Parametrizing topological Ramsey spaces

by

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Abstract

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We prove a general theorem indicating that essentially all infinite-dimensional Ramsey-type theorems proven using topological Ramsey space theory can be parametrized by products of infinitely many perfect sets. This theorem has applications in several known spaces, showing that certain ultrafilters are preserved under both side-by-side and iterated Sacks forcing. In particular, the well-known result of 'selective ultrafilters on the natural numbers are preserved under Sacks forcing' is extended to the corresponding ultrafilters on richer structures. We also characterize ultrafilters in topological Ramsey spaces in an abstract setting. The technique of combinatorial forcing is crucial in the proof of the general parametrization theorem, and ultra-Ramsey theory plays an important role in the applications.

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Contents

Chapter 1

Introduction

On my first day as a PhD student, I met the graduate administrator, Jemima Merisca, and asked if I had a pigeonhole in the department. After repeating the sentence but still seeing question marks popping out of Jemima's head, I realized that the word 'pigeonhole' needed a replacement. To my surprise, the pigeonhole principle is called *the pigeonhole principle.*

The infinite pigeonhole principle states that if infinitely many items are put into finitely many containers then there must be a container with infinitely many items inside. Comparing it to Ramsey's theorem — a minor lemma Frank P. Ramsey [39] proved along the way to his goal of solving a problem in first-order logic — we see that the pigeonhole principle is indeed the 1-dimensional Ramsey's theorem.

Theorem 1.1 (Ramsey, [39])**.** *For every positive integer d and every finite colouring of the family $\mathbb{N}^{[d]}$ of all d-element subsets of the natural numbers, there is an infinite subset M of $\mathbb{N}$ such that the set $M^{[d]}$ of all d-element subsets of M is monochromatic.*

Ramsey's theorem bears the idea that within some sufficiently large systems, however disordered, there must be some order. A great amount of later work in mathematics was fruitfully developed out of Ramsey's theorem, which turned out to be an important early result in Combinatorics. The Ramsey-type theorems on different structures developed out of Ramsey's theorem on the natural numbers are of the following form:

> for a certain colouring of a mathematical structure, there is a monochromatic substructure of a particular type.

Ramsey theory studies the conditions under which Ramsey-type theorems hold. Interestingly, the statement of Ramsey's theorem fails for d being infinite. The first result of infinite-dimensional Ramsey's theory is due to Nash-Williams [35]. Galvin-Prikry [21] showed that the infinite-dimensional Ramsey's theorem holds if the colouring is Borel with respect to the metric topology of $\mathbb{N}^{[\infty]}$. Silver [41] generalized it to analytic colourings. In his new proof of Silver's theorem, Ellentuck [18] improved it to analytic colourings with respect to the exponential (or Vietoris) topology, which we now know as the Ellentuck topology. Ellentuck was the first to use topological notions to describe what is today generally considered the optimal form of this result, thus starting the whole area of topological Ramsey theory.

Topological Ramsey spaces were introduced to facilitate the study of higher-dimensional Ramsey-type theorems, and were axiomatized by Carlson-Simpson [9] and Todorcevic [42]. In such a space, Ramsey-type theorems hold precisely when the colouring partitions the object into sets with the property of Baire relative to the topology associated to the space. In other words, the Ramsey property coincides with the property of Baire with respect to the Ellentuck topology in topological Ramsey spaces. Therefore, the results of Galvin-Prikry, Silver and Ellentuck have their abstract versions. Let $\mathcal{R}$ be a topological Ramsey space.

Theorem 1.2 (Abstract Galvin-Prikry Theorem, [42]). *Every metrically Borel subset of $\mathcal{R}$ is Ramsey.*

Theorem 1.3 (Abstract Silver Theorem, [42]). *Every metrically Souslin-measurable subset of $\mathcal{R}$ is Ramsey.*

Theorem 1.4 (Abstract Ellentuck Theorem, [42]). *Every property of Baire subset (with respect to the Ellentuck topology) of $\mathcal{R}$ is Ramsey, and every meager subset (with respect to the Ellentuck topology) of $\mathcal{R}$ is Ramsey null.*

1.1 Parametrizing Topological Ramsey Spaces

If X is a topological Ramsey space then we have the following Ramsey-type theorem.

Theorem 1.5. *For a certain colouring of X there is a 'large' subset $Y \subseteq X$ such that Y is monochromatic.*

It turns out that the Ramsey-type theorem one obtains from topological Ramsey space theory as above can, in some cases, be strengthened to the following parametrized form.

Theorem 1.6. *For a certain set P and a certain colouring of $X \times P$ there are 'large' subsets $Y \subseteq X$ and $Q \subseteq P$ such that $Y \times P$ is monochromatic.*

The first parametrization of an infinite dimensional Ramsey-type theorem was discovered by Miller [33] and Todorcevic in the 1980s. Since then, many authors ([36, 31, 32]) have parametrized Ramsey-type theorems with perfect sets of real numbers, i.e. $P = \mathbb{R}$ and $Q \subseteq \mathbb{R}$ is a Cantor set. In the case of the Ellentuck space, this work has culminated by a discovery ([13, 10]) of the maximal parametrization with P and Q being infinite products of prescribed sizes. However, not much is known about maximal parametrization of other topological Ramsey spaces.

On the basis of the work of Di Prisco and Todorcevic [13] and using the method of combinatorial forcing ([21, 35]), we proved the following general theorem which implies that essentially all infinite-dimensional Ramsey-type theorems proven using topological Ramsey theory can be parametrized by products of infinitely many perfect sets, provided that a moderate condition **(L4)** holds.

Theorem 1.7 (Moderately-abstract Parametrized Ellentuck theorem). *Let $\mathcal{R}$ be a topological Ramsey space satisfying* **(L4)**. *For every finite Souslin-measurable colouring of $\mathcal{R} \times \mathbb{R}^{\mathbb{N}}$ there exists $A \in \mathcal{R}$ and a sequence $(P_i)_{i<\omega}$ of perfect sets of real numbers such that $[\emptyset, A] \times \prod_{i<\omega} P_i$ is monochromatic.*

This theorem has found applications in the analysis of set-theoretic forcing. Moreover, we expect the idea here can be similarly applied to solve a major open problem in the area: Is maximal parametrization possible for other topological Ramsey spaces, as it is the case for the Ellentuck space?

1.2 Applications of the Parametrized Theorem

In the 1990s, Todorcevic conjectured that many topological Ramsey spaces have ultrafilters associated to them analogous to the way selective ultrafilters are related to the Ellentuck space. In some cases, such ultrafilters have been studied in an independent context. For example, the ultrafilters associated to the Ellentuck space were first considered by Gustave Choquet in the 1960s, motivated by a problem in functional analysis, and were analyzed in great detail by other researchers in the 1970s. Mijares [30] gave a general notion of selective ultrafilters in topological Ramsey Ramsey spaces, which we now know as *weakly selective* ultrafilters. Later on, Di Prisco, Mijares and Nieto [11] defined *selective* ultrafilters in topological Ramsey spaces, and used them to prove complete combinatorics.

A technique extensively used in mathematical logic and recursion theory is *forcing*, which was invented by Paul Cohen in the 1960s to extend the set-theoretic universe and prove consistency and independence results. However, enlarging the universe may destruct the properties of objects in the ground model. Selective ultrafilters in the Ellentuck space have the well-known nice property of being indestructible after Sacks forcing, product Sacks forcing and iterated Sacks forcing ([4, 3]). Also, selective ultrafilters in the Ellentuck space capture the Ramsey property of sets. It is then natural to ask if there exists a notion for ultrafilters in every topological Ramsey space so that the ultrafilters capture the Ramsey property and are preserved under Sacks forcings.

In this thesis, we give a positive answer to this question in the Milliken space $\mathrm{FIN}^{[\infty]}$, a structure richer than the Ellentuck space. It turns out that the notion coincides with *stable ordered-union* ultrafilters Blass [5] introduced when he studied Hindman's theorem. The notion is also a special case of Mijares' *weakly selective* ultrafilters and Di Prisco, Mijares and Nieto's *selective* ultrafilters. Hence we have the following theorem where $\mathcal{P}$ is Sacks forcing, side-by-side (product) Sacks forcing, or iterated Sacks forcing.

Theorem 1.8. *Let $\mathcal{U}$ be a selective ultrafilter on* FIN *in the ground model, and $\dot{\mathcal{V}}$ a name for the upward closure $\{\dot{Y} \subseteq \mathrm{FIN} : \exists [\check{X}] \in \check{\mathcal{U}}\ [\check{X}] \subseteq \dot{Y}\}$ of $\mathcal{U}$. Then $\Vdash_{\mathcal{P}} \dot{\mathcal{V}}$ is a selective ultrafilter.*

The situation becomes more complicated in other topological Ramsey spaces. In [15], Dobrinen and Todorcevic constructed a hierarchy of topological Ramsey spaces $\mathcal{R}_\alpha\ (\alpha < \omega_1)$ in order to classify the Tukey and Rudin-Keisler structure below $\mathcal{U}_\alpha\ (\alpha < \omega_1)$, where $\mathcal{U}_\alpha$ are ultrafilters introduced by Laflamme [24] to obtain different combinatorics and relate Rudin-Keisler ordering. We prove the following theorem and show that *Nash-Williams* ultrafilters in $\mathcal{R}_\alpha$ are preserved under countable-support side-by-side Sacks forcing $\mathcal{P}_\kappa$.

Theorem 1.9. *Let $\mathcal{U}$ be a Nash-Williams ultrafilter in $\mathcal{R}_\alpha$ in the ground model, and $\dot{\mathcal{V}}$ a name for the upward closure $\{Y : (\exists [X] \in \mathcal{U})([X] \subseteq Y)\}$ of $\mathcal{U}$. Then $\Vdash_{\mathcal{P}_\kappa}$ ($\dot{\mathcal{V}}$ is a Nash-Williams ultrafilter in $\mathcal{R}_\alpha$).*

In particular, $\mathcal{U}_\alpha$ are preserved. Dobrinen, Mijares and Trujillo [17] showed that Nash-Williams ultrafilters capture the Ramsey property in a large class of topological Ramsey spaces.

Another example is the hierarchy of high dimensional Ellentuck spaces $\mathcal{E}_k\ (k < \omega)$ Dobrinen [16] constructed to determine the Tukey type of ultrafilters $\mathcal{G}_k$ forced by the analytic quotients $\mathcal{P}(\omega^k)/\mathrm{FIN}^{\otimes k}$, answering a question left open by Blass, Dobrinen and Raghavan [8]. It is known that selective ultrafilters in the Ellentuck space are minimal in the Tukey order, and Dobrinen proved that the initial Tukey type below each $\mathcal{G}_k$ is a chain of size k. This poses the question: are these $\mathcal{G}_k$ preserved under Sacks forcings, as selective ultrafilters are? Again, the answer is positive and we characterize the properties of ultrafilters in an abstract setting. But the exact relation among these properties remains to be seen.

The work in the Milliken space utilises the parametrized Milliken theorem proved by Todorcevic [42], which we strengthen to a local version for selective ultrafilters. The localization of the parametrized theorem uses $\mathcal{U}$-trees introduced by Blass [6] and ultra-Ramsey theory developed in the works of Sirota and Louveau. In the spaces $\mathcal{R}_\alpha$ $(\alpha < \omega_1)$ and $\mathcal{E}_k$ $(k < \omega)$, the localization process was similar, but there was not a parametrized Ramsey-type theorem available. Todorcevic used a richer space, the Hales-Jewett space, to code the product of the Milliken space with $\mathbb{R}^{\mathbb{N}}$ and obtained the parametrized Milliken theorem. However, it seems unlikely that a single topological Ramsey space is powerful enough to code every product space. Theorem 1.7 provides a solution to this problem, paving the way towards a proof of Todorcevic's conjecture mentioned at the beginning of this section.

1.3 Layout

This thesis is divided into seven chapters, including this Introduction. Chapter 2 contains the preliminaries, including the axioms for topological Ramsey spaces, and the definition of Sacks forcing. We also discuss the properties of ultrafilters in topological Ramsey spaces. In Chapter 3 we present the proof of Theorem 1.7. Then in each of the remaining four chapters, we discuss a particular topological Ramsey space. Chapter 4 is a collection of well-known results about the Ellentuck space $\mathbb{N}^{[\infty]}$. In particular, we include various definitions of 'selective' ultrafilters in the space, and show that they are all equivalent in Proposition 4.5. Chapter 5 presents the Local Paramatrized Milliken Theorem, from which we deduce the preservation of selective ultrafilters under (side-by-side and iterated) Sacks forcing. In Chapter 6, similar results regarding Nash-Williams ultrafilters are obtained in the spaces $\mathcal{R}_\alpha$ $(\alpha < \omega_1)$. The final Chapter 7 concerns the high-dimensional Ellentuck spaces $\mathcal{E}_k$.

Most of the materials in Chapters 3 and 7 are included in [48]. The contents in Chapter 5 and Chapter 6 have mostly been published in [46] and [47] respectively.

Chapter 2

Preliminaries

2.1 Topological Ramsey Spaces

In this section, let us recall some definitions and theorems about topological Ramsey spaces from [42].

Consider a triple $(\mathcal{R}, \leq, r)$ where $\mathcal{R}$ is a nonempty set, $\leq$ is a quasi-ordering on $\mathcal{R}$ and the *restriction function* $r : \mathcal{R} \times \omega \to \mathcal{AR}$ which maps an element $X \in \mathcal{R}$ to the sequence $(r_n(X) = r(X, n))$ of finite approximations of X.

The *Ellentuck topology* of $\mathcal{R}$ is the topology generated by basic open sets of the form

$$[a, B] = \{A \in \mathcal{R} : (A \leq B) \wedge (\exists n < \omega)(r_n(A) = a)\}$$

for $a \in \mathcal{AR}$ and $B \in \mathcal{R}$. A set of the form $[a, B]$ is called a *basic set* in $\mathcal{R}$.

Let the first-difference metric $\rho : \mathcal{R} \times \mathcal{R} \to \mathbb{R}$ as follows.

$$\rho(X, Y) = \frac{1}{2^n} \text{ if } r_i(X) = r_i(Y) \text{ if } i < n \text{ and } r_n(X) \neq r_n(Y).$$

The *metric topology* is the topology generated by the first-difference metric ρ on $\mathcal{R}$. Alternatively, it is the product topology when we identify an element $X \in \mathcal{R}$ with its sequence $(r_n(X))_{n<\omega}$ of finite approximations, and consider $\mathcal{R}$ as a subset of $\mathcal{AR}^{\mathbb{N}}$. It has basic open sets of the form

$$[a] = \{A \in \mathcal{R} : (\exists n < \omega)(r_n(A) = a)\}$$

for $a \in \mathcal{AR}$. Clearly from the basic open sets, the Ellentuck topology extends the metric topology. We will be referring to the metric topology unless otherwise specified.

We say $(\mathcal{R}, \leq, r)$ is *closed* if $\mathcal{R}$ is a closed subset of $\mathcal{AR}^{\mathbb{N}}$ when $X \in \mathcal{R}$ is identified with $(r_n(X))_{n<\omega} \in \mathcal{AR}^{\mathbb{N}}$ and $\mathcal{AR}^{\mathbb{N}}$ has the product topology.

Notation. Unless otherwise specified, we use letters A, B, C and X, Y, Z for elements in $\mathcal{R}$; a, b, c for elements in $\mathcal{AR}$; m, n for natural numbers.

Notation. For $a, b \in \mathcal{AR}$, we say a is an *initial segment* of b (or b is an *end-extension* of a), and write $a \sqsubseteq b$, if there exists $B \in \mathcal{R}$ and $n \leq m < \omega$ such that $a = r_n(B)$ and $b = r_m(B)$; $a \sqsubset b$ if $a \sqsubseteq b$ as above and $n < m$. We also write $a \sqsubseteq B$ if $a = r_n(B)$ for some $n < \omega$. The *length* $|a|$ of a is n if there exists

$A \in \mathcal{R}$ such that $a = r_n(A)$. If $|a| < n$, then

$$r_n[a, A] = \{r_n(B) : (a \sqsubseteq B) \wedge (B \leq A)\}.$$

Let

$$\mathcal{AR}[a, A] = \bigcup_{n<\omega} r_n[a, A].$$

For a quasi-ordering $\leq_{\text{fin}}$ as in **(A2)** below, we define

$$\text{depth}_B(a) = \min\{k : a \leq_{\text{fin}} r_k(B)\},$$

where we set $\min \emptyset = \infty$.

We are now ready for the axioms of a topological Ramsey space.

Definition 2.1 ([42])**.** We say $(\mathcal{R}, \leq, r)$ is a *topological Ramsey space* if it is closed and satisfies axioms **(A1)** to **(A4)**.

(A1) (1) $r_0(A) = \emptyset$ for all $A \in \mathcal{R}$.

(2) $A \neq B$ implies $r_n(A) \neq r_n(B)$ for some n.

(3) $r_n(A) = r_m(B)$ implies $n = m$ and $r_k(A) = r_k(B)$ for all $k < n$.

(A2) There is a quasi-ordering $\leq_{\text{fin}}$ on $\mathcal{AR}$ such that

(1) $\{a \in \mathcal{AR} : a \leq_{\text{fin}} b\}$ is finite for all $b \in \mathcal{AR}$,

(2) $A \leq B$ if and only if $(\forall n)(\exists m) r_n(A) \leq_{\text{fin}} r_m(B)$,

(3) $\forall a, b \in \mathcal{AR}\ [a \sqsubseteq b \wedge b \leq_{\text{fin}} c \Rightarrow (\exists d \sqsubseteq c)(a \leq_{\text{fin}} d)]$.

(A3) (1) If $\text{depth}_B(a) < \infty$ then $[a, A] \neq \emptyset$ for all $A \in [\text{depth}_B(a), B]$.

(2) $A \leq B$ and $[a, A] \neq \emptyset$ imply that there is $A' \in [\text{depth}_B(a), B]$ such that $\emptyset \neq [a, A'] \subseteq [a, A]$.

(A4) If $\text{depth}_B(a) < \infty$ and if $\mathcal{O} \subseteq \mathcal{AR}_{|a|+1}$, then there is $A \in [\text{depth}_B(a), B]$ such that $r_{|a|+1}[a, A] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, A] \cap \mathcal{O} = \emptyset$.

Example 1 (Ellentuck space)**.** Let $\mathcal{R}$ be

$$\mathbb{N}^{[\infty]} := \{M \subseteq \mathbb{N} : M \text{ is infintie}\},$$

which is the set of all infinite subsets of $\mathbb{N}$; let $\leq = \subseteq$ be the usual subset relation on $\mathbb{N}^{[\infty]}$; for $A \in \mathbb{N}^{[\infty]}$, let $r_n(A)$ be the set of the smallest n elements of A. Then

$$\mathcal{AR} = \mathbb{N}^{[<\infty]} = \{a \subseteq \mathbb{N} : a \text{ is finite}\}.$$

It can be checked that the Ellentuck space $(\mathbb{N}^{[\infty]}, \subseteq, r)$ is a topological Ramsey space. We will investigate the Ellentuck space in Chapter 4.

Example 2 (Milliken space)**.** Let FIN be the set of all finite nonempty subsets of $\mathbb{N}$, i.e. $\text{FIN} = \mathbb{N}^{[<\infty]} \setminus \{\emptyset\}$. For $x, y \in \text{FIN}$, by $x < y$ we mean $\max(x) < \min(y)$. A *block-sequence* is a sequence

$X = (x_n)_{n<M}$ of elements in FIN where $M \leq \omega$ and $x_n < x_{n+1}$ for all $n < M - 1$. Each x_n is called a *block*. The sequence is an *infinite block-sequence* if $M = \omega$, and it is a *finite block-sequence* if $M < \omega$.

Let $\mathcal{R} = \text{FIN}^{[\infty]}$, which is the set of all infinite block-sequences. Let $\leq$ be given, for $X = (x_n)_{n<\omega}$ and $Y = (y_n)_{n<\omega}$ in $\text{FIN}^{[\infty]}$, by

$$X \leq Y \text{ if } (\forall n < \omega)(\exists k < \omega)(\exists n_0 < \cdots < n_k < \omega)(x_n = y_{n_0} \cup \cdots y_{n_k}).$$

i.e. $X \leq Y$ if every block in X is obtained by taking a finite union of blocks in Y. For $X = (x_i)_{i<\omega} \in \text{FIN}^{[\infty]}$ and $n < \omega$, let $r_n(X) = (x_i)_{i<n}$. Then $\mathcal{AR} = \text{FIN}^{[<\infty]}$, which is the set of all finite block-sequences. It can be checked that the Milliken space $(\text{FIN}^{[\infty]}, \leq, r)$ is a topological Ramsey space. We will investigate the Milliken space in Chapter 5.

To state some important properties of topological Ramsey spaces, let us have more definitions. Let $\mathcal{R}$ be a topological Ramsey space. Recall that a set X in a topological space is *nowhere dense* if every open set has an open subset disjoint from X; a set is *meagre* if it is a countable union of nowhere dense sets.

Definition 2.2 ([42])**.** A subset $\mathcal{X}$ of $\mathcal{R}$ has the Ellentuck *property of Baire* if $\mathcal{X} = \mathcal{O} \triangle \mathcal{M}$ for some Ellentuck open set $\mathcal{O} \subseteq \mathcal{R}$ and Ellentuck meagre set $\mathcal{M} \subseteq \mathcal{R}$.

Definition 2.3 ([42])**.** A subset $\mathcal{X}$ of $\mathcal{R}$ is *Ramsey* if for every nonempty basic set $[a, A]$ there is a $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$. We say $\mathcal{X}$ is *Ramsey null* if it is Ramsey and the second alternative always holds.

Theorem 2.4 (Abstract Ellentuck Theorem, [42])**.** *If $(\mathcal{R}, \leq, r)$ is a topological Ramsey space then every Ellentuck property of Baire subset of $\mathcal{R}$ is Ramsey and every Ellentuck meagre subset is Ramsey null.*

Let us return to the examples of the Ellentuck space $(\mathbb{N}^{[\infty]}, \subseteq, r)$ and the Milliken space $(\text{FIN}^{[\infty]}, \leq, r)$, and see what we could obtain from the Abstract Ellentuck Theorem 2.4.

2.1.1 Example: the Ellentuck space

Applying the Abstract Ellentuck Theorem 2.4 in Example 1 to the Ellentuck space, we obtain the Ellentuck Theorem

Theorem 2.5 (Ellentuck, [18])**.** *Suppose $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ has the Ellentuck property of Baire. Then for every basic set $[a, A]$ there is $B \in [a, A]$ such that $[a, B]$ is either included in or disjoint from $\mathcal{X}$.*

The famous Ramsey's Theorem follows as a corollary.

Corollary 2.6 (Ramsey's Theorem, [39])**.** *Let $k < \omega$. For every finite colouring of the set $\mathbb{N}^{[k]}$ of all k-element subsets of $\mathbb{N}$ there exists $M \in \mathbb{N}^{[\infty]}$ such that the set $M^{[k]}$ of all k-element subsets of M is monochromatic.*

Proof. We prove the statement for a 2-colouring of $\mathbb{N}^{[k]}$. Then general result follows by applying the same argument finitely many times. Let $c : \mathbb{N}^{[k]} \to 2$ be a 2-colouring. Define $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ by

$$X \in \mathcal{X} \text{ if and only if } c\,(r_k(X)) = 0.$$

Since $\mathcal{X}$ is metrically open, it is Ellentuck open, hence it has the Ellentuck property of Baire. Taking $[a, A] = [\emptyset, \mathbb{N}]$ and applying the Ellentuck Theorem 2.5, we obtain $M \in \mathbb{N}^{[\infty]}$ such that $M^{[\infty]} \subseteq \mathcal{X}$ or $M^{[\infty]} \cap \mathcal{X} = \emptyset$. So $M^{[k]}$ is monochromatic of colour 0 or monochromatic of colour 1, respectively. □

2.1.2 Example: the Milliken space

Applying the Abstract Ellentuck Theorem 2.4 to the Milliken Space in Example 2, we obtain Milliken's Theorem.

Theorem 2.7 (Milliken, [34])**.** *Suppose $\mathcal{X} \subseteq \mathrm{FIN}^{[\infty]}$ has the Ellentuck property of Baire. Then for every basic set $[a, A]$ there is $B \in [a, A]$ such that $[a, B]$ is either included in or disjoint from $\mathcal{X}$.*

Hindman's Theorem then follows from Milliken's Theorem in the same way as Ramsey's Theorem follows from the Ellentuck Theorem.

Corollary 2.8 (Hindman's Theorem, [22, 2])**.** *For every finite colouring of* FIN *there is $B = (b_n)_{n<\omega} \in \mathrm{FIN}^{[\infty]}$ such that the set*

$$[B] = \{y \in \mathrm{FIN} : (\exists k < \omega)(\exists n_0 < \cdots < n_k) y = b_{n_0} \cup \cdots \cup b_{n_k}\}$$

is monochromatic.

Proof. Similar to the proof of Ramsey's Theorem 2.6, we consider a 2-colouring $c : \mathrm{FIN} \to 2$. Define $\mathcal{X} \subseteq \mathrm{FIN}^{[\infty]}$ by

$$X \in \mathcal{X} \text{ if and only if } c\,(r_1(X)) = 0.$$

Note that $\mathcal{X}$ is metrically open, so Ellentuck open, hence $\mathcal{X}$ has the Ellentuck property of Baire. Taking $[a, A] = [\emptyset, \{\{n\} : n \in \mathbb{N}\}]$ and applying Milliken's Theorem 2.7, we obtain $B \in \mathrm{FIN}^{[\infty]}$ such that $[\emptyset, B] \subseteq \mathcal{X}$ or $[\emptyset, B] \cap \mathcal{X} = \emptyset$. So $[B]$ is monochromatic of colour 0 or monochromatic of colour 1, respectively. □

Via the mapping $f : \mathrm{FIN} \to \mathbb{N}$ where $f(x) = \sum_{i \in x} 2^i$, the more familiar version of Hindman's Theorem easily follows.

Corollary 2.9 (Hindman's Theorem, [22].)**.** *For every finite colouring of $\mathbb{N}$ there exists $M \in \mathbb{N}^{[\infty]}$ such that the set*

$$\left\{ \sum_{i<k} x_i : (k < \omega) \wedge (\forall i < k)(x_i \in M) \right\}$$

of all finite sums of elements in M is monochromatic.

In the rest of this thesis, we will always refer to the metric topology of a topological Ramsey space unless stated otherwise. We may refer to a particular topological Ramsey space by $\mathcal{R}$ when the quasi-ordering $\leq$ and the restriction function r are understood from the context. In addition to the notation introduced before Definition 2.1, the following is also used.

Notation. Let $\mathcal{R}$ be a topological Ramsey space. For $n < \omega$ and $B \in \mathcal{R}$, $[n, B] = [r_n(B), B]$. For each $n < \omega$, $\mathcal{AR}_n = \{r_n(A) : A \in \mathcal{R}\}$.

2.2 Sacks Forcing

Let 2^ω and $(2^\omega)^\omega$ be equipped with the product topology. We use 2^ω interchangeably with $\mathbb{R}$, and $(2^\omega)^\omega$ interchangeably with $\mathbb{R}^\mathbb{N}$. On the set of all finite 01-sequences $2^{<\omega}$, the symbols "$|\cdot|$", "$\sqsubseteq$" and "$\restriction$" respectively denote *length* of the sequence, *initial segment* and *restriction* to an initial segment of certain length. Two finite 01-sequences are *comparable* if one is an initial segment of the other; otherwise they are *incomparable*.

Definition 2.10 ([3])**.** We call a nonempty set $p \subseteq 2^{<\omega}$ a *tree* if it is $\sqsubseteq$-downwards closed. A tree p is *perfect* if every $s \in p$ has incomparable end-extensions $t, u \in p$. In particular, every perfect tree is infinite.

For a perfect tree p, let $[p] = \{f \in 2^\omega : (\forall n \in \omega)(f \restriction n \in p)\}$ be the set of all infinite branches of p. Then $[p] \subseteq 2^\omega$ is a perfect set.

Definition 2.11 ([40])**.** Sacks forcing $\mathcal{P}$ is the set of all perfect trees, ordered by $p \leq q$ if $p \subseteq q$.

Note that $p \leq q$ if and only if $[p] \subseteq [q]$.

Definition 2.12 ([3])**.** For $p \in \mathcal{P}$ and $s \in p$, let $p|s = \{t \in p : (t \sqsubseteq s) \vee (s \sqsubseteq t)\}$. The number of branchings below s in the tree p is called the *branching level* of s in p, which is

$$|\{i < |s| : (\exists t \in p)((|t| > i) \wedge (t \restriction i = s \restriction i) \wedge (t \restriction (i+1) \neq s \restriction (i+1)))\}|.$$

The *nth branching level* $l(n,p)$ of the tree p is the set of all $s \in p$ which have branching level n and are $\sqsubseteq$-minimal with this property. Note that $l(n,p) \subseteq p$ is a collection of nodes in p, rather than a set of natural numbers. If $p, q \in \mathcal{P}, q \subseteq p, n \in \omega$ and $l(n,q) = l(n,p)$, then we write $q \leq^n p$.

Lemma 2.13 (Fusion 1, [3])**.** *Suppose $(p_k)_{k\in\omega} \subseteq \mathcal{P}$ and $(m_k)_{k\in\omega} \subseteq \omega$ is unbounded and increasing such that $p_{k+1} \leq^{m_k} p_k$ for all $k \in \omega$. Then $q = \bigcap_{k\in\omega} p_k \in \mathcal{P}$ and $q \leq^{m_k} p_k$ for all $k \in \omega$. We call $(p_k)_{k\in\omega}$ a* fusion sequence *and q the* fusion *of the sequence.*

Now we are ready to define countable-support side-by-side Sacks forcing.

Definition 2.14 ([3])**.** Let κ be an infinite cardinal. Let $\mathcal{P}_k$ be the set of all sequences $p = (p^i)_{i<\kappa}$ such that, for every $i < \kappa$, $p^i \in \mathcal{P}$ and for all but countably many $i < \kappa$, $p^i = 2^{<\omega}$. We say p^i is the *ith tree* of p. For $p = (p^i)_{i<\kappa}$ and $q = (q^i)_{i<\kappa}$ in $\mathcal{P}_\kappa$, $p \leq q$ if $p^i \subseteq q^i$ for all $i < \kappa$.

For $p \in \mathcal{P}_\kappa$, let $[p] = \prod_{i<\kappa}[p^i]$. For $\varepsilon \in [p]$ and $i < \kappa$, let ε^i be the ith component in ε, so $\varepsilon^i \in [p^i]$. The *support* of p is $\mathrm{supp}(p) = \{i < \kappa : p^i \neq 2^{<\omega}\}$. So each $p \in \mathcal{P}_\kappa$ has countable support.

Notation. For a set S, $[S]^{<\omega}$ denotes the set of all finite subsets of S.

Definition 2.15. Let κ be an infinite cardinal. Let $n \in \omega$ and $p \in \mathcal{P}_\kappa$. The set $l(n,p)$ is defined as follows.

$$l(n,p) = \prod_{i\in n} l(n,p^i).$$

For $\sigma \in l(n,p)$ and $i \in n$, let σ^i denote the ith component of σ, so $\sigma^i \in p^i$.

For $p, q \in \mathcal{P}_\kappa$, we write $q \leq^n p$ if $q \leq p$ and $q^i \leq^n p^i$ for all $i \in n$.

For n, p, q as above and $\sigma \in l(n, p)$, we write $q \leq_\sigma p$ if $q \leq p$ and $\sigma^i \in q^i$ for every $i \in n$. We also define $p|\sigma$ as follows. For $i < \kappa$,

$$(p|\sigma)^i = \begin{cases} p^i|\sigma^i & \text{if } i \in n; \\ p^i & \text{otherwise.} \end{cases}$$

Moreover, let $\varepsilon \in [p]$ and $\sigma \in l(n, p)$. We say σ is a *pre-initial segment* of ε and ε is a *post-end-extension* of σ, and write $\sigma \sqsubseteq^* \varepsilon$, if $\sigma^i \sqsubseteq \varepsilon^i$ for every $i \in n$.

Lemma 2.16 (Fusion 2, [3])**.** *Let κ be an infinite cardinal. Suppose $(p_k)_{k\in\omega} \subseteq \mathcal{P}_\kappa$. Suppose also that $(m_k)_{k\in\omega} \subseteq \omega$ is unbounded and increasing with $\bigcup_{k\in\omega} m_k \supseteq \bigcup\{\mathrm{supp}(p_k) : k \in \omega\}$. Define $q = (q^i)_{i<\kappa}$ where $q^i = \bigcap_{k\in\omega} p_k^i$ for each $i < \kappa$. Then $q \in \mathcal{P}_\kappa$ and $q \leq^{m_k} p_k$ for all $k \in \omega$.*

Recall that 2^ω has the product topology. So it has basic open sets of the form $[s] = \{f \in 2^\omega : s \sqsubseteq f\}$, for $s \in 2^{<\omega}$. Let κ be an infinite cardinal. The set $(2^\omega)^\kappa$ also has the product topology, with basic open sets of the form $[\sigma] = \{\varepsilon = (\varepsilon^i)_{i\in\kappa} \in (2^\omega)^\kappa : \sigma \sqsubseteq^* \varepsilon\}$, where there is $n < \omega$ such that $\sigma \in l(n, 2^{<\omega})$. If $\kappa = \omega$, we may think of such σ as an element of $(2^{<\omega})^{<\omega}$. For $p \in \mathcal{P}_\kappa$, $[p]$ inherits the subspace topology from $(2^\omega)^\kappa$.

We keep in mind that $\sqsubseteq$ denotes end-extension in different cases: Between finite approximations and elements in $\mathcal{AR} \cup \mathcal{R}$, we use $\sqsubseteq$ to denote end-extension of an element. The symbol is also used to denote end-extensions of a node inside a tree such as $2^{<\omega}$.

2.3 Ultrafilters in Topological Ramsey Spaces

Definition 2.17. Let S be a set. A (non-principal) *ultrafilter* on the base set S is a collection $\mathcal{U}$ of subsets of S with the following properties. For subsets M, N, A, B of S,

(1) $S \in \mathcal{U}$ but $\{x\} \notin \mathcal{U}$ for all $x \in S$,

(2) $M \subseteq N$ and $M \in \mathcal{U}$ implies that $N \in \mathcal{U}$,

(3) $M = A \cup B$ and $M \in \mathcal{U}$ implies that $A \in \mathcal{U}$ or $B \in \mathcal{U}$,

(4) $M \in \mathcal{U}$ and $N \in \mathcal{U}$ implies that $M \cap N \in \mathcal{U}$.

There is also a general definition of ultrafilters in topological Ramsey spaces. We will look at some special ultrafilters in topological Ramsey spaces.

Definition 2.18 ([11])**.** A subset $\mathcal{V} \subseteq \mathcal{R}$ is an *ultrafilter* if the following holds.

(a) $\mathcal{V}$ is a *filter* on $(\mathcal{R}, \leq)$, i.e.

   (i) $\forall A, B \in \mathcal{R}\ ((A \in \mathcal{V}) \wedge (A \leq B) \Rightarrow (B \in \mathcal{V}))$;

   (ii) For every $A, B \in \mathcal{V}$ and $a \in \mathcal{AR}$,

$$(([a, A] \neq \emptyset) \wedge ([a, B] \neq \emptyset) \Rightarrow (\exists C \in \mathcal{V})\,(C \in [a, A] \cap [a, B]))\,.$$

(b) $\mathcal{V}$ is a maximal filter on $(\mathcal{R}, \leq)$: If $\mathcal{V}'$ is a filter on $(\mathcal{R}, \leq)$ and $\mathcal{V} \subseteq \mathcal{V}'$ then $\mathcal{V} = \mathcal{V}'$.

(c) For every $A \in \mathcal{V}$ and $a \in \mathcal{AR}[\emptyset, A]$,

(i) if $B \in [\text{depth}_A(a), A] \cap \mathcal{U}$ then $[a, B] \neq \emptyset$,

(ii) if $B \in \mathcal{V}$, $B \leq A$, and $[a, B] \neq \emptyset$, then there exists $A' \in [\text{depth}_A(a), A] \cap \mathcal{V}$ such that $\emptyset \neq [a, A'] \subseteq [a, B]$.

Definition 2.19 ([42]). A family $\mathcal{F} \subseteq \mathcal{AR}$ is *Nash-Williams* if $a \not\sqsubseteq b$ for every distinct pair $a, b \in \mathcal{F}$. It is *Sperner* if $a \not\leq_{\text{fin}} b$ for every distinct pair $a, b \in \mathcal{F}$.

Every Sperner family is Nash-Williams. The family $\mathcal{AR}_n$ is Nash-Williams for every $n < \omega$.

Notation. For a family $\mathcal{F}$ and an element X, let $\mathcal{F}|X = \{Y \in \mathcal{F} : Y \leq X\}$.

Definition 2.20. Let $\mathcal{R}$ be a topological Ramsey space and $\mathcal{U}$ an ultrafilter in $\mathcal{R}$. We say

- $\mathcal{U}$ is *Nash-Williams* if for every Nash-Williams family $\mathcal{G} \subseteq \mathcal{AR}$ and every partition $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1$ there exists $X \in \mathcal{U}$ and $i \in 2$ such that $\mathcal{G}_i \cap \mathcal{AR}[\emptyset, X] = \emptyset$;

- [30]. $\mathcal{U}$ is *Ramsey* if for all $A \in \mathcal{U}$, $a \in \mathcal{AR}[\emptyset, A]$ and $n \in \omega$, and for every $f : \mathcal{AR}_{|a|+n} \to 2$ there exists $B \in [\text{depth}_A(a), A] \cap \mathcal{U}$ such that f is constant on $r_{|a|+n}[a, B]$;

- [30]. $\mathcal{U}$ is *weakly selective* if for every $A \in \mathcal{U}$ and every $\{A_b\}_{b \in \mathcal{AR}_1} \subseteq \mathcal{U}|A$ with $[b, A_b] \neq \emptyset$ for each $b \in \mathcal{AR}_1(A)$ there exists $B \in \mathcal{U}|A$ such that $[b, B] \subseteq [b, A_b]$ for every $b \in \mathcal{AR}_1(B)$;

- [11]. $\mathcal{U}$ is *selective* if for every $A \in \mathcal{U}$ and every $\{A_a\}_{a \leq A} \subseteq \mathcal{U}|A$ with $[a, A_a] \neq \emptyset$ for each $a \leq A$ there exists $B \in \mathcal{U}|A$ such that $[a, B] \subseteq [a, A_a]$ for every $a \leq B$.

It is clear from the definition that every selective ultrafilter is weakly selective. We prove in Section 4.1.2 that these properties coincide for ultrafilters on $\mathbb{N}$ in the Ellentuck space $\mathbb{N}^{[\infty]}$. However, this is not always the case for topological Ramsey spaces. For example, Trujillo [45] constructed an ultrafilter which is weakly selective but not Ramsey in the space $\mathcal{R}_1$ built by Dobrinen and Todorcevic [14]. We discuss the relations among these properties for ultrafilters in specific spaces in later chapters. For now we are interested in some general results.

