the subject of the forthcoming [LLHLM].

Our method follows that of [HLM17], except we use the new technology of [LLHLM18a] on the Galois side, and our method of dealing with the operators S'_τ, S_τ on the local automorphic side is new. When $f = 1$, our operators contain those considered in [HLM17] (see Remark 2.3.9).

Similar operators also appeared in work of Breuil and Diamond [BD14] for the group $\mathrm{GL}_2(\mathbb{Q}_{p^f})$. It may be interesting to note that some of our operators (the ones used to recover the Fontaine-Laffaille invariants corresponding to the 2-dimensional subquotients of $\bar{r}|_{G_{F_w}}$) have a strong affinity with theirs, though the context is different and there is no obvious relation between the two settings.

The case of $\mathrm{GL}_3(\mathbb{Q}_p)$ when $\bar{r}|_{G_{F_w}}$ is wildly ramified with a 2-dimensional irreducible subquotient was studied by similar means in [LMP16]. Finally, [HLM17] has recently been generalized in a different direction to $\mathrm{GL}_n(\mathbb{Q}_p)$ by Park and Qian in [PQ17], who again use a similar method.

0.1 Notation and conventions

If L is any field, G_L denotes the absolute Galois group $\mathrm{Gal}(\overline{L}/L)$ with respect to a fixed choice of algebraic closure $\overline{L}$. If L is a p-adic field then $W_L \leq G_L$ denotes the Weil group of L and $I_L \leq W_L$ the inertia group. We write $\mathbb{A}_L$ for the ring of adèles of L and $\mathbb{A}_L^\infty$ for the ring of finite adèles.

Throughout the thesis, K_0 denotes an unramified p-adic field of residue degree $f \geq 1$, and k its residue field. We write $q = p^f$. The symbol E denotes a p-adic field serving as the coefficient field. It is always assumed to be finite over $\mathbb{Q}_p$ but large enough to contain the image of all embeddings $\mathrm{Hom}_{\mathbb{Q}_p}(K_0, \overline{\mathbb{Q}}_p)$. The residue field of E is called $\mathbb{F}$.

Let $K = K_0(\pi)$ be the Galois totally ramified extension of K_0 of degree $e := q-1$ generated by (any) eth root π of $-p$. Write $E(u) = u^e + p \in K_0[u]$ for the minimal polynomial of π. With this choice of π there is associated a fundamental character $\omega_{K_0,\pi} : G_{K_0} \to k^\times$ (of niveau f) sending g to $g(\pi)/\pi \mod p$. Its restriction to I_{K_0} does not depend on the choice of π or even of the uniformizer $-p$, so we denote it by simply ω_{K_0}. It is equal to $\mathrm{Art}_{K_0}^{-1}|_{I_{K_0}} \mod p$, where $\mathrm{Art} : K_0^\times \to G_{K_0}^{\mathrm{ab}}$ denotes the reciprocity map of local class field theory normalized in such a way that it takes uniformizers to geometric Frobenius automorphisms.

We make a fixed choice of field embedding $\tau_0 : k \hookrightarrow \mathbb{F}$, and define inductively $\tau_{j+1} = \tau_j^{1/p}$. We then define the character $\omega_j : G_{K_0} \to \mathbb{F}^\times$ to be $\tau_j \circ \omega_{K_0,\pi}$. The notation is cyclic, so $\omega_f = \omega_0$. We will prefer to use the former. Note that with this choice of ordering for our field embeddings, we have

$$\prod_{j=0}^{f-1} \omega_j^{a_j} = \omega_f^{\sum_{j=0}^{f-1} a_{f-j} p^j}$$

for any integers $a_j \in \mathbb{Z}$. We will often consider p-adic expansions of the form $\sum_{j=0}^{f-1} a_{f-j} p^j$; the coefficients are always considered to be indexed modulo f. In order to simplify notation, all sums without any explicit limits written go from 0 to $f-1$.

We often consider modules M over rings like $k \otimes_{\mathbb{F}_p} \mathbb{F}$. Recall that since $\mathbb{F}$ is sufficiently large, $k \otimes_{\mathbb{F}_p} \mathbb{F} \cong \prod_{j=0}^{f-1} \mathbb{F}$ as an $\mathbb{F}$-algebra. Letting $\epsilon_j \in k \otimes_{\mathbb{F}_p} \mathbb{F}$ denote the idempotent corresponding to the jth factor, we have $M = \prod_{j=0}^{f-1} M^j$ where $M^j = \epsilon_j M$ is the submodule $\{m \in M \mid (\alpha \otimes \beta)m = (1 \otimes \tau_j(\alpha)\beta \quad \forall \alpha \in k, \beta \in \mathbb{F}\}$. It will often be the case that M is equipped with a $\phi \otimes 1$-semilinear map $\varphi_M : M \to M$, where ϕ denotes the absolute Frobenius of k (we also use ϕ for the induced Frobenius of $W(k)$). We may therefore view φ_M as a collection of $\mathbb{F}$-linear maps $M^j \to M^{j+1}$. We often do this without comment, including in variants of this situation with rings other than $k \otimes_{\mathbb{F}_p} \mathbb{F}$.

Fix a choice of elements $(p_n)_{n \geq 0}$ in $\bar{K}$ obeying $p_{n+1}^p = p_n$ and $p_0 = -p$. Given this and the choice of π already made, there exist unique elements $(\pi_n)_{n \geq 0}$ in $\bar{K}$ such that $\pi_n^e = p_n$, $\pi_{n+1}^p = \pi_n$, and $\pi_0 = \pi$. We write $(K_0)_\infty = K_0((p_n)_{n \geq 0})$ and $K_\infty = K((\pi_n)_{n \geq 0})$. Observe that restriction defines a canonical isomorphism $\mathrm{Gal}(K_\infty/(K_0)_\infty) = \mathrm{Gal}(K/K_0) =: \Delta$, via which we consider $\omega_{K_0,\pi}$ as a character $\mathrm{Gal}(K_\infty/(K_0)_\infty) \to W(k)^\times$.

An *inertial type* is a representation $\tau : I_{K_0} \to \mathrm{GL}_n(E)$ with open kernel that extends to W_{K_0}. A tamely ramified inertial type factors as a sum of fundamental characters $\widetilde{\omega}_{nf} : I_{K_0} \to \mathcal{O}_E^\times$; we say that τ is a principal series type if it is tamely ramified of niveau f, which is to say $\tau = \bigoplus_{i=0}^{n-1} \widetilde{\omega}_f^{A_i}$ for $A_i \in \mathbb{Z}$. Equivalently, τ factors through Δ.

The convention on Hodge-Tate weights is that the cyclotomic character $\epsilon : G_{\mathbb{Q}_p} \to {\mathbb{Q}_p}^\times$ has Hodge-Tate weight $\{1\}$. We write ω for the cyclotomic character mod p. Since K_0 is unramified, we may canonically identify $\mathrm{Hom}(K_0, E) \cong \mathrm{Hom}(k, \mathbb{F})$. Let HT_j denote the set of Hodge-Tate weights of a representation of G_{K_0} with respect to embedding τ_j. Then our convention means that a character $\psi : G_{K_0} \to E^\times$ is Hodge-Tate with $\mathrm{HT}_j(\psi) = \{a_j\}$ iff ψ agrees with the character $\prod_{j=0}^{f-1}\left(\tau_j \circ \mathrm{Art}_{K_0}^{-1}\right)^{a_j} : I_{K_0} \to E^\times$ on an open neighbourhood of the identity in I_{K_0}, for $a_j \in \mathbb{Z}$. We will be interested in representations having Hodge type $\mu := (2,1,0)_{j=0}^{f-1}$.

The notation nr_λ stands for the unramified character $G_{K_0} \to \mathbb{F}^\times$ taking any geometric Frobenius to $\lambda \in \mathbb{F}^\times$. The symbol δ_* is the Kronecker delta $\in \{0,1\}$.

When dealing with the algebraic group GL_3 we always identify its Weyl group with S_3 the group of permutations of $\{0,1,2\}$ and we embed S_3 inside GL_3 by identifying $\varsigma \in S_3$ with the matrix whose (i,j)-th entry is $\delta_{i=\varsigma(j)}$. Then we have

$$\varsigma \mathrm{Diag}(d_0, d_1, d_2)\varsigma^{-1} = \mathrm{Diag}(d_{\varsigma^{-1}(0)}, d_{\varsigma^{-1}(1)}, d_{\varsigma^{-1}(2)}).$$

We write T_3 and B_3 for the standard diagonal torus and upper-triangular Borel in GL_3, respectively. We identify the character group $X^*(T_3)$ with $\mathbb{Z}^3$ in the usual way and define $\eta = (1,0,-1)$. If v is a weight vector in some algebraic representation, we write $\mathrm{wt}(v)$ for its weight.

If M is a module then $\mathrm{soc}(M)$ and $\mathrm{rad}(M)$ denote the socle and radical of M, respectively. The socle series of M is defined by setting $\mathrm{soc}^i(M) = \mathrm{soc}\left(M/\mathrm{soc}^{i-1}(M)\right)$ inductively, where $\mathrm{soc}^1(M) = \mathrm{soc}(M)$. The cosocle of M is $\mathrm{cosoc}(M) = M/\mathrm{rad}(M)$. We often draw lattices to indicate the submodule structure of a module. It is understood that these are Alperin diagrams [Alp80] with the socle at the bottom and the cosocle at the top.

If $J \subseteq \{0, \dots, f-1\}$ we define ∂J "the boundary of J" to be

$$\partial J = \{j \mid (j \in J \,\text{and}\, j+1 \notin J) \,\text{or}\, (j \notin J \,\text{and}\, j+1 \in J)\}.$$

Note that $|\partial J|$ is always even. We write J^c for the complement $\{0, \dots, f-1\} \setminus J$.

All other notation will be introduced in the body of the thesis.

Chapter 1

Integral p-adic Hodge theory

In this chapter we carry out the calculations on the local Galois side. We begin by specifying our assumptions about the local mod p Galois representation, which throughout the chapter is called $\bar{\rho}$ rather than $\bar{r}_{G_{F_w}}$, and defining the Fontaine-Laffaille invariants mentioned in the Introduction. This comprises Sections 1.1 and 1.2. The main result of this chapter, Theorem 1.7.1, shows how to recover these invariants as the reduction mod p of Frobenius eigenvalues on the weakly admissible filtered φ-module associated to certain well-chosen potentially crystalline lifts of $\bar{\rho}$. In order to establish this result we make use of various categories of integral p-adic Hodge theoretic data that describe lattices inside potentially crystalline representations, so as to make the link between characteristic 0 and p. Our argument blends the approach of [HLM17] using strongly divisible modules with the theory developed in [LLHLM18a], using Kisin modules and the notion of shape. Thus we spend some time explaining the relationships between the categories used in the two approaches in Sections 1.3-1.5, starting with an overview of the argument in Section 1.3. The main calculations and results are then concentrated in Sections 1.6 and 1.7.

1.1 The local mod p Galois representation

This section establishes our basic assumptions and notation concerning the local mod p Galois representation.

Let $\bar{\rho} : G_{K_0} \to \mathrm{GL}_3(\mathbb{F})$ be continuous. We assume that it is maximally nonsplit, meaning that it is uniserial and the graded pieces of its socle filtration are 1-dimensional. Then it may be expressed as

$$\bar{\rho} \sim \begin{pmatrix} \omega_f^{\sum_{j=0}^{f-1}(a^2_{f-j}+1)p^j}\mathrm{nr}_{\mu_2} & *_{21} & * \\ & \omega_f^{\sum_{j=0}^{f-1}(a^1_{f-j}+1)p^j}\mathrm{nr}_{\mu_1} & *_{10} \\ & & \omega_f^{\sum_{j=0}^{f-1}(a^0_{f-j}+1)p^j}\mathrm{nr}_{\mu_0} \end{pmatrix} \tag{1.1.1}$$

where $a_j^i \in \mathbb{Z}, \mu_i \in \mathbb{F}^\times$, and the extension classes $*_{21}$ and $*_{10}$ are nonzero. An essential assumption that will be made throughout this thesis is that $\bar{\rho}$ is Fontaine-Laffaille and δ-generic:

Definition 1.1.2. Let $\delta \geq 0$ be an integer. Let $\bar{\rho} : G_{K_0} \to \mathrm{GL}_3(\mathbb{F})$ be a continuous maximally nonsplit representation as in (1.1.1). We say that $\bar{\rho}$ satisfies (FL_δ), or alternatively $\bar{\rho} \in (\mathrm{FL}_\delta)$, if we can choose the a_j^i so that

$$a_j^2 - a_j^1, a_j^1 - a_j^0, a_j^2 - a_j^0 \in [\delta, p-1-\delta] \tag{1.1.3}$$

for each $0 \leq j \leq f-1$.

Definition 1.1.4. Let $\bar{\rho} \in (\mathrm{FL}_\delta)$ where $\delta \geq 1$. For a presentation of $\bar{\rho}$ as in Definition 1.1.2, we define $A_{00} = \sum_{j=0}^{f-1} (a_{f-j}^0 - 1)p^j$ and set $\bar{\rho}' := \bar{\rho} \otimes \omega_f^{-A_{00}}$. This depends only on $\bar{\rho}$.

Remark 1.1.5. The notation is chosen because $\bar{\rho} \in (\mathrm{FL}_\delta)$ guarantees that $\bar{\rho}'$ is in the essential image of the Fontaine-Laffaille functor if $\delta \geq 3$. See Proposition 1.2.7 below.

1.2 Fontaine-Laffaille modules and invariants

In this section we first recall relevant facts about certain categories of Fontaine-Laffaille modules and their relations with Galois representations. Then we define the Fontaine-Laffaille invariants of a Galois representation $\bar{\rho} \in (\mathrm{FL}_3)$.

Definition 1.2.1. The category $\mathrm{FL}_{\mathrm{fr}}^{[0,p-2]}(\mathcal{O}_E)$ consists of finite free $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$-modules M having the following structures:

- a decreasing filtration $\mathrm{Fil}^i(M)$ by $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$-submodules, each a $W(k)$-direct summand of M, such that $\mathrm{Fil}^0(M) = M$ and $\mathrm{Fil}^{p-1}(M) = 0$;
- ϕ-semilinear and $\mathcal{O}_E$-linear Frobenius maps $\varphi_i : \mathrm{Fil}^i(M) \to M$ obeying $\varphi_i|_{\mathrm{Fil}^{i+1}(M)} = p\varphi_{i+1}$ and $\sum_{i=0}^{p-2} \varphi_i(\mathrm{Fil}^i(M)) = M$.

Morphisms in $\mathrm{FL}_{\mathrm{fr}}^{[0,p-2]}(\mathcal{O}_E)$ are defined to be $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$-linear maps compatible with the filtrations and Frobenii.

The category $\mathrm{FL}^{[0,p-2]}(\mathbb{F})$ consists of finite free $k \otimes_{\mathbb{F}_p} \mathbb{F}$-modules M having the analogous structures: a filtration $\mathrm{Fil}^i(M)$ obeying $\mathrm{Fil}^0(M) = M$ and $\mathrm{Fil}^{p-1}(M) = 0$ and Frobenius maps $\varphi_i : \mathrm{Fil}^i(M) \to M$ obeying $\varphi_i|_{\mathrm{Fil}^{i+1}(M)} = 0$ and $M = \sum_{i=0}^{p-2} \varphi_i(\mathrm{Fil}^i(M))$.

There is an obvious functor $- \otimes \mathbb{F} : \mathrm{FL}_{\mathrm{fr}}^{[0,p-2]}(\mathcal{O}_E) \to \mathrm{FL}^{[0,p-2]}(\mathbb{F})$.

There are functors from these categories of Fontaine-Laffaille modules to categories of Galois representations making use of a period ring A_{cris}, which is a $W(\bar{k})$-algebra admitting a decreasing filtration by ideals $A_{\mathrm{cris}} = \mathrm{Fil}^0(A_{\mathrm{cris}}) \supseteq \mathrm{Fil}^1(A_{\mathrm{cris}}) \supseteq \cdots$, a $\phi_{W(\bar{k})}$-semilinear ring endomorphism φ, and a continuous G_{K_0}-action preserving $\mathrm{Fil}^i(A_{\mathrm{cris}})$ and commuting with φ. For $0 \leq i \leq p-2$ we have $\varphi(\mathrm{Fil}^i(A_{\mathrm{cris}})) \subseteq p^i A_{\mathrm{cris}}$ so we may define $\varphi_i = \varphi/p^i : \mathrm{Fil}^i(A_{\mathrm{cris}}) \to A_{\mathrm{cris}}$. See

[Hat18] for more details. We define contravariant functors $T^*_{\text{cris}} : \mathrm{FL}^{[0,p-2]}_{\text{fr}}(\mathcal{O}_E) \to \mathrm{Rep}_{G_{K_0}}(\mathcal{O}_E)$, where the latter denotes the category of finite free $\mathcal{O}_E$-modules with continuous G_{K_0}-action, by

$$T^*_{\text{cris}}(M) = \mathrm{Hom}_{W(k),\varphi_\bullet,\mathrm{Fil}^\bullet}(M, A_{\text{cris}})$$

and $T^*_{\text{cris}} : \mathrm{FL}^{[0,p-2]}(\mathbb{F}) \to \mathrm{Rep}_{G_{K_0}}(\mathbb{F})$ by

$$T^*_{\text{cris}}(M) = \mathrm{Hom}_{W(k),\varphi_\bullet,\mathrm{Fil}^\bullet}(M, A_{\text{cris}}/p).$$

An object M in $\mathrm{FL}^{[0,p-2]}_{\text{fr}}(\mathcal{O}_E)$ or $\mathrm{FL}^{[0,p-2]}(\mathbb{F})$ decomposes as a product $M = \prod_{j=0}^{f-1} M^j$, where $M^j = \varepsilon_j M$ as explained in the Notation section. Each M^j inherits a filtration by $\mathcal{O}_E$-submodules or $\mathbb{F}$-submodules and we define the set of Fontaine-Laffaille weights of M with respect to j to be the multiset of integers $\mathrm{FL}_j(M) = \{i : \mathrm{gr}^i(M^j) \neq 0\}$ where i is counted with multiplicity equal to $\mathrm{rk}_{\mathcal{O}_E}\mathrm{gr}^i(M^j)$ or $\dim_{\mathbb{F}} \mathrm{gr}^i(M^j)$, respectively. We remark that in the case of $\mathcal{O}_E$-coefficients, it is not hard to see from the definitions that $\mathrm{gr}^i M^j$ is finite free over $\mathcal{O}_E$, so this definition makes sense. Moreover, clearly $\mathrm{FL}_j(M) = \mathrm{FL}_j(M \otimes \mathbb{F})$. We say that M is of regular weight if $\mathrm{FL}_j(M)$ is multiplicity-free for each j.

Proposition 1.2.2. 1. *The categories $\mathrm{FL}^{[0,p-2]}_{\text{fr}}(\mathcal{O}_E)$ and $\mathrm{FL}^{[0,p-2]}(\mathbb{F})$ are abelian. The functors T^*_{cris} are exact and fully faithful and for $M \in \mathrm{FL}^{[0,p-2]}_{\text{fr}}(\mathcal{O}_E)$ we have $T^*_{\text{cris}}(M \otimes \mathbb{F}) = T^*_{\text{cris}}(M) \otimes \mathbb{F}$. Moreover, the functors T^*_{cris} preserve rank in the sense that $\mathrm{rk}_{W(k)\otimes_{\mathbb{Z}_p}\mathcal{O}_E} M = \mathrm{rk}_{\mathcal{O}_E} T^*_{\text{cris}}(M)$ for $M \in \mathrm{FL}^{[0,p-2]}_{\text{fr}}(\mathcal{O}_E)$ and $\mathrm{rk}_{k\otimes_{\mathbb{F}_p}\mathbb{F}} M = \dim_{\mathbb{F}} T^*_{\text{cris}}(M)$ for $M \in \mathrm{FL}^{[0,p-2]}(\mathbb{F})$.*

2. *T^*_{cris} defines an anti-equivalence of categories between $\mathrm{FL}^{[0,p-2]}_{\text{fr}}(\mathcal{O}_E)$ and the category of G_{K_0}-stable $\mathcal{O}_E$-lattices inside crystalline E-representations of G_{K_0} whose Hodge-Tate weights are all in $[0, p-2]$. If M corresponds to a lattice in the crystalline representation ρ, then for $\tau : K_0 \to E$,*
$$\mathrm{HT}_\tau(\rho) = \mathrm{FL}_\tau(M).$$

3. *The essential image of $T^*_{\text{cris}} : \mathrm{FL}^{[0,p-2]}(\mathbb{F}) \to \mathrm{Rep}_{G_{K_0}}(\mathbb{F})$ is closed under taking subquotients.*

4. *If each $a_j \in [0, p-2]$ let $M \in \mathrm{FL}^{[0,p-2]}(\mathbb{F})$ denote a rank 1 object such that $\mathrm{gr}^{a_j}(M^j) \neq 0$. Then $T^*_{\text{cris}}(M)|_{I_{K_0}} = \omega_f^{\sum_j a_{f-j}p^j}$.*

Proof. These facts are well-known but aren't easily available in the form stated above in the literature. For a source that treats $\mathcal{O}_E$- and $\mathbb{F}$-coefficients as we do in slightly more generality, see Section 2.2 of [GL14] and the references therein. The first three claims above may be deduced by following the proof of Theorem 2.2.1 of *loc. cit.*. We remark that the concurrence of Hodge-Tate weights with Fontaine-Laffaille weights uses our chosen normalization $\mathrm{HT}(\epsilon) = +1$; otherwise they would differ by a minus sign. For the fourth claim, see for example [FL82], Section 5. ∎

We make some general definitions concerning the representation of $\varphi_\bullet$ via matrices.

Definition 1.2.3. Let M be an object of $\mathrm{FL}^{[0,p-2]}(\mathbb{F})$ of rank n. A basis $\underline{e}$ of M is equivalent to a choice of $\mathbb{F}$-basis $\underline{e}^j = (e_0^j, \dots, e_{n-1}^j) = \varepsilon_j \underline{e}$ for each M^j. We say that $\underline{e}$ is *adapted to the filtration* if for each $0 \le j \le f-1$ we have

$$\mathrm{Fil}^k(M^j) = \sum_{i \ge i_k} \mathbb{F} e_i^j$$

for some integers $0 \le i_0 \le \cdots \le i_{p-2} \le n-1$.

Lemma 1.2.4. *There always exists a basis adapted to the filtration. If M is of regular weight then an adapted basis $\underline{e}$ is unique up to a transformation of the form $\underline{e}^j \mapsto \underline{e}^j L^j$ where $L^j \in \overline{B}_n(\mathbb{F})$ is lower-triangular.*

Proof. This is clear. □

Definition 1.2.5. Let $\underline{e}$ be a basis adapted to the filtration and assume M is of regular weight. We define $[\varphi_M]_{\underline{e}}^j$ to be the matrix such that $\varphi_k(e_{i_k}^j) = \underline{e}^{j+1} \cdot [\varphi_M]_{\underline{e}}^j$ for k such that $\mathrm{gr}^k(M^j) \neq 0$.

Then $[\varphi_M]_{\underline{e}}^j \in \mathrm{GL}_n(\mathbb{F})$ and for $L^j \in \overline{B}_n(\mathbb{F})$ we have

$$[\varphi_M]_{\underline{e}L}^j = \left(L^{j+1}\right)^{-1} \cdot [\varphi_M]_{\underline{e}}^j \cdot \pi(L^j) \tag{1.2.6}$$

where $\pi : \overline{B}_n(\mathbb{F}) \to T_n(\mathbb{F})$ is the natural projection.

If M is of regular weight then the isomorphism class of $M \in \mathrm{FL}^{[0,p-2]}(\mathbb{F})$ is completely determined by the sets $\mathrm{FL}_j(M)$ and matrices $[\varphi_M]_{\underline{e}}^j$ for $0 \le j \le f-1$.

1.2.1 Fontaine-Laffaille invariants

We now use the results of the previous section to define invariants for $\bar\rho \in (\mathrm{FL}_3)$.

Proposition 1.2.7. *Assume that $\bar\rho \in (\mathrm{FL}_3)$ and define a_i^j as in (1.1.1). Up to isomorphism there exists a unique object M of $\mathrm{FL}^{[0,p-2]}(\mathbb{F})$ such that $T^*_{\mathrm{cris}}(M) \cong \bar\rho'$. Let $0 = \bar\rho_3' \subset \bar\rho_2' \subset \bar\rho_1' \subset \bar\rho'$ denote the socle filtration of $\bar\rho'$. Then M has a unique filtration by subobjects*

$$0 \subset M_0 \subset M_1 \subset M_2 = M$$

such that $T^*_{\mathrm{cris}}(M_i) \cong \bar\rho'/\bar\rho'_{i+1}$ for $0 \le i \le 2$ and the corresponding elements of $\mathrm{Ext}^1(M_1/M_0, M_0)$ and $\mathrm{Ext}^1(M_2/M_1, M_1, M_0)$ are nonzero.

Proof. Because $\bar\rho \in (\mathrm{FL}_3)$ the hypotheses of [GG12] Lemma 3.1.5 are satisfied for $\bar\rho'$. Hence $\bar\rho'$ has an ordinary crystalline lift with Hodge-Tate weights $\mathrm{HT}_j = \{a_j^2 - a_j^0 + 2, a_j^1 - a_j^0 + 2, 2\}$. As $a_j^2 - a_j^0 + 2 \le p-2$ it follows from Proposition 1.2.2(2) that this lift is in the essential image of T^*_{cris} and by (1) that $\bar\rho'$ is too. The other statements now follow from full faithfulness and exactness of T^*_{cris} and the fact that $\bar\rho$ is nonsplit. □

Corollary 1.2.8. *There exists a basis $\underline{e}$ adapted to the filtration on M such that each $[\varphi_M]^j_{\underline{e}}$ is upper-triangular. This basis is unique up to a transformation of the form $\underline{e}^j \mapsto \underline{e}^j D^j$ where $D^j \in T_3(\mathbb{F})$. We have*

$$[\varphi_M]^j_{\underline{e}} = \begin{pmatrix} \alpha_j & x_j & y_j \\ & \beta_j & z_j \\ & & \gamma_j \end{pmatrix} \tag{1.2.9}$$

where $\alpha_j, \beta_j, \gamma_j \in \mathbb{F}^\times$ and $x_j, y_j, z_j \in \mathbb{F}$, and there exist j_0 and j_1 such that $x_{j_0} \neq 0$ and $z_{j_1} \neq 0$.

Proof. Continue with notation as in Proposition 1.2.7. We in particular have $\mathrm{FL}_j(M_i) = \{2, \ldots, a^i_j - a^0_j + 2\}$ and $T^*_{\mathrm{cris}}(M_i/M_{i-1}) \cong \omega_f^{\sum_{j=0}^{f-1} a^i_{f-j} p^j} \mathrm{nr}_{\mu_i}$ for $0 \leq i \leq 2$. Define $\underline{e}$ by letting $\underline{e}^j = (e^j_0, e^j_1, e^j_2)$ where e^j_i is a basis of the 1-dimensional space $\mathrm{Fil}^{a^i_j - a^0_j + 2}(M^j_i)$. Then $\underline{e}$ is a basis of M adapted to the filtration since $\mathrm{Fil}^k(M) \cap M_i = \mathrm{Fil}^k(M_i)$, and we have $M^j = \mathbb{F}e^j_0 + \cdots + \mathbb{F}e^j_i$ so clearly $[\varphi_M]^j_{\underline{e}}$ is upper-triangular. That is has the form in (1.2.9) follows from Proposition 1.2.2(4) and nonsplitness.

Conversely any basis $\underline{f}$ of M adapted to the filtration whose Frobenius matrices are upper triangular defines a filtration by subobjects $0 \subset N_0 \subset N_1 \subset N_2 = M$. There is a unique such filtration, so $\underline{f}^j = \underline{e}^j B^j$ for $B^j \in B_3(\mathbb{F})$. On the other hand, $\underline{f}$ and $\underline{e}$ are both adapted to the filtration so by Lemma 1.2.4 $\underline{f}^j = \underline{e}^j L^j$ for $L^j \in \overline{B}_3(\mathbb{F})$. Hence $B^j = L^j$ is diagonal. □

Remark 1.2.10. The reason we have normalized $\bar{\rho}'$ so that its Fontaine-Laffaille weights start at 2 rather than 0 is for technical reasons which become clear in (1.6.9) below.

Definition 1.2.11. We refer to a basis of M as in Corollary 1.2.8 as a *framed basis*.

We use framed bases to define our Fontaine-Laffaille invariants for $\bar{\rho}$. However, for our invariants to be well-defined elements of $\mathbb{F}^\times$, we require some extra hypotheses.

Definition 1.2.12. Suppose that $\bar{\rho}$ satisfies (FL_δ) with $\delta \geq 3$, and pick a framed basis $\underline{e}$ of M so (1.2.9) holds. We say that $\bar{\rho}$ satisfies (FL^*_δ) if

$$x_j, z_j \neq 0$$

for each $0 \leq j \leq f-1$. We say that $\bar{\rho}$ satisfies (FL^{**}_δ) if

$$x_j, z_j, y_j, y_j\beta_j - x_j z_j \neq 0$$

for each $0 \leq j \leq f-1$.

Remark 1.2.13.

1. It is easy to see using 1.2.6 that these conditions do not depend on the choice of the framed basis $\underline{e}$.

2. It is clear that most representations $\bar{\rho}$ satisfying (FL_δ) will satisfy (FL^{**}_δ), so making this assumption is not very restrictive.

3. In our global argument in Chapter 3, we will show that when $\bar{\rho} \in (\mathrm{FL}_\delta)$ with $\delta \geq 5$, $(\mathrm{FL}_\delta^{**})$ is a consequence of the more basic Assumption (SW) that the global Galois representation has a unique Serre weight. See Definition 3.5.3, Theorem 3.5.6 and Remark 3.5.5 for the precise statements and discussion.

Definition 1.2.14. Assume that $\bar{\rho} \in (\mathrm{FL}_\delta^{**})$ and pick a framed basis of M as in (1.2.9). Let $S \subseteq \{0, \dots, f-1\}$ be any subset. We define quantities

$$\begin{aligned} X_S &= \prod_{j \in S} \alpha_j \cdot \prod_{j+1 \notin S} \beta_j \cdot \prod_{j \in S, j+1 \notin S} x_j^{-1} \cdot \prod_{j \notin S, j+1 \in S} x_j \\ Z_S &= \prod_{j \in S} \beta_j \cdot \prod_{j+1 \notin S} \gamma_j \cdot \prod_{j \in S, j+1 \notin S} z_j^{-1} \cdot \prod_{j \notin S, j+1 \in S} z_j \\ Y_S &= \prod_{j \notin S} \beta_j \cdot \prod_{j \in S} \frac{y_j \beta_j - x_j z_j}{y_j} \end{aligned}$$

lying in $\mathbb{F}^\times$. It is not hard to show using the uniqueness in Corollary 1.2.8 that X_S, Z_S, Y_S do not depend on the choice of framed basis $\underline{e}$ for any S. We call them the *Fontaine-Laffaille invariants of* $\bar{\rho}$.

Together with the set of inertial exponents a_i^j of $\bar{\rho}$, these invariants (over)determine the isomorphism class of $\bar{\rho}$, as is easy to see. However, we will only be able to recover some of them. The next lemma shows that they are still enough to determine $\bar{\rho}$.

Lemma 1.2.15. *The isomorphism class of $\bar{\rho}$ is determined uniquely by the inertial exponents $(a_i^j)_{0 \le i \le 2}^{0 \le j \le f-1}$ together with the collection of invariants X_T, Z_T for any $T \subseteq \{0, \dots, f-1\}$ and Y_S where S is of the form*

$$S = \{j \,|\, (j \in \partial J \text{ and } j+1 \notin K) \text{ or } (j \notin \partial J \text{ and } j+1 \in K)\}$$

for some $J, K \subseteq \{0, \dots, f-1\}$ obeying $J \subseteq K$.

Proof. By making a diagonal change of basis we may assume that $\alpha_j, \beta_j, \gamma_j = 1$ for $0 \le j \le f-2$ and $x_0 = z_0 = 1$. Then $X_{\{0,\dots,f-1\}} = \alpha_{f-1}$, $Z_{\{0,\dots,f-1\}} = \beta_{f-1}$ and $Z_\emptyset = \gamma_{f-1}$. For $1 \le k \le f-2$ we have $X_{\{1,\dots,k\}} = \beta_{f-1} x_k^{-1}$ and $X_{\{1,\dots,f-1\}} = \alpha_{f-1}\beta_{f-1}x_{f-1}^{-1}$. From this and a similar argument using $Z_{\{1,\dots,k\}}$ we see that we may recover all $\alpha_j, \beta_j, \gamma_j, x_j, z_j$ from the invariants X_T and Z_T. It follows that knowing the Y_S is equivalent to knowing all products $\prod_{j \in S} \frac{y_j\beta_j - x_j z_j}{y_j}$. If we take $J = K = \{j_0\}$ then $S = \{j_0\}$ (consider $f = 1$ and $f \ge 2$ separately). So we can recover each individual $\frac{y_j\beta_j - x_j z_j}{y_j}$, and hence each y_j. Finally, we have already remarked earlier that the a_i^j as well as the $\alpha_j, \beta_j, \gamma_j, x_j, y_j, z_j$ constitute all the data of M and hence of $\bar{\rho}$. $\square$

1.3 Summary of categories used

Our calculations make use of a number of categories of p-adic Hodge theoretic data. Figure 1.3.1 below depicts these categories and their interrelations.

$$
\begin{tikzcd}
 & Y^{\mu,\tau^{-1}}(\mathcal{O}_E) \arrow[r, "-\otimes\mathbb{F}"] \arrow[d, hook, "\mathrm{incl}"] & Y^{\mu,\tau^{-1}}(\mathbb{F}) \arrow[d, hook, "\mathrm{incl}"] & \\
\mathrm{S.D.Mod}^{[0,r],\mathrm{dd}}(\mathcal{O}_E)_0 & Y^{[0,r],\mathrm{dd}}(\mathcal{O}_E) \arrow[l, "\Theta_h", "\sim"'] \arrow[r, "-\otimes\mathbb{F}"] \arrow[d] & Y^{[0,r],\mathrm{dd}}(\mathbb{F}) \arrow[d] & \\
 & \varphi\mathrm{Mod}^{\text{ét}}_{K_0}(\mathcal{O}_E) \arrow[r, "-\otimes\mathbb{F}"] \arrow[d, "V^*"', "\wr"] & \varphi\mathrm{Mod}^{\text{ét}}_{K_0}(\mathbb{F}) \arrow[d, "V^*"', "\wr"] & \\
 & \mathrm{Rep}_{G_{(K_0)_\infty}}(\mathcal{O}_E) \arrow[r, "-\otimes\mathbb{F}"] & \mathrm{Rep}_{G_{(K_0)_\infty}}(\mathbb{F}) & \\
\mathrm{Rep}^{K\text{-sst}}_{G_{K_0}}(E) & \mathrm{Rep}^{K\text{-sst}}_{G_{K_0}}(\mathcal{O}_E) \arrow[l, "-\otimes E"'] \arrow[u, "\mathrm{res}"'] \arrow[r, "-\otimes\mathbb{F}"] & \mathrm{Rep}_{G_{K_0}}(\mathbb{F}) \arrow[u, "\mathrm{res}"'] & \mathrm{FL}^{[0,p-2]}(\mathbb{F}) \arrow[l, "T^*_{\mathrm{cris}}"] \arrow[uul, "\mathcal{F}"'] \\
\mathrm{Mod}(\varphi,N)^{\mathrm{dd}}(E) \arrow[u, "T^*_{\mathrm{st}}"'] & \mathrm{S.D.Mod}^{[0,r],\mathrm{dd}}(\mathcal{O}_E) \arrow[l, "\mathcal{S}_0"'] \arrow[u, "T^*_{\mathrm{st}}"'] \arrow[r, "\overline{\mathcal{S}}"] \arrow[uuuul, bend left, "(-)_0"] & \mathrm{BrMod}^{[0,r],\mathrm{dd}}(\mathbb{F}) \arrow[u, "T^*_{\mathrm{st}}"'] \arrow[uuuu, bend right, "\mathcal{M}"'] &
\end{tikzcd}
$$

Figure 1.3.1: Categories of p-adic Hodge theoretic data and functors between them. Hooked arrows denotes fully faithful functors and tildes denote (anti)equivalences of categories. The diagram is organized so that the ring of coefficients changes from E to $\mathcal{O}_E$ to $\mathbb{F}$ as one moves from left to right, and so the bottom two rows relate to representations of G_{K_0} and the top four rows relate to representations of $G_{(K_0)_\infty}$. All descent data (dd) is from K to K_0.

We explain a few of these categories now: $\mathrm{Rep}_{G_{K_0}}(\mathbb{F})$ (resp $\mathrm{Rep}_{G_{(K_0)_\infty}}(\mathbb{F})$ means continuous representations of G_{K_0} (resp. $G_{(K_0)_\infty}$) on finite-dimensional $\mathbb{F}$-vector spaces, $\mathrm{Rep}^{K\text{-sst}}_{G_{K_0}}(E)$ denotes continuous representations of G_{K_0} on finite dimensional E-vector spaces becoming semistable when restricted to G_K, and $\mathrm{Rep}^{K\text{-sst}}_{G_{K_0}}(\mathcal{O}_E)$ is the category of G_{K_0}-stable $\mathcal{O}_E$ lattices inside them. Finally $\mathrm{Rep}_{G_{(K_0)_\infty}}(\mathcal{O}_E)$ is the category of continuous representations of $G_{(K_0)_\infty}$ on finite free $\mathcal{O}_E$-modules. The remaining categories and functors will be defined in Sections 1.4 and 1.5.

We give a brief overview of the argument that establishes the main result Theorem 1.7.1. The goal is to relate data (Frobenius eigenvalues) in the weakly admissible module in $\mathrm{Mod}(\varphi,N)^{\mathrm{dd}}(E)$ associated with a potentially crystalline lift ρ of $\bar{\rho}$ becoming crystalline over K and having Hodge type $\mu = (2,1,0)_{j=0}^{f-1}$ and (a certain) inertial type τ. This lift corresponds to a strongly divisible module $\mathfrak{M} \in \mathrm{S.D.Mod}^{[0,2],\mathrm{dd}}$. It follows from Theorem 1.5.13 that the corresponding object M in $Y^{[0,2],\mathrm{dd}}(\mathcal{O}_E)$ lies in the subcategory $Y^{\mu,\tau^{-1}}(\mathcal{O}_E)$. Equivalence classes of Kisin modules in $Y^{\mu,\tau^{-1}}(\mathbb{F})$ are classified in [LLHLM18a] by their shape. Accordingly we need to compute the shape of $M \otimes \mathbb{F}$. This is achieved in Section 1.6 using the Breuil module $\overline{\mathcal{S}}(\mathfrak{M})$ and the functor

$\mathcal{M}$. Once the shape of $M \otimes \mathbb{F}$ is known, there is also a canonical form of M determined in [LLHLM18a]. Using this, the desired result is obtained by tracing through the diagram back to $\mathrm{Mod}(\varphi, N)^{\mathrm{dd}}(E)$ as well as in the other direction to $\mathrm{FL}^{[0,p-2]}(\mathbb{F})$ using the functor $\mathcal{F}$.

We remark that in many cases we actually use variants of the functors in Figure 1.3.1 having the opposite covariance, in order to most easily use existing results.

1.4 Lattices inside potentially semistable G_{K_0}-representations

In this section we recall three categories that describe potentially semistable representations of G_{K_0}, the integral lattices in these representations, and their reductions mod p respectively. These are the categories of weakly admissible filtered isocrystals with monodromy, of strongly divisible modules, and of Breuil modules, all with descent data and coefficients. Much of this material is taken from [EGH13], to which we refer the reader for more details.

1.4.1 Filtered isocrystals with monodromy and descent data

Let L be a finite extension of $\mathbb{Q}_p$ with maximal unramified subextension L_0. Let L' be a subextension of $L/\mathbb{Q}_p$ such that L/L' is Galois.

Definition 1.4.1. A *filtered isocrystal with monodromy, descent data from L to L′, and E-coefficients*, alternatively referred to as a filtered $(\varphi, N, L/L', E)$-module, is a finite free $L_0 \otimes_{\mathbb{Q}_p} E$-module D together with

- a $\phi \otimes 1$-semilinear bijection $\varphi_D : D \to D$;
- a nilpotent $L_0 \otimes_{\mathbb{Q}_p} E$-linear endomorphism N obeying $N\varphi = p\varphi N$;
- an exhausted and separated decreasing filtration $(\mathrm{Fil}^i D_L)_{i\in\mathbb{Z}}$ on D_L by $L\otimes_{\mathbb{Q}_p} E$-submodules;
- a L_0-semilinear and E-linear action of $\mathrm{Gal}(L/L')$ commuting with φ and N and preserving the filtration on D_L.

We denote this category by $\mathrm{Mod}(\varphi, N, L/L', E)$. An object D is called *weakly admissible* if the underlying $(\varphi, N, L/L, \mathbb{Q}_p)$-module is weakly admissible. The corresponding abelian category of weakly admissible objects is denoted $\mathrm{Mod}^{\mathrm{w.a.}}(\varphi, N, L/L', E)$.

Let $\mathrm{Rep}^{L\text{-sst}}_{G_{L'}}(E)$ denote the category of continuous E-representations of $G_{L'}$ that become semistable upon restriction to G_L. We refer the reader to Section 3.1 of [EGH13] for details about the antiequivalence of abelian tensor categories $\mathrm{Rep}^{L-\mathrm{st}}_{G_{L'}}(E) \xrightarrow{\sim} \mathrm{Mod}^{\mathrm{w.a.}}(\varphi, N, L/L', E)$ defined by

$$D^{*,L'}_{\mathrm{st}}(\rho) = \mathrm{Hom}_{G_{L'}}(\rho, B_{\mathrm{st}})$$

and its inverse functor $T^{*,L'}_{\mathrm{st}}$. For an integer r we also define a covariant version of this equivalence by

$$D^{L',r}_{\mathrm{st}}(\rho) = D^{*,L'}_{\mathrm{st}}(\rho^\vee \otimes \epsilon^r) \cong D^{*,L'}_{\mathrm{st}}(\rho)^\vee(r) \tag{1.4.2}$$

Here we are referring to the duality $D \mapsto D^\vee$ and Tate twist $D \mapsto D(r)$ on the category $\mathrm{Mod}(\varphi, N, L/L', E)$, which are defined as follows. We set $D^\vee = \mathrm{Hom}_{L_0}(D, L_0)$ with Frobenius $\varphi_{D^\vee}(f) = \mathrm{Frob}_{L_0} \circ f \circ \varphi_D^{-1}$, the natural induced mondromy and filtration, and Galois action $g \cdot f = g \circ f \circ g^{-1}$ for $g \in \mathrm{Gal}(L/L'), f \in D^\vee$. The Tate twist $D(r)$ is just D with the same monodromy operator and Galois action but $\varphi_{D(r)} = p^r \varphi_D$ and $\mathrm{Fil}^i(D(r)_L) = \mathrm{Fil}^{i-r}(D_L)$. For all this and the isomorphism in (1.4.2), see for example Section 8.3 of [BC][1].

Remark 1.4.3. Here and throughout this section, we use the notation of [HLM17] rather than that of [EGH13]. For example, the functor $D_{\mathrm{st}}^{*,L'}$ above is denoted $D_{\mathrm{st}}^{L'}$ in [EGH13]. The same is true of the functors T_{st}^* used below.

We will only be interested in the niveau 1 tamely ramified case $L = K$, $L' = K_0$, $\mathrm{Gal}(L/L') = \Delta$. In this case, since $\varphi^j : D^j \to D^{j+1}$ is an $E[\Delta]$-linear isomorphism, there exists a principal series type $\tau = \bigoplus_{i=0}^{n-1} \widetilde{\omega}_f^{A_i}$ such that D^j has an E-basis $\underline{e}^j = (e_0^j, \ldots, e_{n-1}^j)$ with Δ acting on e_i^j by $\widetilde{\omega}_f^{A_i}$.

Definition 1.4.4. In the above situation we say that D is of *type* τ and that $\underline{e}$ is a *framed basis*. Given a choice of framed basis $\underline{e}$ (and its concomitant implicit ordering of τ) we define $\mathrm{Mat}_{\underline{e}}(\varphi)^j \in \mathrm{GL}_n(E)$ to be the matrix such that $\varphi(\underline{e}^j) = \underline{e}^{j+1} \cdot \mathrm{Mat}_{\underline{e}}(\varphi)^j$.

Lemma 1.4.5. *If D has a framed basis $\underline{e}$ of type τ then $D^\vee(r)$ has a framed basis $\underline{e}^\vee$ of type τ^{-1} such that*

$$\mathrm{Mat}_{\underline{e}^\vee}(\varphi_{D^\vee})^j = p^r \cdot {}^t\,\mathrm{Mat}_{\underline{e}}(\varphi_D)^{-1}.$$

Proof. This follows easily from the description of the dual and Tate twist above. □

1.4.2 Strongly divisible modules

Let L and L' be as in the previous subsection and assume from now on that L/L' is tamely ramified with ramification index $e(L/L')$. Let e denote the absolute ramification index of L and write ℓ for the residue field of L. Choose a uniformizer π of L such that $\pi^{e(L/L')} \in L'$ (which can be done since the ramification is tame) and let $E(u) \in L'[u]$ denote its minimal polynomial. For $g \in \mathrm{Gal}(L/L')$ write $g(\pi) = h_g \pi$ where $h_g \in W(\ell)$. We define the Breuil ring S as in Section 3.1 of [EGH13]. It is the p-adic completion of $W(\ell)[u, u^{ie}/i!]_{i\geq 1}$, or more concretely the $W(\ell)$-subalgebra of $L_0[[u]]$ given by

$$S = \left\{ \sum_{i=0}^{\infty} w_i \frac{u^{ie}}{i!} \,\middle|\, w_i \to 0 \text{ in } W(\ell) \right\}.$$

It is $W(\ell)$-flat and we can give it the following structures: the unique continuous ϕ-semilinear map φ_S taking u to u^p and $u^{ie}/i!$ to $u^{ipe}/i!$, a $W(\ell)$-linear monodromy map N_S which for us will be irrelevant, and a decreasing filtration by ideals (depending on the choice of π) $\{\mathrm{Fil}^i(S)\}_{i\geq 0}$

[1]Note the definition of the Tate twist given in Definition 8.3.1 is backwards if we want it to commute with contravariant functors as stated there.

where $\mathrm{Fil}^i(S)$ is defined to be the closure of the ideal generated by $E(u)^j/j!$ for all $j \geq i$ (it is easy to check that $E(u)^j/j!$ does in fact lie in S). If $i \leq p-1$ then $\varphi(\mathrm{Fil}^i(S)) \subseteq p^i S$ and we write φ_i for $\varphi/p^i : \mathrm{Fil}^i(S) \to S$. For $0 \leq i \leq p-2$ we define $c_i = \varphi_r(E(u)^i) \in S^\times$. It is easy to check that $c_i \equiv 1 \mod p$. There is an action of $\mathrm{Gal}(L/L')$ by continuous ring automorphisms $\hat{g}$ defined by $\hat{g}(w\frac{u^i}{[i/e]!}) = g(w)h_g^i\frac{u^i}{[i/e]!}$ for any $i \geq 0$ and $w \in W(\ell)$. It is helpful to observe that S is not noetherian but it is local, with maximal ideal $\mathfrak{m}_S = (p, \mathrm{Fil}^1(S)) = (p, u, \mathrm{Fil}^p(S))$ and residue field ℓ. We will also use that $S/\mathrm{Fil}^p(S)$ is p-torsion free below.

We write $S_{\mathcal{O}_E}$ for $S \otimes_{\mathbb{Z}_p} \mathcal{O}_E$. The data of $\varphi, \varphi_i, N, \mathrm{Fil}^i$ and the action of $\mathrm{Gal}(L/L')$ all extend $\mathcal{O}_E$-linearly to this ring.

Definition 1.4.6. Fix an integer $0 \leq h \leq p-2$. The category $\mathrm{S.D.Mod}^{[0,r],\mathrm{dd}}(\mathcal{O}_E)$ of *strongly divisible modules of height $\leq r$ with $\mathcal{O}_E$-coefficients and descent data from L to L'* is defined on page 13 of [EGH13]. In particular an object $\mathfrak{M}$ is a finite free $S_{\mathcal{O}_E}$-module together with a submodule $\mathfrak{M}_r$, a $S_{\mathcal{O}_E}$-semilinear Frobenius map $\varphi : \mathfrak{M} \to \mathfrak{M}$, an $S_{\mathcal{O}_E}$-linear monodromy map $N : \mathfrak{M} \to \mathfrak{M}$, and a semilinear action of $\mathrm{Gal}(L/L')$ that commutes with φ and N and preserves $\mathfrak{M}_r$. Moreover, $\mathfrak{M}$ must obey

- $\mathfrak{M}_r \supseteq (\mathrm{Fil}^r(S))\mathfrak{M}$;
- $\mathfrak{M}_r \cap \mathfrak{j}\mathfrak{M} = \mathfrak{j}\mathfrak{M}_r$ for each ideal $\mathfrak{j} \leq \mathcal{O}_E$;
- $\varphi(\mathfrak{M}_r)$ is contained in $p^r\mathfrak{M}$ and generates it over $S_{\mathcal{O}_E}$;
- axioms involving N, which for us are irrelevant.

The category $\mathrm{S.D.Mod}^{[0,r],\mathrm{dd}}(\mathcal{O}_E)_0$ of *quasi-strongly divisible modules of height $\leq r$ with $\mathcal{O}_E$-coefficients and descent data from L to L'* is defined as above but we forget all data involving the monodromy operator N (thus the axioms above constitute its complete definition). We write $\mathfrak{M} \mapsto \mathfrak{M}_0$ for the natural forgetful functor.

If $\mathfrak{M}$ is a (quasi-)strongly divisible module, we write φ_r for $\varphi/p^r : \mathfrak{M}_r \to \mathfrak{M}$.

There is a covariant functor $T^*_{\mathrm{st}} : \mathrm{S.D.Mod}^{[0,r],\mathrm{dd}}(\mathcal{O}_E) \to \mathrm{Rep}_{G_{L'}}(\mathcal{O}_E)$ defined as $T^*_{\mathrm{st}}(\mathfrak{M}) = \mathrm{Hom}_{S,(-)_r,\varphi,N}(\mathfrak{M}, \hat{A}_{\mathrm{st}})$ for a certain period ring $\hat{A}_{\mathrm{st}}$ defined in [EGH13] and [Bre99]. This functor induces an equivalence of categories between $\mathrm{S.D.Mod}^{[0,r],\mathrm{dd}}(\mathcal{O}_E)$ and the category of $G_{L'}$-stable $\mathcal{O}_E$-lattices inside potentially semistable E-representations of $G_{L'}$ which become semistable when restricted to G_L and have Hodge-Tate weights in $[0,r]$ (Proposition 3.1.4, [EGH13]). We will use the contravariant version $T^{L',r}_{\mathrm{st}}(\mathfrak{M}) := T^{*,L'}_{\mathrm{st}}(\mathfrak{M})^\vee \otimes \epsilon^r$, which clearly has the same property.

We now specialize to our tamely ramified niveau 1 case $L = K$, $L' = K_0$, $\pi = -p$. There is an $\mathcal{O}_E$-algebra decomposition $S_{\mathcal{O}_E} = \prod_j S^j$ where $S^j = \varepsilon_j S = \mathcal{O}_E[u, u^{ie}/i!]^\wedge_{i\geq 1}$, so $\varphi_{S_{\mathcal{O}_E}} : S^j \to S^{j+1}$ is the $\mathcal{O}_E$-linear map taking $\frac{u^{ie}}{[i/e]!}$ to $\frac{u^{ipe}}{[i/e]!}$. Each S^j is a local $\mathcal{O}_E$-algebra, with maximal ideal $\mathfrak{m}_{S^j} = (\varpi_E, \mathrm{Fil}^1(S^j))$ and residue field $\mathbb{F}$. Accordingly given a (quasi-)strongly divisible module $\mathfrak{M}$ we write $\mathfrak{M} = \prod_j \mathfrak{M}^j$, $\mathfrak{M}_r = \prod_j \mathfrak{M}^j_r$ and $\varphi_r : \mathfrak{M}^j_r \to \mathfrak{M}^{j+1}_r$.

