Local Commutative Algebra and Hochschild Cohomology Through the Lens of Koszul Duality

by

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Abstract

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This thesis splits into two halves, the connecting theme being Koszul duality. The first part concerns local commutative algebra. Koszul duality here manifests in the homotopy Lie algebra. In the second part, which is joint work with Vincent Gélinas, we study Hochschild cohomology and its characteristic action on the derived category.

We begin by defining the homotopy Lie algebra $\pi^*(\phi)$ of a local homomorphism ϕ (or of a ring) in terms of minimal models, slightly generalising a classical theorem of Avramov. Then, starting with work of Félix and Halperin, we introduce a notion of Lusternik-Schnirelmann category for local homomorphisms (and rings). In fact, to ϕ we associate a sequence $\mathrm{cat}_0(\phi) \geq \mathrm{cat}_1(\phi) \geq \mathrm{cat}_2(\phi) \geq \cdots$ each $\mathrm{cat}_i(\phi)$ being either a natural number or infinity. We prove that these numbers characterise weakly regular, complete intersection, and (generalised) Golod homomorphisms. We present examples which demonstrate how they can uncover interesting information about a homomorphism. We give methods for computing these numbers, and in particular prove a positive characteristic version of Félix and Halperin's Mapping Theorem.

A motivating interest in L.S. category is that finiteness of $\mathrm{cat}_2(\phi)$ implies the existence of certain six-term exact sequences of homotopy Lie algebras, following classical work of Avramov. We introduce a variation $\widetilde{\pi}^*(\phi)$ of the homotopy Lie algebra which enjoys long exact sequences in all situations, and construct a comparison $\widetilde{\pi}^*(\phi) \to \pi^*(\phi)$ which is often an isomorphism. This has various consequences; for instance, we use it to characterise quasi-complete intersection homomorphisms entirely in terms of the homotopy Lie algebra.

In the second part of this thesis we introduce a notion of A_∞ centre for minimal A_∞ algebras. If A is an augmented algebra over a field k we show that the image of the natural homomorphism $\chi_k : \mathrm{HH}^*(A,A) \to \mathrm{Ext}^*_A(k,k)$ is exactly the A_∞ centre of A, generalising a theorem of Buchweitz, Green, Snashall and Solberg from the case of a Koszul algebra. This is deduced as a consequence of a much wider enrichment of the entire characteristic action $\chi : \mathrm{HH}^*(A,A) \to \mathsf{Z}(D(A))$. We give a number of representation theoretic applications.

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Contents

Chapter 1

Introduction

This thesis splits roughly into two halves, the connecting theme being Koszul duality. The first part concerns local commutative algebra, which is broadly the study of singularities. Koszul duality here manifests in the homotopy Lie algebra, whose use in commutative algebra has been pioneered by Avramov.

The main character in the second part of this thesis is Hochschild cohomology, and its characteristic action on the derived category. Structures known as A_∞ algebras arise naturally through Koszul duality, and we investigate how these can be used to understand and constrain the characteristic action. This part of the thesis is joint work with Vincent Gélinas, and our contribution here should be considered equal. It is adapted from [34].

Koszul duality is a wide reaching and mysterious phenomenon. In topology it connects a group with its classifying space. In rational homotopy theory it explains the relationship between Sullivan's commutative model and Quillen's Lie algebra model for a space. In algebra it yields unexpected and useful derived equivalences. Rich examples and other Koszul duality like phenomena abound throughout algebra and geometry.

In its simplest algebraic incarnation Koszul duality concerns the relationship between an associative algebra A augmented over a field k, and the Yoneda algebra $E = \mathrm{Ext}^*_A(k,k)$. When A is a Koszul algebra (see section 2.3) this relationship is particularly striking. Perhaps the most beautiful aspect is the symmetry: this hypothesis alone means that E is also Koszul, and that there is a canonical isomorphism $A \cong \mathrm{Ext}^*_E(k,k)$. We will also see that the the representation theory of A is intimately connected to that of E. We say that A and E are *Koszul dual* to each other. Aside from the rich structure arising from this assumption, it turns out that Koszul algebras are extremely common.

A fundamental example is the Koszul duality between the polynomial algebra $S = k[x_0, ..., x_n]$ and the exterior algebra $\Lambda = \bigwedge_k(y_0, ..., y_n)$. It is this example which gives rise to the famous Bernstein-Gel'fand-Gel'fand correspondence [30], which describes an equivalence between bounded complexes of coherent sheaves on projective space $\mathbb{P}^n_k$ and finitely generated Λ modules without free summands.

On the other hand, local commutative algebra concerns singularities. It is by now very well understood that one can use homological methods to unravel the character of a singularity. A fundamental illustration is that the violence of a singularity is reflected in the growth of the Betti numbers

$\beta^i_R(k) = \dim_k \mathrm{Ext}^i_R(k,k)$, where R is the local ring of functions on the singularity and k is its residue field.

These ideas go all the way back to Hilbert's syzygy theorem, which says that if M is a finitely generated module over a polynomial algebra $S = k[x_1, ..., x_n]$ on some field k, then the Betti numbers $\beta^i_S(M) = \dim_k \mathrm{Ext}^i_S(M,k)$ vanish as soon as $i > n$. In modern language, we say that S has global dimension n. In the mid 1950s Auslander, Buchbaum and Serre established a converse: a local ring R has finite global dimension if and only if it is *regular*, that is, if it is the ring of functions at some smooth point. By these means a number of long standing problems in commutative algebra fell under the force of homological algebra.

The importance of the Betti numbers $\beta^i_R(k) = \dim_k \mathrm{Ext}^i_R(k,k)$ suggests that Koszul duality is already at play in local commutative algebra. In fact, the structure of $\mathrm{Ext}^*_R(k,k)$ has been an important focal point in this area for decades. One highlight is the cohomology operators of Gulliksen and Eisenbud. If R is a complete intersection of codimension c, these operators $\chi_1, ..., \chi_c$ are central elements in $\mathrm{Ext}^2_R(k,k)$ which act on the cohomology of any R module. They have turned out to be extraordinarily useful. In short, they control the stable behaviour of modules over R. A good place to start is [15], where the geometric nature of the cohomology operators is exploited.

Miraculously, from the hypothesis that R is a commutative it follows that the Yoneda algebra $\mathrm{Ext}^*_R(k,k)$ is the universal enveloping algebra of a certain graded Lie algebra. This fact is the culmination of work of Milnor-Moore, Löfwall, Sjödin, Levin and Schoeller (see section 3.1). This object bears a close and mysterious resemblance to the Whitehead Lie algebra $\pi_*(\Omega X)$ associated to a simply connected based space X. As such it is known as the homotopy Lie algebra of R, and we denote it $\pi^*(R)$. It will be a central character in this thesis.

In the early 1980s Avramov and Roos began to uncover deep connections between local commutative algebra and rational homotopy theory, centring around the homotopy Lie algebra. This has led to a great deal of collaboration and progress in both areas, some of which is covered in the surveys [9] and [6]. Much of what is now known about $\pi^*(R)$ was inspired by the work of rational homotopy theorists, although it is often much more difficult to establish a theorem in commutative algebra than its cousin in rational homotopy theory (or vice versa). For example, the famous dichotomy theorem says that either R is a complete intersection, in which case $\pi^{\geq 3}(R) = 0$, or the sequence of numbers $\epsilon_i(R) = \dim_k \pi^i(R)$ must grow exponentially. This was proven by Avramov in [13] almost two decades after the dichotomy theorem for finite CW complexes was established by Félix, Halperin and Thomas in [45].

But the homotopy Lie algebra has been important regardless of its connection to topology. For instance, understanding the numbers $\epsilon_i(R)$ can lead to fine control of the Poincaré series of modules over R. As another example, when R is a complete intersection, the cohomology operators themselves come from $\pi^2(R)$. Likely the most famous application of the homotopy Lie algebra is Avramov's use of it in [13] to solve a long standing conjecture of Quillen. In his groundbreaking development of the simplicially defined cotangent (or André-Quillen) homology, Quillen posed the following question: if $\phi : R \to S$ is a ring homomorphism which is essentially of finite type such that S has finite flat dimension as an R module, and such that the *cotangent complex* $\mathcal{L}_\phi$ has finite projective dimension, then must ϕ be a locally-complete intersection? The homotopy Lie algebra is usually much easier to compute than cotangent homology. Avramov was able to pass some of this information through a spectral sequence from the dg to the simplicial world. Via this channel the dichotomy theorem for the homotopy Lie algebra was used to settle Quillen's quastion.

Long before Quillen's foundational work in rational homotopy theory, the connection between commutative algebras and Lie algebras was exploited by Chevalley and Eilenberg. The famous Chevalley-Eilenberg cochain complex $C^*(L;\mathbb{Q})$, which computes the Lie algebra cohomology of L, is an example of a *minimal model* when L is nilpotent. We will say what this means shortly (in the dual, homological context), and much more detail will be given in section 2.6.

After the work of Quillen, Sullivan's use of minimal models in rational homotopy theory was an important break though, both theoretically and computationally. It was once again Avramov who imported minimal models into local commutative algebra. They will be the fundamental objects of study in chapter 3. I'll explain now how they can be used to compute the homotopy Lie algebra, and then go over some highlights from chapter 3.

Let X be a graded set of variables in strictly positive degrees. We denote by $Q[X]$ the free strictly graded commutative Q algebra on X. This is the exterior algebra over Q on odd elements of X tensored with the polynomial algebra on even elements of X. A dg algebra over Q will be called *semi-free* if it has the form $Q[X]$ after forgetting its differential. These are non-linear analogues of free resolutions for modules. In particular, they can be used to resolve local rings.

For us, a *minimal model* is a semi-free dg algebra $A = Q[X]$ over a regular local ring Q which satisfies the minimality condition $\partial(\mathfrak{m}_A) \subseteq \mathfrak{m}_A^2$, where $\mathfrak{m}_A$ is the unique maximal, graded ideal of A. Intuitively, these dg algebras are as close to being free as possible, because regular local rings behave homologically as if they were free.

If R is a local ring then by the Cohen structure theorem there is a regular local ring Q with the same embedding dimension of R, and a surjective homomorphism $Q \to \widehat{R}$ onto the completion of R. It is possible to inductively construct a minimal model $A = Q[X]$ and a surjective quasi-isomorphism $A \xrightarrow{\simeq} \widehat{R}$ of Q algebras.

Note that the indecomposables $\mathfrak{m}_A/\mathfrak{m}_A^2$ of A naturally form graded vector space over $k = A/\mathfrak{m}_A$.

Theorem 1 (theorem 15)**.** *Suppose that $A = Q[X]$ is a minimal model resolving $\widehat{R}$ as above. The shifted and dualised space of indecompables $\pi^*(A) = (\mathfrak{s}\mathfrak{m}_A/\mathfrak{m}_A^2)^\vee$ canonically has the structure of a graded Lie algebra over k, and there is a natural isomorphism $U\pi^*(A) \cong \mathrm{Ext}_R^*(k,k)$.*

This extends a classical theorem of Avramov [9, theorem 4.2] which deals with $\pi^{\geq 2}(A)$. It follows that $\pi^*(A)$ is independent of the construction of A and thus we denote it $\pi^*(R)$. This is the homotopy Lie algebra of R.

Using Avramov, Foxby and Herzog's theory of Cohen factorisations one can also associate a minimal model A to a local homomorphism $\phi : R \to S$. In this way one obtains an analogous graded Lie algebra $\pi^*(A) = \pi^*(\phi)$, which we call the homotopy Lie algebra of ϕ.

In rational homotopy it is necessary to impose conditions on a space to obtain a well-behaved homotopy Lie algebra. Indeed, Quillen [105] has shown that any finite type graded Lie algebra over $\mathbb{Q}$ occurs as $\pi_*(\Omega X)$ for some rational space X, which means no structure theory is possible in general. After the work of Félix and Halperin (and a number of other authors, see the beginning of section 3.2) we know that $\pi_*(\Omega X)$ enjoys some surprising rigidity properties when X has finite *Lusternik-Schnirelmann category*, which simply means that X admits a finite cover by open sets, each of which is contractible in X. These authors also characterised the L.S. category of a rational space algebraically in terms of minimal models [43].

Starting with the work of Félix and Halperin we discuss a notion of L.S. category for local rings and local homomorphisms. In fact, to a local ring R and a local homomorphism ϕ we associate sequences

$$\mathrm{cat}_0(R),\ \mathrm{cat}_1(R),\ \mathrm{cat}_2(R),\ \cdots \quad \text{and} \quad \mathrm{cat}_0(\phi),\ \mathrm{cat}_1(\phi),\ \mathrm{cat}_2(\phi),\ \cdots$$

Each cat_i is either a natural number or infinity, and we'll see that the sequences are in fact weakly decreasing (this is a consequence of the Mapping Theorem). After establishing well-definedness we prove that the vanishing of these numbers characterises regularity and the complete intersection property. We also show that these invariants characterise the Golod property. We also give several examples which demonstrate how the behaviour of the L.S. category can uncover interesting information about a local ring of homomorphism. This part of the thesis especially is just the beginning of what will hopefully become a more complete development of L.S. category in local commutative algebra.

Our main interest in L.S. category is that its finiteness implies that the homotopy Lie algebra $\pi^*(\phi)$ has good structural properties, after the important work of Félix, Halperin, Jacobsson, Löfwall and Thomas [44] (see theorem 18). Using classical work of Avramov one can also build certain six term exact sequences of homotopy Lie algebras under the hypothesis that $\mathrm{cat}_2(\phi)$ is finite. This is the subject of section 3.5.

In [43] Félix and Halperin prove the *Mapping Theorem*, an important tool for calculating L.S. category in rational homotopy theory. The corresponding fact in local commutative algebra has been missing for some time. It turns out their result generalises *verbatim* to the situation of positive residual characteristic (even over a regular local base ring), we give a proof in section 3.3.

Theorem (The Mapping Theorem 20)**.** *If $A \to B$ is a surjective homomorphism of minimal models then* $\mathrm{cat}\, A \geq \mathrm{cat}\, B$.

The proof of the classical mapping theorem uses good properties of the category of dg commutative algebras over $\mathbb{Q}$ which are not available in our context. Thus, the proof we give is necessarily more delicate.

This version of the Mapping theorem generalises theorem 1.2 of [19], among other things. More importantly the theorem can be used to construct new ring homomorphisms of finite L.S. category. In this situation the deep results of [44] then apply, strongly constraining the algebraic structure of $\pi^*(\phi)$. The proof also applies in the dual, cohomological situation of rational homotopy theory, but now over a field of any characteristic. This has potential applications to homotopy theory at large primes, after the work of Anick [4] (but that direction will not be pursued in this thesis).

Jumping ahead to section 3.5, we investigate long exact sequences for the homotopy Lie algebra, and for variations on it. An important construction we make in that section is the following.

Theorem (proposition 12 and theorem30)**.** *To a local homomorphism $\phi : (R, k) \to (S, l)$ one can associate a graded vector space $\widetilde{\pi}^*(\phi)$ with the following properties. If $(Q, u) \xrightarrow{\psi} (R, k) \xrightarrow{\phi} (S, l)$ is a sequence of local homomorphisms there is a long exact sequence*

$$\cdots \to \widetilde{\pi}^{i-1}(\psi) \otimes_k l \to \widetilde{\pi}^i(\phi) \to \widetilde{\pi}^i(\phi\psi) \to \widetilde{\pi}^i(\psi) \otimes_k l \to \widetilde{\pi}^{i+1}(\phi) \to \cdots$$

There is a natural comparison map $\widetilde{\pi}^*(\phi) \to \pi^*(\phi)$ which is an isomorphism whenever $\mathrm{cat}_2(\phi) < \infty$.

One of the reasons for introducing $\widetilde{\pi}^*(\phi)$ is that it also recovers some other known long exact sequences which do not involve $\pi^*(\phi)$, but another variation defined using divided powers instead of free variables. Details on this are given in section 3.5.

We show that vanishing of $\widetilde{\pi}^*(\phi)$ characterises quasi-complete intersection homomorphisms and deduce the following characterisation as a corollary.

Theorem (theorem 33)**.** *Let $\phi : (R,k) \to (S,l)$ be a local homomorphism. Then ϕ is quasi-complete intersection if and only if $\pi_i(R) \otimes_k l \to \pi_i(S)$ is an isomorphism for $i \geq 3$.*

This establishes the converse to a theorem of Avramov, Henriques and Şega [18, theorem 5.3]. It means that the quasi-complete intersection property can in fact be detected by the homotopy Lie algebra.

One of the key technical tools is the *Tate model for the diagonal*, which associates to a minimal model $A = Q[X]$ a minimal dg module of A over $A \otimes_Q A$. This minimal resolution turns out to have a dg algebra structure, and is in fact free as a divided power algebra over $A \otimes_Q A$. This classical construction is recalled in section 2.6, where a few applications are discussed.

The Tate model for the diagonal involves both free and divided power variables. Building homomorphisms between such dg algebras is often a serious technical problem. In section 3.4 we deal with this problem by establishing that the Tate model for the diagonal is in a certain sense functorial. This proves to be very useful, and some applications are given in that section.

Theorem (theorem 31)**.** *Let $\phi : (R,k) \to (S,l)$ be a local homomorphism. If $\phi^i : \pi^i(S) \to \pi^i(R) \otimes_k l$ is surjective for all even i then there are six term exact sequences*

$$0 \longrightarrow \pi^{2i-1}(\phi) \longrightarrow \pi^{2i-1}(S) \longrightarrow \pi^{2i-1}(R) \otimes_k l$$

$$\xrightarrow{\ \delta\ } \pi^{2i}(\phi) \longrightarrow \pi^{2i}(S) \longrightarrow \pi^{2i}(R) \otimes_k l \longrightarrow 0.$$

After [8], the homomorphism ϕ is called *small* if $\phi^* : \pi^*(S) \to \pi^*(R) \otimes_k l$ is surjective. It is known that these for these homomorphisms one obtains three term exact sequences as above (with $\delta = 0$). It is striking that also surjectivity in even degrees is equivalent to the presence of these six term exact sequences, even though there is a priori no long exact sequence at all. As a consequence of this theorem, if $(Q,u) \xrightarrow{\psi} (R,k) \xrightarrow{\phi} (S,l)$ are local homomorphisms such that both $\mathrm{cat}_2(\psi)$ and $\mathrm{cat}_2(\phi)$ are finite, we prove that there is a long exact sequence

$$\cdots \to \pi^{i-1}(\psi) \otimes_k l \to \pi^i(\phi) \to \pi^i(\phi\psi) \to \pi^i(\psi) \otimes_k l \to \pi^{i+1}(\phi) \to \cdots$$

of homotopy Lie algebras (see corollary 6).

In chapter 4 the theme changes dramatically, our central interest from this point on in the characteristic action of the Hochschild cohomology $\mathrm{HH}^*(A,A)$ on the derived category an algebra A. The underlying theme remains Koszul duality. This part of the thesis is adapted from [34], and as such its content is joint work with Vincent Gélinas.

At this point it is wise to remember that Koszul duality originates independently from topology. Here it concerns the relation between a based space X and its loop space $Y = \Omega X$. Keeping track of certain homotopical data, one can recover X, up to homotopy, as a classifying space BY.

In algebra, B and Ω become the *bar* and *cobar* functors. They are indispensable tools for us below. The ideas surrounding them are extremely wide reaching, and Koszul duality manifests in many places through some kind of bar-cobar formalism. Many of these ideas were first codified in [55].

It is through these functors that one arrives at the idea of an A_∞ algebra, a flexible generalisation of associative algebras defined by Stasheff with topological motivation. Roughly, an A_∞ algebra possesses a sequence of n-ary operations for $n \geq 2$, rather than only a binary multiplication, that together describe a structure which is associative up to homotopy. These structures provide us with the framework we need to interpret our results on the characteristic action, as we will explain now.

The Hochschild cohomology $\mathrm{HH}^*(A, A)$ acts centrally on the derived category of A through the *characteristic morphism* $\chi_M : \mathrm{HH}^*(A, A) \to \mathsf{Z}(\mathrm{Ext}^*_A(M, M))$ for each dg module M. This action has a great many uses, controlling much of the homological behaviour of M. To name a few directions, it has been studied in connection with the cohomology operators of Gulliksen [14]; the theory of support varieties [112]; with obstructions to deforming modules; it connects to geometry via the Atiyah-Chern character [37]; and in rational homotopy theory it shows up as an Umkehr map to the loop homology algebra [49].

Despite this, the image of $\chi_M : \mathrm{HH}^*(A, A) \to \mathrm{Ext}^*_A(M, M)$ is not well understood beyond the fact that $\mathrm{im}(\chi_M) \subseteq \mathsf{Z}(\mathrm{Ext}^*_A(M, M))$. We investigate how the image of χ_M is constrained by the natural A_∞ algebra structure on $\mathrm{Ext}^*_A(M, M)$.

To this end we introduced a notion of A_∞ centre for a minimal A_∞ algebra A. In short, this can be defined by setting $\mathsf{Z}_\infty(A)$ to be the image of the canonical projection $\Pi : \mathrm{HH}^*(A, A) \to A$. It is also possible to define $\mathsf{Z}_\infty(A)$ using "higher commutators" for the higher products of A.

This is perhaps best viewed as a computational tool. For example, one whether certain obstructions vanish (see remark 20), or establishing the finite generation conditions which are vital in the theory of support varieties (see for example [112]). Specifically, when is $\mathrm{HH}^*(A, A)$ Noetherian, and when is $\mathrm{Ext}^*_A(k, k)$ module-finite over the image of χ_k?

Once again this goes back to the work of Quillen, who introduced support theory to the modular representation theory of finite groups. Here it is vital to know that the cohomology algebra of a finite group is finitely generated, as established by Venkov and Evens. More generally, Snashall and Solberg conjectured in [112] that for every finite dimension k algebra A the Hochschild cohomology $\mathrm{HH}^*(A, A)/(\mathrm{nil})$ modulo nilpotents should be a finite generated k algebra. This is know for many classes of algebras (see op. cit.), but a counter-example has since been produced by Xu [123]. Aside from computing the image of χ_k, in some cases our methods are helpful in describing the algebra $\mathrm{HH}^*(A, A)/(\mathrm{nil})$ (see corollary 12).

Now we come to the original motivation for this work. Buchweitz, Green, Snashall and Solberg [38] have shown that if A is a Koszul algebra then the image of $\chi_k : \mathrm{HH}^*(A, A) \to \mathrm{Ext}^*_A(k, k)$ is the graded centre $\mathsf{Z}(\mathrm{Ext}^*_A(k, k))$. Their theorem is generalised, and given a more conceptual proof, in the following. The connectedness assumption will be explained in section 2.3.

Theorem (theorem 37)**.** *If A is a strongly connected augmented dg algebra then the image of χ : $\mathrm{HH}^*(A, A) \to \mathrm{Ext}^*_A(k, k)$ is exactly $\mathsf{Z}_\infty(\mathrm{Ext}^*_A(k, k))$.*

Since Koszulity of A can be characterized by formality of $\mathrm{RHom}_A(k, k)$, and since the A_∞ centre coincides with the graded centre for honest graded algebras, the theorem of Buchweitz, Green, Snashall and Solberg is precisely the special case when A is Koszul.

Let us emphasize the computational nature of our definitions. The A_∞-centre is concretely definable in terms of certain *higher commutators*. Moreover, the natural A_∞ structure on $\mathrm{Ext}^*_A(k, k)$ be uncovered algorithmically, and in many cases very explicitly. This is so for the d-Koszul algebras of Berger [27], after the work of He and Lu [66] (see section 4.3), and also for monomial quiver algebras, after Tamaroff

[116]. Knowing this structure, theorem 37 can often be used to compute the image of χ_k completely and explicitly by hand.

Theorem 37 can be generalised to arbitrary dg modules, rather than the augmentation module.

Theorem (corollary 8)**.** *Let M be a right dg module over A, and take $\mathrm{Ext}^*_A(M,M)$ to be a minimal A_∞ model for $\mathrm{RHom}_A(M,M)$. For every dg module M, the characteristic morphism $\chi_M : \mathrm{HH}^*(A,A) \to \mathrm{Ext}_A(M,M)$ lands in the A_∞ centre of $\mathrm{Ext}^*_A(M,M)$. If M is homologically balanced as an $\mathrm{REnd}_A(M)-A$ bimodule then $\mathrm{im}(\chi_M) = \mathsf{Z}_\infty(\mathrm{Ext}^*_A(M,M))$.*

The terminology *homologically balanced* is explained in section 2.1. We recover theorem 37 as the special case that $M = k$ because k is homologically balanced whenever A is strongly connected (see section 2.3). Corollary 8 is deduced from the following theorem.

Theorem. *Let A be a dg algebra augmented over a field k, and let M be a right dg module over A. Denote the derived endomorphism algebra of M by $R = \mathrm{REnd}_A(M,M)$. The characteristic morphism χ_M lifts canonically into the Hochschild cohomology of R:*

$$\begin{tikzcd} \mathrm{HH}^*(A,A) \arrow[rr, "\widetilde{\chi}_M"] \arrow[dr, "\chi_M"'] & & \mathrm{HH}^*(R,R) \arrow[dl, "\Pi_R"] \\ & \mathsf{Z}(\mathrm{Ext}^*_A(M,M)). & \end{tikzcd}$$

Moreover, if M is homologically balanced then $\widetilde{\chi}_M$ is an isomorphism.

This in turn is a consequence of a larger enrichment $\widetilde{\chi}$ for the entire characteristic action at once (theorems 34 and 35). This comes from a detailed investigation of various models for the characteristic action of $\mathrm{HH}^*(A,A)$ on $D(A)$. A consequence is that this action is actually *independent* of A: it depends only on the enhanced derived category $D^{\mathrm{dg}}(A)$ up to quasi-equivalence.

In the final section of this thesis we compute a few examples showing theorem 37 in action. This section 4.3 also contains a few applications to specific classes of algebras. We also return to the question of finite generation of $\mathrm{HH}^*(A,A)/(\mathrm{nil})$. The d-Koszul algebras appearing in the theorem below are a natural non-quadratic generalisation of Koszul algebras, they will be defined in section 37.

Theorem (theorem 40 and corollary 12)**.** *Assume that A is d-Koszul with nilpotent augmentation ideal, then in even cohomological degrees χ surjects onto the graded centre*

$$\chi : \mathrm{HH}^{\mathrm{even}}(A,A) \twoheadrightarrow \mathsf{Z}(\mathrm{Ext}^{\mathrm{even}}_A(k,k))$$

and this induces an isomorphism of graded commutative algebras

$$\mathrm{HH}^*(A,A)/(\mathrm{nil}) \cong \mathsf{Z}(\mathrm{Ext}_A(\Bbbk,\Bbbk))/(\mathrm{nil}).$$

Finally, let us point our another application taken from section 4.2. The proof uses A_∞ structures and the ideas behind the A_∞ centre in an essential way, but we can state it purely in dg language.

Theorem (corollary 10)**.** *Assume k has characteristic zero and let L be a dg Lie algebra over k. Then $\Pi : \mathrm{HH}^*(UL,UL) \to \mathrm{H}_*(UL)$ is surjective if and only if L is formal and quasi-isomorphic to an abelian Lie algebra.*

In the literature these dg Lie algebras have been called quasi-abelian. Understanding when this condition holds is important because it means the associated deformation problem is extremely simple, and in particular unobstructed.

1.1 Overview

The first background section 2.1 is a rapid account of some standard definitions in homological algebra, the most important being the centre of a graded category. It could be skimmed or skipped entirely.

After this, sections 2.2, 2.3, 2.4, and 2.5 contain a more leisurely introduction to the basic concepts surrounding Koszul duality. They should be read in order. Since the content here is relatively standard an expert may also wish to skim or skip these sections.

Section 2.2 is an introduction to twisting cochains and the bar construction which will be relevant throughout this thesis. Section 2.3 contains some highlights from the classical theory of Koszul duality. It is needed for chapter 4 only, but it will likely provide helpful context for both parts of this thesis. Then in sections 2.4 and 2.5 we cover the background on A_∞ algebras and Hochschild cohomology which will be needed in chapter 4.

The material in these background sections is classical and none of it should be considered original, except possibly parts of the presentation of twisted tensor products and of the Hochschild cochain complex, which are adapted from [34, sections 2.1 and 2.3].

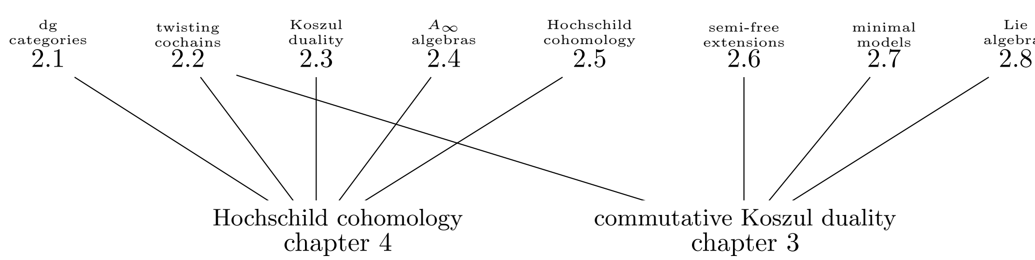
The background sections 2.6, 2.7 and 2.8 concern some important aspects of the homological algebra of local rings, setting the scene for chapter 3. These sections should also be read in order. Much of the content here is non-standard, so these sections should be read before chapter 3.

In section 2.6 we introduce semi-free extensions and semi-free divided power extensions. The background on divided powers is apparently more general than any easily accessible reference, but this generality is largely superficial. This section also contains a quite general form of the famous Tate model for the diagonal, followed by a few applications.

Section 2.7 begins with an introduction to the Cohen Factorisations of [16]. After this the important class of minimal models is introduced. This section also contains a discussion of Golod homomorphisms.

Finally, 2.8 contains background on graded Lie algebras, leading directly on to section 3.1 on the homotopy Lie algebra. The content is standard, except that theorem 13 (and its corollary) most likely does not appear in the literature in this generality.

To summarise, the reader should be aware of the following main dependencies.



In chapter 3 we investigate several related topics in local commutative algebra.

Section 3.1 leads directly on from the background section 2.8. It contains the fundamental constructions and results describing the homotopy Lie algebra of a local ring or local homomorphism.

In section 3.2 we introduce and study the LS category for a local ring or ring homomorphism. This is really the beginning of a full development. We establish a series of interesting properties and give several examples. The proof of well-definedness is technical and delayed until the end of the section. One of the most important ways to calculate LS category is the mapping theorem, which is established in section 3.3.

A key tool throughout this thesis is the Tate model for the diagonal. The purpose of section 3.4 is the proof a technical result which shows that the Tate model for the diagonal can in some sense be made functorial. We use this result a number of times in the thesis. In particular, we explain in this section how it can be used to give a simpler proof of a result of Avramov and Iyengar.

In the last section 3.5 of chapter 3 we discuss in detail some of the long exact exact sequences which can be built from the homotopy Lie algebra, and from variations on it.

Chapter 4 concerns the characteristic action of Hochschild cohomology $\mathrm{HH}^*(A,A)$ on the derived category $D(A)$, and how this action interacts with Koszul duality for A.

The core of this chapter is section 4.1. Here we use a convolution enhancement for the derived category, and the yoga of twisting cochains, to explain how the characteristic action can be enriched in a strong sense.

Section 4.2 is about interpreting the previous section through the lens of A_∞ structures. Here we introduce and begin to study a notion of A_∞ centre. As an application, at the end of the section we give a criterion for when a dg Lie algebra is quasi-abelian.

Finally, the philosophy of this chapter is applied to a few examples of d-Koszul algebras in section 4.3. This section also summarises a few quick applications of the preceding sections.

1.2 Notation and Conventions

We briefly indicate in this section some of the notation and basic objects that will be used throughout the rest of this thesis. These definitions are far from complete, they are here largely to make clear which set of conventions will be adhered to below. We also sneak in a couple of fundamental lemmas.

A graded object in a category C is simply an indexed sequence $X = \{X_i\}_{i\in\mathbb{Z}}$ of objects of C. By default indices are written as subscripts, and we say X is homologically graded. When an object is naturally cohomologically graded we may use superscripts according to the convention $X^i = X_{-i}$. A graded object X has a suspension ΣX defined by the equalities $(\Sigma X)_i = X_{i-1}$. When the objects X_i have an underlying set, elements $x \in X_i$ are said to have degree i and we write $|x| = i$. The corresponding element of $(\Sigma X)_{i+1}$ is written Σx.

If k is a commutative ring, then a complex of k modules is a graded k module M equipped with a morphism $\partial : M \to \Sigma M$ such that $\partial^2 = 0$. The suspension ΣM is itself a complex with $\partial(\Sigma x) = -\Sigma\partial(x)$. We use the notation $\sup M = \sup\{i : \mathrm{H}_i(M) \neq 0\}$. The cone on a chain map $f : M \to N$ is the complex $\mathrm{cone}(f) = N + \Sigma M$ with the differential $\partial(n + \Sigma m) = \partial(n) - \Sigma\partial(m) + f(m)$. A quasi-isomorphism $f : M \to N$ is a chain map such that the induced map in homology $\mathrm{H}_*(f) : \mathrm{H}_*(M) \to \mathrm{H}_*(N)$ is an isomorphism. Throughout, the symbol $\simeq$ will be used to indicate a quasi-isomorphism (in a slight abuse of notation, we do not worry about the fact that this symbol is undirected). The following lemma will be extremely useful for detecting quasi-isomorphisms. It can be proven by hand, or using the Eilenberg-Moore comparison theorem [120, theorem 5.5.11].

Lemma 1. *Let M and N be complexes with complete, exhaustive filtrations, and let $f : M \to N$ be a map of filtered complexes. If the map $\mathrm{gr}(f) : \mathrm{gr}(M) \to \mathrm{gr}(N)$ of associated graded complexes is a quasi-isomorphism (or isomorphism, respectively) then so is f.*

An (associative) dg algebra is a complex A of k modules with a product $m : A \otimes A \to A$ and a unit $k \to A$ which are both chain maps, and which satisfy the usual associativity and unitality axioms. Dually, an (associative) dg coalgebra is a complex C of k modules with a coproduct $\Delta : C \to C \otimes C$ and a counit $C \to k$ which satisfy the dual associativity and unitality axioms. Details can be found for instance in [90, chapter 1]. A right dg comodule over C is a complex M with a coaction $\Delta : M \to M \otimes C$ which is associative and unital. The situation for left comodules and bicomodules is analogous.

Dual to the usual definition for algebras, C is augmented if it is equipped with a homomorphism $k \to C$ of coalgebras which splits the counit $C \to k$. From an augmentation C inherits an increasing filtration defined as follows. Write $\overline{C} = \ker(C \to k)$, since C is augmented there is an induced nonunital coproduct $\overline{\Delta} : \overline{C} \to \overline{C} \otimes \overline{C}$. Using associativity to iterate yields the n-fold coproduct $\overline{\Delta}^{(n)} : \overline{C} \to \overline{C}^{\otimes n}$. With this we define $C_{(n)} = k + \ker(\overline{\Delta}^{(n)})$ for $n \geq 1$. Then $k = C_{(1)} \subseteq C_{(2)} \subseteq \cdots \subseteq C$ is called the *primitive filtration* and we say that C is *cocomplete* if $\bigcup C_{(n)} = C$. This is the case for most coalgebras which come up in practice, and it will be a crucial assumption below.

An important example is the tensor coalgebra $\mathsf{T}^{\mathsf{co}}(V) = \bigoplus_{w \geq 0} V^{\otimes w}$ on a graded k module V. We use the notation $\pi_w : \mathsf{T}^{\mathsf{co}}(V) \to V^{\otimes w}$ for the natural projections. The coproduct is given by $\Delta(v_1 \otimes \cdots \otimes v_w) = \sum_{i=0}^{w} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_w)$. With the projection π_1 it is cofree among cocomplete coalgebras equipped with a linear map to V.

Let M be a bicomodule over a coalgebra C, with right and left coactions $\Delta_r : M \to M \otimes C$ and $\Delta_l : M \to C \otimes M$. The compatibility condition asserts that $(\Delta_l \otimes 1)\Delta_r = (1 \otimes \Delta_r)\Delta_l$ and we denote this by $\Delta^{(3)}$. A k linear map $q : M \to C$ is by definition a *coderivation* if $\Delta q = (q \otimes 1)\Delta_r + (1 \otimes q)\Delta_l$. The space of coderivations from M to C is denoted $\mathrm{coDer}(M, C)$. We will need the following

Lemma 2 (Quillen [107, part II section 3.1])**.** *Let M be a bicomodule over $\mathsf{T}^{\mathsf{co}}(V)$. The corestriction $\mathrm{coDer}(M, \mathsf{T}^{\mathsf{co}}(V)) \to \mathrm{Hom}(M, V)$, $q \mapsto \pi_1 q$ is an isomorphism. Its inverse lifts $\alpha : M \to V$ to the coderivation*

$$q = \sum_{i,j \geq 0} (\pi_i \otimes \alpha \otimes \pi_j)\Delta^{(3)}.$$

Now we list some of our basic commutative algebra conventions.

A local ring is by definition a Noetherian commutative ring with exactly one maximal ideal. We will often say that $(R, \mathfrak{m}, k)$ is a local ring to mean R is a local ring with maximal ideal $\mathfrak{m}$ and residue field k. This shorthand applies as well to $(R, \mathfrak{m})$ or (R, k). At the same time, to avoid coming up with names for maximal ideals we will often automatically denote by $\mathfrak{m}_R$ the maximal ideal of a local ring R.

If (R, k) is a local ring the Poincaré series of a finitely generated R module M is by defintion $P^R_M(t) = \sum_{i \geq 0} \dim_k \mathrm{Tor}^R_i(M, k) t^i$.

If a base ring is not specified then by dg algebra we mean dg algebra over $\mathbb{Z}$ (that is, a dg ring). In chapter 3 most of the dg algebras we deal with will be strictly commutative. This means any two elements satisfy $xy = (-1)^{|x||y|}yx$ and furthermore $x^2 = 0$ whenever $|x|$ is odd[1]. For us, a local dg algebra A is by definition a strictly commutative dg algebra concentrated in nonnegative degrees such that A_0 is local and each A_i is finite as a module over A_0. In particular, A has exactly one maximal ideal. As for local rings, we use the notation $\mathfrak{m}_A$ by default for this ideal.

A homomorphism $A \to B$ of local dg algebras is called *minimal* if $\partial(\mathfrak{m}_B) \subseteq \mathfrak{m}_A B + \mathfrak{m}_B^2$. The dg algebra A will be called *absolutely minimal* if $\partial(\mathfrak{m}_A) \subseteq \mathfrak{m}_A^2$.

[1] In characteristic 2 this kind of commutativity is not even controlled by an operad (let alone a Σ-split operad), making its meaning homotopically unclear. Rather, this condition is a direct translation of the rigid structures which one works with in rational homotopy theory.

Chapter 2

Background

2.1 Graded Categories and Differential Graded Categories

In this short section we list some definitions concerning graded categories and dg categories over a commutative ring k. In particular, we make explicit how some standard definitions for dg algebras and dg modules generalise to the case of many objects. This will be somewhat terse; the coming sections should (hopefully) be more readable.

Undecorated tensor product and Hom's are all taken over k.

A graded k category C consists of a collection of objects and for any two objects x, y a graded k module $\mathrm{Hom}_C(x, y)$, which will often be denoted ${}_yC_x$. We should also have bilinear composition maps ${}_xC_y \otimes {}_yC_z \to {}_xC_z$ and units $k \to {}_xC_x$ which satisfy the usual associativity and unitality conditions.

The underlying ungraded k category C_0 has the same objects as C and $\mathrm{Hom}_{C_0}(x, y) = \mathrm{Hom}_C(x, y)_0$. If there is an equivalence $\Sigma : C_0 \to C_0$ and a natural isomorphism $\mathrm{Hom}_C(x, \Sigma y) \cong \Sigma\mathrm{Hom}_C(x, y)$ then we say that C has an internal suspension. In this way a graded category is essentially the same thing as a k category with a distinguished auto-equivalence. In particular, every triangulated category is naturally a graded category.

By definition the *graded centre* $\mathsf{Z}(C)$ of C is the graded k module of graded natural transformations $1_C \to 1_C$. This means

$$\mathsf{Z}_n(C) = \{\, \text{families } \{\xi_x \in \mathrm{Hom}_C(x, x)_n\}_{x \in C} \;:\; f\xi_x = (-1)^{nm}\xi_y f \text{ for all } f \in \mathrm{Hom}_C(x, y)_m \,\}\,.$$

If C has a single object then C is simply a graded k algebra, and $\mathsf{Z}(C)$ is the usual graded centre of C. On the other hand, if C has an internal suspension Σ then $\mathsf{Z}_n(C)$ can be identified with the set of all natural transformations $\xi : 1_{C_0} \to \Sigma^n$ which satisfy $\xi\Sigma = (-1)^n\Sigma\xi$ (that is, $\xi_{\Sigma x}\Sigma = (-1)^n\Sigma\xi_x : x \to \Sigma^{n+1}x$ for all objects x). Under composition $\mathsf{Z}(C)$ becomes a graded commutative k algebra (not necessarily strictly graded commutative). Krause and Ye discuss graded centres of triangulated categories in [83].

For any object x of C there is a restriction homomorphism $\mathsf{Z}(C) \to \mathsf{Z}(\mathrm{End}_C(x))$. More generally one can restrict along any full subcategory of C (in [79] Keller upgrades this with the observation that one can also do this for the Hochschild cochain complex of any dg category, see section 2.5).

A dg category C over k consists of a collection of objects and a complex ${}_yC_x = \mathrm{Hom}_C(x, y)$ of k modules for any objects x, y. Again C should come with composition maps ${}_xC_y \otimes {}_yC_z \to {}_xC_z$ and units $k \to {}_xC_x$, both of which are chain maps, and which satisfy the usual associativity and unitality conditions.

Any dg category has an underlying graded category by forgetting the differential. More interestingly, taking homology produces a graded category $\mathrm{H}_*(C)$ with the same objects and $\mathrm{Hom}_{\mathrm{H}_*(C)}(x, y) = \mathrm{H}_*(\mathrm{Hom}_C(x, y))$. The underlying ungraded category $\mathrm{H}_0(C)$ is usually called the homotopy category of C.

A functor $F : C \to D$ between dg categories is a functor of the underlying ungraded categories which additionally respects differentials, meaning that $\mathrm{Hom}_C(x, y) \to \mathrm{Hom}_D(F(x), F(y))$ is a chain map for any two objects x and y of C. If the induced functor $\mathrm{H}_*(F) : \mathrm{H}_*(C) \to \mathrm{H}_*(D)$ is an equivalence then F is called a quasi-equivalence.

A right module M over a dg small category C is a complex M_x for every object x of C together with a chain map $M_x \otimes {}_xC_y \to M_y$ for any two objects x and y, and this action should be associative and unital in the obvious sense. Similarly one can define left modules and bimodules. C is canonically a bimodule over itself, and for any object x we have the representable right module ${}_xC$ (or left module C_x). The representable modules play the role of free modules. Given a right module M and a left module N over C, one can form the tensor product $M \otimes_C N$, which is the quotient of $\bigoplus_{x\in C} M_x \otimes {}_xN$ by the identification $m \otimes fn = mf \otimes n$ for $m \in M_x$ and $f \in {}_xC_y$ and $n \in {}_yN$.

The preceding definitions can be put into a formal framework following [80]. A dg quiver Q consists of a set of objects and a complex ${}_yQ_x$ for each pair of objects. The collection of dg quivers over a fixed set of objects S forms a monoidal dg category by setting

$$\mathrm{Hom}(Q, Q') = \prod_{x,y} \mathrm{Hom}({}_yQ_x, {}_yQ'_x) \qquad\qquad {}_y(Q \otimes Q')_x = \bigoplus_z {}_yQ_z \otimes {}_zQ'_x.$$

The unit is the quiver k_S with ${}_x(k_S)_x = k \cdot 1_x$ and ${}_y(k_S)_x = 0$ when $x \neq y$. A dg category with a set of objects S is then essentially the same thing as a monoid with respect to this structure, and modules over dg categories are essentially modules over the corresponding monoids.

A dg category C on a set S of objects is called augmented if it is equipped with a morphism $C \to k_S$ splitting the unit $k_S \to C$.

One reason for introducing this framework is that it reveals the correct definition of dg cocategory: these should be dg quivers with a decomposition $C \to C \otimes C$ and a counit $C \to k_S$ satisfying the usual comonoid axioms. Cocategories can be used to build bar constructions for augmented dg categories.

A fundamental example of a dg category is the enhancement of the derived category of an algebra. To describe this we need some background. These dg categories are never augmented.

Let A be a dg algebra (or small dg category) over k. A right module M over A is called *semi-free* if it admits an exhaustive filtration $0 = M_{(0)} \subseteq M_{(1)} \subseteq M_{(2)} \subseteq \cdots$ such that the quotients $M_{(i)}/M_{(i+1)}$ are all free (or representable) A modules[1].

The objects of $D^{\mathrm{dg}}(A)$ are by definition all semi-free right A modules. The complex ${}_ND^{\mathrm{dg}}(A)_M$ is the usual complex $\mathrm{Hom}_A(M, N)$ of A linear homomorphisms. We also use the notation $\mathrm{RHom}_A(M, N)$ for this complex, and the notation $\mathrm{Ext}^*_A(M, N) = \mathrm{H}_*(\mathrm{RHom}_A(M, N))$ for its homology. The homotopy category $D(A) = \mathrm{H}_0(D^{\mathrm{dg}}(A))$ is by definition the derived category of A. It has the structure of a triangulated category.

[1]The filtration can be indexed by $\mathbb{N}$ or by any ordinal. The two definitions are equivalent as a filtration by an ordinal can always be transformed into one by $\mathbb{N}$ with the same property.

We also write $D^{\mathrm{dg}}_{\mathrm{perf}}(A)$ for the full dg subcategory of $D^{\mathrm{dg}}(A)$ on those dg modules which are isomorphic in $D(A)$ to an object of $D_{\mathrm{perf}}(A)$. Here $D_{\mathrm{perf}}(A) = \mathrm{thick}_{D(A)}(A)$ is the perfect derived category of A, that is, the smallest full, triangulated subcategory of $D(A)$ which contains A and is closed under summands. In general, when M is a set of objects in $D(A)$, we denote by $\mathrm{think}_{D(A)}(M)$ the smallest full, triangulated subcategory of $D(A)$ which contains the objects in M and is closed under summands.

In general, when M and N are not necessarily semi-free, $\mathrm{RHom}_A(M,N)$ is defined to be $\mathrm{Hom}_A(F,N)$ where F is a semi-free right A module with a chosen surjective A-linear quasi-isomorphism $F \xrightarrow{\simeq} M$. The complex $\mathrm{RHom}_A(M,N)$ is defined up to quasi-isomorphism, but often we will have a specific semi-free resolution for M in mind. Moreover $\mathrm{RHom}_A(M,N)$ is naturally a dg algebra using $\mathrm{Hom}_A(F,F)$ as a model with composition as product. This dg algebra depends on F only up to quasi-isomorphism *of dg algebras.*

Suppose that M and N are right and left A modules respectively. The derived tensor product $M \otimes^{\mathsf{L}}_A N$ is similarly defined to be $F \otimes_A N$, where $F \xrightarrow{\simeq} M$ is a semi-free resolution (one could equally well resolve N instead).

Suppose that M is a $B-A$ bimodules and N is a $C-A$ bimodule, for three small dg categories A, B and C. Assume also that each ${}_xM$ and ${}_yN$ are semi-free right A modules. Then $\mathrm{RHom}_A(M,N)$ is naturally a $C-B$ bimodule with ${}_y\mathrm{RHom}_A(M,N)_x = \mathrm{RHom}_A({}_xM, {}_yN)$. In particular when $B=C$ and $M=N$ the quiver $\mathrm{RHom}_A(M,M)$ is naturally a dg category with the same set of objects as B.

When M is a set of right A modules $\mathrm{RHom}_A(M,M)$ is by definition the full dg subcategory of $D^{\mathrm{dg}}(A)$ on M. This canonically makes M into a $\mathrm{RHom}_A(M,M)-A$ bimodule.

We make a definition which will be useful later. Suppose that A and B are dg categories and M is a $B-A$ bimodule. We say that M is *homologically balanced* if the natural functors $B \to \mathrm{REnd}_A(M)$ and $A^{\mathrm{op}} \to \mathrm{REnd}_{B^{\mathrm{op}}}(M)$ are both quasi-equivalences. This condition was used by Keller in [79]. The following is a consequence of the five lemma.

Lemma 3 ([79])**.** *Let M be a $B-A$ bimodule. The natural map $B \to \mathrm{RHom}_A(M,M)$ is a quasi-equivalence if and only if the functor $-\otimes^{\mathsf{L}}_B M : D_{\mathrm{perf}}(B) \to D(A)$ is fully faithful.*

2.2 Twisting Cochains and the Bar Construction

In this section we work with algebras and coalgebras over a field k. Undecorated tensor product and Hom's are taken over k. It is possible to work over a commutative ring instead, but we will not need this generality.

If A is a dg algebra with product $m : A \otimes A \to A$ and C is a dg coalgebra with coproduct $\Delta : C \to C \otimes C$ then the complex $\mathrm{Hom}(C,A)$ becomes a dg algebra with the cup product $f \smile g = m \circ (f \otimes g) \circ \Delta$. This is verified in [90, proposition 2.1.2]. We call this the *convolution algebra* connecting C and A. Using this structure, an element $\tau \in \mathrm{Hom}(C,A)_{-1}$ is called a *twisting cochain* if it satisfies the Maurer-Cartan equation

$$\partial(\tau) + \tau \smile \tau = 0.$$

Assume further that both C and A are augmented. Then a twisting cochain $\tau : C \to A$ is called *augmented* if it vanishes on both augmentations, so the two compositions $k \to C \xrightarrow{\tau} A$ and $C \xrightarrow{\tau} A \to k$ are zero. The set of augmented twisting cochains from C to A is denoted $\mathbf{tw}(C,A)$.

Let **Alg** be the category of augmented dg algebras over k, and let **coAlg** be the category of cocomplete augmented dg coalgebras over k. The set of twisting cochains is natural in both arguments, so we have a functor

$$\mathbf{tw} : \mathbf{coAlg}^{\mathsf{op}} \times \mathbf{Alg} \to \mathbf{Sets}.$$

Theorem 2 ([90, section 2.2])**.** *This functor is representable in both arguments: there are functors $\Omega : \mathbf{coAlg} \to \mathbf{Alg}$ and $B : \mathbf{Alg} \to \mathbf{coAlg}$ and natural isomorphisms*

$$\mathbf{Alg}(\Omega C, A) \cong \mathbf{tw}(C, A) \cong \mathbf{coAlg}(C, BA).$$

The functors B and Ω are called the *bar* and *cobar constructions* respectively. By definition we have an adjunction

$$B : \mathbf{Alg} \rightleftarrows \mathbf{coAlg} : \Omega.$$

According to [90], this can be upgraded to a Quillen equivalence of model categories, this sums up much of the force behind Koszul duality. Classical Koszul duality concerns what can be said about formality on either side of this adjunction. This will be discussed in section 2.3.