2.3.1 Sometimes Ramsey ultrafilters are selective

In [30], Mijares stated an extra axiom **(A8)** which guarantees that every Ramsey ultrafilter is weakly selective.

Definition 2.21 ([30]). For a topological Ramsey space $(\mathcal{R}, \leq, r)$ the axiom **(A8)** holds if for arbitrary $A, B \in \mathcal{R}$, $n < \omega$ and $b \in r_n[\emptyset, B]$,

$$r_{n+1}[b, B] \subseteq r_{n+1}[b, A] \Rightarrow [b, B] \subseteq [b, A]$$

.

Theorem 2.22 ([30]). *If* **(A8)** *holds in $\mathcal{R}$ then every Ramsey ultrafilter in $\mathcal{R}$ is weakly selective.*

Among the topological Ramsey spaces we are concerned with, the Ellentuck space $\mathbb{N}^{[\infty]}$, the Milliken space $\text{FIN}^{[\infty]}$ and the spaces $\mathcal{R}_\alpha$ ($\alpha < \omega_1$) satisfy **(A8)** , while the High-dimensional Ellentuck spaces $\mathcal{E}_k$ ($k \geq 2$) do not satisfy it.

2.3.2 Sometimes Nash-Williams ultrafilters are Ramsey

As in the previous subsection, we define a property of topological Ramsey spaces which guarantees that every Nash-Williams ultrafilter is Ramsey.

Definition 2.23. A topological Ramsey space $\mathcal{R}$ has a *head start* if the following conditions hold:

(1) Every element $X \in \mathcal{R}$ can be written as $X = \bigcup_{i<\omega} X(i)$ where $r_n(X) = \bigcup_{i<n} X(i)$ for every $n < \omega$.

(2) For every ultrafilter $\mathcal{U}$ in $\mathcal{R}$, and for every $B \in \mathcal{U}$ and $a \in \mathcal{AR}$, there exists $A \in \mathcal{U}$ such that $a \sqsubseteq A$ and $\bigcup\{A(i) : i \geq \mathrm{depth}_A(a)\} \subseteq B$.

When the condition (1) holds, we may write $A/a = \{A(i) : i \geq \mathrm{depth}_A(a)\}$.

Again, the Ellentuck space $\mathbb{N}^{[\infty]}$, the Milliken space $\mathrm{FIN}^{[\infty]}$ and the spaces $\mathcal{R}_\alpha$ ($\alpha < \omega_1$) have a head start while the High-dimensional Ellentuck spaces $\mathcal{E}_k$ ($k \geq 2$) do not have a head start.

Theorem 2.24. *If a topological Ramsey space $\mathcal{R}$ has a head start, then every Nash-Williams ultrafilter in $\mathcal{R}$ is Ramsey.*

Proof. Suppose $\mathcal{R}$ has a head start. Let $\mathcal{U}$ be a Nash-Williams ultrafilter in $\mathcal{R}$. Let $A \in \mathcal{U}$, $a \in \mathcal{AR}[\emptyset, A]$ and $n \in \omega$ be given. Let $f : \mathcal{AR}_{|a|+n} \to 2$ be given. We define a partition $\mathcal{AR}_{|a|+n} = \mathcal{G}_0 \sqcup \mathcal{G}_1$ as follows. For $b \in \mathcal{AR}_{|a|+n}$, let $b \in \mathcal{G}_0$ if and only if

$$\exists X \in \mathcal{R} \text{ with } r_{|a|+n}(X) = a \cup (b/a) \text{ and } f(a \cup (b/a)) = 0,$$

where $b/a = \bigcup\{b(n) : n \geq |a|\}$. Since $\mathcal{U}$ is Nash-Williams, there exists $B \in \mathcal{U}$ and $i \in 2$ such that $\mathcal{G}_i|B = \emptyset$. As $\mathcal{R}$ has a head start, there exists $B' \in \mathcal{U}$ with $a \sqsubseteq B'$ and $B'/a \subseteq B$. Then by Definition 2.18 (a)(ii) of ultrafilters in $\mathcal{R}$, because $A, B' \in \mathcal{U}$ and $[a, A], [a, B] \neq \emptyset$, there exists $B'' \in \mathcal{U}$ such that $B'' \in [a, A] \cap [a, B']$.

If $\mathcal{G}_0|B = \emptyset$, then for all $b \in r_{|a|+n}[\emptyset, B]$,

$$\forall X \in \mathcal{R}\ ((r_{|a|+n}(X) = a \cup (b/a)) \Rightarrow (f(a \cup (b/a)) = 1)).$$

Consider $Y \in [a, B'']$. We check $f(r_{|a|+n}(Y)) = 1$. Since $a \sqsubseteq Y$, $Y \leq B'' \leq B'$, and $B'/a \subseteq B$, there exists $b \in r_{|a|+n}(B)$ such that $r_{|a|+n}(Y) = a \cup (b/a)$. So $f(a \cup (b/a)) = 0$. Since $Y \in [a, B'']$ was arbitrary, we have $f \restriction r_{|a|+n}[a, B''] = 1$.

If $\mathcal{G}_1|B = \emptyset$, then for all $b \in r_{|a|+n}[\emptyset, B]$,

$$\exists X \in \mathcal{R}\ ((r_{|a|+n}(X) = a \cup (b/a) \wedge (f(a \cup (b/a)) = 0).$$

In particular, $f(a \cup (b/a)) = 0$ for all $b \in \mathcal{AR}_{|a|+n}(B)$. So $f \restriction r_{|a|+n}[a, B''] = 0$. □

Chapter 3

Parametrized Theorems in Topological Ramsey Spaces

Miller and Todorcevic [33] parametrized the Galvin-Prikry Theorem with perfect sets of real numbers. This is the first parametrization of an infinite-dimensional Ramsey-type theorem. Answering a question of Miller, Pawlikowski [36] extended this result to a parametrization of the Ellentuck Theorem. Thereafter, Mijares [31] parametrized the Abstract Ellentuck Theorem 2.4, for abstract topological Ramsey spaces, with perfect subsets of $\mathbb{R}$.

In this chapter, we impose an extra condition, **(L4)**, and prove in the first two sections that a parametrized version of the Abstract Ellentuck Theorem 2.4 holds for every topological Ramsey space satisfying **(L4)**. We show that **(L4)** is a necessary condition for the theorem to hold in Section 3.3. This is a parametrization of the space with an infinite sequence of perfect subsets of the real numbers. The parametrized theorem can be applied to obtain preservation results related to countable-support side-by-side Sacks forcing, as we will see in later chapters.

The proof of the theorem involves the notion of *barriers*, which was introduced by Nash-Williams [35] in the development of the theory of well-quasi-orderings. The notion has since had many applications far beyond its original use.

Recall the definition of Nash-Williams family and Sperner family from Definition 2.19.

Definition 3.1. For a topological Ramsey space $\mathcal{R}$ we consider a family $\mathcal{F} \subseteq \mathcal{AR}$. We say $\mathcal{F}$ is *Nash-Williams* if $s \not\sqsubseteq t$ for every distinct pair $s, t \in \mathcal{F}$; $\mathcal{F}$ is *Sperner* if $s \not\leq_{\text{fin}} t$ for every distinct pair $s, t \in \mathcal{F}$.

Let $a \in \mathcal{AR}$ and $A \in \mathcal{R}$. We say $\mathcal{F}$ is a *barrier on* $[a, A]$ if $\mathcal{F}$ is Sperner and every $X \in [a, A]$ has an initial segment in $\mathcal{F}$. We say $\mathcal{F}$ is a *barrier on* A if it is a barrier on $[\emptyset, A]$.

Definition 3.2 (Rank of barriers, [42])**.** Let $\mathcal{R}$ be a topological Ramsey space, $a \in \mathcal{AR}$ and $A \in \mathcal{R}$. Let $\mathcal{F}$ be a barrier on $[a, A]$. Consider

$$T(\mathcal{F}) = \{s \in \mathcal{AR} : (a \sqsubseteq s \leq A) \wedge (\exists t \in \mathcal{F})(s \sqsubseteq t)\}$$

as a tree ordered by end-extension $\sqsubseteq$. We define a strictly decreasing map $\rho_{\mathcal{F}}$ as follows.

$$\begin{aligned}\rho_{\mathcal{F}} &: T(\mathcal{F}) \to \mathrm{Ord}\\ &s \to \sup\{\rho_{\mathcal{F}}(t)+1 : (t \in T(\mathcal{F})) \wedge (s \sqsubset t)\}.\end{aligned}$$

The *rank* of $\mathcal{F}$ on $[a, A]$ is $\mathrm{rk}(\mathcal{F}) = \rho_{\mathcal{F}}(a)$.

From now on, we consider a topological Ramsey space $\mathcal{R}$ satisfy the following condition.

(L4) Let $A \in \mathcal{R}$ and $a \in \mathcal{AR}[\emptyset, A]$. Let $\{O_b : b \in \mathcal{AR}_{|a|+1}[a, A]\}$ be a family of open subsets of $(2^\omega)^\omega$. Then there exists $B \in [\mathrm{depth}_A(a), A]$ and $q \in \mathcal{P}_\omega$ such that $O_b \cap [q]$ is constant on $b \in \mathcal{AR}_{|a|+1}[a, B]$.

Our aim in this section is proving the Moderately-Abstract Parametrized Ellentuck Theorem below. The proof is an extension of an adaptation of the results for the Ellentuck space $\mathbb{N}^{[\infty]}$ in [42, §9] to $\mathcal{R}_\alpha$. Instead of parametrization with the infinite product trees $\bigcup_{k\in\omega}\prod_{i<k} H_i$ of finite sets (see [42, §3.3]), we consider parametrization with perfect trees $p \in \mathcal{P}$, and we extend the result to infinite sequences of perfect trees $p \in \mathcal{P}_\kappa$ using the Halpern-Läuchli theorem.

Theorem 3.3 (Moderately-Abstract Parametrized Ellentuck Theorem)**.** *Let $\mathcal{R}$ be a topological Ramsey space satisfying* **(L4)**. *For every Souslin-measurable colouring of $\mathcal{R} \times \mathbb{R}^{\mathbb{N}}$ and for every $A \in \mathcal{R}, a \in \mathcal{AR}[\emptyset, A]$ and $p \in \mathcal{P}_\omega$ there exists $B \in [a, A]$ and $q \le p$ such that $[a, B] \times [q]$ is monochromatic.*

Lemma 3.4. *For $p \in \mathcal{P}_\omega$ and $O \subseteq [p]$ open, there exists $q \le p$ such that $[q] \subseteq O$ or $[q] \cap O = \emptyset$.*

Proof. If there exists $\varepsilon \in (2^\omega)^\omega$ such that $\varepsilon \in O \cap [p]$, then since O is open, there exists a pre-initial segment $\sigma \in (2^{<\omega})^{<\omega}$ of ε such that $[p|\sigma] \subseteq O$. So let $q = p|\sigma$. Otherwise, there exists no such ε hence $[p] \cap O = \emptyset$. □

Lemma 3.5. *Suppose* **(L4)** *holds. Let $p \in \mathcal{P}_\omega$, $A \in \mathcal{R}$, and $a \in \mathcal{AR}[\emptyset, A]$. Let $n \in \omega$. Let $\mathcal{F}$ be a barrier on $[a, A]$ and $O_b\,(b \in \mathcal{F})$ be a family of open subsets of $[p]$. Then there are $B \in [\mathrm{depth}_A(a), A]$, $q \le^n p$ and a clopen set $G \subseteq [q]$ such that $O_b \cap [q] = G$ for all $b \in \mathcal{F} \cap \mathcal{AR}[a, B]$.*

Proof. Induct on $\mathrm{rk}(\mathcal{F})$. The base case $\mathrm{rk}(\mathcal{F}) = 0$ is trivial, and that the set G is clopen follows from Lemma 3.4 applied to $[p|\sigma]$ for every $\sigma \in l(n, p)$.

Suppose $\mathrm{rk}(\mathcal{F}) > 0$. We can write

$$\mathcal{F} = \bigcup\{\mathcal{F}_b : b \in \mathcal{AR}_{|a|+1}[a, A]\},$$

where

$$\mathcal{F}_b = \{c \in \mathcal{F} : r_{|a|+1}(c) = b\}.$$

Whenever $a \sqsubset b \in \mathcal{AR}[a, A]$ and $\mathcal{F}_b \neq \emptyset$, $\mathcal{F}_b$ is a barrier on $[b, A]$ of smaller rank than $\mathcal{F}$.

We build fusion sequences $([n_k, B_k])_{k<\omega}$ and $(p_k)_{k<\omega}$ such that for $k \ge 1$:

(1) $B_0 = A$, $p_0 = p$ and $n_k = k + \mathrm{depth}_A(a)$;

(2) for each $b \in \mathcal{F} \cap \mathcal{AR}[a, A]$ such that $\mathrm{depth}_{B_k}(b) = n_k$, we have that

$$c \mapsto O_c \cap [p_{k+1}] \text{ is constant on } \mathcal{F}_b \cap \mathcal{AR}[b, B_{k+1}];$$

(3) $p_{k+1} \leq^{n+k} p_k$, $B_{k+1} \in [n_k, B_k]$.

Suppose we have constructed B_k, p_k. The set

$$\{b \in \mathcal{AR}[a, A] : \text{depth}_{B_k}(b) = n_k\}$$

is a subset of $\{b \in \mathcal{AR} : b \leq_{\text{fin}} r_{n_k}(B_k)\}$, so it is finite by **(A2)** (1). Enumerate it as $b_0, \dots, b_{l-1}$, and construct $(Y_i)_{i \leq l}$ and $(q_i)_{i \leq l}$ such that

(i) $Y_0 = B_k$, $q_0 = p_k$;

(ii) $Y_{i+1} \in [n_k, Y_i]$, $q_{i+1} \leq^{n+k} q_i$;

(iii) $c \mapsto O_c \cap [q_{i+1}]$ is constant on $\mathcal{F}_{b_i} \cap \mathcal{AR}[b_i, Y_{i+1}]$.

Suppose we have constructed Y_i, q_i for some $i < l$. If $\mathcal{F}_{b_i} = \emptyset$, then let $Y_{i+1} = Y_i$ and $q_{i+1} = q_i$. Now assume $\mathcal{F}_{b_i} \neq \emptyset$. Since $\mathcal{F}_{b_i}$ is a barrier on $[b_i, Y_i]$ of smaller rank than $\mathcal{F}$, by applying the induction hypothesis to b_i, Y_i and q_i we find $Y_{i+1} \in [n_k, Y_i]$, $q_{i+1} \leq^{n+k} q_i$ and $G_{i+1} \subseteq [q_{i+1}]$ clopen such that

$$O_c \cap [q_{i+1}] = G_{i+1} \text{ for all } c \in \mathcal{F}_{b_i} \cap \mathcal{AR}[b_i, Y_{i+1}].$$

This finishes the construction of $(Y_i)_{i \leq l}$ and $(q_i)_{i \leq l}$. Let $B_{k+1} = Y_l$ and $p_{k+1} = q_l$. This in turn finishes the construction of $([n_k, B_k])$ and (p_k). Then let B_∞, p_∞ be the fusions of the sequences.

Claim 3.5.1. *The map $c \mapsto O_c \cap [p_\infty]$ restricted to $\mathcal{F} \cap [a, B_\infty]$ depends only on $r_{|a|+1}(c)$.*

Proof. Suppose $c, c' \in \mathcal{F} \cap \mathcal{AR}[a, B_\infty]$ such that $r_{|a|+1}(c) = b = r_{|a|+1}(c')$. Let $n_k = \text{depth}_{B_\infty}(b)$. So $n_k = \text{depth}_{B_k}(b)$. Therefore

$$c \mapsto O_c \cap [p_{k+1}] \text{ is constant on } \mathcal{F}_b \cap \mathcal{AR}[b, B_{k+1}].$$

Since $c, c' \in \mathcal{F}_b \cap \mathcal{AR}[b, B_{k+1}]$ and $[p_\infty] \subseteq [p_{k+1}]$, the result follows. □

Finally, apply **(L4)** to find $B \leq B_\infty$ and $q \leq p_\infty$ such that $c \mapsto O_c \cap [q]$ is constant on $\mathcal{F} \cap \mathcal{AR}[a, B]$. □

Corollary 3.6 follows easily from Lemma 3.4, **(L4)** and Lemma 3.5.

Corollary 3.6. *Suppose* **(L4)** *holds. Let $A \in \mathcal{R}$ and $a \in \mathcal{AR}[\emptyset, A]$. Let $\mathcal{F}$ be a barrier on $[a, A]$ and O_b $(b \in \mathcal{F})$ be a family of open subsets of $(2^\omega)^\omega$. Then for every $p \in \mathcal{P}$ there exists $q \leq p$, $B \in [\text{depth}_A(a), A]$ such that either $[q] \subseteq O_b$ for every $b \in \mathcal{F} \cap \mathcal{AR}[a, B]$ or $O_b \cap [q] = \emptyset$ for every $b \in \mathcal{F} \cap \mathcal{AR}[a, B]$.*

In order to prove the Moderately-Abstract Parametrized Ellentuck Theorem 3.3, concerning Souslin-measurable subsets of $\mathcal{R} \times \mathbb{R}^{\mathbb{N}}$, we first consider the open subsets of $\mathcal{R} \times \mathbb{R}^{\mathbb{N}}$ in Subsection 3.1, and generalize it to all Souslin-measurable subsets in Subsection 3.2.

3.1 Open Subsets of $\mathcal{R} \times \mathbb{R}^{\mathbb{N}}$

In this section, we fix an open subset $\mathcal{O}$ of $\mathcal{R} \times \mathbb{R}^{\mathbb{N}}$. We use combinatorial forcing to prove Theorem 3.12 below.

Definition 3.7. Let $a \in \mathcal{AR}, A \in \mathcal{R}, p \in \mathcal{P}_\omega$ and $\sigma \in 2^{(<\omega)\times(<\omega)}$. We say (A, p) *accepts* (a, σ) if $[a, A] \times [p|\sigma] \subseteq \mathcal{O}$; (A, p) *rejects* (a, σ) if $[a, A] \times [p|\sigma] \neq \emptyset$ and there does not exist $q \leq_\sigma p$ and $B \in [\mathrm{depth}_A(a), A]$ such that (B, q) accepts (a, σ). We say (A, p) *decides* (a, σ) if it either accepts or rejects (a, σ).

Lemma 3.8. *Let $a \in \mathcal{AR}, A \in \mathcal{R}$, $p \in \mathcal{P}_\omega$ and $\sigma \in 2^{(<\omega)\times(<\omega)}$.*

(1) If (A, p) accepts (a, σ) then (B, q) accepts (a, σ) for every $B \leq A$ and $q \leq p$.

(2) If (A, p) rejects (a, σ), $\mathrm{depth}_B(a) < \infty$, $B \leq A$ and $q \leq_\sigma p$ then (B, q) rejects (a, σ).

(3) If (A, p) decides (a, σ), $B \leq A$ and $q \leq p$, then (B, q) decides (a, σ).

(4) If $[a, A] \times [p|\sigma] \neq \emptyset$, then there exists $B \in [\mathrm{depth}_A(a), A]$ and $q \leq_\sigma p$ such that (B, q) decides (a, σ); otherwise (A, p) accepts (a, σ).

(5) If (A, p) decides (a, τ) where $\emptyset \neq [p|\sigma] \subseteq [p|\tau]$ then (A, p) decides (a, σ) in the same way.

(6) For a finite or infinite sequence $(\tau_i)_{i\in I} \subseteq 2^{(<\omega)\times(<\omega)}$, if $\bigcup_{i\in I} p|\tau_i = p|\sigma$ and (A, p) rejects (a, τ_i) for all $i \in I$, then (A, p) rejects (a, σ).

Proof. It is straightforward to see that (3) follows from (1) and (2), and that (4) follows directly from Definition 3.7. So we prove (1), (2), (5) and (6).

(1) $B \leq A$ and $q \leq p$ imply that $[a, B] \times [q|\sigma]$, which may be empty, is included in $[a, A] \times [p|\sigma]$ as a subset. So the result follows.

(2) Assuming (B, q) does not reject (a, σ) we aim for a contradiction. By assumption, there exists $q' \leq_\sigma q$ and $B' \in [\mathrm{depth}_B(a), B]$ such that $[a, B'] \times [q'|\sigma] \subseteq \mathcal{O}$. Since $B' \leq B \leq A$ and $[a, B'] \neq \emptyset$, by the transitivity of $\leq$ and axiom **(A3)** (2) of topological Ramsey spaces, there exists $B'' \in [\mathrm{depth}_A(a), A]$ such that $\emptyset \neq [a, B''] \subseteq [a, B']$. Thus $[a, B''] \times [q'|\sigma] \subseteq \mathcal{O}$ and hence (B'', q') accepts (a, σ). This contradicts that (A, p) rejects (a, σ).

(5) If (A, p) accepts (a, τ), then clearly (A, p) accepts (a, σ). So we suppose (A, p) rejects (a, τ). Assuming (A, p) does not reject (a, σ), we aim for a contradiction. By assumption, there exists $q \leq_\sigma p$ and $B \in [\mathrm{depth}_A(a), A]$ such that $[a, B] \times [q|\sigma] \subseteq \mathcal{O}$. Note that $q|\sigma \leq_\tau p$: $[p|\sigma] \subseteq [p|\tau]$ implies that $\tau(i) \sqsubseteq \sigma(i)$ or $\sigma(i) \sqsubseteq \tau(i)$ and there is no branching between $\sigma(i)$ and $\tau(i)$, for $i \in \mathrm{domain}(\tau)$. Thus $q \leq_\sigma p$ implies $q \leq_\tau p$ and hence $q|\sigma \leq_\tau p$. So we have $q|\sigma \leq_\tau p$ and $B \in [\mathrm{depth}_A(a), A]$ such that $[a, B] \times [(q|\sigma)|\tau] \subseteq \mathcal{O}$, contradicting that (A, p) rejects (a, τ).

(6) By assumption, $[a, A] \times [p|\sigma] \neq \emptyset$. We assume (A, p) does not reject (a, σ) and aim for a contradiction. Let $B \in [\mathrm{depth}_A(a), A]$, $q \leq_\sigma p$ be such that (B, q) accepts (a, σ). Then $\exists i\ (q|\tau_i) \leq_{\tau_i} p$. So $(B, q|\tau_i)$ accepts (a, τ_i), contradicting that (A, p) rejects (a, τ_i).

∎

Lemma 3.9. *Let $A \in \mathcal{R}$, $a \in \mathcal{AR}[\emptyset, A]$, $p \in \mathcal{P}_\omega$. Then there exists $q \leq^{\mathrm{depth}_A(a)} p$ and $B \in [\mathrm{depth}_A(a), A]$ such that $\forall m < \omega\ \forall \sigma \in l(m, q)$*

$$\forall b \in \mathcal{AR}\ (\mathrm{depth}_A(a) \leq \mathrm{depth}_B(b) \leq m) \Rightarrow (B, q) \textit{ decides } (b, \sigma).$$

Proof. We construct fusion sequences $([n_k, A_k])_{k<\omega}$ and $(p_k)_{k<\omega}$ such that for $k \geq 0$:

(i) $A_0 = A$, $p_0 = p$, and $n_k = \mathrm{depth}_A(a) + k$;

(ii) $p_{k+1} \leq^{n_k} p_k$, $A_{k+1} \in [n_k, A_k]$;

(iii) for every $i \leq k$ and every element (σ, b) in the set

$$S_i = l(n_i, p_k) \times \{b \in \mathcal{AR} : \mathrm{depth}_{A_k}(b) = n_i\}$$

(A_{k+1}, p_{k+1}) decides (b, σ).

Suppose we have A_k, p_k. Note S_k is finite by **(A2)** (1). Apply Lemma 3.8 (4) repeatedly for elements of S_k, we obtain $A_{k+1} \in [n_k, A_k]$ and $p_{k+1} \leq^{n_k} p_k$ such that (A_{k+1}, p_{k+1}) decides every $(b, \sigma) \in S_k$. Then by the induction hypothesis on (A_k, p_k) and Lemma 3.8 (1), (A_{k+1}, p_{k+1}) satisfies (iii).

Now let B, q be the fusions of the sequences. Then $q \leq^{\mathrm{depth}_A(a)} p$ and $B \in [\mathrm{depth}_A(a), A]$. We check that B, q satisfy the lemma: Let $m \in \omega$, $\sigma \in l(m, q)$ and $b \in \mathcal{AR}$ with $\mathrm{depth}_A(a) \leq \mathrm{depth}_B(b) \leq m$. Let $n_k = \mathrm{depth}_B(b)$. So $n_k = \mathrm{depth}_{A_k}(b)$. We can find $\tau \in l(n_k, q)$ such that $[q|\sigma] \subseteq [q|\tau]$. By construction, (A_{k+1}, p_{k+1}) decides (b, τ), so it decides (b, σ) by Lemma 3.8 (5). Then by Lemma 3.8 (3), (B, q) decides (b, σ) as required. □

3.1.1 Digression: Abstract Galvin Lemma

In order to prove Theorem 3.12, we slightly strengthen the Abstract Galvin Lemma [42, Theorem 5.15]. The proof follows easily from that in [42]. We include it here for completeness.

Theorem 3.10 (Abstract Galvin Lemma)**.** *Let $\mathcal{R}$ be a topological Ramsey space. For every family $\mathcal{F} \subseteq \mathcal{AR}$, every $A \in \mathcal{R}$ and every $a \in \mathcal{AR}[\emptyset, A]$ there exists $B \in [\mathrm{depth}_A(a), A]$ such that either $\mathcal{F} \cap \mathcal{AR}[a, B] = \emptyset$ or every $X \in [a, B]$ has an initial segment in $\mathcal{F}$.*

Proof. As in Definition 2.3, we say a subset $\mathcal{X} \subseteq \mathcal{R}$ is *Ramsey* if for every basic set $[a, X] \neq \emptyset$ there exists $Y \in [a, X]$ such that $[a, Y] \subseteq \mathcal{X}$ or $[a, Y] \cap \mathcal{X} = \emptyset$. By Theorem 1.2, every metrically Borel set of $\mathcal{R}$ is Ramsey. Therefore, the open set $\mathcal{O} = \{X \in \mathcal{R} : (\exists n)(r_n(X) \in \mathcal{F})\}$ is Ramsey. So, for the given a and A, there exists $C \in [a, A]$ such that either of the following holds: If $[a, C] \subseteq \mathcal{O}$, then every $X \in [a, C]$ has an initial segment in $\mathcal{F}$. Otherwise $[a, C] \cap \mathcal{O} = \emptyset$, so $\mathcal{F}|[a, C] = \emptyset$. Finally, by axiom **(A3)** (2) of topological Ramsey spaces, we can find $B \in [\mathrm{depth}_A(a), A]$ such that $[a, C] \subseteq [a, B]$. Then B is as required. □

Although we only need the usual version [42, Theorem 5.17], we note that the following slightly strengthened Abstract Nash-Williams Theorem follows easily from Theorem 3.10. Recall that $\mathcal{F} \subseteq \mathcal{AR}$ is a *Nash-Williams family* if $s \not\sqsubseteq t$ for every pair of distinct $s, t \in \mathcal{F}$.

Theorem 3.11 (Abstract Nash-Williams Theorem)**.** *Let $\mathcal{R}$ be a topological Ramsey space. For every Nash-Williams family $\mathcal{F} \subseteq \mathcal{AR}$, every partition $\mathcal{F} = \mathcal{F}_0 \sqcup \mathcal{F}_1$, and every basic set $[a, A] \neq \emptyset$, there exists $B \in [\mathrm{depth}_A(a), A]$ and $i \in 2$ such that $\mathcal{F}_i | [a, B] = \emptyset$.*

The digression ends. We return to open subsets of $\mathcal{R} \times \mathbb{R}^{\mathbb{N}}$ where $\mathcal{R}$ is a topological Ramsey space satisfying **(L4)**.

Theorem 3.12. *Let $\mathcal{O} \subseteq \mathcal{R} \times \mathbb{R}^{\mathbb{N}}$ be an open set. For every $p \in \mathcal{P}_\omega, A \in \mathcal{R}$ and $a \in \mathcal{AR}[\emptyset, A]$ there exists $q \leq p$ and $B \in [\mathrm{depth}_A(a), A]$ such that $[a, B] \times [q] \subseteq \mathcal{O}$ or $[a, B] \times [q] \cap \mathcal{O} = \emptyset$.*

Proof. By Lemma 3.9, shrinking p and A, we may assume for all $b \in \mathcal{AR}$ and $m \in \omega$,

$$\forall \sigma \in l(m, p) \ \left(\mathrm{depth}_A(a) \leq \mathrm{depth}_A(b) \leq m \Rightarrow (A, p) \text{ decides } (b, \sigma)\right). \tag{$*$}$$

In particular,

$$\forall \sigma \in l(\mathrm{depth}_A(a), p) \quad (A, p) \text{ decides } (a, \sigma).$$

If there is $\sigma \in l(\mathrm{depth}_A(a), p)$ such that (A, p) accepts (a, σ), then $[a, A] \times [p|\sigma] \subseteq \mathcal{O}$, and $B = A$, $q = p|\sigma$ satisfy the theorem. So we assume

$$\forall \sigma \in l(\mathrm{depth}_A(a), p) \quad (A, p) \text{ rejects } (a, \sigma).$$

Then by Lemma 3.8 (6),

$$(A, p) \text{ rejects } (a, \emptyset).$$

We aim to find $q \leq p$ and $B \in [\mathrm{depth}_A(a), A]$ such that $[a, B] \times [q] \cap \mathcal{O} = \emptyset$. We achieve this by constructing barriers on $[a, A]$ of higher and higher rank, shrinking A and p correspondingly such that (A, p) rejects $(c, \emptyset)$ for every c in the barriers, as follows.

We construct sequences $(A_k)_{k<\omega} \subseteq \mathcal{R}$, $(p_k)_{k<\omega} \subseteq \mathcal{P}$ and $(\mathcal{F}_k)_{k<\omega}$ such that $\forall k < \omega$,

(1) $a_0 = a$, $A_0 = A$, $p_0 = p$, $\mathcal{F}_0 = \{a\}$;

(2) $n_k = \min\{\mathrm{depth}_{A_k}(b) : b \in \mathcal{F}_k\}$;

(3) $A_{k+1} \in [n_k, A_k]$, $p_{k+1} \leq^{n_k} p_k$;

(4) $\mathcal{F}_{k+1}$ is a barrier on $[a, A_{k+1}]$;

(5) $(\forall b \in \mathcal{F}_k \cap \mathcal{AR}[a, A_{k+1}] \ \exists c \in \mathcal{F}_{k+1} \ b \sqsubset c)$ and
$(\forall c \in \mathcal{F}_{k+1} \cap \mathcal{AR}[a, A_{k+1}] \ \exists b \in \mathcal{F}_k \ b \sqsubset c)$;

(6) (A_k, p_k) rejects $(b, \emptyset)$ for $b \in \mathcal{F}_k \cap \mathcal{AR}[a, A_k]$.

Suppose we have constructed a_k, A_k, p_k and $\mathcal{F}_k$. We construct sequences $(b_i)_{0<i<\omega}$, $(B_i)_{i<\omega}$, $(q_i)_{i<\omega}$ and $(\mathcal{G}_i)_{i<\omega}$ such that $\forall i < \omega$

(i) $B_0 = A_k$, $q_0 = p_k$;

(ii) $B_{i+1} \in [\mathrm{depth}_{B_i}(b_{i+1}), B_i]$, $q_{i+1} \leq^{\mathrm{depth}_{B_i}(b_{i+1})} q_i$;

(iii) $b_{i+1} \in \mathcal{F}_k \setminus \{b_1, \ldots, b_i\}$ such that $\mathrm{depth}_{B_i}(b_{i+1})$ is minimal;

(iv) $\mathcal{G}_{i+1}$ is a barrier on $[b_{i+1}, B_{i+1}]$;

(v) (B_{i+1}, q_{i+1}) rejects $(c, \emptyset)$ for $c \in \mathcal{G}_{i+1} \cap \mathcal{AR}[b_{i+1}, B_{i+1}]$;

(vi) (B_{i+1}, q_{i+1}) rejects $(b, \emptyset)$ for $b \in \mathcal{F}_k \cap \mathcal{AR}[a, B_{i+1}]$.

Suppose we have B_i, q_i. Let b_{i+1} be as in (iii). By the induction hypothesis (6), (A_k, p_k) rejects $(b_{i+1}, \emptyset)$. So by Lemma 3.8 (2),

$$(B_i, q_i) \text{ rejects } (b_{i+1}, \emptyset).$$

For $c \in \mathcal{AR}$, let

$$O_c^{i+1} = \bigcup \{[q_i|\tau] : [c, B_i] \times [q_i|\tau] \subseteq \mathcal{O}\}$$
$$\mathcal{B}_{i+1} = \{c \in \mathcal{AR}[b_{i+1}, B_i] : (b_{i+1} \neq c) \wedge (O_c^{i+1} \neq \emptyset)\}.$$

Applying Theorem 3.10 to $\mathcal{B}_{i+1}$ and b_{i+1}, B_i, we have two cases.

Case 1. $\exists B_{i+1} \in [\mathrm{depth}_{B_i}(b_{i+1}), B_i]$ such that $\mathcal{B}_{i+1} \cap \mathcal{AR}[b_{i+1}, B_{i+1}] = \emptyset$. Then $[b_{i+1}, B_{i+1}] \times [q_i] \cap \mathcal{O} = \emptyset$: Otherwise $\exists (X, \varepsilon) \in [b_{i+1}, B_{i+1}] \times [q_i] \cap \mathcal{O}$. Since $\mathcal{O}$ is open, there is $c \sqsubseteq X$ and $\sigma \sqsubseteq^* \varepsilon$ such that $b_{i+1} \sqsubset c$ and $[c] \times [q_i|\sigma] \subseteq \mathcal{O}$. So $O_c^{i+1} \neq \emptyset$ and hence $c \in \mathcal{B}_{i+1} \cap \mathcal{AR}[b_{i+1}, B_i]$, contradicting $\mathcal{B}_{i+1} \cap \mathcal{AR}[b_{i+1}, B_i] = \emptyset$.

Let $q_{i+1} = q_i$ and $\mathcal{G}_{i+1}$ be an arbitrary barrier on $[b_{i+1}, B_{i+1}]$ but $\mathcal{G}_{i+1} \neq \{b_{i+1}\}$. In particular, (B_{i+1}, q_{i+1}) rejects $(c, \emptyset)$ for all $c \in \mathcal{AR}[b_{i+1}, B_{i+1}]$.

Case 2. $\exists B_{i+1} \in [\mathrm{depth}_{B_i}(b_{i+1}), B_i]$ such that there is a barrier $\mathcal{G}_{i+1} \subseteq \mathcal{B}_{i+1} \cap \mathcal{AR}[b_{i+1}, B_{i+1}]$ on $[b_{i+1}, B_{i+1}]$. By **(L4)** and Lemma 3.5, we further assume that there exist $q_{i+1} \leq^{\mathrm{depth}_{B_i}(b_{i+1})} q_i$ and $G \subseteq [q_{i+1}]$ clopen such that

$$[q_{i+1}] \cap O_c^{i+1} = G \text{ for all } c \in \mathcal{G}_{i+1} \cap \mathcal{AR}[b_{i+1}, B_{i+1}].$$

Claim 3.12.1. $G = \emptyset$.

Proof of Claim 3.12.1. Otherwise by Lemma 3.6, there is $q' \leq q_{i+1}$ such that $[q'] \subseteq G \subseteq O_c^{i+1}$. Then

$$\begin{aligned}[b_{i+1}, B_{i+1}] \times [q'] &\subseteq \bigcup \{[c, B_{i+1}] \times [q] : c \in \mathcal{G}_{i+1} \cap \mathcal{AR}[b_{i+1}, B_{i+1}]\} \\ &\subseteq \bigcup \{[c, B_i] \times [q] : c \in \mathcal{G}_{i+1} \cap \mathcal{AR}[b_{i+1}, B_{i+1}]\} \\ &\subseteq \mathcal{O}.\end{aligned}$$

This contradicts that (B_i, q_i) rejects $(b_{i+1}, \emptyset)$. □

Claim 3.12.2. (B_{i+1}, q_{i+1}) *rejects* $(c, \emptyset)$ *for* $c \in \mathcal{G}_{i+1} \cap \mathcal{AR}[b_{i+1}, B_{i+1}]$.

Proof of Claim 3.12.2. Let $c \in \mathcal{G}_{i+1} \cap \mathcal{AR}[b_{i+1}, B_{i+1}]$. Since $q_{i+1} \leq p$, by $(*)$ and Lemma 3.8 (5), we can find m large enough such that

$$\forall \sigma \in l(m, q_{i+1}) \quad (A, p) \text{ decides } (c, \sigma).$$

Now let $\sigma \in l(m, q_{i+1})$. By Lemma 3.8 (1) and (2),

$$(B_{i+1}, q_{i+1}) \text{ decides } (c, \sigma) \text{ in the same way as } (A, p).$$

If (A, p) accepts (c, σ), then $[c, A] \times [p|\sigma] \subseteq \mathcal{O}$ hence $[q_{i+1}|\sigma] \subseteq O_c^{i+1}$. This contradicts that $[q_{i+1}] \cap O_c^{i+1} = \emptyset$. Therefore, (A, p) and (B_{i+1}, q_{i+1}) must reject (c, σ) for all $\sigma \in l(m, q_{i+1})$. Then by Lemma 3.8 (6), (B_{i+1}, q_{i+1}) rejects $(c, \emptyset)$. □

This finishes the construction of (b_i), (B_i), (q_i) and $(\mathcal{G}_i)$. By construction (i)-(iv) hold. In both Case 1 and Case 2, (v) holds. (vi) holds by the induction hypothesis (6), Lemma 3.8 (2) and the fact that $B_{i+1} \leq A_k$ and $q_{i+1} \leq p_k$.

By (ii), (iii) and Lemma 5.17, $\left(\text{depth}_{B_i}(b_{i+1})\right)_{i<\omega} \subseteq \omega$ is increasing and unbounded, so $\left([\text{depth}_{B_i}(b_{i+1}), B_i]\right)_{i<\omega}$ and $(q_i)_{i<\omega}$ are fusion sequences. Let A_{k+1}, p_{k+1} be the fusions of the sequences. Let

$$\mathcal{F}_{k+1} = \bigcup \{\mathcal{G}_i : b_i \in \mathcal{F}_k \cap \mathcal{AR}[a, A_{k+1}]\}.$$

This finishes the construction of (A_k), (p_k) and $(\mathcal{F}_k)$. Let us check that (1)-(6) hold.(1) and (2) hold by construction

(3) holds by (ii) since $B_0 = A_k$ and $\text{depth}_{B_0}(b_1) = n_k$ by (2) and (iii).

(4) $\mathcal{F}_k$ is a barrier on $[a, A_k]$ by the induction hypothesis (4). Moreover, each $\mathcal{G}_{i+1}$ is a barrier on $[b_i, A_{k+1}]$ by (iv). So $\mathcal{F}_{k+1}$ is a barrier on $[a, A_{k+1}]$.