Definition 1.4.7. Let $\tau = \bigoplus_{i=0}^{n-1} \widetilde{\omega}_f^{A_i}$ be a principal series type. We say that a (quasi-)strongly divisible module $\mathfrak{M}$ is *of type* τ if it is of rank n and there exist S^j-bases $\underline{e}^j$ of $\mathfrak{M}^j$ such that Δ acts on e_i^j via $\widetilde{\omega}_f^{A_i}$ for $0 \le i \le n-1$ and $0 \le j \le f-1$. We refer to any basis with this property as a *framed basis* of $\mathfrak{M}$.

Lemma 1.4.8. *Let $\tau = \bigoplus_{i=0}^{n-1} \widetilde{\omega}_f^{A_i}$ be a principal series type and let $\mathfrak{M}$ be a (quasi-)strongly divisible module. The following are equivalent:*

1. *$\mathfrak{M}$ has type τ;*
2. *$\mathfrak{M}^j/\mathfrak{m}_{S^j} \cong \bar{\tau}$ as an $\mathbb{F}[\Delta]$-module for each $0 \le j \le f-1$;*
3. *for each $0 \le j \le f-1$ there exists a tuple $\underline{f}^j = (f_0^j, \dots, f_{n-1}^j)$ of elements of $\mathfrak{M}_r^j$ which generate $\mathfrak{M}_r^j/(\mathrm{Fil}^p(S^j))\mathfrak{M}^j$ over S^j such that Δ acts on f_i^j by $\widetilde{\omega}_f^{A_i}$.*

Proof. It is obvious that (1) implies (2), so assume that (2) holds. Set $\mathfrak{N}^j = \mathfrak{M}_r^j/(\mathrm{Fil}^p(S^j))\mathfrak{M}^j$. This is a submodule of $\mathfrak{M}^j/\mathrm{Fil}^p(S^j)$, hence is free over $\mathcal{O}_E$. By the second axiom in Definition 1.4.6, we have an inclusion of $\mathbb{F}[u]/(u^{ep})$-modules $\mathfrak{N}^j/p \hookrightarrow \mathfrak{M}^j/(p, \mathrm{Fil}^p(S^j))$. Since $u^{er}\mathfrak{M}^j/(p, \mathrm{Fil}^p(S^j)) \subseteq \mathfrak{N}^j/p$, Lemme 2.2.1.1 (which is stated for $e=1$ but generalizes easily) of [Bre98] applies and we obtain

$$\dim_{\mathbb{F}} \mathfrak{N}^j/(p,u) = \dim_{\mathbb{F}} \mathfrak{M}^j/\mathfrak{m}_{S^j} = \dim_{\mathbb{F}} \mathfrak{M}^{j+1}/\mathfrak{m}_{S^{j+1}}.$$

But φ_r induces an $\mathbb{F}[\Delta]$-linear surjection

$$\mathfrak{N}^j/(p,u) \twoheadrightarrow \mathfrak{M}^{j+1}/\mathfrak{m}_{S^{j+1}}, \tag{1.4.9}$$

which must therefore be an isomorphism. Now $\mathfrak{N}^j/u$ is a free $\mathcal{O}_E[\Delta]$-module whose reduction mod p is $\bar{\tau}$ for each $0 \le j \le f-1$. It follows immediately that $\mathfrak{N}^j/u = \tau$. Moreover, as $u\mathfrak{N}^j$ is an $\mathcal{O}_E[\Delta]$-submodule of $\mathfrak{N}^j$ such that the corresponding quotient is free over $\mathcal{O}_E$, and $p \nmid |\Delta|$, it follows that $u\mathfrak{N}^j$ has an $\mathcal{O}_E[\Delta]$-complement in $\mathfrak{N}^j$. Finally by Nakayama's lemma we see that there exists a generating set of $\mathfrak{N}^j$ of the desired form. Applying the complement argument again, we deduce the existence of a set $\underline{f}^j$ as in (3).

Finally, assume (3). We run the argument above backwards: from the isomorphism (1.4.9) we get $\mathfrak{M}^j/\mathfrak{m}_{S^j} = \bar{\tau}$ for each $0 \le j \le f-1$, hence $\mathfrak{M}^j/(u, \mathrm{Fil}^p(S^j)) = \tau$. Now apply the same argument about a complement and Nakayama's lemma to see that $\mathfrak{M}^j$ has a generating set $\underline{e}^j$ such that Δ acts on e_i^j by $\widetilde{\omega}_f^{A_i}$. As $\mathfrak{M}^j$ is free it follows that $\underline{e}^j$ is a basis as in (1). □

The next corollary to the proof will be used later.

Lemma 1.4.10. *Let $\mathfrak{M}$ be a strongly divisible module of rank n. A tuple $\underline{f}^j = (f_0^j, \dots, f_{n-1}^j) \subset \mathfrak{M}_r^j$ generates $\mathfrak{M}_r^j/(\mathrm{Fil}^p(S^j))\mathfrak{M}^j$ iff $\varphi_r(\underline{f}^j)$ is an S^{j+1}-basis of $\mathfrak{M}^{j+1}$.*

Proof. We use the notation of the proof of the previous lemma. By Nakayama's lemma the isomorphism (1.4.9) shows that $\varphi_r(\underline{f}^j)$ is an S^{j+1}-basis of $\mathfrak{M}^{j+1}$ iff $\underline{f}^j$ is an $\mathbb{F}$-basis of $\mathfrak{N}^j/(p,u)$. But by Nakayama's lemma again this is equivalent to being a generating set of $\mathfrak{N}^j$. $\square$

Definition 1.4.11. Let $\mathfrak{M}$ be an object of $\mathrm{S.D.Mod}^{[0,h],\mathrm{dd}}_{K/K_0}(\mathcal{O}_E)_0$ of type τ. Let $\underline{e}$ be a framed basis. Then there exists a tuple $\underline{f}$ as in Lemma 1.4.8(3). We refer to any such tuple as a *framed system of generators of* $\mathfrak{M}^j_r/(\mathrm{Fil}^p(S^j))\mathfrak{M}^j$. We also refer to the pair $(\underline{e},\underline{f})$ as simply a *framed system of generators* of $\mathfrak{M}$. We define $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathfrak{M}_r)^j$ to be the matrix in $\mathrm{Mat}_n(S^j)$ such that

$$\underline{f}^j = \underline{e}^j \cdot \mathrm{Mat}_{\underline{e},\underline{f}}(\mathfrak{M}_r)^j$$

and $\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r)^j$ to be the matrix in $\mathrm{GL}_n(S^{j+1})$ such that

$$\varphi_r(\underline{f}^j) = \underline{e}^{j+1} \cdot \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r)^j.$$

Remark 1.4.12. In fact, we always have $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathfrak{M}_r)^j \in \mathrm{GL}_n(S^j[1/p])$. To see this, note that by definition we have

$$\begin{aligned}\underline{e}^{j+1} \cdot \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r)^j &= \varphi_r(\underline{e}^j \cdot \mathrm{Mat}_{\underline{e},\underline{f}}(\mathfrak{M}_r)^j) \\ &= \varphi(\underline{e}^j) \cdot p^r \varphi_{S_{\mathcal{O}_E}}\left(\mathrm{Mat}_{\underline{e},\underline{f}}(\mathfrak{M}_r)^j\right)\end{aligned}$$

Writing $\varphi(\underline{e}^j) = \underline{e}^{j+1}B^j$ for some $B^j \in \mathrm{Mat}_n(S^{j+1})$ we deduce that $\varphi_{S_{\mathcal{O}_E}}\left(\mathrm{Mat}_{\underline{e},\underline{f}}(\mathfrak{M}_r)^j\right) \in \mathrm{GL}_n(S^{j+1})$. The claim now follows from the fact that S^j is local.

We now explain the relationship between strongly divisible modules and filtered isocrystals. Let $s_0 : S[1/p] \to K_0$ denote the Galois and Frobenius equivariant map taking $u \mapsto 0$. Following [EGH13] we define a functor

$$\mathcal{S}_0 : \mathrm{S.D.Mod}^{[0,r],\mathrm{dd}}(\mathcal{O}_E) \to \mathrm{Mod}^{\mathrm{w.a.}}(\varphi, N, K/K_0, E)$$

by setting $\mathcal{S}_0(\mathfrak{M}) = D$ where $D = \mathfrak{M}[1/p] \otimes_{S[1/p],s_0} K_0$ with φ_D and the $\mathrm{Gal}(K/K_0)$-action defined as the semilinear extensions of the corresponding structures on $\mathfrak{M}$, N_D defined as the linear extension of $N_{\mathfrak{M}}$, and the filtration on D_K defined via the recipe in the proof of Proposition 3.1.4 of [EGH13]. See *loc. cit.* for more details. It also shown in the proof in question that $T^*_{\mathrm{st}}(\mathfrak{M})$ is naturally a G_{K_0}-stable $\mathcal{O}_E$-lattice in $T^*_{\mathrm{st}}(D)$. Hence we have a natural isomorphism of functors

$$T^*_{\mathrm{st}}(\mathcal{S}_0(\mathfrak{M})) \cong T^*_{\mathrm{st}}(\mathfrak{M}) \otimes_{\mathcal{O}_E} E. \tag{1.4.13}$$

Lemma 1.4.14. *Suppose that ρ is a G_{K_0}-stable $\mathcal{O}_E$-lattice inside a potentially semistable representation of G_{K_0} having Hodge-Tate weights inside $[0,r]$ that becomes semistable when restricted to G_K. Let $\mathfrak{M}$ be a strongly divisible module of type τ^{-1} such that $T^*_{\mathrm{st}}(\mathfrak{M}) = \rho$ and let $(\underline{e},\underline{f})$ be a framed system of generators of $\mathfrak{M}$. Then $D^{K_0,r}_{\mathrm{st}}(\rho)$ is of type τ and has a framed basis $\underline{e}'$ such* *that*

$$\mathrm{Mat}_{\underline{e}'}(\varphi)^j = s_0\left(\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r)^j\right)^{-1}\cdot s_0\left(\mathrm{Mat}_{\underline{e},\underline{f}}(\mathfrak{M}_r)^j\right) \tag{1.4.15}$$

Proof. Let $D = \mathcal{S}_0(\mathfrak{M})$ and let $\underline{e}_0 = \underline{e}\otimes 1 \in D$. Then $\underline{e}_0$ is clearly a basis of D, which we see is of type τ^{-1} by the definition of the Galois action on D. The equation $\varphi_r\left(\underline{e}^j\,\mathrm{Mat}_{\underline{e},\underline{f}}(\mathfrak{M}_r)^j\right) = \underline{e}^{j+1}\cdot\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r)^j$ becomes $\varphi_D(\underline{e}_1^j)\cdot s_0\left(\mathrm{Mat}_{\underline{e},\underline{f}}(\mathfrak{M}_r)^j\right) = \underline{e}_1^{j+1}p^r\,\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r)^j$ in D. By Remark 1.4.12, this implies that

$$\mathrm{Mat}_{\underline{e}_1}(\varphi)^j = p^r\cdot s_0\left({}^t\,\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r)^j\right)\cdot s_0\left({}^t\,\mathrm{Mat}_{\underline{e},\underline{f}}(\mathfrak{M}_r)^j\right)^{-1}.$$

Note that $D_{\mathrm{st}}^{K_0,r}(\rho) \cong D^\vee(r)$ by (1.4.13). The result now follows from Lemma 1.4.5 by taking $\underline{e}'$ to be the basis of $D^\vee(r)$ obtained from $\underline{e}_1$ there. □

1.4.3 Breuil modules

Let L, π and L' be as at the beginning of Section 1.4.2. In this section we recall aspects of the theory of Breuil modules, which are (in some sense) the reductions mod p of the strongly divisible modules in the previous section, and are used to describe the mod p reductions of stable lattices inside potentially semistable Galois representations.

The relevant mod p version of the Breuil ring is $\overline{S} := S/(p, \mathrm{Fil}^p(S)) = \ell[u]/(u^{ep})$, having the Frobenius $\varphi_{\overline{S}}$, monodromy, filtration and $\mathrm{Gal}(L/L')$-action induced from S. We also write $\overline{S}_{\mathbb{F}} = \overline{S}\otimes_{\mathbb{F}_p}\mathbb{F}$, with all structures extended $\mathbb{F}$-linearly.

Definition 1.4.16. Let $0 \le r \le p-2$. The category $\mathrm{BrMod}^{[0,r],\mathrm{dd}}(\mathbb{F})$ of *Breuil modules of height* $\le r$ *with* $\mathbb{F}$*-coefficients and descent data from* L *to* L' is defined on pages 17-18 of [EGH13]. In particular an object is a finite free $\overline{S}_{\mathbb{F}}$-module $\overline{\mathfrak{M}}$ with a submodule $\overline{\mathfrak{M}}_r$, a $\varphi_{\overline{S}_{\mathbb{F}}}$-semilinear map $\varphi_r : \overline{\mathfrak{M}}_r \to \overline{\mathfrak{M}}$, a $\overline{S}_{\mathbb{F}}$-linear monodromy map $N : \overline{\mathfrak{M}} \to \overline{\mathfrak{M}}$, and a $\overline{S}_{\mathbb{F}}$-semilinear action of $\mathrm{Gal}(L/L')$ that commutes with φ_r and N and preserves $\overline{\mathfrak{M}}_r$. Moreover, the following axioms must hold:

- $\overline{\mathfrak{M}}_r \supseteq u^{er}\overline{\mathfrak{M}}$;
- the image of φ_r generates $\overline{\mathfrak{M}}$ over $\overline{S}_{\mathbb{F}}$;
- other conditions on N, which we don't need.

If $\mathfrak{M}$ is in $\mathrm{S.D.Mod}^{[0,r],\mathrm{dd}}(\mathcal{O}_E)$ then $\overline{\mathfrak{M}} := \mathfrak{M}/(\varpi_E, \mathrm{Fil}^p(S_{\mathcal{O}_E}))$ has a natural structure of an object of $\mathrm{BrMod}^{[0,r],\mathrm{dd}}(\mathbb{F})$ by defining $\overline{\mathfrak{M}}_r$ to be the image of $\mathfrak{M}_r$ and giving it the naturally induced monodromy and Galois action. By the second axiom in Definition 1.4.6, $\varphi_r : \mathfrak{M}_r \to \mathfrak{M}$ induces a well-defined map $(\varphi_r)_{\overline{\mathfrak{M}}} : \overline{\mathfrak{M}}_r \to \overline{\mathfrak{M}}$. This defines a functor $\overline{\mathcal{S}} : \mathrm{S.D.Mod}^{[0,r],\mathrm{dd}}(\mathcal{O}_E) \to \mathrm{BrMod}^{[0,r],\mathrm{dd}}(\mathbb{F})$.

In [EGH13], Section 3.2 there is defined a functor $T_{\mathrm{st}}^* : \mathrm{BrMod}^{[0,r],\mathrm{dd}}(\mathbb{F}) \to \mathrm{Rep}_{\mathbb{F}}(G_{L'})$ by $T_{\mathrm{st}}^*(\overline{\mathfrak{M}}) = \mathrm{Hom}_{\mathrm{BrMod}}(\overline{\mathfrak{M}}, \hat{A})$ for the period ring $\widehat{A} = \widehat{A}_{\mathrm{st}}\otimes_S \overline{S}$. Lemma 3.2.2 *op. cit.* shows that

T_{st} is faithful, rank-preserving and compatible with strongly divisible modules in the sense that for any $\mathfrak{M}$ in $\mathrm{S.D.Mod}^{[0,r],\mathrm{dd}}(\mathcal{O}_E)$ we have a natural isomorphism

$$T^*_{\mathrm{st}}(\overline{S}(\mathfrak{M})) \cong T^*_{\mathrm{st}}(\mathfrak{M}) \otimes_{\mathcal{O}_E} \mathbb{F}. \tag{1.4.17}$$

In fact we will use the variant of this functor defined by $T^r_{\mathrm{st}}(\overline{\mathfrak{M}}) = T^*_{\mathrm{st}}(\overline{\mathfrak{M}})^\vee \otimes \omega^r$, for the sole reason that certain results in [HLM17] we wish to quote are stated using this version. We recall that there is a duality theory for the category $\mathrm{BrMod}^{[0,r],\mathrm{dd}}(\mathbb{F})$ which is to say there is a functor $\overline{\mathfrak{M}} \mapsto \overline{\mathfrak{M}}^*$ obeying $\overline{\mathfrak{M}}^{**} \cong \overline{\mathfrak{M}}$ and $T^r_{\mathrm{st}}(\overline{\mathfrak{M}}) \cong T^*_{\mathrm{st}}(\overline{\mathfrak{M}}^*)$. This result is due to Caruso [Car08]; see [EGH13] Definition 3.2.8 for an explicit definition.

We once again restrict to the case $L = K$, $L' = K_0$ and $\pi = -p$, and make some brief remarks and definitions analogous to those made in the previous subsection. We have natural decompositions $\overline{S}_{\mathbb{F}} = \prod_j \overline{S}^j$ and $\overline{\mathfrak{M}} = \prod_j \overline{\mathfrak{M}}^j$ as before.

Definition 1.4.18. Let $\tau = \bigoplus_{i=0}^{n-1} \widetilde{\omega}_f^{A_i}$ be a principal series type. We say that a Breuil module $\overline{\mathfrak{M}}$ in $\mathrm{BrMod}^{[0,r],\mathrm{dd}}_{L/L'}(\mathbb{F})$ is *of type* τ if it is of rank n and there exist $\overline{S}^j$ bases $\underline{e}^j$ of $\overline{\mathfrak{M}}^j$ such that Δ acts on e^j_i by $\omega_f^{A_i}$ for $0 \le i \le n-1$ and $0 \le j \le f-1$. We refer to any basis with this property as a *framed basis* of $\overline{\mathfrak{M}}$.

Lemma 1.4.19. *Let $\tau = \bigoplus_{i=0}^{n-1} \widetilde{\omega}_f^{A_i}$ be a principal series type and $\overline{\mathfrak{M}}$ an object of* $\mathrm{BrMod}^{[0,r],\mathrm{dd}}_{L/L'}(\mathbb{F})$. *The following are equivalent:*

1. *$\overline{\mathfrak{M}}$ has type τ;*
2. *$\overline{\mathfrak{M}}^j/u \cong \bar{\tau}$ as an $\mathbb{F}[\Delta]$-module for each $0 \le j \le f-1$;*
3. *for each $0 \le j \le f-1$ there exists a set of $\overline{S}^j$-generators $\underline{f}^j = (f^j_0, \dots, f^j_{n-1})$ of $\overline{\mathfrak{M}}^j_r$ such that Δ acts on f^j_i by $\omega_f^{A_i}$.*

Proof. This is entirely analogous to Lemma 1.4.8, but easier. □

Definition 1.4.20. Let $\overline{\mathfrak{M}}$ be an object of $\mathrm{BrMod}^{[0,r],\mathrm{dd}}(\mathbb{F})$ of type τ. Let $\underline{e}$ be a framed basis. Then there exists a tuple $\underline{f}$ as in Lemma 1.4.19(3). We refer to any such tuple as a *framed system of generators of* $\overline{\mathfrak{M}}_r$. We define $\mathrm{Mat}_{\underline{e},\underline{f}}(\overline{\mathfrak{M}}_r)^j \in \mathrm{Mat}_n(\overline{S}^j)$ and $\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r)^j \in \mathrm{GL}_n(\overline{S}^{j+1})$ exactly as in Definition 1.4.11.

We close this section with a lemma about change of basis in Breuil modules. It is a slight generalization of Lemma 2.2.8 of [HLM17].

Lemma 1.4.21. *Let $\overline{\mathfrak{M}}$ be an object in* $\mathrm{BrMod}^{[0,r],\mathrm{dd}}_{K/K_0}(\mathbb{F})$ *of type τ, and let $\underline{e}, \underline{f}$ be a framed basis and system of generators respectively. Suppose that there are matrices $V^j \in \mathrm{Mat}_n(\overline{S}^j)$ and $B^j \in \mathrm{GL}_n(\overline{S}^j)$ such that*

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r)^{j-1} V^j \equiv \mathrm{Mat}_{\underline{e},\underline{f}}(\overline{\mathfrak{M}}_r)^j B^j \mod u^{(r+1)e}.$$

for each $0 \leq j \leq f-1$. *Then there exists a framed basis* $\underline{e}'^j$ *and framed system of generators* $\underline{f}'^j$ *of type* τ *such that* $\mathrm{Mat}_{\underline{e}',\underline{f}'}(\overline{\mathfrak{M}}_r)^j = V^j$ *and* $\mathrm{Mat}_{\underline{e}',\underline{f}'}(\varphi_r) = \varphi_{\overline{S}_{\mathbb{F}}}(B^j)$.

Proof. The proof is the same as Lemma 2.2.8 of [HLM17]. We have $\left(\mathrm{Mat}_{\underline{e},\underline{f}}(\overline{\mathfrak{M}}_r)^j + u^{e(r+1)}M^j\right) B^j = \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r)^{j-1}V^j$ for some $M^j \in \mathrm{Mat}_n(\bar{S}^j)$ because B^j is invertible. Then $\underline{f}_1^j := \underline{e}^j(\mathrm{Mat}_{\underline{e},\underline{f}}(\overline{\mathfrak{M}}_r)^j + u^{e(r+1)}M^j)$ is a framed system of generators of $\overline{\mathfrak{M}}_r^j$. As $\underline{f}_1^j = \underline{f}^j + u^e(\text{something in } \overline{\mathfrak{M}}_r^j)$ we see that $\mathrm{Mat}_{\underline{e},\underline{f}_1}(\varphi_r)^j = \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r)^j$. Now set $\underline{e}'^j = \underline{e}^j \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r)^{j-1}$ and $\underline{f}'^j = \underline{e}'^j V^j = \underline{f}_1^j B^j$. Since B^j is invertible, $\underline{f}'^j$ is also a framed system of generators for $\overline{\mathfrak{M}}_r^j$. We finally compute

$$\varphi_r(\underline{f}'^j) = \varphi_r(\underline{f}_1^j)\varphi_{\bar{S}}(B^j) = \underline{e}^{j+1}\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r)^j\varphi_{\bar{S}}(B^j) = \underline{e}'^{j+1}\varphi_{\bar{S}}(B^j)$$

which proves the claim. □

1.5 Kisin modules and φ-modules

The theory in the preceding section dealt with potentially semistable representations having Hodge-Tate weights in the range $[0, r]$. The lifts of $\bar{\rho}$ arising from automorphic representations in Chapter 3 are more specialized: they are potentially crystalline with parallel Hodge-Tate weights $(2, 1, 0)$. In order to capitalize on this fact it is convenient to use the results in [LLHLM18a] which deals with this exact situation. Since the work in question uses the theory of Kisin modules, we begin with a review of the relevant definitions and the closely related φ-modules. Unlike the categories of the previous section, these categories describe representations of $G_{(K_0)_\infty}$.

1.5.1 Étale φ-modules

Let $L/\mathbb{Q}_p$ be an arbitrary extension with residue field ℓ, and ϖ_L a uniformizer. Make a choice of elements $(\varpi_n)_{n\geq 0}$ in $\overline{L}$ such that $\varpi_{n+1}^p = \varpi_n$ and $\varpi_0 = \varpi_L$, and set $L_\infty = L((\varpi_n)_{n\geq 0})$. In this situation (and more general situations) the theory of norm fields associates with L_∞ an "imperfect norm field" that is (noncanonically) isomorphic to $\ell((u_L))$ together with a Cohen ring $\mathcal{O}_{\mathcal{E}_L} \subset W(\mathrm{Frac}(\mathcal{R}))[1/p]$ that can be used to describe representations of G_{L_∞}, constructed as follows. Let $\mathcal{R} = \varprojlim \mathcal{O}_{\overline{L}}/p$ where the transition maps are the Frobenius. Write $\underline{\varpi} = (\varpi_n)_{n\geq 0} \in \mathcal{R}$ and let $[\underline{\varpi}] \in W(\mathcal{R})$ denote the Teichmüller lift. Define $\mathfrak{S}_L = W(\ell)[[u_L]]$ where u_L is a variable and embed $\mathfrak{S} \hookrightarrow W(\mathcal{R})$ by identifying u_L with $[\underline{\varpi}]$. Let $\mathcal{O}_{\mathcal{E},L}$ denote the p-adic completion of $\mathfrak{S}[1/u_L]$, so $\mathcal{O}_{\mathcal{E},L}$ is a discrete valuation ring with residue field $\ell((u_L))$. Concretely,

$$\mathcal{O}_{\mathcal{E},L} = \left\{ \sum_{i=-\infty}^{\infty} w_i u_L^i \;\middle|\; w_i \in W(\ell), \lim_{i\to-\infty} w_i = 0 \right\}.$$

Let $\mathcal{E}_L$ denote its field of fractions. Then the inclusion $\mathfrak{S}_L \hookrightarrow W(\mathcal{R})$ extends to an inclusion $\mathcal{E}_L \hookrightarrow W(\mathrm{Frac}(\mathcal{R}))[1/p]$. Let $\mathcal{E}_L^{\mathrm{un}} \subset W(\mathrm{Frac}(\mathcal{R}))[1/p]$ denote the maximal unramified extension of $\mathcal{E}_L$ contained in $W(\mathrm{Frac}(\mathcal{R}))[1/p]$ and let $\mathcal{O}_{\mathcal{E}^{\mathrm{un}},L} \subset W(\mathrm{Frac}(\mathcal{R}))$ denote its ring of integers.

As $\mathrm{Frac}(\mathcal{R})$ is algebraically closed the residue field $\mathcal{O}_{\mathcal{E}^{\mathrm{un}},L}/p\mathcal{O}_{\mathcal{E}^{\mathrm{un}},L}$ is a separable closure of $\ell((u_L))$. Write $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{un}}},L}$ for the p-adic completion of $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{un}}},L}$ and $\widehat{\mathcal{E}_L^{\mathrm{un}}}$ for its field of fractions. Finally, write $\mathfrak{S}_L^{\mathrm{un}} = \mathcal{O}_{\mathcal{E}^{\mathrm{un}},L} \cap W(R) \subset W(\mathrm{Frac}(\mathcal{R}))$. The ring $W(\mathrm{Frac}(\mathcal{R}))$ has a natural action of G_L. The induced action of G_{L_∞} preserves $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{un}}},L}$.

We endow $\mathfrak{S}_L$ with the ϕ-semilinear Frobenius map taking u_L to u_L^p. This extends uniquely all the way to $\mathcal{O}_{\widehat{\mathcal{E}_L^{\mathrm{un}}}}$. Let R be a complete local noetherian $\mathcal{O}_E$-algebra (for example, $R = \mathcal{O}_E$ or $R = \mathbb{F}$). Then we extend this Frobenius map R-linearly to $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{un}}}_L} \widehat{\otimes}_{\mathbb{Z}_p} R$.

Definition 1.5.1. The category of *φ-modules with coefficients in R* is denoted $\varphi\,\mathrm{Mod}_L(R)$ and consists of finite free $\mathcal{O}_{\mathcal{E},L}\widehat{\otimes}_{\mathbb{Z}_p} R$-modules M equipped with a $\phi\otimes 1$-semilinear map $\varphi_M : M \to M$. We say that M is *étale* if the image of φ generates M (equivalently, the linearization of φ is an isomorphism) and write $\varphi\,\mathrm{Mod}_L^{\mathrm{et}}(R)$ for the full subcategory of étale objects. It is abelian.

The point of these constructions is that we obtain an exact antiequivalence of abelian tensor categories [Fon90]

$$\begin{aligned} \varphi\,\mathrm{Mod}_L^{\mathrm{et}}(R) &\to \mathrm{Rep}_{G_{L_\infty}}(R) \\ M &\mapsto V_L^*(M) := \mathrm{Hom}_{\mathcal{O}_{\mathcal{E},L},\varphi}(M, \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{un}}}_L}) \end{aligned}$$

where $\mathrm{Rep}_{G_{L_\infty}}(R)$ denote the category of p-adically continuous representations of G_{L_∞} on finite free R-modules. This construction is also compatible with base change $\mathcal{O}_E \to \mathbb{F}^\times$.

In certain cases of interest to us it is possible to add descent data to the above construction. Following [CDM14], let L' be a subfield of L such that $L = L'(\varpi_L)$ where $\varpi_{L'} = \varpi_L^m$ is a uniformizer of L', for some integer m prime to p. We also assume that L' contains a primitive mth root of 1 so that L/L' is Galois. Choose $(\varpi_n)_{n\geq 0}$ as before and set $\varpi_n' = \varpi_n^m$. Then we may perform all the constructions above with $u_{L'} = [\underline{\varpi}']$, identifying $\mathfrak{S}_{L'}$ and $\mathcal{O}_{\mathcal{E},L'}$ with the $W(\ell)$-subalgebras of $\mathfrak{S}_L$ and $\mathcal{O}_{\mathcal{E},L}$ generated by $u_{L'} = u_L^m$. Define a $W(\ell)$-semilinear action of $\mathrm{Gal}(L_\infty/L'_\infty) = \mathrm{Gal}(L/L')$ on $\mathcal{O}_{\mathcal{E},L}$ such that $g(u_L) = \omega_{\varpi_L}(g)u_L$, so the fixed subring is $\mathcal{O}_{\mathcal{E},L'}$ (here ω_{ϖ_L} is the character $\mathrm{Gal}(L/L') \to W(\ell)^\times$ such that $g(\varpi_L) = \omega_{\varpi_L}(g)\varpi_L$ which is well defined because we assumed L' contains the mth roots of 1). This leads to the following definition.

Definition 1.5.2. A *φ-module with descent data from L to L' and coefficients in R* is an object M of $\varphi\,\mathrm{Mod}_L(R)$ with an $\mathcal{O}_{\mathcal{E},L}$-semilinear action of $\mathrm{Gal}(L/L')$ that commutes with the Frobenius φ_M. It is *étale* if the underlying object of $\varphi\,\mathrm{Mod}_L(R)$ is. We denote the resulting category by $\varphi\,\mathrm{Mod}_{L/L'}^{\mathrm{et,dd}}(R)$.

There is an exact antiequivalence of abelian tensor categories

$$\begin{aligned} \varphi\,\mathrm{Mod}_{L/L'}^{\mathrm{et,dd}}(R) &\to \mathrm{Rep}_{G_{L'_\infty}}(R) \\ M &\mapsto V_{\mathrm{dd}}^*(M) := \mathrm{Hom}_{\mathcal{O}_{\mathcal{E},L},\varphi}(M, \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{un}}},L'}) \end{aligned}$$

where we define the action of $G_{L'_\infty}$ on the Hom group by $g(f) = g \circ f \circ \bar{g}^{-1}$, where bar denotes projection to $\mathrm{Gal}(L_\infty/L'_\infty)$. Moreover the diagram

$$\begin{tikzcd}
\varphi\,\mathrm{Mod}^{\text{ét,dd}}_{L/L'}(R) \arrow[r, "V^*_{\mathrm{dd}}"] & \mathrm{Rep}_{G_{L'_\infty}}(R) \\
\varphi\,\mathrm{Mod}^{\text{ét}}_{L'}(R) \arrow[u] \arrow[ur, "V^*_{L'}"'] &
\end{tikzcd}$$

where the vertical arrow is the base change $M \mapsto M \otimes_{\mathcal{O}_{\mathcal{E},L'}} \mathcal{O}_{\mathcal{E},L}$ is a commutative diagram of equivalences of categories. One sees that the vertical arrow is an equivalence with inverse given by taking $\mathrm{Gal}(L/L')$-invariants using Galois descent (valid since $p \nmid |\,\mathrm{Gal}(L/L')|$) and that V^*_{dd} is an equivalence by arguing as in Lemma A.2.6 of [HLM17]. See also Section 2.1.3 of [CDM14].

Now that we are done with generalities we specialize to the case $L = K$, $L' = K_0$, $\varpi_{L'} = -p$ and $\varpi_L = \pi$, so $m = e$. We use the notation $u = u_K$ and $v = u_{K_0} = u^e$. As before, given $M \in \varphi\,\mathrm{Mod}_{K_0}(R)$ (resp. $\varphi\,\mathrm{Mod}^{\mathrm{dd}}_{K/K_0}(R)$) of rank n we decompose $M = \prod_{j=0}^{f-1} M^j$ and write $\mathrm{Mat}_{\underline{e}}(\varphi_M)^j$ for the matrix in $\mathrm{GL}_n(R^{j+1})$ such that $\varphi(\underline{e}^j) = \underline{e}^{j+1} \cdot \mathrm{Mat}_{\underline{e}}(\varphi_M)^j$, where $\underline{e}$ is a basis (resp. framed basis) of M. Here the notion of framed basis with respect to a principal series type has the obvious definition.

We now explain the functor $\mathcal{F} : \mathrm{FL}^{[0,p-2]}(\mathbb{F}) \to \varphi\,\mathrm{Mod}^{\mathrm{et}}_{K_0}(\mathbb{F})$ in Figure 1.3.1. We define $\mathcal{F}$ exactly as it appears in Appendix A.1 of [HLM17]. The next lemma describes its key properties.

Lemma 1.5.3. *1. $\mathcal{F}$ is fully faithful.*

2. Let $M \in \mathrm{FL}^{[0,p-2]}(\mathbb{F})$ be of rank n, having regular Fontaine-Laffaille weights $\mathrm{FL}_j(M) = \{m_0^j < m_1^j < \cdots < m_{n-1}^j\}$. If $\underline{e}_0$ is any basis of M adapted to the filtration then $\mathcal{F}(M)$ has a basis $\underline{e}$ such that

$$\varphi_{\mathcal{F}(M)}(\underline{e}^j) = \underline{e}^{j+1} \cdot \mathrm{Diag}(v^{m_0^{j+1}}, \ldots, v^{m_{n-1}^{j+1}})\,\mathrm{Mat}_{\underline{e}_0}(\varphi_M)^j$$

3. The diagram

$$\begin{tikzcd}
\mathrm{Rep}_{G_{(K_0)_\infty}} & \varphi\,\mathrm{Mod}^{\mathrm{et}}_{K_0}(\mathbb{F}) \arrow[l, "V^*"'] \\
\mathrm{Rep}_{G_{K_0}}(\mathbb{F}) \arrow[u, "\mathrm{res}"] & \mathrm{FL}^{[0,p-2]}(\mathbb{F}) \arrow[l, "T^*_{\mathrm{cris}}"] \arrow[u, "\mathcal{F}"']
\end{tikzcd}$$

commutes up to equivalence.

Proof. The fact that $\mathcal{F}$ is fully faithful is proved during its construction in Appendix A.1 of [HLM17]. The second part is a mild generalization of Lemma 2.2.7 *op. cit.* and is proved in exactly the same way. The third statement is shown in the proof of Lemma 2.2.8 *op. cit.*. $\square$

1.5.2 Kisin modules with descent data

The theory of Kisin modules is a refinement of that of φ-modules used to study integral structures in $G_{(K_0)_\infty}$-representations. We will use the variant involving descent data developed in [LLHLM18a, CL18]. In this section we specialize to the case $L = K$, $L' = K_0$. Let R be a complete local noetherian $\mathcal{O}_E$-algebra. We also fix a principal series type $\tau = \bigoplus_{i=0}^{n-1} \widetilde{\omega}_f^{A_i}$.

Definition 1.5.4. A *Kisin module with height in* $[0, r]$ *with coefficients in* R *and descent data from* K *to* K_0 is a finite free $\mathfrak{S}\widehat{\otimes}_{\mathbb{Z}_p} R$-module M together with a $\varphi_{\mathfrak{S}} \otimes 1$-semilinear Frobenius map $\varphi : M \to M$ such that the cokernel of the linearization $\varphi^* M \to M$ is killed by $E(u)^r$, and also a semilinear action of Δ that commutes with φ. We refer to any such basis as a *framed basis* of M. These objects form a category $Y^{[0,r],\mathrm{dd}}(R)$.

If for each $0 \le j \le f-1$ there exists a basis $\underline{e}^j$ of M^j such that Δ acts on e_i^j by $\widetilde{\omega}_f^{A_i} \otimes R$ then we say that M is *of type* τ. The full subcategory of objects of type τ is denoted $Y^{[0,r],\tau}(R)$.

Exactly as in Lemma 1.4.8, saying that M is of type τ is equivalent to asking that $M^j/u \cong \tau \otimes_{\mathcal{O}_E} R$ as $R[\Delta]$-modules for each $0 \le j \le f-1$.

Definition 1.5.5. If $\underline{e}$ is a framed basis of M then $\mathrm{Mat}_{\underline{e}}(\varphi)^j$ is the matrix in $\mathrm{Mat}_n(R[[u]])$ such that $\varphi(\underline{e}^j) = \underline{e}^{j+1} \cdot \mathrm{Mat}_{\underline{e}}(\varphi)^j$ for $0 \le j \le f-1$.

There is a functor $Y^{[0,r],\mathrm{dd}}(R) \to \varphi\,\mathrm{Mod}^{\text{ét},\mathrm{dd}}_{K/K_0}(R)$ taking M to $M \otimes_{\mathfrak{S}} \mathcal{O}_{\widehat{\mathcal{E}},K_0}$. We write T^*_{dd} for the resulting functor $Y^{[0,r],\mathrm{dd}}(R) \to \mathrm{Rep}_{G_{(K_0)_\infty}}(R)$ obtained by composing with $V^*((-)^{\Delta=1})$.

We now explain the connection between Kisin modules and quasi-strongly divisible modules with descent data. In [Car08] it is shown that there is an equivalence of categories $\Theta_r : Y^{[0,r]}(\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{S.D.Mod}^{[0,r]}(\mathbb{Z}_p)_0$ where the notation means the categories analogous to the ones we have defined but with $\mathbb{Z}_p$-coefficients and no descent data. We now update this result to include this extra data.

Following [Car08] we define a functor

$$\begin{aligned} \Theta_r : Y^{[0,r],\mathrm{dd}}(\mathcal{O}_E) \to& \mathrm{S.D.Mod}^{[0,r],\mathrm{dd}}(\mathcal{O}_E)_0 \\ M \mapsto& \Theta_r(M) := S \otimes_{\varphi_S, \mathfrak{S}} M \end{aligned}$$

where the notation means we take the tensor product along the map $\mathfrak{S} \hookrightarrow S \xrightarrow{\varphi_S} S$. The extra structures on $\mathfrak{M} = \Theta_r(M)$ are given by setting

$$\mathfrak{M}_r = \{x \in \mathfrak{M} : (1 \otimes \varphi_M)(x) \in \mathrm{Fil}^r(S) \otimes_{\mathfrak{S}} M \subseteq S \otimes_{\mathfrak{S}} M\}$$

where we note that $1 \otimes \varphi_M$ defines an S-linear map $\mathfrak{M} \to S \otimes_{\mathfrak{S}} M$. We let $(\varphi_{\mathfrak{M}})_r : \mathfrak{M}_r \to \mathfrak{M}$ be the composition

$$\mathfrak{M}_r \xrightarrow{1 \otimes \varphi_M} \mathrm{Fil}^r(S) \otimes_{\mathfrak{S}} M \xrightarrow{(\varphi_S)_r \otimes 1} \mathfrak{M}.$$

We define $\varphi_{\mathfrak{M}}(x) = \frac{1}{c_1^r}(\varphi_{\mathfrak{M}})_r(E(u)^r x)$. Finally the Δ-action on $\mathfrak{M}$ is the obvious S-semilinear one. It is easy to check that $\mathfrak{M}$ is an object of $\mathrm{S.D.Mod}^{[0,r],\mathrm{dd}}(\mathcal{O}_E)_0$ (see [Car08]).

Proposition 1.5.6. *Let $0 \leq r \leq p-2$. The functor Θ_r defined above is an equivalence of categories.*

Proof. In the case $\mathcal{O}_E = \mathbb{Z}_p$ and trivial descent data this is Theorem 2.2.1 of [Car08]. We simply verify that it continues to hold more generally. Given $\mathfrak{M}$ with $\mathcal{O}_E$-coefficients, by the result of Caruso we know there exists a Kisin module $M \in Y^{[0,r]}(\mathbb{Z}_p)$ such that $\Theta_r(M) \cong \mathfrak{M}$, and since Θ_r induces an bijection between the $\mathbb{Z}_p$-endomorphism rings of M and $\mathfrak{M}$, it follows that M has an $\mathcal{O}_E$-action compatible with its other structures. The only non-formal thing is to verify that M is free over $\mathfrak{S}\hat{\otimes}_{\mathbb{Z}_p}\mathcal{O}_E$. To do this we use a trick from Lemma 1.2.5 of [Kis09]: note that if $\mathfrak{M}$ has rank d over $S \otimes \mathcal{O}_E$ then since $W(k) \otimes_S \mathfrak{M} = W(k) \otimes_{\phi,\mathfrak{S}} M$ and $\phi \otimes 1$ is an automorphism of $W(k) \otimes \mathcal{O}_E$ it follows that $W(k) \otimes_{\mathfrak{S}} M = M/uM$ is free of rank d over $W(k) \otimes \mathcal{O}_E$ (where we view $W(k)$ as an S- and $\mathfrak{S}$-module via $u \mapsto 0$). By Nakayama's lemma we can find a surjection $(\mathfrak{S} \otimes \mathcal{O}_E)^d \twoheadrightarrow M$. The left hand side is free of rank dk over $\mathfrak{S}$, where $k = \operatorname{rank}_{\mathbb{Z}_p} \mathcal{O}_E$. But we also have $\operatorname{rank}_{\mathfrak{S}} M = \operatorname{rank}_S \mathfrak{M} = dk$. Hence the map is an isomorphism. This shows that Θ_r is essentially surjective between the categories with $\mathcal{O}_E$-coefficients. The proof of full faithfulness is formal.

We now add descent data. Note, for example, that if $M \in Y^{[0,r]}(\mathcal{O}_E)$ then for $g \in \Delta$ we can define $M^{(g)} = \mathfrak{S} \otimes_{g,\mathfrak{S}} M$ with Frobenius $\varphi_{\mathfrak{S}} \otimes \varphi_M$, and this is again an object of $Y^{[0,r]}(\mathcal{O}_E)$. To say that M has a descent datum is equivalent to asking that there exist morphisms $\hat{g} : M^{(g)} \to M$ in $Y^{[0,r]}(\mathcal{O}_E)$ for each $g \in \Delta$ obeying the cocycle relations $\widehat{gh} = \hat{g} \circ \hat{h}^{(g)}$ (note $M^{(gh)} = (M^{(h)})^{(g)}$). Similar remark are true on the strongly divisible module side, and one checks that easily that $\Theta_r(M^{(g)}) \cong \Theta_r(M)^{(g)}$. It is now formal to show that Θ_r induces an equivalence of categories when descent data is taken into account. $\square$

Corollary 1.5.7. *Let M be an object of $Y^{[0,r],\tau}(\mathcal{O}_E)$ and let $\mathfrak{M} = \Theta_r(M)$. If $\underline{\hat{e}}$ is a framed basis of M then $\mathfrak{M}$ has a framed system of generators $(\underline{e}, \underline{f})$ of type τ such that $\operatorname{Mat}_{\underline{e},\underline{f}}(\mathfrak{M}_r)^j$ is the unique matrix V^j in $\operatorname{Mat}_n(\mathcal{O}_E[[u]]) \subseteq \operatorname{Mat}_n(S^j)$ obeying $\operatorname{Mat}_{\underline{\hat{e}}}(\varphi)^{j-1} V^j = E(u)^r \cdot \mathrm{id}$, and $\operatorname{Mat}_{\underline{e}}(\varphi_r)^j = c_r \cdot \mathrm{id}$.*

Proof. Let us abbreviate $\operatorname{Mat}_{\underline{\hat{e}}}(\varphi_M)^j = A^j$. Since $E(u)^r M \subseteq \langle \operatorname{im}(\varphi_M) \rangle$ it follows that there exists a matrix $V^j \in \operatorname{Mat}_n(\mathcal{O}_E[[u]])$ such that $A^j V^j = V^j A^j = E(u)^r \cdot \mathrm{id}$. Set $\underline{e}^j = 1 \otimes \underline{\hat{e}}^{j-1} \in S^j \otimes_{\varphi_S, \mathfrak{S}^{j-1}} M^{j-1} = \mathfrak{M}^j$ and define $\underline{f}^j = \underline{e}^j V^j$. Then a calculation shows that $(1 \otimes \varphi_M)(\underline{f}^j) = E(u)^r \underline{e}^{j+1}$, so $\underline{f}^j$ lies in $\mathfrak{M}_r^j$. Moreover it follows that $\varphi_r(\underline{f}^j) = (\varphi_S)_r(E(u)^r)\underline{e}^{j+1} = c_r \underline{e}^{j+1}$. From Lemma 1.4.10 we see that $(\underline{e}, \underline{f})$ is a system of generators, and it is easy to check from the definitions that it is framed. This proves the claim. $\square$

Lemma 1.5.8. *The diagram of functors*

$$\begin{tikzcd}
\mathrm{S.D.Mod}^{[0,r],\mathrm{dd}}(\mathcal{O}_E)_0 & Y^{[0,r],\mathrm{dd}}(\mathcal{O}_E) \arrow[l, "\Theta_r"'] \arrow[d, "\left(-\otimes \mathcal{O}_{\widehat{\mathcal{E}},K_0}\right)^{\Delta=1}"] \\
\mathrm{S.D.Mod}^{[0,r],\mathrm{dd}}(\mathcal{O}_E) \arrow[u, "(-)_0"] \arrow[d, "T^*_{\mathrm{st}}"] & \varphi\,\mathrm{Mod}^{\mathrm{et}}_{K_0}(\mathcal{O}_E) \arrow[d, "V^*_{K_0}"] \\
\mathrm{Rep}_{G_{K_0}}(\mathcal{O}_E) \arrow[r, "\mathrm{res}"] & \mathrm{Rep}_{G_{(K_0)_\infty}}(\mathcal{O}_E)
\end{tikzcd}$$

commutes up to equivalence.

Proof. Let $\mathfrak{M}$ be an object of $\mathrm{S.D.Mod}^{[0,r],\mathrm{dd}}_{K/K_0}(\mathcal{O}_E)$ and M in $Y^{[0,r],\mathrm{dd}}(\mathcal{O}_E)$. The result can be extracted from the literature as follows: by Lemma 3.3.4 of [CL09] that there is a natural bijection

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(M, \mathfrak{S}^{\mathrm{un}}) = \mathrm{Hom}_{S,\varphi}(\Theta_r(M), A_{\mathrm{cris}})$$

and from (3.1.5) in the same paper that

$$\mathrm{Hom}_{S,\varphi}(\mathfrak{M}_0, A_{\mathrm{cris}}) = T^*_{\mathrm{st}}(\mathfrak{M}).$$

The result now follows from Corollary 2.1.4 of [Kis06] which says that $\mathrm{Hom}_{\mathfrak{S},\varphi}(M, \mathfrak{S}^{\mathrm{un}}) = T^*_{\mathrm{dd}}(M)$ and the fact that Θ_r is an equivalence of categories. Note that the references above do not deal with descent data or coefficients exactly as we do, but it doesn't matter because the functors are the same. □

We now define the functor $\mathcal{M} : \mathrm{BrMod}^{[0,r],\mathrm{dd}}(\mathbb{F}) \to Y^{[0,r],\mathrm{dd}}(\mathbb{F})$ appearing in Figure 1.3.1. A closely related functor is defined in Appendix A.1 [HLM17], where it is called $\underline{M}_{k((u))}$ and takes values in the category $\varphi\,\mathrm{Mod}^{\mathrm{et},\mathrm{dd}}(\mathbb{F})$. Since $\underline{M}_{k((u))}$ factors through $Y^{[0,r],\mathrm{dd}}(\mathbb{F})$ by its definition, we let $\mathcal{M}$ be $\underline{M}_{k((u))}$ but having target $\varphi^{[0,r],\mathrm{dd}}(\mathbb{F})$.

Lemma 1.5.9. *1. If $\overline{\mathfrak{M}}$ is of type τ with framed system of generators $(\underline{e}, \underline{f})$ then $\mathcal{M}(\overline{\mathfrak{M}}^*)$ has a basis $\underline{e}'$ of type τ^{-1} such that*

$$\varphi(\underline{e}'^j) = \underline{e}'^{j+1} \cdot {}^t\,\mathrm{Mat}_{\underline{e},\underline{f}}(\overline{\mathfrak{M}}_r)^{j+1} \left(\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_r)^j\right)^{-1}.$$

2. The diagram

$$\begin{tikzcd}
\mathrm{Rep}_{G_{K_0}}(\mathbb{F}) \arrow[r] & \mathrm{Rep}_{G_{(K_0)_\infty}}(\mathbb{F}) \\
\mathrm{BrMod}^{[0,r],\mathrm{dd}}(\mathbb{F}) \arrow[u, "T^*_{\mathrm{st}}"] \arrow[r, "\mathcal{M}"] & Y^{[0,r],\mathrm{dd}}(\mathbb{F}) \arrow[u, "T^*_{\mathrm{dd}}"']
\end{tikzcd}$$

commutes up to equivalence.

Proof. The proof of the first claim is a mild generalization of the proof of Lemma 2.2.6, [HLM17]. The second follows from Lemma A.2.7 *op. cit.*. □

1.5.3 Kisin modules of Hodge type μ

Let τ be a principal series type and $\mu \in ([0,r])^{\mathrm{Hom}(K_0,E)}$ a p-adic Hodge type. In [CL18], inspired by the theory of local Shimura varieties, the authors construct subcategories $Y^{\mu,\tau}(R) \subseteq Y^{[0,r],\tau}(R)$ consisiting of Kisin modules of height $\leq r$ and of type τ which morally speaking are supposed to correspond to Galois representations of Hodge type μ. Unfortunately the construction of these categories is somewhat abstruse and to recall details would take us astray. So in this section we will only explain why these categories are useful for us and recall the results we need. On the other hand, when R is $\mathcal{O}_E$-flat and reduced one can show *a posteriori* that conditions for membership in $Y^{\mu,\tau}(R)$ are fairly concrete. See for example [LLHLM18a] Proposition 4.18.

We will specialize to the case $n = 3$ and $\mu = (2,1,0)_{j=0}^{f-1}$. In [LLHLM18a], under the assumption that τ is 3-generic (Definition 1.5.10 below), the authors manage to give a classification of points of $Y^{\mu,\tau}(\mathbb{F})$ in terms of an associated datum called the *shape* of an object of $Y^{\mu,\tau}(\mathbb{F})$, which we recall below. They also compute the universal framed deformation of a Kisin module of a given shape and give a necessary and sufficient monodromy criterion for the associated $G_{(K_0)_\infty}$-representation to come from a potentially crystalline G_{K_0}-representation. The utility of this for us is that these constructions are all explicit and we can make use of the "canonical forms" in the tables of [LLHLM18a].

In order to go further we now recall some basic definitions and notation from [LLHLM18a].

Definition 1.5.10. Let $\tau = \bigoplus_{i=0}^{2} \widetilde{\omega}_f^{A_i}$ be a principal series type where $A_i = -\sum_{j=0}^{f-1} b^i_{f-j} p^j$ for some $b^i_j \in [0, p-1]$ (note the minus sign). We say that τ is δ-*generic* if

$$b^i_j - b^{i'}_j \in [\delta, p-1-\delta] \mod p-1$$

for each $0 \leq i \neq i' \leq 2$ and $0 \leq j \leq f-1$. Write $b^i_{(j_0)} = \sum_{j=0}^{f-1} b^i_{f-j+j_0} p^j$ so that $b^i_{(j_0)} \equiv p^{j_0} b^i_{(0)}$ mod e and $A_i = -b^i_{(0)}$ by definition.

If τ is 1-generic (which implies that the b^i_j are unique) we define the *orientation* of τ to be the element $(s_j)_{j=0}^{f-1} \in S_3^f$ such that

$$b^{s_j(0)}_{j+1} > b^{s_j(1)}_{j+1} > b^{s_j(2)}_{j+1}.$$

This is equivalent to asking that $b^{s_j(0)}_{(j)} > b^{s_j(1)}_{(j)} > b^{s_j(2)}_{(j)}$.