The bar construction goes back to [42] while the cobar construction goes back to [1]. Both originate in algebraic topology, and they remain indispensable there and in algebra. We will quickly explain how to build the bar construction of an augmented algebra now, since this is all we need below. Many more details, including the cobar construction, can be found in [90, section 2.2].

Let A be an augmented dg algebra. Denote $\overline{A} = \operatorname{coker}(k \to A)$, then using the augmentation there is an induced non-unital product $\overline{m} : \overline{A} \otimes \overline{A} \to \overline{A}$. As a graded coalgebra BA is the tensor coalgebra $\mathsf{T}^{\mathsf{co}}(\Sigma\overline{A})$. It is traditional to write elements of BA using the notation $\Sigma a_1 \otimes \cdots \otimes \Sigma a_w = [a_1|\cdots|a_w]$. We also write π_w for the projection $BA \to B_w A = (\Sigma\overline{A})^{\otimes w}$, and define two degree -1 maps b_1 and b_2 by the commutativity of the following two diagrams

$$\begin{tikzcd} \overline{A} \arrow[r, "\partial"] \arrow[d, "\Sigma"'] & \overline{A} \arrow[d, "\Sigma"] \\ B_1 A \arrow[r, "-b_1"] & B_1 A \end{tikzcd} \qquad \begin{tikzcd} \overline{A}^{\otimes 2} \arrow[r, "\overline{m}"] \arrow[d, "\Sigma\otimes\Sigma"'] & \overline{A} \arrow[d, "\Sigma"] \\ B_2 A \arrow[r, "b_2"] & B_1 A. \end{tikzcd}$$

Finally, because BA is cofree on B_1A, there is a unique coderivation $b : BA \to BA$ such that $\pi_1 b = b_1\pi_1 + b_2\pi_2$ by lemma 2. Explicitly, b is given by

$$b[a_1|\cdots|a_w] = \sum_{i=1}^{w}(-1)^{\epsilon_i}[a_1|\cdots|\partial(a_i)|\cdots|a_w] + \sum_{i=2}^{w}(-1)^{\epsilon_i}[a_1|\cdots|m(a_{i-1}, a_i)|\cdots|a_w]$$

where $\epsilon_i = i + |a_1| + ... + |a_{i-1}|$. The condition $b^2 = 0$ is a consequence of the fact that A is an associative dg algebra. Thus BA with b as a differential is a cocomplete dg coalgebra. This is the bar construction of A. The universal twisting cochain into A is $\pi : BA \xrightarrow{\pi_1} \Sigma\overline{A} \hookrightarrow \Sigma A$. The bijection in the above theorem takes a coalgebra map $\phi : C \to BA$ to the twisting cochain $\pi\phi$, and it follows that every other twisting cochain into A factors uniquely through π.

The construction of ΩC is linearly dual: as an algebra $\Omega C = \mathsf{T}(\Sigma^{-1}\overline{C})$ and the differential encodes precisely the coproduct and differential on C. When $\overline{C}$ is concentrated in degrees ≥ 2, or in degrees ≤ 0, there is a canonical isomorphism $B(C^\vee) \cong (\Omega C)^\vee$. By the theorem there is also a universal twisting cochain from C, which we denote $\iota : C \to \varepsilon\Omega C$.

Quasi-isomorphisms of dg coalgebras are poorly behaved. Instead, a homomorphism $\phi : C \to C'$ between cocomplete dg coalgebras is called a *weak equivalence* if the induced algebra homomorphism $\Omega\phi : \Omega C \to \Omega C'$ is a quasi-isomorphism. In [85] it is proven that a weak equivalence is a quasi-isomorphism, but this notion is stronger. Since we will not make serious use of this notion, we refer to [90] or [85] for discussion.

Now we explain how twisting cochains can be used to build resolutions and compute derived functors. These twisted tensor products and twisted convolution algebras will be important throughout this thesis.

If $\tau : C \to A$ is a twisting cochain we may perturb the usual differential on $\mathrm{Hom}(C, A)$ to $\partial + [\tau, -]$. The condition $(\partial + [\tau, -])^2 = 0$ is equivalent to the Maurer-Cartan equation. Equipped with this differential $\mathrm{Hom}(C, A)$ is called the *twisted convolution algebra* associated to τ, and it is denoted $\mathrm{Hom}^\tau(C, A)$. Naturality of twisting cochains for algebra and coalgebra maps extends to naturality of twisted convolution algebras. For later reference we need the following lemma.

Lemma 4. *If $\tau : C \to A$ is an augmented twisting cochain from a cocomplete dg coalgebra, and $\phi : A \to A'$ is a quasi-isomorphism dg algebras, then the induced map*

$$\phi_* : \mathrm{Hom}^\tau(C, A) \to \mathrm{Hom}^{\phi\tau}(C, A')$$

is also a quasi-isomorphism of dg algebras.

The lemma follows easily from lemma 1 using the filtration on twisted convolution algebras induced from the the primitive filtration on C, which is complete since C is cocomplete. The twists $[\tau, -]$ and $[\phi\tau, -]$ decrease the primitive filtration, so they vanish after passing to the associated graded complexes and then then result is a standard fact about hom complexes.

Let M be a right C comodule and let N be a left A-module. The convolution algebra $\mathrm{Hom}(C, A)$ acts on the left of $M \otimes N$ via the cap product

$$\alpha \frown - : M \otimes N \xrightarrow{\Delta\otimes 1} N \otimes C \otimes M \xrightarrow{1\otimes\alpha\otimes 1} N \otimes A \otimes M \xrightarrow{1\otimes m} N \otimes M.$$

If $\tau : C \to A$ is a twisting cochain the $\partial_M \otimes 1 + 1 \otimes \partial_N + (\tau \frown -)$ is a square zero differential on $M \otimes N$. With this differential, it is easy to verify that the *twisted tensor product* $M \otimes^\tau N$ becomes a left dg module over the twisted convolution algebra $\mathrm{Hom}^\tau(C, A)$.

Similarly, if M is a right A module and N is a left comodule, the right action of $\mathrm{Hom}(C, A)$ on $M \otimes N$ allows us to equip it with a differential $\partial_M \otimes 1 + 1 \otimes \partial_N - (- \frown \tau)$. This twisted tensor product $M \otimes^\tau N$ is a right dg module over $\mathrm{Hom}^\tau(C, A)$.

The following theorem is fundamental in the theory of twisting cochains.

Theorem 3 ([72, 85, 90])**.** *If $\tau : C \to A$ is an augmented twisting cochain the following are equivalent:*

1. *$\Omega C \to A$ is a quasi-isomorphism;*
2. *$C \to BA$ is a weak equivalence;*
3. *using the counit of C, the natural A bimodule map $A \otimes^\tau C \otimes^\tau A \to A$ is a quasi-isomorphism.*

We say that τ is *acyclic* if it satisfies the equivalent conditions of the theorem. Note that according to the theorem, the universal twisting cochains $C \xrightarrow{\iota} \Omega C$ and $BA \xrightarrow{\pi} A$ are both acyclic.

Note that the twisted part of the differential on $A \otimes^{\tau} C \otimes^{\tau} A$ decreases the primitive filtration on C. Since C is cocomplete, this filtration witnesses the fact that $A \otimes^{\tau} C \otimes^{\tau} A \to A$ is a semi-free dg module resolution over $A^{\mathsf{op}} \otimes A$.

In the thesis of Lefèvre-Hasegawa the conditions of the theorem are also shown to be equivalent to having a canonical equivalence of derived categories $- \otimes^{\tau} A : D^{\mathsf{co}}(C) \leftrightarrows D(A) : - \otimes^{\tau} C$, see [85] for details.

Theorem 4. *If $\tau : C \to A$ is an acyclic twisting cochain then there is a canonical quasi-isomorphism*

$$C^{\vee} \xrightarrow{\simeq} \mathrm{RHom}_A(k, k).$$

Proof. By the previous theorem we have a semi-free bimodule resolution $A \otimes^{\tau} C \otimes^{\tau} A \xrightarrow{\simeq} A$. Tensoring down on the left produces a semi-free resolution of right dg modules $C \otimes^{\tau} A \xrightarrow{\simeq} k$ (this is sometimes called the *Koszul complex* of A).

There is a natural left action $C^{\vee}$ on $C \otimes^{\tau} A$ using the coproduct of C. This produces a homomorphism of dg algebras $\phi : C^{\vee} \to \mathrm{Hom}_A(C \otimes^{\tau} A, C \otimes^{\tau} A)$. Composing ϕ with the quasi-isomorphism $\mathrm{Hom}_A(C \otimes^{\tau} A, C \otimes^{\tau} A) \xrightarrow{\simeq} \mathrm{Hom}_A(C \otimes^{\tau} A, k) \cong C^{\vee}$ yields the identity on $C^{\vee}$, and therefore ϕ is a quasi-isomorphism. $\square$

Example 1. Let V be a graded vector space which is degree-wise finite dimensional and concentrated in non-negative degrees. Let $S = k[\![V]\!] = \widehat{\mathrm{Sym}}(V)$ be the free complete strictly graded commutative algebra on V. Also let $\Gamma = \mathrm{Sym}^{\mathrm{co}}(\varepsilon V)$ be the cofree strictly graded commutative coalgebra on εV, which may be defined by dualising $\Gamma \cong (\mathrm{Sym}((\varepsilon V)^{\vee}))^{\vee}$. This coalgebra is naturally isomorphic to the free divided power algebra $k\langle \varepsilon V \rangle$ on εV, which we will discuss in section 2.6.

The composition $\tau : \Gamma \twoheadrightarrow \varepsilon V \hookrightarrow \varepsilon S$ is an acyclic twisting cochain. The Maurer-Cartan equation is easy to check, and the twisted tensor product $S \otimes^{\tau} \Gamma \otimes^{\tau} S$ is isomorphic to

$$k[\![V, V]\!]\langle \varepsilon V \rangle \quad \text{with} \quad \partial(\varepsilon v) = v \otimes 1 - 1 \otimes v \quad \text{for} \ \ v \in V.$$

This is the classical Tate model for the diagonal which will be discussed in section 2.6. It is well known that $k[\![V, V]\!]\langle \varepsilon V \rangle \to k[\![V]\!]$ is a quasi-isomorphism (see for example [44, proposition 1.9]). This example will be generalised by theorem 13.

2.3 Koszul Duality

In this section we discuss some aspects of Koszul Duality for associative algebras. Of course, much more can be said, and we will have to completely omit many important aspects of this theory. Continuing from the previous section we work over a field k. All algebras are k algebras, and undecorated tensor product and Hom's taken over k.

In its simplest algebraic incarnation Koszul duality concerns the relation between an associative, augmented (possibly graded or dg) algebra A, and the Yoneda algebra $\mathrm{Ext}^*_A(k, k)$. Theorem 4 provides the connection with twisting cochains: if $\tau : C \to A$ is an acyclic twisting cochain then there is a canonical isomorphism $H_*(C^\vee) \cong \mathrm{Ext}^*_A(k,k)$ of graded algebras. It is no secret that passing to homology (usually) throws away information. With this in mind we say that the dg algebra $\mathrm{RHom}_A(k,k)$ is *Koszul dual* to A. We use the notation $A^! = \mathrm{RHom}_A(k,k)$ with the understanding that this is only defined up to quasi-isomorphism. Since $\pi : BA \to A$ is an acyclic twisting cochain $(BA)^\vee$ functorially provides a model for the Koszul dual dg algebra to A. Note that this model is augmented: we should really consider $A^!$ to be an augmented dg algebra defined up to augmented quasi-isomorphism.

Koszul duality has many algebraic and geometric incarnations, and is best motivated through examples. For instance Priddy's original work [104] discusses in detail the Steenrod algebra from algebraic topology. But there are two basic reasons why one should be interested in $A^!$ if one is interested in A. Firstly, the representation theory of A is closely related to that of $A^!$. This goes back at least to [26], and will be explained in part below. Secondly, under mild finiteness conditions it is possible to recover A from $A^!$. In fact there is a canonical quasi-isomorphism $A^{!!} \simeq A$. We will sketch the reason for this now, much more detail can be found in [90] or [85].

The price of working with algebras is that repeated dualisation necessitates some finiteness conditions. The theory of twisting cochains suggests a version of Koszul duality using coalgebras which avoids dualising unnecessarily. Details of this picture are worked out in the thesis of Lefèvre-Hasegawa [85]. For readability we mostly stick to the algebra side in this thesis. Thus, let us say that an augmented dg algebra A is *strongly connected* if $\mathrm{H}_*(A)$ is degree-wise finite dimensional and it is either homologically connected or cohomologically simply-connected, which means $\mathrm{H}_*(\overline{A})$ must be concentrated in degrees ≥ 1 or in degrees ≤ -2 respectively.

Theorem 5. *If A is a strongly connected then $A^{!!}$ is canonically quasi-isomorphic to A. Thus, A can be recovered from $A^!$ up to quasi-isomorphism.*

A inductive argument shows that up to quasi-isomorphism we can arrange that $\overline{A}$ itself is degree-wise finite dimensional and concentrated in degrees ≥ 1 or in degrees ≤ -2 (a more intuitive way of doing this is to replace A with a minimal A_∞ model using theorem 8, this will be explained in the next section). Having done this, there is a canonical isomorphism $(B(BA)^\vee)^\vee \cong (\Omega BA)^{\vee\vee} \cong \Omega BA$. One can show that the natural homomorphism $\Omega BA \to A$ is always a quasi-isomorphism by writing down an explicit contraction as in [85, lemme 1.3.2.3] (note that this is also a consequence of theorem 3). Since $(B(BA)^\vee)^\vee$ is a model for the double Koszul dual to A, this produces a quasi-isomorphism $A^{!!} \simeq A$.

Since $A^! \to \mathrm{RHom}_A(k,k)$ is a quasi-isomorphism by definition, the theorem may be restated in the terminology of section 2.1.

Corollary 1. *If A is a strongly connected then k is homologically balanced as an $A^! - A$ bimodule.*

The following is perhaps the most basic derived equivalence which Koszul duality yields. These ideas go back to [26]. With appropriate definitions the theorem extends to the A_∞ setting, worked out in [92]. Lefèvre-Hasegawa establishes a much larger and more generally applicable derived equivalence in [85], relating A and its Koszul dual dg coalgebra (the source of any acyclic cochain into A).

Recall that $D_{\mathrm{perf}}(A) = \mathrm{thick}_{D(A)}(A)$ is the prefect derived category of A. We also write $D_{\mathrm{fd}}(A)$ for the full subcategory of $D(A)$ on those objects with finite dimensional homology.

Theorem 6. *If A is strongly connected we have natural equivalences of triangulated categories*

$$D_{\mathrm{fd}}(A) \simeq D_{\mathrm{perf}}(A^!) \qquad \textit{and} \qquad D_{\mathrm{perf}}(A) \simeq D_{\mathrm{fd}}(A^!).$$

Proof. The first equivalence is quite formal using the techniques of Keller [75], and strong connectedness is not needed.

First note that $D_{\mathrm{fd}}(A) = \mathrm{thick}_{D(A)}(k)$. Let $K \xrightarrow{\simeq} k$ be a semi-free right A module resolution. Following [75] we work with the dg enhancement $D^{\mathrm{dg}}_{\mathrm{fd}}(A)$ of semi-free dg modules with finite dimensional cohomology. Since k is generator for $D_{\mathrm{fd}}(A)$ the functor

$$\mathrm{RHom}_A(k,-) = \mathrm{Hom}_A(K,-) : D^{\mathrm{dg}}_{\mathrm{fd}}(A) \to D^{\mathrm{dg}}_{\mathrm{perf}}(\mathrm{End}_A(K))$$

is a quasi-equivalence by the five lemma. Then since $A^! \simeq \mathrm{End}_A(K)$ we also have an equivalence $D_{\mathrm{perf}}(\mathrm{End}_A(K)) \simeq D_{\mathrm{perf}}(A^!)$.

The second equivalence then follows by reversing the roles of A and $A^!$ using theorem 5. □

Classical Quadratic Koszul Duality

Starting with an augmented graded algebra A, it would be desirable to be able to replace the large dg algebras $\mathrm{RHom}_A(k,k)$ or $(BA)^\vee$ with the simpler graded algebra $\mathrm{Ext}^*_A(k,k)$ in theorems 5 and 6. With this in mind, we define A to be *Koszul* if A is strongly connected and $A^!$ is formal, which mean there is a chain of quasi-isomorphisms connecting $A^!$ to $\mathrm{H}_*(A^!) = \mathrm{Ext}^*_A(k,k)$. We will explain here how this relates to the more classical picture of Koszul duality, which concerns algebras and coalgebras defined by quadratic data, and small resolutions over them. These ideas go back to the seminal work of Priddy [104].

Miraculously, the assumption that A is Koszul leads (with the data of a presentation of A) to a simple, canonical quasi-isomorphism $(BA)^\vee \xrightarrow{\simeq} \mathrm{Ext}^*_A(k,k)$, or more fundamentally $\mathrm{Tor}^A_*(k,k) \xrightarrow{\simeq} BA$.

So, let A be an augmented graded algebra and suppose we have a graded vector space V with a map $V \to A$ such that $\mathsf{T}(V) \to A$ is surjective. Assume that V is minimal, meaning that the kernel of this map lies in $\mathsf{T}^{\geq 2}(V)$. Choose also a minimal space of relations $R \subseteq \mathsf{T}^{\geq 2}(V)$, so that A is the quotient of $\mathsf{T}(V)$ by the two-sided ideal (R) generated by R. Minimality means here that $(V \cdot R + R \cdot V) \cap R = 0$. [2]

By choosing a space of generators any semi-free dg module can be written in the form $M = (U \otimes A, \partial)$ where $U = \bigcup U_{(i)}$ is a filtered vector space such that $U_{(0)} = 0$ and $\partial(U_i) \subseteq U_{i-1} \otimes A$. We say that M (equipped with this space of generators) is *linear* if $\partial(U) \subseteq U \otimes V$. The minimal semi-free resolution of the augmentation module k begins

$$\cdots \to \varepsilon^2 R \otimes A \to \varepsilon V \otimes A \to A \to 0.$$

And therefore $\mathrm{Tor}^A_*(k,k)$ will begin $(k + \varepsilon V + \varepsilon^2 R + ...)$. Note that for this resolution to be linear R must be quadratic.

Aside from the algebra $A = A(V,R) = \mathsf{T}(V)/(R)$, there is an equally natural way to define a coalgebra from quadratic data of V and $R \subseteq V^{\otimes 2}$. We can take the cocomplete, augmented coalgebra $C(V,R)$ which is cofree on V subject to the constraint that $C(V,R) \to C(V,R) \otimes C(V,R) \to V \otimes V$ factors

[2] With this data we can build the dg algebra $\mathsf{T}(V, \varepsilon R)$ with $\partial(\varepsilon r) = r \in \mathsf{T}(V)$. This is the first step in the non-commutative Tate resolution of $A = \mathsf{T}(V)/(R)$: as in section 2.6 one inductively constructs a dg algebra resolution $\mathsf{T}(V_0, V_1, V_2, ...) \xrightarrow{\simeq} A$ with $V_0 = V$ and $V_1 = \varepsilon R$ and V_i in degree i. We can tautologically interpret $\mathsf{T}(V_0, V_1, ...) = \Omega C$ as the cobar construction of an A_∞ coalgebra, which comes with an acyclic twisting cochain $C \to A$ (A_∞ structures will be discussed shortly). It follows from this discussion that there is a canonical isomorphism $C = (k + \varepsilon V_0 + \varepsilon V_1 + ...) \cong \mathrm{Tor}^A_*(k,k)$. If A is Koszul, we can construct C with no higher coproducts. In other words the differential $\mathsf{T}(V_0, V_1, ...)$ must be quadratic, and in particular we must be able to choose the relations R quadratic.

through $R \hookrightarrow V \otimes V$. Concretely, $C(V,R)$ may be presented as the sub-coalgebra of $\mathsf{T}^{\mathsf{co}}(V)$ whose weight w part is $C_w = \bigcap_{p+2+q=w} V^{\otimes p} \otimes R \otimes V^{\otimes q}$. It is easy to check that there is a twisting cochain $\tau : C(\Sigma V, \Sigma^2 R) \twoheadrightarrow \Sigma V \hookrightarrow \Sigma A$. This factors through the universal one via the inclusion $C(\Sigma V, \Sigma^2 R) \hookrightarrow \mathsf{T}^{\mathsf{co}}(\Sigma V) \hookrightarrow BA$.

Theorem 7. *Let $A = \mathsf{T}(V)/(R)$ be as above, and assume A is strongly connected. The following are equivalent:*

1. *the dg algebra $A^! = \mathrm{RHom}_A(k,k)$ is formal;*
2. *the augmentation module k admits a linear resolution;*
3. *R can be chosen quadratic and $C(\Sigma V, \Sigma^2 R) \hookrightarrow BA$ is a quasi-isomorphism, so that $C(\Sigma V, \Sigma^2 R) \cong \mathrm{Tor}^A_*(k,k)$.*
4. *R can be chosen quadratic and the twisting cochain $\tau : C(\Sigma V, \Sigma^2 R) \to A$ is acyclic.*

An algebra satisfying these equivalent conditions is called *Koszul*.

The equivalence of 3 with 2 is in [26]. Under the strongly connected assumption 4 and 3 are easily seen to be equivalent. The obtained resolution $C(\Sigma V, \Sigma^2 R) \otimes^\tau A \xrightarrow{\sim} k$ is called the *Koszul complex* by Beilinson, Ginzburg and Soergel. It generalises the classical Koszul complex, when A is a polynomial algebra (see example 1). These results ultimately trace back to Priddy [104]. The connection to 1 is in Keller's [77], but see also earlier the work of Gugenheim and May [62].

2.4 A_∞ Algebras

This section contains a quick introduction to the theory of A_∞ algebras. The survey [76] is a good place to start for more intuition, and much more detail can be found in [90]. We continue with the notation of the previous section, working implicitly over a field k.

A_∞ algebras (and A_∞ spaces) were invented by Stasheff [113] in the early 60s as a tool to understand loop spaces and their relation to general H-spaces. In short, he used them to develop a fundamental recognition principle for loop spaces. They have since found many important applications in connection with algebra, geometry and mathematical physics (see [78] for some references in this direction). We will attempt to motivative this structure at a basic algebraic level as a continuation of the previous sections.

Outside of the Koszul case working with $A^!$ can be complicated. Most models, such as $(BA)^\vee$, are extremely large and contain more information than necessary. But since $A^!$ is not formal the graded algebra $\mathrm{Ext}^*_A(k,k)$ does not contain enough information. Koszul algebras are surprisingly plentiful, but non-Koszul algebras certainly arise - the quadratic condition being the main restriction.

It turns out that $\mathrm{Ext}^*_A(k,k)$ is naturally an A_∞ algebra: it has a series of operations $m_n : \mathrm{Ext}^*_A(k,k)^{\otimes n} \to \mathrm{Ext}^*_A(k,k)$ (unique up to isomorphism) which together satisfy certain higher associativity rules. This structure solves the information problem: it is precisely enough to functorially recover A. Thus we think of it as an A_∞ model for $A^!$. Having done this, we also solve the problem of unwieldy models. The issue that remains is actually computing these higher products.

More motivation in a similar vein comes from looking for small resolutions. As we have have just seen, if A is a Koszul there is an acyclic twisting cochain $\tau : \mathrm{Tor}^A_*(k,k) \to A$. Consequently we obtain a (potentially) small, functorial resolution $M \otimes^\tau \mathrm{Tor}^A_*(k,k) \otimes^\tau A$ of any right A module M. In fact this is always possible. In general $\mathrm{Tor}^A_*(k,k)$ will be an A_∞ coalgebra, and knowing this structure one can form the twisted tensor product $M \otimes^\tau \mathrm{Tor}^A_*(k,k) \otimes^\tau A$ resolving any right A module. Since we won't need these resolutions in this thesis, we refer the reader to [90] for information on twisted tensor products in the A_∞ setting.

An A_∞ algebra is a complex A with a sequence of degree $n-2$ operations $m_n : A^{\otimes n} \to A$ for $n \geq 2$. These should satisfy the quadratic Stasheff identities

$$\partial(m_n) + \sum_{r+s+t=n} (-1)^{rs+t} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0,$$

as well as the strict unitality conditions $m_n(-,...,1,...,-) = 0$ for $n \geq 3$ and $m_2(-,1) = m_2(1,-) = \mathrm{id}_A$. We say that A is augmented if it is equipped with a splitting $A = \overline{A} + k$ and all of the operations preserve $\overline{A}$. An A_∞ algebra is called *minimal* if its differential vanishes.

These operations together describe a structure which is associative up to homotopy: the higher associator $\sum(-1)^{rs+t} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t})$ is a boundary in $\mathrm{Hom}(A^{\otimes n}, A)$. In particular, the second Stasheff identity says $\partial(m_2) = 0$ and the third says that $m_2(m_2 \otimes 1 - 1 \otimes m_2) = \partial(m_3)$, therefore $\mathrm{H}_*(A)$ is canonically an associative algebra. The sequence of operations can be thought of as a coherent reason for $\mathrm{H}_*(A)$ to be associative. Even when A is minimal and no reason seems necessary, the operations can witness associativity in a nontrivial way, carrying important information.

Note that differential graded algebras are exactly A_∞-algebras for which $m_n = 0$ when $n \geq 3$.

An augmented A_∞ algebra possesses exactly enough structure to build a bar construction. The underlying graded coalgebra of BA is $\mathsf{T}^{\mathrm{co}}(\Sigma\overline{A})$, with $B_w A = (\Sigma\overline{A})^{\otimes w}$, just as in section 2.2. And as in that section $b_1 = -\Sigma\partial\Sigma^{-1}$ is the differential naturally induced on $B_1 A$, while $b_n : B_n A \to B_1 A$ is defined by the commutativity of the following diagram

$$\begin{tikzcd}
\overline{A}^{\otimes n} \arrow[r, "\overline{m}_n"] \arrow[d, "\Sigma^{\otimes n}"'] & \overline{A} \arrow[d, "\Sigma"] \\
B_n A \arrow[r, "(-1)^n b_n"] & B_1 A.
\end{tikzcd}$$

By lemma 2 there is a unique coderivation $b : BA \to BA$ such that $\pi_1 b = b_1\pi_1 + b_2\pi_2 + b_3\pi_3 + ...$ The condition $b^2 = 0$ is equivalent to the Stasheff identities. Thus the bar construction BA is dg coalgebra. We think of it as being Koszul dual to A. In particular the linear dual is a dg algebra and we may write $A^! = (BA)^\vee$. Through this construction augmented A_∞ algebras are essentially the same thing as cofree cocomplete augmented dg coalgebras[3].

An A_∞ coalgebra C may be defined dually in terms of higher coproducts $\Delta_n : C \to C^{\otimes n}$ satisfying dual Stasheff relations. We skip this discussion and refer to [90] for details.

Alternatively, an augmented A_∞ coalgebra structure on $C = \overline{C} + k$ can be specified by an augmented differential d on the tensor algebra $\mathsf{T}(\Sigma^{-1}\overline{C})$. Tautologically, $\mathsf{T}(\Sigma^{-1}\overline{C})$ equipped with this differential is the cobar construction ΩC. Again, minimality means that the differential of C vanishes. Equivalently, d strictly increases weight. A dg coalgebra is then an A_∞ coalgebra such that the differential of ΩC increases weight by no more than one.

The important assumption that C is cocomplete translates to the assumption that ΩC should be semi-free: C must admit an exhaustive filtration $0 = \overline{C}_{(0)} \subseteq \overline{C}_{(1)} \subseteq ... \subseteq \overline{C}$ such that $d(\Sigma^{-1}\overline{C}_{(i)}) \subseteq$

[3] It is not possible to simply drop the word "augmented" here. Removing this assumption from one side introduces curvature on the other. This was discovered by Positselski [103].

$\mathsf{T}(\Sigma^{-1}\overline{C}_{(i-1)})$ for all i. For dg coalgebras this notion of cocompleteness is equivalent to the usual one.

A_∞ coalgebras are often produced using a noncommutative version of the Tate construction (the commutative version will be discussed in detail in section 2.6). In short, given a graded algebra A of the form $\mathsf{T}(V)/(R)$ with $R \subseteq \mathsf{T}^{\geq 2}(V)$, it is possible to inductively construct a semi-free resolution $\mathsf{T}(W) \xrightarrow{\simeq} A$ with some differential d which presents $\mathsf{T}(W)$ as the cobar construction of a cocomplete A_∞ coalgebra $C = k + \Sigma W$. Some details on this construction can be found in [25].

Having a general enough theory of twisting cochains (as developed in [90]) allows one to assert that the composition $C \to \Omega C \to A$ is an acyclic twisting cochain. The dual $C^\vee$ is an A_∞ algebra and we may write $A^! = C^\vee$. To explain why it is acceptable to write this last equality we at least need to know what is a morphism of A_∞ algebras.

An A_∞ morphism $\phi : A \to A'$ consists of a sequence of maps $\phi_n : A^{\otimes n} \to A'$ for $n \geq 1$ having degree $1 - n$ respectively. They must satisfy the quadratic identities

$$\sum_{r+s+t=n} (-1)^{rs+t} \phi_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) \; - \sum_{i_1+\ldots+i_k=n} (-1)^u m_k(\phi_{i_1} \otimes \ldots \otimes \phi_{i_k}) = \partial(\phi_n)$$

where $u = (k-1)(i_1-1) + (k-2)(i_2-1) + \cdots + 2(i_{k-2}-1) + (i_{k-1}-1)$. We also impose the strict unitality conditions $\phi_n(-,...,-,1,-,...,-) = 0$ for $n \geq 2$ and $\phi_1(1) = 1$. When A and A' are augmented, we say that ϕ is augmented if it takes each $\overline{A}^{\otimes n}$ into $\overline{A'}$.

The first identity asserts that ϕ_1 is a chain map. The second asserts that $\phi_1 m_2 - m_2(\phi_1 \otimes \phi_1) = \partial(\phi_2)$. Therefore ϕ induces a homomorphism $\mathrm{H}_*(\phi) : \mathrm{H}_*(A) \to \mathrm{H}_*(A')$ of associative algebras. The higher components should be thought of as a coherent reason for this to be the case. We say that ϕ is *strict* if $\phi_n = 0$ for $n \geq 2$.

We compose A_∞ morphisms $\phi : A \to A'$ and $\psi : A' \to A''$ according to the rule

$$(\psi\phi)_n = \sum_{i_1+\ldots+i_k=n} (-1)^u \psi_k(\phi_{i_1} \otimes \ldots \otimes \phi_{i_k})$$

with the same sign u as above. It is not difficult to check that ϕ is an isomorphism (that is, it admits a two-sided inverse) if and only if ϕ_1 is an isomorphism of complexes. We say that ϕ is a *quasi-isomorphism* if $\mathrm{H}_*(\phi)$ is an isomorphism.

Let ϕ be an augmented morphism of augmented A_∞ algebras. By cofreeness the shifted maps $\Sigma\phi_n(\Sigma^{-1})^{\otimes n} : B_n A \to B_1 A'$ assemble uniquely into a coalgebra morphism $B(\phi) : BA \to BA'$. Being careful of signs, one can check that the identities defining an A_∞ morphism are equivalent to the assertion that $B(\phi)$ is a chain map. Thus through the bar construction the category of augmented A_∞ algebras is equivalent to the category of cofree cocomplete dg coalgebras. Strict A_∞ morphisms correspond to coalgebra morphisms which preserve weight.

The bar-cobar formalism easily allows one to show that every A_∞ algebra is quasi-isomorphic to a dg algebra, and that any quasi-isomorphism of A_∞ algebras can be transferred to strict quasi-isomorphism between the corresponding dg algebras (there is a canonical quasi-isomorphism $A \to \Omega BA$ of A_∞ algebras). Thus A_∞ algebras are homotopically no more general than dg algebras. They simply provide a more flexible presentation for the same homotopical information carried by dg algebras. This perspective is especially compelling with the following important theorem of Kadeishvilli in mind.

Theorem 8 (Kadeishivili's Minimal Model Theorem [73])**.** *Let A be an augmented A_∞ algebra. The associative algebra $\mathrm{H}_*(A)$ admits the structure of a minimal augmented A_∞ algebra extending its natural* *product, in such a way that there is a quasi-isomorphism*

$$\mathrm{H}_*(A) \xrightarrow{\simeq} A$$

which induces the identity map in homology. Moreover, this structure is unique up to isomorphism of A_∞ algebras.

One caveat is that an isomorphism of A_∞ algebras can be a relatively complicated object.

The theorem is extremely important even when applied to a dg algebra. Starting with a graded algebra A, it produces a minimal A_∞ algebra quasi-isomorphic to the Koszul dual $(BA)^\vee$. Thus there is a natural A_∞ structure on $\mathrm{Ext}^*_A(k,k)$, and we call this the *minimal model* for $A^!$. When A is strongly connected (using the same definition as in section 2.3) a similar argument to the proof of theorem 5 produces a canonical quasi-isomorphism $(B\mathrm{Ext}^*_A(k,k))^\vee \xrightarrow{\simeq} A$. Thus from the A_∞ algebra $\mathrm{Ext}^*_A(k,k)$ we can functorially recover A.

If we build a minimal non-commutative Tate model $\Omega C \xrightarrow{\simeq} A$ as sketched above, then uniqueness of minimal semi-free dg algebras allows us the make the identification $\Omega C \cong (B\mathrm{Ext}^*_A(k,k))^\vee$ and conclude that there is an isomorphism $C^\vee \cong \mathrm{Ext}^*_A(k,k)$ of A_∞ algebras.

We end this section with a very general example showing how to compute a part of the natural A_∞ structure on $\mathrm{Ext}^*_A(k,k)$. If $A = \mathsf{T}(V)/(R)$ with $R \subseteq \mathsf{T}^{\geq 2}(V)$ as above, then examining the first two steps in the minimal non-commutative Tate construction readily yields the following theorem of Keller.

Theorem 9 (Keller [77, proposition 2])**.** *There are canonical isomorphisms* $\mathrm{Ext}^1_A(k,k) = \Sigma^{-1}V^\vee$ *and* $\mathrm{Ext}^2_A(k,k) = \Sigma^{-2}R^\vee$*, and for every n the following diagram commutes*

$$\begin{tikzcd}
\mathrm{Ext}^1_A(k,k)^{\otimes n} \arrow[r, "m_n"] \arrow[d, "\Sigma^{\otimes n}"'] & \mathrm{Ext}^2_A(k,k) \arrow[d, "\Sigma"] \\
(V^\vee)^{\otimes n} \arrow[r, "(-1)^n \iota_n^\vee"] & R_n^\vee.
\end{tikzcd}$$

In the statement R_n is the weight n part of R and $\iota_n : R_n \hookrightarrow V^{\otimes n}$ is the natural inclusion.

For algebras of global dimension 2 this completely describes the Koszul dual A_∞ algebra $\mathrm{Ext}^*_A(k,k)$.

2.5 Models for the Hochschild Cochain Complex

In this section we will discuss models for the Hochschild cochain complex in various settings, as well the structure that these models enjoy.

To begin with let A be an associative algebra (possibly graded or dg) over a commutative ring k. We denote the enveloping algebra by $A^{\mathsf{ev}} = A^{\mathsf{op}} \otimes_k A$. Bimodules over A are equivalent to left or right A^{ev} modules, and in particular A is naturally both a left and right A^{ev} module.

Hochschild homology and cohomology were originally defined in terms of the bar construction. When A is flat as a k module we may instead define Hochschild homology as

$$\mathrm{HH}_*(A/k, A) = \mathrm{Tor}^{A^{\mathsf{ev}}}_*(A, A).$$

When A is projective over k we may define Hochschild cohomology as

$$\mathrm{HH}^*(A/k, A) = \mathrm{Ext}^*_{A^{\mathsf{ev}}}(A, A).$$

We will use these definitions briefly in section 2.6 (example 3 and remark 2). However, in the rest of this section we will discuss much more structural models for Hochschild cohomology which will be important in chapter 4.

We continue with the conventions of the last few sections, so k is once again a field and everything is assumed to be k linear. From this point on we elide k from our notation for Hochschild cohomology.

Let $\tau : C \to A$ be a twisting cochain from an augmented cocomplete dg coalgebra to an augmented dg algebra. Recall that twisted tensor products and the twisted convolution algebra $\mathrm{Hom}^\tau(C, A)$ were defined in section 2.2. In that section we also observed that $A \otimes C \otimes A$ is a bimodule over $\mathrm{Hom}(C, A)$ using the so-called cap product. The twisted tensor product $A \otimes^\tau C \otimes^\tau A$ by definition has the differential

$$\partial_A \otimes 1 \otimes 1 + 1 \otimes \partial_C \otimes 1 + 1 \otimes 1 \otimes \partial_A + (\tau \frown -) - (- \frown \tau).$$

It is easy to check that this makes $A \otimes^\tau C \otimes^\tau A$ a dg bimodule over over $\mathrm{Hom}^\tau(C, A)$. It is also clear that this action commutes with the A bimodule structure. In particular, the left action gives us a homomorphism $\mathrm{Hom}^\tau(C, A) \to \mathrm{End}_{A^{\mathsf{ev}}}(A \otimes^\tau C \otimes^\tau A)$ of dg algebras.

At this point we can make an argument similar to the proof of theorem 4. There is a natural bimodule homomorphism $A \otimes^\tau C \otimes^\tau A \to A$ which gives rise to the second map below

$$\mathrm{Hom}^\tau(C, A) \to \mathrm{End}_{A^{\mathsf{ev}}}(A \otimes^\tau C \otimes^\tau A) \to \mathrm{Hom}_{A^{\mathsf{ev}}}(A \otimes^\tau C \otimes^\tau A, A).$$

Note that the composition from left to right is an isomorphism of complexes (this is clear on the level of vector spaces and the differentials are automatically compatible, no calculation necessary).

Finally, if we assume τ is acyclic then according to theorem 3 $A \otimes^\tau C \otimes^\tau A \to A$ is a quasi-isomorphism. Then $\mathrm{End}_{A^{\mathsf{ev}}}(A \otimes^\tau C \otimes^\tau A) \to \mathrm{Hom}_{A^{\mathsf{ev}}}(A \otimes^\tau C \otimes^\tau A, A)$ is a quasi-isomorphism as well. Cocompleteness of C is essential here because this guarantees that $A \otimes^\tau C \otimes^\tau A$ is semi-free as a dg bimodule. Thus, we have constructed a canonical dg algebra quasi-isomorphism

$$\mathrm{Hom}^\tau(C, A) \xrightarrow{\simeq} \mathrm{End}_{A^{\mathsf{ev}}}(A \otimes^\tau C \otimes^\tau A) = \mathrm{RHom}_{A^{\mathsf{ev}}}(A, A).$$

In particular, it is a consequence of theorem 3 that the Hochschild cohomology $\mathrm{HH}^*(A, A)$ can be computed as $\mathrm{H}_*(\mathrm{Hom}^\tau(C, A))$ using any acyclic twisting cochain into A[4]. This description of Hochschild cohomology is essentially in [90] or [85]. Negron has used this description to compute Hochschild cohomology in [102] (the author also deals more generally with non-augmented algebras and curved dg coalgebras). Similar remarks apply to Hochschild cohomology with coefficients in a bimodule; details on this can also be found in op. cit. This description of Hochschild cohomology has been used as well in topology by Berglund and Börjeson [29].

Example 2. An obvious application is the computation of Hochschild cohomology for Koszul algebras.

[4]Note that if we had used the right cap action of $\mathrm{Hom}^\tau(C, A)$ on $A \otimes^\tau C \otimes^\tau A$ then the same argument would have produced an anti-isomorphism $\mathrm{H}_*(\mathrm{Hom}^\tau(C, A)) \cong \mathrm{HH}^*(A, A)^{\mathsf{op}}$. Both induce the same map on cohomology since they are equalised by $\mathrm{End}_{A^{\mathsf{ev}}}(A \otimes^\tau C \otimes^\tau A) \to \mathrm{Hom}_{A^{\mathsf{ev}}}(A \otimes^\tau C \otimes^\tau A, A)$. So, we have a funny proof that Hochschild cohomology is commutative.

If $A = \mathsf{T}(V)/(R)$ is a Koszul algebra as in section 2.3 and $C = C(\varepsilon V, \varepsilon^2 R)$ then we obtain a small model $\mathrm{Hom}^{\tau}(C, R)$ for the Hochschild cochain complex of A using theorem 7.

For example, let $S = \mathrm{Sym}(V)$ (assumed strongly connected to fit into the framework of section 2.3, otherwise we should use the completion of S) and $\Gamma = \mathrm{Sym}^{\mathrm{co}}(\varepsilon V)$ be as in example 1. Then $\tau : \Gamma \to \varepsilon V \to \varepsilon S$ is an acyclic twisting cochain. Since S and Γ are commutative and cocommutative (this is extremely rare, it can essentially only occur in this example) the twist on $\mathrm{Hom}^{\tau}(\Gamma, S) = \mathrm{Hom}(\Gamma, S)$ vanishes. We obtain a graded version of the Hochschild-Kostant-Rosenburg theorem $\mathrm{HH}^*(S, S) \cong \mathrm{Hom}(\Gamma, S)$. The original version of the Hochschild-Kostant-Rosenburg theorem for smooth algebras will appear in remark 2 below.

Let A be an augmented dg algebra. The universal twisting cochain $\pi : BA \to A$ is acyclic according to theorem 3, and we set

$$C^*(A, A) = \mathrm{Hom}^{\pi}(BA, A).$$

This is exactly the classical reduced Hochschild cochain complex: it has a weight decomposition $C^w(A, A) \cong \mathrm{Hom}(\overline{A}^{\otimes w}, A)$ and unraveling the definitions recovers Hochschild's original differential. The Hochschild cohomology of A is by definition $\mathrm{HH}^*(A, A) = \mathrm{H}_*(C^*(A, A))$. The convolution product is exactly the classical cup product. We will discuss shortly how to see Gerstenhaber's Lie bracket in terms of the bar construction.

The Hochschild cochain complex has a natural decreasing filtration which we call called the *weight filtration*

$$C^{(w)}(A, A) = \mathrm{Hom}^{\pi}(B_{\geq w} A, A).$$

Note that this filtration is complete since BA is cocomplete. The induced weight filtration on Hochschild cohomology is denoted $\mathrm{HH}^{(n)}(A, A)$.

Hochschild Cohomology for A_∞ Algebras

Hochschild cohomology for A_∞ algebras can be defined using an A_∞ analogue of the twisted convolution algebra. This construction is essentially contained in [85, chapitre 8], but we follow the approach of [34, section 2.3], where more detail can be found.

Let C be an augmented cocomplete coalgebra and A an augmented A_∞ algebra. The complex $\mathrm{Hom}(C, A)$ inherits a sequence of convolution operations

$$M_n(f_1, ..., f_n) = m_n(f_1 \otimes ... \otimes f_n)\Delta^{(n)}.$$

One verifies as in [85, lemma 8.1.1.4] that these operations make $\mathrm{Hom}(C, A)$ an A_∞ algebra. Moreover, the augmentations of C and A canonically make $\mathrm{Hom}(C, A)$ an augmented A_∞ algebra.

A twisting cochain $C \to A$ is a degree -1 map $\tau : C \to A$ which satisfies the higher Maurer-Cartan equation

$$\partial(\tau) + \sum_{n \geq 2} M_n(\tau, ..., \tau) = 0$$

The sum makes sense since C is cocomplete. We say that τ is augmented if it vanishes on both augmentations, exactly as in section 2.2.

By cofreeness of the bar construction there is an augmented coalgebra homomorphism $\phi_\tau : C \to BA$ lifting τ. By construction the higher Maurer-Cartan equation holds if and only if ϕ_τ respects the differentials of these coalgebras. It follows that there is a natural bijection

$$\mathbf{tw}(C, A) \cong \mathbf{coAlg}(C, BA)$$

where $\mathbf{tw}(C, A)$ is the set of augmented twisting cochains from C to A. In particular the projection $\pi : BA \to A$ is an augmented twisting cochain, and all other augmented twisting cochains into A factor uniquely through π.

From an augmented twisting cochain $\tau : C \to A$ one can form the twisted convolution A_∞ algebra $\mathrm{Hom}^\tau(C, A)$. The reader should be warned that all of the operations on $\mathrm{Hom}(C, A)$ should be twisted to build this structure, not only the differential. This is done in [85, chapitre 8] using twists of topological A_∞ algebras. A simpler construction is given in [34, section 2.3].

For simplicity we present here only the formula which shows how to twist the differential. We use the *higher commutators* $[-;-]_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A$. These are the degree $2-(p+q)$ maps given by

$$[v_1, ..., v_p; v_{p+1}, ..., v_{p+q}]_{p,q} = \sum_{\sigma \in \mathrm{sh}(p,q)} (-1)^{|\sigma|}(-1)^{|\sigma;v|} m_{p+q}(v_{\sigma^{-1}(1)} \otimes ... \otimes v_{\sigma^{-1}(p+q)})$$

where $\mathrm{sh}(p, q)$ is the set of permutations in S_{p+q} which separately preserve the order of $\{1, ..., p\}$ and of $\{p + 1, ..., p + q\}$. Note that $[-;-]_{1,1}$ is the usual commutator for m_2. Note also that $[-;-]_{p,q}\mathtt{t} = (-1)^{pq}[-;-]_{q,p}$, where $\mathtt{t}$ is the natural isomorphism $A^{\otimes p} \otimes A^{\otimes q} \to A^{\otimes q} \otimes A^{\otimes p}$ which acts according to the Koszul sign rule.

The twisted differential on $\mathrm{Hom}(C, A)$ is defined using higher commutators for the convolution operations. It is given by

$$\partial^\tau = \partial + \sum_{n \geq 1} [\tau, ..., \tau; -]_{n,1}.$$

Unraveling this to an explicit formula yields

$$\partial(f) + \sum_{i,j} (-1)^{(n+1)j} m_{i+1+j}(\tau^{\otimes i} \otimes f \otimes \tau^{\otimes j})\Delta^{(i+1+j)}$$

for the boundary of an element f in $\mathrm{Hom}^\tau(C, A)_n$. The verification that $(\partial^\tau)^2 = 0$ is contained in [34, section 2.3]. It is easiest to first connect this definition the the coderivation model introduced below.

Finally, the Hochschild cochain complex of A is $C^*(A, A) = \mathrm{Hom}^\pi(BA, A)$. Its homology is the Hochschild cohomology $\mathrm{HH}^*(A, A)$. In this context we also have a complete weight filtration with the same definition

$$C^{(w)}(A, A) = \mathrm{Hom}^\pi(B_{\geq w}A, A).$$

The induced weight filtration on Hochschild cohomology is denoted $\mathrm{HH}^{(n)}(A, A)$ as before. A construction of the natural A_∞ algebra structure on $C^*(A, A)$ can be found in [34, section 2.3].

The Gerstenhaber Lie algebra

In his fundamental investigations into the deformation theory of associative algebras [52], Gerstenhaber introduced a degree 1 Lie bracket on $\mathrm{HH}^*(A, A)$. This structure together with the cup product (and their compatibility) is now summed up by saying that $\mathrm{HH}^*(A, A)$ is a *Gerstenhaber algebra*.

In this section we discuss another description of the twisted hom complex which makes the definition of Gerstenhaber's bracket transparent. This description goes back to Getzler and Jones [54] and Stasheff [114].

A morphism $\xi : C' \to C$ of dg coalgebras naturally makes C' into a bicomodule over C, so we may consider the space $\mathrm{coDer}(C', C)$ of coderivations from C' into C, which was introduced in section 1.2. This is the space of linear maps $C' \to C$ which satisfy the (dual) Leibniz identity $\Delta q = (\xi \otimes q + q \otimes \xi)\Delta$. It has a natural differential as a sub-complex of $\mathrm{Hom}(C', C)$.

Now let $\tau : C \to A$ be an augmented twisting cochain from an augmented cocomplete dg coalgebra to an augmented A_∞ algebra, and let $\phi = \phi_\tau$ be its lift to a coalgebra morphism $C \to BA$.

By lemma 2, there is a canonical isomorphism of graded vector spaces

$$\mathrm{Hom}(C, A) \cong \Sigma^{-1}\mathrm{Hom}(C, \Sigma\overline{A}) \oplus C^\vee \cong \Sigma^{-1}\mathrm{coDer}(C, BA) \oplus C^\vee.$$

At the same time, there is a natural chain map $\mathrm{ad} : C^\vee \to \mathrm{coDer}(C, BA)$ which sends $f : C \to k$ to the inner coderivation

$$\mathrm{ad}(f) : C \xrightarrow{\Delta} C \otimes C \xrightarrow{\phi\otimes f - f\otimes\phi} BA.$$

The cone of this map is denoted

$$\mathrm{cone}(\mathrm{ad}) = \Sigma^{-1}\mathrm{coDer}(C, BA) \rtimes C^\vee.$$

As promised above, the verification that $(\partial^\tau)^2 = 0$ is achieved as follows.

Proposition 1. *The isomorphism* $\mathrm{Hom}^\tau(C, A) \cong \Sigma^{-1}\mathrm{coDer}(C, BA) \rtimes C^*$ *is one of complexes, and in particular* $(\partial^\tau)^2 = 0$*.*

This is essentially due to Getzler and Jones, and in the dg situation Stasheff [114]. The computation can be found for instance in [34, proposition 2.23].

In the case of the identity homomorphism $C \to C$ the complex $\mathrm{coDer}(C, C)$ is naturally a dg Lie algebra. The bracket is the graded commutator $[p, q] = pq - (-1)^{|p||q|}qp$, making $\mathrm{coDer}(C, C)$ a sub Lie algebra of the associated Lie algebra $\mathrm{End}(C)^{\mathsf{Lie}}$.[5]

The natural action of $\mathrm{coDer}(C, C)$ on C makes $\mathrm{coDer}(C, C) \rtimes \Sigma C^\vee$ a dg Lie algebra as follows. The bracket on $\mathrm{coDer}(C, C)$ has just been defined, while if $f, g \in C^\vee$ and $q \in \mathrm{coDer}(C, C)$ then

$$[\Sigma f, q] = -(-1)^{|q|(|f|+1)}[q, \Sigma f] = \Sigma(fq) \quad \text{and} \quad [\Sigma f, \Sigma g] = 0.$$

In particular, if A is an augmented A_∞ algebra then

$$\Sigma C^*(A, A) = \Sigma\mathrm{Hom}^\pi(BA, A) \cong \mathrm{coDer}(BA, BA) \rtimes \Sigma(BA)^\vee$$

is naturally a dg Lie algebra.

Proposition 2. *This bracket on* $\Sigma C^*(A, A)$ *canonically gives* $\mathrm{HH}^*(A, A)$ *the structure of a Gerstenhaber algebra. It coincides with the classical one introduced by Gerstenhaber.*

[5] This is a graded Lie algebra (with compatible differential) in the full sense of section 2.8. The reduced square is given by $q^{[2]} = q^2$ on odd elements.

The appropriate chain level statement is that $C^*(A,A)$ is a B_∞ algebra (also known a brace algebra, or E_2 algebra, or G_∞ algebra), this is a version of the so-called Deligne conjecture (now proven by a number of authors [115, 96, 119]). For information on the B_∞ structure of $C^*(A,A)$ see [79]. The first statement of the proposition is a consequence of this fact. The second statement is a calculation contained in [54].