(5) holds since each $\mathcal{G}_{i+1} \neq \{b_{i+1}\}$.

(6) follows from (vi) and Lemma 3.8 (2).

By (2) and (3), $(n_k)_{k<\omega}$ is increasing in k. Again Lemma 5.17 gives that $(n_k)_{k<\omega}$ is unbounded, so $([n_k, A_k])_{k<\omega}$ and $(p_k)_{k<\omega}$ are fusion sequences. Let B, q be the fusions of the sequences. Then by (6) and Lemma 3.8 (2),

$$\forall k < \omega \; \forall c \in \mathcal{F}_k \cap \mathcal{AR}[a, B] \quad (B, q) \text{ rejects } (c, \emptyset).$$

Claim 3.12.3. *$[a, B] \times [q] \cap \mathcal{O} = \emptyset$.*

Proof of Claim 3.12.3. Suppose otherwise and aim for a contradiction. If $(X, \varepsilon) \in [a, B] \times [q] \cap \mathcal{O}$, then there exist $b \sqsubseteq X$ and $\sigma \sqsubseteq^* \varepsilon$ such that

$$a \sqsubset b \wedge [b, B] \times [q|\sigma] \subseteq \mathcal{O}.$$

By (5), we may find k large enough such that

$$\exists c \in \mathcal{F}_k \cap \mathcal{AR}[a, B] \quad b \sqsubseteq c.$$

So $[c, B] \times [q|\sigma] \subseteq \mathcal{O}$ contradicting that (B, q) rejects $(c, \emptyset)$. □

□

3.2 Souslin-measurable Subsets of $\mathcal{R} \times \mathbb{R}^{\mathbb{N}}$

In this subsection, we extend the result in the previous subsection from open subsets to all Souslin-measurable subsets, by adapting the results in [42, §9] to abstract topological Ramsey spaces parametrized by infinite sequences of perfect trees.

Definition 3.13. A subset $\mathcal{X} \subseteq \mathcal{R} \times \mathbb{R}^{\mathbb{N}}$ is *perfectly Ramsey* if for every $A \in \mathcal{R}, a \in \mathcal{AR}[\emptyset, A]$ and $p \in \mathcal{P}_\omega$ there exists $B \in [\mathrm{depth}_A(a), A]$ and $q \leq p$ such that $[a, B] \times [q] \subseteq \mathcal{X}$ or $[a, B] \times [q] \cap \mathcal{X} = \emptyset$.

So we can rephrase Theorem 3.12 as follows.

Theorem 3.14. *Every open subset of $\mathcal{R} \times \mathbb{R}^{\mathbb{N}}$ is perfectly Ramsey.*

Lemma 3.15. *The perfectly Ramsey subsets of $\mathcal{R} \times \mathbb{R}^{\mathbb{N}}$ form a σ-field.*

Proof. It is straightforward to check that the collection of perfectly Ramsey subsets form a field. We check that it is closed under countable union. Suppose $(\mathcal{X}_m)_{m<\omega}$ is a sequence of perfectly Ramsey sets. Let $\mathcal{X} = \bigcup_{m<\omega} \mathcal{X}_m$. Without loss of generality, $\mathcal{X}_m \subseteq \mathcal{X}_{m+1}$. Let $p \in \mathcal{P}_\omega$, $A \in \mathcal{R}$, and $a \in \mathcal{AR}[\emptyset, A]$. We aim to find $B \in [\mathrm{depth}_A(a), A]$ and $q \leq p$ such that $[a, B] \times [q]$ is included in or disjoint from $\mathcal{X}$.

We construct fusion sequences $([n_k, A_k])_{k<\omega}$ and $(p_k)_{k<\omega}$ such that for $k \geq 0$:

(i) $A_0 = A$, $p_0 = p$, and $n_k = \mathrm{depth}_A(a) + k$;

(ii) $p_{k+1} \leq^k p_k$, $A_{k+1} \in [n_k, A_k]$;

(iii) for every element (σ, b) in the set

$$S_k = l(k, p_k) \times \{b \in \mathcal{AR} : \mathrm{depth}_{A_k}(b) = n_k\}$$

either $[b, A_{k+1}] \times [p_{k+1}|\sigma] \subseteq \mathcal{X}_k$ or $[b, A_{k+1}] \times [p_{k+1}|\sigma] \cap \mathcal{X}_k = \emptyset$.

Suppose we have p_k, A_k. Since $\mathcal{X}_k$ is perfectly Ramsey, and the set S_k is finite by **(A2)**(1), we can shrink p_k, A_k finitely many times to obtain p_{k+1}, A_{k+1}.

Let $A_\infty = \lim A_m$ and $p_\infty = \bigcap_{m<\omega} p_m$. Thus $A_\infty \in [\mathrm{depth}_A(a), A]$ and $p_\infty \leq p$. Moreover, for every $m < \omega$, $\sigma \in l(m, p_\infty)$ and b with $\mathrm{depth}_{A_\infty}(b) = \mathrm{depth}_A(a) + m$,

$$[b, A_\infty] \times [p_\infty|\sigma] \subseteq [b, A_{m+1}] \times [p_{m+1}|\sigma],$$

which is included in or disjoint from $\mathcal{X}_m$.

By construction, the set $\mathcal{X} \cap [a, A_\infty] \times [p_\infty]$ is open in $[a, A_\infty] \times [p_\infty]$ with respect to the subspace topology. Therefore, there exists an open subset $\mathcal{O} \subseteq \mathcal{R} \times \mathbb{R}^{\mathbb{N}}$ such that $\mathcal{X} \cap [a, A_\infty] \times [p_\infty] = \mathcal{O} \cap [a, A_\infty] \times [p_\infty]$. By Theorem 3.14, $\mathcal{O}$ is perfectly Ramsey. Hence we can find $B \in [\mathrm{depth}_A(a), A_\infty]$ and $q \leq p_\infty$ such that $[a, B] \times [q] \subseteq \mathcal{O}$ or $[a, B] \times [q] \cap \mathcal{O} = \emptyset$. On the other hand, $[a, B] \times [q] \subseteq [a, A_\infty] \times [p_\infty]$, so we must have $[a, B] \times [q] \subseteq \mathcal{X}$ or $[a, B] \times [q] \cap \mathcal{X} = \emptyset$. □

Theorem 3.16. *The field of perfectly Ramsey subsets of $\mathcal{R} \times \mathbb{R}^{\mathbb{N}}$ is closed under the Souslin operation.*

Proof. Let $\mathcal{X}_v (v \in [\omega]^{<\omega})$ be a given Souslin scheme of perfectly Ramsey subsets of $\mathcal{R} \times \mathbb{R}^{\mathbb{N}}$. Without loss of generality, we identify $[\omega]^{<\omega}$ with the set of finite strictly increasing sequences, $[\omega]^{\omega}$ with the set of infinite strictly increasing sequences, and assume $\mathcal{X}_u \subseteq \mathcal{X}_v$ whenever $v \sqsubseteq u$. Let $p \in \mathcal{P}_\omega$, $A \in \mathcal{R}$, and

$a \in \mathcal{AR}[\emptyset, A]$. Let $\mathcal{X} = \bigcup_{f\in[\omega]^\omega} \bigcap_{n\in\omega} \mathcal{X}_{f\restriction n}$. We aim to find $\bar{B} \in [\mathrm{depth}_A(a), A]$ and $\bar{q} \le p$ such that $[a, \bar{B}] \times [\bar{q}] \subseteq \mathcal{X}$ or $[a, \bar{B}] \times [\bar{q}] \cap \mathcal{X} = \emptyset$, thus showing that $\mathcal{X}$ is also perfectly Ramsey.

For $v \in [\omega]^{<\omega}$, let

$$\mathcal{X}_v^* = \bigcup_{f \sqsupseteq v} \bigcap_{n\in\omega} \mathcal{X}_{f\restriction n}.$$

So $\mathcal{X}_v^* \subseteq \mathcal{X}_v$ and $\mathcal{X}_u^* \subseteq \mathcal{X}_v^*$ whenever $v \sqsubseteq u$. We build fusion sequences $([n_k, A_k])_{k<\omega}$ and $(p_k)_{k<\omega}$ such that for $k \ge 0$:

1. $A_0 = A$, $p_0 = p$, and $n_k = \mathrm{depth}_A(a) + k$;
2. $p_{k+1} \le^k p_k$, $A_{k+1} \in [n_k, A_k]$;
3. for every element (σ, v, b) in the set

$$S_k = l(k, p_k) \times [k]^{<\omega} \times \{b \sqsupseteq a : \mathrm{depth}_{A_k}(b) = n_k\},$$

either

$(1)_k$ $[b, A_{k+1}] \times [p_{k+1}|\sigma] \cap \mathcal{X}_v^* = \emptyset$; or

$(2)_k$ there does not exist $q \le_\sigma p_{k+1}$ and $B \in [n_k, A_{k+1}]$ with $[b, B] \times [q|\sigma] \cap \mathcal{X}_v^* = \emptyset$.

Suppose we have constructed A_k, p_k. Let S_k be enumerated as $(\sigma_l, v_l, b_l)_{l<L}$ where $L < \omega$ by **(A2)** (1). We can construct sequences $([n_k, B_l])_{l\le L}$ and $(q_l)_{l\le L}$ such that for $l < L$:

(i) $B_0 = A_k$ and $q_0 = p_k$;

(ii) $q_{l+1} \le^k q_l$, $B_{l+1} \in [n_k, B_l]$;

(iii) either $[b_l, B_{l+1}] \times [q_{l+1}|\sigma_l] \cap \mathcal{X}_{v_l}^* = \emptyset$, or $B_{l+1} = B_l$, $q_{l+1} = q_l$ and there does not exist $q \le q_l|\sigma_l$ and $B \in [n_k, B_l]$ such that $[b_{l+1}, B] \times [q] \cap \mathcal{X}_{v_l}^* = \emptyset$.

Let $A_{k+1} = B_L$ and $p_{k+1} = q_L$. This finishes the construction of $([n_k, A_k])_{k<\omega}$ and $(p_k)_{k<\omega}$. Then let A_∞ and p_∞ be the fusions.

For $v \in [\omega]^{<\omega}$, let

$$\begin{aligned}\Psi(\mathcal{X}_v^*) &= \bigcup \{[b, A_\infty] \times [p_\infty|\sigma] : [b, A_\infty] \times [p_\infty|\sigma] \cap \mathcal{X}_v^* = \emptyset \wedge a \sqsubseteq b\},\\ \Phi(\mathcal{X}_v^*) &= (\mathcal{X}_v \cap [a, A_\infty] \times [p_\infty]) \setminus \Psi(\mathcal{X}_v^*), \text{ and}\\ \mathcal{M}_v &= \Phi(\mathcal{X}_v^*) \setminus \bigcup_{l>\max v} \Phi(\mathcal{X}_{v^\frown l}^*).\end{aligned}$$

In particular, $\mathcal{M}_v \cap \Psi(\mathcal{X}_v^*) = \emptyset$,

$$\begin{aligned}\mathcal{X}_v^* &= \bigcup_{l>\max v} \mathcal{X}_{v^\frown l}^* \subseteq \bigcup_{l>\max v} \mathcal{X}_{v^\frown l}, \text{ and}\\ \mathcal{X}_v^* \cap [a, A_\infty] \times [p_\infty] &\subseteq \bigcup_{l>\max v} (\mathcal{X}_{v^\frown l} \cap [a, A_\infty] \times [p_\infty])\\ &\subseteq \bigcup_{l>\max v} \Phi(\mathcal{X}_{v^\frown l}^*).\end{aligned}$$

So

$$\mathcal{M}_v \subseteq \Phi(\mathcal{X}_v^*) \setminus (\mathcal{X}_v^* \cap [a, A_\infty] \times [p_\infty]) = \Phi(\mathcal{X}_v^*) \setminus \mathcal{X}_v^*.$$

Note that $\Psi(\mathcal{X}_v^*)$ is open in $[a, A_\infty] \times [p_\infty]$ with respect to the subspace topology, so as in the proof of Lemma 3.15, we can find an open (and hence perfectly Ramsey, by Theorem 3.14) set $\mathcal{O} \subseteq \mathcal{R} \times \mathbb{R}^{\mathbb{N}}$ such that $\Psi(\mathcal{X}_v^*) = \mathcal{O} \cap ([a, A_\infty] \times [p_\infty])$. Let $\mathcal{O}^c = \mathcal{R} \times \mathbb{R}^{\mathbb{N}} \setminus \mathcal{O}$. Therefore $\Phi(\mathcal{X}_v^*) = (\mathcal{X}_v \cap \mathcal{O}^c) \cap [a, A_\infty] \times [p_\infty]$, where $\mathcal{X}_v \cap \mathcal{O}^c$ is perfectly Ramsey.

We say a set $\mathcal{Y} = [a, A_\infty] \times [p_\infty]$ is *perfectly Ramsey inside* $[a, A_\infty] \times [p_\infty]$ if it is the intersection of a perfectly Ramsey set with $[a, A_\infty] \times [p_\infty]$. The definition of *perfectly Ramsey null inside* is similar. So by the arguments above, $\Psi(\mathcal{X}_v^*)$ and $\Phi(\mathcal{X}_v^*)$ are both perfectly Ramsey inside $[a, A_\infty] \times [p_\infty]$. Since the perfectly Ramsey sets form a σ-field (Lemma 3.15), $\mathcal{M}_v$ is also perfectly Ramsey inside $[a, A_\infty] \times [p_\infty]$.

Claim 3.16.1. *For every $v \in [\omega]^{<\omega}$, $\mathcal{M}_v$ is perfectly Ramsey null inside $[a, A_\infty] \times [p_\infty]$.*

Proof. Suppose $\emptyset \neq [b, B] \times [q] \subseteq [a, A_\infty] \times [p_\infty] \cap \mathcal{M}_v$ for some $v \in [\omega]^{<\omega}$, $b \in \mathcal{AR}$ and $B \in \mathcal{R}$. We aim for a contradiction.

Pick $Y \in [b, B]$, so $b \sqsubseteq Y \leq B \leq A_\infty$. By **(A2)** (1), we may pick $l \geq |b|$ large enough such that, for $b' = r_l(Y)$,

$$\mathrm{depth}_{A_\infty}(b') = n_k,$$

where k is in turn large enough such that $v \in [k]^{<\omega}$. Thus $[b', Y] \neq \emptyset$ and $Y \leq A_\infty$, so by **(A3)** (2) there exists $Y' \in [n_k, A_\infty]$ such that $[b', Y'] \subseteq [b', Y]$. So

$$[b', Y'] \times [q] \subseteq [b', Y] \times [q] \subseteq [b, B] \times [q] \subseteq \mathcal{M}_v,$$

which is disjoint from $\mathcal{X}_v^*$. Pick $\sigma \in l(k, p_\infty)$ such that $\sigma(i) \in q(i)$ for all $i < k$, so $q \leq_\sigma p_\infty$. Thus, for $\sigma \in l(k, p_\infty)$, $v \in [k]^{<\omega}$ and $b' \sqsupseteq a$ with $\mathrm{depth}_{A_\infty}(b') = n_k$, we have $q \leq_\sigma p_\infty$ and $Y' \in [n_k, A_\infty]$ such that $[b', Y'] \times [q|\sigma] \cap \mathcal{X}_v^* = \emptyset$. By the construction of A_∞ and p_∞, $(1)_k$ must hold. So $[b', A_\infty] \times [p_\infty|\sigma] \cap \mathcal{X}_v^* = \emptyset$. Thus

$$[b', Y'] \times [q|\sigma] \subseteq \mathcal{M}_v \cap \Psi(\mathcal{X}_v^*),$$

contradicting $\mathcal{M}_v \cap \Psi(\mathcal{X}_v^*) = \emptyset$. □

Therefore we can build fusion sequences $([n_k, B_k])_{k<\omega}$ and $(q_k)_{k<\omega}$ such that for $k \geq 0$,

(i) $B_0 = A_\infty, q_0 = p_\infty$ and $n_k = \mathrm{depth}_A(a) + k$;

(ii) $q_{k+1} \leq^k q_k$, $B_{k+1} \in [n_k, B_k]$;

(iii) $\forall \tau \in l(k, q_k) \, \forall v \in [k]^{<\omega} \, \forall b \sqsupseteq a$ with $\mathrm{depth}_{B_k}(b) = n_k$, $[b, B_{k+1}] \times [q_{k+1}|\tau] \cap \mathcal{M}_v = \emptyset$;

Let B_∞ and q_∞ be the fusions of the sequences. We check that $[a, B_\infty] \times [q_\infty] \cap \mathcal{M}_v = \emptyset$ for all $v \in [\omega]^{<\omega}$: Suppose $(X, \varepsilon) \in [a, B_\infty] \times [q_\infty]$ and $v \in [\omega]^{<\omega}$. We can find $k > \max v$ such that there exist $b \sqsubseteq X$ and $\sigma \sqsubseteq^* \varepsilon$ with

$$\mathrm{depth}_{B_\infty}(b) = n_k \ \wedge \ \sigma \in l(k, q_\infty).$$

As $[b, B_{k+1}] \times [q_{k+1}|\sigma] \cap \mathcal{M}_v = \emptyset$ by construction, we have $(X, \varepsilon) \notin \mathcal{M}_v$.

Claim 3.16.2. $[a, B_\infty] \times [q_\infty] \cap \mathcal{X}_\emptyset^* = [a, B_\infty] \times [q_\infty] \cap \Phi(\mathcal{X}_\emptyset^*)$.

Proof of Claim 3.16.2. Since $\mathcal{X}^*_\emptyset \subseteq \Phi(\mathcal{X}^*_\emptyset)$, it is clear that $[a, B_\infty] \times [q_\infty] \cap \mathcal{X}^*_\emptyset \subseteq [a, B_\infty] \times [q_\infty] \cap \Phi(\mathcal{X}^*_\emptyset)$. To prove the other inclusion, we pick $\bar{x} \in [a, B_\infty] \times [q_\infty] \cap \Phi(\mathcal{X}^*_\emptyset)$. Recall that $[a, B_\infty] \times [q_\infty] \cap \mathcal{M}_\emptyset = \emptyset$ and $\mathcal{M}_\emptyset = \Phi(\mathcal{X}^*_\emptyset) \setminus \bigcup_{l<\omega} \Phi(\mathcal{X}^*_l)$. So there exists $l_0 \in \omega$ such that $\bar{x} \in \Phi(\mathcal{X}^*_{\{l_0\}})$. By repeating this process, we find $v = \{l_0 < l_1 < \cdots\} \in [\omega]^{<\omega}$ such that $\bar{x} \in \Phi(\mathcal{X}^*_{v|k}) \subseteq \mathcal{X}_{v|k}$ for all $k < \omega$. Hence $\bar{x} \in \bigcap_{m<\omega} \mathcal{X}_{v|k} \subseteq \mathcal{X} = \mathcal{X}^*_\emptyset$. ☐

As $\Phi(\mathcal{X}^*_\emptyset)$ is perfectly Ramsey inside $[a, A_\infty] \times [p_\infty]$, $B_\infty \in [\text{depth}_A(a), A_\infty]$ and $q_\infty \leq p_\infty$, there exists $\bar{B} \in [\text{depth}_A(a), B_\infty] \subseteq [\text{depth}_A(a), A]$ and $\bar{q} \leq q_\infty \leq p$ such that $[a, \bar{B}] \times [\bar{q}] \subseteq \Phi(\mathcal{X}^*_\emptyset)$ or $[a, \bar{B}] \times [\bar{q}] \cap \Phi(\mathcal{X}^*_\emptyset) = \emptyset$. As $\mathcal{X} = \mathcal{X}^*_\emptyset$, by Claim 3.16.2, we have $[a, \bar{B}] \times [\bar{q}] \subseteq \mathcal{X}$ or $[a, \bar{B}] \times [\bar{q}] \cap \mathcal{X} = \emptyset$, as required. ☐

This finishes the proof of Theorem 3.3. In fact, by **(A3)** (2), in the conclusion of Theorem 1.7, we may choose $B \in [\text{depth}_A(a), A]$ and obtain the following.

Corollary 3.17. *Suppose $\mathcal{R}$ is a topological Ramsey space satisfying* **(L4)**. *For every finite Souslin-measurable colouring of $\mathcal{R} \times \mathbb{R}^{\mathbb{N}}$, for every $A \in \mathcal{R}$, $a \in \mathcal{AR}[\emptyset, A]$ and $p \in \mathcal{P}_\omega$ there exists $B \in [\text{depth}_A(a), A]$ and $q \leq p$ such that $[a, B] \times [q]$ is monochromatic.*

3.3 Necessity of (L4)

In this section, we show that **(L4)** is a necessary condition for Theorem 1.7 to hold.

Lemma 3.18. *Let $p \in \mathcal{P}_\omega$ and $n \in \omega$. Then for every open set $O \subseteq [p]$ there exists $q \leq^n p$ such that $O \cap [q]$ depends only on $l(n, q)$, i.e. for every $\sigma \in l(n, q)$, $[q|\sigma]$ is either included or disjoint from O.*

Proof. Apply Lemma 3.4 to shrink each $p|\sigma$. ☐

Lemma 3.18 and the method of fusion gives following corollary.

Corollary 3.19. *Suppose $p \in \mathcal{P}_\omega$, $n \in \omega$, and $O_l\,(l \in \omega)$ is a family of open subsets of $(2^\omega)^\omega$. Suppose also that $(n_l)_{l<\omega}$ is a strictly increasing sequence of natural numbers above n. Then there exists $q \leq^n p$ such that for every $l \in \omega$, $O_l \cap [q]$ depends only on $l(n_l, q)$.*

Theorem 3.20. *Assume that for every finite Souslin-measurable colouring of $\mathcal{R} \times \mathbb{R}^{\mathbb{N}}$ and for every $A \in \mathcal{R}, a \in \mathcal{AR}[\emptyset, A]$ and $p \in \mathcal{P}_\omega$ there exists $B \in [a, A]$ and $q \leq p$ such that $[a, B] \times [q]$ is monochromatic. Then* **(L4)** *must hold.*

Proof. Let $p \in \mathcal{P}_\omega$, $A \in \mathcal{R}$ and $\{O_b : b \in \mathcal{AR}_{|a|+1}[a, A]\}$ be a family of open subsets of $[p]$. We aim to find $B \in [a, A]$ and $q \leq p$ such that $b \mapsto O_b \cap [q]$ is constant on $\mathcal{AR}_{|a|+1}[a, B]$.

By **(A2)** (1), for every $l < \omega$, the set $\{b \in \mathcal{AR} : \text{depth}_A(b) = l\}$ is finite. Then by Lemma 3.18 and fusion, similar to Corollary 3.19, we may assume that

$$\forall b \in \mathcal{AR}_{|a|+1}[a, A] \quad O_b \cap [p] \text{ depends only on } l(\text{depth}_A(b), p), \tag{3.1}$$

i.e.

$$\forall b \in \mathcal{AR}|a| + 1[a, A]\ \forall \sigma \in l(\text{depth}_A(b), p) \quad [p|\sigma] \subseteq O_b \text{ or } [p|\sigma] \cap O_b = \emptyset. \tag{3.2}$$

Let us define a colouring $c : [a, A] \times [p] \to 2$. For $(X, \varepsilon) \in [a, A] \times [p]$, let $b = r_{|a|+1}(X)$ and $\sigma \in l(\mathrm{depth}_A(b), p)$ be such that $\sigma \sqsubseteq^* \varepsilon$. Define

$$c(X, \varepsilon) = \begin{cases} 1 & \text{iff } [p|\sigma] \subseteq O_b \\ 0 & \text{iff } [p|\sigma] \subseteq O_b^c. \end{cases}$$

Thus c is continuous, hence Souslin-measurable. By the assumption APET, there is $B \in [a, A]$ and $q \le p$ such that $[a, B] \times [q]$ is monochromatic.

Suppose, without loss of generality, $c \restriction [a, B] \times [q] = 1$. Consider $b \in \mathcal{AR}_{|a|+1}[a, B]$. Let $X \in [a, B]$ be such that $b = r_{|a|+1}(X)$. Then for every $\varepsilon \in [q]$, $c(X, \varepsilon) = 1$. Hence

$$\forall \sigma \in l(\mathrm{depth}_A(b), q) \quad [p|\sigma] \subseteq O_b.$$

(Note that $\sigma \in l(\mathrm{depth}_A(b), q)$ are extensions of elements in $l(\mathrm{depth}_A(b), p)$.) So $[q] \subseteq O_b$.

Similarly, $[q] \subseteq O_b^c$ if $c \restriction [a, b] \times [q] = 0$. □

Chapter 4

Ellentuck Space $\mathbb{N}^{[\infty]}$

The Ellentuck spaces is a prototype example of topological Ramsey spaces. Its simple yet rich structure has been studied before in different contexts. In this chapter, we collect the different properties defined in literature for ultrafilters in the Ellentuck space and check if they are equivalent. We also briefly discuss the well-known result regarding preservation of selective ultrafilters on the natural numbers under Sacks forcing.

A lot of the symbols used in abstract topological Ramsey spaces have their more familiar forms in the Ellentuck space. An element $X \in \mathbb{N}^{[\infty]}$ is just an infinite subset of the natural numbers. The set $[\emptyset, X]$ is the same as the set $[X]^{[\infty]}$ of all infinite subsets of X, and $\mathcal{AR}[\emptyset, X]$ is the set $[X]^{[<\infty]}$ of finite subsets of X. Similarly, $r_n[\emptyset, X]$ is the set $[X]^n$ of subsets of X which are of size n. A finite set $a \in \mathbb{N}^{[<\infty]}$ is the $\mathcal{AR}[\emptyset, X]$ if and only if $a \subseteq X$. The order $\leq$ on $\mathbb{N}^{[\infty]}$ coincides with the subset relation $\subseteq$.

4.1 Ultrafilters in the Ellentuck Space

In this section, we consider ultrafilters on the base set ω. Clearly, such an ultrafilter is an ultrafilter in the Ellentuck space $\mathbb{N}^{[\infty]}$, in the sense of Definition 2.18.

4.1.1 Ultrafilter quantifiers

A useful notational device we utilize is *filter quantifiers* introduced by Blass in [7].

Notation. Let $\mathcal{F}$ be an ultrafilter on ω. We define a quantifier $\mathcal{F}$ as follows. Let $\varphi(x)$ be a formula. By $\mathcal{F}x\, \varphi(x)$ we mean "almost all (in the sense of $\mathcal{F}$)" x have the property $\varphi(x)$, i.e.

$$\mathcal{F}x\, \varphi(x) \text{ if and only if } \{x \in \omega : \varphi(x)\} \in \mathcal{F}.$$

For example, $\mathcal{F}x\, \mathcal{F}y\, \varphi(x, y)$ means $\{x \in \omega : \{y \in \omega : \varphi(x, y)\} \in \mathcal{F}\} \in \mathcal{F}$.

Equivalently, we have

$$\mathcal{F}x\, \varphi(x) \text{ if and only if } (\exists X \in \mathcal{F})(\forall x \in X)\, \varphi(x).$$

Similarly, $\mathcal{F}x\, \mathcal{F}y\, \varphi(x, y)$ if and only if $(\exists X \in \mathcal{F})(\forall x \in X)(\exists Y \in \mathcal{F})(\forall y \in Y)\, \varphi(x, y)$.

Example 3. When $\mathcal{F}$ is the *Fréchet filter*, i.e.

$$\mathcal{F} = \{A \subseteq \omega : \omega \setminus A \text{ is finie}\},$$

$\mathcal{F}x$ becomes the commonly used quantifier $\forall^\infty x$, which means "for all but finitely many x".

When $\mathcal{U}$ is an ultrafilter on ω, the quantifier $\mathcal{U}x$ has the following nice properties: Let $\varphi(x), \psi(x)$ be formulas.

- $\neg\,\mathcal{U}x\,\varphi(x)$ if and only if $\mathcal{U}x\,\neg\,\varphi(x)$;
- $\mathcal{U}x\,\varphi(x) \wedge \mathcal{U}x\,\psi(x)$ if and only if $\mathcal{U}x\,(\varphi(x) \wedge \psi(x))$;
- $\mathcal{U}x\,\varphi(x) \vee \mathcal{U}x\,\psi(x)$ if and only if $\mathcal{U}x\,(\varphi(x) \vee \psi(x))$.

Definition 4.1 ([42])**.** Let $\mathcal{U}$ be an ultrafilter on ω and $k > 0$ be an integer. We define $\mathcal{U}^k$ to be a subset of $[\omega]^k$ such that for $X \in [\omega]^k$,

$$X \in \mathcal{U}^k \quad \text{if} \quad \mathcal{U}x_0 \cdots \mathcal{U}x_{k-1}\,\{x_0, \ldots, x_{k-1}\} \in X.$$

It is straightforward to check that $\mathcal{U}^k$ is an ultrafilter on $[\omega]^k$.

Proposition 4.2. *Let $\mathcal{U}$ be an ultrafilter on ω, $k \geq 2$ be an integer. For each $a \in [\omega]^{k-1}$, let $X_a = \{x \in \omega : a \cup \{x\} \in X\}$. Then $X \in \mathcal{U}^k$ if and only if $\{a : X_a \in \mathcal{U}\} \in \mathcal{U}^{k-1}$.*

Proof. By the definition of $\mathcal{U}^k$, $X \in \mathcal{U}^k$

$$\begin{aligned}
&\Leftrightarrow \mathcal{U}x_0 \cdots \mathcal{U}x_{k-2}\mathcal{U}y\,\{x_0, \ldots, x_{k-2}, y\} \in X\\
&\Leftrightarrow \mathcal{U}x_0 \cdots \mathcal{U}x_{k-2}\,\{y \in \omega : \{x_0, \ldots, x_{k-2}\} \cup \{y\} \in X\} \in \mathcal{U} \quad \text{by the definition of } \mathcal{U}y\\
&\Leftrightarrow \mathcal{U}x_0 \cdots \mathcal{U}x_{k-2}\,X_{\{x_0,\ldots,x_{k-2}\}} \in \mathcal{U} \quad \text{by the definition of } X_{\{x_0,\ldots,x_{k-2}\}}\\
&\Leftrightarrow \mathcal{U}x_0 \cdots \mathcal{U}x_{k-2}\,\{x_0, \ldots, x_{k-2}\} \in \{a \in [\omega]^k : X_a \in \mathcal{U}\}\\
&\Leftrightarrow \{a \in [\omega]^k : X_a \in \mathcal{U}\} \in \mathcal{U}^{k-1} \quad \text{by the definition of } \mathcal{U}^{k-1}.
\end{aligned}$$

□

4.1.2 Equivalent properties for ultrafilters on ω

In this subsection we show that the properties of an ultrafilter on ω being Nash-Williams, Ramsey, weakly selective and selective are all equivalent.

It is straight forward to check that the Ellentuck space $\mathbb{N}^{[\infty]}$ has a head start (Definition 2.23) and satisfies **(A8)** (Definition 2.21). By Theorem 2.24 and Theorem 2.22, every Nash-Williams ultrafilter is Ramsey and every Ramsey ultrafilter is weakly selective. We already know that every selective ultrafilter is weakly selective. So it remains to check that every weakly selective ultrafilter is selective as well as Nash-Williams.

Note that, by identifying each $\{n\} \in \mathbb{N}^{[1]}$ with $n \in \mathbb{N}$, an ultrafilter on ω is weakly selective if and only if the following holds: for every $A \in \mathcal{U}$ and every $\{A_n\}_{n<\omega} \subseteq \mathcal{U}|A$ there exists $B \in \mathcal{U}|A$ such that $B/n \subseteq A_n$ for every $n \in B$. Recall from Section 2.3 that for a family $\mathcal{F}$ and an element X, $\mathcal{F}|X = \{Y \in \mathcal{F} : Y \leq X\}$.

Notation. Let $B \in \mathbb{N}^{[\infty]}$, $a \in \mathbb{N}^{[<\infty]}$ and $n \in \omega$. We define $B/n = \{m \in B : n < m\}$ and $B/a = \{m \in B : \max a < m\}$.

Proposition 4.3. *Every weakly selective ultrafilter on ω is selective.*

Proof. Let $\mathcal{U}$ be a weakly selective ultrafilter on ω, i.e. for every $A \in \mathcal{U}$ and $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{U}|A$ there exists $B \in \mathcal{U}|A$ such that $B/n \subseteq A_n$ for all $n \in B$.

Suppose we are given $X \in \mathcal{U}$ and $\{X_a\}_{a\in\mathbb{N}^{[<\infty]}} \subseteq \mathcal{U}|X$ with $a \subseteq X_a$ for all $a \subseteq A$. We find $B \in \mathcal{U}|X$ such that $B/a \subseteq X_a/a$ for all $a \subseteq B$. For $n \in \omega$, define $A_n = \bigcap\{X_a : \max(a) \leq n\}$. By weak selectivity of $\mathcal{U}$, there exists $B \in \mathcal{U}|X$ such that $B/n \subseteq A_n/n$ for all $n \in B$. We check that B satisfies the statement. Let $a \subseteq B$. Then $B/a = B/\max(a) \subseteq A_{\max(a)}$. But by the definition of $A_{\max(a)}$, $A_{\max(a)} \subseteq X_a$. Hence $B/a \subseteq X_a/a$ as required. ☐

To prove that weakly selective implies Nash-Williams, we induct on the rank of Nash-Williams families. Recall Definition 3.2 of the rank of barriers. The same definition apply to Nash-Williams families: Let $\mathcal{G} \subseteq \mathbb{N}^{[<\infty]}$ be a Nash-Williams family, i.e. $s \not\sqsubseteq t$ for distinct elements $s, t \in \mathcal{G}$. Consider $T(\mathcal{G}) = \{s \in \mathbb{N}^{[\infty]} : (\exists t \in \mathcal{G})(s \sqsubseteq t)\}$ as a tree under $\sqsubseteq$. Then define $\rho_{\mathcal{G}} : T(\mathcal{G}) \to \mathrm{Ord}$ by $\rho_{\mathcal{G}}(s) = \sup\{\rho_{\mathcal{G}}(t) + 1 : t \in T(\mathcal{G}) \wedge t \sqsupseteq s\}$. Then the *rank* of $\mathcal{G}$ is $\mathrm{rk}(\mathcal{G}) = \rho(\emptyset)$.

Proposition 4.4. *Every weakly selective ultrafilter on ω is Nash-Williams.*

Proof. Let $\mathcal{U}$ be a weakly selective ultrafilter. Let $\mathcal{G} \subseteq \mathbb{N}^{[<\infty]}$ be a Nash-Williams family and $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1$ be a partition. We find $X \in \mathcal{U}$ such that $\mathcal{G}_i \cap [X]^{[<\infty]} = \emptyset$ for some $i \in 2$.

We induct on $\mathrm{rk}(\mathcal{G})$. When $\mathrm{rk}(\mathcal{G}) = 0$, $\mathcal{G} = \{\emptyset\}$ and the case is trivial. When $\mathrm{rk}(\mathcal{G}) = 1$, $\mathcal{G} \subseteq \mathbb{N}^{[1]}$. The partition $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1$ corresponds to a partition of $\mathbb{N}$. Since $\mathcal{U}$ is an ultrafilter on ω, there exists $B \in \mathcal{U}$ such that $\{n\} \in \mathcal{G}'$ for all $n \in B$, where $\mathcal{G}'$ is one of $\mathcal{G}_0, \mathcal{G}_1$ and $\mathbb{N}^{[1]} \setminus \mathcal{G}$.

Now we assume $\mathrm{rk}(\mathcal{G}) \geq 2$. For $n \in \omega$, let

$$\mathcal{G}_{\{n\}} = \{s \in \mathbb{N}^{[<\infty]} : (\{n\} < s) \wedge (s \cup \{n\}) \in \mathcal{G}\}.$$

Then $\mathcal{G}_{\{n\}}$ is a Nash-Williams family of a smaller rank than $\mathcal{G}$. Consider the partition $\mathcal{G}_{\{n\}} = \mathcal{G}^0_{\{n\}} \sqcup \mathcal{G}^1_{\{n\}}$ where, for $i \in 2$,

$$s \in \mathcal{G}^i_{\{n\}} \text{ if and only if } s \cup \{n\} \in \mathcal{G}_i.$$

By the induction hypothesis, for each $n \in \omega$ there exists $X_n \in \mathcal{U}$ and $i_n \in 2$ such that $\mathcal{G}^{i_n}_{\{n\}} \cap [X_n]^{[<\infty]} = \emptyset$, i.e. $s \cup \{n\} \notin \mathcal{G}_{i_n}$ for every $s \leq X_n/n$. Since $\mathcal{U}$ is weakly selective, and $\{\{n\} \cup X_n : n \in \omega\} \subseteq \mathcal{U}$ there exists $X \in \mathcal{U}$ such that $X/n \subseteq X_n/n$ for every $n \in X$. Thus for every $n \in X$ and every $s \leq X/n$, $s \cup \{n\} \notin \mathcal{G}_{i_n}$. Note

$$X = \{n \in X : i_n = 0\} \cup \{n \in X : i_n = 1\} \in \mathcal{U}.$$

Since $\mathcal{U}$ is an ultrafilter on ω, one of the two parts of X is in $\mathcal{U}$. So there exists $Y \in \mathcal{U}|X$ and $i \in 2$ such that for every $n \in Y$ and $s \leq Y/n$, $s \cup \{n\} \notin \mathcal{G}_i$. Therefore, for every $a \leq Y$, $a \notin \mathcal{G}_i$. So $\mathcal{G}_i \cap [Y]^{[<\infty]} = \emptyset$. ☐

There have been various definitions for selective and Ramsey ultrafilters on ω. We list some of them here, and show that they are indeed equivalent.

Proposition 4.5 (Folklore)**.** *Let $\mathcal{U}$ be an ultrafilter on ω. The following are equivalent.*

(1) $\mathcal{U}$ *is weakly selective, i.e. for every* $A \in \mathcal{U}$ *and* $\{A_n\}_{n<\omega} \subseteq \mathcal{U}|A$ *there exists* $B \in \mathcal{U}|A$ *such that* $B/n \subseteq A_n$ *for every* $n \in B$.

(2) [23]. *For every partition* $\{A_n : n \in \omega\}$ *of* ω *with each* $A_n \notin \mathcal{U}$ *there exists* $X \in \mathcal{U}$ *such that* $X \cap A_n$ *is a singleton for all* $n < \omega$.

(3) For every $f : \omega \to \omega$ *there exists* $X \in \mathcal{U}$ *such that* $f \restriction X$ *is 1-1 or constant.*

(4) For every $l, k \in \omega$ *and every colouring* $c : [\omega]^k \to l$ *there exists* $X \in \mathcal{U}$ *such that* $[X]^k$ *is monochromatic.*

(5) [42]. *For every* $k > 0$, $\mathcal{U}^k$ *is generated by sets* $[X]^k$ *where* $X \in \mathcal{U}$.

Proof. We prove (1)⇒(5)⇒(4)⇒(3)⇒(2)⇒(1).