Definition 1.5.11. Let M be an object of $Y^{[0,2],\tau}(R)$ of type τ with eigenbasis $\underline{e}$. We define associated matrices $A^j_{\underline{e}} \in \mathrm{GL}_3(R[[v]])$ (where $v = u^e$) for $0 \leq j \leq f-1$ by the formula

$$A^j_{\underline{e}} = \mathrm{Diag}_i(u^{-b^{s_{j+1}(i)}_{(j+1)}}) s^{-1}_{j+1} \mathrm{Mat}_{\underline{e}}(\varphi)^j s_{j+1} \mathrm{Diag}_i(u^{b^{s_{j+1}(i)}_{(j+1)}})$$

Remark 1.5.12. It is easy to check that this matrix does actually lie in $\mathrm{GL}_3(R[[v]])$ by the definition of the orientation. The purpose of this definition is simply that the tables of [LLHLM18a] are given in terms of the matrices $A^j_{\underline{e}}$.

We now explain the notion of the shape of a Kisin module. Let $\mathcal{I}(\mathbb{F}) \leq \mathrm{GL}_3(\mathbb{F}[[v]])$ denote the subgroup of matrices that are upper-triangular mod v, and let $\widetilde{W} = N_{\mathrm{GL}_3}(T_3)(\mathbb{F}((v)))/T_3(\mathbb{F}[[v]])$ denote the extended affine Weyl group of GL_3 which is equal to the group of matrices $\mathrm{Diag}(v^a, v^b, v^c)\varsigma$ for $\varsigma \in S_3$. Bruhat-Tits theory gives a double coset decomposition

$$\mathrm{GL}_3(\mathbb{F}((v))) = \bigsqcup_{\widetilde{w}\in\widetilde{W}} \mathcal{I}(\mathbb{F})\widetilde{w}\mathcal{I}(\mathbb{F}).$$

Following [LLHLM18a], we say that $\overline{M} \in Y^{[0,2],\tau}(\mathbb{F})$ has *shape* $(\widetilde{w}_j)_{j=0}^{f-1} \in \widetilde{W}^f$ if the matrix $A^j_{\underline{e}}$ of Definition 1.5.11 lies in $\mathcal{I}(\mathbb{F})\widetilde{w}_j\mathcal{I}(\mathbb{F})$ for $0 \leq j \leq f-1$ for any choice of framed basis $\underline{e}$.

If τ is 3-generic and $\bar{\rho} : G_{K_0} \to \mathrm{GL}_3(\mathbb{F})$ is continuous then the proof of Theorem 3.2 in [LLHLM18a] shows that *if* there exists $\overline{M} \in Y^{[0,2],\tau}(\mathbb{F})$ such that $T^*_{\mathrm{dd}}(\overline{M}) \cong \bar{\rho}|_{G_{(K_0)_\infty}}$ then $\overline{M}$ is unique up to isomorphism. This allows us to define the shape of the pair $w(\bar{\rho},\tau) = (w_j(\bar{\rho},\tau))_{j=0}^{f-1}$ to be the shape of $\overline{M}$.

The next theorem collects the main results from [LLHLM18a] that we need.

Theorem 1.5.13. *Suppose that ρ is a G_{K_0}-stable $\mathcal{O}_E$-lattice inside an n-dimensional potentially crystalline E-representation having inertial type τ and Hodge type $\mu = (2,1,0)_{j=0}^{f-1}$. If $\mathfrak{M}$ is a strongly divisible module in* $\mathrm{S.D.Mod}^{[0,2],\mathrm{dd}}(\mathcal{O}_E)$ *such that $T^*_{\mathrm{st}}(\mathfrak{M}) \cong \rho$ then there exists M in $Y^{\mu,\tau^{-1}}(\mathcal{O}_E)$ of rank n such that $\Theta_r(M) \cong \mathfrak{M}_0$.*

In particular there exists $\overline{M} \in Y^{[0,2],\tau^{-1}}(\mathbb{F})$ such that $T^*_{\mathrm{dd}}(\overline{M}) \cong \bar{\rho}|_{G_{(K_0)_\infty}}$ so if τ is 3*-generic the shape $w(\bar{\rho},\tau^{-1})$ is defined.*

Moreover M has a framed basis $\underline{e}$ such that $A^j_{\underline{e}}$ is given by the matrix in the final column and row corresponding to shape $w_j(\bar{\rho},\tau^{-1}) \in \widetilde{W}$ of Table 5 of [LLHLM18a], subject to the conditions there. In particular $\overline{A}^j_{\underline{e}} = A^j_{\underline{e}} \mod \varpi_E$ must be the matrix given in the first column of the table. Finally, the monodromy condition in the first column of Table 7 loc. cit. *holds on the entries of $\overline{A}^j_{\underline{e}}$.*

Proof. The first paragraph follows from the uniqueness in the proof of Proposition 5.17 of [CL18], the fact that Θ_2 is an equivalence of categories, and Lemma 1.5.8. The second paragraph follows immediately (see also [CL18] Corollary 5.18). The statement about $A^j_{\underline{e}}$ follows from the computation of the universal framed deformation of $\overline{M}$ to $Y^{\mu,\tau^{-1}}(R)$ for R flat over $\mathcal{O}_E$ in Theorem 4.17 of [LLHLM18a]. The final statement regarding monodromy follows from Theorem 5.1 and Theorem 5.6 of [LLHLM18a] (see also the definition of the "monodromy condition" in Definition 5.7. Table 6 presents one of the equations coming from this monodromy condition; the monodromy condition in Table 7 is a consequence of the vanishing of the expression in Table 6). ∎

Remark 1.5.14. Table 7 in [LLHLM18a] only contains results for some of the possible shapes. However, all the shapes we compute in Section 1.6 belong to this table.

1.6 Shape computations

Let $\bar{\rho} : G_{K_0} \to \mathrm{GL}_3(\mathbb{F})$ be a Galois representation satisfying (FL_δ) and choose a_j^i as in Definition 1.1.2. For the remainder of the thesis we will interested in the principal series types τ of Definition 1.6.1 whose choice is explained later in Remark 2.3.7. The goal in this section is to compute the shapes $\mathbf{w}(\bar{\rho}, \tau^{-1})$. The strategy is to use Breuil modules following a method simlar to that of [HLM17].

Definition 1.6.1. Let $\sigma = (\sigma_j)_{j=0}^{f-1} \in S_3^f$ and $K \subseteq \{0, \dots, f-1\}$. We define a principal series type $\tau = \tau(\sigma, K)$ by setting $\tau = \bigoplus_{i=0}^{2} \widetilde{\omega}_f^{A_i}$ where

$$A_i := \sum_{k=0}^{f-1} \left(a_{f-k}^{\sigma_{f-k}^{-1}(i)} + \delta_{f-k+1 \notin K}(1 - \sigma_{f-k+1}^{-1}(i)) \right) p^k.$$

The main result of this section is the computation of $\mathbf{w}(\bar{\rho}, \tau^{-1})$ in the next proposition.

Proposition 1.6.2. *Let $\bar{\rho} : G_{K_0} \to \mathrm{GL}_3(\mathbb{F})$ be a Galois representation satisfying (FL_δ) with $\delta \geq 5$. Assume that $\bar{\rho}$ is the reduction mod p of a G_{K_0}-stable $\mathcal{O}_E$-lattice inside a potentially crystalline E-representation of inertial type $\tau = \tau(\sigma, K)$ and parallel Hodge type $(2,1,0)_{j=0}^{f-1}$ (so the shape $w(\bar{\rho}, \tau^{-1})$ is defined according to Theorem 1.5.13). Let M be a Fontaine-Laffaille module in $\mathrm{FL}^{[0,p-2]}(\mathbb{F})$ such that $T^*_{\mathrm{cris}}(M) = \bar{\rho}$ and define $x_j, y_j, z_j \in \mathbb{F}$ as in Corollary 1.2.8.*

1. *If $\bar{\rho}$ satisfies $(\mathrm{FL}_\delta^{**})$ then*

$$\mathbf{w}_j(\bar{\rho}, \tau^{-1}) = \begin{cases} w_0 v & \text{if } j+2 \in K \\ \mathrm{Diag}(v^2, v, 1) & \text{if } j+2 \notin K. \end{cases}$$

2. *Assume that $K = \emptyset$ and $\sigma_j = \begin{cases} \mathrm{id} & j \in J \\ (01) & j \notin J \end{cases}$ for some $J \subseteq \{0, \dots, f-1\}$. Then*

$$\mathbf{w}_j(\bar{\rho}, \tau^{-1}) = \begin{cases} s_\beta \mathrm{Diag}(v^2, v, 1) & \text{if } j+1 \in \partial J \text{ and } x_{j+1} = 0 \\ \mathrm{Diag}(v^2, v, 1) & \text{else.} \end{cases}$$

3. *Assume that* $K = \emptyset$ *and* $\sigma_j = \begin{cases} \mathrm{id} & j \in J \\ (12) & j \notin J \end{cases}$ *for some* $J \subseteq \{0, \dots, f-1\}$. *Then*

$$\mathbf{w}_j(\bar{\rho}, \tau^{-1}) = \begin{cases} s_\alpha \mathrm{Diag}(v^2, v, 1) & \text{if } j+1 \in \partial J \text{ and } z_{j+1} = 0 \\ \mathrm{Diag}(v^2, v, 1) & \text{else.} \end{cases}$$

4. *Assume that* $\bar{\rho}$ *satisfies* (FL^*_δ), *that* $\sigma_j = \begin{cases} \mathrm{id} & j \in J \\ (02) & j \notin J \end{cases}$ *for some* $J \subseteq \{0, \dots, f-1\}$ *and that* $\partial J \subseteq K - 1$. *Then*

$$\mathbf{w}_j(\bar{\rho}, \tau^{-1}) = \begin{cases} w_0 v & \text{if } j+1 \in \partial J \\ s_\alpha s_\beta v & \text{if } j+1 \notin \partial J, j+2 \in K \text{ and } y_{j+1} = 0 \\ s_\beta s_\alpha v & \text{if } j+1 \notin \partial J, j+2 \in K \text{ and } y_{j+1}\beta_{j+1} - x_{j+1}z_{j+1} = 0 \\ \mathrm{Diag}(v^2, v, 1) & \text{if } j+2 \notin K. \end{cases}$$

Proof. Write ρ for the stable $\mathcal{O}_E$-lattice in the statement of the proposition and let $\mathfrak{M} \in \mathrm{S.D.Mod}^{[0,2],\mathrm{dd}}(\mathcal{O}_E)$ be a strongly divisible module of type τ such that $T_{\mathrm{st}}^{K_0,2}(\mathfrak{M}) \cong \rho$. Then setting $\overline{\mathfrak{M}} = \overline{\mathcal{S}}(\mathfrak{M})$ we have $T_{\mathrm{st}}^{K_0,2}(\overline{\mathfrak{M}}) \cong \bar{\rho}$ by (1.4.17). We refer to [HLM17] Section 2.3 for definitions concerning the notion of Breuil submodule. The key fact we want to use is Proposition 2.3.5 *loc. cit.*, which implies that since $\bar{\rho}$ is upper-triangular, we may find a filtration $0 \subset \overline{\mathfrak{M}}_2 \subset \overline{\mathfrak{M}}_1 \subset \overline{\mathfrak{M}}_0 = \overline{\mathfrak{M}}$ by Breuil submodules such that $T_{\mathrm{st}}^{K_0,2}(\overline{\mathfrak{M}}_i/\overline{\mathfrak{M}}_{i+1})|_{I_{K_0}} = \omega_f^{\sum_j (a^i_{f-j}+1)p^j}$ for $0 \leq i \leq 2$ (the notion of a quotient of Breuil submodules is explained in the reference cited).

Since $\overline{\mathfrak{M}}_i/\overline{\mathfrak{M}}_{i+1}$ is in particular a rank 1 object of $\mathrm{BrMod}^{[0,2],\mathrm{dd}}(\mathbb{F})$, the classification of rank 1 Breuil modules in Lemma 3.3.2 of [EGH13] together with the inclusion $\overline{\mathfrak{M}}^j_i/u \hookrightarrow \overline{\mathfrak{M}}^j/u$ and Lemma 1.4.19 imply that there exist for each $0 \leq j \leq f-1$ and $0 \leq i \leq 2$ integers $r^j_i \in [0, 2e]$ and $\psi_j(i) \in \{0, 1, 2\}$ such that $(\overline{\mathfrak{M}}^j_i/\overline{\mathfrak{M}}^j_{i+1})_2 = u^{r^j_i}(\overline{\mathfrak{M}}^j_i/\overline{\mathfrak{M}}^j_{i+1})$, and $\overline{\mathfrak{M}}^j_i/\overline{\mathfrak{M}}^j_{i+1}$ has a $\overline{S}^j$-basis upon which Δ acts by $\omega_f^{A_{\psi_j(i)}}$, and we have

$$r^j_i \equiv p^j \left(A_{\psi_{j+1}(i)} - A_{\psi_j(i)} \right) \mod e \tag{1.6.3}$$

and

$$\sum_j (a^i_{f-j}+1)p^j \equiv A_{\psi_0(i)} + \frac{1}{e} \sum_j r^j_i p^{f-j} \mod e. \tag{1.6.4}$$

Observe that despite the notation, ψ_j is not *a priori* a permutation. We now exploit these two congruences. Observe that by (1.6.3) we may write

$$r_i^j = \delta_i^j e + \sum_k \Big(a_{f-k+j}^{\sigma_{f-k+j}^{-1}(\psi_{j+1}(i))} - a_{f-k+j}^{\sigma_{f-k+j}^{-1}(\psi_j(i))} + \delta_{f-k+j+1\notin K}(\sigma_{f-k+j+1}^{-1}(\psi_j(i)) - \sigma_{f-k+j+1}^{-1}(\psi_{j+1}(i)))\Big) p^k \tag{1.6.5}$$

where $\delta_i^j \in [0,2]$. A rather unpleasant computation now shows that

$$A_{\psi_0(i)} + \frac{1}{e}\sum_j r_i^j p^{f-j} = \sum_j \Big(a_{f-j}^{\sigma_{f-j}^{-1}(\psi_{f-j}(i))} + \delta_{f-j+1\notin K}(1 - \sigma_{f-j+1}^{-1}(\psi_{f-j}(i)))\Big)p^j + \sum_j \delta_i^j p^{f-j}.$$

Hence by (1.6.4) we obtain

$$\sum_j (a_{f-j}^i + 1)p^j \equiv \sum_j \Big(a_{f-j}^{\sigma_{f-j}^{-1}(\psi_{f-j}(i))} + \delta_i^{f-j} + \delta_{f-j+1\notin K}(1 - \sigma_{f-j+1}^{-1}(\psi_{f-j}(i))\Big)p^j \mod e.$$

By 3-genericity we deduce that $\psi_j = \sigma_j$ for each $0 \le j \le f-1$, and also

$$\delta_i^j = \begin{cases} 1 & \text{if } j+1 \in K \\ \sigma_{j+1}^{-1}\sigma_j(i) & \text{if } j+1 \notin K. \end{cases}$$

At this point we pause to introduce some new notation. For $0 \le i, i' \le 2$ define

$$d_{ii'}^j = \sum_k \Big(a_{f-k+j}^{\sigma_{f-k+j}^{-1}(\sigma_{j+1}(i))} - a_{f-k+j}^{\sigma_{f-k+j}^{-1}(\sigma_j(i'))})p^k + \delta_{f-k+j+1\notin K}(\sigma_{f-k+j+1}^{-1}(\sigma_j(i')) - \sigma_{f-k+j+1}^{-1}(\sigma_{j+1}(i))\Big)p^k$$

and

$$b_{ii'}^j = \sum_k \Big(a_{f-k+j}^{\sigma_{f-k+j}^{-1}(\sigma_j(i))} - a_{f-k+j}^{\sigma_{f-k+j}^{-1}(\sigma_j(i'))})p^k + \delta_{f-k+j+1\notin K}(\sigma_{f-k+j+1}^{-1}(\sigma_j(i')) - \sigma_{f-k+j+1}^{-1}(\sigma_j(i'))\Big)p^k.$$

Then one checks that $\Big[p^j(A_{\sigma_{j+1}(i)} - A_{\sigma_j(i')})\Big] = u_{ii'}^j e + d_{ii'}^j$ and $\Big[p^j(A_{\sigma_j(i)} - A_{\sigma_j(i')})\Big] = v_{ii'}^j e + b_{ii'}^j$, where we define

$$u_{ii'}^j = \begin{cases} 0 & \text{if } \sigma_{j+1}^{-1}\sigma_j(i') \le i \\ 1 & \text{if } \sigma_{j+1}^{-1}\sigma_j(i') > i \end{cases}$$

and

$$v_{ii'}^j = \begin{cases} 0 & \text{if } \sigma_{j+1}^{-1}\sigma_j(i') \le \sigma_{j+1}^{-1}\sigma_j(i) \\ 1 & \text{if } \sigma_{j+1}^{-1}\sigma_j(i') > \sigma_{j+1}^{-1}\sigma_j(i). \end{cases}$$

Note that in this new notation (1.6.5) becomes

$$r_i^j = \delta_i^j e + d_{ii}^j.$$

We deduce that $\overline{\mathfrak{M}}^j$ has a basis $\underline{e}^j = (e_0^j, e_1^j, e_2^j)$ of type $\omega_f^{A_{\sigma_j(0)}} \oplus \omega_f^{A_{\sigma_j(1)}} \oplus \omega_f^{A_{\sigma_j(2)}}$ and $\overline{\mathfrak{M}}_2^j$ has a system of generators $\underline{f}^j = (f_0^j, f_1^j, f_2^j)$ of type $\omega_f^{A_{\sigma_{j+1}(0)}} \oplus \omega_f^{A_{\sigma_{j+1}(1)}} \oplus \omega_f^{A_{\sigma_{j+1}(2)}}$ such that

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\overline{\mathfrak{M}}_2)^j = \begin{pmatrix} u^{\delta_0^j e + d_{00}^j} & & \\ x_{01}^j u^{eu_{01}^j + d_{01}^j} & u^{\delta_1^j e + d_{11}^j} & \\ x_{02}^j u^{eu_{02}^j + d_{02}^j} & x_{12}^j u^{eu_{12}^j + d_{12}^j} & u^{\delta_2^j e + d_{22}^j} \end{pmatrix}$$

and

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_2)^j = \begin{pmatrix} \alpha_{00}^{j+1} & & \\ \alpha_{01}^{j+1} u^{ev_{01}^{j+1} + b_{01}^{j+1}} & \alpha_{11}^{j+1} & \\ \alpha_{02}^{j+1} u^{ev_{02}^{j+1} + b_{02}^{j+1}} & \alpha_{12}^{j+1} u^{ev_{12}^{j+1} + b_{12}^{j+1}} & \alpha_{22}^{j+1} \end{pmatrix}$$

where $x_{ii'}^j, \alpha_{ii'}^{j+1} \in \mathbb{F}[u^e]/(u^{ep})$ and $\alpha_{ii}^{j+1} \in (\mathbb{F}[u^e]/(u^{ep}))^\times$ (because $\varphi_2(\overline{\mathfrak{M}}_2^j)$ generates $\overline{\mathfrak{M}}^{j+1}$). This concludes the preliminary analysis of the structure of $\overline{\mathfrak{M}}$. We next do a base change to refine these matrices.

For the next claim, note that $\delta_{i'}^j - u_{ii'}^j \in [0,1]$ if $i \neq 2$.

Claim 1.6.6. We may assume that $x_{ii'}^j \in \mathbb{F} + \mathbb{F}u^{e(\delta_{i'}^j - u_{ii'}^j)}$ and that $\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_2)^j$ is diagonal with entries $\alpha_{ii}^{j+1} \in \mathbb{F}^\times$ for each $0 \leq j \leq f-1$.

Proof. Set

$$V^j = \begin{pmatrix} u^{\delta_0^j e + d_{00}^j} & & \\ x_{01}'^j u^{eu_{01}^j + d_{01}^j} & u^{\delta_1^j e + d_{11}^j} & \\ x_{02}'^j u^{eu_{02}^j + d_{02}^j} & x_{12}'^j u^{eu_{12}^j + d_{12}^j} & u^{\delta_2^j e + d_{22}^j} \end{pmatrix}$$

and

$$B^j = \begin{pmatrix} \alpha_{00}^j & & \\ \beta_{01}^j & \alpha_{11}^j & \\ \beta_{02}^j & \beta_{12}^j & \alpha_{22}^j \end{pmatrix}.$$

where $x_{ii'}'^j, \beta_{ii'}^j \in \mathbb{F}[u^e]/(u^{ep})$ are unknowns. The equation $\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_2)^{j-1} V^j = \mathrm{Mat}_{\underline{e},\underline{f}}(\overline{\mathfrak{M}}_2)^j B^j$ reduces to the equations

1. $\beta_{01}^j u^{\delta_1^j e + d_{11}^j} = u^{eu_{01}^j + d_{01}^j}[\alpha_{01}^j u^{e(\delta_0^j + v_{01}^j - u_{01}^j)} + \alpha_{11}^j x_{01}'^j - \alpha_{00}^j x_{01}^j]$

2. $\beta_{12}^j u^{\delta_2^j e + d_{22}^j} = u^{eu_{12}^j + d_{12}^j}[\alpha_{12}^j u^{e(\delta_1^j + v_{12}^j - u_{12}^j)} + \alpha_{22}^j x_{12}'^j - \alpha_{11}^j x_{12}^j]$

3. $\beta_{02}^j u^{\delta_2^j + d_{22}^j} = u^{eu_{02}^j + d_{02}^j}[\alpha_{02}^j u^{e(\delta_0^j + v_{02}^j - u_{02}^j)} + \alpha_{12}^j x_{01}'^j u^{e(v_{12}^j + u_{01}^j - u_{02}^j)} + \alpha_{22}^j x_{02}'^j - \alpha_{00}^j x_{02}^j] - \beta_{01}^j x_{12}^j u^{eu_{12}^j + d_{12}^j}$.

using that $b_{ii'}^j + d_{ki}^j - d_{ki'}^j = 0$ for any $0 \leq i, i', k \leq 2$. One checks easily that $\delta_i^j + v_{ii'}^j - u_{ii'}^j \geq 0$

for $i \neq i'$ and $v^j_{12} + u^j_{01} - u^j_{02} \geq 0$ so these equations make sense. For $i < i'$ define

$$\begin{aligned} c^j_{ii'} &= d^j_{i'i'} - d_{ii'} \\ &= \sum_k \Big(a_{f-k+j}^{\sigma^{-1}_{f-k+j}(\sigma_{j+1}(i'))} - a_{f-k+j}^{\sigma^{-1}_{f-k+j}(\sigma^{-1}_{j+1}(i))} \\ &\quad + \delta_{f-k+j+1 \notin K}(\sigma^{-1}_{f-k+j+1}\sigma_{j+1}(i) - \sigma^{-1}_{f-k+j+1}\sigma_{j+1}(i'))\Big) p^k. \end{aligned}$$

This lies in $(0, e)$.

Since α^j_{11} is a unit we can pick $x'^j_{01} \in \mathbb{F} + \mathbb{F}u^{e(\delta^j_1 - u^j_{01})}$ so that the expression in square brackets in the first equation is divisible by $u^{e(1+\delta_1 - u^j_{01})}$. Then we can choose $\beta^j_{01} \in u^{e-c^j_{01}}\mathbb{F}[u^e]/(u^{ep})$ satisfying the equation. Similarly we can pick $x'^j_{12} \in \mathbb{F} + \mathbb{F}u^{e(\delta^j_2 - u^j_{12})}$ and $\beta^j_{12} \in u^{e-c^j_{12}}\mathbb{F}[u^e]/(u^{ep})$ satisfying the second equation. Write $\beta^j_{01} = u^{e-c_{01}}\tilde{\beta}^j_{01}$. As $d^j_{12} - c^j_{01} - d^j_{02} = 0$ the third equation becomes

$$\begin{aligned} \beta^j_{02} u^{\delta^j_2 + d^j_{22}} = u^{eu^j_{02} + d^j_{02}}[\alpha^j_{02} u^{e(\delta^j_0 + v^j_{02} - u^j_{02})} + \alpha^j_{12} x'^j_{01} u^{e(v^j_{12} + u^j_{01} - u^j_{02})} \\ + \alpha^j_{22} x'^j_{02} - \alpha^j_{00} x^j_{02} - \tilde{\beta}^j_{01} x^j_{12} u^{e(1+u^j_{12} - u^j_{02})}]. \end{aligned}$$

Since α^j_{22} is a unit we can pick $x'^j_{02} \in \mathbb{F} + \mathbb{F}u^{e(\delta^j_2 - u^j_{02})}$ so that the expression in square brackets is divisible by $u^{e(1+\delta^j_2 - u^j_{02})}$. Then we can pick $\beta^j_{02} \in u^{e-c^j_{02}}\mathbb{F}[u^e]/(u^{ep})$ satisfying the equation.

Now note that for $i < i'$ we have

$$\begin{aligned} p(e - c^j_{ii'}) = e(p - 1 - a^{i'}_{j+1} + a^i_{j+1} + \delta_{j+2 \notin K}(\sigma^{-1}_{j+2}\sigma_{j+1}(i') - \sigma^{-1}_{j+2}\sigma_{j+1}(i))) \\ + \sum_k \Big(p - 1 - a_{f-k+j+1}^{\sigma^{-1}_{f-k+j+1}\sigma_{j+1}(i')} + a_{f-k+j+1}^{\sigma^{-1}_{f-k+j+1}\sigma_{j+1}(i)} \\ + \delta_{f-k+j+2 \notin K}(\sigma^{-1}_{f-k+j+2}\sigma_{j+1}(i') - \sigma^{-1}_{f-k+j+2}\sigma_{j+1}(i))\Big) p^k. \end{aligned}$$

This is $\geq 3e$ by 5-genericity. Thus $\varphi_{\overline{S}^j}(B^j)$ is diagonal with entries in $\mathbb{F}^\times$ modulo u^{3e}. Thus performing the change of basis in Lemma 1.4.21 allows us to *reset all notation* and assume from the beginning that $x^j_{ii'} \in \mathbb{F} + \mathbb{F}u^{(\delta^j_{i'} - u^j_{ii'})e}$ and $A^j \equiv \mathrm{Diag}(\alpha^{j+1}_{00}, \alpha^{j+1}_{11}, \alpha^{j+1}_{22}) \mod u^{3e}$ where $\alpha^{j+1}_{ii} \in \mathbb{F}^\times$. We now perform a second change of basis. This time let

$$V^j = \begin{pmatrix} u^{\delta^j_0 e + d^j_{00}} & & \\ \frac{\alpha^j_{00}}{\alpha^j_{11}} x^j_{01} u^{eu^j_{01} + d^j_{01}} & u^{\delta^j_1 e + d^j_{11}} & \\ \frac{\alpha^j_{00}}{\alpha^j_{22}} x^j_{02} u^{eu^j_{02} + d^j_{02}} & \frac{\alpha^j_{11}}{\alpha^j_{22}} x^j_{12} u^{eu^j_{12} + d^j_{12}} & u^{\delta^j_2 e + d^j_{22}} \end{pmatrix}$$

and

$$B^j = \mathrm{Diag}(\alpha^j_{00}, \alpha^j_{11}, \alpha^j_{22}).$$

Then clearly we have $\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_2)^{j-1} V^j \equiv \mathrm{Mat}_{\underline{e},\underline{f}}(\overline{\mathfrak{M}}_2)^j B^j \mod u^{3e}$ and making the change of basis in Lemma 1.4.21 proves the claim. $\square$

Claim 1.6.7. Building on Claim 1.6.6 we can in fact take $x^j_{ii'} = y^j_{ii'}u^{e(\delta^j_{i'}-u^j_{ii'})}$ with $y^j_{ii'} \in \mathbb{F}$ obeying

$$\begin{cases} y^j_{01} \in \mathbb{F}^\times x_j \\ y^j_{02} \in \mathbb{F}^\times y_j \\ y^j_{12} \in \mathbb{F}^\times z^j \\ y^j_{02} - y^j_{01}y^j_{12} \in \mathbb{F}^\times(y_j\beta_j - x_jz_j) \end{cases} \tag{1.6.8}$$

Proof. Let $\overline{M}$ denote the twist of $\mathcal{M}(\overline{\mathfrak{M}}^*)$ such that $V^*(M) \cong \bar\rho'|_{G_{(K_0)_\infty}}$. It follows from Lemma 1.5.9 that $\overline{M}$, has a basis $\underline{e}^j$ of type $\omega_f^{A_{00}-A_0} \oplus \omega_f^{A_{00}-A_1} \oplus \omega_f^{A_{00}-A_2}$ such that $\varphi_{\overline{M}}(\underline{e}^j) = \underline{e}^{j+1}\sigma_{j+2}({}^tV^{j+1})({}^tA^j)^{-1}\sigma^{-1}_{j+1}$. Hence $\overline{M}' := \overline{M}^{\Delta=1}$ has a basis $\underline{e}'^j = \underline{e}^j\mathrm{Diag}_i(u^{[p^j(A_i-A_{00})]})$ such that

$$\begin{aligned}\varphi_{\overline{M}'}(\underline{e}'^j) &= \underline{e}'^{j+1}\cdot \mathrm{Diag}_i(u^{-[p^{j+1}(A_i-A_{00})]})\sigma_{j+2}({}^tV^{j+1})({}^tA^j)^{-1}\sigma^{-1}_{j+1}\mathrm{Diag}_i(u^{p[p^j(A_i-A_{00})]})\\ &= \underline{e}'^{j+1}\cdot\sigma_{j+2}\mathrm{Diag}_i(u^{-[p^{j+1}(A_{\sigma_{j+2}(i)}-A_{00})]})\\ &\quad\cdot\begin{pmatrix}(\alpha^{j+1}_{00})^{-1}u^{\delta^{j+1}_0e+d^{j+1}_{00}} & (\alpha^{j+1}_{11})^{-1}x^{j+1}_{01}u^{eu^{j+1}_{01}+d^{j+1}_{01}} & (\alpha^{j+1}_{22})^{-1}x^{j+1}_{02}u^{eu^{j+1}_{02}+d^{j+1}_{02}}\\ & (\alpha^{j+1}_{11})^{-1}u^{\delta^{j+1}_1e+d^{j+1}_{11}} & (\alpha^{j+1}_{22})^{-1}x^{j+1}_{12}u^{eu^{j+1}_{12}+d^{j+1}_{12}}\\ & & (\alpha^{j+1}_{22})^{-1}u^{\delta^{j+1}_2e+d^{j+1}_{22}}\end{pmatrix}\\ &\quad\cdot\mathrm{Diag}_i(u^{p[p^j(A_{\sigma_{j+1}(i)}-A_{00})]})\sigma^{-1}_{j+1}\end{aligned}$$

Now using the fact that

$$\begin{aligned}p[p^j(A_{\sigma_{j+1}(i')}-A_{00})]+d^{j+1}_{ii'} - [p^{j+1}(A_{\sigma_{j+2}(i)}-A_{00})]\\ = e(a^{i'}_{j+1} - a^0_{j+1} + 1 + \delta_{j+2\notin K}(1-\sigma^{-1}_{j+2}\sigma_{j+1}(i')))\end{aligned} \tag{1.6.9}$$

as well as the fact that $\delta^{j+1}_i + 1 + \delta_{j+2\notin K}(1-\sigma^{-1}_{j+2}\sigma_{j+1}(i)) = 2$ for each $0 \le i \le 2$ the equation above becomes

$$\varphi_{\overline{M}'}(\underline{e}'^j) = \underline{e}'^{j+1}\cdot\sigma_{j+2}\begin{pmatrix}1 & x^{j+1}_{01}v^{u^{j+1}_{01}-\delta^{j+1}_1} & x^{j+1}_{02}v^{u^{j+1}_{02}-\delta^{j+1}_2}\\ & 1 & x^{j+1}_{12}v^{u^{j+1}_{12}-\delta^{j+1}_2}\\ & & 1\end{pmatrix}\mathrm{Diag}_i\left((\alpha^{j+1}_{ii})^{-1}v^{a^i_{j+1}-a^0_{j+1}+2}\right)\sigma^{-1}_{j+1}.$$

Making the change of basis

$$\underline{e}''^j = \underline{e}'^j\sigma_{j+1}\begin{pmatrix}1 & x^j_{01}v^{u^j_{01}-\delta^j_1} & x^j_{02}v^{u^j_{02}-\delta^j_2}\\ & 1 & x^j_{12}v^{u^j_{12}-\delta^j_2}\\ & & 1\end{pmatrix}$$

we get

$$\varphi_{\overline{M}'}(\underline{e}''^j) = \underline{e}''^{j+1} \cdot \mathrm{Diag}_i\left((\alpha_{ii}^{j+1})^{-1} v^{a^i_{j+1}-a^0_{j+1}+2}\right)\begin{pmatrix} 1 & \phi(x^j_{01} v^{u^j_{01}-\delta^j_1}) & \phi(x^j_{02} v^{u^j_{02}-\delta^j_2}) \\ & 1 & \phi(x^j_{12} v^{u^j_{12}-\delta^j_2}) \\ & & 1 \end{pmatrix}.$$

Note by Claim 1.6.6 $g^j_{ii'} := \phi(x^j_{ii'} v^{u^j_{ii'}-\delta^j_{i'}})$ lies in $\mathbb{F}v^{-p} + \mathbb{F}$. On the other hand, we know from Lemma 1.5.3 that there must exist matrices $X^j \in \mathrm{GL}_3(\mathbb{F}((v)))$ for $0 \le j \le f-1$ such that

$$\begin{aligned}\begin{pmatrix} (\alpha_{00}^{j+1})^{-1}v^2 & (\alpha_{00}^{j+1})^{-1} g^j_{01} v^2 & (\alpha_{00}^{j+1})^{-1} g^j_{02} v^2 \\ & (\alpha_{11}^{j+1})^{-1} v^{a^1_{j+1}-a^0_{j+1}+2} & (\alpha_{11}^{j+1})^{-1} g^j_{12} v^{a^1_{j+1}-a^0_{j+1}+2} \\ & & (\alpha_{22}^{j+1})^{-1} v^{a^2_{j+1}-a^0_{j+1}+2} \end{pmatrix} \phi(X^j) \\ = X^{j+1}\begin{pmatrix} \alpha^j v^2 & x^j v^2 & y^j v^2 \\ & \beta^j v^{a^1_{j+1}-a^0_{j+1}+2} & z^j v^{a^1_{j+1}-a^0_{j+1}+2} \\ & & \gamma^j v^{a^2_{j+1}-a^0_{j+1}+2} \end{pmatrix}.\end{aligned} \tag{1.6.10}$$

We now show that (1.6.10) implies $X^j \in T_3(\mathbb{F})$ and $g^j_{ii'} \in \mathbb{F}$. This is enough to prove Claim 1.6.7. Write $X^j = (f^j_{ii'})_{0 \le i,i' \le 2}$ for $f^j_{ii'} \in \mathbb{F}((v))$. First one shows that X^j is upper-triangular. For example, from the bottom left entry of (1.6.10) we obtain the equations

$$f^{j+1}_{20} = \left(\alpha_{22}^{j+1}\alpha_j\right)^{-1} v^{a^2_{j+1}-a^0_{j+1}} \phi(f^j_{20})$$

which imply each $f^j_{20} = 0$ by looking at the first nonzero term of positive degree and the term of lowest degree. Once it has been shown that X^j is upper-triangular, it is easy to see that its diagonal coefficients lie in $\mathbb{F}^\times$ by looking at the diagonal entries of (1.6.10). Finally the claim will then follow from arguments like that in Example 1.6.11 by considering the remaining entries of (1.6.10).

□

Example 1.6.11. Let us show that if $f_j \in \mathbb{F}((v))$ for $0 \le j \le f-1$ obey the equations $f_{j+1} = v^{-d_j}[c_j\phi(f_j) + \zeta_j v^{-p} + \theta_j]$ for some $d_j \in [2, p-2], c_j \in \mathbb{F}^\times$ and $\zeta_j, \theta_j \in \mathbb{F}$ then each $f_j, \zeta_j, \theta_j = 0$.

If f_0 has terms of positive degree and the minimal such degree is $n_0 > 0$, then the minimal positive degree term of f_j is n_j defined recursively by $n_{j+1} = pn_j - d_j$. But then the equation $n_0 = n_f$ is impossible since each $d_j \le p-2$. This shows that each f_j contains terms of nonpositive degree only. If the smallest term of f_0 is of degree $m_0 \le -2$, then the smallest term of f_j will be of degree m_j where $m_{j+1} = pm_j - d_j$. But again the equation $m_0 = m_f$ is impossible. This implies that each $f_j \in \mathbb{F}v^{-1} + \mathbb{F}$. But since $d_j \ge 2$ it is now easy to see that this implies $\zeta_j, \theta_j, f_j = 0$. □

From Claim 1.6.7 we conclude that in the original Breuil module $\overline{\mathfrak{M}}$ with $\underline{e}, \underline{f}$ as before we actually have

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\overline{\mathfrak{M}}_2)^j = \begin{pmatrix} u^{\delta_0^j e + d_{00}^j} & & \\ y_{01}^j u^{\delta_1^j e + d_{01}^j} & u^{\delta_1^j e + d_{11}^j} & \\ y_{02}^j u^{\delta_2^j e + d_{02}^j} & y_{12}^j u^{\delta_2^j e + d_{12}^j} & u^{\delta_2^j e + d_{22}^j} \end{pmatrix} \text{ and } \mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_2)^j = \mathrm{Diag}(\alpha_{00}^{j+1}, \alpha_{11}^{j+1}, \alpha_{22}^{j+1})$$

where $y_{ii'}^j \in \mathbb{F}$ satisfy (1.6.8). Define $\overline{M} = \mathcal{M}(\overline{\mathfrak{M}}^*) \in Y^{[0,2],\tau^{-1}}$. It follows from Lemma 1.5.9 that $\overline{M}$ has a framed basis $\underline{e}$ of type τ^{-1} such that

$$\varphi(\underline{e}^j) = \underline{e}^{j+1} \cdot \sigma_{j+2} \begin{pmatrix} (\alpha_{00}^{j+1})^{-1} u^{\delta_0^{j+1} e + d_{00}^{j+1}} & (\alpha_{11}^{j+1})^{-1} y_{01}^{j+1} u^{\delta_1^{j+1} e + d_{01}^{j+1}} & (\alpha_{22}^{j+1})^{-1} y_{02}^{j+1} u^{\delta_2^{j+1} e + d_{02}^{j+1}} \\ & (\alpha_{11}^{j+1})^{-1} u^{\delta_1^{j+1} e + d_{11}^{j+1}} & (\alpha_{22}^{j+1})^{-1} y_{12}^{j+1} u^{\delta_2^{j+1} e + d_{12}^{j+1}} \\ & & (\alpha_{22}^{j+1})^{-1} u^{\delta_2^{j+1} e + d_{22}^{j+1}} \end{pmatrix} \sigma_{j+1}^{-1}.$$

In the notation of Definition 1.5.10 we have $b_i^j = a_j^i + \delta_{j+1 \notin K}(1 - \sigma_{j+1}^{-1}(i))$ and it is easy to see that the orientation is $s_j = \sigma_{j+1} w_0$. (Technically, we defined $b_i^j \in [0, p-1]$ following [LLHLM18a] but this doesn't matter because we can shift each a_j^i by the same multiple of $p-1$ to achieve this.) Applying Definition 1.5.11 we compute

$$A_{\underline{e}}^j = w_0 \mathrm{Diag}_i \left(u^{-b_{(j+1)}^{\sigma_{j+2}(i)}} \right) \mathrm{Mat}_{\underline{e}}(\varphi_{\overline{M}})^j \mathrm{Diag}_i \left(u^{b_{(j+1)}^{\sigma_{j+1}(i)}} \right) \sigma_{j+1}^{-1} \sigma_{j+2} w_0$$

and using the definition $d_{ii'}^{j+1} = b_{(j+1)}^{\sigma_{j+2}(i)} - b_{(j+1)}^{\sigma_{j+1}(i')}$ this becomes

$$A_{\underline{e}}^j = \begin{pmatrix} (\alpha_{22}^{j+1})^{-1} & & \\ (\alpha_{22}^{j+1})^{-1} y_{12}^{j+1} & (\alpha_{11}^{j+1})^{-1} & \\ (\alpha_{22}^{j+1})^{-1} y_{02}^{j+1} & (\alpha_{11}^{j+1})^{-1} y_{01}^{j+1} & (\alpha_{00}^{j+1})^{-1} \end{pmatrix} w_0 \begin{pmatrix} v^{\delta_0^{j+1}} & & \\ & v^{\delta_1^{j+1}} & \\ & & v^{\delta_2^{j+1}} \end{pmatrix} \sigma_{j+1}^{-1} \sigma_{j+2} w_0.$$

If $j+2 \in K$ then the diagonal matrix is constant $= v$. If $j+2 \notin K$ then this is equal to

$$\begin{pmatrix} (\alpha_{22}^{j+1})^{-1} & & \\ (\alpha_{22}^{j+1})^{-1} y_{12}^{j+1} & (\alpha_{11}^{j+1})^{-1} & \\ (\alpha_{22}^{j+1})^{-1} y_{02}^{j+1} & (\alpha_{11}^{j+1})^{-1} y_{01}^{j+1} & (\alpha_{00}^{j+1})^{-1} \end{pmatrix} w_0 \sigma_{j+1}^{-1} \sigma_{j+2} w_0 \begin{pmatrix} v^2 & & \\ & v & \\ & & 1 \end{pmatrix}.$$

Since $T_{\mathrm{dd}}^*(\overline{M}) \cong T_{\mathrm{st}}^*(\overline{\mathfrak{M}}^*) \cong T_{\mathrm{st}}^{K_0,2}(\overline{\mathfrak{M}}) \cong \bar{\rho}$ using Lemma 1.5.9(2), the four statements of Proposition 1.6.2 now follow from (1.6.8) and the cases in Lemma 1.6.12. This completes the proof of Proposition 1.6.2. $\square$

The next lemma was used above. Its proof is an elementary computation.

Lemma 1.6.12. *Let* $A = \begin{pmatrix} a & & \\ u & b & \\ t & w & c \end{pmatrix} w_0 \varsigma w_0 \in \mathrm{GL}_3(\mathbb{F})$ *where* $\varsigma \in S_3$. *Write* $[\cdot]$ *for the image in the double coset space* $B_3(\mathbb{F}) \backslash \mathrm{GL}_3(\mathbb{F}) / B_3(\mathbb{F})$.

1. *If $uw - bt, u, w, t \neq 0$ then $[A] = [w_0]$ for any ς. If $\varsigma = w_0$ then the same is always true.*

2. *Assume that $\varsigma = \mathrm{id}$ and $u, w \neq 0$. If $t = 0$ then $[A] = [s_\alpha s_\beta]$. If $uw - bt = 0$ then $[A] = [s_\beta s_\alpha]$.*

Now let $B = \begin{pmatrix} a & & \\ u & b & \\ t & w & c \end{pmatrix} w_0 \varsigma w_0 \mathrm{Diag}(v^2, v, 1) \in \mathrm{GL}_3(\mathbb{F}((v)))$ where $\varsigma \in S_3$ and $a, b, c, u, w, t \in \mathbb{F}$. Write $[\cdot]$ for the image in the double coset space $\mathcal{I}(\mathbb{F}) \backslash \mathrm{GL}_3(\mathbb{F}((v)))/\mathcal{I}(\mathbb{F})$.

1. *We have $[B] = [\mathrm{Diag}(v^2, v, 1)]$ in any of the following cases:*
   - $\varsigma = \mathrm{id}$,
   - $\varsigma = s_\alpha$ *and* $w \neq 0$,
   - $\varsigma = s_\beta$ *and* $u \neq 0$,
   - ς *is arbitrary and* $u, w, t, uw - bt \neq 0$.

2. *If $\varsigma = s_\alpha$ and $w = 0$ then $[B] = [s_\beta \mathrm{Diag}(v^2, v, 1)]$. If $\varsigma = s_\beta$ and $u = 0$ then $[B] = s_\alpha[\mathrm{Diag}(v^2, v, 1)]$.*

□

1.7 Main results

Let $\bar{\rho} : G_{K_0} \to \mathrm{GL}_3(\mathbb{F})$ be a continuous representation satisfying (FL_δ) and let $\tau = \tau(\sigma, K)$ denote an inertial types from Definition 1.6.1. In this section we show our main results relating the Fontaine-Laffaille invariants of Section 1.2 to Frobenius eigenvalues of $D_{\mathrm{st}}^{K_0,2}(\rho)$ for potentially crystalline lifts ρ of Hodge type μ and inertial type τ. Given the shape computations of the previous section, the proofs of these results are simply a matter of combining the tables of [LLHLM18a] with the calculations of Sections 1.4-1.5.

Theorem 1.7.1. *Assume $\bar{\rho} \in (\mathrm{FL}_\delta^{**})$ with $\delta \geq 5$ and let ρ be a potentially crystalline of $\bar{\rho}$ of Hodge type μ and inertial type $\tau = \tau(\sigma, K)$. In particular the invariants of Definition 1.2.14 make sense. Write $D = D_{\mathrm{st}}^{K_0,2}(\rho)$ and let $\lambda_i \in E$ denote the eigenvalue of φ_D^f on the isotypic component $D^{\Delta = \widetilde{\omega}_f^{A_i}}$.*

1. *Assume that*
$$\sigma_j = \begin{cases} \mathrm{id} & j \in J \\ (02) & j \notin J \end{cases} \tag{1.7.2}$$
for some $J \subseteq \{0, \dots, f-1\}$. Then
$$\overline{p^f \lambda_1^{-1}} = (-1)^{|S|} Y_S$$

where

$$S = \{j \,|\, (j \in \partial J \text{ and } j+1 \notin K) \text{ or } (j \notin \partial J \text{ and } j+1 \in K)\}\,. \tag{1.7.3}$$

2. *Assume that*

$$\sigma_j = \begin{cases} \mathrm{id} & j \in J \\ (01) & j \notin J \end{cases} \tag{1.7.4}$$

for some $J \subseteq \{0, \dots, f-1\}$ *and* $K = \emptyset$. *Then*

$$\overline{p^{2f-|J|}\lambda_1^{-1}} = (-1)^{f-|J|+\frac{1}{2}|\partial J|} X_{J^c}.$$

3. *Assume that*

$$\sigma_j = \begin{cases} \mathrm{id} & j \in J \\ (12) & j \notin J \end{cases} \tag{1.7.5}$$

for some $J \subseteq \{0, \dots, f-1\}$ *and* $K = \emptyset$. *Then*

$$\overline{p^{f-|J|}\lambda_2^{-1}} = (-1)^{|J|+\frac{1}{2}|\partial J|} Z_{J^c}.$$

Notation 1.7.6. For any of the types τ appearing in this theorem, we define $m_\tau \in \mathbb{Z}, i_\tau \in \{0,1,2\}$ and $\mathrm{FL}_\tau \in \mathbb{F}^\times$ so that the conclusion of the theorem in each case is that

$$\overline{p^{m_\tau}\lambda_{i_\tau}^{-1}} = \mathrm{FL}_\tau.$$

Proof. We will do the calculation at first without any assumptions on σ or K. Let $\mathfrak{M}$ be a strongly divisible module of type τ^{-1} such that $T^*_{\mathrm{st}}(\mathfrak{M}) \cong \rho$. By Theorem 1.5.13 there exists $M \in Y^{\mu,\tau^{-1}}(\mathcal{O}_E)$ such that $\Theta_2(M) \cong \mathfrak{M}_0$. By Proposition 1.6.2, we know that M has a framed basis $\underline{e}$ such that $A^j_{\underline{e}}$ is given by the final column of Table 5 in [LLHLM18a] in row $w_0 v$ if $j+2 \in K$ and $\mathrm{Diag}(v^2, v, 1)$ if $j+2 \notin K$. That is

- if $j+2 \in K$ we have

$$A^j_{\underline{e}} = \begin{pmatrix} c^j_{00} & c^j_{00}c^j_{21}(c^{*j}_{20})^{-1} & c^j_{02} + (u^e+p)c^{*j}_{02} \\ 0 & (u^e+p)c^{*j}_{11} & (u^e+p)c^j_{12} \\ u^e c^{*j}_{20} & u^e c^j_{21} & c^j_{22} + (u^e+p)c'^j_{22} \end{pmatrix}; \quad \bar{A}^j_{\underline{e}} = \begin{pmatrix} & & u^e\bar{c}^{*j}_{02} \\ & u^e\bar{c}^{*j}_{11} & u^e\bar{c}^j_{12} \\ u^e\bar{c}^{*j}_{20} & u^e\bar{c}^j_{21} & u^e\bar{c}'^j_{22} \end{pmatrix} \tag{1.7.7}$$

  for $c^j_{ii'} \in \mathcal{O}_E, c^{*j}_{ii'} \in \mathcal{O}^\times_E$ subject to all condition coming from the requirement that the reduction mod ϖ_E of $A^j_{\underline{e}}$ be the matrix $\overline{A}^j_{\underline{e}}$ given, as well as the equations $c^j_{00}c^j_{22} = -pc^j_{02}c^{*j}_{20}$ and $c^j_{00}c'^j_{22} - c^{*j}_{02}c^{*j}_{20} = -pc^{*j}_{02}c^{*j}_{20}$. In particular $\mathrm{ord}_p(c^j_{00}), \mathrm{ord}_p(c^j_{22}) > 0$.

- if $j+2 \notin K$ we have

$$A^j_{\underline{e}} = \begin{pmatrix} (u^e+p)^2 b^{*j}_{00} & 0 & 0 \\ u^e(u^e+p)b^j_{10} & (u^e+p)b^{*j}_{11} & 0 \\ u^e(b^j_{20}+(u^e+p)b'^j_{20}) & u^e b^j_{21} & b^{*j}_{22} \end{pmatrix}; \quad \overline{A}^j_{\underline{e}} = \begin{pmatrix} u^{2e}\overline{b}^{*j}_{00} & 0 & 0 \\ u^{2e}\overline{b}^j_{21} & u^e\overline{b}^{*j}_{11} & 0 \\ u^{2e}\overline{b}'^j_{20} & u^e\overline{b}^j_{21} & \overline{b}^{*j}_{22} \end{pmatrix} \quad (1.7.8)$$

for $b^j_{ii'} \in \mathcal{O}_E, b^{*,j}_{ii'} \in \mathcal{O}_E^\times$ subject to all equations coming from the requirement that the reduction mod ϖ_E of $A^j_{\underline{e}}$ is the matrix $\overline{A}^j_{\underline{e}}$ given. Note that we have applied the monodromy equation $\overline{b}^j_{20} = 0$ in accordance with Theorem 1.5.13.

According to Definition 1.5.11 and Corollary 1.5.7, $\mathfrak{M}$ has a framed system of generators $(\underline{e},\underline{f})$ such that $\mathrm{Mat}_{\underline{e},\underline{f}}(\varphi_2)^j = c_2 \cdot \mathrm{id}$ and

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\mathfrak{M}_2)^j = s_j \mathrm{Diag}_i\left(u^{-b^{s_j(i)}_{(j)}}\right)\frac{E(u)^2}{\det(A^{j-1}_{\underline{e}})}(A^{j-1})^{\mathrm{adj}}\mathrm{Diag}_i\left(u^{b^{s_j(i)}_{(j)}}\right)s_j^{-1}$$

where $s_j = \sigma_{j+1}w_0$ and $b^i_j = a_j^{\sigma_j^{-1}(i)} + \delta_{j+1\notin K}(1-\sigma^{-1}_{j+1}(i))$ is the notation of Section 1.5, and adj denotes the adjugate matrix. By Lemma 1.4.14, D has a framed basis $\underline{e}'$ of type τ such that

$$\mathrm{Mat}_{\underline{e}'}(\varphi_D)^j = s_0(c_2)^{-1}\cdot s_0\left(\mathrm{Mat}_{\underline{e},\underline{f}}(\mathfrak{M}_2)^j\right).$$

[Note the conflict between two uses of s_0 here.] One easily computes that the right hand side of this formula is

$$\begin{cases} s_0(c_2)^{-1}\cdot\sigma^{-1}_{j+1}\mathrm{Diag}\left(\frac{-c^{j-1}_{00}}{c^{*(j-1)}_{02}c^{*(j-1)}_{20}}, \frac{p}{c^{*(j-1)}_{11}}, \frac{p^2}{c^{j-1}_{00}}\right)\sigma_{j+1} & j+1\in K \\ s_0(c_2)^{-1}\cdot\sigma^{-1}_{j+1}\mathrm{Diag}\left(\frac{-p^2}{b^{*(j-1)}_{22}}, \frac{p}{b^{*(j-1)}_{11}}, \frac{-1}{b^{*(j-1)}_{00}}\right)\sigma_{j+1} & j+1\notin K. \end{cases} \quad (1.7.9)$$

In particular we now easily calculate that

- if (1.7.2) holds then $\lambda_1 = s_0(c_2)^{-f}p^f\prod_{j+1\in K}(c^{*(j-1)}_{11})^{-1}\prod_{j+1\notin K}(b^{*(j-1)}_{11})^{-1}$
- if (1.7.4) holds and $K=\emptyset$ then $\lambda_1 = s_0(c_2)^{-f}p^{2f-|J|}(-1)^{f-|J|}\prod_{j+1\in J}(b^{*j}_{11})^{-1}\prod_{j+1\notin J}(b^{*j}_{22})^{-1}$
- if (1.7.5) holds and $K=\emptyset$ then $\lambda_2 = s_0(c_2)^{-f}p^{f-|J|}(-1)^{|J|}\prod_{j+1\in J}(b^{*j}_{00})^{-1}\prod_{j+1\notin J}(b^{*j}_{11})^{-1}$.

