CONSTRUCTION SCHEMES AND THEIR APPLICATIONS

by

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Abstract

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We study *capturing construction schemes*, a new combinatorial tool introduced by Todorčević to build uncountable structures. It consists of a ranked family of finite sets that provides a framework to do recursive constructions of uncountable objects by working with finite amalgamations of finite isomorphic substructures, the uncountable substructures of the final object can be further study using capturing.

In this Thesis we study the consistency of capturing construction schemes, and related definitions, we prove results of consistency, and give several applications of this tool both to infinite combinatorics and Banach space theory. For example, we show weaker forms of capturing, such as n-capturing, form a strict hierarchy which is related to the m-Knaster Hierarchy. We also show how capturing construction schemes can be used in constructing Suslin trees and Hausdorff gaps of a special kind in an intuitive manner. And give some applications to the theory of nonseparable Banach spaces.

Acknowledgements

Contents

Chapter 1

Introduction and Preliminaries

The aim of this Thesis is to study a class of ranked families called *Construction Schemes* and their applications. The motivation for this work is the application of set-theoretic and combinatorial methods to certain areas of Functional Analysis that deal with nonseparable normed spaces. In mathematics when dealing with objects that have metric or topological structure one usually puts a separability condition with the hope of avoiding pathologies. Thus, there is a deep and rich theory of, say, separable Banach spaces while the theory of arbitrary Banach spaces is much less explored. When one analyzes this from a set-theoretic point of view one sees that, while the nonseparable theory is indeed much more influenced by additional axioms of set theory, what remains unexplored is in fact finite-dimensional amalgamation techniques that are much more relevant to the nonseparable than separable theory.

Construction schemes can be used to build complex structures in a recursive manner in such a way that one has some control over the uncountable substructures, thus making them easier to understand and to study. Construction schemes serve as unification of completeness theorems from model theory (see [Kei70]) and Forcing from set theory ([Coh66]). My work follows that of Todorčević [Tod17] where Construction Schemes are introduced and used to build several Banach spaces relevant to certain well-known problems from nonseparable Functional Analysis. It should be noted that similar spaces had previously been built using forcing by López-Abad & Todorčević [LAT11] , and Bell, Ginsburg, Todorčević [BGT82]. The spaces build with the use of Construction Schemes are more intuitive which makes them accessible to experts in other fields that have no background on the technique of forcing. Moreover they open the possibility of reformulating open problems from nonseparable functional analysis as problems about finite-dimensional amalgamations that could be accessible to a wider spectrum of mathematicians.

We illustrate in this Thesis how construction schemes can be used to recursively build complex mathematical structures, and perhaps more important, is that the constructed structures are rather canonical and their properties could be analyzed in a similar intuitive manner. To analyze these structures Todorčević [Tod17] introduced the concept of *capturing* and showed that *Capturing Construction Schemes* exist using Jensen's Diamond Principle $\diamondsuit$.

The study of Construction Schemes presented in this work is done through three different

perspectives which I will now list very broadly.

Consistency Strength of Construction Schemes: During our study of construction schemes we noticed that some combinatorial constructions only require a weak version of capturing, whereas the Banach space applications do not reflect this. That motivates the study of the hierarchy of n-capturing construction schemes. In Kalajdzievski & Lopez [KL17] we study this hierarchy and show its relationship with the m-Knaster hierarchy from the theory of Forcing Axioms. We also show that adding ω_1 Cohen reals adds fully capturing construction schemes. This implies in particular that many standard techniques on Set Theory have important consequences in the theory of non separable Banach spaces.

Combinatorial Applications: Several interesting objects well known to set theorist can be built using capturing Construction schemes. In [LT17] we show how the existence of a Capturing Construction Scheme imply the existence of a Suslin tree and a destructible gap. Furthermore we show that a stronger form of destructible gap, named T-gaps, can also be constructed with the same techniques. An interesting feature of these theorems is that we only need a weaker form of capturing to carry out the constructions. Namely, we only need 3-capturing Construction Schemes. This is not the case on the applications to Banach Space Theory where the full strength of capturing seems to be necessary.

Banach Space Applications: The theory of non separable Banach spaces is in recent years becoming an interesting area for applications of methods of Set Theory. The main interest is to see how much of the deep separable theory can be extended to the context of nonseparable Banach spaces. Many examples of non separable Banach spaces relevant to this question appear in [LAT11], such as for example Banach spaces with no uncountable biorthogonal systems but with uncountable ε-biorthogonal systems for $0 < \varepsilon < 1$, or Banach spaces with uncountable Schauder basis of basis constant K that do not have uncountable $(< K)$-basic sequences for K an arbitrary constant bigger than 1. These examples show a striking discrepancy between the theory of separable and non-separable Banach spaces since, for example, any separable infinite-dimensional Banach space has an infinite K-basic sequence for K an arbitrary constant bigger than 1. Our work [LÍ7] has shown that we can find similar examples using capturing construction schemes. Our spaces, however, seem to be more interesting than those of [LAT11] since they can be further analyzed and therefore used to attack some other problems from the area.

The Thesis is organized as follows.   In Chapter 1 we introduce the background definitions and results necessary for the rest of the Thesis. In particular we mention classical axioms of Set Theory such as Jensen's diamond $\diamondsuit$, and Martin Axiom for ω_1 dense sets, MA_{ω_1}. We give a brief introduction of important objects of combinatorics that will be important in future chapters: Suslin trees, gaps, destructible gaps, and T-gaps. Since we will be concerned with

applications to Functional Analysis we also mention the necessary concepts from the Banach Space Theory, mainly concepts such as: Schauder basis, basic sequences, biorthogonal Systems and their relation to the geometry of Banach Spaces: Mazur Intersection Property. We then introduce the main concept of the Thesis: the Construction Schemes and immediately illustrate their application to the construction of classical objects, such as Aronzajn trees and Hausdorf gaps. It's important to notice that this section requires no extra axioms in ZFC so it is not possible to generalize this first examples towards the construction of Suslin trees or destructible gaps without the use of extra axioms. To get such application we introduce the notion of capturing construction schemes, and different variations on the idea of capturing, and show that they are incompatible with MA_{ω_1}.

In Chapter 2 we present all of the results about the consistency of construction schemes that we know at the moment. We begin by showing the existence of construction schemes in ZFC.

Theorem 2.1. *For any given type $(m_k, n_k, r_k)_{k<\omega}$ there is a construction scheme $\mathcal{F}$ of that type.*

Another framework to build uncountable objects using finite approximations in such a way that one can control the uncountable substructures was developed by Shelah [She85] using $\diamondsuit$. The existence of capturing construction schemes also follows from $\diamondsuit$ (see [Tod17]). We show that adding $\aleph_1$ Cohen reals implies there are capturing construction schemes.

Theorem 2.2. *Adding $\kappa \geq \aleph_1$ Cohen reals also adds a fully capturing construction scheme.*

Thus, capturing construction schemes are added in many finite support iterations. This means that some of the most common techniques in Set Theory have relevant consequences in Functional Analysis. We move on to study weaker forms of capturing. We study the Hierarchy of n-capturing construction schemes show that there is a relation between the m-Knaster Hierarchy and n-capturing construction schemes.

Theorem 2.3. $MA_{\omega_1}(K_m)$ *and n-capturing are independent if $n \leq m$ and they are incompatible if $n > m$. Also $MA_{\omega_1}(precaliber \aleph_1)$ is independent of capturing.*

We also prove equivalent results for fully $\vec{P}$-capturing construction schemes and n-$\vec{P}$-capturing construction schemes. We finish the Chapter with a summary of all known consistency results at the moment.

In Chapter 3 we explore applications of capturing construction schemes to the classical problems of Set Theory. In particular, we construct a Suslin tree and a Hausdorff gap, using 3-capturing construction schemes, this will serve as illustration of how to use capturing construction schemes to build uncountable structures.

Theorem 3.1. *Assume there is a Construction Scheme that is 3-capturing. Then there is a Suslin tree.*

Theorem 3.3. *Assume there is a 3-capturing construction scheme. Then there is a Hausdorff (ω_1, ω_1)-gap that is a T-gap.*

Using partitions to capture we can reduce the level of capturing needed to construct Suslin trees and T-gaps. We do not know if n-capturing implies n-$\vec{P}$-capturing.

Theorem 3.4. *Let $\omega = \bigcup_{i<\omega} P_i$, with P_i infinite, and let $\vec{P} = (P_i : i < \omega)$. Assume there are 2-$\vec{P}$-capturing construction schemes, then there is Suslin tree and Hausdorff T-gap.*

We finish Chapter 3 by showing that the notion of Hausdorff T-gap is stronger than the standard notion of a destructible Hausdorff gap. We do this with a forcing iteration that gives us a model with destructible gaps but no T-gaps.

Theorem 3.5. *There is a model of set theory in which there is a destructible Hausdorff (ω_1, ω_1)-gap but with no T-gaps.*

In Chapter 4 we apply capturing construction schemes to Banach Spaces. The motivation for this Chapter are the following Theorems of López-Abadand Todorčević [LAT11]

Theorem 1.1 (Theorem 4.5 of [LAT11])**.** *For every $\varepsilon > 0$ rational, there is a forcing notion $\mathbb{P}_\varepsilon$ which forces a Banach space $\mathcal{Y}_\varepsilon$ with an uncountable ε-biorthogonal system and such that for every $0 \leq \tau < \frac{\varepsilon}{1+\varepsilon}$, $\mathcal{Y}_\varepsilon$ has no uncountable τ-biorthogonal system.*

Theorem 1.2 (Theorem 6.4 of [LAT11])**.** *For every constant $K > 1$ there is a forcing notion $\mathbb{P}_K$ which forces a Banach space $\mathcal{Y}_K$ with an uncountable K-basis yet for every $1 \leq K' < K$, $\mathcal{Y}_K$ has no uncountable K'-basic sequences.*

One feature of the constructions in this Chapter is that they can be understood without much of a background in Set Theory. We give first a general overview of the constructions, then we construct the following Banach spaces of density $\aleph_1$.

Theorem 4.4. *Assume there is a capturing construction scheme $\mathcal{F}$. Then for every $\varepsilon \in (0, 1) \cap \mathbb{Q}$, there is a Banach space $\mathcal{X}_\varepsilon$ with an uncountable ε-biorthogonal system but no uncountable τ-biorthogonal system for every $0 \leq \tau < \frac{\varepsilon}{1+\varepsilon}$.*

The notion of an uncountable ε-biorthogonal is related to the Mazur Intersection Property, more concretely we also show:

Theorem 4.4. *The Banach Space $\mathcal{X}_\varepsilon$ does not have the Mazur Intersection Property, is polyhedral and it's norm depends only on finitely many coordinates (see below for definition).*

Theorem 4.5. *Assume there is a capturing construction scheme $\mathcal{F}$. Then for every constant $K > 1$, there is a Banach space $\mathcal{X}_K$ with a K-basis of length ω_1 but no uncountable K'-basic sequence for every $1 \leq K' < K$.*

1.1   Preliminaries

We follow the standard notations in combinatorics and Set Theory, for more background see Kunen [Kun80] or Jech [Jec02]. For background in the Banach Space Theory, see Lindenstrauss & Tzafriri [LT77] or Hajek et al. [HMSVZ08].

We write $[\omega]^\omega$ for the collection of infinite subsets of ω. If $a, b \subset \omega$ are infinite sets we say a is *almost contained* in b, and write $a \subseteq^* b$, provided $a \setminus b$ is finite. Analogously, we denote $[\omega_1]^{<\omega}$ for the collection of finite subsets of ω_1. For bounded subsets $A, B \subset \omega_1$ we write $A < B$ if for every $a \in A$ and $b \in B$ we have $a < b$. By $A \sqsubset B$ we mean that A is an initial segment of B, meaning that $A \subset B$ and if $a \in A$, $b \in B$ and $b < a$ then $b \in A$.

Definition 1.4. For an uncountable set X and $s_\alpha \subset X$ for $\alpha < \gamma$. We say $(s_\alpha)_{\alpha<\gamma}$ forms a Δ-*system* if there exists $s \subset X$ such that $s_\alpha \cap s_\beta = s$ for every $\alpha < \beta < \gamma$.

We will work with a special kind of Δ-systems.

Definition 1.5. For $X = \omega_1$ and $s_\alpha \subset \omega_1$, we say $(s_\alpha)_{\alpha<\gamma}$ is an *increasing Δ-system* if it is a Δ-system and $s_\alpha < s_\beta$ for every $\alpha < \beta < \gamma$.

The following is a classical result of combinatorics known as the Δ-system Lemma or Shanin's Lemma.

Lemma 1.6. *For X uncountable and $s_\alpha \subset X$ finite for $\alpha < \omega_1$. There is $\Gamma \subset \omega_1$ uncountable and $s \subset X$ such that $(s_\alpha : \alpha \in \Gamma)$ forms an increasing Δ-system.*

For the duration of this work we will assume that all Δ-systems are increasing without mention.

For $A, B \subset \omega_1$ finite, there is a unique order-preserving bijection between A and B. We denote this bijection by $\varphi_{A,B} : A \to B$, and use it to transport structures on A to structures on B. In particular:

- If $f : A \to X$ is a function on A into some set X, we denote $\varphi_{A,B}(f) : B \to X$ the map $f \circ \varphi_{A,B}^{-1}$.

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & X \\
{\scriptstyle \varphi_{A,B}}\Big\downarrow & \nearrow {\scriptstyle \varphi_{A,B}(f)} & \\
B & &
\end{array}
$$

- If $\mathcal{S} \subset \mathcal{P}(A)$ is a family of subsets of A, we define

$$\varphi_{A,B}(\mathcal{S}) = \{\varphi_{A,B}(S) : S \in \mathcal{S}\}$$

We mention forcing results and forcing axioms throughout this work. It is assumed the reader is familiar with the classical results and principles of Set Theory. We recall the most relevant definitions, the reader is refer to [Jec02] for background.

By a forcing notion $\mathbb{P}$ we mean a partial order $(\mathbb{P}, \leq)$.

Definition 1.7. Let $\mathbb{P}$ be a forcing notion.

1. A set $D \subset \mathbb{P}$ is *dense* if for every $p \in \mathbb{P}$ there is $q \in D$ such that $q \leq p$.

2. Given $p, q \in \mathbb{P}$ we say p and q are *incompatible* and write $p \perp q$ if there is no $r \in \mathbb{P}$ such that $r \leq p, q$. Otherwise we say p and q are *compatible* and write $p \not\perp q$.

3. Let $\mathcal{D}$ be a family of dense sets of $\mathbb{P}$. A set $G \subset \mathbb{P}$ is a $\mathcal{D}$-*generic filter*, if

   (a) for every $p, q \in G$, we have $p \not\perp q$,

   (b) for every $r \in \mathbb{P}$ and $p \in G$, then $p \leq r$ implies $r \in G$,

   (c) for every $D \in \mathcal{D}$ we have $D \cap G \neq \emptyset$.

4. We say $A \subset P$ is an antichain if every two elements in A are incompatible, i.e, for all $p, q \in A$ we have $p \perp q$.

5. We say $\mathbb{P}$ has the *countable chain condition*, or $\mathbb{P}$ is *ccc*, if every antichain of $\mathbb{P}$ is countable.

   We are in conditions to define the Martin's Axiom.

   MA_λ: For every ccc forcing notion $\mathbb{P}$ and every family $\mathcal{D}$ of dense sets of $\mathbb{P}$ with $|\mathcal{D}| \leq \lambda$, there is a $\mathcal{D}$-generic filter G.

The notation refers to Martin Axiom for λ dense sets. Martin's Axiom is the statement MA_λ holds for all $\lambda < \mathfrak{c}$. It is independent of ZFC, as a note MA_λ implies $\lambda < \mathfrak{c}$ therefore it implies the negation of the continuum hypothesis. We will be mostly interested in MA_{ω_1} and weaker versions of this axiom.

Definition 1.8. Let $\mathbb{P}$ be a forcing notion. We say that $\mathbb{P}$ has *precaliber* $\aleph_1$ if for every $W \subset \mathbb{P}$ uncountable, we can find $W_0 \subset W$ also uncountable such that for all $m < \omega$ and all $p_0, \ldots, p_{m-1} \in W_0$ there is $p \in \mathbb{P}$ with $p \leq p_0, \ldots, p_{m-1}$.

We say $\mathbb{P}$ is *m-Knaster* for $m \geq 2$, denoted as K_m if for every $W \subset \mathbb{P}$ uncountable, we can find $W_0 \subset W$ uncountable such that for all $p_0, \ldots, p_{m-1} \in W_0$ there is $p \in \mathbb{P}$ with $p \leq p_0, \ldots, p_{m-1}$.

Note first that having precaliber $\aleph_1$ implies K_m, and K_m implies K_n for $n \leq m$. Also, a forcing notion which is K_m or has precaliber $\aleph_1$ is clearly ccc. Thus, we have the following implications:

$$\mathrm{ccc} \Leftarrow \mathrm{K}_2 \Leftarrow \ldots \Leftarrow \mathrm{K}_m \Leftarrow \mathrm{K}_{m+1} \Leftarrow \ldots \Leftarrow \text{precaliber } \aleph_1$$

We can now define the following forcing axioms which are weaker than MA_λ:

MA$_\lambda$(K$_m$): For every K$_m$ forcing notion $\mathbb{P}$ and every family $\mathcal{D}$ of dense sets of $\mathbb{P}$ with $|\mathcal{D}| \leq \lambda$, there is a $\mathcal{D}$-generic filter G.

MA$_\lambda$(precaliber $\aleph_1$): For every forcing notion $\mathbb{P}$ with precaliber $\aleph_1$ and every family $\mathcal{D}$ of dense sets of $\mathbb{P}$ with $|\mathcal{D}| \leq \lambda$, there is a $\mathcal{D}$-generic filter G.

The axioms MA(K$_m$) and MA(precaliber $\aleph_1$) are defined similarly as MA. It is clear that we have the following implications

$$
\begin{array}{l}
\text{MA}_\lambda \\
\Downarrow \\
\text{MA}_\lambda(\text{K}_2) \Longrightarrow \cdots \Longrightarrow \text{MA}_\lambda(\text{K}_m) \Longrightarrow \text{MA}_\lambda(\text{K}_{m+1}) \Rightarrow \cdots \Rightarrow \text{MA}_\lambda(\text{precaliber } \aleph_1)
\end{array}
$$

None of the implications above can be reverse. To see this note that MA$_{\omega_1}$ kills Suslin trees (see definitions below) yet MA$_{\omega_1}$(K$_2$) preserves Suslin trees. Therefore, if we start with a model V with Suslin trees and we force with a K$_2$ forcing such that MA$_{\omega_1}$(K$_2$) holds, we obtain a model $V[G]$ where MA$_{\omega_1}$(K$_2$) holds but MA$_{\omega_1}$ fails. To see none of the other implications can be reversed we refer the reader to Barnet [Bar92] where it is shown that if you start with a model V then add a Cohen real, and force with a K$_m$ poset that forces MA(K$_m$), the resulting model satisfies MA(K$_m$) and MA$_{\omega_1}$(K$_{m+1}$) fails. In Chapter 5 we give an alternative prove that MA$_\lambda$(K$_m$) $\not\Leftarrow$ MA$_\lambda$(K$_{m+1}$) using consctruction schemes.

The following cardinal numbers are related to the continuum and they play an important role in combinatorial arguments. We will come back to them in Chapter 5.

Definition 1.9. We say that $\mathcal{A} \subset [\omega]^\omega$ has the *finite intersection property* (f.i.p.) if for every $A_0, \ldots, A_n \in \mathcal{A}$ their intersection $A_0 \cap \ldots \cap A_n$ is infinite.

Let $\mathcal{A}$ be a family with the f.i.p. and $P \subset \omega$ infinite. We say P is a *pseudo intersection* for $\mathcal{A}$ if for every $A \in \mathcal{A}$, we have $P \subseteq^* A$.

We say $\mathcal{B} \subset \omega^\omega$ is *unbounded* if for every $h \in \omega^\omega$ there is $f \in \mathcal{B}$ such that $f \not<^* g$.

Finally, let

$$\mathfrak{p} = \min\{|\mathcal{A}| : \mathcal{A} \text{ has the f.i.p and it has no pseudo intersection.}\}$$
$$\mathfrak{b} = \min\{|\mathcal{B}| : \mathcal{B} \text{ is unbounded.}\}$$

It is easy to see that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{c}$. If we assume MA$_{\omega_1}$(K$_m$) or MA$_{\omega_1}$(precaliber $\aleph_1$) then $\omega_1 < \mathfrak{p}$. Actually there is a close relation between cardinal invariants and some forcing axioms.

1.1.1 Gaps in the quotient algebra $\mathcal{P}(\omega)/\text{Fin}$

Hausdorff's (ω_1, ω_1)-gaps in the quotient algebra $\mathcal{P}(\omega)/\text{Fin}$ are important set theoretic tools that naturally show up in a wide range of applications in set theory and related areas (see, for example,[DW87]). It turns out that there are numerous analogies between (ω_1, ω_1)-gaps

and Aronszajn trees (see definition below for definitions), for example [LT01]. Suslin trees, which play an important role in this work, are a very specific kind of Aronszajn trees since they may admit uncountable branches in ω_1-preserving forcing extensions of the set-theoretic universe. Analogously, as it is well known, some (ω_1, ω_1)-gaps may be filled in ω_1-preserving forcing extensions of the universe, so this sort of gaps are sometimes called destructible gaps, or *Suslin gaps*. We present this notions here and the proof of the basic results for the convenience of the reader.

The study of gaps in $\mathcal{P}(\omega)/\text{Fin}$ leads to a Ramsey-theoretic characterization of destructible gaps, which further strengthens the analogy between Suslin trees and destructible gaps. Furthermore, it points out to a natural variation of the notion of destructible gap, a *T-gap*, a notion that we introduce below and we will come back to in Chapter 3.

We recall the definition of gap in $[\omega]^\omega$ as well as some well known results.

Definition 1.10. We say $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$, with $a_\alpha, b_\alpha \subset \omega$ infinite, is a *pre-gap* if for every $\alpha < \beta < \omega_1$

1. $a_\alpha \cap b_\alpha = \emptyset$.

2. $a_\alpha \subseteq^* a_\beta$ and $b_\alpha \subseteq^* b_\beta$.

3. $a_\delta \cap b\gamma$ is finite for every $\delta, \gamma < \omega_1$.

   We say that $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is a *gap* if it is a pre-gap and

4. there is no infinite $c \subset \omega$ such that

   (a) $a_\alpha \subseteq^* c$ for every $\alpha < \omega_1$.
   (b) $b_\alpha \cap c$ is finite for every $\alpha < \omega_1$.

The existence of (ω_1, ω_1)-gaps is due to Hausdorff [Hau36]. It is easy to see that every pre-gap $(a_\alpha, b_\alpha)_{\alpha < \gamma}$ for $\gamma < \omega_1$ is not a gap.

The following Ramsey property of gaps is going to be useful in the rest of the Thesis since it makes constructions of gaps with different properties more intuitive.

Proposition 1.11. *A pre-gap $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ form an (ω_1, ω_1)-gap if and only if for every uncountable $\Gamma \subset \omega_1$ there are $\alpha < \beta$ in Γ such that $a_\alpha \cap b_\beta \neq \emptyset$.*

Definition 1.12. We say a gap $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is a destructible gap if for every uncountable $\Gamma \subset \omega_1$ there are $\alpha < \beta$ in Γ such that $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset$.

In [Dow95] a destructible gap is constructed using $\diamondsuit$, we present a natural construction in Chapter 3. The next proposition implies that under MA_{ω_1} all gaps are indestructible, meaning there are no destructible gaps (see e.g. [TF95]). Thus, we have that the existence of a destructible gaps is independent of ZFC.

Proposition 1.13. *The following are equivalent:*

1. *There is an ω_1-preserving forcing notion that splits $(a_\alpha, b_\alpha)_{\alpha<\omega_1}$.*

2. *The forcing notion defined by $p \in \mathbb{P} = [\omega_1]^{<\omega}$ iff $a_\alpha \cap b_\beta = \emptyset$ for all $\alpha \neq \beta$ in p ordered by extension has the ccc.*

3. *For every uncountable $\Gamma \subset \omega_1$ there are $\alpha < \beta$ in Γ such that $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset$.*

In the literature, (ω_1, ω_1)-gaps with these properties are called 'destructible gaps', 'fillable gaps', 'Souslin gaps' or 'S-gaps', we will refer to them as S-gaps or destructible gaps. This definition leads us to the following natural strengthening.

Definition 1.14. We say a gap $(a_\alpha, b_\alpha)_{\alpha<\omega_1}$ is a *tower gap* or a *T-gap* if for every uncountable $\Gamma \subset \omega_1$ there are $\alpha < \beta$ such that $a_\alpha \subseteq a_\beta$ and $b_\alpha \subseteq b_\beta$.

Theorem 3.5 asserts there is a model with a destructible gap but no T-gaps. In other words, it is consistent that there are destructible gaps but no T-gaps, therefore the concept of T-gaps is stronger than just destructible gap. See Chapter 3 for the prove of this fact.

There is another interesting fact, even though T-gaps are destructible by a ccc forcing. They are not destructible by a K_2 forcing, i.e, a forcing such that for every $W \subset \mathbb{P}$ uncountable, there is $W_0 \subset W$ uncountable with $p \not\perp q$ for every $p, q \in W_0$.

Proposition 1.15. *Let $(a_\alpha, b_\beta)_{\alpha<\omega_1}$ be a T-gap, and $\mathbb{P}$ a K_2 forcing notion. Then $\mathbb{P}$ does not destroy the gap $(a_\alpha, b_\beta)_{\alpha<\omega_1}$. In other words, $(a_\alpha, b_\beta)_{\alpha<\omega_1}$ is still a T-gap in $V[G]$, where G is a generic filter for $\mathbb{P}$.*

We give the proofs of this propositions.

Proof of Proposition 1.11. Suppose $(a_\alpha, b_\alpha)_{\alpha<\omega_1}$ is not a gap and let $c \subset \omega$ witness this. There is $n < \omega$ and uncountable $\Gamma \subset \omega_1$ such that $a_\alpha \setminus c \subset n$ and $b_\alpha \cap c \subset n$ for all $\alpha \in \Gamma$. We can also assume that there are $s, t \subset n$ such that for every $\alpha \in \Gamma$ $a_\alpha \cap n = s$ and $b_\alpha \cap n = t$. The condition $a_\alpha \cap b_\alpha = \emptyset$ implies that $s \cap t = \emptyset$.

For every $\alpha < \beta$ in Γ we have

$$a_\alpha \cap b_\beta = (a_\alpha \cap n) \cap (b_\beta \cap n) = s \cap t = \emptyset$$

Suppose now that there is $\Gamma \subset \omega_1$ uncountable such that $a_\alpha \cap b_\beta = \emptyset$ for every $\alpha < \beta$ in Γ. Define

$$c = \bigcup_{\alpha \in \Gamma} a_\alpha$$

is clear that $a_\alpha \subset^* c$ for every $\alpha < \omega_1$. We just have to check that $c \cap b_\gamma$ is finite for all $\gamma < \omega_1$. Let $\gamma < \omega_1$. Since $a_\alpha \cap b_\gamma$ is finite, if $c \cap b_\gamma$ is infinite there must be some $\delta \in \Gamma$ limit in Γ, $\gamma < \delta$ such that

$$\bigcup_{\alpha \in \Gamma \cap \delta} a_\alpha \cap b_\gamma \quad \text{is infinite}$$

but $b_\gamma \setminus b_\delta$ is finite and $\bigcup_{\alpha \in \Gamma \cap \delta} a_\alpha \cap b_\delta = \emptyset$, contradiction.

$\square$

Proof of Proposition 1.13. First we see $(3) \Rightarrow (2) \Rightarrow (1)$. Let $\mathbb{P}$ be as in (2). Notice that $\mathbb{P}$ forces $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ to split by forcing $\Gamma \subset \omega_1$ without the property of Proposition 1.11. We see that $\mathbb{P}$ is ccc hence ω_1-preserving.

Let $(p_\alpha)_{\alpha < \omega_1}$ in $\mathbb{P}$. There is uncountable $\Gamma \subset \omega_1$ such that:

(i) $(p_\alpha)_{\alpha \in \Gamma}$ forms a Δ-system with $|p_\alpha| = k$.

(ii) If $p_\alpha = \{\delta_1^\alpha < \ldots < \delta_k^\alpha\}$ there is $n < \omega$ such that $a_{\delta_i^\alpha} \setminus n \subset a_{\delta_k^\alpha}$ and the same for $b_{\delta_i^\alpha}$.

(iii) There are $s_i, t_i \subset n$ for $i = 1, \ldots, k$ such that $a_{\delta_i^\alpha} \cap n = s_i$ and $b_{\delta_i^\alpha} \cap n = t_i$.

Note that $s_i \cap t_j = \emptyset$. Consider $\{\delta_k^\alpha\}_{\alpha \in \Gamma}$ by hypothesis

$$\text{there are } \alpha < \beta \text{ in } \Gamma \text{ such that } (a_{\delta_k^\alpha} \cap b_{\delta_k^\beta}) \cup (a_{\delta_k^\beta} \cap b_{\delta_k^\alpha}) = \emptyset.$$

By (iii) we have $(a_{\delta_i^\alpha} \cap b_{\delta_j^\beta}) \cup (a_{\delta_j^\beta} \cap b_{\delta_i^\alpha}) \cap n = \emptyset$, by (ii) we have

$$(a_{\delta_i^\alpha} \cap b_{\delta_j^\beta}) \cup (a_{\delta_j^\beta} \cap b_{\delta_i^\alpha}) \setminus n \subset (a_{\delta_k^\alpha} \cap b_{\delta_k^\beta}) \cup (a_{\delta_k^\beta} \cap b_{\delta_k^\alpha}) = \emptyset$$

and $p_\alpha \cup p_\beta \in \mathbb{P}$.

$(2) \Rightarrow (3)$ Let Γ be an uncountable subset of ω_1. Take $(p_\alpha = \{\alpha\})_{\alpha \in \Gamma}$ since $\mathbb{P}$ has the ccc there is $\alpha < \beta$ in Γ such that $p_\alpha \not\perp p_\beta$ but this implies $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset$ as we wanted.

$(1) \Rightarrow (2)$ Let $\mathbb{Q}$ be a forcing notion ω_1-preserving that splits $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$. By the proof of Proposition 1.11 for every $\dot{\Gamma}_0 \subset \omega_1$ uncountable we can find $\dot{\Gamma}$ such that

$$\mathbb{Q} \Vdash \dot{\Gamma} \subset \dot{\Gamma}_0 \text{ uncountable.}$$
$$\mathbb{Q} \Vdash \text{``for every } \alpha < \beta \text{ in } \dot{\Gamma}, \ (a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset.\text{''}$$

Applying $(2) \Leftrightarrow (3)$, which we already proved, $\mathbb{Q} \Vdash$ "$\mathbb{P}$ has the ccc". If $\mathbb{P}$ has an uncountable antichain on the ground model it has an uncountable antichain on $V^{\mathbb{Q}}$ because $\mathbb{Q}$ is ω_1-preserving. Thus $\mathbb{P}$ is ccc and we finish the proof.

$\square$

Proof of Proposition 1.15. Start with a T-gap $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$, and a K_2 forcing notion $\mathbb{P}$, as above. Since $\mathbb{P}$ is ccc, ω_1 is preserved, therefore $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is a pre-gap in $V[G]$.

Let $\dot{\Gamma}$ be a name for an uncountable subset of ω_1 in $V[G]$. Take $W \subset \omega_1$ uncountable, and $(p_\alpha : \alpha \in W) \subset \mathbb{P}$ such that

$$\text{For all } \alpha \in W, \ p_\alpha \Vdash \alpha \in \dot{\Gamma}. \tag{1.1}$$

Since $\mathbb{P}$ is K_2 there is W_0 such that for every $\alpha < \beta$ in W_0, there is $q \leq p_\alpha, p_\beta$. Apply the gap condition from Proposition 1.11 to obtain $\alpha < \beta$ such that $a_\alpha \cap b_\beta \neq \emptyset$. Also, by the T-gap condition, there are $\alpha' < \beta'$ in W_0 such that $a_{\alpha'} \subset a_{\beta'}$ and $b_{\alpha'} \subset b_{\beta'}$.