(1)⇒(5): Suppose $\mathcal{U}$ is weakly selective, and hence selective by Proposition 4.3. We need to show for every $W \in \mathcal{U}^k$ there is $X \in \mathcal{U}$ such that $[X]^k \subseteq W$. We induct on k. When $k = 1$, suppose $W \in \mathcal{U}^1$, i.e. $\mathcal{U}x\,\{x\} \in W$. Equivalently,

$$\exists X \in \mathcal{U}\ \forall x \in X\ \{x\} \in W.$$

Therefore $[X]^1 \subseteq W$ and the statement holds for $k = 1$.

Now suppose $k > 1$ and $W \in \mathcal{U}^k$. For $a \in [\omega]^{k-1}$, let

$$W_a = \{b \in [\omega]^1 : (a < b) \wedge (a \cup b) \in W\}.$$

By Proposition 4.2, $\{a \in [\omega]^{k-1} : W_a \in \mathcal{U}\} \in \mathcal{U}^{k-1}$. Then by the induction hypothesis, there exists $Y \in \mathcal{U}$ such that $[Y]^{k-1} \subseteq \{a \in [\omega]^{k-1} : W_a \in \mathcal{U}\}$. Thus, for all $a \in [Y]^{k-1}$, $W_a \in \mathcal{U}$. We define A_a $(a \in \mathbb{N}^{[<\infty]})$ as follows.

$$A_a = \begin{cases} (W_a \cap Y) \cup a & \text{if } a \in [Y]^{k-1}; \\ Y & \text{otherwise.} \end{cases}$$

Then by selectivity of $\mathcal{U}$, there exists $B \in \mathcal{U}|Y$ such that $B/a \subseteq W_a \cap Y$ for all $a \in [B]^{k-1}$. Let us check $[B]^k \subseteq W$: Suppose $b \in [B]^k$. Let $a = b \setminus \max(b)$. Then $a \in [B]^{k-1}$. So $\max(b) \in B/a \subseteq W_a \cap Y$. By the definition of W_a, $b = a \cup \max(b) \in W$. Thus $[B]^k \subseteq W$.

(5)⇒(4): Let $c : [\omega]^k \to l$ be a given colouring. Then $[\omega]^k = \bigcup_{i<l} c^{-1}\{i\}$. Since $\mathcal{U}^k$ is an ultrafilter on $[\omega]^k$, there exists an $i < l$ such that $c^{-1}\{i\} \in \mathcal{U}^k$. By (5), there exists $X \in \mathcal{U}$ such that $[X]^k \subseteq c^{-1}\{i\}$. Thus $[X]^k$ is monochromatic.

(4)⇒(3): Given $f : \omega \to \omega$, define $c : [\omega]^2 \to 2$ by

$$c(\{x, y\}) = 0 \text{ if and only if } f(x) = f(y).$$

By (4) there exists $X \in \mathcal{U}$ such that $c \restriction [X]^2$ is constant. If $c \restriction [X]^2 \equiv 0$ then $f \restriction X$ is constant; otherwise $c \restriction [X]^2 \equiv 1$ and $f \restriction X$ is 1-1.

(3)⇒(2): Let $\{A_n : n \in \omega\}$ be a partition of ω as in (2). We define $f : \omega \to \omega$ as follows. For $m \in \omega$, let $f(m) = n$ if $m \in A_n$. By (3) there exists $X \in \mathcal{U}$ such that $f \restriction X$ is 1-1 or constant. If there exists n such that $f \restriction X \equiv n$ then $X \in \mathcal{U}$ and $X \subseteq A_n$. This contradicts that $A_n \notin \mathcal{U}$. So f must be 1-1 on X. Therefore, $X \cap A_n$ is a singleton for all $n < \omega$.

(2)⇒(1): Let $A, \{A_n\}_{n<\omega}$ be given as in (1). Let $A_{-1} = A$. By taking finite intersections, we may assume $\{A_n\}_{n<\omega}$ is decreasing. Let $B_n = A_{n-1} \setminus A_n$ for each $n < \omega$. So $A \setminus \bigcup_{i<n} B_n = A_n$. Then

$\{B_n, \omega \setminus A : n < \omega\}$ form a partition of ω such that none of the pieces of the partition is in $\mathcal{U}$. So by (2), there exists $Y \in \mathcal{U}$ such that $|Y \cap B_n| = 1$ for all n. We find an increasing sequence $(y_j)_{j<\omega} \subset Y$ as follows. Let

$$y_0 = \min Y$$
$$y_{j+1} = \min\{y \in Y/y_j : Y/y \cap B_{y_j} = \emptyset\}.$$

Note that for $j \in \omega$, $Y/y_{j+1} \cap \bigcup_{i<y_j} B_i = \emptyset$ by construction. Since $(y_j)_{j<\omega}$ is increasing, the set $\{[0, y_0), [y_j, y_{j+1}) : j < \omega\}$ form a partition of ω. So by (2) again, there exists $X = (x_i)_{i<\omega} \in \mathcal{U}$ such that the intersection of X with each piece of the partition is a singleton. So

$$\forall i < \omega \, \exists j < \omega \quad x_i < y_j \leq x_{i+1}.$$

Since $\mathcal{U}$ is an ultrafilter, either $(x_{2i})_{i<\omega}$ or $(x_{2i+1})_{i<\omega}$ is in $\mathcal{U}$. We relabel the one in $\mathcal{U}$ as $X' = (x'_i)_{i<\omega}$. Thus

$$\forall i < \omega \, \exists j < \omega \quad x'_i < y_j < x'_{i+1}.$$

Now let $B = X' \cap A \cap Y$. So $B \in \mathcal{U}$. We check that B satisfies (1): Suppose $x'_i \in B$. Then $B/x'_i \subseteq X'/x'_i \cap Y/x'_i$. Since $x'_i < y_j < x'_{i+1}$ for some j, $A_{x'_i} \supseteq A_{y_j}$ and $B/x'_i \subseteq Y/y_j \cap A$. But Y/y_j is disjoint from $\bigcup_{i<y_j} B_i$, so

$$B/x'_i \subseteq A \setminus \bigcup_{i<y_j} B_i = A_{y_j} \subseteq A_{x'_i}.$$

Thus $B/n \subseteq A_n$ for every $n \in B$ as required. □

4.2 Preservation of Ultrafilters under Sacks Forcing

In their 1979 paper [4], Baumgartner and Laver proved the following theorem showing that selective ultrafilters on ω are preserved under iterated Sacks forcing.

Theorem 4.6 (Baumgartner-Laver, [4]). *Let $\mathcal{U}$ be a selective ultrafilter on ω. Then for any $\alpha \geq 1$,*

$$\Vdash_\alpha \mathcal{U} \textit{generates a selective ultrafilter,}$$

where $\Vdash_\alpha$ is the α-iterated Sacks forcing relation.

More results on this topic followed. In 1981 Halpern and Pincus [37, Theorem 6] showed that selective ultrafilters on ω in the constructible universe L are preserved after adding finitely many Sacks reals side by side. Then in 1984 Laver [25] constructed a selective ultrafilter on ω that is preserved by $\mathcal{P}_\kappa$ for arbitrary cardinals κ. From Laver's result, Todorcevic immediately observed the more general fact that selective ultrafilters on ω are preserved by $\mathcal{P}_\kappa$ for arbitrary cardinals κ. This observation was first stated and proved in lectures on Forcing given by Todorcevic in 1991, and appeared in [43].

Todorcevic proved using large cardinals that selective ultrafilters are generic over $L(\mathbb{R})$ for the poset $\mathcal{P}(\mathbb{N})/\text{FIN}$ [12], and conjectured that many topological Ramsey spaces have an ultrafilter associated to them analogous to the way selective ultrafilters on ω are related to the Ellentuck space in the 1990s. In the following chapters, we aim to prove such results in various topological Ramsey spaces.

Chapter 5

Milliken Space $\mathrm{FIN}^{[\infty]}$

The Milliken space is another prototype example of topological Ramsey spaces. As in the previous chapter for the Ellentuck space, we discuss the different properties of ultrafilters in the Milliken space. We then localize the Parametrized Milliken Theorem 5.6 ([42]) to selective ultrafilters in the space. The localized theorem then leads us to the preservation of selective ultrafilters in the Milliken space under both side-by-side and iterated Sacks forcing.

In the Milliken space, we often write A/a for the set $\{b \in A : b < a\}$, where $b < a$ if and only if $\max(b) < \min(a)$. Similarly, $A/n = \{b \in A : n < \min(b)\}$, for natural numbers n. We write $\mathrm{FIN}^{[<\infty]}$ for the set of all finite block-sequences, and $\mathrm{FIN}^{[n]}$ for the set of all block-sequences of length n. As in the Ellentuck space, we may write $[X]^{[<\infty]}$ for the set of all finite block-subsequences of X, i.e. $[X]^{[<\infty]} = \mathcal{AR}[\emptyset, X]$; we write $[X]^n$ for the set of all n-block-subsequences of X, i.e. $[X]^n = r_n[\emptyset, X]$.

5.1 Ultrafilters in the Milliken Space

We consider ultrafilters on FIN generated by $[X]$ where $X \in \mathrm{FIN}^{[\infty]}$, or in other words, the *ordered-union* ultrafilters, as defined by Blass. When we write $[X] \in \mathcal{U}$ for some ultrafilter $\mathcal{U}$, we tacitly assume $X \in \mathrm{FIN}^{[\infty]}$.

It can be shown that the notion of *ordered-union ultrafilters on* FIN (Definition 2.17) coincides with that of *ultrafilters in* $\mathrm{FIN}^{[\infty]}$ (Definition 2.18) via the following correspondence: for an ordered-union ultrafilter $\mathcal{U}$ on the base set FIN, let

$$\mathcal{V}_{\mathcal{U}} = \{A \in \mathrm{FIN}^{[\infty]} : [A] \in \mathcal{U}\};$$

for an ultrafilter $\mathcal{V} \subseteq \mathrm{FIN}^{[\infty]}$, let $\mathcal{U}_{\mathcal{V}}$ be the ultrafilter on FIN generated by the set $\{[A] : A \in \mathcal{F}\}$. Thus, although *Nash-Williams*, *Ramsey*, *weak selectivity*, and *selectivity* are properties defined for ultrafilters in $\mathrm{FIN}^{[\infty]}$, it makes sense to discuss these properties for ordered-union ultrafilters on FIN as well.

Definition 5.1. [5]. An ultrafilter $\mathcal{U}$ on FIN is an *ordered-union* ultrafilter if it is generated by elements of the form $[X]$ where $X \in \mathrm{FIN}^{[\infty]}$. Furthermore, we say it is *stable* if for every $\{[X_n] : n < \omega\} \subseteq \mathcal{U}$ there exists $[X] \in \mathcal{U}$ such that for every $n < \omega$ there exists $m < \omega$ such that $X/m \subseteq [X_n]$.

We say an ultrafilter $\mathcal{U}$ has the *Ramsey property* if for every $n < \omega$ and every finite colouring of $\mathrm{FIN}^{[n]}$ there exists $[X] \in \mathcal{U}$ such that $[X]^n$ is monochromatic.

Theorem 5.2 ([5]). *For every ordered-union ultrafilter $\mathcal{U}$, stability is equivalent to the Ramsey property.*

Proposition 5.3. *For an ultrafilter on* FIN *the Ramsey property defined above is equivalent to being Ramsey as defined in Definition 2.20, namely for every $[A] \in \mathcal{U}$, $a \leq A$, and $n < \omega$ and for every finite colouring of $\mathrm{FIN}^{[|a|+n]}$ there exists $B \in [\mathrm{depth}_A(a), A]$ such that $[B] \in \mathcal{U}$ and $r_{|a|+n}[a, B]$ is monochromatic.*

Proof. It is clear from the definitions that Ramsey implies the Ramsey property. Now suppose $\mathcal{U}$ has the Ramsey property. Let A, a, n be given as in the statement, and let $f : \mathrm{FIN}^{[|a|+n]} \to l$ be a given finite colouring. We define $c : \mathrm{FIN}^{[n]} \to l + 1$ as follows.

$$c(b) = \begin{cases} f(a \cup b) & \text{if } a < b; \\ l & \text{otherwise.} \end{cases}$$

Then by the Ramsey property, there exists $[X] \in \mathcal{U}$ such that $c \restriction [X]^n$ is constant. If $c \restriction [X]^n \equiv l$, then $a \not< b$ for every $b \in [X]^n$, which is impossible. So there exists $i < l$ such that $f(a \cup b) = i$ for every $b \in [X]^n$. In particular, $a < \min X$. Let $B = a \cup X$. Then B satisfies the statement. □

Note that for a ordered-union ultrafilter $\mathcal{U}$, if $[X] \in \mathcal{U}$, then $[X/a] \in \mathcal{U}$ for every $a \in \mathrm{FIN}^{[<\infty]}$. We also note that every selective ordered-union ultrafilter in $\mathcal{U}$ is an idempotent with respect to the operation $\cup$ in the following sense.

Lemma 5.4. *Given a selective ordered-union ultrafilter $\mathcal{U}$ on* FIN, *let $\mathcal{U} \cup \mathcal{U} = \{A \subseteq \mathrm{FIN} : \{x \in \mathrm{FIN} : \{y \in \mathrm{FIN} : x \cup y \in A\} \in \mathcal{U}\} \in \mathcal{U}\}$. Then $\mathcal{U} = \mathcal{U} \cup \mathcal{U}$.*

Proof. We prove that $\mathcal{U} \subseteq \mathcal{U} \cup \mathcal{U}$. Equality follows since $\mathcal{U} \cup \mathcal{U}$ is also an ultrafilter. Consider $[B] \in \mathcal{U}$, and let $B = (b_n)_{n<\omega}$. We show that $[B] \in \mathcal{U} \cup \mathcal{U}$. Let $X = \{x : \{y : x \cup y \in [B]\} \in \mathcal{U}\}$. It is sufficient to prove that $X \in \mathcal{U}$. We show this by proving that $[B] \subseteq X$ and using the property that $\mathcal{U}$ is closed under superset.

For an arbitrary $x \in [B]$, let $x = b_{n_0} \cup \cdots \cup b_{n_l}$ with $n_0 < \cdots < n_l$. Then $[B/\{x\}] \subseteq \{y : x \cup y \in [B]\}$. $[B] \in \mathcal{U}$, so $[B/\{x\}] \in \mathcal{U}$, and hence $\{y : x \cup y \in [B]\} \in \mathcal{U}$, that is $x \in X$. As $x \in [B]$ was chosen arbitrary, we have $[B] \subseteq X$ as required. □

5.1.1 Equivalent properties for ordered-union ultrafilters

As in the case of $\mathbb{N}^{[\infty]}$, we show that the properties of stability, Nash-Williams, Ramsey, weak selectivity and selectivity are all equivalent for ordered-union ultrafilters on FIN.

Proposition 5.5. *Let $\mathcal{U}$ be a ordered-union ultrafilter on* FIN. *The following properties are equivalent.*

(1) Nash-Williams,

(2) Ramsey,

(3) weak selectivity,

(4) selectivity,

(5) stability.

Proof. (1)⇒(2) and (2)⇒(3) hold by Theorem 2.24 and Theorem 2.22 as it is straightforward to check that the Milliken space $\mathrm{FIN}^{[\infty]}$ has a head start (Definition 2.23) and satisfies **(A8)** (Definition 2.21). (4)⇒(3) holds by definition. (2)⇔(5) holds by Theorem 5.2 and Proposition 5.3. So we check only (3)⇒(1) and (3)⇒(4).

(3)⇒(1): The proof is very similar to that of Proposition 4.4. Let $\mathcal{G} \subseteq \mathrm{FIN}^{[<\infty]}$ be a Nash-Williams family and $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1$ be a given partition. We find $[X] \in \mathcal{U}$ such that $\mathcal{G}_i \cap [X]^{[<\infty]} = \emptyset$ for some $i \in 2$. We induct on the rank of $\mathcal{G}$. When $\mathrm{rk}(\mathcal{G}) = 0$, $\mathcal{G} = \{\emptyset\}$ is a singleton, so the case is trivial. When $\mathrm{rk}(\mathcal{G}) = 1$, $\mathcal{G} \subseteq \mathrm{FIN}^{[1]}$. The statement follows since $\mathcal{U}$ is an ordered-union ultrafilter on FIN as in the proof of Proposition 4.4.

Now assume $\mathrm{rk}(\mathcal{G}) \geq 2$. For $b \in \mathrm{FIN}$, define

$$\mathcal{G}_b^i = \{a \in \mathrm{FIN}^{[<\infty]} : (b < a) \wedge (\{b\} \cup a \in \mathcal{G}_i)\}.$$

Then $\mathcal{G}_b$ is a Nash-Williams family with a smaller rank. By the induction hypothesis, there exists $[X_b] \in \mathcal{U}$ and $i_b \in 2$ such that $\mathcal{G}_b^{i_b} \cap [X_b]^{[<\infty]} = \emptyset$, i.e.

$$\{b\} \cup a \notin \mathcal{G}_{i_b} \text{ for every } a \in [X_b/b]^{[<\infty]}.$$

So for every $b \in \mathrm{FIN}$ there exists $[X_b] \in \mathcal{U}$ and $i_b \in 2$ such that $\mathcal{G}_b^{i_b}|X_b = \emptyset$. By the base case, there exists $[H] \in \mathcal{U}$ and $i \in 2$ such that $i_b = i$ for every $b \in [H]$. Since $\mathcal{U}$ is weakly selective, there exists $[X'] \in \mathcal{U}$ such that $X'/b \subseteq X_b$ for every $b \in [X']$. Let $[X] \in \mathcal{U}$ be such that $[X] \subseteq [X'] \cap [H]$. Then for every $b \in [X]$ and every $a \in [X/b]^{[<\infty]}$, we have $a \leq X_b'/b$, so $\{b\} \cup a \notin \mathcal{G}_i$. Therefore $a \notin \mathcal{G}_i$ for every $a \in [X]^{[<\infty]}$. Thus $\mathcal{G}_i \cap [X]^{[<\infty]} = \emptyset$ as required.

(3)⇒(4): Let $[A] \in \mathcal{U}$ and $\{[A_a] : a \in \mathrm{FIN}^{[<\infty]}\} \subseteq \mathcal{U}$ be given. We define $\{[X_b] : b \in \mathrm{FIN}\} \subseteq \mathcal{U}$: let $[X_b] \in \mathcal{U}$ be such that $[X_b] \subseteq \bigcap\{[A_a] : \max(a) = b\}$. Note $\max(a) = b$ means that b is the last block in a, so there are only finitely many such a. Then by weak selectivity, there exists $[X] \in \mathcal{U}$ such that $X/b \subseteq X_b$ for every $b \in [X]$. We check that $[X]$ satisfies the theorem. For $a \in [X]^{[<\infty]}$, let $b = \max(a)$. Then $b \in [X]$ and $X/a = X/b \subseteq X_b \subseteq A_a$. □

5.2 Parametrized Milliken Theorem

Instead of trying to prove **(L4)** for the Milliken space, we quote the Parametrized Milliken Theorem from [42], where Todorcevic proved the theorem by coding the product space $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ with the Hales-Jewett space.

Theorem 5.6 (Parametrized Milliken Theorem, [42]). *For every finite Souslin-measurable colouring of the product* $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ *there exists* $B \in \mathrm{FIN}^{[\infty]}$ *and* $p \in \mathcal{P}_\omega$ *such that* $[B]^{[\infty]} \times [p]$ *is monochromatic.*

5.3 Local Parametrized Milliken Theorem

In this section, let us fix a selective ordered-union ultrafilter $\mathcal{U}$ on FIN and show it localizes the Parametrized Milliken Theorem 5.6, i.e. we prove the following theorem.

Theorem 5.7 (Local Parametrized Milliken Theorem). *Let* $\mathcal{U}$ *be a selective ordered-union ultrafilter on* FIN. *For every finite Souslin-measurable colouring of the product* $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ *there exists* $B \in \mathcal{U}$

and $p \in \mathcal{P}_\omega$ such that $[B]^{[\infty]} \times [p]$ is monochromatic.

We first consider open subsets of $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ in subsection 5.3.1, then generalize the result to Souslin-measurable subsets in subsection 5.3.2. For the rest of this section, we fix an arbitrary selective ordered-union ultrafilter $\mathcal{U}$ on FIN.

5.3.1 Open subsets of $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$

Ultra-Ramsey theory plays a role in the proof. So we adapt definitions in [42, Section 7.2] for the space $\mathrm{FIN}^{[<\infty]}$. Consider $\mathrm{FIN}^{[<\infty]}$ as a tree ordered by end-extension $\sqsubseteq$ with root $\emptyset$. From now on in this section, by a *tree* we mean a subtree of $\mathrm{FIN}^{[<\infty]}$ that is *downward closed*, namely, if the subtree contains an element a, then it contains every initial segment $b \sqsubseteq a$.

Notation. For a tree T, we denote the collection of all infinite branches of T by $[T]$, that is,

$$[T] = \left\{ A = (a_i)_{i=1}^{\infty} \in \mathrm{FIN}^{[\infty]} : \{a_1, \dots, a_n\} \in T \text{ for all } n \in \omega \right\}.$$

Let the maximal node of T comparable with every other node of T be denoted by $\mathrm{st}(T)$, and call it the *stem* of T.

Definition 5.8 ([42])**.** A $\mathcal{U}$*-tree* T is a tree such that, for all $t \in T$ with $\mathrm{st}(T) \sqsubseteq t$, $\{x \in \mathrm{FIN} : t \sqsubseteq t \cup \{x\} \in T\} \in \mathcal{U}$. For two $\mathcal{U}$-trees T and T', we say T' is a *pure refinement* of T, and write $T' \leq^0 T$, if $T' \subseteq T$ and $\mathrm{st}(T') = \mathrm{st}(T)$.

Definition 5.9 ([42])**.** A subset $\mathcal{G}$ of $\mathrm{FIN}^{[\infty]}$ is $\mathcal{U}$*-open* if for every $A \in \mathcal{G}$ there is a $\mathcal{U}$-tree T such that $A \in [T]$ and $[T] \subseteq \mathcal{G}$.

The collection of all $\mathcal{U}$-open subsets of $\mathrm{FIN}^{[\infty]}$ is a topology on $\mathrm{FIN}^{[\infty]}$ with basis $[T]$ for $\mathcal{U}$-trees T. This $\mathcal{U}$*-topology* includes the metric topology on $\mathrm{FIN}^{[\infty]}$.

Definition 5.10 ([42])**.** A subset $\mathcal{X}$ of $\mathrm{FIN}^{[\infty]}$ is $\mathcal{U}$*-Ramsey* if for every $\mathcal{U}$-tree T there is a pure refinement $T' \leq^0 T$ such that $[T'] \subseteq \mathcal{X}$ or $[T'] \cap \mathcal{X} = \emptyset$.

The proof of Theorem 7.42 in [42] can be readily adapted to give us the following.

Lemma 5.11. *If $\mathcal{X} \subseteq \mathrm{FIN}^{[\infty]}$ is (metrically) Souslin-measurable, then $\mathcal{X}$ is $\mathcal{U}$-Ramsey.*

Lemma 5.12. *For a $\mathcal{U}$-tree T with stem a, there exists $[X] \in \mathcal{U}$ with $a \sqsubseteq X$ such that $[a, X] \subseteq [T]$. Conversely, for $a \in [X]^{[<\infty]}$ with $[X] \in \mathcal{U}$, there exists a $\mathcal{U}$-tree T with stem a such that $[T] \subseteq [a, X]$.*

Proof. First we assume T is a $\mathcal{U}$-tree with stem a. Since T is a $\mathcal{U}$-tree, for each $b > a$ or $b = \emptyset$ with $a \cup b \in T$ the set $\mathcal{A}_b = \{x \in \mathrm{FIN} : a \cup b \sqsubset a \cup b \cup \{x\} \in T\}$ is in $\mathcal{U}$. Since $\mathcal{U}$ is ordered-union, there exists $[X_b] \in \mathcal{U}$ with $[X_b] \subseteq \mathcal{A}_b$. On the other hand, for each $b \in \mathrm{FIN}^{[<\infty]} \setminus \{b' : a \sqsubseteq a \cup b' \in T\}$, let $[X_b] = \mathrm{FIN}^{[<\infty]}$. Now we have $\{[X_b] : b \in \mathrm{FIN}^{[<\infty]}\} \subseteq \mathcal{U}$. By selectivity of $\mathcal{U}$, there exists an $[X_\infty] \in \mathcal{U}$ such that $X_\infty / b \subseteq X_b$ for every $b \leq X_\infty$.

Let $X = a \cup (X_\infty / a)$. Then $[X] \in \mathcal{U}$. We check that $[a, X] \subseteq [T]$. Let $Y = a \cup (y_n)_{n<\omega} \in [a, X]$. We show $a \cup \{y_1, \dots, y_n\} \in T$ for every $n \in \omega$ by induction. For $n = 0$, it is a, the stem of T. Suppose $n = k + 1$. By the induction hypothesis, $a \cup \{y_1, \dots, y_k\} \in T$. Then $y_{k+1} \in [X/(a \cup \{y_1, \dots, y_k\})] = [X_\infty / \{y_1, \dots, y_k\}] \subseteq [X_{\{y_1,\dots,y_k\}}] \subseteq \mathcal{A}_{\{y_1,\dots,y_k\}}$. So by the definition of $\mathcal{A}_{\{y_1,\dots,y_k\}}$, $a \cup \{y_1, \dots, y_k, y_{k+1}\} \in T$. Thus, if $Y \in [a, X]$ then $Y \in [T]$. Hence $[a, X] \subseteq [T]$ as required.

Conversely, let $[X] \in \mathcal{U}$ and $a \in [X]^{[<\infty]}$. The set $T := \{b \in \mathrm{FIN}^{[<\infty]} : b \sqsubseteq a\} \cup \{b \in \mathrm{FIN}^{[<\infty]} : a \sqsubseteq b \leq X\}$ is a $\mathcal{U}$-tree with stem a. This is because, for every $b \in T$ with $a \sqsubseteq b$, $\{x \in \mathrm{FIN} : b \sqsubseteq b \cup \{x\} \in T\} = [X/b] \in \mathcal{U}$. □

Definition 5.13. For a $\mathcal{U}$-tree T and $n \in \omega$, let $l_n(T)$ be the n^{th} level of the tree T above its stem:

$$l_n(T) = \{b \in T : |b| = |\mathrm{st}(T)| + n\}.$$

For $[X] \in \mathcal{U}$ and $a \leq X$, let $l_n[a, X] = \{b \in \mathrm{FIN}^{[|a|+n]} : a \sqsubseteq b \leq X\}$.

Theorem 5.14. *For every (metrically) Souslin-measurable subset $\mathcal{X}$ of $\mathrm{FIN}^{[\infty]}$, there exists $[X] \in \mathcal{U}$ such that $[\emptyset, X] \subseteq \mathcal{X}$ or $[\emptyset, X] \cap \mathcal{X} = \emptyset$.*

Proof. By Lemma 5.11, $\mathcal{X}$ is $\mathcal{U}$-Ramsey. So for the $\mathcal{U}$-tree $\mathrm{FIN}^{[<\infty]}$, there exists a pure refinement $T \leq^0 \mathrm{FIN}^{[<\infty]}$ such that $[T] \subseteq \mathcal{X}$ or $[T] \cap \mathcal{X} = \emptyset$. By Lemma 5.12, there exists $[X] \in \mathcal{U}$ with $[\emptyset, X] \subseteq [T]$. □

Now we aim to prove that for an open set $\mathcal{O}$ of $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ in the product topology, the set $\{X \in \mathrm{FIN}^{[\infty]} : \exists p \in \mathcal{P}_\omega\ ([\emptyset, X] \times [p] \subseteq \mathcal{O})$ or $([\emptyset, X] \times [p] \cap \mathcal{O} = \emptyset)\}$ is Souslin-measurable, after which we could apply Theorem 5.14. For this, we define uniform families in $\mathrm{FIN}^{[\infty]}$. It is analogous to Pudlák and Rödl's definition of uniform family in the Ellentuck space in [38].

Notation. For $\mathcal{S} \subseteq \mathrm{FIN}^{[<\infty]}$ and $a \in \mathrm{FIN}^{[<\infty]}$, let $\mathcal{S}_{[a]} = \{y : a \sqsubseteq y \in \mathcal{S}\}$.

Definition 5.15 ([38])**.** Let γ be a countable ordinal and $\mathcal{S} \subseteq \mathrm{FIN}^{[<\infty]}$. Let $X \in \mathrm{FIN}^{[\infty]}$ and $b \leq X$. The notion of γ-uniform is defined recursively as follows. The family $\mathcal{S}$ is γ-*uniform* on $[b, X]$ if

- $\gamma = 0$ and $\mathcal{S} = \{b\}$;
- $\gamma = \beta + 1$, $b \notin \mathcal{S}$ and for each $a \in l_1[b, X]$, $\mathcal{S}_{[a]}$ is β-uniform on $[a, X]$; or
- γ is a limit ordinal, $b \notin \mathcal{S}$ and there exists a sequence $(\gamma_a)_{a \in l_1[b,X]}$ of ordinals with $\bigcup_{a \in l_1[b,X]} \gamma_a = \gamma$ such that for each $a \in l_1[b, X]$, $\mathcal{S}_{[a]}$ is γ_a-uniform on $[a, X]$.

We say $\mathcal{S}$ is *uniform* on $[b, X]$ if it is γ-uniform on $[b, X]$ for some γ.

For example, for $n < \omega$, the only n-uniform family on $[b, X]$ is $l_n[b, X]$. On the other hand, for each $k < \omega$, the family $\mathcal{S} = \{y = (y_n) \in \mathrm{FIN}^{[<\infty]} : (b \sqsubseteq y \leq X) \wedge (|y| = \max(y_{|b|+1}) + k)\}$ is ω-uniform on $[b, X]$, and $\mathcal{T} = \{y = (y_n) \in \mathrm{FIN}^{[<\infty]} : (b \sqsubseteq y \leq X) \wedge (|y| = \max(y_{|b|+2}) + k)\}$ is $(\omega+1)$-uniform on $[b, X]$.

Recall from [42, Def. 1.12] that a family $\mathcal{F}$ of finite block-sequences is *Nash-Williams* if $x \not\sqsubseteq y$ for every distinct pair $x, y \in \mathcal{F}$. The family $\mathcal{F}$ is a *front* on $[b, X]$ if it is Nash-Williams and every element of $[b, X]$ has an initial segment in $\mathcal{F}$. The definition of rank of a barrier in [42, Def. 1.24] can be used for fronts as well.

Definition 5.16 ([42])**.** Let $\mathcal{F}$ be a front on $[b, X]$. The set $T(\mathcal{F}) = \{x : \exists y \in \mathcal{F}\ x \sqsubseteq y\}$ is the $\sqsubseteq$-downward closure of $\mathcal{F}$. Let $\rho_{\mathcal{F}}(x) = \sup\{\rho_{\mathcal{F}}(s) + 1 : (x \sqsubset s) \wedge (s \in T(\mathcal{F}))\}$, where $\sup \emptyset$ is defined to be 0. Then the rank of $\mathcal{F}$ on $[b, X]$ is $\rho(\mathcal{F}) = \rho_{\mathcal{F}}(b)$.

Lemma 5.17. *If $\mathcal{S}$ is uniform on $[b, X]$, then it is a front on $[b, X]$.*

Proof. Let $\mathcal{S}$ be γ-uniform on $[b, X]$. We prove by induction on γ. If $\gamma = 0$, then $\mathcal{S} = \{b\}$ and the conclusion holds. So we assume $\gamma > 0$. By the definition of γ-uniform, for each $a \in l_1[b, X]$, $\mathcal{S}_{[a]}$ is β-uniform for some $\beta < \gamma$. So by the induction hypothesis, $\mathcal{S}_{[a]}$ is a front on $[a, X]$. Then $\mathcal{S} = \bigcup_{a \in l_1[b,X]} \mathcal{S}_{[a]}$ is a front on $[b, X]$. □

Lemma 5.18. *If $\mathcal{F}$ is a front on $[b, X]$, then there exists a uniform family $\mathcal{S}$ on $[b, X]$ such that every $s \in \mathcal{S}$ has an initial segment in $\mathcal{F}$.*

Proof. We induct on the rank of $\mathcal{F}$. If $\rho(\mathcal{F}) = 0$ then $\mathcal{F} = \{b\}$ itself is a uniform family on $[b, X]$. So we assume $\rho(\mathcal{F}) > 0$. For each $a \in l_1[b, X]$, the family $\mathcal{F}_{[a]} = \{s : a \sqsubseteq s \in \mathcal{F}\}$ is a front on $[a, X]$ of a smaller rank. By the induction hypothesis, there exists a countable ordinal γ_a and a γ_a-uniform family $\mathcal{S}_a$ on $[a, X]$ such that every element in $\mathcal{S}_a$ has an initial segment in $\mathcal{F}_{[a]}$. Then the family $\mathcal{S} = \bigcup_{a \in l_1[b,X]} \mathcal{S}_a$ is $(\bigcup_{a \in l_1[b,X]} \gamma_a)$-uniform on $[b, X]$ and every element in $\mathcal{S}$ has an initial segment in $\mathcal{F}$. □

Theorem 5.19. *For every open set $\mathcal{O} \subseteq \mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ there exist $[X] \in \mathcal{U}$ and $p \in \mathcal{P}_\omega$ such that $[\emptyset, X] \times [p] \subseteq \mathcal{O}$ or $([\emptyset, X] \times [p]) \cap \mathcal{O} = \emptyset$.*

Proof. Let

$$\mathcal{X}_0 = \{X \in \mathrm{FIN}^{[\infty]} : (\exists p \in \mathcal{P}_\omega)\,(([\emptyset, X] \times [p]) \cap \mathcal{O} = \emptyset)\}, \text{ and}$$
$$\mathcal{X}_1 = \{X \in \mathrm{FIN}^{[\infty]} : (\exists p \in \mathcal{P}_\omega)\,([\emptyset, X] \times [p] \subseteq \mathcal{O})\}.$$

Let $\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_1$. We claim that $\mathcal{X}_0$ and $\mathcal{X}_1$ are Souslin-measurable, hence so is $\mathcal{X}$. Then applying Theorem 5.14 to the set $\mathcal{X} \subseteq \mathrm{FIN}^{[\infty]}$, we get $[X] \in \mathcal{U}$ such that $[\emptyset, X] \subseteq \mathcal{X}$ or $[\emptyset, X] \cap \mathcal{X} = \emptyset$. If $[\emptyset, X] \subseteq \mathcal{X}$ then $X \in \mathcal{X}$, so, by the definition of $\mathcal{X}$, there exists $p \in \mathcal{P}_\omega$ such that X and p satisfies the theorem. Otherwise, $[\emptyset, X] \cap \mathcal{X} = \emptyset$, and we would have the following contradiction. The set $\mathcal{O} \cap ([\emptyset, X] \times (2^\omega)^\omega)$ is open in $[\emptyset, X] \times (2^\omega)^\omega$ in the subspace topology. By the Parametrized Milliken Theorem 5.6, there exist $Y \in [\emptyset, X]$ and $q \in \mathcal{P}_\omega$ such that $[\emptyset, Y] \times [p] \subseteq \mathcal{O}$ or $([\emptyset, Y] \times [p]) \cap \mathcal{O} = \emptyset$. Then $Y \in \mathcal{X}$, contradicting $Y \in [\emptyset, X]$ and $[\emptyset, X] \cap \mathcal{X} = \emptyset$.

We first prove that $\mathcal{X}_0$ is Souslin-measurable. Since $\mathcal{O}$ is open, for each element (X, ϵ) of $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$, $(X, \epsilon) \in \mathcal{O}$ if and only if there exists $a \sqsubseteq X$ and a pre-initial segment σ of ϵ such that $[a] \times [\sigma] \subseteq \mathcal{O}$. So $\mathcal{X}_0$ can be written as

$$\mathcal{X}_0 = \{X \in \mathrm{FIN}^{[\infty]} : (\exists p \in \mathcal{P}_\omega)(\forall a \leq X)(\forall F \subseteq \omega \text{ finite})(\forall \sigma \in \prod_{i \in F} p(i))[a] \times [\sigma] \nsubseteq \mathcal{O}\}.$$

Therefore, $\mathcal{X}_0$ is analytic, hence Souslin-measurable.

Lastly, we show that $\mathcal{X}_1$ is also analytic. Let $Y \in \mathrm{FIN}^{[\infty]}$ and $p \in \mathcal{P}_\omega$. Assume $\{Y\} \times [p] \subseteq \mathcal{O}$. As above, this implies that for every $\epsilon \in [p]$ there exists $a_\epsilon \sqsubseteq Y$ and a pre-initial segment σ_ϵ of ϵ such that $[a_\epsilon] \times [\sigma_\epsilon] \subseteq \mathcal{O}$. Thus $\{[\sigma_\epsilon] : \epsilon \in [p]\}$ is an open cover of $[p]$. Since $(2^\omega)^\omega$ is compact, the closed subset $[p]$ is also compact. So $[p]$ has a finite subcover $\{[\tau_1], \ldots, [\tau_k]\} \subseteq \{[\sigma_\epsilon] : \epsilon \in [p]\}$. Correspondingly, there are $a_1, \ldots, a_k \sqsubseteq Y$ such that $[a_i] \times [\tau_i] \subseteq \mathcal{O}$ for every $i \leq k$. We can find $a \sqsubseteq Y$ such that $a_i \sqsubseteq a$ for every $i \leq k$. So $[a] \times [p] \subseteq \mathcal{O}$. Thus we have proved that if $\{Y\} \times [p] \subseteq \mathcal{O}$ then Y has an initial segment a such that $[a] \times [p] \subseteq \mathcal{O}$. Note that the converse of this also holds. Therefore, for $X \in \mathrm{FIN}^{[\infty]}$ and $p \in \mathcal{P}_\omega$, $[\emptyset, X] \times [p] \subseteq \mathcal{O}$ if and only if every $Y \leq X$ has an initial segment in the set $\mathcal{F} = \{a \leq X : [a] \times [p] \subseteq \mathcal{O}\}$. By shrinking $\mathcal{F}$ to a front $\mathcal{F}'$ on $[\emptyset, X]$ and using Lemma 5.17 and Lemma 5.18, we have the following.

For $X \in \mathrm{FIN}^{[\infty]}$, $X \in \mathcal{X}_1$ if and only if

$$(\exists p \in \mathcal{P}_\omega)(\exists \mathcal{S} \subseteq \mathrm{FIN}^{[<\infty]})((\mathcal{S} \text{ is uniform on } [\emptyset, X]) \wedge ((\forall a \in \mathcal{S})(\exists b \in \mathcal{F}')(b \sqsubseteq a))),$$

if and only if

$$\begin{aligned}&(\exists p \in \mathcal{P}_\omega)(\exists \mathcal{S} \subseteq \mathrm{FIN}^{[<\infty]})\\ &\quad((\mathcal{S} \text{ is uniform on } [\emptyset, X]) \wedge ((\forall a \in \mathcal{S})(\exists b \leq X)((b \sqsubseteq a) \wedge ([b] \times [p] \subseteq \mathcal{O}))),\end{aligned}$$

Thus $\mathcal{X}_1$ is also analytic. □

5.3.2 Perfectly $\mathcal{U}$-Ramsey sets

Recall that we have fixed an arbitrary selective ultrafilter $\mathcal{U}$ on FIN throughout this Section 5.3. Unless otherwise stated, we are working with the metric topology on $\mathrm{FIN}^{[\infty]}$, and the product topology on $(2^\omega)^\omega$ and $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$. In this subsection, we generalizes Theorem 5.19 to all Souslin-measurable subsets of $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$, thus proving that every selective ultrafilter on FIN is localizing.