The next step is to compute the Fontaine-Laffaille module of $\bar{\rho}'$ in terms of these matrix entries. Write $\overline{A}^j_{\underline{e}} = F^jD^j$ where $F^j \in \mathrm{GL}_3(\mathbb{F})$ and

$$D^j := \begin{cases} u^e & j+2\in K \\ \mathrm{Diag}(u^{2e},u^e,1) & j+2\notin K. \end{cases}$$

Letting $\underline{e}$ also denote the framed basis of $\overline{M}$, a basis of $\overline{M}^{\Delta=1}$ is given by $\underline{e}'^j := \underline{e}^j\mathrm{Diag}(u^{b^0_{(j)}}, u^{b^1_{(j)}}, u^{b^2_{(j)}})$

with respect to which we calculate

$$\varphi_{\overline{M}'}(\underline{e}'^{j}) = \underline{e}'^{j+1} \cdot s_{j+1}F^{j}w_0\sigma_{j+2}^{-1}\sigma_{j+1}\mathrm{Diag}(u^{e(a^{i}_{j+1}+1)})\sigma_{j+1}^{-1}$$

in a manner similar to the computations surrounding (1.6.9). With respect to the basis $\underline{e}''^{j} = \underline{e}'^{j}s_jF^{j-1}s_j^{-1}s_{j-1}w_0$ we have

$$\mathrm{Mat}_{\underline{e}''}(\varphi_{\overline{M}'})^{j} = \mathrm{Diag}_i(u^{e(1+a^{i}_{j+1})})w_0F^{j-1}w_0\sigma_{j+1}^{-1}\sigma_j. \tag{1.7.10}$$

Let M_0 denote the Fontaine-Laffaille module such that $T^{*}_{\mathrm{cris}}(M_0) \cong \bar{\rho}$. By the full faithfulness of $\mathcal{F}$, Lemma 1.5.3(2) and (1.7.10) we deduce that M_0 has a basis $\underline{e}_0$ adapted to the filtration such that $\mathrm{Mat}_{\underline{e}_0}(\varphi)^{j} = w_0F^{j-1}w_0\sigma_{j+1}^{-1}\sigma_j$. By Lemma 1.2.4 and (1.2.5) there exists a unique lower-unipotent matrix L^{j} with coefficients in $\mathbb{F}$ such that $\widetilde{F}^{j} := L^{j}w_0F^{j-1}w_0\sigma_{j+1}^{-1}\sigma_j$ is the upper-triangular matrix of Corollary 1.2.9. Table 1.7.11 computes $\widetilde{F}^{j+1}$.

Table 1.7.11: The Fontaine-Laffaille matrix of Corollary 1.2.8 for $\bar{\rho}$.

$\sigma_{j+2}^{-1}\sigma_{j+1}$	$\widetilde{F}^{j+1}$ if $j+2 \in K$	$\widetilde{F}^{j+1}$ if $j+2 \notin K$
id	$\begin{pmatrix} \bar{c}'^{j}_{22} & \bar{c}^{j}_{21} & \bar{c}^{*j}_{20} \\ & \bar{c}^{*j}_{11}-\frac{\bar{c}^{j}_{12}\bar{c}^{j}_{21}}{\bar{c}'^{j}_{22}} & \frac{-\bar{c}^{j}_{12}\bar{c}^{*j}_{20}}{\bar{c}'^{j}_{22}} \\ & & \frac{\bar{c}^{*j}_{02}\bar{c}^{*j}_{11}\bar{c}^{*j}_{20}}{\bar{c}^{*j}_{11}\bar{c}'^{j}_{22}-\bar{c}^{j}_{21}\bar{c}^{j}_{12}} \end{pmatrix}$	$\begin{pmatrix} \bar{b}^{*j}_{22} & \bar{b}^{j}_{21} & \bar{b}'^{j}_{20} \\ & \bar{b}^{*j}_{11} & \bar{b}^{j}_{10} \\ & & \bar{b}^{*j}_{00} \end{pmatrix}$
(01)	$\begin{pmatrix} \bar{c}'^{j}_{22} & \bar{c}^{*j}_{20} & \bar{c}^{j}_{21} \\ & \frac{-\bar{c}^{*j}_{20}\bar{c}^{j}_{12}}{\bar{c}'^{j}_{22}} & \bar{c}^{*j}_{11}-\frac{\bar{c}^{j}_{12}\bar{c}^{j}_{21}}{\bar{c}'^{j}_{22}} \\ & & \frac{-\bar{c}^{*j}_{02}\bar{c}^{*j}_{11}}{\bar{c}^{j}_{12}} \end{pmatrix}$	$\begin{pmatrix} \bar{b}^{j}_{21} & \bar{b}^{*j}_{22} & \bar{b}'^{j}_{20} \\ & \frac{-\bar{b}^{*j}_{22}\bar{b}^{*j}_{11}}{\bar{b}^{j}_{21}} & \bar{b}^{j}_{10}-\frac{\bar{b}'^{j}_{20}\bar{b}^{*j}_{11}}{\bar{b}^{j}_{21}} \\ & & \bar{b}^{*j}_{00} \end{pmatrix}$
(12)	$\begin{pmatrix} \bar{c}^{j}_{21} & \bar{c}'^{j}_{22} & \bar{c}^{*j}_{20} \\ & \bar{c}^{j}_{12}-\frac{\bar{c}^{*j}_{11}\bar{c}'^{j}_{22}}{\bar{c}^{j}_{21}} & \frac{-\bar{c}^{*j}_{11}\bar{c}^{*j}_{20}}{\bar{c}^{j}_{21}} \\ & & \frac{-\bar{c}^{*j}_{02}\bar{c}^{*j}_{11}\bar{c}^{*j}_{20}}{\bar{c}^{*j}_{11}\bar{c}'^{j}_{22}-\bar{c}^{j}_{21}\bar{c}^{j}_{12}} \end{pmatrix}$	$\begin{pmatrix} \bar{b}^{*j}_{22} & \bar{b}'^{j}_{20} & \bar{b}^{j}_{21} \\ & \bar{b}^{j}_{10} & \bar{b}^{*j}_{11} \\ & & \frac{-\bar{b}^{*j}_{11}\bar{b}^{*j}_{00}}{\bar{b}^{j}_{10}} \end{pmatrix}$
(012)	$\begin{pmatrix} \bar{c}^{*j}_{20} & \bar{c}'^{j}_{22} & \bar{c}^{j}_{21} \\ & \bar{c}^{j}_{12} & \bar{c}^{*j}_{11} \\ & & \frac{-\bar{c}^{*j}_{02}\bar{c}^{*j}_{11}}{\bar{c}^{j}_{12}} \end{pmatrix}$	$\begin{pmatrix} \bar{b}'^{j}_{20} & \bar{b}^{*j}_{22} & \bar{b}^{j}_{21} \\ & \frac{-\bar{b}^{j}_{10}\bar{b}^{*j}_{22}}{\bar{b}'^{j}_{20}} & \bar{b}^{*j}_{11}-\frac{\bar{b}^{j}_{10}\bar{b}^{j}_{21}}{\bar{b}'^{j}_{20}} \\ & & \frac{-\bar{b}^{*j}_{00}\bar{b}^{*j}_{11}}{\bar{b}^{j}_{10}} \end{pmatrix}$
(021)	$\begin{pmatrix} \bar{c}^{j}_{21} & \bar{c}^{*j}_{20} & \bar{c}'^{j}_{22} \\ & \frac{-\bar{c}^{*j}_{11}\bar{c}^{*j}_{20}}{\bar{c}^{j}_{21}} & \bar{c}^{j}_{12}-\frac{\bar{c}'^{j}_{22}\bar{c}^{*j}_{11}}{\bar{c}^{j}_{21}} \\ & & \bar{c}^{j}_{02} \end{pmatrix}$	$\begin{pmatrix} \bar{b}^{j}_{21} & \bar{b}'^{j}_{20} & \bar{b}^{*j}_{22} \\ & \bar{b}^{j}_{10}-\frac{\bar{b}^{*j}_{11}\bar{b}'^{j}_{20}}{\bar{b}^{j}_{21}} & \frac{-\bar{b}^{*j}_{11}\bar{b}^{*j}_{22}}{\bar{b}^{j}_{21}} \\ & & \frac{-\bar{b}^{*j}_{00}\bar{b}^{*j}_{11}\bar{b}^{*j}_{22}}{\bar{b}_{11}\bar{b}'^{j}_{20}-\bar{b}^{j}_{10}\bar{b}^{j}_{21}} \end{pmatrix}$
(02)	$\begin{pmatrix} \bar{c}^{*j}_{20} & \bar{c}^{j}_{21} & \bar{c}'^{j}_{22} \\ & \bar{c}^{*j}_{11} & \bar{c}^{j}_{12} \\ & & \bar{c}^{*j}_{02} \end{pmatrix}$	$\begin{pmatrix} \bar{b}^{j}_{20} & \bar{b}^{j}_{21} & \bar{b}^{*j}_{22} \\ & \bar{b}^{*j}_{11}-\frac{\bar{b}^{j}_{10}\bar{b}^{j}_{21}}{\bar{b}'^{j}_{20}} & \frac{-\bar{b}^{j}_{10}\bar{b}^{*j}_{22}}{\bar{b}'^{j}_{20}} \\ & & \frac{\bar{b}^{*j}_{00}\bar{b}^{*j}_{11}\bar{b}^{*j}_{22}}{\bar{b}^{j}_{21}\bar{b}^{j}_{10}-\bar{b}'^{j}_{20}\bar{b}^{*j}_{11}} \end{pmatrix}$

From the matrices in this table it is possible to compute the Fontaine-Laffaille invariants of $\bar{\rho}$

directly. We obtain

- if (1.7.2) holds then defining S as in (1.7.3) we have $Y_S = (-1)^{f-|S|}\prod_{j+2\in K}\bar{c}_{11}^{*j}\prod_{j+2\notin K}\bar{b}_{11}^{*j}$.
- if (1.7.4) holds and $K=\emptyset$ then $X_{J^c} = (-1)^{\frac{1}{2}|\partial J|}\prod_{j+2\in J}\bar{b}_{11}^{*j}\prod_{j+2\notin J}\bar{b}_{22}^{*j}$.
- if (1.7.5) holds and $K=\emptyset$ then $Z_{J^c} = (-1)^{\frac{1}{2}|\partial J|}\prod_{j+2\in J}\bar{b}_{00}^{*j}\prod_{j+2\notin J}\bar{b}_{11}^{*j}$.

By comparing these items with the Frobenius eigenvalues computed just after (1.7.9) we complete the proof of Theorem 1.7.1. □

Proposition 1.7.12. *Assume $\bar{\rho}\in(\mathrm{FL}_\delta)$ with $\delta\geq 5$ and let ρ be a potentially crystalline lift of $\bar{\rho}$ of Hodge type μ and inertial type $\tau=\tau(\sigma,K)$. Write $D=D_{\mathrm{st}}^{K_0,2}(\rho)$ and let $\lambda_i\in E$ denote the eigenvalue of φ_D^f on the isotypic component $D^{\Delta=\widetilde{\omega}_f^{A_i}}$.*

1. *Assume that (1.7.4) holds and $K=\emptyset$. Then*
$$\mathrm{ord}_p(\lambda_1) = f + \sum_{\substack{j\notin J\\ j+1\notin J}} 1 + \sum_{\substack{j\in J\\ j+1\notin J\\ x_j\neq 0}} 1 + \sum_{\substack{j\in\partial J\\ x_j=0}}\epsilon_j$$
where $\epsilon_j\in(0,1)$.
2. *Assume that (1.7.5) holds and $K=\emptyset$. Then*
$$\mathrm{ord}_p(\lambda_2) = \sum_{\substack{j\notin J\\ j+1\notin J}} 1 + \sum_{\substack{j\in J\\ j+1\notin J\\ z_j\neq 0}} 1 + \sum_{\substack{j\in\partial J\\ z_j=0}}\epsilon_j$$
where $\epsilon_j\in(0,1)$.
3. *Assume that $\bar{\rho}\in(\mathrm{FL}_\delta^*)$ and (1.7.2) holds and $\partial J\subseteq K-1$. Then*
$$\mathrm{ord}_p(\lambda_1) = f + \sum_{\substack{j\notin\partial J\\ j+1\in K\\ y_j=0}}\epsilon_j - \sum_{\substack{j\notin\partial J\\ j+1\in K\\ y_j\beta_j-x_jz_j=0}}\epsilon'_j$$
where $\epsilon_j,\epsilon'_j\in(0,1)$.

Proof. The method of proof is the same as the previous theorem, the difference being that the vanishing of the various Fontaine-Laffaille parameters implies that $w(\bar{\rho},\tau)$ will take different values according to Proposition 1.6.2. We will show part 3 only, since the others are similar. Choose $\mathfrak{M}$ and M as in the proof of Theorem 1.7.1. By Proposition 1.6.2(4) and the corresponding column of Table 5 in [LLHLM18a] we can choose $\underline{e}$ and $A_{\underline{e}}^j$ such that

- If $j+1\in\partial J$ then $A_{\underline{e}}^j$ is exactly as in (1.7.7).

- If $j+2 \notin K$ then $A_{\underline{e}}^j$ is exactly as in (1.7.8).

- If $j+1 \notin \partial J$ and $y_{j+1}=0$ then

$$A_{\underline{e}}^j = \begin{pmatrix} c_{20}^j c_{01}^j (c_{21}^{*j})^{-1} & c_{01}^j & c_{02}^j + (u^e+p)c_{02}^{*j} \\ u^e c_{10}^{*j} & c_{11}^j & c_{12}^j + (u^e+p)c_{12}'^j \\ u^e c_{20}^j & u^e c_{21}^{*j} & c_{20}^j c_{12}^j (c_{10}^{*j})^{-1} + (u^e+p)c_{22}'^j \end{pmatrix}; \quad \bar{A}_{\underline{e}}^j = \begin{pmatrix} 0 & 0 & u^e \bar{c}_{02}^{*j} \\ u^e \bar{c}_{10}^{*j} & 0 & u^e \bar{c}_{12}'^j \\ 0 & u^e \bar{c}_{21}^{*j} & u^e \bar{c}_{22}'^j \end{pmatrix}$$

  satisfying the equations $c_{11}^j c_{20}^j = -pc_{10}^{*j} c_{21}^{*j}$, $c_{01}^j c_{12}^j = c_{11}^j c_{02}^j$, and $c_{21}^{*j} c_{02}^j - c_{22}'^j c_{01}^j = pc_{21}^{*j} c_{02}^{*j}$. In particular $\mathrm{ord}_p(c_{20}^j) \in (0,1)$.

- If $j+1 \notin \partial J$ and $y_{j+1}\beta_{j+1} - x_{j+1}z_{j+1} = 0$ then

$$A^j = \begin{pmatrix} c_{00}^j & (c_{20}^{*j})^{-1} c_{00}^j c_{21}^j + (u^e+p)c_{01}^{*j} & c_{02}^j \\ 0 & (u^e+p)c_{11}'^j & (u^e+p)c_{12}^{*j} \\ u^e c_{20}^{*j} & u^e c_{21}^j & c_{22}^j + (u^e+p)c_{22}'^j \end{pmatrix}; \quad \bar{A}^j = \begin{pmatrix} 0 & u^e \bar{c}_{01}^{*j} & 0 \\ 0 & 0 & u^e \bar{c}_{12}^{*j} \\ u^e \bar{c}_{20}^{*j} & u^e \bar{c}_{21}^j & u^e \bar{c}_{22}'^j \end{pmatrix}$$

  satisfying the equations $c_{00}^j c_{22}^j = -pc_{20}^{*j} c_{02}^j$ and $c_{11}'^j c_{00}^j c_{22}'^j - c_{11}'^j c_{02}^j c_{20}^{*j} = pc_{12}^{*j} c_{01}^{*j} c_{20}^{*j}$. In particular $\mathrm{ord}_p(c_{11}'^j) \in (0,1)$.

Exactly as in (1.7.9) we compute the Frobenius on D as

$$s_0(c_2) \cdot \mathrm{Mat}_{\underline{e}}(\varphi_D)^j =$$
$$\begin{cases} \cdot {}^t\sigma_{j+1}^{-1} \mathrm{Diag}\left(\frac{-c_{00}^{(}j-1)}{c_{02}^{*(j-1)} c_{20}^{*(j-1)}}, \frac{p}{c_{11}^{*(j-1)}}, \frac{p^2}{c_{00}^{(}j-1)} \right) {}^t\sigma_{j+1} & j \in \partial J \\ \cdot {}^t\sigma_{j+1}^{-1} \mathrm{Diag}\left(\frac{-p^2}{b_{22}^{*(j-1)}}, \frac{p}{b_{11}^{*(j-1)}}, \frac{-1}{b_{00}^{*(j-1)}} \right) {}^t\sigma_{j+1} & j+1 \notin K \\ \cdot {}^t\sigma_{j+1}^{-1} \mathrm{Diag}\left(\frac{-pc_{10}^{*(j-1)} c_{02}^{j-1}}{c_{12}^{j-1} c_{20}^{j-1} c_{02}^{*(j-1)}}, \frac{-pc_{20}^{j-1}}{c_{10}^{*(j-1)} c_{21}^{*(j-1)}}, \frac{pc_{12}^{j-1}}{c_{02}^{j-1} c_{10}^{*(j-1)}} \right) {}^t\sigma_{j+1} & j \notin \partial J, j+1 \in K \text{ and } y_j = 0 \\ \cdot {}^t\sigma_j^{-1} \mathrm{Diag}\left(\frac{-pc_{02}^{j-1} c_{11}'^{(j-1)}}{c_{22}^{j-1} c_{01}^{*(j-1)} c_{12}^{*(j-1)}}, \frac{p}{c_{11}'^{(j-1)}}, \frac{pc_{22}^{j-1}}{c_{02}^{j-1} c_{20}^{*(j-1)}} \right) {}^t\sigma_{j+1} & j \notin \partial J, j+1 \in K \text{ and } y_j\beta_j - x_jz_j = 0. \end{cases}$$

This proves part 3 of the proposition. □

Corollary 1.7.13. *Let $\bar{\rho}$ and ρ be as in the previous proposition. If there exists j_0 such that $x_{j_0} = 0$ then it is possible to choose J as in (1.7.4) and $K = \emptyset$ so that $\mathrm{ord}_p(\lambda_1) > 2f - |J|$. If there exists j_0 such that $z_{j_0} = 0$ then it is possible to choose J as in (1.7.5) and $K = \emptyset$ so that $\mathrm{ord}_p(\lambda_2) > f - |J|$. If $\bar{\rho} \in (\mathrm{FL}_\delta^*)$ and there exists j_0 such that either $y_{j_0} = 0$ or $y_{j_0}\beta_{j_0} - x_{j_0}z_{j_0} = 0$ then it is possible to choose J as in (1.7.2) and $J \subseteq K$ so that $\mathrm{ord}_p(\lambda_1) \neq f$.*

Proof. If there exists j_0 such that $x_{j_0} = 0$, then since $\bar{\rho}$ is nonsplit there must exist j_1 such that $x_{j_1} \neq 0$. Choose J so that $\partial J = \{j_0, j_1\}$ and $j_1 \in J$. Then from Proposition 1.7.12(1) we obtain $\mathrm{ord}_p(\lambda_1) = f + |J^c| + \epsilon_{j_0} > 2f - |J|$ and the claim follows. The proof of the second statement is similar using Proposition 1.7.12(2).

If $\bar{\rho} \in (\mathrm{FL}_{\delta}^{*})$ then pick $J = \emptyset$ so $\partial J = \emptyset$. If there exists j_0 such that either $y_{j_0} = 0$ or $y_{j_0}\beta_{j_0} - x_{j_0}z_{j_0} = 0$, define $K = \{j_0 + 1\}$. Then we deduce the claim from Proposition 1.7.12(3). $\square$

Chapter 2

Modular representation theory of GL_3

In this chapter we are mainly interested in modular representations of the finite group $\mathrm{GL}_n(\mathbb{F}_q)$ when $n = 3$. Representations of these groups behave differently depending on the characteristic ℓ of the coefficient field. When $\ell \neq p$ (referred to as the cross-characteristic case), they are closely related to the characteristic 0 theory and studied by way of Deligne-Lusztig characters. This case is better understood than the alternative $\ell = p$, which is called the defining characteristic case and which is the topic of this chapter. In this case the theory is closely tied to the representation theory of the algebraic group GL_n, since we can view $\mathrm{GL}_n(\mathbb{F}_q)$ as the set of fixed points under the Frobenius morphism of the algebraic group. For example, one of the possible classifications of irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_n(\mathbb{F}_q)$ (Proposition 2.2.1 below) makes use of this connection.

The main result of this chapter is a statement about the nonvanishing of certain group algebra operators on mod p principal series representations of $\mathrm{GL}_3(\mathbb{F}_q)$ (Theorem 2.3.2). The proof makes heavy use of representations of algebraic groups. Accordingly we begin in Section 2.1 with an overview of some background on the representation theory of reductive algebraic groups (fairly general ones, despite the title of this chapter). Then in Section 2.2 we generalize a result first due to Pillen [Pil97] about restricting algebraic representations of reductive groups to finite subgroups. Although this result turned out to not be necessary for the remainder of the chapter (see Remark 2.5.6), it is closely related to the method of proof of our main result. For this reason, and because it may be of independent interest, we decided to leave it in.

Section 2.3 we state the main result Theorem 2.3.2 of this chapter. Since the proof is long and technical, we also include in this section an overview of the strategy. In Section 2.4, we review the necessary background material and then procede with the proof in Section 2.5.

The final section is in a somewhat different vein than the others. In it we study how lifts of the group algebra operators to characteristic 0 behave on smooth principal series representations of $\mathrm{GL}(\mathbb{Q}_{p^f})$.

Throughout this section let $\overline{\mathbb{F}}_p$ be an algebraic closure of the prime field $\mathbb{F}_p$, and fix a choice of embedding $\omega_f : \mathbb{F}_q \hookrightarrow \overline{\mathbb{F}}_p$ via which we consider $\mathbb{F}_q$ as a subfield of $\overline{\mathbb{F}}_p$. When we speak of a representations V of an algebraic group G defined over $\overline{\mathbb{F}}_p$, we in particular envision V as an $\overline{\mathbb{F}}_p$-module with an action of $G(\overline{\mathbb{F}}_p)$. A choice of $\mathbb{F}_q$-structure via ω_f on G then induces an action of $G(\mathbb{F}_q)$ on V.

2.1 Representations of reductive groups

In this section we recall some important facts about representations of split reductive groups. All references in this section are to [Jan03] unless otherwise noted. Let G be a reductive group defined and split over a field k, which for the moment is arbitrary. In fact all such groups arise via base change from $\mathbb{Z}$. Let $T \subseteq G$ be a k-split maximal torus. This choice determines a root datum $(X(T), R, Y(T), R^\vee, \langle\, ,\, \rangle)$ where $R \subset X(T)$ is the set of roots together with a Weyl group W acting on $X(T)$. Choose a positive system $R^+ \subset R$ with associated simple roots S. This determines a Borel subgroup $B = TU$ (whose roots are R^+, note this is opposite the convention of [Jan03]). The irreducible G-modules may be classified in terms of T and R^+: let $X(T)^+ = \{\lambda \in X(T) : \langle \lambda, \alpha^\vee \rangle \geq 0 \quad \forall \alpha \in R^+\}$ be the set of dominant weights with respect to R^+. The isomorphism classes of irreducible G-modules are in bijection with $X(T)^+$. We write $L(\lambda)$ for the module corresponding to $\lambda \in X(T)^+$. It is uniquely determined among the irreducible modules by the fact that it has a (unique up to scalar) weight λ vector killed by U (II.2.7). We call a vector with this property (in any G-module) a *highest weight vector* of weight λ. For any $\lambda \in X(T)^+$ there exists a G-module $V(\lambda)$ universal for the property that it is generated by a highest weight vector of weight λ (II.2.13). Hence $L(\lambda) = \mathrm{cosoc}(V(\lambda))$. It is called a *Weyl module* of highest weight λ. Its character (as a T-module) is given by the well-known Weyl character formula involving $\rho \in X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ which is defined to be half the sum of the positive roots (II.5.10). We define the dot action of W on $X(T)$ by $w \cdot \lambda = w(\lambda + \rho) - \rho$. If $\mathrm{char}(k) = 0$ then $V(\lambda) = L(\lambda)$ is irreducible.

Now suppose that k is a perfect field of characteristic p. Since G arises via base change from $\mathbb{Z}$, it also arises via base change from $\mathbb{F}_p$ and hence has a Frobenius morphism $F : G \to G$. This gives rise to the Frobenius kernels $G_n := \ker(F^n) \leq G$. We get a restriction functor $\mathrm{Rep}(G) \to \mathrm{Rep}(G_n)$; G-modules are often studied this way. We also get a notion of Frobenius twist: if V is a G-module then $V^{[n]}$ denotes the twist of V by F^n. An important result is *Steinberg's factorization theorem* (II.3.16), which says that if $\lambda \in X_n(T)$ and $\mu \in X(T)^+$ then $L(\lambda + p^n\mu) \cong L(\lambda) \otimes L(\mu)^{[n]}$. Here, for any positive integer n we define $X_n(T) = \{\lambda \in X(T)^+ : \langle \lambda, \alpha^\vee \rangle < p^n \quad \forall \alpha \in S\}$.

In particular by induction we have

$$L\left(\sum_{j=0}^{n} \lambda_j p^j\right) \cong \bigotimes_{j=0}^{n} L(\lambda_j)^{[j]}$$

for $\lambda_1, \ldots, \lambda_n \in X_1(T)$.

The irreducible G_nT-modules are in bijection with $X(T)$ (II.9.6). For $\lambda \in X(T)$ we write $\widehat{L}_n(\lambda)$ for the associated irreducible. If $\lambda \in X_n(T)$ then $L(\lambda)|_{G_nT} \cong \widehat{L}_n(\lambda)$ (*loc. cit.*). We remark that if $X_n(T)$ is a system of representatives for $X(T)/p^nX(T)$ (which is the case below) then for $\mu \in X(T)^+$ and $\lambda \in X(T)$

$$L(\mu)|_{G_nT} \cong \widehat{L}_n(\lambda) \Rightarrow \mu - \lambda \in p^nX(T). \tag{2.1.1}$$

To see this, write $\mu = \mu_0 + p^n\mu_1$ with $\mu_0 \in X_n(T)$, $\mu_1 \in X(T)^+$. Note that $L(\mu_1)^{[n]}|_{G_nT} = \bigoplus_{\nu \in L(\mu_1)} p^n\nu$, so by Steinberg's factorization theorem and II.9.6(f) we have

$$L(\mu)|_{G_nT} \cong \bigoplus_{\nu \in L(\mu_1)} \widehat{L}_n(\mu_0 + p^n\nu)$$

and (2.1.1) follows.

We write $\widehat{Q}_n(\lambda)$ for the injective hull of $\widehat{L}_n(\lambda)$ in the category of G_nT-representations. If $p \geq 2h - 2$, where h is the Coxeter number of G (defined below) and $\lambda \in X_n(T)$ then $\widehat{Q}(\lambda)$ has a unique G-module structure, which in fact makes it the injective hull of $L(\lambda)$ in the full subcategory of p^n-bounded G-modules (this term is also defined below). In particular $\mathrm{soc}_G(\widehat{Q}(\lambda)) = L(\lambda)$. See II.11.11 for these results.

Many results in the representation theory of G make use of the geometry of the system of facets and alcoves in $X(T)$ under the dot action of the p-affine Weyl group $W_p := pR \rtimes W$. An *alcove* is a set of the form

$$\{\lambda \in X(T) : (n_\alpha - 1)p < \langle \lambda + \rho, \alpha^\vee \rangle < n_\alpha p \quad \forall \alpha \in R^+\}$$

for some collection of integers $(n_\alpha)_{\alpha \in R^+}$. The closure of any alcove (i.e. replace the $<$ by $\leq$ in the definition) is a fundamental domain for the action of W_p on $X(T)$ and W_p permutes the alcoves simply transitively (II.6.1(5)). Any two weights in the same W_p-orbit are said to be *linked*. The *linkage principle* (II.6.17) says that if $\mathrm{Ext}^1_G(L(\mu), L(\lambda)) \neq 0$ then λ and μ are linked. Since it is also true that irreducible modules exhibit no self-extensions (II.2.12(1)), it follows that $\mathrm{Ext}^1_G(L(\mu), L(\lambda)) = 0$ for any μ, λ lying in the same alcove.

2.2 Restriction of algebraic representations to finite subgroups

The goal of this section is to prove a theorem about restricting algebraic representations of a reductive group $G_{/\overline{\mathbb{F}}_p}$ to a finite subgroup Γ (Proposition 2.2.2), generalizing work of Pillen ([Pil97], Lemma 3.1) who showed it in the split case. Since the main result of this section won't be used in the sequel, the reader won't lose anything by skipping its proof. However, we do use some of the preparatory material in this section later on. The reason for our interest in this problem is best illustrated by referring to the summary of proof of the main result in

Section 2.3. There we make use of the device of embedding a representation of $\Gamma = \mathrm{GL}_3(\mathbb{F}_q)$ into a representation of the algebraic group $G = \prod_{j=0}^{f-1} \mathrm{GL}_{3/\overline{\mathbb{F}}_p}$. Now it is clear that Γ can be thought of as the set of fixed points of the Frobenius morphism Frob^f on $\mathrm{GL}_{3/\overline{\mathbb{F}}_p}$ coming from the standard $\mathbb{F}_q$-structure on the latter. In this case, when the finite subgroup is the fixed points of a Frobenius morphism coming from an $\mathbb{F}_q$-structure on G with respect to which G is split, Pillen's theorem applies. But for technical reasons, we really do want to embed Γ into the product group G which involves thinking of Γ as the set of fixed points for the Frobenius coming from the nonsplit $\mathbb{F}_p$-structure on G given by $\mathrm{Res}_{\mathbb{F}_q/\mathbb{F}_p}(\mathrm{GL}_3)$. In fact the ideas in Pillen's proof work in this generality and the differences are largely technical, but there are some small modifications to his argument. Moreover Pillen only considers the case of G semisimple and simply connected, so at the same time we extend the result to the case when G is reductive with simply connected derived subgroup. We remark that [LLHLM18b] also extends Pillen's original result to the case of split $\mathrm{GL}_{3/\mathbb{F}_q}$ via a different method. Again in this section all references are to [Jan03] unless otherwise noted.

For the remainder of the section let G be a connected reductive algebraic group defined and split over $\mathbb{F}_p$ such that the derived group G' is simply connected. Let T, R, R^+, S, W be as in Section 2.1. Then $T_1 := T \cap G'$ is a split maximal torus of G'. The restriction map $X(T) \to X(T_1)$ is surjective and identifies the roots R and the Weyl groups W of G and G'. It will be denoted by a bar: $\mu \mapsto \bar{\mu}$. Its kernel is $X^0(T) = \{\mu \in X(T) : \langle \mu, \alpha^\vee \rangle = 0 \ \forall \alpha \in R\} = X(T)^W$. Since G' is simply connected, for any simple root $\alpha \in S$ there exists $\omega'_\alpha \in X(T)$ satisfying $\langle \omega'_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta}$ for all $\beta \in S$. This element is any choice of lift of the fundamental weight ω_α for G', and is therefore unique up to an element of $X^0(T)$. Define $\rho' = \sum_{\alpha \in S} \omega'_\alpha$. We define the dot action of the extended p-affine Weyl group $\widetilde{W}_p := pX(T) \rtimes W$ on $X(T)$ by $w \cdot \mu = w(\mu + \rho') - \rho'$. Note that this is independent of the choice of lifts ω'_α.

Decompose $R = \bigsqcup_{j=0}^{f-1} R_j$ into irreducible components, and let $\alpha_{0,j}^\vee$ denote the longest positive coroot of R_j. Then $\lambda \leq \mu$ in $X(T)$ implies $\langle \lambda, \alpha_{0,j}^\vee \rangle \leq \langle \mu, \alpha_{0,j}^\vee \rangle$ for each j and for $\lambda \in X(T)^+ := \{\lambda \in X(T) \,|\, \langle \lambda, \alpha^\vee \rangle \geq 0 \ \forall \alpha \in S\}$ we have $\langle \lambda, \alpha_{0,j}^\vee \rangle \geq 0$. Moreover if $\lambda \in X(T)^+$ then $\langle \lambda, \alpha^\vee \rangle \leq \max_{0 \leq j \leq f-1} \langle \lambda, \alpha_{0,j}^\vee \rangle$ for each $\alpha \in R^+$, and $\sum_{j=0}^{f-1} \langle \lambda, \alpha_{0,j}^\vee \rangle = 0$ iff $\lambda \in X^0(T)$. Let $h = \max\{\langle \rho', \alpha^\vee \rangle + 1 \,|\, \alpha \in R^+\}$ be the maximum among the Coxeter numbers of the R_j. For example, in the case $G = \prod_{j=0}^{f-1} \mathrm{GL}_n$, $h = n$.

We say that a weight $\lambda \in X(T)$ is *δ-deep in its alcove*, or just is *δ-deep*, if $\langle \lambda + \rho', \alpha^\vee \rangle \in [\delta, p - \delta] \mod p$ for all $\alpha \in R^+$. A G-module is said to be δ-deep if the highest weights of all of its irreducible factors are δ-deep. We call a G-module *p^n-bounded* if all the highest weights of its irreducible factors μ satisfy $\langle \mu, \alpha_{0,j}^\vee \rangle < 2p^n(h-1)$ for each $0 \leq j \leq f-1$. This implies that G is p^n-bounded in the sense of [Jan03] p.333.

Since G' is semisimple and simply connected, each $\lambda \in X(T)^+$ may be written as $\mu_0 + p^n \mu_1$ where $\mu_0 \in X_n(T)$ and $\mu_1 \in X(T)^+$ (II.3.15, Remark (2)). In particular (2.1.1) holds.

Let π be a finite-order automorphism of the based root datum of G given by the choice of R^+. Since it preserves $\langle \, , \rangle$ and the set of (simple) roots by definition, it permutes the $\alpha_{0,j}^\vee$ and preserves $X^0(T)$. It moreover permutes the ω'_α and preserves ρ', both up to elements of $X^0(T)$, and hence preserves the dot action. It permutes W, and preserves $\leq$. If we fix a pinning $(x_\alpha)_{\alpha\in R}$ of G as in II.1.2, then by Theorem II.1.15 it follows that π may be lifted to an automorphism of (G, B, T, x_α) over $\mathbb{F}_p$ such that, using π for both maps, we have $\pi(\lambda) = \lambda \circ \pi^{-1}|_T$. It necessarily commutes with the Frobenius endomorphism of G, since $\mathrm{Frob}^n \circ \pi$ and $\pi \circ \mathrm{Frob}^n$ induce the same map $U_\alpha \to U_{\pi(\alpha)}$ and by II.1.3(10). Identifying G with its $\overline{\mathbb{F}}_p$-points, we let $F = \mathrm{Frob}^n \circ \pi = \pi \circ \mathrm{Frob}^n$ and set $\Gamma_n = G^F$. This is a finite group, which is the set of $\mathbb{F}_{p^n}$-points of a form of G defined over $\mathbb{F}_{p^n}$ (see Proposition 1.4.27 in [GM]). In fact the map taking π to this form induces a bijection between conjugacy classes of finite-order automorphisms of the based root datum and forms of G defined over $\mathbb{F}_{p^n}$. This is seen using the Galois cohomology classification of forms of G. Any G-module becomes a Γ_n-module by restriction. The next result is Proposition 1.3 of the Appendix in [Her09].

Proposition 2.2.1. *If $\lambda \in X_n(T)$ then $L(\lambda)$ restricts to an irreducible representation of Γ_n and $L(\lambda)|_{\Gamma_n} \cong L(\lambda')|_{\Gamma_n}$ iff $\lambda - \lambda' \in (p^n - \pi)X^0(T)$. All irreducible $\overline{\mathbb{F}}_p$-representations of Γ_n arise this way.* □

We write $F(\lambda) = L(\lambda)|_{\Gamma_n}$ and let $U_n(\lambda)$ be the injective hull of $F(\lambda)$ in the category of $\overline{\mathbb{F}}_p[\Gamma_n]$-modules.

The main result of this section is the next proposition.

Proposition 2.2.2. *Assume that $h \geq 2$ and let $\delta \in (0, p-h)$ be an integer. Let V be a G-module such that each irreducible constituent $L(\mu)$ of V obeys $\max_{0\leq j\leq f-1}\langle \mu, \alpha^\vee_{0,j}\rangle < \delta p^n$. If V is $(\delta + h - 1)$-deep, $\mathrm{soc}^i_G(V) = \mathrm{soc}^i_{\Gamma_n}(V)$ for all $i \geq 1$.*

The proof of this proposition closely follows the proof of Lemma 3.1 of [Pil97], which is the case where G is semisimple and $\pi = \mathrm{id}$. However, since the proof therein references multiple sources and because there are some small but nontrivial modifications to be made as well as a small gap to be filled, we have opted to present the complete argument.

The first part of the next result is often known as the "translation principle" or the "tensor product theorem".

Proposition 2.2.3. *Suppose that $\lambda \in X(T)^+$ lies in the interior of a p-alcove and that $\nu \in X(T)^+$ is small enough so that $\lambda + \nu'$ lies in the same alcove as λ for all weights ν' of $L(\nu)$. Then*

$$L(\lambda) \otimes L(\nu) = \bigoplus_{\nu' \in L(\nu)} \dim(L(\nu)_{\nu'}) \cdot L(\lambda + \nu').$$

Moreover if $\lambda \in X_n(T)$ then

$$\widehat{Q}_n(\lambda) \otimes L(\nu) = \bigoplus_{\nu' \in L(\nu)} \dim(L(\nu)_{\nu'}) \cdot \widehat{Q}_n(\lambda + \nu')$$

as G_nT-modules, and if $p \geq 2h - 2$ then also as G-modules.

Proof. The first isomorphism is well-known in this generality: see [Hum06] Proposition 6.4. The second isomorphism of G_nT-modules follows exactly as in [Pil93], Lemma 5.1. In order to upgrade it to an isomorphism of G-modules, we proceed as follows: the right hand side is injective in the full subcategory of p^n-bounded G-modules. The G-socle of the right-hand side is $\bigoplus_{\nu' \in L(v)} \dim(L(\nu)_{\nu'}) \cdot L(\lambda+\nu')$ and this injects into the left hand side by the first isomorphism. Since $\widehat{Q}_n(\lambda) \otimes L(\nu)$ is p^n-bounded (II.11.11), injectivity gives an inclusion of G-modules from the right-hand side to the left; by comparing dimensions it must be an isomorphism. $\square$

We start with the easier inclusion, whose proof is essentially the same as Lemma 2.3 of [Pil97].

Lemma 2.2.4. *Let V be a G-module and $\delta \geq 1$ an integer such that each irreducible constituent $L(\mu)$ of V obeys $\langle \mu, \alpha_{0,j}^\vee \rangle < \delta p^n$ for each $0 \leq j \leq f-1$. If V is δ-deep then $\mathrm{soc}^i_G(V) \subseteq \mathrm{soc}^i_{\Gamma_n}(V)$ for all $i \geq 1$.*

Proof. Since the hypotheses on V are preserved under taking quotients, it suffices to show that $\mathrm{soc}_G(V)$ is completely reducible as a Γ_n-module. The G-socle of V is a direct sum of modules $L(\mu_0 + p^n\mu_1) \cong L(\mu_0) \otimes L(\mu_1)^{[n]}$ where $\mu_0 \in X_n(T)$ and $\mu_1 \in X(T)^+$. As a Γ_n-module this is isomorphic to $L(\mu_0) \otimes L(\pi(\mu_1))$. Since V is δ-deep, μ_0 has distance $\geq \delta$ from any alcove wall and by assumption we must have $\langle \mu_1, \alpha^\vee \rangle < \delta$ for all $\alpha \in R^+$. Since π preserves the positive roots, it follows that $\langle \pi(\mu_1), \alpha^\vee \rangle < \delta$ for all $\alpha \in R^+$. Now Proposition 2.2.3 implies that $L(\mu_0) \otimes L(\pi(\mu_1))$ is a direct sum of irreducible algebraic representations with highest weights in the same alcove as that of μ_0. In particular they are in $X_n(T)$. Therefore $L(\mu_0) \otimes L(\pi(\mu_1))$ is a semisimple Γ_n-representation by Proposition 2.2.1. $\square$

Proposition 2.2.5. *Let $\lambda \in X_n(T)$. Let p be an odd prime. Assume that $p \geq 2h-2$ so that $\widehat{Q}_n(\lambda)$ has a unique G-module structure. If λ is δ-deep in its alcove, where $1 \leq \delta := \max_{0 \leq j \leq f-1} \langle \rho', \alpha_{0,j}^\vee \rangle = h-1$ then $\widehat{Q}_n(\lambda)|_{\Gamma_n} = U_n(\lambda)$.*

Proof. By [Jan81], 2.10, $U_n(\lambda)$ occurs with multiplicity 1 inside $\widehat{Q}_n(\lambda)$ and it is enough to show that if $\mu \in X_n(T)$ is such that $\mu - \lambda \notin (p^n - \pi)X^0(T)$ then we have $[L(\mu) \otimes L(\pi(\nu)) : L(p^n\nu + \lambda)]_G = 0$ for all $\nu \in X(T)^+$. If this was nonzero we would have $p^n\nu + \lambda \leq \mu + \pi(\nu)$, hence $(p^n - \pi)\nu \leq \mu - \lambda$. Choose j such that $\langle \nu, \alpha_{0,j}^\vee \rangle$ is maximal. Then

$$(p^n - 1)\langle \nu, \alpha_{0,j}^\vee \rangle \leq \langle (p^n - \pi)\nu, \alpha_{0,j}^\vee \rangle \leq \langle \mu - \lambda, \alpha_{0,j}^\vee \rangle < (p^n - 1)\langle \rho', \alpha_{0,j}^\vee \rangle$$

where the last inequality is because $\mu, \lambda \in X_n(T)$ and λ is 1-deep in its alcove. Thus $\max_j \langle \nu, \alpha_{0,j}^\vee \rangle < \delta$. It follows that $\langle \pi\nu, \alpha^\vee \rangle \in [0, \delta)$ for each $\alpha \in R^+$.

Now if $[L(\mu) \otimes L(\pi(\nu)) : L(p^n\nu + \lambda)]_G \neq 0$ then $[L(\mu) \otimes L(\pi(\nu)) : p^n\nu \otimes L(\lambda)]_{G_nT} \neq 0$. Since $\widehat{Q}_n(\lambda)$ is the projective envelope of $L(\lambda)$ in the category of G_nT-modules (II.11.5(3)), we get a nonzero map $\widehat{Q}_n(\lambda) \to L(\mu - p^n\nu) \otimes L(\pi\nu)$. Therefore

$$\mathrm{Hom}_{G_nT}(\widehat{Q}_n(\lambda) \otimes L(\pi\nu)^\vee, L(\mu - p^n\nu)) \neq 0.$$

Since λ is δ-deep, we get by the translation principle

$$\mathrm{Hom}_{G_nT}(\widehat{Q}_n(\lambda+\nu'), L(\mu-p^n\nu)) \neq 0$$

for some weight $\nu' \in L(\pi\nu)^\vee$. By (2.1.1) this implies that μ and $\lambda+\nu'$ differ by an element of $p^nX(T)$. But since $\mu, \lambda+\nu' \in X_n(T)$, they actually differ by an element of $p^nX^0(T)$, say $\mu = \lambda+\nu'+p^nz$. It follows that $p^n\nu+\lambda \leq \mu+\pi\nu = \lambda+\nu'+\pi\nu+p^nz$, hence $(p^n-\pi)\nu \leq \nu'+p^nz$. Picking j as before we have

$$(p^n-1)\langle \nu, \alpha_{0,j}^\vee\rangle \leq \langle \nu'+p^nz, \alpha_{0,j}^\vee\rangle = \langle \nu', \alpha_{0,j}^\vee\rangle \leq \langle -w_0\pi\nu, \alpha_{0,j}^\vee\rangle \leq \langle \nu, \alpha_{0,j}^\vee\rangle.$$

Since $p \neq 2$ it follows that $\langle \nu, \alpha_{0,j}^\vee\rangle = 0$ which implies by the choice of j that $\nu \in X^0(T)$. But then $L(\mu)\otimes L(\pi(\nu)) = L(\mu+\pi\nu)$ so we obtain $\lambda+p^n\nu = \mu+\pi\nu$, hence $\mu-\lambda \in (p^n-\pi)X^0(T)$, which is a contradiction. □

Now we prove Proposition 2.2.2. Note since V is $(h-1)$-deep, we automatically have $p \geq 2h-2$.

Proof. Since the hypotheses on V are preserved under taking quotients, we just need to show that the socles coincide. Assume that $\mathrm{soc}_G(V) = \bigoplus L(\lambda_0+p^n\lambda_1)$ where each $\lambda_0 \in X_n(T)$ and $\lambda_1 \in X(T)^+$. Note that $L(\lambda_0+p^n\lambda_1) = L(\lambda_0)\otimes L(p^n\lambda_1) \hookrightarrow \widehat{Q}_n(\lambda_0)\otimes L(p^n\lambda_1)$. The first step is to show that the embedding $\mathrm{soc}_G(V) \hookrightarrow N := \bigoplus \widehat{Q}_n(\lambda_0)\otimes L(p^n\lambda_1)$ extends to all of V. It is enough to show that for each composition factor $L(\mu) = L(\mu_0+p^n\mu_1)$ of V that $\mathrm{Ext}^1_G(L(\mu), \widehat{Q}_n(\lambda_0)\otimes L(p^n\lambda_1)) = 0$ because of the exact sequence

$$\cdots \to \mathrm{Hom}_G(V,N) \to \mathrm{Hom}_G(\mathrm{soc}_G V, N) \to \mathrm{Ext}^1_G\left(\frac{V}{\mathrm{soc}_G V}, N\right) = 0.$$

Since the $\widehat{Q}_n(\cdot)$ are injective G_n-modules, so are the $\widehat{Q}_n(\lambda_0)\otimes L(p^n\lambda_1)$ by I.3.10(c) and one has for $\mu = \mu_0+p^n\mu_1$ that

$$\mathrm{Ext}^1_G(L(\mu), \widehat{Q}_n(\lambda_0)\otimes L(p^n\lambda_1)) = \mathrm{Ext}^1_G(L(\mu_1)^{[n]}, \mathrm{Hom}(L(\mu_0), \widehat{Q}_n(\lambda_0)\otimes L(\lambda_1)^{[n]}))$$

since this is a general property of Ext. Since G_n acts trivially on $L(\mu_1)^{[n]}$ we have $\mathrm{Hom}_G(L(\mu_1)^{[n]}, -) = \mathrm{Hom}_{G/G_n}(L(\mu_1), (-)^{G_n})$ so passing this equality to the derived functors we obtain that the above is equal to

$$\begin{aligned}&\mathrm{Ext}^1_{G/G_n}(L(\mu_1)^{[n]}, \mathrm{Hom}_{G_n}(L(\mu_0), \widehat{Q}_n(\lambda_0)\otimes L(\lambda_1)^{[n]}))\\ &\qquad = \mathrm{Ext}^1_{G/G_n}(L(\mu_1)^{[n]}, \mathrm{Hom}_{G_n}(L(\mu_0), \widehat{Q}_n(\lambda_0))\otimes L(\lambda_1)^{[n]}).\end{aligned}$$

Now since $\mathrm{soc}_{G_n}(\widehat{Q}_n(\lambda_0)) = L(\lambda_0)|_{G_n}$, if $\mathrm{Hom}_{G_n}(L(\mu_0), \widehat{Q}_n(\lambda_0))$ is nonzero we must have $\mu_0-\lambda_0 = p^n\nu \in p^nX^0(T)$. Then the Hom space is 1-dimensional by II.3.10(3). Since it contains

$\mathrm{Hom}_G(L(\mu_0), \widehat{Q}_n(\lambda_0))$, which is nonzero, the two must be equal and hence $\mathrm{Hom}_{G_n}(L(\mu_0), \widehat{Q}_n(\lambda_0))$ is the trivial G-module. So in this case the Ext space reduces to $\mathrm{Ext}^1_{G/G_n}(L(\mu_1)^{[n]}, L(\lambda_1)^{[n]}) = \mathrm{Ext}^1_G(L(\mu_1), L(\lambda_1))$. The boundedness condition on V implies that $\langle \lambda_1, \alpha^\vee \rangle, \langle \mu_1, \alpha^\vee \rangle \in [0, \delta)$ for all $\alpha \in R^+$. Since $\delta < p - h$ both λ_1 and μ_1 lie in the lowest alcove and hence this space of extensions is 0 as desired.

We know from Proposition 2.2.4 that the Γ_n-socle contains the G-socle of V. Since $\bigoplus L(\lambda_0) \otimes L(p^n\lambda_1) = \mathrm{soc}_G V \subseteq \mathrm{soc}_G N \subseteq \mathrm{soc}_{\Gamma_n} N$, if we can show that $\bigoplus L(\lambda_0) \otimes L(p^n\lambda_1)$ is equal to the Γ_n-socle of N then we will be done because $\mathrm{soc}_{\Gamma_n} V$ sits between $\mathrm{soc}_G V$ and $\mathrm{soc}_{\Gamma_n} N$. Therefore we just need to show that the Γ_n-socle of $\widehat{Q}_n(\lambda_0) \otimes L(p^n\lambda_1)$ is equal to $L(\lambda_0) \otimes L(p^n\lambda_1)$.

As a Γ_n-module, $\widehat{Q}_n(\lambda_0) \otimes L(p^n\lambda_1)$ is equal to $\widehat{Q}_n(\lambda_0) \otimes L(\pi(\lambda_1))$. We have $\langle \pi(\lambda_1), \alpha^\vee \rangle \in [0, \delta)$ for all $\alpha \in R^+$, and the deepness on V implies that λ_0 is $(\delta + h - 1)$-deep. It follows by the translation principle that as a G-module we have

$$\widehat{Q}_n(\lambda_0) \otimes L(\pi(\lambda_1)) = \bigoplus_{\nu' \in L(\pi\lambda_1)} \dim(L(\pi\lambda_1)_{\nu'}) \cdot \widehat{Q}_n(\lambda_0 + \nu').$$

All the weights $\lambda_0 + \nu'$ are still $(h-1)$-deep. By Proposition 2.2.5, restricting this to Γ_n we get

$$\bigoplus_{\nu' \in L(\pi\lambda_1)} \dim(L(\pi\lambda_1)_{\nu'}) \cdot U_n(\lambda_0 + \nu')$$

which has Γ_n-socle equal to $\bigoplus_{\nu' \in L(\pi\lambda_1)} \dim(L(\pi\lambda_1)_{\nu'}) \cdot L(\lambda_0 + \nu')$. By the translation principle, this is the restriction to Γ_n of $L(\lambda_0) \otimes L(\pi\lambda_1)$ which is the same as the restriction to Γ_n of $L(\lambda_0) \otimes L(p^n\lambda_1)$. This proves the claim. $\square$

2.3 Summary of main results

From now on we specialize the notation of the previous section to the case where $G = \prod_{j=0}^{f-1} \mathrm{GL}_3$ and T and B are the standard diagonal torus and upper-triangular Borel respectively. We identify $X(T)$ with $(\mathbb{Z}^3)^f$ and $L(\lambda)$ with $\boxtimes_{j=0}^{f-1} L(\lambda_j)$, adopting the convention that when $\lambda_j \in \mathbb{Z}^3$, $L(\lambda_j)$ refers to the irreducible representation of the jth factor GL_3 of highest weight λ_j. Let $\pi : G \to G$ be the left shift automorphism $(g_j)_{j=0}^{f-1} \mapsto (g_{j+1})_{j=0}^{f-1}$. The map

$$g \mapsto (\mathrm{Frob}^f(\omega_f(g)), \mathrm{Frob}^{f-1}(\omega_f(g)), \ldots, \mathrm{Frob}(\omega_f(g))) \tag{2.3.1}$$

is an isomorphism $\mathrm{GL}_3(\mathbb{F}_q) \xrightarrow{\sim} \Gamma_1$ and we let $F(\lambda)$ denote the irreducible representation of $\mathrm{GL}_3(\mathbb{F}_q)$ coming from Proposition 2.2.1 via this isomorphism. Thus $F(\lambda)$ is characterized by the fact that it has a $B_3(\mathbb{F}_q)$-invariant vector on which $T_3(\mathbb{F}_q)$ acts by the character $\bigotimes_{i=0}^2 \omega_f^{\sum_j \lambda^i_{f-j} p^j}$.

