The Lindelöf class of L-functions

by

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Abstract

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Meromorphic functions, called L-functions, play a vital role in number theory. In 1989, Selberg defined a class of L-functions that serves as an axiomatic model for L-functions arising from geometry and arithmetic. Even though the Selberg class successfully captures many characteristics common to most L-functions, it fails to be closed under addition. This creates obstructions, in particular, not allowing us to interpolate between L-functions. To overcome this limitation, V. K. Murty defined a general class of L-functions based on their growth rather than functional equation and Euler product. This class, which is called the Lindelöf class of L-functions, is endowed with the structure of a ring.

In this thesis, we study further properties of this class, specifically, its ring structure and topological structure. We also study the zero distribution and the a-value distribution of elements in this class and prove certain uniqueness results, showing that distinct elements cannot share complex values and L-functions in this class cannot share two distinct values with any other meromorphic function. We also establish the value distribution theory for this class with respect to the universality property, which states that every holomorphic function is approximated infinitely often by vertical shifts of an L-function. In this context, we precisely formulate and give some evidence towards the Linnik-Ibragimov conjecture.

To my parents.

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Chapter 1

Introduction

The Riemann zeta-function is defined on $\Re(s) > 1$ by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{1.1}$$

The Dirichlet series and the Euler product of the zeta-function converge absolutely on the right half plane $\Re(s) > 1$ and uniformly on each compact set in this plane. Euler, who studied $\zeta(s)$ as a function of a real variable was the first to notice this Euler product, which can be regarded as the analytic version of the unique factorization of natural numbers. Additionally, this product also captures information regarding the distribution of primes in natural numbers. As Euler observed, the infinitude of primes follows from the fact that the product diverges at $s = 1$. Riemann realized deeper connections between the zeta function and the distribution of prime numbers by studying $\zeta(s)$ as a function of a complex variable. Earlier, Gauss had predicted that if $\pi(x)$ denotes the number of primes less than x, then asymptotically $\pi(x) \sim x/\log x$. More precisely, he conjectured

$$\pi(x) \sim Li(x) := \int_2^x \frac{\log t}{t}\, dt$$

as $x \to \infty$.

In 1896, using ideas of Riemann, Jacques Hadamard and Charles Jean de la Vallée-Poussin independently proved this conjecture, which is known today as the prime number theorem. It turns out that this is a consequence of $\zeta(s)$ having a simple pole at $s = 1$ and being non-zero on the vertical line $\Re(s) = 1$.

This motivated a careful study of functions of the type (1.1). For instance, Dirichlet proved the prime number theorem for an arithmetic progression $a \pmod{q}$, with $(a, q) = 1$, using similar techniques by studying the function

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \tag{1.2}$$

where χ is a Dirichlet character modulo q, defined as a group homomorphism $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ extended to $\chi : \mathbb{Z} \to \mathbb{C}$ by periodicity and setting $\chi(n) = 0$ if $(n, q) > 1$.

Building on ideas of Riemann, connections were established between the zeroes of the zeta-function and the distribution of prime numbers. In 1901, Helge von Koch observed that the Riemann hypothesis implies a much stronger error term for the prime number theorem.

In this thesis, we study functions similar to the zeta-function, which are called L-functions. We introduce a class of L-functions and study their analytic properties, with emphasis on the value distribution theory of such functions.

1.1 The Riemann zeta-function

In this section, we recall a few basic properties of the Riemann zeta-function $\zeta(s)$ and motivate the definition of the Selberg class. The zeta-function given by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

is absolutely convergent on $\Re(s) > 1$.

A series of the type $\sum_n a_n/n^s$ is called a Dirichlet series. For any Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

the abscissa of absolute convergence, denoted as $\sigma_0(F)$ is defined as the infimum of all σ, such that the Dirichlet series is absolutely convergent on $\Re(s) > \sigma$. We also define the abscissa of conditional convergence, denoted $\sigma_c(F)$ to be the supremum of all σ such that the Dirichlet series is divergent for $\Re(s) < \sigma$.

The zeta-function has abscissa of absolute convergence $\sigma_0(\zeta) = 1$ and abscissa of conditional convergence at $\sigma_c(\zeta) = 0$. Although, $\zeta(s)$ as a function in one complex variable was introduced by Riemann, its special values at positive integers were already studied by Euler.