Finally, we remark that there is also a dual picture for the Hochschild cohain complex in terms of derivations out of the cobar construction. We refer to [34, section 2.3] for details.

Naturality of the Hocschild Cochain Complex

Naturality for twisting cochains extends readily to the A_∞ situation. Let $\tau : C \to A$ be an augmented twisting cochain. Any dg coalgebra morphism $\xi : C' \to C$ yields a twisting cochain $\tau\xi : C' \to A$ in the obvious way, while a morphism $\psi : A \to A'$ of A_∞ algebras yields a twisting cochain $\psi\tau : C \to A'$ by composing

$$C \xrightarrow{\phi_\tau} BA \xrightarrow{B(\psi)} BA' \xrightarrow{\pi} A'.$$

Now suppose that $\phi : A \to A'$ and $\psi : A' \to A''$ is a sequence of A_∞ morphisms. Denote the corresponding twisting cochains by $\tau = \pi B(\phi)$ and $\sigma = \pi B(\psi)$, and note that $\psi\tau = \sigma B(\phi)$. We obtain two maps

$$\mathrm{Hom}^\tau(BA, A') \xrightarrow{\psi_*} \mathrm{Hom}^{\psi\tau}(BA, A'') \quad \text{and} \quad \mathrm{Hom}^\sigma(BA', A'') \xrightarrow{\phi^*} \mathrm{Hom}^{\sigma B(\phi)}(BA, A'')$$

defined as follows. Firstly $\phi^* = \mathrm{Hom}(\phi, A')$ is the obvious map. We define ψ_* in terms of coderivations:

$$\Sigma^{-1}\mathrm{coDer}(BA, BA') \rtimes (BA)^\vee \xrightarrow{\psi_*} \Sigma^{-1}\mathrm{coDer}(BA, BA'') \rtimes (BA)^\vee$$

is given by $\psi_* = \Sigma^{-1}\mathrm{coDer}(BA, B(\psi)) \rtimes (BA)^\vee$. We state a generalisation of lemma 4.

Proposition 3. *Both ϕ^* and ψ_* are chain maps. Furthermore, ϕ^* and ψ_* are quasi-isomorphisms whenever ϕ and ψ are respectively.*

The proof comes down to another application of lemma 1, the key point being that the twisted part of the differential on $\mathrm{Hom}^\tau(BA, A')$ decreases the weight filtration, and that $\mathrm{gr}(\phi) : \mathrm{gr}(BA) \to \mathrm{gr}(BA')$ is a quasi-isomorphism whenever ϕ is by the Künneth theorem. Details can be found in [34, proposition 2.27].

Suppose that $\phi : A \to A'$ is a quasi-isomorphism of A_∞-algebras corresponding to a twisting cochain $\tau = \pi B(\phi)$. An immediate consequence of the proposition is that we have a canonical chain of quasi-isomorphisms

$$C^*(A', A') \xrightarrow{\phi^*} \mathrm{Hom}^\tau(A, A') \xleftarrow{\phi_*} C^*(A, A).$$

And in particular a canonical isomorphism $\mathrm{HH}^*(A', A') \cong \mathrm{HH}^*(A, A)$.

If ϕ and ψ are strict morphisms of dg algebras then ϕ^* and ψ_* are easily seen to be morphisms of dg algebras. In general, the twisted convolution algebras inherit A_∞ structures and ϕ^* is a strict morphism of A_∞ algebras, while ψ_* extends canonically to a possibly non-strict morphism of A_∞ algebras (see [34, section 2.3]).

The Projection and Shearing Morphisms

On the Hochschild cohomology of any dg or A_∞-algebra A we have two well known maps

$$\Pi : \mathrm{HH}^*(A,A) \to \mathrm{H}(A) \quad \text{ and } \quad \chi : \mathrm{HH}^*(A,A) \to \mathrm{H}(A^!) = \mathrm{Ext}_A(k,k)$$

which we call the *projection* and *shearing* morphisms respectively. On the chain level the definitions are very simple: $\Pi : \mathrm{Hom}^\pi(BA, A) \to \mathrm{Hom}(k, A) = A$ is defined by composing with the coaugmentation $k \to BA$ while $\chi : \mathrm{Hom}^\pi(BA, A) \to \mathrm{Hom}(BA, k) = A^!$ is defined by composing with the augmentation $A \to k$.

For ordinary graded algebras, the image of the projection map $\mathrm{HH}^*(A,A) \to A$ is exactly the graded centre $\mathsf{Z}(A)$. Section 4.2 will be devoted to discussing the meaning of the image in general. Shamir [110] discusses the shearing map in detail.

Note that the projection morphism is the quotient by the positive part of the weight filtration. In [34, sections 2.3 and 3.4] it is explained that Hochschild cohomogy naturally has another filtration which is in some sense Koszul dual to the weight filtration. We call this the *shearing filtration* because the shearing morphism is the quotient by the positive part of the shearing filtration.

As a special case of the naturality from proposition 3, we get naturality of the projection and shearing maps for quasi-isomorphisms of A_∞-algebras. That is, if $\phi : A \to A'$ is a quasi-isomorphism of A_∞-algebras, then we have a diagram

$$\begin{tikzcd} (BA)^\vee & C^*(A,A) \arrow[l, "\chi"'] \arrow[r, "\Pi"] \arrow[d] & A \arrow[dd, "\simeq"', "\phi"] \\ & \bullet \arrow[u, phantom, "\simeq" left] & \\ (BA')^\vee \arrow[uu, "B(\phi)^\vee", "\simeq"'] & C^*(A',A') \arrow[l, "\chi"'] \arrow[r, "\Pi"] \arrow[u] & A' \end{tikzcd} \tag{2.1}$$

which commutes after taking homology.

In terms of coderivations the shearing map is the obvious quotient

$$C^*(A,A) = s^{-1}\mathrm{coDer}(BA, BA) \rtimes (BA)^\vee \to (BA)^\vee,$$

and there is a similar interpretation of the projection map in terms of derivations.

Hochschild Cohomology for (Non-Augmented) Categories and Functoriality

Hochschild cohomology is also defined for small dg and A_∞ categories (this is sometimes called Hochschild-Mitchell cohomology after [100]). If A is augmented, the definition $C^*(A,A) = \mathrm{Hom}^\pi(BA, A)$ continues to make perfect sense. In this situation the bar construction BA is cofree as a cocomplete cocategory with the same objects as A. The composition gives rise to a differential on BA and the contents of this section so far above generalises verbatim to this situation. Rather than explain this properly we refer the reader to [80].

However, when working with categories the non-augmented situation will often arise by force, since module categories are never augmented. So, we collect here some definitions for later use, restricting to the dg case for simplicity.

Fix a small dg category A, and consider its unreduced bar construction

$$B_{\mathsf{unr}}A = \mathsf{T}^{\mathsf{co}}(\Sigma A) = \bigoplus_{w\geq 0}(\Sigma A)^{\otimes w} = \bigoplus_{\substack{w\geq 0\\ x_0,\dots,x_w}} \Sigma({}_{x_w}A_{x_{w-1}})\otimes\cdots\otimes\Sigma({}_{x_1}A_{x_0}),$$

the cofree graded cocategory on the graded quiver ΣA. The obvious "bar" dg structure on $B_{\mathsf{unr}}A$ is problematic (it is contractible), but we can still make sense of the differential on the A-bimodule $A\otimes^{\pi} B_{\mathsf{unr}}A\otimes^{\pi} A$. We will be content cheating and simply defining it by the following formula: the boundary $\partial^{\pi}(f)$ of an element $f = g\otimes[f_1|...|f_w]\otimes h$ is the sum

$$\begin{aligned}\partial(f) \quad &+ \quad (-1)^{|g|+1} g f_1\otimes[f_2|...|f_w]\otimes h\\ &+ \quad \sum(-1)^{\epsilon_i+|f_i|} g\otimes[f_1|...|f_if_{i+1}|...|f_w]\otimes h\\ &+ \quad (-1)^{\epsilon_n} g\otimes[f_1|...|f_{w-1}]\otimes f_w h\end{aligned}$$

where ∂ is the usual tensor product differential on $A\otimes\mathsf{T}^{\mathsf{co}}(\Sigma A)\otimes A$, and where $\epsilon_i = (i-1)+|f_1|+...+|f_{i-1}|$. The A-bilinear map $A\otimes^{\pi} B_{\mathsf{unr}}A\otimes^{\pi} A \xrightarrow{\sim} A$ is a quasi-isomorphism.

As a graded vector space the unreduced Hochschild cochain complex $C^*_{\mathsf{unr}}(A,A)$ is

$$\mathrm{Hom}(B_{\mathsf{unr}}A, A) = \prod_{\substack{w\geq 0\\ x_0,\dots,x_w}} \mathrm{Hom}\big(\Sigma({}_{x_w}A_{x_{w-1}})\otimes\cdots\otimes\Sigma({}_{x_1}A_{x_0}), {}_{x_w}A_{x_0}\big).$$

The differential may be defined through the identification $C^*_{\mathsf{unr}}(A,A)\cong \mathrm{Hom}_{A-A}(A\otimes^{\pi} B_{\mathsf{unr}}A\otimes^{\pi} A, A)$. It is given by the classical formula of Hochschild: the boundary $\partial^{\pi}(\phi)$ of $\phi\in C^*_{\mathsf{unr}}(A,A)_n$ is determined by setting $\partial^{\pi}(\phi)([f_1|...|f_w])$ equal to

$$\begin{aligned}\partial(\phi)([f_1|...|f_w]) \quad &+ \quad (-1)^{|f_1|n+1} f_1\phi([f_2|...|f_w])\\ &+ \quad \sum(-1)^{\epsilon_i+|f_i|}\phi([f_1|...|f_if_{i+1}|...|f_w])\\ &+ \quad (-1)^{\epsilon_n}\phi([f_1|...|f_{w-1}])f_w.\end{aligned}$$

And the Hochschild cohomology of A is by definition $\mathrm{HH}^*(A,A) = \mathrm{H}_*(C^*_{\mathsf{unr}}(A,A))$. If A is augmented then the canonical restriction $C^*_{\mathsf{unr}}(A,A)\to C^*(A,A)$ would be a quasi-isomorphism, so this is consistent with our previous definitions.

As Keller remarks in [79] if $A\to A'$ is a fully faithful embedding of dg categories then one has a restriction morphism $C^*_{\mathsf{unr}}(A',A')\to C^*_{\mathsf{unr}}(A,A)$ by forgetting those objects which do not come from A. This fact is visible from the above formulas. In particular there is a canonical homomorphism $\mathrm{HH}^*(A',A')\to \mathrm{HH}^*(A,A)$. This functoriality will be extremely useful to us later in chapter 4.

2.6 Semi-Free Extensions and Acyclic closures

This section contains background of semi-free dg algebras and semi-free divided power algebras. Much more detail can be found in [12].

Let A be a strictly graded commutative ring, and let X be a graded set. The free strictly graded commutative A algebra on X is denoted $A[X]$, see [12, section 2.1] for an explicit description. To emphasise a different point of view we may also use the notation $\mathrm{Sym}_A(X)$, or $\mathrm{Sym}(X)$ when A is understood.

If A is a dg algebra and $z \in A_i$ is a cycle and x is a variable of degree $i+1$, then $A[x]$ has a unique dg algebra structure for which $A \to A[x]$ is a morphism of dg algebras and $\partial(x) = z$. We may write $A[x] = A[x : \partial(x) = z]$ to be explicit.

The morphism $A \to A[x]$ is an example of a *semi-free extension.* In general, we define a semi-free extension to be any morphism $A \to A[X]$ obtained by repeatedly adjoining variables in this way. That is, X should be a graded set with an increasing filtration $\varnothing = X_{(0)} \subseteq X_{(1)} \subseteq \cdots$ and $A[X] = \bigcup_n A[X_{(n)}]$ with $\partial(X_{(n)}) \subseteq A[X_{(n-1)}]$. In this thesis we work always with positively graded dg algebras, so we can and will take $X_{(n)} = X_{\leq n}$ without comment. In the dual, cohomological situation encountered in rational homotopy theory this doesn't work, and the filtration on X should not be neglected. The following lifting property for semi-free extensions is fundamental. It is proven by induction on the filtration $X_{\leq n}$ as in [12, proposition 2.1.9].

Lemma 5. *Let $A \to A[X]$ be a semi-free extension. If $\phi : B \to C$ is a surjective quasi-isomorphism of A algebras, then any morphism of A algebras $A[X] \to B$ lifts along ϕ.*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, two heads, "\simeq"] \\ A[X] \arrow[r] \arrow[ur, dashed] & C \end{tikzcd}$$

That is, a morphism exists as above, making the diagram commute.

Now we assume A is a local dg algebra. A semi-free extension $A \to A[X]$ is called minimal if it is minimal as a homomorphism of local dg algebras, which means $\partial(\mathfrak{m}_{A[X]}) \subseteq \mathfrak{m}_A + \mathfrak{m}^2_{A[X]}$.

Proposition 4. *Any surjection $A \to B$ of local dg algebras can be factored as $A \to A[X] \to B$, where $A \to A[X]$ is a minimal semi-free extension and $A[X] \to B$ is a surjective quasi-isomorphism. Any two such factorisations are isomorphic. The set X is degree-wise finite and concentrated in positive degrees.*

Such a factorisation is called a *minimal model* for the surjection $A \to B$. We will discuss in the next section how Cohen factorisations are used to deal with the non-surjective case. The first and last sentences of the proposition can be found in [12, proposition 2.1.10]. By lemma 5 any two minimal models are connected by a quasi-isomorphism $A[X] \to A[X']$ (compatible with A and B). By Nakayama's lemma it suffices to show that $k[X] \to k[X']$ is an isomorphism, and this is dealt with by [12, lemma 7.2.3].

Divided Powers

Now we discuss divided power structures. Let A be a strictly graded commutative algebra with an ideal I. A divided power structure on I is a collection of functions defined on even elements of I

$$x^{(0)} = 1 \quad \text{and} \quad x \mapsto x^{(n)} : I_{2i} \to I_{2ni} \quad \text{for } x \geq 1,$$

these functions should satisfy the well-known axioms which can be found for instance in [64, section 1.7]. We will call (A, I) a *divided power algebra.* A homomorphism of divided power algebras $\phi : (A, I) \to (A', I')$ should satisfy $\phi(I) \subseteq I'$ and $\phi(x^{(n)}) = \phi(x)^{(n)}$.

If A contains the field of rational numbers then any proper ideal supports a unique divided power structure given by $x^{(n)} = \frac{1}{n!}x^n$.

A key example is the free divided power algebra on a variable u of degree i. Let A be any strictly graded commutative algebra. If i is odd then $A\langle u\rangle = A[u]$ is the usual exterior algebra $A + uA$ with $u^2 = 0$. If i is even then by definition $A\langle u\rangle = A + uA + u^{(2)}A + ...$ where each $u^{(n)}$ formally generates a rank one free module with $|u^{(n)}| = ni$. Multiplication is given by the rule

$$u^{(m)}u^{(n)} = \tbinom{m+n}{m}u^{(m+n)}$$

where $\binom{m+n}{m} = \frac{(m+n)!}{m!n!}$. Alternatively one can define $\mathbb{Z}\langle u\rangle$ to be the $\mathbb{Z}$ subalgebra of $\mathbb{Q}[u]$ generated by $u^n/n!$ for $n \geq 0$, and then set $A\langle u\rangle = A \otimes_{\mathbb{Z}} \mathbb{Z}\langle u\rangle$.

More generally, if U is a degree-wise finite non-negatively graded set then by definition $A\langle U_{\leq n}\rangle$ is the tensor product $\bigotimes_A A\langle u\rangle$ over $u \in U_{\leq n}$, and $A\langle U\rangle = \bigcup_n A\langle U_{\leq n}\rangle$ along the natural embeddings $A\langle U_{\leq n}\rangle \hookrightarrow A\langle U_{\leq n+1}\rangle$. An alternative notation for $A\langle U\rangle$ is $\Gamma_A(U)$, or $\Gamma(U)$ when A is understood.

Throughout this thesis we will use the notation $(U)^{(n)}$ for the ideal in $A\langle U\rangle$ generated by all monomials $u_1^{(n_1)} \cdots u_k^{(n_k)}$ with $n_1 + ... + n_k \geq n$ and $u_i \in U$.

Lemma 6. *The graded algebra $A\langle U\rangle$ has a unique divided power structure on $(U)^{(1)}$ extending the functions $u \mapsto u^{(n)}$ for $u \in U$.*

This is more general than [64, proposition 1.7.6], but the proof there can be adapted to work in this situation. The free divided power algebra $A\langle U\rangle$ is characterised by the following important property.

Lemma 7. *Let $A \to B$ be a homomorphism of strictly graded commutative algebras, and suppose that B has a divided power structure on an ideal I. If U is a degree-wise finite non-negatively graded set then any graded function $U \to I$ extends uniquely to a homomorphism of divided power algebras $(A\langle U\rangle, (U)^{(1)}) \to (B, I)$*

Again this is more general than [64, lemma 1.7.8] but the proof is similar.

Now we introduce some notation which will be useful later. Let B be a dg algebra with a divided power structure on a semi-ideal I. The indecomposables of B (with respect to this structure) are denoted $\mathrm{ind}^{\gamma}(B) = \mathrm{ind}^{\gamma}(B, I) = \mathfrak{m}_B/(\mathfrak{m}_B^2 + I^{(2)})$.

A homomorphism $A \to A[X]\langle U\rangle$ will be called a mixed free extension. We can combine lemma 7 and the usual universal property for free extensions to obtain a universal property for mixed extensions. It says that homomorphisms from $A[X]\langle U\rangle$ to an A algebra B with divided powers on I correspond bijectively to graded functions $X \to B$ and $U \to I$.

Lemma 8. *Let (A, k) be a local graded algebra with two mixed extensions $A[X]\langle U\rangle$ and $A[Y]\langle V\rangle$, with X, U, Y and V all degree-wise finite and in strictly positive degrees. Suppose we have a morphism*

$$\phi : A[X]\langle U\rangle \to A[Y]\langle V\rangle$$

of A algebras with divided powers on $(U)^{(1)}$ and $(V)^{(1)}$ respectively. Then ϕ is an isomorphism if and only if $\mathrm{ind}^{\gamma}(\phi \otimes_A k) : \mathrm{ind}^{\gamma}(k[X]\langle U\rangle) \to \mathrm{ind}^{\gamma}(k[Y]\langle V\rangle)$ is an isomorphism.

Proof. The proof is almost the same as [64, lemma 1.8.7]. It is only necessary to show ϕ is an isomorphism assuming that $\mathrm{ind}^{\gamma}(\phi \otimes_A k)$ is.

This assumption and the hypotheses imply that X and Y, and U and V respectively, have the same (graded) cardinality. It follows that each $\phi_i : A[X]\langle U\rangle_i \to A[Y]\langle V\rangle_i$ is a map of free A_0 modules of equal finite rank. Since A_0 is Noetherian it therefore suffices to show that ϕ is surjective.

Assume towards a proof by induction that we know $\phi_{<i}$ is surjective. Since $\mathrm{ind}^\gamma(\phi\otimes_A k)_i$ is surjective

$$0 = \frac{A[Y]\langle V\rangle_i}{(\mathfrak{m}_A, \mathrm{im}(\phi)_i, (Y)^2, (V)^{(2)})_i} = \frac{k[Y]\langle V\rangle_i}{\mathfrak{m}_{A_0}\cdot A[Y]\langle V\rangle_i + (A_{>0}\cdot A[Y]\langle V\rangle_{<i})_i + \mathrm{im}(\phi)_i + (Y)_i^2 + (V)_i^{(2)}}.$$

By the induction hypothesis $(A_{>0}\cdot A[Y]\langle V\rangle_{<i})_i \subseteq \mathrm{im}(\phi)_i$ and $(Y)_i^2 + (V)_i^{(2)} \subseteq \mathrm{im}(\phi)_i$. Therefore $A[Y]\langle V\rangle_i/(\mathfrak{m}_{A_0}\cdot A[Y]\langle V\rangle_i + \mathrm{im}(\phi)_i) = 0$ and by Nakayama's lemma ϕ_i is surjective. □

Next we discuss how this structure is used in the dg situation. Let A be a strictly commutative dg algebra with a semi-ideal I, that is, we assume I is an ideal in the underlying graded algebra of A but we do not impose the condition $\partial(I) \subseteq I$. A divided power structure on I is a divided power structure in the graded sense above which additionally satisfies $\partial(x^{(n)}) = \partial(x)x^{(n-1)}$ for all $n \geq 1$.

Let A be dg algebra with divided power structure on I, and let $z \in A_i$ be a cycle. If u is a variable of degree $i+1$ then $A\langle u\rangle$ is admits a unique differential making it a dg algebra with divided power structure on $(z)^{(1)} + I$, for which $A \to A\langle u\rangle$ is a morphism of dg algebras with divided powers and $\partial(u) = z$. We may write $A\langle u\rangle = A\langle u : \partial(x) = z\rangle$ to be explicit.

The morphism $A \to A\langle u\rangle$ is an example of a *Tate extension*, or a *semi-free divided power extension* (often also called a semi-free Γ extension). In general a Tate extension is any morphism $A \to A\langle U\rangle$ obtained as a colimit of such extentions, so that $A\langle U\rangle = \bigcup_n A\langle u_1, ..., u_n\rangle$ with $\partial(u_n) \in A\langle u_1, ..., u_{n-1}\rangle$, where $U = \{u_1, u_2, ...\}$ is a graded set of variables.

Then analogous lifting property for Tate extensions is similar to lemma 5 (with almost the same proof), but delicate to state. We will not need it below, so we skip it (but note that we will make the same lifting argument by hand to prove theorem 21).

A Tate extension $A \to A\langle U\rangle$ is called minimal if $\partial(\mathfrak{m}_{A\langle U\rangle}) \subseteq \mathfrak{m}_A + (U)^{(2)}$. Analogous to lemma 4, every surjection of local dg algebras $A \to B$ can be factored as a minimal Tate extension $A \to A\langle U\rangle$ followed by a surjective quasi-isomorphism $A\langle U\rangle \to B$. This factorisation is call a *minimal Tate model* or an *acyclic closure* of the homomorphism $A \to B$.

The Tate Model for the Diagonal

We end this section by discussing the classical *Tate model for the diagonal*. Inspired by rational homotopy theory, Tate used semi-free divided power extensions to build dg algebra resolutions [117]. In particular, Tate wrote down the minimal resolution of the residue field of a complete intersection local ring (see [64, proposition 1.5.4]). The theorem below is a structural version of his construction which summarises and slightly extends several results from the literature. Similar statements can be found in [44, proposition 1.9], [19, theorem 1.1] and [12, proposition 7.2.9]. Because the statement here is more general than can be found in the literature, we will sketch a proof. In section 3.4 we will explain how this construction can be made functorial.

We say that Q is a smooth K algebra if we are given a homomorphism of commutative Noetherian rings $K \to Q$ which is essentially of finite type, and such that the Kernel I of the multiplication map $\mu : Q \otimes_K Q \to Q$ is locally generated by a regular sequence. This means that $I_\mathfrak{p}$ is generated by a regular sequence for every prime $\mathfrak{p}$ of the enveloping algebra $Q^{\mathsf{ev}} = Q \otimes_K Q$. See for example [20, section 1] for discussion and other definitions of smoothness. In particular if Q is local with maximal ideal $\mathfrak{m}$ and $\mathfrak{n} = \mu^{-1}(\mathfrak{m})$ then $I_{\mathfrak{n}}$ is generated by a regular sequence $\underline{x}$. We continue to use this notation in the theorem below. Note that Q^{ev} is Noetherian since $K \to Q$ is essentially of finite type.

We will often apply the theorem in the situation $K = Q$, then smoothness is trivial and our notation becomes less complicated (we will make this simplifying assumption in section 3.4). Another important case is when K is a local ring and Q is $K[x_1, ..., x_n]$ localised at $(\mathfrak{m}_K, x_1, ..., x_n)$.

If $Q \to Q[X]$ is a semi-free extension, the double $Q[X] \otimes_K Q[X]$ is simply denoted $Q^{\mathsf{ev}}[X, X]$ or even $Q[X]^{\mathsf{ev}}$. For geometric reasons, the multiplication map $Q^{\mathsf{ev}}[X, X] \to Q[X]$ is referred to as the diagonal.

Theorem 10. *Let $(Q, \mathfrak{m}, k)$ be a local ring which is also a smooth K algebra as above, and let $Q \to Q[X]$ be a semi-free extension with X degree-wise finite and concentrated in strictly positive degrees. The multiplication map $Q^{\mathsf{ev}}_{\mathfrak{n}}[X, X] \to Q[X]$ has a Tate model of the form*

$$Q^{\mathsf{ev}}_{\mathfrak{n}}[X, X] \to Q^{\mathsf{ev}}_{\mathfrak{n}}[X, X]\langle \Sigma\underline{x}, \Sigma X\rangle \to Q[X].$$

The differential satisfies $\partial(\Sigma x) = x$ for $x \in \underline{x}$ and $\partial(\Sigma x) = 1 \otimes x - x \otimes 1 - \Sigma\partial_1(x)$ modulo decomposables in $\mathfrak{m}^2_{Q[X]^{\mathsf{ev}}_{\mathfrak{n}}\langle \Sigma\underline{x}, \Sigma X\rangle} + (\Sigma X)^{(2)}$ for $x \in X$.

If moreover $Q \to Q[X]$ is minimal as a semi-free extension then $Q^{\mathsf{ev}}_{\mathfrak{n}}[X, X]\langle \Sigma\underline{x}, \Sigma X\rangle$ is minimal as a dg module over $Q^{\mathsf{ev}}_{\mathfrak{n}}[X, X]$, that is $\partial(\Sigma X) \subseteq (\mathfrak{m}_{Q[X]^{\mathsf{ev}}_{\mathfrak{n}}})Q[X, X]\langle \Sigma X\rangle$.

In the statement of the theorem $\partial_1 : XQ \to XQ$ is the strict part of the differential. That is, the part which preserves the weight decomposition $Q[X] = \bigoplus_w \mathrm{Sym}^w_Q(X)$. The $\partial_1(x)$ in the statement is really some lift of $\partial_1(x)$ to XQ^{ev}. Any lift is equally good, and we continue with this abuse of notation below.

Lemma 9. *Any map $Q^{\mathsf{ev}}_{\mathfrak{n}}[X, X]\langle \Sigma\underline{x}, \Sigma X\rangle \to Q[X]$ whose differential has the form described in the theorem above is a quasi-isomorphism.*

Because $\underline{x}$ is a regular sequence there is an obvious quasi-isomorphism $Q^{\mathsf{ev}}_{\mathfrak{n}}[X, X]\langle \Sigma\underline{x}, \Sigma X\rangle \to Q[X, X]\langle \Sigma X\rangle$. Then by a homological version of Nakayama's lemma it suffices to apply $- \otimes^{\mathsf{L}}_Q k$ and show that $k[X, X]\langle \Sigma X\rangle \to k[X]$ is a quasi-isomorphism. Finally this is essentially contained in [44, proposition 1.9] (the proof is an application of lemma 1).

Proof of theorem 10. Assume inductively that we have constructed a factorisation

$$Q^{\mathsf{ev}}_{\mathfrak{n}}[X, X] \to Q^{\mathsf{ev}}_{\mathfrak{n}}[X, X]\langle \Sigma\underline{x}, X_{<n}\rangle \to Q[X]$$

whose differential has the above form. Given $x \in X_n$ we must define $\partial(\Sigma x)$. Automatically the differential restricts to $Q^{\mathsf{ev}}_{\mathfrak{n}}[X_{<n}, X_{<n}]\langle \Sigma\underline{x}, \Sigma X_{<n}\rangle$. Consider $\partial[1 \otimes x - x \otimes 1 - \Sigma\partial_1(x)]$, by the induction hypothesis this a cycle in the kernel $I \to Q^{\mathsf{ev}}_{\mathfrak{n}}[X_{<n}, X_{<n}]\langle \Sigma\underline{x}, \Sigma X_{<n}\rangle \to Q[X_{<n}]$. By the previous lemma I is acyclic, so there is $y \in I_n$ such that $\partial(y) = \partial[1 \otimes x - x \otimes 1 - \Sigma\partial_1(x)]$. Automatically, y is decomposable, because otherwise for degree reasons we could write $y = a\Sigma x' + p$ with p a decomposable in $m^2_{Q[X]^{\mathsf{ev}}_{\mathfrak{n}}\langle \Sigma\underline{x}, \Sigma X\rangle} + (\Sigma X)^{(2)}$ and $x' \in X_{n-1}$ and $a \in Q^{\mathsf{ev}}_{\mathfrak{n}} \smallsetminus \mathfrak{n}Q^{\mathsf{ev}}_{\mathfrak{n}}$. But then by the induction hypothesis we would have $\partial(y) = a \otimes x' - x' \otimes a - a\Sigma\partial_1(x') + q$ with q also decomposable. This is not the case since $\partial[1 \otimes x - x \otimes 1 - \Sigma\partial_1(x)]$ involves no terms of the form $a\Sigma\partial_1(x')$, using the fact that $\partial_1^2 \otimes_Q k = 0$. So setting $\partial(\Sigma x) = 1 \otimes x - x \otimes 1 - \Sigma\partial_1(x) - y$ finishes the induction. Taking the colimit builds the Tate model for the diagonal.

For the final statement, first note that minimality of $Q \to Q[X]$ implies $\partial_1 \otimes_Q k = 0$, and therefore by the description of the differential $Q^{\mathsf{ev}}_{\mathfrak{n}}[X,X] \to Q^{\mathsf{ev}}_{\mathfrak{n}}[X,X]\langle \varsigma\underline{x}, \varsigma X\rangle$ is a minimal Tate extension. In other words, $Q^{\mathsf{ev}}_{\mathfrak{n}}[X,X] \to Q^{\mathsf{ev}}_{\mathfrak{n}}[X,X]\langle \varsigma\underline{x}, \varsigma X\rangle \to Q[X]$ is an acyclic closure.

Applying $- \otimes_{Q^{\mathsf{ev}}_{\mathfrak{n}}[X]\langle \varsigma\underline{x}\rangle} k$ produces an acyclic closure $k[X]\langle \varsigma X\rangle \to k$. It suffices to show that this is a minimal dg module resolution, that is, $\partial(\varsigma X) \subseteq (X)k[X]\langle \varsigma X\rangle$. This is [12, theorem 6.3.4]. □

Remark 1. For simplicity, assume $K = Q$. One can also construct the Tate model for the diagonal $Q[X,X]\langle \varsigma X\rangle$ through Koszul duality methods. In short $Q[X] = \mathcal{C}_Q L$ is the cobar construction of a curved Lie_∞ coalgebra L over Q (see section 2.8 for some details in this direction). Its universal envelope UL is a curved A_∞ coalgebra, and there is an acyclic twisting cochain $\tau : UL \to \mathcal{C}_Q L$ over Q. One can use τ to build a twisted tensor product and an isomorphism $\mathcal{C}_Q L \otimes^\tau_Q UL \otimes^\tau_Q \mathcal{C}_Q L \cong Q[X,X]\langle \varsigma X\rangle$. In particular, the appearance of divided powers here can be explained by the Poincaré-Birkhoff-Witt theorem. Some of these words are explained (in much less generality) in section 2.8. See especially the proof of theorem 13. This construction will be explained properly somewhere else. This is also connected to joint work with Vincent Gélinas on explicit universal envelopes of Lie_∞ algebras, currently in preparation [33].

The case of a Koszul algebra over a field (when there is no higher structure) is a relatively easy verification. This is essentially lemma 13 below.

Example 3. The situation for complete intersections is classical. Let K be a commutative ring and set $Q = K[x_1, ..., x_n]$ (we won't bother to localise as this wasn't important in theorem 10). The kernel of $\mu : Q \otimes_K Q \to Q$ is generated by the regular sequence $\underline{x} = \{x_i \otimes 1 - 1 \otimes x_i\}$. Now take a regular sequence $f_1, ..., f_c$ in Q and set $R = Q/(f_1, ..., f_c)$. As above we write $Q^{\mathsf{ev}} = Q \otimes_K Q$ and $R^{\mathsf{ev}} = R \otimes_K R$.

The Koszul complex $Q \to Q[\underline{\eta}] \xrightarrow{\simeq} R$ is a semi-free model for R, where the $\underline{\eta} = \eta_1, ...\eta_c$ are degree one variables with $\partial(\eta_i) = f_i$.

According to theorem 10 we can build an acyclic closure $Q^{\mathsf{ev}}[\underline{\eta}, \underline{\eta}] \to Q^{\mathsf{ev}}[\underline{\eta}, \underline{\eta}]\langle \varsigma\underline{x}, \varsigma\underline{\eta}\rangle \xrightarrow{\simeq} Q[\underline{\eta}]$. Tensoring down to R we obtain an acyclic closure $R^{\mathsf{ev}} \to R^{\mathsf{ev}}\langle \varsigma\underline{x}, \varsigma\underline{\eta}\rangle \xrightarrow{\simeq} R$. This is the minimal R^{ev} free resolution of R. This construction goes back to Tate (he used it to resolve the residue field of R, see [64, proposition 1.5.4]).

Guccione and Guccione [60] use a similar construction to compute the Hochschild homology and cohomology of R over K. Assume that $K \to R$ is flat (otherwise we compute Shukla homology instead). In short, $\mathrm{HH}_*(R/K, R)$ is the homology of

$$(R^{\mathsf{ev}}\langle \varsigma\underline{x}, \varsigma\underline{\eta}\rangle) \otimes_{R^{\mathsf{ev}}} R \cong R\langle \varsigma\underline{x}, \varsigma\underline{\eta}\rangle.$$

In characteristic zero this appears already in the work of Wolffhardt [122], and implicitly in the work of Quillen [106]. Note that in this dg algebra $\partial(\underline{x}) = 0$. From this we see that $R\langle \varsigma\underline{x}, \varsigma\underline{\eta}\rangle$, and hence $\mathrm{HH}_*(R/K, R)$, splits into a direct sum of components $R\langle \varsigma\underline{x}, \varsigma\underline{\eta}\rangle^{(w)} = \sum_{i+j=w} \Gamma^i_R(\varsigma\underline{\eta}) \otimes_R \bigwedge^j_R(\varsigma\underline{x})$. This is the Hodge decomposition established in [60].

Similarly if R is projective over K then $\mathrm{HH}^*(R/K, R)$ is the homology of $\mathrm{Hom}_{R^{\mathsf{ev}}}(R^{\mathsf{ev}}\langle \varsigma\underline{x}, \varsigma\underline{\eta}\rangle, R) \cong R[y, \xi]$ where $y = (\varsigma\underline{x})^\vee$ and $\xi = (\varsigma\underline{\eta})^\vee$. Hochschild cohomology also obtains a Hodge decomposition by these considerations.

The same argument gives a short proof of the more general result in [61]: all that was required was for the map $K \to Q$ to be smooth.

Finally, note that we obtain resolutions $M \otimes_K R\langle \varsigma\underline{x}, \varsigma\underline{\eta}\rangle \xrightarrow{\simeq} M$ for any R module M. In particular if $K = k$ is the residue field of R (so that R is an honest complete intersection ring) we have a resolution

$R\langle \varepsilon\underline{x}, \varepsilon\underline{\eta}\rangle \xrightarrow{\simeq} k$, but this time $R\langle \varepsilon\underline{x}, \varepsilon\underline{\eta}\rangle$ has a different differential (it can be written as a double complex in such a way that the vertical part computes Hochschild homology).

Remark 2. Theorem 10 can be used to compute Hochschild homology (and cohomology) much more generally. Let Q be a local ring which is a smooth K algebra and let $R = Q/I$ be a factor ring of Q. Assume that $K \to R$ is flat (otherwise we compute Shukla homology instead). Denote $R^{\mathsf{ev}} = R \otimes_K R$ and $Q^{\mathsf{ev}} = Q \otimes_K Q$, with $\mathfrak{n}$ and $\underline{x}$ as above as well.

The theorem allows us to compute $\mathrm{HH}_*(R/K, R)$ from a minimal model $Q \to Q[X] \xrightarrow{\simeq} R$. We first need the observation that there is a canonical quasi-isomorphism $R \otimes^{\mathsf{L}}_{R^{\mathsf{ev}}} R \simeq R \otimes^{\mathsf{L}}_{R^{\mathsf{ev}}_{\mathfrak{n}}} R$, because R is already $\mathfrak{n}$-local. From the resolution $Q^{\mathsf{ev}}_{\mathfrak{n}}[X, X]\langle \varepsilon\underline{x}, \varepsilon X\rangle \to Q[X]$ we obtain a resolution $R^{\mathsf{ev}}_{\mathfrak{n}}\langle \varepsilon\underline{x}, \varepsilon X\rangle \to R$. Hence $R \otimes^{\mathsf{L}}_{R^{\mathsf{ev}}_{\mathfrak{n}}} R \simeq (R^{\mathsf{ev}}_{\mathfrak{n}}\langle \varepsilon\underline{x}, \varepsilon X\rangle) \otimes_{R^{\mathsf{ev}}_{\mathfrak{n}}} R \cong R\langle \varepsilon\underline{x}, \varepsilon X\rangle$.

Therefore $\mathrm{HH}_*(R/K, R)$ is the homology of the semi-free divided power algebra $R\langle \varepsilon\underline{x}, \varepsilon X\rangle$. Similarly, we can produce a dual model for $\mathrm{HH}^*(R/K, R)$ using theorem 10.

Note that in the cases $Q = R$ we recover the famous Hochschild-Kostant-Rosenberg theorem.

Remark 3. The Tate model for the diagonal also easily reproduces a result of Fiorenza and Manetti on mapping cones of Lie_∞ algebras [50]. This will be explained in section 3.5 (see corollary 5).

2.7 Cohen Factorisations and Minimal Models

In this section we present the basic facts we need about the theory of Cohen factorisations, introduced by Avramov, Foxby and Herzog [16]. We will also introduce an important class of dg algebras which we call minimal models.

Let $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local homomorphism. By definition ϕ is *weakly regular* if it is flat and the fibre $S/\mathfrak{m}S$ is a regular ring. A *regular factorisation* of ϕ is a factorisation of ϕ into local homomorphisms

$$\begin{tikzcd} & R' \arrow[dr] & \\ R \arrow[ur] \arrow[rr, "\phi"] & & S \end{tikzcd}$$

such that $R \to R'$ is weakly regular and $R' \to S$ is surjective.

For example, ϕ is said to be *essentially of finite type* if for some finite sequence $\underline{x} = x_1, ..., x_n$ of variables ϕ can be factored as an extension $R \to R[\underline{x}]_{(\underline{x})}$ followed by a surjection $R[\underline{x}]_{(\underline{x})} \to S$. Largely, these are the homomorphisms of interest in algebraic geometry. Since $R \to R[\underline{x}]_{(\underline{x})}$ is weakly regular, such a homomorphism admits a regular factorisation by definition.

A regular factorisation $R \to R' \to S$ in which R' is complete is called a *Cohen factorisation*. It is an important theorem of Avramov, Foxby and Herzog that any local homomorphism $R \to S$ in which S is complete admits a Cohen factorisation [16, (1.1)]. By this means, these authors initiated a program for investigating the local properties of ring homomorphisms, using Cohen factorisations to reduce to properties of surjective homomorphisms.

The composition $R \to S \to \widehat{S}$ of ϕ with the natural embedding of S into its completion is known as the semi-completion of ϕ, and is generally denoted $\grave{\phi}$. By Avramov, Foxby and Herzog's theorem $\grave{\phi} : R \to \widehat{S}$ always admits a Cohen factorisation.

A morphism between two Cohen factorisations $R \to R' \to S$ and $R \to R'' \to S$ is a commutative diagram

$$\begin{tikzcd}[row sep=small] & R' \arrow[dd, "\psi"] \arrow[dr] & \\ R \arrow[ur] \arrow[dr] & & S \\ & R'' \arrow[ur] & \end{tikzcd}$$

of local homomorphisms. The factorisation $R \to R' \to S$ is said to be a deformation of $R \to R'' \to S$ (and $R \to R'' \to S$ a reduction of $R \to R' \to S$) if ψ is surjective. In this situation the kernel of ψ is automatically generated by a regular sequence $\underline{x}$. Moreover, if I denotes the kernel of $R' \to S$, then $\underline{x}$ descends to an l vector space basis for $(I+\mathfrak{m}R'+\mathfrak{m}_{R'}^2)/(\mathfrak{m}R'+\mathfrak{m}_{R'}^2)$. Conversely, any lift $\underline{x} \subseteq I$ of a basis for $(I+\mathfrak{m}R'+\mathfrak{m}_{R'}^2)/(\mathfrak{m}R'+\mathfrak{m}_{R'}^2)$ is a regular sequence in R' and reducing modulo $\underline{x}$ produces a Cohen factorisation $R \to R'' = R'/(\underline{x}) \to S$ which is a reduction of $R \to R' \to S$. These facts can be found in [16, (1.5)], where it is also observed that $\dim_l(I+\mathfrak{m}R'+\mathfrak{m}_{R'}^2)/(\mathfrak{m}R'+\mathfrak{m}_{R'}^2) = \operatorname{edim} R'/\mathfrak{m}R' - \operatorname{edim} S/\mathfrak{m}S$. With this in mind a Cohen factorisation $R \to R' \to S$ of ϕ is said to be *minimal* if $\operatorname{edim} R'/\mathfrak{m}R' = \operatorname{edim} S/\mathfrak{m}S$, or equivalently, if it admits no proper reduction.

While Cohen factorisations cannot necessarily be compared directly, by [16, (1.2)] any two Cohen factorisations $R \to R' \to S$ and $R \to R'' \to S$ admit a common deformation

$$\begin{tikzcd} & R' \arrow[dr] & \\ R \arrow[ur] \arrow[r] \arrow[dr] & A \arrow[u] \arrow[r] \arrow[d] & S \\ & R''. \arrow[ur] & \end{tikzcd}$$

So, the horizontal row is a Cohen factorisation and the two vertical homomorphisms are surjective. Moreover, any two Cohen factorisations of ϕ can be connected by a sequence of codimension one deformations. This observation is often useful to simplify arguments which compare Cohen factorisations.

Complete Intersections and Quasi-Complete Intersection

Now we discuss some classes of homomorphisms which have been introduced using Cohen factorisations.

A surjective homomorphism is called complete intersection if its kernel is generated by a regular sequence. In general a local homomorphism $R \to S$ is complete intersection if in some Cohen factorisation $R \to R' \to \widehat{S}$ the homomorphism $R' \to \widehat{S}$ is complete intersection. This definition was introduced by Avramov in [13], where it was used to settle some long-standing conjectures of Quillen.

It follows readily from the definition that any complete intersection homomorphism has finite flat dimension. The next definition is an analogue which allows for infinite flat dimension.

A surjective local homomorphism $R \to S$ is called a quasi-complete intersection (qci) if it admits an acyclic closure $R \to R\langle U\rangle \xrightarrow{\simeq} S$ in which U is concentrated in degrees 1 and 2. These homomorphisms were introduced by Blanco, Majadas and Rodicio in [31], where they were characterised in terms of vanishing of André-Quillen cohomology. In [18, section 7] this class is extended to include non-surjective homomorphisms: $R \to S$ is qci if in some Cohen factorisation $R \to R' \to \widehat{S}$ the homomorphism $R' \to \widehat{S}$ is qci.

In both cases the property is independent of the choice of Cohen factorisation (the arguments can be found in the references given).

Minimal Models

Now we explain the construction of *minimal models*, which will be extremely important throughout this thesis.

Let R be a local ring. The classical Cohen structure theorem is the absolute analogue of the existence of Cohen factorisations. It says that there is a complete, regular local ring Q and a surjection $Q \to \widehat{R}$. We can and will assume this presentation is minimal in the sense that $\ker(Q \to R) \subseteq \mathfrak{m}_Q^2$. By proposition 4 there is a minimal model $Q \to Q[X] \to \widehat{R}$. Note that $Q[X]$ is in fact absolutely minimal, so $\partial(\mathfrak{m}_{Q[X]}) \subseteq \mathfrak{m}_{Q[X]}$. The dg algebra $Q[X]$ will be called a *minimal Cohen model* for R.

We can use the theory of Cohen factorisations to make a relative version of this construction. Let $\phi : (R, k) \to (S, l)$ be a local homomorphism. The semi-completion of ϕ admits a minimal Cohen factorisation $R \to R' \to \widehat{S}$. By proposition 4 there is a minimal model $R' \to R'[X] \to \widehat{S}$. The factorisation $R \to R'[X] \to \widehat{S}$ will be called a *minimal Cohen model* for ϕ. If $Q = k \otimes_R R'$, then the fibre $k \otimes_R R'[X] = Q[X]$ is a dg algebra model for the completion of the derived fibre $k \otimes^{\mathsf{L}}_R S$. Note that $k \to Q[X]$ is a minimal dg algebra morphism, and therefore $Q[X]$ is absolutely minimal.

In both situations $Q[X]$ is a semi-free extension of a regular local ring which is also absolutely minimal. In an abuse of terminology, we will call a local dg algebra of this form a *minimal model*. These dg algebras will be very important in what follows.

In section 2.4 minimal A_∞ algebras were also called minimal models. Needless to say these are two very separate (but related) notions. The definition given here will be used exclusively in chapter 3, while the term minimal model is reserved for minimal A_∞ algebras in chapter 4.

Golod Homomorphims

Another class of homomorphisms for which Cohen factorisations are elucidating are Golod homomorphisms. However, these do not fit into the same basic pattern as complete intersection and quasi-complete intersection homomorphisms.

Let $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a finite local homomorphism. We recall the classical bound

$$P^S_l(t) \leq \frac{P^R_k(t)}{(1+t) - tP^R_S(t)}$$

which is a coefficient-wise inequality of formal power series. The bound appears in [57] where it is attributed to Serre. In loc. cit. Golod characterised those surjective ring homomorphisms for which this bound is attained in terms of trivial Massey operations on $\mathrm{Tor}^R_*(S, k)$.

Classically ϕ has been called Golod if the Serre bound above is an equality. However, in certain exceptional cases this bound can be improved and Avramov gives a definition in [10] which takes this into account. We give an equivalent definition now.

Let $R \to R\langle U\rangle \xrightarrow{\simeq} k$ be an acyclic closure, so that $S\langle U\rangle \simeq S \otimes^{\mathsf{L}}_R k$. Avramov defines ϕ to be Golod if $S\langle U\rangle$ is a Golod dg algebra in the sense of [10, definition (2.2)]. Take also a minimal Cohen model $R \to R'[X] \xrightarrow{\simeq} \widehat{S}$ and set $A = k \otimes_R R'[X]$. It is easy to see that the Golod condition for dg algebras is invariant under completion and quasi-isomorphisms, therefore A is Golod if and only if $S\langle U\rangle$ is so. Finally, it is most convenient to use the characterisation [10, theorem 3.4 (3)] for our definition: we say that ϕ is Golod if the map $\mathfrak{m}_A \to \mathfrak{m}_A/\mathfrak{m}_A^2$ induces an injection $\mathrm{H}_*(\mathfrak{m}_A) \hookrightarrow \mathrm{H}_*(\mathfrak{m}_A/\mathfrak{m}_A^2) = \mathfrak{m}_A/\mathfrak{m}_A^2$. The relation to the homotopy Lie algebra will be discussed in example 4.

Note that this definition implies that the fibre $\overline{S} = S/\mathfrak{m}S$ is Artinian with $\mathfrak{m}_{\overline{S}}^2 = 0$. In particular $R \to S$ is automatically module finite. To account for the non-finite case one might want to extend the definition using Cohen factorisations. The next proposition suggests such a generalisation. We present it only as an example of how Cohen factorisations are used to introduce new properties of ring homomorphisms.

Let us emphasise that the proposition is for the most part a consequence of the results of Avramov in [10], which builds on work of Golod [57] and Levin [87].

Following [10, (4.4)], a surjective homomorphism $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is called *exceptional* if its kernel is generated by a single non zero-divisor in $\mathfrak{m} \smallsetminus \mathfrak{m}^2$, otherwise it is *standard*. Exceptional homomorphisms are Golod, but as the name suggests they are not representative of the typical situation.

Proposition 5. *If $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ is a local homomorphism then the following are equivalent:*

1. *for some Cohen factorisation $R \to R' \to \widehat{S}$ the equality*

$$P_l^S(t) = \frac{P_l^{R'}(t)}{(1+t)^{d+1} - tP_{\widehat{S}}^{R'}(t)}$$

holds, where $d = \mathrm{edim}(R'/\mathfrak{m}R') - \mathrm{edim}(S/\mathfrak{m}S)$;

2. *the above equality holds for any Cohen factorisation;*
3. *in some minimal Cohen factorisation $R \to R' \to \widehat{S}$ the map $R' \to \widehat{S}$ is standard and Golod;*
4. *in any minimal Cohen factorisation $R \to R' \to \widehat{S}$ the map $R' \to \widehat{S}$ is standard and Golod.*

The proposition will not be important to us later, and it is partly superseded by proposition 9, so we sketch the proof only. Proposition 9 will characterise the equivalent conditions here in terms of L.S. category. With that result in mind, a homomorphism satisfying the conditions above will be called generalised Golod of level 1.

We will need a standard lemma.

Lemma 10. *If $(R, \mathfrak{m}, k)$ is a local ring and x is a non zero divisor in $\mathfrak{m} \smallsetminus \mathfrak{m}^2$ then $P_M^R(t) = (1+t)P_M^{R/x}(t)$ for any R module M such that $xM = 0$.*

The proof uses the simple calculation $M \otimes_R^{\mathsf{L}} k \simeq M \otimes_{R/x}^{\mathsf{L}} k + \Sigma M \otimes_{R/x}^{\mathsf{L}} k$.

Proof of proposition 5. Since $d = 0$ if and only if the Cohen factorisation is minimal, the implications $2 \implies 4 \implies 3 \implies 1$ are clear using the characterisation [10, theorem (4.6)]. Thus we must show that show that $1 \implies 2$.

For this we appeal to the fact that any two Cohen factorisations $R \to R' \to \widehat{S}$ and $R \to R'' \to \widehat{S}$ are connected by a sequence of deformations. So we can assume we have a homomorphism $R'' \to R'$ (compatible with R and $\widehat{S}$) whose kernel is generated by a nonzero divisor x in $\mathfrak{m}_{R''} \smallsetminus (\mathfrak{m}_{R''}^2 + \mathfrak{m}R'')$, and we must show that the formula in part 1 holds for R' if and only if it holds for R''.

Let $e = \mathrm{edim}(R''/\mathfrak{m}R'') - \mathrm{edim}(S/\mathfrak{m}S)$ and $d = \mathrm{edim}(R'/\mathfrak{m}R') - \mathrm{edim}(S/\mathfrak{m}S)$, and note that $e = d+1$ since $\mathrm{edim}(R''/\mathfrak{m}R'') = \mathrm{edim}(R'/\mathfrak{m}R') + 1$. Now we use lemma 10 to make the calculation

$$\frac{P_l^{R''}(t)}{(1+t)^{e+1} - tP_{\widehat{S}}^{R''}(t)} = \frac{(1+t)P_l^{R'}(t)}{(1+t)^{e+1} - (1+t)tP_{\widehat{S}}^{R'}(t)} = \frac{P_l^{R'}(t)}{(1+t)^{d+1} - tP_{\widehat{S}}^{R'}(t)}.$$

This concludes the proof. □

Note that in the exceptional cases we have the improved bound

$$P_l^S(t) = \frac{P_l^{R'}(t)}{(1+t)^{d+2} - tP_{\widehat{S}}^{R'}(t)}.$$

Remark 4. The Serre bound above is attained by the famous bar resolution. Details on this construction and the A_∞ structure which underlies it are given by Burke in [39].