For a $\mathcal{U}$-tree T and $x \in T$, let $T|x = \{y \in T : y \sqsubseteq x \text{ or } x \sqsubseteq y\}$. Recall that $l_n(T)$ denotes the n^{th} level of T above its stem. Recall also that by $T' \leq^0 T$ we mean $T' \subseteq T$ and $\mathrm{st}(T') = \mathrm{st}(T)$. This notion is extended as follows.

Definition 5.20 ([42]). For two $\mathcal{U}$-trees T' and T, and $n \in \omega$, we write $T' \leq^n T$ if $T' \subseteq T$ and $l_n(T') = l_n(T)$.

Definition 5.21 ([42]). A sequence $(T_n)_{n<\omega}$ of $\mathcal{U}$-trees is a *fusion sequence* if $T_{n+1} \leq^n T_n$ for every $n \in \omega$. In this case, the set $T_\infty = \bigcap_{n<\omega} T_n$ is called the *fusion* of the sequence. Note that T_∞ is also a $\mathcal{U}$-tree.

Definition 5.22. A subset $\mathcal{B}$ of $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ is *perfectly $\mathcal{U}$-Ramsey* if for every $\mathcal{U}$-tree T and every $p \in \mathcal{P}_\omega$ there exist $T' \leq^0 T$ and $p' \leq p$ such that $[T'] \times [p'] \subseteq \mathcal{B}$ or $([T'] \times [p']) \cap \mathcal{B} = \emptyset$. We say $\mathcal{B}$ is *perfectly $\mathcal{U}$-Ramsey null* if the second alternative always holds.

By Lemma 5.12, the following lemma follows from Theorem 5.19.

Lemma 5.23. *Every open subset of $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ is perfectly $\mathcal{U}$-Ramsey.*

Now we aim to show that the collection of perfectly $\mathcal{U}$-Ramsey sets is closed under the Souslin operation, through a series of lemmas. The following lemma is immediate from the definition of perfectly $\mathcal{U}$-Ramsey.

Lemma 5.24. *A subset $\mathcal{B}$ of $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ is perfectly $\mathcal{U}$-Ramsey if and only if for every $\mathcal{U}$-tree T, $p \in \mathcal{P}_\omega$, and $n < \omega$, there exist $T' \leq^0 T$ and $p' \leq^n p$ such that*

$$\forall \sigma \in l(n,p) \quad [T'] \times [p'|\sigma] \subseteq \mathcal{B} \text{ or } ([T'] \times [p'|\sigma]) \cap \mathcal{B} = \emptyset.$$

Lemma 5.25. *The set of all perfectly $\mathcal{U}$-Ramsey null subsets of $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ is a σ-ideal.*

Proof. Closure under subsets and finite union is clear. We show that it is closed under countable union. By Lemma 5.24 we notice that a set $\mathcal{B}$ is perfectly $\mathcal{U}$-Ramsey null if and only if for every n, T, p given there exist $T' \leq^0 T$ and $p' \leq^n p$ such that $([T'] \times [p']) \cap \mathcal{B} = \emptyset$. Now suppose $\mathcal{Y}_k$ is perfectly $\mathcal{U}$-Ramsey null for every $k < \omega$. We prove that so is $\bigcup_{k<\omega} \mathcal{Y}_k$. Given T, p we construct fusion sequences $(T_k)_{k<\omega}$ and $(p_k)_{k<\omega}$ recursively.

Let $T_0 = T$ and $p_0 = p$. At stage $k+1$, enumerate $l_k(T_k)$ as $(x_j)_{j<\omega}$. Let $q_{-1} = p_k$. For each $j \in \omega$, as $\mathcal{Y}_k$ is perfectly $\mathcal{U}$-Ramsey null, there exist $S_j \leq^0 T_k|x_j$ and $q_j \leq^{k+j} q_{j-1}$ such that $([S_j] \times [q_j]) \cap \mathcal{Y}_k = \emptyset$. Thus $(q_j)_{j<\omega}$ is a fusion sequence. Then let $T_{k+1} = \bigcup_{j\in\omega} S_j$ and $p_{k+1} = \bigcap_{j<\omega} q_j$.

By construction, T_{k+1} is a $\mathcal{U}$-tree, $p_{k+1} \in \mathcal{P}_\omega$, $T_{k+1} \leq^k T_k$, $p_{k+1} \leq^k p_k$ and $([T_{k+1}] \times [p_{k+1}]) \cap \mathcal{Y}_k = \emptyset$. We can take the fusions $T_\infty = \bigcap_{k<\omega} T_k$ and $p_\infty = \bigcap_{k<\omega} p_k$. Then $[T_\infty] \times [p_\infty]$ is disjoint from $\mathcal{Y}_k$ for each $k < \omega$, so it is disjoint from $\bigcup_{k<\omega} \mathcal{Y}_k$. So $\bigcup_{k<\omega} \mathcal{Y}_k$ is perfectly $\mathcal{U}$-Ramsey null. □

Lemma 5.26. *The set of all perfectly $\mathcal{U}$-Ramsey subsets of $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ is a σ-field.*

Proof. Similarly we only show that it is closed under countable union. Suppose $\mathcal{Y}_k$ is perfectly $\mathcal{U}$-Ramsey for every $k < \omega$. We check that so is $\mathcal{Y} = \bigcup_{k<\omega} \mathcal{Y}_k$. Given T, p we construct fusion sequences $(T_k)_{k<\omega}$ and $(p_k)_{k<\omega}$ recursively.

Let $T_0 = T$ and $p_0 = p$. At stage $k+1$, let $l_k(T_k) = (x_j)_{j<\omega}$. Let $q_{-1} = p_k$. For each $j \in \omega$, by Lemma 5.24, there exist $S_j \leq^0 T_k|x_j$ and $q_j \leq^{k+j} q_{j-1}$ such that for every $\sigma \in l(k+j, q_j)$ we have $[S_j] \times [q_j|\sigma] \subseteq \mathcal{Y}_k$ or $([S_j] \times [q_j|\sigma]) \cap \mathcal{Y}_k = \emptyset$. Let $T_{k+1} = \bigcup_{j<\omega} S_j$ and $p_{k+1} = \bigcap_{j<\omega} q_j$. Then T_{k+1} is a $\mathcal{U}$-tree with $T_{k+1} \leq^k T_k$ and $p_{k+1} \in \mathcal{P}_\omega$.

Let $T_\infty = \bigcap_{k<\omega} T_k$ and $p_\infty = \bigcap_{k<\omega} p_k$. Then $T_\infty \leq^0 T$, $p_\infty \leq p$ and for every $k < \omega$ and $x \in l_k(T_\infty)$,

$$\forall \sigma \in l(k+1, p_\infty) \quad [T_\infty|x] \times [p_\infty|\sigma] \subseteq \mathcal{Y}_k \text{ or } ([T_\infty|x] \times [p_\infty|\sigma]) \cap \mathcal{Y}_k = \emptyset.$$

Thus $\mathcal{Y} \cap ([T_\infty] \times [p_\infty])$ is open in $[T_\infty] \times [p_\infty]$ with the subspace topology. Then by Lemma 5.23, there exist $T' \leq^0 T_\infty$ and $p' \leq p_\infty$ such that $[T'] \times [p'] \subseteq \mathcal{Y}$ or $([T'] \times [p']) \cap \mathcal{Y} = \emptyset$. In particular, $T' \leq^0 T$ and $p' \leq p$ as required. Therefore $\mathcal{Y} = \bigcup_{k<\omega} \mathcal{Y}_k$ is perfectly $\mathcal{U}$-Ramsey. □

We use combinatorial forcing ([35, 21]) to prove that the field of perfectly $\mathcal{U}$-Ramsey subsets of $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ is closed under the Souslin operation. Recall that by $\sigma \in 2^{(<\omega)\times(<\omega)}$ we mean a function $\sigma : F \to 2^{<\omega}$ where $F \subseteq \omega$ is finite.

Definition 5.27. Consider an arbitrary $\mathcal{U}$-tree T, $p \in \mathcal{P}_\omega$, $\mathcal{X} \subseteq \mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ and $(a, \sigma) \in \mathrm{FIN}^{[<\infty]} \times 2^{(<\omega)\times(<\omega)}$. We say (T, p) *$\mathcal{X}$-accepts* (a, σ) if $[T|a] \times [p|\sigma] \subseteq \mathcal{X}$; (T, p) *$\mathcal{X}$-rejects* (a, σ) if $[T|a] \times [p|\sigma] \neq \emptyset$ and there does not exist $T' \leq^0 T|a$ and $p' \leq p|\sigma$ such that (T', p') $\mathcal{X}$-accepts (a, σ); (T, p) *$\mathcal{X}$-decides* (a, σ) if it either $\mathcal{X}$-accepts or $\mathcal{X}$-rejects (a, σ).

The following lemma is immediate from the definition.

Lemma 5.28. *Consider an arbitrary $\mathcal{U}$-tree T, $p \in \mathcal{P}_\omega$, $\mathcal{X} \subseteq \mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ and $(a, \sigma) \in \mathrm{FIN}^{[<\infty]} \times 2^{(<\omega)\times(<\omega)}$.*

(1) *(T, p) $\mathcal{X}$-accepts every (a, σ) such that $[T|a] \times [p|\sigma] = \emptyset$.*

(2) *If (T, p) $\mathcal{X}$-decides (a, σ) and $[T'] \times [p'] \subseteq [T] \times [p]$, then (T', p') $\mathcal{X}$-decides (a, σ).*

(3) *If $a \in T$ and $\sigma(n) \in p(n)$ for every $n \in \mathrm{domain}(\sigma)$, then there exist $T' \leq^0 T|a$ and $p' \leq p|\sigma$ such that (T', p') $\mathcal{X}$-decides (a, σ).*

Lemma 5.29. *Given a Souslin scheme $\langle \mathcal{X}_s : s \in \omega^{<\omega} \rangle$ of subsets of $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$, a $\mathcal{U}$-tree T and $p \in \mathcal{P}$, there exist $T_\infty \leq^0 T$, $p_\infty \leq p$ and an enumeration $(b_n)_{n<\omega}$ of $\{b \in T_\infty : \mathrm{st}(T_\infty) \sqsubseteq b\}$ such that for every $s \in \omega^{<\omega}$ and every $k \geq \max(s)$, if $\tau \in l(k, p_\infty)$, then (T_∞, p_∞) $\mathcal{X}_s$-decides (b_k, τ). Without loss of generality, we restrict $\omega^{<\omega}$ to the set of all finite strictly increasing sequences.*

Proof. Let $\{b \in T : \mathrm{st}(T) \sqsubseteq b\}$ be enumerated as $(b_n)_{n<\omega}$ such that $b_m \not\sqsubseteq b_n$ when $n < m$. In particular, $b_0 = \mathrm{st}(T)$. We recursively shrink T and p "cone by cone", starting from $T_{-1} = T$ and $p_{-1} = p$. Let $l(k+1, p_k) \times \{s \in \omega^{<\omega} : \max(s) \leq k+1\}$ be enumerated as $(\tau_l, s_l)_{l \leq m}$ where m is a finite number. Let $q_{-1} = p_k$. By Lemma 5.28 (3), for every $l \leq m$ there exists $S_l \leq^0 T_k|b_{k+1}$ and $q_l \leq^{k+1} q_{l-1}$ with $q_l|\tau_l \leq q_{l-1}|\tau_l$ such that (S_l, q_l) $\mathcal{X}_{s_l}$-decides (b_{k+1}, τ_l). Let $p_{k+1} = q_m$, then $p_{k+1} \leq^{k+1} p_k$ and $p_{k+1}|\tau_l \leq q_l|\tau_l$ for every $l \leq m$. Let $T_{k+1} = (T_k \setminus T_k|b_{k+1}) \cup \bigcap_{l \leq m} S_l$. Note that T_{k+1} is a $\mathcal{U}$-tree. Then (T_{k+1}, p_{k+1}) $\mathcal{X}_s$-decides (b_{k+1}, τ) for every $\tau \in l(F_{k+1}, k+1, p_k)$ and every $s \in \omega^{<\omega}$ with $\max(s) \leq k+1$. Re-enumerate $T_{k+1} \setminus \{b_0, \ldots, b_{k+1}\}$ as $(b_{k+1+n})_{n<\omega}$ such that $b_m \not\sqsubseteq b_n$ when $n < m$.

Let $T_\infty = \bigcap_{k<\omega} T_k$ and $p_\infty = \bigcap_{k<\omega} p_k$. Since $p_{k+1} \leq^{k+1} p_k$ by construction, $(p_k)_{k<\omega}$ is a fusion sequence and $p_\infty \in \mathcal{P}_\omega$. We check that T_∞ is a $\mathcal{U}$-tree. Let $b \in T_\infty$. There exists n such that b is enumerated as b_{n+1} since stage n. Then, as T_{n+1} is a $\mathcal{U}$-tree, $\{x \in \mathrm{FIN} : b \sqsubseteq b \cup \{x\} \in T_\infty\} = \{x \in \mathrm{FIN} : b \sqsubseteq b \cup \{x\} \in T_{n+1}\} \in \mathcal{U}$. Then T_∞ and p_∞ satisfy the theorem by Lemma 5.28 (2). ☐

Theorem 5.30. *The field of perfectly $\mathcal{U}$-Ramsey subsets of $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ is closed under the Souslin operation.*

Proof. Let $\langle \mathcal{Y}_s : s \in \omega^{<\omega} \rangle$ be a Souslin scheme of perfectly $\mathcal{U}$-Ramsey sets. Assume without loss of generality that $\mathcal{Y}_t \subseteq \mathcal{Y}_s$ whenever $s \sqsubseteq t$. Let $\mathcal{Y} = \bigcup_{f \in \omega^\omega} \bigcap_{n<\omega} \mathcal{Y}_{f\restriction n}$. For a set $\mathcal{X} \subseteq \mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$, let $\mathcal{X}^c$ denote its complement $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega \setminus \mathcal{X}$. Given T, p, we aim to show that there exist $T' \leq^0 T$ and $p' \leq p$ such that $[T'] \times [p'] \subseteq \mathcal{Y}$ or $\mathcal{Y}^c$.

Let $\mathcal{Y}^*_s = \bigcup_{f \sqsupset s} \bigcap_{n<\omega} \mathcal{Y}_{f\restriction n}$. Note that $\mathcal{Y}^*_s \subseteq \mathcal{Y}_s$ and $\mathcal{Y}^*_\emptyset = \mathcal{Y}$. By Lemma 5.29, we can find $T_\infty \leq^0 T$ and $p_\infty \leq p$ where the elements in T_∞ above the stem are enumerated as $(b_n)_{n<\omega}$ with $b_m \not\sqsubseteq b_n$ when $n < m$ such that for every $s \in \omega^{<\omega}$ and every $k \geq \max(s)$, if $\tau \in l(k, p_\infty)$, then (T_∞, p_∞) $(\mathcal{Y}^*_s)^c$-decides (b_k, τ). For each $s \in \omega^{<\omega}$, let $\mathcal{O}_s = \{(b, \tau) : [T_\infty|b] \times [p_\infty|\tau] \subseteq (\mathcal{Y}^*_s)^c\}$. We define an envelope for each $\mathcal{Y}^*_s$ by

$$\Phi(\mathcal{Y}^*_s) = ([T_\infty] \times [p_\infty] \cap \mathcal{Y}_s) \setminus \bigcup_{(b,\tau) \in \mathcal{O}_s} [T_\infty|b] \times [p_\infty|\tau].$$

Since $[T_\infty] \times [p_\infty]$, $\mathcal{Y}_s$ and $[T_\infty|b] \times [p_\infty|\tau]$ are perfectly $\mathcal{U}$-Ramsey, $\Phi(\mathcal{Y}^*_s)$ is also perfectly $\mathcal{U}$-Ramsey by Lemma 5.26. Clearly, $[T_\infty] \times [p_\infty] \cap \mathcal{Y}^*_s \subseteq \Phi(\mathcal{Y}^*_s) \subseteq [T_\infty] \times [p_\infty] \cap \mathcal{Y}_s$.

We say a set $\mathcal{B}$ is *perfectly $\mathcal{U}$-Ramsey null inside* $[T_\infty] \times [p_\infty]$ if for every $\mathcal{U}$-tree S and $q \in \mathcal{P}_\omega$ with $[S] \times [q] \subseteq [T_\infty] \times [p_\infty]$ there exist $S' \leq^0 S$ and $q' \leq q$ such that $[S'] \times [q'] \subseteq \mathcal{B}^c$.

Claim 5.30.1. *For $s \in \omega^{<\omega}$, if $\mathcal{B} \subseteq \Phi(\mathcal{Y}^*_s) \setminus \mathcal{Y}^*_s$ is perfectly $\mathcal{U}$-Ramsey, then $\mathcal{B}$ is perfectly $\mathcal{U}$-Ramsey null inside $[T_\infty] \times [p_\infty]$.*

Suppose $\mathcal{B} \subseteq \Phi(\mathcal{Y}^*_s) \setminus \mathcal{Y}^*_s$ is perfectly $\mathcal{U}$-Ramsey. Given $[S] \times [q] \subseteq [T_\infty] \times [p_\infty]$, we can find $S' \leq^0 S$ and $q' \leq q$ such that $[S'] \times [q'] \subseteq \mathcal{B}$ or $\mathcal{B}^c$. Assuming $[S'] \times [q'] \subseteq \mathcal{B}$, we aim for a contradiction. Since $[S'] \times [q'] \subseteq [T_\infty] \times [p_\infty]$, we can pick $k \geq \max(s)$, $b_k \in S'$ and $\tau_k \in l(k, p_\infty)$ with $\tau_k(j) \in q'(j)$ for all $j \in \mathrm{domain}(\tau_k)$. Since we chose T_∞ and p_∞ to satisfy the conclusion of Lemma 5.29, (T_∞, p_∞) $(\mathcal{Y}^*_s)^c$-decides (b_k, τ_k). By Lemma 5.28(2), (S', q') also $(\mathcal{Y}^*_s)^c$-decides (b_k, τ_k). But $[S'] \times [q'] \subseteq \mathcal{B}$ and $\mathcal{B}$

is disjoint from $\mathcal{Y}_s^*$, we must have that (S', q') $(\mathcal{Y}_s^*)^c$-accepts (b_k, τ_k). Thus (T_∞, p_∞) must also $(\mathcal{Y}_s^*)^c$-accept (b_k, τ_k). Hence $(b_k, \tau_k) \in \mathcal{O}_s$ and $[S'|b_k] \times [q'|\tau_k] \subseteq \bigcup_{(b,\tau)\in\mathcal{O}_s} [T_\infty|b] \times [p_\infty|\tau]$. This contradicts the assumption that $[S'] \times [q'] \subseteq \mathcal{B}$ which is disjoint from $\bigcup_{(b,\tau)\in\mathcal{O}_s} [T_\infty|b] \times [p_\infty|\tau]$. Therefore $[S'] \times [q'] \subseteq \mathcal{B}^c$. This finishes the proof of Claim 5.30.1.

For $s \in \omega^{<\omega}$, let $\mathcal{M}_s = \Phi(\mathcal{Y}_s^*) \setminus \bigcup_{n<\omega} \Phi(\mathcal{Y}_{s^\frown n}^*)$. Note that $\mathcal{Y}_s^* = \bigcup_{n<\omega} \mathcal{Y}_{s^\frown n}^*$ and $\Phi(\mathcal{Y}_{s^\frown n}^*) \supseteq [T_\infty] \times [p_\infty] \cap \mathcal{Y}_{s^\frown n}^*$. So $[T_\infty] \times [p_\infty] \cap \mathcal{Y}_s^* \subseteq \bigcup_{n<\omega} \Phi(\mathcal{Y}_{s^\frown n}^*)$. Hence $\mathcal{M}_s \subseteq \Phi(\mathcal{Y}_s^*) \setminus ([T_\infty] \times [p_\infty] \cap \mathcal{Y}_s^*) = \Phi(\mathcal{Y}_s^*) \setminus \mathcal{Y}_s^*$ since $\Phi(\mathcal{Y}_s^*) \subseteq [T_\infty] \times [p_\infty]$. As $\mathcal{M}_s$ is perfectly $\mathcal{U}$-Ramsey, by Claim 5.30.1 $\mathcal{M}_s$ is perfectly $\mathcal{U}$-Ramsey null inside $[T_\infty] \times [p_\infty]$. The set of all perfectly $\mathcal{U}$-Ramsey null sets form a σ-ideal (Lemma 5.25), so similarly $\bigcup_{s\in\omega^{<\omega}} \mathcal{M}_s$ is also perfectly $\mathcal{U}$-Ramsey null inside $[T_\infty] \times [p_\infty]$. Hence there exist $T' \leq^0 T_\infty$ and $p' \leq p_\infty$ such that $[T'] \times [p'] \cap \bigcup_{s\in\omega^{<\omega}} \mathcal{M}_s = \emptyset$.

Claim 5.30.2. $([T'] \times [p']) \cap \Phi(\mathcal{Y}_\emptyset^*) = ([T'] \times [p']) \cap \mathcal{Y}_\emptyset^*$.

The left inclusion is clear since $\Phi(\mathcal{Y}_\emptyset^*) \supseteq \mathcal{Y}_\emptyset^*$. To prove the right inclusion, pick $\bar{x} \in [T'] \times [p'] \cap \Phi(\mathcal{Y}_\emptyset^*)$. $[T'] \times [p'] \cap \mathcal{M}_\emptyset = \emptyset$, so there exists n_0 such that $\bar{x} \in \Phi(\mathcal{Y}_{\langle n_0\rangle}^*)$. $[T'] \times [p'] \cap \mathcal{M}_{\langle n_0\rangle} = \emptyset$, so there exists n_1 such that $\bar{x} \in \Phi(\mathcal{Y}_{\langle n_0,n_1\rangle}^*)$. By repeating this process we find $f = \langle n_k\rangle_k \in \omega^\omega$ such that $\bar{x} \in \Phi(\mathcal{Y}_{f\restriction k}^*) \subseteq \mathcal{Y}_{f\restriction k}$ for all $k < \omega$. So $\bar{x} \in \bigcap_{k<\omega} \mathcal{Y}_{f\restriction k} \subseteq \mathcal{Y} = \mathcal{Y}_\emptyset^*$. This finishes the proof of Claim 5.30.2.

Since $[T'] \times [p'] \cap \Phi(\mathcal{Y}_\emptyset^*)$ is perfectly $\mathcal{U}$-Ramsey, there exist $T'' \leq^0 T$ and $p'' \leq p$ such that

$$[T''] \times [p''] \subseteq ([T'] \times [p']) \cap \Phi(\mathcal{Y}_\emptyset^*) \text{ or } (([T'] \times [p']) \cap \Phi(\mathcal{Y}_\emptyset^*))^c.$$

By Claim 5.30.2, $([T'] \times [p']) \cap \Phi(\mathcal{Y}_\emptyset^*) = ([T'] \times [p']) \cap \mathcal{Y}_\emptyset^* = ([T'] \times [p']) \cap \mathcal{Y}$. So, in particular, we have $T'' \leq^0 T, p'' \leq p$ and $[T''] \times [p''] \subseteq \mathcal{Y}$ or $\mathcal{Y}^c$. Therefore $\mathcal{Y}$ is perfectly $\mathcal{U}$-Ramsey. Thus we have proved that the field of perfectly $\mathcal{U}$-Ramsey subsets of $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ is closed under the Souslin operation. □

Corollary 5.31. *Every Souslin-measurable subset of* $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ *is perfectly* $\mathcal{U}$*-Ramsey.*

Thus, we have proved that every selective ultrafilter on FIN is localizing. Therefore, as promised in the introduction, we see that a selective ultrafilter localizes the Parametrised Milliken Theorem 5.6 as follows.

Theorem 5.7 (Local Parametrized Milliken Theorem)**.** *Let* $\mathcal{U}$ *be a selective ordered-union ultrafilter on* FIN*. For every finite Souslin-measurable colouring of the product* $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ *there exists* $B \in \mathcal{U}$ *and* $p \in \mathcal{P}_\omega$ *such that* $[B]^{[\infty]} \times [p]$ *is monochromatic.*

5.4 Selectivity Preserved under Side-by-side Sacks Forcing

Let $\mathcal{U}$ be an arbitrary selective ordered-union ultrafilter on FIN. We fix an arbitrary infinite cardinal κ, and prove that $\mathcal{U}$ is preserved after adding κ Sacks reals using countable-support side-by-side Sacks forcing $\mathcal{P}_\kappa$. Recall that $\mathcal{P}_\kappa$ is the set of all functions $p : \kappa \to \mathcal{P}$ with countable support, ordered by $p \leq q$ if $p(\alpha) \subseteq q(\alpha)$ for every $\alpha < \kappa$. Forcing with $\mathcal{P}_\kappa$ adds a Sacks real for each $\alpha \in \kappa$. From now on in this subsection, $\Vdash$ is used to denote the forcing relation with respect to $\mathcal{P}_\kappa$.

Lemma 5.32 ([3])**.** *Suppose* $p \in \mathcal{P}_\kappa$ *and* $n \in \omega$*. If* $q \leq p$ *then there exists* $\sigma \in l(n, p)$ *such that* q *and* $p|\sigma$ *are compatible.*

Corollary 5.33 ([3]). *Suppose $p \in \mathcal{P}_\kappa$ and $n \in \omega$. If $p \Vdash \dot{\xi} \in V$ then there exists $q \leq^n p$ such that for each $\sigma \in l(n, q)$ there exists $a_\sigma \in V$ such that $q|\sigma \Vdash \dot{\xi} = a_\sigma$.*

If q and $\dot{\xi}$ are as in this Corollary then we say q *determines $\dot{\xi}$ relative to n*. We say q *determines $\dot{\xi}$* if there exist n such that q determines $\dot{\xi}$ relative to n. For $q \Vdash \dot{\tau} : \mathrm{FIN} \to V$, we say q *determines $\dot{\tau}$* if q determines $\dot{\tau}(x)$ for every $x \in \mathrm{FIN}$.

As FIN is countable, we enumerate it as $\mathrm{FIN} = \{x_n : n \in \omega\}$.

Corollary 5.34. *If $p \in \mathcal{P}_\kappa$ and $p \Vdash \dot{\tau} : \mathrm{FIN} \to 2$ then there exists $q \leq p$ such that q determines $\dot{\tau}$.*

Proof. We construct a fusion sequence $\langle p_n : n < \omega \rangle$ recursively, starting from $p_0 = p$ and $F_0 = \emptyset$. Since each p_n in the sequence has countable support, the set $\bigcup_{n<\omega} \mathrm{supp}(p_n)$ is countable. Enumerate $\bigcup_{n<\omega} \mathrm{supp}(p_n)$ as $\{k_n : n < \omega\}$. For $n < \omega$, by Corollary 5.33, there exists $p_{n+1} \leq^{k_n} p_n$ such that for each $\sigma \in l(k_n, p_n)$, $p_{n+1}|\sigma$ decides $\dot{\tau}(x_n)$. Then the fusion $q = \bigcap_{n<\omega} p_n$ satisfies the statement. □

Let $\dot{\mathcal{V}}$ be a name with respect to $\mathcal{P}_\kappa$ for the upward closure of $\mathcal{U}$. Our aim is to prove that, forcing with $\mathcal{P}_\kappa$, $\dot{\mathcal{V}}$ is a selective ultrafilter in the extension. It is straightforward to check the properties of $\dot{\mathcal{V}}$ for being a selective ultrafilter except for the following.

Theorem 5.35. *If $p \in \mathcal{P}_\kappa$ and $p \Vdash \dot{\tau} \subseteq \mathrm{FIN}$, then there exist $q \leq p$ and $[B] \in \mathcal{U}$ such that $q \Vdash [B] \subseteq \dot{\tau}$ or $q \Vdash [B] \cap \dot{\tau} = \emptyset$.*

Proof. We abuse notation and use $\dot{\tau}$ to denote the characteristic function of the set $\dot{\tau}$, so $p \Vdash \dot{\tau} : \mathrm{FIN} \to 2$. We will find $q \leq p$ and $[B] \in \mathcal{U}$ such that $q \Vdash$ "$\dot{\tau}$ is constant on $[B]$". By the proof of Corollary 5.34 we can construct a fusion sequence $\{p_n : n \in \omega\}$ with $\{k_n : n \in \omega\}$ such that the fusion p_∞ of the sequence satisfies the following condition.

$$\forall n < \omega \ \forall \sigma \in l(k_n, p_\infty) \ \exists i_\sigma \in 2 \text{ such that } p_\infty|\sigma \Vdash \dot{\tau}(x_n) = i_\sigma.$$

Note that $p_\infty \in \mathcal{P}_\kappa$, so $\mathrm{supp}(p_\infty)$ is countable. We may assume without loss of generality that $\mathrm{supp}(p_\infty) \subseteq \omega$. Then $[p'] \subseteq (2^\omega)^\omega$. We consider the following subset $\mathcal{F}$ of $\mathrm{FIN} \times [p_\infty]$:

$$\mathcal{F} = \{(x_n, \epsilon) : (\exists \sigma \in l(k_n, p_\infty))(\forall i \in k_n)((\sigma(i) \in (\epsilon(j,i))_{j<\omega}) \wedge (p_\infty|\sigma \Vdash \dot{\tau}(x_n) = 0))\}.$$

Let $\mathcal{X} = \{((y_n)_{n=1}^\infty, \epsilon) \in \mathrm{FIN}^{[\infty]} \times [p_\infty] : (y_1, \epsilon) \in \mathcal{F}\}$. Then $\mathcal{X}$ is open in $\mathrm{FIN}^{[\infty]} \times [p_\infty]$ where $\mathrm{FIN}^{[\infty]} \times [p_\infty] \subseteq \mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ has the subspace topology. Since $\mathcal{U}$ is localizing, there exists $[B] \in \mathcal{U}$ and $q \leq p_\infty$ such that $[\emptyset, B] \times [q] \subseteq \mathcal{X}$ or $([\emptyset, B] \times [q]) \cap \mathcal{X} = \emptyset$, hence $[B] \times [q] \subseteq \mathcal{F}$ or $([B] \times [q]) \cap \mathcal{F} = \emptyset$.

We check that q and $[B]$ satisfy the theorem. Firstly, suppose $[B] \times [q] \subseteq \mathcal{F}$. We prove that if $x_n \in [B]$, then $q \Vdash \dot{\tau}(x_n) = 0$. Assuming there exists $r \leq q$ with $r \Vdash \dot{\tau}(x_n) = 1$, we aim for a contradiction. By Lemma 5.32, there exists $\sigma \in l(k_n, q)$ such that r is compatible to $q|\sigma$. Let $\epsilon \in [q]$ be such that σ is a pre-initial segment of ϵ, that is, $\sigma(i) \in (\epsilon(j,i))_{j<\omega}$ for every $i \in k_n$. Then, as $(x_n, \epsilon) \in \mathcal{F}$ and by the definition of $\mathcal{F}$, there exists a pre-initial segment $\sigma' \in l(k_n, p_\infty)$ of ϵ such that $p_\infty|\sigma' \Vdash \dot{\tau}(x_n) = 0$. Since $q \leq p_\infty$, we have $\sigma'(i) \sqsubseteq \sigma(i)$ for every $i \in k_n$, so $q|\sigma \leq p_\infty|\sigma'$. Therefore $q|\sigma \Vdash \dot{\tau}(x_n) = 0$. This contradicts that $r \Vdash \dot{\tau}(x_n) = 1$ and r, $q|\sigma$ are compatible. If $[B] \times [q] \cap \mathcal{F} = \emptyset$, then we similarly have $q \Vdash \dot{\tau}(x_n) = 1$ for every $x_n \in [B]$. □

This completes the proof of the following theorem.

Theorem 5.36. *Let κ be an infinite cardinal, and $\mathcal{P}_\kappa$ be countable-support side-by-side Sacks forcing adding κ Sacks reals. Let $\mathcal{U}$ be a selective ultrafilter on* FIN *in the ground model, and $\dot{\mathcal{V}}$ a name for the upward closure $\{\dot{Y} \subseteq \mathrm{FIN} : \exists[\check{X}] \in \check{\mathcal{U}}\ [\check{X}] \subseteq \dot{Y}\}$ of $\mathcal{U}$. Then $\Vdash_{\mathcal{P}_\kappa} \dot{\mathcal{V}}$ is a selective ultrafilter on* FIN.

5.5 Selectivity Preserved under Iterated Sacks Forcing

In [20], Fernández-Bretón and Hrušák established the existence of selective ordered-union ultrafilters in the iterated Sacks model. They also suggested that the argument in the previous section may be modified to obtain a corresponding result for iterated Sacks forcing, hence obtaining the same result as in [20]. In this section we show that selective ultrafilters are preserved under countable-support iterated Sacks forcing.

Let $\mathcal{U}$ be a selective ordered-union ultrafilter on FIN. Let α be an arbitrary ordinal.

Definition 5.37. Let $\mathcal{P}_\alpha$ be the α-iterated Sacks forcing with countable support. Equivalently, $p \in \mathcal{P}_\alpha$ if and only if

$$\mathrm{domain}(p) \subseteq \alpha \text{ is countable and } \forall\beta \in \mathrm{domain}(p)\ \Vdash_\beta p(\beta) \in \mathcal{P}.$$

For $p, q \in \mathcal{P}_\alpha$, $p \leq q$ if and only if

$$\mathrm{domain}(p) \supseteq \mathrm{domain}(q) \text{ and } \forall\beta \in \mathrm{domain}(q)\ p\restriction\beta \Vdash_\beta p(\beta) \leq q(\beta).$$

Definition 5.38 ([4])**.** For $p, q \in \mathcal{P}$ and $m, n \in \omega$, let $(p, m) \leq (q, n)$ if $p \leq q$, $m > n$, and

$$(\forall s \in q \cap 2^n)(\exists t, u \in p \cap 2^m)\ s \subseteq t, u \text{ and } t \neq u.$$

For $p, q \in \mathcal{P}_\alpha$, $m, n \in \omega$ and $F \subseteq \mathrm{domain}(q)$ finite, let $(p, m) \leq_F (q, n)$ if $p \leq q$, $m > n$ and

$$\forall\beta \in F \quad p\restriction\beta \Vdash_\beta (p(\beta), m) \leq (q(\beta), n).$$

Note that if $(p, m) \leq (q, n)$ then $p \cap 2^n = q \cap 2^n$.

Also note that if $k \geq m$ and $(p, m) \leq_F (q, n)$ then $(p, k) \leq_F (q, n)$.

Definition 5.39 ([4])**.** Let $p \in \mathcal{P}_\alpha$, $F \subseteq \mathrm{domain}(p)$ and $n < \omega$. We say $\sigma : F \to 2^n$ is *consistent* with $p \in \mathcal{P}_\alpha$ if $p|\sigma \in \mathcal{P}_\alpha$, or equivalently,

$$\forall\beta \in F \quad (p|\sigma)\restriction\beta \Vdash_\beta \sigma(\beta) \in p(\beta).$$

We say p is (F, n)-*determined* if $\forall\sigma \in F \to 2^n$, either σ is consistent with p, or

$$\exists\beta \in F \quad (\sigma\restriction(F \cap \beta) \text{ is consistent with } p) \wedge ((p|\sigma)\restriction\beta \Vdash_\beta \sigma(\beta) \notin p(\beta)).$$

Lemma 5.40 (Fusion Lemma, [4])**.** *Suppose $\langle(p_i, n_i, F_i) : i \in \omega\rangle$ is a sequence such that $p_i \in \mathcal{P}_\alpha$, $n_i \in \omega$, $F_i \subseteq F_{i+1}$, $\bigcup\{F_i : i \in \omega\} = \bigcup\{\mathrm{domain}(p_i) : i \in \omega\}$, and for each i, $(p_{i+1}, n_{i+1}) \leq_{F_i} (p_i, n_i)$. Define p so that* $\mathrm{domain}(p) = \bigcup\{\mathrm{domain}(p_i) : i \in \omega\}$ *and for all* $\beta \in \mathrm{domain}(p)$

$$p(\beta) = \bigcap\{p_i(\beta) : i \in \omega, \beta \in \mathrm{domain}(p_i)\}.$$

Then $p \in \mathcal{P}_\alpha$ and $p \leq p_i$ for each i.

Lemma 5.41. [4, Lemma 2.3]. *Let $\alpha \geq 1$. If $p \in \mathcal{P}_\alpha$, $n \in \omega$, $p \Vdash_\alpha \dot{a} \in V$ and F is a finite subset of* domain(p)*, then there exists (q, m) such that $(q, m) \leq_F (p, n)$, q is (F, n)-determined, and $\forall \sigma : F \to 2^n$, if σ is consistent with q then there exists a_σ such that $q|\sigma \Vdash_\alpha \dot{a} = a_\sigma$.*

Recall Corollary 5.31 from Section 5.3.2.

Corollary 5.31. *Every Souslin-measurable subset of* $\mathrm{FIN}^{[\infty]} \times (2^\omega)^\omega$ *is perfectly $\mathcal{U}$-Ramsey.*

Theorem 5.42. *Let $\mathcal{U}$ be a selective ultrafilter on* FIN*. If $p \in \mathcal{P}_\alpha$ and $p \Vdash_\alpha \dot{\tau} : \mathrm{FIN} \to 2$ then there exists $q \leq p$, $[B] \in \mathcal{U}$ and $i \in 2$ such that $q \Vdash_\alpha \dot{\tau} \restriction [B] \equiv i$.*

Proof. Let $\mathrm{FIN} = \{x_k : k < \omega\}$. Let $p_{-1} = p$, $n_{-1} = 1$ and $F_{-1} = 1 = \{0\}$. By Lemma 5.41, as $p_{-1} \Vdash_\alpha \dot{\tau}(x_0) \in 2$, there exists $(p_0, n_0) \leq_{F_{-1}} (p_{-1}, n_{-1})$ such that p_0 is (F_{-1}, n_{-1})-determined and $\forall \sigma : F_{-1} \to 2^{n_{-1}}$

$$\text{if } \sigma \text{ is consistent with } p_0 \text{ then } \exists i_\sigma \in 2\ p_0|\sigma \Vdash_\alpha \dot{\tau}(x_0) = i_\sigma.$$

Recursively using Lemma 5.41 we can choose $(F_k)_{k<\omega}$ and $(n_k)_{k<\omega}$ increasing with $\bigcup\{F_k : k < \omega\} = \bigcup\{\mathrm{domain}(p_k) : k < \omega\}$ where $(p_k)_{k<\omega}$ are constructed such that $(p_{k+1}, n_{k+1}) \leq_{F_k} (p_k, n_k)$, p_{k+1} is (F_k, n_k)-determined, and $\forall \sigma : F_k \to 2^{n_k}$

$$\text{if } \sigma \text{ is consistent with } p_{k+1} \text{ then } \exists i_\sigma \in 2\ p_{k+1}|\sigma \Vdash_\alpha \dot{\tau}(x_k) = i_\sigma.$$

By Lemma 5.40, we can take the fusion p_∞ of the sequence.