By the way W_0 was obtained we can find $q \leq p_\alpha, p_\beta$ and $q' \leq p_{\alpha'}, p_{\beta'}$. By equation (1.1) we have

$$q \Vdash \exists \alpha < \beta \text{ in } \dot{\Gamma}, a_\alpha \cap b_\beta \neq \emptyset$$

This implies $(a_\alpha, b_\beta)_{\alpha < \omega_1}$ is a gap in $V[G]$ since $\dot{\Gamma}$ arbitrary. Similarly,

$$q' \Vdash \exists \alpha' < \beta' \text{ in } \dot{\Gamma}, a_{\alpha'} \subset a_{\beta'} \text{ and } b_{\alpha'} \subset b_{\beta'}$$

this shows $(a_\alpha, b_\beta)_{\alpha < \omega_1}$ is a T-gap in $V[G]$. □

1.1.2 Trees and other combinatorial objects

One of the most common objects in combinatorics are trees. The reason is that many problems can be coded as problems on trees. For example, perfect sets can be seen as branches of perfect trees which has multiple applications in Descriptive Set Theory. This makes the study of trees a classical part of Set Theory.

Definition 1.16. We say a partially ordered set $(T, <)$ is a *tree* if for each $t \in T$ the set $\{s \in T : s < t\}$ of predecessors of t is well-ordered by $<$.

We can then consider the ordered type of $\{s \in T : s < t\}$ and say that t is on the level α and denote it by $\text{Lev}(t) = \alpha$ if the set of its predecessors has well-ordered type α under $<$. When we talk about level α of $(T, <)$ we mean $T_\alpha = \{t \in T : \text{Lev}(t) = \alpha\}$. We say $(T, <)$ has height μ if $\mu = \sup\{\text{Lev}(t) + 1 : t \in T\}$. For simplicity we will write T to refer to $(T, <)$.

Example 1.1 (Binary Trees)**.** Consider the collection of finite sequences of 0's and 1's, and denote it by $2^{<\omega}$. We represent elements of $2^{<\omega}$ by letters $t, s, \ldots$ and t_i denotes the ith element of t. For a sequence $t \in 2^{<\omega}$ we denote by $|t|$ the length of t, thus $t = (t_i : i < |t|)$. For $n < |t|$ we let $t \restriction n = (t_i : i < n)$, notice that $t \restriction n \in 2^{<\omega}$ for every $n < \omega$. If $t, s \in 2^{<\omega}$, we say that $s < t$ if $|s| < |t|$ and $t \restriction |s| = s$.

It is clear that $(2^{<\omega}, <)$ as defined above is a tree of height ω.

In the same way we can define $2^{<\alpha}$ for any ordinal α we call this tree the binary tree of height α.

Definition 1.17. Let T be a tree, and $B, A \subset T$. We say B is a *branch* of T if it is maximal and linearly ordered. We say A is an *antichain* if for different $t, s \in A$ neither $t < s$ nor $t > s$, we say that t and s are incompatible and we denote it by $t \perp s$.

We are now in condition to define what is an Aronszajn tree and a Suslin tree.

Definition 1.18. We way a tree T is an *Aronszajn tree* if T has height ω_1, it has countable levels and it doesn't have uncountable branches.

We say a tree T is a *Suslin tree* if T has height ω_1, and it has neither uncountable branches nor uncountable antichains.

A classical result of Set Theory is that there is an Aronszajn tree in ZFC, however the existence of a Suslin tree is independent of ZFC. To see this, let $(T, <)$ is a Suslin tree and consider the forcing notion $\mathbb{P} = (T, >)$ then we can force an uncountable branch to T. Since T is a Suslin tree implies $\mathbb{P}$ is ccc, this means that MA_{ω_1} implies there are no Suslin trees. Saharon Shelah [She84] showed that adding one Cohen real forces a Suslin tree, Stevo Todorčević showed that modifying a coherent map by a Cohen real results in a Suslin tree. We provide a new construction of a Suslin tree in Chapter 3.

To define Jensen's diamond principle we need some definitions

Definition 1.19. We say $C \subset \omega_1$ is a *club* if it is closed and unbounded in ω_1.

We say $S \subset \omega_1$ is *stationary* if for every club C, we have $C \cap S \neq \emptyset$.

Jensen's diamond principle:

$\Diamond$ There is a sequence $(S_\alpha : \alpha < \omega_1)$ such that

    1. For every $\alpha < \omega_1$, $S_\alpha \subseteq \alpha$.

    2. For every $\Gamma \subset \omega_1$ the set $\{\alpha < \omega_1 : \Gamma \cap \alpha = S_\alpha\}$ is stationary.

we call the sequence $(S_\alpha : \alpha < \omega_1)$ a $\Diamond$-sequence.

The $\Diamond$-sequence $(S_\alpha : \alpha < \omega_1)$ contains all subsets of ω therefore $\Diamond$ implies the continuum hypothesis. Recall $\mathrm{MA}_{\omega_1}(\text{precaliber } \aleph_1)$ implies $\omega_1 < \mathfrak{p} \leq \mathfrak{c}$ therefore $\mathrm{MA}_{\omega_1}(\text{precaliber } \aleph_1)$ is incompatible with $\Diamond$. It is well known that $\Diamond$ implies the existence of a Suslin tree (see for example [Jec02] or Chapter 3) but the other direction does not hold. For example, adding a Cohen real to a model with $\omega_1 < \mathfrak{c}$ will force a model with a Suslin tree but CH is false therefore $\Diamond$ is false as well.

1.1.3   Banach spaces

We give some preliminaries of the theory of Banach spaces. We follow standard notation (see, for example, [LT77] and [HMSVZ08]).

Definition 1.20. A *Banach space* $(\mathcal{X}, \|\cdot\|)$ is a complete normed space in $\mathbb{R}$, we refer to it as $\mathcal{X}$. The *unit ball* $B_{\mathcal{X}}$ is the collection of $x \in \mathcal{X}$ such that $\|x\| \leq 1$.

The *dual space* $\mathcal{X}^*$, is the Banach space form by the maps $x^* : \mathcal{X} \to \mathbb{R}$ which are linear and bounded equip with the supremum norm $\|x^*\| = \sup\{|x^*(x)| : x \in B_{\mathcal{X}}\}$.

We recall some other notions of Banach space theory relevant to this work.

Definition 1.21. Let $\mathcal{X}$ be a Banach space and $(y_\alpha, y_\alpha^*)_{\alpha < \omega_1}$ a sequence in $\mathcal{X} \times \mathcal{X}^*$. For $\varepsilon \geq 0$, we say that $(y_\alpha, y_\alpha^*)_{\alpha < \omega_1}$ forms an *ε-biorthogonal system* if $y_\alpha^*(y_\alpha) = 1$ for every $\alpha < \omega_1$, and $|y_\alpha^*(y_\beta)| \leq \varepsilon$ for every $\alpha \neq \beta$. If $\varepsilon = 0$ we say $(y_\alpha)_{\alpha < \omega_1}$ forms a *biorthogonal system*.

A Banach space $\mathcal{X}$ have the *Mazur intersection property* (MIP) if every closed convex subset of $\mathcal{X}$ is the intersection of closed balls. The following result relates the above algebraic definition with the more geometric Mazur intersection property.

Theorem 1.22 ([SM97]). *Let $\mathcal{X}$ be a Banach space.*

1. *If $(y_\alpha, y_\alpha^*)_{\alpha<\kappa}$ forms a biorthogonal system with $\{y_\alpha^*\}_{\alpha<\kappa}$ dense in $\mathcal{X}^*$ then $\mathcal{X}$ admits an equivalent norm with the MIP.*

2. *If $\mathcal{X}$ is nonseparable and has an equivalent norm with the MIP, then $\mathcal{X}$ has an uncountable ε-biorthogonal system for some $0 \leq \varepsilon < 1$.*

One of the most important tools of Linear Algebra are Hamel basis of vector spaces. It is natural then to look for a Banach space equivalent to Hamel Basis.

Definition 1.23. Let $\mathcal{X}$ be a Banach Space, and let $(x_n)_{n<\omega}$ be a sequence in $\mathcal{X}$. We say $(x_n)_{n<\omega}$ is a *Schauder basis* of $\mathcal{X}$ if for every $x \in \mathcal{X}$ there is a unique sequence of scalars $(a_n)_{n<\omega}$ such that $\sum_n a_n x_n$ is absolutely convergent and

$$x = \sum_{n<\omega} a_n x_n$$

We say $(x_n)_{n<\omega}$ is a *basic sequence*, if it is a Schauder basis of $\overline{\text{span}}(x_n : n < \omega)$.

The following result is useful to check if a sequence is a Schauder basis.

Proposition 1.24. *Let $\mathcal{X}$ be a Banach space and $(x_n)_{n<\omega}$ a sequence of nonzero vectors on $\mathcal{X}$. Then $(x_n)_{n<\omega}$ is a Schauder sequence if and only if the following conditions hold:*

1. *The linear span of $(x_n)_{n<\omega}$ is dense on $\mathcal{X}$.*

2. *There is a constant K such that, for every sequence of scalars $(a_i)_{i<\omega}$, and $n < m$*

$$\left\| \sum_{i=0}^{n} a_i x_i \right\| \leq K \left\| \sum_{i=0}^{m} a_i x_i \right\|$$

We call the minimum constant K that makes 2 above hold, the basis constant.

Note that a sequence $(x_n)_{n<\omega}$ of nonzero vectors of $\mathcal{X}$ is a basic sequence if and only if condition 2 above holds.

Theorem 1.25 (Theorem 1.a.5 of [LT17]). *Every infinite dimensional Banach space has a basic sequence. Furthermore, given $\varepsilon > 0$ we can take the basic sequence to have basic constant $(1 + \varepsilon)$.*

First we prove the following Lemma.

Lemma 1.26. *Let $\mathcal{X}$ be an infinite dimensional Banach space and $Z \subset \mathcal{X}$ is a finite dimensional subspace. For every $\varepsilon > 0$ there is $x \in \mathcal{X}$ with $\|x\| = 1$ such that*

$$\|y\| \leq (1 + \varepsilon)\|y + \lambda x\|$$

for every $y \in Z$ and every $\lambda \in \mathbb{R}$.

Proof. Let $Z \subset \mathcal{X}$ be finite dimensional, and $0 < \varepsilon < 1$ be given. Apply Heine-Borel to conclude that,

$$\mathbb{S}_Z = \{y \in Z : \|y\| = 1\} \text{ is totally bounded.}$$

Thus, we can find a finite sequence $(y_i)_{i < n}$ in $\mathbb{S}_Z$ such that for every $y \in \mathbb{S}_Z$ there is $i_0 < n$ with $\|y - y_i\| < \varepsilon/2$.

Take y_i^* on $\mathcal{X}^*$ such that $\|y_i^*\| = 1$ and $y_i^*(y_i) = 1$. Since $\mathcal{X}$ is infinite dimensional there is $x \in \mathcal{X}$ such that $\|x\| = 1$ and $y_i(x) = 0$ for all $i < n$. We check that this x works.

Let $y \in Z$ and $\lambda \in \mathbb{R}$ be given. Without loss of generality we can assume that $\|y\| = 1$. There is $i < n$ such that

$$\|y_i - y\| < \frac{\varepsilon}{2}$$

Then we have

$$\|y + \lambda x\| \geq |y_i^*(y + \lambda x)| \geq |y_i^*(y)|$$

using the linearity of y_i^* we have,

$$1 = |y_i^*(y_i)| \leq |y_i^*(y)| + |y_i^*(y_i - y)| \leq |y_i^*(y)| + \frac{\varepsilon}{2}$$

Therefore

$$|y_i^*(y)| \geq 1 - \frac{\varepsilon}{2} \geq \frac{1}{1 + \varepsilon}$$

and the Lemma follows. $\qquad\qquad\square$

Proof of Theorem 1.25. Let $\varepsilon > 0$ be given. Take a sequence $0 < \varepsilon_n < 1$ such that

$$\prod_{n < \omega} (1 + \varepsilon_n) \leq 1 + \varepsilon$$

Take some $x_1 \in \mathcal{X}$ with $\|x_1\| = 1$, and let Z_1 be the span of x_1. Now use the Lemma above to find $x_2 \in \mathcal{X}$ with $\|x_2\| = 1$ such that

$$\|y\| \leq (1 + \varepsilon_1)\|y + \lambda x_2\|$$

for every $y \in Z_1$ and every $\lambda \in \mathbb{R}$. Denote the span of x_1 and x_2 by Z_2.

We can continue to use the Lemma to find a sequence $(x_n)_n$ of normalized vectors of $\mathcal{X}$,

$Z_n = \text{span}\{x_1, \ldots, x_n\}$ such that

$$\|y\| \le (1 + \varepsilon_n)\|y + \lambda x_{n+1}\|$$

for every $y \in Z_n$ and every $\lambda \in \mathbb{R}$.

Let $Y = \overline{\text{span}}\{x_1, \ldots, x_n, \ldots\}$. Then $(x_n)_n$ forms a Schauder basis of Y of basic constant $(1 + \varepsilon)$. To see this, let $a_1, \ldots, a_n, \ldots$ be a sequence of scalars, and $n < m$ be given. Then

$$\left\|\sum_{i=1}^{n} a_i x_i\right\| \le \prod_{i=n+1}^{m} (1 + \varepsilon_i) \left\|\sum_{i=1}^{m} a_i x_i\right\| \le (1 + \varepsilon) \left\|\sum_{i=1}^{m} a_i x_i\right\|$$

$\square$

If the Banach space $\mathcal{X}$ on which we are working is not separable it cannot have a Schauder sequence. Also, basic sequences of $\mathcal{X}$ will only give us information about separable subspaces of $\mathcal{X}$. Since the nonseparable subspaces of $\mathcal{X}$ can have behavior that differs from separable subspaces, we need to strengthen the definition to work with uncountable sequences. In this work we are only concerned with Banach spaces of density $\aleph_1$. This is because the study of nonsparable Banach spaces is much different that the separable counterpart, and already for spaces of density $\aleph_1$ we can see a big difference between the separable and nonseparable theories.

Definition 1.27. We say that a sequence $(y_\alpha)_{\alpha < \omega_1}$ in a Banach space $\mathcal{X}$ is an *uncountable Schauder basis of constant K* for $K \ge 1$, if the two conditions hold:

1. $\mathcal{X} = \overline{\text{span}}\{y_\alpha : \alpha < \omega_1\}$, and

2. For every $\lambda < \omega_1$ and every sequence of reals $(a_\alpha)_{\alpha < \omega_1}$ we have

$$\left\|\sum_{\alpha < \lambda} a_\alpha y_\alpha\right\| \le K \left\|\sum_{\alpha < \omega_1} a_\alpha y_\alpha\right\|$$

We say that $(y_\alpha)_{\alpha < \omega_1}$ is an *uncountable K-basic sequence* if condition 2 above holds. Equivalently, $(y_\alpha)_{\alpha < \omega_1}$ is a Schauder basis of constant K on $\overline{\text{span}}\{y_\alpha : \alpha < \omega_1\}$.

The first thing we want is a Theorem that says *every Banach space of density $\aleph_1$ has an uncountable $(1 + \varepsilon)$-basic sequence* unfortunately that is no possible, we cannot even guarantee that a Banach space of density $\aleph_1$ has an uncountable biorthogonal system. However we have the following result of 2006 from Todorčević,

Theorem 1.28 (Todorčević [Tod06])**.** *Assume* MA_{ω_1} *and* PID. *Every Banach space of density $\aleph_1$ has a quotient with an uncoutable Schauder basis of constant $K = 1$. In particular every Banach space of density $\aleph_1$ has an uncountable biorthogonal system.*

Recall that PID is the P-ideal dichotomy stating that for any P-ideal $\mathcal{I}$ of countable subsets of some index set S, either S can be partitioned into countably many subsets orthogonal to $\mathcal{I}$

or there is an uncountable subset of S all of whose countable subsets belong to $\mathcal{I}$. For more about this sort of dichotomies the reader is referred to [Tod11].

In Chapter 4 we will show that positive results in this direction require some extra axioms. We do this by constructing Banach spaces of density $\aleph_1$ that do not have uncountable biorthogonal systems, even if they have some uncountable ε-biorthogonal system. We also construct other spaces with uncountable Schauder basis of constant $K > 1$ but no uncountable L-basic sequence for $1 \leq L < K$. This results are done with a capturing construction scheme. Thus we get the following results:

Corollary 1.29. *Assume MA_{ω_1} and PID.*

There are no capturing construction schemes.

See below for the definition of a capturing construction scheme.

We will see in chapter 3 that there is a Suslin tree provided there is a 3-capturing construction scheme, therefore we get:

Corollary 1.30. *Assume MA_{ω_1}.*

There are no 3-capturing construction schemes.

Some Technical comments: We introduce here the techniques that will appear in Chapter 4 without mention. Let $c_{00}(\omega_1)$ be the vector space of functions $x : \omega_1 \to \mathbb{R}$ with finite support. Where the support of x is defined by

$$\mathrm{supp}(x) = \overline{\{\alpha < \omega_1 : x(\alpha) \neq 0\}}$$

For $\gamma < \omega_1$ we let

$$e_\gamma(\alpha) = \begin{cases} 0 & \alpha \neq \gamma \\ 1 & \alpha = \gamma \end{cases}$$

be the unit basis vector of $c_{00}(\omega_1)$.

If F is a finite subset of ω_1 and $h : F \to \mathbb{R}$, we consider the extension of h in $c_{00}(\omega_1)$ to be zero outside of F and still refer to it as h without risk of confusion.

If $h, x \in c_{00}(\omega_1)$ we denote

$$\langle h, x \rangle = \sum_{\alpha < \omega_1} h(\alpha)x(\alpha) \tag{1.2}$$

which is well-defined because h and x have finite support.

In order to make counting arguments we work most of the time on $c_{00}(\omega_1, \mathbb{Q})$, meaning we consider functions in $c_{00}(\omega_1)$ that only take values in $\mathbb{Q}$.

1.2   Construction schemes

In this section, we introduce the notion of a construction scheme.

Capturing construction schemes were introduced in Todorčević [Tod17], where they were used to construct compact spaces and non-separable normed spaces with considerable control on their non-separable structure. In section 5 of [Tod17] a general framework to construct Banach spaces using construction schemes is introduced. This framework, together with the forcing amalgamations of [LAT11], constitute the technology behind the proofs of Theorem 4.4 and Theorem 4.5.

The key feature of this scheme is that it provides a family $\mathcal{F}$ of finite subsets of ω_1 which allow us to construct uncountable structures such us trees, gaps or norming sets. The way this works is the following; we use the elements F of $\mathcal{F}$ to "approximate" an uncountable structure in ω_1 and use the canonical decomposition (see below) for the recursive construction, for this we want all approximations of the same rank k to be "isomorphic". The recursive step is done by amalgamating *many* isomorphic structures of lower rank. These amalgamations will determine the behavior of uncountable substructures of the limit structure via an appropriate property of *capturing* of the construction scheme.

We explore this ideas in greater detail.

Definition 1.31. Let $(m_k)_{k<\omega}, (n_k)_{1\leq k<\omega}$ and $(r_k)_{1\leq k<\omega}$ be sequences of natural numbers such that $m_0 = 1$, $m_{k-1} > r_k$ for all $k > 0$, $n_k \geq 2$ and for every $r < \omega$ there are infinitely many k's with $r_k = r$. If for every $k > 0$ we have

$$m_k = n_k(m_{k-1} - r_k) + r_k$$

we say that $(m_k, n_k, r_k)_{k<\omega}$ forms a *type*.

Definition 1.32. We say that $\mathcal{F}$ is a *construction scheme of type* $(m_k, n_k, r_k)_{k<\omega}$ if $\mathcal{F} \subset [\omega_1]^{<\omega}$, a family of finite subsets of ω_1, we can partition $\mathcal{F} = \bigcup_{k<\omega} \mathcal{F}_k$ and for every $F \in \mathcal{F}$ there is $R(F) \sqsubset F$, such that

1. For every $A \subset \omega_1$ finite, there is $F \in \mathcal{F}$ such that $A \subset F$.

2. $\forall F \in \mathcal{F}_k$, $|F| = m_k$ and $|R(F)| = r_k$.

3. For all $F, E \in \mathcal{F}_k$, $E \cap F \sqsubset F, E$.

4. $\forall F \in \mathcal{F}_k$, there are unique $F_0, \ldots, F_{n-1} \in \mathcal{F}_{k-1}$ with

$$F = \bigcup_{i<n} F_i$$

Furthermore $n = n_k$ and $(F_i)_{i<n_k}$ forms an increasing Δ-system with root $R(F)$, i.e.,

$$R(F) < F_0 \setminus R(F) < \ldots < F_{n_k-1} \setminus R(F)$$

We call the sequence $(F_i)_{i<n_k}$ of (4) the *canonical decomposition* of F.
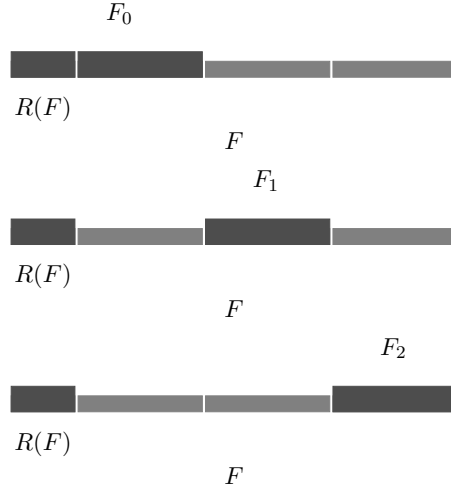
Figure 1.1: Canonical decomposition of F into F_0, F_1, and F_2 with root $R(F)$.

It is implicitly proved in [Tod17] that for any type $(m_k, n_k, r_k)_{k<\omega}$ there is a construction scheme with that type.

To avoid confusion we will use m_k, n_k and r_k as above and we will omit reference to the type of a construction scheme. For $F \in \mathcal{F}$ and $F = \bigcup_{i<n_k} F_i$ the canonical decomposition of F. We simplify the notation and write $\varphi_i : F_0 \to F_i$ for the unique order-preserving bijection φ_{F_0,F_i}. Analogously, if f is a function on F_0 we can define the function $\varphi_i(f)$ in F_i by $\gamma \mapsto f(\varphi_i^{-1}(\gamma))$.

The following lemma tells us more about the structure of a construction scheme

Lemma 1.33. *For $F \in \mathcal{F}_k$, $E \in \mathcal{F}_l$, with $l \leq k$ we have $E \cap F \sqsubseteq E$.*

Proof. We prove the lemma by induction on k and l. If $l = k$ the result follows by the properties of $\mathcal{F}$. It's enough to show that: if it holds for $l \leq k - 1$, it holds for l and k as well. Let F as above and let $F = \bigcup_{i<n_k} F_i$ be its canonical decomposition. Since the F_i's are in $\mathcal{F}_{k-1}$ we can apply our hypothesis and $E \cap F_i \sqsubseteq E$ for every $i < n_k$. If $E \cap (F \setminus R(F)) = \emptyset$ then the result follows, otherwise let $i < n_k$ be minimal such that $E \cap (F_i \setminus R(F)) \neq \emptyset$ then $E \cap F = E \cap F_i$. Because if not, there is $i < j < n_{k+1}$ with $E \cap F_j \not\sqsubseteq E$. Thus we have $E \cap F = E \cap F_i \sqsubset E$ and the result follows. $\square$

Corollary 1.34. *For $F \in \mathcal{F}_k$, $E \in \mathcal{F}_l$ and $F = \bigcup_{i<n_k} F_i$ the canonical decomposition of F. If $E \subset F$ and $l < k$ then there is some $i < n_k$ with $E \subset F_i$. In particular, if $l = k - 1$ we have $E = F_i$.*

Corollary 1.35. *Let $E, F \in \mathcal{F}_k$, then $\varphi_{E,F}(\mathcal{F} \restriction E) = \mathcal{F} \restriction F$. Where $\mathcal{F} \restriction F = \{L \in \mathcal{F} : L \subset F\}$.*

Lemma 1.36. *For $F \in \mathcal{F}_k$, $E \in \mathcal{F}_l$ and $E \subset F$ (in particular $l \leq k$). For every $\mu \in E$ there is a copy E^* of E in F such that*

  1. $E^* \cap (\mu + 1) = E \cap (\mu + 1)$.

2. $E^* \setminus \mu$ is an interval of F with $\mu \in E^*$.

Proof. We prove the lemma by induction on k and l. The result follows for $l = k$. Suppose the result hold for l and $k - 1$ and $l < k$. Take $F = \bigcup_{i<n_k} F_i$, the canonical decomposition of F. By Corollary 1.34 there is $i < n_k$ such that $E \subset F_i$. By the induction hypothesis there is E^{**} a copy of E in F_i such that the conclusion holds. If $\mu \notin R(F)$ then $E^* = E^{**}$ works.

Otherwise, let $E^* = \varphi_{F_i,F_0}(E^{**})$ by Corollary 1.35, E^* is a copy of E and $E^* \setminus \mu$ is an interval of F_0. Since $\mu \in R(E)$ then (1) holds, and (2) holds because F_0 is an interval of F. □

At this point, the reader is probably expecting to see a proof that construction schemes exists. This expectation is warranted, but we will not do it in this Chapter. Instead, we postpone the proof of existence to Chapter 2 and dedicate the next section to illustrate why we are interested in construction schemes. We invite the reader to see Theorem 2.1 for a proof that construction schemes of any reasonable type exists in ZFC.

1.2.1 First Applications

Let us see now how the construction scheme $\mathcal{F}$ can be used to recursively construct classical combinatorial structures in a natural and intuitive way. We start with a Hausdorff gap in ω^ω. We have already seen gaps in $[\omega]^\omega$. Turns out every (ω_1, ω_1)-gap in ω^ω can be transformed into an (ω_1, ω_1)-gap of $[\omega]^\omega$. The reason why we choose this example as a first application is because it illustrates how to apply construction schemes.

Example 1.2 (Hausdorff gap)**.** Fix a construction scheme $\mathcal{F}$.

Our aim is to construct a pre-gap $(a_\alpha, b_\alpha)_{\alpha<\omega_1}$ with the property of Proposition 1.11. We do this by constructing an increasing sequence $(N_k)_{k<\omega}$, and approximations $(a_\alpha^F, b_\alpha^F : \alpha \in F)$, for all $F \in \mathcal{F}_k$, such that

(*i*) for every $F, E \in \mathcal{F}_k$, and $\alpha \in F$ and $\beta \in E$ such that $\beta = \varphi_{F,E}(\alpha)$ where $\varphi_{F,E}$ is the increasing bijection from F onto E. Then we have $a_\beta^E = \varphi_{F,E}(a_\alpha^F)$, and $b_\beta^E = \varphi_{F,E}(b_\alpha^F)$

(*ii*) for every $F \in \mathcal{F}_k$ and $\alpha \in F$, $a_\alpha^F, b_\alpha^F \subset N_k$,

(*iii*) For every $l < k$, $E \in \mathcal{F}_l$, $F \in \mathcal{F}_k$, with $E \subset F$, and $\alpha, \beta \in E$, we have:

    (*a*) $a_\alpha^F \cap N_l = a_\alpha^E$, and $b_\alpha^F \cap N_l = b_\alpha^E$,

    (*b*) $a_\alpha \setminus a_\beta \subset N_l$, and $b_\alpha \setminus b_\beta \subset N_l$, and

    (*c*) $a_\alpha \cap b_\beta \subset N_l$.

Let us proceed with the construction of $(a_\alpha^F, b_\beta^F : \alpha \in F)$ and $F \in \mathcal{F}_k$, with conditions (*i*) to (*iii*). Start with $N_0 = 2$. Take $F \in \mathcal{F}_0$, we have $F = \{\alpha\}$ for some $\alpha < \omega_1$. Let $a_\alpha^F = \{0\}$ and $b_\alpha^F = \{1\}$.

Now suppose we have $(a_\alpha^E, b_\alpha^E : \alpha \in E)$ for all $E \in \mathcal{F}_l$, $l < k$, satisfying (i) to (iii). Take $F \in \mathcal{F}_k$ and let $F = \bigcup_{i < n_k} F_i$ be the canonical decomposition of F. Since $F_i \in \mathcal{F}_{k-1}$ we have $(a_\alpha^{F_i}, b_\alpha^{F_i} : \alpha \in F_i)$ defined.

Let $(P_i)_{i < n_k}$ such that $P_0 \geq N_{k-1} + r_k$, and $P_{j+1} - P_j \geq m_k$, for all $j < n_k - 1$. And let $Q < \omega$ such that $Q \geq P_j + m_k$ for $j = n_k - 1$. Take $N_k = Q + m_k$.

Enumerate $R(F) = \{\alpha_0 < \ldots < \alpha_{r_k - 1}\}$, then

$$a_{\alpha_i}^F = a_{\alpha_i}^{F_0} \cup \{N_{k-1}, \ldots, N_{k-1} + i\}$$
$$b_{\alpha_i}^F = b_{\alpha_i}^{F_0} \cup \{Q, \ldots, Q + i\}$$

Note that condition (i) implies $a_{\alpha_i}^{F_0} = a_{\alpha_i}^{F_j}$ and $b_{\alpha_i}^{F_0} = b_{\alpha_i}^{F_j}$ for any $j < n_k$.

Enumerate $F_0 \setminus R(F) = \{\gamma_{r_k} < \ldots < \gamma_{m_k - 1}\}$, then

$$a_{\gamma_i}^F = a_{\gamma_i}^{F_0} \cup \{N_{k-1}, \ldots, N_{k-1} + r_k - 1\} \cup \{P_0, \ldots, P_0 + i\}$$
$$b_{\gamma_i}^F = b_{\gamma_i}^{F_0} \cup \{Q, \ldots, Q + i\}$$

Now pick $0 < j < n_k$ and let $\delta_i = \varphi_j(\gamma_i)$. Recall φ_j is the increasing bijection on F_0 onto F_j, let

$$a_{\delta_i}^F = a_{\gamma_i}^{F_j} \cup \{N_{k-1}, \ldots, N_{k-1} + r_k - 1\} \cup \{P_j, \ldots, P_j + i\}$$
$$b_{\delta_i}^F = b_{\gamma_i}^{F_j} \cup \{Q, \ldots, Q + m_k - 1\} \cup \bigcup_{l < j}\{P_l, \ldots, P_l + i\}$$

It is clear from the construction that $(a_\alpha^F, b_\alpha^F : \alpha \in F)$ satisfies conditions (i) to (iii).

Now let

$$a_\alpha = \bigcup_{\substack{F \in \mathcal{F} \\ \alpha \in F}} a_\alpha^F \qquad b_\alpha = \bigcup_{\substack{F \in \mathcal{F} \\ \alpha \in F}} b_\alpha^F$$

it is clear that $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is a pre-gap, by conditions (i)–(iii). To see it is a gap. Let $(\xi_n)_{n < \omega}$ be an increasing sequence of ordinals and $\sup_n \xi_n \leq \xi < \omega_1$. Pick $E \in \mathcal{F}$ with $\xi_0, \xi \in E$. Since E is finite there is $\xi_N \notin E$. We know there is $F \in \mathcal{F}_k$ with $E \subset F$ and $\xi_N \in F$. Furthermore, pick F such that $k < \omega$ is the first $k < \omega$ with this property, i.e, if $F \in \mathcal{F}_l$ and $l < \omega$, then either $E \not\subset F$ or $\xi_N \notin F$. Let $F = \bigcup_{i < n_k} F_i$ be the canonical decomposition of F. There are $i < j < n_k$ such that $\xi_N \in F_i$ and $\xi \in F_j$, this is where we used that $k < \omega$ was the first that $E \cup \{\xi\} \subset F$. Let $\alpha = \xi_N$ and $\beta = \xi$, then $P_i \in (a_\alpha^F \cap b_\beta^F)$ therefore $a_\alpha \cap b_\beta \neq \emptyset$. By Proposition 1.11, $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is an (ω_1, ω_1)-gap. ∎

Note that the conditions (i)–(iii) are very intuitive if you want to approximate a pre-gap. We turn our attention now to Aronsajn trees.

We want to use a construction scheme to build an Aronsajn tree in a natural way. The idea is to construct a tree $T \subset \omega^{<\omega_1}$, such that every branch of T is one-to-one, therefore T does

not have uncountable branches. To do this, we construct a map $\rho : [\omega_1]^2 \to \omega$, such that for every $\alpha < \omega_1$, the map $\rho_\alpha : \alpha \to \omega$ defined by $\rho_\alpha(\xi) = \rho(\xi, \alpha)$ is one-to-one and

$$\text{for every } \alpha < \beta, \ \{\xi < \alpha : \rho_\alpha(\xi) \neq \rho_\beta(\xi)\} \text{ is finite.} \tag{1.3}$$

a map with this property is called a *coherent map*. Given a Coherent map $(\rho_\alpha : \alpha < \omega_1)$ as before we let

$$T_\alpha = \Big\{\sigma \in \omega^\alpha : \{\xi < \alpha : \sigma(\xi) \neq \rho_\alpha(\xi)\} \text{ is finite and } \sigma \text{ is one-to-one}\Big\}$$
$$T = \bigcup_{\alpha < \omega_1} T_\alpha \tag{1.4}$$

The tree T has the induced order by extension from $\omega^{<\omega_1}$. Therefore, if we construct a coherent map $(\rho_\alpha : \alpha < \omega_1)$ we can construct an Aronsajn tree.