For a real number x, let $\{x\}$ denote the fractional part of x, i.e., $\{x\} := x - [x]$. By partial summation, we obtain that for $\Re(s) > 1$,

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + s\int_1^{\infty} \frac{\frac{1}{2} - \{x\}}{x^{s+1}}\, dx. \tag{1.3}$$

The integral in (1.3) is absolutely convergent for $\Re(s) > 0$ and uniformly convergent for $\Re(s) > \epsilon$ for any $\epsilon > 0$. Hence, (1.3) gives the analytic continuation of $\zeta(s)$ in the region $\Re(s) > 0$ with a simple pole at $s = 1$ with residue 1.

Denote by $\Gamma(s)$ the Euler gamma-function, which is defined for $\Re(s) > 0$ by

$$\Gamma(s) := \int_0^{\infty} e^{-x} x^{s-1}\, dx.$$

Using the functional equation

$$\Gamma(s+1) = s\Gamma(s),$$

$\Gamma(s)$ is meromorphically continuable to the whole of $\mathbb{C}$ except for simple poles at non-positive integers $s = 0, -1, -2, \cdots$, with residues given by

$$Res_{s=-n}\Gamma(s) = \frac{(-1)^n}{n!}.$$

This Euler gamma-function plays a crucial role in the functional equation of the zeta-function, which enables us to meromorphically continue the zeta-function to $\mathbb{C}$.

Consider the function

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

This is often referred to as the completed zeta-function. Riemann established the following functional equation for $\zeta(s)$, given by

Proposition 1.1.1. $\xi(s) = \xi(1-s)$.

In other words,

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{1.4}$$

This gives the analytic continuation of $\zeta(s)$ in the region $\Re(s) < 1/2$.

Since $\Gamma(s)$ has poles at non-positive integers, using (1.4), we immediately conclude that $\zeta(s)$ has simple zeroes at all negative even integers $s = -2, -4, -6, \cdots$. These are called the trivial zeroes of the zeta-function. Moreover, the Euler product

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \text{ for } \Re(s) > 1$$

implies that $\zeta(s)$ does not have any zeroes in the region $\Re(s) > 1$. Now, using the functional equation, we also conclude that $\zeta(s)$ has no zeroes other than the trivial zeroes in the region $\Re(s) < 0$.

The strip $0 < \Re(s) < 1$ is rather mysterious and the zeroes of $\zeta(s)$ in this region are referred to as the non-trivial zeroes. Observing the symmetric nature of the functional equation (1.4) of $\zeta(s)$, Riemann predicted that all the non-trivial zeroes of $\zeta(s)$ lie on the vertical line $\Re(s) = 1/2$. This is known as the famous Riemann hypothesis.

Conjecture 1 (Riemann hypothesis)**.** *If* $\zeta(s) = 0$ *for* $0 < \Re(s) < 1$*, then* $\Re(s) = 1/2$.

It is not very difficult to show that $\zeta(s)$ has no zeroes on the vertical line $\Re(s) = 1$. This already results in the asymptotic formula for the number of primes less than x for a large number x.

Similar approaches help us to prove the analogues of prime number theorem in different settings. The prime number theorem in an arithmetic progression where the terms are co-prime to the common difference is given by studying the Dirichlet L-functions. In the number field setting, the analogue for the prime number theorem is given by studying the Dedekind zeta-functions $\zeta_K(s)$. In each of these cases, the corresponding "zeta" function, which is defined using some Euler product, captures the distribution of primes in the respective underlying structure. There are further intricate connections between primes and the zeroes of these functions. This raises the curiosity to study these functions as a family and also better understand its zero-distribution.

1.2 Organization of thesis

Selberg initiated the study of a family of L-functions which satisfy the basic properties of the Riemann zeta-function, called the Selberg class.

In chapter 2, we discuss some basic properties of the Selberg class. The emphasis of this thesis is on the value distribution theory of L-functions. In chapter 2, we provide some known results on the value distribution theory of L-functions in the Selberg class. We also discuss Selberg's orthogonality conjectures and some of its consequences.

In chapter 3, we introduce a larger family of L-functions, which overcomes some limitations of the Selberg class. This new class $\mathbb{M}$, based on the growth of L-functions forms a ring and has a rather rich algebraic and geometric structure. It was defined by V. K. Murty [34]. In particular, it forms a differential ring and has a natural topology attached to it. We also study the ideal theory of this ring and show that it is non-Noetherian. We address the analogous question to the degree conjecture in the context of this new class and also provide the classification of elements with $c_F^* = 0$. This chapter forms a major part of [16], which is joint work with V. K. Murty.