Let $(a_j^2, a_j^1, a_j^0)_{j=0}^{f-1} \in (\mathbb{Z}^3_+)^f$ be a tuple of integers satisfying (1.1.3). Throughout this chapter and the next, we are interested in a particular Serre weight

$$F_0 := F((-a_j^0 - 1, -a_j^1, -a_j^2 + 1)_{j=0}^{f-1}).$$

The next theorem is the main result of this chapter. It is easy to see that for any *distinct* characters $\chi_i : \mathbb{F}_q \to \overline{\mathbb{F}}_p^\times$ the space of $B_3(\mathbb{F}_q)$-invariant vectors upon which $T_3(\mathbb{F}_q)$ acts by $\chi_0 \otimes \chi_1 \otimes \chi_2$ in the principal series representation $\pi = \mathrm{Ind}_{B_3(\mathbb{F}_q)}^{\mathrm{GL}_3(\mathbb{F}_q)}(\chi_0 \otimes \chi_1 \otimes \chi_2)$ is 1-dimensional. We refer to any nonzero element of this subspace (somewhat abusively) as a *highest weight vector* of π.

Theorem 2.3.2. *For $\sigma \in S_3^f$ and $K \subseteq \{0, \ldots, f-1\}$ define a principal series type $\tau = \tau(\sigma, K) = \bigoplus_{i=0}^2 \widetilde{\omega}_f^{A_i}$ as in Definition 1.6.1. Assume that the integers a_j^i satisfy (1.1.3) with $\delta \geq 15$. In each case below, we define operators $S_\tau, S'_\tau \in \overline{\mathbb{F}}_p[\mathrm{GL}_3(\mathbb{F}_q)]$.*

1. *Assume that*

$$\sigma_j = \begin{cases} \mathrm{id} & \text{if } j \in J \\ (02) & \text{if } j \notin J \end{cases} \tag{2.3.3}$$

for some $J \subseteq \{0, \ldots, f-1\}$ and that $J \subseteq K$ and $K \neq \emptyset$. Define

$$S_\tau = \sum_{x,y,z \in \mathbb{F}_q} \omega_f(x)^{\sum_j (p - a_{f-j}^{\sigma_{f-j}^{-1}(2)} + a_{f-j}^0 - \delta_{f-j+1 \notin K})p^j} \cdot \omega_f(z)^{\sum_j (p - a_{f-j}^2 + a_{f-j}^1)p^j} \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

and

$$S'_\tau = \sum_{x,y,z \in \mathbb{F}_q} \omega_f(x)^{\sum_j (p - a_{f-j}^1 + a_{f-j}^0)p^j} \cdot \omega_f(z)^{\sum_j (p - a_{f-j}^2 + a_{f-j}^{\sigma_{f-j}^{-1}(0)} - \delta_{f-j+1 \notin K})p^j} \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

Let $\pi = \mathrm{Ind}_{B_3(\mathbb{F}_q)}^{\mathrm{GL}_3(\mathbb{F}_q)}\left(\omega_f^{-A_1} \otimes \omega_f^{-A_0} \otimes \omega_f^{-A_2}\right)$ and $\pi' = \mathrm{Ind}_{B_3(\mathbb{F}_q)}^{\mathrm{GL}_3(\mathbb{F}_q)}\left(\omega_f^{-A_0} \otimes \omega_f^{-A_2} \otimes \omega_f^{-A_1}\right)$ and write v, v' for their highest weight vectors respectively.

Then Sv and $S'v'$ are nonzero in the unique quotients of π and π' with socle F_0, respectively.

2. *Assume that*

$$\sigma_j = \begin{cases} \mathrm{id} & \text{if } j \in J \\ (01) & \text{if } j \notin J \end{cases} \tag{2.3.4}$$

and that $K=\emptyset$. Define

$$S_\tau = \sum_{x,y\in\mathbb{F}_q} \omega_f(x)^{\sum_{f-j\in J}(p-a^1_{f-j}+a^0_{f-j})p^j} \begin{pmatrix} 1 & x & y \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$$

and

$$S'_\tau = \sum_{x\in\mathbb{F}_q} \omega_f(x)^{\sum_{f-j\notin J}(p-a^1_{f-j}+a^0_{f-j})p^j} \begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}.$$

Write $\pi = \mathrm{Ind}_{B_3(\mathbb{F}_q)}^{\mathrm{GL}_3(\mathbb{F}_q)}\left(\omega_f^{-A_0}\otimes\omega_f^{-A_2}\otimes\omega_f^{-A_1}\right)$ and $\pi' = \mathrm{Ind}_{B_3(\mathbb{F}_q)}^{\mathrm{GL}_3(\mathbb{F}_q)}\left(\omega_f^{-A_1}\otimes\omega_f^{-A_0}\otimes\omega_f^{-A_2}\right)$. Then the images of Sv and $S'v'$ are nonzero in the unique quotient of π and π' with socle F_0 respectively.

3. *Assume that*

$$\sigma_j = \begin{cases} \mathrm{id} & \text{if } j\in J \\ (12) & \text{if } j\notin J \end{cases} \tag{2.3.5}$$

and that $K=\emptyset$. Define

$$S_\tau = \sum_{y,z\in\mathbb{F}_q} \omega_f(z)^{\sum_{f-j\notin J}(p-a^2_{f-j}+a^1_{fj})p^j} \begin{pmatrix} 1 & & y \\ & 1 & z \\ & & 1 \end{pmatrix} \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}.$$

and

$$S'_\tau = \sum_{z\in\mathbb{F}_q} \omega_f(z)^{\sum_{f-j\in J}(p-a^2_{f-j}+a^1_{f-j})p^j} \begin{pmatrix} 1 & & \\ & 1 & z \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}.$$

Let $\pi = \mathrm{Ind}_{B_3(\mathbb{F}_q)}^{\mathrm{GL}_3(\mathbb{F}_q)}\left(\omega_f^{-A_2}\otimes\omega_f^{-A_0}\otimes\omega_f^{-A_1}\right)$ and $\pi' = \mathrm{Ind}_{B_3(\mathbb{F}_q)}^{\mathrm{GL}_3(\mathbb{F}_q)}\left(\omega_f^{-A_0}\otimes\omega_f^{-A_1}\otimes\omega_f^{-A_2}\right)$ with highest weight vectors v, v', respectively. Then Sv and $S'v'$ are nonzero in the unique quotients of π and π' with socle F_0, respectively.

Notation 2.3.6. In each of the cases considered in this theorem, we define $\varsigma=\varsigma(\tau)\in S_3$ to be the permutation such that $\pi = \mathrm{Ind}_{B_3(\mathbb{F}_q)}^{\mathrm{GL}_3(\mathbb{F}_q)}\left(\omega_f^{-A_{\varsigma(0)}}\otimes\omega_f^{-A_{\varsigma(1)}}\otimes\omega_f^{-A_{\varsigma(2)}}\right)$.

Remark 2.3.7. It is not obvious that F_0 is actually an irreducible constituent of any of these principal series, but it may be deduced from either [GHS], Proposition 10.1.7, or the results of [LLHLM18b] on the submodule structure of principal series for $\mathrm{GL}_3(\mathbb{F}_q)$. However, we don't need these results because it also follows from the proof of the theorem.

In fact, the choice of types τ in Definition 1.6.1 was motivated by the fact that the principal series representations $\mathrm{Ind}_{B_3(\mathbb{F}_q)}^{\mathrm{GL}_3(\mathbb{F}_q)}\left(\omega_f^{A_0}\otimes\omega_f^{A_1}\otimes\omega_f^{A_2}\right)$ are the only ones containing F_0 as a Jordan-Hölder factor, as can be checked using [GHS] Proposition 10.1.7.

Remark 2.3.8. If the integers a_i^j satisfy (1.1.3) when $\delta \geq 15$, then Theorem 1.3.1 of [LLHLM18b] implies that for any of the principal series representations π, π' in Theorem 2.3.2 all Jordan-Hölder constituents occur with multiplicity 1. In particular, given that F_0 is constituent, there exists a unique quotient with irreducible socle F_0.

Remark 2.3.9. Taking $f = 1$, τ as in (1.7.2) and $K = \{0\}$, the operators S_τ and S'_τ recover the operators denoted S and S' in [HLM17].

Remark 2.3.10. While we have been working with algebraically closed coefficients, the irreducible representations $F(\lambda)$ may all be defined over $\mathbb{F}_p$ so the Theorem is equally valid over a finite subfield $\mathbb{F} \subseteq \overline{\mathbb{F}}_p$ containing the image of ω_f.

The method of proof of this theorem is to embed π and π' into well-chosen Weyl modules $V = \boxtimes_{j=0}^{f-1} V(\lambda_j)$ for G, and then interpret the action of the operators S_τ and S'_τ by way of the distribution algebra of G. Since the structure of Weyl modules for GL_3 is known in the cases we need, this allows us to do explicit computations to prove nonvanishing of the group algebra operators. This technique was inspired by [Pil97] and [LLHLM18b], which both embed principal series representations into algebraic representations, the latter deducing the entire submodule structure of (sufficiently generic) principal series representations of $\mathrm{GL}_3(\mathbb{F}_q)$. However, [LLHLM18b] embed principal series into Weyl modules for split $\mathrm{GL}_{3/\mathbb{F}_q}$. Using the product group G with the nonsplit $\mathbb{F}_p$-structure above allows for simplifications to their set-up and in particular makes the kind of computations we want to do feasible. Our method of computation was also inspired by [Irv86], which determines the submodule structure of certain Weyl modules for SL_3.

We need the following general lemmas.

Lemma 2.3.11. *Let G_1 and G_2 be split reductive groups over a field k. If L_i is a simple module for G_i ($i = 1, 2$), then $L_1 \boxtimes L_2$ is a simple module for $G_1 \times G_2$. Moreover, every simple module for $G_1 \times G_2$ is of this form for unique L_1, L_2.*

Proof. If $N \subseteq L_1 \boxtimes L_2$ is a $G_1 \times G_2$-submodule, then picking a k-basis of L_2, we have $N \subseteq L_1 \boxtimes L_2 \cong L_1^n$ as a G_1-module. The image of N under this isomorphism must be a sub-direct sum of L_1^n by [Jan03] II.2.8, so we must have $N = L_1 \boxtimes W$ for some k-subspace $W \subseteq L_2$. If $\phi : L_1 \to k$ is any nonzero linear functional, then the image of $N \subseteq L_1 \boxtimes L_2 \xrightarrow{\phi\otimes 1} L_2$ is W. On the other hand, this map is G_2-equivariant. It follows that W is a G_2-submodule of L_2 and hence $W = L_2$. This proves that $L_1 \boxtimes L_2$ is simple.

If $T_i \subset G_i$ are split maximal tori then $T_1 \times T_2$ is a split maximal torus of $G_1 \times G_2$. Identifying $X(T_1 \times T_2)^+ = X(T_1)^+ \times X(T_2)^+$ for some choice of positive roots, it is clear that $L(\lambda_1) \boxtimes L(\lambda_2)$ has highest weight (λ_1, λ_2), so the other statements follow. $\square$

Corollary 2.3.12. *In the situation of the previous Lemma let M_i be a finite dimensional G_i-module, and for simplicity assume that the Jordan-Hölder constituents of M_i all occur with multiplicity 1 ($i = 1, 2$). Then the Jordan-Hölder constituents of $M_1 \boxtimes M_2$ are exactly the*

$L_1 \boxtimes L_2$ where L_i is a constituent of M_i each occuring with multiplicity 1, and the submodule lattice of $M_1 \boxtimes M_2$ is the product lattice of the submodule lattices of M_1 and M_2.

Proof. The first claim about the Jordan-Hölder constituents is easy to prove by induction on the length of M_1 and M_2, using the Lemma (which is the length 1 case). For the second, it now suffices to show that for each constituent $L_1 \boxtimes L_2$, if Q_i denotes the unique quotient of M_i having socle L_i, then $Q_1 \boxtimes Q_2$ is the unique quotient of $M_1 \boxtimes M_2$ having socle $L_1 \boxtimes L_2$. It is clear that $Q_1 \boxtimes Q_2$ is a quotient of $M_1 \boxtimes M_2$ having $L_1 \boxtimes L_2$ in its socle, so it suffices to prove that the socle is irreducible. But if $L_1' \boxtimes L_2' \hookrightarrow Q_1 \boxtimes Q_2$ for some irreducibles L_i', then considering the G_1 module structure on both sides we obtain $L_1'^n \hookrightarrow Q_1^m$ for some $m, n \geq 1$, which implies $L_1' = L_1$. Similarly $L_2' = L_2$ and we are done. $\square$

2.4 Weyl modules and the distribution algebra

In this section we recall facts about Weyl modules for the algebraic group $\mathrm{GL}_{3/\overline{\mathbb{F}}_p}$ and their $\mathbb{Z}$-forms. Let $G_0 = \mathrm{GL}_{3/\mathbb{Z}}$, and let $G_0' = \mathrm{SL}_{3/\mathbb{Z}}$ be the derived subgroup. We consider the standard diagonal torus T_3 and upper-triangular Borel B_3 in G_0, with root system R. Identify $X(T_3)$ with $\mathbb{Z}^3$, and let $\alpha = (1,-1,0)$ and $\beta = (0,1,-1)$ denote the simple positive roots and $\gamma = (1,0,-1)$ the nonsimple positive root. For $\delta \in R^+$ write s_δ for the corresponding reflection in the Weyl group and let $w_0 = s_\alpha s_\beta s_\alpha$ denote the longest element. For each $\delta \in R$ let $U_\delta \leq G_0'$ denote the standard root subgroup; we fix a pinning $(x_\delta : \mathbb{G}_{a/\mathbb{Z}} \xrightarrow{\sim} U_\delta)_{\delta \in R}$ as in [Jan03] II.1.2 such that $x_\delta(1)$ is a matrix having all entries equal to 0 or 1. This choice gives rise to raising and lowering operators $X_{\pm\alpha}, X_{\pm\beta}, X_{\pm\gamma}$ and weight operators $H_\alpha, H_\beta, H_\gamma = H_\alpha + H_\beta$ inside the distribution algebra $\mathfrak{U} := \mathrm{Dist}(G_0')$. We refer to [Jan03] for the definition of the distribution algebra. By [Jan03] II.1.12, $\mathfrak{U}$ is the $\mathbb{Z}$-subalgebra of the universal enveloping $\mathbb{C}$-algebra of $\mathrm{Lie}(G_0')$ generated by elements $X_{\pm\delta}^{(n)} := X_{\pm\delta}^n/n!$ and $H_{\alpha,m}, H_{\beta,m}$ with $n, m \in \mathbb{N}$, $\delta \in \{\alpha, \beta, \gamma\}$, where $H_{\delta,m} := \binom{H_\delta}{m}$. It is free over $\mathbb{Z}$ with basis given by

$$X_\alpha^{(n_\alpha)} X_\gamma^{(n_\gamma)} X_\beta^{(n_\beta)} H_{\alpha,m_\alpha} H_{\beta,m_\beta} X_{-\alpha}^{(n_{-\alpha})} X_{-\gamma}^{(n_{-\gamma})} X_{-\beta}^{(n_{-\beta})}$$

for $n_{\pm\delta}, m_\delta \geq 0$. We let $\mathfrak{U}^+$ denote the subalgebra of $\mathfrak{U}$ generated by $X_\delta^{(n_\delta)}$ with $\delta \in R^+$, $n_\delta \geq 1$ and similarly for $\mathfrak{U}^-$ with $\delta \in -R^+$. Moreover we have $\mathrm{Dist}((G_0')_{\overline{\mathbb{F}}_p}) = \mathfrak{U} \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p$.

By definition we have $[X_\delta, X_{-\delta}] = H_\delta$ for $\delta \in R^+$. With our choice of pinning, we also have for example

$$[X_\beta, X_{-\gamma}] = X_{-\alpha} \text{ and } [X_{-\beta}, X_{-\alpha}] = X_{-\gamma}. \tag{2.4.1}$$

The following elementary lemma applies to $\mathfrak{U}$ and therefore also to $\mathfrak{U}_{\overline{\mathbb{F}}_p}$ by noting that X_{δ_1} and X_{δ_2} commute whenever $\delta_1 + \delta_2$ is not a root and nonzero. We will use it heavily in the calculations of Section 2.5.

Lemma 2.4.2. *Let A be a ring of characteristic 0. For $x \in A$, write $x^{(n)} = x^n/n!$. If $a, b \in A$*

and $[a, b] = \epsilon\gamma$, where $\epsilon = \pm 1$ and γ commutes with a and b, then for all $m, n \geq 1$ we have

$$a^{(m)}b^{(n)} = \sum_{\ell=0}^{\min(m,n)} \epsilon^{\ell} b^{(n-\ell)} a^{(m-\ell)} \gamma^{(\ell)}.$$

Proof. By induction. □

Any G_0-module (resp. $(G_0)_{\overline{\mathbb{F}}_p}$-module) has the structure of a $\mathfrak{U}$-module (resp. $\mathfrak{U}_{\overline{\mathbb{F}}_p}$-module) by restriction. If V is a G_0-module and λ a weight, then for $\delta \in R^+$, $H_{\delta,m}$ acts by the scalar $\binom{\langle\lambda,\delta^\vee\rangle}{m}$ on V_λ. Moreover, $X_{\pm\delta}^{(n)}$ maps $V_\lambda \to V_{\lambda\pm n\delta}$. The same is true over $\overline{\mathbb{F}}_p$. This implies the next lemma.

Lemma 2.4.3. *If V is a G_0- or $(G_0)_{\overline{\mathbb{F}}_p}$-module and $v \in V$ is a weight vector, then*

$$X_\delta X_{-\delta}^{(n)} v = X_{-\delta}^{(n)} X_\delta v + [\langle v, \delta^\vee\rangle - (n-1)] X_{-\delta}^{(n-1)} v$$

for any $\delta \in \{\alpha, \beta, \gamma\}$.

Proof. By induction using the above remarks. □

The formula in the next lemma may be thought of as a Taylor expansion.

Lemma 2.4.4. *If V is a $(G_0)_{\overline{\mathbb{F}}_p}$-module and $v \in V$ then for $\delta \in R$ and $u \in \overline{\mathbb{F}}_p$ we have*

$$x_\delta(u)v = \sum_{i\geq 0} u^i X_\delta^{(i)} v.$$

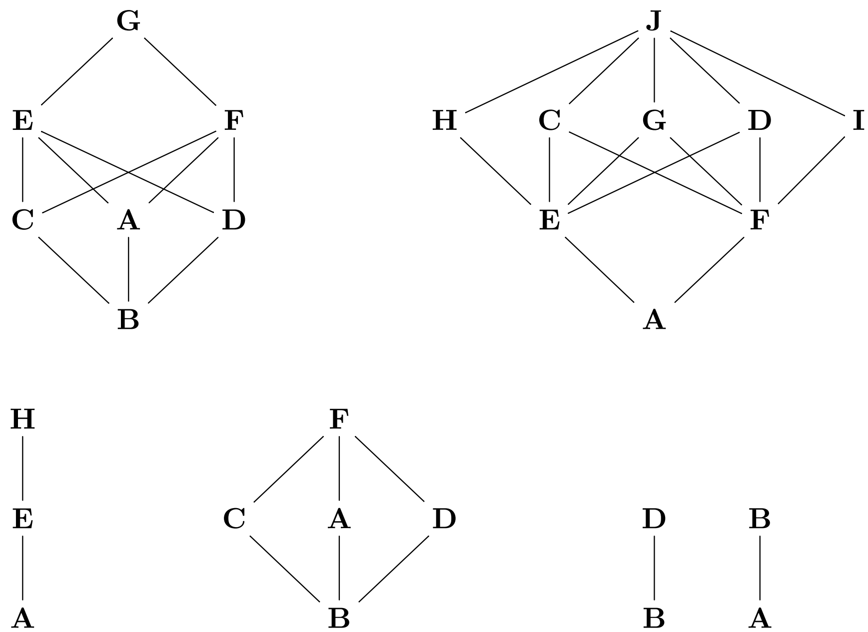
Proof. This is II.1.19(6) in [Jan03]. □

Remark 2.4.5. In the formula above, 0^0 is interpreted as 1. This plays a role below.

Next we recall some facts about Weyl modules. If $\lambda \in \mathbb{Z}^3$ is a dominant weight let $V(\lambda)_{\mathbb{Z}}$ denote the integral Weyl module for G_0 of highest weight λ, defined in [Jan03] II.8.2, and set $V(\lambda) := V(\lambda)_{\mathbb{Z}} \otimes \overline{\mathbb{F}}_p$. Briefly, $V(\lambda)_{\mathbb{Z}}$ is defined to be the finitely generated $\mathbb{Z}$-module generated by $\mathrm{Dist}(G_0)v$ where v is a highest weight vector inside the Weyl module of $\mathrm{GL}_{3/\mathbb{Q}}$ of highest weight λ. This is G_0-stable and it is well known (*loc. cit.*) that $V(\lambda)$ is then the Weyl module of highest weight λ for $(G_0)_{\overline{\mathbb{F}}_p}$ defined by the same symbol in Section 2.1. We note that $V(\lambda)|_{G_0'}$ is equal to the corresponding Weyl module for G_0'. This is important because some of the references below only work with semisimple groups and this fact allows us to extend results from SL_3 to GL_3. For example there is a similar construction of the weight λ integral Weyl module for G_0' in Section 27 of [Hum72]. The submodule structure of $V(\lambda)$ is completely known for small λ (at least when λ lies in the interior of a p-alcove, which is the only case we need). In order to express it, we use Humphreys' labelling **A-J** of the 10 lowest dominant p-alcoves [Hum06] of $(G_0)_{\overline{\mathbb{F}}_p}$. In order to be explicit, we list the interiors of these alcoves in the table below.

$\lambda \in$	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**
$\langle \lambda + \eta, \alpha^\vee \rangle \in$	$(0,p)$	$(0,p)$	$(0,p)$	$(p,2p)$	$(0,p)$	$(p,2p)$	$(p,2p)$	$(0,p)$	$(2p,3p)$	$(p,2p)$
$\langle \lambda + \eta, \beta^\vee \rangle \in$	$(0,p)$	$(0,p)$	$(p,2p)$	$(0,p)$	$(p,2p)$	$(0,p)$	$(p,2p)$	$(2p,3p)$	$(0,p)$	$(p,2p)$
$\langle \lambda + \eta, \gamma^\vee \rangle \in$	$(0,p)$	$(p,2p)$	$(p,2p)$	$(p,2p)$	$(2p,3p)$	$(2p,3p)$	$(2p,3p)$	$(2p,3p)$	$(2p,3p)$	$(3p,4p)$

Proposition 2.4.6. *Suppose that $p \geq 5$. Let λ be a weight in the interior of one the alcoves* **A**-**J**. *The submodule structure of $V(\lambda)$ is given by the diagram:*



*For example, if $\lambda \in$ **G** then the submodule structure is given by the top left diagram, where each alcove label corresponds to the unique irreducible representation $L(\mu)$ with μ lying in that alcove and linked to λ. The cases* **I**,**E**,**C** *not pictured are symmetric with* **H**,**F**,**D**, *respectively. If $\lambda \in$ **A** then $V(\lambda) = L(\lambda)$ is already irreducible.*

Proof. This proposition is taken directly from [BDM15], Section 4. Note in that paper they work with SL_3. Alternatively, it is not hard to rederive these structures using direct calculations as in Lemma 2.4.13 below. □

Remark 2.4.7. In the previous proposition, the alcove **J** is far enough from the boundary of the dominant region to exhibit Jantzen's generic decomposition pattern [Jan77b] for GL_3.

We now make some further remarks on the structure of $V(\lambda)$ as a $\mathfrak{U}_{\overline{\mathbb{F}}_p}$-module, with special attention to the case when λ lies in alcove **G** or **J**, since these are the cases that figure in the proofs of Section 2.5. The next definition is taken from [Jan77a].

Definition 2.4.8. A *primitive element* in a $(G_0)_{\overline{\mathbb{F}}_p}$-module is a nonzero weight vector killed by $X_\delta^{(n)}$ for all $\delta \in R^+$ and $n \geq 1$.

For example, a highest weight vector inside Weyl module is a primitive element.

Lemma 2.4.9. *A primitive element in a $(G_0)_{\overline{\mathbb{F}}_p}$-module generates a quotient of $V(\lambda)$.*

Proof. This follows from the universal property of Weyl modules, since $X_\delta^{(n)}v = 0$ for all $\delta \in R^+$ and $n \geq 1$ implies v is U-invariant by Lemma 2.4.4. □

Lemma 2.4.10. *Let λ be a dominant weight and let $\mathfrak{J}_\lambda$ denote the left ideal of $(\mathfrak{U})_{\overline{\mathbb{F}}_p}$ generated by $X_\delta^{(n)}$, $H_{\alpha,n} - \binom{\langle\lambda,\alpha^\vee\rangle}{n}\cdot 1$ and $H_{\beta,n} - \binom{\langle\lambda,\beta^\vee\rangle}{n}\cdot 1$ for $\delta \in R^+$ and $n \geq 1$, as well as $X_{-\alpha}^{(m_\alpha)}, X_{-\beta}^{(m_\beta)}$ for all $m_\alpha \geq \langle\lambda,\alpha^\vee\rangle + 1$ and $m_\beta \geq \langle\lambda,\beta^\vee\rangle + 1$. Then $V(\lambda) \cong (\mathfrak{U})_{\overline{\mathbb{F}}_p}/\mathfrak{J}_\lambda$ as a $(\mathfrak{U})_{\overline{\mathbb{F}}_p}$-module.*

Proof. This follows from Satz 2 of [Jan77a], which is the analogous result for the integral Weyl module $V(\lambda)_{\mathbb{Z}}$. □

Lemma 2.4.11. *Let v be a highest weight vector of $V(\lambda)$ for some dominant weight λ. If $b + c \geq \langle\lambda,\gamma^\vee\rangle + 1$ then $X_{-\gamma}^{(c)}X_{-\beta}^{(b)}v = 0$. Similarly if $a + c \geq \langle\lambda,\gamma^\vee\rangle + 1$ then $X_{-\alpha}^{(a)}X_{-\gamma}^{(c)}v = 0$.*

Proof. This follows from the well-known fact that all weights of $V(\lambda)$ satisfy $w_0\lambda \leq \mu \leq \lambda$. □

Lemma 2.4.12. *Let λ be a dominant weight and let $\xi = \lambda - r\alpha - s\beta$ where $0 \leq r \leq \langle\lambda,\alpha^\vee\rangle$ and $0 \leq s$. A basis of the $V(\lambda)_\xi$ is given by the elements $X_{-\alpha}^{(r-i)}X_{-\gamma}^{(i)}X_{-\beta}^{(s-i)}v$ for $\max(0, s - \langle\lambda,\beta^\vee\rangle) \leq i \leq \min(r,s)$, where v is a highest weight vector in $V(\lambda)$. In particular these elements are nonzero.*

Proof. It is clear that $V(\lambda)_\xi$ is spanned by the vectors $X_{-\alpha}^{(r-i)}X_{-\gamma}^{(i)}X_{-\beta}^{(s-i)}v$ for $0 \leq i \leq \min(r,s)$. But this vector is 0 by Lemma 2.4.10 if $s - i \geq \langle\lambda,\beta^\vee\rangle + 1$. Then one checks via the Kostant multiplicity formula that $\dim(V(\lambda)_\xi) = \min(r,s) - \max(0, s - \langle\lambda,\beta^\vee\rangle) + 1$, and the result follows. □

Lemma 2.4.13. *Let v denote a highest weight vector in $V(\lambda)$.*

1. *If $\lambda \in \mathbf{G}$ then $v_{\mathbf{F}} = X_{-\beta}^{(\langle\lambda,\beta^\vee\rangle - p + 1)}v$, $v_{\mathbf{E}} = X_{-\alpha}^{(\langle\lambda,\alpha^\vee\rangle - p + 1)}v$, $v_{\mathbf{C}} = X_{-\alpha}^{(\langle\lambda,\gamma^\vee\rangle - 2p + 2)}X_{-\beta}^{(\langle\lambda,\beta^\vee\rangle - p + 1)}v$ and $v_{\mathbf{D}} = X_{-\beta}^{(\langle\lambda,\gamma^\vee\rangle - 2p + 2)}X_{-\alpha}^{(\langle\lambda,\alpha^\vee\rangle - p + 1)}v$ are primitive and generate the unique submodule of $V(\lambda)$ having irreducible cosocle with highest weight in alcove **F**,**E**,**C**,**D**, respectively.*

2. *If $\lambda \in \mathbf{J}$ then $v_{\mathbf{I}} = X_{-\beta}^{(\langle\lambda,\beta^\vee\rangle - p + 1)}v$ and $v_{\mathbf{H}} = X_{-\alpha}^{(\langle\lambda,\alpha^\vee\rangle - p + 1)}v$ are primitive and generate the unique submodule of $V(\lambda)$ having irreducible cosocle with highest weight in alcove **I** and **H** respectively.*

Proof. That these elements are primitive is shown via direct calculation. Once this is known, by Lemma 2.4.9 we know they must generate a quotient of the Weyl module of the appropriate weight and the other claims follow by examining the structures in Proposition 2.4.6. We include two calculations as an example. Suppose $\lambda \in \mathbf{G}$. First note $v_{\mathbf{F}} \neq 0$ by Lemma 2.4.12. Since X_α commutes with $X_{-\beta}$, is it immediate that $X_\alpha^{(n)}v_{\mathbf{F}} = 0$ for all $n \geq 1$. And using Lemma 2.4.3 we compute $X_\beta v_{\mathbf{F}} = X_{-\beta}^{(\langle\lambda,\beta^\vee\rangle - p + 1)}X_\beta v + pX_{-\beta}^{(\langle\lambda,\beta^\vee\rangle - p)}v = 0$. This proves $X_{-\beta}^{(n)}v_{\mathbf{F}} = 0$ for all $n < p$; for $n \geq p$ it follows from weight considerations. Since $X_\gamma = [X_\alpha, X_\beta]$ this is enough to prove $v_{\mathbf{F}}$ primitive and hence the submodule generated by $v_{\mathbf{F}}$ is a quotient of a Weyl module in alcove $\mathbf{F}$. Using this, together with Lemma 2.4.12 again, we deduce that $v_{\mathbf{C}} \neq 0$, and then similar calculations prove that $v_{\mathbf{C}}$ is primitive. $\square$

Corollary 2.4.14. *If $\lambda \in \mathbf{G}$ then the submodules of $V(\lambda)$ having irreducible cosocle $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{F}$ respectively are themselves Weyl modules. If $\lambda \in \mathbf{J}$ then the same is true of $\mathbf{A}$, $\mathbf{H}$, $\mathbf{I}$.*

Proof. This follows from Lemma 2.4.9, Lemma 2.4.13 and Proposition 2.4.6. $\square$

Finally in the remainder of this section we collect some important results allowing us to completely describe the $\mathfrak{U}_{\overline{\mathbb{F}}_p}$-module structure of the irreducible algebraic representations $L(\lambda)$, where $\lambda \in \mathbb{Z}^3$ is a dominant weight. In any case, $L(\lambda) = \operatorname{cosoc} V(\lambda)$ so Lemmas 2.4.10 and 2.4.11 apply. But we can say more.

As we know from Section 2.1, $L(\lambda)$ decomposes via the Steinberg tensor product theorem. It is easy to check directly from the definitions that the Frobenius endomorphism of $(G_0)_{\overline{\mathbb{F}}_p}$ induces an $\overline{\mathbb{F}}_p$-algebra endomorphism Frob^* of $\mathfrak{U}_{\overline{\mathbb{F}}_p}$ taking

$$X_\delta^{(n)} \mapsto \begin{cases} X_\delta^{(n/p)} & p|n \\ 0 & p \nmid n \end{cases} \tag{2.4.15}$$

for $\delta \in R$. Moreover, if L_0 and L_1 are $\mathfrak{U}$-modules then by definition of the Hopf algebra structure on $\mathfrak{U}$, $\mathfrak{U}$ acts on $L_0 \otimes L_1$ via the formula

$$X_\delta^{(n)}(v_0 \otimes v_1) = \sum_{\ell=0}^{n} (X_\delta^{(n-\ell)} v_0) \otimes (X_\delta^{(\ell)} v_1) \tag{2.4.16}$$

for $v_i \in L_i$.

Given these statements we are reduced to understanding the $\mathfrak{U}_{\overline{\mathbb{F}}_p}$-module structure of $L(\lambda)$ for $\lambda \in X_1(T_3)$. In this case there is a result due to Nanhua Xi [Xi96] giving a complete description of $L(\lambda)$. In order to state it, we first define a certain monomial in $\mathfrak{U}^-_{\overline{\mathbb{F}}_p}$.

Definition 2.4.17. Given $\lambda \in X_1(T_3)$ we define its Xi monomial to be the element of $\mathfrak{U}^-_{\overline{\mathbb{F}}_p}$ given by

$$\Xi(\lambda) = X_{-\beta}^{(\langle\theta,\beta^\vee\rangle)} X_{-\alpha}^{(\langle s_\beta\theta,\alpha^\vee\rangle)} X_{-\beta}^{(\langle s_\alpha s_\beta\theta,\beta^\vee\rangle)}$$

where $\theta = (p-1)\eta - \lambda$. An equivalent expression for it is:

$$\Xi(\lambda) = X_{-\alpha}^{(\langle\theta,\alpha^\vee\rangle)} X_{-\beta}^{(\langle s_\alpha\theta,\beta^\vee\rangle)} X_{-\alpha}^{(\langle s_\beta s_\alpha\theta,\alpha^\vee\rangle)}.$$

Unless otherwise stated, we use the first expression in our calculations.

Let $(\mathfrak{U}^-_{\overline{\mathbb{F}}_p})_1$ denote the subalgebra of $\mathfrak{U}_{\overline{\mathbb{F}}_p}$ generated by $X_{-\delta}^{(n)}$ for $0 \le n \le p-1$ and $\delta \in R^+$. In fact $\Xi(\lambda)$ lies in $(\mathfrak{U}^-_{\overline{\mathbb{F}}})_1$ ([Xi96] Theorem 6.5(i)) but we don't need this.

Proposition 2.4.18. *Let $\lambda \in X_1(T_3)$ and let $v \in L(\lambda)$ be a highest weight vector. For $\mathfrak{u} \in (\mathfrak{U}^-_{\overline{\mathbb{F}}_p})_1$ we have $\mathfrak{u}v = 0$ in $L(\lambda)$ iff $\mathfrak{u} \cdot \Xi(\lambda) = 0$ in $\mathfrak{U}_{\overline{\mathbb{F}}_p}$.*

Proof. This follows from Theorem 6.7(i) in [Xi96], if we observe that the intersection with $(\mathfrak{U}^-_{\overline{\mathbb{F}}_p})_1$ of the ideal there denoted $\mathfrak{I}'_\lambda$ is precisely

$$\mathfrak{n} = \{\mathfrak{u} \in (\mathfrak{U}^-_{\overline{\mathbb{F}}_p})_1 \,|\, \mathfrak{u} \cdot \Xi(\lambda) = 0\}.$$

To see this, consider the map $(\mathfrak{U}^-_{\overline{\mathbb{F}}_p})_1 \to L(\lambda)$ sending $\mathfrak{u}$ to $\mathfrak{u}v$, which is surjective because $L(\lambda)$ is an irreducible representation of the Frobenius kernel of $(G_0)_{\overline{\mathbb{F}}_p}$. Its kernel is $(\mathfrak{U}^-_{\overline{\mathbb{F}}_p})_1 \cap \mathfrak{I}'_\lambda$, which contains $\mathfrak{n}$. On the other hand, Theorem 6.7(iii) of *loc. cit.* says that $\dim(L(\lambda)) = \dim(\mathfrak{U}^-_{\overline{\mathbb{F}}_p})_1 - \dim(\mathfrak{n})$, which proves the claim. $\square$

2.5 Proof of Theorem 2.3.2

This section contains the proofs of Theorem 2.3.2 following the approach explained in Section 2.3. Throughout the computations of this section we often use without comment, in addition to the lemmas in the previous section, the following elementary observations about $\mathfrak{U}_{\overline{\mathbb{F}}_p}$:

- $X_{-\gamma}$ commutes with $X_{-\beta}$ and $X_{-\alpha}$ in $\mathfrak{U}$ so $X^{(c)}_{-\gamma}$ commutes with $X^{(b)}_{-\beta}$ and $X^{(a)}_{-\alpha}$ in $\mathfrak{U}_{\overline{\mathbb{F}}_p}$ for any $a, b, c \geq 0$;
- $X^{(n)}_* X^{(m)}_* = \binom{n+m}{n} X^{(n+m)}_*$ is 0 if $\lfloor \frac{n}{p} \rfloor + \lfloor \frac{m}{p} \rfloor < \lfloor \frac{n+m}{p} \rfloor$;
- as a corollary to the previous point, if $0 \leq n < p$ then $X^{(ap+n)}_* = X^{(ap)}_* X^{(n)}_*$ for any $a \geq 0$ since $\binom{ap+n}{ap} \equiv 1 \mod p$;
- similarly if $0 \leq n \leq p-3$ then we have $\binom{p-1}{n} \equiv (-1)^n$, $\binom{p-2}{n} \equiv (-1)^n(n+1)$, and $\binom{p-3}{n} \equiv (-1)^n \frac{(n+1)(n+2)}{2} \mod p$.

The reader should keep these in mind in order to follow the calculations.

Proof of Theorem 2.3.2(1)

By applying the automorphism

$$g \mapsto \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} {}^t g^{-1} \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$$

of $\mathrm{GL}_3(\mathbb{F}_q)$ and replacing (a^2_j, a^1_j, a^0_j) with $(-a^0_j, -a^1_j, -a^2_j)$ we deduce the second part of Theorem 2.3.2(1) (the statement about S'_τ) from the first. For the remainder of the proof, let π, σ,J,K,ς be as in the first part of the theorem and write $S = S_\tau$.

The first step is to embed π into a Weyl module $V = \boxtimes_{j=0}^{f-1} V(\lambda_j)$ for the algebraic group $\prod_{j=0}^{f-1} \mathrm{GL}_{3/\overline{\mathbb{F}}_p}$. To do this, for $0 \le i \le 2$ write $-A_{\varsigma(i)} \equiv \sum_j \xi^i_{f-j} p^j \mod e$ where $(\xi^0_j, \xi^1_j, \xi^2_j) \in X_1(T_3)$ for each $0 \le j \le f-1$. We may take

$$\xi^i_j = -a_j^{\sigma_j^{-1}(\varsigma(i))} + p\epsilon^i_j - \epsilon^i_{j+1} - \delta_{j+1 \notin K}(1 - \sigma^{-1}_{j+1}\varsigma(i))$$

where

$$\epsilon_j = \begin{cases} (1,0,0) & j \in J \\ (0,0,-1) & j \notin J. \end{cases}$$

Define $\lambda_j = \xi_j + (p-1)\eta$. Then it is easy to check that

$$\xi_j \in \mathbf{B} \text{ and } \lambda_j \in \mathbf{J} \iff j \in J$$
$$\xi_j \in \mathbf{A} \text{ and } \lambda_j \in \mathbf{G} \iff j \notin J.$$

Make a choice of highest weight vector $v = \otimes_j v_j$ in V. It is clear that under the embedding in (2.3.1), $T_3(\mathbb{F}_q)$ acts on v via the character $\bigotimes_{i=0}^{2} \omega_f^{\sum_j \xi^i_{f-j} p^j}$. Hence by Frobenius reciprocity, we obtain a nonzero map of $\mathrm{GL}_3(\mathbb{F}_q)$-representations

$$\pi \to V$$

sending the highest weight vector of π to v. In fact, this map is injective as one sees by comparing socles, using Proposition 2.2.2 with $n = 1$ and the fact that π is multiplicity-free. We won't need this, however.

It will be helpful for the reader to consult the diagrams of the submodule structures of the $V(\lambda_j)$ in Figures 2.5.2 and 2.5.3.

Notation 2.5.1. We make the following convention: if $\mathbf{X}$ is the name of an alcove, then $L_j(\mathbf{X})$ denotes the (unique) Jordan-Hölder constituent of $V(\lambda_j)$ having highest weight in $\mathbf{X}$. We let $\Xi_j(\mathbf{X})$ denote the Xi monomial of the first tensor factor of $L_j(\mathbf{X})$ under the Steinberg decomposition. We define $Q_j(\mathbf{X})$ to be the unique (algebraic) quotient of $V(\lambda_j)$ having socle $L_j(\mathbf{X})$. Moreover, $v_{\mathbf{X},j}$ denotes the primitive element of $V(\lambda_j)$ corresponding to alcove $\mathbf{X}$ in Lemma 2.4.13.

There is a particular quotient of V in which we are interested in because it contains F_0 in its socle. This is $Q := \boxtimes_{j=0}^{f-1} Q_j(\mathbf{X}_j)$ where

$$\mathbf{X}_j = \begin{cases} \mathbf{D} & j \in J \\ \mathbf{C} & j \notin J \end{cases}.$$

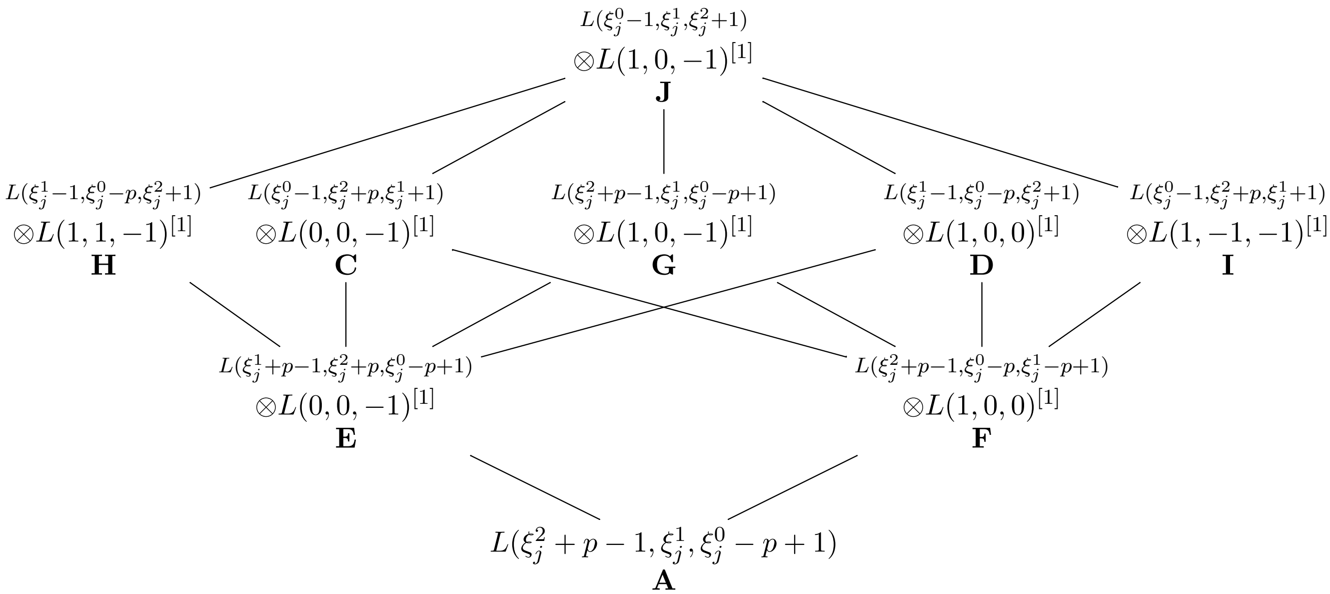


Figure 2.5.2: The submodule structure of $V(\lambda_j)$ when $\lambda_j \in \mathbf{J}$.

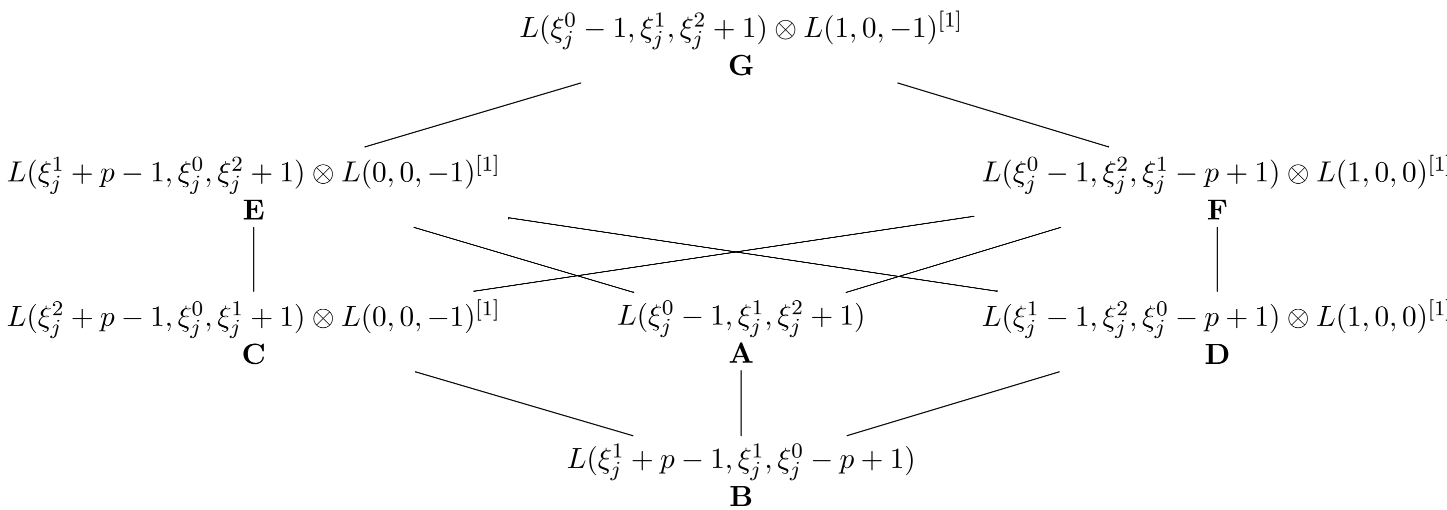


Figure 2.5.3: The submodule structure of $V(\lambda_j)$ when $\lambda_j \in \mathbf{G}$.

We also write $Q_j = Q_j(\mathbf{X}_j)$. By Corollary 2.3.12, the (algebraic) socle of Q is

$$\mathrm{soc}(Q) = \boxtimes_{j=0}^{f-1} L((-a_j^0 - 1, -a_j^1, -a_j^2 + 1) - \zeta_{j,j+1}) \otimes L(\epsilon_j)^{[1]}$$

where $\zeta_{j,j+1}$ is given in the following table.

$\zeta_{j,j+1} =$	$j+1 \in K$	$j+1 \notin K$
$j, j+1 \in J$	$(0,1,0)$	
$j \in J, j+1 \notin J$	$(0,0,-1)$	$(-1,0,0)$
$j \notin J, j+1 \in J$	$(0,1,0)$	
$j, j+1 \notin J$	$(-1,0,0)$	$(0,0,-1)$

The unfilled entries in the table don't occur because we assumed $J \subseteq K$. Now we have

$$\begin{aligned}
\mathrm{soc}(Q)|_{\mathrm{GL}_3(\mathbb{F}_q)} &= \bigotimes_j L((-a_j^0 - 1, -a_j^1, -a_j^2 + 1)_j - \zeta_{j,j+1})^{[f-j]}|_{\mathrm{GL}_3(\mathbb{F}_q)} \otimes L(\epsilon_j)^{[f-j+1]}|_{\mathrm{GL}_3(\mathbb{F}_q)} \\
&= \bigotimes_j \left[L((-a_j^0 - 1, -a_j^1, -a_j^2 + 1)_j - \zeta_{j,j+1}) \otimes L(\epsilon_{j+1})\right]^{[f-j]}|_{\mathrm{GL}_3(\mathbb{F}_q)} \qquad (2.5.4) \\
&= \bigoplus_{\{(v_j)_j \,|\, v_j \in L(\epsilon_j)\}} \left[\prod_j \dim\left(L(\epsilon_j)_{v_j}\right)\right] F((-a_j^0 - 1, -a_j^1, -a_j^2 + 1)_j - \zeta_{j,j+1} + v_{j+1})
\end{aligned}$$

where the last step holds by the translation principle. Since $\zeta_{j,j+1}$ is a permutation of ϵ_{j+1} it is clear that this sum contains F_0 with multiplicity 1. In fact, note that each weight of $L(\epsilon_j)$ occurs with multiplicity 1 so the multiplier in (2.5.4) is equal to 1. Our goal is now to show two things.

1. The image of Sv in Q lies in $\mathrm{soc}(Q)$, and
2. it is nonzero under the projection $\mathrm{soc}(Q) \twoheadrightarrow F_0$.

Claim 2.5.5. These two statements together imply Theorem 2.3.2(1).

Proof. Let π' denote the (nonzero) image of π under the $\mathrm{GL}_3(\mathbb{F}_q)$-equivariant morphism $\pi \to V \twoheadrightarrow Q$. Write v for the highest weight vector of π as well as that of V. Then by statement 1, the image of Sv in Q lies in $\pi' \cap \mathrm{soc}_{\mathrm{GL}_3(\mathbb{F}_q)}(Q) = \mathrm{soc}(\pi')$ (observing that $\mathrm{soc}(Q)|_{\mathrm{GL}_3(\mathbb{F}_q)}$ is visibly semisimple due to the computation above, hence lies inside $\mathrm{soc}_{\mathrm{GL}_3(\mathbb{F}_q)}(Q)$), and by statement 2, there is a map $\mathrm{soc}(\pi') \to F_0$ under which the image of Sv is nonzero (that is, the map $\mathrm{soc}(\pi') \hookrightarrow \mathrm{soc}_{\mathrm{GL}_3(\mathbb{F}_q)}(Q) \twoheadrightarrow \mathrm{soc}(Q)|_{\mathrm{GL}_3(\mathbb{F}_q)} \twoheadrightarrow F_0$, where the second map is the projection using semisimplicity). In particular F_0 is a constituent of $\mathrm{soc}(\pi')$ and the claim follows. $\square$

Remark 2.5.6. The argument above could have been made slightly simpler if we had applied Proposition 2.2.2 but there was no need. We did use the easy direction of Proposition 2.2.2, but did not have to quote the result since its conclusion was obvious due to the explicitness of (2.5.4).