Example 1.3 (Coherent Map). We want to approximate $(\rho_\alpha : \alpha \to \omega : \alpha < \omega_1)$ such that ρ_α is one-to-one, and the condition (1.3) holds.

Fix a construction scheme $\mathcal{F}$, we construct $(N_k)_{k<\omega}$ increasing, and $(\rho_\alpha^F : (F \cap \alpha) \to N_k : \alpha \in F)$ for every $F \in \mathcal{F}_k$, such that

(i) for $F, E \in \mathcal{F}_k$, $\alpha \in F$, and $\beta \in E$ with $\beta = \varphi_{F,E}(\alpha)$, we have $\rho_\beta^E = \varphi_{F,E}(\rho_\alpha^F)$,

(ii) ρ_α^F is one-to-one.

(iii) for every $l < k$, $E \in \mathcal{F}_l$, $F \in \mathcal{F}_k$, with $E \subset F$, and $\alpha \in E$, we have:

(a) $\rho_\alpha^F \restriction E = \rho_\alpha^E$,

(b) for $\xi \in (F \cap \alpha)$, $\beta \in F$, with $\alpha < \beta$, if $\rho_\alpha^F(\xi) \neq \rho_\beta^F(\xi)$ then $\xi \in E$.

Note that part (b) of (ii), will imply property 1.3 and condition (ii) will imply ρ_α is one-to-one, conditions (i) and (iii)–(a) make sure that ρ_α will be well defined.

Suppose we have $(\rho_\alpha^E : (E \cap \alpha) \to N_l : F \in \mathcal{F}_l)$ with $l < k$. Take $F \in \mathcal{F}_k$, and $F = \bigcup_{i<n_k} F_i$ be the canonical decomposition of F.

For $\alpha \in F_0$ let $\rho_\alpha^{F_0} = \rho_\alpha^F$.

Now enumerate $F = \{\gamma_0 < \gamma_1 < \ldots, \gamma_{m_k-1}\}$, Pick $\gamma_i \in F_j \setminus R(F)$ for some $j > 0$, for $\xi \in (F_j \cap \gamma_j)$ define

$$\rho_{\gamma_i}^F(\xi) = \begin{cases} \rho_{\gamma_i}^{F_j}(\xi) & \xi \in F_j \\ N_{k-1} + \sigma & \xi \notin F_j \text{ and } \xi = \gamma_\sigma \end{cases}$$

Note that ρ_α^F is well defined and conditions (i)–(iii) hold.

For every $\xi < \alpha < \omega_1$ let $F \in \mathcal{F}$ with $\alpha, \xi \in F$, define

$$\rho_\alpha(\xi) = \rho_\alpha^F(\xi)$$

note that this is well defined by (iii).

Also, $\rho_\alpha : \alpha \to \omega$ is one-to-one, otherwise there are $\xi_0 < \xi_1 < \alpha$, and $F \in \mathcal{F}$ with $\xi_0, \xi_1, \alpha \in F$ and $\rho_\alpha^F(\xi_0) = \rho_\alpha^F(\xi_1)$ which contradicts (iii) above. To see $(\rho_\alpha : \alpha < \omega_1)$ has property (1.3), let $\alpha < \beta < \omega_1$ and suppose we have and increasing sequence $\xi_0 < \xi_1 < \ldots < \alpha$ such that

$$\rho_\alpha(\xi_n) \neq \rho_\beta(\xi_n) \text{ for every } n < \omega. \tag{1.5}$$

Take $E \in \mathcal{F}$ with $\xi_0, \alpha, \beta \in E$. Since E is finite there is $N < \omega$ such that $\xi_N \notin E$. Pick $F \in \mathcal{F}_k$ such that $E \subset F$, and $\xi_N \in F$. Furthermore, we take F so that k is the first with this property. Let $F = \bigcup_{i<n_k} F_i$ be the canonical decomposition of F. Then $\xi_N \in F_i$, $\alpha \in F_j$, and $\beta \in F_{j*}$ for some $i < j \leq j^* < n_k$. By the construction we have

$$\rho_\alpha^F(\xi_N) = \rho_\beta^F(\xi_N)$$

This contradicts (1.5) therefore (1.3) holds and $(\rho_\alpha : \alpha < \omega_1)$ is as we wanted.

Let $T \subset \omega^{<\omega_1}$ be defined as in (1.4), then T is an Aronsajn tree. ∎

No doubt the reader would have noted that our construction of ρ_α^F is the minimum work one would have to do to construct a one-to-one function with property (1.3).

We hope this examples motivate the reader to continue the study of construction schemes. In the next section we study a property that makes construction schemes all the more useful, allowing for constructions beyond ZFC.

1.2.2 Capturing

As we have seen already, construction schemes are useful tools for buiding uncountable structures. However, in order to control the behavior of the family of uncountable subsets of the structure under construction we require an extra property of the construction scheme. In this section we introduce the concept of a capturing construction scheme of Todorcevic [Tod17]. They form the main tool behind the constructions of Lopez & Todorcevic [LT17] and Lopez [Lí7] which we present latter in this Thesis.

Given a construction scheme $\mathcal{F}$ of any type. The idea is that for every uncountable Δ-system, we can find $F \in \mathcal{F}$ such that the canonical decomposition of F witnesses a finite part of the Δ-System. More precisely,

Definition 1.37. Let $\mathcal{F}$ be a construction scheme. We say that $\mathcal{F}$ is *n-capturing* if for every uncountable Δ-system $(s_\xi)_{\xi<\omega_1}$ of finite subsets of ω_1 with root s there are $\xi_0 < \ldots < \xi_{n-1} < \omega_1$, and $F \in \mathcal{F}$ with canonical decomposition $F = \bigcup_{i<n_k} F_i$, such that

$$s \subset R(F)$$
$$\text{for every } i < n, \quad s_{\xi_i} \setminus s \subset F_i \setminus R(F),$$
$$\text{for every } i < n, \quad \varphi_i(s_{\xi_0}) = s_{\xi_i}.$$

We say that $\mathcal{F}$ is *capturing* if $\mathcal{F}$ is n-capturing for every $n < \omega$.

Remark 1.1. Note that a construction scheme $\mathcal{F}$ which is n-capturing must have type $(m_k, n_k, r_k)_{k<\omega}$ with $n_k \geq n$ for infinitely many k's. This should be contrasted with previous methods to construct uncountable objects via amalgamations of finite substructures (see [Vel84] or [She85]) where only two amalgamations were considered. This is relevant because it is consistent to have n-capturing construction schemes but no $(n+1)$-capturing construction schemes, see Theorem 2.3.

Remark 1.2. Suppose $\mathcal{F}$ is a 2-capturing construction scheme of type $(m_k, 2, r_k)_{k<\omega}$, in other words $n_k = 2$ for every $k < \omega$. It is easy to see that $\mathcal{F}$ is an $(\omega, 1)$-morass in the sense of Veleman [Vel84], this is clear with his (equivalent) definition of *expanded simplified* $(\omega, 1)$-*morass*. since morasses do not have an equivalent definition of capturing, capturing construction schemes generalize morasses in a strong sense

The use of $(\kappa, 1)$-morasses for regular κ is well known in Set Theory. This suggest a hypothetical generalization of capturing construction schemes to higher cardinals would have interesting consequence in Set Theory.

We will see later in Chapter 3 that 3-capturing is enough to construct interesting combinatorial objects. For now we have the following example.

Example 1.4. Suppose $\mathcal{F}$ is 3-capturing. Consider the forcing $\mathbb{P}$ of all P finite subsets of ω_1 such that for every $\xi_0 < \xi_1 < \xi_2$ in P, $\mathcal{F}$ does not 3-captures $\big\{\{\xi_i\} : i < 3\big\}$.

The ordering $P \leq Q$ in $\mathbb{P}$ means $Q \subset P$.

We check that $\mathbb{P}$ is ccc. Fix $(P_\alpha : \alpha < \omega_1) \subset \mathbb{P}$. We can assume, by going to a subsequence, it is a Δ-System. We apply 3-capturing and get $\alpha_0 < \alpha_1 < \alpha_2$ and $F \in \mathcal{F}$ capturing $(P_{\alpha_i} : i = 0, 1, 2)$. We take $P = P_{\alpha_0} \cup P_{\alpha_2}$ and we have $P \leq P_{\alpha_0}, P_{\alpha_2}$ and $P \in \mathbb{P}$. Therefore $\mathbb{P}$ is ccc.

Now let $\alpha < \omega_1$, consider $\mathscr{D}_\alpha = \{P \in \mathbb{P} : P \setminus \alpha \neq \emptyset\}$.

Claim 1.38. *For every $\alpha < \omega_1$, $\mathscr{D}_\alpha$ is dense.*

Proof. Let $P \in \mathbb{P}$, and $\alpha < \omega_1$ be given. If $P \setminus \alpha \neq \emptyset$ we are done. Otherwise, let $E \in \mathcal{F}$, such that $P \cup \{\alpha\} \in E$. Now take $F \in \mathcal{F}_k$ such that $E \subset F$ and $r_k = 0$. This implies $R(F) = \emptyset$. If $F = \bigcup_{i<n_k} F_i$ is the canonical decomposition of F. There is $i < n_k$ with $E \subset F_i$. Let $i < j < n_k$, then $F_j > \alpha$. Take

$$Q = P \cup \varphi_{F_i, F_j}(P)$$

Then $Q \in \mathbb{P}$, because $\mathcal{F}$ does not capture any three elements of P, therefore it does not captures any three elements of $\varphi_{F_i, F_j}(P)$. Finally, if some $F^* \in \mathcal{F}_\ell$ captures $\big\{\{\xi_i\} : i < 3\big\}$ where $\xi_0 \in P$ and $\xi_2 \in \varphi_{F_i, F_j}(P)$, then $F^* \cap F_j$ is not an initial segment of F^* (we assume here that $\ell < k$ the other case is analogous). This contradicts Lemma 1.33. Therefore $Q \in \mathbb{P}$ and $Q \leq P$. It is clear that $Q \in \mathscr{D}_\alpha$ because $\varphi_{F_i, F_j}(P) \subset Q \setminus \alpha$. $\square$

Let $\mathcal{D} = \{\mathscr{D}_\alpha : \alpha < \omega_1\}$. Apply MA_{ω_1} to find G, a $\mathcal{D}$-generic filter. Let

$$\Gamma = \bigcup_{P \in G} P$$

Then Γ is uncountable and $\mathcal{F}$ does not 3-captures $\{\{\xi\} : \xi \in \Gamma\}$. Indeed, suppose there are $\xi_0 < \xi_1 < \xi_2$ in Γ and $F \in \mathcal{F}$ 3-capturing the corresponding Δ-System. There is $P \in G$ with $\xi_0, \xi_1, \xi_2 \in P$, but then P is 3-captured by F which contradicts the fact that $P \in \mathbb{P}$. $\blacksquare$

The previous example has as a consequence,

Corollary 1.30. *Assume MA_{ω_1}.*

There are no 3-capturing construction schemes.

We will see other proofs of this result in latter Chapters. It is clear that n-capturing implies m-capturing for $m \leq n$. Thus, we have the following hierarchy:

$$2\text{-capturing} \Longleftarrow \ldots \Longleftarrow n\text{-capturing} \Longleftarrow (n+1)\text{-capturing} \Longleftarrow \ldots \Longleftarrow \text{capturing}$$

We will show in Chapter 5 that none of the implications above can be reversed. There is a generalization of capturing that proves useful in some examples of Todorcevic [Tod17]. We present it here for completeness.

Definition 1.39. Let $\mathcal{F}$ be a construction scheme. We say that $\mathcal{F}$ is *fully capturing* if for every uncountable Δ-system $(s_\xi)_{\xi < \omega_1}$ of finite subsets of ω_1 with root s, and every $k^* < \omega$ there are $F \in \mathcal{F}_k$ with $k > k^*$, canonical decomposition $F = \bigcup_{i < n_k} F_i$, and $\xi_0 < \ldots < \xi_{n_k-1} < \omega_1$, such that

$$s \subset R(F)$$
$$\text{for every } i < n_k, \quad s_{\xi_i} \setminus s \subset F_i \setminus R(F),$$
$$\text{for every } i < n_k, \quad \varphi_i(s_{\xi_0}) = s_{\xi_i}.$$

Definition 1.40. Let $\omega = \bigcup_{\ell < \omega} P_\ell$ be a partition of ω into infinite components and let $\vec{P} = (P_\ell : \ell < \omega)$. Suppose (m_k, n_k, r_k) forms a type such that for every $\ell < \omega$, and every $r < \omega$ there are infinitely many k's in P_ℓ with $r_k = r$. Then we say $(m_k, n_k, r_k)_k$ forms a $\vec{P}$-*type*.

Definition 1.41. Let $\mathcal{F}$ be a construction scheme with type $(m_k, n_k, r_k)_k$, and $2 \geq n$. We say $\mathcal{F}$ is n-$\vec{P}$-*capturing* if $(m_k, n_k, r_k)_k$ forms a $\vec{P}$-type, and for every uncountable Δ-system $(s_\xi)_{\xi < \omega_1}$ of finite subsets of ω_1 with root s, and every $\ell < \omega$, there are $\xi_0 < \ldots < \xi_{n-1} < \omega_1$, $k \in P_\ell$ and $F \in \mathcal{F}_k$ with canonical decomposition $F = \bigcup_{i < n_k} F_i$, such that

$$s \subset R(F)$$
$$\text{for every } i < n, \quad s_{\xi_i} \setminus s \subset F_i \setminus R(F),$$
$$\text{for every } i < n, \quad \varphi_i(s_{\xi_0}) = s_{\xi_i}.$$

We say $\mathcal{F}$ is $\vec{P}$-*capturing* if $\mathcal{F}$ is n-$\vec{P}$-capturing for every $n < \omega$.

What makes this version interesting is that it allows for different amalgamations. For example, the existence of a 2-$\vec{P}$-capturing construction scheme implies there are Suslin trees and T-gaps. We can also define $\vec{P}$-*fully capturing* in the obvious manner.

Chapter 2

Consistency of Capturing Construction Schemes

This Chapter is dedicated to prove results about the existence of capturing construction schemes and to explore the relation between the different forms of capturing. Our main goal for this Chapter is to show that construction schemes exists in ZFC and that capturing construction schemes are consistent with ZFC. We do this on Section 2.1 and Section 2.2 respectively. The rest of the Chapter is dedicated to study the relation between capturing and the weaker forms of capturing: n-capturing, this is relevant because the combinatorial consequences of capturing only require 3-capturing, see Chapter 3. Yet applications to Banach spaces seems to demand capturing, or $\vec{P}$-capturing, see Chapter 4.

The first consistency result about construction schemes can be found in Todorčević [Tod17], where $\diamondsuit$ is used to show existence of fully capturing construction schemes. Implicit in the proof of Theorem 2.3 of [Tod17] is the result that construction schemes exists in ZFC.

We dedicate Section 2.1 to the proof of the following Theorem which we consider of independent interest (see definitions below).

Theorem 2.1. *For any given type $(m_k, n_k, r_k)_{k<\omega}$ there is a construction scheme $\mathcal{F}$ of that type.*

We present the proof of existence of construction schemes on ZFC first because construction schemes are useful in providing a framework to build classical combinatorial structures in an intuitive way. The reader will also benefit by reading the proof of construction schemes in ZFC as the technique is analogous to the argument adding ω_1 Cohen Reals.

Section 2.2 of this Chapter is dedicated to the proof of the consistency of capturing construction schemes. We present a result from Kalajdzievski and Lopez [KL17], adding ω_1 Cohen reals also adds fully capturing construction schemes. We already know by Corollary 1.30 that capturing construction schemes cannot be shown to exists in ZFC alone, so it is necessary to assume extra axioms or use forcing to prove the existence of capturing construction schemes.

Theorem 2.2. *Adding $\kappa \geq \aleph_1$ Cohen reals also adds a fully capturing construction scheme.*

In Section 2.3 we study the relation between capturing and n-capturing. We provide a detailed analysis of the consistency of n-capturing construction schemes. We conclude that it is consistent to have n-capturing constrution schemes, but no $(n + 1)$-capturing. Furthermore, we show that n-capturing implies $\mathrm{MA}_{\omega_1}(\mathrm{K}_{n+1})$ fails.

Theorem 2.3. *$MA_{\omega_1}(K_m)$ and n-capturing are independent if $n \leq m$ and they are incompatible if $n > m$. Also $MA_{\omega_1}(precaliber \aleph_1)$ is independent of capturing.*

On Section 2.4 we extend the previous results to other forms of capturing, such as full $\vec{P}$-capturing, and n-$\vec{P}$-capturing.

We finish the Chapter with a summary, Section 2.5, of set theoretic axioms that are consistent or inconsistent with the existence of a capturing construction scheme. Some of this results have already been proved in the previous sections but we find it useful to include a summary of consistency results here.

2.1   Construction Schemes on ZFC

The result of this section is implicit in the proof of Todorcevic [Tod17]. Before going into the prove let us recall what a construction scheme is:

Definition 2.4. Let $(m_k)_{k<\omega}, (n_k)_{1\leq k<\omega}$ and $(r_k)_{1\leq k<\omega}$ be sequences of natural numbers such that $m_0 = 1$, $m_{k-1} > r_k$ for all $k > 0$, $n_k \geq 2$ and for every $r < \omega$ there are infinitely many k's with $r_k = r$. If for every $k > 0$ we have

$$m_k = n_k(m_{k-1} - r_k) + r_k$$

we say that $(m_k, n_k, r_k)_{k<\omega}$ forms a *type*.

Definition 2.5. We say that $\mathcal{F}$ is a *construction scheme of type* $(m_k, n_k, r_k)_{k<\omega}$ if $\mathcal{F} \subset [\omega_1]^{<\omega}$, a family of finite subsets of ω_1, we can partition $\mathcal{F} = \bigcup_{k<\omega} \mathcal{F}_k$ and for every $F \in \mathcal{F}$ there is $R(F) \sqsubset F$, such that

1. For every $A \subset \omega_1$ finite, there is $F \in \mathcal{F}$ such that $A \subset F$.

2. $\forall F \in \mathcal{F}_k$, $|F| = m_k$ and $|R(F)| = r_k$.

3. For all $F, E \in \mathcal{F}_k$, $E \cap F \sqsubset F, E$.

4. $\forall F \in \mathcal{F}_k$, there are unique $F_0, \ldots, F_{n-1} \in \mathcal{F}_{k-1}$ with

$$F = \bigcup_{i<n} F_i$$

Furthermore $n = n_k$ and $(F_i)_{i < n_k}$ forms an increasing Δ-system with root $R(F)$, i.e.,

$$R(F) < F_0 \setminus R(F) < \ldots < F_{n_k - 1} \setminus R(F)$$

We are now in conditions to state the Theorem.

Theorem 2.1. *Let $(m_k, n_k, r_k)_{k < \omega}$ be a type, there is a construction scheme $\mathcal{F}$ of type $(m_k, n_k, r_k)_{k < \omega}$*

Proof. Let a type $(m_k, n_k, r_k)_{k < \omega}$ be given, we fix this type for the rest of the proof. The idea of the proof is to define a construction scheme on $\mathcal{F}^\beta$ on β, by induction on $\beta < \omega_1$ limit.

We start by showing there is $\mathcal{F}^\omega$ which is a construction scheme in ω, i.e, $\mathcal{F}^\omega$ is a family of finite subsets of ω such that for every $A \subset \omega$ finite, there is $F \in \mathcal{F}^\omega$ with $A \subset F$, and also conditions (2)–(4) from the definition of a construction scheme hold.

We want $\{0, 1, \ldots, m_k - 1\} \in \mathcal{F}^\omega_k$. With this in mind we define first collections $\mathbf{F}(k, i) \subset 2^{m_k}$ for every $k < \omega$, $i \leq k$. Start with

$$\mathbf{F}(1, 0) = \bigcup_{i < m_1} \{i\}$$
$$\mathbf{F}(1, 1) = \Big\{ \{0, 1, \ldots, m_1 - 1\} \Big\}$$

we must have $r_1 = 0$ therefore condition (4) holds by decomposing $\{0, \ldots, m_1\}$ into singletons.

Let $k > 1$, and suppose we have constructed $\mathbf{F}(l, i)$, for all $l < k$, and $i < m_l$, such that conditions (2)–(4) of a construction scheme hold. For $i < n_k$ define

$$F_i = \{0, \ldots, r_k - 1, r_k + i(m_{k-1} - r_k), \ldots, r_k + (i+1)(m_{k-1} - r_k)\}$$

and consider $\varphi_i : F_0 \to F_i$ the order preserving bijection between F_0 and F_i, note that both F_0 and F_i have the same size, m_{k-1}, and φ_0 is the identity on $F_0 = \{0, 1, \ldots, m_{k-1}\}$. Then,

$$\mathbf{F}(k, j) = \bigcup_{i < n_k} \varphi_i \Big(\mathbf{F}(k-1, j) \Big) \text{ for all } j < k$$
$$\mathbf{F}(k, k) = \Big\{ \{0, 1, \ldots, m_k - 1\} \Big\}$$

Note that $\mathbf{F}(k, k-1) = \{F_i : i < n_k\}$. We check that conditions (2)–(4) hold.

Let $F = \{0, 1, \ldots, m_k - 1\}$, define $R(F) = \{0, 1, \ldots, r_k - 1\}$ and let $F = \bigcup_{i < n_k} F_i$ be the canonical decomposition of F. By the way $\mathbf{F}(k, k-1)$ has been defined, this is the only possible decomposition for F. Thus (4) works at level k. It is also clear by the definition of $\mathbf{F}(k, k)$ and $\mathbf{F}(k, k-1)$, that (3) holds at levels k and $k-1$. Now let $E_0, E_1 \in \mathbf{F}(k, j)$ with $j < k-1$, by definition, there are $C_0, C_1 \in \mathbf{F}(k-1, j)$, and $a_0, a_1 < n_k$, such that $E_\tau = \varphi_{a_\tau}(C_\tau)$, $\tau = 0, 1$. If $a_0 \neq a_1$ then (3) holds because $E_0 \cap E_1 \subset R(F)$ and then $E_0 \cap E_1 = \varphi_{a_0}(C_0 \cap C_1)$ which is an initial segment of E_0 becuase $C_0 \cap C_1$ is an initial segment of C_0, and the same for C_1. If $a_0 = a_1$ then (3) holds for E_0, E_1 because it holds for C_0, C_1.

Now for every $E \in \mathbf{F}(k,j)$ for $j < k$ we have $E = \varphi_a(C)$ for some $a < n_k$ and $C \in \mathbf{F}(k-1,j)$ and since (2) holds for C it holds for E. Also C has a canonical decomposition $C = \bigcup_{i<n_j} C_i$ with $C_i \in \mathbf{F}(k-1,j-1)$. Since φ_a is a bijection, we have $E = \bigcup_{i<n_j} E_i$ with $E_i = \varphi_a(C_i) \in \mathbf{F}(k,j-1)$. We want to see this decomposition is unique. Note that $E \subset F_a$ and by definition any decomposition of E will have to be contained in F_a. If we have two different decompositions of E we can pull them via φ_a into different decompositions of C which is a contradition. Thus (2)–(4) holds in $\mathbf{F}(k,i)$, $i \leq k$.

We are now in conditions to define $\mathcal{F}^\omega$,

$$\mathcal{F}_i^\omega = \bigcup_{k \geq i} \mathbf{F}(k,i)$$

$$\mathcal{F}^\omega = \bigcup_{i<\omega} \mathcal{F}_i^\omega$$

By the definition of the $\mathbf{F}(k,i)$'s is clear that $\mathcal{F}^\omega$ satisfies conditions (2)–(4) of a construction scheme. To check (1), let $A \subset \omega$ and take k big enough such that $A \subset \{0,1,\ldots,m_k\} \in \mathcal{F}^\omega$. This finish the first step of the induction.

Let $\delta < \omega_1$ limit, and suppose we have constructed $\mathcal{F}^\beta$ for every $\beta < \delta$ limit, such that

⊠$_0^\beta$ For every $\gamma \leq \beta$ limit, $\mathcal{F}^\gamma$ is a construction scheme on γ of type $(m_k, n_k, r_k)_{k<\omega}$ and $\mathcal{F}^\gamma \subset \mathcal{F}^\beta$.

⊠$_1^\beta$ For every finite $A \subset \beta$ and $\alpha < \beta$ we can find $F \in \mathcal{F}$ with canonical decomposition $F = \bigcup_i F_i$, such that $A \subseteq F_0$, $R(F) = F_0 \cap \alpha$.

Note that $\mathcal{F}^\omega$ as defined above satisfies ⊠$_1^\omega$, indeed let $A \subset \omega$ and $\alpha < \omega$, there is $k < \omega$ big enough such that $A \subset \{0,1,\ldots,m_k\} \in \mathcal{F}^\omega$, and $r_k = \alpha$, here it is used that $r_k = \alpha$ for infinitely many values of k. If δ is a limit of limits we can find $\delta_n < \delta$ increasing with $\delta = \sup_n \delta_n$, such that $\mathcal{F}^{\delta_n}$ satisfies ⊠$_0^{\delta_n}$ and ⊠$_1^{\delta_n}$, then

$$\mathcal{F}_k^\delta = \bigcup_{n<\omega} \mathcal{F}_k^{\delta_n}$$

$$\mathcal{F}^\delta = \bigcup_{k<\omega} \mathcal{F}_k^\delta$$

satisfies ⊠$_0^\delta$ and ⊠$_1^\delta$.

The only case left is when δ is a successor, i.e, $\delta = \beta + \omega$ for $\beta < \omega_1$ limit.

For $A \subset \omega_1$ finite, we will use the notation, $\mathcal{F}^\beta \upharpoonright A = \{F \in \mathcal{F}^\beta : F \subset A\}$, we also define $\mathcal{F}_i^\beta \upharpoonright A = \{F \in \mathcal{F}_i^\beta : F \subset A\}$ for all $i < \omega$.

First we construct a sequence $(k_i, D_i, W_i, \mu_i)_{i<\omega}$ with nice properties. We do this by induction. Start by fixing a bijection $f : \omega \to \beta$. Pick $k_0 > 0$ and $D_0 \in \mathcal{F}_{k_0}^\beta$ such that $f(0) \in D_0$. Pick also $\mu_0 \in D_0$.

Now suppose we have constructed an increasing sequence $(k_i)_{i<j}$ in ω, $D_i \in \mathcal{F}^\beta_{k_i}$ for $i < j$, $W_i \in \mathcal{F}_{k_i}$ for $i < j - 1$, and $\mu_i \in D_i$ for $i < j$. Such that

1. $f(i) \in D_i$ for all $i < j$,

2. $D_i \subset \mu_{i+1}$ for all $i < j - 1$,

3. $W_i \cap \mu_{i+1} = D_i \cap \mu_i$ for all $i < j$, and

4. $W_i \setminus \mu_{i+1}$ is an interval of D_{i+1} such that $\mu_{i+1} \in W_i$, for all $i < j$.

5. For every $A \subset D_i$ and $\alpha \in D_i$ there is $F \in \mathcal{F}^\beta \restriction D_{i+1}$ with canonical decomposition $F = \bigcup_{a<n_k} F_a$, such that $A \subset F_0$ and $R(F) = F_0 \cap \alpha$.

We can apply $\boxtimes^\beta_1$ finitely many times (at most $m_{k_{j-1}} 2^{m_{k_{j-1}}}$ times) to obtain $k > k_{j-1}$, and $D \in \mathcal{F}^\beta_k$ such that condition (5) works on $\mathcal{F}^\beta \restriction D$. Apply $\boxtimes^\beta_1$ one more time to find $k_j \geq k$ and $D_j \in \mathcal{F}^\beta_{k_j}$ with canonical decomposition

$$D_j = \bigcup_{i<n_{k_j}} D_j(i),$$

such that $D_{j-1} \cup \{f(j)\} \cup D \subset D_j(0)$, and $R(D_j) = D_j(0) \cap \mu_{j-1}$.

Since $\mu_{j-1} \in D_{j-1} \subset D_j(0)$ we can apply Lemma 1.36 and obtain $W \in \mathcal{F}^\beta_{j-1}$ such that $W \setminus \mu_{j-1}$ is an interval of $D_j(0)$, and $W \cap (\mu_{j-1} + 1) = D_j \cap (\mu_{j-1} + 1)$. Let $\mu_j = \min(D_j(1))$. If $\varphi_i : D_j(0) \to D_j(i)$ is the increasing bijection between $D_j(0)$ and $D_j(i)$, then let $W_{j-1} = \varphi_1(D_{j-1})$. Note that $\mu_j = \varphi_1(\mu_{j-1})$ and then $W_{j-1} \setminus m_j$ is an interval of $D_j(1)$, therefore $W_{j-1} \setminus \mu_j$ is an interval of D_j since $R(D_j) \subset \mu_j$, . It is easy to check that k_j, D_j, μ_j, and W_{j-1} satisfy conditions (1) to (5). This finishes the construction of $(k_i, D_i, W_i, \mu_i)_{i<\omega}$.

Now for every $j < \omega$, let

$$F_j = (D_j \cap \mu_j) \cup \{\beta, \beta + 1, \ldots, \beta + |D_j \setminus \beta| - 1\},$$

and let $\Phi_j : D_j \to F_j$ be the increasing bijection between D_j and F_j. Note that condition (3) and (4) imply

$$\Phi_j = \Phi_{j+1} \circ \varphi_{D_j, W_j} \tag{2.1}$$

Now define

$$\mathbf{F}(j, i) = \Phi_j \left(\mathcal{F}^\beta_i \restriction D_j \right)$$
$$\mathcal{F}^{\beta+\omega}_i = \bigcup_{k_j > i} \mathbf{F}(j, i)$$
$$\mathcal{F}^{\beta+\omega} = \bigcup_{i<\omega} \mathcal{F}^{\beta+\omega}_i$$
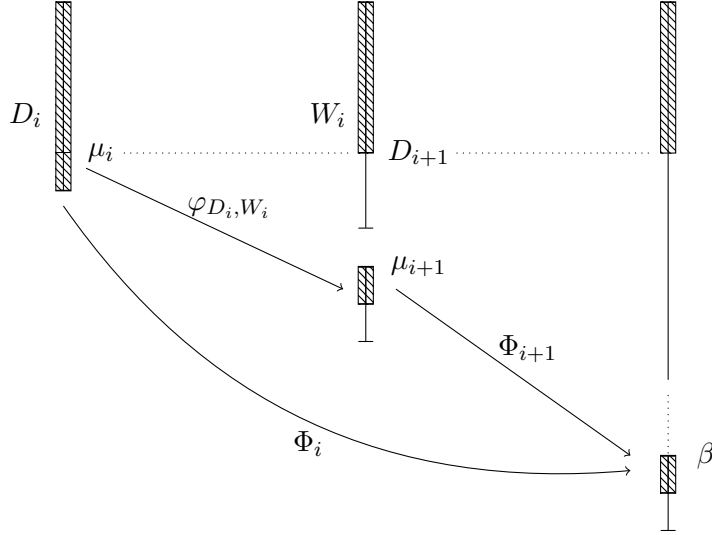
Figure 2.1: The diagram is commutative, $\Phi_i = \Phi_{i+1} \circ \varphi_{D_i, W_i}$.

Note that (2.1) imply that $\mathcal{F}^{\beta+\omega}$ is well defined, W_j is a witness to $\mathbf{F}(j,i) \subset \mathbf{F}(j+1,i)$ since $\mathcal{F}^\beta \upharpoonright D_j$ is isomorphic to $\mathcal{F}^\beta \upharpoonright W_j$. Condition (1) implies $\mathcal{F}^\beta \subset \mathcal{F}^{\beta+\omega}$. Indeed, let $F \in \mathcal{F}_k^\beta$, there is $j < \omega$ big enough so that $k > k_j$, and for every $\alpha \in F$ there is $i < j$ with $f(i) = \alpha$. Then $F \subset D_{j+1}$, therefore $F \in \mathbf{F}(j+1, k)$

We check that $\mathcal{F}^{\beta+\omega}$ works. We have already checked $\boxtimes_0^{\beta+\omega}$. To see $\boxtimes_1^{\beta+\omega}$ let $A \subset \beta + \omega$ finite and $\alpha < \beta + \omega$. Take $j < \omega$ big enough so that for every $a \in (A \cup \{\alpha\}) \cap \beta$ there is $i < j$ with $f(i) = a$, and $A \cup \{\alpha\} \subset \beta + j$. Then, by condition (1) on the induction we have $(A \cap \beta) \subset D_j$, therefore $(A \cap \beta) \subset (D_{j+1} \cap \mu_{j+1})$, which implies $A \cup \{\alpha\} \subset F_{j+1}$.