Claim 5.42.1. *For every $k < \omega$, p_∞ is (F_k, n_k)-determined in the same way as p_k.*

Proof. By the definition of $(p_{k+1}, n_{k+1}) \leq_{F_k} (p_k, n_k)$, we have, for all $k < \omega$,

$$\forall \beta \in F_k \quad p_{k+1} \restriction \beta \Vdash_\beta (p_{k+1}(\beta), n_{k+1}) \leq (p_k(\beta), n_k),$$

so

$$p_{k+1} \restriction \beta \Vdash_\beta p_{k+1}(\beta) \cap 2^{n_k} = p_k(\beta) \cap 2^{n_k}.$$

Since $p_i(\beta) \cap 2^{n_j} \subseteq p_i(\beta) \cap 2^{n_{j+1}}$ for arbitrary $i, j \in \omega$, we have

$$\forall k < \omega\ \forall \beta \in F_k\ \forall i > k \quad p_i \restriction \beta \Vdash_\beta p_i(\beta) \cap 2^{n_k} = p_k(\beta) \cap 2^{n_k}.$$

Thus

$$p_\infty \restriction \beta \Vdash_\beta p_\infty(\beta) \cap 2^{n_k} = p_k(\beta) \cap 2^{n_k}.$$

Suppose $\sigma : F_k \to 2^{n_k}$. Then either of the following holds.

(1) σ is consistent with p_{k+1}, i.e.

$$\forall \beta \in F_k \quad (p_{k+1}|\sigma) \restriction \beta \Vdash_\beta \sigma(\beta) \in p_{k+1}(\beta).$$

So $(p_\infty|\sigma) \restriction \beta \Vdash_\beta \sigma(\beta) \in p_\infty(\beta)$, hence σ is consistent with p_∞.

(2) There exists $\beta \in F_k$ such that $\sigma \restriction (F \cap \beta)$ is consistent with p_{k+1}, (so it is consistent with p_∞,) but $(p_{k+1}|\sigma) \restriction \beta \Vdash \sigma(\beta) \notin p_{k+1}(\beta)$. So $(p_\infty|\sigma) \restriction \beta \Vdash_\beta \sigma(\beta) \notin p_\infty(\beta)$.

□

Now for $\beta \in \mathrm{domain}(p_\infty)$, consider the set of $\mathcal{P}_\beta$-names

$$[p_\infty(\beta)] = \{\dot f : \Vdash_\beta (\dot f \in 2^\omega) \wedge (\forall n < \omega)(\dot f \cap 2^n \in p_\infty(\beta))\}.$$

Equip $[p_\infty(\beta)]$ with the topology generated by

$$[\dot s] = \{\dot f \in [p_\infty(\beta)] : \Vdash_\beta \dot s \sqsubseteq \dot f\},$$

where $\dot s$ is a $\mathcal{P}_\beta$-name such that $\Vdash_\beta \dot s \in p_\infty(\beta)$.

Claim 5.42.2. *For $\beta \in \mathrm{domain}(p_\infty)$, $[p_\infty(\beta)]$ is perfect.*

Proof. Consider $\dot f \in [p_\infty(\beta)]$ and $[\dot s]$ an open neighbourhood of $\dot f$, i.e.

$$\Vdash_\beta (\dot f \in 2^\omega) \wedge (\forall n < \omega)(\dot f \cap 2^n \in p_\infty(\beta));$$
$$\Vdash_\beta (\dot s \in p_\infty(\beta)) \wedge (\dot s \sqsubseteq \dot f).$$

Since $\Vdash_\beta\ p_\infty(\beta) \in \mathcal{P}$, we have

$$\Vdash_\beta \text{ “}\{f : (\forall n < \omega)(f \cap 2^n \in p_\infty(\beta))\} \text{ form a perfect set”}$$

i.e.

$$\begin{aligned}\Vdash_\beta\ (\forall f \in 2^\omega)(\forall s \in 2^{<\omega})[(\forall n < \omega)(f \cap 2^n) \in p_\infty(\beta) \wedge (s \sqsubseteq f)\\ \Rightarrow (\exists g \neq f)(g \in 2^\omega) \wedge (\forall n < \omega)(g \cap 2^n \in p_\infty(\beta)) \wedge (s \sqsubseteq g)].\end{aligned}$$

Now we have $\mathcal{P}_\beta$-names $\dot f, \dot s$ as above, so there is a $\mathcal{P}_\beta$-name $\dot g$ such that

$$\Vdash_\beta\ (\dot g \neq \dot f) \wedge (\dot g \in 2^\omega) \wedge (\forall n < \omega)(\dot g \cap 2^n \in p_\infty(\beta)) \wedge (\dot s \sqsubseteq \dot g).$$

Thus, we have $\dot g \neq \dot f$ in $[p_\infty(\beta)] \cap [\dot s]$. Moreover, $\Vdash_\beta\ \dot g \neq \dot f$. □

Define $[p_\infty] = \prod_{\beta \in \mathrm{domain}(p_\infty)} [p_\infty(\beta)]$ and equip it with the product topology. We assume without loss of generality that the set $\mathrm{domain}(p_\infty)$ is infinite. So $[p_\infty] \cong (2^\omega)^\omega$.

Consider the subsets $\mathcal{F} \subseteq \mathrm{FIN} \times [p_\infty]$ and $\mathcal{X} \subseteq \mathrm{FIN}^{[\infty]} \times [p_\infty]$ defined by

$$\begin{aligned}\mathcal{F} = \{(x_k, \varepsilon) : (\exists \sigma : F_k \to 2^{n_k} \text{ consistent with } p_\infty)\\ (\forall \beta \in F_k)(\Vdash_\beta \sigma(\beta) \sqsubseteq \varepsilon(\beta)) \wedge (p_\infty|\sigma \Vdash_\alpha \dot\tau(x_k) = 0)\};\\ \mathcal{X} = \{\ ((y_n)_{n=1}^\infty, \varepsilon) : (y_1, \varepsilon) \in \mathcal{F}\}.\end{aligned}$$

Then $\mathcal{X}$ is an open subset of $\mathrm{FIN}^{[\infty]} \times [p_\infty]$. By Theorem **??**, there exists $[B] \in \mathcal{U}$ and $Q_\beta\,(\beta \in \mathrm{domain}(p_\infty))$ such that every Q_β is a perfect subset of $[p_\infty(\beta)]$ and $[\emptyset, B] \times \prod_{\beta \in \mathrm{domain}(p_\infty)} Q_\beta \subseteq \mathcal{X}$ or $\mathcal{X}^c$, hence $[B] \times \prod_{\beta \in \mathrm{domain}(p_\infty)} Q_\beta \subseteq \mathcal{F}$ or $\mathcal{F}^c$.

Note, for $\beta \in \text{domain}(p_\infty)$,

$$Q_\beta \subseteq [p_\infty(\beta)] = \{\dot{f} : \Vdash_\beta (\dot{f} \in 2^\omega) \wedge (\forall n < \omega)(f \cap 2^n \in p_\infty(\beta))\}.$$

We define $q \in \mathcal{P}_\alpha$ as follows. Let $\text{domain}(q) = \text{domain}(p_\infty)$. For $\beta \in \text{domain}(q)$, let $q(\beta)$ be a $\mathcal{P}_\beta$-name for the set

$$\bigcup_{n<\omega} \left\{ s \in 2^n : \left(\exists \dot{f} \in Q_\beta\right) \left(\Vdash_\beta \dot{f} \cap 2^n = s\right)\right\}.$$

Claim 5.42.3. $q \in \mathcal{P}_\alpha$.

Proof. Let $\beta \in \text{domain}(q)$. We show $\Vdash_\beta q(\beta) \in \mathcal{P}$. Suppose $\dot{s}$ is a $\mathcal{P}_\beta$-name and $\Vdash_\beta \dot{s} \in q(\beta)$. We look for $\mathcal{P}_\beta$-names $\dot{t}, \dot{u}$ such that

$$\Vdash_\beta (\dot{t} \sqsupseteq \dot{s}) \wedge (\dot{u} \sqsupseteq \dot{s}) \wedge (\dot{t} \not\sqsubseteq \dot{u}) \wedge (\dot{u} \not\sqsubseteq \dot{t}).$$

Since $\Vdash_\beta \dot{s} \in q(\beta)$, there exists $\dot{f} \in Q_\beta$ such that $\Vdash_\beta \dot{f} \cap 2^n = \dot{s}$. As Q_β is a perfect subset of $[p_\infty(\beta)]$ there exists $\dot{g} \in Q_\beta$ such that

$$\Vdash_\beta (\dot{g} \neq \dot{f}) \wedge (\forall n < \omega)(\dot{g} \cap 2^n \in p_\infty(\beta)) \wedge (\dot{s} \sqsubseteq \dot{g}).$$

In particular, there exists $n' < \omega$ such that $\Vdash_\beta \dot{g} \cap 2^{n'} \neq \dot{f} \cap 2^{n'}$. Let $m = \max\{n, n'\} + 1$ and $\dot{t}, \dot{u}$ be $\mathcal{P}_\beta$-names for $\dot{g} \cap 2^m$ and $\dot{f} \cap 2^m$ respectively. Then $\dot{t}, \dot{u}$ satisfy the statement. □

Claim 5.42.4. $q \leq p_\infty$.

Proof. By construction, $\text{domain}(q) = \text{domain}(p_\infty)$, so we only check $q \restriction \beta \Vdash_\beta q(\beta) \leq p_\infty(\beta)$ for $\beta \in \text{domain}(p_\infty)$. In fact, we check $\Vdash_\beta q(\beta) \leq p_\infty(\beta)$. Suppose $\Vdash_\beta \dot{s} \in q(\beta)$ for some $\mathcal{P}_\beta$-name $\dot{s}$. By the definition of $q(\beta)$, there exists $\dot{f} \in Q_\beta$ and $n < \omega$ such that $\Vdash_\beta \dot{f} \cap 2^n = \dot{s}$. Equivalently, there exist $n < \omega$ and $\dot{f} \in Q_\beta$ such that

$$\Vdash_\beta (\dot{f} \in 2^\omega) \wedge (\forall m < \omega)(f \cap 2^m \in p_\infty(\beta)) \wedge (\dot{f} \cap 2^n = \dot{s}),$$

so $\Vdash_\beta \dot{s} \in p_\infty(\beta)$. □

Suppose $[B] \times \prod_{\beta \in \text{domain}(p_\infty)} Q_\beta \subseteq \mathcal{F}$. Let $x_k \in [B]$. We show $q \Vdash_\alpha \dot{\tau}(x_k) = 0$. Assuming there exists $r \leq q$ such that $r \Vdash_\alpha \dot{\tau}(x_k) = 1$, we aim for a contradiction.

Claim 5.42.5. *There exists $\sigma : F_k \to 2^{n_k}$ such that $r \not\perp q|\sigma$.*

Proof. Since $r \leq q$, we have $r \restriction \beta \Vdash_\beta r(\beta) \leq q(\beta)$ for all $\beta \in \text{domain}(q)$. Recall $F_k \subseteq \bigcup \text{domain}(p_k) = \text{domain}(p_\infty) = \text{domain}(q)$. Let $F_k = \{\beta_1, \ldots, \beta_l\} \subseteq \alpha$. We construct σ recursively, starting with $r_0 = r$ and $\sigma_0 = \emptyset$. Given r_i and σ_i for some $i < l$, we find $r' \leq r_i \restriction \beta_{i+1}$ and $s \in 2^{n_k}$ such that

$$r' \Vdash_{\beta_{i+1}} s \in r_i(\beta_{i+1}) \text{ and } \Vdash_{\beta_{i+1}} s \in q(\beta_{i+1}).$$

Let $\sigma_{i+1} = \sigma_i \cup \{(\beta_{i+1}, s)\}$ and $r_{i+1} \in \mathcal{P}_\alpha$ be such that $r_{i+1} \restriction \beta_{i+1} = r'$ and $r_{i+1}(\gamma) = r_i(\gamma)$ for $\gamma \in \text{domain}(r) \setminus \beta_{i+1}$.

Then let $\sigma = \sigma_l$. Thus, for all $\beta \in F_k$, $r_l \restriction \beta \Vdash_\beta \sigma(\beta) \in r_l(\beta)$. Therefore $r_l|\sigma = r_l$, $r_l \leq r$ and $r_l|\sigma \leq q|\sigma$. □

Now pick $\varepsilon \in [q|\sigma]$ where $[q|\sigma] = \prod_{\beta \in \text{domain}(q)}[(q|\sigma)(\beta)]$ and

$$[(q|\sigma)(\beta)] = \begin{cases} \{\dot{f} \in Q_\beta : \Vdash_\beta \sigma(\beta) \sqsubseteq \dot{f}\} & \text{if } \beta \in F_k; \\ [q(\beta)] & \text{otherwise.} \end{cases}$$

Then $(x_k, \varepsilon) \in \mathcal{F}$, and by the definition of $\mathcal{F}$, $p_\infty|\sigma \Vdash_\alpha \dot{\tau}(x_k) = 0$. So $q|\sigma \leq p_\infty|\sigma$ implies that $q|\sigma \Vdash_\alpha \dot{\tau}(x_k) = 0$, contradicting $q|\sigma \not\perp r$ and $r \Vdash_\alpha \dot{\tau}(x_k) = 1$.

Suppose $[B] \times \prod_{\beta \in \text{domain}(p_\infty)} Q_\beta \cap \mathcal{F} = \emptyset$. Similarly suppose $x_k \in [B]$, $r \leq q$ and $r \Vdash_\alpha \dot{\tau}(x_k) = 0$. We aim for a contradiction in order to conclude that $q \Vdash_\alpha \dot{\tau} \restriction [B] \equiv 1$. As in Claim 5.42.5, we can find $\sigma : F_k \to 2^{n_k}$ and $r_l = r_l|\sigma$ witnessing $r \not\perp q|\sigma$. Since $q \leq p_\infty$ and p_∞ is (F_k, n_k)-determined, σ must be consistent with p_∞. Hence by the construction of p_∞, $\exists i \in 2$ $p_\infty|\sigma \Vdash_\alpha \dot{\tau}(x_k) = i$. On the other had, similar to the previous case, by the definition of $\mathcal{F}$, we have $p_\infty|\sigma \not\Vdash_\alpha \dot{\tau}(x_k) = 0$. Thus $p_\infty|\sigma \Vdash_\alpha \dot{\tau}(x_k) = 1$, contradiction that $q|\sigma \leq p_\infty|\sigma$, $r \not\perp q|\sigma$ and $r \Vdash_\alpha \dot{\tau}(x_k) = 0$. □

Chapter 6

Spaces $\mathcal{R}_\alpha\,(\alpha < \omega_1)$

In [24] Laflamme constructed forcings $\mathbb{P}_\alpha$ to add the ultrafilters $\mathcal{U}_\alpha$ for $\alpha < \omega_1$ in order to obtain different combinatorics and related Rudin-Keisler ordering. These ultrafilters satisfy certain partition properties: $\mathcal{U}_1$ is weakly Ramsey; $\mathcal{U}_n$ $(n < \omega)$ is n-Ramsey; $\mathcal{U}_\alpha$ $(\omega \le \alpha < \omega_1)$ satisfies analogous Ramsey partition properties. Inspired by the work of Laflamme, Dobrinen and Todorcevic [14, 15] constructed a new hierarchy of topological Ramsey spaces $\mathcal{R}_\alpha(\alpha < \omega_1)$, which are modified versions of dense subsets of $\mathbb{P}_\alpha$, and proved extensions of the Pudlak-Rödl Theorem, canonizing equivalent relations on barriers of these spaces. This enabled their complete classification of the structure of the Tukey (resp. Rudin-Keisler) of ultrafilters Tukey (resp. Rudin-Keisler) reducible to $\mathcal{U}_\alpha$, as well as the Rudin-Keisler structure of ultrafilters Tukey reducible to $\mathcal{U}_\alpha$.

Among other properties, each $\mathcal{U}_\alpha$ is Nash-Williams in the corresponding space $\mathcal{R}_\alpha$. We would like to show that $\mathcal{U}_\alpha$, and in fact every Nash-Williams ultrafilter, is preserved under countable-support side-by-side Sacks forcing: the upward closure of the ultrafilter is still a Nash-Williams ultrafilter in the forcing extension.

The procedure is similar to that in Chapter 5. We first prove the Parametrized $\mathcal{R}_\alpha$ Theorem, and then localize it.

6.1 Topological Ramsey Spaces $\mathcal{R}_\alpha$

Intuitively, an element in $\mathcal{R}_\alpha$ is a subtree of $\mathbb{T}_\alpha$ which has the same shape as $\mathbb{T}_\alpha$. For example, $\mathbb{T}_0 = \{\langle\rangle\} \cup \{\langle n\rangle : n < \omega\}$ and $\mathcal{R}_0$ is the Ellentuck space $\mathbb{N}^{[\infty]}$. We have $\mathbb{T}_1$ as another example.

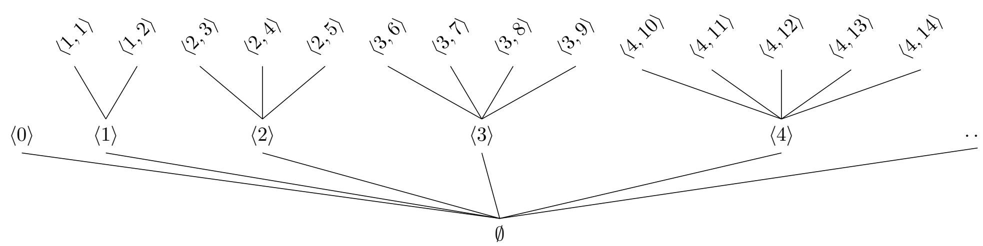

Figure 6.1: $\mathbb{T}_1$

Definition 6.1 ($\mathcal{R}_1$, [14])**.** Let

$$\mathbb{T}_1 = \{\langle\rangle\} \cup \{\langle n\rangle : n < \omega\} \cup \bigcup_{n>0} \{\langle n, i\rangle : \tfrac{1}{2}n(n+1) \le i < \tfrac{1}{2}n(n+1) + (n+1)\}.$$

We may think of $\mathbb{T}_1$ as an infinite sequence of finite trees of height 2, where the *nth subtree* of $\mathbb{T}_1$ is $\mathbb{T}_1(n) = \{\langle\rangle, \langle n\rangle, \langle n, i\rangle : \frac{1}{2}n(n+1) \le i < \frac{1}{2}n(n+1) + (n+1)\}$ for $n > 0$ and the *0th subtree* is $\mathbb{T}_1(0) = \{\langle\rangle, \langle 0\rangle\}$. A subtree $X \subseteq \mathbb{T}_1$ is in $\mathcal{R}_1$ if and only if there is a strictly increasing sequence $(k_n)_{n<\omega} \subseteq \omega$ such that $X \cap \mathbb{T}_1(k_n) \cong \mathbb{T}_1(n)$ for each $n < \omega$ and $X \cap \mathbb{T}_1(j) = \emptyset$ for $j \in \omega \setminus (k_n)_{n\in\omega}$.

In general, for $\alpha < \omega_1$, $\mathbb{T}_\alpha$ is always a union of an infinite sequence of finite trees: $\mathbb{T}_\alpha = \bigcup\{\mathbb{T}_\alpha(n) : n < \omega\}$, and $\mathbb{T}_\alpha(n)$ is called the *nth subtree* of $\mathbb{T}_\alpha$ for $n < \omega$. A subtree $X \subseteq \mathbb{T}_\alpha$ is in $\mathcal{R}_\alpha$ if and only if there is a strictly increasing sequence $(k_n)_{n<\omega} \subseteq \omega$ such that $X \cap \mathbb{T}_\alpha(k_n) \cong \mathbb{T}_\alpha(n)$ for each $n < \omega$ and $X \cap \mathbb{T}_\alpha(j) = \emptyset$ for $j \in \omega \setminus (k_n)_{n<\omega}$. We refer to [15, Section 2] for a detailed construction of $\mathbb{T}_\alpha$ and a rigorous definition of the relation $\cong$ involving a function ψ_α and an auxiliary structure $\mathbb{S}_\alpha$. However, the materials about $\mathcal{R}_\alpha$ presented here should be sufficient for the purpose of this thesis.

Definition 6.2 ([15])**.** Let $\alpha < \omega_1$, $X \in \mathcal{R}_\alpha$ and $(k_n)_{n<\omega} \subseteq \omega$ be the sequence associated with X as described above. For $n < \omega$, the *nth tree* of X is $X(n) = X \cap \mathbb{T}_\alpha(k_n)$. The *nth approximation* of X is $X \restriction n = \bigcup_{i<n} X(i)$. The *set of all nth approximations* is $\mathcal{AR}_n^\alpha = \{X \restriction n : X \in \mathcal{R}_\alpha\}$ and the *set of all finite approximations* is $\mathcal{AR}_\alpha = \bigcup_{n<\omega} \mathcal{AR}_n^\alpha$.

If $a = X \restriction m$ for some $m < \omega$, then we say a is an *initial segment* of X, and write $a \sqsubseteq X$. In this case, the *length* of a, denoted by $|a|$, is m. For $n < m$, the *nth tree* of a is $a(n) = X(n)$. Similarly, if $b \in \mathcal{AR}_\alpha$ satisfies $|a| < |b|$ and $\forall n < |a|\ a(n) = b(n)$, then we write $a \sqsubseteq b$.

Since $\mathbb{T}_\alpha$ is fixed for each α, an element $X \in \mathcal{R}_\alpha$ is completely determined by the set of all maximal numbers in the top nodes of the tree X. For $X \in \mathcal{R}_\alpha \cup \mathcal{AR}_\alpha$, let $[X]$ denote the collection of all maximal numbers in the $\sqsubseteq$-maximal nodes in X, where we use $\sqsubseteq$ to denote end-extension of nodes in a tree. It is also useful to know which of the subtrees $\mathbb{T}_\alpha(n)$ a node in X belongs to. So for a node $t \in X \setminus \{\emptyset\}$, let $\|t\| = n$ if $t \in \mathbb{T}_\alpha(n)$. Similarly, for $i < \omega$, $\|X(i)\| = n$ if $X(i) \subseteq \mathbb{T}_\alpha(n)$ and $\|X\| = \{\|t\| : t \in X\}$.

In the following example of $X \in \mathcal{R}_2$ (Figure 2), $\|\langle 3, 6, 23\rangle\| = 3$, $\|X\| = \{1, 2, 3, \dots\}$, and $[X] = \{2, 3, 5, 16, 18, 19, 21, 23, 25, 27, \dots\}$.

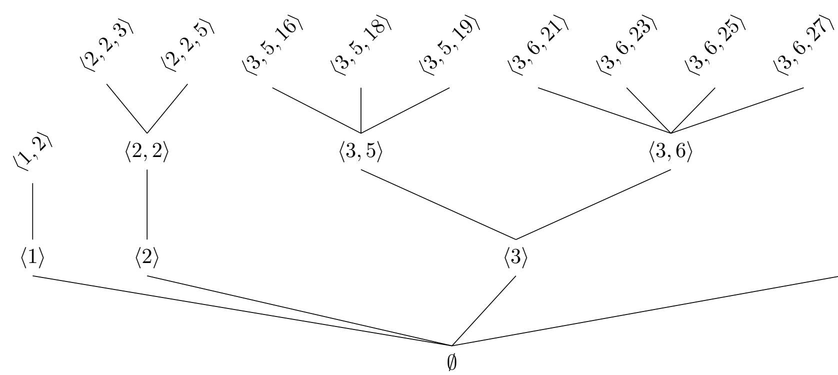

Figure 6.2: $X \in \mathcal{R}_2$

Definition 6.3. If $\alpha < \omega_1$ and $X, Y \in \mathcal{R}_\alpha \cup \mathcal{AR}_\alpha$, we write $X \leq Y$ if $X \subseteq Y$. For $a \in \mathcal{AR}_\alpha$ and $X \in \mathcal{R}_\alpha$, $[a, X] = \{Y \in \mathcal{R}_\alpha : (a \sqsubseteq Y) \wedge (Y \subseteq a \cup X)\}$, and $X/a = \bigcup\{X(n) : (n \in \omega) \wedge (\max \|a\| < \|X(n)\|)\}$. If $k \in \omega$, then $X/k = \bigcup\{X(n) : \|X(n)\| > k\}$.

The following notation is also used. Let $\alpha < \omega_1$, $X \in \mathcal{R}_\alpha$ and $a \in \mathcal{AR}_\alpha$. The set $\mathcal{AR}_\alpha(X) = \{b \in \mathcal{AR}_\alpha : b \leq X\}$ is the collection of all finite approximations of subtrees $Y \in \mathcal{R}_\alpha$ of X. We further define $\mathcal{AR}_\alpha[a, X] = \{b \in \mathcal{AR}_\alpha : a \sqsubseteq b \leq X\}$. Let $\mathcal{R}_\alpha(n) = \{X(n) : X \in \mathcal{R}_\alpha\}$ be the set of all finite subtrees of $\mathbb{T}_\alpha$ having the same shape as $\mathbb{T}_\alpha(n)$.

From now on we assume $\alpha < \omega_1$ and may omit the subscription α in $\mathbb{T}_\alpha$ when there is no confusion. Unless otherwise stated, we equip $\mathcal{R}_\alpha$ with the topology induced by the first-difference metric ρ , where ρ is defined as in Section 2.1: For $X, Y \in \mathcal{R}_\alpha$, $\rho(X, Y) = \frac{1}{k}$ where $k = \min\{n < \omega : X(n) \neq Y(n)\}$. Therefore the basic open subsets of $\mathcal{R}_\alpha$ are of the form

$$[a, \mathbb{T}] = \{X \in \mathcal{R}_\alpha : a \sqsubseteq X\} \quad \text{for } a \in \mathcal{AR}_\alpha.$$

This metric topology is coarser than the Ellentuck topology usually associated with the spaces $\mathcal{R}_\alpha$.

6.2 Ultrafilters in $\mathcal{R}_\alpha$

We consider ultrafilters on $[\mathbb{T}]$ generated by elements of the form $[X]$ where $X \in \mathcal{R}_\alpha$. As in the case of ordered-union ultrafilters on FIN, it is routine to check that such ultrafilters are equivalent to the ultrafilters in $\mathcal{R}_\alpha$ as defined in Definition 2.18.

Theorem 6.4. *Let $\alpha < \omega_1$ and $\mathcal{U}$ be an ultrafilter on $[\mathbb{T}]$ generated by elements of the form $[X]$ $(X \in \mathcal{R}_\alpha)$. The following relations hold among properties for $\mathcal{U}$.*

Nash-Williams $\Rightarrow$ Ramsey $\Rightarrow$ weak selectivity $\Leftrightarrow$ selectivity.

Moreover, Ramsey $\not\Leftarrow$ weak selectivity.

Proof. Again it is straightforward to check that the spaces $\mathcal{R}_\alpha$ has a head start (Definition 2.23) and satisfies **(A8)** (Definition 2.21), so by Theorems 2.24 and 2.22, the first two implications hold. It follows from the definitions that selectivity implies weak selectivity.

Let $\mathcal{U}$ be weakly selective. We show it is selective. Let $\{[X_a] : a \in \mathcal{AR}_\alpha\} \subseteq \mathcal{U}$. We aim to find $[X] \in \mathcal{U}$ such that $X/a \subseteq X_a$ for all $a \leq X$. We recursively construct a $\leq$-decreasing sequence $([Y_n])_{n<\omega} \subseteq \mathcal{U}$. For $n = 0$, consider the set $S_0 = \{b \in \mathcal{AR}_\alpha : \max[b] = 0\}$. Since S_0 is finite and $\mathcal{U}$ is an ultrafilter, we can find $[Y_0] \in \mathcal{U}$ such that $[Y_0] \subseteq \bigcap_{b \in S_0}[X_b]$. Suppose we have constructed $Y_0, \ldots, Y_n$. The set $S_{n+1} = \{b \in \mathcal{AR}_\alpha : \max[b] = n+1\}$ is finite. We can find $[Y_{n+1}] \in \mathcal{U}$ such that $[Y_{n+1}] \subseteq \bigcap_{b \in S_{n+1}}[X_b] \cap [Y_n]$. This finishes the construction of the sequence $([Y_n])_{n<\omega}$. In particular, for all $n \in \omega$, $[Y_n] \in \mathcal{U}$ and $Y_n \leq X_b$ for all $b \in \mathcal{AR}_\alpha$ with $\max[b] = n$. In order to apply the property that $\mathcal{U}$ is weakly selective, for each $b \in \mathcal{AR}_1^\alpha$, let $A_b = Y_n$ where $n = \max[b]$. Then there exists $[X] \in \mathcal{U}$ such that $X/b \subseteq A_b$ for all $b \in \mathcal{AR}_1^\alpha(X)$. Let us check that X is a witness to the selectivity of $\mathcal{U}$. Suppose $a \leq X$. Let $b \in \mathcal{AR}_1^\alpha$ be such that $\max[b] = \max[a]$. So $b \leq X$. Thus

$$X/a = X/b \subseteq A_b = Y_{\max[b]} = Y_{\max[a]} \leq X_a$$

as required.

The last statement of the theorem holds as Trujillo constructed a weakly selective ultrafilter in $\mathcal{R}_1$ which is not Ramsey. See [45, Theorem 19] and [44, Theorem 2.5.1]. □

6.3 Parametrized $\mathcal{R}_\alpha$ Theorem

In this section we aim to prove the Parametrized $\mathcal{R}_\alpha$ Theorem.

Theorem 6.5 (Parametrized $\mathcal{R}_\alpha$ Theorem)**.** *Let $\alpha < \omega_1$. For every finite Souslin-measurable colouring of $\mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ there exists $X \in \mathcal{R}_\alpha$ and $p \in \mathcal{P}_\omega$ such that $[\emptyset, X] \times [p]$ is monochromatic.*

By the Moderately-Abstract Parametrized Ellentuck Theorem 1.7, it is sufficient to prove that $\mathcal{R}_\alpha$ satisfies **(L4)**. First recall Lemma 3.4.

Lemma 3.4. *For $p \in \mathcal{P}_\omega$ and $O \subseteq [p]$ open, there exists $q \leq p$ such that $[q] \subseteq O$ or $[q] \cap O = \emptyset$.*

Definition 6.6. Let $p \in \mathcal{P}_\omega, k \in \omega$, and $S \subseteq (2^\omega)^\omega$. Let Θ be a set of pre-initial segments of elements in $[p]$. We say $S \cap [p]$ *depends only on* Θ if, for each $\sigma \in \Theta$, either all or none of the post-end-extensions of σ are in S, i.e. either $[\sigma] \subseteq S$ or $[\sigma] \cap S = \emptyset$.

A finite number of applications of Lemma 3.4 gives the following.

Lemma 6.7. *Let $p \in \mathcal{P}_\omega$ and $n \in \omega$. Then for every open set $O \subseteq [p]$ there exists $q \leq^n p$ such that $O \cap [q]$ depends only on $l(n, q)$.*

Applying Lemma 6.7 and the method of fusion we have the following corollary.

Corollary 6.8. *Suppose $p \in \mathcal{P}_\omega, n \in \omega$ and O_l $(l \in \omega)$ is a family of open subsets of $(2^\omega)^\omega$. Suppose also that the sequence (n_l) is unbounded and strictly increasing. Then there exists $q \leq^n p$ such that for every $l \in \omega$, $O_l \cap [q]$ depends only on $l(n_l, q)$.*

We will be using the infinite version of the Halpern-Läuchli theorem and its immediate corollary below.

Notation. For $p \in \mathcal{P}$ and $n \in \omega$, by $p(n)$ we denote the set of nodes in p with length n, i.e. $p(n) = \{s \in p : |s| = n\}$.

Theorem 6.9 (HL_ω, [25])**.** *If $p = (p^i)_{i\in\omega} \in \mathcal{P}_\omega$ and $\bigcup_{n<\omega} \prod_{i<\omega} p^i(n) = G_0 \cup G_1$, then there exists $j \in 2, A \in [\omega]^\omega$ and for each $i < \omega$ there exists $q^i \leq p^i$ such that $\bigcup_{n\in A} \prod_{i<\omega} q^i(n) \subseteq G_j$.*

Corollary 6.10. *For $p \in \mathcal{P}_\omega, n \in \omega$, and $\bigcup_{n<\omega} \prod_{i<\omega} p^i(n) = G_0 \cup G_1$ there exists $A \in [\omega]^\omega$ and $q \leq^n p$ such that*

$$\forall \sigma \in l(n, p)\ \exists j \in 2\ \bigcup_{n\in A} \prod_{i<\omega} (q|\sigma)^i(n) \subseteq G_j.$$

Lemma 6.11. *Suppose $M \in \mathbb{N}^{[\infty]}$ and O_l $(l \in M)$ is a family of open subsets of $(2^\omega)^\omega$. Then for every $p \in \mathcal{P}_\omega$ and $n \in \omega$ there exists $q \leq^n p$, an infinite subset $N \subseteq M$ and a clopen subset $G \subseteq [q]$ such that for every $l \in N$, $O_l \cap [q] = G$.*

Proof. Let $(n_l)_{l\in\omega} \subseteq \{m \in \omega : n < m\}$ be an increasing sequence. Applying Corollary 6.8 to shrink p, we may assume that

$$\forall l \in \omega \quad O_l \cap [p] \text{ depends only on } l(n_l, p).$$

By increasing each n_l if necessary, we may assume that

$$\forall l \in \omega \quad O_l \cap [p] \text{ depends only on } \prod_{i\in n_l} \{t \in p' : |t| = n_l\}.$$

We define a colouring $c : p \to 2$ as follows. Let $s = (s^i) \in p = (p^i)_{i<\omega}$. If there does not exist $k \in M$ such that $n_k \leq |s^i|$ for each $i \in n_k$ then let $c(s) = 0$. Otherwise, let $k(s) = \max\{k : (\forall i \in n_k)(n_k \leq |s^i|)\}$ and define

$$\bar{s} = (\bar{s}^i)_{i<\omega} \quad \text{ where } \bar{s}^i = \begin{cases} s_i \restriction n_{k(s)} & \text{if } i \in n_{k(s)} \\ s_i & \text{if } i \in \omega \setminus n_{k(s)}. \end{cases}$$

Since $O_{k(s)} \cap [p]$ depends only on $\prod_{i\in n_{k(s)}}\{t \in p^i : |t| = n_{k(s)}\}$, either $[p|\bar{s}] \subseteq O_{k(s)}$ or $[p|\bar{s}] \cap O_{k(s)} = \emptyset$. We let $c(s) = 1$ if and only if $[p|\bar{s}] \subseteq O_{k(s)}$.

Take $\bar{n} \in M/n$. By Corollary 6.10, there exist $N \in [M]^\omega$ and $q \leq^{\bar{n}} p$ such that c is constant on $\bigcup_{m\in N} \prod_{i<\omega} (q|\sigma)^i(m)$ for every $\sigma \in l(\bar{n}, q)$. Without loss of generality, we may assume $N \in [\{m \in \omega : \bar{n} < m\}]^\omega$. Now we check that the map $l \mapsto O_l \cap [q]$ is constant on N, hence q, N satisfy the lemma: Let $l, l' \in N$. Suppose $\varepsilon \in O_l \cap [q]$. Let $t = (\varepsilon^i \restriction n_l)_{i\in\omega}$. As $\varepsilon \in O_l$, by the definition of c, $c(t) = 1$. There exists a unique $\sigma \in l(\bar{n}, q)$ with $\sigma \sqsubseteq^* \varepsilon$. Since $t \in \prod_{i<\omega}(q|\sigma)^i(n_l)$ and $c(t) = 1$, it must be the case that c is constantly 1 on $\bigcup_{m\in N} \prod_{i<\omega}(q|\sigma)^i(m)$. Therefore, as $l' \in N$, $c \restriction \prod_{i<\omega}(q|\sigma)^i(l') \equiv 1$. Let $t' = (\varepsilon^i \restriction n_{l'})_{i<\omega}$. Then $c(t') = 1$, so $\varepsilon \in O_{l'}$. Thus we have proved that, for $l, l' \in N$ and $\varepsilon \in [q]$, if $\varepsilon \in O_l$ then $\varepsilon \in O_{l'}$. Hence by symmetry, $O_l \cap [q] = O_{l'} \cap [q]$ as required. Moreover, since $O_l \cap [q]$ depends only on $l(n_l, q)$, the set $O_l \cap [q]$ is clopen. □

Recall from Section 2.3 the term $\mathcal{F}|X$ for a family $\mathcal{F}$ and a set X. Here we use a slightly different definition: let $\mathcal{F}|X = \{Y \in \mathcal{F} : Y \subseteq X\}$.

Theorem 6.12 (Finite version of the pigeonhole principle for $\mathcal{R}_\alpha(n)$, [15])**.** *Let $n \leq k < \omega$ and $X \in \mathcal{R}_\alpha$ be given. Then there is an l such that for each 2-colouring $f : \mathcal{R}_\alpha(n)|X(l) \to 2$ there is a $\zeta \in \mathcal{R}_\alpha(k)|X(l)$ such that f is monochromatic on $\mathcal{R}_\alpha(n)|\zeta$.*

To prove **(L4)** in $\mathcal{R}_\alpha$, it is sufficient to assume $a = \emptyset$. So we prove the following lemma.

Lemma 6.13. *Let $m \in \omega$ and $A \in \mathcal{R}_\alpha$. Let O_b ($b \in \mathcal{R}_\alpha(m)|A$) be a family of open subsets of $(2^\omega)^\omega$. Then there exist $q \in \mathcal{P}_\omega$, $B \leq A$ and a clopen subset $G \subseteq [q]$ such that $O_b \cap [q] = G$ for every $b \in \mathcal{R}_\alpha(m)|B$.*

Proof. For $k \in \omega$, the set $\mathcal{R}_\alpha(m)|A(m + k)$ is finite. So by Corollary 6.8, we may assume

$$\forall k \in \omega\ \forall b \in \mathcal{R}_\alpha(m)|A(m + k)\ \exists n \in \omega \quad O_b \text{ depends only on } l(n, p). \tag{6.1}$$

Starting from $A_0 = A$ and $p_0 = (2^{<\omega})^{<\omega}$, we construct decreasing sequences $(A_k)_{k\in\omega}$ and $(p_k)_{k\in\omega}$ together with $(m_k)_{k\in\omega} \subseteq \omega$ such that for every $k \in \omega$,

(i) $A_{k+1} \restriction (m + k + 1) = A_k \restriction (m + k + 1)$ and $p^i_{k+1} \leq^{k-i} p^i_k$ for every $i \leq k$; and

(ii) the mapping $b \mapsto O_b \cap [p_k]$ is constant on $\mathcal{R}_\alpha(m)|A_k(m + k)$ and the constant value is a clopen subset of $[p_k]$.