Let $Q \to R$ be a standard, surjective, Golod homomorphism, let $A \to R$ be the minimal Q free resolution of R, and let $K \to k$ be the minimal Q free resolution of k. Burke constructs an A_∞ algebra structure on A making $A \to R$ a quasi-isomorphism of A_∞ algebras, and an A_∞ A module structure on K compatible with the R module structure of k. From this data one constructs the twisted tensor product $K \otimes_Q BA \otimes_Q R$, and Burke shows that the Golod property implies that $K \otimes_Q BA \otimes_Q R \to k$ is a minimal R free resolution.

Now suppose that $R \to S$ is generalised Golod of level 1. Let $R \to R' \to \widehat{S}$ be a minimal Cohen factorisation, let $A \to S$ be the minimal R' free resolution of S, and let $K \to k$ be the minimal R' free resolution of k, note that these are both R flat resolutions. Then $R' \to S$ is Golod so using the constructions of Burke we obtain a minimal S free resolution $K \otimes_{R'} \otimes BA \otimes_{R'} S \to k$.

Gulliksen and Schoeller's Theorem

To end this section we record the the following important consequence of theorem 10. It was proven independently by Gulliksen and Schoeller.

Theorem 11 (Gulliksen [64], Schoeller [109])**.** *If $Q[X]$ is a minimal model with residue field k and X_0 is a minimal generating set for $\mathfrak{m}_Q$ then we have an acyclic closure $Q[X]\langle t, \mathtt{s}X\rangle \xrightarrow{\simeq} k$ where t is a set of variables in degree 1 with $\partial(t) = X_0$. Moreover $Q[X]\langle t, \mathtt{s}X\rangle$ is minimal as a dg module over $Q[X]$.*

In particular if $Q[X] \to R$ is a minimal Cohen model we have an acyclic closure $R\langle t, \mathtt{s}X\rangle \xrightarrow{\simeq} k$ which is minimal as a complex. Even if R does not admit a minimal Cohen model (without completing), the acyclic closure of $R \to k$ is minimal as a complex.

If $Q \to Q\langle t\rangle \xrightarrow{\simeq} k$ is the Koszul complex which resolves k over Q, then lemma 5 produces a Q algebra morphism $Q[X] \to Q\langle t\rangle$. We obtain a quasi-isomorphism $Q[X]\langle t, \mathtt{s}X\rangle = Q[X, X]\langle \mathtt{s}X\rangle \otimes_{Q[X]} Q\langle t\rangle \xrightarrow{\simeq} Q\langle t\rangle \xrightarrow{\simeq} k$. When $Q[X]$ is a minimal Cohen model for R the quasi-isomorphism $R\langle t, \mathtt{s}X\rangle \xrightarrow{\simeq} k$ is given by adjunction. The last statement follows using the faithfully flat base change $- \otimes_R \widehat{R}$ and uniqueness of acyclic closures.

2.8 Lie Algebras and Lie Coalgebras

In this section we will quickly define graded Lie algebras and graded Lie coalgebras, and explain why both of these objects are equivalent to certain semi-free dg algebras. Everything in this section takes place over a fixed field k, about which we make no additional assumptions. The content of this section is classical.

We define graded Lie algebras in terms of operations (bracket and reduced square) following [12, section 10], where more details can be found. This approach leads to a complicated definition, but one which should be at least partially familiar to all readers.

A graded Lie algebra is a graded vector space L over k equipped with a bilinear bracket $[\ ,\ ] : L \otimes L \to L$ and a reduced square defined on odd elements $(\ )^{[2]} : L^{2i+1} \to L^{4i+2}$ which is quadratic, so $(\alpha u)^{[2]} = \alpha^2 u^{[2]}$ for any odd u in L and α in k. These operations should satisfy the following axioms.

- Compatibility: for any odd elements u, v in L

$$(u+v)^{[2]} = u^{[2]} + [u,v] + v^{[2]}.$$

- Anti-symmetry: for any x, y, z in L with z even

$$[x,y] + (-1)^{|x||y|}[y,x] = 0 \quad \text{and} \quad [z,z] = 0.$$

- Jacobi identity: for any x, y, z, u, v in L with u odd

$$[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]], \quad [u^{[2]}, v] = [u,[u,v]] \quad \text{and} \quad [u,[u,u]] = 0.$$

For example, any graded associative algebra A over k gives rise to a graded Lie algebra A^{Lie} with the same underlying graded vector space. The bracket is given by the graded commutator $[x,y] = xy - (-1)^{|x||y|}yx$ while the reduced square is simply $u^{[2]} = u^2$ for any x, y, u in A with u odd.

A morphism of graded Lie algebras is a map of graded vector spaces $\phi : L \to L'$ which respects the two operations, so that $\phi[x,y] = [\phi(x), \phi(y)]$ and $\phi(u^{[2]}) = (\phi(u))^{[2]}$ for any x, y, u in L with u odd. The category of graded Lie algebras over k is denoted $\mathbf{Lie}^{\mathbf{gr}}$, leaving k implicit. We further assume that the objects of $\mathbf{Lie}^{\mathbf{gr}}$ are degree-wise finite dimensional and concentrated in strictly positive degrees. Gradings here will be written cohomologically.

In characteristic zero a Lie coalgebra may be defined as a vector space L with a cobracket $L \to L \otimes L$ which is anti-symmetric and satisfies the Jacobi condition laid out for instance in [108, section 1]. In general, the symmetry condition is more subtle and it is easiest to define Lie coalgebras as follows.

First, if V is a graded vector space then the exterior algebra $\bigwedge^*(V)$ is the quotient of the tensor algebra $\mathsf{T}(V)$ by the two sided ideal $\big(u \otimes v + (-1)^{|u||v|} v \otimes u,\ w \otimes w\ :\ u, v, w \in V \text{ with } w \text{ even}\big)$. Then $\bigwedge^*(V)$ has an internal grading inherited from V and an additional weight grading, with $\bigwedge^w(V) = (v_1 \cdots v_w\ :\ v_i \in V)$ spanned as a vector space by products of length w.

A graded Lie coalgebra over k is firstly a graded vector space L together with a cobracket $\Delta : L \to \bigwedge^2 L$. This gives rise to a map $\Delta \wedge 1 - 1 \wedge \Delta : \bigwedge^2 L \to \bigwedge^3 L$, and we impose the condition that $(\Delta \wedge 1 - 1 \wedge \Delta)\Delta = 0$. How this relates to the Jacobi identity is explained in the proof of theorem 12 below (which we sketch only).

A Lie_∞ coalgebra is the structure one gets by allowing the cobracket to take arbitrary positive weight $\Delta = \Delta^1 + \Delta^2 + \Delta^3 + ... : L \to \bigwedge^{>0} L$. For details in characteristic zero see [84] or [90], and in arbitrary characteristic there is some discussion in [28].

Any graded associative coalgebra C has an associated Lie coalgebra C^{Lie} with the same underlying graded vector space. The cobracket is simply the coproduct $C \to C \otimes C$ composed with the natural projection $C \otimes C \to \bigwedge^2 C$.

A morphism of Lie coalgebras is a map of graded vector spaces $\phi : L \to L'$ making the obvious diagram commute:

$$\begin{tikzcd} L \arrow[r, "\Delta"] \arrow[d, "\phi"'] & \bigwedge^2 L \arrow[d, "\bigwedge^2 \phi"] \\ L' \arrow[r, "\Delta"] & \bigwedge^2 L'. \end{tikzcd}$$

By definition $\mathbf{coLie_{gr}}$ is the category of degree-wise finite dimensional graded Lie coalgebras which are concentrated in strictly positive degrees. Gradings here will be written homologically.

Finally, we introduce an important category of *quadratic* semi-free algebras over k. Once again k will be left implicit in our notation. The objects of **q-semi-free** are bigraded algebras of the form $A = k[V] = \mathrm{Sym}(V)$ where V is a degree-wise finite dimensional graded vector space with $V = V_{\geq 0}$. They should be graded cohomologically by weight $A^w = \mathrm{Sym}^w(V)$ and homologically by the internal degree $A_i = k[V]_i = \mathrm{Sym}(V)_i$ induced from V, so that A is strictly graded commutative with respect to its lower grading. The differential must satisfy $\partial : A^w_i \to A^{w+1}_{i-1}$. Morphisms in **q-semi-free** are local homomorphisms of dg algebras which respect both gradings[6].

These dg algebras are called quadratic because the differential applied to the generators V takes values in the quadratics forms $\mathrm{Sym}^2(V)$. The relationship between this category and the category of minimal models will be made clear in section 3.1.

The following theorem is fundamental in the Koszul duality between Lie algebras and commutative algebras. It is essentially due to Quillen [105, appendix B] in characteristic zero and Avramov [9, theorem 4.2] in positive characteristic.

Theorem 12. *We have equivalences of categories*

$$\begin{tikzcd} \textbf{q-semi-free} \arrow[r, shift left, "\pi_*"] & \mathbf{coLie_{gr}} \arrow[l, shift left, "\mathcal{C}"] \arrow[r, shift left, "(-)^\vee"] & \mathbf{Lie^{gr}}. \arrow[l, shift left, "(-)^\vee"] \end{tikzcd}$$

The proof is a series of verifications, we sketch it only. The functors themselves will be defined as we go. The key point for the two functors on the left is that there is a natural isomorphism $\bigwedge^w(\varepsilon V) \cong \varepsilon^w \mathrm{Sym}^w(V)$, where we use ε to denote the usual suspension acting on the internal degree only.

If A is an object of **q-semi-free** we define $\pi_*(A) = \varepsilon A^1$. Writing $A = \mathrm{Sym}(V)$, the graded Lie coalgebra structure on $\varepsilon A^1 = \varepsilon V$ is defined by commutativity of the left square in the following diagram

$$\begin{tikzcd} V_{i-1} \arrow[r, "\partial"] \arrow[d, "\cong"] & \mathrm{Sym}^2(V)_{i-2} \arrow[r, "\partial"] \arrow[d, "\cong"] & \mathrm{Sym}^3(V)_{i-3} \arrow[d, "\cong"] \\ (\varepsilon V)_i \arrow[r, "\Delta"] & \bigwedge^2(\varepsilon V)_i \arrow[r, "\Delta\wedge 1 - 1\wedge\Delta"] & \bigwedge^3(\varepsilon V)_i. \end{tikzcd}$$

Since $\partial^2 = 0$, the condition $(\Delta \wedge 1 - 1 \wedge \Delta)\Delta = 0$ follows after checking that the right square above commutes.

This process is reversible: if L is an object of $\mathbf{coLie_{gr}}$ then $\mathcal{C}L = \mathrm{Sym}(\varepsilon^{-1}L)$ is a quadratic semi-free dg algebra with differential defined on $V = \varepsilon^{-1}L$ by the left square in the above diagram. Then ∂

[6] These correspond to *strict* morphisms of graded Lie algebras. Working with *non-strict* morphisms, which may increase the weight filtration, gives extra flexibility which can be useful. Non-strict morphisms are homotopically no more general than strict morphisms, and we won't need them, so we refer to [90] for more information.

extends uniquely to a derivation on $\mathcal{C}L$, and the right square above implies $\partial^2 = 0$.

The remaining functors in the theorem are just given by taking the k linear graded dual.

Let L be a graded Lie coalgebra. The projection $\mathsf{T}(L) \to \bigwedge(L)$ is dual to an embedding $\iota : (\bigwedge L)^\vee \to \mathsf{T}(L^\vee)$, and in weight 2 the image may be described as the linear span

$$\iota(\textstyle\bigwedge^2 L)^\vee = \Big(x \otimes y - (-1)^{|x||y|} y \otimes x,\ u \otimes u \ :\ x, y, u \in L^\vee \text{ with } u \text{ odd} \Big).$$

Using this fact, one may define the operations on the dual $L^\vee$ as follows. The cobracket $\Delta : L \to \bigwedge^2 L$ dualises to a linear map $\Delta^* : \iota(\bigwedge^2 L)^\vee \xrightarrow{\cong} (\bigwedge^2 L)^\vee \xrightarrow{\Delta^\vee} L^\vee$ and we set

$$[x, y] = \Delta^*(x \otimes y - (-1)^{|x||y|} y \otimes x) \quad \text{and} \quad u^{[2]} = \Delta^*(u \otimes u).$$

One could also phrase this in terms of the divided power algebra structure on $(\mathrm{Sym}(\varepsilon^{-1} L))^\vee \cong \Gamma(\varepsilon L^\vee)$. The compatibility and anti-symmetry conditions are quite clear already on the level of $\mathsf{T}^2(L^\vee)$, because these identities hold for the tensors $x \otimes y - (-1)^{|x||y|} y \otimes x$ and $u \otimes u$. The Jacobi conditions are equivalent to $(\Delta \wedge 1 - 1 \wedge \Delta)\Delta = 0$. To see this one works in $\iota(\bigwedge^3 L)^\vee$, which is similarly spanned by certain anti-symmetric tensors in $L^\vee$ (of three types, corresponding to the three Jacobi identities). We can sketch the classical Jacobi identity for x, y, x in $L^\vee$. Denote the anti-symmetrisation of a tensor $x_1 \otimes \cdots \otimes x_n$ by $\overline{x_1 \otimes \cdots \otimes x_n}$, so for instance $\overline{x \otimes y} = x \otimes y - (-1)^{|x||y|} y \otimes x$. The key identity is

$$\overline{x \otimes y \otimes z} = \overline{x \otimes y} \otimes z + (-1)^{|z|(|x|+|y|)} \overline{z \otimes x} \otimes y + (-1)^{|x|(|y|+|z|)} \overline{y \otimes z} \otimes x.$$

With this, one can check easily that

$$(\overline{x \otimes y \otimes z})\iota(\Delta \wedge 1 - 1 \wedge \Delta)\Delta = [[x, y], z] + (-1)^{|z|(|x|+|y|)}[[z, x], y] + (-1)^{|x|(|y|+|z|)}[[y, z], x]$$

and this vanishes since $(\Delta \wedge 1 - 1 \wedge \Delta)\Delta = 0$. This establishes the classical Jacobi identity, the other two are similar. Having checked this, it follows that a Lie coalgebra structure on L is equivalent to a Lie algebra structure on $L^\vee$, and this finishes the proof of the theorem.

Remark 5. If L is a graded Lie algebra then the composition $\mathcal{C}(L^\vee)$ is the well-known Chevalley-Eilenberg dg algebra of L.

Remark 6. At the same time, the functor $\mathcal{C} : \mathbf{coLie_{gr}} \to \mathbf{q\text{-}semi\text{-}free}$ is an example of a *cobar construction* in the sense of [90]. It can be extended to the category of dg Lie coalgebras (and ultimately, Lie_∞ coalgebras). There is a corresponding bar construction $L : \mathbf{Com_{dg}} \to \mathbf{coLie_{dg}}$ which takes a strictly commutative augmented dg algebra A to the cofree Lie coalgebra $\mathrm{coLie}(\varepsilon\overline{A})$ on the suspension of the augmentation ideal $\overline{A} = \ker(A \to k)$. Just as for $\mathcal{C}L$, the Lie coalgebra LA is equipped with a differential which encodes exactly the product on A. Good properties of the adjunction $(\mathcal{C}, L)$ underlie the Koszul duality between commutative and Lie algebras, see [105] or [70] in characteristic zero. In any characteristic there is a natural quasi-isomorphism $\mathcal{C}LA \to A$, see [51]. On the other hand, $L \to L\mathcal{C}L$ need not be a quasi-isomorphism outside of characteristic zero, but it is a weak-equivalence in a certain sense (unfortunately, not the fully-fledged model category sense). This point is delicate, and we choose to side-step these difficulties by working almost exclusively with minimal dg algebras, which means focusing our attention on the left adjoint $\mathcal{C}$.

At this point we will very quickly discuss universal envelopes of Lie algebras and coalgebras. In short, the two functors $A \mapsto A^{\mathsf{Lie}}$ and $C \mapsto C^{\mathsf{Lie}}$ have left and right adjoints respectively.

In terms of explicit universal properties this means the following. Let L be a graded Lie algebra. There is a graded associative algebra UL equipped with a map of Lie algebras $L \to (UL)^{\mathsf{Lie}}$, and for any graded associative algebra A, every morphism $L \to A^{\mathsf{Lie}}$ factors as $L \to (UL)^{\mathsf{Lie}} \to A^{\mathsf{Lie}}$ for a unique algebra map $UL \to A$. The algebra UL is the *universal enveloping algebra* of L. More details can be found in [12, section 10].

Similarly, every graded Lie coalgebra L has a *universal enveloping coalgebra*, which we will denote in the same way as UL. It enjoys the dual universal property: UL comes with a map of Lie coalgebras $(UL)^{\mathsf{Lie}} \to L$, and for any graded associative coalgebra C, every morphism $C^{\mathsf{Lie}} \to L$ factors as $C^{\mathsf{Lie}} \to (UL)^{\mathsf{Lie}} \to L$ for a unique coalgebra map $C \to UL$.

Remark 7. Contemplating the construction in the proof of theorem 12 one concludes that the functors $A \mapsto A^{\mathsf{Lie}}$ and $C \mapsto C^{\mathsf{Lie}}$ are linearly dual to each other, and therefore the two universal properties above are linearly dual as well. It follows that for any graded Lie coalgebra L in $\mathbf{coLie_{gr}}$ there is a canonical isomorphism $(UL)^\vee \cong U(L^\vee)$.

As we will see in the proof below, it will actually be better for us to work with the completed cobar construction $\widehat{\mathcal{C}}L = \prod_w \mathcal{C}^w L$. Note that this is the graded product: it only makes a difference if L is non-zero in degree 1. These completed cobar constructions could be added to the equivalence of theorem 12.

Theorem 13. *If L is a graded Lie coalgebra in $\mathbf{coLie_{gr}}$ then the composition $\tau : UL \to L \hookrightarrow \Sigma\widehat{\mathcal{C}}L$ is an acyclic twisting cochain.*

Proof. To check that τ is a twisting cochain we only need the fact that $(UL)^{\mathsf{Lie}} \to L$ is a Lie coalgebra morphism. This makes the upper left part of the diagram below commute

$$\begin{tikzcd}
 & UL \otimes UL \arrow[r] & L \otimes L \arrow[r] \arrow[dd] & \Sigma^2\widehat{\mathcal{C}}L \otimes \widehat{\mathcal{C}}L \arrow[dr] \arrow[d, phantom, "-"] & \\
UL \arrow[ur] \arrow[ddr] & & & {} & \Sigma^2\widehat{\mathcal{C}}L. \\
 & & \bigwedge^2 L \arrow[dr, "\cong"] & & \\
 & L \arrow[ur, "\Delta"] \arrow[dr, "\cong"'] & & \Sigma^2\mathcal{C}^2 L \arrow[uur] & \\
 & & \Sigma\mathcal{C}^1 L \arrow[ur, "\partial"'] & &
\end{tikzcd}$$

The lower square commutes by definition and the upper right part anti-commutes. Comparing the upper and lower paths yields the equality $\tau \smile \tau + \partial(\tau) = 0$.

To see that τ is acyclic we verify that the bimodule homomorphism $\widehat{\mathcal{C}}L \otimes^\tau UL \otimes^\tau \widehat{\mathcal{C}}L \to \widehat{\mathcal{C}}L$ is a quasi-isomorphism. This is easily seen by reduction to the abelian case, as follows.

Firstly, UL admits a decreasing filtration $U^{(w)}L = \bigcap_{v<w} \ker(UL \to UL^{\otimes v} \to L^{\otimes v})$. Since L is concentrated in strictly positive degrees this filtration is complete. We also filter $\widehat{\mathcal{C}}L$ by weight $\widehat{\mathcal{C}}^{(w)}L = \prod_{v \geq w} \mathcal{C}^v L$. The effect of passing to the associated graded vector spaces is to replace L with the abelian Lie coalgebra L^{ab} on the same space but with cobracket equal to zero. In other words $\mathrm{gr}(\widehat{\mathcal{C}}L) \cong \mathcal{C}L^{\mathsf{ab}} \cong k[\Sigma^{-1}L]$ and by the Poincaré-Birkhoff-Witt theorem $\mathrm{gr}(UL) \cong UL^{\mathsf{ab}} \cong k\langle L \rangle$. We also give $\widehat{\mathcal{C}}L \otimes^\tau UL \otimes^\tau \widehat{\mathcal{C}}L$ the total weight filtration. By lemma 1 it suffices to show that $\mathrm{gr}(\widehat{\mathcal{C}}L \otimes^\tau UL \otimes^\tau \widehat{\mathcal{C}}L) \to \mathrm{gr}(\widehat{\mathcal{C}}L)$ is a quasi-

isomorphism. This is exactly the Tate resolution of the diagonal

$$k[\Sigma^{-1}L, \Sigma^{-1}L]\langle L\rangle \to k[\Sigma^{-1}L] \quad \text{with} \quad \partial(x) = \Sigma^{-1}x \otimes 1 - 1 \otimes \Sigma^{-1}x \quad \text{for} \quad x \in L,$$

which is well-known to be a quasi-isomorphism. $\square$

By theorem 4 and remark 7 we obtain

Corollary 2. *If L is a graded Lie coalgebra in* $\mathbf{coLie_{gr}}$ *there is a canonical quasi-isomorphism* $UL^\vee \xrightarrow{\simeq} \mathrm{RHom}_{\widehat{\mathcal{C}}L}(k,k)$.

We end this section by stating a fact from [44, section 1], which is in a sense Koszul dual to the previous corollary. It can be proven similarly using theorem 13.

Theorem 14. *If L is in* $\mathbf{coLie_{gr}}$ *there is a canonical quasi-isomorphism* $\widehat{\mathcal{C}}L \xrightarrow{\simeq} \mathrm{RHom}_{UL^\vee}(k,k)$. *This results in a bigraded algebra isomorphism* $\mathrm{H}^p(\mathcal{C}L)_q \cong \mathrm{Ext}^p_{UL^\vee}(k,k)_q$.

Chapter 3

Koszul Duality in Local Commutative Algebra

As we discussed in the introduction, the behaviour of the homotopy Lie algebra $\pi^*(\phi)$ reflects the character of the singularity defined by a local homomorphism $\phi : R \to S$. In quite a broad sense, anything about the homotopy Lie algebra is for us something to do with Koszul Duality. In this chapter we investigate a few ideas in this direction. However, many of directions remain to be properly explored, and some important avenues will be completely ignored for now.

3.1 The Homotopy Lie algebra

In this section we explain how to assign a graded Lie algebra to a local ring R or a local homomorphism $\phi : R \to S$, using the background from section 2.8.

Recall that we are calling a dg algebra A a *minimal model* if it is a semi-free extension $Q[X]$ of a regular local ring Q with X in positive degrees, and A is absolutely minimal, so $\partial(\mathfrak{m}_A) \subseteq \mathfrak{m}_A^2$. Whenever A is a minimal model we define $\pi_*(A) = \mathfrak{s}(\mathfrak{m}_A/\mathfrak{m}_A^2)$ and $\pi^*(A) = \pi_*(A)^\vee$. Using the constructions in section 2.7, minimal models come up for us in two ways:

1. In the absolute situation we take a minimal Cohen model $Q[X] \xrightarrow{\simeq} \widehat{R}$, and then $A = Q[X]$ is by definition a minimal model. We define $\pi_*(R) = \pi_*(A)$ and $\pi^*(R) = \pi^*(A)$.
2. In the relative situation one builds a minimal Cohen model $R \to R'[X] \xrightarrow{\simeq} \widehat{S}$ for a homomorphism $\phi : (R, k) \to (S, l)$. The fibre $A = k \otimes_R R'[X]$ is then a minimal model, and we define $\pi_*(\phi) = \pi_*(A)$ and $\pi^*(\phi) = \pi^*(A)$.

These definitions depend a-priori on the construction of A. Independence is dealt with by the main theorem of this section, which we state now. It extends a classical theorem of Avramov [9, theorem 4.2], which only deals with $\pi^{\geq 2}(R)$.

Theorem 15. *If A is a minimal model with residue field k then $\pi^*(A)$ is canonically a graded Lie algebra over k, and there is a natural isomorphism $U\pi^*(A) \cong \mathrm{Ext}^*_A(k, k)$ of Hopf algebras with codivided powers. In particular:*

1. *If (R, k) is a local ring then $U\pi^*(R) \cong \mathrm{Ext}^*_R(k, k)$.*

2. *If $\phi : (R, k) \to (S, l)$ is a local homomorphism then $U\pi^*(\phi) \cong \mathrm{Ext}^*_F(l, l)$ where $F \to l$ is any augmented dg algebra model for $k \otimes^{\mathsf{L}}_R S \to l$.*

The assertion about Hopf algebras with codivided powers will be explained later in this section. We call $\pi^*(R)$ and $\pi^*(\phi)$ the *homotopy Lie algebra* of R and of ϕ respectively. The theorem states that our definition agrees with the classical one from, for example, [9].

Originally the homotopy Lie algebra was uncovered though $\mathrm{Ext}^*_R(k, k)$. It was shown by Levin [89] and Shoeller [109] that $\mathrm{Ext}^*_R(k, k)$ is a Hopf algebra whose dual $\mathrm{Tor}^R_*(k, k)$ is commutative with divided powers in positive degrees. At the point the famous structure theorem of Milnor and Moore [99], and André and Sjödin in positive characteristic [111], shows that $\mathrm{Ext}^*_R(k, k)$ is the universal envelope of a uniquely defined graded Lie algebra. Our approach is side step this structure theory and build $\pi^*(R)$ directly. From theorem 15 is follows a fortiori that $\mathrm{Ext}^*_R(k, k)$ is a Hopf algebra with codivided powers. We do not invoke the work of Levin or Schoeller.

To an absolutely minimal local dg algebra A we can associate a bigraded dg algebra $\mathrm{gr}(A)$ by setting $\mathrm{gr}^w A_i = (\mathfrak{m}^w_A/\mathfrak{m}^{w+1}_A)_i$. The upper grading is referred to as *weight*, the lower grading is the *internal degree.* Using the fact that A is minimal there is an induced differential $\partial : (\mathfrak{m}^w_A/\mathfrak{m}^{w+1}_A)_i \to (\mathfrak{m}^{w+1}_A/\mathfrak{m}^{w+2}_A)_{i-1}$.

It is convenient to fix a field k over which we will work implicitly from this point on. Let **min-mod** be the full subcategory of minimal models with residue field k. And recall that the category **q-semi-free** of quadratic semi-free dg algebras was defined in section 2.8. Since the associated graded algebra of a regular local ring is a polynomial algebra, we obtain a functor

$$\mathrm{gr} : \textbf{min-mod} \longrightarrow \textbf{q-semi-free}.$$

Through this, the fact that $\pi^*(A) = (\varepsilon\mathfrak{m}_A/\mathfrak{m}^2_A)^\vee = (\varepsilon\mathrm{gr}^1 A)^\vee$ is a graded Lie algebra is dealt with by theorem 12.

The functor grA should be thought of roughly as taking the underlying graded Lie algebra of a minimal Lie_∞ algebra. That is, at this point we throw away some homotopical information which might allow us to recover A from its homotopy Lie algebra.

Note that $\mathrm{gr}A = \mathcal{C}(\pi_*(A))$. As in section 2.8 the functor we really care about is the completion $\widehat{\mathrm{gr}}A = \prod_w \mathrm{gr}^w A$. Then $\widehat{\mathrm{gr}}A = \widehat{\mathcal{C}}(\pi_*(A))$ and following lemma combined with corollary 2 finishes the first part of theorem 15.

Lemma 11. *If $A = Q[X]$ is a minimal model then there is a canonical isomorphism of graded algebras*

$$\mathrm{Ext}^*_A(k, k) \cong \mathrm{Ext}^*_{\widehat{\mathrm{gr}}A}(k, k).$$

Proof. The key point is that the hypotheses guarantee that if $F \to k$ is the minimal dg A module resolution, then $\widehat{\mathrm{gr}}F \to k$ is the minimal dg $\widehat{\mathrm{gr}}A$ module resolution. This can be seen from theorem 11 above, which says we may take F to be the acyclic closure $Q[X]\langle t, \varepsilon X\rangle$ of k, with a differential that satisfies $\partial(\varepsilon x) = x$ modulo $(\mathfrak{m}^2_A + \mathfrak{m}_A(t, \varepsilon X)^{(1)})$, and $\partial(t) = X_0$ where X_0 is a minimal set of generators for $\mathfrak{m}_Q$. Then $\widehat{\mathrm{gr}}A = k[\![X_0, X]\!]$ and by minimality $\widehat{\mathrm{gr}}F = k[\![X_0, X]\!]\langle t, \varepsilon X\rangle$ obtains a differential $\partial : \mathfrak{m}^w_A/\mathfrak{m}^{w+1}_A F_i \to \mathfrak{m}^{w+1}_A/\mathfrak{m}^{w+2}_A F_{i-1}$ making it a graded dg module over $\widehat{\mathrm{gr}}A$. The differential of $\widehat{\mathrm{gr}}F$

still satisfies the conditions $\partial(\varepsilon x) = x$ modulo $(\mathfrak{m}_{\widehat{\mathrm{gr}}A}(t, \varepsilon X)^{(1)})$ and $\partial(t) = X_0$, and it's well known that this makes $\widehat{\mathrm{gr}}F$ acyclic (see the proof of lemma 9).

We obtain a sequence of isomorphisms of graded vector spaces

$$\mathrm{Ext}^*_A(k,k) \cong \mathrm{Hom}_A(F,k) \cong \mathrm{Hom}_{\widehat{\mathrm{gr}}A}(\widehat{\mathrm{gr}}F,k) \cong \mathrm{Ext}^*_{\widehat{\mathrm{gr}}A}(k,k).$$

It remains to understand why this isomorphism $\mathrm{Ext}^*_A(k,k) \cong \mathrm{Ext}^*_{\widehat{\mathrm{gr}}A}(k,k)$ respects the algebra structure. For this we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(F,k) \arrow[r, "\cong"] & \mathrm{Hom}_{\widehat{\mathrm{gr}}A}(\widehat{\mathrm{gr}}F,k) \\
\mathrm{Hom}_A(F,F) \arrow[u, "\simeq"] \arrow[r, "\phi \mapsto \widehat{\mathrm{gr}}\phi"] & \mathrm{Hom}_{\widehat{\mathrm{gr}}A}(\widehat{\mathrm{gr}}F,\widehat{\mathrm{gr}}F). \arrow[u, "\simeq"]
\end{tikzcd}$$

The map along the bottom of the diagram is *not* a chain map. However, it is easily seen to be an algebra map which takes cycles to cycles. The product of $\mathrm{Ext}^*_A(k,k)$ may be computed by lifting two elements to cycles in $\mathrm{Hom}_A(F,F)$ and composing them there, similarly for $\mathrm{Ext}^*_{\widehat{\mathrm{gr}}A}(k,k)$. With this in mind, the map $\phi \mapsto \widehat{\mathrm{gr}}\phi$ is good enough to witness the fact that the isomorphism above is one of algebras. $\square$

As is evident from the proof, the isomorphism $\mathrm{Ext}^*_A(k,k) \cong \mathrm{Ext}^*_{\widehat{\mathrm{gr}}A}(k,k)$ need not lift to quasi-isomorphism between $\mathrm{RHom}_A(k,k)$ and $\mathrm{RHom}_{\widehat{\mathrm{gr}}A}(k,k)$. In fact, we've seen already that $\mathrm{RHom}_{\widehat{\mathrm{gr}}A}(k,k)$ is always formal, while $\mathrm{RHom}_A(k,k)$ is usually not so.

Both 1 and 2 in theorem 15 are dealt with by

Lemma 12. *Let $A \xrightarrow{\simeq} B$ be a quasi-isomorphism of local dg algebras with common residue field k. There are canonical isomorphisms of graded algebras*

$$\mathrm{Ext}^*_A(k,k) \cong \mathrm{Ext}^*_B(k,k) \cong \mathrm{Ext}^*_{\widehat{B}}(k,k).$$

The proof of the lemma is standard. With this 1 is established using the quasi-isomorphism $A = Q[X] \xrightarrow{\simeq} \widehat{R}$. For 2, if F is a model for the derived fibre $k \otimes^{\mathsf{L}}_R S$ then $\widehat{F}$ is connected to A by a chain of quasi-isomorphisms, where $A = k \otimes_R R'[X]$ is the minimal model arising from a minimal Cohen model $R \to R'[X] \to \widehat{S}$ for $R \to \widehat{S}$. This is because $R'[X]$ is a flat resolution of $\widehat{S}$ over R. Therefore $\mathrm{Ext}^*_A(k,k) \cong \mathrm{Ext}^*_F(k,k)$ by the lemma.

It remains to explain the assertion in theorem 15 about Hopf algebras with codivided powers.

$\mathrm{Tor}^A_*(k,k)$ has the structure a graded coalgebra, dual to the Yoneda algebra structure on $\mathrm{Ext}^*_A(k,k)$. Taking an acyclic closure $A \to A\langle U\rangle \xrightarrow{\simeq} k$ we see that $\mathrm{Tor}^A_*(k,k) \cong k\langle U\rangle$ by Gulliksen and Schoeller's minimality theorem 11. Thus $\mathrm{Tor}^A_*(k,k)$ also has the structure of a (free) divided power algebra, with divided powers supported on the ideal of positive degree elements. It is classical that this makes $\mathrm{Tor}^A_*(k,k)$ into a Hopf algebra with divided powers[1], that is, the coproduct $\Delta : \mathrm{Tor}^A_*(k,k) \to \mathrm{Tor}^A_*(k,k)^{\otimes 2}$ is a homomorphism of divided power algebras. We say that the dual $\mathrm{Ext}^*_A(k,k) = \mathrm{Tor}^A_*(k,k)^\vee$ is a Hopf algebra with codivided powers.

[1]This generalises the work of Assmus, Levin and Schoeller [5, 87, 109] from the case that A is a local ring. The general cases can be shown (for example) by using lemma 11 to reduce to the equicharacteristic case and then using lemma 13 to present $U\pi_*(A)$ as a model for $\mathrm{Tor}^{\mathrm{gr}(A)}_*(k,k)$ with this structure.

The isomorphism constructed in lemma 11 clearly preserves this structure. That is, it was explicitly dual to an isomorphism $\mathrm{Tor}^A_*(k,k) \cong k\langle U\rangle \cong \mathrm{Tor}^{\widehat{\mathrm{gr}}(A)}_*(k,k)$ of divided power algebras.

The map from theorem 4 which was used to build the isomorphism in theorem 15 made use of the twisted tensor product $\widehat{\mathcal{C}}\pi_*(A) \otimes^\tau U\pi_*(A) \simeq k$.

Lemma 13. *Let L be a graded Lie coalgebra in $\mathbf{coLie_{gr}}$, and let $\tau : UL \to \widehat{\mathcal{C}}L$ be the acyclic twisting cochain of theorem 13.*

The natural divided paper structure is compatible with the twist by τ. That is, $\widehat{\mathcal{C}}L \otimes^\tau UL$ is a divided power algebra with divided powers supported on the ideal $\left(\overline{UL}\right)$ and thus $\widehat{\mathcal{C}}L \to \widehat{\mathcal{C}}L \otimes^\tau UL \to k$ is an acyclic closure.

Remark 8. The lemma works just as well for the bimodule resolution $\widehat{\mathcal{C}}L \otimes^\tau UL \otimes^\tau \widehat{\mathcal{C}}L$.

Proof. It is a classical theorem the universal envelope of a graded Lie coalgebra is a Hopf algebra with divided powers (this is a structural version of the Poincaré-Birkhoff-Witt theorem, it is explicit in [32, proposition 4.2]).

Let u be a primitive element of UL, with $\Delta(u) = 1 \otimes u + u \otimes 1$. This implies $\Delta(u^{(n)}) = \sum_{i+j=n} u^{(i)} \otimes u^{(j)}$, so

$$\begin{aligned}
\partial(u^{(n)}) &= (m \otimes 1)(1 \otimes \tau \otimes 1)(1 \otimes \Delta)(1 \otimes u)\\
&= \sum_{i+j=n} (m \otimes 1)(1 \otimes \tau \otimes 1)(1 \otimes u^{(i)} \otimes u^{(j)})\\
&= \tau(u) \otimes u^{(n-1)}\\
&= \partial(u)u^{(n-1)}
\end{aligned}$$

since $\tau(u^{(i)}) = 0$ unless $i = 1$. Since $\left(\overline{UL}\right)$ is generated by primitives as a divided power ideal it follows quite formally that $\partial(u^{(n)}) = \partial(u)u^{(n-1)}$ for any $u \in \left(\overline{UL}\right)$ (this involves short a calculation for each of the divided power axioms).

Note that through the augmentation there is an algebra homomorphism $\epsilon : UL \to k \to \widehat{\mathcal{C}}L$. Note that τ is actually a derivation with respect to ϵ. It follows that the twisted differential satisfies

$$\begin{aligned}
\partial(uv) &= (-1)^{|v'||u''|}\tau(u'v')u''v''\\
&= (-1)^{|v'||u''|}(\epsilon(u')\tau(v')u''v'' + \tau(u')\epsilon(v')u''v'')\\
&= (-1)^{|v'||u|}\tau(v')uv'' + \tau(u')u''v\\
&= u\partial(v) + \partial(u)v
\end{aligned}$$

where we have used an informal Sweedler notation $\Delta(u) = u' \otimes u''$ and $\Delta(v) = v' \otimes v''$. (Actually τ is a divided power derivation for the trivial divided power structure on the zero ideal of $\widehat{\mathcal{C}}\pi_*(A)$, so a similar trick can be used to simplify the verification above.)

In all, this shows that $\widehat{\mathcal{C}}L \otimes^\tau UL$ is a dg divided power algebra. Since $\widehat{\mathcal{C}}L \otimes^\tau UL$ is acyclic and minimal as a complex, the last assertion is clear. □

It follows from the lemma that the natural divided power structure on $U\pi_*(R)$ gives rise to the divided power structure of $\mathrm{Tor}^{\widehat{\mathcal{C}}\pi_*(A)}_*(k,k) \cong k \otimes_{\widehat{\mathcal{C}}\pi_*(A)} (\widehat{\mathcal{C}}\pi_*(A) \otimes^\tau U\pi_*(A)) \cong U\pi_*(R)$. This isomorphism is dual to the one given by theorem 4. We have finished the proof of theorem 15.

Example 4. Recall that Golod homomorphisms were discussed in section 2.7. Knowing from theorem 15 that our definition of the homotopy Lie algebra agrees with the usual one, it follows from [10, theorem (3.4)] that $\phi : R \to S$ is Golod if and only if $\pi^*(\phi)$ is free as a graded Lie algebra.

Slightly extending the notion introduced by Avramov in [11] we also say that ϕ is generalised Golod of level n if the graded Lie algebra $\pi^{>n}(\phi)$ is free. Thus, the Golod condition coincides with generalised Golod of level 0. It is not difficult to see that the conditions in proposition 5 are equivalent to being generalised Golod of level 1 (see proposition 9 for more detail).

Remark 9. A local ring R is Koszul if its residue field has finite linearity defect, in the sense of [67]. Equivalently, the graded algebra $\mathrm{gr}(R)$ is Koszul in the classical sense of section 2.3 (see remark 1.10 of op. cit.). It is also equivalent that $\pi^*(R)$ is generated in degree 1 as a graded Lie algebra.

One can see using the proof of lemma 11 that R is Koszul if and only if $\mathrm{gr}(A) \to \mathrm{gr}(R)$ is a quasi-isomorphism whenever $A \xrightarrow{\simeq} R$ is a minimal Cohen model. In particular, $\mathrm{gr}(R)$ can be recovered directly from the homotopy Lie algebra $\pi^*(R)$ by taking $\mathrm{H}_0(\mathcal{C}\pi_*(R))$.

Deviations of a Local Ring or Homomorphism

Let $R \to R\langle U\rangle \to k$ be an acyclic closure of the residue field of R. Classically, the *deviations* of R have been defined as $\epsilon_n(R) = \mathrm{card}(U_n)$. By Gulliksen's theorem 11 above there is an isomorphism $k\langle U\rangle \cong \mathrm{Tor}^R_*(k,k)$. It follows that the deviations are uniquely determined by the well-known formula

$$P^R_k(t) = \frac{\prod_{i\geq 1}(1+t^{2i-1})^{\epsilon_{2i-1}(R)}}{\prod_{i\geq 1}(1-t^{2i})^{\epsilon_{2i}(R)}},$$

see [12, remark 7.1.1]. By theorem 15 the same formula characterises the numbers $\dim_k \pi_i(R)$. Taking a minimal model $Q[X] \xrightarrow{\simeq} R$ and choosing a minimal set X_0 of generators for $\mathfrak{m}_Q$, we recover another theorem of Avramov that $\epsilon_n(R) = \dim_k \pi_n(R) = \mathrm{card}(X_{n-1})$, see [9] or [12, proposition 7.2.3].

In [13, section 3] the deviations of a local homomorphism $\phi : (R,k) \to (S,l)$ are (essentially) defined as $\epsilon_n(\phi) = \dim_l \pi_n(\phi)$. Theorem 15 allows one to express these invariants in terms of the Poincaré series of the dg algebra $k \otimes^{\mathbf{L}}_R S$ using exactly the same formula.

A great deal can be said about the behaviour of deviations, especially their growth. We will not discuss this rich subject here. The reader may consult [65, 12, 13, 47, 63] for a beginning.

Functoriality of the Homotopy Lie Algebra

We end this section by describing briefly in what sense the homotopy Lie algebra is functorial. This is essentially classical, but (except for the absolute homotopy Lie algebra) details are difficult to find in the literature. So we sketch the arguments.

Suppose we have a commutative diagram of local homomorphisms

$$\begin{array}{ccc} (R,k) & \xrightarrow{\phi} & (S,l) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ (T,t) & \xrightarrow{\psi} & (U,u). \end{array}$$

Following [16] we take a minimal Cohen factorisation $T \to T' \to \widehat{U}$, then we form the fibre product $T' \times_{\widehat{U}} \widehat{S}$. There is a natural homomorphism $R \to T' \times_{\widehat{U}} \widehat{S}$, and we can take a Cohen factorisation of this to obtain $R \to \widetilde{R}' \to T' \times_{\widehat{U}} \widehat{S}$. Finally, if the induced factorisation $R \to \widetilde{R}' \to \widehat{S}$ is not minimal, we can factor $R' = \widetilde{R}'/(\underline{x})$ by a regular sequence $\underline{x} \subseteq \mathfrak{m}_{\widetilde{R}'}/\mathfrak{m}^2_{\widetilde{R}'}$ to obtain a minimal Cohen factorisation $R \to R' \to \widehat{S}$.

We can then extend this construction to a diagram of minimal models

$$\begin{tikzcd} & R'[X] \arrow[dr, "\simeq"] & \\ R \arrow[ur] \arrow[r] \arrow[d] & \widetilde{R}'[\Sigma\underline{x}, X] \arrow[u, "\simeq"'] \arrow[r, "\simeq"] \arrow[d] & \widehat{S} \arrow[d] \\ T \arrow[r] & T'[Y] \arrow[r, "\simeq"] & \widehat{U}. \end{tikzcd}$$

Having done this we set $A = k \otimes_R R'[X]$ and $\widetilde{A} = k \otimes_R \widetilde{R}'[X]$ and $B = t \otimes_T T'[Y]$.

Theorem 16. *The induced map*

$$\pi_*(\phi) = \Sigma\mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_l u \xleftarrow{\cong} \mathrm{H}_*(\Sigma\mathfrak{m}_{\widetilde{A}}/\mathfrak{m}^2_{\widetilde{A}}) \otimes_l u \longrightarrow \Sigma\mathfrak{m}_B/\mathfrak{m}_B^2 = \pi_*(\psi)$$

is a homomorphism of Lie coalgebras over u. It is independent of all choices made, and we denote it by $(\alpha,\beta)_* : \pi_*(\phi) \otimes_l u \to \pi_*(\psi)$. By dualising we obtain a homomorphism $(\alpha,\beta)^* : \pi^*(\psi) \to \pi^*(\phi) \otimes_l u$ of Lie algebras.

If we have another diagram of local homomorphisms

$$\begin{tikzcd} (T,t) \arrow[r, "\psi"] \arrow[d, "\alpha'"] & (U,u) \arrow[d, "\beta'"] \\ (T',t') \arrow[r, "\psi'"] & (U',u'), \end{tikzcd}$$

then $((\alpha',\beta')_* \otimes_u u')(\alpha,\beta)_* = ((\alpha'\alpha),(\beta'\beta))_* : \pi_*(\phi) \otimes_l u' \to \pi_*(\psi')$.

In many situations we can directly build compatible minimal Cohen models, see [101]. In this case it is clear from theorem 12 that $(\alpha,\beta)_*$ is a homomorphism of Lie coalgebras.

The theorem can be proven by invoking theorem 15 to work instead with divided power Hopf algebras, after which this is essentially a classical theorem of Avramov [9, theorem 2.1].

We sketch some details now. Let $R \to R\langle U\rangle \xrightarrow{\simeq} k$ and $T \to T\langle V\rangle \xrightarrow{\simeq} t$ be acyclic closures. The classical lifting property [64, lemma 1.8.6] allows us to lift $k \to t$ to $R\langle V\rangle \to T\langle W\rangle$. By tensoring these homomorphisms together we get a diagram of dg algebras

$$\begin{tikzcd} A & R'[X]\langle U\rangle \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \widehat{S}\langle V\rangle \\ \widetilde{A} \arrow[u] \arrow[d] & \widetilde{R}'[\Sigma\underline{x}, X]\langle U\rangle \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[u] \arrow[d] & \widehat{S}\langle V\rangle \arrow[u, equal] \arrow[d] \\ B & T'[Y]\langle W\rangle \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \widehat{U}\langle W\rangle. \end{tikzcd}$$

with quasi-isomorphisms as indicated. Now we use the fact that $\mathrm{Tor}^C_*(c,u)$ is a functor from local dg algebras (C, c) with a morphism $c \to u$ to the category of Hopf algebras over u with divided powers, and this functor takes quasi-isomorphisms to isomorphisms. The right column of the diagram is independent of the choice of Cohen factorisations and maps between them, while the left column is independent of the choice of acyclic closure and lift between them. Thus applying this functor establishes independence and functoriality.

It remains to explain why the map described in the theorem is the same as the one induced by the functor $\mathrm{Tor}_*^{(-)}(-, u)$ (after taking divided power indecomposables to recover π_*). This is a technical fact which follows from the functoriality of the Tate model for the diagonal, proven in section 3.4. The result we need is theorem 22 from that section, which is really a version of [21, Theorem 3.4].

A local homomorphism $\phi : (R, k) \to (S, l)$ also gives rise functorially to a homomorphism of Lie coalgebras which we will denote $\phi_* : \pi_*(R) \otimes_k l \to \pi_*(S)$. It is dual to a homomorphism of Lie algebras $\phi^* : \pi^*(S) \to \pi^*(R) \otimes_k l$.

3.2 The L.S. Category of a Ring Homomorphism

To a local homomorphism $\phi : R \to S$ we will associate a sequence of numbers $\mathrm{cat}_i(\phi)$ for $i = 0, 1, 2...$, which, in brief, can be thought of as the Lusternik-Schnirelmann category of the i-connected cover of the derived fibre of ϕ. The i-category can be a natural number, or it can be infinite. In this section we prove that this invariant is well-defined and we establish some of its basic properties.

The ideas in this section are old. However, the definitions given below do not seem to appear elsewhere in this generality, so the results are at least superficially new. Except for well-definedness of cat_0 and cat_1, most of these these results follow without a great deal of fuss from known facts in the literature.

First, let us briefly describe the history of the L.S. category, beginning with topology. Much more detail can be found in [46] or [68].

Let X be a topological space. A subspace U of X is called contractible in X if the inclusion $U \to X$ is homotopic to a constant map. We then define the *L.S. category* of X to be the least integer m such that X can be covered by $m+1$ open sets, each of which is contractible in X. This number is denoted $\mathrm{cat}(X) = m$. If no such m exists then X has infinite L.S. category. The normalisation is chosen so that $\mathrm{cat}(X) = 0$ precisely if X is contractible. Note also that $\mathrm{cat}(S^n) = 1$ for the n-dimensional sphere S^n.

This invariant was introduced by Lusternik and Schnirelmann in [93], where it is shown that if X is a closed, smooth manifold then any smooth function on X has strictly more than $\mathrm{cat}(X)$ critical points. The L.S. category was imported to rational homotopy theory in the thesis of Lemaire [86]. After the ground-breaking work of Félix and Halperin in [43], the central position of L.S. category in rational homotopy theory was cemented. In short, a simply connected space X is called *rational* if all of the homotopy groups $\pi_i(X)$ are rational vector spaces, and crucially rational spaces can be represented algebraically by their Sullivan models, which are certain semi-free dg commutative algebras over $\mathbb{Q}$. Félix and Halperin showed that for rational spaces L.S. category can be computed algebraically in terms of their Sullivan models. The definition given below for ring homomorphisms is a direct transcription of their characterisation. From this they were able to deduce the Mapping Theorem: if $X \to Y$ is a map of simply connected rational spaces such that each $\pi_i(X) \to \pi_i(Y)$ is injective, then $\mathrm{cat}(X) \leq \mathrm{cat}(Y)$. Further, they used the Mapping Theorem to prove the beautiful Dichotomy Theorem: if X is a simply connected rational space of finite category then the sequence of numbers $\dim_{\mathbb{Q}} \pi_i(X)$ is either eventually zero, or it must grow exponentially.

Around the time of [43] Avramov and Roos began to uncover deep connections between local commutative algebra and rational homotopy theory. This has led to a great deal of collaboration and progress in both areas, some of which is described in the surveys [9] and [6]. As an example, Avramov in [13] has since established the local commutative algebra analogue of the Dichotomy Theorem: if $\phi : R \to S$ is a local homomorphism of finite category, then either ϕ is complete intersection or the deviations $\epsilon_i(\phi)$ grow exponentially (see also [21, corollary 5.5]). Another highlight is the so-called Five Author Paper of Félix, Halperin, Jacobsson, Löfwall and Thomas [44], in which the authors use L.S. category to constrain the structure of the homotopy Lie algebra in both rational homotopy theory and local commutative algebra (see theorem 18 below).

The Definition of L.S. Category

We now define the i-category of a local homomorphism $\phi : R \to S$ in stages, starting with category for dg algebras.

If $(A, \mathfrak{m}_A, k)$ is a local dg algebra then the *category* of A is the least integer m such that the quotient $A \to A/\mathfrak{m}_A^{m+1}$ factors as a split monomorphism followed by a surjective quasi-isomorphism. In other words, $\mathrm{cat}(A)$ is the least m for which a diagram of local dg algebras exists

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[dr] & B \arrow[l, bend right, "\rho"'] \arrow[d, two heads, "\simeq"] \\ & A/\mathfrak{m}_A^{m+1} \end{tikzcd}$$

in which the triangle commutes and $\rho\iota = 1$. If no such m exists then by definition A has infinite category.