Let $A^* = \Phi_{j+1}^{-1}(A)$ and $\alpha^* = \Phi_{j+1}^{-1}(\alpha)$. By condition (5), there is $F^* \in \mathcal{F}^\beta \upharpoonright D_{j+2}$ with canonical decomposition $F^* = \bigcup_{i<n_k} F_i^*$ such that $A^* \subset F_0$ and $R(F) = F_0 \cap \alpha^*$. Now let $A^{**} = \varphi_{D_{j+1}, W_{j+1}}(A^*)$, $\alpha^{**} = \varphi_{D_{j+1}, W_{j+1}}(\alpha^*)$, and $F^{**} = \varphi_{D_{j+1}, W_{j+1}}(F^*)$. Then F^{**} has canonical decomposition $F^{**} = \bigcup_{i<n_k} F_i^{**}$ and $A^{**} \subset F_0^{**}$, and $R(F^{**}) = F_0^{**} \cap \alpha^{**}$. If we take $F = \Phi_{i+2}(F^{**})$, then $F \in \mathbf{F}(j+2, k)$ and it has a canonical decomposition $F = \bigcup_{i<n_k} F(i)$. By (2.1) we have $\Phi_{j+2}(A^{**}) = A \subset F(0)$ and $R(F) = F(0) \cap \Phi_{j+2}(\alpha^{**})$ but $\Phi_{j+2}(\alpha^{**}) = \alpha$, also by (2.1). Therefore $\boxtimes_1^{\beta+\omega}$ holds.

Let now

$$\mathcal{F}_k = \bigcup_{\substack{\beta < \omega_1 \\ \beta \text{ limit}}} \mathcal{F}_k^\beta$$

$$\mathcal{F} = \bigcup_{k<\omega} \mathcal{F}_k$$

Let us see that $\mathcal{F}$ is as we wanted.

Claim 2.6. *$\mathcal{F}$ is a construction scheme of type $(m_k, n_k, r_k)_{k<\omega}$.*

Proof. To see $\mathcal{F}$ is a construction scheme, consider $A \subset \omega_1$ finite, $E, F \in \mathcal{F}_k$ for some $k < \omega$. There is $\beta < \omega_1$ limit such that $A, E, F \subset \beta$. Then $\boxtimes_0^\beta$ implies $E, F \in \mathcal{F}_k^\beta$, therefore $E \cap F$ is an initial segment of E and F. Also $|F| = m_k$ and $R(F) = r_k$, and there is $D \in \mathcal{F}^\beta$ such that $A \subset D$, by construction $D \in \mathcal{F}$. This shows (1)–(3) in the definition of construction scheme.

To finish the proof, suppose there are two decompositions of F,

$$F = \bigcup_{i<n^0} F_i^0 = \bigcup_{j<n^1} F_j^1$$

where $F_i^0, F_j^1 \in \mathcal{F}_{k-1}$ for $i < n^0, j < n^1$. By the construction there is some $\delta < \omega_1$ limit with $F, F_i^0, F_j^1 \in \mathcal{F}^\delta$, for $i < n^0, j < n^1$ (we can argue $F_i^0, F_j^1 \in \mathcal{F}^\beta$ for $i < n^0, j < n^1$ but we do not need this to reach a contradiction). This contradicts $\boxtimes_0^\delta$. Therefore $\mathcal{F}$ is a construction scheme of type $(m_k, n_k, r_k)_{k<\omega}$ $\qquad\square$

Thus, we constructed $\mathcal{F}$ a construction scheme of type $(m_k, n_k, r_k)_{k<\omega}$ which is what we wanted to prove. $\qquad\square$

We know then that there are construction schemes in ZFC as we have already proved. Using a construction scheme $\mathcal{F}$, we can build different uncountable structures but we are not always able to prove that this structures have interesting combinatorial properties without capturing. On the next section we focus our efforts on the existence of capturing construction schemes.

2.2 Consistency of capturing $\mathcal{F}$

In this section we show that, given any type $(m_k, n_k, r_k)_k$ there is a fully capturing construction scheme $\mathcal{F}$ of type $(m_k, n_k, r_k)_k$. The proof will be by iterated forcing. We show that adding $\kappa \geq \aleph_1$ Cohen reals forces the result above. This is shown by starting with a construction scheme $\mathcal{F}^\omega$ constructed as in Section 2.1, and then we build a name for $\mathcal{F}$ in $V[G]$ which fully captures every uncountable Δ-System of the form

$$(\{\xi\} : \xi \in \Gamma) \text{ where } \Gamma \subset \omega_1 \text{ is uncountable.} \qquad (\bigstar)$$

Let us show first recall the definition of capturing, and fully capturing construction scheme.

Definition 2.7. Let $\mathcal{F}$ be a construction scheme. We say that $\mathcal{F}$ is *fully capturing* if for every uncountable Δ-system $(s_\xi)_{\xi<\omega_1}$ of finite subsets of ω_1 with root s, and every $k^* < \omega$ there are $F \in \mathcal{F}_k$ with $k > k^*$, canonical decomposition $F = \bigcup_{i<n_k} F_i$, and $\xi_0 < \ldots < \xi_{n_k-1} < \omega_1$, such that

$$s \subset R(F)$$
$$\text{for every } i < n_k, \quad s_{\xi_i} \setminus s \subset F_i \setminus R(F),$$
$$\text{for every } i < n_k, \quad \varphi_i(s_{\xi_0}) = s_{\xi_i}.$$

We have to show first, that if $\mathcal{F}$ fully captures Δ-Systems of the form ($\bigstar$) then it is fully capturing.

2.2.1　Capturing Δ-Systems of the form ($\bigstar$) implies capturing

Recall that on Example 1.4 we kill 3-capturing by killing all capturing of Δ-Systems of the form ($\bigstar$). In this subsection we show that the reverse is also true. Namely we have

Lemma 2.8. *Suppose $\mathcal{F}$ is a construction scheme which fully captures Δ-Systems of the form ($\bigstar$), then $\mathcal{F}$ is fully capturing.*

Proof. Let $\mathcal{F}$ be a construction scheme and suppose $\mathcal{F}$ fully captures all Δ-Systems of the form $(\{\xi\} : \xi \in \Gamma)$ where Γ is an uncountable subset of ω_1. Let $(D_\alpha)_{\alpha<\omega_1}$ be an uncountable Δ-System with root D and $k^* < \omega$. We want to show there are $F \in \mathcal{F}_k$ with $k > k^*$, and $\alpha_0 < \ldots < \alpha_{n_k-1}$ such that

$$D_{\alpha_i} \subset F_i, \quad i < n_k$$
$$\varphi_i(D_{\alpha_0}) = D_{\alpha_i}, \quad i < n_k$$

For every D_α we are going to define the closure of D_α on $\mathcal{F}$, $\overline{D}_\alpha$. Let

$$k_\alpha = \min\{\ell < \omega : \exists F \in \mathcal{F}_\ell, \quad D_\alpha \subset F\}$$

Now pick $F \in \mathcal{F}_{k_\alpha}$ with $D_\alpha \subset F$, and let

$$\overline{D}_\alpha = F \cap (\max(D_\alpha) + 1)$$

Let us check that $\overline{D}_\alpha$ is well defined since it depends only on D_α and $\mathcal{F}$, and not on the choice of $F \in \mathcal{F}_{k_\alpha}$ we made above. Suppose we pick different $F, F^* \in \mathcal{F}_{k_\alpha}$, with $D_\alpha \subset F, F^*$ we have $D_\alpha \subset F \cap F^*$. Since $F \cap F^*$ is an initial segment of both F and F^* we have $F \cap \max(D_\alpha) = F^* \cap \max(D_\alpha)$.

Take $S \subset \omega_1$ uncountable, and $k_0, d < \omega$, such that $k_\alpha = k_0$ and $|\overline{D}_\alpha| = d$ for all $\alpha \in S$. Furthermore, given $\alpha < \beta$ in S, then $\varphi_{\overline{D}_\alpha, \overline{D}_\beta}(D_\alpha) = D_\beta$.

Let $\xi_\alpha = \max(\overline{D}_\alpha)$ for all $\alpha \in S$. Consider $\{\{\xi_\alpha\} : \alpha \in S\}$, since $\mathcal{F}$ fully captures Δ-Systems of the form ($\bigstar$) there are $\alpha_0 < \ldots < \alpha_{n_k-1}$ in S, and $F \in \mathcal{F}_k$ with $k > k_0, k^*$, such that $F = \bigcup_{i<n_k} F_i$ is the canonical decomposition of F and

$$\forall i < n_k, \quad \xi_{\alpha_i} \in F_i \setminus R(F)$$
$$\forall i < n_k, \quad \varphi_i(\xi_{\alpha_0}) = \xi_{\alpha_i}$$

Pick now $E \in \mathcal{F}_{k_0}$ with $D_{\alpha_0} \subset E$, then $E \cap F_0$ is an initial segment of E. Recall $\xi_{\alpha_0} \in E \cap F_0$, therefore $\overline{D}_{\alpha_0} = E \cap (\xi_{\alpha_0} + 1) \subset F_0$. Arguing the same way we find $\overline{D}_{\alpha_i} \subset F_i$ for $i < n_k$. Also

$\varphi_i(\xi_{\alpha_0}) = \xi_{\alpha_i}$ therefore $\varphi_i(\overline{D}_{\alpha_0}) = \overline{D}_{\alpha_i}$ which implies

$$\varphi_i(D_{\alpha_0}) = D_{\alpha_i}$$

which in turn implies

$$\forall i < n_k, \quad D_{\alpha_i} \setminus D \in F_i \setminus R(F)$$

This shows that $\mathcal{F}$ fully captures $(D_\alpha)_{\alpha < \omega_1}$ as we wanted. $\qquad\square$

We are now in conditions to show the main result of the Chapter.

2.2.2 Adding Cohen reals and Capturing

In this section we show that adding $\aleph_1$ Cohen reals also adds a fully capturing construction scheme. Recall the following result Lemma from Chapter 1.

Lemma 1.36. *For $F \in \mathcal{F}_k$, $E \in \mathcal{F}_l$ and $E \subset F$ (in particular $l \leq k$). For every $\mu \in E$ there is a copy E^* of E in F such that*

1. *$E^* \cap (\mu + 1) = E \cap (\mu + 1)$.*

2. *$E^* \setminus \mu$ is an interval of F with $\mu \in E^*$.*

If we add $\aleph_1$ Cohen reals, then we force a fully capturing construction scheme.

Theorem 2.2. *Adding $\kappa \geq \aleph_1$ Cohen reals also adds a fully capturing construction scheme.*

Proof. Assume first that $\kappa = \aleph_1$. And let $(m_k, n_k, r_k)_k$ be a type on the ground model. We start by fixing $\mathcal{F}^\omega$, a construction scheme on ω build as in Section 2.1, with the following property:

For every $A \subset \omega$ finite and $a < \omega$ there is $F \in \mathcal{F}$ with canonical decomposition $\bigcup_{i < n_k} F_i$, such that $A \subset F_0$ and $R(F) = F_0 \cap a$. $\qquad$ (2.2)

Definition 2.9. Let $\vec{p} \in \mathbb{P}$ if and only if $\text{supp}(\vec{p}) \subset \omega_1$ finite, for every $\delta \in \text{supp}(\vec{p})$, δ is limit, $\vec{p}(\delta) = (D_\delta^p, a_\delta^p)$ where $D_\delta^p \in \mathcal{F}^\omega$, $a_\delta^p \in D_\delta^p$, and for every $\delta_0 < \delta_1$ in $\text{supp}(\vec{p})$

1. $D_{\delta_0}^p \subseteq D_{\delta_1}^p$, and

2. $a_{\delta_0}^p < a_{\delta_1}^p$

We say $\vec{p} \leq \vec{q}$ if $\text{supp}(\vec{q}) \subset \text{supp}(\vec{p})$,

(i) for every $\delta < \delta' \in \text{supp}(\vec{q})$, $a_{\delta'}^p - a_\delta^p \geq a_{\delta'}^q - a_\delta^q$, and

(ii) for every $\delta \in \text{supp}(\vec{q})$ with $D_\delta^q \in \mathcal{F}_k$, there is $W \in \mathcal{F}_k$, $W \subseteq D_\delta^p$, with $W \cap a_\delta^p$ having the same size that $D_\delta^q \cap a_\delta^q$, and $W \setminus a_\delta^p$ is an interval of D_δ^p with $a_\delta^p \in W$.

We say $\vec{p} \sim \vec{q}$ if $\vec{p} \leq \vec{q}$ and $\vec{q} \leq \vec{p}$.

Note that $\mathbb{P}$ is equivalent to the forcing $\mathbb{C}_{\omega_1}$ for adding ω_1 Cohen reals. To see this, notice that $\mathbb{P}$ is a dense suborder of the partial order which is defined as above minus conditions 1, 2, (i), and (ii), and this partial order is a finite support product of countable partial orders.

Now for $a < \delta < \omega_1$, define the function $\phi_{a,\delta} : \omega_1 \to \omega_1$ by

$$\phi_{a,\delta}(\alpha) = \begin{cases} \alpha & \alpha < a \\ \delta + i & \alpha = a + i \end{cases}$$

Note that, for $\varphi : \omega \to \omega$ increasing we have

$$\phi_{\varphi(a),\delta} \circ \varphi = \phi_{a,\delta} \tag{2.3}$$

Now let $\vec{p} \in \mathbb{P}$, with $\mathrm{supp}(\vec{p}) = (\delta_0 < \ldots < \delta_n)$ suppose $\vec{p}(\delta_i) = (D_i, a_i)$. We define $\Phi^p : D_n \to \omega_1$ as:

$$\Phi^q(x) = \begin{cases} x & x < a_0 \\ \phi_{a_i,\delta_i}(x) & a_i \le x < a_{i+1} \\ \phi_{a_n,\delta_n}(x) & x \ge a_n \end{cases}$$

Finally, for $\vec{p}$ as above we define

$$\mathbf{F}_p = \Phi^p\Big(\mathcal{F}^\omega \upharpoonright D_n\Big)$$

Let $G \subset \mathbb{P}$ be a generic filter, we define $\mathcal{F}$ in $V[G]$ as

$$\mathcal{F} = \bigcup_{p \in G} \mathbf{F}_p$$

Claim 2.10. *Let $\xi < \omega_1$ and $\vec{p} \in \mathbb{P}$. There is $\vec{q} \le \vec{p}$ and $x < \omega$ such that $\xi = \Phi^q(x)$. In particular there is $\delta \in \mathrm{supp}(\vec{q})$ with $x \in D_\delta^q$.*

Proof. Let $\xi < \omega_1$ and $\vec{p} \in \mathbb{P}$ be given, we want to find $\vec{q} \le \vec{p}$ and $x < \omega$ such that $\xi = \Phi^q(x)$.

Take $\delta < \omega_1$ limit such that $\delta \le \xi < \delta + \omega$. We write $\xi = \delta + \ell$ where $\ell < \omega$. Consider $\mathrm{supp}(\vec{p}) = \{\delta_0 < \ldots < \delta_n\}$ and $\vec{p}(\delta_i) = (D_i, a_i)$ with $D_i \in \mathcal{F}_{k_i}^\omega$.

Case 1: $\delta = \delta_j$ for some $j \le n$, Pick $a > D_n$ and find $F \in \mathcal{F}^\omega$ with canonical decomposition $\bigcup_{i<n_k} F_i$, $D_n \cup \{a_j, a_j + 1, \ldots, a_j + \ell\} \subset F_0$ and $R(F) = F_0 \cap a_j$. Apply Lemma 1.36 to find $W_i^* \in \mathcal{F}_{k_i}^\omega$ for $i > j$, such that $|W_i^* \cap a_i| = |D_i \cap a_i|$, $(W_i^* \setminus a_i)$ is an interval of F with $a_i \in W_i^*$, for $i > j$. Now let $c_i = \varphi_1(a_i)$, and $W_i = \varphi_1(W_i^*)$ for $i > j$.

Define $\vec{q} \in \mathbb{P}$ with $\operatorname{supp}(\vec{q}) = \operatorname{supp}(\vec{p})$ such that

$$\vec{q}(\gamma) = \begin{cases} (D_i, a_i) & \text{for } \gamma = \delta_i, i \leq j. \\ (F, c_i) & \text{for } \gamma = \delta_i, i > j. \end{cases}$$

Note that the W_i's witness $\vec{q} \leq \vec{p}$. By construction $\Phi^q(a_i + \ell) = \xi$.

Case 2: $\delta_j < \delta < \delta_{j+1}$ for some $j < n$. Assume then there is $j < n$ with $\delta_j < \delta < \delta_{j+1}$. Pick $a > D_n$ and find $F \in \mathcal{F}^\omega$ with canonical decomposition $\bigcup_{i<n_k} F_i$, $D_n \cup \{a, a+1, \ldots, a+\ell\} \subset F_0$ and $R(F) = F_0 \cap a_{j+1}$. Apply Lemma 1.36 to find $W_i^* \in \mathcal{F}_{k_i}^\omega$ for $i > j$, such that $|W_i^* \cap a_i| = |D_i \cap a_i|$, $(W_i^* \setminus a_i)$ is an interval of F with $a_i \in W_i^*$, for $i > j$. Now let $c_i = \varphi_1(a_i)$, and $W_i = \varphi_1(W_i^*)$ for $i > j$.

Define $\vec{q} \in \mathbb{P}$ with $\operatorname{supp}(\vec{q}) = \operatorname{supp}(\vec{p}) \cup \{\delta\}$ such that

$$\vec{q}(\gamma) = \begin{cases} (D_i, a_i) & \text{for } \gamma = \delta_i, i \leq j. \\ (F, a) & \text{for } \gamma = \delta, \\ (F, c_i) & \text{for } \gamma = \delta_i, i > j. \end{cases}$$

It's clear that $\vec{q} \leq \vec{p}$ (this is witness by the W_i's) and $\Phi^q(a + \ell) = \xi$ by construction.

Case 3: $\delta < \delta_0$. Take $a > D_n$ and find $F \in \mathcal{F}^\omega$ with canonical decomposition $\bigcup_{i<n_k} F_i$, $D_n \cup \{a, a+1, \ldots, a+\ell\} \subset F_0$ and $R(F) = \emptyset$. Apply Lemma 1.36 to find $W_i^* \in \mathcal{F}_{k_i}^\omega$ for $i \leq n$, such that $|W_i^* \cap a_i| = |D_i \cap a_i|$, $(W_i^* \setminus a_i)$ is an interval of F with $a_i \in W_i^*$, for $i \leq n$. Now let $c_i = \varphi_1(a_i)$, and $W_i = \varphi_1(W_i^*)$ for $i \leq n$.

Define $\vec{q} \in \mathbb{P}$ with $\operatorname{supp}(\vec{q}) = \{\delta\} \cup \operatorname{supp}(\vec{p})$ such that

$$\vec{q}(\gamma) = \begin{cases} (F, a) & \text{for } \gamma = \delta, \\ (F, c_i) & \text{for } \gamma = \delta_i, i \leq n. \end{cases}$$

The construction shows that $\vec{q} \leq \vec{p}$ (by the pick of the W_i's) and $\Phi^q(a + \ell) = \xi$ by construction.

Case 4: $\delta > \delta_n$. Pick $a > D_n$ and let $F \in \mathcal{F}^\omega$ such that $D_n \cup \{a, a+1, \ldots, a+\ell\} \subset F$. Apply Lemma 1.36 to find $W_i \in \mathcal{F}_{k_i}^\omega$, such that $|W_i \cap a_i| = |D_i \cap a_i|$, $(W_i \setminus a_i)$ is an interval of F with $a_i \in W_i$.

Define $\vec{q} \in \mathbb{P}$ with $\operatorname{supp}(\vec{q}) = \operatorname{supp}(\vec{p}) \cup \{\delta\}$ such that

$$\vec{q}(\gamma) = \begin{cases} (D_i, a_i) & \text{for } \gamma = \delta_i, i \leq n. \\ (F, a) & \text{for } \gamma = \delta. \end{cases}$$

By the construction and the choice of W_i's, we have $\vec{q} \leq \vec{p}$ and $\Phi^q(a + \ell) = \xi$.

This finishes all of the cases. $\qquad\square$

We see now, arguing as in Section 2.1, that $\mathcal{F}$ is a Construction Scheme.

Claim 2.11. $\mathcal{F}$ *as above is a construction scheme on* $V[G]$.

Proof. Now let $A \subset \omega_1$ finite and $\vec{p} \in \mathbb{P}$. Write $A = \{\xi_1 < \xi_2 < \ldots < \xi_n\}$. We can apply Claim 2.10 to find $\vec{q_1} \leq \vec{p}$ and $x_1 < \omega$ finite such that $\Phi^{q_1}(x_1) = \xi_1$. Inductively we can find $\vec{q_n} \leq \ldots \leq \vec{q_1} \leq p$ and $x_1, x_2, \ldots, x_n < \omega$ such that $\Phi^{q_n}(x_i) = \xi_i$ for all $i = 1, 2, \ldots, n$.

If $\mathrm{supp}(\vec{q_n}) = \{\delta_0 < \ldots < \delta_n\}$ and $\vec{q}(\delta_i) = (D_i, a_i)$, then $F = \Phi^{q_n}(D_n)$ is such that $A \subset F$ and

$$\vec{q_n} \Vdash F \in \mathcal{F}$$

This shows property 1 of a Construction Scheme.

To see 2 note that $F \in \mathcal{F}_k$ if there is some $\vec{p} \in \mathbb{P}$ and $F \subset \omega_1$ finite such that $F \in \Phi^p\left(\mathcal{F}_k^\omega \upharpoonright D_n\right)$, where $\mathrm{supp}(\vec{p}) = \{\delta_1, \ldots, \delta_n\}$ and $\vec{p}(\delta_i) = (D_i, a_i)$. Thus there is $D \in \mathcal{F}_k^\omega$ such that $F = \Phi^p(D)$. We have $|F| = |D| = m_k$ and $|R(F)| = |R(D)| = r_k$ because Φ^p is a bijection.

To simplify the notation, when we take $\vec{p} \in \mathbb{P}$, we assume $\mathrm{supp}(\vec{p}) = \{\delta_1, \ldots, \delta_n\}$, and $\vec{p}(\delta_i) = (D_i, a_i)$, and we consider $\mathcal{F}^\omega \upharpoonright D_n$. In other words when we write $\mathcal{F}^\omega$ we mean $\mathcal{F}^\omega \upharpoonright D_n$. This way we are free to use the symbols D, and n without confusion.

We check property 3. Let $\check{F}, \check{E} \in \mathcal{F}_k$ in $V[G]$. Then there is some $\vec{p} \in \mathbb{P}$, and $D_0, D_1 \in \mathcal{F}_k^\omega$ such that $F = \Phi^p(D_0)$, and $E = \Phi^p(D_1)$. Since Φ^p is an increasing bijection we have $F \cap E = \Phi^p(D_0 \cap D_1) \sqsubseteq \Phi^p(D_0), \Phi^p(D_1)$. Which shows $F \cap E \sqsubseteq F, E$ as we wanted to show.

We prove property 4 by contradiction. Let $\check{F} \in \mathcal{F}_k$ such that there are two different decompositions of $\check{F}$ in $V[G]$.

$$\check{F} = \bigcup_{i<n} \check{F_i} = \bigcup_{j<n'} \check{F_j'}$$

There is $\vec{p} \in \mathbb{P}$ such that $F \in \Phi^p(\mathcal{F}_k^\omega)$, and $F_i, F_j' \in \Phi^p(\mathcal{F}_{k-1}^\omega)$, for $i < n$ and $j < n'$. By definition there are $D \in \mathcal{F}_k^\omega$ and $D_i, D_j' \in \mathcal{F}_{k-1}^\omega$ for $i < n$, $j < n'$, such that $F = \Phi^p(D)$, $F_i = \Phi^p(D_i)$ for $i < n$, and $F_j' = \Phi^p(D_j')$ for $j < n'$. Since Φ^p is a bijection we have

$$D = \bigcup_{i<n} D_i = \bigcup_{j<n'} D_j'$$

this contradicts the uniqueness of decomposition of $\mathcal{F}^\omega$.

And so $\mathcal{F}$ is a construction scheme on $V[G]$ as we wanted to show. $\square$

We have $\mathcal{F}$ on $V[G]$ a construction scheme on $V[G]$. To show $\mathcal{F}$ is fully capturing, let $\dot{\Gamma}$ be a name for an uncountable subset of ω_1 which defines a Δ-System of the form ($\bigstar$), and $k^* < \omega$ be given. Take $\Omega \subset \omega_1$ uncountable and $\vec{p}_\alpha \in \mathbb{P}$ for $\alpha \in \Omega$ such that

$$\vec{p}_\alpha \Vdash \alpha \in \dot{\Gamma} \tag{2.4}$$

By Claim 2.10 above, we can assume without loss of generality that there is $\delta \in \mathrm{supp}(\vec{p}_\alpha)$ such that $\vec{p}_\alpha(\delta) = (D, a)$, and $\alpha \in \phi_{a,\delta}(D)$.

Find $\Omega_0 \subset \Omega$ uncountable, $\delta_{\alpha,0} < \ldots < \delta_{\alpha,d-1} < \omega_1$ limit, $D_i \in \mathcal{F}^\omega_{k_i}$ for $i < d$, $a_0 < \ldots < a_{d-1}$, and $x < \omega$ such that:

1. $(\mathrm{supp}(\vec{p}_\alpha) : \alpha \in \Omega_0)$ form a Δ-System with root $\{\delta_{\alpha,0}, \ldots, \delta_{\alpha,r-1}\}$,

2. $\mathrm{supp}(\vec{p}_\alpha) = \{\delta_{\alpha,0}, \ldots, \delta_{\alpha,d-1}\}$,

3. $\vec{p}_\alpha(\delta_{\alpha,i}) = (D_i, a_i)$ for every $i < d$,

4. $x \in D_{d-1}$ and $\Phi^{p_\alpha}(x) = \alpha$.

Take $j_0 = d - 1$ if $x \geq a_{d-1}$, or $j_0 < d$ such that $a_{j_0} \leq x < a_{j_0+1}$.
Pick $k > k^*, k_{d-1}$, and $\alpha_0 < \ldots < \alpha_{n_k-1}$ in Ω_0. We want to find $\vec{q} \in \mathbb{P}$ such that

$$\vec{q} \Vdash \alpha_i \in \dot{\Gamma}, \dot{\mathcal{F}} \text{ captures } \alpha_0, \ldots, \alpha_{n_k-1}. \tag{2.5}$$

Take $F^* \in \mathcal{F}^\omega_k$ such that, $F^* = \bigcup_{i<n_k} F^*_i$ is the canonical decomposition of F^*, $D_{d-1} \subset F^*_0$, and $R(F^*) = F^*_0 \cap a_r$.

For $i < d$, note $a_i \in D_i \subset F^*_0$, therefore we can apply Lemma 1.36 to find $W_{0i} \in \mathcal{F}^\omega_{k_i}$ with $|W_{0i} \cap a_i| = |D_i \cap a_i|$ and $W_{0i} \setminus a_i$ an interval of F^*_0 with $a_i \in W_{0i}$. Let $\varphi_i : F^*_0 \to F^*_i$ be the increasing bijection between F^*_0 and F^*_i. Define $W_{ij} = \varphi_i(W_{0j})$, and $a_{ij} = \varphi_{D_j, W_{ij}}(a_j)$ for $i < n_k$, $j < d$, and $x_i = \varphi_{D_{j_0}, W_{ij_0}}(x)$ for $i < n_k$.

Since φ_i is a bijection we have

$$W_{ij} \in \mathcal{F}^\omega_{k_j}, |W_{ij} \cap a_{ij}| = |D_j \cap a_j|, \text{ and } W_{ij} \setminus a_{ij} \text{ is an interval of } F^* \text{ with } a_{ij} \in W_{ij} \tag{2.6}$$

and by equation (2.3) we have

$$\phi_{a_{ij_0}, \delta_{j_0}}(x_i) = \alpha_i \tag{2.7}$$

We define $\vec{q} \in \mathbb{P}$ with $\mathrm{supp}(\vec{q}) = \{\delta_{\alpha_i, j} : i < n_k, j < d\}$. Note now that $\delta_{\alpha,i}$ does not depend on α for $i < r$.

$$\vec{q}(\delta_{\alpha_i, j}) = \begin{cases} (D_j, a_j) & \text{for } j < r \\ (F^*, a_{ij}) & \text{for } j \geq r \end{cases}$$

With this definition, we have $\Phi^q(x_i) = \phi_{a_{ij}, \delta_{j_0}}(x_i) = \alpha_i$ by (2.7). Also, $\vec{q} \leq \vec{p}_{\alpha_i}$ for $i < n_k$. Indeed $\mathrm{supp}(\vec{q}) \subset \mathrm{supp}(\vec{p}_{\alpha_i})$, and condition (i) of Definition 2.9 holds. Given $\delta_{ij} \in \mathrm{supp}(\vec{p}_{\alpha_i})$. If $i < r$ then $\vec{p}_{\alpha_i}(\delta_{ij}) = (D_j, a_j) = \vec{q}(\delta_{ij})$. If $i \geq r$, by (2.6), we have that $W_{ij} \cap a_{ij}$ has the same size that $D_j \cap a_j$ and $(W_{ij} \setminus a_{ij})$ is an interval of F^*. This shows condition (ii) of Definition 2.9 Therefore $\vec{q} \leq \vec{p}_{\alpha_i}$. Which implies $\vec{q} \Vdash \alpha_i \in \dot{\Gamma}$ for every $i < n_k$ because of (2.4).

Finally, let $F = \Phi^q(F^*)$. Then $\vec{q} \Vdash F \in \dot{\mathcal{F}}$, and by the construction of $(x_i : i < n_k)$ and (2.7) we have $\vec{q}$ forces F captures $\alpha_0, \ldots, \alpha_{n_k-1}$. Therefore (2.5) holds which is what we wanted to prove.

To see that in $V[G]$ there are fully capturing construction schemes of any type suppose we are given a name $\dot{t}$ for a type on $V[G]$, i.e, a name for a sequence of integers that forms a type.

For $A \subset \omega_1$, let $\mathbb{P}_A$ be the collection of conditions $p \in \mathbb{P}$ with $\text{supp}(p) \subset A$. Take $\mathscr{A}_0 \subset \mathbb{P}$ a maximal antichain such that, every $p \in \mathscr{A}_0$ decides $\dot{t} \upharpoonright 2$. This means p decides the first two elements in the sequence $\dot{t}$. Suppose we continue this way, and find $\mathscr{A}_\ell$. For every $p \in \mathscr{A}_\ell$ we find $\mathscr{B}_\ell$ a maximal antichain below p such that, every $q \in \mathscr{B}_{p,\ell}$ decides $\dot{t} \upharpoonright (\ell + 2)$. Let

$$\mathscr{A}_{\ell+1} = \bigcup_{p \in \mathscr{A}_\ell} \mathscr{B}_{p,\ell}$$
$$\mathscr{A}_\omega = \bigcup_{\ell < \omega} \mathscr{A}_\ell$$

Since $\mathbb{P}$ has the ccc, every $\mathscr{A}_\ell$ is countable and therefore $\mathscr{A}_\omega$ is countable. Also, every $p \in \mathscr{A}_\omega$ has finite support, thus there is some $\alpha < \omega_1$ such that $\mathscr{A}_\omega \subset \mathbb{P}_\alpha$. This implies $\dot{t}$ is in $V[G_\alpha]$ where $G_\alpha = G \cap \mathbb{P}_\alpha$.

It is well known (see for example Theorem 8.2.1 of [Kun80]) that $\mathbb{P} = \mathbb{P}_\alpha * \mathbb{P}_{\omega \setminus \alpha}$. We can consider $V[G_\alpha]$ as the ground model, then $\dot{t}$ is a type on the ground model and forcing with $\mathbb{P}_{\omega_1 \setminus \alpha}$ is equivalent to adding $\aleph_1$ Cohen reals, therefore it adds a fully capturing construction scheme of type $\dot{t}$.

Suppose now $\kappa > \aleph_1$. Let $\mathbb{C}_\kappa$ be the forcing for adding κ Cohen reals. We know that $\mathbb{C}_{\omega_1}$ adds capturing construction schemes, by Lemma 2.12, forcing with $\mathbb{C}_{\kappa \setminus \omega_1}$ preserves capturing since it has precaliber $\aleph_1$. Therefore forcing with $\mathbb{C}_\kappa$ adds capturing construction schemes. $\square$

2.3   The hierarchies of n-capturing construction schemes and m-Knaster

We say a forcing notion $\mathbb{P}$ is K_m if for every uncountable sequence $(p_\alpha)_\alpha$ of $\mathbb{P}$ we can find an uncountable subsequence $(p_{\alpha_\gamma})_\gamma$, such that every $\gamma_1 < \ldots < \gamma_m < \omega_1$ we have $p_{\alpha_{\gamma_1}}, \ldots, p_{\alpha_{\gamma_m}}$ have a common extension.