We now embark on proving statements 1 and 2 above. To begin, we use Lemma 2.4.4 together with the fact that

$$\begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & z \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & y \\ & 1 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}.$$

To ease notation, let us temporarily write

$$S = \sum_{x,y,z\in\mathbb{F}_q} \omega_f(x)^B \omega_f(z)^C \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} w_0$$

We compute:

$$\begin{aligned} Sv &= \sum_{x,y,z\in\mathbb{F}_q} \omega_f(x)^B \omega_f(z)^C \bigotimes_{j=0}^{f-1} \begin{pmatrix} 1 & \omega_j(x) & \omega_j(y) \\ & 1 & \omega_j(z) \\ & & 1 \end{pmatrix} w_0 v_j \\ &= \sum_{(i_j,\ell_j,k_j)\geq 0} \left[\sum_{x,y,z\in\mathbb{F}_q} \omega_f(x)^{B+\sum_j i_{f-j}p^j} \omega_f(y)^{\sum_j \ell_{f-j}p^j} \omega_f(z)^{C+\sum_j k_{f-j}p^j} \right] \\ &\qquad\qquad \cdot \bigotimes_{j=0}^{f-1} X_\beta^{(k_j)} X_\gamma^{(\ell_j)} X_\alpha^{(i_j)} w_0 v_j \\ &= -w_0 \sum_{(i_j,\ell_j,k_j)_{j=0}^{f-1}\in\mathfrak{D}} \bigotimes_{j=0}^{f-1} X_{-\alpha}^{(k_j)} X_{-\gamma}^{(\ell_j)} X_{-\beta}^{(i_j)} v_j \end{aligned} \tag{2.5.7}$$

where in the last step we used [Jan03] I.7.18(1) as well as the fact that

$$\sum_{a\in\mathbb{F}_q} a^n = \begin{cases} -1 & \text{if } q|n \text{ and } n\neq 0 \\ 0 & \text{else} \end{cases}$$

(recall Remark 2.4.5). Here $\mathfrak{D}$ is defined to be the set of tuples $(i_j,\ell_j,k_j)_{j=0}^{f-1}$ obeying

$$\begin{aligned} \sum_j \left(p - a_{f-j}^{\sigma_{f-j}^{-1}(2)} + a_{f-j}^0 - \delta_{f-j+1\notin K} + i_{f-j}\right) p^j &\equiv 0 \mod e \text{ and } \neq 0 \\ \sum_j \ell_{f-j} p^j &\equiv 0 \mod e \text{ and } \neq 0 \\ \sum_j \left(p - a_{f-j}^2 + a_{f-j}^1 + k_{f-j}\right) p^j &\equiv 0 \mod e \text{ and } \neq 0 \end{aligned} \tag{2.5.8}$$

as well as

$$i_j, \ell_j, k_j \geq 0. \tag{2.5.9}$$

For any tuple in $\mathfrak{D}$ we may write

$$\begin{aligned} i_j =&(r_j - 1)p + a_j^{\sigma_j^{-1}(2)} - a_j^0 + \delta_{j+1 \notin K} - r_{j+1} \\ \ell_j =&t_j p - t_{j+1} \\ k_j =&(u_j - 1)p + a_j^2 - a_j^1 - u_{j+1} \end{aligned} \tag{2.5.10}$$

for some integers $r_j, t_j, u_j \in \mathbb{Z}$. It is easy to see that (2.5.9) plus the nonzero condition in (2.5.8) imply $r_j, t_j, u_j \geq 1$ for all $0 \leq j \leq f-1$. Now observe that if $j \notin J, j+1 \in K$ then $r_j \geq 2$. And if $j \notin J, j+1 \notin K$ and $r_j = 1$ then we must have $r_{j+1} = 1$. From these statements we see the implication

$$j \notin J \text{ and } r_j = 1 \Rightarrow r_{j+1} = 1 \text{ and } j+1 \notin K.$$

But this means if there is a single j_0 such that $j_0 \notin J$ and $r_{j_0} = 1$ then *all* $j \notin K$, that is $K = \emptyset$. Since we assumed this was not the case, we must have

$$r_j \geq 2 \text{ if } j \notin J.$$

Let $\mathfrak{E}$ denote the set of tuples $(r_j, t_j, u_j)_{j=0}^{f-1}$ of integers satisfying the conditions just found, namely $r_j, u_j, t_j \geq 1$ and $r_j \geq 2$ if $j \notin J$. It is easy to check that the correspondence just outlined between $\mathfrak{D}$ and $\mathfrak{E}$ is a bijection. Since it suffices to prove statement 1 and 2 for $-w_0 S v$, our strategy is to now consider each of the terms appearing in

$$\begin{aligned} -w_0 S v = \sum_{(r_j,t_j,u_j)_{j=0}^{f-1} \in \mathfrak{E}} \bigotimes_{j=0}^{f-1} & X_{-\alpha}^{((u_j-1)p + a_j^2 - a_j^1 - u_{j+1})} X_{-\gamma}^{(t_j p - t_{j+1})} \\ & \cdot X_{-\beta}^{((r_j-1)p + a_j^{\sigma_j^{-1}(2)} - a_j^0 + \delta_{j+1 \notin K} - r_{j+1})} v_j \end{aligned} \tag{2.5.11}$$

individually and analyze their images in the quotient Q. We write

$$\Theta_j(u,t,r) = X_{-\alpha}^{((u_j-1)p + a_j^2 - a_j^1 - u_{j+1})} X_{-\gamma}^{(t_j p - t_{j+1})} X_{-\beta}^{((r_j-1)p + a_j^{\sigma_j^{-1}(2)} - a_j^0 + \delta_{j+1 \notin K} - r_{j+1})}.$$

Lemma 2.5.12. *Only terms such that $r_j, t_j \leq 2$, $u_j \leq 4$ for each j may be nonzero in V.*

Proof. By Lemma 2.4.10 we must have

$$i_j \leq \langle \lambda_j, \beta^\vee \rangle \leq \begin{cases} p + a_j^2 - a_j^0 - 1 & j \in J \\ 2p - a_j^2 + a_j^0 - 1 & j \notin J \end{cases}$$

which implies $r_j \le 2$. Similarly, by Lemma 2.4.11 we have

$$i_j + \ell_j \le \langle \lambda_j, \gamma^\vee \rangle \le \begin{cases} 3p + a_j^2 - a_j^1 - 2 & j \in J \\ 3p - a_j^1 + a_j^0 - 2 & j \notin J \end{cases}$$

which now implies $t_j \le 2$ in all cases. Finally, note that we must have $k_j \le \langle \lambda_j, \gamma^\vee \rangle$ or else $\mathrm{wt}(\Theta_j(u,t,r)v_j) \not\le w_0\lambda_j$. Hence the bounds above also imply that $u_j \le 4$ in all cases. $\square$

The purpose of this lemma is that since r_j, t_j, u_j are now known to be constrained in a small interval, we may essentially forget about them in many arguments by genericity. From now on we often do this without comment.

So far in our computation we have not used anything about the σ_j, but starting soon we must divide into cases because the behaviour of the terms in (2.5.11) genuinely varies depending on both j and $j+1$.

Claim 2.5.13. All terms in (2.5.11) except those such that

$$r_j = \begin{cases} 1 & j \in J \\ 2 & j \notin J \end{cases}$$

and $t_j = 1$ if $j \notin J$ vanish in Q.

Proof. For $j \in J$, observe that if λ_j' is the highest weight of $L_j(\mathbf{J})$ then we have

$$\langle \lambda_j', \gamma^\vee \rangle = \begin{cases} p + a_j^2 - a_j^1 - 3 & j+1 \in K \\ p + a_j^2 - a_j^1 - 2 & j+1 \notin K. \end{cases}$$

From Lemma 2.4.11, we deduce that

$$X_{-\gamma}^{(p-a_j^1+a_j^0+r_{j+1}-2)} X_{-\beta}^{(a_j^2-a_j^0+\delta_{j+1\notin K}-r_{j+1})} v_j \in \mathrm{rad}(V(\lambda_j)).$$

In particular the image of this element in Q_j is in $\mathrm{soc}(Q_j)$ (as Q_j has length 2). But $X_{-\beta}^{(p)}$ kills $\mathrm{soc}(Q_j)$ by weight considerations, so in order for $\Theta_j(u,t,r)v_j$ to have nonzero image we must have $r_j = 1$. Since we already know $r_j = 2$ if $j \notin J$ by Lemma 2.5.12, we have shown the first claim.

In order to prove the second, assume $j \notin J$. Observe that $v_{\mathbf{F},j} = X_{-\beta}^{(p-a_j^2+a_j^0+\epsilon_j')} v_j$ where we temporarily define

$$\epsilon_j' = \begin{cases} 0 & j+1 \in J \\ -1 & j+1 \in K \setminus J \\ 1 & j+1 \notin K. \end{cases}$$

Hence $\Theta_j(u,r,t)v_j$ is a nonzero multiple of

$$X_{-\alpha}^{((u_j-1)p+a_j^2-a_j^1-u_{j+1})}X_{-\gamma}^{(t_jp-t_{j+1})}X_{-\beta}^{(a_j^2-a_j^0+\delta_{j+1\notin K}-\epsilon'_j)}v_{\mathbf{F},j}.$$

Since $\langle \mathrm{wt}(v_{\mathbf{F},j}),\gamma^\vee\rangle \leq 2p+a_j^2-a_j^1-2$ it now follows from Corollary 2.4.14 and Lemma 2.4.11 that $t_j=1$ as desired. $\square$

Lemma 2.5.14. *Consider the alcove* **J** *Weyl module $V(\lambda_j)$ of Figure 2.5.2. Let v'_j denote a choice of highest weight vector of* $\mathrm{soc}(Q_j)$ *in Q_j. Then*

1. *the image of $X_{-\gamma}^{(\xi_j^0-\xi_j^1)}X_{-\beta}^{(\xi_j^1-\xi_j^2-1)}v_j$ in Q_j is a nonzero multiple of $X_{-\beta}^{(\xi_j^0-\xi_j^2-p-1)}v'_j$;*

2. *the image of $X_{-\gamma}^{(\xi_j^0-\xi_j^1+1)}X_{-\beta}^{(\xi_j^1-\xi_j^2-2)}v_j$ in Q_j is a nonzero multiple of*

$$\left[X_{-\gamma}X_{-\beta}^{(\xi_j^0-\xi_j^2-p-2)}-(\xi_j^0-\xi_j^1)X_{-\alpha}X_{-\beta}^{(\xi_j^0-\xi_j^2-p-1)}\right]v'_j.$$

Proof. We will prove the second statement since it is similar to the first but harder. The goal is to find an explicit element $\mathcal{Q}$ whose image in Q_j is v'_j. Let $\mu=\lambda_j-(\xi_j^0-\xi_j^1)\alpha-p\beta$ be the highest weight of $\mathrm{soc}(Q_j)$ and let $M=\langle v_{\mathbf{H},j},v_{\mathbf{I},j}\rangle$. By Lemma 2.4.12 and Lemma 2.4.13 we find the following bases:

$$\begin{aligned}
V(\lambda_j)_\mu &= \mathrm{Span}\left\{X_{-\alpha}^{(\xi_j^0-\xi_j^1-\ell)}X_{-\gamma}^{(\ell)}X_{-\beta}^{(p-\ell)}v_j \,|\, 0\leq \ell\leq \xi_j^0-\xi_j^1\right\}\\
\langle v_{\mathbf{I},j}\rangle_\mu &= \mathrm{Span}\left\{X_{-\alpha}^{(\xi_j^0-\xi_j^1-\ell)}X_{-\gamma}^{(\ell)}X_{-\beta}^{(p-\ell)} \,|\, 1\leq \ell\leq p-\xi_j^1+\xi_j^2\right\}\\
\langle v_{\mathbf{H},j}\rangle_\mu &= \mathrm{Span}\left\{\sum_{\ell=0}^{\xi_j^0-\xi_j^1}X_{-\alpha}^{(\xi_j^0-\xi_j^1-\ell)}X_{-\gamma}^{(\ell)}X_{-\beta}^{(p-\ell)}\right\}.
\end{aligned}$$

Consider $\theta := X_{-\alpha}^{(\xi_j^0-\xi_j^1)}X_{-\beta}^{(p)}v_j$. By the above, $\theta\notin M$. Moreover, $X_{-\alpha}^{(\xi_j^0-\xi_j^1)}X_{-\beta}^{(p)}$ kills the highest weight vector of cosoc $V(\lambda_j)$, so the image of θ in $V(\lambda_j)/M$ is a nonzero element of the radical, which is equal to the socle. It is easy to check that $\mathrm{soc}(V(\lambda_j)/M))_\mu$ is 1-dimensional. This implies that the image of θ in Q_j is a highest weight vector. Since we are free to modify θ by elements of M_μ without changing this statement, we therefore take

$$\mathcal{Q} := \sum_{\ell=p-\xi_j^1+\xi_j^2+1}^{\xi_j^0-\xi_j^1}X_{-\alpha}^{(\xi_j^0-\xi_j^1-\ell)}X_{-\gamma}^{(\ell)}X_{-\beta}^{(p-\ell)}v_j.$$

We now prove the lemma. We compute

$$X_{-\alpha}X_{-\beta}^{(\xi_j^0-\xi_j^2-p-1)}\mathcal{Q} \equiv \left[\sum_{i=p-\xi_j^1+\xi_j^2+1}^{\xi_j^0-\xi_j^1}\binom{\xi_j^0-\xi_j^1}{i}\binom{\xi_j^1-\xi_j^2-1}{p-i}\right]X_{-\alpha}X_{-\gamma}^{(\xi_j^0-\xi_j^1)}X_{-\beta}^{(\xi_j^1-\xi_j^2-1)}v_j \mod M$$
$$= X_{-\alpha}X_{-\gamma}^{(\xi_j^0-\xi_j^1)}X_{-\beta}^{(\xi_j^1-\xi_j^2-1)}v_j \mod M$$

where the second step follows from the fact that for any $m \leq n$ in $(0,p)$ we have

$$\sum_{i=m}^{n}\binom{n}{i}\binom{p-m}{p-i} \equiv 1 \mod p.$$

Similarly

$$X_{-\gamma}X_{-\beta}^{(\xi_j^0-\xi_j^2-p-2)}\mathcal{Q} \equiv (\xi_j^0-\xi_j^1)X_{-\alpha}X_{-\gamma}^{(\xi_j^0-\xi_j^1)}X_{-\beta}^{(\xi_j^1-\xi_j^2-1)}v_j + (\xi_j^0-\xi_j^1+1)X_{-\gamma}^{(\xi_j^0-\xi_j^1+1)}X_{-\beta}^{(\xi_j^1-\xi_j^2-2)}v_j \mod M.$$

The claim now follows immediately. □

Lemma 2.5.15. *Consider the alcove G Weyl module $V(\lambda_j)$ of Figure 2.5.3. Let v_j' denote a choice of highest weight vector of* $\mathrm{soc}(Q_j)$ *in Q_j. Then*

1. *The image of $X_{-\gamma}^{(\xi_j^0-\xi_j^2)}X_{-\beta}^{(p-1)}v_j$ in Q_j is a nonzero multiple of $X_{-\beta}^{(p+\xi_j^0-\xi_j^1-1)}v_j'$.*
2. *The image of $X_{-\gamma}^{(\xi_j^0-\xi_j^2+1)}X_{-\beta}^{(p-2)}v_j$ in Q_j is a nonzero multiple of*

$$\left[X_{-\gamma}X_{-\beta}^{(p+\xi_j^0-\xi_j^1-2)} - (\xi_j^0-\xi_j^2)X_{-\alpha}X_{-\beta}^{(p+\xi_j^0-\xi_j^1-1)}\right]v_j'.$$

Proof. We prove the second statement, since the proof of the first is similar but easier. Observe that the given element of $V(\lambda_j)$ is clearly a nonzero multiple of $X_{-\gamma}^{(\xi_j^0-\xi_j^2+1)}X_{-\beta}^{(p-\xi_j^1+\xi_j^2-2)}v_{\mathbf{F},j}$. Moreover the image of $v_{\mathbf{C},j} = X_{-\alpha}^{(\xi_j^0-\xi_j^2)}v_{\mathbf{F},j}$ in Q_j is a nonzero multiple of v_j' by Lemma 2.4.13. We may therefore directly compute

$$X_{-\gamma}X_{-\beta}^{(p+\xi_j^0-\xi_j^1-2)}v_{\mathbf{C},j} = \sum_{\ell=0}^{\xi_j^0-\xi_j^2}(\ell+1)X_{-\alpha}^{(\xi_j^0-\xi_j^2-\ell)}X_{-\gamma}^{(\ell+1)}X_{-\beta}^{(p+\xi_j^0-\xi_j^1-2-\ell)}v_{\mathbf{F},j}$$
$$= (\xi_j^0-\xi_j^2)X_{-\alpha}X_{-\gamma}^{(\xi_j^0-\xi_j^2)}X_{-\beta}^{(p-\xi_j^1+\xi_j^2-1)}v_{\mathbf{F},j} + (\xi_j^0-\xi_j^2+1)X_{-\gamma}^{(\xi_j^0+\xi_j^2+1)}X_{-\beta}^{(p-\xi_j^1+\xi_j^2-2)}v_{\mathbf{F},j}$$

where the second equality follows from Lemma 2.4.10. Similarly we compute directly that

$$X_{-\alpha}X_{-\beta}^{(p+\xi_j^0-\xi_j^1-1)}v_{\mathbf{C},j} = X_{-\alpha}X_{-\gamma}^{(\xi_j^0-\xi_j^2)}X_{-\beta}^{(p-\xi_j^1+\xi_j^2-1)}v_{\mathbf{F},j}.$$

The desired equation now follows. □

With these two lemmas we now study the image of $\Theta_j(u,t,r)v_j$ in Q_j on a case-by-case basis. From now on we fix for each $0 \leq j \leq f-1$ a highest weight vector $v'_j \in \mathrm{soc}(Q_j)$, and decomposing $\mathrm{soc}(Q_j)$ via the Steinberg tensor product theorem we in fact write $v'_j = v_j^0 \otimes v_j^1$ where v_j^0 is a highest weight vector for $L((-a_j^0-1,-a_j^1,-a_j^2+1)-z_{j,j+1})$ and v_j^1 is a highest weight vector for $L(\epsilon_j)^{[1]}$.

Case I. If $j, j+1 \in J$ and $j+1 \in K$ then by Lemma 2.5.14 we see that the image of $\Theta_j(u,t,r)v_j$ in Q_j is equal to

$$\binom{p-t_{j+1}}{a_j^1-a_j^0+1-t_{j+1}}^{-1} X_{-\alpha}^{(a_j^2-a_j^1-u_{j+1})} X_{-\gamma}^{(a_j^1-a_j^0+1-t_{j+1})} X_{-\beta}^{(a_j^2-a_j^1-2)} v_j^0 \otimes X_{-\alpha}^{(u_j-1)} X_{-\gamma}^{(t_j-1)} v_j^1.$$

Case II. If $j \in J, j+1 \notin J, j+1 \in K$ then by Lemma 2.5.14 we see that the image of $\Theta_j(u,t,r)v_j$ in Q_j is equal to

$$(-1)^{a_j^1-a_j^0} X_{-\alpha}^{(a_j^2-a_j^1-u_{j+1})} X_{-\gamma}^{(a_j^1-a_j^0-1)} X_{-\beta}^{(a_j^2-a_j^1-2)} v_j^0 \otimes X_{-\alpha}^{(u_j-1)} X_{-\gamma}^{(t_j-1)} v_j^1.$$

Case III. If $j \in J, j+1 \notin J, j+1 \notin K$ then by Lemma 2.5.14 we see that the image of $\Theta_j(u,t,r)v_j$ in Q_j is equal to

$$\begin{aligned}(-1)^{a_j^1-a_j^0} \Big[&(a_j^1-a_j^0) X_{-\alpha}^{(a_j^2-a_j^1-u_{j+1})} X_{-\gamma}^{(a_j^1-a_j^0)} X_{-\beta}^{(a_j^2-a_j^0-2)} \\ &+ (a_j^1-a_j^0+1)(a_j^2-a_j^1+1-u_{j+1}) X_{-\alpha}^{(a_j^2-a_j^1+1-u_{j+1})} X_{-\gamma}^{(a_j^1-a_j^0-1)} X_{-\beta}^{(a_j^2-a_j^1-1)} \Big] v_j^0 \\ &\otimes X_{-\alpha}^{(u_j-1)} X_{-\gamma}^{(t_j-1)} v_j^1.\end{aligned}$$

Case IV. If $j \notin J, j+1 \in J, j+1 \in K$ then by Lemma 2.5.15 we see that the image of $\Theta_j(u,t,r)v_j$ in Q_j is equal to

$$\binom{p-t_{j+1}}{a_j^1-a_j^0+1-t_{j+1}}^{-1} X_{-\alpha}^{(a_j^2-a_j^1-u_{j+1})} X_{-\gamma}^{(a_j^1-a_j^0+1-t_{j+1})} X_{-\beta}^{(a_j^2-a_j^0-2)} v_j^0 \otimes X_{-\alpha}^{(u_j-1)} X_{-\beta} v_j^1.$$

Case V. If $j, j+1 \notin J, j+1 \in K$ then by Lemma 2.5.15 we see that the image of $\Theta_j(u,t,r)v_j$

in Q_j is equal to

$$(-1)^{a_j^1-a_j^0}\Big[(a_j^1-a_j^0)X_{-\alpha}^{(a_j^2-a_j^1-u_{j+1})}X_{-\gamma}^{(a_j^1-a_j^0)}X_{-\beta}^{(a_j^2-a_j^0-2)} \\ +(a_j^1-a_j^0+1)(a_j^2-a_j^1+1-u_{j+1})X_{-\alpha}^{(a_j^2-a_j^1+1-u_{j+1})}X_{-\gamma}^{(a_j^1-a_j^0-1)}X_{-\beta}^{(a_j^2-a_j^1-1)}\Big]v_j^0 \\ \otimes X_{-\alpha}^{(u_j-1)}X_{-\beta}v_j^1.$$

Case VI. If $j, j+1 \notin J, j+1 \notin K$ then by Lemma 2.5.15 we see that the image of $\Theta_j(u,t,r)v_j$ in Q_j is equal to

$$(-1)^{a_j^1-a_j^0}X_{-\alpha}^{(a_j^2-a_j^1-u_{j+1})}X_{-\gamma}^{(a_j^1-a_j^0-1)}X_{-\beta}^{(a_j^2-a_j^1-2)}v_j^0 \otimes X_{-\alpha}^{(u_j-1)}X_{-\beta}v_j^1.$$

Observe that the first tensor factors are the same in Cases II and VI, I and IV, and III and V. This is significant because $z_{j,j+1}$ is also the same in these three pairs of cases, which allows us to cut down the amount of work needed. From now on, we write the expressions appearing in Cases I - VI above as

$$\text{image of } \Theta_j(u,t,r)v_j \text{ in } Q_j = \theta_j(u_{j+1},t_{j+1})v_j^0 \otimes \eta_j(u_j,t_j)v_j^1.$$

Incidentally, we have already shown the truth of statement 1 by these calculations.

Lemma 2.5.16. *The image of $\Theta_j(u,t,r)v_j$ is only possibly nonzero in Q_j if*

$$(u_j,t_j) \in \begin{cases} \{(1,2),(2,1)\} & j \in J \\ \{(1,1),(2,1)\} & j \notin J \end{cases}$$

Proof. If $j \in J$ then v_j^1 is a highest weight vector of $L(1,0,0)$ and from cases I - III above we have $\eta_j(u_j,t_j)v_j^1 = X_{-\alpha}^{(u_j-1)}X_{-\gamma}^{(t_j-1)}v_j^1$ which is 0 if $u_j+t_j > 3$. If $j \notin J$ then v_j^1 is a highest weight vector of $L(0,0,-1)$ and we have $\eta_j(u_j,t_j)v_j^1 = X_{-\alpha}^{(u_j-1)}X_{-\beta}v_j^1$, which is 0 if $u_j > 2$. From these two cases we deduce that $u_j \le 2$ in all cases, and $(u_j,t_j) \ne (2,2)$.

In cases I and IV (which are the same), we easily compute that

$$\theta_j(u_{j+1},t_{j+1})\Xi(\mathrm{wt}(v_j^0)) = 0$$

if $u_{j+1} = t_{j+1} = 1$. Together with the previous paragraph, this proves that $j \in J \Rightarrow (u_j,t_j) \in \{(1,2),(2,1)\}$ and the lemma follows. $\square$

This lemma allows us to more simply write

$$\theta_j(u_{j+1},t_{j+1})v_j^0 \otimes \eta_j(u_j,t_j)v_j^1 = \theta_j(u_{j+1})v_j^0 \otimes \eta_j(u_j)v_j^1$$

by taking $t_{j+1} = \begin{cases} 3 - u_{j+1} & j \in J \\ 1 & j \notin J \end{cases}$. From these calculations and (2.5.11), we deduce that the image of $-w_0 S v$ inside Q, under the identification

$$\mathrm{soc}(Q)|_{\mathrm{GL}_3(\mathbb{F}_q)} = \bigotimes_j \left[L((-a_j^0 - 1, -a_j^1, -a_j^2 + 1)_j - \zeta_{j,j+1}) \otimes L(\epsilon_{j+1})\right]^{[f-j]} |_{\mathrm{GL}_3(\mathbb{F}_q)}$$

from (2.5.4) is equal to

$$\sum_{(u_j)_{j=0}^{f-1} \in \{1,2\}^f} \bigotimes_{j=0}^{f-1} [\theta_j(u_{j+1}) v_j^0 \otimes \eta_{j+1}(u_{j+1}) v_{j+1}^1]. \tag{2.5.17}$$

We therefore need to compute the image of $\theta_j(u_{j+1})v_j^0 \otimes \eta_{j+1}(u_{j+1})v_{j+1}^1$ under the projection

$$\pi_j : L((-a_j^0 - 1, -a_j^1, -a_j^2 + 1) - \zeta_{j,j+1}) \otimes L(\epsilon_{j+1}) \twoheadrightarrow L(-a_j^0 - 1, -a_j^1, -a_j^2 + 1).$$

This is accomplished in Lemma 2.5.18 below, which shows that

$$\pi_j(\theta_j(1) v_j^0 \otimes \eta_{j+1}(1) v_{j+1}^1) = c_j \pi_j(\theta_j(2) v_j^0 \otimes \eta_{j+1}(2) v_j^1)$$

for some constant c_j such that $1 + c_j \neq 0$. To be precise, we have

$$c_j = \begin{cases} 0 & \text{in cases II and IV by Lemma 2.5.18(1)} \\ \frac{1}{a_j^1 - a_j^0} & \text{in cases I and IV by Lemma 2.5.18(2)} \\ \frac{-(a_j^2 - a_j^0)}{a_j^2 - a_j^1 - 1} & \text{in cases III and V by Lemma 2.5.18(3).} \end{cases}$$

Given this result, we deduce from (2.5.17) that the image of $-w_0 S v$ under the projection $\mathrm{soc}(Q) \twoheadrightarrow F_0$ is equal to

$$\begin{aligned} \sum_{(u_j)_{j=0}^{f-1} \in \{1,2\}^f} \left[\prod_{\{j : u_{j+1} = 1\}} c_j\right] \bigotimes_j \pi_j(\theta_j(2) v_j^0 \otimes \eta_{j+1}(2) v_j^1) \\ = \left(\prod_j (1 + c_j)\right) \bigotimes_j \pi_j(\theta_j(2) v_j^0 \otimes \eta_{j+1}(2) v_j^1). \end{aligned}$$

Since Lemma 2.5.18 also shows that $\pi_j(\theta_j(2) v_j^0 \otimes \eta_{j+1}(2) v_j^1) \neq 0$, this finishes the proof of Theorem 2.3.2(1). □

Lemma 2.5.18. *Let $(a, b, c) \in \mathbb{Z}^3$ be a 5-deep weight inside the lowest alcove for* GL_3*. Let $u = 1, 2$. In each item below v_0 and v_1 denote highest weight vectors of the corresponding tensor factors.*

1. *Let* $\theta(u) = X_{-\alpha}^{(a-b-u)} X_{-\gamma}^{(b-c-1)} X_{-\beta}^{(a-b-2)} v_0 \otimes X_{-\alpha}^{(u-1)} X_{-\beta} v_1$ *inside* $L(-c-1,-b,-a+2) \otimes L(0,0,-1)$. *If* π *denotes the projection to* $L(-c-1,-b,-a+1)$ *then* $\pi(\theta(2)) \neq 0$ *and* $\pi(\theta(1)) = 0$.

2. *Let* $\theta'(u) = \binom{p-3+u}{b-c-2+u}^{-1} X_{-\alpha}^{(a-b-u)} X_{-\gamma}^{(b-c-2+u)} X_{-\beta}^{(a-b-2)} v_0 \otimes X_{-\alpha}^{(u-1)} X_{-\gamma}^{(2-u)} v_1$ *inside* $L(-c-1,-b-1,-a+1) \otimes L(1,0,0)$. *If* π *denotes the projection to* $L(-c-1,-b,-a+1)$ *then* $\pi(\theta'(1)) = \frac{1}{b-c}\pi(\theta'(2)) \neq 0$

3. *Let* $\theta''(u) = [(b-c)X_{-\alpha}^{(a-b-u)} X_{-\gamma}^{(b-c)} X_{-\beta}^{(a-b-2)} + (b-c+1)(a-b+1-u) X_{-\alpha}^{(a-b+1-u)} X_{-\gamma}^{(b-c-1)} X_{-\beta}^{(a-b-1)}] v_0 \otimes X_{-\alpha}^{(u-1)} X_{-\beta} v_1$ *inside* $L(-c,-b,-a+1) \otimes L(0,0,-1)$. *If* π *denotes the projection to* $L(-c-1,-b,-a+1)$ *then* $\pi(\theta''(1)) = \frac{-(a-c)}{a-b-1}\pi(\theta''(2)) \neq 0$.

Proof. We first remark that $\theta(u), \theta'(u), \theta''(u)$ are all of weight $w_0\lambda_0 := (-a+1,-b,-c+1)$. The statements are given in increasing order of difficulty of calculation. We will give the most details for the third statement and only summarize the proofs of the others. From the tensor product theorem the $w_0\lambda_0$ weight space in $L(-c,-b,-a+1) \otimes L(0,0,-1)$ is 4-dimensional, equal to

$$L(-c,-b,-a)_{w_0\lambda_0} \oplus L(-c,-b-1,-a+1)_{w_0\lambda_0} \oplus L(-c-1,-b,-a+1)_{w_0\lambda_0}, \tag{2.5.19}$$

where the first summand has dimension 2. A basis of this weight space is given by

$$\begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} X_{-\alpha}^{(a-b-2)} X_{-\gamma}^{(b-c)} X_{-\beta}^{(a-b-2)} v_0 \otimes X_{-\gamma} v_1 \\ X_{-\alpha}^{(a-b-1)} X_{-\gamma}^{(b-c-1)} X_{-\beta}^{(a-b-1)} v_0 \otimes X_{-\gamma} v_1 \\ X_{-\alpha}^{(a-b)} X_{-\gamma}^{(b-c+1)} X_{-\beta}^{(a-b-1)} v_0 \otimes X_{-\beta} v_1 \\ X_{-\alpha}^{(a-b-1)} X_{-\gamma}^{(b-c)} X_{-\beta}^{(a-b-1)} v_0 \otimes v_1 \end{pmatrix}. \tag{2.5.20}$$

To see this one uses the Xi monomial to check that the given vectors are all nonzero and the first two are linearly independent. Some of these calculation are carried out in Example 2.5.23(1). For the remainder of the proof, all vectors will be expressed with respect to this basis. Using the Xi monomial, one sees that the vector $v' := X_{-\alpha}^{(a-b-1)} X_{-\gamma}^{(b-c)} X_{-\beta}^{(a-b-1)} v_0 \otimes X_{-\gamma} v_1$ is nonzero and therefore must be a lowest weight vector for the summand $L(-c,-b,-a) \subseteq L(-c,-b,-a+1) \otimes L(0,0,-1)$. It follows that a basis of the first direct summand in 2.5.19 is $\{X_\beta X_\alpha v', X_\alpha X_\beta v'\}$. After a great deal of computation (see Example 2.5.23(2) where one of these is carried out), we find that

$$X_\beta X_\alpha v' = \begin{pmatrix} 1 \\ a-b-1 \\ -1 \\ -1 \end{pmatrix} \quad \text{and } X_\alpha X_\beta v' = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}. \tag{2.5.21}$$

In order to find a basis vector for the second direct summand in (2.5.19), we observe first that $v'' = (a-b-1)v_0 \otimes X_{-\beta} v_1 - X_{-\beta} v_0 \otimes v_1$ is killed by X_α and X_β and is therefore a highest weight vector for the summand $L(-c,-b-1,-a+1) \subseteq L(-c,-b,-a+1) \otimes L(0,0,-1)$. Thus a lowest weight vector for this factor is given by $X_{-\alpha}^{(a-b-2)} X_{-\gamma}^{(b-c+1)} X_{-\beta}^{(a-b-2)} v''$ and a basis of the second summand in (2.5.19) by $X_\beta X_{-\alpha}^{(a-b-2)} X_{-\gamma}^{(b-c+1)} X_{-\beta}^{(a-b-2)} v''$. Calculating in the same manner as before, this ends up being equal to

$$\begin{pmatrix} -1 \\ -(a-b-1) \\ -(a-b-1) \\ -(a-b-1) \end{pmatrix}.$$

Now finally one checks that

$$\theta(1) = \begin{pmatrix} 0 \\ 0 \\ a-c \\ 0 \end{pmatrix} \quad \theta(2) = \begin{pmatrix} -(b-c) \\ -(b-c+1)(a-b-1) \\ 0 \\ 0 \end{pmatrix}$$

and the unique constant μ such that

$$\theta(1) - \mu\theta(2) \in \operatorname{span}\left\langle \begin{pmatrix} 1 \\ a-b-1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -(a-b-1) \\ -(a-b-1) \\ -(a-b-1) \end{pmatrix} \right\rangle$$

is $\mu = \frac{-(a-c)}{a-b-1}$ which proves the claim.

Now we prove the second statement. In this case the $w_0\lambda_0$ weight space is 2-dimensional, equal to

$$L(-c,-b-1,-a+1)_{w_0\lambda_0} \oplus L(-a-1,-b,-a+1)_{w_0\lambda_0}. \tag{2.5.22}$$

A highest weight vector for $L(-c,-b-1,-a+1) \subseteq L(-c-1,-b-1,-a+1) \otimes L(1,0,0)$ is given by $v_0 \otimes v_1$. Using the Xi monomial (to check nonvanishing) one sees that a lowest weight vector for $L(-c,-b-1,-a+1)$ is given by $X_{-\alpha}^{(a-b-2)} X_{-\gamma}^{(b-c)} X_{-\beta}^{(a-b-2)} v_0 \otimes X_{-\gamma} v_1$. Then a basis for the first direct summand in 2.5.22 is given by X_β times this, which works out to be

$$X_{-\alpha}^{(a-b-1)} X_{-\gamma}^{(b-c-1)} X_{-\beta}^{(a-b-2)} v_0 \otimes X_{-\gamma} v_1 + X_{-\alpha}^{(a-b-2)} X_{-\gamma}^{(b-c)} X_{-\beta}^{(a-b-2)} v_0 \otimes X_{-\alpha} v_1.$$

This is $(-1)^{b-c}[\theta'(1) - (b-c)\theta'(2)]$ and it follows that $\pi(\theta'(1)) = (b-c)\pi(\theta'(2))$ as desired.

Finally, the first statement is easy because one checks directly using the Xi monomial that $\theta(1) = 0$ and $\theta(2) \neq 0$. Since the $w_0\lambda_0$ weight space of $L(-c-1,-b,-a+2) \otimes L(0,0,-1)$ is 1-dimensional by the tensor product theorem the result follows. $\square$

Example 2.5.23. We do two example computations with the Xi monomial that were used in the above proof.

1. We show that the first two vectors in (2.5.20) are nonzero and linearly independent. It suffices to show that $X_{-\alpha}^{(a-b-2)}X_{-\gamma}^{(b-c)}X_{-\beta}^{(a-b-2)}v_0$ and $X_{-\alpha}^{(a-b-1)}X_{-\gamma}^{(b-c-1)}X_{-\beta}^{(a-b-1)}v_0$ are linearly independent inside $L(-c,-b,-a+1)$. The Xi monomial of this representation is $\Xi = \Xi(-c,-b,-a+1) = X_{-\beta}^{(p-a+b)}X_{-\alpha}^{(2p-a+c-1)}X_{-\beta}^{(p-b+c-1)}$. We compute

$$\begin{aligned} X_{-\alpha}^{(a-b-2)}X_{-\gamma}^{(b-c)}X_{-\beta}^{(a-b-2)}\Xi &= \binom{p-2}{a-b-2}X_{-\alpha}^{(a-b-2)}X_{-\gamma}^{(b-c)}X_{-\beta}^{(p-2)}X_{-\alpha}^{(2p-a+b-1)}X_{-\beta}^{(p-b+c-1)} \\ &= (-1)^{a-b}(a-b-1)X_{-\alpha}^{(a-b-2)}X_{-\gamma}^{(b-c)} \\ &\qquad \sum_{\ell=0}^{p-2}X_{-\alpha}^{(2p-a+c-1-\ell)}X_{-\gamma}^{(\ell)}X_{-\beta}^{(p-2-\ell)}X_{-\beta}^{(p-b+c-1)}. \end{aligned}$$

The ℓth term here is zero if either $b-c+\ell \geq p$ or $p-2-\ell+p-b+c-1 \geq p$. So only two terms $\ell = p-b+c-2, p-b+c-1$ survive and we are left with

$$= (-1)^{b-c}(a-b-1)(b-c+1)X_{-\alpha}^{(p-1)}X_{-\gamma}^{(p-2)}X_{-\beta}^{(p-1)} + (a-b-1)(b-c)X_{-\alpha}^{(p-1)}X_{-\gamma}^{(p-1)}X_{-\beta}^{(p-2)}.$$

On the other hand we similarly calculate

$$X_{-\alpha}^{(a-b-1)}X_{-\gamma}^{(b-c-1)}X_{-\beta}^{(a-b-1)}\Xi = (b-c)X_{-\alpha}^{(p-1)}X_{-\gamma}^{(p-2)}X_{-\beta}^{(p-1)} + (a-b)(b-c)X_{-\alpha}^{(p-2)}X_{-\gamma}^{(p-1)}X_{-\beta}^{(p-2)}.$$

Since these two elements of $\mathfrak{U}_{\mathbb{F}_p}^1$ are linearly independent the result follows from Proposition 2.4.18.

2. We calculate the element $X_\alpha X_\beta v'$ in (2.5.21):

$$\begin{aligned} X_\beta v' &= X_{-\alpha}^{(a-b-1)}X_\beta X_{-\gamma}^{(b-c)}X_{-\beta}^{(a-b-1)}v_0 \otimes X_{-\gamma}v_1 \\ &= X_{-\alpha}^{(a-b-1)}X_{-\gamma}^{(b-c)}X_{-\beta}^{(a-b-2)}v_0 \otimes X_{-\gamma}v_1 + (a-b)X_{-\alpha}^{(a-b)}X_{-\gamma}^{(b-c-1)}X_{-\beta}^{(a-b-1)}v_0 \otimes X_{-\gamma}v_1. \end{aligned}$$

Applying X_α we obtain

$$\begin{aligned} X_{-\alpha}^{(a-b-1)}X_{-\gamma}^{(b-c-1)}X_{-\beta}^{(a-b-1)}v_0 \otimes X_{-\gamma}v_1 - X_{-\alpha}^{(a-b-1)}X_{-\gamma}^{(b-c)}X_{-\beta}^{(a-b-2)}v_0 \otimes X_{-\beta}v_1 \\ - (a-b)X_{-\alpha}^{(a-b)}X_{-\gamma}^{(b-c-1)}X_{-\beta}^{(a-b-1)}v_0 \otimes X_{-\beta}v_1. \end{aligned}$$

Finally, it is easy to check that $X_{-\alpha}^{(a-b-1)}X_{-\gamma}^{(b-c)}X_{-\beta}^{(a-b-2)} - (a-b-1)X_{-\alpha}^{(a-b)}X_{-\gamma}^{(b-c-1)}X_{-\gamma}^{(a-b-1)}$ kills Ξ, so we get (2.5.21).

Proof of Theorem 2.3.2(2) and (3), part I

We now move on to the proof of the other parts of Theorem 2.3.2. Unfortunately, the method of proof is the same - embed the principal series inside a Weyl module and compute directly, and the specifics of this case differ enough to necessitate full details. On the other hand, the computations turn out to be simpler.

First, by applying the automorphism $g \mapsto w_0{}^t g^{-1} w_0$ of $\mathrm{GL}_3(\mathbb{F}_q)$ we easily deduce Theorem 2.3.2(2) from part (3), so we may focus on proving the two halves of part (3). Thus, from now on let $\sigma, \tau, J, K = \emptyset, \pi, \pi'$ be as in the statement of part (3). We write $S = S_\tau$ and $S' = S'_\tau$. The statement about S and π is proved in this subsection, and the statement about S' and π' in the next.

As before, we first embed π into a Weyl module $V = \boxtimes_j V(\lambda_j)$. We set

$$\xi_j^i = -a_j^{\sigma_j^{-1}\varsigma(i)} + (p-1)\epsilon^i - 1 + \sigma_{j+1}^{-1}\varsigma(i)$$

where $\epsilon = (1,0,0)$ so that $(\xi_j^0, \xi_j^1, \xi_j^2) \in X_1(T_3)$ for each $0 \le j \le f-1$, and set $\lambda_j = \xi_j + (p-1)\eta$. Then we have

$$\begin{aligned} \xi_j \in \mathbf{A} \text{ and } \lambda_j \in \mathbf{G} &\iff j \in J \\ \xi_j \in \mathbf{B} \text{ and } \lambda_j \in \mathbf{J} &\iff j \notin J. \end{aligned}$$

Though the notation has changed, the submodule structure of the $V(\lambda_j)$ may still be read off of Figures 2.5.2 and 2.5.3.

In the same way as before, after picking a highest weight vector $v = \otimes_j v_j \in V$ we obtain a nonzero $\mathrm{GL}_3(\mathbb{F}_q)$-equivariant map $\pi \to V$ by Frobenius reciprocity. This time we define $Q = \boxtimes_{j=0}^{f-1} = \boxtimes_{j=0}^{f-1} Q_j(\mathbf{X}_j)$ where

$$\mathbf{X}_j = \begin{cases} \mathbf{D} & j \in J \\ \mathbf{H} & j \notin J. \end{cases}$$

Then we have

$$\mathrm{soc}(Q) = \boxtimes_{j=0}^{f-1} L((-a_j^0 - 1, -a_j^1, -a_j^2 + 1) - \zeta_{j,j+1}) \otimes L(\epsilon_j)^{[1]}$$

where

$$\epsilon_j = \begin{cases} (1,0,0) & j \in J \\ (1,-1,-1) & j \notin J \end{cases}$$

and $z_{j,j+1}$ is determined by the following table.

	$\zeta_{j,j+1} =$
$j, j+1 \in J$	$(1,0,0)$
$j \in J, j+1 \notin J$	$(1,-1,1)$
$j \notin J, j+1 \in J$	$(1,0,0)$
$j, j+1 \notin J$	$(1,1,-1)$

The equation (2.5.4) holds *verbatim* with the new notation (and note that all weights in

$L(\epsilon_j)$ still occur with multiplicity 1, so the multiplier is again equal to 1). And since $z_{j,j+1}$ is a permutation of ϵ_{j+1} we see that $\mathrm{soc}(Q)_{\mathrm{GL}_3(\mathbb{F}_q)}$ contains F_0 with multiplicity 1. As in Claim 2.5.5 , our goal is to show that the image of Sv in Q lies in $\mathrm{soc}(Q)$ and is nonzero under the projection $\mathrm{soc}(Q) \twoheadrightarrow F_0$.

Computing as in (2.5.7) we deduce that in V we have

$$Sv = -s_\alpha s_\beta \sum_{(\ell_j,k_j)_{j=0}^{f-1}\in\mathfrak{D}} \bigotimes_j X_{-\alpha}^{(\ell_j)} X_{-\gamma}^{(k_j)} v_j \tag{2.5.24}$$

where $\mathfrak{D}$ denotes the set of tuples $(\ell_j,k_j)_{j=0}^{f-1}$ such that

$$\ell_j, k_j \geq 0 \tag{2.5.25}$$

and

$$\sum_j \ell_{f-j} p^j \equiv 0 \mod e \text{ and } \neq 0 \tag{2.5.26}$$

$$\sum_j (k_{f-j} + \delta_{f-j\notin J}(p - a_{f-j}^2 + a_{f-j}^1))p^j \equiv 0 \mod e \text{ and } \neq 0.$$

In particular we may write

$$\begin{aligned} \ell_j &= t_j p - t_{j+1} \\ k_j &= \begin{cases} u_j p - u_{j+1} & j \in J \\ (u_j - 1)p + a_j^2 - a_j^1 - u_{j+1} & j \notin J \end{cases} \end{aligned}$$

for some integers $t_j, u_j \in \mathbb{Z}$. By (2.5.25) and (2.5.26) we must have $t_j, u_j \geq 1$ for each $0 \leq j \leq f-1$.

Lemma 2.5.27. *The only terms in (2.5.24) that are possibly nonzero in V obey $t_j = 1$ for $0 \leq j \leq f-1$ and*

$$u_j = \begin{cases} 1 & j \in J \\ 1,2,3 & j \notin J. \end{cases}$$

Proof. By Lemma 2.4.10 we must have $\ell_j \leq \langle \lambda_j, \alpha^\vee \rangle \leq 2p - a_j^1 + a_j^0$ in all cases, which implies $t_j = 1$. Similarly we must have

$$\ell_j + k_j \leq \langle \lambda_j, \gamma^\vee \rangle \leq \begin{cases} 3p - a_j^2 + a_j^1 - 2 & j \in J \\ 3p + a_j^2 - a_j^1 - 2 & j \notin J \end{cases}$$

which implies the second claim if we use that $\ell_j = p - 1$. $\square$

Like before, this lemma shows that we can consider u_j to be small enough to ignore by genericity in the remainder of our arguments. Let us now write $\Theta_j(u) = X_{-\alpha}^{(\ell_j)} X_{-\gamma}^{(k_j)} v_j = X_{-\alpha}^{(p-1)} X_{-\gamma}^{(k_j)}$. The next step is to analyze the image of $\Theta_j(u)v_j$ in Q_j on a case-by-case basis. Let $v_j^0 \otimes v_j^1 \in \mathrm{soc}(Q_j)$ denote a fixed choice of highest weight vector, where v_j^0 is a highest weight vector of $L((-a_j^0 - 1, -a_j^1, -a_j^2 + 1) - \zeta_{j,j+1})$ and v_j^1 is a highest weight vector of $L(\epsilon_j)^{[1]}$. In each case below, we make the choice of v_j^0 and v_j^1 such that the final equation is true.

Case I. Assume that $j, j+1 \in J$. Then clearly $\Theta_j(u)v_j = X_{-\alpha}^{(p-1)} X_{-\gamma}^{(p-1)} v_j$ is a nonzero multiple of $X_{-\alpha}^{(a_j^2 - a_j^0 - 2)} X_{-\gamma}^{(p-1)} v_{\mathbf{E},j}$. One checks that $X_{-\alpha}^{(a_j^2 - a_j^0 - 2)} X_{-\gamma}^{(p - a_j^2 + a_j^1)} \Xi_j(\mathbf{E}) = 0$, so $X_j := X_{-\alpha}^{(a_j^2 - a_j^0 - 2)} X_{-\gamma}^{(p - a_j^2 + a_j^1)} v_{\mathbf{E},j}$ lies in the radical of the sub-Weyl module generated by $v_{\mathbf{E},j}$. Moreover, X_j is nonzero in V by Lemma 2.4.12, and one can easily check that its weight is not a possible weight of the irreducible constituent of V with highest weight lying in $\mathbf{A}$ or $\mathbf{C}$. Hence it lies in the sub-Weyl module of alcove $\mathbf{D}$. As $\mathrm{wt}(X_j) = \mathrm{wt}(v_{\mathbf{D},j}) - (p + a_j^1 - a_j^0 - 2)\alpha$, we deduce from Lemma 2.4.12 that X_j is a nonzero multiple of $X_{-\alpha}^{(p + a_j^1 - a_j^0 - 2)} v_{\mathbf{D},j}$. Thus we see that the image of $\Theta_j(u)v_j$ in Q_j is (after rescaling)

$$X_{-\alpha}^{(a_j^1 - a_j^0 - 2)} X_{-\gamma}^{(a_j^2 - a_j^1 - 1)} v_j^0 \otimes X_{-\alpha} v_j^1.$$

Case II. Assume that $j \in J, j+1 \notin J$. Then arguing exactly as in the previous case, we see that (after rescaling) the image of $\Theta_j(u)v_j = X_{-\alpha}^{(p-1)} X_{-\gamma}^{(p - u_{j+1})}$ in Q_j is

$$\binom{p - u_{j+1}}{a_j^2 - a_j^1 + 2 - u_{j+1}}^{-1} X_{-\alpha}^{(a_j^1 - a_j^0 - 3)} X_{-\gamma}^{(a_j^2 - a_j^1 + 2 - u_{j+1})} v_j^0 \otimes X_{-\alpha} v_j^1.$$

Case III. Assume that $j \notin J, j+1 \in J$. Since $v_{\mathbf{H},j} = X_{-\alpha}^{(p - a_j^1 + a_j^0 + 1)}$ in this case we obviously have

$$\begin{aligned} \Theta_j(u)v_j &= X_{-\gamma}^{((u_j - 1)p + a_j^2 - a_j^1 - 1)} X_{-\alpha}^{(p-1)} v_j \\ &= (-1)^{a_j^1 - a_j^0} X_{-\gamma}^{((u_j - 1)p)} X_{-\gamma}^{(a_j^2 - a_j^1 - 1)} X_{-\alpha}^{(a_j^1 - a_j^0 - 2)} v_{\mathbf{H},j} \end{aligned}$$

and thus the image of $\Theta_j(u)v_j$ in Q_j is (after rescaling)

$$X_{-\alpha}^{(a_j^1 - a_j^0 - 2)} X_{-\gamma}^{(a_j^2 - a_j^1 - 1)} v_j^0 \otimes X_{-\gamma}^{(u_j - 1)} v_j^1.$$

Case IV. Assume that $j, j+1 \notin J$. One checks in a manner very similar to the previous case that the image of $\Theta_j(u)v_j$ in Q_j is

$$X_{-\alpha}^{(a_j^1 - a_j^0 - 1)} X_{-\gamma}^{(a_j^2 - a_j^1 - u_{j+1})} v_j^0 \otimes X_{-\gamma}^{(u_j - 1)} v_j^1.$$

The conclusion in each of the four cases just discussed was that the image of $\Theta_j(u)v_j$ in Q_j is equal to an element of the form

$$\theta_j(u_{j+1})v_j^0 \otimes \eta_j(u_j)v_j^1.$$

In particular, we have already shown that the image of Sv in Q lies in $\mathrm{soc}(Q)$. Now in (2.5.17), we deduce that the image of $-s_\beta s_\alpha Sv$ inside Q under the identification

$$\mathrm{soc}(Q)|_{\mathrm{GL}_3(\mathbb{F}_q)} = \bigotimes_{j=0}^{f-1} \left[L((-a_j^0-1,-a_j^1,-a_j^2+1)_j - \zeta_{j,j+1}) \otimes L(\epsilon_{j+1})\right]^{[f-j]}|_{\mathrm{GL}_3(\mathbb{F}_q)}$$

is equal to

$$\sum_{(u_j)_{j=0}^{f-1}\in\{1,2,3\}^f} \bigotimes_{j=0}^{f-1} [\theta_j(u_{j+1})v_j^0 \otimes \eta_{j+1}(u_{j+1})v_{j+1}^1]. \tag{2.5.28}$$

(In this sum by definition let us take only terms with $u_{j+1} = 1$ whenever $j+1 \in J$ to be nonzero). Let

$$\pi_j : L((-a_j^0-1,-a_j^1,-a_j^2+1) - \zeta_{j,j+1}) \otimes L(\epsilon_{j+1}) \twoheadrightarrow L(-a_j^0-1,-a_j^1,-a_j^2+1)$$

denote the projection. Lemma 2.5.29 below shows that if $j+1 \notin J$ then

$$\pi_j(\theta_j(2)v_j^0 \otimes \eta_{j+1}(2)v_{j+1}^1) = \mu_j^2 \pi_j(\theta_j(3)v_j^0 \otimes \eta_{j+1}(3)v_{j+1}^1)$$
$$\pi_j(\theta_j(1)v_j^0 \otimes \eta_{j+1}(1)v_{j+1}^1) = \mu_j^1 \pi_j(\theta_j(3)v_j^0 \otimes \eta_{j+1}(3)v_{j+1}^1)$$

where μ_j^1, μ_j^2 are constants obeying $1+\mu_j^1+\mu_j^2 \neq 0$. Precisely, we have

$$\mu_j^1 = \begin{cases} \frac{-(a_j^2-a_j^1-1)(a_j^2-a_j^1)(a_j^2-a_j^1+1)}{2(a_j^2-a_j^1+3)} & j \in J \\ 0 & j \notin J \end{cases} \quad \text{and } \mu_j^2 = \begin{cases} \frac{2(a_j^2-a_j^1)}{(a_j^2-a_j^1+3)} & j \in J \\ 0 & j \notin J. \end{cases}$$

If we write simply θ for the term of (2.5.28) such that $u_j = \begin{cases} 1 & j \in J \\ 3 & j \notin J \end{cases}$ then we have shown that the image of $-s_\alpha s_\beta Sv$ under the projection $\mathrm{proj} : \mathrm{soc}(Q) \twoheadrightarrow F_0$ is equal to

$$\sum_{(u_j)\in\{1,2,3\}^f} \left(\prod_{j+1\notin J u_{j+1}=1} \mu_j^1\right)\left(\prod_{j+1\notin J, u_{j+1}=2} \mu_j^2\right) \mathrm{proj}(\theta) = \left[\prod_{j+1\notin J} (1+\mu_j^1+\mu_j^2)\right] \mathrm{proj}(\theta).$$

Since Lemma 2.5.29 also shows that $\mathrm{proj}(\theta) \neq 0$, we have completed the proof of the first half of Theorem 2.3.2(3). $\square$

Lemma 2.5.29. *Let $(a,b,c) \in \mathbb{Z}^3$ be a 5-deep weight in the lowest alcove for* GL_3. *Let* $u =$

$1,2,3$. *In each item below v_0 and v_1 denote highest weight vectors of the corresponding tensor factors.*

1. *The image of $X_{-\alpha}^{(b-c-1)}X_{-\gamma}^{(a-b-u)}v_0\otimes X_{-\gamma}^{(u-1)}v_1$ under the projection $L(-c-2,-b-1,-a+2)\otimes L(1,1,-1)\twoheadrightarrow L(-c-1,-b,-a+1)$ is 0 if $u=1,2$ and nonzero if $u=3$.*

2. *The image of $X_{-\alpha}^{(b-c-2)}X_{-\gamma}^{(a-b-1)}v_0\otimes X_{-\alpha}v_1$ under the projection $L(-c-2,-b,-a+1)\otimes L(1,0,0)\twoheadrightarrow L(-c-1,-b,-a+1)$ is nonzero.*

3. *Let $\theta(u)=\binom{p-u}{a-b+2-u}^{-1}X_{-\alpha}^{(b-c-3)}X_{-\gamma}^{(a-b+2-u)}v_0\otimes X_{-\gamma}^{(u-1)}v_1$ inside $L(-c-2,-b+1,-a)\otimes L(1,1,-1)$. If π denotes the projection onto $L(-c-1,-b,-a+1)$ then we have*
$$\begin{aligned}\pi(\theta(1))&=\frac{-(a-b-1)(a-b)(a-b+1)}{2(a-b+3)}\pi(\theta(3)),\\ \pi(\theta(2))&=\frac{2(a-b)}{a-b+3}\pi(\theta(3))\end{aligned}$$
and $\pi(\theta(3))\neq 0$.