While the definition makes sense for any local dg algebra, we will usually only apply it when A is a minimal model. That is, when $\partial(\mathfrak{m}_A) \subseteq \mathfrak{m}_A^2$ and A has the form $Q[X]$ for a regular local ring Q and a graded set X in strictly positive degrees. It is these dg algebras for which category has the best properties. However, the extra generality is necessary and we will make no such assumption below unless explicitly mentioned.

Remark 10. Note that if $\mathrm{cat}(A) \leq m$ then the maximal ideal $\mathfrak{m}_{\mathrm{H}_*(A)}$ of $\mathrm{H}_*(A)$ satisfies $\mathfrak{m}_{\mathrm{H}_*(A)}^{m+1} = 0$, and in particular the product of more than m positive degree elements must vanish. It is useful to have a name for this nilpotency: if B is a local graded algebra then the *Loewey length* of B is the (possibly infinite) number

$$\ell\ell(B) = \inf\{i \geq 0 : \mathfrak{m}_B^i = 0\}.$$

So, the observation above may be rewritten as $\mathrm{cat}(A) \geq \ell\ell(\mathrm{H}_*(A)) - 1$. However, finiteness of category is a much stronger and more structural nilpotence condition than this. It says that, up to homotopy, A is a retract of $A/\mathfrak{m}_A^{m+1}$.

Remark 11. There is by proposition 4 a minimal semi-free model for the quotient $A \to A[Y] \xrightarrow{\simeq} A/\mathfrak{m}_A^{m+1}$, and this model is unique up to isomorphism. If $\mathrm{cat}(A) = m$ then in fact this extension $A \to A[Y]$ admits a retract. This is because by the lifting property of lemma 5 there is an A algebra lift

$A[Y] \to B$, which makes $A \to A[Y]$ split via B. Hence, to establish the inequality $\operatorname{cat}(A) \le m$ we will always take *the* minimal model $A \to A[Y] \xrightarrow{\simeq} A/\mathfrak{m}_A^{m+1}$ and attempt to construct a splitting $A \leftarrow A[Y]$.

Remark 12. Let $m \le n$ be two natural numbers and suppose that a retract exists as in the above diagram, demonstrating that $\operatorname{cat}(A) \le m$. Then if $A \to A[Y] \xrightarrow{\simeq} A/\mathfrak{m}_A^{n+1}$ is a minimal model there is also a retract $A \leftarrow A[Y]$. This follows readily from the existence of the A algebra lift in the diagram

$$\begin{tikzcd} A[Y] \arrow[r, dashed] \arrow[d, "\simeq"] & B \arrow[d, "\simeq"] \\ A/\mathfrak{m}_A^{n+1} \arrow[r] & A/\mathfrak{m}_A^{m+1}. \end{tikzcd}$$

Let A be a minimal model. We define the *i-connected cover* of A to be the dg algebra

$$A^{(i)} = A/([\mathfrak{m}_A]_{<i})$$

where $([\mathfrak{m}_A]_{<i})$ is the ideal in A generated by all elements of $\mathfrak{m}_A$ with degree less than i.

The name i-connected cover comes by analogy with topology. The homomorphism $A \to A^{(i)}$ induces a map $\pi^*(A^{(i)}) \to \pi^*(A)$ of graded Lie algebras, and by construction this will give an isomorphism $\pi^*(A^{(i)}) \cong \pi^{>i}(A)$. In other words, the effect of factoring by $([\mathfrak{m}_A]_{\le i})$ is to remove the homotopy groups $\pi^{<i}(A)$.

Now suppose that $(R, \mathfrak{m}, k)$ and $(S, \mathfrak{n}, l)$ are local rings, and let $\phi : R \to S$ be a local homomorphism. Let $R \to R'[X] \to \widehat{S}$ be a minimal Cohen model of ϕ, as in section 2.7. In particular, the fibre $Q = R'/\mathfrak{m}R'$ is regular and $A = Q[X] = k \otimes_R R'[X]$ is a model for the derived fibre $k \otimes^{\mathsf{L}}_R S$. For a natural number i we define the *i-category* of ϕ to be

$$\operatorname{cat}_i(\phi) = \operatorname{cat}(A^{(i)})$$

In other words, $\operatorname{cat}_0(\phi)$ is the category of $Q[X]$, while $\operatorname{cat}_i(\phi)$ is the category of $l[X_{\ge i}] = l[X]/(X_{<i})$ for any $i > 0$.

We also define the *absolute i-category* of a local ring by taking a minimal model $A = Q[X] \xrightarrow{\simeq} \widehat{R}$. Then exactly as before $\operatorname{cat}_i(R) = \operatorname{cat}(A^{(i)})$ by definition. Thus, the inequality $\operatorname{cat}_i(R) \le m$ says intuitively that the i-connected cover of R satisfies the strong $(m+1)$-nilpotence condition described above.

If R is equicharacteristic we recover the absolute situation by considering a coefficient field $k \to R$. That is, we have $\operatorname{cat}_i(R) = \operatorname{cat}_i(k \to R)$ for all i. Conversely, if $\phi : R \to S$ is a flat local homomorphism with fibre $F = k \otimes_R S$ then it is easy to see that $\operatorname{cat}_i(\phi) = \operatorname{cat}_i(R)$ for all i.

Properties of L.S. Category and Examples

Recall that when the extension $k \to l$ is separable any two minimal Cohen factorisations of $R \to \widehat{S}$ are isomorphic, and hence A is determined up to isomorphism. In the general case, we need to establish that the numbers $\operatorname{cat}_i(R)$ and $\operatorname{cat}_i(\phi)$ are well-defined.

Theorem 17. *For all i the numbers $\operatorname{cat}_i(R)$ and $\operatorname{cat}_i(\phi)$ are independent of the choice of minimal models. If R is a quotient of a regular local ring Q then the i-category may be computed using a minimal model $Q[X] \xrightarrow{\simeq} R$ (without completing R). Similarly, if ϕ admits a minimal regular factorisation $R \to T \to S$ then the i-category may be computed using a minimal model $T[Y] \xrightarrow{\simeq} S$ (without completing S).*

Note that the last statement applies in particular to surjective homomorphisms. To be more precise, the assertion is that $\mathrm{cat}_i(\phi) = \mathrm{cat}(\overline{T}[Y]^{(i)})$ for all i, where $\overline{T}$ is the fibre $T/\mathfrak{m}T$, even though $\mathrm{cat}_i(\phi)$ is computed using a Cohen factorisation of the semi-completion $\grave{\phi} : R \to \widehat{S}$.

Let us point out that well-definedness of cat_i for $i \geq 1$ is essentially proven by Avramov and Iyengar in [21, proposition 2.4]. The main difficulty will be in dealing with cat_0 and cat_1. Since the proof of the theorem is long we delay it until the end of this section. Instead we start by discussing some properties of this invariant.

Firstly, the following fact is an immediate consequence of the mapping theorem, which will be stated later in this section (and proven in section 3.3).

Proposition 6. *For any local homomorphism $\phi : R \to S$, the i-category decreases as i increases*

$$\mathrm{cat}_0(\phi) \geq \mathrm{cat}_1(\phi) \geq \mathrm{cat}_2(\phi) \geq \mathrm{cat}_3(\phi) \cdots$$

Remark 13. In [21] Avramov and Iyengar consider the *weak category* of a local homomorphism $\phi : R \to S$, which they define as follows

$$\mathrm{wcat}(\phi) = \inf \left\{ m \in \mathbb{N} \;:\; \ell\ell(\mathrm{H}_*(A^{(i)}) \leq m+1 \text{ for all } i \geq 2 \right\},$$

where $A = k \otimes_R R'[X]$ is the dg algebra constructed as above from a minimal Cohen model $R \to R'[X] \to \widehat{S}$. By the previous proposition $\mathrm{wcat}(\phi) \leq \mathrm{cat}_2(\phi)$. Avramov and Iyengar show that finiteness of weak category suffices to establish good properties of the homotopy Lie algebra $\pi^*(\phi)$, and they prove that almost small homorphisms have finite weak category. In fact, it will be a consequence of the mapping theorem that almost small homomorphisms have finite 1-category. See corollary 3 for details and the definition of almost small homomorphisms.

Unsurprisingly, category and Loewy length are closely connected invariants. We have already observed that $\mathrm{cat}(A) \geq \ell\ell(\mathrm{H}_*(A)) - 1$. In the other direction, the following lemma is extremely useful for bounding the i-category of a homomorphism.

If B is a local dg algebra then $\ell\ell(B)$ is by definition the Loewy length of the underlying graded algebra. This is certainly not a quasi-ismorphism invariant. Rather, it is a property of a particular model for B.

Lemma 14. *Let A and B be local dg algebras.*

1. *If there is a surjective quasi-isomorphism $A \to B$ then* $\mathrm{cat}(A) \leq \ell\ell(B) - 1$.
2. *If A is a retract of B then* $\mathrm{cat}(A) \leq \mathrm{cat}(B)$.

Proof. For 1 assume that $\ell\ell(B) = m+1$ is finite. Let $A \to A[Y] \xrightarrow{\simeq} A/\mathfrak{m}_A^{m+1}$ be a minimal model. Because $\mathfrak{m}_B^{m+1} = 0$ the homomorphism $A \to B$ factors through $A/\mathfrak{m}_A^{m+1}$, as in the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[dr] & A[Y] \arrow[l, dashed, bend right] \arrow[d, "\simeq"] \arrow[dr] & \\
 & A/\mathfrak{m}_A^{m+1} \arrow[r] & B.
\end{tikzcd}$$

Since $A \to B$ is a surjective quasi-isomorphism the standard lifting property from lemma 5 produces an A algebra splitting as showing in the diagram. A similar lifting argument easily establishes 2. $\square$

Note that $\mathrm{cat}_0(\phi) = \mathrm{cat}_1(\phi)$ for surjective homomorphisms. In general, $\mathrm{cat}_0(\phi)$ is will be infinite unless the fibre $\overline{S} = S/\mathfrak{m}S$ is Artinian, because $\ell\ell(\overline{S}) \leq \mathrm{cat}_0(\phi) + 1$.

Example 5. Consider the homomorphism $f : k[\![f_1, ..., f_c]\!] \hookrightarrow k[\![x_1, ..., x_n]\!]$ where $f_1, ..., f_c$ is a regular sequence in $(x_1, ..., x_n)^2$. Then f is flat and the fibre $A = k[\![x_1, ..., x_n]\!]/(f_1, ..., f_c)$ is a complete intersection of codimension c. One can easily build a minimal Cohen factorisation

$$k[\![f_1, ..., f_c]\!] \to k[\![f_1, ..., f_c, y_1, ..., y_n]\!] \xrightarrow{y_i \mapsto x_i} k[\![x_1, ..., x_n]\!].$$

Since the kernel of $k[\![f_1, ..., f_c, y_1, ..., y_n]\!] \to k[\![x_1, ..., x_n]\!]$ is generated by the regular sequence $(f_1(y) - f_1), ..., (f_c(y) - f_c)$ the homomorphism f is complete intersection in the sense of [13]. For a minimal model we have the Koszul complex $k[\![f_1, ..., f_c, y_1, ..., y_n]\!][Z] \xrightarrow{\simeq} k[\![x_1, ..., x_n]\!]$ where $Z = \{z_1, ..., z_c\}$ is a set of degree one variables with $\partial(z_i) = f_i(y) - f_i$. Hence

$$\mathrm{cat}_0(f) = \mathrm{cat}(k[\![y_1, ..., y_n]\!][Z]), \quad \mathrm{cat}_1(f) = \mathrm{cat}(k[Z]) \quad \text{and} \quad \mathrm{cat}_{\geq 2}(f) = \mathrm{cat}(k).$$

Thus $\mathrm{cat}_{\geq 2}(f) = 0$, and since $k[Z]$ is an exterior algebra on c generators $\mathrm{cat}_1(f) = c$ is the codimension of A. I claim that $\mathrm{cat}_0(f) = \ell\ell(A) - 1$. Suppose that $m + 1 = \ell\ell(A)$ is finite (this is equivalent to $n = c$). Then applying lemma 14 to the surjective quasi-isomorphism $k[\![y_1, ..., y_n]\!][Z] \to A$ shows that $\mathrm{cat}_0(f) \leq \ell\ell(A) - 1$. As we have already mentioned the reverse inequality always holds, so this concludes the calculation of $\mathrm{cat}_i(f)$ for all i. Note that if the f_i come from homogeneous forms in the standard graded polynomial algebra $k[x_1, ..., x_n]$ then (assuming $n = c$) the number $\mathrm{cat}_0(f) = \ell\ell(A) - 1$ is the socle degree of A, which is equal to the product $\deg(f_1) \cdots \deg(f_c)$.

On the other hand $\mathrm{cat}_1(\phi)$ is often finite even when the fibre $\overline{S}$ is not Artinian. As a consequence of the next proposition, this is always the case if R is regular.

Proposition 7. *For any minimal model A such that A_0 is a field we have* $\mathrm{cat}(A) \leq \sup(A)$.

For a local ring R we have $\mathrm{cat}_0(R) = \ell\ell(R) - 1$ *and* $\mathrm{cat}_1(R) \leq \mathrm{codepth}(R)$. *In particular* $\mathrm{cat}_i(R)$ *is always finite for $i > 0$.*

For a local homomorphism $\phi : R \to S$ we have $\mathrm{cat}_1(\phi) \leq \mathrm{fd}_R S + \mathrm{edim}\, S/\mathfrak{m}S$, *where* $\mathrm{fd}_R S$ *is the flat dimension of S as an R module.*

In the statement we have used the notation $\mathrm{codepth}(R) = \mathrm{edim}(R) - \mathrm{depth}(R)$.

The analogous statement in rational homotopy theory is that a simply connected space X with finite dimensional cohomology $\mathrm{H}^*(X; \mathbb{Q})$ has finite rational category. This local algebra version is also essentially well known, compare [13, lemma 3.8] or [21, theorem 5.7].

Proof. Recall that $m = \sup(A)$ is by definition $\sup\{i \in \mathbb{N} \ : \ \mathrm{H}_i(A) \neq 0\}$. Define a graded subspace $I \subseteq A$ by setting $I_i = A_i$ when $i > m$ and $I_m = \partial(A_{m+1})$ and $I_i = 0$ when $i < m$. By construction I is an acyclic ideal of A and hence $A \to A/I$ is a surjective quasi-isomorphism. But $\mathfrak{m}_{A/I}^{m+1} = (A/I)_{>0}^{m+1} \subseteq (A/I)_{\geq m+1} = 0$ so by lemma 14 $\mathrm{cat}(A) \leq \ell\ell(A/I) \leq m$.

For the second assertion we take a minimal model $Q[X] \to \widehat{R}$. Then by lemma 14 $\mathrm{cat}_0(R) = \mathrm{cat}(Q[X]) \leq \ell\ell(R) - 1$, and the other inequality is clear. Using the previous part $\mathrm{cat}_1(R) = \mathrm{cat}(k[X]) \leq \sup(k[X]) = \mathrm{pd}_Q(R) = \mathrm{codepth}(R)$.

For the last assertion, take a minimal Cohen model $R \to R'[X] \xrightarrow{\simeq} \widehat{S}$ and write $Q = k \otimes_R R'$. By definition of minimality $\dim Q = \mathrm{edim}\, S/\mathfrak{m}S$. Since $Q[X]$ is a model for $k \otimes^{\mathsf{L}}_R S$ we have $\sup Q[X] = \mathrm{fd}_R S$.

Now the 1-category of ϕ is $\mathrm{cat}(l[X])$, which is bounded above by $\sup(l[X])$. If K is the Koszul complex over Q on a minimal set of generators for $\mathfrak{m}_Q$, then there is a quasi-isomorphism of complexes $K \otimes_Q Q[X] \xrightarrow{\simeq} l[X]$. A simple spectral sequence argument now shows that $\sup(k[X]) = \sup(K \otimes_Q Q[X]) \leq \mathrm{fd}_R S + \dim Q = \mathrm{fd}_R S + \mathrm{edim}\, S/\mathfrak{m}S$. This establishes the inequality $\mathrm{cat}_1(\phi) \leq \mathrm{fd}_R S + \mathrm{edim}\, S/\mathfrak{m}S$. □

Remark 14. The bound $\mathrm{cat}_1(\phi) \leq \mathrm{fd}_R S + \mathrm{edim}\, S/\mathfrak{m}S$ can be improved dramatically, because the estimate of $\sup(l[X])$ in the proof above was very crude. The actual computation of $\sup(l[X])$ is doable, but since the argument is long we leave it for now and content ourselves with the bound given in the proposition.

The extent to which $\mathrm{cat}(A)$ only depends only on the Lie algebra $\pi^*(A)$ is an interesting question. Example 5 shows that in general $\mathrm{cat}(A)$ cannot be computed from $\pi^*(A)$: in the notation there $\pi^*(f)$ only depends on the quadratic part of f while $\mathrm{cat}_0(f)$ does not. However a similar argument to the previous proposition yields the following result from the Five Author Paper [44].

Proposition 8. *Let A be a complete minimal model and let $L = \pi^*(A)$ be its homotopy Lie algebra. Then $\mathrm{cat}(A) \leq \mathrm{gldim}(UL)$ where $\mathrm{gldim}(UL)$ is the global dimension $\sup\left\{i : \mathrm{Ext}^i_{UL}(k,k)_* \neq 0\right\}$.*

Proof. We sketch the argument in [44]. If $\mathrm{Ext}^{>g}_{UL}(k,k)_* = 0$ then by theorem 14 also $H^{>g}(\mathrm{gr}A)_* = 0$. Let $V \subset \mathfrak{m}^g_A$ be any Q submodule whose image in $\mathfrak{m}^g_A/\mathfrak{m}^{g+1}_A$ is a complement to the kernel of $\partial : \mathrm{gr}^g(A) \to \mathrm{gr}^{g+1}(A)$. Then $I = V + \mathfrak{m}^{g+1}_A$ is an ideal in A and $\mathrm{gr}(I)$ is acyclic by construction. By lemma 1 I itself is acyclic, and just as in the proof above this forces $\mathrm{cat}(A) \leq g$. □

The authors of [44] also show that $\mathrm{depth}(UL) \leq \mathrm{cat}(A)$, and that if this is an equality then so is $\mathrm{cat}(A) = \mathrm{gldim}(UL)$.

An important theme in what follows is that it is desirable to have a homomorphism of finite i-category, and thus it is desirable to have theorems which bound category from above. This theme came up in the historical discussion above. For example, in the Five Author Paper it is shown that there are interesting consequences for the structure of the homotopy Lie algebra of a minimal model with finite category.

Theorem 18 (Félix-Halperin-Jacobsson-Löfwall-Thomas [44, theorems B and C])**.** *If $A = k[X]$ is a minimal semi-free dg algebra over a field k such that $\mathrm{cat}(A)$ is finite, then the homotopy Lie algebra $\pi^*(A)$ has finite dimensional radical, and*

$$\dim(\mathrm{rad}(\pi^*(A))^{\mathrm{even}}) \leq \mathrm{cat}(A).$$

In particular this bounds the dimension of the centre of $\pi^*(A)$. Let R be a local ring. By results of Avramov and Sun [23, theorem 5.3] and Avramov, Gasharov and Peeva [17, theorem 5.3] the complexity $\mathrm{cx}_R(M)$ of any module M with finite complete intersection dimension is bounded above by the dimension of the degree 2 part of the centre of $\pi^*(R)$. It follows in particular that $\mathrm{cx}_R(M) \leq \mathrm{cat}_0(R)$. Hence results which bound category above are potentially useful. A relative version of this fact involving $\pi_*(\phi)$ is also conceivable.

Another reason why it is desirable to have a homomorphism of finite category is that it allows one to construct long exact sequences of homotopy Lie algebras (see theorem 25 and the exact sequence (3.2) below it). This is proven in unpublished work of Avramov, see the discussion in [9] and [6]. Details for the case of a flat homomorphism are in [7]. Some details are also given below in section 3.5, where we discuss how to obtain long exact sequences in more general situations.

Aside from the question of finiteness, the values taken by i-category are interesting in their own right. The following proposition summarises in a few cases what it means for a homomorphism to have small i-category.

Recall that complete intersection homomorphisms, quasi-complete intersection (qci) homomorphisms and Golod homomorphisms were defined in section 2.7. The generalised Golod conditions were defined in example 4. In short, ϕ is generalised Golod of level i if the graded Lie algebra $\pi^{>i}(\phi)$ is free.

Proposition 9. *Let $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local homomorphism, then*

1. $\mathrm{cat}_0(\phi) = 0$ *if and only if ϕ is flat and its fibre is a field.*
2. $\mathrm{cat}_1(\phi) = 0$ *if and only if ϕ is weakly regular.*
3. $\mathrm{cat}_2(\phi) = 0$ *if and only if ϕ is complete intersection.*
4. $\mathrm{cat}_3(\phi) = 0$ *if and only if ϕ is quasi-complete intersection and either ϕ is a complete intersection or k has characteristic zero.*
5. $\mathrm{cat}_0(\phi) \leq 1$ *if and only if ϕ is Golod.*
6. *More generally* $\mathrm{cat}_i(\phi) \leq 1$ *if and only if ϕ is generalised Golod of level i.*

Note that if the extension $k \to l$ is separable 1 means that ϕ is formally étale, and if $k \to l$ is an isomorphism this means the completion $\widehat{\phi} : \widehat{R} \to \widehat{S}$ is an isomorphism. Similarly, in the separable case 2 means that ϕ is formally smooth.

Proof. The equivalences 1, 2 and 3 are all easy consequences of the definitions (but see [21, corollary 5.5] for details relevant 3).

For 4, assume that $\mathrm{char}(k) = 0$ and that ϕ is a quasi-complete intersection. Then we have a minimal Tate model $R \to R'\langle X\rangle \xrightarrow{\simeq} \widehat{S}$ with $X = X_{\leq 2}$. Since $\mathrm{char}(k) = 0$ this is also a minimal Cohen model, so $\mathrm{cat}_3(\phi) = 0$.

Conversely, suppose that $\mathrm{cat}_3(\phi) = 0$ and ϕ is not complete intersection. Then we have a minimal Cohen model $R \to R'[X] \xrightarrow{\simeq} \widehat{S}$ with $X = X_{\leq 2}$. The key point is that the hypothesis guarantees that $R'[X]$ is minimal as a complex of R' modules. Suppose towards a contradiction that k has characteristic $p > 0$. Since ϕ is not complete intersection X_2 contains some element x. If R' is equicharacteristic then we easily arrive at the contradiction that $[x^p]$ is a nontrivial class in $H_{2p}(R'[X])$, even though $H_*(R'[X]) \cong \widehat{S}$. If R' is not equicharacteristic, use Cohen's theorem to obtain a surjection $Q \to R'$ from a regular local ring Q. Build a minimal model $Q \to Q[Y] \xrightarrow{\simeq} R'$. By uniqueness of minimal models, this extends to a minimal model $Q[Y] \to Q[Y, X] \xrightarrow{\simeq} \widehat{S}$ which is a lift of $R'[X]$. Now x^p is a cycle in $(Q/p)[Y, X]_{2p}$, and it cannot be a boundary since it maps to x^p in $(R'/p)[X]_{2p}$, which is nontrivial since $(R'/p)[X]_{2p}$ is minimal as a complex. This is a contradiction since $(Q/p)[Y, X] \simeq (Q/p) \otimes^{\mathsf{L}}_Q \widehat{S}$ can have homology only in degrees 1 and 2. It follows that $\mathrm{char}(k) = 0$, and the Tate model $R'[X] = R'\langle X\rangle$ shows that ϕ is a quasi-complete intersection.

It remains to establish the equivalences 5 and 6. These will be straightforward corollaries of the results in [10]. We treat 6 since 5 is a special case.

Let $R \to R'[X] \xrightarrow{\simeq} \widehat{S}$ be a minimal model and define $A = k \otimes_R R'[X]^{(i)}$, so that $\operatorname{cat}_i(\phi) = \operatorname{cat}(A)$. According to [10, theorem 3.4] A is a Golod dg algebra if and only $\pi^*(A) \cong \pi^{>i}(\phi)$ is free, if and only if $\mathrm{H}_*(\mathfrak{m}_A) \to \mathrm{H}_*(\mathfrak{m}_A/\mathfrak{m}_A^2) = \mathfrak{m}_A/\mathfrak{m}_A^2 = \pi_*(A)$ is injective (this map may be thought of as dual to the Hurwicz of A). We will show that this last condition is equivalent to $\operatorname{cat}(A) \leq 1$. By definition if $\operatorname{cat}(A) \leq 1$ then $\mathrm{H}_*(\mathfrak{m}_A)$ is a retract of $\mathrm{H}_*(\mathfrak{m}_A/\mathfrak{m}_A^2) = \mathfrak{m}_A/\mathfrak{m}_A^2$, so this direction is clear. If $\mathrm{H}_*(\mathfrak{m}_A) \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is injective, chose any l vector space compliment Z in $\mathfrak{m}_A/\mathfrak{m}_A^2$ to the image of $\mathrm{H}_*(\mathfrak{m}_A)$. The composition $A \to A/\mathfrak{m}_A^2 \to A/(\mathfrak{m}_A^2, Z)$ is then a surjective quasi-isomorphism, so $\operatorname{cat}(\phi) \leq 1$ by lemma 14. $\square$

The proposition should give some intuition on the behaviour of category. It may be instructive to display the conditions $\operatorname{cat}_i(\phi) = j$ in an array:

	0	1	$\cdots$
cat_0	flat with field fibre	Golod	
cat_1	weakly regular	gen. Golod of level 1	
cat_2	complete intersection	gen. Golod of level 2	
cat_3	qci with $\operatorname{char}(k) = 0$	gen. Golod of level 3	
$\vdots$			$\ddots$

As we move downwards we see increasingly relaxed regularity conditions. However, even going down the first column we immediately run into the conjectures stated by Quillen in [106]. The first of these conjectures has been settled by Avramov [13], and using these ideas Avramov and Iyengar prove [21, corollary 5.5], showing that the first column of the table stabilises almost immediately under the assumption $\operatorname{wcat}(\phi) < \infty$: the conditions $\operatorname{cat}_i(\phi) = 0$ all imply that ϕ is complete intersection.

Moving to the right in the table one obtains increasingly relaxed Golod-like conditions. Intuition from Koszul duality is useful here: a trivial commutative algebra is Koszul dual to a free Lie algebra. As the derived fibre of ϕ becomes less trivial its category increases, and the Koszul dual Lie algebra $\pi^*(\phi)$ becomes less free.

We leave the absolute version of the proposition to the reader. In short, $\operatorname{cat}_0(R) = 0$ only when R is a field; $\operatorname{cat}_1(R) = 0$ means that R is regular; and $\operatorname{cat}_2(R) = 0$ means that R is a complete intersection. The Golod terminology is slightly miss-matched: the strong condition $\operatorname{cat}_0(R) \leq 1$ is satisfied only when the multiplication of R is trivial, that is, when $\mathfrak{m}_R^2 = 0$. Instead, R is Golod if and only if $\operatorname{cat}_1(R) \leq 1$.

Using a result of Wiebe [121] one can also characterise complete intersections using cat_1.

Proposition 10. *A local ring $(R, \mathfrak{m}, k)$ is a complete intersection if and only if* $\operatorname{cat}_1(R) = \operatorname{codepth}(R)$.

Since in general $0 \leq \operatorname{cat}_1(R) \leq \operatorname{codepth}(R)$, the proposition puts regular rings and complete intersections at opposite ends of a scale with the different values of $\operatorname{cat}_1(R)$ interpolating. On this scale the Golod condition is as close to regular as possible. The proposition was suggested as a possibility by Luchezar Avramov.

Proof. Let $c = \operatorname{codepth}(R)$. If R is a complete intersection the computation of $\operatorname{cat}_1(R)$ is as in example 5. Let $Q \to \widehat{R}$ be a minimal model. The kernel is generated by a regular sequence $\underline{f} = f_1, ..., f_c$ in Q. Thus there is a minimal model $Q[X] \xrightarrow{\simeq} \widehat{R}$ with $X = X_1$ and $\partial(X_1) = \underline{f}$. From here $\operatorname{cat}_1(R) = \operatorname{cat}(k[X]) = c$ since $k[X]$ is an exterior algebra on c generators.

For the converse it is convenient to use the following version of Wiebe's criterion, taken from [35, theorem 2.3.15]. The ring R is a complete intersection if and only if $\mathrm{H}_{>0}(K)^c \neq 0$, where K is the Koszul complex over R on a minimal generating set for $\mathfrak{m}$.

Assume R is not a complete intersection and let $Q[X] \xrightarrow{\simeq} \widehat{R}$ be a minimal model. Then $\mathrm{cat}_1(R) = \mathrm{cat}(k[X])$. Note that $\sup(k[X]) = \mathrm{pd}_Q(R) = c$. If we let I be the ideal in $k[X]$ with $I_i = k[X]_i$ for $i > c$ and $I_c = \partial(k[X]_{c+1})$, then $k[X] \to k[X]/I$ will be a surjective quasi-isomorphism. I claim that $\ell\ell(k[X]/I) - 1 < c$, using part 1 of lemma 14 this will show that $\mathrm{cat}(k[X]) < c$ and complete the proof of the proposition.

Let $z_1, ..., z_c$ be in $\mathfrak{m}_{k[X]/I}$. To establish the claim we must show that the product $z_1...z_c$ is zero. For degree reasons it is clear that $z_1...z_c = 0$ if any of the z_i are in $k[X]_{\geq 2}$. So we assume all the z_i have degree 1. In particular they are automatically cycles.

There is a chain of quasi-isomorphisms $k[X] \simeq K$, so $\mathrm{H}_*(k[X]) \cong \mathrm{H}_*(K)$. Since R is not a complete intersection the class $[z_1]...[z_c]$ vanishes in $\mathrm{H}_*(k[X])$ by Wiebe's criterion. Then $z_1...z_c$ is a boundary in $k[X]$, and by definition this means $z_1...z_c$ is in I_c. Therefore $z_1...z_c = 0$ in $k[X]/I$, and we are done. □

Example 6. The generic short local ring is generalised Golod of level 2. Building on work of Conca [40], Avramov, Iyengar and Şega [22] show that for a generic local ring $(R, \mathfrak{m}, k)$ satisfying $\mathfrak{m}^3 = 0$ there is a quadratic hypersurface ring $(Q, \mathfrak{n}, k)$ and a surjective Golod homomorphism $\phi : Q \to R$ whose kernel is contained in $\mathfrak{n}^2$. Since ϕ is Golod it is small [8, theorem 3.5], and it follows that there is a short exact sequence of homotopy Lie algebras

$$0 \to \pi^*(\phi) \to \pi^*(R) \to \pi^*(Q) \to 0$$

(this will be explained in section 3.5). Note that $\pi^*(Q)$ is $\mathrm{edim}(Q) = \mathrm{edim}(R)$ dimensional in degree 1 and one dimensional in degree 2, after which it vanishes. Now the fact that a sub-Lie algebra of a free Lie algebra is free implies $\pi^{>2}(R)$ is free (see [86, proposition A.1.10] and remark 16 below). Generically $\mathfrak{m}^2 \neq 0$ so ϕ is not an isomorphism, and it follows that $\pi^{>1}(R)$ is not free. At this point we can completely calculate the sequence $\mathrm{cat}_i(R)$:

$$\mathrm{cat}_0(R) = 2, \quad \mathrm{cat}_1(R) = 2 \quad \text{and} \quad \mathrm{cat}_i(R) = 1 \quad \text{for all } i \geq 2.$$

On the other hand by results of Anick [3] and Avramov [11] short local rings exist which are not generalised Golod. These rings satisfy $\mathrm{cat}_i(R) = 2$ for all i.

At the beginning of this section we alluded to the fact that theorems in rational homotopy theory have often produced analogous theorems in local commutative algebra, and vice versa. The local commutative algebra version of the Mapping Theorem has apparently been missing for some time. We will give a proof in this thesis.

Theorem (theorem 20 below)**.** *If $\phi : A \to B$ is a surjective homomorphism of minimal models then*

$$\mathrm{cat}\, A \geq \mathrm{cat}\, B.$$

Keeping in mind the inequalities $\sup(k[X]) \geq \mathrm{cat}(k[X]) \geq \ell\ell H_*(k[X]) + 1$, this generalises a theorem of Avramov and Iyengar, which may be stated as follows.

Theorem 19 (Avramov-Iyengar [19, theorem 1.2])**.** *If $k[X] \to k[Y]$ is a surjection of minimal semi-free dg algebras over a field k, then*

$$\sup k[X] \geq \ell\ell H_*(k[Y]) + 1.$$

Proof of Well-Definedness

To finish this section we will return to the proof of theorem 17, which says that the numbers $\mathrm{cat}_i(\phi)$ associated to a local homomorphism $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ are well-defined.

The main point is the following lemma, which we will set up now. If $Q \to P$ is a surjection of regular local rings, then the kernel is automatically generated by a regular sequence $\underline{x}$ in $\mathfrak{m}_Q \setminus \mathfrak{m}_Q^2$. Let $Q \to Q[t] \to P$ be the Koszul complex which resolves P over Q, where t is a set of variables in degree 1 with $\partial(t) = \underline{x}$. Given a minimal semi-free extension $Q[t] \to Q[t, X]$, we can apply $- \otimes_{Q[t]} P$ to get a (quasi-isomorphic) minimal semi-free extension $P \to P[X]$.

Lemma 15. *In the situation just described,* $\mathrm{cat}\, Q[t, X] = \mathrm{cat}\, P[X]$*.*

To establish this we first separate out the key technical observation.

Lemma 16. *The natural surjection $Q[t, X]/\mathfrak{m}_{Q[t,X]}^{m+1} \to P[X]/\mathfrak{m}_{P[X]}^{m+1}$ is a quasi-isomorphism*

Proof. We can simplify the situation by noting that it suffices to treat the codimension one case. In other words, we may assume that $P = Q/(x)$ for a single non-zero-divisor x in $\mathfrak{m}_Q \setminus \mathfrak{m}_Q^2$. In particular the Koszul complex $Q[t]$ looks like $tQ \xrightarrow{x\partial_t} Q$. As a Q module $Q[t, X]/\mathfrak{m}_{Q[t,X]}^{m+1}$ is the direct sum $t(Q[X]/\mathfrak{m}_{Q[X]}^{m}) + (Q[X]/\mathfrak{m}_{Q[X]}^{m+1})$, but the differential has components crossing in both directions. To fix this we filter by setting $F_{(i)}(Q[X]/\mathfrak{m}_{Q[X]}^{m+1}) = t(Q[X]/\mathfrak{m}_{Q[X]}^{m})_{\leq i+1} + (Q[X]/\mathfrak{m}_{Q[X]}^{m+1})_{\leq i}$. We also filter $P[X]/\mathfrak{m}_{P[X]}^{m+1}$ by degree, so $F_{(i)} = (P[X]/\mathfrak{m}_{P[X]}^{m+1})_{\leq i}$. By lemma 1 it suffices to show that $\mathrm{gr}(\pi) : \mathrm{gr}(Q[t, X]/\mathfrak{m}_{Q[t,X]}^{m+1}) \to \mathrm{gr}(P[X]/\mathfrak{m}_{P[X]}^{m+1})$ is a quasi-isomorphism. But $\mathrm{gr}(Q[t, X]/\mathfrak{m}_{Q[t,X]}^{m+1})$ looks like

$$t(Q[X]/\mathfrak{m}_{Q[X]}^{m}) \xrightarrow{x\partial_t} (Q[X]/\mathfrak{m}_{Q[X]}^{m+1})$$

with only the differential $x\partial_t$ as indicated, and $\mathrm{gr}(P[X]/\mathfrak{m}_{P[X]}^{m+1})$ has no differential at all. It is clear that the cokernel of $x\partial_t$ is $P[X]/\mathfrak{m}_{P[X]}^{m+1}$. Thus to show that $\mathrm{gr}(\gamma)$ is a quasi-isomorphism we must show that multiplication $Q[X]/\mathfrak{m}_{Q[X]}^{m} \xrightarrow{x} Q[X]/\mathfrak{m}_{Q[X]}^{m+1}$ is injective. This map diagonalises: both sides split into a direct sum of Q modules generated by monomials in X, and multiplication by x respects this decomposition. So for a monomial M of weight $i < m$ in $\mathrm{Sym}_Q^i(X) \cong (X)^i/(X)^{i+1}$, we must show that $MQ/M\mathfrak{m}_Q^j \xrightarrow{x} MQ/M\mathfrak{m}_Q^{j+1}$ is injective, where $i + j = m$. But this is isomorphic to $Q/\mathfrak{m}_Q^j \xrightarrow{x} Q/\mathfrak{m}_Q^{j+1}$. Finally, this is injective because x is a non-zerodivisor in the associated graded ring $\mathrm{gr}_{\mathfrak{m}}Q$, which is a polynomial algebra since Q is regular. □

Proof of lemma 15. First we deal with the inequality $\mathrm{cat}\, Q[t, X] \leq \mathrm{cat}\, P[X]$. This is formal and applies to any sequence of extensions $Q \to Q[t] \to Q[t, X]$ with a quasi-isomorphism $Q[t] \to P$, with no need for regularity. Suppose that $\mathrm{cat}\, P[X] \leq m$ is finite, and construct minimal models $Q[t, X] \to Q[t, X, Y] \xrightarrow{\simeq} Q[t, X]/\mathfrak{m}_{Q[t,X]}^{m+1}$ and $P[X] \to P[X, Z] \xrightarrow{\simeq} P[X]/\mathfrak{m}_{P[X]}^{m+1}$. By the lifting property of semi-free extensions there is a $Q[t, X]$ algebra map $\alpha : Q[t, X, Y] \to P[X, Z]$ as in the following commutative diagram

$$\begin{tikzcd}
Q[t, X, Y] \arrow[rr, "\alpha"] \arrow[dr] & & P[X, Z] \arrow[dl, two heads, "\simeq"] \\
& P[X]/\mathfrak{m}_{P[X]}^{m+1}. &
\end{tikzcd}$$

By definition of category the map $P[X] \to P[X, Z]$ is split by some $\beta : P[X, Z] \to P[X]$. Then we use the lifting property again on the following diagram of $Q[t, X]$ algebra homomorphisms

$$\begin{tikzcd}
Q[t, X] \arrow[rr] \arrow[dr, two heads, "\simeq"] & & Q[t, X, Y] \arrow[dl, "\beta\alpha"] \\
& P[X], &
\end{tikzcd}$$

to produce a splitting $Q[t, X, Y] \to Q[t, X]$. This means that $\mathrm{cat}(Q[t, X]) \leq m$.

The inequality $\mathrm{cat}\, Q[t, X] \geq \mathrm{cat}\, P[X]$ relies more on the specific situation. So suppose $\mathrm{cat}\, Q[t, X] \leq m$ is finite. First observe that using lemma 16 and the commutative square

$$\begin{tikzcd}
Q[t, X, Y] \arrow[r, "\simeq"] \arrow[d, "\simeq"] & Q[t, X]/\mathfrak{m}_{Q[t,X]}^{m+1} \arrow[d, "\simeq"] \\
P[X, Y] \arrow[r] & P[X]/\mathfrak{m}_{P[X]}^{m+1},
\end{tikzcd}$$

the homomorphism $P[X, Y] \to P[X]/\mathfrak{m}_{P[X]}^{m+1}$ is a quasi-isomorphism. Now take a minimal model $Q[t, X] \to Q[t, X, Y] \xrightarrow{\simeq} Q[t, X]/\mathfrak{m}_{Q[t,X]}^{m+1}$ with a splitting $Q[t, X, Y] \to Q[t, X]$. We apply $- \otimes_{Q[t]} P$ to get the diagram

$$\begin{tikzcd}
P[X] \arrow[r] \arrow[dr] & P[X, Y] \arrow[l, bend right] \arrow[d, "\simeq"] \\
& P[X]/\mathfrak{m}_{P[X]}^{m+1},
\end{tikzcd}$$

and this is witness to the inequality $\mathrm{cat}\, P[X] \leq m$. $\square$

Proof of theorem 17. We treat only the relative situation, for a local homomorphism $\phi : R \to S$. The absolute statement is similar and slightly simpler.

Assume that we have two minimal Cohen factorisations connected by a deformation as in the diagram

$$\begin{tikzcd}
& R' \arrow[dr] & \\
R \arrow[ur] \arrow[r] \arrow[dr] & A \arrow[u, two heads] \arrow[r] \arrow[d, two heads] & S \\
& R''. \arrow[ur] &
\end{tikzcd}$$

The fibres $\overline{R'} = R'/\mathfrak{m}R'$, $\overline{R''} = R''/\mathfrak{m}R''$, and $\overline{A} = A/\mathfrak{m}A$ are all regular local rings.

Recall from the discussion in section 2.7 that up to isomorphism we may assume $R' = A/(\underline{x})$ and

$R'' = A/(\underline{y})$, where $\underline{x}$ and $\underline{y}$ are regular sequences in A which both descend to bases for $(I + \mathfrak{m}A + \mathfrak{m}_A^2)/(\mathfrak{m}A + \mathfrak{m}_A^2)$, with I being the kernel of $A \to S$ (see also [16, (1.5)]). Note that $\underline{x}$ and $\underline{y}$ can be extended in the same way to a minimal generating set for I, so we may build a minimal model for $A \to S$ of the form $A \to A[s, X] = A[t, X] \xrightarrow{\simeq} S$, where s and t are sequences of variables in degree 1 with $\partial(s) = \underline{x}$ and $\partial(t) = \underline{y}$, and where $s \cup X$ and $t \cup X$ both freely generate $A[s, X] = A[t, X]$. We obtain minimal models

$$R'[X] = A[s, X]/(\underline{x}, s) \xleftarrow{\simeq} A[s, X] = A[t, X] \xrightarrow{\simeq} A[t, X]/(\underline{y}, t) = R''[X].$$

For $i \geq 2$ we have identifications $R'[X]/(\mathfrak{m}_{R'}, X_{<i}) = A[s, X]/(\mathfrak{m}_A, s, X_{<i}) = A[t, X]/(\mathfrak{m}_A, t, X_{<i}) = R''[X]/(\mathfrak{m}_{R''}, X_{<i})$. Hence in this case the dg algebras used to compute $\mathrm{cat}_i(\phi)$ are all isomorphic. This argument appears already in [21, proposition 2.4].

For $i = 0$ we pass to fibres

$$\overline{R'}[X] \xleftarrow{\simeq} \overline{A}[s, X] = \overline{A}[t, X] \xrightarrow{\simeq} \overline{R''}[X].$$

Then the lemma says that $\mathrm{cat}(\overline{R'}[X]) = \mathrm{cat}(\overline{A}[s, X]) = \mathrm{cat}(\overline{A}[t, X]) = \mathrm{cat}(\overline{R''}[X])$. This shows that $\mathrm{cat}_0(\phi)$ is well-defined.

To deal with $i = 1$, let $\underline{z}$ be a regular sequence in $\overline{A}$ whose image in $\mathfrak{m}_{\overline{A}}/\mathfrak{m}_{\overline{A}}^2$ is a basis for a complement to the inclusion $(I + \mathfrak{m}A + \mathfrak{m}_A^2)/(\mathfrak{m}A + \mathfrak{m}_A^2) \hookrightarrow \mathfrak{m}_{\overline{A}}/\mathfrak{m}_{\overline{A}}^2$. By construction $(\underline{z})\overline{R'} = \mathfrak{m}_{\overline{R'}}$ and $(\underline{z})\overline{R''} = \mathfrak{m}_{\overline{R''}}$. We obtain quasi-isomorphisms

$$l[s, X]/(s) \xleftarrow{\simeq} (\overline{A}/\underline{z})[s, X] = (\overline{A}/\underline{z})[t, X] \xrightarrow{\simeq} l[t, X]/(t).$$

At this point our notation could become misleading: $l[X] = l[s, X]/(s)$ and $l[X] = l[t, X]/(t)$ are isomorphic as algebras but their differentials will be different in general. In any case, we can use the lemma again to conclude that $\mathrm{cat}(l[s, X]/(s)) = \mathrm{cat}((\overline{A}/\underline{z})[s, X]) = \mathrm{cat}((\overline{A}/\underline{z})[t, X]) = \mathrm{cat}(l[t, X]/(t))$.

This establishes well-definedness of i-category for all i.

Knowing this, the final part of the theorem is dealt with by the following proposition. $\square$

Proposition 11. *If A is a minimal model then* $\mathrm{cat}(A) = \mathrm{cat}(\widehat{A})$.

Proof. Say $A = Q[X]$ for some regular local ring Q, so that $\widehat{A} = \widehat{Q}[X]$.

Note that if Q is a field then $A = \widehat{A}$ and the proposition is obvious. This is the case when A is being used to compute $\mathrm{cat}_i(\phi)$ or $\mathrm{cat}_i(R)$ for any $i > 0$. The difficultly is in dealing with $i = 0$.

If $\mathrm{cat}(A) \leq m$ then one can build a minimal model $A \to A[Y] \xrightarrow{\simeq} A/\mathfrak{m}_A^{m+1}$ with a splitting $A[Y] \to A$. Simply applying $- \otimes_Q \widehat{Q}$ shows that $\mathrm{cat}(\widehat{A}) \leq m$. Hence $\mathrm{cat}(A) \geq \mathrm{cat}(\widehat{A})$.

The key point for the reverse inequality is that if $\mathrm{cat}(\widehat{Q}[X]) \leq m$ is finite then $\mathrm{H}_0(\widehat{A}) = \widehat{Q}/(\partial(X_1)) = (Q/(\partial(X_1)))\widehat{\ }$ is Artinian, and hence so is $\mathrm{H}_0(A) = Q/(\partial(X_1))$, which means $\mathrm{H}_0(A)$ is already complete. It follows that $A \to \widehat{A}$ is a quasi-isomorphism. This is because $\mathrm{H}_*(\widehat{A}) \cong \mathrm{H}_*(A) \otimes_Q \widehat{Q} \cong \mathrm{H}_*(Q[Y])$ since each $\mathrm{H}_i(Q[Y])$ is a finitely generated $\mathrm{H}_0(A) = \mathrm{H}_0(A)\widehat{\ }$ module. Next, since $\widehat{Q}/(\partial(X_1))$ is Artinian there is a system of parameters $\underline{x}$ for $\widehat{Q}$ which lies in the kernel of $\widehat{Q} \to S/\mathfrak{m}S$. We can find a subset $z \subseteq X_1\widehat{Q}$ such that $\partial(z) = \underline{x}$. Since $\widehat{Q}$ is Cohen-Macaulay $\underline{x}$ is automatically a regular sequence, so the homomorphism $\widehat{Q}[z] \to \widehat{Q}/(\underline{x})$ is a quasi-isomorphism. Note that the composition $Q \to \widehat{Q} \to \widehat{Q}/(\underline{x})$ is surjective since $\widehat{Q}/(\underline{x})$ is Artinian.

Now, if $A \to A[Y] \xrightarrow{\simeq} A/\mathfrak{m}_A^{m+1}$ is a minimal model, and $\alpha : \widehat{A}[Y] \to \widehat{A}$ is a splitting, then we can build the following diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\simeq"'] & A[Y] \arrow[l, bend right, dashed, "\beta"'] \arrow[dl, "\alpha"'] \arrow[ddl] \\
\widehat{A} \arrow[d, "\simeq"'] & \\
\widehat{A}/(\underline{x}, z). &
\end{tikzcd}$$

Since the composition $A \to \widehat{A} \to \widehat{A}/(\underline{x}, z)$ is a surjective quasi-isomorphism, an A algebra lift β exists as shown in the diagram, by lemma 5. This shows that $\mathrm{cat}(A) \leq \mathrm{cat}(\widehat{A})$ and concludes the proof. $\square$

3.3 The Mapping Theorem

The purpose of this section is to prove the following theorem, which has been discussed already in the previous section.

Theorem 20. *If $\phi : A \to B$ is a surjective homomorphism of minimal models then*

$$\mathrm{cat}\, A \geq \mathrm{cat}\, B.$$

Remark 15. Suppose that $\phi : A \to B$ is a morphism of minimal models such that $\phi_* : \pi_*(A) \to \pi_*(B)$ is surjective. Then $\mathrm{gr}(A) \to \mathrm{gr}(B)$ is surjective, and it follows that $\widehat{\phi} : \widehat{A} \to \widehat{B}$ is surjective. According to the theorem $\mathrm{cat}\, \widehat{A} \geq \mathrm{cat}\, \widehat{B}$, and by proposition 11 we can conclude that $\mathrm{cat}\, A \geq \mathrm{cat}\, B$. Therefore the hypotheses of the mapping theorem can be weakened to surjectivity of $\phi_* : \pi_*(A) \to \pi_*(B)$.

From now on we take $A = Q[X]$ and $B = P[Y]$.

Both P and Q are regular local rings, so the kernel of $Q \to P$ is generated by a regular sequence $\underline{x}$ in $\mathfrak{m}_Q \smallsetminus \mathfrak{m}_Q^2$. Thus the Koszul complex $Q \to Q[t] \xrightarrow{\simeq} P$ with $t = \varepsilon x$ is a model for P. We need a theorem of Avramov and Iyengar which generalises Gulliksen and Schoeller's theorem 11 from section 2.7, it explains how to extend this Koszul complex to an acyclic closure of ϕ.

Theorem (Avramov-Iyengar [19, theorem 1.1])**.** *There are sets $\widetilde{Y}, Z \subseteq Q[X]$ which together freely generate $Q[X] = Q[\widetilde{Y}, Z]$, such that the kernel of ϕ is $(\underline{x}, Z)$ and ϕ maps $\widetilde{Y}$ bijectively onto Y. Moreover ϕ admits an acyclic closure of the form*

$$Q[X] \to Q[X, t]\langle U \rangle \xrightarrow{\simeq} P[Y]$$

where U is a graded set with a bijection $\varepsilon Z \cong U$, $\varepsilon z \mapsto u_z$, such that

$$\partial(u_z) = z \quad \textit{modulo} \quad \left(\mathfrak{m}_{Q[X]}^2, \mathfrak{m}_{Q[X]}(U)^{(1)}\right) \quad \textit{for all } z \textit{ in } Z.$$

We are treating t as a set of free exterior variables, but these are the same as odd divided power variables, so can freely switch between the two notations $Q[X, t]\langle U \rangle = Q[X]\langle t, U \rangle$.

Since $P \to P[Y]$ is a semi-free extension and $P[X]\langle U\rangle \to P[Y]$ is a surjective quasi-isomorphism there is a P algebra splitting $\alpha : P[Y] \to P[X]\langle U\rangle$.

Lemma 17. *In the above situation, the splitting satisfies* $\alpha(Y) \subseteq (\mathfrak{m}_{P[X]})P[X]\langle U\rangle$.

Proof. We use induction on degree. Assume $\alpha(Y_{<n}) \subseteq (\mathfrak{m}_{P[X]})P[X]\langle U\rangle$. Our goal is to show that if y is in Y_n then $\alpha(y)$ is in $(\mathfrak{m}_{P[X]})P[X]\langle U\rangle$ as well. By minimality $\partial(y) \in \mathfrak{m}^2_{P[Y]}$ so $\partial(\alpha(y)) = \alpha(\partial(y))$ is in $(\mathfrak{m}^2_{P[X]})P[X]\langle U\rangle$ by assumption. But we will show that this is impossible unless $f = \alpha(y)$ is in $(\mathfrak{m}_{P[X]})P[X]\langle U\rangle$.