Throughout this section we will fix some $n, m \geq 2$. Recall that $\text{MA}_{\omega_1}(K_m)$ implies $\text{MA}_{\omega_1}(K_n)$ for every $m \leq n$, whereas n-capturing implies m-capturing for every $m \leq n$. Thus, we have the following two hierarchies:

$$\text{MA}_{\omega_1}(K_2) \Longrightarrow \cdots \Longrightarrow \text{MA}_{\omega_1}(K_n) \Longrightarrow \text{MA}_{\omega_1}(K_{n+1}) \Longrightarrow \cdots \Longrightarrow \text{MA}_{\omega_1}(\text{precaliber } \aleph_1)$$

$$\text{2-capturing} \Longleftarrow \cdots \Longleftarrow n\text{-capturing} \Longleftarrow (n+1)\text{-capturing} \Longleftarrow \cdots \Longleftarrow \text{capturing}$$

The main result of this section give us a relation between this two types of axioms and shows that none of the implications above can be reversed. In particular it is consistent that there are n-capturing construction schemes but no m-capturing construction schemes for $m > n$.

Theorem 2.3. *$MA_{\omega_1}(K_m)$ and n-capturing are independent if $n \leq m$ and they are incompatible if $n > m$. Also $MA_{\omega_1}(\text{precaliber } \aleph_1)$ is independent of capturing.*

We start the analysis of n-capturing with the following preservation lemma.

Lemma 2.12. *Capturing is preserved by K_n forcing notions. Let $\mathbb{P}$ be a K_n forcing notion and let $\mathcal{F}$ be a n-capturing construction scheme on V. If $G \subset \mathbb{P}$ is a generic filter for $\mathbb{P}$, then $\check{\mathcal{F}}$ is a n-capturing construction scheme on $V[G]$. In particular capturing is preserved by precaliber $\aleph_1$ forcing notions.*

Proof. Let $\mathbb{P}$ be a K_n forcing notion and $\dot{\Gamma}$ a $\mathbb{P}$-name for an uncountable subset of ω_1. Let $W \subset \omega_1$ and $p_\alpha \in \mathbb{P}$, $\alpha \in W$ such that

$$p_\alpha \Vdash \alpha \in \dot{\Gamma}$$

for every $\alpha \in W$. Since $\mathbb{P}$ is K_n there is n-linked $W_0 \subset W$ uncountable. Recall $\mathcal{F}$ is n-capturing in V, therefore there are $\alpha_0 < \ldots < \alpha_{n-1}$ in W_0 which are captured by $\mathcal{F}$. We find now $q \in \mathbb{P}$ with $q \le p_0, \ldots, p_{n-1}$, then

$$q \Vdash \alpha_0, \ldots, \alpha_{n-1} \in \dot{\Gamma}, \text{ and they are captured by } \check{\mathcal{F}}.$$

$\square$

The following result is well known but we give a detailed proof for the convenience of the reader.

Theorem 2.13. *Let $\kappa > \aleph_1$ be a regular cardinal such that $\kappa^{\omega_1} = \kappa$. Then:*

1. *There is a focing notion with precaliber $\aleph_1$ which forces $MA_{\omega_1}(\text{precaliber } \aleph_1)$.*

2. *There is a K_m forcing notion which forces $MA_{\omega_1}(K_m)$.*

Proof. We will construct $\mathbb{P}$ with precaliber $\aleph_1$ as an iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \kappa)$ where for every α, $\dot{\mathbb{Q}}_\alpha = \{\emptyset\}$ or

$$\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha \text{ has precaliber } \aleph_1.$$

It is clear that we can repeat the same argument with K_m forcings instead of precaliber $\aleph_1$. We start by fixing a coding function $\varphi : \kappa \to \kappa \times \kappa$ such that φ is surjective and $\varphi(\alpha) = (\beta, \gamma)$ implies $\beta \le \alpha$.

Notice there are 2^{ω_1} many non-isomorphic forcing notions of size $\le \omega_1$ which have precaliber $\aleph_1$. We use the axiom of Choice to make an exhaustive list of them $(W_{0,\gamma} : \gamma < \kappa)$. We can do this because $|2^{\omega_1}| \le \kappa^{\omega_1} = \kappa$ by the hypothesis of the Theorem. Now we take $\mathbb{P}_0 = W_{0,\delta}$ where $\delta < \kappa$ is such that $\varphi(0) = (0, \delta)$.

Suppose we have $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$, such that for every $\alpha < \delta$,

(i) $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$,

(ii) $|\mathbb{P}_\alpha| < \kappa$,

(iii) $\mathbb{P}_\alpha$ has precaliber $\aleph_1$, and

(iv) $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a forcing notion of size at most ω_1 which has precaliber $\aleph_1$ i.e,

$$\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha \text{ is a forcing notion that has precaliber } \aleph_1.$$

we want to find $\mathbb{P}_\delta$ and $\dot{\mathbb{Q}}_\delta$.

If δ is limit we let $\mathbb{P}_\delta$ be the countable support iteration of $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$. Then $\mathbb{P}_\delta$ satisfies conditions (ii) and (iii) because having precaliber $\aleph_1$ is preserved by finite support products (condition (i) and (iv) are void in this case).

If δ is a successor, then $\delta = \gamma + 1$ for some $\gamma < \delta$, also we have $\dot{\mathbb{Q}}_\gamma$ is a $\mathbb{P}_\gamma$-name for a forcing notion with precaliber $\aleph_1$ by (iv) above. Consider $\mathbb{P}_\delta = \mathbb{P}_\gamma * \dot{\mathbb{Q}}_\gamma$. Then conditions (i)–(iii) are satisfied because having precaliber $\aleph_1$ is preserved by finite products.

To complete the forcing we have to find a $\mathbb{P}_\delta$-name $\dot{\mathbb{Q}}_\delta$ for a forcing notion of size $\leq \omega_1$ which has precaliber $\aleph_1$.

We consider, as before, all possible names for forcings of size $\leq \omega_1$ which has precaliber $\aleph_1$ (note that ω_1 is preserved because $\mathbb{P}_\delta$ is ccc). We know the size of $\mathbb{P}_\delta$ is $< \kappa$ because $\mathbb{P}_\delta = \mathbb{P}_\gamma * \dot{\mathbb{Q}}_\gamma$ and, by (ii) we have $|\mathbb{P}_\gamma| < \kappa$, and by (iv), $\dot{\mathbb{Q}}_\gamma$ is a name for a forcing notion of size $\leq \omega_1$. Thus, there are at most $\kappa^{\omega_1} = \kappa$ many $\mathbb{P}_\delta$-names for a focing notion of size ω_1. We can write an exhaustive list with all of the names for forcings of size $\leq \omega_1$ which have precaliber $\aleph_1$, $(W_{\delta,\alpha} : \alpha < \kappa)$.

Now we consider $\varphi(\delta) = (\beta, \eta)$ and look at $W_{\beta,\eta}$. In other words, consider the ηth $\mathbb{P}_\beta$-name for a forcing notion of size $\leq \omega_1$ that has precaliber $\aleph_1$. Since $W_{\beta,\eta}$ is a $\mathbb{P}_\beta$-name, it is also a $\mathbb{P}_\delta$-name. If $\mathbb{P}_\delta$ forces that $W_{\beta,\eta}$ has precaliber $\aleph_1$ then we let $\dot{\mathbb{Q}}_\delta = W_{\beta,\eta}$, where $(\beta, \eta) = \varphi(\delta)$ as before. Otherwise, we let $\dot{\mathbb{Q}}_\delta = \{\emptyset\}$. This gives us $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \leq \delta)$ with properties (i)–(iv) as we wanted.

Let $\mathbb{P}_\kappa$ be the finite support iteration of $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \kappa)$. Since precaliber $\aleph_1$ is preserved by finite support iterations we have $\mathbb{P}_\kappa$ has precaliber $\aleph_1$. We show that $\mathbb{P}_\kappa$ forces $MA_{\omega_1}(\text{precaliber } \aleph_1)$. This finishes the proof of the Theorem. $\qquad\square$

Claim 2.14. $\mathbb{P}_\kappa$ *forces* $MA_{\omega_1}(\text{precaliber } \aleph_1)$.

Proof. Fix G_α, a generic filter for $\mathbb{P}_\alpha$. For simplicity of notation we will represent the forcing notion of $\mathbb{P}_\alpha$ as $\Vdash_\alpha$ instead of $\Vdash_{\mathbb{P}_\alpha}$.

Let $\dot{W} \in V[G_\kappa]$ be a forcing notion of size $\leq \omega_1$ which has precaliber $\aleph_1$, without loss of generality we can assume $\dot{W}$ is $(\omega_1, \dot{\preceq})$, and let $\dot{\mathscr{D}} = (\dot{D}_\alpha : \alpha < \omega) \in V[G_\kappa]$ be a collection of dense sets of $\dot{W}$.

For every $\alpha, \beta < \omega_1$, pick two maximal antichains, $\mathcal{A}_{\alpha,\beta}$ and $\mathcal{B}_{\alpha,\beta}$ such that:

1. for every $q \in \mathcal{A}_{\alpha,\beta}$, q decides whether $\alpha \dot{\preceq} \beta$.

2. for every $q \in \mathcal{B}_{\alpha,\beta}$, q decides whether $\alpha \in \dot{D}_\beta$.

i.e, the antichains contain all relevant information about $\dot{W}$ and $\dot{\mathcal{D}}$.

Now consider

$$\mathcal{A} = \bigcup_{\alpha, \beta < \omega_1} \mathcal{A}_{\alpha, \beta} \cup \mathcal{B}_{\alpha, \beta}$$

Since $\mathbb{P}_\kappa$ has finite support, every $q \in \mathcal{A}$ is in $\mathbb{P}_{\alpha_q}$ for some $\alpha_q < \kappa$. Also note $|\mathcal{A}| = \aleph_0 \cdot \omega_1 < \kappa$. Thus there is $\lambda = \sup_q \alpha_q < \kappa$ such that $\mathbb{P}_\lambda$ contains all of the information about $\dot{W}$ and $\dot{\mathcal{D}}$. This means that $\dot{W}$ and $\dot{\mathcal{D}}$ are in $V[G_\lambda]$, therefore there is a $\mathbb{P}_\lambda$-name for $\dot{W}$. This name will be somewhere on the list $(W_{\lambda, \alpha} : \alpha < \kappa)$ we constructed above. Say $W_{\lambda, \eta}$ is the $\mathbb{P}_\lambda$-name for $\dot{W}$.

Recall we constructed $\mathbb{P}_\kappa$ with $\varphi : \kappa \to \kappa \times \kappa$ surjective, so there is some stage $\delta \geq \lambda$ such that $\varphi(\delta) = (\lambda, \eta)$. Since $\dot{W}$ and $\dot{\mathcal{D}}$ are in $V[G_\lambda]$ they will be in $V[G_\delta]$, and $W_{\lambda, \eta} = \dot{\mathbb{Q}}_\delta$ by the choice of φ. Therefore $G_{\delta+1}$ is a $\mathcal{D}$-generic filter and $G_{\delta+1}$ is in $V[G_\kappa]$ and this finishes the proof. $\qquad\square$

Consider the following property

($\bigstar$)$_m$ For every $\Gamma \subset \omega^\omega$ there is $\Gamma_0 \subset \Gamma$ uncountable such that Γ_0 has no $g_0, \ldots, g_m$ and $k < \omega$ with $g_0 \upharpoonright k = \ldots = g_m \upharpoonright k$, and $|\{g_0(k), \ldots, g_m(k)\}| = m + 1$.

Recall the following result of Todorčević implicit in [Tod89]

Theorem 2.15 (Todorčević [Tod85], see also [Tod89]). *$MA_{\omega_1}(K_m)$ implies ($\bigstar$)$_m$.*

The following result proves the first half of Theorem 2.3

Theorem 2.16. *Let $\mathcal{F}$ be a $(m + 1)$-capturing construction scheme. Then ($\bigstar$)$_m$ fails.*

Proof. Let $\mathcal{F}$ be as above. For every $F \in \mathcal{F}_l$ we construct, inductively on l, $(f_\alpha^F : (l + 1) \to N_l)_{\alpha < \omega_1}$ such that

1. for $E, F \in \mathcal{F}_l$ and $\varphi : E \to F$ the increasing bijection between E and F, for every $\alpha \in E$, if $\beta = \varphi(\alpha)$ then $f_\beta^F = f_\alpha^E$.

2. for $E \in \mathcal{F}_{l_0}$ and $F \in \mathcal{F}_{l_1}$, $l_0 < l_1$, if $\alpha \in E \cap F$ then $f_\alpha^F \upharpoonright (l_0 + 1) = f_\alpha^E$.

Let $F \in \mathcal{F}_k$ with canonical decomposition $F = \bigcup_{i < n_k} F_i$ and suppose $(f_\alpha^{F_i} : \alpha \in F_i)$ is defined for all $i < n_k$ satisfying (1) and (2) above. Let $f_\alpha^F = \emptyset$ if $\alpha \notin F$.

For $\alpha \in R(F)$ let $f_\alpha^F(k) = N_{k-1}$ and $f_\alpha^F \upharpoonright k = f_\alpha^{F_0}$.

For $\alpha_0 \in F_0 \setminus R(F)$ and $\alpha_i = \varphi_i(\alpha)$, $i < n_k$. We let $f_{\alpha_i}^F \upharpoonright k = f_{\alpha_i}^{F_i}$ and

$$f_{\alpha_i}^F = N_{k-1} + i + 1$$

And let $N_k = N_{k-1} + n_k + 1$.

It is easy to see that (1) and (2) hold, and so $f_\alpha = \bigcup_{F \in \mathcal{F}} f_\alpha^F$ is a well defined function. Then $\Gamma = \{f_\alpha : \alpha < \omega_1\}$ is a witness to the failure of ($\bigstar$)$_m$. To see this suppose $\Gamma_0 = \{f_\alpha : \alpha \in W\}$

where $W \subset \omega_1$ is uncountable. Since $\mathcal{F}$ is $(m+1)$-capturing there are $\xi_0 < \ldots < \xi_m$ in W captured by some $F \in \mathcal{F}_k$. This implies $f_{\xi_0} \restriction k = \ldots = f_{\xi_m} \restriction k$ and $|\{f_{\xi_0}(k), \ldots, f_{\xi_m}(k)\}| = m+1$, and hence $(\bigstar)_m$ fails as we wanted to show. $\qquad\square$

Proof of Theorem 2.3. Start by assuming $n \leq m$. To see n-capturing is independent of $\mathrm{MA}_{\omega_1}(\mathrm{K}_m)$, note that any model of MA_{ω_1} is also a model of $\mathrm{MA}_{\omega_1}(\mathrm{K}_m)$ and contains no n-capturing construction scheme for any $2 \leq n < \omega$ (see [LT17] for $n > 2$, and see Proposition 2.19 of this paper for $n = 2$). Thus, it is consistent to have $\mathrm{MA}_{\omega_1}(\mathrm{K}_m)$ and no n-capturing construction schemes. To show the other direction, start with a model V that has a capturing construction scheme $\mathcal{F}$. Let $\mathbb{K}_m$ be the K_m poset that forces $\mathrm{MA}_{\omega_1}(\mathrm{K}_m)$. Then $\mathcal{F}$ remains m-capturing on the extension by Lemma 2.12 hence it is n-capturing provided $n \leq m$.

Suppose now $n > m$ and V is a model of $\mathrm{MA}_{\omega_1}(\mathrm{K}_m)$, then $(\bigstar)_m$ holds on V. By Theorem 2.16 we know V contains no $(m+1)$-capturing construction scheme, otherwise $(\bigstar)_m$ fails which is a contradiction. Thus V has no n-capturing construction scheme for $n > m$, as we wanted to show.

To see $\mathrm{MA}_{\omega_1}(\text{precaliber } \aleph_1)$ and capturing are independent we proceed in the same manner. Any model of MA_{ω_1} satisfies $\mathrm{MA}_{\omega_1}(\text{precaliber } \aleph_1)$ and has no capturing construction scheme. Finally, let V be a model that contains a capturing construction scheme. Let $\mathbb{K}$ be a forcing notion with precaliber $\aleph_1$ that forces $\mathrm{MA}_{\omega_1}(\text{precaliber } \aleph_1)$. Since $\mathbb{K}$ has precaliber $\aleph_1$, $\mathcal{F}$ remains capturing in the extension. This finishes the proof. $\qquad\square$

It is interesting to find a K_n forcing notion that kills $(n+1)$-capturing in an obvious way. Suppose $\mathcal{F}$ is a capturing construction scheme. Let $\mathcal{F}$ be fixed.

Definition 2.17. Let $P \in \mathbb{P}_n$ if $\mathcal{F}$ does not capture $\big\{\{\xi_i\} : i \leq n\big\}$ for any $\xi_0 < \ldots < \xi_n$ in P. We say $P \leq Q$ if $Q \subset P$.

Lemma 2.18. $\mathbb{P}_n$ *defined as above is* K_n.

Proof. Take $(P_\alpha : \alpha < \omega_1) \subset \mathbb{P}_n$. We can find $D_\alpha \in \mathcal{F}_{k_\alpha}$ such that $P_\alpha \subset D_\alpha$.

Find $\Gamma \subset \omega_1$ uncountable, and $k < \omega$ such that

1. $(D_\alpha : \alpha \in \Gamma)$ forms a Δ-System,

2. $k_\alpha = k$ for all $\alpha \in \Gamma$, and

3. for every $\alpha < \beta$ in Γ, we have $\varphi_{D_\alpha, D_\beta}(P_\alpha) = P_\beta$.

Note that (2) and (3) imply that for all $\alpha < \beta$, $\xi \in D_\alpha \cap D_\beta$, then $\xi \in P_\alpha$ if and only if $\xi \in P_\beta$.

We show $(P_\alpha : \alpha \in \Gamma)$ is n-linked. Take $\alpha_0 < \ldots < \alpha_{n-1}$ in Γ. Let $Q = \bigcup_{i<n} P_i$. Suppose $\xi_0 < \ldots < \xi_n$ are in Q and $F \in \mathcal{F}_\ell$ captures $\big\{\{\xi_i\} : i \leq n\big\}$. Take $F = \bigcup_{i<n_\ell} F_i$ the canonical

decomposition of F. We must have

$$\xi_i \in F_i \setminus R(F)$$
$$\left(F_i \setminus R(F) : i < n_l \right) \text{ are pairwise disjoint} \tag{2.8}$$

Let us get a contradiction.

Case $l \leq k$: Let $j \leq n$ with $\xi_n \in P_{\alpha_j}$. Applying Proposition 1.33, $F \cap D_{\alpha_j} \sqsubseteq F$. Therefore $\xi_0, \ldots, \xi_n \in D_{\alpha_j}$ which implies $\xi_0, \ldots, \xi_n \in P_{\alpha_j}$. But $\mathcal{F}$ captures $\{\{\xi_i\} : i \leq n\}$ and this is a contradiction because $P_{\alpha_j} \in \mathbb{P}_n$.

Case $l > k$: There is some $j < n$ and $i_0 < i_1 \leq n$ such that $\xi_{i_0}, \xi_{i_1} \in P_{\alpha_j}$. Then $F_{i_1} \in \mathcal{F}_{\ell-1}$, and $F_{i_1} \cap D_{\alpha_j} \sqsubseteq D_{\alpha_j}$ by Proposition 1.33, but this implies $\xi_{i_0} \in F_{i_1}$. This contradicts (2.8)

We conclude that for every $\xi_0 < \ldots \xi_n$ in Q, $\mathcal{F}$ does not capture $\{\{\xi_i\} : i \leq n\}$. Hence $Q \in \mathbb{P}_n$. It is clear that $Q \leq P_{\alpha_i}$ for $i < n$. This finishes the proof. $\square$

It is clear that $\mathbb{P}_n$ kills $(n+1)$-capturing, thus we have an explicit proof that $\mathrm{MA}_{\aleph_1}(\mathrm{K}_n)$ is incompatible with m-capturing for $m > n$.

Assume $m > 2$ and note that the model obtained in the proof of Theorem 2.3, which starts with a capturing construction scheme and then forces $\mathrm{MA}_{\omega_1}(\mathrm{K}_m)$, shows the consistency of

$$\mathrm{MA}_{\omega_1}(\mathrm{K}_m) + m\text{-capturing} + \neg(m+1)\text{-capturing} + \neg\mathrm{MA}_{\omega_1}(\mathrm{K}_{m-1})$$

this gives us an alternative proof of $\mathrm{MA}_{\omega_1}(\mathrm{K}_m) \not\Leftarrow \mathrm{MA}_{\omega_1}(\mathrm{K}_{m+1})$ showing that the hierarchy of m-Knaster forcing axioms is strict.

To get that MA_{ω_1} implies there are no 2-capturing construction schemes, we prove the following:

Proposition 2.19. *If $\mathcal{F}$ is 2-capturing, then $\mathbb{P}_1$ is c.c.c.*

Proof. Suppose $(P_\alpha : \alpha < \omega_1) \subset \mathbb{P}_1$ forms an uncountable antichain, and refine this family so that it forms a Δ-system. Since $\mathcal{F}$ is 2-capturing, we can recursively construct a family $(D_\alpha : \alpha \in \Gamma) \subset \mathcal{F}$ and refine it so that $(D_\alpha : \alpha \in \Gamma) \subset \mathcal{F}_k$ forms an uncountable Δ-System, and for $\alpha \in \Gamma$, D_α captures some $(P_{\alpha'}, P_{\alpha''})$. Again, since $\mathcal{F}$ is 2-capturing, there are some $F \in \mathcal{F}$, $\alpha < \beta \in \Gamma$, such that F captures (D_α, D_β).

We claim that $P_{\alpha'} \cup P_{\beta''} \in \mathbb{P}_1$, which finishes the proof with a contradiction. Suppose $\xi_0 < \xi_1 \in P_{\alpha'} \cup P_{\beta''}$ are captured by some $E \in \mathcal{F}_l$. Note that since $P_{\alpha'}, P_{\beta''} \in \mathbb{P}_1$, $\xi_0 \in P_{\alpha'} \setminus P_{\beta''}$, $\xi_1 \in P_{\beta''} \setminus P_{\alpha'}$, and so $\xi_0 \in D_\alpha \setminus D_\beta$, $\xi_1 \in D_\beta \setminus D_\alpha$. Let $E = \bigcup_{i<n_l} E_i$, $D_\beta = \bigcup_{i<n_k}(D_\beta)_i$, $D_\alpha = \bigcup_{i<n_k}(D_\alpha)_i$ be the respective canonical decompositions.

Case $l \leq k$: Applying Proposition 1.33, $E \cap D_\beta \sqsubseteq E$. Therefore $\xi_1 \in E \cap D_\beta$ gives $\xi_0 \in D_\beta$, and this is a contradiction.

Case $l > k$: Recall that $\phi_{E_0, E_1}(\mathcal{F} \restriction E_0) = \mathcal{F} \restriction E_1$, and D_α capturing $(P_{\alpha'}, P_{\alpha''})$ implies $\xi_0 \in (D_\alpha)_0 \setminus R(D_\alpha)$. Since there is some $E' \subset E$ with $\xi_0 \in E' \in \mathcal{F}_k$, and $\xi_0 \in (D_\alpha)_0 \in \mathcal{F}_{k-1}$, we get that ξ_0 must be in the 0'th component of the canonical decomposition of E', and hence

$\phi_{E_0,E_1}(\xi_0) = \xi_1$ must be in the 0'th component of the canonical decomposition of some element in $\mathcal{F}_k \upharpoonright E_1$, which contradicts $\xi_1 \in (D_\beta)_1 \setminus R(D_\beta)$.

$\square$

2.4 Other forms of capturing

Recall the definition of $\vec{P}$-capturing construction scheme from [Tod17].

Definition 2.20. Let $\omega = \bigcup_{\ell < \omega} P_\ell$ be a partition of ω into infinite components and let $\vec{P} = (P_\ell : \ell < \omega)$. Suppose (m_k, n_k, r_k) forms a type such that for every $\ell < \omega$, and every $r < \omega$ there are infinitely many k's in P_ℓ with $r_k = r$. Then we say $(m_k, n_k, r_k)_k$ forms a $\vec{P}$-*type*.

Definition 2.21. Let $\mathcal{F}$ be a construction scheme with type $(m_k, n_k, r_k)_k$, and $2 \geq n$. We say $\mathcal{F}$ is n-$\vec{P}$-*capturing* if $(m_k, n_k, r_k)_k$ forms a $\vec{P}$-type, and for every uncountable Δ-system $(s_\xi)_{\xi < \omega_1}$ of finite subsets of ω_1 with root s, and every $\ell < \omega$, there are $\xi_0 < \ldots < \xi_{n-1} < \omega_1$, $k \in P_\ell$ and $F \in \mathcal{F}_k$ with canonical decomposition $F = \bigcup_{i < n_k} F_i$, such that

$$s \subset R(F)$$
$$\text{for every } i < n, \quad s_{\xi_i} \setminus s \subset F_i \setminus R(F),$$
$$\text{for every } i < n, \quad \varphi_i(s_{\xi_0}) = s_{\xi_i}.$$

We say $\mathcal{F}$ is $\vec{P}$-*capturing* if $\mathcal{F}$ is n-$\vec{P}$-capturing for every $n < \omega$.

We prove the following Theorem about the consistency of other forms of capturing.

Theorem 2.22. *Adding $\kappa \geq \aleph_1$ Cohen reals implies there are $\vec{P}$-capturing construction schemes, and fully $\vec{P}$-capturing construction schemes.*

Proof. The proof is an adjustment of the proof of Theorem 2.2 therefore we only give a sketch for a fully $\vec{P}$-capturing construction scheme.

Let $\vec{P}$ be a partition of ω and let $(m_k, n_k, r_k)_{k < \omega}$ be a given $\vec{P}$-type on the ground model.

It is easy to see, using the fact that $(m_k, n_k, r_k)_{k < \omega}$ is a $\vec{P}$-type, that there is a Construction Scheme $\mathcal{F}^\omega$ on ω such that:

For every $\ell < \omega$, $A \subset \omega$ finite, and $a < \omega$, there is $k \in P_\ell$ and $F \in \mathcal{F}_k$ with canonical decomposition $\bigcup_{i < n_k} F_i$, such that $A \subset F_0$ and $R(F) = F_0 \cap a$. (2.9)

Suppose now $\dot{\mathcal{F}}$ is defined as in Theorem 2.2 and $\dot{\Gamma}$ is a name for an uncountable subset of ω_1 which defines a Δ-System of the form ($\bigstar$). Let $\ell < \omega$ and $k^* < \omega$ be given.

Find $\Omega \subset \omega_1$ uncountable and $p_\alpha \in \mathbb{P}$ for $\alpha \in \Omega$ such that

$$p_\alpha \Vdash \alpha \in \dot{\Gamma} \tag{2.10}$$

And there is $\delta \in \text{supp}(p_\alpha)$ such that $p_\alpha(\delta) = (D, a)$, and $\alpha \in \phi_{a,\delta}(D)$. And $\delta_{\alpha,0} < \ldots < \delta_{\alpha,d-1} < \omega_1$ limit, $D_i \in \mathcal{F}^\omega_{k_i}$ for $i < d$, $a_0 < \ldots < a_{d-1}$, and $x < \omega$ such that:

1. $(\text{supp}(p_\alpha) : \alpha \in \Omega_0)$ form a Δ-System with root $\{\delta_{\alpha,0}, \ldots, \delta_{\alpha,r-1}\}$,

2. $\text{supp}(p_\alpha) = \{\delta_{\alpha,0}, \ldots, \delta_{\alpha,d-1}\}$,

3. $p_\alpha(\delta_{\alpha,i}) = (D_i, a_i)$ for every $i < d$,

4. For $x \in D_{d-1}$ with $\Phi^{p_\alpha}(x) = \alpha$, there is fixed j_0 with: $j_0 = d-1$ if $x \geq a_{d-1}$, or $j_0 < d-1$ and is such that $a_{j_0} \leq x < a_{j_0+1}$.

Apply (2.9) to find $k \in P_\ell$ with $k > k^*$, and $F^* \in \mathcal{F}^\omega_k$ such that $k > k_{d-1}$, $F^* = \bigcup_{i<n_k} F^*_i$ is the canonical decomposition of F^*, $D_{d-1} \subset F^*_0$, and $R(F^*) = F^*_0 \cap a_r$.

Pick arbitrary $\alpha_0 < \ldots < \alpha_{n_k-1}$ in Ω. We construct $q \in \mathbb{P}$, such that

$$q \Vdash \alpha_i \in \dot{\Gamma}, \exists F \in \dot{\mathcal{F}}_k \text{ captures } \alpha_0, \ldots, \alpha_{n_k-1}. \tag{2.11}$$

For $i < d$, note $a_i \in D_i \subset F^*_0$, therefore we can apply Lemma 1.36 to find $W_{0,i} \in \mathcal{F}^\omega_{k_i}$ with $W_{0,i} \cap a_i = D_i \cap a_i$ and $W_{0,i} \setminus a_i$ an interval of F^*_0 with $a_i \in W_{0,i}$. Let $\varphi_i : F^*_0 \to F^*_i$ be the increasing bijection between F^*_0 and F^*_i. Define $W_{i,j} = \varphi_i(W_{0,j})$, and $a_{i,j} = \varphi_{D_j,W_{i,j}}(a_j)$ for $i < n$, $j < d$, and $x_i = \varphi_{D_{j_0},W_{i,j_0}}(x)$ for $i < n_k$.

It is easy to check that

$$W_{i,j} \in \mathcal{F}^\omega_{k_j}, |W_{i,j} \cap a_{i,j}| = |D_j \cap a_j|, \text{ and } W_{i,j} \setminus a_{i,j} \text{ is an interval of } F^* \text{ with } a_{i,j} \in W_{i,j} \tag{2.12}$$

and as before we have

$$\phi_{a_{i,j_0}, \delta_{\alpha_i,j_0}}(x_i) = \alpha_i \tag{2.13}$$

We define $q \in \mathbb{P}$ with $\text{supp}(q) = \{\delta_{\alpha_i,j} : i < n, j < d\}$. Note now that $\delta_{\alpha,i}$ does not depend on α for $i < r$.

$$q(\delta_{\alpha_i,j}) = \begin{cases} (D_j, a_j) & \text{for } j < r \\ (F^*, a_{i,j}) & \text{for } j \geq r \end{cases}$$

With this definition, we have $\Phi^q(x_i) = \phi_{a_{i,j_0}, \delta_{\alpha_i,j_0}}(x_i) = \alpha_i$ by (2.7), and $(W_{i,j} : r \leq j < d)$ is a witness to $q \leq p_{\alpha_i}$ for every $i < n_k$, by (2.12). This implies $q \Vdash \alpha_i \in \dot{\Gamma}$ for every $i < n_k$ because of (2.10).

Finally, let $F = \Phi^q(F^*)$. Then $q \Vdash F \in \dot{\mathcal{F}}$, and by the construction of $(x_i : i < n_k)$ and (2.13) we have q forces F captures $\alpha_0, \ldots, \alpha_{n_k-1}$. Therefore (2.11) holds which is what we wanted to prove.

$\square$

We also have the following results related to the consistency of n-$\vec{P}$-capturing. The proof is analogous to the arguments in Section 2.3.

Theorem 2.23. *Let $\vec{P}$ be a partition of ω as above. Then n-$\vec{P}$-capturing and $MA_{\omega_1}(K_m)$ are independent if $n \leq m$ and they are incompatible if $n > m$. Also $\vec{P}$-capturing, $\vec{P}$-fully capturing, and fully capturing are all independent of $MA_{\omega_1}(precaliber \; \aleph_1)$.*

It is clear that n-$\vec{P}$-capturing implies n-capturing and $\vec{P}$-capturing implies capturing, however we do not know if any of the implications can be reversed. Analogously, fully capturing implies capturing but we do not know if it is consistent to have capturing without fully capturing.

2.5   Summary of Consistency Results

We finish the Chapter with a list of all of the consistency results about capturing construction schemes that we know at this moment. The proofs of this results will be given in later Chapters of this Thesis.

For the sake of simplicity, when we say *there is a capturing construction scheme $\mathcal{F}$*, or when we talk about *consistency of capturing, consistency of n-capturing*, or a variant of the above, what we mean is that for any given type $(m_k, n_k, r_k)_{k<\omega}$, there is a capturing ($n$-capturing) construction scheme $\mathcal{F}$ of type $(m_k, n_k, r_k)_{k<\omega}$. A summary of all results can be seen in Table 2.1.