Proof. The proof is similar to that of Lemma 2.5.18 so we give the main steps only. For the first statement it is easy to check that $X_{-\alpha}^{(b-c-1)}X_{-\gamma}^{(a-b-u)}\Xi(-c-2,-b-1,-a+2)=0$ if $u=1,2$ and is nonzero if $u=3$. By considering the weight we see that $X_{-\alpha}^{(b-c-1)}X_{-\gamma}^{(a-b-3)}v_0\otimes X_{-\gamma}^{(2)}v_1$ is a nonzero multiple of $s_\alpha s_\beta v_0\otimes w_0 v_1 = s_\alpha s_\beta(v_0\otimes v_1)$ and the result follows. The second statement is very similar.

We now prove the third part of the lemma. Note $\mathrm{wt}(\theta(u))=(-a+1,-c-1,-b)$ and this weight space in $L(-c-2,-b+1,-a)\otimes L(1,1,-1)$ has dimension 3 equal to

$$\begin{aligned}L(-c-1,-b+2,-a-1)_{\mathrm{wt}(\theta(u))}&\oplus L(-c-1,-b,-a+1)_{\mathrm{wt}(\theta(u))}\\ &\oplus L(-c-1,-b+1,-a)_{\mathrm{wt}(\theta(u))}\end{aligned}\tag{2.5.30}$$

by the translation principle. We find all constants μ_i such that the projection of $\theta(i)-\mu_i\theta(3)$ to the second summand is zero, for $i=1,2$. Let

$$\begin{pmatrix}e_0\\ e_1\\ e_2\end{pmatrix}=\begin{pmatrix}X_{-\alpha}^{(b-c-2)}X_{-\gamma}^{(a-b)}X_{-\beta}v_0\otimes v_1\\ X_{-\alpha}^{(b-c-3)}X_{-\gamma}^{(a-b)}v_0\otimes X_{-\gamma}v_1\\ X_{-\alpha}^{(b-c-2)}X_{-\gamma}^{(a-b-2)}X_{-\beta}v_0\otimes X_{-\gamma}^{(2)}v_1\end{pmatrix}.$$

This is a basis of the $\mathrm{wt}(\theta(u))$ weight space with respect to which we express elements from now on. That these elements are nonzero can be checked using the Xi monomial in a manner similar to Example 2.5.23. Starting from the fact that $v_0\otimes v_1$ is a highest weight vector for the summand $L(-c-1,-b+2,-a-1)$ one computes that bases of the first and third summand in (2.5.30) are

$$\begin{pmatrix} \frac{-1}{a-b+1} \\ 1 \\ \frac{-1}{a-b-1} \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ \frac{-(a-b-1)}{2} \\ 1 \end{pmatrix}$$

respectively. Since $\theta(1) = \frac{(-1)^{a-b}}{a-b+1}e_0$, $\theta(2) = \frac{(-1)^{a-b}}{a-b+1}e_1$ and $\theta(3) = \frac{(-1)^{a-b}2}{(a-b-1)(a-b)(a-b+1)}e_2$ we now easily calculate that there are unique solutions $\mu_2 = \frac{2(a-b)}{a-b+3}$ and $\mu_1 = \frac{-(a-b-1)(a-b)(a-b+1)}{2(a-b+3)}$ which proves the lemma.

□

Proof of Theorem 2.3.2(3), part 2

We now finish the proof of Theorem 2.3.2 by addressing the second part of statement (3) regarding S' and π'. The proof uses the same method, but fortunately the calculations in this case are easiest of all.

This time, define

$$\xi_j^i = -a_j^{\sigma_j^{-1}(i)} + p\epsilon_j'^i - \epsilon_{j+1}'^i - 1 + \sigma_{j+1}^{-1}(i)$$

where

$$\epsilon_j' := \begin{cases} (0,0,0) & j \in J \\ (0,0,-1) & j \notin J. \end{cases}$$

In the same way as before, we obtain a nonzero map $\pi' \to V := \boxtimes_{j=0}^{f-1} V(\lambda_j)$ where $\lambda_j = \xi_j + (p-1)\eta$ via the action of $T_3(\mathbb{F}_q)$ on a choice of highest weight vector $v \in V$, and in fact this map is an embedding, though we don't need this result. Then it is clear that

$$\xi_j \in \mathbf{A} \text{ and } \lambda_j \in \mathbf{G} \iff j \in J$$
$$\xi_j \in \mathbf{B} \text{ and } \lambda_j \in \mathbf{J} \iff j \notin J.$$

As before, the structure of $V(\lambda_j)$ may be seen in Figures 2.5.2 and 2.5.3. We define the quotient of V by $Q := \boxtimes_{j=0}^{f-1} Q_j = \boxtimes_{j=0}^{f-1} Q_j(\mathbf{X}_j)$ where

$$\mathbf{X}_j = \begin{cases} \mathbf{G} & j \in J \\ \mathbf{I} & j \notin J. \end{cases}$$

Then we observe that

$$\mathrm{soc}(Q) = \boxtimes_{j=0}^{f-1} L((-a_j^0 - 1, -a_j^1, -a_j^2 + 1) - \zeta_{j,j+1}) \otimes L(\epsilon_j)^{[1]}$$

where

$$\epsilon_j = \begin{cases} (1,0,-1) & j \in J \\ (1,-1,-1) & j \notin J \end{cases}$$

and $\zeta_{j,j+1}$ is determined by the following table.

	$\zeta_{j,j+1} =$
$j, j+1 \in J$	$(1,0,-1)$
$j \in J, j+1 \notin J$	$(1,-1,-1)$
$j \notin J, j+1 \in J$	$(1,-1,0)$
$j, j+1 \notin J$	$(1,-1,-1)$

Note that $L(1,0,-1)$ is 8-dimensional, having as weights the 6 permutations of $(1,0,-1)$ occurring with multiplicity 1 and $(0,0,0)$ occurring with multiplicity 2 (this is easy to see, since $(0,0,0)$ is the only other allowed weight). Then the equation (2.5.4) holds *verbatim* and since $z_{j,j+1}$ always occurs in $L(\epsilon_{j+1})$ with multiplicity 1 we observe that F_0 is a constituent of $\mathrm{soc}(Q)|_{\mathrm{GL}_3(\mathbb{F}_q)}$ occurring with multiplicity 1. As in Claim 2.5.5, if we can show that the image of $S'v$ in Q lies in $\mathrm{soc}(Q)$ and is nonzero under the projection $\mathrm{soc}(Q) \twoheadrightarrow F_0$ then we will have proved Theorem 2.3.2(3), part 2.

Much as in (2.5.7) we compute

$$S'v = -s_\beta \sum_{(k_j)_{j=0}^{f-1} \in \mathfrak{D}} \bigotimes_j X_{-\beta}^{(k_j)} v_j \tag{2.5.31}$$

where $\mathfrak{D}$ is defined to be the set of tuples $(k_j)_{j=0}^{f-1}$ of integers such that

$$k_j \geq 0 \tag{2.5.32}$$

and

$$\sum_j \left(k_{f-j} + \delta_{f-j \in J}(p - a_{f-j}^2 + a_{f-j}^1)\right) p^j \equiv 0 \mod e \text{ and } \neq 0$$

From this last equation it is clear that we can write

$$k_j = \begin{cases} (u_j - 1)p + a_j^2 - a_j^1 - u_{j+1} & j \in J \\ u_j p - u_{j+1} & j \notin J \end{cases} \tag{2.5.33}$$

for some choice of integers $(u_j)_{j=0}^{f-1}$. Then by (2.5.32) and genericity we clearly must have $u_j \geq 1$ if $j \in J$ and $u_j \geq 0$ if $j \notin J$. It then follows that if $u_j = 0$ we must have $j \notin J$ and $u_{j+1} = 0$, so by the requirement that the sum in (2.5.33) be nonzero we deduce that $u_j \geq 1$ for all $0 \leq j \leq f-1$.

Lemma 2.5.34. *Only terms in (2.5.31) obeying $u_j \in [1,2]$ for each $0 \leq j \leq f-1$ may be nonzero in V.*

Proof. We have already shown that $u_j \geq 1$. And it is clear that

$$\langle \lambda_j, \beta^\vee \rangle \leq \begin{cases} p + a_j^2 - a_j^1 - 1 & j \in J \\ 2p - a_j^2 + a_j^2 - 1 & j \notin J \end{cases}$$

so the result follows from Lemma 2.4.10. □

Let us write $\Theta_j(u) = X_{-\beta}^{(k_j)}$. We now analyze the image of $\Theta_j(u)v_j$ inside Q_j on a case-by-case basis. In each case below, we make the choice of highest weight vector $v_j^0 \otimes v_j^1 \in \mathrm{soc}(Q_j)$ so that the final equation is true.

Case I. Assume that $j, j+1 \in J$. Then $\Theta_j(u)v_j = X_{-\beta}^{((u_j-1)p)} X_{-\beta}^{(a_j^2 - a_j^1 - u_{j+1})} v_j$. If $u_{j+1} = 1$ then this clearly lies in $\langle v_{\mathbb{F}} \rangle$ and hence has image 0 inside Q_j. If $u_{j+1} = 2$ then the image in Q_j is

$$X_{-\beta}^{(a_j^2 - a_j^1 - 2)} v_j^0 \otimes X_{-\beta}^{(u_j - 1)} v_j^1.$$

Case II. Assume that $j \in J, j+1 \notin J$. The image of $\Theta_j(u)v_j$ in Q_j is just

$$X_{-\beta}^{(a_j^2 - a_j^1 - u_{j+1})} v_j^0 \otimes X_{-\beta}^{(u_j - 1)} v_j^1.$$

Case III. Assume that $j \notin J, j+1 \in J$. Then $\Theta_j(u)v_j = \binom{p-u_{j+1}}{a_j^2 - a_j^1 + 1 - u_{j+1}}^{-1} X_{-\beta}^{((u_j-1)p + a_j^2 - a_j^1 + 1 - u_{j+1})} v_{\mathbf{I}}$ so its image inside Q_j is

$$\binom{p - u_{j+1}}{a_j^2 - a_j^1 + 1 - u_{j+1}}^{-1} X_{-\beta}^{(a_j^2 - a_j^1 + 1 - u_{j+1})} v_j^0 \otimes X_{-\beta}^{(u_j - 1)} v_j^1.$$

Note that if $u_j = 2$ then this is zero.

Case IV. Assume that $j, j+1 \notin J$. Exactly the same as the previous case, the image of $\Theta_j(u)v_j$ in Q_j is

$$\binom{p - u_{j+1}}{a_j^2 - a_j^1 + 1 - u_{j+1}}^{-1} X_{-\beta}^{(a_j^2 - a_j^1 + 1 - u_{j+1})} v_j^0 \otimes X_{-\beta}^{(u_j - 1)} v_j^1.$$

Again this is zero if $u_j = 2$.

From Cases III and IV we see that $j \notin J \Rightarrow u_j = 1$. Following the notation of earlier computations, we write the image found in each of the four cases above as $\theta_j(u_{j+1})v_j^0 \otimes \eta_j(u_j)v_j^1$. In particular, we have already shown that the image of $S'v$ in Q lies in $\mathrm{soc}(Q)$. Now from (2.5.31) the image of $-s_\beta S'v$ inside Q, under the identification

$$\mathrm{soc}(Q)|_{\mathrm{GL}_3(\mathbb{F}_q)} = \bigotimes_j \left[L((-a_j^0-1,-a_j^1,-a_j^2+1)_j - \zeta_{j,j+1}) \otimes L(\epsilon_{j+1})\right]|^{[f-j]}_{\mathrm{GL}_3(\mathbb{F}_q)}$$

is equal to

$$\sum_{(u_j)_{j=0}^{f-1}\in\{1,2\}^f} \bigotimes [\theta_j(u_{j+1})v_j^0 \otimes \eta_{j+1}(u_{j+1})v_{j+1}^1] \tag{2.5.35}$$

where all terms with $j+1\notin J$ and $u_{j+1}=2$ are zero. Letting

$$\pi_j : L((-a_j^0-1,-a_j^1,-a_j^2+1)-\zeta_{j,j+1})\otimes L(\epsilon_{j+1}) \twoheadrightarrow L(-a_j^0-1,-a_j^1,-a_j^2+1)$$

denote the projection map, Lemma 2.5.36 below shows that if $j+1\in J$ then

$$\pi_j(\theta_j(1)v_j^0\otimes\eta_{j+1}(1)v_j^1) = c_j(pi_j(\theta_j(2)v_j^0\otimes\eta_{j+1}(2)v_j^1)$$

for some constant c_j obeying $1+c_j\neq 0$. To be precise, we have

$$c_j = \begin{cases} 0 & j\in J \\ a_j^2 - a_j^1 & j\notin J \end{cases}.$$

The same lemma shows moreover that the term of (2.5.35) corresponding to the tuple $u_j = \begin{cases} 2 & j\in J \\ 1 & j\notin J \end{cases}$, which we call θ, obeys $\mathrm{proj}(\theta)\neq 0$, where $\mathrm{proj} : \mathrm{soc}(Q)\twoheadrightarrow F_0$ is the projection. We conclude that the image of $-s_\beta S'v$ under the projection $\mathrm{soc}(Q)\twoheadrightarrow F_0$ is

$$\sum_{(u_j)_{j=0}^{f-1}\in\{1,2\}^f}\left(\prod_{j+1\in J, u_{j+1}=1} c_j\right)\mathrm{proj}(\theta) = \left[\prod_{j+1\in J}(1+c_j)\right]\mathrm{proj}(\theta)\neq 0$$

This completes the proof of Theorem 2.3.2. □

Lemma 2.5.36. *Let $(a,b,c)\in\mathbb{Z}^3$ be a 5-deep weight in the lowest alcove for* GL_3*. In each case below v_0 and v_1 denote highest weight vectors for the corresponding tensor factors.*

1. *The image of $X_{-\beta}^{(a-b-2)}v_0\otimes X_{-\beta}v_1 \in L(-c-2,-b,-a+2)\otimes L(1,0,-1)$ under the projection to $L(-c-1,-b,-a+1)$ is nonzero.*

2. *The image of $X_{-\beta}^{(a-b-1)}v_0\otimes v_1 \in L(-c-2,-b+1,-a+2)\otimes L(1,-1,-1)$ under the projection to $L(-c-1,-b,-a+1)$ is nonzero.*

3. *For $u=1,2$ let $\theta(u) = \binom{p-u}{a-b+1-u}^{-1} X_{-\beta}^{(a-b+1-u)}v_0\otimes X_{-\beta}^{(u-1)}v_1$ inside $L(-c-2,-b+1,-a+1)\otimes L(1,0,-1)$. If π denotes the projection to $L(1,-1,-1)$ then $\pi(\theta(1)) = (a-b)\pi(\theta(2))\neq 0$.*

Proof. In the first statement, the given vector is a nonzero multiple of $s_\beta v_0 \otimes s_\beta v_1$ so the result is immediate. In the second statement, again the given vector is a nonzero multiple of $s_\alpha(v_0 \otimes v_1)$ so the result follows.

In the third statement, $\mathrm{wt}(\theta(u)) = (-c-1, -a+1, -b)$ and this weight space is 2-dimensional, equal to

$$L(-c-1, -b+1, -a)_{\mathrm{wt}(\theta(u))} \oplus L(-c-1, -b, -a+1)_{(\mathrm{wt}(\theta(u))}.$$

Using the Xi monomial, one checks that a basis for the first direct summand above is

$$\begin{aligned} X_{-\beta}^{(a-b)}[v_0 \otimes v_1] &= X_{-\beta}^{(a-b)} v_0 \otimes v_1 + X_{-\beta}^{(a-b-1)} v_0 \otimes X_{-\beta} v_1 \\ &= (-1)^{a-b}\theta(1) + (-1)^{a-b-1}(a-b)\theta(2) \end{aligned}$$

so the claim follows. $\square$

2.6 Characteristic 0 principal series

In this section we prove some results in a somewhat different direction, working in characteristic 0. We continue to make use of inertial types $\tau = \tau(\sigma, K)$ as in Definition 1.6.1. We are interested in the action of natural lifts of the operators defined in Section 2.3 on certain smooth principal series representations. Let F_w denote an unramified extension of $\mathbb{Q}_p$ of degree f and identify its residue field with $\mathbb{F}_q$ (the subscript w is irrelevant in this section; we are simply anticipating the case that F_w arises as the completion of a number field F at the place w that occurs later). Let $\overline{\mathbb{Q}}_p$ denote an algebraic closure of $\mathbb{Q}_p$ whose residue field is identified with $\overline{\mathbb{F}}_p$. Let the Teichmüller lift $\overline{\mathbb{F}}_p \to \overline{\mathbb{Q}}_p$ be denoted by a tilde $x \mapsto \widetilde{x}$. It induces a function $\overline{\mathbb{F}}_p[\mathrm{GL}_3(\mathbb{F}_q)] \to \overline{\mathbb{Q}}_p[\mathrm{GL}_3(\mathbb{F}_q)]$. If S_* is any of the operators defined in Section 2.3, we let $\widetilde{S}_*$ denote its image under this function.

Let $K = \mathrm{GL}_3(\mathcal{O}_{F_w})$ denote the standard maximal compact subgroup of $\mathrm{GL}_3(F_w)$ (not to be confused with $K \subseteq \{0, \ldots, f-1\}$ or the field K of Chapter 1) and Iw the standard Iwahori subgroup of $\mathrm{GL}_3(F_w)$, which is the preimage of the upper-triangular matrices under the map $K \twoheadrightarrow \mathrm{GL}_3(\mathbb{F}_q)$. We also write $K(1)$ for the kernel of this map. If $(a, b, c) \in \mathbb{Z}^3$ is a tuple of integers and π_w is a smooth $\overline{\mathbb{Q}}_p$-representation of $\mathrm{GL}_3(F_w)$ then we define the Iwahori eigenspace $\pi_w^{\mathrm{Iw}=(a,b,c)}$ to be the subspace of π_w on which Iw acts via the character

$$\mathrm{Iw} \longrightarrow B_3(\mathbb{F}_q) \longrightarrow T_3(\mathbb{F}_q) \xrightarrow{\widetilde{\omega}_f^a \otimes \widetilde{\omega}_f^b \otimes \widetilde{\omega}_f^c} \overline{\mathbb{Q}}_p^\times .$$

The normalizer of the Iwahori subgroup in $\mathrm{GL}_3(F_w)$ is generated by Iw, $F_w^\times$, and the matrix

$$\Pi = \begin{pmatrix} & 1 & \\ & & 1 \\ p & & \end{pmatrix}.$$

Observe that Π has order 3 modulo $F_w^\times$ and that it maps $\pi_w^{\mathrm{Iw}=(a,b,c)} \mapsto \pi_w^{\mathrm{Iw}=(b,c,a)}$. We write Iw_1 for the maximal pro-p subgroup of Iw, which is the preimage of the upper-triangular unipotent matrices in $\mathrm{GL}_3(\mathbb{F}_q)$.

The operators $\widetilde{S}_*$ have a well defined action on $\pi_w^{\mathrm{Iw}=(a,b,c)}$ since the latter is contained in $\pi_w^{K(1)}$, and it is easy to check that they translate between different Iwahori eigenspaces.

From now on, we prefer to work with a non-algebraically closed coefficient field. In the statement of the proposition, $E \subseteq \overline{\mathbb{Q}}_p$ is any subfield finite over $\mathbb{Q}_p$ large enough to contain the Teichmüller lifts of all elements in the image of ω_f.

Proposition 2.6.1. *Write $\tau(\sigma, K) = \bigoplus_{i=0}^2 \widetilde{\omega}_f^{A_i}$ as in Definition 1.6.1 and suppose that $\psi_i : F_w \to \overline{\mathbb{Q}}_p^\times$ is a smooth character obeying $\psi_i|_{\mathcal{O}_{F_w}^\times} = \widetilde{\omega}_f^{-A_i}$. We also set*

$$(-B_0, -B_1, -B_2) = \left(\sum_j (-a_j^0 - 1)p^j, \sum_j -a_j^1 p^j, \sum_j (-a_j^2 + 1)p^j \right).$$

1. *Assume that (2.3.3) holds and that $J \subseteq K$. Let $\pi_w = \mathrm{Ind}_{B_3(F_w)}^{\mathrm{GL}_3(F_w)} (\psi_1 \otimes \psi_0 \otimes \psi_2)$. Then we have an equality of operators*

$$\widetilde{S}'_Y \Pi = p^f \kappa_Y \psi_1(p) \widetilde{S}_Y$$

between the 1-dimensional spaces $\pi_w^{\mathrm{Iw}=(-A_1,-A_0,-A_2)} \to \pi_w^{\mathrm{Iw}=(-B_0,-B_1,-B_2)}$, where $\kappa_Y \in \mathcal{O}_E^\times$ obeys

$$\kappa_Y \equiv (-1)^{A_0 - A_2 + f - 1} \frac{\prod_j m_j! n_j! m'_j! n'_j!}{\prod_j (p - 1 - m_j - n_j)!(m'_j + n'_j)!} \mod \varpi_E \tag{2.6.2}$$

where

$$\begin{aligned} m_j &= a_j^1 - a_j^{\sigma_j^{-1}(2)} + t_j p - t_{j+1} - \delta_{j+1 \notin K} \\ n_j &= p - a_j^1 + a_j^0 \\ m'_j &= a_j^1 - a_j^{\sigma_j^{-1}(0)} + s_j p - s_{j+1} + \delta_{j+1 \notin K} \\ n'_j &= p - a_j^2 + a_j^{\sigma_j^{-1}(0)} - s_j p + s_{j+1} - \delta_{j+1 \notin K} \end{aligned}$$

with

$$t_j = \begin{cases} 1 & j \in J \\ 0 & j \notin J \end{cases} \text{ and } s_j = 1 - t_j.$$

2. *Assume that (2.3.4) holds and that $K = \emptyset$. Let $\pi_w = \mathrm{Ind}_{B_3(F_w)}^{\mathrm{GL}_3(F_w)} (\psi_0 \otimes \psi_2 \otimes \psi_1)$. Then*

we have an equality of operators

$$\widetilde{S}'_X\Pi^{-1} = p^{|J|}\kappa_X\psi_1(p)^{-1}\widetilde{S}_X$$

between the 1*-dimensional spaces* $\pi_w^{\mathrm{Iw}=(-A_0,-A_2,-A_1)} \to \pi_w^{\mathrm{Iw}=(-B_0,-B_1,-B_2)}$*, where* $\kappa_X \in \mathcal{O}_E^\times$ *obeys*

$$\kappa_X \equiv (-1)^{A_1+f-|J|+1}\prod_j \frac{m_j!n_j!}{(p-a_j^1+a_j^0-n_j)!} \mod \varpi_E$$

where

$$m_j = \begin{cases} p-a_j^1+a_j^0 & j,j+1\in J\\ p-a_j^1+a_j^0-1 & j\in J, j+1\notin J\\ a_j^1-a_j^0 & j\notin J, j+1\in J\\ a_j^1-a_j^0-1 & j,j+1\notin J\end{cases} \text{ and } n_j = \begin{cases} 0 & j\in J\\ p-a_j^1+a_j^0 & j\notin J.\end{cases}$$

3. *Assume that (2.3.5) holds and that* $K=\emptyset$*. Let* $\pi_w = \mathrm{Ind}_{B_3(F_w)}^{\mathrm{GL}_3(F_w)}(\psi_2\otimes\psi_0\otimes\psi_1)$*. Then we have an equality of operators*

$$\widetilde{S}'_Z\Pi = p^{f-|J|}\kappa_Z\psi_2(p)\widetilde{S}_Z$$

between the 1*-dimensional spaces* $\pi_w^{\mathrm{Iw}=(-A_2,-A_0,-A_1)} \to \pi_w^{\mathrm{Iw}=(-B_0,-B_1,-B_2)}$*, where* $\kappa_Z \in \mathcal{O}_E^\times$ *obeys*

$$\kappa_Z \equiv (-1)^{A_2+|J|-1}\prod_j \frac{m_j!n_j!}{(p-a_j^2-a_j^1-n_j)!} \mod \varpi_E$$

where

$$m_j = \begin{cases} a_j^2-a_j^1-1 & j,j_1\in J\\ a_j^2-a_j^1 & j\in J, j+1\notin J\\ p-a_j^2+a_j^1-1 & j\notin J, j+1\in J\\ p-a_j^2+a_j^1 & j,j+1\notin J\end{cases} \text{ and } n_j = \begin{cases} p-a_j^2+a_j^1 & j\in J\\ 0 & j\notin J.\end{cases}$$

Remark 2.6.3. To see that the Iwahori eigenspaces in this proposition are actually 1-dimensional as claimed, use the fact that if $\pi_w = \mathrm{Ind}_{B_3(F_w)}^{\mathrm{GL}_3(F_w)}(\psi_a\otimes\psi_b\otimes\psi_c)$ where $\psi_i|_{\mathcal{O}_{F_w}^\times} = \widetilde{\omega}_f^i$ then we have $\pi_w^{K(1)} = \mathrm{Ind}_{B_3(\mathbb{F}_q)}^{\mathrm{GL}_3(\mathbb{F}_q)}(\omega_f^a\otimes\omega_f^b\otimes\omega_f^c)$. As $\pi_w^{\mathrm{Iw}=(a',b',c')} = \left(\pi_w^{K(1)}\right)^{B_3(\mathbb{F}_q)=(a',b',c')}$, with obvious notation. The claim now follows from [Her11], Lemma 2.3 and the fact that the principal series for $\mathrm{GL}_3(\mathbb{F}_q)$ are multiplicity-free. It also follows easily using this observation that a basis for $\pi_w^{\mathrm{Iw}=(a,b,c)}$ is given by the unique function $v : \mathrm{GL}_3(K_0)\to\overline{\mathbb{Q}}_p^\times$ with support $B_3(K_0)\cdot\mathrm{Iw}$ such that $v(1)=1$.

Notation 2.6.4. For any of the types τ considered in the theorem, we define $n_\tau \in \mathbb{Z}, \epsilon_\tau \in \{\pm 1\}$ and $\kappa_\tau \in \mathcal{O}_E^\times$ so that the conclusion of the theorem in each case is

$$\widetilde{S}'_\tau \Pi^{\epsilon_\tau} = p^{n_\tau} \kappa_\tau \psi_{i_\tau}(p)^{\epsilon_\tau} \widetilde{S}_\tau.$$

Also note that with the definition of ς in Notation 2.3.6 we have $\pi_w = \mathrm{Ind}_{B_3(F_w)}^{\mathrm{GL}_3(F_w)}(\psi_{\varsigma(0)} \otimes \psi_{\varsigma(1)} \otimes \psi_{\varsigma(2)})$ in each case.

For the proof of the proposition we will need Theorem 2.5.1 of [BD14] (Stickelberger's theorem), which is reproduced here for convenience in slightly different notation. It gives the dominant p-adic term of certain Jacobi sums.

Theorem 2.6.5. *1. Let $m, n \in (0, e]$ such that $e \nmid m + n$. Write*

$$m = \sum_{j=0}^{f-1} m_{f-j} p^j$$
$$n = \sum_{j=0}^{f-1} n_{f-j} p^j$$
$$m + n = \sum_{j=0}^{f-1} (m+n)_{f-j} p^j$$

where $m_j, n_j, (m+n)_j \in [0, p-1]$. Then in $\mathcal{O}_E$ we have

$$\sum_{s \in \mathbb{F}_q} \widetilde{\omega}_f(s)^m \widetilde{\omega}_f(1-s)^n = \kappa p^u + C$$

where $u = \frac{1}{p-1} \sum_{j=0}^{f-1} (p - 1 - (m_j + n_j - (m+n)_j)) \in \mathbb{Z}^{\geq 0}$,

$$\kappa = (-1)^{f-1-u} \frac{\prod_{j=0}^{f-1} m_j! n_j!}{\prod_{j=0}^{f-1} (m+n)_j!} \in \mathbb{Z}_p^\times$$

and the p-valuation of C is $> u$.

2. If $a \in (0, e)$ then $\sum_{s \in \mathbb{F}_q} \widetilde{\omega}_f(s)^a \widetilde{\omega}_f(1-s)^{e-a} = (-1)^{a+1}$.

Proof. The first statement is Theorem 2.5.1 of [BD14]. The second statement is easy; just note that $s \mapsto s/(1-s)$ defines a bijection $\mathbb{F}_q \setminus \{0, 1\} \to \mathbb{F}_q \setminus \{0, -1\}$. □

Proof. The proof strategy is the same as [HLM17], Proposition 3.2.2. For part 1 of the theorem, the proof is in fact essentially the same. And since the proof of part 3 is similar to that of part 2 we will concentrate on proving part 2 only.

Let v denote the basis vector of $\pi_w^{\mathrm{Iw}=(-A_0,-A_2,-A_1)}$ mentioned in Remark 2.6.3. We first remark that

$$\Pi^{-1}v = \psi_2(p)^{-1}\sum_{\lambda,\mu\in\mathbb{F}_q}\begin{pmatrix}\lambda & \mu & 1\\ 1 & & \\ & 1 & \end{pmatrix}v. \tag{2.6.6}$$

To see this, one checks easily that $\theta := \sum_{\lambda,\mu\in\mathbb{F}_q}\begin{pmatrix}\lambda & \mu & 1\\ 1 & & \\ & 1 & \end{pmatrix}$ sends $\pi_w^{\mathrm{Iw}=(a,b,c)}$ to $\pi_w^{\mathrm{Iw}=(c,a,b)}$, so $\Pi\theta v$ must be a multiple of v. Then compute

$$(\Pi\theta v)(1) = \sum_{\lambda,\mu\in\mathbb{F}_q} v\left(\begin{pmatrix}1 & & \\ & 1 & \\ & & p\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ \lambda & \mu & 1\end{pmatrix}\right) = \psi_2(p)^{-1}$$

as the matrix in the centre lies in the support of v iff $\lambda=\mu=0$. Now from (2.6.6) we find

$$\widetilde{S}'\Pi^{-1}v = \psi_2(p)^{-1}\sum_{x,\lambda,\mu\in\mathbb{F}_q}\widetilde{\omega}_f(x)^{\sum_{f-j\notin J}(p-a^1_{f-j}+a^0_{f-j})p^j}A_{\lambda,\mu,x} \tag{2.6.7}$$

where

$$A_{\lambda,\mu,x} = \begin{pmatrix}1+x\lambda & x\mu & x\\ \lambda & \mu & 1\\ & 1 & \end{pmatrix}.$$

Note that

$$A_{0,\mu,x} = \begin{pmatrix}1 & & \\ & \mu & 1\\ & 1 & \end{pmatrix}\begin{pmatrix}1 & x\mu & x\\ & 1 & \\ & & 1\end{pmatrix}.$$

As v is Iw_1-invariant, the sum over terms with $\mu=0$ in (2.6.7) is 0 since we can pull out the coefficient of x (this happens even if $J^c\neq\emptyset$). If $\lambda\neq 0$, then

$$A_{\lambda,\mu,x} = \begin{pmatrix}1 & x+\lambda^{-1} & -\mu\lambda^{-1}\\ & 1 & 0\\ & & 1\end{pmatrix}\begin{pmatrix} & & 1\\ 1 & & \\ & 1 & \end{pmatrix}\begin{pmatrix}\lambda & \mu & 1\\ & 1 & 0\\ & & -\lambda^{-1}\end{pmatrix}.$$

Hence (2.6.7) is equal to

$$(-1)^{A_1}\psi_2(p)^{-1}\sum_{x,\mu\in\mathbb{F}_q}\sum_{\lambda\in\mathbb{F}_q^\times}\widetilde{\omega}_f(\lambda)^{A_1-A_0}\widetilde{\omega}_f(x)^{\sum_{f-j\notin J}(p-a^1_{f-j}+a^0_{f-j})p^j}$$
$$\cdot\begin{pmatrix}1 & x+\lambda^{-1} & -\mu\lambda^{-1}\\ & 1 & 0\\ & & 1\end{pmatrix}\begin{pmatrix} & & 1\\ 1 & & \\ & 1 & \end{pmatrix}v.$$

Making the change of variables $(\lambda', \mu', x') = (\lambda^{-1}, -\mu\lambda^{-1}, x + \lambda^{-1})$ we get

$$\widetilde{S}'\Pi^{-1}v = (-1)^{A_1}\psi_2(p)^{-1} \sum_{x',\mu' \in \mathbb{F}_q} \sum_{\lambda' \in \mathbb{F}_q{}^\times} \widetilde{\omega}_f(\lambda')^{A_0-A_1}\widetilde{\omega}_f(x'-\lambda')^{\sum_{f-j\notin J}(p-a^1_{f-j}+a^0_{f-j})p^j}$$
$$\cdot \begin{pmatrix} 1 & x' & \mu' \\ & 1 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} v. \tag{2.6.8}$$

Observe that $A_0 - A_1 + \sum_{f-j\notin J}(p - a^1_{f-j} + a^0_{f-j})p^j \equiv \sum_{f-j\in J}(p - a^1_{f-j} + a^0_{f-j})p^j \mod e$. If $J \neq \emptyset$, this implies that the contribution of the terms with $x' = 0$ above is zero. Let us make the assumptions $J, J^c \neq \emptyset$ for now; we will deal with the remaining cases after.

The sum (2.6.8) running over $x' \in \mathbb{F}_q{}^\times$, make the change of variables

$$(\lambda, \mu, x) = (\lambda' x'^{-1}, \mu', x')$$

(these are new variables) to get

$$(-1)^{A_1}\psi_2(p)^{-1} \sum_{\mu \in \mathbb{F}_q} \sum_{x,\lambda \in \mathbb{F}_q{}^\times} \widetilde{\omega}_f(\lambda)^{A_0-A_1}\widetilde{\omega}_f(1-\lambda)^{\sum_{f-j\notin J}(p-a^1_{f-j}+a^0_{f-j})p^j}$$
$$\widetilde{\omega}_f(x)^{\sum_{f-j\in J}(p-a^1_{f-j}+a^0_{f-j})p^j} \cdot \begin{pmatrix} 1 & x & \mu \\ & 1 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} v.$$

which is clearly equal to

$$(-1)^{A_1}\psi_2(p)^{-1} \left[\sum_{\lambda \in \mathbb{F}_q} \widetilde{\omega}_f(\lambda)^{A_0-A_1}\widetilde{\omega}_f(1-\lambda)^{\sum_{f-j\notin J}(p-a^1_{f-j}+a^0_{f-j})p^j} \right] \widetilde{S}v.$$

It remains to evaluate the Jacobi sum in square brackets; using the notation of Theorem 2.6.5(1) we may take

$$m_j = \begin{cases} p - a^1_j + a^0_j & j, j+1 \in J \\ p - a^1_j + a^0_j - 1 & j \in J, j+1 \notin J \\ a^1_j - a^0_j & j \notin J, j+1 \in J \\ a^1_j - a^0_j - 1 & j, j+1 \notin J \end{cases} \text{ and } n_j = \begin{cases} 0 & j \in J \\ p - a^1_j + a^0_j & j \notin J \end{cases}$$

and $(m+n)_j = p - a^1_j + a^0_j - n_j$. We easily calculate that $u = |J|$ and κ is as claimed in the proposition.

Now we consider the case $J = \emptyset$. Beginning from (2.6.8), we may do the exact same change of variables on the terms of the sum with $x' \neq 0$, this time using Theorem 2.6.5(2) to simplify the Jacobi sum. We find that the contribution of these terms is

$$(-1)^{A_1+1+\sum_j(p-a^1_{f-j}+a^0_{f-j})p^j}\psi_2(p)^{-1}\sum_{\mu\in\mathbb{F}_q}\sum_{x\in\mathbb{F}_q{}^\times}\begin{pmatrix}1 & x & \mu\\ & 1 & 0\\ & & 1\end{pmatrix}\begin{pmatrix} & & 1\\ 1 & & \\ & 1 & \end{pmatrix}v.$$

On the other hand, the contribution of the terms with $x' = 0$ is directly calculated to be (renaming $\mu' = \mu$)

$$\begin{aligned}&(-1)^{A_1}\psi_2(p)^{-1}\sum_{\mu\in\mathbb{F}_q}\sum_{\lambda'\in\mathbb{F}_q{}^\times}\widetilde{\omega}_f(\lambda')^0(-1)^{\sum_j(p-a^1_{f-j}+a^0_{f-j})p^j}\begin{pmatrix}1 & 0 & \mu\\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} & & 1\\ 1 & & \\ & 1 & \end{pmatrix}\\ &\quad= (-1)^{A_1+1+\sum_j(p-a^1_{f-j}+a^0_{f-j})p^j}\psi_2(p)^{-1}\sum_{\mu\in\mathbb{F}_q}\begin{pmatrix}1 & 0 & \mu\\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} & & 1\\ 1 & & \\ & 1 & \end{pmatrix}\end{aligned}$$

Adding these together, we get exactly what is claimed in the theorem statement. The case $J^c = \emptyset$ follows similarly.

□

Chapter 3

Mod p local-global compatibility

In this chapter we put the results on the Galois side from Chapter 1 and the results on the local automorphic side from Chapter 2 together to get our main local-global compatibility theorem (Theorem 3.5.6) in the cohomology of definite unitary groups.

3.1 Definite unitary groups

We first review the global set-up of definite unitary groups of rank 3 considered in the papers [EGH13, HLM17, LLHL17]. Let F be a totally imaginary CM number field with maximal totally real subfield[1] $F^+ \neq \mathbb{Q}$. Let c be the nonidentity element of $\mathrm{Gal}(F/F^+)$ and for simplicity assume that all finite primes of F^+ are unramified in F. Moreover assume that all primes of F^+ dividing p split in F. We write Σ_p^+ (resp. Σ_p) for the places of F^+ (resp. F) lying above p. For each place $v \in \Sigma_p^+$, choose a place $w \in \Sigma_p$ lying above it, so that v splits as ww^c in F. Write $F_p^+ = F \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $\mathcal{O}_{F^+,p} = \mathcal{O}_{F^+} \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Let G be a definite unitary group over F^+ quasi-split at all finite places and let $\mathcal{G}$ be a model of G defined over $\mathcal{O}_{F^+}$. That is, $G(F_v^+) \cong U_3(\mathbb{R})$ is compact for each $v|\infty$ and G is an outer form of GL_3 that splits over F. Section 2.1 of [Ger10] explains how to concretely construct such a group and its model by considering an involution of the second kind on $\mathrm{Mat}_n(F)$ and an order in $\mathrm{Mat}_n(F)$ preserved by this involution. The details of the construction are not important and it would be possible to simply specify a list of axioms G needs to satisfy (for example, see [EGH13], Section 7.1). We will review all relevant properties of G below.

If v is a place of F^+ splitting as ww^c in F, then the isomorphism $F_v^+ \cong F_w$ induces an isomorphism $\iota_w : G(F_v^+) \xrightarrow{\sim} \mathrm{GL}_3(F_w)$ which restricts to an isomorphism $\iota_w : \mathcal{G}(\mathcal{O}_{F_v^+}) \xrightarrow{\sim} \mathrm{GL}_3(\mathcal{O}_{F_w})$. This isomorphism depends on the choice of w, and [Ger10] shows how we can ensure that $\iota_{w^c} = {}^t(\iota_w)^c$.

[1] This is required for technical reasons in [Lab11]. Without it, Theorem 3.3.1 might not hold.

3.2 p-adic, mod p and classical automorphic forms

For general reductive groups G there is currently no notion of "p-adic" or "mod p" automorphic form to play a role in the hypothetical p-adic and mod p Langlands correspondences. Nevertheless a good substitute seems to be the (completed) cohomology of arithmetic quotients of the locally symmetric space associated to G having p-adic or mod p coefficients (see [CE12, Eme14]). When G is compact at infinity (which is the case here) there is also a related notion of algebraic automorphic form ([Gro99]). In this section we define the relevant spaces for the group G and explain their connections with classical automorphic forms on G.

Definition 3.2.1. If W is a finitely generated $\mathcal{O}_E$-module with an action of $\mathcal{G}(\mathcal{O}_{F^+,p})$ and $U \leq G(\mathbb{A}_{F^+}^{\infty,p}) \times \mathcal{G}(\mathcal{O}_{F^+,p})$ is a compact open subgroup we define the space of algebraic automorphic forms on G with level U and coefficients in W to be the $\mathcal{O}_E$-module

$$H^0(U,W) := \{f : G(F^+)\backslash G(\mathbb{A}_{F^+}^{\infty}) \to W \mid f(gu) = u_p^{-1} \cdot f(g) \quad \forall g \in G(\mathbb{A}_{F^+}^{\infty}), u \in U\}$$

where $u \mapsto u_p$ denotes the projection $U \to \mathcal{G}(\mathcal{O}_{F^+,p})$. The reason for the notation H^0 (from a more general viewpoint than we really need) is that this module may be interpreted as the 0th cohomology of the local system defined by W on the arithmetic quotient of level U associated to G; see [Eme14] 2.1.3. This arithmetic quotient is finite, since G is compact at infinity and so the degree 0 cohomology is the only nonzero one.

When W is the Weyl module $V_{\mathcal{O}_E}(\lambda)$ defined below, $H^0(U,W)$ is an example of a space of algebraic modular forms in the sense of [Gro99].

For any compact open subgroup U as above we may write $G(\mathbb{A}_{F^+}^{\infty}) = \coprod_{i\in\Lambda} G(F^+)t_iU$ for finitely many $t_i \in G(\mathbb{A}_{F^+}^{\infty})$. It is easy to check that there is an isomorphism of $\mathcal{O}_E$-modules

$$H^0(U,W) \xrightarrow{\sim} \bigoplus_{i\in\Lambda} W^{U\cap t_i^{-1}G(F^+)t_i}$$

given by $f \mapsto (f(t_i))_{i\in\Lambda}$. In particular $H^0(U,W)$ is finitely generated over $\mathcal{O}_E$. Note that $\varinjlim_U H^0(U,W)$ where U runs over all compact open subgroups of $G(\mathbb{A}_{F^+}^{\infty,p}) \times \mathcal{G}(\mathcal{O}_{F^+,p})$ has a natural $G(\mathbb{A}_{F^+}^{\infty})$-action given by $(g\cdot f)(x) = g_p \cdot f(xg)$. The result just stated then shows that this is an admissible $G(\mathbb{A}_{F^+}^{\infty})$-module over $\mathcal{O}_E$.

We say that U is *sufficiently small* if for some finite place v of F^+ the projection of U to $G(F_v^+)$ contains no element of finite order other than the identity. By compactness of G at ∞, the groups $U \cap t_i^{-1}G(F^+)t_i$ are finite (see Proposition 1.4 of [Gro99]), so if U is sufficiently small then $U \cap t_i^{-1}G(F^+)t_i = \{1\}$ and for any W and any $\mathcal{O}_E$-algebra A we have

$$H^0(U,W) \otimes_{\mathcal{O}_E} A \xrightarrow{\sim} H^0(U, W \otimes_{\mathcal{O}_E} A).$$

There is a Hecke action on the space $H^0(U,W)$. Let us say that the level U is *unramified* at a place v of F^+ if $U = \mathcal{G}(\mathcal{O}_{F_v^+}) \times U^v$ for some compact open $U^v \leq G(\mathbb{A}_{F^+}^{\infty,v})$. Let $\mathcal{P}_U$ denote the set of finite places w of F such that $v := w|_{F^+}$ splits in F, $v \nmid p$, and U is unramified at v. If $\mathcal{P} \subseteq \mathcal{P}_U$ is a subset of finite complement then we define the abstract Hecke algebra $\mathbb{T}^{\mathcal{P}} = \mathcal{O}_E[T_w^{(i)} : w \in \mathcal{P}, 0 \leq i \leq 3]$ where the Hecke operator $T_w^{(i)}$ acts on $H^0(U, W)$ via the double coset operator

$$\iota_w^{-1}\left[\mathrm{GL}_3(\mathcal{O}_{F_w})\begin{pmatrix}\varpi_w \mathrm{id}_i & \\ & \mathrm{id}_{3-i}\end{pmatrix}\mathrm{GL}_3(\mathcal{O}_{F_w})\right],$$

where ϖ_w is a uniformizer of F_w (the operator is independent of ϖ_w).

Next we make the connection with spaces of classical automorphic forms on G. Let $\mathcal{A}$ denote the space of automorphic forms on $G(\mathbb{A}_{F^+})$. In the compact case this means the space of smooth functions $G(F^+)\backslash G(\mathbb{A}_{F^+}) \to \mathbb{C}$ that are $G(F^+_\infty)$-finite and right invariant under some compact open subgroup of $G(\mathbb{A}^\infty_{F^+})$. It is a pre-unitary representation of $G(\mathbb{A}_{F^+})$ by right translation, which preserves the natural inner product. Let Irr_∞ denote the set of isomorphism classes of continuous complex representations of $G(F^+_\infty)$. Compactness implies that all $W \in \mathrm{Irr}_\infty$ are finite-dimensional and the action of $G(F^+_\infty)$ is completely reducible. Hence the $G(\mathbb{A}_{F^+})$-representation $\mathcal{A}$ decomposes into a direct sum of $G(\mathbb{A}_{F^+})$-modules as

$$\mathcal{A} = \bigoplus_{W \in \mathrm{Irr}_\infty} W \otimes_{\mathbb{C}} \mathrm{Hom}_{G(F^+_\infty)}(W, \mathcal{A}). \tag{3.2.2}$$

For $W \in \mathrm{Irr}_\infty$ write S_W for the space of functions $f : G(\mathbb{A}^\infty_{F^+}) \to W^\vee$ right invariant under some compact open subgroup of $G(\mathbb{A}^\infty_{F^+})$ obeying $f(\gamma g) = \gamma_\infty f(g)$ for all $g \in G(\mathbb{A}^\infty_{F^+})$ and $\gamma \in G(F^+)$. This space is a $G(\mathbb{A}^\infty_{F^+})$-module by right translation. Then the map

$$f \mapsto (w \mapsto (g \mapsto (g_\infty^{-1} f(g^\infty))(w)))$$

is an isomorphism of $G(\mathbb{A}^\infty_{F^+})$-modules $S_W \xrightarrow{\sim} \mathrm{Hom}_{G(F^+_\infty)}(W, \mathcal{A})$.

Now fix an algebraic closure $\overline{\mathbb{Q}}_p \supset E$ and an isomorphism of fields $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. Let $(\mathbb{Z}^{3,+})_c^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_p)}$ denote the set of elements $\lambda = (\lambda_\tau)_\tau$ in $(\mathbb{Z}^{3,+})^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_p)}$ such that $\lambda_{\tau,i} = -\lambda_{\tau^c,2-i}$ for each $\tau : F \hookrightarrow \overline{\mathbb{Q}}_p$ and $0 \leq i \leq 2$. Identifying $\mathrm{Hom}(F, \overline{\mathbb{Q}}_p) = \bigsqcup_{w \in \Sigma_p} \mathrm{Hom}(F_w, \overline{\mathbb{Q}}_p)$, we write λ_w, for the projection of λ to $(\mathbb{Z}^{3,+})^{\mathrm{Hom}(F_w,\overline{\mathbb{Q}}_p)}$. Given λ_w, we define a representation $V_E(\lambda_w)$ of $G(F^+_v)$ over E, where $v = w|_{F^+}$, as follows: for $\tau : F_w \hookrightarrow \overline{\mathbb{Q}}_p$ let $V_{F_w}(\lambda_{w,\tau})$ denote the Weyl module of highest weight $\lambda_{w,\tau}$ for GL_{3/F_w} defined in Section 2.1 and give it an action of $G(F^+_v)$ via ι_w. Then set

$$V_E(\lambda_w) = \bigotimes_{\tau : F_w \hookrightarrow \overline{\mathbb{Q}}_p} V_{F_w}(\lambda_{w,\tau}) \otimes_{F_w,\tau} E.$$

We also define $V_{\mathcal{O}_E}(\lambda_w)$ similarly, by taking $V_{\mathcal{O}_{F_w}}(\lambda_{w,\tau})$ to be the integral Weyl module of highest weight $\lambda_{w,\tau}$ for $\mathrm{GL}_{3/\mathcal{O}_{F_w}}$ ([Jan03], II.8.2) and giving it an action of $\mathcal{G}(\mathcal{O}_{F^+_v})$ via ι_w. So $V_{\mathcal{O}_E}(\lambda_w)$ is an $\mathcal{O}_E$-representation of $\mathcal{G}(\mathcal{O}_{F^+_v})$ and we have $V_{\mathcal{O}_E}(\lambda_w) \otimes_{\mathcal{O}_E} E \cong V_E(\lambda_w)$.

Since $\lambda \in (\mathbb{Z}^{3,+})_c^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_p)}$, the representations of $\mathcal{G}(\mathcal{O}_{F^+_v})$ given by $V_{\mathcal{O}_E}(\lambda_w)$ and $V_{\mathcal{O}_E}(\lambda_{w^c})$

are isomorphic (and similarly with $V_E(\lambda_w)$ and $V_E(\lambda_{w^c})$ as $G(F_v^+)$-representations). To see this, note that twisting by the inverse transpose automorphism of GL_3 changes the Weyl module $V(\mu)$ to $V(-w_0\mu)$, by the universal property for example. We denote them by $V_{\mathcal{O}_E}(\lambda_v)$ and $V_E(\lambda_v)$ respectively. Finally, we define a representation of $G(F_p^+)$ over E by

$$V_E(\lambda) := \bigotimes_{v\in\Sigma_p^+} V_E(\lambda_v)$$

and an integral model over $\mathcal{O}_E$ having an action of $\mathcal{G}(\mathcal{O}_{F^+,p})$ by $V_{\mathcal{O}_E}(\lambda) = \bigotimes_{v\in\Sigma_p^+} V_{\mathcal{O}_E}(\lambda_v)$.