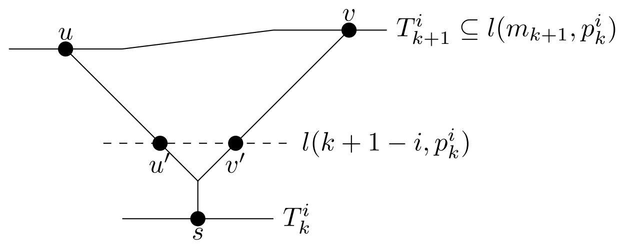

Figure 6.3: Constructing T^i_{k+1} from T^i_k for $i \leq k$

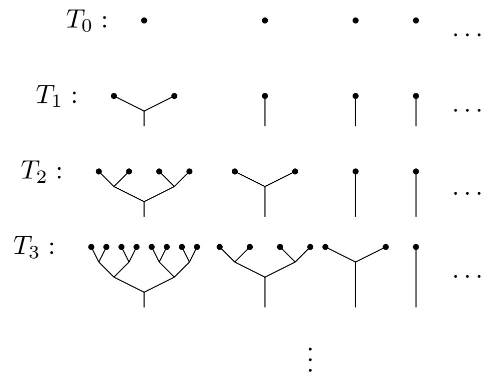

Figure 6.4: The sequence (T_k)

We also define auxiliary sets T_k at each step k, where $T_k = \prod_{i<\omega} T^i_k$ and $T^i_k \subseteq p^i_k$ for every $i < \omega$. Moreover, we require that, for $i \leq k$ and $s \in T^i_k$ there are exactly $k-i$ branchings below s in p^i_k, and $|T^i_k| = 2^{k-i}$. For $i \geq k$, T^i_k is a singleton.

First notice that (ii) holds for $k = 0$ since $\mathcal{R}_\alpha(m)|A_0(m)$ is a singleton. Let $m_0 = 0$ and $T_0 = \prod_{i<\omega}\{\emptyset\}$. Suppose we have constructed A_k, p_k and m_k. By Theorem 6.12, there exists $l \in \omega$ such that for every colouring $\mathcal{R}_\alpha(m)|A_k(l) \to 2^{2^{\frac{1}{2}(k+1)(k+2)}}$ there exists $\zeta \in \mathcal{R}_\alpha(m+k+1)|A_k(l)$ with $\mathcal{R}_\alpha(m)|\zeta$ monochromatic. By assumption (1), we can find $m_{k+1} > m_k$ such that

$$\forall b \in \mathcal{R}_\alpha(m)|A_k(l) \quad O_b \text{ depends only on } l(m_{k+1}, p_k).$$

We define $T_{k+1} = \prod_{i<\omega} T^i_{k+1}$ as follows:

- For $i \leq k$, by construction, every $s \in T^i_k$ has exactly $k-i$ branchings below it in p^i_k. For each $s \in T^i_k$, choose $u, v \in l(m_{k+1}, p^i_k)$ such that u, v are respectively end-extensions of two distinct elements $u', v' \in l(k+1-i, p^i_k)$ end-extending s, as shown in Figure 6.3. Thus, T^i_{k+1} is a set of end-extensions of elements in T^i_k and $|T^i_{k+1}| = 2|T^i_k|$. The idea is to construct a sequence (T_k) as shown in Figure 6.4.

- If $i > k$ then $T^i_k = \{s\}$ for some $s \in p^i_k$. Pick an arbitrary $t \in l(m_{k+1}, p^i_k)$ end-extending s, and let $T^i_{k+1} = \{t\}$. So $|T^i_{k+1}| = 1$.

Then

$$\left|\prod_{i\in m_{k+1}} T^i_{k+1}\right| = \prod_{i\leq k} 2|T^i_k| = \prod_{i<k+1} 2^{k+1-i} = 2^{\frac{1}{2}(k+1)(k+2)}.$$

By the choice of m_{k+1}, for every $b \in \mathcal{R}_\alpha(m)|A_k(l)$ and every $\sigma \in \prod_{i\in m_{k+1}} T^i_{k+1}$, either $[p_k|\sigma] \subseteq O_b$ or $[p_k|\sigma] \cap O_b = \emptyset$. For each b there are $2^{2^{\frac{1}{2}(k+1)(k+2)}}$ possibilities for $[p_k|\sigma]$ ($\sigma \in \prod_{i\in m_{k+1}} T^i_{k+1}$) to be inside or disjoint from O_b. Thus, by the choice of l, there exists $\zeta \in \mathcal{R}_\alpha(m+k+1)|A_k(l)$ such that for each $\sigma \in \prod_{i\in m_{k+1}} T^i_{k+1}$,

$$\text{either } \forall b \in \mathcal{R}_\alpha(m)|\zeta \quad [p_k|\sigma] \subseteq O_b, \text{ or } \forall b \in \mathcal{R}_\alpha(m)|\zeta \quad [p_k|\sigma] \cap O_b = \emptyset.$$

Now let $A_{k+1} \leq A_k$ be such that $A_{k+1}\restriction(m+k+1) = A_k\restriction(m+k+1)$ and $A_{k+1}(m+k+1) = \zeta$. For $i \in \omega$, let $p^i_{k+1} = \bigcup_{s\in T^i_{k+1}} p^i_k|s$, so $p^i_{k+1} \leq^{k-i} p^i_k$ for $i \leq k$. Let $p_{k+1} = (p^i_{k+1})_{i<\omega}$. Then the mapping $b \mapsto O_b \cap [p_{k+1}]$ is constant on $\mathcal{R}_\alpha(m)|A_{k+1}(m+k+1)$ and the constant value is a clopen subset of $[p_{k+1}]$, hence (ii) is satisfied.

This finishes the construction of the sequences $(A_k)_{k\in\omega}$ and $(p_k)_{k\in\omega}$. Let

$$A_\infty = A\restriction m \cup \bigcup_{k\in\omega} A_k(m+k), \text{ and } p_\infty = (p^i_\infty)_{i<\omega} = \left(\bigcap_{k'<\omega} p^i_{i+k'}\right)_{i<\omega}.$$

Clearly, $A_\infty \in \mathcal{R}_\alpha$. By (i), for every $i, k' \in \omega$, $p^i_{i+k'+1} \leq^{k'} p^i_{i+k'}$, so $\bigcap_{k'<\omega} p^i_{i+k'} \in \mathcal{P}$. Hence $p_\infty \in \mathcal{P}_\omega$. Then for $k \in \omega$, the mapping $b \mapsto O_b \cap [p_\infty]$ is constant on $\mathcal{R}_\alpha(m)|A_\infty(m+k)$. Let the constant value be denoted by $O^*_{\|A_\infty(m+k)\|}$. Note that $O^*_{\|A_\infty(m+k)\|}$ is clopen in $[p_\infty]$. Now we have $\|A_\infty\| \in \mathbb{N}^{[\infty]}$ and a family O^*_j ($j \in \|A_\infty\|$) of open subsets of $[p_\infty]$. By Lemma 6.11, there exists an infinite $N \subseteq \|A_\infty\|$, $q \leq p_\infty$ and a clopen $G \subseteq [q]$ such that $O^*_j \cap [q] = G$ for every $j \in N$. Then we can find $B \in \mathcal{R}_\alpha$ with $B \leq A_\infty$ and $\|B\| = N$. Thus $O_b \cap [q] = G$ for all $b \in \mathcal{R}_\alpha(m)|B$ as required. □

Thus we finish the proof of Theorem 6.5.

6.4 Local Parametrized $\mathcal{R}_\alpha$ Theorem

In this section we aim to prove that Nash-Williams ultrafilters $\mathcal{U}$ on $[\mathbb{T}]$ generated by elements of the form $[X]$ ($X \in \mathcal{R}_\alpha$) are localizing:

Theorem 6.14 (Local Parametrized $\mathcal{R}_\alpha$ Theorem)**.** *Let $\alpha < \omega_1$ and $\mathcal{U}$ be a Nash-Williams ultrafilter on $[\mathbb{T}]$ generated by elements of the form $[X]$ ($X \in \mathcal{R}_\alpha$). For every finite Souslin-measurable colouring of $\mathcal{R}_\alpha \times \mathbb{R}^\mathbb{N}$ there exists $X \in \mathcal{R}_\alpha$ with $[X] \in \mathcal{U}$ and $p \in \mathcal{P}_\omega$ such that $[\emptyset, X] \times [p]$ is monochromatic.*

From now on in this section, we fix a Nash-Williams ultrafilter $\mathcal{U}$ on $[\mathbb{T}]$ generated by elements of the form $[X]$ ($X \in \mathcal{R}_\alpha$).

6.4.1 Open subsets of $\mathcal{R}_\alpha \times \mathbb{R}^\mathbb{N}$

Firstly we relativise the ultra-Ramsey theory. The definitions in this subsection are adapted to $\mathcal{R}_\alpha$ from those in [42, §7]. Consider $\mathcal{AR}_\alpha$ as a tree ordered by $\sqsubseteq$ with root $\emptyset$. From now on by a *tree* we mean a downward closed subtree of $\mathcal{AR}_\alpha$.

Notation. For a tree T, let $[T]$ denote the set of all infinite branches of T, that is, $[T] = \{X \in \mathcal{R}_\alpha : X\restriction n \in T \text{ for all } n \in \omega\}$. Let the *stem of* T, denoted by $\mathrm{st}(T)$, be the maximal node of T that is $\sqsubseteq$-comparable with every node in T. For $s \in T$, by T/s we denote the set of nodes in T above s, i.e. $T/s = \{t \in T : s \sqsubseteq t\}$.

Definition 6.15. A *$\mathcal{U}$-tree* T is a tree such that for all $t \in T$ with $\mathrm{st}(T) \sqsubseteq t$ there exists $[X] \in \mathcal{U}$ such that $t \cup a \in T$ for every $a \in \mathcal{R}_\alpha(|t|)|X$. For two $\mathcal{U}$-trees T and T', we say T' is a *pure refinement* of T, and write $T' \leq^0 T$, if $T' \leq T$ and $\mathrm{st}(T') = \mathrm{st}(T)$. Similarly, $T' \leq^n T$ if $T' \leq^0 T$ and that T and T' agree on the first n levels above the common stem.

In [14, 15], $\mathcal{U}$-trees for the spaces $\mathcal{R}_\alpha$ are seen in well-founded form to determine the Rudin-Keisler structure inside the Tukey type of $\mathcal{U}$.

Definition 6.16. A sequence (T_n) of $\mathcal{U}$-trees is a *fusion sequence* if $T_{n+1} \leq^n T_n$ for all $n \in \omega$. In this case $T_\infty := \bigcap_{n\in\omega} T_n$ is also a $\mathcal{U}$-tree and is called the *fusion* of the sequence.

Lemma 6.17. *Suppose T, T' are $\mathcal{U}$-trees such that $\mathrm{st}(T') \in T/\mathrm{st}(T)$. Then $T \cap T'$ is also a $\mathcal{U}$-tree.*

Proof. Let $t \sqsupseteq \mathrm{st}(T') \sqsupseteq \mathrm{st}(T)$ be such that $t \in T \cap T'$. By assumption, there exist $[X]$ and $[X']$ in $\mathcal{U}$ such that $t \cup a \in T$ for every $a \in \mathcal{R}_\alpha(|t|)|X$ and $t \cup a \in T'$ for every $a \in \mathcal{R}_\alpha(|t|)|X'$. Since $[X], [X'] \in \mathcal{U}$ and $\mathcal{U}$ is an ultrafilter, there exists $[Y] \in \mathcal{U}$ such that $[Y] \subseteq [X] \cap [X']$. Then $t \cup a \in T \cap T'$ for every $a \in \mathcal{R}_\alpha(|t|)|Y$ as required. □

Definition 6.18. A subset $G \subseteq \mathcal{AR}_\alpha$ is *$\mathcal{U}$-open* if for every $t \in G$ there exists a $\mathcal{U}$-tree T such that $\mathrm{st}(T) = t$ and $T/t \subseteq G$.

Lemma 6.19. *For every subset $G \subseteq \mathcal{AR}_\alpha$ and every $s \in \mathcal{AR}_\alpha$, since $\mathcal{U}$ is* Nash-Williams*, either*

$$(\exists [X] \in \mathcal{U})(\forall a \in \mathcal{R}_\alpha(|s|)|X)(s \cup a \in G) \quad \textit{or} \quad (\exists [X] \in \mathcal{U})(\forall a \in \mathcal{R}_\alpha(|s|)|X)(s \cup a \notin G).$$

Proof. For $G \subseteq \mathcal{AR}_\alpha$, consider $\mathcal{G} = \{b \in \mathcal{AR}^\alpha_{|s|+1} : s \sqsubseteq b\} \subseteq \mathcal{AR}_\alpha$ and the partition $\mathcal{G} = G_0 \cup G_1$ where $G_0 = G$. Since $\mathcal{U}$ is *Nash-Williams*, there exist $[X] \in \mathcal{U}$ and $i \in 2$ such that $G_i|X = \emptyset$. Thus, if $i = 1$, then the first alternative in the lemma holds; if $i = 0$, then the second alternative holds. □

The lemma below follows immediately from the definition of $\mathcal{U}$-open sets and Lemma 6.19.

Lemma 6.20. *A subset $G \subseteq \mathcal{AR}_\alpha$ is $\mathcal{U}$-open if and only if $(\exists [X] \in \mathcal{U})(\forall a \in \mathcal{R}_\alpha(|s|)|X)(s \cup a \in G)$ holds for every $s \in G$.*

Definition 6.21. A subset $\mathcal{G} \subseteq \mathcal{R}_\alpha$ is *$\mathcal{U}$-open* if for every $X \in \mathcal{G}$ there exists a $\mathcal{U}$-tree T such that $X \in [T] \subseteq \mathcal{G}$.

By Lemma 6.17, the basic $\mathcal{U}$-open sets $[T]$ (T a $\mathcal{U}$-tree) generates a topology on $\mathcal{R}_\alpha$, containing the metric topology.

Definition 6.22. A subset $\mathcal{X} \subseteq \mathcal{R}_\alpha$ is *$\mathcal{U}$-Ramsey* if for every $\mathcal{U}$-tree T there exists $T' \leq^0 T$ such that $[T'] \subseteq \mathcal{X}$ or $[T'] \cap \mathcal{X} = \emptyset$. It is *$\mathcal{U}$-Ramsey null* if the second alternative always holds.

Lemma 6.23. *Every $\mathcal{U}$-open set is $\mathcal{U}$-Ramsey.*

Proof. Let $\mathcal{X} \subseteq \mathcal{R}_\alpha$ be a given $\mathcal{U}$-open set. Let $G = \{s \in \mathcal{AR}_\alpha : (\exists \mathcal{U}\text{-tree } T)((\mathrm{st}(T) = s) \wedge ([T] \subseteq \mathcal{X}))\}$. Note G is a $\mathcal{U}$-open subset of $\mathcal{AR}_\alpha$. Also note that if $s \notin G$ then there does not exist $[X] \in \mathcal{U}$ such that $s \cup a \in G$ for all $a \in \mathcal{R}_\alpha(|s|)|X$: If every $s \cup a$ has a $\mathcal{U}$-tree in $\mathcal{X}$ above it, then put them together we would get a $\mathcal{U}$-tree in $\mathcal{X}$ above s. By Lemma 6.19, $s \notin G$ implies that $(\exists [X] \in \mathcal{U})(\forall a \in \mathcal{R}_\alpha(|s|)|X)(s \cup a \notin G)$. Let $F = \mathcal{AR}_\alpha \setminus G$, so F satisfies the criterion of Lemma 6.20 for being $\mathcal{U}$-open. It follows that F is $\mathcal{U}$-clopen.

We prove that for every $t \in F$ there exists a $\mathcal{U}$-tree T' with stem t such that $[T'] \cap \mathcal{X} = \emptyset$. Then given T with stem s: either $s \in G$, so there exists $\mathcal{U}$-tree T' such that $[T' \cap T] \subseteq \mathcal{X}$ by the definition of G; or $s \in F$, so there exists T' such that $[T' \cap T] \cap \mathcal{X} = \emptyset$.

Claim 6.23.1. *If T is a $\mathcal{U}$-tree with* $\mathrm{st}(T) = t$ *such that $T/t \subseteq F$, then $[T] \cap \mathcal{X} = \emptyset$.*

First note that such tree exists since F is $\mathcal{U}$-open. Suppose $X \in [T] \cap \mathcal{X}$. Since $\mathcal{X}$ is $\mathcal{U}$-open, there exists T' such that $X \in [T'] \subseteq \mathcal{X}$. Let $s = \mathrm{st}(T')$. Then $s \in G$. But this contradicts that $s \in T/t \subseteq F$. $\square$

The proofs of the following three lemmas below closely follow from those of Lemma 7.40, 7.41 and Theorem 7.42 in [42], respectively. The corollary then easily follows.

Lemma 6.24. *Every $\mathcal{U}$-nowhere dense sets is $\mathcal{U}$-Ramsey null.*

Lemma 6.25. *The $\mathcal{U}$-Ramsey null sets form a σ-field.*

Lemma 6.26. *A subset of $\mathcal{R}_\alpha$ has the property of Baire with respect to the $\mathcal{U}$-topology if and only if it is $\mathcal{U}$-Ramsey.*

Corollary 6.27. *If $\mathcal{X} \subseteq \mathcal{R}_\alpha$ is (metrically) Souslin-measurable, then it is $\mathcal{U}$-Ramsey.*

This finishes the relativization of ultra-Ramsey theory. Now we show that selectivity helps us obtain a $\mathcal{U}$-tree from an element $[X] \in \mathcal{U}$, and vice versa.

Lemma 6.28. *For a $\mathcal{U}$-tree T with stem s there is $[X] \in \mathcal{U}$ such that $[s, X] \subseteq [T]$. Conversely for $s \sqsubseteq X$ with $[X] \in \mathcal{U}$ there is a $\mathcal{U}$-tree T with stem s such that $[T] \subseteq [s, X]$.*

Proof. For $t \in T/s$, there exists $[X_t] \in \mathcal{U}$ such that $t \cup a \in T$ for every $a \in \mathcal{R}_\alpha(|t|)|X_t$. For $t \in \mathcal{AR}_\alpha \setminus (T/s)$, let $X_t = \mathbb{T}$. (Note that $[\mathbb{T}] \in \mathcal{U}$ since $\mathcal{U}$ is an ultrafilter on the base set $[\mathbb{T}]$.) Apply *selectivity* of $\mathcal{U}$ to $\{[X_t] : t \in \mathcal{AR}_\alpha\} \subseteq \mathcal{U}$, we get $[X] \in \mathcal{U}$ such that $X/t \subseteq X_t$ for all $t \leq X$. Making finite changes to X, we may assume $s \sqsubseteq X$.

We check that $[s, X] \subseteq [T]$: Suppose $A \in [s, X]$, we prove $A \restriction n \in T$ for all n by induction. Firstly, $A \restriction (|s| + 1) = s \in T$ by assumption. Now consider $A \restriction (n + 1) = A \restriction n \cup A(n)$. By the induction hypothesis, $A \restriction n \in T$. As $A \leq X$, $A(n) \in \mathcal{R}_\alpha(n)|(X/(A \restriction n)) \subseteq \mathcal{R}_\alpha(n)|X_{A\restriction n}$ since $X/(A \restriction n) \subseteq X_{A\restriction n}$. Then by the definition of $X_{A\restriction n}$, $A \restriction n \cup A(n) \in T$, that is, $A \restriction (n + 1) \in T$ as required.

Now suppose $s \sqsubseteq X$ and $[X] \in \mathcal{U}$. Starting with the stem s we construct T recursively. For $t \in T$, let the set of the immediate descendants of t in T be $\{t \cup b : b \in \mathcal{R}_\alpha(|t|)|X\}$. Note that $[X] \in \mathcal{U}$ implies that T is a $\mathcal{U}$-tree. $\square$

Theorem 6.29. *For every finite Souslin-measurable colouring of $\mathcal{R}_\alpha$ there exists $[X] \in \mathcal{U}$ such that $[\emptyset, X]$ is monochromatic.*

Proof. Without loss of generality, consider a two colouring of $\mathcal{R}_\alpha$ given by $\mathcal{R}_\alpha = \mathcal{X} \cup \mathcal{X}^c$ where $\mathcal{X}$ is Souslin-measurable. By Corollary 6.27, $\mathcal{X}$ is $\mathcal{U}$-Ramsey, so there is a $\mathcal{U}$-tree T such that $[T] \subseteq \mathcal{X}$ or $\mathcal{X}^c$. Then by Lemma 6.28, there exists $[X] \in \mathcal{U}$ such that $[\emptyset, X] \subseteq [T] \subseteq \mathcal{X}$ or $\mathcal{X}^c$ as required. $\square$

Now, as in Section 5.3 for the Milliken space, we define uniform families in $\mathcal{R}_\alpha$, in order to obtain Theorem 6.32 below.

Notation. For $\mathcal{S} \subseteq \mathcal{AR}_\alpha$ and $a \in \mathcal{AR}_\alpha$, let $\mathcal{S}_{[a]} = \{y : a \sqsubseteq y \in \mathcal{S}\}$.

Definition 6.30. Let γ be a countable ordinal and $\mathcal{S} \subseteq \mathcal{AR}_\alpha$. Let $X \in \mathcal{R}_\alpha$ and $b \in \mathcal{AR}_\alpha$. We say $\mathcal{S}$ is a γ-*uniform family* on $[b, X]$ if

- $\gamma = 0$ and $\mathcal{S} = \{b\}$; or
- $\gamma = \beta + 1$, $b \notin \mathcal{S}$ and $\mathcal{S}_{[a]}$ is β-uniform on $[a, X]$ for every $a \in \mathcal{AR}^\alpha_{|b|+1}[b, X]$; or
- γ is a limit ordinal, $b \notin \mathcal{S}$ and there exists a sequence $(\gamma_a)_{a \in \mathcal{AR}^\alpha_{|b|+1}[b,X]}$ of ordinals, with $\bigcup\{\gamma_a : a \in \mathcal{AR}^\alpha_{|b|+1}[b, X]\} = \gamma$, such that $\mathcal{S}_{[a]}$ is γ_a-uniform on $[a, X]$ for every $a \in \mathcal{AR}^\alpha_{|b|+1}[b, X]$.

We say $\mathcal{S}$ is a *uniform family* on $[b, X]$ if it is γ-uniform on $[b, X]$ for some $\gamma < \omega_1$.

For example, if $n < \omega$, the only n-uniform family on $[b, X]$ is $\mathcal{AR}^\alpha_{|b|+n}[b, X]$. For every $k \in \omega$, the family $\mathcal{S} = \{y : (b \sqsubseteq y \leq X) \wedge (|y| = \|y(|b| + 1)\| + k)\}$ is ω-uniform, and the family $\mathcal{T} = \{y : (b \sqsubseteq y \leq X) \wedge (|y| = \|y(|b| + 2)\| + k)\}$ is $(\omega + 1)$-uniform.

As Lemma 5.17 and Lemma 5.18 in the Milliken space, the following lemma holds in $\mathcal{R}_\alpha$.

Lemma 6.31. *Let $b \in \mathcal{AR}_\alpha$ and $X \in \mathcal{R}_\alpha$. If $\mathcal{S}$ is a uniform family on $[b, X]$, then it is a front on $[b, X]$. Conversely, if $\mathcal{F}$ is a front on $[b, X]$, then there exists a uniform family $\mathcal{S}$ on $[b, X]$ such that every $s \in \mathcal{S}$ as an initial segment in $\mathcal{F}$.*

A proof similar to that of Theorem 5.19 gives the following theorem.

Theorem 6.32. *For every open set $\mathcal{O} \subseteq \mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ there exist $[X] \in \mathcal{U}$ and $p \in \mathcal{P}_\omega$ such that $[\emptyset, X] \times [p] \subseteq \mathcal{O}$ or $[\emptyset, X] \times [p] \cap \mathcal{O} = \emptyset$.*

6.4.2 Perfectly $\mathcal{U}$-Ramsey sets

In this subsection, we strengthen Theorem 6.32 from open sets $\mathcal{O}$ to all Souslin-measurable sets $\mathcal{B}$. The procedure is similar to that in Section 5.3.2.

Definition 6.33. A subset $\mathcal{B} \subseteq \mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ is *perfectly $\mathcal{U}$-Ramsey* if for all $\mathcal{U}$-tree T and $p \in \mathcal{P}_\omega$ there exist $T' \leq^0 T$ and $p' \leq p$ such that $[T'] \times [p'] \subseteq \mathcal{B}$ or $[T'] \times [p'] \cap \mathcal{B} = \emptyset$. We say $\mathcal{B}$ is *perfectly $\mathcal{U}$-Ramsey null* if the second alternative always holds.

The following lemma is immediate.

Lemma 6.34. *A subset $\mathcal{B} \subseteq \mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ is perfectly $\mathcal{U}$-Ramsey if and only if for arbitrary $n \in \omega$, $\mathcal{U}$-tree T and $p \in \mathcal{P}_\omega$,*

$$\exists T' \leq^0 T\ \exists p' \leq^n p\ \forall \sigma \in l(n, p') \quad [T'] \times [p'|\sigma] \subseteq \mathcal{B} \text{ or } [T'] \times [p'|\sigma] \cap \mathcal{B} = \emptyset.$$

For $a \in \mathcal{AR}_\alpha$, define $\mathcal{R}^{(a)}_\alpha$ to be the space consisting of elements in $[a, \mathbb{T}]$ with their stem a chopped off. Recall that for $a \in \mathcal{AR}_\alpha$ $|a|$ is the length of a, i.e. $|a| = m$ if $a = X \restriction m$ for some $X \in \mathcal{R}_\alpha$. Note that $|a|$ is different from $\|a\|$.

Definition 6.35. For $a \in \mathcal{AR}_\alpha$, let $\mathcal{R}^{(a)}_\alpha = \{Y \setminus a : Y \in [a, \mathbb{T}]\}$. Let $k = \max \|a\| + 1$. If $X = Y \setminus a \in \mathcal{R}^{(a)}_\alpha$ and $n \in \omega$, then we define $X(n) = Y(n) - k$.

More precisely, we define the space $\mathcal{R}^{(a)}_\alpha$ as follows. Let $\mathbb{T}^{(k)} = \bigcup_{n \geq k} \mathbb{T}_\alpha(n)$. Then the *nth subtree of* $\mathbb{T}^{(k)}$ is $\mathbb{T}^{(k)}(n) = \mathbb{T}_\alpha(n+k)$. The members of $\mathcal{R}^{(a)}_\alpha$ are infinite subtrees X of $\mathbb{T}^{(k)}$ with the same structure as $\mathbb{T}^{(|a|)}$, that is, $X \in \mathcal{R}_\alpha^{(a)}$ if $X \subseteq \mathbb{T}^{(k)}$ such that there exists a strictly increasing sequence $(k_n)_{n<\omega}$ such that $X \cap \mathbb{T}^{(k)}(k_n) \cong \mathbb{T}^{(|a|)}(n)$ for all $n \in \omega$, and $X \cap \mathbb{T}^{(k)}(j) = \emptyset$ for all $j \in \omega \setminus (k_n)_{n\in\omega}$. Let the *nth tree of X* be $X(n) = X \cap \mathbb{T}^{(k)}(k_n)$. For $n < \omega, \restriction, \mathcal{AR}_\alpha^{(a)}, \mathcal{AR}_n^{\alpha\,(a)}, \leq^{(a)}$ are defined in the same way as those for $\mathcal{R}_\alpha$. Basic open sets of $\mathcal{R}_\alpha^{(a)}$ are of the form $[b, \mathbb{T}^{(k)}]$ with $b \in \mathcal{AR}_\alpha^{(a)}$.

It follows from the fact that $(\mathcal{R}_\alpha, \leq, r)$ is a topological Ramsey space that $(\mathcal{R}_\alpha^{(a)}, \leq^{(a)}, r)$ is also a topological Ramsey space.

Theorem 6.36. *The space $(\mathcal{R}_\alpha^{(a)}, \leq^{(a)}, r)$ is a topological Ramsey space.*

Definition 6.37. An ultrafilter $\mathcal{U}_a$ on the base set $[\mathbb{T}^{(k)}]$ is *selective* if it is generated by sets of the form $[X]$ with $X \in \mathcal{R}_\alpha^{(a)}$ such that for every family $\{[X_b] : b \in \mathcal{AR}_\alpha^{(a)}\} \subseteq \mathcal{U}_a$ there exists $[X] \in \mathcal{U}_a$ such that $X/b \subseteq X_b$ for all $b \leq^{(a)} X$. Moreover, it is *Nash-Williams* if, in addition, for every Nash-Williams subset $\mathcal{G} \subseteq \mathcal{AR}_\alpha^{(a)}$ and partition $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1$ there exist $[X] \in \mathcal{U}_a$ and $i \in 2$ such that $\mathcal{G}_i|X = \emptyset$.

Theorem 6.38. *If $\mathcal{U}_a$ is a Nash-Williams ultrafilter on $[\mathbb{T}^{(k)}]$ then for every open set $\mathcal{B} \subseteq \mathcal{R}_\alpha^{(a)} \times \mathbb{R}^{\mathbb{N}}$ there exist $[X] \in \mathcal{U}_a$ and $p \in \mathcal{P}_\omega$ such that $[\emptyset, X] \times [p] \subseteq \mathcal{B}$ or $[\emptyset, X] \times [p] \cap \mathcal{B} = \emptyset$.*

The proof follows closely from that in Section 6.4.1 leading to Theorem 6.32.

Lemma 6.39. *If $\mathcal{B} \subseteq \mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ is open, then $\mathcal{B}$ is perfectly $\mathcal{U}$-Ramsey.*

Proof. Let a $\mathcal{U}$-tree T and $p \in \mathcal{P}_\omega$ be given. Let $a = \mathrm{st}(T)$. Let $\mathcal{B}_a = \{(X/a, \varepsilon) : (a \sqsubseteq X) \wedge ((X, \varepsilon) \in \mathcal{B})\} \subseteq \mathcal{R}_\alpha^{(a)} \times \mathbb{R}^{\mathbb{N}}$, and $\mathcal{U}_a$ be the ultrafilter generated by $\{[X/a] : (a \sqsubseteq X) \wedge ([X] \in \mathcal{U})\}$. Note that $\mathcal{B}_a$ is open in $\mathcal{R}_\alpha^{(a)} \times \mathbb{R}^{\mathbb{N}}$. We check that $\mathcal{U}_a$ is Nash-Williams: Let $\mathcal{G} \subseteq \mathcal{AR}_\alpha^{(a)}$ be Nash-Williams and $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1$. For $i = 0, 1$, let $\mathcal{G}'_i = \{a \cup y : y \in \mathcal{G}_i\}$, and similarly for $\mathcal{G}'$. We obtain a Nash-Williams subset $\mathcal{G}' \subseteq \mathcal{AR}_\alpha$ and a partition $\mathcal{G}' = \mathcal{G}'_0 \sqcup \mathcal{G}'_1$. Since $\mathcal{U}$ is Nash-Williams, there exists $[X] \in \mathcal{U}$ and $i \in 2$ such that $\mathcal{G}'_i|X = \emptyset$. So $[X/a] \in \mathcal{U}_a$ and $\mathcal{G}_i|(X/a) = \emptyset$.

So by Theorem 6.38 there exist $[X_a] \in \mathcal{U}_a$ and $p' \leq p$ such that $[\emptyset, X_a] \times [p'] \subseteq \mathcal{B}_a$ or $\mathcal{B}_a^c$. Thus $[a, a \cup X_a] \times [p'] \subseteq \mathcal{B}$ or $[a, a \cup X_a] \times [p'] \cap \mathcal{B} = \emptyset$. Hence by Lemma 6.28 we can find a $\mathcal{U}$-tree T' with stem a such that $[T'] \times [p'] \subseteq \mathcal{B}$ or $\mathcal{B}^c$. ☐

Then by a standard procedure (see [46, §2.2]), we have the following.

Theorem 6.40. *The field of perfectly $\mathcal{U}$-Ramsey subsets of $\mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ is closed under the Souslin operation.*

Corollary 6.41. *Every Souslin-measurable subset of $\mathcal{R}_\alpha \times \mathbb{R}^{\mathbb{N}}$ is perfectly $\mathcal{U}$-Ramsey.*

Restating the above corollary, we obtain the Local Parametrized $\mathcal{R}_\alpha$ Theorem 6.14.

6.5 Preservation under Sacks Forcing

Exactly the same as in Section 5.4, using Theorem 6.14, we have the following theorem saying that the upward closure of $\mathcal{U}$ is still "ultra" in the extension.

Theorem 6.42. *If $p \in \mathcal{P}_\kappa$ and $p \Vdash \tau \subseteq [\mathbb{T}_\alpha]$, then there exist $q \leq p$ and $[B] \in \mathcal{U}$ such that $q \Vdash ([B] \subseteq \tau)$ or $q \Vdash ([B] \cap \tau = \emptyset)$.*

It is then straightforward to show that the upward closure is selective in the extension. Now we check that it is also Nash-Williams in the extension.

Lemma 5.32 and Corollary 5.33 will again be used. We restate them here.

Lemma 5.32 ([3]). *Suppose $p \in \mathcal{P}_\kappa$ and $n \in \omega$. If $q \leq p$ then there exists $\sigma \in l(n,p)$ such that q and $p|\sigma$ are compatible.*

Corollary 5.33 ([3]). *Suppose $p \in \mathcal{P}_\kappa$ and $n \in \omega$. If $p \Vdash \dot{\xi} \in V$ then there exists $q \leq^n p$ such that for each $\sigma \in l(n,q)$ there exists $a_\sigma \in V$ such that $q|\sigma \Vdash \dot{\xi} = a_\sigma$.*

Theorem 6.43. *Suppose $p \in \mathcal{P}_\kappa$ and $p \Vdash ((\mathcal{G} \subseteq \mathcal{AR}_\alpha$ is Nash-Williams$) \wedge (\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1))$. Then there exists $q \leq p$, $i < 2$ and $[X] \in \mathcal{U}$ such that $q \Vdash (\mathcal{G}_i|X = \emptyset)$.*

Proof. Let $p \in \mathcal{P}_\kappa$ as in the statement of the theorem. We may consider $\mathcal{G} \subseteq \mathcal{AR}_\alpha$ with $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1$ as a function $g : \mathcal{AR}_\alpha \to 3$ given by the following formula. For $a \in \mathcal{AR}_\alpha$,

$$g(a) = \begin{cases} 0 & \text{if } a \in \mathcal{G}_0; \\ 1 & \text{if } a \in \mathcal{G}_1; \\ 2 & \text{if } a \in \mathcal{AR}_\alpha \setminus \mathcal{G}. \end{cases}$$

Since $\mathcal{AR}_\alpha$ is countable, we enumerate it as $\mathcal{AR}_\alpha = \{a_k : k \in \omega\}$.

We construct $(p_k)_k$ recursively as follows, starting with $p_{-1} = p$. Suppose we have $p_k \leq p$. By Corollary 5.33, there exists $p_{k+1} \leq^{k+1} p_k$ such that for every $\sigma \in l(k+1, p_k)$ there is $i_\sigma \in 3$ such that $p_{k+1} \Vdash (g(a_{k+1}) = i_\sigma)$. Let $p_\infty = \left(\bigcap_{k\in\omega} p_k^i\right)_{i<\omega}$. Then $p_\infty \in \mathcal{P}_\kappa$ and for each $k < \omega$,

$$\forall \sigma \in l(k, p_\infty)\ \exists i_\sigma \in 3 \quad p_\infty \Vdash (g(a_k) = i_\sigma).$$

Let

$$\begin{aligned} \mathcal{F} &= \{(a_k, \varepsilon) : (\exists \sigma \in l(k, p_\infty))((\forall i < k)(\sigma^i \in \varepsilon^i) \wedge (p_\infty|\sigma \Vdash g(a_k) = 0 \text{ or } 2))\} \\ &\subseteq \mathcal{AR}_\alpha \times [p_\infty], \text{ and} \\ \mathcal{X} &= \{(X, \varepsilon) : (\forall a_k \sqsubseteq X)((a_k, \varepsilon) \in \mathcal{F})\} \subseteq \mathcal{R}_\alpha \times [p_\infty]. \end{aligned}$$

Note that $\mathcal{X}^c = \{(X, \varepsilon) : (\exists a_k \sqsubseteq X)((a_k, \varepsilon) \notin \mathcal{F})\}$ is an open subset of $\mathcal{R}_\alpha \times [p_\infty]$. Then by Theorem 6.14, there exists $[X] \in \mathcal{U}$ and $q \leq p_\infty$ such that $[\emptyset, X] \times [q] \subseteq \mathcal{X}$ or $[\emptyset, X] \times [q] \cap \mathcal{X} = \emptyset$.

Suppose first $[\emptyset, X] \times [q] \subseteq \mathcal{X}$. We show that $a_k \leq X$ implies $q \Vdash (g(a_k) = 0 \text{ or } 2)$, so $q \Vdash (\mathcal{G}_1|X = \emptyset)$: Assume $r \leq q$ is such that $r \Vdash (g(a_k) = 1)$. We aim for a contradiction. By Lemma 5.32, there is $\sigma \in l(k, k, p_\infty)$ such that r is compatible with $q|\sigma$ and $q|\sigma \neq \emptyset$. Let $\varepsilon \in [q]$ be such that $\sigma \sqsubseteq^* \varepsilon$, and let $Y \leq X$ be such that $a_k \sqsubseteq Y$. As $(Y, \varepsilon) \in [\emptyset, X] \times [q] \subseteq \mathcal{X}$, $(a_k, \varepsilon) \in \mathcal{F}$. Hence $q|\sigma \leq p_\infty|\sigma \Vdash (g(a_k) = 0 \text{ or } 2)$, contradicting that $r \Vdash g(a_k) = 1$ and r is compatible with $q|\sigma$.

Now suppose $[\emptyset, X] \times [q] \subseteq \mathcal{X}^c$. We check that $a_k \leq X$ implies $q \Vdash (\exists a_l \leq X)(((a_l \sqsubseteq a_k) \vee (a_l \sqsupseteq a_k)) \wedge (g(a_l) = 1))$. Then, as $q \Vdash (\mathcal{G}$ is Nash-Williams$)$, $q \Vdash (\mathcal{G}_0|X = \emptyset)$. Assume $r \leq q$ is such that $r \Vdash ((\forall a_l \leq X)((a_l \sqsubseteq a_k \text{ or } a_l \sqsupseteq a_k) \to (g(a_l) = 0 \text{ or } 2))$. We aim for a contradiction. Let $\varepsilon \in [r]$ and $Y \leq X$ be such that $a_k \sqsubseteq Y$. Then $(Y, \varepsilon) \in \mathcal{X}^c$. So there is $a_l \sqsubseteq Y$ such that $(a_l, \varepsilon) \notin \mathcal{F}$. Let $\tau \in l(l, p_\infty)$ be such that $\tau \sqsubseteq^* \varepsilon$. Then $p_\infty|\tau \Vdash (g(a_l) = 1)$. This contradicts that $r|\tau \leq p_\infty|\tau$ and $r|\tau \leq r \Vdash ((\forall a_l \leq X)((a_l \sqsubseteq a_k \text{ or } a_l \sqsupseteq a_k) \to (g(a_l) = 0 \text{ or } 2))$. □

Thus we have shown that every Nash-Williams ultrafilter in $\mathcal{R}_\alpha$ is preserved under countable-support side-by-side Sacks forcing:

Theorem 6.44. *Let α be a countable ordinal, κ be an infinite cardinal and $\mathcal{P}_\kappa$ be countable-support side-by-side Sacks forcing adding κ Sacks reals. Let $\mathcal{U}$ be a Nash-Williams ultrafilter in $\mathcal{R}_\alpha$ in the ground model, and $\dot{\mathcal{V}}$ a name for the upward closure $\{Y : (\exists[X] \in \mathcal{U})([X] \subseteq Y)\}$ of $\mathcal{U}$. Then $\Vdash_{\mathcal{P}_\kappa}$ ($\dot{\mathcal{V}}$ is a Nash-Williams ultrafilter in $\mathcal{R}_\alpha$).*

6.5.1 Necessity of Nash-Williams

In this subsection, we show that the ultrafilter being Nash-Williams is a necessary condition for the Local Parametrized $\mathcal{R}_\alpha$ Theorem 6.14 to hold.