Axiom	Relationship to capturing $\mathcal{F}$	Notes
Suslin Tree	Consequence of 3-capturing	Theorem 3.1
Destructible gap	Consequence of 3-capturing	Theorem 3.3
$\diamondsuit$	Implies capturing	Theorem 2.3 of [Tod17]
CH	Independent	Theorem 2.28
$\mathfrak{p} = \omega_1$	Independent	Theorem 2.24
PID	Incompatible with 3-capturing	Corollary 2.27
MA_{ω_1}	Incompatible with 3-capturing	Corollary 1.30
PFA	Incompatible with 3-capturing	Theorem 1.29
$MA_{\omega_1}(\text{precalliber } \aleph_1)$	Independent	Theorem 2.3
$MA_{\omega_1}(\sigma\text{-centered})$	Independent	Theorem 2.24
$MA_{\omega_1}(K_m)$	Indep. of n-capturing if $n \leq m$, incomp. if $n > m$.	Theorem 2.3

Table 2.1: Relationship of different axioms with the existence of a capturing construction scheme $\mathcal{F}$.

Theorem 2.24. *Let $\kappa > \aleph_1$, with $\kappa = \kappa^{<\kappa}$. Then it is consistent that there are fully $\vec{P}$-capturing construction schemes and $\mathrm{MA}_\kappa(\sigma\text{-centered})$ holds. In particular $\mathfrak{p} = \kappa$.*

Also, it is consistent that there are fully $\vec{P}$-capturing construction schemes and $\mathfrak{p} = \omega_1$.

Proof. Following the proof of Theorem 2.13 we can show that there is a forcing notion that has precaliber $\aleph_1$ and forces $\mathrm{MA}_\kappa(\sigma\text{-centered})$. It is well known that finite support iterations add Cohen reals at limit stages, therefore there are fully $\vec{P}$-capturing construction schemes.

The second part follows from a result of Bell.

Theorem 2.25 (Bell [Bel81]). *Let $\kappa = \min\{\lambda : \mathrm{MA}_\lambda(\sigma\text{-centered}) \text{ fails}\}$. Then $\mathfrak{p} = \kappa$.*

To finish the proof let V be our ground model. Then adding ω_1 Cohen reals implies there are fully $\vec{P}$-capturing construction schemes. We just have to show $\mathfrak{p} = \omega_1$ after adding ω_1 Cohen reals. We are going to show that $\mathfrak{b} = \omega_1$.

Consider the family $\mathscr{B}$ given by the sequence $(f_\alpha : \alpha < \omega_1)$ added by $\mathbb{C}_{\omega_1}$. We show this sequence is unbounded on ω^ω.

Let $M = V[G_\kappa]$ be our ground model and let G_λ be a generic filter for $\mathbb{C}_\lambda$. Take $g : \omega \to \omega$ in $M[G_{\omega_1}]$ and let $\dot{g}$ be a name for g on M. Since $\mathbb{C}_{\omega_1}$ has the ccc, we can pick a maximal antichain $\mathscr{A}_k$ deciding the value of $g(k)$. Recall $\mathbb{C}_{\omega_1}$ is a finite support iteration, therefore for every $p \in \mathscr{A}_k$ there is $\alpha_p < \omega_1$ such that $p \in \mathbb{C}_{\alpha_p}$. Let

$$\mathscr{A} = \bigcup_{k<\omega} \mathscr{A}_k$$

and take $\lambda = \sup\{\alpha_p : p \in \mathscr{A}\}$. Then $g \in V[G_\lambda]$.

Now we prove that $f_{\lambda+1} \not<^* g$. Without loss of generality we can assume $g \in M$ and we force with $\mathbb{C}_1$ adding a single Cohen real f. Let $p \in \mathbb{C}_1$ and $n < \omega_1$ be given. Take $k > n$ such that $\mathrm{supp}(p) < k$. Let q be equal to p on $\mathrm{supp}(p)$ and $q(k) = g(k) + 1$, then $q \leq p$ and

$$q \Vdash f(k) > g(k)$$

This implies $f \not<^* g$ and finishes the proof that $\mathfrak{b} = \omega_1$ hence $\mathfrak{p} = \omega_1$ as we wanted to see.

$\square$

We would like to remark that it is known (see Roitman [Roi79]) that adding a Cohen real to a model of $\mathrm{MA}_{\omega_1}(\sigma\text{-centered})$ preserves $\mathrm{MA}_{\omega_1}(\sigma\text{-centered})$. Also, adding a Cohen real to a model of $\mathrm{MA}_{\omega_1}(\text{precaliber } \aleph_1)$ gives a model where $\mathrm{MA}_{\omega_1}(\sigma\text{-centered})$ holds but $\mathrm{MA}_{\omega_1}(\text{precaliber } \aleph_1)$ fails. Therefore,

Corollary 2.26. *It is consistent that there are capturing construction schemes, $MA(\sigma\text{-centered})$ holds, but $MA_{\omega_1}(\text{precaliber } \aleph_1)$ does not holds.*

This previous results are interesting when we take into account Theorem 3.1 from Chapter 3,

Theorem 3.1. *Assume there is a 3-capturing construction scheme, then there is a Suslin tree.*

Now we have the following immediate corollaries

Corollary 1.30. MA_{ω_1} *implies there is no 3-capturing construction scheme.*

Recall that there are no Suslin trees in a model of PID, i.e, PID implies Suslin Hypothesis.

Corollary 2.27. *PID implies there is no capturing construction scheme.*

It is a classical result of Todorcevic [Tod06] that MA_{ω_1} and PID imply that every Banach space of density ω_1 has an uncountable Biorthogonal System. In Chapter 4 we will show that capturing construction schemes imply there are Banach spaces of density ω_1 without uncountable Biorthogonal Systems. This leads to the following Corollary which also follows from the previous Corollary since PFA implies PID (see [Tod00] and [Tod11])

Corollary 1.29. *PFA implies there is no capturing construction scheme.*

This is a consequence of the discussion above and the fact that PFA implies MA_{ω_1} and PID. PFA is the forcing axiom for proper forcings, the definition of proper forcing is tangential to this work. The interesting reader is refer to Shelah [She98] or the monograph of Baumgartner [Bau84], for the reader interested on the applications of PFA we refer to Todorcevic [Tod14].

The following Theorem summarizes all of the positive results about existence of capturing construction schemes $\mathcal{F}$.

Theorem 2.28. *Let $\kappa > \aleph_1$ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. The following statements are consistent:*

(i)   *There is a fully $\vec{P}$-capturing construction scheme and $\diamondsuit$ holds. In particular CH holds.*

(ii) *There is a fully $\vec{P}$-capturing construction scheme and $MA_{\omega_1}(\text{precaliber } \aleph_1)$ holds.*

(iii) *There is a n-capturing construction scheme and $MA_{\omega_1}(K_n)$ holds.*

(iv) *There is a fully $\vec{P}$-capturing construction scheme, $\mathfrak{b} = \aleph_1$, $\mathfrak{c} = \kappa$.*

(v)   *Let $\mathbb{P} = (\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \kappa)$ be a finite support iteration. If there is $\eta \leq \kappa$ of cofinality ω_1 and for every $\eta < \alpha < \kappa$,*

$$\Vdash_\alpha \dot{\mathbb{Q}}_\alpha \text{ has precaliber } \aleph_1$$

then $\mathbb{P}$ forces there are fully $\vec{P}$-capturing construction schemes.

(vi) *Let $\mathbb{P} = (\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \kappa)$ be a finite support iteration. If there is $\eta \leq \kappa$ of cofinality ω_1 and for every $\eta < \alpha < \kappa$,*

$$\Vdash_\alpha \dot{\mathbb{Q}}_\alpha \text{ is } K_m$$

then $\mathbb{P}$ forces there are m-$\vec{P}$-capturing construction schemes.

The only part that needs futher explanation are part (v) and (vi). It is a well known result that finite support iterations add Cohen reals at limit stages therefore, if $\mathbb{P}$ is as in part (v) above, then $\mathbb{P}_\eta$ will had ω_1 Cohen reals and there are fully $\vec{P}$-capturing construction schemes. This construction schemes are preserved by the iteration because of Lemma 2.12. An analogous argument shows (vi).

Chapter 3

Trees and Gaps

We turn our focus now to applications of capturing construction schemes. Within Set Theory the study of trees and gaps is interesting as they are relatively simple combinatorial objects that appear relatively often. We show that there is a natural construction of a Suslin tree and a Hausdorff T-gap provided there is a capturing construction scheme.

We start the Chapter showing the following result.

Theorem 3.1. *Assume there is a Construction Scheme that is 3-capturing. Then there is a Suslin tree.*

We then use the same idea to prove that there are T-gaps in every model with a 3-capturing construction scheme. Remember the definitions of Hausdorff (ω_1, ω_1)-gap, destructible gap, and T-gap.

Definition 3.2.   1. A pre-gap $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ form a Hausdorff (ω_1, ω_1)-gap if for every uncountable $\Gamma \subset \omega_1$ there are $\alpha < \beta$ in Γ such that $a_\alpha \cap b_\beta \neq \emptyset$.

   2. We say a gap $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is a destructible gap if for every uncountable $\Gamma \subset \omega_1$ there are $\alpha < \beta$ in Γ such that $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset$.

   3. We say a gap $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is a *T-gap* if for every uncountable $\Gamma \subset \omega_1$ there are $\alpha < \beta$ such that $a_\alpha \subseteq a_\beta$ and $b_\alpha \subseteq b_\beta$.

We show the following result.

Theorem 3.3. *Assume there is a 3-capturing construction scheme. Then there is a Hausdorff (ω_1, ω_1)-gap that is a T-gap.*

The proof of the Theorem contains a natural example of a T-gap. This becomes more interesting when this construction is compared with previous known examples of T-gaps. For a construction of a Hausdorff T-gap using $\diamondsuit$ the reader is refer to Alan Dow [Dow95].

We do not know if the previous results can be improved to the case of 2-capturing construction schemes. We can say more about this question if we use partitions to capture.

Theorem 3.4. *Let $\omega = \bigcup_{i<\omega} P_i$, with P_i infinite, and let $\vec{P} = (P_i : i < \omega)$. Assume there are 2-$\vec{P}$-capturing construction schemes, then there is Suslin tree and Hausdorff T-gap.*

We finish the Chapter studying the relation between T-gaps and destructible gaps, or S-gaps. Recall that every T-gap can be filled by a ccc forcing notion, Proposition 1.13. Therefore, the existence of a 3-capturing construction scheme implies there is a destructible (ω_1, ω_1)-gap. Every T-gap is destructible but the converse need not be true. More precisely, we have the following result

Theorem 3.5. *There is a model of set theory in which there is a destructible Hausdorff (ω_1, ω_1)-gap but with no T-gaps.*

3.1   Suslin trees

We dedicate this section to show that 3-capturing implies there are Suslin trees. We also talk about some of the consequences of this result. More concretely, we want to prove the following result.

Theorem 3.1. *Assume there is a Construction Scheme that is 3-capturing. Then there is a Suslin tree.*

This Theorem gives us another proof of the following Corollary.

Corollary 1.30. *Assume MA_{ω_1}. There are no 3-capturing construction schemes.*

Recal that Corollary 1.30 was proved in Chapter 1 directly. In Chapter 2 we saw a different proof: If there are 3-capturing construction schemes then $MA_{\omega_1}(K_2)$ fails. We present another proof viaTheorem 3.1 and the following well known fact.

Lemma 3.6. *Assume MA_{ω_1}. There are no Suslin trees.*

Proof. Let $(\mathbb{S}, \preceq)$ be a tree of height ω_1 such that for every $t \in \mathbb{S}$ and every $\alpha < \omega$, there is some $s \in \mathbb{S}$ with $Lev(s) > \alpha$ and $t \preceq s$. Every Suslin tree contains a Suslin tree with this property.

Let $\mathbb{P} = (\mathbb{S}, \succeq)$. We show that forcing with $\mathbb{P}$ adds an uncountable chain to $\mathbb{S}$. Define $\mathscr{D}_\alpha = \{t \in \mathbb{S} : Lev(t) > \alpha\}$. The property of $\mathbb{S}$ shows that $\mathscr{D}_\alpha$ is dense for every $\alpha < \omega_1$. Therefore, any $\{\mathscr{D}_\alpha : \alpha < \omega_1\}$-generic filter G will be an uncoutable chain of $\mathbb{S}$.

If $\mathbb{S}$ is a Suslin tree, then $\mathbb{P}$ has the ccc, otherwise $\mathbb{S}$ contains an uncountable antichain, and MA_{ω_1} forces an uncountable chain for $\mathbb{S}$. $\square$

Let us go into the idea behind the proof of Theorem 3.1 before we go into the details.

3.1.1 Outline of the construction

We start with $\mathcal{F}$, a 3-capturing construction scheme, and we want to construct $\mathbb{S} \subset \{0,1\}^{\omega_1}$ a Suslin tree. We do this by recursively defining finite approximations on $\{0,1\}^F$ for every $F \in \mathcal{F}$.

More precisely; for every $F \in \mathcal{F}$ and every $\alpha \in F$, we construct functions $f_\alpha^F, g_\alpha^F : F \to \{0,1\}$ such that

1. $f_\alpha^F \upharpoonright \alpha = g_\alpha^F \upharpoonright \alpha$

2. $f_\alpha^F(\alpha) = 0$, $g_\alpha^F(\alpha) = 1$.

We want the functions to be isomorphic and coherent:

3. If $E, F \in \mathcal{F}_k$, $\alpha \in E$ and $\bar{\alpha} = \varphi_{E,F}(\alpha)$ then, $f_{\bar{\alpha}}^F = \varphi_{E,F}(f_\alpha^E)$ and $g_{\bar{\alpha}}^F = \varphi_{E,F}(g_\alpha^E)$.

4. If $E \subset F$, then for every $\alpha \in E$ we have

$$f_\alpha^E \subset f_\alpha^F \quad \text{and} \quad g_\alpha^E \subset g_\alpha^F$$

We can now define $h_\alpha = \bigcup_{F \in \mathcal{F}, \alpha \in F} f_\alpha \upharpoonright \alpha = \bigcup_{F \in \mathcal{F}, \alpha \in F} g_\alpha \upharpoonright \alpha$, and then $(h_\alpha : \alpha < \omega)$ is such that

$$h_\alpha \upharpoonright F = f_\alpha^F \upharpoonright (\alpha \cap F) = g_\alpha^F \upharpoonright (\alpha \cap F) \text{ for every } F \in \mathcal{F} \text{ with } \alpha \in F \qquad (3.1)$$

Note that $h_\alpha : \alpha \to \{0,1\}$ and is well defined by the properties of a construction scheme (Definition 1.32 and Lemma 1.33), and (1)–(4) above. Now let

$$\mathbb{S} = (h_\alpha \upharpoonright \delta : \delta \leq \alpha < \omega_1) \qquad (3.2)$$

then $\mathbb{S}$ is our candidate for Suslin tree.

To recall, we have to construct $(f_\alpha^F, g_\alpha^F : \alpha \in F, F \in \mathcal{F})$ with properties (1)–(4), and show that $\mathbb{S}$ defined as above is a Suslin tree. We do that now.

3.1.2 Proof of Theorem 3.1

We construct $(f_\alpha^F, g_\alpha^F : \alpha \in F, F \in \mathcal{F})$ by recursion on $\mathcal{F}$.

For $F \in \mathcal{F}_0$ we have $F = \{\alpha\}$ and we let $f_\alpha^F(\alpha) = 0$ and $g_\alpha^F(\alpha) = 1$.

Let $F \in \mathcal{F}_k$ with $k > 0$, $R(F) = R$. Suppose $F = \bigcup_{i<n_k} F_i$, the canonical decomposition of F, and for all $i < n_k$, $f_\alpha^{F_i}, g_\alpha^{F_i}$ are defined for all $\alpha \in F_i$ satisfying (1)–(4). Let $\varphi_i : F_0 \to F_i$ be the increasing bijection between F_0 and F_i.

For $\alpha \in R$, let $f_\alpha^F = \bigcup_{i<n_k} \varphi_i(f_\alpha^{F_0})$ and $g_\alpha^F = \bigcup_{i<n_k} \varphi_i(g_\alpha^{F_0})$.

For $\delta \in F_{2i} \setminus R$ and $\delta = \varphi_{2i}(\alpha)$ for some $\alpha \in F_0$ let

$$f_\delta^F = \bigcup_{j \leq 2i} \varphi_j(f_\alpha^{F_0}) \cup \bigcup_{2i < j < n_k} \varphi_j(g_\alpha^{F_0})$$

$$g_\delta^F = \bigcup_{j < 2i} \varphi_j(f_\alpha^{F_0}) \cup \bigcup_{2i \leq j < n_k} \varphi_j(g_\alpha^{F_0})$$

For $\delta \in F_{2i+1} \setminus R$ and $\delta = \varphi_{2i+1}(\alpha)$ for some $\alpha \in F_0$ let

$$f_\delta^F = \bigcup_{j < 2i+1} \varphi_j(g_\alpha^{F_0}) \cup \bigcup_{2i+1 \leq j < n_k} \varphi_j(f_\alpha^{F_0})$$

$$g_\delta^F = \bigcup_{j \leq 2i+1} \varphi_j(g_\alpha^{F_0}) \cup \bigcup_{2i+1 < j < n_k} \varphi_j(f_\alpha^{F_0})$$

By the construction it follows that for every $i < n_k$ and every $\alpha \in F_i$, $f_\delta^{F_i} \subset f_\delta^F$ and $g_\alpha^{F_i} \subset g_\alpha^F$. Also, if $F, E \in \mathcal{F}_k$, $F = \bigcup_{i < n_k} F_i$, and $E = \bigcup_{i < n_k} E_i$ are the canonical decompositions of F and E respectively. Then, by hypothesis, if $\alpha \in E_i$ and $\bar{\alpha} = \varphi_{E_i, F_i}(\alpha)$ we have $f_{\bar{\alpha}}^{F_i} = \varphi_{E_i, F_i}(f_\alpha^{E_i})$, and the same for $g_\alpha^{E_i}$ and $g_{\bar{\alpha}}^{F_i}$. Then for $\alpha \in E$, $\bar{\alpha} = \varphi_{E,F}$ we have $f_{\bar{\alpha}}^F = \varphi_{E,F}(f_\alpha^E)$ and $g_{\bar{\alpha}}^F = \varphi_{E,F}(g_\alpha^E)$. So conditions (1)–(4) are satisfied. This finishes the recursion.

Define $h_\alpha : \alpha \to \{0, 1\}$ by $h_\alpha = \bigcup_{F \in \mathcal{F}, \alpha \in F} f_\alpha \upharpoonright \alpha = \bigcup_{F \in \mathcal{F}, \alpha \in F} g_\alpha \upharpoonright \alpha$. Then $(h_\alpha : \alpha < \omega_1)$ satisfies (3.1). So we are in position to define $\mathbb{S} \subset 2^{<\omega_1}$ as in (3.2). Now $\mathbb{S}$ is a Suslin tree.

Claim 3.7. *If $\mathcal{F}$ is a 3-capturing construction scheme, then $\mathbb{S}$ is a Suslin tree.*

Proof. It is clear that $\mathbb{S}$ has height ω_1 since for every $\alpha < \omega_1$, $h_\alpha \in \mathbb{S}$. Next we see that $\mathbb{S}$ has neither uncountable antichains not uncountable chains.

Let $W = (h_\alpha \upharpoonright \delta_\alpha : \delta_\alpha \leq \alpha, \ \alpha \in \Gamma) \subset \mathbb{S}$ with $\Gamma \subset \omega_1$ uncountable.

There are $\alpha < \beta$ in Γ and $F \in \mathcal{F}$ such that F captures α and β. In particular $\beta = \varphi_1(\alpha)$ and then $h_\alpha \subset h_\beta$ which implies $(h_\alpha \upharpoonright \delta_\alpha) \not\perp (h_\beta \upharpoonright \delta_\beta)$. This implies $\mathbb{S}$ has no uncountable antichains.

In particular, the levels of $\mathbb{S}$ are countable and we can find an uncountable $\Gamma_0 \subset \Gamma$ such that for every $\alpha < \beta$ in Γ_0, $\alpha < \delta_\beta$. Let $F \in \mathcal{F}$, 3-capture Γ_0. Thus there are $\alpha_0 < \alpha_1 < \alpha_2$ in Γ_0 captured by $F = \bigcup_{i < n_k} F_i$. By equation (3.1) we have that $h_{\alpha_1}(\alpha_0) = g_{\alpha_0}^{F_0}(\alpha_0) = 1$ and $h_{\alpha_2}(\alpha_0) = f_{\alpha_0}^{F_0}(\alpha_0) = 0$ and since $\alpha_0 < \delta_{\alpha_1}, \delta_{\alpha_2}$ then $h_{\alpha_1} \perp h_{\alpha_2}$. Thus $\mathbb{S}$ does not have uncountable chains. $\square$

We showed that $\mathbb{S}$ is a Suslin tree. This finishes the proof of Theorem 3.1 $\square$

3.2   A Haudorff T-gap

In this section we construct a T-gap by recursion on $\mathcal{F}$. The structure of the proof is similar to the construction of a Suslin tree on the previous section. We will use $\mathcal{F}$ to recursively define

finite approximations to our T-gap. After the construction is complete we will use 3-capturing
to show it has the T-gap property.

Theorem 3.3. *Assume there is a 3-capturing Construction Scheme. Then there is a* (ω_1, ω_1)-
gap that is a T-gap and so, in particular there is (ω_1, ω_1)-*gap that can be filled in a forcing
extension over a partially ordered set satisfying the countable chain condition.*

Proof of Theorem 3.3. Let $\mathcal{F}$ be a 3-capturing construction scheme. We define a sequence
$(N_k)_{k<\omega}$ in ω and $(a_\alpha^F, b_\alpha^F : \alpha \in F)$ such that

1. For $F \in \mathcal{F}_k$ and every $\alpha \in F$, $a_\alpha^F, b_\alpha^F \subset N_k$ and $a_\alpha^F \cap b_\alpha^F = \emptyset$.

2. For $E, F \in \mathcal{F}_k$, if $\alpha \in E$ and $\bar{\alpha} = \varphi_{E,F}(\alpha)$ then

$$a_\alpha^E = a_{\bar{\alpha}}^F$$
$$b_\alpha^E = b_{\bar{\alpha}}^F$$

3. If $E \subset F$ with $E \in \mathcal{F}_l$, $F \in \mathcal{F}_k$ and $l \leq k$, then

   (a) For every $\alpha \in E$, $\quad a_\alpha^F \cap N_l = a_\alpha^E$ and $b_\alpha^F \cap N_l = b_\alpha^E$.

   (b) For every $\alpha < \beta$ in E, $\quad a_\alpha^F \setminus N_l \subset a_\beta^F$ and $b_\alpha^F \setminus N_l \subset b_\beta^F$.

   (c) For every $\alpha, \beta \in E$, $\quad a_\alpha^F \cap b_\beta^F \subset N_l$.

The construction is as follows. For $F \in \mathcal{F}_0$ we have $F = \{\alpha\}$ for some $\alpha < \omega_1$, let $a_\alpha^F = \{0\}$
and $b_\alpha^F = \{1\}$ and $N_0 = 2$.

Suppose that $(a_\alpha^E, b_\alpha^E : \alpha \in E, E \in \mathcal{F}_l, l < k)$ satisfies (1)–(3). For $F \in \mathcal{F}_k$, if

$$F = \bigcup_{i<n} F_i \text{ is the canonical decomposition of } F.$$

We define $(a_\alpha^F, b_\alpha^F : \alpha \in F)$ as follows

For $\alpha \in R(F)$ let $a_\alpha^F = a_\alpha^{F_0}$ and $b_\alpha^F = b_\alpha^{F_0}$.

For $\delta \in F_{2i} \setminus R(F)$ and $\delta = \varphi_{2i}(\alpha)$ for some $\alpha \in F_0$ let

$$a_\delta^F = a_\alpha^{F_0} \cup \{N_{k-1}\}$$
$$b_\delta^F = b_\alpha^{F_0} \cup \{N_{k-1} + 1\}$$

For $\delta \in F_{2i+1} \setminus R(F)$ and $\delta = \varphi_{2i+1}(\alpha)$ for some $\alpha \in F_0$ let

$$a_\delta^F = a_\alpha^{F_0} \cup \{N_{k-1} + 1\}$$
$$b_\delta^F = b_\alpha^{F_0} \cup \{N_{k-1}\}$$

Finally let $N_k = N_{k-1} + 2$.

It is clear that $a_\alpha^F, b_\alpha^F \subset N_k$ and $a_\alpha^F \cap b_\alpha^F = \emptyset$ so (1) holds. If $E \in \mathcal{F}_k$ with canonical decomposition $E = \bigcup_{i < n_k} E_i$, then for every $\alpha \in E_i$ and $\bar{\alpha} = \varphi_{E_i, F_i}(\alpha)$, we have

$$a_\alpha^{E_i} = a_{\bar{\alpha}}^{F_i} \text{ and } b_\alpha^{E_i} = b_{\bar{\alpha}}^{F_i}$$

and then (2) holds as well. Notice that $a_\alpha^F \cap N_{k-1} = a_\alpha^{F_i}$ for some $i < n_k$, and for every $\alpha < \beta \in F_i$ we have $a_\alpha^F \setminus N_{k-1} = a_\beta^F \setminus N_{k-1}$ and the same for b_α^F. So property (3) holds. This finishes the recursion.

For $\alpha < \omega_1$ let

$$a_\alpha = \bigcup_{F \in \mathcal{F}, \alpha \in F} a_\alpha^F \qquad b_\alpha = \bigcup_{F \in \mathcal{F}, \alpha \in F} b_\alpha^F$$

Conditions (1)–(3) imply that for every $\alpha < \omega_1$, $a_\alpha \cap b_\alpha = \emptyset$, for every $\alpha < \beta$, if $k < \omega$ is large enough (meaning there is $F \in \mathcal{F}_k$ with $\alpha, \beta \in F$) then $a_\alpha \setminus N_k \subset a_\beta$ and $b_\alpha \setminus N_k \subset b_\beta$. Also, for $\alpha, \beta < \omega_1$ and $k < \omega$ large enough $a_\alpha \cap b_\beta \subset N_k$. This shows that $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is a pre-gap.

We use Definition 3.2 to see that $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is a T-gap. In orther words we want to show that, given $\Gamma \subset \omega_1$ uncountable we can find $\alpha_i < \beta_i$ in Γ for $i = 0, 1$, such that $a_{\alpha_0} \cap b_{\beta_0} \neq \emptyset$, and $a_{\alpha_1} \subset a_{\beta_1}$, $b_{\alpha_1} \subset b_{\beta_1}$.

Let $\Gamma \subset \omega_1$ uncountable. Since $\mathcal{F}$ is 3-capturing there is $F \in \mathcal{F}_k$ and $\xi_0 < \xi_1 < \xi_2$ in Γ captured by F i.e, $\xi_i \in F_i \setminus R(F)$ for $i < 3$ and $\xi_j = \varphi_j(\xi_0)$ for $j = 1, 2$. By the construction of a_{ξ_i}, b_{ξ_i}, $i = 0, 1, 2$, we have that $a_{\xi_i} \cap N_k = a_{\xi_i}^F$ and $b_{\xi_i} \cap N_k = b_{\xi_i}^F$. This and (b) of (3) give

$$a_{\xi_0} \cap b_{\xi_1} \neq \emptyset \tag{3.3}$$

$$a_{\xi_0} \subset a_{\xi_2} \quad \text{and} \quad b_{\xi_0} \subset b_{\xi_2} \tag{3.4}$$

Thus we can take $\alpha_i = \xi_0$ for $i = 0, 1$ and $\beta_0 = \xi_1$ and $\beta_1 = \xi_2$. And so equation (3.3) implies $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is a gap and by (3.4) it is a T-gap as we wanted to see. $\square$

3.3   Using Partitions to Capture

It turns out that we can improve the results in the previous sections if we use the partition version of capturing. We construct first a Suslin Tree, however instead of needing 3-capturing as in the previous section, all we need now is 2-$\vec{P}$-capturing. We also present analogous results for T-gaps, given a 2-$\vec{P}$-capturing construction scheme we can construct a T-gap.

Let us recall the definition of partition capturing before we apply it.

Definition 3.8. Let $\omega = \bigcup_{\ell < \omega} P_\ell$ be a partition of ω into infinite components and let $\vec{P} = (P_\ell : \ell < \omega)$. Suppose (m_k, n_k, r_k) forms a type such that for every $\ell < \omega$, and every $r < \omega$ there are infinitely many k's in P_ℓ with $r_k = r$. Then we say $(m_k, n_k, r_k)_k$ forms a $\vec{P}$-type.

Definition 3.9. Let $\mathcal{F}$ be a construction scheme with type $(m_k, n_k, r_k)_k$, and $2 \geq n$. We say $\mathcal{F}$ is n-$\vec{P}$-*capturing* if $(m_k, n_k, r_k)_k$ forms a $\vec{P}$-type, and for every uncountable Δ-system $(s_\xi)_{\xi < \omega_1}$ of finite subsets of ω_1 with root s, and every $\ell < \omega$, there are $\xi_0 < \ldots < \xi_{n-1} < \omega_1$, $k \in P_\ell$ and $F \in \mathcal{F}_k$ with canonical decomposition $F = \bigcup_{i < n_k} F_i$, such that

$$s \subset R(F)$$
$$\text{for every } i < n, \quad s_{\xi_i} \setminus s \subset F_i \setminus R(F),$$
$$\text{for every } i < n, \quad \varphi_i(s_{\xi_0}) = s_{\xi_i}.$$

We say $\mathcal{F}$ is $\vec{P}$-*capturing* if $\mathcal{F}$ is n-$\vec{P}$-capturing for every $n < \omega$.

We construct first a Suslin tree.

Theorem 3.10. *Let $\omega = \bigcup_{\ell < \omega} P_\ell$, be a partition of ω into infinite pieces, and let $\vec{P} = (P_\ell : \ell < \omega)$. Assume there is a Construction Scheme that is 2-$\vec{P}$-capturing, then there is a Suslin tree.*

Proof. We construct $(f_\alpha^F, g_\alpha^F : \alpha \in F, F \in \mathcal{F})$ as above.

For $F \in \mathcal{F}_0$ we have $F = \{\alpha\}$ and we let $f_\alpha^F(\alpha) = 0$ and $g_\alpha^F(\alpha) = 1$.

Let $k \in P_\ell$, and $F \in \mathcal{F}_k$ with $k > 0$. Suppose $F = \bigcup_{i < n_k} F_i$, the canonical decomposition of F, and for all $i < n_k$, $f_\alpha^{F_i}, g_\alpha^{F_i}$ are defined for all $\alpha \in F_i$ satisfying (1)–(4) from Theorem 3.1. Let $\varphi_i : F_0 \to F_i$ be the increasing bijection between F_0 and F_i.

For $\alpha \in R(F)$, let $f_\alpha^F = \bigcup_{i < n_k} \varphi_i(f_\alpha^{F_0})$ and $g_\alpha^F = \bigcup_{i < n_k} \varphi_i(g_\alpha^{F_0})$.

If ℓ is even then

If $\delta \in F_{2i} \setminus R$ and $\delta = \varphi_{2i}(\alpha)$ for some $\alpha \in F_0$ let

$$f_\delta^F = \bigcup_{j \leq 2i} \varphi_j(f_\alpha^{F_0}) \cup \bigcup_{2i < j < n_k} \varphi_j(g_\alpha^{F_0})$$
$$g_\delta^F = \bigcup_{j < 2i} \varphi_j(f_\alpha^{F_0}) \cup \bigcup_{2i \leq j < n_k} \varphi_j(g_\alpha^{F_0})$$

For $\delta \in F_{2i+1} \setminus R$ and $\delta = \varphi_{2i+1}(\alpha)$ for some $\alpha \in F_0$ let

$$f_\delta^F = \bigcup_{j < 2i+1} \varphi_j(g_\alpha^{F_0}) \cup \bigcup_{2i+1 \leq j < n_k} \varphi_j(f_\alpha^{F_0})$$
$$g_\delta^F = \bigcup_{j \leq 2i+1} \varphi_j(g_\alpha^{F_0}) \cup \bigcup_{2i+1 < j < n_k} \varphi_j(f_\alpha^{F_0})$$

Otherwise, ℓ is odd, then,

If $\delta \in F_i \setminus R$ and $\delta = \varphi_i(\alpha)$ for some $\alpha \in F_0$ let

$$f_\delta^F = \bigcup_{j \leq i} \varphi_j(f_\alpha^{F_0}) \cup \bigcup_{i < j < n_k} \varphi_j(g_\alpha^{F_0})$$

$$g_\delta^F = \bigcup_{j < i} \varphi_j(f_\alpha^{F_0}) \cup \bigcup_{i \leq j < n_k} \varphi_j(g_\alpha^{F_0})$$

Now, instead of using equation (3.1) to define $h_\alpha : \alpha \to 2$. We let $h_\alpha : \alpha + 1 \to 2$.

$$h_\alpha = f_\alpha \restriction (\alpha + 1) \text{ for every } F \in \mathcal{F} \text{ with } \alpha \in F.$$

Now we let

$$\mathbb{S} = \{h_\alpha \restriction \delta : \delta \leq \alpha + 1 < \omega_1\}$$

This defines $\mathbb{S}$. To see that $\mathbb{S}$ is a Suslin tree we proceed as in Theorem 3.1.