$P[X]\langle U\rangle$ has a P module basis of monomials in X and U with divided powers of elements of U, and an element is in $(\mathfrak{m}_{P[X]})P[X]\langle U\rangle$ precisely when all its monomials which are purely divided power monomials in U have coefficients from $\mathfrak{m}_P$. So, if f is not in $(\mathfrak{m}_{P[X]})P[X]\langle U\rangle$ we may consider an element z in Z of maximal degree i such that u_z appears in a divided power monomial with unit coefficient in f. Denote $Z' = Z_i \smallsetminus \{z\}$. By considering each monomial in f separately, and using the precise description of the differential described in Avramov and Iyengar's theorem, we see that $\partial(f) = zf'$ modulo $(\mathfrak{m}_P, X_{<n}, Z', \mathfrak{m}^2_{P[X]})P[X]\langle U\rangle$, where f' is the sum of the divided power monomials in f involving Σz, each with the divided power exponent of Σz reduced by 1. But zf' is not in $(\mathfrak{m}_P, X_{<n}, Z', \mathfrak{m}^2_{P[X]})P[X]\langle U\rangle$, so we reach the contradiction that $\partial(f)$ is not in $(\mathfrak{m}^2_{P[X]})P[X]\langle U\rangle$. This completes the induction step. $\square$

Proof of theorem 20. If we assume that $\mathrm{cat}(Q[X]) = m$ is finite, then by definition we have a diagram of local dg algebras

$$\begin{tikzcd} Q[X] \arrow[r] \arrow[dr] & C \arrow[l, bend right] \arrow[d, two heads, "\simeq"] \\ & Q[X]/\mathfrak{m}^{m+1}_{Q[X]} \end{tikzcd}$$

making $Q[X]$ a retract of C. The only assumption we need to make about C is that $\underline{x}$ is regular on it, so that $C \otimes_Q Q[t] \to (C/\underline{x})$ is a quasi-isomorphism. This is easy to arrange.

We obtain a dg algebra $C\langle t, U\rangle = C \otimes_{Q[X]} Q[X]\langle t, U\rangle$ and a quasi-isomorphism $C\langle t, U\rangle \xrightarrow{\simeq} (C/\underline{x})\langle U\rangle$. Consider the two ideals $I = (\mathfrak{m}_{C/\underline{x}})(C/\underline{x})\langle U\rangle$ and $J = (\mathfrak{m}_{P[X]})P[X]\langle U\rangle/(\mathfrak{m}^{m+1}_{P[X]})$. From these we obtain subalgebras $A = P + I \subseteq (C/\underline{x})\langle U\rangle$ and $B = P + J \subseteq P[X]\langle U\rangle/(\mathfrak{m}^{m+1}_{P[X]})$ (not the same A and B as in the original statement).

With this setup, we separate the proof into two claims.

Claim 1: $\mathrm{cat}(A) \leq m$. **Claim 2:** $P[Y]$ is a retract of A.

Having established these we will be done by lemma 14.

We start with claim 1. Apply $- \otimes_{Q[X]} Q[X]\langle t, U\rangle$ to the quasi-isomorphism $C \xrightarrow{\simeq} Q[X]/\mathfrak{m}^{m+1}_{Q[X]}$ to get another one $C\langle t, U\rangle \xrightarrow{\simeq} Q[t, X]\langle U\rangle/(\mathfrak{m}^{m+1}_{Q[X]})$. We have a commutative diagram of dg algebras

$$\begin{tikzcd} C\langle t, U\rangle \arrow[r, "\simeq"] \arrow[d, "\simeq"] & Q[t, X]\langle U\rangle/(\mathfrak{m}^{m+1}_{Q[X]}) \arrow[d, "\simeq"] \\ (C/\underline{x})\langle U\rangle \arrow[r] & P[X]\langle U\rangle/(\mathfrak{m}^{m+1}_{Q[X]}). \end{tikzcd}$$

The left vertical arrow is a quasi-isomorphism by our assumption on C, and the right vertical arrow is a quasi-isomorphism by lemma 16. Hence the lower arrow is a quasi-isomorphism. Now we have two short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d] & (C/\underline{x})\langle U\rangle \arrow[r] \arrow[d, two heads, "\simeq"] & k\langle U\rangle \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J \arrow[r] & P[X]\langle U\rangle/(\mathfrak{m}_{Q[X]}^{m+1}) \arrow[r] & k\langle U\rangle \arrow[r] & 0.
\end{tikzcd}$$

By the five lemma the left-most map $I \to J$ is a surjective quasi-isomorphism. It follows that the homomorphism $A \to B$ is also a surjective quasi-isomorphism. Note also that $\ell\ell(B) \leq m+1$ since by construction $J^{m+1} = 0$. Therefore $\operatorname{cat}(A) \leq m$ by lemma 14.

It remains to establish claim 2. If we apply $- \otimes_{Q[X]} P[X]\langle U\rangle$ to the split monomorphism $Q[X] \to C$ we obtain a split monomorphism $\beta : P[X]\langle U\rangle \to (C/\underline{x})\langle U\rangle$. In the discussion before lemma 17 we remarked that $P[Y]$ is a P algebra retract of $P[X]\langle U\rangle$ via α. The composition $\beta\alpha$ is a split monomorphism as well, and by lemma 17 we have $\beta\alpha(Y) \subseteq \beta((\mathfrak{m}_{P[X]})P[X]\langle U\rangle) \subseteq (\mathfrak{m}_{C/\underline{x}})(C/\underline{x})\langle U\rangle$. Therefore $\beta\alpha$ factors through the subalgebra A:

$$\begin{tikzcd}
P[Y] \arrow[r, "\beta\alpha"'] \arrow[dr, dashed] & (C/\underline{x})\langle U\rangle \arrow[l, bend right] \\
& A \arrow[u, hook]
\end{tikzcd}$$

making $P[Y]$ a retract of A. □

Remark 16. The mapping theorem can be seen as a generalisation of the fact that a sub-Lie algebra of a free Lie algebra is itself free (see [86, proposition A.1.10]). Indeed, suppose that L' is a sub-algebra of a free graded Lie algebra L (concentrated in strictly positive degrees). Then one may construct minimal models $A = \widehat{\mathcal{C}}(L^\vee)$ and $B = \widehat{\mathcal{C}}(L'^\vee)$ with $\pi^*(A) = L$ and $\pi^*(B) = L'$, and a morphism $A \to B$ inducing the inclusion $L' \to L$ on homotopy Lie algebras. By the argument in the proof of proposition 9 we have $\operatorname{cat}(A) \leq 1$. Since $L' \to L$ is injective $A \to B$ is surjective. Hence by the mapping theorem $\operatorname{cat}(B) \leq 1$ and therefore L' is also free. Note however that Avramov does make use of this fact about Lie algebras for his characterisation of Golod dg algebras in [11].

Example 7 (Large homomorphisms)**.** Levin [88] defined a surjective homomorphism $\phi : (R, k) \to (S, k)$ to be *large* if the induced map $\operatorname{Tor}_*^R(k,k) \to \operatorname{Tor}_*^S(k,k)$ is surjective. This is closely related to the hypothesis of the mapping theorem. By theorem 3.4 in [21] (which appears with a simpler proof as corollary 4 below), ϕ is large precisely if $\pi_*(R) \to \pi_*(S)$ is surjective. Taking compatible minimal models $Q[X] \xrightarrow{\simeq} \widehat{R}$ and $P[Y] \xrightarrow{\simeq} \widehat{S}$ we see that $R \to S$ is large if and only the induced homomorphism $Q[X] \to P[Y]$ is surjective. Hence by the mapping theorem $\operatorname{cat}_i(R) \geq \operatorname{cat}_i(S)$ for all i.

In general, given a sequence of local homomorphisms $\psi : A \to R$ and $\phi : R \to S$ one might say that ϕ is *large relative to* A if $\phi_* : \pi_*(\psi) \to \pi_*(\phi\psi)$ is surjective. In this case the mapping theorem says that $\operatorname{cat}_i(\psi) \geq \operatorname{cat}_i(\phi\psi)$ for all i.

The following corollary strengthens [21, theorem 5.6]. It is a consequence of the mapping theorem and another theorem of Avramov and Iyengar.

Corollary 3. *If* $\phi : R \to S$ *is an almost small homomorphism then* $\operatorname{cat}_1(\phi) \leq \operatorname{edim}(R) - \operatorname{depth}(S)$ *and* $\operatorname{cat}_2(\phi) \leq \operatorname{edim}(S) - \operatorname{depth}(S)$.

Theorem (Avramov-Iyengar [21, theorem 4.11])**.** *Let $\phi : R \to S$ be a surjective local map. Assume R has a minimal regular presentation $Q \to R$, and choose a minimal set $\underline{x} \subseteq Q$ such that $P = Q/\underline{x} \to S$ is a minimal regular presentation. Let $Q[t]$ be the Koszul complex with $t = \Sigma\underline{x}$ resolving P over Q. If ϕ is almost small, one can build the following commutative diagram in which every row, column and diagonal is a minimal model:*

$$\begin{tikzcd}
Q \arrow[r] \arrow[d] \arrow[dr] & Q[t] \arrow[r, "\simeq"] \arrow[d] & P \arrow[d] \\
Q[X] \arrow[r] \arrow[d, "\simeq"] & Q[t,X,T] \arrow[r, "\simeq"] \arrow[d, "\simeq"] \arrow[dr, "\simeq"] & P[X,Y] \arrow[d, "\simeq"] \\
R \arrow[r] & R[t,Y] \arrow[r, "\simeq"] & S.
\end{tikzcd}$$

In [21] minimality of the diagonal is not stated, but the proof explicitly shows that $\partial(t, X, Y) \subseteq (\mathfrak{m}_Q) + (t, X, Y)^2$.

Proof of corollary 3. Let $R \to R' \to \widehat{S}$ be a minimal Cohen factorisation. By [21, proposition 4.8] the surjective homomorphism $R' \to \widehat{S}$ is almost small.

In the notation of the theorem applied to $R' \to \widehat{S}$, we have a surjection $k[t, X, Y] \to k[t, Y]$ of minimal semi-free dg algebras over k. By the mapping theorem and proposition 7

$$\mathrm{cat}_1(\phi) = \mathrm{cat}\, k[t, Y] \leq \mathrm{cat}\, k[t, X, Y] \leq \sup(k[t, X, Y]) = \mathrm{pd}_Q(\widehat{S})$$

and $\mathrm{pd}_Q(\widehat{S}) = \mathrm{edim}(R') - \mathrm{depth}(\widehat{S}) = \mathrm{edim}(R) + \mathrm{edim}(R'/\mathfrak{m}R') - \mathrm{depth}(\widehat{S})$, with $\mathrm{edim}(R'/\mathfrak{m}R') = \mathrm{edim}(\widehat{S}/\mathfrak{m}\widehat{S})$ by minimality of the Cohen factorisation. Similarly

$$\mathrm{cat}_2(\phi) = \mathrm{cat}\, k[Y]/(Y_1) \leq \mathrm{cat}\, k[X, Y] \leq \sup(k[X, Y]) = \mathrm{pd}_P(\widehat{S})$$

and $\mathrm{pd}_P(\widehat{S}) = \mathrm{edim}(\widehat{S}) - \mathrm{depth}(\widehat{S})$. $\square$

3.4 Functoriality of the Tate model for the diagonal

The main result of this section is theorem 21, which concerns functoriality of the Tate model for the diagonal.

In this section Q is a local ring and $Q[X]$ is a semi-free Q algebra with X in positive degrees. Later on we will take Q to be regular, but we make no such assumption now. The Tate model for the diagonal was constructed in theorem 10. We state a slightly simplified version here:

Theorem. *Let $Q \to Q[X]$ be a semi-free extension of a local ring (Q, k) with X degree-wise finite and concentrated in strictly positive degrees. The multiplication map $Q[X, X] \to Q[X]$ has a Tate model of the form*

$$Q[X, X] \to Q[X, X]\langle \Sigma X\rangle \to Q[X].$$

The differential satisfies $\partial(\Sigma x) = 1 \otimes x - x \otimes 1 - \Sigma\partial_1(x)$ modulo decomposables in $\mathfrak{m}^2_{Q[X,X]\langle \Sigma X\rangle} + (\Sigma X)^{(2)}$.

If moreover $Q \to Q[X]$ is minimal as a semi-free extension then $Q[X, X]\langle \Sigma X\rangle$ is minimal as a dg module over $Q[X, X]$.

Recall that in remark 1 it was briefly indicated that the Tate model for the diagonal can be constructed using twisting cochains, and from this perspective its form is a reflection of the Poincaré-Birkhoff-Witt theorem. Details on that construction will appear elsewhere.

Example 8. Let (R, k) be a local ring with a minimal regular presentation $Q \to R$, and extend this to a minimal model $Q[X] \to R$. Let X_0 denote some choice of minimal generating set for $\mathfrak{m}_Q$. One may also build an acyclic closure $R \to R\langle U\rangle \to k$ of the residue field of R. The deviations of R are by definition the numbers $\epsilon_n(R) = \mathrm{card}(U_n)$. Recall from section 3.1 that Avramov has shown that we have $\mathrm{card}(X_{n-1}) = \epsilon_n(R)$ for all $n \geq 1$. A proof was sketched in section section 3.1. But one may also deduce this from theorem 10, which produces a minimal Tate model $Q[X,X] \to Q[X,X]\langle \varepsilon X\rangle \to Q[X]$. First, if $K = Q\langle \varepsilon X_0\rangle$ is the Koszul complex with $\partial(\varepsilon X_0) = X_0$, then the lifting property 5 applied to $Q[X] \to k \xleftarrow{\simeq} K$ provides a dg algebra morphism $Q[X] \to K$. One may then apply $- \otimes_{Q[X]} K$ to the Tate model for the diagonal to obtain quasi-isomorphisms $Q[X]\langle \varepsilon X_0, \varepsilon X\rangle \xrightarrow{\simeq} K \xrightarrow{\simeq} k$, which by adjunction gives a quasi-isomorphism $R\langle \varepsilon X_0, \varepsilon X\rangle \xrightarrow{\simeq} k$. This is an acyclic closure which is witness to Avramov's equalities.

Theorem 21. *Let Q be a complete local ring and let $\xi : Q[X] \to Q[Y]$ be a homomorphism of minimal semi-free Q algebras. Both $Q[X]$ and $Q[Y]$ have Tate models as in theorem 10, and ξ extends to a homomorphism of divided power algebras, as in the commutative diagram below*

$$\begin{tikzcd}
Q[X,X] \arrow[r, "\xi\otimes\xi"] \arrow[d] & Q[Y,Y] \arrow[d] \\
Q[X,X]\langle \varepsilon X\rangle \arrow[r, "\widetilde{\xi}"] \arrow[d] & Q[Y,Y]\langle \varepsilon Y\rangle \arrow[d] \\
Q[X] \arrow[r, "\xi"] & Q[Y].
\end{tikzcd}$$

Moreover we can build $\widetilde{\xi}$ so that $\widetilde{\xi}(\varepsilon x) = \varepsilon\xi_1(x)$ modulo decomposables $\mathfrak{m}^2_{Q[Y,Y]\langle \Sigma Y\rangle} + (\varepsilon Y)^{(2)}$, for $x \in X$.

In the statement of the theorem $\xi_1 : XQ \to YQ$ is the strict part of ξ. That is, the part which preserves the weight decomposition $Q[X] = \bigoplus_w \mathrm{Sym}^w_Q(X)$ and $Q[Y] = \bigoplus_w \mathrm{Sym}^w_Q(Y)$.

Remark 17. The analogous theorem for a morphism of semi-free dg algebras $Q[X] \to P[Y]$ over a homomorphism $Q \to P$ follows from theorem 21 by adjunction.

Let us introduce some notation to use in the next lemma. We will denote by K the kernel of $Q[X,X]\langle \varepsilon X\rangle \to Q[X]$, and $K(n)$ is the ideal generated by $(x \otimes 1 - 1 \otimes x)$ for all $x \in X_{\leq n}$, and all the divided powers $(\varepsilon X)^{(1)}$. Note that $K = \bigcup K(n)$. Because of the form of the differential in the Tate model for the diagonal we have for any $x \in X_n$ that $\partial(\varepsilon x) = x \otimes 1 - 1 \otimes x + q$ where $q \in K(n-1) + \mathfrak{m}_Q K(n)$. Below we will also use the notation $\mathbf{Dec} = \mathfrak{m}^2_{Q[X,X]\langle \Sigma X\rangle} + (\varepsilon X)^{(2)}$ for decomposables, and $\mathbf{sDec} = (X \otimes 1, 1 \otimes X, \varepsilon X)^2 + \mathfrak{m}_Q(\varepsilon X)^{(1)} + (\varepsilon X)^{(2)}$ for strongly decomposables.

Lemma 18.

1. *Any element t in $\mathbf{Dec} \cap K$ can be written as $t = \partial(u) + v$ with $u \in \mathfrak{m}_Q(\varepsilon X)^{(1)}$ and $v \in \mathbf{sDec}$.*

2. *Any element f in K can be written as $f = \partial(g) + h$ with both $g, h \in (\varepsilon X)^{(1)}$.*

Proof. For 1, assume that t has degree n. Then we can write $t = \sum_{x \in X_n} a_1(x \otimes 1 - 1 \otimes x) + v_0 = \sum (a_1 \partial(\varepsilon x) + a_1 t_1) + v_0$ with $a_1 \in \mathfrak{m}_Q$ and $t_1 \in \mathbf{Dec}$ and $v_0 \in \mathbf{sDec}$ (we are suppressing the indexes in these summations). I will show by induction that for all i we can write

$$t = \sum (a_1 \partial(\varepsilon x_1) + \ldots + a_i \partial(\varepsilon x_i) + a_i t_i + v_0 + a_1 v_1 + \ldots + a_{i-1} v_{i-1})$$

where $a_j \in \mathfrak{m}_Q^i$ and $t_j \in \mathbf{Dec} \cap K$ and $v_j \in \mathbf{sDec}$ and all the x_i are in X_n. The base case has been dealt with. Next one writes $t_i = \sum a(x \otimes 1 - 1 \otimes x) + v_i = \sum (a \partial(\varepsilon x) + a t_{i+1}) + v_i$ with $a \in \mathfrak{m}_Q$ and $t_{i+1} \in \mathbf{Dec}$ and $v_i \in \mathbf{sDec}$ as before. Substituting this into the previous expression for t finishes the induction step. Part 1 is then established by setting $u = a_1 \varepsilon x_1 + a_2 \varepsilon x_2 + \ldots$ and $v = v_0 + a_1 v_1 + \ldots$ using completeness of Q.

For 2 it suffices to show the following instead: *Any element f in K can be written as $f = \partial(g) + h$ with $g \in (\varepsilon X)^{(1)}$ and $h \in (\varepsilon X)^{(1)} + \mathfrak{m}_Q K$.* Using completeness of Q a similar induction to part 1 will finish the proof.

Towards this we take $f \in K(n)$, and proceed by induction on n. So write $f = r + s$ with $r \in (X_{\leq n} \otimes 1 - 1 \otimes X_{\leq n})$ and $s \in (\varepsilon X)^{(1)}$. It suffices to prove the lemma for r.

Write $r = \sum_{x \in X_{\leq n}} (x \otimes 1 - 1 \otimes x) a = \sum \partial(\varepsilon x) a - \sum q a$, where the $q \in K(n-1) + \mathfrak{m}_Q K(n)$ are as in the paragraph before the lemma, and $a \in Q[X, X]\langle \varepsilon X \rangle$. The first summation differs from $\sum (\varepsilon x) \partial(a)$ by $\partial(\sum (\varepsilon x) a)$, while the second summation is in $K(n-1) + \mathfrak{m}_Q K$, so we are done by induction. ☐

Proof of theorem 21. We just need to define $\widetilde{\xi}$ on εX, which we do inductively. So assume we already have $\widetilde{\xi}_{<n} : Q[X, X]\langle \varepsilon X_{<n} \rangle \to Q[Y, Y]\langle \varepsilon Y_{<n} \rangle$. Fix $x \in X_n$. By lemmas 6 and 7, in order to define $\widetilde{\xi}(\varepsilon x)$ we must find an element $y \in (\varepsilon Y)^{(1)}_n$ whose boundary is $\widetilde{\xi}_{<n}(\partial(\varepsilon x))$, and which has the form $y = \varepsilon \xi_1(x)$ modulo $\mathbf{Dec}$.

Recall that $\partial(\varepsilon x) = x \otimes 1 - 1 \otimes x$ modulo $\mathbf{Dec}$. This means $\widetilde{\xi}_{<n}(\partial(\varepsilon x)) = \xi_1(x) \otimes 1 - 1 \otimes \xi_1(x) + s = \partial(\varepsilon \xi_1(x)) + t$, where s and t are in $\mathbf{Dec}$.

Since t is a decomposable element in the kernel of $Q[Y, Y]\langle \varepsilon Y \rangle \to Q[Y]$ we can write $t = \partial(u) + v$ as in part 1 of lemma 18. Note that v is automatically in the kernel K of $Q[Y_{<n}, Y_{<n}]\langle \varepsilon Y_{<n} \rangle \to Q[Y_{<n}]$. Since t is a cycle so v. But K is acyclic so $v = \partial(f)$ for some f in K. We apply part 2 of the lemma to get $v = \partial(f - \partial(g))$ with $h = f - \partial(g)$ in $(\varepsilon Y_{<n})^{(1)}$. Having degree $n+1$, h is automatically decomposable. Thus $y = \varepsilon \xi_1(x) + u + h$ has the desired properties, and we set $\widetilde{\xi}(x) = y$. ☐

The Functoriality of Avramov and Iyengar

In [21] Avramov and Iyengar state a theorem which they call a functorial enhancement of the equalities $\mathrm{card}(X_{n-1}) = \epsilon_n(R)$. Their proof is extremely technical and spans more than six pages. Theorem 21 will allow us to give a shorter proof of their result, which we state after some setup.

Let $\phi : (R, k) \to (S, l)$ be a local homomorphism of complete local rings. Then R and S admit compatible, minimal Cohen presentations as in the diagram below

$$\begin{array}{ccc} Q & \longrightarrow & P \\ \downarrow & & \downarrow \\ R & \longrightarrow & S. \end{array}$$

By the lifting property of lemma 5 one may extend this to a morphism of minimal models

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & P \arrow[d] \\ Q[X] \arrow[r, "\xi"] \arrow[d, "\simeq"] & P[Y] \arrow[d, "\simeq"'] \\ R \arrow[r] & S. \end{tikzcd}$$

And hence a morphism of minimal semifree dg algebras $\phi^{\mathsf{sf}} : l[X] \to l[Y]$. At the same time, by a standard lifting property analogous to proposition 5 there is a morphism of acyclic closures

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ R\langle U\rangle \arrow[r] \arrow[d, "\simeq"] & S\langle V\rangle \arrow[d, "\simeq"'] \\ k \arrow[r] & l. \end{tikzcd}$$

And hence a morphism of free divided power algebras

$$\begin{tikzcd} \mathrm{Tor}^R(k,l) \arrow[r] \arrow[d, equal] & \mathrm{Tor}^S(l,l) \arrow[d, equal] \\ l\langle U\rangle \arrow[r, "\phi^\gamma"] & l\langle V\rangle. \end{tikzcd}$$

Corollary 4 (Avramov-Iyengar [21, Theorem 3.4])**.** *In the above above situation, for every $n \geq 2$ there is a a commutative diagram of homomorphisms of l vector spaces*

$$\begin{tikzcd} \mathrm{ind}_{n-1}(l[X]) \arrow[r, "\mathrm{ind}_{n-1}(\phi^{\mathsf{sf}})"] \arrow[d, "\cong"] & \mathrm{ind}_{n-1}(l[Y]) \arrow[d, "\cong"] \\ \mathrm{ind}^\gamma_n(l\langle U\rangle) \arrow[r, "\mathrm{ind}^\gamma_n(\phi^\gamma)"] & \mathrm{ind}^\gamma_n(l\langle V\rangle) \end{tikzcd}$$

in which the vertical maps are isomorphisms.

Let $\widetilde{\xi} : P[X,X]\langle \Sigma X\rangle \to Q[Y,Y]\langle \Sigma Y\rangle$ be the morphism of Tate models for the diagonal given by Remark 17. By uniqueness, we are free to use the construction of acyclic closures explained in example 8. So let $\underline{s}$ and $\underline{t}$ be minimal generating sets for $\mathfrak{m}_P$ and $\mathfrak{m}_Q$ respectively. Proposition 5 produces a commutative diagram of dg algebras

$$\begin{tikzcd} Q[X] \arrow[r] \arrow[d] & P[Y] \arrow[d] \\ Q\langle \Sigma\underline{t}\rangle \arrow[r] \arrow[d, "\simeq"] & P\langle \Sigma\underline{t}\rangle \arrow[d, "\simeq"'] \\ k \arrow[r] & l. \end{tikzcd}$$

From this one easily produces a morphism $R\langle \Sigma\underline{t}, \Sigma X\rangle \to S\langle \Sigma\underline{s}, \Sigma Y\rangle$ of acyclic closures constructed as in example 8. It is now clear from the description $\widetilde{\xi}(\Sigma x) = \Sigma\xi_1(x)$ modulo decomposables in theorem 21 that the following diagram commutes

$$\begin{tikzcd}
\mathrm{ind}_{n-1}(l[X]) \arrow[r, equal] & lX_{n-1} \arrow[r] \arrow[d, equal] & lY_{n-1} \arrow[r, equal] \arrow[d, equal] & \mathrm{ind}_{n-1}(l[Y]) \\
\mathrm{ind}^{\gamma}_{n}(l\langle \underline{t}, \Sigma X\rangle) \arrow[r, equal] & l\Sigma X_n \arrow[r] & l\Sigma Y_n \arrow[r, equal] & \mathrm{ind}^{\gamma}_{n}(l\langle \underline{s}, \Sigma Y\rangle),
\end{tikzcd}$$

thus we have deduced Avramov-Iyengar's theorem from theorem 21.

Finally, we state a more general version of Avramov-Iyengar's theorem, which can be established using the same proof. This version can be applied to the homotopy Lie algebra of a homomorphism, and in particular we needed it to establish theorem 16.

Let $(P, \mathfrak{m}, k)$ and $(Q, \mathfrak{n}, l)$ be complete regular local rings, and let $Q \to A = Q[X]$ and $P \to B = Q[Y]$ be minimal semi-free extensions (we do not assume A or B are absolutely minimal). If we take a minimal generating set a for $\mathfrak{m}$ and b for $\mathfrak{n}$ then we can build acyclic closures $A \to A\langle \Sigma a, \Sigma X\rangle \xrightarrow{\simeq} k$ and $B \to B\langle \Sigma b, \Sigma Y\rangle \xrightarrow{\simeq} l$ using theorem 10.

Theorem 22. *In the notation above, a local homomorphism $\xi : A \to B$ induces a uniquely defined homomorphism $\mathrm{Tor}^{\xi} : \mathrm{Tor}^A_*(k, l) \to \mathrm{Tor}^B_*(l, l)$ of divided power Hopf algebras over l, and there is a commutative diagram*

$$\begin{tikzcd}
\mathrm{H}_*(\Sigma\mathfrak{m}_A/\mathfrak{m}_A^2) \otimes_k l \arrow[r, "\Sigma\mathrm{ind}(\xi)"] \arrow[d, "\cong"'] & \mathrm{H}_*(\Sigma\mathfrak{m}_B/\mathfrak{m}_B^2) \arrow[d, "\cong"] \\
\mathrm{ind}^{\gamma}\mathrm{Tor}^A_*(k, l) \arrow[r, "\mathrm{ind}^{\gamma}(\mathrm{Tor}^{\xi})"] & \mathrm{ind}^{\gamma}\mathrm{Tor}^B_*(l, l).
\end{tikzcd}$$

Non-minimality makes little difference since $\Sigma\mathfrak{m}_A/\mathfrak{m}_A^2$ can only have a differential at the first step. We include this because it was necessary for theorem 16.

The theorem follows from the existence of a homomorphism $\widetilde{\xi} : A\langle \Sigma a, \Sigma X\rangle \to A\langle \Sigma a, \Sigma X\rangle \to B\langle \Sigma b, \Sigma Y\rangle$ extending ξ, with the property that $\widetilde{\xi}(\Sigma x) = \Sigma\xi(x)$ modulo decomposables. This is produced by theorem 21.

3.5 Some Long Exact Sequences

In this section we discuss long exact sequences for the homotopy Lie algebra (or coalgebra). In short, to a sequence $(Q, u) \xrightarrow{\psi} (R, k) \xrightarrow{\phi} (S, l)$ of local homomorphisms one would like to associate a *Jacobi-Zariski exact sequence*

$$\cdots \to \pi_{i+1}(\phi) \to \pi_i(\psi) \otimes_k l \to \pi_i(\phi\psi) \to \pi_i(\phi) \to \pi_{i-1}(\psi) \otimes_k l \to \cdots$$

This is always possible when the residual characteristic is zero. In general however it is necessary to impose additional hypotheses on ϕ and ψ. We will introduce a variation $\widetilde{\pi}_*(-)$ which produces an analogous long exact sequence in all situations. Then we discuss the question of calculating this variation, and in particular of when $\widetilde{\pi}_*(-)$ agrees with $\pi_*(-)$.

Gulliksen Minimality and Avramov's Six-Term Exact Sequences

We first explain how Avramov established the existence of a long exact sequence of homotopy Lie algebras (in some circumstances) by building on work of Gulliksen. This subsection is essentially classical, if somewhat revisionist.

We begin with some definitions abstracted from [64, section 3]. We say that a semi-free extension $A \to A[X]$ is *Gulliksen minimal* if $\partial(A[X]_{\mathrm{even}}) \subseteq \mathfrak{m}_A B + \mathfrak{m}^2_{A[X]}$. With divided powers the condition is reversed: a Tate extension $A \to A\langle U \rangle$ is called *Gulliksen minimal* if $\partial(A\langle U \rangle_{\mathrm{odd}}) \subseteq \mathfrak{m}_A B + (U)^{(2)}$.

Theorem 23 (Gulliksen [64, lemma 3.2.1 and theorem 3.2.3])**.** *Let $R \to R\langle U \rangle$ be a Gulliksen minimal Tate extension of a local ring R. There is a minimal Tate extension $R \to R\langle V \rangle$ and a surjective quasi-isomorphism $R\langle U \rangle \xrightarrow{\simeq} R\langle V \rangle$ of divided power algebras which induces a quasi-isomorphism of indecomposables*

$$\mathrm{ind}^{\gamma}(R\langle U \rangle) \xrightarrow{\simeq} \mathrm{ind}^{\gamma}(R\langle V \rangle).$$

Essentially the same proof shows the following semi-free version:

Theorem 24. *Let $R \to R[X]$ be a Gulliksen minimal semi-free extension of a local ring R. There is a minimal semi-free extension $R \to R[Y]$ and a surjective quasi-isomorphism $R[X] \xrightarrow{\simeq} R[Y]$ of divided power algebras which induces a quasi-isomorphism of indecomposables*

$$\mathfrak{m}_{R[X]}/\mathfrak{m}^2_{R[X]} \xrightarrow{\simeq} \mathfrak{m}_{R[Y]}/\mathfrak{m}^2_{R[Y]}.$$

Let $\phi : (R, k) \to (S, l)$ be a local homomorphism, and build a minimal model $Q[X] \xrightarrow{\simeq} \widehat{R}$. At this point we can take a minimal Cohen factorisation of the $Q \to Q' \to \widehat{S}$ of the composition $Q \to \widehat{R} \to \widehat{S}$. Note that Q' is automatically regular. Then we apply proposition 4 to the surjection $Q'[X] \to \widehat{S}$ to obtain a minimal Cohen model $Q[X] \to Q'[X, Z] \xrightarrow{\simeq} \widehat{S}$. Crucially, $Q'[X, Z]$ can and typically does fail to be absolutely minimal, since the differential $\partial(z)$ of $z \in Z$ may involve indecomposables from $Q'[X]$. In any case, tensoring down produces a minimal model $\widehat{R} \to R'[Z] = R \otimes_{Q[X]} Q[X, Z] \xrightarrow{\simeq} \widehat{S}$. The following diagram summarises the situation

$$\begin{tikzcd}
 & Q'[X,Z] \arrow[dr, "\simeq"] & & \\
Q[X] \arrow[ur] \arrow[dr, "\simeq"'] & & R'[Z] \arrow[dr, "\simeq"] & \\
 & \widehat{R} \arrow[ur] \arrow[rr, "\phi"] & & \widehat{S}.
\end{tikzcd}$$

In particular, we obtain a *fibre sequence* of semi-free dg algebras

$$Q[X] \longrightarrow Q'[X, Z] \longrightarrow \overline{Q}[Z]$$

where $\overline{Q}[Z] = k \otimes_{Q[X]} Q'[X, Z] = k \otimes_R R'[Z]$. By taking indecomposables we get a short exact sequence of chain complexes

$$0 \longrightarrow \frac{\mathfrak{m}_{Q[X]}}{\mathfrak{m}^2_{Q[X]}} \otimes_k l \longrightarrow \frac{\mathfrak{m}_{Q'[X,Z]}}{\mathfrak{m}^2_{Q'[X,Z]}} \longrightarrow \frac{\mathfrak{m}_{\overline{Q}[Z]}}{\mathfrak{m}^2_{\overline{Q}[Z]}} \longrightarrow 0. \tag{3.1}$$

Theorem 25 (Avramov)**.** *If $\mathrm{wcat}(\phi)$ is finite then the extension $Q' \to Q'[X, Z]$ is Gulliksen minimal.*

A published proof of this is difficult to find. However, the case that ϕ is flat is in [7, theorem 1.1], and the proof there can be adapted to general case.

Having established Gulliksen's condition, one constructs Avramov's long exact sequence as follows. It follows from theorem 24 that there is a minimal semi-free extension $Q' \to Q'[Y']$ and surjective quasi-isomorphisms

$$Q'[X,Z] \xrightarrow{\simeq} Q'[Y'] \xrightarrow{\simeq} \widehat{S},$$

with the property that $\mathfrak{m}_{Q'[X,Z]}/\mathfrak{m}^2_{Q'[X,Z]} \to \mathfrak{m}_{Q'[Y']}/\mathfrak{m}^2_{Q'[Y']}$ is a quasi-isomorphism as well. If the homomorphism $Q' \to \widehat{S}$ fails to be a minimal Cohen presentation then we may find a regular sequence $\underline{q} \subseteq \mathfrak{m}_{Q'} \smallsetminus \mathfrak{m}^2_{Q'}$ such that $P = Q'/\underline{q} \to \widehat{S}$ is. After possibly making a change of basis, there is a subset $\underline{y} \subseteq Y'_1$ with $\partial(\underline{y}) = \underline{q}$. We set $Y = Y' \smallsetminus \underline{y}$, and then we have a minimal Cohen model $P[Y] = Q'[Y']/(\underline{q},\underline{y}) \xrightarrow{\simeq} \widehat{S}$ and it is easy to see that the induced map of indecomposables

$$\frac{\mathfrak{m}_{Q'[Y']}}{\mathfrak{m}^2_{Q'[Y']}} \xrightarrow{\simeq} \frac{\mathfrak{m}_{P[Y]}}{\mathfrak{m}^2_{P[Y]}}$$

is a quasi-isomorphism as well (it is in isomorphism except in degrees 1 and 2, where we collapse a contractible subcomplex). Putting all this together, there is a natural isomorphism $\Sigma \mathrm{H}_*(\mathfrak{m}_{Q'[X,Z]}/\mathfrak{m}^2_{Q'[X,Z]}) \cong \Sigma(\mathfrak{m}_{P[Y]}/\mathfrak{m}^2_{P[Y]}) \cong \pi_*(S)$.

Observe that the Gulliksen minimality condition implies that the differential of $\mathfrak{m}_{Q'[X,Z]}/\mathfrak{m}^2_{Q'[X,Z]}$ vanishes on even elements. Therefore half of the connecting homomorphisms vanish in the long exact homology sequence associated to 3.1. After applying Σ this looks like

$$\begin{array}{l} 0 \longrightarrow \pi_{2i}(R)\otimes_k l \longrightarrow \pi_{2i}(S) \longrightarrow \pi_{2i}(\phi) \\ \xrightarrow{\delta} \pi_{2i-1}(R)\otimes_k l \longrightarrow \pi_{2i-1}(S) \longrightarrow \pi_{2i-1}(\phi) \longrightarrow 0. \end{array} \tag{3.2}$$

These are the six-term exact sequences of Avramov.

Remark 18. When $\mathrm{char}(k) = 0$ the conclusion of theorem 24 holds without any minimality hypothesis (this follows from [64, lemma 3.2.1]). In other words, taking indecomposables is an exact functor in characteristic zero. It follows that these six term exact sequences exist without any additional hypotheses on ϕ.

How one can patch these sequences together into a Jacobi-Zariski long exact sequence of relative homotopy Lie coalgebras is dealt with by proposition 12 and theorem 30.

Translating Between Semi-Free Models And Tate Models

The above sketch is historically somewhat apocryphal, since Avramov and Gulliksen both worked on the divided power side. We will describe now how the two approaches are directly connected.

First we recall that if $R \to R\langle U\rangle \xrightarrow{\simeq} k$ is an acyclic closure then there is a canonical isomorphsim $\pi_i(R) \cong \mathrm{ind}^\gamma(k\langle U\rangle)_i$; see the discussion in section 3.1 or corollary 4, or one may deduce this from theorem 10 as we do now.

First recall the construction of the dg algebra $Q'[X,Y]$ as above, starting from the local homomorphism $\phi : R \to S$. According to theorem 10 we have a Tate model for the diagonal

$$Q'[X,Z,X,Z]\langle \Sigma X, \Sigma Z\rangle \xrightarrow{\simeq} Q'[X,Z] \xrightarrow{\simeq} R'[Z] \xrightarrow{\simeq} \widehat{S}.$$

Let $Q'[t] \xrightarrow{\simeq} l$ be the Koszul complex resolving l over the regular ring Q'. The standard lifting property of lemma 5 produces a Q' algebra map $Q'[X,Z] \to Q'[t]$. Now we can apply $Q'[t] \otimes_{Q'[X,Z]} -$ to obtain a quasi-isomorphism $Q'[t,X,Z]\langle \Sigma X, \Sigma Z\rangle \xrightarrow{\simeq} Q'[t] \xrightarrow{\simeq} l$, and by adjunction we get a quasi-isomorphism

$$\widehat{S}\langle t, \Sigma X, \Sigma Z\rangle \xrightarrow{\simeq} l.$$

This is a Tate model for $\widehat{S} \to l$. By the description of the differential in theorem 10 there is a canonical isomorphism

$$\Sigma \left(\frac{\mathfrak{m}_{Q'[X,Z]}}{\mathfrak{m}^2_{Q'[X,Z]}} \right) \cong \mathrm{ind}^\gamma(l\langle t, \Sigma X, \Sigma Z\rangle).$$

Therefore the semi-free extension $Q' \to Q'[X,Z]$ is Gulliksen minimal if and only if the Tate extension $S \to S\langle t, \Sigma X, \Sigma Z\rangle$ is Gulliksen minimal. This explains the shift between the two definitions of Gulliksen minimality.

We use the same construction to build an acyclic closure $\widehat{R} \to \widehat{R}\langle s, \Sigma X\rangle \xrightarrow{\simeq} k$, starting from the Koszul complex $Q[s] \xrightarrow{\simeq} k$. We use theorem 21 (and remark 17) to produce a morphism of divided power algebras $\xi : Q[X,X]\langle \Sigma X\rangle \to Q'[X,Z,X,Z]\langle \Sigma X, \Sigma Z\rangle$ compatible with the inclusion $Q[X] \to Q'[X,Z]$ such that $\xi(\Sigma x) = \Sigma x$ for $x \in X$. This is enough to witness the Tate extension $\widehat{S} \to \widehat{S}\langle t, \Sigma X, \Sigma Z\rangle$ as the composition of two minimal Tate extensions

$$\widehat{S} \to \widehat{S}\langle s, \Sigma X\rangle \to \widehat{S}\langle s, r, \Sigma X, \Sigma Z\rangle$$

with $t = s \cup r$. The result is a fibre sequence of divided power algebras

$$l\langle s, \Sigma X\rangle \to l\langle s, r, \Sigma X, \Sigma Z\rangle \to l\langle r, \Sigma Z\rangle.$$

Much as in the semi-free case one obtains a short exact sequence of indecomposables, and from here one can hope to obtain a long exact sequence of homotopy Lie coalgebras. Avramov argues directly in [7, theorem 1.1] (in the case that ϕ is flat) that $l\langle s, r, \Sigma X, \Sigma Z\rangle$ is Gulliksen minimal, and obtains his six-term exact sequences by these means.

We have shown that by going through the Tate model for the diagonal one can translate between semi-free models and Tate models, with Gulliksen minimality being equivalent on either side. Both sides have advantages, but it seems that the semi-free side is more intuitive.

The Six-Term Exact Sequences for $\pi^\gamma_*(\phi)$

When $\phi : R \to S$ is surjective one can take an acyclic closure $R \to R\langle U\rangle \xrightarrow{\simeq} S$. Then we can take the fibre $k\langle U\rangle = k \otimes_R R\langle U\rangle \simeq k \otimes^{\mathsf{L}}_R S$. Thinking of this as a divided power analogue of the minimal model for S over R, we define

$$\pi^\gamma_*(\phi) = \Sigma\mathrm{ind}^\gamma k\langle U\rangle$$

The uniqueness of acyclic closures makes $\pi^\gamma_*(\phi)$ well-defined. Certainly nothing like theorem 15 holds for this object, but the following considerations make $\pi^\gamma_*(\phi)$ a useful computational tool.

We continue $R\langle U\rangle$ to an acyclic closure $R\langle U\rangle \to R\langle U, V\rangle \xrightarrow{\simeq} k$. Base changing reveals an acyclic closure $S\langle V\rangle \xrightarrow{\simeq} k$. In order to compare $\pi^\gamma_*(\phi)$ with $\pi_*(S)$ and $\pi_*(R)$ we need to understand the extent to which the Tate model $R \to R\langle U, V\rangle \xrightarrow{\simeq} k$ can fail to be minimal. Avramov remarks that his proof of

[7, theorem 1.1] shows the following theorem.

Theorem 26 (Avramov [7, remark 1.10])**.** *If $R \to S$ admits an acyclic closure which is minimal as a complex of R modules, then $R \to R\langle U, V\rangle$ is Gulliksen minimal. Therefore from the short exact sequence*

$$0 \longrightarrow \mathrm{ind}^\gamma(k\langle U\rangle) \longrightarrow \mathrm{ind}^\gamma(k\langle U,V\rangle) \longrightarrow \mathrm{ind}^\gamma(k\langle V\rangle) \longrightarrow 0$$

we obtain a long exact homology sequence which splits into six-term exact sequences

$$\begin{aligned} 0 \longrightarrow \pi^\gamma_{2i+1}(\phi) \longrightarrow \pi_{2i}(R) \longrightarrow \pi_{2i}(S) \\ \xrightarrow{\delta} \pi^\gamma_{2i}(\phi) \longrightarrow \pi_{2i-1}(R) \longrightarrow \pi_{2i-1}(S) \longrightarrow 0. \end{aligned} \tag{3.3}$$

The theorem applies in particular to large homomorphisms by [88, theorem 2.5] (large homomorphisms were independently proven to have this property by Avramov and Rahbar-Rochandel, according to loc. cit.)

Example 9. Recall from section 2.7 that a surjective homomorphism $\phi : R \to S$ with kernel I is a quasi-complete intersection if its two-step Tate model is acyclic. In other words, it admits an acyclic closure $R \to R\langle U_1, U_2\rangle \xrightarrow{\simeq} S$. In this case minimality forces $R\langle U_1, U_2\rangle$ to be minimal as a complex, therefore theorem 26 applies in this situation. It is clear that $\pi^\gamma_1(\phi) = 0$, $\pi^\gamma_2(\phi) \cong kU_1 \cong k \otimes_S I/I^2$ and $\pi^\gamma_3(\phi) \cong kU_2 \cong k \otimes_S \mathrm{H}_1$ where $\mathrm{H}_1 = \mathrm{H}_1(R\langle U_1\rangle)$ is the first Koszul homology of I (see [31]). In fact, it follows readily from the definitions that quasi-complete intersections are characterised by the vanishing of $\pi^\gamma_{\geq 4}(\phi)$, see theorem 33 for a variation on this. We obtain an exact sequence

$$0 \longrightarrow k \otimes_S \mathrm{H}_1 \xrightarrow{\phi_2} \pi_2(R) \longrightarrow \pi_2(S) \longrightarrow k \otimes_S I/I^2 \longrightarrow \pi_1(R) \xrightarrow{\phi_1} \pi_1(S) \longrightarrow 0,$$

and isomorphisms $\phi_i : \pi_i(R) \cong \pi_i(S)$ for $i \geq 3$. This appears as [18, theorem 5.3]. The case of a map of embedded deformations (which are natural examples of qci homomorphisms, see [18]) appears already in [7, remark 1.10] (but note that qci homomorphisms had not been defined at the time of op. cit.)

The lifting property of lemma 5 means that a minimal model and an acyclic closure can always be compared:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R\langle U\rangle \arrow[d, "\simeq"] \\ R[X] \arrow[r, "\simeq"] \arrow[ur, dashed] & S. \end{tikzcd}$$

Passing to fibres produces a homomorphism $k[X] \to k\langle U\rangle$, and then taking indecomposables builds the *comparison map* $\pi_*(\phi) \to \pi^\gamma(\phi)$. This map will be an isomorphism in characteristic zero (by uniqueness of minimal models) or when ϕ is complete intersection, But in general it will fail dramatically to be so. Below we factor this comparison through another variation $\widetilde{\pi}_*(\phi)$ which has the advantage of producing exact sequences in all situations (and the disadvantage of being slightly more mysterious).

The Mixed Version of Gulliksen's Theorem

Now we state a mixed version of the two theorems 23 and 24. It could easily be expanded to include these theorems as special cases, but for notational convenience we refrain from doing so.

Theorem 27. *Let $R \to R[X]$ be a minimal semi-free extension of a local ring (R, k), and let $R[X] \to R[X]\langle U\rangle$ be a minimal Tate extension. There is a semi-free mixed extension $R \to R[Y]\langle V\rangle$ such that*

1. *the extension is minimal in the sense that the differential of* $\operatorname{ind}^\gamma(k[Y]\langle V\rangle)$ *vanishes;*
2. *there is a surjective quasi-isomorphism $\phi : R[X]\langle U\rangle \xrightarrow{\simeq} R[Y]\langle V\rangle$ of dg R algebras with divided powers on $(U)^{(1)}$ and $(V)^{(1)}$ respectively;*
3. *and ϕ induces a quasi-isomorphism*

$$\operatorname{ind}^\gamma(k \otimes_R \phi) : \operatorname{ind}^\gamma(k[X]\langle U\rangle) \xrightarrow{\simeq} \operatorname{ind}^\gamma(k[Y]\langle V\rangle).$$

Our notation could be ambiguous, so let us emphasise that we *do not* claim that the reduced mixed extension $R \to R[Y]\langle V\rangle$ can be obtained as a composition $R \to R[Y] \to R[Y]\langle V\rangle$ (i.e. $\partial(Y)$ could involve terms from W). The proof doesn't show this, and we don't use it below.

We need the following lemma, which is a direct consequence of lemma 8.

Lemma 19. *Let $R[X]\langle U\rangle$ be a mixed extension of a local ring (R, k) and suppose $X' \to R[X]\langle U\rangle$ and $U' \to (U)^{(1)}$ are two graded functions from positively graded sets. Then the induced map $R[X']\langle U'\rangle \to R[X]\langle U\rangle$ is an isomorphism if and only if the induced function*

$$X' \cup U' \to \operatorname{ind}^\gamma(k[X]\langle U\rangle)$$

makes $X' \cup U'$ a basis of the k vector space $\operatorname{ind}^\gamma(k[X]\langle U\rangle)$.

Proof of theorem 27. As a graded vector space $\operatorname{ind}^\gamma(k[X]\langle U\rangle) \cong kX + kU$. Since the two extensions $k \to k[X]$ and $k[X] \to k[X]\langle U\rangle$ are minimal the differential looks like $kX \xleftarrow{\operatorname{ind}^\gamma(\partial)} kU$ (i.e. all other components of $\operatorname{ind}^\gamma(\partial)$ vanish with respect to this decomposition). Let $U' \subseteq kU$ be a basis for the kernel of $\operatorname{ind}^\gamma(\partial) : kU \to kX$ and let $V \subseteq kU$ be a basis for any k linear complement to kU'. Choose a lift $f : U' \cup V \to (U)^{(1)}$ and set $X' = \partial(f(U')) \subseteq R[X]\langle U\rangle$. Note that the image of X' in $\operatorname{ind}^\gamma(k[X]\langle U\rangle)$ is precisely $\operatorname{ind}^\gamma(\partial)(U') \subseteq kX$. Finally, choose a subset $Y \subseteq kX$ which freely spans a complement to $k(\operatorname{ind}^\gamma(\partial)(U'))$, and a lift $g : Y \to R[X]\langle U\rangle$.

Together f and g induce a homomorphism of graded algebras $R[X', Y]\langle U', V\rangle \to R[X]\langle U\rangle$ (forgetting the differential or now). By lemma 19 this homomorphism is an isomorphism.

It follows that the composition

$$\phi : R[Y]\langle V\rangle \hookrightarrow R[X', Y]\langle U', V\rangle \to R[X]\langle U\rangle \to R[X]\langle U\rangle/(X', (f(U))^{(1)})$$

is an isomorphism as well. But now note that by construction the ideal $(X', (f(U))^{(1)})$ is closed under the differential of $R[X]\langle U\rangle$. In this way $R[Y]\langle V\rangle$ obtains a well-defined differential making it a dg algebra with divided powers supported on $(V)^{(1)}$, and there is a surjective homomorphism $\phi : R[X]\langle U\rangle \to R[Y]\langle V\rangle$ of dg algebras with divided powers.

By construction the differential of $\operatorname{ind}^\gamma(k[Y]\langle V\rangle) \cong kY + kV$ vanishes, so condition 1 is clear.

It is also clear that $\operatorname{ind}^\gamma(k[X]\langle U\rangle) \to \operatorname{ind}^\gamma(k[Y]\langle V\rangle)$ is a quasi-isomorphism, because by construction this map $kX + kU \to kY + kV$ simply collapses the contractible subcomplex $kX' + kU'$, so 3 holds also.