Fix a set of embeddings $\{\tilde{\tau} : F \hookrightarrow \overline{\mathbb{Q}}_p\}$ such that $\mathrm{Hom}(F, \overline{\mathbb{Q}}_p) = \{\widetilde{\tau}\} \cup \{\widetilde{\tau} \circ c\}$. Since the Weyl modules in question are all defined over F (or even $\mathbb{Z}$, as we used in Chapter 2), the space $V_{\mathcal{O}_E}(\lambda) \otimes_{\mathcal{O}_E,\iota} \mathbb{C} = V_E(\lambda) \otimes_{E,\iota} \mathbb{C} \cong \bigotimes_{\widetilde{\tau}} V_F(\lambda_{\widetilde{\tau}}) \otimes_{F,\widetilde{\tau}} \mathbb{C}$ has a natural action of $G(F_\infty^+)$ where for $v|\infty$, $G(F_v^+)$ acts via $G(F_v^+) \to G(\mathbb{C}) \cong \mathrm{GL}_3(\mathbb{C})$, both of these maps being induced by the (unique) embedding $\iota \circ \widetilde{\tau}$ defining v. For $\gamma \in G(F^+), v \in V_{\mathcal{O}_E}(\lambda) \otimes_{\mathcal{O}_E,\iota} \mathbb{C}$ this new action obeys $\gamma_\infty \cdot v = \gamma_p \cdot v$. Finally, one checks that there is an isomorphism of $G(\mathbb{A}_{F^+}^\infty)$-modules

$$\begin{aligned} \varinjlim_U H^0(U, V_{\mathcal{O}_E}(\lambda)) \otimes_{\mathcal{O}_E,\iota} \mathbb{C} &\xrightarrow{\sim} S_{(V_{\mathcal{O}_E}(\lambda)\otimes_{\mathcal{O}_E,\iota}\mathbb{C})^\vee} \\ f &\mapsto (g \mapsto g_p \cdot f(g)). \end{aligned} \tag{3.2.3}$$

In particular, since we have already seen the left hand side is admissible we deduce the well-known result that $\mathcal{A}$ is an admissible $G(\mathbb{A}_{F^+})$-module. Together with the fact that $\mathcal{A}$ is pre-unitary, this implies that $\mathcal{A}$ decomposes as a direct sum

$$\mathcal{A} = \bigoplus_\pi m(\pi)\pi$$

where π runs over isomorphism classes of automorphic representations of $G(\mathbb{A}_{F^+})$ and $m(\pi) \in \mathbb{N}$ is a finite multiplicity.

3.3 Galois representations and local-global compatibility

It is known how to attach p-adic Galois representations to automorphic representations of G, and that these representations satisfy local-global compatility at all places. In this section we recall these facts. We continue to make use of our fixed isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$.

Let ℓ be a prime, possibly equal to p. Recall that the local Langlands correspondence for GL_n over an ℓ-adic field L is a family of bijections

$$\mathrm{rec}_{L,\mathbb{C}} : \mathrm{Irr}_{\mathbb{C}}(\mathrm{GL}_n(L)) \to \mathrm{WD}_n^{\mathrm{F-ss}}(L)$$

between isomorphism classes of irreducible admissible $\mathbb{C}$-representations of $\mathrm{GL}_n(L)$ and n-dimensional Frobenius-semisimple Weil-Deligne representations of W_L over $\mathbb{C}$, where W_L denotes the Weil group of L. The correspondence is natural in the sense that it is the unique map satisfying the list of natural properties in the Introduction of [HT01]. The only specific thing we need to know about the correspondence is that $\mathrm{rec}_{L,\mathbb{C}}$ sends $\mathrm{Ind}_{B_3(L)}^{\mathrm{GL}_3(L)}(\psi_0 \otimes \psi_1 \otimes \psi_2)$ to the Weil-Deligne representation $\psi_0^{-1}|\cdot|^2 \oplus \psi_1^{-1}|\cdot| \oplus \psi_2^{-1}$ having trivial monodromy for any smooth characters $\psi_i : L^\times \to \mathbb{C}^\times$ such that $\psi_i\psi_{i+1}^{-1} \neq |\cdot|^{\pm 1}$ (note this is an unnormalized principal series representation).

We define the local Langlands correspondence over $\overline{\mathbb{Q}}_p$, denoted rec_L, by $\iota \circ \mathrm{rec}_L = \mathrm{rec}_{L,\mathbb{C}} \circ \iota$. In the theorem below, $|\cdot| : F^\times \to \overline{\mathbb{Q}}_p^\times$ denotes $\iota^{-1} \circ |\cdot|$. We recall that if (σ, N) is a Weil-Deligne representation then $(\sigma, N)^{\mathrm{F-ss}}$ means $(\sigma^{\mathrm{ss}}, N)$.

If $\ell \neq p$, recall that Grothendieck's ℓ-adic monodromy theorem gives an equivalence of categories between continuous representations $\rho : G_L \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ and bounded Weil-Deligne representations of L over $\overline{\mathbb{Q}}_p$, which we denote $\rho \mapsto \mathrm{WD}(\rho)$ (see [Gee], Proposition 2.17). On the other hand, if $\ell = p$ then there is a construction allowing us to associate a Weil-Deligne representation $\mathrm{WD}(\rho)$ over E with any potentially semistable (equivalently, de Rham) representation $G_L \to \mathrm{GL}_n(E)$, for $E \subset \overline{\mathbb{Q}}_p$ sufficiently large and finite over $\mathbb{Q}_p$. The construction is defined for example in Appendix B of [CDT99]; we briefly review the details since it will be used below. Suppose that ρ becomes semistable when restricted to G_K for some finite Galois extension $K \supseteq L$. Then we have the associated $(\varphi, N, \mathrm{Fil}, K/L, E)$-module $D = D_{\mathrm{st}}^*(\rho^\vee)$ of Section 1.4. We define an action of $g \in W_L$ on D by making g act as the product of the commuting operators given by the image of g in $\mathrm{Gal}(K/L)$ and φ_D^{-n}, where the image of g on the residue field of W_L is the nth power of *absolute* Frobenius. This makes D into a finite free $L_0 \otimes_{\mathbb{Q}_p} E$-module with a continuous action of W_L. Then we define $\mathrm{WD}(\rho) = D \otimes_{L_0 \otimes_{\mathbb{Q}_p} E, \tau_0} \overline{\mathbb{Q}}_p$ for any choice of embedding $\tau_0 : L_0 \hookrightarrow E$. This is independent of τ_0. These constructions are used in the statement of local-global compability for Galois representations associated with cuspidal automorphic representations of G.

Theorem 3.3.1. *If π is an automorphic representation of G contributing to $\varinjlim_U H^0(U, V_{\mathcal{O}_E}(\lambda)) \otimes_{\mathcal{O}_E,\iota} \mathbb{C}$ in (3.2.3) for some $\lambda \in (\mathbb{Z}^{3,+})_c^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_p)}$, then there exists a unique continuous semisimple representation*

$$r_\pi : G_F \to \mathrm{GL}_3(\overline{\mathbb{Q}}_p)$$

such that

1. *$r_\pi^c \cong r_\pi \otimes \epsilon^{-2}$,*

2. *r_π is de Rham at all places dividing p and crystalline if π has level prime to p. If $\tau : F \hookrightarrow \overline{\mathbb{Q}}_p$ then $\mathrm{HT}_\tau(r_\pi) = -w_0\lambda_\tau + (2,1,0)$.*

3. *if v is a place of F^+ which splits as $v = ww^c$ in F (including possibly $v|p$) then*

$$\mathrm{WD}(r_\pi|_{G_{F_w}})^{\mathrm{F-ss}} \cong \mathrm{rec}_{F_w}((\pi_v \circ \iota_w^{-1}) \otimes |\cdot|^{-1}).$$

Proof. This is Theorem 7.2.1 of [EGH13]; see the list of references in the proof there. The only thing to note in addition is that item (3) holds without semisimplification at places dividing p by [Car14]. $\square$

Remark 3.3.2. Note that a Galois representation of G_F is uniquely determined by its local factors at F_w for places w of F lying above a place of F^+ that is split in F. This is by the Chebotarev density theorem because the set of such places has Dirichlet density 1 in F.

Remark 3.3.3. It is not actually the most natural that we should have our Galois representations attached to π taking values in GL_3 from the point of view of general philosophy concerning the association of Galois representations to automorphic representations. Rather, they should take values in the C-group of G, which is described in Section 8.3 of [BG14]. And indeed it is known that the Galois representations above extend to the C-group (*loc. cit.*). But since we are only interested in studying local-global compatibility for GL_3 anyway, we don't need this extra structure.

3.4 Serre weights and modularity

Definition 3.4.1. A *Serre weight for* $\mathcal{G}$ is an isomorphism class of an (absolutely) irreducible $\mathbb{F}$-representation of $\mathcal{G}(\mathcal{O}_{F^+,p})$.

The Serre weights are parametrized by the set $\prod_{w\in\Sigma_p}(\mathbb{Z}^3_+)^{\mathrm{Hom}(k_w,\overline{\mathbb{F}}_p)}_{\mathrm{res}}$ of p-restricted tuples (λ_w) obeying $\lambda_{\tau,i} = -\lambda_{\tau^c,2-i}$ for any $\tau : k_w \hookrightarrow \overline{\mathbb{F}}_p$. Namely, let $F(\lambda_w)$ be the representation of $\mathrm{GL}_3(k_w)$ so denoted in Section 2.2. Then for $v \in \Sigma_p^+$, $F(\lambda_v) = F(\lambda_w)\circ\iota_w$ is independent of the choice of w dividing v and we define the Serre weight $F(\lambda) = \bigotimes_{v\in\Sigma_p^+} F(\lambda_v)$.

If λ_w lies in the lowest alcove for each $w \in \Sigma_p$ then using the fact that $\mathrm{Hom}(F_w, \overline{\mathbb{Q}}_p)$ and $\mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)$ are in natural bijection (since each F_w is unramified), it makes sense to write $V_{\mathcal{O}_E}(\lambda)$, and we have $V_{\mathcal{O}_E}(\lambda)\otimes_{\mathcal{O}_E}\mathbb{F} \cong F(\lambda)$ by Proposition 2.4.6.

Definition 3.4.2. We now consider a continuous irreducible representation $\bar{r} : G_F \to \mathrm{GL}_3(\mathbb{F})$. Let U be a compact open subgroup of $G(\mathbb{A}^{\infty;p}_{F^+}) \times \mathcal{G}(\mathcal{O}_{F^+,p})$ and $\mathcal{P} \subseteq \mathcal{P}_U$ a subset as in Section 3.2, and assume that $\bar{r}$ is unramified at all places in $\mathcal{P}$. Then $\bar{r}$ gives rise to a maximal ideal $\mathfrak{m}_{\bar{r}} \leq \mathbb{T}_{\mathcal{P}}$ by defining the reduction mod $\mathfrak{m}_{\bar{r}}$ of $T^{(i)}_w$ so the equation

$$\det(1-\bar{r}^\vee(\mathrm{Frob}_w)X) = \sum_{j=0}^{3}(-1)^j p^{\binom{j}{2}}(T^{(j)}_w \mod \mathfrak{m}_{\bar{r}})X^j$$

in $\mathbb{F}[X]$ is satisfied for each $w \in \mathcal{P}$.

Let V be a Serre weight for $\mathcal{G}$. We say that $\bar{r}$ is *automorphic of weight* V, or that V is a *modular weight* of $\bar{r}$, if there exists a compact open U unramified at p, and a set $\mathcal{P}$ as above such that $\bar{r}$ is unramified at all places in $\mathcal{P}$, such that $H^0(U,V)_{\mathfrak{m}_{\bar{r}}} \neq 0$ (equivalently $H^0(U,V)[\mathfrak{m}_{\bar{r}}] \neq 0$ by commutative algebra). The set of weights of $\bar{r}$ is denoted $W(\bar{r})$. We say that $\bar{r}$ itself is automorphic if $W(\bar{r}) \neq \emptyset$.

Remark 3.4.3. As one might expect from the terminology, $\bar{r}$ being automorphic is equivalent to $\bar{r}$ being the reduction mod p of one of the p-adic representations r_π of Theorem 3.3.1. This follows from an argument like in (3.5.8) below.

From now on, fix a choice of preferred place $w|p$ in F and let $v = w|_{F^+}$ and $q = p^f = |k_w|$. We write $k_w = \mathbb{F}_q$ so the notation in what follows agrees with that of Chapter 2. In order to be able to most easily focus on the single place v in what follows, we assume that $\bar{r}$ is a Galois representation automorphic of weight $V = F(\lambda)$ where $\lambda_{v'}$ lies in the lowest alcove for all $v' \neq v$. Write $V' = \bigotimes_{v'|p, v' \neq v} F(\lambda_{v'})$ and $\widetilde{V}' = \bigotimes_{v'|p, v'\neq v} V_{\mathcal{O}_E}(\lambda_{v'})$. Then by the remarks above we have $\widetilde{V}' \otimes_{\mathcal{O}_E} \mathbb{F} = V'$.

We fix a choice of a compact open subgroup $U \leq G(\mathbb{A}_{F^+}^{\infty,p}) \times \mathcal{G}(\mathcal{O}_{F^+,p})$ that is sufficiently small and unramified at all places dividing p, and fix a subset $\mathcal{P} \subseteq \mathcal{P}_U$ as above such that $H^0(U, V)_{\mathfrak{m}_{\bar{r}}} \neq 0$. We define

$$W(\bar{r}|_{G_{F_w}}) = \{\text{Serre weights } F \text{ of } \mathrm{GL}_n(\mathbb{F}_q) \mid H^0(U, (F \circ \iota_w) \otimes V')[\mathfrak{m}_{\bar{r}}] \neq 0\}.$$

(Despite the notation, it is not yet known that this set depends only on $\bar{r}|_{G_{F_w}}$, or that it is independent of V'.)

3.5 Main result

In this section we prove our main result about recovering the Galois representation from cohomology. Having fixed U and V' at the end of the previous section, we define $H^0(U^v, V') = \varinjlim_{U_v} H^0(U^v U_v, V')$ where the limit runs over compact open subgroups $U_v \leq \mathcal{G}(\mathcal{O}_{F_v^+})$. This space has a smooth action of $G(F_v^+)$, and therefore of $\mathrm{GL}_3(F_w) = \mathrm{GL}_3(\mathbb{Q}_{p^f})$ via ι_w. Via the analogue of mod p local-global compatibility for $\mathrm{GL}_2(\mathbb{Q}_p)$ described in the introduction, we expect $H^0(U^v, V')[\mathfrak{m}_{\bar{r}}]$ to depend only on $\bar{r}|_{G_{F_w}}$.

Lemma 3.5.1. *$W(\bar{r}|_{G_{F_w}})$ is the set of Serre weights F for $\mathrm{GL}_3(\mathbb{F}_q)$ such that*

$$\mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Z}_{p^f})}(F^\vee, H^0(U^v, V')[\mathfrak{m}_{\bar{r}}] \neq 0.$$

Proof. This follows from the proof of Lemma 7.4.3 of [EGH13], which in fact shows that for any finite-dimensional $\mathbb{F}$-module V_v with a smooth U_v-action, where $U_v \leq G(F_v^+)$ is compact open, we have

$$\mathrm{Hom}_{U_v}(V_v^\vee, H^0(U^v, V')) \cong H^0(U^v U_v, V_v \otimes V') \tag{3.5.2}$$

as $\mathbb{T}^{\mathcal{P}}$-modules. □

For our main result we will make the following assumption.

Definition 3.5.3. Let $\bar{r}, U, V'$ be as above. We say that $\bar{r}$ satisfies Assumption (SW) if $\bar{r}|_{G_{F_w}} \in (\mathrm{FL}_\delta)$ with $\delta \geq 3$, so we can pick a presentation of $\bar{r}|_{G_{F_w}}$ as in (1.1.1) and $W(\bar{r}|_{G_{F_w}}) = \{F_0\}$ in the notation of Section 2.3.

Remark 3.5.4. Note that F_0 is the unique ordinary weight of $\bar{r}|_{G_{F_w}}$ in the terminology of [BH15]. With certain technical assumptions on U and $\bar{r}$ and if $\bar{r}|_{G_{F_{w'}}}$ is Fontaine-Laffaille for each place $w'|p$ it follows from Theorem 4.4.1 of [BLGGT14] and Lemma 3.1.5 of [GG12] that (if $\bar{r}$ is automorphic) then F_0 always lies in $W(\bar{r}|_{G_{F_w}})$ (for some choice of V'). So in these cases the above assumption is equivalent to asking that $W(\bar{r}|_{G_{F_w}})$ be a singleton. This should be true more generally.

Remark 3.5.5. Based on forthcoming work [LLHLM] that determines $W(\bar{r})$ in cases we are considering, it should be the true that "most" $\bar{r}$ such that $\bar{r}|_{G_{F_w}} \in (\mathrm{FL}_\delta)$ (for sufficiently large δ) obey Assumption (SW), so this assumption again represents the most generic case. We show below that Assumption (SW) implies that $\bar{r}|_{G_{F_w}} \in (\mathrm{FL}_\delta^{**})$ in certain cases. The converse statement is not true, but there should be a slightly more complicated condition on the Fontaine-Laffaille invariants of $\bar{r}|_{G_{F_w}}$ which is necessary and sufficient for $W(\bar{r}|_{G_{F_w}})$ to be a singleton.

Theorem 3.5.6. *Let $\bar{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be a continuous representation satisfying Assumption (SW) for a choice of U, V' as above. Assume that (1.1.3) holds with $\delta \geq 15$. Let τ be any of the principal series types of Theorem 2.3.2 and write $\widetilde{\eta} = \omega_f^{A_{\varsigma(0)}} \otimes \omega_f^{A_{\varsigma(1)}} \otimes \omega_f^{A_{\varsigma(2)}}$ where $\varsigma = \varsigma(\tau) \in S_3$ is defined in Notation 2.6.4. Let $\mathbb{T}$ denote the image of $\mathbb{T}^{\mathcal{P}}$ inside $\mathrm{End}_{\mathcal{O}_E}(H^0(U^v, V')[\mathfrak{m}_{\bar{r}}]^{\mathrm{Iw}=\eta^{-1}}$. Assume that the $\mathcal{O}_E$-dual of $H^0(U^v, V')[\mathfrak{m}_{\bar{r}}]^{\mathrm{Iw}=\eta^{-1}}$ is free over $\mathbb{T}$ for each τ. Then*

1. *$\bar{r}|_{G_{F_w}} \in (\mathrm{FL}_\delta^{**})$, and*

2. *For each τ there is an equality*
$$S'_\tau \Pi^{\epsilon_\tau} = \bar{\kappa}_\tau \mathrm{FL}_\tau S_\tau \tag{3.5.7}$$
of maps $H^0(U^v, V')[\mathfrak{m}_{\bar{r}}]^{\mathrm{Iw}=\eta^{-1}} \to H^0(U^v, V')[\mathfrak{m}_{\bar{r}}]^{\mathrm{Iw}=\eta_0^{-1}}$, where $\kappa_\tau \in \mathbb{F}^\times$,$\epsilon_\tau \in \{\pm 1\}$ are defined in Notation 2.6.4 and $\mathrm{FL}_\tau \in \mathbb{F}^\times$ is defined in Notation 1.7.6. Moreover, these maps are injective with nonzero domain.

Proof. We prove both parts at the same time. Let τ and η be as in the statement of the theorem. In addition to $\varsigma, \kappa_\tau, \mathrm{FL}_\tau$ we make use of $n_\tau, m_\tau \in \mathbb{Z}$ and $i_\tau \in \{0,1,2\}$ defined in Notations 1.7.6 and 2.6.4. Let

$$M := S(U^v, \widetilde{V}')_{\mathfrak{m}_{\bar{r}}}^{\mathrm{Iw}=\widetilde{\eta}^{-1}} \cong S(U^v \cdot \mathrm{Iw}, \widetilde{\eta} \otimes \widetilde{V}')_{\mathfrak{m}_{\bar{r}}}.$$

(the second isomorphism by (3.5.2). Then $M \neq 0$ because $F_0^\vee$ is a Jordan-Hölder factor of $\mathrm{Ind}_{\mathrm{Iw}}^{\mathrm{GL}_3(F_w)}(\widetilde{\eta}) \otimes_{\mathcal{O}_E} \mathbb{F} = \mathrm{Ind}_{B_3(k_w)}^{\mathrm{GL}_3(k_w)}(\eta)$ and F_0 is a modular weight of $\bar{r}|_{G_{F_w}}$ by Assumption (SW). Moreover M is a finite free $\mathcal{O}_E$-module. Using the fixed isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ we obtain from (3.2.3) and (3.2.2) a decomposition

$$M_{\overline{\mathbb{Q}}_p} \cong \bigoplus_\pi m(\pi) \cdot \pi_v^{\mathrm{Iw}=\widetilde{\eta}^{-1}} \otimes (\pi^{\infty,v})^{U^v} \tag{3.5.8}$$

where π runs over the irreducible $\overline{\mathbb{Q}}_p$-representations $\pi_\infty \otimes \pi_v \otimes \pi^v$ of $G(\mathbb{A}_{F^+})$ such that $\pi \otimes_\iota \mathbb{C}$ is a cuspidal automorphic representation of G of multiplicity $m(\pi)$ contributing to S_λ where λ is determined by the highest weight of $\widetilde{V}'$ via the recipe in Section 3.2, and moreover whose associated Galois representation r_π (which is absolutely irreducible) lifts $\bar{r}^\vee$ (by local-global compatibility in Theorem 3.3.1).

For any π contributing to (3.5.8) we must have

- $\pi_v \cong \mathrm{Ind}_{B_3(F_w)}^{\mathrm{GL}_3(F_w)} \left(\psi_{\varsigma(0)} \otimes \psi_{\varsigma(1)} \otimes \psi_{\varsigma(2)}\right)$ where $\psi_i : F_w^\times \to \overline{\mathbb{Q}}_p^\times$ is a smooth character (depending on π) such that $\psi_i|_{\mathcal{O}_{F_w}^\times} = \widetilde{\eta}_i^{-1}$, and

- $r_\pi|_{G_{F_w}}^\vee$ is a potentially crystalline lift of $\bar{r}|_{G_{F_w}}$ having parallel Hodge-Tate weights $(2,1,0)$ and
$$\mathrm{WD}(r_\pi|_{G_{F_w}}^\vee)^{\mathrm{F-ss}} \cong \psi_{\varsigma(0)}^{-1}|\cdot|^2 \oplus \psi_{\varsigma(1)}^{-1}|\cdot| \oplus \psi_{\varsigma(2)}^{-1}.$$

The first point follows from Frobenius reciprocity and the fact that the principal series type τ is the inertial type associated to π_v via the inertial local Langlands correspondence (see [EGH13], Theorem 2.4.1). The second point follows from local global compatibility in Theorem 3.3.1 at places dividing p as well as the special case of the local Langlands correspondence mentioned near the beginning of Section 3.3. From the second point and the description of $\mathrm{WD}(r_\pi|_{G_{F_w}})$ in Section 3.3 we immediately see that the φ^f-eigenvalue on
$$D_{\mathrm{st}}^{F_w,2}(r_\pi|_{G_{F_w}}^\vee)^{I_{F_w}=\widetilde{\eta}_i}$$
is equal to
$$p^{\varsigma^{-1}(i)f}\psi_i(p)^{-1}. \tag{3.5.9}$$

We have $M_E = \bigoplus_{\mathfrak{p}} M_E[\mathfrak{p}_E]$ where $\mathfrak{p}$ runs over the minimal primes of $\mathbb{T}_E$. Extending coefficients, $M_E[\mathfrak{p}_E]$ is a direct summand of (3.5.8). By the Chebotarev density theorem (see Remark 3.3.2), the π_v contributing to $M_E[\mathfrak{p}_E]$ are all *identical*. Then by Proposition 2.6.1 the equation
$$\widetilde{S}'_\tau \Pi^{\epsilon_\tau} = p^{n_\tau}\kappa_\tau \psi_{i_\tau}(p)^{\epsilon_\tau}\widetilde{S}_\tau$$
holds on $M[\mathfrak{p}] \subseteq M_E[\mathfrak{p}_E]$. Hence the equation
$$S'_\tau \Pi^{\epsilon_\tau} = \overline{p^{n_\tau}\kappa_\tau \psi_{i_\tau}(p)^{\epsilon_\tau}} S_\tau \tag{3.5.10}$$
holds on $M[\mathfrak{p}]_{\mathbb{F}} \subseteq M_{\mathbb{F}}[\mathfrak{m}] = H^0(U^v, V')[\mathfrak{m}_{\bar{r}}]^{\mathrm{Iw}=\widetilde{\eta}^{-1}}$.

The assumption of freeness over the Hecke algebra implies that this last inclusion is an equality. Indeed, observe that

- $M[\mathfrak{p}] = \mathrm{Hom}_{\mathcal{O}_E}(M^{\mathrm{d}}/\mathfrak{p}, \mathcal{O}_E)$ since M is $\mathcal{O}_E$-free, hence $M[\mathfrak{p}]_{\mathbb{F}} = \mathrm{Hom}_{\mathcal{O}_E}(M^{\mathrm{d}}/\mathfrak{p}, \mathcal{O}_E)_{\mathbb{F}}$,

- the natural inclusion

$$\mathrm{Hom}_{\mathcal{O}_E}(M^{\mathrm{d}}/\mathfrak{p}, \mathcal{O}_E)_{\mathbb{F}} \hookrightarrow \mathrm{Hom}_{\mathbb{F}}(M^{(\mathrm{d}}/\mathfrak{p})_{\mathbb{F}}, \mathbb{F})$$

  is an isomorphism if $M^{\mathrm{d}}/\mathfrak{p}$ is free over $\mathcal{O}_E$, and

- $M^{\mathrm{d}}/\mathfrak{p}$ is free over $\mathcal{O}_E$ since M^{d} is free over $\mathbb{T}$ and $\mathbb{T}/\mathfrak{p} = \mathcal{O}_E$.

We deduce that (3.5.10) holds on $H^0(U^v, V')[\mathfrak{m}_{\bar{r}}]^{\mathrm{Iw}=\widetilde{\eta}^{-1}}$. By Assumption (SW) and Frobenius reciprocity, the $\mathrm{GL}_3(\mathcal{O}_{F_w})$-representation generated by any vector $v \in H^0(U^v, V')[\mathfrak{m}_{\bar{r}}]^{\mathrm{Iw}=\widetilde{\eta}^{-1}}$ is the unique quotient of $\mathrm{Ind}_{B_3(k_w)}^{\mathrm{GL}_3(k_w)}\left(\bigotimes_{i=0}^{2}\omega_f^{A_{\varsigma(i)}}\right)$ with socle F_0. From Theorem 2.3.2, we know that both $S'_\tau\Pi^{\epsilon_\tau}v$ and $S_\tau v$ are nonzero in the equation

$$S'_\tau\Pi^{\epsilon_\tau}v = \overline{p^{n_\tau}\kappa_\tau\psi_{i_\tau}(p)^{\epsilon_\tau}}S_\tau v.$$

In particular we must have $\mathrm{ord}_p(\psi_{i_\tau}(p)) = -\epsilon_\tau n_\tau$ or equivalently by (3.5.9)

$$\mathrm{ord}_p(\lambda_{i_\tau}) = \varsigma^{-1}(i_\tau)f + \epsilon_\tau n_\tau. \tag{3.5.11}$$

But observe that $m_\tau = \varsigma^{-1}(i_\tau)f + \epsilon_\tau n_\tau$ in all cases. By Corollary 1.7.13, if $\bar{r}|_{G_{F_w}} \notin (\mathrm{FL}_\delta^{**})$ we can therefore choose τ so that (3.5.11) is false. This proves $\bar{r}|_{G_{F_w}} \in (\mathrm{FL}_\delta^{**})$. Now Theorem 1.7.1 implies that (3.5.7) holds. This proves Theorem 3.5.6. $\square$

3.6 Freeness over the Hecke algebra

The assumption of freeness over the Hecke algebra is important in the proof of Theorem 3.5.6. In this section we illustrate that if $\bar{r}$ satisfies certain technical assumptions it is possible to choose U, V' so that Hecke freeness holds under Assumption (SW), using the Taylor-Wiles patching method of [CEG+16]. The method used here follows very closely the techniques used to prove the analogous result in Section 5.3 of [HLM17], so we only give a summary. In fact, our argument is simpler because of (SW): we don't need to consider any extra operators like the U_p-operators of *op. cit.* in order to deal with the possible presence of extra Serre weights.

Let F/F^+ be a CM field such that

(i) F/F^+ is unramified at all finite places and all places of F^+ above p split in F.

We fix a place $w|p$ of F and let $v = w|_{F^+}$. Let $\bar{r} : G_F \to \mathrm{GL}_3(\mathbb{F})$ be a continuous Galois representation such that

(ii) $\bar{r}$ is unramified outside p

(iii) $\bar{r}$ is Fontaine-Laffaille with regular weights at all places dividing p

(iv) $\bar{r}$ has image containing $\mathrm{GL}_3(k)$ for some $k \subseteq \mathbb{F}$ with $|k| > 9$

(v) $\overline{F}^{\ker(\mathrm{ad}(\bar{r}))}$ does not contain $F(\zeta_p)$.

As in [HLM17] the assumption (v) allows us to choose a finite place v_1 of F^+ prime to p satisfying

- v_1 splits in F as $v_1 = w_1 w_1^c$;
- v_1 does not split completely in $F(\zeta_p)$;
- $\bar{r}(\mathrm{Frob}_{w_1})$ has distinct $\mathbb{F}$-rational eigenvalues, no two of which have ratio $(\mathbb{N}v_1)^{\pm 1}$.

We choose a unitary group G/F^+ and a model $\mathcal{G}/\mathcal{O}_{F^+}$ as in Section 3.1. Let $U^v = \prod_{v' \neq v} U_{v'} \leq G(\mathbb{A}_{F^+}^{\infty,v})$ be a compact open subgroup such that

(vi) $U_{v'} = \mathcal{G}(\mathcal{O}_{F^+_{v'}})$ for all places v' of F^+ that split in F, except for v_1 and the places dividing p;

(vii) U_{v_1} is the preimage of the upper-triangular matrices under the isomorphism

$$\mathcal{G}(\mathcal{O}_{F^+_{v_1}}) \to \mathcal{G}(k_{v_1}) \cong \mathrm{GL}_3(k_{w_1})$$

induced by ι_{w_1};

(viii) $U_{v'}$ is a hyperspecial maximal compact subgroup of $G(F^+_{v'})$ if v' is inert in F.

The choice of U_{v_1} implies that $U^v U_v$ is sufficiently small in the sense of Section 3.2 for any compact open $U_v \leq G(F_v^+)$.

Assume there exists a choice of λ defining V' and $\widetilde{V}'$ as in Section 3.4, so in particular $\widetilde{V}' \otimes_{\mathcal{O}_E} \mathbb{F} \cong V'$. For $w \in S_p$ let ψ_w denote $-w_0\lambda_w$. Let $\mathcal{P}$ denote the set of finite places w' of F lying over a place of F^+ that splits in F, and which do not divide p or v_1, and define $\mathfrak{m}_{\bar{r}} \leq \mathbb{T}^{\mathcal{P}}$ as in Definition 3.4.2. We make the crucial automorphy assumption that

(ix) $S(U^v, V')_{\mathfrak{m}_{\bar{r}}} \neq 0$.

By the choice of U^v and v_1 we remark that this implies (iii), and by the choice of V' this also implies (iv). We finally assume that

(x) $\bar{r}$ obeys Assumption (SW).

As in the proof of Theorem 3.5.6, this implies that $S(U^v, V')_{\mathfrak{m}_{\bar{r}}}^{\mathrm{Iw}=\tilde{\eta}^{-1}} \neq 0$ for any of the characters $\widetilde{\eta}$ used in the statement of that theorem.

Let $\widetilde{\eta}$ be any of the characters defined in Theorem 3.5.6. Let $\mathbb{T}$ denote the $\mathcal{O}_E$ subalgebra generated by $\mathbb{T}^{\mathcal{P}}$ inside $\mathrm{End}_{\mathcal{O}_E}(H^0(U^v, \widetilde{V}')_{\mathfrak{m}_{\bar{r}}}^{\mathrm{Iw}=\widetilde{\eta}^{-1}}$.

Theorem 3.6.1. *Let $\bar{r} : G_F \to \mathrm{GL}_3(\mathbb{F})$ be a continuous Galois representation and $U^v \leq G(\mathbb{A}_{F^+}^{\infty,v})$, $\widetilde{V}'$ as above satisfy (i)-(x). Then*

$$\mathrm{Hom}_{\mathcal{O}_E}(H^0(U^v, \widetilde{V}')_{\mathfrak{m}_{\bar{r}}}^{\mathrm{Iw}=\widetilde{\eta}^{-1}}, \mathcal{O}_E)$$

is free over $\mathbb{T}$.

Proof. The proof uses a modification due to [HLM17] of the Taylor-Wiles patching method developed in [CEG$^+$16] in order to construct a patching functor M_∞, along with the Diamond-Fujiwara trick to deduce freeness over the Hecke algebra. Since the argument we use is essentially the same as the one in [HLM17] (but simpler), we give a summary of the method and refer the reader to [HLM17] for more details.

By Lemma 5.1.2 of [HLM17], we may assume that E is large enough so that $\mathbb{T}[1/p]$ is reduced.

Let S be the set of places of F^+ dividing p together with v_1. For each $v' \in S$ fix a choice of place $\tilde{v}'$ of F lying over it, and take $\tilde{v} = w$. Write $\tilde{S}$ for the set of these places. Let δ_{F/F^+} denote the quadratic character of F/F^+. We consider the deformation problem

$$\mathcal{S} = (F/F^+, S, \tilde{S}, \mathcal{O}_E, \bar{r}^\vee, \epsilon^{-2}\delta_{F/F^+}, \{R^\square_{\tilde{v}_1}\} \cup \{R^{\square,\psi_{\tilde{v}}}_{\tilde{v}'}\}_{v'|p, v' \neq v} \cup \{R^\square_{\tilde{v}}\})$$

in the terminology of [CHT08]. Here for a place w' of F we write $R^\square_{w'}$ for the universal $\mathcal{O}_E$-lifting ring of $\bar{r}^\vee|_{G_{F_w}}$, and $R^{\square,\psi_{w'}}_{w'}$ for the universal crystalline lifting ring of Hodge type $\psi_{w'} + (2,1,0)$ if w' lies over p.

Set

$$R^{\mathrm{loc}} = R^\square_{\tilde{v}_1} \hat{\otimes} R^\square_{\tilde{v}} \hat{\otimes} \hat{\bigotimes}_{v'|p, v' \neq v} R^{\square,\psi_{\tilde{v}}}_{\tilde{v}'}.$$

Let $q \geq 3[F^+ : \mathbb{Q}]$ be an integer, and set

$$R_\infty = R^{\mathrm{loc}}[[x_1, \ldots, x_{q-3[F^+:\mathbb{Q}]}]].$$

If $\widetilde{\eta}$ denotes any of the characters used in the proof of Theorem 3.5.6 let τ° denote $\mathrm{Ind}_{\mathrm{Iw}}^{\mathrm{GL}_3(\mathcal{O}_{F_w})}(\widetilde{\eta})$ denote the corresponding choice of $\mathcal{O}_E$-lattice inside the principal series type over E. The construction of [CEG$^+$16], slightly modified, provides us with a local map

$$S_\infty := \mathcal{O}_E[[z_1, \ldots, z_{9\#S}, y_1, \ldots, y_q]] \to R_\infty$$

and an $R_\infty[[\mathrm{GL}_3(\mathcal{O}_{F_w})]]$-module M_∞ having a compatible $\mathrm{GL}_3(F_w)$-action with the properties (i)-(iv) listed in Theorem 5.2.1 of [HLM17], except that in (iii) and (iv) we have a surjection $R_\infty \twoheadrightarrow \mathbb{T}$ with kernel containing $\mathfrak{a}R_\infty$ and equivariant identifications

$$(M_\infty(\tau^\circ)/\mathfrak{a})^{\mathrm{d}} \cong H^0(U^v, \tilde{V}')^{\mathrm{Iw}=\tilde{\eta}^{-1}}_{\mathfrak{m}_{\bar{r}}}$$

and

$$\begin{aligned}(M_\infty(\tau^\circ)/\mathfrak{p})^{\mathrm{d}} &\cong H^0(U^v, \widetilde{V}')[\mathfrak{p}]^{\mathrm{Iw}=\widetilde{\eta}^{-1}} \\ (M_\infty(\tau^\circ)/\mathfrak{m}_{\bar{r}})^\vee &\cong H^0(U^v, V')[\mathfrak{m}_{\bar{r}}]^{\mathrm{Iw}=\eta^{-1}}\end{aligned}$$

respectively where $\mathfrak{p}$ is any $\mathcal{O}_E$-point of Spec($\mathbb{T}$) (the point being that this holds for each η of interest). The only difference to the proof of Theorem 5.2.1 of [HLM17] in our case is that we are allowing places above p to be unramified, and this has no effect on the constructions of [CEG$^+$16]. For the remainder of the proof we freely use the notation of [HLM17] and [CEG$^+$16].

Note that $R(\tau^\circ)$ by definition is the image of R_∞ inside $\mathrm{End}_{\mathcal{O}_E}(M_\infty(\tau^\circ)$. By (iii) the action of $R_\infty/\mathfrak{a}$ on $M_\infty(\tau^\circ)/\mathfrak{a}$ factors through $\mathbb{T}$. Hence the action of $R_\infty(\tau^\circ)/\mathfrak{a}$ on $M_\infty(\tau^\circ)$ factors through $\mathbb{T}$, which gives rise to a surjection $R_\infty(\tau^\circ)/\mathfrak{a} \twoheadrightarrow \mathbb{T}$. As $R_\infty(\tau^\circ)$ acts faithfully on $M_\infty(\tau^\circ)$, the $R_\infty(\tau^\circ)/\mathfrak{a}$-module $M_\infty(\tau^\circ)/\mathfrak{a}$ has full support by Lemma 2.2(1) of [Tay08]. Since $\mathbb{T}$ is reduced, we get an induced isomorphism $(R_\infty(\tau^\circ))_{\mathrm{red}} \xrightarrow{\sim} \mathbb{T}$.

It suffices to prove that $M_\infty(\tau^\circ)$ is free over $R_\infty(\tau^\circ)$, for then $M_\infty(\tau^\circ)/\mathfrak{a}$ is free over $R_\infty(\tau^\circ)/\mathfrak{a}$ and the map $R_\infty(\tau^\circ)/\mathfrak{a} \twoheadrightarrow \mathbb{T}$ must be an isomorphism.

In order to show this we now follow the proof of Theorem 5.2.3 in [HLM17]. Let $n = \dim_{\mathbb{F}}(M_\infty(F_0)/\mathfrak{m}_{\bar{r}})$. We have $M_\infty(\tau^\circ) \twoheadrightarrow M_\infty(\overline{\tau}^\circ) \cong M_\infty(F_0)$, where the isomorphism is by Assumption (SW) and Proposition 5.1.1 of [Le]. Hence by Nakayama's lemma, $M_\infty(\tau^\circ)$ is generated over $R_\infty(\tau^\circ)$ by n elements. We may therefore fix a surjection $R_\infty(\tau^\circ)^n \twoheadrightarrow M_\infty(\tau^\circ)$ with kernel P, and get an exact sequence

$$0 \to P[1/p] \to R_\infty(\tau^\circ)[1/p]^n \to M_\infty(\tau^\circ)[1/p] \to 0.$$

As $M_\infty(\tau^\circ)[1/p]$ is projective over $R_\infty(\tau^\circ)[1/p]$ by (ii), it now suffices to show that it is projective of constant rank n, since then $P[1/p]$ is projective of rank 0, hence equal to 0 and this implies $P = 0$ because $R_\infty(\tau^\circ)$ is p-torsion free.

But the argument in the third paragraph of the proof of Theorem 5.2.3 in [HLM17] shows that each irreducible component of Spec$(R_\infty(\tau^\circ)[1/p]$ has an E-point in the closed subscheme Spec($\mathbb{T}$). Therefore it suffices to observe that for any $\mathcal{O}_E$-point of Spec($\mathbb{T}$) with corresponding prime $\mathfrak{p}$, $M_\infty(\tau^\circ)/\mathfrak{p}$ is $\mathcal{O}_E$-torsion free by (iv) and has rank n modulo ϖ_E by definition, hence has generic rank $= n$. This proves the claim.

□

Remark 3.6.2. For completeness, we should also make some remarks to the effect that the assumptions (i)-(x) in Theorem 3.6.1 are not overly restrictive in the sense that there exists Galois representations $\bar{r}$ to which Theorem 3.6.1 and hence Theorem 3.5.6 apply. If one had conditions on the local representation $\bar{r}|_{G_{F_w}}$ guaranteeing Assumption (SW) then it would not be hard to deduce a result of this form from the globalization argument of [HLM17] Theorem 5.3.7. But since this local condition won't be known until [LLHL17] is available, we have not pursued this.

Bibliography

[AHHV17] N. Abe, G. Henniart, F. Herzig, and M.-F. Vignéras, *A classification of irreducible admissible* mod p *representations of p-adic reductive groups*, J. Amer. Math. Soc. **30** (2017), no. 2, 495–559. MR 3600042

[Alp80] J. L. Alperin, *Diagrams for modules*, J. Pure Appl. Algebra **16** (1980), no. 2, 111–119. MR 556154

[BC] Olivier Brinon and Brian Conrad, *Cmi summer school notes on p-adic hodge theory*, unpublished notes.

[BD14] Christophe Breuil and Fred Diamond, *Formes modulaires de Hilbert modulo p et valeurs d'extensions entre caractères galoisiens*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 5, 905–974. MR 3294620

[BDM15] C. Bowman, S. R. Doty, and S. Martin, *Decomposition of tensor products of modular irreducible representations for* SL_3*: the* $p \geq 5$ *case*, Int. Electron. J. Algebra **17** (2015), 105–138. MR 3310689

[BG14] Kevin Buzzard and Toby Gee, *The conjectural connections between automorphic representations and Galois representations*, Automorphic forms and Galois representations. Vol. 1, London Math. Soc. Lecture Note Ser., vol. 414, Cambridge Univ. Press, Cambridge, 2014, pp. 135–187. MR 3444225

[BH15] Christophe Breuil and Florian Herzig, *Ordinary representations of* $G(\mathbb{Q}_p)$ *and fundamental algebraic representations*, Duke Math. J. **164** (2015), no. 7, 1271–1352. MR 3347316

[BLGGT14] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor, *Potential automorphy and change of weight*, Ann. of Math. (2) **179** (2014), no. 2, 501–609. MR 3152941

[BP12] Christophe Breuil and Vytautas Paˇskūnas, *Towards a modulo p Langlands correspondence for* GL_2, Mem. Amer. Math. Soc. **216** (2012), no. 1016, vi+114. MR 2931521

[Bre98] Christophe Breuil, *Construction de représentations p-adiques semi-stables*, Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 3, 281–327. MR 1621389

[Bre99] ______, *Représentations semi-stables et modules fortement divisibles*, Invent. Math. **136** (1999), no. 1, 89–122. MR 1681105

[Bre10] ______, *The emerging p-adic Langlands programme*, Proceedings of the International Congress of Mathematicians. Volume II, Hindustan Book Agency, New Delhi, 2010, pp. 203–230. MR 2827792

[Bre12] ______, *Correspondance de Langlands p-adique, compatibilité local-global et applications [d'après Colmez, Emerton, Kisin, ...]*, Astérisque (2012), no. 348, Exp. No. 1031, viii, 119–147, Séminaire Bourbaki: Vol. 2010/2011. Exposés 1027–1042. MR 3050714

[Car08] Xavier Caruso, *Conjecture de l'inertie modérée de Serre*, Invent. Math. **171** (2008), no. 3, 629–699. MR 2372809

[Car14] Ana Caraiani, *Monodromy and local-global compatibility for $l = p$*, Algebra Number Theory **8** (2014), no. 7, 1597–1646. MR 3272276

[CDM14] Xavier Caruso, Agnès David, and Ariane Mézard, *Un calcul d'anneaux de déformations potentiellement barsotti-tate*, preprint, 2014.

[CDPs14] Pierre Colmez, Gabriel Dospinescu, and Vytautas Paˇ skūnas, *The p-adic local Langlands correspondence for* $\mathrm{GL}_2(\mathbb{Q}_p)$, Camb. J. Math. **2** (2014), no. 1, 1–47. MR 3272011

[CDT99] Brian Conrad, Fred Diamond, and Richard Taylor, *Modularity of certain potentially Barsotti-Tate Galois representations*, J. Amer. Math. Soc. **12** (1999), no. 2, 521–567. MR 1639612

[CE12] Frank Calegari and Matthew Emerton, *Completed cohomology—a survey*, Nonabelian fundamental groups and Iwasawa theory, London Math. Soc. Lecture Note Ser., vol. 393, Cambridge Univ. Press, Cambridge, 2012, pp. 239–257. MR 2905536

[CEG+16] Ana Caraiani, Matthew Emerton, Toby Gee, David Geraghty, Vytautas Paˇ skūnas, and Sug Woo Shin, *Patching and the p-adic local Langlands correspondence*, Camb. J. Math. **4** (2016), no. 2, 197–287. MR 3529394

[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Publ. Math. Inst. Hautes Études Sci. (2008), no. 108, 1–181, With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras. MR 2470687

[CL09] Xavier Caruso and Tong Liu, *Quasi-semi-stable representations*, Bull. Soc. Math. France **137** (2009), no. 2, 185–223. MR 2543474

[CL18] Ana Caraiani and Brandon Levin, *Kisin modules with descent data and parahoric local models*, Ann. Sci. Éc. Norm. Supér. (4) **51** (2018), no. 1, 181–213. MR 3764041

[EGH13] Matthew Emerton, Toby Gee, and Florian Herzig, *Weight cycling and Serre-type conjectures for unitary groups*, Duke Math. J. **162** (2013), no. 9, 1649–1722. MR 3079258

[Eme06] Matthew Emerton, *A local-global compatibility conjecture in the p-adic Langlands programme for* $\mathrm{GL}_{2/\mathbb{Q}}$, Pure Appl. Math. Q. **2** (2006), no. 2, Special Issue: In honor of John H. Coates. Part 2, 279–393. MR 2251474

[Eme11] ———, *Local-global compatibility in the p-adic langlands programme for* $\mathrm{GL}_2(\mathbb{Q}_p)$, preprint, 2011.

[Eme14] ———, *Completed cohomology and the p-adic Langlands program*, Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II, Kyung Moon Sa, Seoul, 2014, pp. 319–342. MR 3728617

[Enn17] John Enns, *On weight elimination for* $\mathrm{GL}_n(\mathbb{Q}_{p^f})$, preprint, 2017.

[FL82] Jean-Marc Fontaine and Guy Laffaille, *Construction de représentations p-adiques*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 4, 547–608 (1983). MR 707328

[Fon90] Jean-Marc Fontaine, *Représentations p-adiques des corps locaux. I*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 249–309. MR 1106901

[Gee] Toby Gee, *Modularity lifting theorems*, notes for a course at the 2013 Arizona Winter School.

[Ger10] David James Geraghty, *Modularity lifting theorems for ordinary Galois representations*, ProQuest LLC, Ann Arbor, MI, 2010, Thesis (Ph.D.)–Harvard University. MR 2941425

[GG12] Toby Gee and David Geraghty, *Companion forms for unitary and symplectic groups*, Duke Math. J. **161** (2012), no. 2, 247–303. MR 2876931

[GHS] Toby Gee, Florian Herzig, and David Savitt, *General serre weight conjectures*, to appear in J. Eur. Math. Soc.

[GL14] Hui Gao and Tong Liu, *A note on potential diagonalizability of crystalline representations*, Math. Ann. **360** (2014), no. 1-2, 481–487. MR 3263170

[GM] Meinolf Geck and Gunter Malle, *Reductive groups and steinberg maps.*

[Gro99] Benedict H. Gross, *Algebraic modular forms*, Israel J. Math. **113** (1999), 61–93. MR 1729443

[Hat18] Shin Hattori, *Integral p-adic hodge theory and ramification of crystalline representations*, preprint, 2018.

[Her09] Florian Herzig, *The weight in a Serre-type conjecture for tame n-dimensional Galois representations*, Duke Math. J. **149** (2009), no. 1, 37–116. MR 2541127

[Her11] ———, *The classification of irreducible admissible mod p representations of a p-adic* GL_n, Invent. Math. **186** (2011), no. 2, 373–434. MR 2845621

[HLM17] Florian Herzig, Daniel Le, and Stefano Morra, *On* mod *p local-global compatibility for* GL_3 *in the ordinary case*, Compos. Math. **153** (2017), no. 11, 2215–2286. MR 3705291

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich. MR 1876802

[Hum72] James E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York-Berlin, 1972, Graduate Texts in Mathematics, Vol. 9. MR 0323842

[Hum06] ———, *Modular representations of finite groups of Lie type*, London Mathematical Society Lecture Note Series, vol. 326, Cambridge University Press, Cambridge, 2006. MR 2199819

[Irv86] Ronald S. Irving, *The structure of certain highest weight modules for* SL_3, J. Algebra **99** (1986), no. 2, 438–457. MR 837554

[Jan77a] Jens C. Jantzen, *Darstellungen halbeinfacher Gruppen und kontravariante Formen*, J. Reine Angew. Math. **290** (1977), 117–141. MR 0432775

[Jan77b] ———, *über das Dekompositionsverhalten gewisser modularer Darstellungen halbeinfacher Gruppen und ihrer Lie-Algebren*, J. Algebra **49** (1977), no. 2, 441–469. MR 0486093

[Jan81] Jens Carsten Jantzen, *Zur Reduktion modulo p der Charaktere von Deligne und Lusztig*, J. Algebra **70** (1981), no. 2, 452–474. MR 623819

[Jan03] ———, *Representations of algebraic groups*, second ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003. MR 2015057

[Kis06] Mark Kisin, *Crystalline representations and F-crystals*, Algebraic geometry and number theory, Progr. Math., vol. 253, Birkhäuser Boston, Boston, MA, 2006, pp. 459–496. MR 2263197

[Kis09] ———, *Moduli of finite flat group schemes, and modularity*, Ann. of Math. (2) **170** (2009), no. 3, 1085–1180. MR 2600871

[Lab11] J.-P. Labesse, *Changement de base CM et séries discrètes*, On the stabilization of the trace formula, Stab. Trace Formula Shimura Var. Arith. Appl., vol. 1, Int. Press, Somerville, MA, 2011, pp. 429–470. MR 2856380

[Le] Daniel Le, *Lattices in the cohomology of $U(3)$ arithmetic manifolds*, to appear in Math. Annalen.

[LLHL17] Daniel Le, Bao V. Le Hung, and Brandon Levin, *Weight elimination in serre-type conjectures*, 2017.

[LLHLM] Daniel Le, Bao V. Le Hung, Brandon Levin, and Stefano Morra, *Serre weights for wildly ramified three-dimensional representations*, in preparation.

[LLHLM18a] ———, *Potentially crystalline deformation rings and Serre weight conjectures: shapes and shadows*, Invent. Math. **212** (2018), no. 1, 1–107. MR 3773788

[LLHLM18b] ———, *Serre weights and breuil's lattice conjecture in dimension three*, preprint, 2018.

[LMP16] Daniel Le, Stefano Morra, and Chol Park, *On* mod p *local-global compatibility for* $\mathrm{GL}_3(\mathbb{Q}_p)$ *in the non-ordinary case*, preprint, 2016.

[Pil93] Cornelius Pillen, *Reduction modulo p of some Deligne-Lusztig characters*, Arch. Math. (Basel) **61** (1993), no. 5, 421–433. MR 1241047

[Pil97] ———, *Loewy series for principal series representations of finite Chevalley groups*, J. Algebra **189** (1997), no. 1, 101–124. MR 1432367

[PQ17] Chol Park and Zicheng Qian, *On* mod p *local-global compatibility for* $\mathrm{GL}_n(\mathbb{Q}_p)$ *in the ordinary case*, preprint, 2017.

[Tay08] Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations. II*, Publ. Math. Inst. Hautes Études Sci. (2008), no. 108, 183–239. MR 2470688

[Xi96] Nanhua Xi, *Irreducible modules of quantized enveloping algebras at roots of* 1, Publ. Res. Inst. Math. Sci. **32** (1996), no. 2, 235–276. MR 1382803