Let $W = (h_\alpha \restriction \delta_\alpha : \delta_\alpha \leq \alpha + 1, \alpha \in \Gamma)$, with $\Gamma \subset \omega_1$ uncountable, be an uncountable subset of $\mathbb{S}$. We use 2-$\vec{P}$-capturing to find $\alpha_0 < \alpha_1$ in Γ and $F \in \mathcal{F}_k$ with $k \in P_1$, such that F captures α_0 and α_1. Since $\ell = 1$ is odd, we have $h_{\alpha_0} \subset h_{\alpha_1}$ by definition therefore $(h_{\alpha_0} \restriction \delta_{\alpha_0}) \not\perp (h_{\alpha_1} \restriction \delta_{\alpha_1})$.

Thus $\mathbb{S}$ has no uncountable antichains, therefore the levels of $\mathbb{S}$ are countable. This means we can assume that $\alpha < \delta_\beta$ for all $\alpha < \beta$ in Γ. Also, without loss of generality, we can assume that $\delta_\alpha = \alpha + 1$ for every $\alpha \in \Gamma$. Indeed suppose $\delta_\alpha < \alpha + 1$ for every $\alpha \in \Gamma$, by the Pressing Down Lemma we can find an uncountable $\Gamma_0 \subset \Gamma$ such that $\delta_\alpha = \delta_\beta$ for all $\alpha, \beta \in \Gamma_0$. But then we can find $\alpha < \beta$ in Γ_0 such that $\alpha \geq \delta_\beta$, which is a contradiction.

So we assume $\delta_\alpha = \alpha + 1$ for every $\alpha \in \Gamma$. Now we use 2-$\vec{P}$-capturing to find $\xi_0 < \xi_1$ in Γ and $F \in \mathcal{F}_k$ with $k \in P_2$, such that F captures $\xi_0 < \xi_1$. Since $\ell = 2$ is even, then we have $h_{\xi_0}(\xi_0) = f_{\xi_0}^F(\xi_0) = 0$, and $h_{\xi_1}(\xi_0) = f_{\xi_1}^F(\xi_0) = g_{\xi_0}^{F_0}(\xi_0) = 1$. Therefore W cannot be an uncountable chain.

This finishes the proof.   □

As a Corollary we also have the following result.

Corollary 3.11. *Assume MA_{ω_1}. There are no 2-$\vec{P}$-capturing construction schemes for any partition $\vec{P}$.*

Now we show how to construct Hausdorff T-gaps from a 2-$\vec{P}$-capturing construction scheme.

Theorem 3.12. *Let $\omega = \bigcup_{\ell < \omega} P_\ell$ be a partition of ω into infinite components, and set $\vec{P} = (P_n : n < \omega)$. Assume there is a Construction Scheme that is 2-$\vec{P}$-capturing, then there is a T-gap.*

Proof. We follow the previous construction. For $F \in \mathcal{F}_0$ we have $F = \{\alpha\}$ for some $\alpha < \omega_1$, let $a_\alpha^F = \{0\}$ and $b_\alpha^F = \{1\}$ and $N_0 = 2$.

Suppose that $(a_\alpha^E, b_\alpha^E : \alpha \in E, E \in \mathcal{F}_l)$ for $l < k$ satisfies (1)–(3) from the proof of Theorem 3.3. Take $\ell < \omega$ such that $k \in P_\ell$, and let $F \in \mathcal{F}_k$ be given, if

$$F = \bigcup_{i<n} F_i \text{ is the canonical decomposition of } F.$$

If ℓ is even, define $(a_\alpha^F, b_\alpha^F : \alpha \in F)$ as follows

For $\alpha \in R(F)$ let $a_\alpha^F = a_\alpha^{F_0}$ and $b_\alpha^F = b_\alpha^{F_0}$.

For $\delta \in F_{2i} \setminus R(F)$ and $\delta = \varphi_{2i}(\alpha)$ for some $\alpha \in F_0$ let

$$a_\delta^F = a_\alpha^{F_0} \cup \{N_{k-1}\}$$
$$b_\delta^F = b_\alpha^{F_0} \cup \{N_{k-1} + 1\}$$

For $\delta \in F_{2i+1} \setminus R(F)$ and $\delta = \varphi_{2i+1}(\alpha)$ for some $\alpha \in F_0$ let

$$a_\delta^F = a_\alpha^{F_0} \cup \{N_{k-1} + 1\}$$
$$b_\delta^F = b_\alpha^{F_0} \cup \{N_{k-1}\}$$

Otherwise, ℓ is odd, and we define $(a_\alpha^F, b_\alpha^F : \alpha \in F)$ as

For $\alpha \in R(F)$ let $a_\alpha^F = a_\alpha^{F_0}$ and $b_\alpha^F = b_\alpha^{F_0}$.

For $\delta \in F_i \setminus R(F)$ and $\delta = \varphi_i(\alpha)$ for some $\alpha \in F_0$ let

$$a_\delta^F = a_\alpha^{F_0} \cup \{N_{k-1}\}$$
$$b_\delta^F = b_\alpha^{F_0} \cup \{N_{k-1} + 1\}$$

Finally let $N_k = N_{k-1} + 2$.

Given $\Gamma \subset \omega_1$ uncountable. We can find $\xi_0 < \xi_1$ in Γ and $F \in \mathcal{F}_k$ with $k \in P_2$ such that F captures ξ_0 and ξ_1. Since $\ell = 2$ is even, we have $N_{k-1} \in a_{\xi_0}^F \cap b_{\xi_1}^F$. Thus $(a_\alpha, b_\alpha : \alpha < \omega_1)$ is a gap.

On the other hand, we can also find $\alpha_0 < \alpha_1$ in Γ and $F \in \mathcal{F}_k$ with $k \in P_1$ such that F captures α_0 and α_1. Then, since $\ell = 1$ is odd, we have $a_{\alpha_0} \subset a_{\alpha_1}$ and $b_{\alpha_0} \subset b_{\alpha_1}$. Therefore $(a_\alpha, b_\alpha : \omega_1)$ is a T-gap. $\square$

3.4   Hausdorff T-gaps versus Hausdorff S-gaps

Recall that an S-gap is a destructible gap, i.e, a gap which can be split by a ccc forcing. The purpose of this Section is to prove the following.

Theorem 3.5. *There is a model in which there is an S-gap but which does not have any T-gaps.*

Proof. We start with a ground model in which GCH holds and has an S-gap.

Let $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ be a gap with the property that $a_\beta \not\subset a_\alpha$ for any $\alpha < \beta < \omega_1$. It is clear that every gap is equivalent to a gap with this property. Let $\mathbf{A} = (a_\alpha)_{\alpha < \omega_1}$ and consider the following forcing notion

$$\mathbb{P}_{\mathbf{A}} = \{p \in [\mathbf{A}]^{<\omega} : (\forall x \neq y \in p)\ x \not\subset y \text{ and } y \not\subset x\}$$

ordered by reversed inclusion.

Claim 3.13. $\mathbb{P}_{\mathbf{A}}$ *is ccc.*

Proof. Let $(p_\alpha)_{\alpha < \omega_1}$. Applying the Δ-system Lemma we can assume that the p_α's are a disjoint with $|p_\alpha| = n$ and $p_\alpha = (x_{\alpha,i})_{i<n}$ for every $\alpha < \omega_1$ where we preserved the natural order in $\mathbf{A}$. This implies that $x_{\beta,j} \not\subset x_{\alpha,i}$ for $\alpha < \beta$ and $i, j < n$.

Let M be a countable elementary submodel of $H_{\mathfrak{c}^+}$ and $\gamma = \omega_1 \cap M$.

Take $\beta > \gamma$ and fix $k < \omega$ such that

$$x_{\beta,i} \cap k \not\subset x_{\gamma,i} \quad \forall i < n. \tag{3.5}$$

Consider $\Gamma = \{\alpha < \omega_1 : x_{\alpha,i} \cap k = x_{\beta,i} \cap k \quad \forall i < n\}$, then $\Gamma \in M$ and $\beta \in \Gamma$. Therefore Γ is uncountable. Take $\alpha \in M \cap \Gamma$, by (3.5)

$$x_{\alpha,i} \not\subset x_{\gamma,i} \quad \forall i < n$$

and $p_\alpha \cup p_\gamma \in \mathbb{P}_{\mathbf{A}}$ witness $p_\alpha \not\perp p_\gamma$. $\qquad\qquad\square$

We will force a model where MA_{ω_1} holds for a forcing of the form $\mathbb{P}_{\mathbf{A}}$. First, fix a bijective mapping $\pi : \omega_2 \to \omega_2 \times \omega_2$ where $\pi(\alpha) = (\beta, \gamma)$ with $\beta \leq \alpha$. This is the usual book keeping mapping. Suppose we have $\mathbb{P}_\lambda = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \lambda \rangle$ a finite support iteration with

$$\mathbb{P}_\alpha \Vdash \text{``}\dot{\mathbb{Q}}_\alpha = \mathbb{P}_{\dot{\mathbf{A}}} \text{ if } \dot{\mathbf{A}} \text{ is a gap''}.$$

for some $\dot{\mathbf{A}} \in V^{\mathbb{P}_\alpha}$. Then, in $V^{\mathbb{P}_\lambda}$ there are $\aleph_2$ many names for gaps (by GCH), and we can fix a well-ordering of them. If $\pi(\lambda) = (\beta, \gamma)$, let $\dot{\mathbf{A}}$ be the γth name for a gap in $V^{\mathbb{P}_\beta}$. If $\dot{\mathbf{A}}$ is a gap in $V^{\mathbb{P}_\lambda}$ then let $\dot{\mathbb{Q}}_\lambda = \mathbb{P}_{\dot{\mathbf{A}}}$.

Claim 3.14. *The finite support iteration $\mathbb{P}_{\omega_2}$ is ccc and forces MA_{ω_1} for orderings of the form $\mathbb{P}_{\mathbf{A}}$.*

Proof. Let $\mathbf{A}$ and $\vec{\mathcal{D}} = (D_\alpha : \alpha < \omega_1)$ be a gap and a collection of dense sets of $\mathbb{P}_{\dot{\mathbf{A}}}$ in $V[G_{\omega_2}]$ respectively. Then, there is $\lambda < \omega_2$ such that both $\mathbf{A}$ and $\vec{\mathcal{D}}$ are in $V[G_\lambda]$. Since $\mathbf{A}$ is a gap in $V[G_{\omega_2}]$ then is a gap in $V[G_\lambda]$ and there is $\xi \geq \lambda$ such that $\pi(\xi) = (\lambda, \gamma)$ and the γth name in

$V^{\mathbb{P}_\lambda}$ is a name for $\mathbf{A}$. It follows that there is a $\vec{\mathcal{D}}$-generic filter in $V[G_{\xi+1}] \subset V[G_{\omega_2}]$ and the proof is finished. $\qquad \square$

This applied to a gap $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ forces $\Gamma \subset \omega_1$ uncountable without the property in Definition 3.3. This shows that there are no T-gaps. Thus, the proof is finished once we show the following.

Claim 3.15. *Forcing with $\mathbb{P}_\mathbf{A}$ preserves S-gaps.*

Proof. Suppose that one $\mathbb{P}_\mathbf{A}$ kills an S-gap $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$.

Then $\mathbb{P}_\mathbf{A}$ forces $\dot{\Gamma} \subset \omega_1$ uncountable without property (3) of Proposition 1.13 i.e, for every $\alpha < \beta$

$$\mathbb{P}_\mathbf{A} \Vdash \alpha, \beta \in \dot{\Gamma} \Rightarrow (a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \emptyset$$

Since $\dot{\Gamma}$ is uncountable we can find (in the ground model) $\Gamma \subset \omega_1$ uncountable and $(p_\alpha : \alpha \in \Gamma) \subset \mathbb{P}_\mathbf{A}$ such that

$$p_\alpha \Vdash \alpha \in \dot{\Gamma}$$

In particular, we have

$$\forall \alpha < \beta \in \Gamma \quad \left((a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset \Longrightarrow p_\alpha \cup p_\beta \notin \mathbb{P}_\mathbf{A} \right) \tag{3.6}$$

We may assume that the p_γ's are disjoint and that they all have some fixed size n and $p_\alpha = (x_{\alpha,i})_{i < n}$ preserves the natural order in $\mathbf{A}$.

Choose a countable elementary sub-model M of H_{c^+} containing all these objects and let $\gamma = \min(\Gamma \setminus M)$.

Since $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is an S-gap, the elementarity of M gives us the existence of a $\beta \in \Gamma$ above γ such that

$$a_\beta \cap b_\gamma = \emptyset \text{ and } a_\gamma \cap b_\beta = \emptyset \tag{3.7}$$

Choose $k < \omega$ such that

$$a_\gamma \setminus k \subseteq a_\beta \text{ and } b_\gamma \setminus k \subseteq b_\beta \tag{3.8}$$

$$\forall x \in p_\gamma \quad \forall y \in p_\beta \quad y \cap k \nsubseteq x \cap k \tag{3.9}$$

Let $s = a_\beta \cap k$, $t = b_\beta \cap k$ and

$$\Gamma_0 = \{\alpha \in \Gamma : a_\alpha \cap k = s \quad b_\alpha \cap k = t \quad x_{\alpha,i} \cap k = x_{\beta,i} \cap k (i < n)\}$$

Then $\Gamma_0 \in M$ and $\beta \in \Gamma_0 \setminus M$ so Γ_0 is an uncountable subset of Γ. Since $(a_\alpha, b_\alpha)_{\alpha < \omega_1}$ is a S-gap there must exist $\alpha \in \Gamma_0 \cap M$ such that

$$a_\alpha \cap b_\beta = \emptyset \qquad a_\beta \cap b_\alpha = \emptyset \tag{3.10}$$

Combining equations $(3.7),(3.8)$ and (3.10) we obtain that

$$a_\alpha \cap b_\gamma = \emptyset \qquad a_\gamma \cap b_\alpha = \emptyset \tag{3.11}$$

Form the fact that $\alpha \in \Gamma_0$ and by (3.9) we conclude that

$$\forall x \in p_\gamma \quad \forall y \in p_\alpha \quad y \not\subseteq x \tag{3.12}$$

Thus $p_\alpha \cup p_\gamma \in \mathbb{P}_W$, contradicting (3.6). $\square$

The previous claim also implies that if $\mathbb{P}_\alpha$ preserves S-gaps, then so does $\mathbb{P}_{\alpha+1}$. Suppose now $\mathbb{P}_\alpha$ preserves S-gaps for every $\alpha < \lambda \le \omega_2$, with λ limit. If $\mathbb{P}_\lambda$ kills an S-gap, applying the Δ-system lemma (or a counting argument in case α has countable cofinality) we find $\eta < \lambda$ such that $\mathbb{P}_\eta$ kills an S-gap. Contradiction, thus $\mathbb{P}_\lambda$ also preserves S-gaps.

This shows that $V[G_{\omega_2}]$ contains an S-gap, since V does and $\mathbb{P}_{\omega_2}$ preserves it, and there are no T-gaps in $V[G_{\omega_2}]$ which finish the proof. $\square$

Remark 3.1. The method above answers a particular case of Problem 59 of [BNC15]. Particularly, it produces a model of with no Suslin towers but destructible gaps.

It can be seen that the model $V[G_{\omega_2}]$ also contains Suslin trees and it has no 3-capturing construction schemes.

Chapter 4

Constructions of Banach spaces

The purpose of this Chapter is to apply the construction scheme to the theory of nonseparable Banach spaces inspired by the forcing constructions of Bell, Ginsburg and Todorčević [BGT82], and López-Abadand Todorčević [LAT11]. The class of nonseparable Banach spaces exhibit phenomena which are not present in the more studied class of separable Banach spaces. Some of the most striking differences appear on J. López-Abadand S. Todorčević[LAT11] where they developed forcing constructions of Banach spaces via finite-dimensional approximations. Here are two examples from [LAT11]

Theorem 4.1 (Theorem 4.5 of [LAT11])**.** *For every $\varepsilon > 0$ rational, there is a poset $\mathbb{P}_\varepsilon$ which forces a Banach space $\mathcal{Y}_\varepsilon$ with an uncountable ε-biorthogonal system and such that for every $0 \leq \tau < \frac{\varepsilon}{1+\varepsilon}$, $\mathcal{Y}_\varepsilon$ has no uncountable τ-biorthogonal system.*

Theorem 4.2 (Theorem 6.4 of [LAT11])**.** *For every constant $K > 1$ there is a poset $\mathbb{P}_K$ which forces a Banach space $\mathcal{Y}_K$ with an uncountable K-basis yet for every $1 \leq K' < K$, $\mathcal{Y}_K$ has no uncountable K'-basic sequences.*

Recall that none of these two phenomena can happen in the class of separable Banach spaces when, of course, we replace 'uncountable' by 'infinite' since every infinite dimensional space has a basic sequence, hence it has a biorthogonal system. Also, the previous spaces, or other spaces with the same properties cannot exists on a model of PFA by the following result of Todorčević.

Theorem 4.3 (Todorčević [Tod06])**.** *Assume PFA. Every Banach space $\mathcal{X}$ of density $\aleph_1$ has an uncountable Biorthogonal System.*

We use capturing construction schemes to build *norming* sets and define nonseparable Banach spaces. The main results of this Chapter are the following Theorems.

Theorem 4.4. *Assume there is a capturing construction scheme $\mathcal{F}$. Then for every $\varepsilon \in (0,1) \cap \mathbb{Q}$, there is a Banach space $\mathcal{X}_\varepsilon$ with an uncountable ε-biorthogonal system but no uncountable τ-biorthogonal system for every $0 \leq \tau < \frac{\varepsilon}{1+\varepsilon}$.*

Theorem 4.5. *Assume there is a capturing construction scheme $\mathcal{F}$. Then for every constant $K > 1$, there is a Banach space $\mathcal{X}_K$ with a K-basis of length ω_1 but no uncountable K'-basic sequence for every $1 \leq K' < K$.*

In each case the construction is based on a single rule of multiple amalgamation of a family of finite-dimensional Banach spaces indexed by $\mathcal{F}$. This adds not only to the clarity over the corresponding forcing constructions but it also gives us Banach spaces that could be further easily analyzed. In fact neither the construction nor the analysis of the corresponding examples require any expertise outside the Banach space geometry.

It is interesting to compare our examples with the corresponding examples in [LAT11]. Given an uncountable sequence of forcing conditions, take an uncountable Δ-subsequence where all conditions are isomorphic and find a condition which amalgamates finitely many of these forcing the desired inequality. Thus, the use of forcing allows us to amalgamate a posteriori since the generic filter G takes care of all the possible Δ-systems whose roots belong to G. However in our recursive construction the amalgamations must be done a priori which limits the class of possible amalgamations. In fact since we do a single amalgamation at any given level of $\mathcal{F}$, our spaces tend to be considerably more homogeneous and therefore much easier to analyze.

This Chapter is structured in the following way: We start with a result of Todorčević [Tod17] that illustrates how capturing construction schemes can be used to construct Banach spaces. The use of a $\vec{P}$-capturing construction scheme is used to build a $C(K)$ Banach space that has no uncountable biorthogonal systems.

On Section 4.2 we give an outline of the constructions of $\mathcal{X}_\varepsilon$ and $\mathcal{X}_K$. The aim is to point out how the construction of Banach spaces with properties independent of ZFC can be systematize, and the Set Theory necessary to carry out this constructions is not a lot.

In Section 4.3 we give a proof of Theorem 4.4 and study some of the geometric properties of $\mathcal{X}_\varepsilon$.

We finish the Chapter with Section 4.4, where we prove Theorem 4.5.

4.1   First Applications to Banach spaces

We want to present the first application of capturing construction schemes to Banach spaces due to Todorčević [Tod17]. It illustrates the method that we will use to prove Theorem 4.4 and Theorem 4.5. Our aim is to convince the reader of the flexibility of this method to construct nonseparable Banach spaces.

The following concept was introduced by Rolewicz [Rol78]

Definition 4.6. Let $C \subset \mathcal{X}$ be nonempty, convex, and closed. We say C is a *support set* if for every point $x \in C$ there is a $x^* \in \mathcal{X}^*$ such that $x^*(x) = \inf\{x^*(y) : y \in C\} < \sup\{x^*(z) : z \in C\}$.

There is a relation between support set and uncountable biorthogonal systems. To make this relation precise we need a new definition.

Definition 4.7. Let $\mathcal{X}$ be a Banach space, a sequence $(y_\alpha, y_\alpha^*)_{\alpha < \omega_1}$ is called a *semibiorthogonal system* if the following conditions hold:

1. For every $\alpha < \omega_1$, $y_\alpha^*(y_\alpha) = 1$,

2. for every $\alpha < \beta < \omega_1$, $y_\beta^*(y_\alpha) = 0$, and

3. for every $\alpha < \beta < \omega_1$, $y_\alpha^*(y_\beta) \geq 0$.

We have the following relation.

Theorem 4.8 (Borwein and Vanderwerff [BV96]). *Let $\mathcal{X}$ be a Banach space, then $\mathcal{X}$ has a support set if and only if it has an uncountable semi-biorthogonal system.*

Theorem 4.9 (Todorčević [Tod17]). *Assume there is a $\vec{P}$-capturing construction scheme $\mathcal{F}$. There is a compact space K such that $C(K)$ does not have uncountable semi-biorthogonal sequences. In particular $C(K)$ contains no supported closed convex subsets.*

Proof. Fix $\mathcal{F}$, a $\vec{P}$-capturing construction scheme. We construct K as a subset of $\{0,1\}^{\omega_1}$ and endow K with the induced product topology. For every $F \in \mathcal{F}$ we define $K_F \subset \{0,1\}^F$ and $(f_\alpha^F, g_\alpha^F : \alpha \in F)$ with the following properties:

(i) For $\alpha \in F$ we have $f_\alpha^F \restriction \alpha = g_\alpha^F \restriction \alpha$, $f_\alpha^F(\alpha) = 0$, and $g_\alpha^F(\alpha) = 1$,

1. For $F, E \in \mathcal{F}_k$ the spaces K_F and K_E are homeomorphic via the mapping $f \in K_F \mapsto f \circ \varphi_{F,E}^{-1}$, and

2. For $F, E \in \mathcal{F}$ with $E \subset F$ we have $K_E \subset K_F$.

Now we define K_F by recursion on $F \in \mathcal{F}$. For $\alpha < \omega_1$ and $F = \{\alpha\} \in \mathcal{F}_0$, we define $K_{\{\alpha\}} = \{f_\alpha^F, g_\alpha^F\}$. Where $f_\alpha^F(\alpha) = 0$ and $g_\alpha^F(\alpha) = 1$.

Suppose now $F \in \mathcal{F}_k$ with canonical decomposition $F = \bigcup_{i < n_k} F_i$ and we have constructed $K_{F_i} = \{f_\alpha^{F_i}, g_\alpha^{F_i} : \alpha \in F_i\}$ for $i < n_k$, so they satisfy (i)–(iii).

Suppose $k \in P_\ell$ with $n_k < 3\ell$. Then we let K_F contain the following functions.

1. For $\alpha \in F_0$, define

$$f_\alpha^F := f_\alpha^{F_0} + \sum_{0 < i < n_k} \varphi_i(f_\alpha^{F_0} \restriction (F_i \setminus F_0))$$

$$g_\alpha^F := g_\alpha^{F_0} + \sum_{0 < i < n_k} \varphi_i(g_\alpha^{F_0} \restriction (F_i \setminus F_0))$$

2. For $\alpha \in F_i \setminus R(F)$ with $0 < i \leq n_k$, let $\delta \in F_0 \setminus R(F)$ with $\varphi_i(\delta) = \alpha$, then

$$f_\alpha^F := f_\delta^{F_0} + f_\alpha^{F_i} + \sum_{\substack{0 < j < n_k \\ j \neq i}} \varphi_j(f_\delta^{F_0}) \restriction (F_j \setminus F_0)$$

$$g_\alpha^F := f_\delta^{F_0} + g_\alpha^{F_i} + \sum_{\substack{0 < j < n_k \\ j \neq i}} \varphi_j(f_\delta^{F_0}) \restriction (F_j \setminus F_0)$$

Suppose otherwise $k \in P_\ell$ with $n_k \geq 3\ell$. Then we let K_F contain the following functions.

1. For $\alpha \in F_0$, define

$$f_\alpha^F := f_\alpha^{F_0} + \sum_{0 < i < n_k} \varphi_i(f_\alpha^{F_0} \upharpoonright (F_i \setminus F_0))$$

$$g_\alpha^F := g_\alpha^{F_0} + \sum_{0 < i < n_k} \varphi_i(g_\alpha^{F_0} \upharpoonright (F_i \setminus F_0))$$

2. For $\alpha \in F_i \setminus R(F)$ with $0 < i \leq n_k$ and $i \neq 2\ell$, let $\delta \in F_0 \setminus R(F)$ with $\varphi_i(\delta) = \alpha$, then

$$f_\alpha^F := f_\delta^{F_0} + f_\alpha^{F_i} + \sum_{\substack{0 < j < n_k \\ j \neq i}} \varphi_j(f_\delta^{F_0}) \upharpoonright (F_j \setminus F_0)$$

$$g_\alpha^F := f_\delta^{F_0} + g_\alpha^{F_i} + \sum_{\substack{0 < j < n_k \\ j \neq i}} \varphi_j(f_\delta^{F_0}) \upharpoonright (F_j \setminus F_0)$$

3. For $\alpha \in F_{2\ell} \setminus R(F)$, let $\delta \in F_0 \setminus R(F)$ with $\varphi_{2\ell}(\delta) = \alpha$, then

$$f_\alpha^F := \left(f_\delta^{F_0} + \varphi_1(g_\delta^{F_0})\right) + \sum_{j < \ell} \left(\left(\varphi_{2j}(f_\delta^{F_0}) \upharpoonright (F_j \setminus F_0)\right) + \left(\varphi_{2j+1}(g_\delta^{F_0}) \upharpoonright (F_j \setminus F_0)\right)\right)$$

$$+ f_\alpha^{F_{2\ell}} + \sum_{j > 2\ell} \varphi_j(g_\delta^{F_0}) \upharpoonright (F_j \setminus F_0)$$

$$g_\alpha^F := \left(f_\delta^{F_0} + \varphi_1(g_\delta^{F_0})\right) + \sum_{j < \ell} \left(\left(\varphi_{2j}(f_\delta^{F_0}) \upharpoonright (F_j \setminus F_0)\right) + \left(\varphi_{2j+1}(g_\delta^{F_0}) \upharpoonright (F_j \setminus F_0)\right)\right)$$

$$+ g_\alpha^{F_{2\ell}} + \sum_{j > 2\ell} \varphi_j(f_\delta^{F_0}) \upharpoonright (F_j \setminus F_0)$$

It is clear by the construction that $K_F = \{f_\alpha^F, g_\alpha^F : \alpha \in F\}$ defined as above satisfy conditions (i)–(iii). This finishes the construction of K.

Suppose $C(K)$ has an uncountable semi-biorthogonal sequence $(y_\alpha, \mu_\alpha)_{\alpha < \omega_1}$. We can assume the y_α's are normalized. Then μ_α are operators on $C(K)$ such that:

1. $\int y_\alpha d\mu_\alpha = 1$ for every $\alpha < \omega_1$,

2. $\int y_\alpha d\mu_\beta = 0$ for every $\alpha < \beta < \omega_1$, and

3. $\int y_\beta d\mu_\alpha \geq 0$ for every $\alpha < \beta < \omega_1$.

Lemma 4.10. *There is $\Gamma \subset \omega_1$ uncountable and $n < \omega$, such that for every $\alpha_0 < \ldots \alpha_{3n}$ in Γ we have:*

$$\left\| -\sum_{i=0}^{2n-1} y_{\alpha_i} + n y_{\alpha_{2n}} + \sum_{i=2n+1}^{3n} y_{\alpha_i} \right\| \geq 4$$

Proof. Let $N < \omega$ and $\Gamma \subset \omega_1$ uncountable such that

$$\sup_{\alpha \in \Gamma} \|\mu_\alpha\| \leq N$$

Then

$$\left\| -\sum_{i=0}^{2n-1} y_{\alpha_i} + n y_{\alpha_{2n}} + \sum_{i=2n+1}^{3n} y_{\alpha_i} \right\| \geq \left| \frac{1}{N} \int \left(-\sum_{i=0}^{2n-1} y_{\alpha_i} + n y_{\alpha_{2n}} + \sum_{i=2n+1}^{3n} y_{\alpha_i} \right) d\mu_{\alpha_{2n}} \right|$$

$$= \frac{1}{N} \int \left(-\sum_{i=0}^{2n-1} y_{\alpha_i} \right) d\mu_{\alpha_{2n}} + \frac{n}{N} \int y_{\alpha_{2n}} d\mu_{\alpha_{2n}} + \frac{1}{N} \int \left(\sum_{i=2n+1}^{3n} y_{\alpha_i} \right) d\mu_{\alpha_{2n}}$$

$$\geq 0 + \frac{n}{N} + 0 = \frac{n}{N}$$

Because of the properties of $\mu_{\alpha_{2n}}$. Take $n \geq 4N$ and we get the result. $\square$

We finish the proof by showing that

Lemma 4.11. *For every normalized uncountable sequence $(y_\alpha)_{\alpha \in \Gamma}$ in $C(K)$ and $n < \omega$. There are $\alpha_0 < \ldots < \alpha_{3n}$ in Γ such that*

$$\left\| -\sum_{i=0}^{2n-1} y_{\alpha_i} + n y_{\alpha_{2n}} + \sum_{i=2n+1}^{3n} y_{\alpha_i} \right\| \leq 3$$

Let $\mathscr{D}$ be the algebra of all functions in $C(K)$ generated by the constant function, and the functions of the form $\delta_\alpha(h) = h(\alpha)$, where $h \in K$ and $\alpha < \omega_1$. Then $\mathscr{D}$ is dense in $C(K)$ by the Stone-Weierestrass Theorem. For $\alpha \in \Gamma$ let $x_\alpha \in \mathscr{D}$ normalized with rational coefficients such that

$$\|x_\alpha - y_\alpha\| < \frac{1}{4n}$$

Then for every $\alpha_0 < \ldots < \alpha_{3n}$ in Γ we have

$$\left\| -\sum_{i=0}^{2n-1} y_{\alpha_i} + n y_{\alpha_{2n}} + \sum_{i=2n+1}^{3n} y_{\alpha_i} \right\| \leq \left\| -\sum_{i=0}^{2n-1} x_{\alpha_i} + n x_{\alpha_{2n}} + \sum_{i=2n+1}^{3n} x_{\alpha_i} \right\| + \sum_{\substack{i \leq 3n \\ i \neq 2n}} \|y_{\alpha_i} - x_{\alpha_i}\| + n\|y_{\alpha_{2n}} - x_{\alpha_{2n}}\|$$

$$< \left\| -\sum_{i=0}^{2n-1} x_{\alpha_i} + n x_{\alpha_{2n}} + \sum_{i=2n+1}^{3n} x_{\alpha_i} \right\| + \frac{3n}{4n} + \frac{n}{4n}$$

$$= \left\| -\sum_{i=0}^{2n-1} x_{\alpha_i} + n x_{\alpha_{2n}} + \sum_{i=2n+1}^{3n} x_{\alpha_i} \right\| + 1$$

Thus is enough to show:

$$\left\| -\sum_{i=0}^{2n-1} x_{\alpha_i} + n x_{\alpha_{2n}} + \sum_{i=2n+1}^{3n} x_{\alpha_i} \right\| \leq 2$$

To see this is true, consider $(\mathrm{supp}(x_\alpha) : \alpha < \omega_1)$. By going to an uncountable subsequence, we can assume that:

1. The sequence $(\mathrm{supp}(x_\alpha) : \alpha < \omega_1)$ forms an increasing Δ-System and

2. For every $\alpha < \beta$, x_α is isomorphic to x_β. This means that, if $\varphi_{\alpha\beta} : \mathrm{supp}(x_\alpha) \to \mathrm{supp}(x_\beta)$ is the increasing bijection. Then $x_\beta = x_\alpha \circ \varphi_{\alpha\beta}^{-1}$.