It remains to show that ϕ is a quasi-isomorphism. By the derived Nakayama lemma it suffices to show that $k \otimes \phi : k[X]\langle U\rangle \to k[Y]\langle V\rangle$ is a quasi-isomorphism. This is another application of lemma 1. We use the weight filtration with $k[X]\langle U\rangle_{(n)} = \sum_{i+j=n}(X)^i(U)^{(j)}$ and $k[Y]\langle V\rangle_{(n)} = \sum_{i+j=n}(Y)^i(V)^{(j)}$. After passing to the associated graded algebras the differential of $k[Y]\langle V\rangle$ vanishes, while $\mathrm{gr}(k[X]\langle U\rangle)$ becomes isomorphic to $k[Y]\langle V\rangle \otimes k[X']\langle U' : \partial(U') = X'\rangle$. It is well-known that $k[X']\langle U' : \partial(U') = X'\rangle \simeq k$ is acyclic (see for example [44, proposition 1.9]), and part 2 follows. □

Definition of $\widetilde{\pi}_*(\phi)$ and Long Exact Sequences

In general a local homomorphism will fail dramatically to induce a long exact sequence of homotopy Lie coalgebras like 3.2 (see for example [9]). In this section we present a modification of the homotopy Lie coalgebra which will produce a long exact sequence in all situations.

It is well-known that the André-Quillen (co)homology functors also repair this defect. By simplicial methods André and Quillen independently associated a graded Lie algebra $D^*(\phi; l)$ to a local homomorphism $\phi : (R, k) \to (S, l)$ (and more generally). The excellent formal properties of these functors is the foundation for many important results in commutative algebra and algebraic geometry.

There is a natural homomorphism of graded Lie algebras $D^*(\phi; k) \to \pi^*(\phi)$ (see [2] for the absolute situation). This comparison map is an isomorphism when $\mathrm{char}(l) = 0$ and always in low degrees (see [106]).

However $D^*(\phi; l)$ can be very difficult to compute, and it often produce strange results from the perspective of Koszul duality. For instance, if k is the field with two elements and ϕ is the inclusion $k \to k[x, y]/(x, y)^2$ then the difficult calculation of $D^*(\phi; k)$ is performed by Goerss in [56]. On the other hand $\pi^*(\phi)$ is the free Lie algebra on two elements of degree 1, as predicted by Koszul duality.

The variation $\widetilde{\pi}^*(\phi)$ we present in this section shares some of the good properties of $D^*(\phi, l)$ while being much more computable, and while giving the expected Koszul duality results in some important cases. For example $\widetilde{\pi}^*(\phi) \cong \pi^*(\phi)$ will be a free Lie algebra whenever ϕ is Golod. Homotopically $\widetilde{\pi}^*(\phi)$ does not appear to have any significant meaning (there is no analogue of theorem 15), but it seems to be computationally quite useful.

We can give some intuition as follows. $\pi_*(\phi)$ only depends on the derived fibre $k \otimes^{\mathsf{L}}_R S$, and somehow this doesn't seem to be enough data to extract a long exact sequence (except in characteristic zero). We will construct a mixed model for $k \otimes^{\mathsf{L}}_R S$, and $\widetilde{\pi}_*(\phi)$ will be given by the (shifted) indecomposables of this mixed model. The way the semi-free and divided power structures are made to interact uses slightly more information about the homomorphism ϕ, and results in a long exact sequence in all situations. Roughly, we will see that $\widetilde{\pi}_*(\phi)$ interpolates between $\pi_*(\phi)$ and $\pi^{\gamma}_*(\phi)$.

All this said, we start with an extremely simple definition of $\widetilde{\pi}_*(-)$ which may make it appear unlikely to be useful. If $\phi : (R, k) \to (S, l)$ is a local homomorphism then we set

$$\widetilde{\pi}_*(\phi) = \mathrm{H}_*(\mathrm{cone}(\phi_* : \pi_*(R) \otimes_k l \to \pi_*(S))).$$

One can equally well define $\widetilde{\pi}^*(\phi) = \widetilde{\pi}_*(\phi)^\vee$, but we will work homologically in this section. Almost tautologically there is a long exact sequence

$$\cdots \to \widetilde{\pi}_{i+1}(\phi) \to \pi_i(R) \otimes_k l \to \pi_i(S) \to \widetilde{\pi}_i(\phi) \to \pi_{i-1}(R) \otimes_k l \to \cdots \tag{3.4}$$

associated to any local homomorphism. We can slightly generalise this as follows:

Proposition 12. *Let* $(Q,u) \xrightarrow{\psi} (R,k) \xrightarrow{\phi} (S,l)$ *be a sequence of local homomorphisms. There is a long exact sequence*

$$\cdots \to \widetilde{\pi}_{i+1}(\phi) \to \widetilde{\pi}_i(\psi) \otimes_k l \to \widetilde{\pi}_i(\phi\psi) \to \widetilde{\pi}_i(\phi) \to \widetilde{\pi}_{i-1}(\psi) \otimes_k l \to \cdots$$

Proof. This is an immediate consequence of the Octahedral axiom (which certainly holds for complexes of l vector spaces). As stated for example in [95], it says that the two unraveled triangles (long exact sequences from 3.4) below can be interlaced with a third, indicated by the dashed arrows

$$\begin{tikzcd}
\pi_*(Q) \otimes_u l \arrow[dr] \arrow[rr, bend left] & & \pi_*(S) \arrow[dr] \arrow[rr, bend left] & & \widetilde{\pi}_*(\phi) \arrow[dr] \arrow[rr, bend left, dashed] & & \Sigma\widetilde{\pi}_*(\psi) \otimes_k l \\
\cdots & \pi_*(R) \otimes_k l \arrow[ur] \arrow[dr] & & \widetilde{\pi}_*(\phi\psi) \arrow[ur, dashed] \arrow[dr] & & \Sigma\pi_*(R) \otimes_k l \arrow[ur] \arrow[dr] & \cdots \\
\Sigma^{-1}\widetilde{\pi}_*(\phi) \arrow[ur] \arrow[rr, bend right, dashed] & & \widetilde{\pi}_*(\psi) \otimes_k l \arrow[ur, dashed] \arrow[rr, bend right] & & \Sigma\pi_*(Q) \otimes_u l \arrow[ur] \arrow[rr, bend right] & & \Sigma\pi_*(S).
\end{tikzcd}$$

□

The real interest of this proposition comes from the fact that there is a natural map $\pi_*(\phi) \to \widetilde{\pi}_*(\phi)$, and that this map is often an isomorphism. This is our strategy for uncovering Jacobi-Zariski exact sequences for the homotopy Lie algebra. By these means $\widetilde{\pi}_*(\phi)$ is a potentially useful computational tool.

An Alternative Construction for $\widetilde{\pi}_*(\phi)$

As before we start with a local homomorphism $\phi : (R,k) \to (S,l)$. The key to calculating $\widetilde{\pi}_*(\phi)$ is an alternative construction which explains how it interpolates between $\pi_*(\phi)$ and $\pi^\gamma_*(\phi)$ (see also theorem 30 specifically).

We show in particular that now there is a natural model F for the (completed) fibre $k \otimes^{\mathbf{L}}_{\widehat{R}} \widehat{S}$ such that $\mathrm{ind}^\gamma(F) = \widetilde{\pi}_*(\phi)$. As above we repeatedly use the Tate model for diagonal to mediate between free extensions and divided powers.

As before we take a minimal Cohen model $Q[X] \xrightarrow{\simeq} \widehat{R}$. Then using a minimal Cohen factorisation $Q \to Q' \to \widehat{S}$ we obtain a minimal Cohen presentation $P = Q'/\underline{q} \to \widehat{S}$ (the regular sequence $\underline{q}$ is as above, in the construction of Avramov's six-term exact sequences). Then we extend to a minimal model $P \to P[Y] \xrightarrow{\simeq} \widehat{S}$. The basic lifting property of lemma 5 produces a diagram

$$\begin{tikzcd}
Q[X] \arrow[r, "\widetilde{\phi}"] \arrow[d, "\simeq"'] & P[Y] \arrow[d, "\simeq"] \\
\widehat{R} \arrow[r, "\widehat{\phi}"] & \widehat{S}
\end{tikzcd}$$

which can be used to define $\phi_* : \pi_*(R) \to \pi_*(S)$. Similarly, if $Q[t] \xrightarrow{\simeq} k$ is the Koszul complex on a minimal set $\underline{x}$ of generators for $\mathfrak{m}_Q$, then there is a homomorphism of Q algebras $Q[X] \to Q[t]$ lifting the quotient homomorphism $Q[X] \to k$.

We use theorem 10 to build $Q[X, X]\langle \varepsilon X\rangle \xrightarrow{\simeq} Q[X]$ with $\partial(\varepsilon x) = x$ modulo decomposables, for $x \in X$. From this we apply $Q[t] \otimes_{Q[X]} -$ to obtain a quasi-isomorphism $Q[t, X]\langle \varepsilon X\rangle \xrightarrow{\simeq} Q[t]$, and finally we use $\widetilde{\phi}$ to apply $- \otimes_{Q[X]} P[Y]$ and build $P[Y]\langle t, \varepsilon X\rangle$.

Note that through $Q[X] \xrightarrow{\simeq} \widehat{R}$ there are natural quasi-isomorphsims $Q[t, X]\langle \varepsilon X\rangle \xrightarrow{\simeq} \widehat{R}\langle t, \varepsilon X\rangle \xrightarrow{\simeq} k$. Through $P[Y] \xrightarrow{\simeq} \widehat{S}$ is also a natural quasi-isomorphism $P[Y]\langle t, \varepsilon X\rangle \to \widehat{S}\langle t, \varepsilon X\rangle \cong \widehat{S} \otimes_{\widehat{R}} \widehat{R}\langle t, \varepsilon X\rangle$. Hence the dg algebra $P[Y]\langle t, \varepsilon X\rangle$ is a model for the derived tensor product $\widehat{S} \otimes^{\mathsf{L}}_{\widehat{R}} k$.

Finally, we examine the differential of $P[Y]\langle t, \varepsilon X\rangle$. Firstly $\partial(t)$ is determined by the image of $\underline{x}$ under $Q \to P$. Next, for $x \in X$ we have $\partial(\varepsilon x) = \widetilde{\phi}(x)$ modulo decomposable elements, thanks to the description of the differential in theorem 10. Putting all these facts together one obtains the following

Theorem 28. *Let $(R, k) \to (S, l)$ be a local homomorphism. The dg algebra $P[Y]\langle t, \varepsilon X\rangle$ constructed above is naturally a model for $\widehat{S} \otimes^{\mathsf{L}}_{\widehat{R}} k$. There is a canonical isomorphism*

$$\varepsilon \operatorname{ind}^\gamma(P[Y]\langle t, \varepsilon X\rangle) \cong \varepsilon\left((\frac{\mathfrak{m}_P}{\mathfrak{m}_P^2} + lY) \xleftarrow{\operatorname{ind}^\gamma(\partial)} (l \otimes_k \frac{\mathfrak{m}_Q}{\mathfrak{m}_Q^2} + lX) \right) \cong \operatorname{cone}(\phi_* : \pi_*(R) \otimes_k l \to \pi_*(S)),$$

and therefore $\widetilde{\pi}_*(\phi) \cong \varepsilon \mathrm{H}_*(\operatorname{ind}^\gamma(P[Y]\langle t, \varepsilon X\rangle))$.

We use this description of $\widetilde{\pi}_*(\phi)$ to build the comparison maps in the following theorem. In it we continue to use the notation from the constructions made above. Note also that by base-changing the minimal Cohen factorisation $Q \to Q' \to \widehat{S}$ along $Q \to \widehat{R}$ we obtain a minimal Cohen factorisation $\widehat{R} \to R' \to \widehat{S}$. We extend this to a minimal Cohen model $R \to R'[Z] \xrightarrow{\simeq} \widehat{S}$. The fibre $k \otimes_Q Q' \cong k \otimes_{\widehat{R}} R'$ will be denoted $\overline{Q}$. The dg algebra $\overline{Q}[Z]$ is a minimal model and by definition $\pi_*(\phi) = \pi_*(\overline{Q}[Z])$.

Theorem 29. *There are quasi-isomorphisms of dg algebras*

$$\overline{Q}[Z] \xleftarrow{\simeq} Q'[t, Z] \xrightarrow{\simeq} P[Y]\langle t, \varepsilon X\rangle$$

which induce the identity map on $\operatorname{Tor}^{\widehat{R}}_*(\widehat{S}, k)$. Taking indecomposables produces a comparison map

$$\pi_*(\phi) \longrightarrow \widetilde{\pi}_*(\phi).$$

Assume that ϕ is surjective and take an acyclic closure $R\langle U\rangle \xrightarrow{\simeq} S$. There is a quasi-isomorphism

$$P[Y]\langle t, \varepsilon X\rangle \xrightarrow{\simeq} k\langle U\rangle$$

of dg algebras with divided powers supported on $(t, \varepsilon X)^{(1)}$ and $(U)^{(1)}$ respectively, which again induces the identity on $\operatorname{Tor}^{\widehat{R}}_*(\widehat{S}, k)$. Taking indecomposables produces a comparison map

$$\widetilde{\pi}_*(\phi) \longrightarrow \pi^\gamma_*(\phi).$$

Proof. We may take a minimal model $Q'[X] \to Q'[X, Z] \xrightarrow{\simeq} \widehat{S}$ and then obtain $R'[Z]$ as $R \otimes_{Q[X]} Q'[X, Z]$.

Recall that $Q[t] \xrightarrow{\simeq} k$ is the Koszul complex of Q and that we have a homomorphism $Q[X] \to Q[t]$ of Q algebras. It extends to a homomorphism $Q'[X] \to Q'[t] \xrightarrow{\simeq} k \otimes_Q Q'$. Note that there is a natural quasi-isomorphism

$$Q'[t,Z] = Q'[t] \otimes_{Q'[X]} Q'[X,Z] \xrightarrow{\simeq} (k \otimes_Q Q') \otimes_{Q'[X]} Q'[X,Z] = \overline{Q}[Z].$$

This constructs the first quasi-isomorphism in the theorem. It is easy to see that this homomorphism induces a quasi-isomorphism upon taking indecomposables:

$$\frac{\mathfrak{m}_{Q'[t,Z]}}{\mathfrak{m}^2_{Q'[t,Z]}} \xrightarrow{\simeq} \frac{\mathfrak{m}_{\overline{Q}[Z]}}{\mathfrak{m}^2_{\overline{Q}[Z]}}.$$

Because this map is an isomorphism except in degrees 0 and 1 where it collapses the contractible subcomplex $l \cdot (\underline{x}) \xleftarrow{\partial} l \cdot (t)$.

Intermediate to the construction of the mixed model $P[Y]\langle t, \Sigma X\rangle$ we can build the dg algebra $P[X,Y]\langle \Sigma X\rangle = Q[X,X]\langle \Sigma X\rangle \otimes_{Q[X]} P[Y]$. It comes with a homomorphism from $Q'[X]$ (using the remaining copy of X on the left and the surjection $Q' \to P$) and a natural surjective quasi-isomorphism $P[X,Y]\langle \Sigma X\rangle \xrightarrow{\simeq} \widehat{S}$.

Now lemma 5 produces a lift

$$\begin{tikzcd}
Q'[X] \arrow[r] \arrow[d] & P[X,Y]\langle \Sigma X\rangle \arrow[d, "\simeq"] \\
Q'[X,Z] \arrow[r, "\simeq"] \arrow[ur, dashed] & \widehat{S}.
\end{tikzcd} \tag{3.5}$$

Applying $Q'[t] \otimes_{Q'[X]} -$ to this lift produces the desired map $Q'[t,Z] \to P[Y]\langle t, \Sigma X\rangle$. It is a quasi-isomorphism because $Q'[X,Z]$ is semi-free as a dg $Q'[X]$ module and $P[X,Y]$ is semi-free as a dg $P[X]$ module. Taking indecomposables and shifting produces the desired map

$$\pi_*(\phi) = \Sigma\left(\frac{\mathfrak{m}_{\overline{Q}[Z]}}{\mathfrak{m}^2_{\overline{Q}[Z]}}\right) \xleftarrow{\cong} \Sigma\mathrm{H}_*\left(\frac{\mathfrak{m}_{Q'[t,Z]}}{\mathfrak{m}^2_{Q'[t,Z]}}\right) \longrightarrow \Sigma\mathrm{H}_*(\mathrm{ind}^\gamma(P[Y]\langle t, \Sigma X\rangle)) \cong \widetilde{\pi}_*(\phi).$$

It remains to establish the second part of the theorem, in which we assume that ϕ is surjective. We continue with the same notation as before, but now we may assume that $Q = Q'$ and $k = l$.

We complete our acyclic closure to obtain $\widehat{R} \to \widehat{R}\langle U\rangle \xrightarrow{\simeq} \widehat{S}$.

We can build a minimal model $Q \to Q[s,Y] \xrightarrow{\simeq} \widehat{S}$ lifting the model $P \to P[Y] \xrightarrow{\simeq} \widehat{S}$. Here s is a degree one set of variables with $\partial(s) = \underline{q}$, so that $Q[s] \simeq P$.

Similarly to above the Tate model for the diagonal builds a surjective quasi-isomorphism $Q[X,s,Y]\langle \Sigma X\rangle \xrightarrow{\simeq} \widehat{S}$. We now have a commutative diagram

$$\begin{tikzcd}
Q \arrow[r] \arrow[d] & \widehat{R}\langle U\rangle \arrow[d, "\simeq"] \\
Q[X,s,Y]\langle \Sigma X\rangle \arrow[r, "\simeq"] \arrow[ur, dashed, "\alpha"] & \widehat{S}.
\end{tikzcd}$$

There are no problems in using the analogue of lemma 5 for this mixed extension, since $\widehat{R}\langle U\rangle$ has divided powers supported on the ideal of all positive degree elements. Therefore a homomorphism of α dg Q algebras with divided powers exists, as indacted by the dashed arrows. We tensor this homomorphism with the Koszul resolution $\beta : Q[t] \xrightarrow{\simeq} k$ to obtain a quasi-issomorphism

$$Q[t,s,Y]\langle \Sigma X\rangle = Q[t]\otimes_{Q[X]} Q[X,s,Y]\langle \Sigma X\rangle \xrightarrow{\beta\otimes_{Q[X]}\alpha} k\otimes_{Q[X]} \widehat{R}\langle U\rangle = k\langle U\rangle$$

Finally, this descends along the quasi-isomorphism $Q[t,s,Y]\langle \Sigma X\rangle \xrightarrow{\simeq} P[t,Y]\langle \Sigma X\rangle$ to give the desired homomorphism, as stated in the theorem. $\square$

An Application to Lie_∞ Algebras

Before moving on we briefly point out that in characteristic zero theorem 28 actually recovers one of the main results in a paper of Fiorenza and Manetti [50]. Our proof is simpler because it entirely avoids the perturbation methods of op. cit. The reader may wish to skip this small subsection.

Some background of Lie_∞ algebras is given at the end of section 4.2. We will only say that a Lie_∞ algebra structure on a degree-wise finite dimensional graded vector space L is specified by a differential making $k[\Sigma^{-1}L^\vee]$ into a semi-free dg algebra. A morphism $L \to L'$ of Lie_∞ algebras is a morphism $k[\Sigma^{-1}L'^\vee] \to k[\Sigma^{-1}L^\vee]$ of semi-free dg algebras.

Corollary 5. *Let k be a field of characteristic zero and let $\chi : L \to L'$ be a morphism of cohomologically-positively graded* Lie_∞ *algebras over k. The shifted cone $\Sigma^{-1}\mathrm{cone}(\chi)$ naturally admits the structure of an* Lie_∞ *algebra in such a way that the projection $\Sigma^{-1}\mathrm{cone}(\chi) \to L$ naturally extends to a* Lie_∞ *morphism.*

The connectedness assumptions are unnecessary, we include them only because of our conventions on semi-free dg algebras. The constructions go through without issue in the general situation.

The point of this result is that even when L and L' are simply dg Lie algebras, the shifted cone $\Sigma^{-1}\mathrm{cone}(\chi)$ does not necessarily admit a dg Lie algebra structure. That is, higher operations must be taken into account to make this construction. Fiorenza and Manetti also prove that the Maurer-Cartan functor associated to $\Sigma^{-1}\mathrm{cone}(\chi)$ controls the deformations of χ.

Proof. The dual notion of a Lie_∞ coalgebra structure on L is simply a differential making $k[\Sigma L]$ a semi-freee dg algebra. We establish the dual result: to a morphism $\chi : L' \to L$ of Lie_∞ coalgebras there is a Lie_∞ coalgebra structure on $\mathrm{cone}(\chi)$ such that the inclusion $L \to \mathrm{cone}(\chi)$ is one of Lie_∞ coalgebras.

In terms of semi-free dg algebras, we are given $k[\Sigma^{-1}L'] \to k[\Sigma^{-1}L]$ and we must construct an extension $k[\Sigma^{-1}L] \to k[\Sigma^{-1}L, L']$ such that the linear part of the differential of $k[\Sigma^{-1}L, L']$ recovers the shifted cone $\Sigma^{-1}\mathrm{cone}(\chi)$. Since the characteristic of k is zero there is non-difference between $k[\Sigma^{-1}L, L']$ and $k[\Sigma^{-1}L]\langle L'\rangle$ and this was exactly the construction we made in theorem 29 (in the proof of that theorem we assumed that $k[\Sigma^{-1}L]$ and $k[\Sigma^{-1}L']$ were minimal, but this assumption was not used). $\square$

Properties of $\widetilde{\pi}_*(\phi)$

Theorem 30. *Let $\phi : R \to S$ be a local homomorphism. We can identify $\widetilde{\pi}_*(\phi)$ in the following cases:*

1. *If $\mathrm{wcat}(\phi)$ is finite then the comparison $\pi_*(\phi) \to \widetilde{\pi}_*(\phi)$ is an isomorphism. This applies for example if ϕ has finite flat dimension, or is Golod, or more generally when ϕ is almost small.*

2. *If ϕ is surjective and admits an acyclic closure $R \to R\langle U\rangle \xrightarrow{\simeq} S$ which is minimal as a complex then $\widetilde{\pi}_*(\phi) \to \pi^\gamma_*(\phi)$ is an isomorphism. This applies for example if ϕ is large or quasi-complete intersection.*

3. *If the residual characteristic of S is zero then $\pi_*(\phi) \to \widetilde{\pi}_*(\phi) \to \pi^\gamma_*(\phi)$ are both isomorphisms.*

Proof. Part 3 follows from remark 18 and the fact that all algebras have divided powers in characteristic zero. Part 1 uses Avramov's six term exact sequences 3.2 while part 2 uses the divided power version 3.3. Since the proofs are similar we only treat 1.

The point is that the dg algebra morphisms from theorem 29 allow us to construct a commutative diagram of dg algebras

$$\begin{tikzcd} Q[X] \arrow[r] \arrow[d, equal] & Q'[X,Z] \arrow[r] \arrow[d, "\simeq"] & Q'[Z,t] \arrow[d, "\simeq"] \\ Q[X] \arrow[r] & P[Y] \arrow[r] & P[Y]\langle t, \Sigma X\rangle. \end{tikzcd}$$

Here we have used the notation from theorem 29. Then we take indecomposables, shift and pass to homology. By theorems 24, 25 and 28 this results in two long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & \pi_i(R)\otimes_k l \arrow[r] \arrow[d, equal] & \pi_i(S) \arrow[r] \arrow[d, equal] & \pi_i(\phi) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & \pi_i(R)\otimes_k l \arrow[r] & \pi_i(S) \arrow[r] & \widetilde{\pi}_*(\phi) \arrow[r] & \cdots \end{tikzcd}$$

We conclude that the comparison $\pi_*(\phi) \to \widetilde{\pi}_*(\phi)$ is an isomorphism using the five lemma. □

Combining this theorem with proposition 12 produces a Jacobi-Zariski long exact sequence for the homotopy Lie algebra (or for $\pi^\gamma_*(-)$) in a variety of situations.

Example 10. Suppose that the local homomorphisms $(Q,u) \xrightarrow{\psi} (R,k) \xrightarrow{\phi} (S,l)$ are both Golod. By [8, theorem 3.5] they are also small, so their composition is small, and therefore all three of ϕ,ψ and $\phi\psi$ have finite 0-category by corollary 3. By theorem 30 and (the linear dual of) proposition 12 there is a long exact sequence

$$\cdots \to \pi^{i+1}(\phi) \leftarrow \pi^i(\psi)\otimes_k l \leftarrow \pi^i(\phi\psi) \leftarrow \pi^i(\phi) \leftarrow \pi^{i-1}(\psi)\otimes_k l \leftarrow \cdots$$

sandwiching $\pi_*(\phi\psi)$ between the two free Lie algebras $\pi^*(\phi)$ and $\pi^*(\psi)$.

We can generalise the situation for small and large homomorphisms (and give a shorter, independent proof that does not use the long exact sequences of Avramov or the Gulliksen minimality conditions).

Theorem 31. *Let $(R,k) \to (S,l)$ be a local homomorphism.*

1. *If $\phi_i : \pi_i(R)\otimes_k l \to \pi_i(S)$ is injective for all even i then $\pi_*(\phi) \to \widetilde{\pi}_*(\phi)$ is an isomorphism.*
2. *If ϕ is surjective and $\phi_i : \pi_i(R)\otimes_k l \to \pi_i(S)$ is surjective for all odd i then $\widetilde{\pi}_*(\phi) \to \pi^\gamma_*(\phi)$ is an isomorphism.*

This is quite surprising. Only the injectivity of ϕ_{even} is enough to guarantee that we have the full six term exact sequences 3.2. And only the surjectivity of ϕ_{odd} is enough to guarantee that we have the full six term exact sequences 3.3.

Proof. We start with 1. We use the constructions and notation of theorem 29 and its proof.

Recall that the homomorphism $Q' \to \widehat{S}$ factors through the minimal Cohen presentation $Q' \to P = Q/\underline{q} \to \widehat{S}$. Recall that we also constructed a quasi-isomorphism $Q'[X] \to R'$ and a semi-free model

$Q'[X] \to Q'[X, Z] \xrightarrow{\simeq} \widehat{S}$. Since $\mathrm{edim}(Q') = \mathrm{edim}(R')$ it follows (possibly making a change of basis) that we can assume $Z = Z' \cup v$ for some $v \subseteq Z_1$ with $\partial(v) = \underline{q}$.

We need to slightly improve the construction in the proof of theorem 29: when we built the lift in diagram 3.5 we could have first passed to the quotient $Q'[X, Z] \xrightarrow{\simeq} P[X, Z'] = Q'[X, Z]/(\underline{q}, v)$ and done this via a lift $P[X, Z'] \to P[X, Y]\langle \varepsilon X\rangle$.

The upshot is that we may assume that the homomorphism $Q'[t, Z] \to P[Y]\langle t, \varepsilon X\rangle$ of that theorem factors as

$$Q'[t, Z] \xrightarrow{\simeq} P[t, Z'] \xrightarrow{\simeq} P[Y]\langle t, \varepsilon X\rangle.$$

Having made this technical assumption, part 1 is now a relatively painless consequence of theorem 27. That theorem says we can reduce $P[Y]\langle t, \varepsilon X\rangle$ to a quasi-isomorphic mixed extension $P[W]\langle V\rangle$ which is minimal in the sense that $\mathrm{ind}^\gamma(l[W]\langle V\rangle)$ has vanishing differential, and moreover the induces chain map $\mathrm{ind}^\gamma(l[Y]\langle t, \varepsilon X\rangle) \to \mathrm{ind}^\gamma(l[W]\langle V\rangle)$ is a quasi-isomorphism:

$$\begin{tikzcd} P[t, Z'] \arrow[r, "\simeq"] \arrow[dr, "\alpha"'] & P[Y]\langle t, \varepsilon X\rangle \arrow[d, "\simeq"] \\ & P[W]\langle V\rangle. \end{tikzcd}$$

The key point is that the hypothesis that $\phi_i : \pi_i(R) \otimes_k l \to \pi_i(S)$ is injective for even i implies that the construction in theorem 27 must remove all the even parts of εX. In other words, V is concentrated in odd degrees. Therefore $P[W]\langle V\rangle = P[W, V]$ is actually semi-free. Since $\mathrm{ind}^\gamma(l[W]\langle V\rangle) = \mathrm{ind}(l[W, V])$ has no differential the extension $P \to P[W, V]$ is minimal. Therefore $\alpha : P[t, Z'] \to P[W, V]$ is a quasi-isomorphism or minimal semi-free P algebras. By [12, lemma 7.2.3] α is an isomorphism. In particular it induces an isomorphism on indecomposables. We have canonical isomorphisms

$$\pi_*(\phi) \cong \varepsilon \mathrm{H}_*\left(\frac{\mathfrak{m}_{P[t,Z']}}{\mathfrak{m}^2_{P[t,Z']}}\right) \cong \varepsilon \mathrm{H}_*\left(\frac{\mathfrak{m}_{P[W,V]}}{\mathfrak{m}^2_{P[W,V]}}\right) \cong \varepsilon \mathrm{H}_*(\mathrm{ind}^\gamma(P[Y]\langle t, \varepsilon X\rangle)) \cong \widetilde{\pi}_*(\phi).$$

This establishes part 1.

The proof of part 2 is similar (and simpler), using theorem 27 to produce a model for $P[Y]\langle t, \varepsilon X\rangle$ which involves only divided power variables. We omit it. ☐

Besides it being striking that injectivity of ϕ_{even} is equivalent to having the full six term exact sequences 3.2, the theorem is useful because this condition is clearly stable under composition.

Corollary 6. *Let $(Q, u) \xrightarrow{\psi} (R, k) \xrightarrow{\phi} (S, l)$ be two local homomorphisms such that $\mathrm{cat}_2(\psi)$ and $\mathrm{cat}_2(\phi)$ are both finite. Then there is a Jacobi-Zariski long exact sequence of homotopy Lie coalgebras*

$$\cdots \to \pi_{i+1}(\phi) \to \pi_i(\psi) \otimes_k l \to \pi_i(\phi\psi) \to \pi_i(\phi) \to \pi_{i-1}(\psi) \otimes_k l \to \cdots$$

Proof. Since $\mathrm{wcat}_2(\psi)$ and $\mathrm{wcat}_2(\phi)$ are both finite we know that $\psi_{2i} : \pi_{2i}(Q) \otimes_u k \to \pi_{2i}(R)$ and $\phi_{2i} : \pi_{2i}(R) \otimes_k l \to \pi_{2i}(S)$ are all injective by Avramov's six term exaxct sequences (3.2). It follows that $(\phi\psi)_{2i} : \pi_{2i}(Q) \otimes_u l \to \pi_{2i}(S)$ are injective as well. By part 1 of theorem 31 there are isomorphisms $\pi_*(\psi) \cong \widetilde{\pi}_*(\psi)$ and $\pi_*(\phi) \cong \widetilde{\pi}_*(\phi)$ and $\pi_*(\phi\psi) \cong \widetilde{\pi}_*(\phi\psi)$. After this the desired long exact sequence comes from proposition 12. ☐

In the theorem below parts 1 and 2 are true and well-known for the homotopy Lie algebra, while part 3 fails except in characteristic zero. One of the purposes of $\widetilde{\pi}_*(-)$ is to extend a step further to 3.

Theorem 32. *A local map $\phi : R \to S$ is*

1. *weakly regular if and only if $\widetilde{\pi}_{\geq 2}(\phi) = 0$;*
2. *complete intersection if and only if $\widetilde{\pi}_{\geq 3}(\phi) = 0$;*
3. *quasi-complete intersection if and only if $\widetilde{\pi}_{\geq 4}(\phi) = 0$.*

Since the interesting part of theorem 32 is 3 we separate it and elaborate slightly in the following theorem. All three parts are consequences of theorem 27, so we omit the proofs of 1 and 2. One direction of the theorem below is due to Avramov, Henriques and Şega [18, theorem 5.3] (this was discussed in example 9), we establish the converse.

Theorem 33. *If $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ is a local homomorphism then the following are equivalent:*

1. *ϕ is quasi-complete intersection;*
2. *$\widetilde{\pi}_{\geq 4}(\phi) = 0$;*
3. *$\pi_{\geq 3}(R) \otimes_k l \to \pi_{\geq 3}(S)$ is an isomorphism.*

And in this case there is an exact sequence in low degrees

$$0 \to \pi_3(\phi) \to \pi_2(R) \otimes_k l \to \pi_2(S) \to \pi_2(\phi) \to \pi_1(R) \otimes_k l \to \pi_1(S) \to \pi_1(\phi) \to 0.$$

Moreover if $R \to R' \to S$ is a minimal Cohen factorisation then

$$\pi_1(\phi) = (\mathfrak{n}/\mathfrak{m}S) \otimes_S l, \quad \pi_2(\phi) = (I/I^2) \otimes_S l \quad \text{and} \quad \pi_3(R) = H_1(K^I) \otimes_S l$$

where I is the kernel of $R' \to S$ and K^I is the Koszul complex on a minimal generating set for I.

Because of long exact sequence ... $\to \pi_*(R) \to \pi_*(S) \to \widetilde{\pi}_*(\phi) \to$ ... the condition $\widetilde{\pi}_{\geq 4}(\phi) = 0$ is equivalent to $\pi_i(R) \to \pi_i(S)$ being an isomorphism for $i \geq 4$ and $\pi_3(\phi)$ being injective. Note that exactly same statement holds for André-Quillen homology, by the characterisation of Blanco, Majadas and Rodicio [31]. Thus qci homomorphisms behave as if they were characterised by vanishing of $\pi_*(-)$, if only there were a long exact sequence of homotopy Lie algebras. This is exactly the role that $\widetilde{\pi}_*(-)$ plays. Part 3 of the theorem says that in this situation $\pi_3(R) \otimes_k l \to \pi_3(S)$ is automatically an isomorphism as well.

Proof. Let $R \to R' \xrightarrow{\phi'} \widehat{S}$ be a minimal Cohen factorisation of ϕ. By the Jacobi-Zariski exact sequence (proposition 12) associated to this factorisation, and part 1 of theorem 32, all of the conditions 1, 2 and 3 are unchanged if we replace ϕ with ϕ'. Therefore we may assume that ϕ is surjective.

The implications $1 \Rightarrow 3 \Rightarrow 2$ are all clear from example 9 and the basic long exact sequence ... $\to$ $\pi_*(R) \to \pi_*(S) \to \widetilde{\pi}_*(\phi) \to$ ... So we must show $2 \Rightarrow 1$.

We apply theorem 27 to a dg algebra similar to the one constucted in theorem 28. Let $Q[X] \xrightarrow{\simeq} \widehat{R}$ be a minimal model, and also build a minimal model $Q \to Q[Y] \xrightarrow{\simeq} \widehat{S}$ (we do not worry about the fact that $Q \to \widehat{S}$ may not be minimal, and so Y_1 may be "too large"). Lemma 5 produces a lift

$Q[X] \to Q[Y]$ of dg Q algebras. We use theorem 10 to build a model $Q[X, X]\langle \varepsilon X\rangle \xrightarrow{\simeq} Q[X]$, then finally we apply $\widehat{R} \otimes_{Q[X]} - \otimes_{Q[X]} Q[Y]$ to build a surjective quasi-isomorphism $\widehat{R}[Y]\langle \varepsilon X\rangle \xrightarrow{\simeq} \widehat{S}$. In low degrees we were lazy about minimality, but the description of the differential in theorem 10 allows us to identify $\mathrm{ind}^{\gamma}(\widehat{R}[Y]\langle \varepsilon X\rangle)_{\geq 2} \cong \varepsilon^{-1}\mathrm{cone}(\pi_{\geq 2}(R) \to \pi_{\geq 3}(S))$. Therefore the hypothesis 2 implies that $\mathrm{ind}^{\gamma}(P[Y]\langle t, \varepsilon X\rangle)$ has homology in degrees 1 and 2 only. Theorem 27 produces a surjective quasi-isomorphism $\widehat{R}[Y]\langle \varepsilon X\rangle \xrightarrow{\simeq} \widehat{R}[Z]\langle U\rangle$ with Z and U both concentrated in degrees 1 and 2 only.

In characteristic zero $\widehat{R}[Z]\langle U\rangle = \widehat{R}\langle Z, U\rangle$ and the augmentation $\widehat{R}\langle Z, U\rangle \xrightarrow{\simeq} \widehat{S}$ is witness to that fact that ϕ is qci. If $\mathrm{char}(k) = p$ is positive we must show that $Z_2 = \varnothing$. For degree reasons $\widehat{R}[Z]\langle U\rangle$ is automatically minimal as a complex, therefore $k[Z]\langle U\rangle \cong \mathrm{Tor}_*^{\widehat{R}}(\widehat{S}, k)$. But if $z \in Z_2$ this contradicts the fact that $\mathrm{Tor}_*^{\widehat{R}}(\widehat{S}, k)$ has a natural divided power algebra structure supported on the ideal of positive degree elements, since this means $z^p \neq 0$ should be impossible. Therefore the two step Tate model $\widehat{R}\langle Z_1, U\rangle \xrightarrow{\simeq} \widehat{S}$ is witness to that fact that ϕ is qci.

Finally, remembering again that ϕ might not be surjective, we must establish that there is a low degree exact sequence as shown. There is always such an exact sequence using $\widetilde{\pi}_*(\phi)$ in place of $\pi_*(\phi)$. The calculation that $\widetilde{\pi}_i(\phi) = \pi_i(\phi)$ is as described for $i = 1, 2, 3$ is relatively simple, and already essentially part of [18, theorem 5.3] (where surjectivity is assumed), so we leave it to the reader. Details can be found in the proof of loc. cit. and in example 9 above. □

Finally, the conjectures stated by Quillen in [106] for André-Quillen homology (and extended by Avramov in [13]) have an obvious analogue for $\widetilde{\pi}_*(-)$. If $\phi : (R, k) \to (S, l)$ is a local homomorphism then $\widetilde{\pi}_i(\phi) = 0$ for large i if and only if $\phi_i : \pi_i(R) \otimes_k l \to \pi_i(S)$ is an isomorphism for large i. So we can ask:

Question. If there is an n such that $\phi_i : \pi_i(R) \otimes_k l \to \pi_i(S)$ is an isomorphism for all $i > n$ then must ϕ be a quasi-complete intersection?

Chapter 4

The Characteristic Action of Hochschild Cohomology

This chapter contains some highlights from the preprint [34]. An overview of its contents was given in the introduction. Everything here is joint work with Vincent Gélinas, and our contribution to this work should be considered equal.

4.1 Enriching the Characteristic Action

This section is adapted from [34, section 3.1].

Let A be a dg algebra which is augmented over a field k[1]. This section concerns the characteristic action of Hochschild cohomology on the derived category, which is a homomorphism $\chi : \mathrm{HH}^*(A, A) \to \mathsf{Z}(D(A))$ (the graded centre $\mathsf{Z}(D(A))$ was defined in section 2.1. This map is important in many areas of mathematics. In representation theory alone, it has been used in connection with the classical cohomology operators of Gulliksen, the theory of support varieties and with the obstruction theory for deforming modules, to name a few applications. As such, it is important to understand the image of the components $\chi_M : \mathrm{HH}^*(A, A) \to \mathsf{Z}(\mathrm{Ext}^*_A(M, M))$. Towards this, we will show that the homomorphisms χ_M can be enriched to land in $\mathrm{HH}^*(R, R)$, where $R = \mathrm{RHom}_A(M, M)$. This implies that an endomorphism in the image of χ_M must satisfy some strong homotopy centrality conditions. We also show that for certain dg modules the image of χ_M is exactly the image of $\Pi : \mathrm{HH}^*(R, R) \to \mathsf{Z}(\mathrm{Ext}^*_A(M, M))$.

As explained in section 2.1 $D^{\mathrm{dg}}(A)$ is dg category of all semi-free dg modules over A. It's homotopy category $D(A) = \mathrm{H}^0(D^{\mathrm{dg}}(A))$ is the full derived category of A. We use the same notation for the graded category $D(A) = \mathrm{H}^*(D^{\mathrm{dg}}(A))$. Similarly $D^{\mathrm{dg}}_{\mathrm{perf}}(A)$ is our notation for the dg category of perfect semi-free dg modules, and $\mathrm{perf}(A) = \mathrm{H}_0(D^{\mathrm{dg}}_{\mathrm{perf}}(A))$ is the perfect derived category of A.

It is well known that the restriction $C^*_{\mathsf{unr}}(D^{\mathrm{dg}}_{\mathrm{perf}}(A), D^{\mathrm{dg}}_{\mathrm{perf}}(A)) \to C^*_{\mathsf{unr}}(A, A)$ is a quasi-isomorphism (this is easily proven with Keller's theorem in [75], for example). Less well known, it follows from the work of Toën [118] that the restriction $C^*_{\mathsf{unr}}(D^{\mathrm{dg}}(A), D^{\mathrm{dg}}(A)) \to C^*_{\mathsf{unr}}(A, A)$ is a quasi-isomorphism (see also [81, section 4.3]). This section contains a proof of this fact. The main theorem is

[1] The augmentation is not important for this section because we could work with the unreduced bar construction $B_{\mathsf{unr}}A$ instead of BA. Since we are most interested in applying this section to the augmentation module k_A we will continue to make this assumption anyway.

Theorem 34. *The restriction map $C^*_{\mathsf{unr}}(D^{\mathrm{dg}}(A), D^{\mathrm{dg}}(A)) \to C^*_{\mathsf{unr}}(A, A)$ is a quasi-isomorphism. Through the induced isomorphism in cohomology, the projection map from $\mathrm{HH}^*(D^{\mathrm{dg}}A, D^{\mathrm{dg}}A)$ and the characteristic action of $\mathrm{HH}^*(A, A)$ coincide. That is, the following diagram commutes*

$$\begin{tikzcd}
\mathrm{HH}^*(D^{\mathrm{dg}}A, D^{\mathrm{dg}}A) \arrow[rr, "\cong", "\mathrm{res}"'] \arrow[dr, "\Pi"'] & & \mathrm{HH}^*(A, A) \arrow[dl, "\chi"] \\
& \mathsf{Z}(D(A)). &
\end{tikzcd}$$

Given a morphism $\zeta : A \to \Sigma^n A$ in $D(A^e)$ we get a family of morphisms $\{M \otimes^{\mathsf{L}}_A \zeta : M \to \Sigma^n M\}$ in $D(A)$ which define a natural transformation $1_{D(A)} \to \Sigma^n$. This determines a map $\chi : \mathrm{HH}^*(A, A) \cong \mathrm{Ext}^*_{A^e}(A, A) \to \mathsf{Z}(D(A))$, which is a homomorphism of graded k algebras. We call this the *characteristic action* of $\mathrm{HH}^*(A, A)$ on $D(A)$. The two most important components are for the modules A and k:

$$\begin{tikzcd}
& \mathrm{HH}^*(A, A) \arrow[dl, "\chi_A"'] \arrow[d] \arrow[dr, "\chi_k"] & \\
\mathrm{H}(A) & \mathsf{Z}(D(A)) \arrow[l, "\mathrm{res}"'] \arrow[r, "\mathrm{res}"] & \mathrm{Ext}_A(k, k).
\end{tikzcd}$$

It is not difficult to check that χ_A and χ_k are the projection and shearing morphisms respectively, which were introduced in section 2.5. In that section χ_k was denoted by χ alone, in this section we reserve χ for the full characteristic action.

Let us be more explicit about this construction. A Hochschild cocycle $\xi : BA \to A$ canonically lifts to a bilinear map ${}_A\xi_A : A \otimes^\pi BA \otimes^\pi A \to A$, and for any dg module M we get an A-linear map

$${}_M\xi_A = M \otimes_A {}_A\xi_A \ : \ M \otimes^\pi BA \otimes^\pi A \longrightarrow M.$$

Since $M \otimes^\pi BA \otimes^\pi A$ is a semi-free resolution of M, this ${}_M\xi_A$ represents an element of $\mathrm{Ext}_A(M, M)$. Naturality up to homotopy is easy to see, so the family $\{{}_M\xi_A\}$ defines an element of $\mathsf{Z}(D(A))$. This explicit description suggests a further enhancement of the characteristic action. First, note that $\mathrm{Hom}^\pi(M \otimes^\pi BA, M) \cong \mathrm{Hom}_A(M \otimes^\pi BA \otimes^\pi A, M)$ has a *convolution product* given by

$$\phi \smile \psi \ : \ M \otimes BA \to M \otimes BA \otimes BA \xrightarrow{\psi \otimes BA} M \otimes BA \xrightarrow{\phi} M,$$

for $\phi, \psi : M \otimes BA \to M$. With this product the natural inclusion $\mathrm{Hom}_A(M, M) \to \mathrm{Hom}^\pi(M \otimes^\pi BA, M)$ becomes a quasi-isomorphism of dg algebras (for semi-free M). In fact, this extends to a *convolution enhancement* $D^{\mathrm{conv}}(A)$ of the derived category $D(A)$, with ${}_N D^{\mathrm{conv}}(A)_M = \mathrm{Hom}^\pi(M \otimes^\pi BA, N)$. The obvious embedding $D^{\mathrm{dg}}(A) \to D^{\mathrm{conv}}(A)$ is then a quasi-equivalence of pre-triangulated dg categories.

The point is that the characteristic action χ_M now lifts to a homomorphism of dg algebras

$$C^*(A, A) \to \mathrm{Hom}^\pi(M \otimes^\pi BA, M) \qquad \xi : BA \to A \quad \mapsto \quad {}_M\xi : M \otimes BA \xrightarrow{M \otimes \xi} M \otimes A \to M.$$

In fact it lifts further, all the way to the Hochschild cochain complex of $\mathrm{RHom}_A(M, M)$, as long as one is willing to work with two enhancements at once. More generally, one can do this for any set of objects together.

Proposition 13. *Let M be a set of objects in $D(A)$ and let $R = \mathrm{REnd}_A(M)$ be the full dg subcategory of $D^{\mathrm{dg}}(A)$ on M. Denote also by R^{conv} the corresponding full dg subcategory of $D^{\mathrm{conv}}(A)$. The characteristic action of $\mathrm{HH}^*(A,A)$ on $\mathrm{H}^*(R)$ lifts canonically to map $\widetilde{\chi}$ of dg algebras, as in the following commutative diagram:*

$$\begin{tikzcd} C^*_{\mathsf{unr}}(A,A) \arrow[r, "\widetilde{\chi}"] \arrow[dr, "\chi"'] & C^*_{\mathsf{unr}}(R,R^{\mathrm{conv}}) \arrow[d, "\Pi"] & C^*_{\mathsf{unr}}(R,R) \arrow[l, "\simeq"'] \arrow[d, "\Pi"] \\ & \widetilde{\mathsf{Z}}(R^{\mathrm{conv}}) & \widetilde{\mathsf{Z}}(R) \arrow[l, "\simeq"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{HH}^*(A,A) \arrow[r, "\widetilde{\chi}"] \arrow[dr, "\chi"'] & \mathrm{HH}^*(R,R) \arrow[d, "\Pi"] \\ & \mathsf{Z}(\mathrm{H}^*(R)). \end{tikzcd}$$

In the proposition $\widetilde{\mathsf{Z}}(R)$ denotes the 'pre-centre' $\prod_{m\in M} \mathrm{REnd}_A(m)$, and similarly for R^{conv}.

The lift $\widetilde{\chi}$ takes a cochain $\xi : BA \to A$ to the composition $B_{\mathsf{unr}}R \xrightarrow{\eta} k_M \xrightarrow{\alpha} R^{\mathrm{conv}}$, where α takes 1_m to $(-)\cdot\xi$ in $\mathrm{Hom}^\pi(m \otimes^\pi BA, m)$. It is not difficult to verify that this is an anti-homomorphism of dg algebras (i.e. reversing the order of composition, but in particular respecting the differential), but we can make things clearer by writing them in a more symmetrical way. We can think of M as an $R-A$ bimodule. Just rewriting things through the tensor-hom adjunction, the top row in the diagram of proposition 13 becomes[2]

$$\mathrm{Hom}^\pi(BA, A) \to \mathrm{Hom}^\pi(B_{\mathsf{unr}}R \otimes^\pi M \otimes^\pi BA, M) \leftarrow \mathrm{Hom}^\pi(B_{\mathsf{unr}}R, R)$$

$$\xi \;\mapsto\; \eta\cdot(-)\cdot\xi \qquad\qquad \zeta\cdot(-)\cdot\eta \;\mapsfrom\; \zeta$$

where η is being used for the counits of both BA and $B_{\mathsf{unr}}R$.

This symmetry suggests that the homologically balanced condition from section 2.1 may play a role.

Theorem 35. *Let M be a set of objects in $D(A)$ and let $R = REnd_A(M)$ be the full dg subcategory of $D^{\mathrm{dg}}(A)$ on M. If the bimodule ${}_RM_A$ is homologically balanced then $\widetilde{\chi} : C^*_{\mathsf{unr}}(A,A) \to C^*_{\mathsf{unr}}(R,R^{\mathrm{conv}})$ is a quasi-isomorphism.*

In the case that M contains the free module A, the restriction map $C^*_{\mathsf{unr}}(R,R) \to C^*_{\mathsf{unr}}(A,A)$ is a quasi-isomorphism, inverse in cohomology to $\widetilde{\chi}$.

The advantage of the last statement is that the restriction map is one of B_∞-algebras (as Keller points out in [79]), and also that it respects the weight filtration on Hochschild cohomology.

Note that we only need to check half of the definition of homologically balanced: in this context the map $R \to \mathrm{REnd}_A(M)$ is a quasi-equivalence by definition.

Proof. In light of the above discussion we just need to show that

$$\Phi : \mathrm{Hom}^\pi(BA, A) \to \mathrm{Hom}^\pi(B_{\mathsf{unr}}R \otimes^\pi M \otimes^\pi BA, M)$$

is a quasi-isomorphism. But using the tensor-hom adjunction on the other side we have $\mathrm{Hom}^\pi(B_{\mathsf{unr}}R \otimes^\pi M \otimes^\pi BA, M) \cong \mathrm{Hom}^\pi(BA, \mathrm{Hom}^\pi(B_{\mathsf{unr}}R \otimes^\pi M, M))$ (as usual checking that the various twists match up). Note though that $\mathrm{Hom}^\pi(B_{\mathsf{unr}}R \otimes^\pi M, M)$ is simply a convolution model for $\mathrm{REnd}_{R^{\mathrm{op}}}(M)$, that is $\mathrm{Hom}^\pi(B_{\mathsf{unr}}R \otimes^\pi M, M) \cong \mathrm{Hom}_R(R \otimes^\pi B_{\mathsf{unr}}R \otimes^\pi M, M) \leftarrow \mathrm{Hom}_R(M,M)$ is a quasi-isomorphism. Hence by

[2] The twist on $\mathrm{Hom}^\pi(B_{\mathsf{unr}}R \otimes^\pi M \otimes^\pi BA, M)$ is the one which makes it isomorphic to $\mathrm{Hom}_{R-A}(R \otimes^\pi B_{\mathsf{unr}}R \otimes^\pi M \otimes^\pi BA \otimes^\pi A, M)$.

assumption the map $A^{\mathrm{op}} \to \mathrm{Hom}^{\pi}(B_{\mathsf{unr}} R \otimes^{\pi} M, M)$ is a quasi-isomorphism, and by lemma 4 this makes Φ a quasi-isomophism as well.