For a subset $\mathcal{X} \subseteq \mathcal{R}_\alpha$, we say $\mathcal{X}$ is *weakly $\mathcal{U}$-Ramsey* if for every $[X] \in \mathcal{U}$ there exists $[Y] \in \mathcal{U}$ with $Y \leq X$ such that either $[\emptyset, Y] \subseteq \mathcal{X}$ or $[\emptyset, Y] \cap \mathcal{X} = \emptyset$. Note that, if $\mathcal{U}$ is selective, then by Lemma 6.28, every $\mathcal{U}$-Ramsey set is weakly $\mathcal{U}$-Ramsey.

Lemma 6.45. *If every open subset of $\mathcal{R}_\alpha$ is weakly $\mathcal{U}$-Ramsey, then $\mathcal{U}$ is Nash-Williams. Namely,*

$$(\forall \mathcal{X} \subseteq \mathcal{R}_\alpha \text{ open})(\forall [X] \in \mathcal{U})(\exists [Y] \in \mathcal{U})((Y \leq X) \wedge ([\emptyset, Y] \subseteq \mathcal{X}) \vee ([\emptyset, Y] \cap \mathcal{X} = \emptyset))$$
$$\Rightarrow (\forall \mathcal{G} \subseteq \mathcal{AR}_\alpha \text{ Nash-Williams})(\forall \mathcal{G}_0 \sqcup \mathcal{G}_1 = \mathcal{G})(\exists [X] \in \mathcal{U})(\exists i \in 2)(\mathcal{G}_i | X = \emptyset).$$

Proof. Let $\mathcal{G} \subseteq \mathcal{AR}_\alpha$ be Nash-Williams, $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1$. Then the set $\mathcal{X} = \bigcup_{a \in \mathcal{G}_0} [a, \mathbb{T}]$ is open in $\mathcal{R}_\alpha$. As every open set is weakly $\mathcal{U}$-Ramsey, there is $[Y] \in \mathcal{U}$ such that $Y \leq X$ and $[\emptyset, Y] \subseteq \mathcal{X}$ or $[\emptyset, Y] \cap \mathcal{X} = \emptyset$.

In the first case, $[\emptyset, Y] \subseteq \bigcup_{a \in \mathcal{F}_0} [a, \mathbb{T}]$. So for every $Z \leq Y$, Z has an initial segment in $\mathcal{G}_0$. But $\mathcal{G}$ is Nash-Williams so Z cannot have initial segment in $\mathcal{G}_1$. i.e. $a \notin \mathcal{G}_1$ for all $a \leq Y$, so $\mathcal{G}_1 | Y = \emptyset$. In the second case, $[\emptyset, Y] \cap \left(\bigcup_{a \in \mathcal{G}_0} [a, \mathbb{T}]\right) = \emptyset$, so $a \notin \mathcal{G}_0$ for all $a \leq Y$, i.e. $\mathcal{G}_0 | Y = \emptyset$. □

From the above lemma, we know that if $\mathcal{U}$ is selective and every open subset of $\mathcal{R}_\alpha$ is $\mathcal{U}$-Ramsey, then $\mathcal{U}$ must be Nash-Williams. Moreover, it follows from the Local Parametrized $\mathcal{R}_\alpha$ Theorem 6.14 that every open subset of $\mathcal{R}_\alpha$ is weakly $\mathcal{U}$-Ramsey. So for Theorem 6.14 to hold for a selective ultrafilter, it is necessary that the ultrafilter is Nash-Williams.

Chapter 7

High-dimensional Ellentuck Spaces

In this chapter, we look into High-dimensional Ellentuck spaces. These are spaces generalizing the Ellentuck space. Motivated by the result in [8] that the generic ultrafilter $\mathcal{G}_2$ forced by $\mathcal{P}(\omega^2)/\left(\mathrm{FIN}^{\otimes 2}\right)$ is neither maximum nor minimum in the Tukey order of ultrafilters, Dobrinen [16] proved that the collection of ultrafilters Tukey reducible to $\mathcal{G}_2$ forms a chain of length two under the partial ordering of Tukey reducibiity. Dobrinen [16] further constructed the topological Ramsey spaces $\mathcal{E}_k$ ($k \geq 2$), each being a dense subset of $(\mathrm{FIN}^{\otimes k})^+$, to show that the ultrafilters Tukey reducible to $\mathcal{G}_k$ form a chain of length k, where $\mathcal{G}_k$ are generic ultrafilters forced by $\mathcal{P}(\omega^k)/\mathrm{FIN}^{\otimes k}$. It is also claimed that these forcings have complete combinatorics ([16]).

The term $\mathrm{FIN}^{\otimes k}$ is defined as follows for $k \geq 2$. Let $\mathcal{F}$ be the Fréchet filter consisting of cofinite subsets of ω,

$$\mathcal{F} = \{B \subseteq \omega : \omega \setminus B \text{ is finite}\}.$$

Then a non-decreasing sequence A of length k is in $\mathrm{FIN}^{\otimes k}$ if and only if

$$\mathcal{F}n_1 \cdots \mathcal{F}n_k \quad (n_1, \ldots, n_k) \in A.$$

Two sets A, B satisfies $A \subseteq^{\mathrm{FIN}^{\otimes k}} B$ if and only if $A \setminus B \in \mathrm{FIN}^{\otimes k}$.

As the High-dimensional Ellentuck spaces generalizes the Ellentuck space, the following Parametrized High-dimensional Ellentuck Theorem extends the Parametrized Ellentuck Theorem [42, Thm. 4.43], which is the parametrization of the Ellentuck Theorem with countable sequences of perfect sets.

Theorem 7.1 (Parametrized High-dimensional Ellentuck Theorem)**.** *For every finite Souslin-measurable colouring of $\mathcal{E}_k \times \mathbb{R}^{\mathbb{N}}$ and for every $A \in \mathcal{E}_k, a \in \mathcal{AR}[\emptyset, A]$ and $p \in \mathcal{P}_\omega$ there exists $B \in [a, A]$ and $q \leq p$ such that $[a, B] \times [q]$ is monochromatic.*

Theorem 7.1 is in turn applied to obtain the preservation of the generic ultrafilter $\mathcal{G}_k$ forced by $\mathcal{P}(\omega^k)/\mathrm{FIN}^{\otimes k}$ under countable-support side-by-side Sacks forcing. Since $\mathcal{P}(\omega^k)/\mathrm{FIN}^{\otimes k}$ is forcing equivalent to $(\mathcal{E}_k, \subseteq^{\mathrm{FIN}^{\otimes k}})$ ([16]), we limit our scope to $\mathcal{B}_k := \mathcal{G}_k \cap \mathcal{E}_k$.

Theorem 7.2. *Let $k \in \omega$, κ be an infinite cardinal and $\mathcal{P}_\kappa$ be countable-support side-by-side Sacks forcing adding κ Sacks reals. Let $\mathcal{B}_k$ be a generic filter for $(\mathcal{E}_k, \subseteq^{\mathrm{FIN}^{\otimes k}})$, and $\dot{\mathcal{V}}$ a name for the upward closure $\{Y : (\exists X \in \mathcal{B}_k)(X \leq Y)\}$ of $\mathcal{B}_k$. Then $\Vdash_{\mathcal{P}_\kappa}$ ($\dot{\mathcal{V}}$ is a Nash-Williams ultrafilter in $\mathcal{E}_k$).*

In the previous chapters, we saw the preservation of ultrafilters under Sacks forcing in the Ellentuck space, the Milliken space, and the spaces $\mathcal{R}_\alpha$ $(\alpha < \omega_1)$. In these cases, the ultrafilters are p-points, whereas the ultrafilter $\mathcal{G}_k$ in the High-dimensional Ellentuck spaces $\mathcal{E}_k$ $(k < \omega)$ are not p-points.

7.1 High-dimensional Ellentuck Spaces $\mathcal{E}_k$

In [16] Dobrinen presented a hierarchy $(\mathcal{E}_k)_{k<\omega}$ of topological Ramsey spaces. The Ellentuck space naturally fits in the hierarchy as $\mathcal{E}_1$. The members of $\mathcal{E}_{k+1}$ look like ω many copies of the members of $\mathcal{E}_k$. Therefore, $\mathcal{E}_k$ are called *High-dimensional Ellentuck spaces*. Let us recall the structure of $\mathcal{E}_k$ from [16], for an arbitrarily fixed integer $k \geq 2$.

Let $\omega^{\not\downarrow k}$ (resp. $\omega^{\not\downarrow \leq k}$) be the set of non-decreasing sequences of members of ω with length k (resp. $\leq k$). The set $\omega^{\not\downarrow \leq k}$ is equipped with a well-ordering $\prec$ of order-type ω.

Definition 7.3. For $\vec{j} = (j_0, \ldots, j_{p-1})$ and $\vec{l} = (l_0, \ldots, l_{q-1})$ in $\omega^{\not\downarrow \leq k}$, we write $\vec{j} \leq_{\text{lex}} \vec{l}$ if $\vec{j} \sqsubseteq \vec{l}$ or $\vec{j}$ precedes $\vec{l}$ in the lexicographical ordering; $\vec{j} <_{\text{lex}} \vec{l}$ if $\vec{j} \leq_{\text{lex}} \vec{l}$ and $\vec{j} \neq \vec{l}$; $\vec{j} \prec \vec{l}$ if either $j_{p-1} < l_{q-1}$ or $j_{p-1} = l_{q-1}$ and $\vec{j} <_{\text{lex}} \vec{l}$.

Note that we use $\sqsubseteq$ as initial segments between sequences here. The first few elements in $(\omega^{\not\downarrow \leq 2}, \prec)$ are as follows:

$$(0), (0,0), (0,1), (1), (1,1), (0,2), (1,2), (2), (2,2), (0,3), \ldots.$$

The definitions below from [1] are equivalent to their original forms in [16].

Definition 7.4 ([1])**.** A function $\hat{X} : \omega^{\not\downarrow \leq k} \to \omega^{\not\downarrow \leq k}$ is an $\mathcal{E}_k$*-tree* if it preserves the well-ordering $\prec$ and initial segments $\sqsubseteq$ on $\omega^{\not\downarrow \leq k}$. For an $\mathcal{E}_k$-tree $\hat{X}$, let X be the restriction of $\hat{X}$ to $\omega^{\not\downarrow k}$. We may identify X with its range and enumerate its elements $\prec$-increasingly.

Definition 7.5 (Space $(\mathcal{E}_k, \leq, r)$)**.** Let $\mathcal{E}_k$ be the set of all X such that $\hat{X}$ is an $\mathcal{E}_k$-tree. For $X, Y \in \mathcal{E}_k$, $Y \leq X$ if (the range of) Y is a subset of (the range of) X. If $X \in \mathcal{E}_k$ is enumerated $\prec$-increasingly as $X = (\vec{v_i})_{i<\omega}$ then $X \subseteq \omega^{\not\downarrow k} \subseteq [\omega]^k$ and we write

$$[X] = \{\max(\vec{v_i}) : i < \omega\} \subseteq \omega;$$

for $n < \omega$, let

$$r_n(X) = \{\vec{v_1}, \ldots, \vec{v_n}\}.$$

We write $\mathcal{AE}_n^k$ for the set $\mathcal{AR}_n$ when $\mathcal{R} = \mathcal{E}_k$.

Note that $\omega^{\not\downarrow k}$ is the greatest element in $\mathcal{E}_k$. Similar to $[X]$, for $S \subseteq [\omega]^k$, let

$$[S] = \{\max(\zeta) : \zeta \in S\}.$$

Definition 7.6. For $\vec{j} \in \omega^{\not\downarrow \leq k}$, let $|\vec{j}|$ denote the length of the sequence $\vec{j}$. For $l < |\vec{j}|$, we defined the map π_l as follows so that it projects members of $\omega^{\not\downarrow \leq k}$ to their initial segments of length l: if $\vec{j} = (j_m)_{m<|\vec{j}|}$, then let

$$\pi_l(\vec{j}) = (j_m)_{m<l}.$$

Enumerate $\omega^{\not\downarrow k}$ $\prec$-increasingly as $(i_n)_{n<\omega}$. For $l<k$, we define the set $N_l^k \subseteq \omega$ by setting $n \in N_l^k$ if and only if

$$l = \max\{l' : (\exists m < n)(\pi_{l'}(\vec{i}_{n+1} \sqsubseteq \vec{i}_m))\}.$$

Since the elements of $\mathcal{E}_k$ preserves $\prec$ and $\sqsubseteq$, $n \in N_l^k$ if and only if for an nth approximation a in $\mathcal{AE}_n^k$, every 1-extension of a has an extra branch, branching off from the lth level in the tree a. (We draw trees upwards, with the root $\emptyset$ at the bottom, being level 0.)

7.2 Ultrafilters in $\mathcal{E}_k$

Unfortunately the method we used in previous chapters to prove the relations among ultrafilters does not apply to High-dimensional Ellentuck Spaces $\mathcal{E}_k$ for $k \geq 2$. We know that every selective ultrafilter is weakly selective from their definitions. But the relations among other properties are not clear, since $\mathcal{E}_k$, for $k \geq 2$, does not satisfy **(A8)** as Example 4 shows. It also remains to be seen whether $\mathcal{E}_k$ has a head start or not. Hence in High-dimensional Ellentuck spaces, we do not yet know if every Ramsey ultrafilter is weakly selective, or if every Nash-Williams ultrafilter is Ramsey.

To better illustrate the example, we use the upper triangular representation of $\omega \not\downarrow 2$ introduced in [1]. The idea of this representation is visualised through the Figures 7.1, 7.2 and Tables 7.1, 7.2.

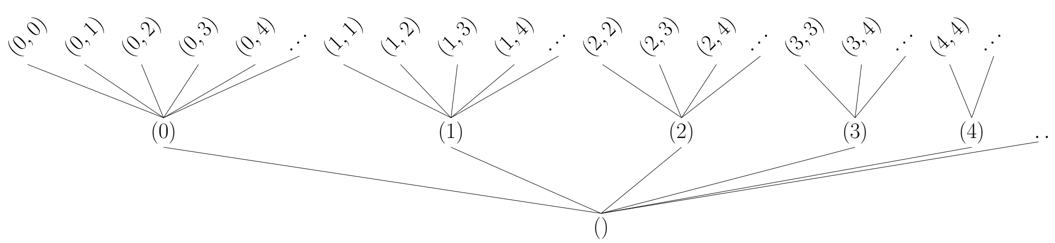

Figure 7.1: $\mathcal{E}_2$-tree $\omega^{\not\downarrow \leq 2}$

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	⋯
	(1,1)	(1,2)	(1,3)	(1,4)	⋯
		(2,2)	(2,3)	(2,4)	⋯
			(3,3)	(3,4)	⋯
				(4,4)	⋯
					⋱

Table 7.1: Upper triangular representation of $\omega^{\not\downarrow 2}$

Example 4. Recall that **(A8)** is the following property.

(A8) For arbitrary $n \in \omega$, $A, B \in \mathcal{R}$ and $b \in \mathcal{AR}_n(B)$, if $r_{n+1}[b,B] \subseteq r_{n+1}[b,A]$ then $[b,B] \subseteq [b,A]$.

Let $n=1$. We give an example of $A, B \in \mathcal{E}_2$, $b \in \mathcal{AE}_1^2(B)$ with $r_2[b,B] \subseteq r_2[b,A]$ but $[b,B] \not\subseteq [b,A]$. Let

$$\begin{aligned}
b &= r_1(\omega^{\not\downarrow 2}) = \{(0,0)\}, \\
A &= \{(2m,2n) : 0 \leq m \leq n < \omega\}, \\
B &= \{(0,2n) : 0 \leq n < \omega\} \cup \{(2m+1,2n+1) : 0 \leq m \leq n < \omega\}.
\end{aligned}$$

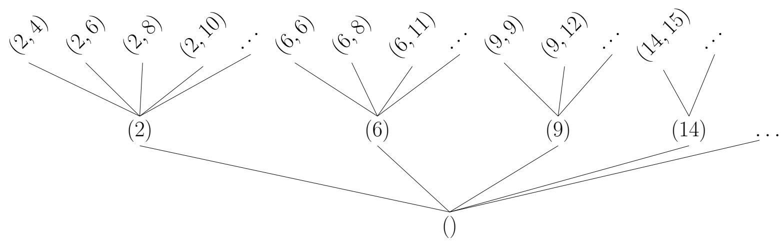

Figure 7.2: An example $\hat{X}$ of $\mathcal{E}_2$-tree, where

$$X = \{(2,4),(2,6),(6,6),(2,8),(6,8),(9,9),(2,10),(6,11),(9,12),(14,15),\dots\}$$

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)	(0,12)	(0,13)	(0,14)	(0,15)	⋯
	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)	(1,10)	(1,11)	(1,12)	(1,13)	(1,14)	(1,15)	⋯
		(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)	(2,10)	(2,11)	(2,12)	(2,13)	(2,14)	(2,15)	⋯
			(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	(3,9)	(3,10)	(3,11)	(3,12)	(3,13)	(3,14)	(3,15)	⋯
				(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	(4,9)	(4,10)	(4,11)	(4,12)	(4,13)	(4,14)	(4,15)	⋯
					(5,5)	(5,6)	(5,7)	(5,8)	(5,9)	(5,10)	(5,11)	(5,12)	(5,13)	(5,14)	(5,15)	⋯
						(6,6)	(6,7)	(6,8)	(6,9)	(6,10)	(6,11)	(6,12)	(6,13)	(6,14)	(6,15)	⋯
							(7,7)	(7,8)	(7,9)	(7,10)	(7,11)	(7,12)	(7,13)	(7,14)	(7,15)	⋯
								(8,8)	(8,9)	(8,10)	(8,11)	(8,12)	(8,13)	(8,14)	(8,15)	⋯
									(9,9)	(9,10)	(9,11)	(9,12)	(9,13)	(9,14)	(9,15)	⋯
										(10,10)	(10,11)	(10,12)	(10,13)	(10,14)	(10,15)	⋯
											(11,11)	(11,12)	(11,13)	(11,14)	(11,15)	⋯
												(12,12)	(12,13)	(12,14)	(12,15)	⋯
													(13,13)	(13,14)	(13,15)	⋯
														(14,14)	(14,15)	⋯
															(15,15)	⋯
																⋱

Table 7.2: Upper triangular representation of X, with members of X shaded, where

$$X = \{(2,4),(2,6),(6,6),(2,8),(6,8),(9,9),(2,10),(6,11),(9,12),(14,15),\dots\}$$

The elements A and B are illustrated in Figures 7.3 and 7.4 with members of A and B shaded.

Unlike the case in the spaces discussed in previous chapters, the notion of ultrafilter on $\omega^{\not{\downarrow} k}$ (Definition 2.17) generated by elements of the form $X \in \mathcal{E}_k$ does not correspond to that of ultrafilters in $\mathcal{E}_k$ (Definition 2.18). Suppose $\mathcal{V}$ is an ultrafilter in $\mathcal{E}_k$. We can find an ultrafilter $\mathcal{U}$ on $\omega^{\not{\downarrow} k}$ generated by $\mathcal{V}$. But conversely, it is not necessarily the case: Given an ultrafilter $\mathcal{U}$ on $\omega^{\not{\downarrow} k}$ generated by elements of the form $X \in \mathcal{E}_k$, let $\mathcal{V} = \mathcal{U} \cap \mathcal{E}_k$. We check that $\mathcal{V}$ does not always satisfy (a) (ii) in Definition 2.18, that is, even if $A, B \in \mathcal{V}$ and $a \in \mathcal{AE}_k$ are such that $[a, A]$ and $[a, B]$ are nonempty, we may not have $C \in \mathcal{F}$ with $C \in [a, A] \cap [a, B]$ as the following example shows.

Example 5. Let $\mathcal{U}$ be an ultrafilter on $\omega^{\not{\downarrow} 2}$ generated by elements of the form $X \in \mathcal{E}_2$. Since

$$\omega^{\not{\downarrow} 2} = \{(m, 2n) : 0 \leq m \leq 2n < \omega\} \cup \{(m, 2n_1) : 0 \leq m \leq 2n + 1 < \omega\},$$

we may assume without loss of generality that

$$A = \{(m, 2n) : 0 \leq m \leq 2n < \omega\}$$

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	$\cdots$
	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	$\cdots$
		(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	$\cdots$
			(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	$\cdots$
				(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	$\cdots$
					(5,5)	(5,6)	(5,7)	(5,8)	$\cdots$
						(6,6)	(6,7)	(6,8)	$\cdots$
							(7,7)	(7,8)	$\cdots$
								(8,8)	$\cdots$
									$\ddots$

Table 7.3: Element A in Example 4

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	$\cdots$
	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	$\cdots$
		(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	$\cdots$
			(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	$\cdots$
				(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	$\cdots$
					(5,5)	(5,6)	(5,7)	(5,8)	$\cdots$
						(6,6)	(6,7)	(6,8)	$\cdots$
							(7,7)	(7,8)	$\cdots$
								(8,8)	$\cdots$
									$\ddots$

Table 7.4: Element B in Example 4

is in $\mathcal{U}$. Let

$$B = A \setminus \{(0, 2n) : 1 < n < \omega\} \cup \{(0, 2n-1) : 1 < n < \omega\}, \text{ and}$$
$$a = \{(0,0)\}$$

Note $A, B \in \mathcal{E}_2$. Also, as it does not include any element of $\mathcal{E}_2$, the set $\{(0, 2n) : 1 < n < \omega\}$ is not in $\mathcal{U}$. Hence $B \in \mathcal{U}$. The upper triangular representation of A and B are shown in Tables 7.5 and 7.6. Clearly, $[a, A]$ and $[a, B]$ are nonempty. However, any $C \in [a, A]$ must have infinitely many members of the form $(0, k)$ $(k < \omega)$ from A, and cannot be an element of $[a, B]$.

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	⋯
	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)	⋯
		(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)	⋯
			(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	(3,9)	⋯
				(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	(4,9)	⋯
					(5,5)	(5,6)	(5,7)	(5,8)	(5,9)	⋯
						(6,6)	(6,7)	(6,8)	(6,9)	⋯
							(7,7)	(7,8)	(7,9)	⋯
								(8,8)	(8,9)	⋯
									(9,9)	⋯
										⋱

Table 7.5: Element A in Example 4

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	⋯
	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)	⋯
		(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)	⋯
			(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	(3,9)	⋯
				(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	(4,9)	⋯
					(5,5)	(5,6)	(5,7)	(5,8)	(5,9)	⋯
						(6,6)	(6,7)	(6,8)	(6,9)	⋯
							(7,7)	(7,8)	(7,9)	⋯
								(8,8)	(8,9)	⋯
									(9,9)	⋯
										⋱

Table 7.6: Element B in Example 4

7.3 Parametrized High-dimensional Ellentuck Theorem

In order to prove the Parametrized High-dimensional Ellentuck Theorem 7.1 , we show that the High-dimensional Ellentuck spaces satisfy **(L4)**. We again use methods developed in [42, §9] and the Halpern-Läuchli theorem.

From now on in this section, we fix an arbitrary integer $k \geq 2$.

Definition 7.7 ([16])**.** Let $l < k$. A subset $U \subseteq \omega^{\not\downarrow k}$ is *isomorphic to a member of* $\mathcal{E}_{k-l}$ if its structure is the same as $\omega^{\not\downarrow (k-l)}$. More precisely, U is *isomorphic to a member of* $\mathcal{E}_{k-l}$ if there is a $\prec$-and-$\sqsubseteq$-preserving bijection

$$\theta : \{\vec{t} \in \hat{U} : \mathrm{st}(\hat{U}) \sqsubseteq \vec{t}\} \to \omega^{\not\downarrow(k-l)},$$

where $\hat{U}$ is the $\sqsubseteq$-downward closure of U and

$$\mathrm{st}(\hat{U}) = \sqsubseteq\text{-}\max\{\vec{s} \in \omega^{\not\downarrow \leq k} : \forall \vec{u} \in U\ \vec{s} \sqsubseteq \vec{u}\}.$$

Fact 7.8 ([16])**.** Let $l < k$, $n \in N_l^k$, $X \in \mathcal{E}_k$ and $a \in r_n[\emptyset, X]$ be given.

(1) Suppose $l \geq 1$. Let $V \subseteq r_{n+1}[a, X]$ be such that $U := \{b \setminus a : b \in V\}$ is isomorphic to a member of $\mathcal{E}_{k-l}$. Then there is a $Y \in [a, X]$ such that $r_{n+1}[a, Y] \subseteq V$.

(2) Suppose $l = 0$. Let $\omega^{\not\downarrow k} = (\vec{i}_j)_{j<\omega}$ and $X = (\vec{v}_j)_{j<\omega}$ be $\prec$-increasing enumerations. Let $I \subseteq \{m \geq n : m \in N_0^k\}$ be an infinite subset such that

   (a) for every distinct elements $m, m' \in I$, $\pi_1(\vec{i}_m) \neq \pi_1(\vec{i}_{m'})$, and

   (b) for $m \in I$, there is $U_m \subseteq \{\vec{v}_j \in X : \pi_1(\vec{i}_j) = \pi_1(\vec{i}_m)\}$ such that U_m is isomorphic to a member of $\mathcal{E}_{k-l}$.

   Then there is a $Y \in [a, X]$ such that $\mathcal{AE}^k_{n+1}[a, Y] \subseteq \bigcup_{m \in I} U_m$.

The following fact then follows.

Fact 7.9. If $S \subseteq \omega^{\not\downarrow k}$ and there exists infinitely many $n \in \pi_1(S)$ such that $\{M \in [\{m \in \omega : n < m\}]^{k-1} : \{n\} \cup M \in S\}$ is isomorphic to an element of $\mathcal{E}_{k-1}$ then there is $X \in \mathcal{E}_k$ such that $X \subseteq S$.

We will be using the infinite Halpern-Läuchli Theorem 6.9 as well as Lemma 6.11. We restate Lemma 6.11 below.

Lemma 6.11. *Suppose $M \in \mathbb{N}^{[\infty]}$ and O_l $(l \in M)$ is a family of open subsets of $(2^\omega)^\omega$. Then for every $p \in \mathcal{P}_\omega$ and $n \in \omega$ there exists $q \leq^n p$, an infinite subset $N \subseteq M$ and a clopen subset $G \subseteq [q]$ such that for every $l \in N$, $O_l \cap [q] = G$.*

Recall, for $X \in \mathcal{E}_k$, $[X] = \{\max(\vec{v}) : \vec{v} \in X\} \subseteq \mathbb{N}$. Similarly, for every $S \subseteq \omega^{\not\downarrow k}$, $[S] = \{\max(\zeta) : \zeta \in S\} \subseteq \mathbb{N}$.

Lemma 7.10. *Suppose $A \in \mathcal{E}_k$ and O_l $(l \in [A])$ is a family of open subsets of $(2^\omega)^\omega$. Then for every $p \in \mathcal{P}_\omega$ there exists $q \leq p$ and $B \leq A$ such that $O_l \cap [q]$ is constant for every $l \in [B]$.*

Proof. We prove by induction on k. When $k = 1$, the result is given by Lemma 6.11. So we assume $k \geq 2$. Let A, O_l, p be given as in the lemma. For $m \in \omega$, let

$$S_m = \{\vec{v} \in A : (m) \sqsubseteq \vec{v}\} \text{ and } M = \{m \in \omega : S_m \neq \emptyset\}.$$

Then for every $m \in M$, S_m is isomorphic to an element of $\mathcal{E}_{k-1}$.

We construct a fusion sequence $(p_n)_{n<\omega}$ and sequences $\{G_m \subseteq (2^\omega)^\omega : m \in M\}$ and $\{B_m \subseteq \omega^{\not\downarrow k} : m \in M\}$ such that for $m < \omega$:

(1) $p_{-1} = p$, $p_m \leq^m p_{m-1}$;

and for $m \in M$:

(2) $G_m \subseteq [p_m]$ is open;

(3) $B_m \subseteq S_m$ is isomorphic to an element of $\mathcal{E}_{k-1}$; and

(4) $\forall l \in [B_m] \quad O_l \cap [p_m] = G_m$.

If $m \notin M$, let $p_m = p_{m-1}$; otherwise, p_m, G_m, and B_m exist by the induction hypothesis. This finishes the construction of $(p_m)_{m<\omega}$, $(B_m)_{m\in M}$ and $(G_m)_{m\in M}$.

Let $p_\infty = \bigcap_{m<\omega} p_m$. Then for every $m \in M$ and $l \in [B_m]$, $O_l \cap [p_\infty] = G'_m$ where $G'_m = G_m \cap [p_\infty]$ which is open in $[p_\infty]$.

Now we have a family $G'_m (m \in M)$ of open subsets of $[p_\infty]$. By Lemma 6.11 again, there exists $q \leq p_\infty$, $N \in [M]^\omega$ and $G \subseteq [q]$ such that $G'_m \cap [q] = G$ for all $m \in N$. By Fact 7.9, we can find $B \in \mathcal{E}_k$ such that $B \subseteq \bigcup_{m\in N} B_m$. Then every $l \in [B]$ is in $[B_m]$ for some m, so $O_l \cap [q] = G'_m \cap [q] = G$, as required. □

Lemma 7.11. *Let $p \in \mathcal{P}_\omega$, $A \in \mathcal{E}_k$ and $a \in \mathcal{AE}_k[\emptyset, A]$. Let $\{O_b : b \in \mathcal{AE}^k_{|a|+1}[a, A]\}$ be a family of open subsets of $[p]$. Then there exists $B \in [\mathrm{depth}_A(a), A]$ and $q \leq p$ such that $O_b \cap [q]$ is constant on $b \in \mathcal{AE}^k_{|a|+1}[a, B]$.*

Proof. Let l be such that $0 \leq l < k$ and $|a| \in N^k_l$.

Suppose $l > 0$. Let $U = \{b \setminus a : b \in \mathcal{AE}^k_{|a|+1}[a, A]\}$. Then U is isomorphic to an element of $\mathcal{E}_{k-l}$. Relabel the sets O_b ($b \in \mathcal{AE}^k_{|a|+1}[a, A]$) as P_n ($n \in [U]$) where $P_{[b\setminus a]} = O_b$. Then by Lemma 7.10, there exists $V \subseteq U$ isomorphic to an element of $\mathcal{E}_{k-l}$, $q \leq p$ and $G \subseteq [q]$ such that $P_n \cap [q] = G$ for all $n \in [V]$. So $O_b \cap [q] = G$ for all $b \in \mathcal{AE}^k_{|a|+1}[a, A]$ with $b \setminus a \in V$. By Fact 7.8 (1) and **(A3)** (2), we can find $B \in [\mathrm{depth}_A(a), A]$ such that $\mathcal{AE}^k_{|a|+1}[a, B] \subseteq V$. Then $O_b \cap [q] = G$ for each $b \in \mathcal{AE}^k_{|a|+1}[a, B]$.

If $l = 0$, the result similarly follows by Lemma 7.10 and Fact 7.8 (2). □

Thus, we have proved that the High-dimensional Ellentuck spaces $\mathcal{E}_k$ ($k \geq 2$) satisfy **(L4)**. Therefore, by Theorem 1.7 we obtain the Parametrized High-dimensional Ellentuck Theorem 7.1.

Theorem 7.1 (Parametrized High-dimensional Ellentuck Theorem)**.** *For every finite Souslin-measurable colouring of $\mathcal{E}_k \times \mathbb{R}^{\mathbb{N}}$ and for every $A \in \mathcal{E}_k, a \in \mathcal{AR}[\emptyset, A]$ and $p \in \mathcal{P}_\omega$ there exists $B \in [a, A]$ and $q \leq p$ such that $[a, B] \times [q]$ is monochromatic.*

7.4 The Ultrafilter $\mathcal{B}_k$

In this section, we apply Theorem 7.1 to obtain the preservation of the generic function $\mathcal{G}_k$ forced by $\mathcal{P}(\omega^k)/\mathrm{FIN}^{\otimes k}$ under countable-support side-by-side Sacks forcing. As mentioned at the beginning of the chapter, $\mathcal{P}(\omega^k)/\mathrm{FIN}^{\otimes k}$ is forcing equivalent to $(\mathcal{E}_k, \subseteq^{\mathrm{FIN}^{\otimes k}})$, so we prove the relevant theorems for $\mathcal{B}_k = \mathcal{G}_k \cap \mathcal{E}_k$.

Lemma 7.12. *Let $\mathcal{G} \subseteq \mathcal{AE}_k$ be a Nash-Williams family and $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1$ be a partition. Then there exists $X \in \mathcal{B}_k$ and $i \in 2$ such that $\mathcal{G}_0 | X = \emptyset$.*

Proof. Given $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1$ as in the lemma, let

$$\mathcal{D}_{\mathcal{G}} = \{X \in \mathcal{E}_k : (\exists i \in 2)(\mathcal{G}_i | X = \emptyset)\}.$$

We check that $\mathcal{D}_{\mathcal{G}}$ is dense in $(\mathcal{E}_k, \subseteq^{\mathrm{FIN}^{\otimes k}})$. By the Abstract Nash-Williams Theorem 3.11, for every $X \in \mathcal{E}_k$ there exists $Y \leq X$ and $i \in 2$ such that $\mathcal{G}_i|Y = \emptyset$. Therefore, for every $X \in \mathcal{E}_k$ there exists $Y \subseteq^{\mathrm{FIN}^{\otimes k}} X$ such that $Y \in \mathcal{D}_{\mathcal{G}}$. Then $\mathcal{D}_{\mathcal{G}}$ is dense. By genericity of $\mathcal{B}_k$, there exists $X \in \mathcal{B}_k \cap \mathcal{D}_{\mathcal{G}}$, so $X \in \mathcal{B}_k$ and $\mathcal{G}_i|X = \emptyset$ for some $i \in 2$. □

Now we check that $\mathcal{B}_k$ is preserved as a Nash-Williams ultrafilter after countable-support side-by-side Sacks forcing. As in Chapters 5 and 6, this is done by proving that $\mathcal{B}_k$ is *localizing*, i.e. the following theorem holds.

Theorem 7.13 (Local Parametrized High-dimensional Ellentuck Theorem)**.** *For every finite Souslin-measurable colouring of $\mathcal{E}_k \times \mathbb{R}^{\mathbb{N}}$ there exists $B \in \mathcal{B}_k$ and $p \in \mathcal{P}_\omega$ such that $[\emptyset, B] \times [p]$ is monochromatic.*

Proof. Fix a Souslin-measurable set $\mathcal{X} \subseteq \mathcal{E}_k \times \mathbb{R}^{\mathbb{N}}$. By Theorem 7.1, for every $X \in \mathcal{E}_k$ there exists $Y \leq X$ (so $Y \subseteq^{\mathrm{FIN}^{\otimes k}} X$) and $p \in \mathcal{P}_\omega$ such that $[\emptyset, Y] \times [p] \subseteq \mathcal{X}$ or $[\emptyset, Y] \times [p] \cap \mathcal{X} = \emptyset$. Therefore, the set

$$\mathcal{D}_{\mathcal{X}} = \{Y \in \mathcal{E}_k : (\exists p \in \mathcal{P}_\omega)(([\emptyset, Y] \times [p] \subseteq \mathcal{X}) \vee ([\emptyset, Y] \times [p] \cap \mathcal{X} = \emptyset))\}$$

is dense in $(\mathcal{E}_k, \subseteq^{\mathrm{FIN}^{\otimes k}})$. Hence by the genericity of $\mathcal{B}_k$ we can find $B \in \mathcal{B}_k \cap \mathcal{D}_{\mathcal{X}}$ satisfying the theorem. □

Let κ be an arbitrary cardinal and $\mathcal{P}_\kappa$ be the corresponding countable-support side-by-side Sacks forcing. The proofs of the lemmas below use Theorem 7.13 and closely follow those of Theorem 5.35 and Theorem 6.43, respectively.

Lemma 7.14. *If $p \in \mathcal{P}_\kappa$ and $p \Vdash \tau \subseteq \mathbb{W}_k$ then there exists $q \leq p$ and $X \in \mathcal{B}_k$ such that $q \Vdash X \subseteq \tau$ or $q \Vdash X \cap \tau = \emptyset$.*

It follows that the upward closure of $\mathcal{B}_k$ after forcing by $\mathcal{P}_k$ is still an ultrafilter.

Lemma 7.15. *Suppose $p \in \mathcal{P}_\kappa$ and $p \Vdash ((\mathcal{G} \subseteq \mathcal{AE}_k$ is Nash-Williams$) \wedge (\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1))$. Then there exists $q \leq p$, $X \in \mathcal{B}_k$ and $i \in 2$ such that $q \Vdash \mathcal{F}_i|X = \emptyset$.*

Therefore we have the following theorem.

Theorem 7.16. *Let $k \in \omega$, κ be an infinite cardinal and $\mathcal{P}_\kappa$ be countable-support side-by-side Sacks forcing adding κ Sacks reals. Let $\mathcal{B}_k$ be a generic filter for $(\mathcal{E}_k, \subseteq^{\mathrm{FIN}^{\otimes k}})$, and $\dot{\mathcal{V}}$ a name for the upward closure $\{Y : (\exists X \in \mathcal{B}_k)(X \leq Y)\}$ of $\mathcal{B}_k$. Then $\Vdash_{\mathcal{P}_\kappa} (\dot{\mathcal{V}}$ is a Nash-Williams ultrafilter in $\mathcal{E}_k)$.*

In this thesis, we proved a version of the Parametrized Ellentuck Theorem which is only moderately abstract — we request the topological Ramsey space to satisfy an extra condition **(L4)** . Moreover, we showed that **(L4)** is in fact a necessary and sufficient condition. However, it is unclear how one proves **(L4)** directly in the Milliken space, or how one proves (or disproves) **(L4)** in the Hales-Jewett space, for example. It would be interesting to know if **(L4)** holds in every topological Ramsey space.

We have also seen that the High-dimensional Ellentuck spaces are in some sense counter intuitive — our usual methods of showing the relations among ultrafilter properties fail in these spaces. It would also be interesting to see how the exact relations among ultrafilters can be determined in the High-dimensional Ellentuck Spaces, and, in general, in topological Ramsey spaces.

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