Since $\mathcal{F}$ is a $\vec{P}$-capturing construction scheme, we can find $k \in \mathbb{P}_{n_0}$, $F \in \mathcal{F}_k$, and $\alpha_0 < \ldots < \alpha_{3n_0} < \omega_1$ such that F captures $(\mathrm{supp}(x_{\alpha_i}) : i \leq 3n_0)$. Let

$$\mathbf{w} = -\sum_{i=0}^{2n_0-1} x_{\alpha_i} + n_0 x_{\alpha_{2n_0}} + \sum_{i=2n_0+1}^{3n_0} x_{\alpha_i}$$

To finish the proof we have to show that $|\mathbf{w}(h)| \leq 2$ for every $h \in K$. Since $\mathrm{supp}(\mathbf{w}) \subset F$ it is enough to show that $|\mathbf{w}(h) \leq 2$ for every $h \in K_F$. Note that $F \in \mathcal{F}_k$, $k \in P_{n_0}$, and $n_k > 3n_0$ since F captured the Δ-System of lenght $3n_0$. Thus, going back to the corresponding definitions of $h \in K_F$ we have that, if $f_\alpha^F \in K_F$ is of the form 1 then

$$\mathbf{w}(f_\alpha)^F = (-2n_0 + n_0 + n_0)x_{\alpha_0}(f_\alpha^{F_0}) = 0$$

the case for g_α^F is analogous. If $f_\alpha^F \in K_F$ is of the form 3, then

$$\mathbf{w}(f_\alpha^F) = (-n_0 + n_0)x_{\alpha_0}(f_\delta^{F_0}) + (-n_0 + n_0)x_{\alpha_0}(g_\delta^{F_0}) = 0$$

If $f_\alpha^F \in K_F$ is of the form 2, then $\mathbf{w}(f_\alpha^F) = 0$. Finally for $g_\alpha \in K_F$ of the form 2, then

$$|\mathbf{w}(g_\alpha^F)| = |x_{\alpha_0}(f_\delta^{F_0} - g_\delta^{F_0})| \leq 2$$

since x_{α_0} has norm 1. This finishes the proof.

$\square$

4.2   Outline of the proofs

The construction of the Banach spaces $\mathcal{X}_\varepsilon$ and $\mathcal{X}_K$ will follow an abstract approach for producing nonseparable Banach structures.

We start with a capturing construction scheme $\mathcal{F}$. First, we construct (recursively) a family $\mathcal{H} = \bigcup_{F \in \mathcal{F}} \mathcal{H}_F$ where $\mathcal{H}_F$ are functions $f : F \to [0,1] \cap \mathbb{Q}$. For $\mathcal{X}_\varepsilon$ we will have $\mathcal{H}_F = \{h_\alpha^F : \alpha \in F\}$. To guarantee nonseparability we want to have the following condition

$$h_\alpha^F \upharpoonright \alpha = 0 \qquad h_\alpha^F(\alpha) = 1 \tag{4.1}$$

The role of $\mathcal{H}$ is to be a norming set, for that we need the following coherence conditions

$$\forall F, E \in \mathcal{F} \quad \text{if } E \subset F \text{ then} \quad h_\alpha^F \restriction E = h_\alpha^F \quad \forall \alpha \in E \tag{4.2}$$

$$\forall F, E \in \mathcal{F} \quad \text{if } E \subset F \text{ then} \quad f \restriction E \in \text{conv}(\pm\mathcal{H}_E) \quad \forall f \in \mathcal{H}_F \tag{4.3}$$

Let $\mathcal{H}_k = \bigcup_{i<k, F \in \mathcal{F}_i} \mathcal{H}_F$. Suppose $\mathcal{H}_k$ has been defined and $F \in \mathcal{F}_k$. Let $F = \bigcup_{i<n_k} F_i$ the the canonical decomposition of F. We will define $\mathcal{H}_F$ by amalgamating the elements of $\mathcal{H}_{F_i}(i < n_k)$ in such a way that (4.1), (4.2) and (4.3) holds for $\mathcal{X}_\varepsilon$ and (4.3) for $\mathcal{X}_K$.

This concludes the construction of $\mathcal{H}$. Next, we will define $\|\cdot\|$ in $c_{00}(\omega_1)$

$$\|x\| = \max\{|\langle f, x\rangle| : f \in \mathcal{H}\} \tag{4.4}$$

Note that $\|\cdot\|$ is well defined by (4.2) and (4.3) and by (4.1) we have $\|x\| = 0$ if and only if $x = 0$ (this for the construction of $\mathcal{X}_\varepsilon$, for $\mathcal{X}_K$ the vectors $e_\alpha \in \mathcal{H}$ for every $\alpha < \omega_1$) so it defines a norm on $c_{00}(\omega_1)$. The respective Banach space $\mathcal{X}$ will be the completion of $(c_{00}(\omega_1), \|\cdot\|)$.

To prove that $\mathcal{X}$ has indeed the properties that we want we will use the capturing of $\mathcal{F}$. Arguing by contradiction we take an uncountable sequence $(y_\alpha)_{\alpha<\omega_1}$ in $\mathcal{X}$ with a certain property. We show (following [LAT11]) that there is an inequality that uncountably many y_α's satisfy.

Take $(x_\alpha)_{\alpha<\omega_1}$ in $c_{00}(\omega_1, \mathbb{Q})$ approximating the y_α's and apply the Δ-System lemma and a counting argument (this is why we take $\mathbb{Q}$ instead of $\mathbb{R}$) to obtain $\Gamma \subset \omega_1$ uncountable such that

1. $(\text{supp}(x_\alpha) : \alpha \in \Gamma)$ forms a Δ-System and

2. the x_α's are "isomorphic" in some manner.

Finding $F \in \mathcal{F}$ capturing enough x_α's we can construct vectors that contradict the inequality.

4.3 Proof of Theorem 4.4

Let us recall the following result.

Theorem 4.4. *Assume there is a capturing construction scheme $\mathcal{F}$. Then for every $\varepsilon \in (0, 1) \cap \mathbb{Q}$, there is a Banach space $\mathcal{X}_\varepsilon$ with an uncountable ε-biorthogonal system but no uncountable τ-biorthogonal system for every $0 \leq \tau < \frac{\varepsilon}{1+\varepsilon}$.*

Let $\mathcal{F}$ be a capturing construction scheme and $0 < \varepsilon < 1$ rational. We construct $\mathcal{H}$ as outlined in 4.2.

We start with $\mathcal{H}_1$, which is formed by $h_\alpha^{\{\alpha\}}$ taking values in $\{\alpha\}$ and sending $\alpha \mapsto 1$.

Suppose $\mathcal{H}_k$ has been built satisfying (4.1), (4.2) and (4.3). Let $F \in \mathcal{F}_k$ and $F = \bigcup_{i<n_k} F_i$ the canonical decomposition of F. Then, we let $\mathcal{H}_F = \{h_\alpha^F : \alpha \in F\}$ where h_α^F is define in the following way

1. For $\alpha \in R$, define $h_\alpha^F := h_0^{F_0} + \sum_{0 < i < n_k} \varphi_i(h_\alpha^{F_0}) \upharpoonright (F_i \setminus F_0)$.

2. For $\alpha \in F_0 \setminus R$, define

$$h_\alpha^F := h_\alpha^{F_0} + \varepsilon \sum_{2 \leq i < n_k} (-1)^i \varphi_i(h_\alpha^{F_0}) \upharpoonright (F_i \setminus F_0).$$

3. For $\delta \in F_1 \setminus R$, and $\alpha \in F_0 \setminus R$ with $\varphi_1(\alpha) = \delta$, define

$$h_\delta^F := \varphi_1(h_\alpha^{F_0}) + \varepsilon \sum_{2 \leq i < n_k} (-1)^{i+1} \varphi_i(h_\alpha^{F_0}) \upharpoonright (F_i \setminus F_0).$$

4. For $\alpha \in F_j \setminus R$ with $2 \leq j < n_k$, define $h_\alpha^F = h_\alpha^{F_j}$.

It is clear that $\mathcal{H}_{k+1}$ satisfies (4.1) and (4.2). Note that if $E \in \mathcal{F}$ is contained in F and $\alpha \in F$, there is $f \in \mathcal{H}_E$ such that $h_\alpha^F(\gamma)$ equals either $f(\gamma)$ or $\varepsilon f(\gamma)$ for every $\gamma \in E$. This shows that (4.3) holds for $\mathcal{H}_{k+1}$. The same observation shows

$$|h_\alpha^F(e_\beta)| \leq \varepsilon \tag{4.5}$$

for all $\alpha \neq \beta$ in F.

This finishes the construction of $\mathcal{H}$.

Define the norm $\| \cdot \|_\varepsilon$ as in (4.4) and let $\mathcal{X}_\varepsilon$ be the completion of $(c_{00}(\omega_1), \| \cdot \|_\varepsilon)$.

We check that $\mathcal{X}_\varepsilon$ is as we wanted. Define h_α to be the union of all $(h_\alpha^F : F \in \mathcal{F})$ which is well defined by (4.2). By (4.5) the sequence $(e_\alpha, h_\alpha)_{\alpha < \omega_1}$ forms an un uncountable ε-biorthogonal system.

Suppose $(y_\alpha, y_\alpha^*)_{\alpha < \omega_1}$ is a τ-biorthogonal system for $0 \leq \tau < \frac{\varepsilon}{1+\varepsilon}$. We can assume that the y_α's are normalized.

Lemma 4.12. *There is $\Gamma \subset \omega_1$ uncountable and $\delta > 0$ such that, for every $n, m < \omega$ with $\frac{m}{2n} = \varepsilon$ and every $\alpha_0 < \ldots < \alpha_{2n+1}$ we have,*

$$\left\| (y_{\alpha_0} - y_{\alpha_1}) - \frac{1}{m} \sum_{i=1}^{n} (y_{\alpha_{2i}} - y_{\alpha_{2i+1}}) \right\|_\varepsilon \geq \delta \tag{4.6}$$

Proof. Let $N < \omega$ and $\Gamma \subset \omega_1$ uncountable such that

$$\sup_{\alpha \in \Gamma} \|y_\alpha^*\| \leq N$$

Then

$$\left\| (y_{\alpha_0} - y_{\alpha_1}) - \frac{1}{m} \sum_{i=1}^{n} (y_{\alpha_{2i}} - y_{\alpha_{2i+1}}) \right\|_\varepsilon \geq \left| \frac{f_{\alpha_1}}{N} \left((y_{\alpha_0} - y_{\alpha_1}) - \frac{1}{m} \sum_{i=1}^{n} (y_{\alpha_{2i}} - y_{\alpha_{2i+1}}) \right) \right|$$

$$\geq \frac{1}{N} \left(1 - \tau - \frac{1}{m}(2n\tau) \right) = \frac{1}{N} \left(1 - \tau(1 + \frac{2n}{m}) \right)$$

Taking $\delta = \frac{1}{N}(1 - \tau(1 + \frac{2n}{m})) = \frac{1}{N}(1 - \tau\frac{1+\varepsilon}{\varepsilon}) > 0$ we obtain the result. $\square$

Theorem 4.4 follows if we show that

Lemma 4.13. *For every normalized* $(y_\alpha)_{\alpha \in \Gamma}$ *in* $\mathcal{X}_\varepsilon$*, there is* $m, n < \omega$ *with* $\frac{m}{2n} = \varepsilon$ *and* $\alpha_0 < \ldots < \alpha_{2n+1}$ *such that*

$$\left\| (y_{\alpha_0} - y_{\alpha_1}) - \frac{1}{m} \sum_{i=1}^{n} (y_{\alpha_{2i}} - y_{\alpha_{2i+1}}) \right\|_\varepsilon < \delta$$

Proof. Let m and n, big enough so that $1/m < \delta/2$ and $m/2n = \varepsilon$.

Let $x_\alpha \in c_{00}(\omega_1, \mathbb{Q})$ for $\alpha \in \Gamma$ normalized such that

$$\|y_\alpha - x_\alpha\|_\varepsilon < \frac{\delta}{4(n+1)} \quad \text{for every } \alpha \in \Gamma.$$

Note that

$$\left\| (y_{\alpha_0} - y_{\alpha_1}) - \frac{1}{m} \sum_{i=1}^{n} (y_{\alpha_{2i}} - y_{\alpha_{2i+1}}) \right\|_\varepsilon \leq$$

$$\leq \left\| (x_{\alpha_0} - x_{\alpha_1}) - \frac{1}{m} \sum_{i=1}^{n} (x_{\alpha_{2i}} - x_{\alpha_{2i+1}}) \right\|_\varepsilon + \sum_{i=0}^{2n+1} \|y_\alpha - x_\alpha\|_\varepsilon$$

$$\leq \left\| (x_{\alpha_0} - x_{\alpha_1}) - \frac{1}{m} \sum_{i=1}^{n} (x_{\alpha_{2i}} - x_{\alpha_{2i+1}}) \right\|_\varepsilon + \frac{\delta}{2}$$

thus, it is enough to find $\alpha_0 < \alpha_1 < \ldots < \alpha_{2n+1}$ in Γ such that

$$\left\| (x_{\alpha_0} - x_{\alpha_1}) - \frac{1}{m} \sum_{i=1}^{n} (x_{\alpha_{2i}} - x_{\alpha_{2i+1}}) \right\|_\varepsilon < \frac{\delta}{2} \tag{4.7}$$

Apply the Δ-System lemma and a counting argument to find $\Gamma_0 \subset \Gamma$ uncountable such that

1. Let $D_\alpha = \text{supp}(x_\alpha)$, then the collection $(D_\alpha : \alpha \in \Gamma_0)$ form a Δ-System with $|D_\alpha| = |D_\beta| = d$ for every $\alpha, \beta \in \Gamma_0$.

2. For $\alpha, \beta \in \Gamma_0$ and $\varphi_{\alpha,\beta} : D_\alpha \to D_\beta$ an increasing bijection then $x_\beta = \varphi_{\alpha,\beta}(x_\alpha)$.

Since $\mathcal{F}$ is capturing there is $F \in \mathcal{F}$ and some $\alpha_0 < \ldots < \alpha_{2n+1}$ in Γ_0, such that F captures $(D_{\alpha_i} : i \leq 2n + 1)$. This means that $F \in \mathcal{F}_k$ for some $k < \omega$ (it will have to be such that $n_k \geq 2n + 1$), and for $F = \bigcup_{i < n_k} F_i$ is the canonical decomposition of F we have

$$D_{\alpha_i} \subset F_i, \ i \leq 2n + 1$$
$$\varphi_i(D_0) = D_i, \ i \leq 2n + 1 \tag{4.8}$$

Let

$$w = (x_{\alpha_0} - x_{\alpha_1}) - \frac{1}{m} \sum_{i=1}^{n} (x_{\alpha_{2i}} - x_{\alpha_{2i+1}})$$

Note that $w \upharpoonright R(F)$ is identically zero by (4.8). We show that $\|w\| < \delta/2$. Let $f \in \mathcal{H}_F$.

If f is of the form (1) it is clear that $\langle f, w \rangle = 0$.

If f is of the form (2) then $f = h_\alpha^F$ for some $\alpha \in F$ and

$$\langle f, w \rangle = h_\alpha^{F_0}(x_{\alpha_0}) - \frac{\varepsilon}{m} \sum_{i=2}^{n} (h_\alpha^{F_0}(x_{\alpha_0}) + h_\alpha^{F_0}(x_{\alpha_0})) = h_\alpha^{F_0}(x_{\alpha_0})\left(1 - \varepsilon\frac{2n}{m}\right) = 0$$

because the amalgamation for f nullifies the term in α_1 and changes the sign of the other odd terms.

If f is of the form (3) then

$$\langle f, w \rangle = -h_\alpha^{F_1}(x_{\alpha_1}) + \frac{\varepsilon}{m} \sum_{i=2}^{n} (h_\alpha^{F_1}(x_{\alpha_1}) + h_\alpha^{F_1}(x_{\alpha_1})) = h_\alpha^{F_1}(x_{\alpha_1})\left(\varepsilon\frac{2n}{m} - 1\right) = 0$$

because the amalgamation for f nullifies the term in α_0 and changes the sign the other even terms

Finally if f is of the form (4) then $|\langle f, w \rangle| = |\frac{1}{m}\langle h_\alpha^{F_j}, \varphi_{\alpha_j}(z)\rangle| \leq \frac{1}{m} < \delta/2$ as we wanted to show. Thus, w witnesses (4.7) contradicting Lemma 4.12 and finishing the proof.　　□

4.3.1　Some geometric properties of the space $\mathcal{X}_\varepsilon$

We study other properties of the space $\mathcal{X}_\varepsilon$. Recall that a Banach space has the Mazur Intersection Property if every closed convex subset is the intersection of closed balls. The following properties are also relevant to the geometry of a Banach space.

Definition 4.14. We say a Banach space $\mathcal{X}$ is *polyhedral* if for every finite dimensional $V \subset \mathcal{X}$, the unit ball of V has finitely many extremal points.

We say the norm of $\mathcal{X}$ *depends on finitely many coordinates* if for every $x \in \mathcal{X} \setminus \{0\}$ there is $\eta > 0$ and $(h_i)_{i<n} \subset S_{\mathcal{X}^*}$ such that for every $w, z \in \eta B_X$ if $h_i(x + w) = h_i(x + z)$ for every $i < n$ then $\|x + w\| = \|x + z\|$.

For separable Banach spaces both properties agree.

Theorem 4.15 ([Fon90])**.** *Let $\mathcal{X}$ be a separable Banach space. Then $\mathcal{X}$ has an equivalent norm that makes it polyhedral if and only if $\mathcal{X}$ has an equivalent norm that depends on finitely many coordinates.*

We show the following.

Theorem 4.16. *The space $\mathcal{X}_\varepsilon$ has the following properties:*

　　1. *$\mathcal{X}_\varepsilon$ does not have the Mazur intersection property.*

   2. $\mathcal{X}_\varepsilon$ is polyhedral.

   3. $\|\cdot\|_\varepsilon$ depends only on finitely many coordinates.

Proof. We follow the proof of Theorem 4.14 of [LAT11]. Note that $\{h_\alpha\}_{\alpha<\omega_1}$ is a norming set of $B_{\mathcal{X}_\varepsilon^*}$ which is not dense on the sphere of $\mathcal{X}_\varepsilon^*$ so $\mathcal{X}_\varepsilon$ does not have the Mazur intersection property by a result of [GGS78].

   Fix $\mu > 0$ such that

$$\mu + \varepsilon(1 + \mu) < 1$$

we will use the following

Claim 4.17. *Let $x \in \mathcal{X}_\varepsilon$, $y \in c_{00}(\omega_1)$ and $F \in \mathcal{F}$ with $\mathrm{supp}(y) \subseteq F$. If $\|x - y\|_\epsilon < \mu\|x\|_\epsilon$ then the norm of x is determined by $\{h_\alpha : \alpha \in F\}$.*

Proof. Let x, y and F as above and let $\gamma \notin F$ by the structure of h_γ we have

$$|h_\gamma(y)| \leq \varepsilon\|y\|_\varepsilon < \varepsilon(1 + \mu)\|x\|_\varepsilon$$

and then

$$|h_\gamma(x)| \leq \|x - y\|_\varepsilon + |h_\gamma(y)| < \|x\|_\varepsilon$$

$\square$

   Suppose now $V \subset \mathcal{X}_\varepsilon$ is a finite dimensional subspace and fix a normalized basis $(x_i)_{i<n}$. Now let $y_i \in c_{00}(\omega_1, \mathbb{Q})$ such that $\|x - y\|_\varepsilon < \eta/n$.

   Then for every $x = \sum_{i<n} a_i x_i$ we have

$$\left\| \sum_{i<n} a_i x_i - \sum_{i<n} a_i y_i \right\|_\varepsilon \leq \max_i a_i \sum_{i<n} \|x_i - y_i\|_\varepsilon < \eta\|x\|_\varepsilon$$

   If we take $F \in \mathcal{F}$ such that $\mathrm{supp}(y_i) \subset F$ for every $i < n$, then $\{h_\alpha : \alpha \in F\}$ contains the extreme points of B_V by Claim 4.17.

   We show $\|\cdot\|_\varepsilon$ depends on finitely many coordinates. Suppose $x \in \mathcal{X}_\varepsilon \setminus \{0\}$. Find $0 < \lambda < 1$ such that $\lambda\mu < 1/3$ and $y \in c_{00}(\omega_1, \mathbb{Q})$ such that $\|x - y\|_\varepsilon < \lambda\mu\|x\|_\epsilon$. If $F \in \mathcal{F}$ is such that $\mathrm{supp}(y) \subseteq F$ then $\eta = \lambda\mu\|x\|_\varepsilon$ and $\{h_\alpha : \alpha \in F\}$ works. To see this note that for every $w \in \mathcal{X}_\varepsilon$, if $\|w\|_\varepsilon < \eta$ then $\|x + w - y\|_\varepsilon < \mu\|x + w\|_\varepsilon$ and Claim 4.17 gives the result. $\square$

4.4   Proof of Theorem 4.5

Out aim in this section is to construct a space that would prove Theorem 4.5

Theorem 4.5. *Assume there is a capturing construction scheme $\mathcal{F}$. Then for every constant $K > 1$, there is a Banach space $\mathcal{X}_K$ with a K-basis of length ω_1 but no uncountable K'-basic sequence for every $1 \leq K' < K$.*

Proof. The idea of the construction follows the outline of Section 4.2. We construct $\mathcal{H}$ by recursion. For $\mathcal{X}_K$ the collection $\mathcal{H}_F$ will have the following *closure property*:

$$\forall f \in \mathcal{H}_F, \ \delta \in F \quad K^{-1}(f \restriction \delta) \in \mathcal{H}_F \tag{4.9}$$

Let $\mathcal{F}$ be a capturing construction scheme and let $K > 1$. $\mathcal{H}_1$ is the set of all functions of the form $K^{-n}e_\alpha$ for every $\alpha < \omega_1$ and $n < \omega$.

Suppose $\mathcal{H}_k$ has been constructed satisfying (4.9) and (4.3). Let $F \in \mathcal{F}_k$ and $F = \bigcup_{i<n_k} F_i$ be the canonical decomposition of F. Then, we let $\mathcal{H}_F$ be the collection of functions of the following type:

1. e_α, for $\alpha \in F$.

2. $\sum_{i<n_k} \varphi_i(f) \restriction (F_i \setminus F_0)$ for every $f \in \mathcal{H}_{F_0}$.

3. $\frac{1}{K^n} \left(\sum_{i<n_k} \varphi_i(f) \restriction (F_i \setminus F_0) \right) \restriction \delta$ for every $f \in \mathcal{H}_{F_0}$, every $\delta \in F$ and $n = 1, 2 \ldots$

   It is clear that (4.9) and (4.3) holds for $\mathcal{H}_{k+1}$. This finishes the construction of $\mathcal{H}$.

   Define $\| \cdot \|_K$ as in (4.4) and let $\mathcal{X}_K$ be the completion of $(c_{00}(\omega_1), \| \cdot \|_K)$.

   We see that $\mathcal{X}_K$ is as we wanted. We first show that $\mathcal{X}_K$ has an uncountable K-basic sequence. Let $(e_\alpha)_{\alpha<\omega_1}$ be the canonical unit vector basis.

Lemma 4.18. *The vectors $(e_\alpha)_{\alpha<\omega_1}$ form a normalized K-basis of $\mathcal{X}_K$. In particular $\mathcal{X}_K$ is not separable.*

Proof. It is clear that the e_α's are normalized. To see they are a K-basic sequence let $n < m < \omega$, $\alpha_1 < \ldots < \alpha_m < \omega_1$ and $(a_i)_{i=1}^m \in \mathbb{R}^m$. Let $F \in \mathcal{F}$ such that $\alpha_i \in F$ for $i = 1, \ldots, m$. Take $\delta = \alpha_{n+1}$ and $f \in \mathcal{H}_F$ such that $\sum_{i=1}^n a_i e_{\alpha_i}$ attains the norm at f, i.e,

$$f\left(\sum_{i=1}^n a_i e_{\alpha_i}\right) = \left\|\sum_{i=1}^n a_i e_{\alpha_i}\right\|_K$$

If f is of the form (1) then $f \restriction \delta = Kg$ for some $g \in \mathcal{H}_F$ and if f is of the form (2) then $f = g$ for some $g \in \mathcal{H}_F$. Thus,

$$\left\|\sum_{i=1}^n a_i e_{\alpha_i}\right\|_K = \left|\left\langle f, \sum_{i=1}^n a_i e_{\alpha_i}\right\rangle\right| = \left|\left\langle f \restriction \delta, \sum_{i=1}^m a_i e_{\alpha_i}\right\rangle\right|$$

$$\leq K \left|\left\langle g, \sum_{i=1}^n a_i e_{\alpha_i}\right\rangle\right| \leq K \left\|\sum_{i=1}^m a_i e_{\alpha_i}\right\|_K$$

by the closure property (4.9). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We proceed by contradiction. Suppose now that $(y_\alpha)_{\alpha<\omega_1}$ is a K'-basic sequence with $1 \le K' < K$. Fix $K' < L < K$ and let $n < \omega$ such that

$$\frac{1}{K} + \frac{1}{n} < \frac{1}{L} \tag{4.10}$$

Take a normalized sequence $(x_\alpha)_{\alpha<\omega_1}$ in $c_{00}(\omega_1, \mathbb{Q})$ such that

$$\|x_\alpha - y_\alpha\|_K < \min\left\{\frac{1}{4K'n}, \frac{L - K'}{8(K')^2 n}\right\} \quad \text{for every } \alpha < \omega_1.$$

The following lemma plays the same role of Lemma 4.12 in Theorem 4.4

Lemma 4.19. *For every* $\alpha_1 < \ldots < \alpha_{2n} < \omega_1$

$$\left\|\sum_{i=1}^{n} x_{\alpha_i}\right\|_K \le L \left\|\sum_{i=1}^{n} x_{\alpha_i} - \sum_{i=n+1}^{2n} x_{\alpha_i}\right\|_K$$

Proof. Note first that $\|\sum_{i=1}^{n} x_{\alpha_i} - \sum_{i=n+1}^{2n} x_{\alpha_i}\|_K \ge 1/2K'$. Indeed, suppose otherwise then

$$\begin{aligned}
1 = \|y_{\alpha_1}\|_K &\le K' \left\|\sum_{i=1}^{n} y_{\alpha_i} - \sum_{i=n+1}^{2n} y_{\alpha_i}\right\|_K \\
&\le K' \left\|\sum_{i=1}^{n} x_{\alpha_i} - \sum_{i=n+1}^{2n} x_{\alpha_i}\right\|_K + K' \sum_{i=1}^{2n} \|y_{\alpha_i} - x_{\alpha_i}\|_K \\
&< K' \left(\frac{1}{2K'} + \frac{2n}{4K'n}\right) = 1
\end{aligned}$$

Now

$$\begin{aligned}
\left\|\sum_{i=1}^{n} x_{\alpha_i}\right\|_K &\le \left\|\sum_{i=1}^{n} y_{\alpha_i}\right\|_K + \sum_{i=1}^{n} \|x_{\alpha_i} - y_{\alpha_i}\|_K \\
&\le K' \left\|\sum_{i=1}^{n} y_{\alpha_i} - \sum_{i=n+1}^{2n} y_{\alpha_i}\right\|_K + \sum_{i=1}^{n} \|x_{\alpha_i} - y_{\alpha_i}\|_K \\
&\le K' \left\|\sum_{i=1}^{n} x_{\alpha_i} - \sum_{i=n+1}^{2n} x_{\alpha_i}\right\|_K + 2K' \sum_{i=1}^{2n} \|x_{\alpha_i} - y_{\alpha_i}\|_K \\
&\le K' \left\|\sum_{i=1}^{n} x_{\alpha_i} - \sum_{i=n+1}^{2n} x_{\alpha_i}\right\|_K + 4K'n \frac{L - K'}{8(K')^2 n} \\
&\le K' \left\|\sum_{i=1}^{n} x_{\alpha_i} - \sum_{i=n+1}^{2n} x_{\alpha_i}\right\|_K + (L - K') \left\|\sum_{i=1}^{n} x_{\alpha_i} - \sum_{i=n+1}^{2n} x_{\alpha_i}\right\|_K
\end{aligned}$$

which is what we wanted to prove. $\qquad\square$

We want to use the capturing of $\mathcal{F}$ to contradict the lemma above.

We proceed as before and find $\Gamma \subset \omega_1$ uncountable such that

1. If $D_\alpha = \text{supp}(x_\alpha)$, then the collection $(D_\alpha : \alpha \in \Gamma)$ form a Δ-System with $|D_\alpha| = |D_\beta| = d$ for every $\alpha, \beta \in \Gamma$.

2. There is a function $z : d \to \mathbb{Q}$ such that, if $\varphi_\alpha : d \to D_\alpha$ is the unique order increasing bijection, then $x_\alpha = \varphi_\alpha(z)$

Since $\mathcal{F}$ is capturing, we can find $F \in \mathcal{F}$ and some $\alpha_1 < \ldots < \alpha_{2n} < \omega_1$ in Γ such that F captures $(D_{\alpha_i} : 1 \le i \le 2n)$. This means that $F \in \mathcal{F}_k$ for some $k < \omega$ (it will have to be such that $n_k \ge 2n$), and for $F = \bigcup_{i < n_k} F_i$ is the canonical decomposition of F we have

$$D_{\alpha_{i+1}} \subset F_i, \ i < 2n$$
$$\varphi_i(D_1) = D_{i+1}, \ i < 2n$$

Let

$$v = \sum_{i=1}^{n} x_{\alpha_i} \quad \text{and} \quad w = \sum_{i=1}^{n} x_{\alpha_i} - \sum_{i=n+1}^{2n} x_{\alpha_i}$$

We show that $\|v\|_K > L\|w\|_K$. Since the x_{α_i}'s are normalized there is $h \in \mathcal{H}_{F_0}$ such that $|\langle h, x_{\alpha_1} \rangle| = 1$. Taking $f = \sum_{i < n_k} \varphi_i(h)$ we get $|\langle f, v \rangle| = n$. Thus $\|v\|_K \ge n$.

Take now $f \in \mathcal{H}_F$.

If f is of the form (1) then, $|\langle f, w \rangle| = 0$

If f is of the form (2) then, $f = (1/K) \sum_{i < n_k} \varphi_i(h) \upharpoonright \delta$ for some $\delta \in F$ and $h \in \mathcal{H}_{F_0}$. If $\delta \in R(F)$ then $|\langle f, w \rangle| = 0$. Suppose $\delta \in F_j \setminus R(F)$ and $\eta \in F_0$ is such that $\varphi_j(\eta) = \delta$

Suppose $j < n$ then

$$|\langle f, w \rangle| \le \left| \frac{1}{K} \langle \sum_{i < j} \varphi_i(h), w \rangle \right| + \frac{1}{K} |\langle h \upharpoonright \eta, x_{\alpha_1} \rangle|$$
$$\le \frac{n-1}{K} + \|x_{\alpha_0}\|_K = \frac{n-1}{K} + 1 < \frac{n}{L} \le \frac{1}{L}\|v\|_K$$

by (4.10).

Suppose now $j \ge n$. Then

$$|\langle f, w \rangle| \le \frac{1}{K} \left| \sum_{i < n-1} \langle \varphi_i(h), x_{\alpha_i} \rangle + \langle \varphi_{n-1}(h), x_{\alpha_{n-1}} \rangle - \sum_{n \ge i < j} \langle \varphi_i(h), x_{\alpha_i} \rangle - \langle \varphi_j(h) \upharpoonright \delta, x_{\alpha_j} \rangle \right|$$
$$\le \frac{1}{K} |(n-1) + \langle h, x_{\alpha_0} \rangle - (j - n) - \langle h \upharpoonright \eta, x_{\alpha_0} \rangle$$
$$\le \frac{n-1}{K} + \frac{\|x_{\alpha_0}\|_K + \|x_{\alpha_0} \upharpoonright \eta\|_K}{K} \le \frac{n}{K} + 1 < \frac{n}{L} \le \frac{1}{L}\|v\|_K$$

If f is of the form (3) then $|\langle f, w \rangle| \le 1 \le \frac{n}{K} + 1 < \frac{n}{L} \le \frac{1}{L}\|v\|_K$.

We conclude that $\|w\|_K < \frac{1}{L}\|v\|_K$ but this contradicts Lemma 4.19 and thus $\mathcal{X}_K$ is as we wanted.

$\square$

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