To make the last statement more precise, we mean that the upper triangle in the following diagram commutes after taking cohomology

$$\begin{tikzcd}
\mathrm{Hom}^{\pi}(B_{\mathsf{unr}} R, R) \arrow[r, "\simeq"] \arrow[d, "\mathrm{res}"] & \mathrm{Hom}^{\pi}(B_{\mathsf{unr}} R, R^{\mathrm{conv}}) \arrow[d, "\mathrm{res}"] & \\
\mathrm{Hom}^{\pi}(BA, A) \arrow[r, "\simeq"] \arrow[ur, "\widetilde{\chi}"] & \mathrm{Hom}^{\pi}(BA, A^{\mathrm{conv}}) \arrow[r, "\cong"] & \mathrm{Hom}^{\pi}(BA \otimes^{\pi} A \otimes^{\pi} BA, A).
\end{tikzcd}$$

The outer rectangle clearly commutes. The lower triangle commutes in cohomology because the two paths are equalised by a quasi-isomorphism, as follows:

$$\mathrm{Hom}^{\pi}(BA, A) \rightrightarrows \mathrm{Hom}^{\pi}(BA \otimes^{\pi} A \otimes^{\pi} BA, A) \xrightarrow{\sim} \mathrm{Hom}^{\pi}(BA, A).$$

The two maps on the left are the left and right actions $\xi \mapsto \xi \cdot (-) \cdot \eta$ and $\eta \cdot (-) \cdot \xi$ respectively. The final map is obtained by pulling back along the natural bicomodule map $BA \to BA \otimes^{\pi} A \otimes^{\pi} BA$. Either composition from left to right is the identity. The statement follows. $\square$

Now we will deduce theorem 34 by taking M to be the collection of all dg modules in $D(A)$. Those uneasy about the size of $D(A)$ can just use a sufficiently large set instead, but this subtlety will not be important here.

Proof of theorem 34. Because of theorem 35, we just need to show that $D = D^{\mathrm{dg}} A$ is homologically balanced as a $D - A$ bimodule. We have a sequence of quasi-fully-faithful functors

$$\begin{tikzcd}
D^{\mathrm{dg}}_{\mathrm{perf}}(A^{\mathrm{op}}) \arrow[r, "\mathrm{RHom}_{A^{\mathrm{op}}}(-,A)", "\simeq"'] \arrow[rrr, bend right, "D \otimes^{\mathsf{L}}_{A} -"'] & (D^{\mathrm{dg}}_{\mathrm{perf}}(A))^{\mathrm{op}} \arrow[r, hook] & D^{\mathrm{op}} \arrow[r, "\mathbf{y}"] & D^{\mathrm{dg}}_{\mathrm{perf}}(D^{\mathrm{op}}),
\end{tikzcd}$$

where the Yoneda embedding $\mathbf{y}$ is quasi-fully-faithful by the Yoneda lemma (note that dg modules in the image of $\mathbf{y}$ are automatically perfect and semi-free). One can check that the diagram commutes (up to a natural quasi-isomorphism). Now since $D \otimes^{\mathsf{L}}_{A} -$ is quasi-fully-faithful it follows from lemma 3 that $A^{\mathrm{op}} \to \mathrm{REnd}_{D^{\mathrm{op}}}(D)$ is a quasi-isomorphism. $\square$

Remark 19. The proof works for any dg subcategory between $D^{\mathrm{dg}}_{\mathrm{perf}}(A)$ and $D^{\mathrm{dg}}(A)$.

Finally, theorem 35 allows us to give a conceptual proof of the following theorem of Buchweitz, Green, Snashall and Solberg.

Theorem 36 (Buchweitz-Green-Snashall-Solberg [38])**.** *If A is a Koszul algebra then $\chi_k : \mathrm{HH}^*(A, A) \to \mathsf{Z}(\mathrm{Ext}_A(k, k))$ is surjective.*

Proof. Since A is Koszul there is a quasi-isomorphism $A^! \simeq \mathrm{RHom}_A(k, k) \simeq \mathrm{Ext}^*_A(k, k)$ respecting the augmentation module k. By corollary 1 k is homologically balanced as an $A^! - A$ bimodule. Therefore by theorem 35 we have a commutative diagram

$$\begin{tikzcd} \mathrm{HH}^*(A,A) \arrow[rr, "\widetilde{\chi}_k", "\cong"'] \arrow[dr, "\chi_k"'] & & \mathrm{HH}^*(\mathrm{Ext}^*_A(k,k), \mathrm{Ext}^*_A(k,k)) \arrow[dl, "\Pi"] \\ & \mathsf{Z}(\mathrm{Ext}^*_A(k,k)). & \end{tikzcd}$$

Therefore $\mathrm{im}(\chi_k) = \mathrm{im}(\Pi) = \mathsf{Z}(\mathrm{Ext}^*_A(k,k))$. □

In the next section we will explain the analogous statement in the non-Koszul case in terms of A_∞ centres.

4.2 The A_∞ centre

In this section we introduce and begin to study a notion of A_∞ centre for minimal A_∞ algebras. Let me remind the reader everything here is joint work with Vincent Gélinas.

To start with, take A to be a strongly augmented dg algebra. From now on we return to the notation $\chi = \chi_k$ and $\widetilde{\chi} = \widetilde{\chi}_k$. Most of the proof of theorem 36 goes through without any Koszul assumption, and we get

Corollary 7. *If A is a strongly connected augmented dg algebra then there is a canonical isomorphism making the following diagram commute:*

$$\begin{tikzcd} \mathrm{HH}^*(A,A) \arrow[rr, "\cong", "\widetilde{\chi}_k"'] \arrow[dr, "\chi"'] & & \mathrm{HH}^*(A^!, A^!) \arrow[dl, "\Pi"] \\ & \mathsf{Z}(\mathrm{H}_*(A^!)). & \end{tikzcd}$$

In the Koszul case the isomorphism of algebras $\mathrm{HH}^*(A,A) \cong \mathrm{HH}^*(A^!, A^!)$ was first established by Buchweitz [36]. For dg algebras it is due to Félix, Menichi and Thomas [48], who proved further that it can be upgraded to an isomorphism of Gerstenhaber algebras.

Our construction is very similar to the one used by Keller in [79], where he recasts things in terms of the restriction maps (as in theorem 35) to see that the isomorphism lifts to one in the homotopy category of B_∞-algebras. However, the purpose of our approach is to make it clear that the isomorphism $\widetilde{\chi}$ interacts well with the characteristic action.

After corollary 7, it remains to understand the image of $\Pi : \mathrm{HH}^*(A^!, A^!) \to \mathsf{Z}(\mathrm{H}_*(A^!))$. In general there is no reason for this to be surjective. Instead the answer will be a kind of A_∞ centre, which can be calculated using a minimal A_∞ model for $A^!$.

Motivation and Definition of the A_∞ Centre

There are several possible notions of centre, and of commutativity, for A_∞ algebras, all generalising the usual concepts for graded algebras. A common answer is that a commutative A_∞ algebra is a C_∞ algebra (introduced by Kadeishvili in [74]). In characteristic zero certain aspects of commutative algebra generalise well to C_∞ algebras. However, this does not obviously give rise to a notion of centre for A_∞ algebras. A more immediate disadvantage is that this property is not even invariant under isomorphisms of A_∞ algebras.

Ultimately, E_∞ algebras encapsulate the right notion of commutative algebras up to homotopy, as far as generalising commutative algebra goes. Commutativity at this level is an extra structure on, rather than a property of, an A_∞ algebra. The commutativity property we introduce below is a completely different and much weaker notion.

Let A be an A_∞ algebra. The space of *homotopy derivations* of A is by definition the positive weight part of the suspended Hochschild cochain complex

$$\mathrm{hoDer}(A,A) = \varepsilon\mathrm{Hom}^\pi(\overline{BA},A) = \varepsilon C^{(1)}(A,A).$$

It is a sub-Lie algebra of $\varepsilon C^*(A,A)$. We can interpret the projection morphism as the obvious map

$$\Pi\ :\ C^*(A,A) = \varepsilon^{-1}\mathrm{hoDer}(A,A) \rtimes A \longrightarrow A,$$

just as we saw shearing morphism in terms of coderivations in section 2.5.

The reason for this notation is the following proposition, which goes back to Stasheff and Schlessinger [108], and ultimately Quillen. A proof is in [69], or more explicitly in [48, Lemma 4.2].

Proposition 14. *The dg Lie algebra* $\mathrm{hoDer}(A,A)$ *is quasi-isomorphism invariant, and there is a canonical chain of quasi-isomorphisms*

$$\mathrm{hoDer}(A,A) \xrightarrow{\sim} \mathrm{Der}(\mathcal{A},\mathcal{A})$$

of dg Lie algebras, where $\mathcal{A} = \Omega BA$. *Or more generally,* $\mathcal{A}$ *can be any semi-free dg algebra quasi-isomorphic to* A.

Now, from the short exact sequence

$$0 \to \varepsilon^{-1}\mathrm{hoDer}(A,A) \to C^*(A,A) \to A \to 0$$

we get a connecting homomorphism $\mathrm{H}_*(A) \xrightarrow{\mathrm{ad}} \mathrm{H}_*(\mathrm{hoDer}(A,A))$. In fact, this canonically lifts to the chain level: the assignment

$$\mathrm{ad} : A \longrightarrow \mathrm{hoDer}(A,A) \cong \prod_{n\geq 1} \varepsilon^{1-n}\mathrm{Hom}(\overline{A}^{\otimes n},A) \qquad a \mapsto \sum_{n\geq 1}(-1)^{|a|}\varepsilon^{1-n}[a;-]_{1,n}.$$

is a chain map giving rise to the above connecting homomorphism in homology[3]. Here we have used the higher commutators against a from section 2.5. Under the quasi-isomorphism of proposition 14 this corresponds to the classical adjoint homomorphism

$$\mathrm{ad} : \mathcal{A} \longrightarrow \mathrm{der}(\mathcal{A},\mathcal{A}) \quad a \mapsto [a,-].$$

With this is mind, we define the A_∞ centre of a minimal A_∞ algebra A to be

$$\mathsf{Z}_\infty(A) = \ker\left(A \xrightarrow{\mathrm{ad}} \mathrm{H}_*(\mathrm{hoDer}(A,A))\right) = \mathrm{im}\left(\mathrm{HH}^*(A,A) \xrightarrow{\Pi} A\right).$$

So, a is A_∞-central if there exists $p \in \mathrm{hoDer}(A,A)$ with $\mathrm{ad}(a) = \partial(p)$. Writing this out explicitly in

[3] We were relaxed about signs here because we didn't specify the isomorphism $\mathrm{hoDer}(A,A) \cong \prod_{n\geq 1}\varepsilon^{1-n}\mathrm{Hom}(\overline{A}^{\otimes n},A)$. The map can be written precisely using the shuffle product on BA as in [34, section 3.2]. However, the the signs in (4.1) should be correct, and that expression is what we use to make calculations.

terms of the higher multiplications, this means that a is central if for $n \geq 2$ the higher commutators $[a;-]_{1,n} : A^{\otimes n} \to A$ vanish together up to a sequence of 'homotopies' $p_i : A^{\otimes i} \to A$ of degree $|a| - i$ for $i \geq 1$, meaning precisely that $[a;-]_{1,n} =$

$$\sum_{r+s+t=n} (-1)^{r(|a|+s)+t(|a|+1)} m_{r+1+s}(1^{\otimes r} \otimes p_s \otimes 1^{\otimes t}) - (-1)^{|a|}(-1)^{rs+t} p_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}). \quad (4.1)$$

In particular the usual commutator $[a;-]_{1,1}$ vanishes, and so $\mathsf{Z}_\infty(A) \subseteq \mathsf{Z}(A)$. Being the image of a map of graded algebras, $\mathsf{Z}_\infty(A)$ is itself a graded commutative subalgebra of A.

The intuition is that keeping track of the homotopy p for which $\mathrm{ad}(a) = \partial(p)$ gives the full derived centre, i.e. Hochschild cohomology

$$\mathrm{HH}^*(A, A) = \{\ (a, p)\ :\ \mathrm{ad}(a) = \partial(p) \text{ where } a \in A \text{ and } p \in \mathrm{hoder}(A, A)/(\text{boundaries})\ \},$$

and forgetting p is exactly the projection morphism.

The reason for introducing $\mathsf{Z}_\infty(A)$ in terms of $\mathrm{hoDer}(A, A)$ is that this perspective will be computationally useful. In particular, examples show that the homotopies p_i for the higher commutators in (4.1) can sometimes be ignored.

Properties of $\mathsf{Z}_\infty(A)$

If A and A' are isomorphic minimal A_∞ algebras, it follows from naturality of the projection morphism (diagram 2.1 in section 2.5) that $\mathsf{Z}_\infty(A) \cong \mathsf{Z}_\infty(A')$ as graded algebras. In particular, if A is a dg algebra, then the A_∞ centre $\mathsf{Z}_\infty(\mathrm{H}_*(A))$ of a minimal model $\mathrm{H}_*(A)$ for A is an invariant of the quasi-isomorphism type of A. In fact $\mathsf{Z}_\infty(\mathrm{H}_*(A))$ is just the image of $\Pi : \mathrm{HH}^*(A, A) \to \mathrm{H}_*(A)$.

If A is formal (isomorphic to a minimal A_∞ algebra with vanishing higher structure) then $\mathsf{Z}_\infty(A) = \mathsf{Z}(A)$, but in general A_∞ centre will be much smaller than the graded centre.

From theorem 7 we can deduce

Theorem 37. *If A is a strongly connected augmented dg algebra then the image of $\chi : \mathrm{HH}^*(A, A) \to \mathrm{Ext}^*_A(k, k)$ is exactly $\mathsf{Z}_\infty(\mathrm{Ext}^*_A(k, k))$.*

This is a direct consequence of the definitions and the diagram of theorem 7, along with naturality (2.1) of the projection morphism from section 2.5). However, it will become useful once we have methods for computing A_∞ centres. Note also that theorem 36 of Buchweitz, Green, Snashall and Solberg is precisely the special case that A and $A^!$ are both formal. It can be generalised to other modules[4].

Corollary 8. *Let M be a right dg module over A, and take $\mathrm{Ext}^*_A(M, M)$ to be a minimal A_∞ model for $\mathrm{RHom}_A(M, M)$. For every dg module M, the characteristic morphism $\chi_M : \mathrm{HH}^*(A, A) \to \mathrm{Ext}_A(M, M)$ lands in the A_∞ centre of $\mathrm{Ext}^*_A(M, M)$. If M is homologically balanced as an $\mathrm{REnd}_A(M) - A$ bimodule then $\mathrm{im}(\chi_M) = \mathsf{Z}_\infty(\mathrm{Ext}^*_A(M, M))$.*

This time the corollary is deduced from proposition 13 and theorem 35. Aside from the augmentation module k, any generator for $D_{\mathrm{perf}}(A)$ is homologically balanced, for example.

[4]Technically we have not discussed minimal models for non-augmented dg algebras like $\mathrm{RHom}_A(M, M)$, but there is no problem with Kadeishivili's minimal model theorem [73] in this generality. The necessary naturality for Hochschild cohomology is also fine in the non-augmented situation.

Remark 20. According to the classical work of Gerstenhaber [52], a first order deformation of a graded algebra A is classified by a weight 2 element $\xi \in \mathrm{HH}^2(A, A)$. The obstruction to continuing a module M along such a deformation is $\mathrm{Char}(\xi) = [{}_M\xi_A] \in \mathrm{Ext}^2_A(M, M)$. The corollary tells us that the obstruction must actually be in $\mathsf{Z}_\infty(\mathrm{Ext}_A(M, M))$. Lowen [91] has extended this picture to deformations of abelian categories. Here the image of the characteristic morphism contains the obstruction to deforming objects along a given deformation of an abelian category. This is an important motivation for us to compute A_∞ centres.

In section 4.3 we will give examples of interesting A_∞ centres, explaining how one can compute them with the philosophy of this section. For now, let us note down a few examples with $\mathsf{Z}_\infty(A) = A$.

We say that a minimal A_∞ algebra A is A_∞ commutative if $\mathsf{Z}_\infty(A) = A$, or equivalently if the projection map $\Pi : \mathrm{HH}^*(A, A) \to A$ is surjective. At the end of this section we make note of one interesting consequence of this condition.

Any algebra with the homotopy type of a commutative dg algebra will be A_∞ commutative, but our condition is much weaker than this.

The shearing map for is split surjective for any Hopf algebra. By theorem 37 this means the Koszul dual to any strongly connected Hopf algebra is always A_∞ commutative.

The algebra $C^*(X; k)$ of cochains on a space X is always A_∞ commutative. More generally, E_∞ algebras satisfy this condition, because of the next paragraph.

One can see (e.g [53, theorem 7]) from the defining formulas that for any algebra over the “brace operad”, denoted $\mathcal{S}_2$ in [97], the projection morphism is split surjective. Alternatively it is shown in [124] that the bar construction of such an algebra is a Hopf algebra. It is proven in [96] that this brace operad $\mathcal{S}_2$ is an E_2 operad, see also [97]. Since this operad is Σ-split, by [69] any E_2 algebra has a model which is a brace algebra, and hence any E_2 algebra is A_∞ commutative.

The Associated Lie_∞ Algebra and A_∞ Commutativity

For the rest of this section we assume that the based field k has characteristic zero. Lie_∞ algebras were discussed tangentially in section 2.8. Here we rapidly record the details we need to establish corollary 9, more information can be found in [90].

To start with we need the symmetric coalgebra and the shuffle product.

Let V be a graded vector space and define $\mathrm{Sym}^{\mathsf{co}}(V) = {}^{\mathrm{Sym}}\mathsf{T}^{\mathsf{co}}(V)$ to be the subspace of symmetric elements with ${}^{\mathrm{Sym}}\mathsf{T}^{\mathsf{co}}_w(V) = {}^{S_w}(V^{\otimes w})$. This makes $\mathrm{Sym}^{\mathsf{co}}(V)$ a sub-coalgebra of $\mathsf{T}^{\mathsf{co}}(V)$. In fact it is the cofree commutative cocomplete coalgebra on V.

The tensor coalgebra is naturally a Hopf algebra with the shuffle product

$$(v_1 \otimes ... \otimes v_p) \Cup (v_{p+1} \otimes ... \otimes v_{p+q}) = \sum_{\sigma \in \mathrm{sh}(p,q)} (-1)^{|\sigma; v|} v_{\sigma^{-1}(1)} \otimes ... \otimes v_{\sigma^{-1}(p+q)},$$

where $\mathrm{sh}(p, q)$ is the set of permutations in S_{p+q} which separately preserve the order of $\{1, ..., p\}$ and of $\{p + 1, ..., p + q\}$. This is the unique coalgebra morphism $m : \mathsf{T}^{\mathsf{co}}(V) \otimes \mathsf{T}^{\mathsf{co}}(V) \to \mathsf{T}^{\mathsf{co}}(V)$ such that the corresponding maps $m_{pq} : \mathsf{T}^{\mathsf{co}}_p(V) \otimes \mathsf{T}^{\mathsf{co}}_q(V) \to V$ vanish unless $p + q = 1$, in which case our hand is forced by unitality. With this product $\mathrm{Sym}^{\mathsf{co}}(V)$ is in fact a sub Hopf algebra of $\mathsf{T}^{\mathsf{co}}(V)$.

Since k has characteristic zero $\mathrm{Sym}^{\mathsf{co}}(V)$ is exactly the subalgebra of $\mathsf{T}^{\mathsf{co}}(V)$ generated by V under the shuffle product.

Now we define a Lie_∞ algebra structure on a graded vector space L to be specified by a degree -1 coderivation b on $\mathrm{Sym}^{\mathsf{co}}(\varepsilon L)$ which is augmented and square zero. By definition the cobar construction $\mathcal{C}^{\mathsf{co}}L$ of L is the dg commutative coalgebra $\mathrm{Sym}^{\mathsf{co}}(\varepsilon L)$ equipped with this differential.

Analogously to the associative case, one can extract from the components $b_n : \mathcal{C}^{\mathsf{co}}_n L \to L$ a sequence of anti-linear maps $l_n : L^{\otimes n} \to L$ for $n \geq 2$. These can be thought of as higher brackets. The condition that $b^2 = 0$ becomes a sequence of Stasheff-like quadratic identities in the l_n generalising the Jacobi identity. These can be found in [84].

It follows from the discussion in section 2.8 that a graded Lie algebra is precisely a Lie_∞ algebra such that the bar differential b decreases weight by exactly 1. More generally, a dg Lie algebra is a Lie_∞ algebra such that b decreases weight by no more than 1. In the other direction, we say that L is minimal if its bar differential strictly decreases weight. In other words, the underlying differential of L vanishes. We say that a minimal Lie_∞ algebra L is *completely abelian* if the differential of $\mathcal{C}^{co}L$ vanishes entirely. Equivalently, all of the operation l_n vanish for $n \geq 2$.

Lemma 20. *Every coderivation $p : \mathsf{T}^{\mathsf{co}}(V) \to \mathsf{T}^{\mathsf{co}}(V)$ restricts to $\mathrm{Sym}^{\mathsf{co}}(V)$.*

The lemma follows immediately from the fact that $\mathsf{T}^{\mathsf{co}}(V)$ and $\mathrm{Sym}^{\mathsf{co}}(V)$ share the same universal property for coderivations. Or it can be proven by examining Quillen's formula for p given in lemma 2.

Consequently, if A is an augmented A_∞ algebra then bar differential of BA automatically restricts to $\mathrm{Sym}^{\mathsf{co}}(\varepsilon\overline{A})$. This defines a Lie_∞ algebra with bar construction $\mathcal{C}^{\mathsf{co}}A^{\mathrm{Lie}} = (\mathrm{Sym}^{\mathsf{co}}(\varepsilon\overline{A}), b)$, which we call the associated Lie_∞ algebra.

Theorem 38. *Assume k has characteristic zero. If A is a minimal A_∞ algebra which is A_∞ commutative then A^{Lie} is completely abelian.*

Proof. By induction on weight we show that the bar differential b vanishes on the symmetric tensors $\mathcal{C}^{\mathsf{co}}(A^{\mathrm{Lie}}) \subseteq BA$.

Since k has characteristic zero, any weight $w+1$ symmetric element can be written as a linear combination of elements of the form $[a] \uplus x$, where a is in A and x is symmetric of weight w. Since a is in the image of the projection map there is a coderivation p which (weakly) decreases weight such that $[b, [a] \uplus -] = [b, p]$. Thus

$$b([a] \uplus x) = bp(x) + (-1)^{|a|} pb(x) - (-1)^{|a|} [a] \uplus b(x).$$

Since p decreases weight and preserves $\mathcal{C}^{\mathsf{co}}(A^{\mathrm{Lie}})$ by lemma 20, this formula and the inductive hypotheses on b means $b([a] \uplus x) = 0$. □

We can rephrase this entirely in terms of dg algebras:

Corollary 9. *Assume k has characteristic zero and let A be an augmented dg algebra such that $\Pi : \mathrm{HH}^*(A, A) \to \mathrm{H}_*(A)$ is surjective. Then the associated dg Lie algebra A^{Lie} is formal and quasi-isomorphic to an abelian Lie algebra.*

The corollary follows by taking a minimal model for A. One only needs to observe that $(-)^{\mathrm{Lie}}$ is functorial even for non-strict morphisms and preserves quasi-isomorphisms.

A dg algebra A satisfying the hypotheses of the corollary need not be quasi-isomorphic to a commutative algebra, so this is an example of what subtler forms of commutativity can be deduced from A_∞-commutativity of the minimal model $\mathrm{H}_*(A)$.

In the literature these dg Lie algebras have been called quasi-abelian. Understanding when this condition holds is important because it means the associated deformation problem is extremely simple, and in particular unobstructed.

A variation on corollary 9 gives a necessary and sufficient condition for deciding when a given Lie algebra is quasi-abelian.

Corollary 10. *Assume k has characteristic zero and let L be a dg Lie algebra over k. Then* $\Pi : \mathrm{HH}^*(UL, UL) \to \mathrm{H}_*(UL)$ *is surjective if and only if L is quasi-abelian.*

This is an immediate consequence of corollary 9 and the following lemma.

Lemma 21. *Let L be a dg Lie algebra. If $(UL)^{\mathsf{Lie}}$ is quasi-abelian then so is L.*

Proof. It follows from the work of Baranovsky [24] that there is a minimal Lie_∞ algebra L^{min} and a quasi-isomorphism $L^{\mathsf{min}} \xrightarrow{\simeq} L$, and there is a universal enveloping A_∞ algebra $U(L^{\mathsf{min}})$ which is also minimal, and a quasi-isomorphism $U(L^{\mathsf{min}}) \xrightarrow{\simeq} UL$. Moreover if L^{min} happens to be an honest graded Lie algebra then Baraonovsky's construction $U(L^{\mathsf{min}})$ coincides with the usual universal envelope. And finally, there is a canonical strict inclusion of Lie_∞ algebras $L^{\mathsf{min}} \hookrightarrow U(L^{\mathsf{min}})^{\mathsf{Lie}}$.

Now, if $(UL)^{\mathsf{Lie}}$ is quasi-abelian then $U(L^{\mathsf{min}})^{\mathsf{Lie}}$ is completely abelian. But from the strict embedding $L^{\mathsf{min}} \hookrightarrow U(L^{\mathsf{min}})^{\mathsf{Lie}}$ it follows that L^{min} is completely abelian as well. But then the quasi-isomorphism $L^{\mathsf{min}} \xrightarrow{\simeq} L$ means that L is quasi-abelian by definition. □

Finally, the following corollary is an algebraic generalisation of a result of Félix, Thomas and Vigué-Poirrier in rational homotopy theory [49] (see [34, section 5] for more discussion on this).

Corollary 11. *Assume k has characteristic zero and let A be a strongly connected dg commutative algebra such that* $\chi : \mathrm{HH}^*(A, A) \to \mathrm{Ext}^*_A(k, k)$ *is surjective. Then A is formal and in fact there is a quasi-isomorphism* $\mathrm{Sym}_k(V) \xrightarrow{\simeq} A$ *for some graded vector space V. In other words, A is free up to homotopy.*

Proof. Since A is commutative its bar construction BA is the universal envelope of a unique dg Lie coalgebra $\mathcal{L}^{\mathsf{co}}A$ (see for example [90, section 13.1.10]). Write $A^! = (BA)^\vee = U(\mathcal{L}^{\mathsf{co}}A)^\vee$. By corollary 7 the projection $\Pi : \mathrm{HH}^*(A^!, A^!) \to \mathrm{H}_*(A^!)$ is surjective, so $(\mathcal{L}^{\mathsf{co}}A)^\vee$ is quasi-abelian by corollary 10. So there is a quasi-isomorphism $L \simeq (\mathcal{L}^{\mathsf{co}}A)^\vee$ of Lie coalgebras, when L is an abelian Lie algebra. Since k has characteristic zero it follows that there is a quasi-isomorphism $\mathrm{Sym}_k(L) \simeq A^!$. By theorem 5 $A \simeq A^{!!} \simeq \mathrm{Sym}_k(L)^! \simeq \mathrm{Sym}_k(\Sigma^{-1}L^\vee)$. By freeness this can be upgraded to a direct quasi-isomorphism $\mathrm{Sym}_k(\Sigma^{-1}L^\vee) \xrightarrow{\simeq} A$ as in the statement. □

4.3 Examples and Applications

This section is adapted from [34, section 4]. As such it is also joint work with Vincent Gélinas.

First we admit that, for the sake of writing down more interesting examples, we work here with a (superficially) more general setup than in the rest of the paper. Instead of working with algebras augmented over a field k, we work over a semi-simple base $\Bbbk$ isomorphic to a finite product of copies of k. Instead of vector spaces we work in this section with (graded or dg) $\Bbbk$ bimodules on which k acts centrally. It is equivalent to work with (graded or dg) quivers whose set of objects S is indexed by the factors of $\Bbbk \cong k^{\times S}$. Undecorated tensor products and homs are taken over $\Bbbk$, and in particular $(-)^\vee$ is the $\Bbbk$ linear duality $\mathrm{Hom}(-, \Bbbk)$.

An augmented $\Bbbk$ algebra is by definition a retraction $\Bbbk \to A \to \Bbbk$ of k algebras. In particular A is a (possibly non-symmetric) $\Bbbk$ bimodule and there is a canonical decomposition of $\Bbbk$ bimodules $A = \Bbbk + \overline{A}$. It is equivalent to think of A as a k linear category whose objects are indexed by the factors of $\Bbbk$. In particular the bar construction can be thought of as a cofree cocategory on the same set of objects (as was hinted at in section 2.1).

We hope the reader accepts in good faith that all the definitions and results of the sections leading up to this one generalise immediately to this context (simply replace the word 'algebra' with 'category' throughout).

We need to explain one more notational convention for this section. So far we have not been explicit about the meaning of the asterisk in $\mathrm{HH}^*(A, A)$ or in $\mathrm{Ext}^*_A(\Bbbk, \Bbbk)$, except that we used $\mathrm{HH}^{(w)}(A, A)$ to denote the weight filtration. This also passes to a weight filtration $\mathrm{Ext}^{(w)}_A(\Bbbk, \Bbbk)$ on $\mathrm{Ext}^*_A(\Bbbk, \Bbbk)$ through the shearing morphism. When A is a graded algebra (that is, when its differential vanishes) the weight filtration splits into a weight *grading*. This is also known as the *cohomological* or *resolution* degree. Thus Hochschild cohomology is bigraded with $\mathrm{HH}^w(A, A)_i$ coming from $\mathrm{Hom}^\pi(B_w A, A)_i$. The same goes for $\mathrm{Ext}^w_A(\Bbbk, \Bbbk)_i$. If $x \in \mathrm{Ext}^w_A(\Bbbk, \Bbbk)_i$ then we will write $\mathrm{wt}(x) = w$ and $|x| = i$.

When A is simply an algebra concentrated in degree zero the internal degree coincides with weight, so $\mathrm{HH}^w(A, A)_i = 0$ and $\mathrm{Ext}^w_A(\Bbbk, \Bbbk)_i = 0$ unless $w + i = 0$. More interestingly, when A is generated in degree 1 as a $\Bbbk$ algebra, it follows from theorem 7 that A is Koszul if and only if $\mathrm{Ext}^*_A(\Bbbk, \Bbbk)$ is concentrated along the diagonal according to the bigrading $(\mathrm{wt}(x), -|x| - \mathrm{wt}(x))$ (that is, $\mathrm{Ext}^w_A(\Bbbk, \Bbbk)_i \neq 0$ implies $2w + i = 0$).

For us, interesting and relevant A_∞ algebras arise for the most part as the Koszul duals to actual graded algebras (but this is by no means the only way; for instance, symplectic topology is a notable source of examples).

The A_∞ structure on Kadeishvili's minimal model $A^! = \mathrm{Ext}^*_A(\Bbbk, \Bbbk)$ can be produced by homological perturbation methods (possibly first exploited by Huebschmann beginning with [71], and made explicit by Merkulov in [98]). This approach has some theoretical advantages, but it is typically easiest to produce a minimal model for $A^!$ by building the noncommutative Tate model $\Omega C \xrightarrow{\simeq} A$ and setting $A^! = C^\vee$. This was briefly sketched in section 2.4, but see [25] for details on the construction.

From the Tate model one deduces Keller's theorem 9. This fundamental calculation is the beginning of understanding the higher structure on $A^!$. Explicitly understanding the higher structure in full is only possible in special cases[5].

Our main source of examples will be the d-Koszul algebras discussed below. They were introduced by Berger in [27], and the Koszul dual A_∞ structure was computed completely by He and Lu in [66].

Another case where this structure can be understood completely is that of *monomial algebras*, when $A = kQ/(R)$ for some quiver Q and some set R of paths in Q. This has been done recently by Tamaroff in [116] using the noncommutative Tate construction. The same calculation was made independently in unpublished work of Chuang and King.

[5]This depends on whether one accepts as explicit the iterative perturbation formulas in [98], or its interpretation in terms of trees in [82]. In any case, these formulas depend of a choice of contraction onto $\mathrm{Ext}^*_A(\Bbbk, \Bbbk)$ from some known model for $\mathrm{RHom}_A(\Bbbk, \Bbbk)$.

Algebras of finite global dimension

Assume that $\mathrm{gldim}(A) = g$ is finite, so that $\mathrm{Ext}_A^{>g}(\Bbbk, \Bbbk) = 0$. The conditions (4.1) defining a top degree element $\xi \in \mathsf{Z}_\infty(\mathrm{Ext}_A^g(\Bbbk, \Bbbk))$ are vacuous aside from $[\xi, s] = 0$ for $s \in \mathrm{Ext}_A^0(\Bbbk, \Bbbk) = \Bbbk$, since the non-trivial higher products against ξ strictly increase weight. The condition $[\xi, s] = 0$ cuts out the symmetric part of the $\Bbbk$ bimodule $\mathrm{Ext}_A^g(\Bbbk, \Bbbk)$, which we denote $\mathrm{Ext}_A^g(\Bbbk, \Bbbk)^{\mathrm{cyc}}$ since these will correspond to cycles in the quiver path algebra case. So

Proposition 15. *If* $\mathrm{gldim}(A) = g$ *is finite then the image of* $\chi : \mathrm{HH}^g(A, A) \to \mathrm{Ext}_A^g(\Bbbk, \Bbbk)$ *is* $\mathrm{Ext}_A^g(\Bbbk, \Bbbk)^{\mathrm{cyc}}$.

Restricting now to $\mathrm{gldim}(A) = 2$ theorem 9 determines the entire A_∞-algebra $\mathrm{Ext}_A^*(\Bbbk, \Bbbk)$, since all other products vanish for strict unitality or degree reasons. We are left to determine $\mathsf{Z}_\infty(\mathrm{Ext}_A^1(\Bbbk, \Bbbk))$.

Proposition 16. *Assume A is finite dimensional with global dimension 2. Then the image of χ : $\mathrm{HH}^*(A, A) \to \mathrm{Ext}_A^*(\Bbbk, \Bbbk)$ is the graded centre $\mathsf{Z}(\mathrm{Ext}_A^*(\Bbbk, \Bbbk))$.*

Proof. The result follows from proposition 15 because $\mathsf{Z}(\mathrm{Ext}^1(\Bbbk, \Bbbk)) = 0$ by the No Loop Conjecture (established in the global dimension 2 case in [59]). ☐

The proposition means we will need to look to algebras of global dimension at least 3 for interesting examples.

d-Koszul algebras

As we saw is section 2.3 a Koszul algebra is necessarily quadratic. Nevertheless, classical Koszul duality can be generalised naturally to d-homogeneous algebras. A d-homogeneous algebra is by definition an algebra with a presentation $A = \mathsf{T}(V)/(R)$ for some $\Bbbk$ bimodule V and some sub-bimodule $R \subseteq V^{\otimes d}$.

If $d > 2$ then according to theorem 7 the Koszul to such an A cannot be formal. That is, the minimal model $A^! = \mathrm{Ext}_A^*(\Bbbk, \Bbbk)$ must have some non-trivial higher products. Rather, a d-Koszul algebra will be one whose A_∞ structure is as simple as possible subject to the contraints of theorem 9.

These algebras were defined by Berger in [27] in terms of acyclicity of an explicit candidate Koszul complex. For our definition we can use the following characterisation of Green, Marcos, Martínez-Villa and Zhang from [58]: a strongly connected d-homogeneous algebra A is d-Koszul if and only if $\mathrm{Ext}_A^*(\Bbbk, \Bbbk)$ is generated as a $\Bbbk$ algebra in weight 1 and 2.

All we need to know about them is the structure theorem of He and Lu [66], and the fact, courtesy of Green, Marcos, Martínez-Villa and Zhang, that our examples are themselves d-Koszul.

First, we mention that some intuition is given by the work Dotsenko and Valette in [41]. They prove that when A is d-Koszul the A_∞ algebra $A^!$ satisfies a natural universal property. We sketch this now.

A minimal A_∞ coalgebra is called $(2, d)$-reduced if its higher coproducts Δ_n are zero except for $n = 2, d$ and, if $d \neq 2$, composing Δ_d with itself in any entry is zero.

If we are given the d-homogeneous data (V, R), so V is a $\Bbbk$ bimodule and $R \subseteq V^{\otimes d}$, then we can associate the algebra $A = A(V, R) = \mathsf{T}(V)/(R)$.

At the same time, to the d-homogeneous data (W, S) we can associate an A_∞ coalgebra as follows. By definition $C = C(W, S)$ is the universal $(2, d)$-reduced A_∞ coalgebra equipped with a map $C \to W$ such that $C \xrightarrow{\Delta_d} C^{\otimes d} \to W^{\otimes d}$ factors through $S \hookrightarrow W^{\otimes d}$, and, if $d \neq 2$, such that then the composition $C \xrightarrow{\Delta_2} C^{\otimes 2} \to W^{\otimes 2}$ is zero. For any other $(2, d)$-reduced A_∞ coalgebra satisfying these conditions there is a unique strict morphism $C' \to C$ over W.

There is a twisting cochain $C(\Sigma V, \Sigma^d R) \to V \to A = A(V, R)$ and Dotsenko and Valette establish that this twisting cochain is acyclic if and only if A is d-Koszul. This makes precise the statement that $A^! = C(\Sigma V, \Sigma^d R)^\vee$ should be as simple as possible subject to theorem 9.

By dualising the above universal properties and exhibiting an explicit construction one recovers the structure theorem of He and Lu, which we state after some setup.

The d-homogeneous dual of $A = A(V, R)$ is by definition the algebra $A_\perp = A(V^\vee, R^\perp)$, where $R^\perp$ is the orthogonal complement to R in $(V^\vee)^{\otimes d} \cong (V^{\otimes d})^\vee$. The weight w part is denoted $A_\perp^w$.

Theorem 39 (He-Lu [66])**.** *If A is a d-Koszul algebra then there are isomorphisms*

$$\mathrm{Ext}_A^{2i}(\Bbbk, \Bbbk) \cong \Sigma^{-2i} A_\perp^{id} \quad \textit{and} \quad \mathrm{Ext}_A^{2i+1}(\Bbbk, \Bbbk) \cong \Sigma^{-2i-1} A_\perp^{id+1}.$$

The products m_n are zero if $n \neq 2, d$. Under the the above identification m_2 is given by

$$\Sigma^{-2i} A_\perp^{di} \otimes \Sigma^{-2j} A_\perp^{dj} \longrightarrow \Sigma^{-2(i+j)} A_\perp^{d(i+j)}$$

$$\Sigma^{-2i-1} A_\perp^{di+1} \otimes \Sigma^{-2j} A_\perp^{dj} \longrightarrow \Sigma^{-2(i+j)-1} A_\perp^{d(i+j)+1}$$

$$\Sigma^{-2i} A_\perp^{id} \otimes \Sigma^{-2j-1} A_\perp^{dj+1} \longrightarrow \Sigma^{-2n} A_\perp^{(i+j)d+1}$$

all of these maps are induced by the standard product on $A_\perp$, after moving all shifts to the front according to the Koszul sign rule. If $d \neq 2$ then m_2 vanishes on $\Sigma^{-2i-1} A_\perp^{id+1} \otimes \Sigma^{-2j-1} A_\perp^{dj+1}$. Finally, the operation m_d vanishes if any of its inputs have even weight, and on odd weight it is the map

$$\Sigma^{-2i_1-1} A_\perp^{i_1 d+1} \otimes \cdots \otimes \Sigma^{-2i_d-1} A_\perp^{jd+1} \longrightarrow \Sigma^{-2(i_1+\ldots+i_d+1)} A_\perp^{d(i_1+\ldots+i_d+1)}$$

which is induced from the iterated product $m_2^{(d)}$ on $A_\perp$.

The theorem has some immediate consequences:

Theorem 40. *Let A be d-Koszul. Then in even weight the A_∞ centre of $\mathrm{Ext}_A^*(\Bbbk, \Bbbk)$ coincides with its graded centre*

$$\mathsf{Z}_\infty(\mathrm{Ext}_A^{\mathrm{even}}(\Bbbk, \Bbbk)) = \mathsf{Z}(\mathrm{Ext}_A^{\mathrm{even}}(\Bbbk, \Bbbk)).$$

This follows since the higher products vanish on even inputs.

Corollary 12. *Let A be d-koszul and assume that $\overline{A}$ is nilpotent. Then the shearing map χ induces an isomorphism of graded algebras*

$$\mathrm{HH}^*(A, A)/(\mathrm{nil}) \cong \mathsf{Z}(\mathrm{Ext}_A^*(\Bbbk, \Bbbk))/(\mathrm{nil})$$

where (nil) *is the nilradical on either side.*

Indeed since $m^{(n)} : \overline{A}^{\otimes n} \to \overline{A}$ is 0 for $n \gg 0$ one easily sees that elements in $\mathrm{Hom}^\pi(BA, \overline{A})$ are nilpotent since then $f^{\smile n} = m^{(n)}(f \otimes ... \otimes f)\Delta^{(n)} = 0$ for $n \gg 0$, and thus ker χ consists of nilpotent elements. If $d > 2$ then odd weight elements in $\mathrm{Ext}^*_A(\Bbbk, \Bbbk)$ are nilpotent, while for $d = 2$ there is no higher structure, and the result follows.

Examples

We end by using theorem 39 to calculate a few interesting examples of A_∞ centres.

If Q is a quiver with a set of objects Q_0 we will take $\Bbbk = k^{\times Q_0}$. Then the path algebra kQ is by definition the tensor algebra $\mathsf{T}_\Bbbk(Q)$. The set Q_w of paths in Q of length w is a basis for the weight w part of kQ. We will always place Q in homological degree 1. So weight coincides with internal degree and all our examples are strongly connected.

When drawing a quiver Q presenting an algebra $A = kQ/(R)$ by a minimal set of relations $R \subseteq kQ_{\geq 2}$, a dashed arrow will indicate a relation connecting two objects. In other words, a dashed arrow from x to y means that $yRx \neq 0$.

Our examples will be d-Koszul thanks to the following theorem.

Theorem 41 (Green-Marcos-Martínez-Villa-Zhang [58])**.** *Assume that $A = kQ/(R)$ where R is a set of paths in Q_d. Then A is d-Koszul if and only if R has the d-covering property: for any three composable paths p, q, r of length at least 1, if both pq ands qr are in R then every subpath of pqr of length d is in ρ. In particular the d-truncated path algebra $kQ/(Q_d)$ is d-Koszul.*

The first example is at this point almost classical. It seems that Madsen was the first to compute it [94].

Example 11. Let $A = k[x]/(x^d)$ with $|x| = s$ even and $d > 2$. Then $A^! = \mathrm{Ext}^*_A(k, k) \cong k[\partial_x, \eta]$ with $|\partial_x| = -s-1$ and $|\eta| = -sd-2$. Note that as an algebra $A^!$ is strictly graded commutative. However we will show that $A^!$ typically isn't A_∞ commutative (depending on the characteristic of k).

The non-trivial higher products are of the form

$$m_d(\eta^{i_1}\partial_x \otimes ... \otimes \eta^{i_d}\partial_x) = \eta^{(\sum_{m=1}^d i_m)+1}$$

Let us calculate the higher commutators. From this formula we see that $\mathrm{ad}(\eta^j) = 0$, while $\mathrm{ad}(\eta^j\partial_x)(\partial_x^{\otimes d-1})$ is given by

$$\mathrm{ad}(\eta^j\partial_x)(\partial_x, ..., \partial_x) = \sum_{i=0}^{d-1}(-1)^i(-1)^i m_d(\partial_x^{\otimes i} \otimes \eta^j\partial_x \otimes \partial_x^{\otimes d-1-i}) = d \cdot \eta^{j+1}$$

This is 0 if and only if $\mathrm{char}(k) \mid d$. When $\mathrm{char}(k) \nmid d$ this cannot be the image of a coboundary $\partial_\pi(p)$ for any $p \in \mathrm{hoDer}(A^!, A^!) = \mathrm{Hom}^\pi(\overline{BA}^!, A^!)$. Indeed, writing $(\mathrm{s}\partial_x)^{\otimes d-1} = [\partial_x|...|\partial_x]$, we have

$$\begin{aligned}
\partial_\pi(p)\big([\partial_x|...|\partial_x]\big) &= \partial(p)\big([\partial_x|...|\partial_x]\big) + \sum_{n=1}^{d-2}[\pi, ..., \pi; p]_{n,1}\big([\partial_x|...|\partial_x]\big) \\
&= 0 + [\pi, p]\big([\partial_x|...|\partial_x]\big) \\
&= \pm\partial_x \cdot p([\partial_x|...|\partial_x]) + \pm p([\partial_x|...|\partial_x]) \cdot \partial_x
\end{aligned}$$

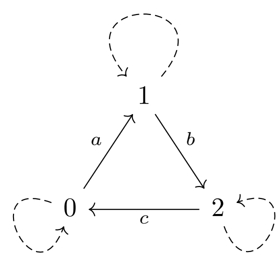
and this is never of the form $d \cdot \eta^{j+1}$ for any $p : \overline{BA}^! \to A^!$. It follows that $[ad(\partial_x\eta^j)] \neq 0$ in

$\mathrm{H}_*(\mathrm{hoDer}(A^!, A^!))$, and so

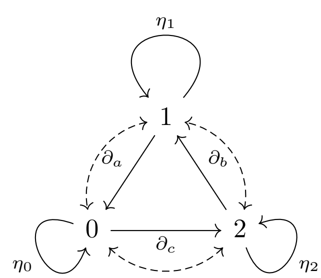
$$\mathsf{Z}_\infty(\mathrm{Ext}_A(k,k)) = \begin{cases} k[\eta] & \text{if } \mathrm{char}(k) \nmid d \\ k[\partial_x, \eta] & \text{if } \mathrm{char}(k) \mid d. \end{cases}$$

Note that for $d = p^r$ and char $k = p$ surjectivity is expected since $k[x]/(x^d)$ is isomorphic to the Hopf algebra kC_d, where C_d is the cyclic group or order d.

Example 12. Let $A = kQ/(Q_3)$, where Q is the quiver



bound by relations $Q_3 = \{cba, acb, bac\}$. The 3-homogeneous dual is simply $A_\perp = kQ^{op}$. Using theorem 39, as an algebra $A^! = \mathrm{Ext}_A(\Bbbk, \Bbbk)$ is given by kQ'/I where Q' is



where $\partial_a := \mathrm{s}^{-1}a^*$, $\eta_0 = \mathrm{s}^{-2}(a^*b^*c^*)$ comes from the 3-cycle in $A_\perp$, and similarly for the rest. The relations generating I are given by

$$\partial_a\partial_b = 0, \quad \partial_b\partial_c = 0, \quad \partial_c\partial_a = 0,$$

and

$$\eta_0\partial_a = \partial_a\eta_1, \quad \eta_1\partial_b = \partial_b\eta_2, \quad \eta_2\partial_c = \partial_c\eta_0,$$

all inherited from $A_\perp$. The degree 2 class $\eta := \eta_0 + \eta_1 + \eta_2$ is then central and generates a polynomial subalgebra $k[\eta] \subseteq \mathsf{Z}(\mathrm{Ext}_A(\Bbbk, \Bbbk))$. The only higher products are given by

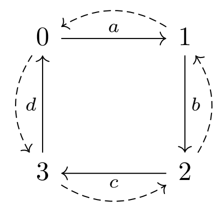
$$m_3(\partial_a \otimes \partial_b \otimes \partial_c) = \eta_0$$

and its cyclic variants, all extended to be multi-linear over $k[\eta]$.

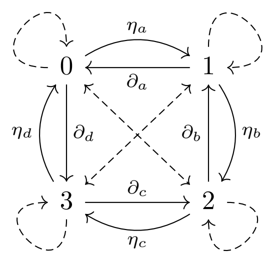
The lack of odd oriented cycle shows that η generates $\mathsf{Z}(\mathrm{Ext}_A(\Bbbk, \Bbbk))$. Since $|\eta|$ is even theorem 40 shows that it is in the image of χ. It follows that $\mathsf{Z}_\infty(\mathrm{Ext}_A(\Bbbk, \Bbbk)) = k[\eta]$ and therefore

$$\mathrm{HH}^*(A, A)/(\mathrm{nil}) \cong k[\eta].$$

Example 13. Let $A = kQ/(Q_3)$ where Q is the quiver



bound by relations $Q_3 = \{dcb, cda, dab, abc\}$. The 3-homogeneous dual is simply $A_\perp = kQ^{op}$. The algebra $A^! = \mathrm{Ext}_A(\Bbbk, \Bbbk)$ is given by kQ'/I where Q' is



with $\partial_a = \mathrm{s}^{-1}a^*$, $\eta_a = \mathrm{s}^{-2}b^*c^*d^*$ as in example 12, and similarly for all the cyclic variations $\partial_b, \partial_c, \partial_d, \eta_b, \eta_c, \eta_d$. The relations are given by

$$\partial_a\partial_b = 0, \quad \partial_b\partial_c = 0, \quad \partial_c\partial_d = 0, \quad \partial_d\partial_a,$$

and

$$\partial_a\eta_a = \eta_d\partial_d, \quad \partial_d\eta_d = \eta_c\partial_c, \quad \partial_c\eta_c = \eta_b\partial_b, \quad \partial_b\eta_b = \eta_a\partial_a,$$

all inherited from $A_\perp$ as in example 12.

Let $\gamma_0 := \partial_a\eta_a = \eta_d\partial_d$ be the simple cycle at 0, and let $\eta_0 := \eta_d\eta_c\eta_b\eta_a$ be the long cycle at 0. We define $\gamma_1, \gamma_2, \gamma_3, \eta_1, \eta_2, \eta_3$ symetrically. Then let $\gamma = \sum_{i=0}^3 \gamma_i$ and $\eta = \sum_{i=0}^3 \eta_i$, noting that $|\gamma| = 3$ and $|\eta| = 8$. Tedious but straightforward calculations in $A_\perp$ show that γ, η generate $\mathsf{Z}(\mathrm{Ext}_A(\Bbbk, \Bbbk)) \cong k[\gamma, \eta]$. Since $|\eta|$ is even, we know that η is in the image of χ, with only classes of the form $\eta^r\gamma$ left to determine.

To compute the higher products we change notation slightly, denoting by $\partial_{j\leftarrow i}$ the degree 1 arrow from i to j. We also denote by $\eta^r_{j\leftarrow i}$ the monomial path in $\{\eta_0, \eta_1, \eta_2, \eta_3\}$ going from i to j of length r. Theorem 39 shows that all higher products are sums of products of the form

$$m_3(\eta^r_{o\leftarrow n}\ \partial_{n\leftarrow m} \otimes \eta^s_{m\leftarrow l}\ \partial_{l\leftarrow k} \otimes \eta^t_{k\leftarrow j}\ \partial_{j\leftarrow i}) = \eta^{r+s+t+1}_{o\leftarrow i}$$

It follows that $\mathrm{ad}(\eta^n\gamma)(\eta^s_{m\leftarrow l}\partial_{l\leftarrow k} \otimes \eta^t_{k\leftarrow j}\partial_{j\leftarrow i}) = 3 \cdot \eta^{4n+s+t+1}_{m\leftarrow i}$. This is 0 if and only if $\mathrm{char}(k) \mid 3$, and when $\mathrm{char}(k) \nmid 3$ similar calculations to example 11 show that $[\mathrm{ad}(\eta^n\gamma)] \neq 0$ in $\mathrm{H}_*(\mathrm{hoDer}(A^!, A^!))$. We have shown

$$Z_\infty \mathrm{Ext}_A(\Bbbk, \Bbbk) = \begin{cases} k[\eta] & \text{if } \mathrm{char}(k) \neq 3, \\ k[\gamma, \eta] & \text{if } \mathrm{char}(k) = 3. \end{cases}$$

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