In chapter 4, we study the a-value distribution of functions in the class $\mathbb{M}$. We first study the number of zeroes of L-functions in $\mathbb{M}$, and give some upper and lower bounds for the number of zeroes in a disc of radius r. Using this and further tools from Nevanlinna theory, we prove some uniqueness results. The main theorems in this chapter are Theorem 4.5.1, 4.5.2, 4.5.3 and 4.5.4. They also appear in [14].

In chapter 5, we study the problem of universality of L-functions in $\mathbb{M}$. The universality property of an L-function in a strip D vaguely claims that any holomorphic function is approximated by infinitely many vertical shifts of the L-function in D. This universality property is known for the elements in the Selberg class and also a number of other naturally occurring L-functions. In view of this, Yu V. Linnik and I. A. Ibragimov conjectured that all Dirichlet series with analytic continuation, satisfying certain "growth conditions" must be universal in "some region". In this context, we explicitly characterize the growth condition for the L-function and describe the region where we expect the function to be universal. Thus, we formulate the Linnik-Ibragimov conjecture more explicitly and also provide partial results in this direction. Our main theorems in this chapter are Theorem 5.3.1 and 5.3.2. These results will appear in [15].

Chapter 2

The Selberg class

In 1989, Selberg [40] defined a general class of L-functions having Euler product, analytic continuation and functional equation of the similar type as the Riemann zeta-function. He formulated some fundamental conjectures on the elements of this class. Ever since, this class of L-functions has been extensively studied and these conjectures have been known to have far reaching consequences.

Definition 2.0.1. *The Selberg class $\mathbb{S}$ consists of meromorphic functions $F(s)$ satisfying the following properties.*

1. **Dirichlet series**- *It can be expressed as a Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s},$$

which is absolutely convergent in the region $Re(s) > 1$. We also normalize the leading coefficient as $a_F(1) = 1$.

2. **Analytic continuation** - *There exists a non-negative integer k, such that $(s-1)^k F(s)$ is an entire function of finite order.*

3. **Functional equation** - *There exist real numbers $Q > 0$ and $\alpha_i > 0$, complex*

numbers β_i *and* $w \in \mathbb{C}$, *with* $Re(\beta_i) \geq 0$ *and* $|w| = 1$, *such that*

$$\Phi(s) := Q^s \prod_i \Gamma(\alpha_i s + \beta_i) F(s) \tag{2.1}$$

satisfies the functional equation

$$\Phi(s) = w\bar{\Phi}(1 - \bar{s}).$$

4. **Euler product** - *There is an Euler product of the form*

$$F(s) = \prod_{p \ prime} F_p(s), \tag{2.2}$$

where

$$\log F_p(s) = \sum_{k=1}^{\infty} \frac{b_{p^k}}{p^{ks}}$$

with $b_{p^k} = O(p^{k\theta})$ *for some* $\theta < 1/2$.

5. **Ramanujan hypothesis** - *For any* $\epsilon > 0$,

$$|a_F(n)| = O_\epsilon(n^\epsilon). \tag{2.3}$$

The Euler product implies that the coefficients $a_F(n)$ are multiplicative, i.e., $a_F(mn) = a_F(m)a_F(n)$ when $(m, n) = 1$. Moreover, each Euler factor also has a Dirichlet series representation

$$F_p(s) = \sum_{k=0}^{\infty} \frac{a_F(p^k)}{p^{ks}},$$

which is absolutely convergent on $\Re(s) > 0$ and non-vanishing on $\Re(s) > \theta$, where θ is as in (2.2).

2.1 Examples

1. The Riemann zeta-function $\zeta(s) \in \mathbb{S}$.

2. Dirichlet L-functions $L(s,\chi)$ and their vertical shifts $L(s+i\theta,\chi)$ are in $\mathbb{S}$, where χ is a primitive Dirichlet character and $\theta \in \mathbb{R}$. Note that $\zeta(s+i\theta) \notin \mathbb{S}$ for $\theta \neq 0$ since it has a pole at $s = 1 - i\theta$.

3. Dedekind zeta functions $\zeta_K(s)$ defined as

$$\zeta_K(s) := \sum_{\mathfrak{a} \subseteq \mathcal{O}_K} \frac{1}{(N_{K/\mathbb{Q}}(\mathfrak{a}))^s},$$

   where $\mathcal{O}_K$ denotes the ring of integers of K and $\mathfrak{a}$ runs over all ideals of $\mathcal{O}_K$, are also elements of the Selberg class.

4. Let L/K be a Galois extension of number fields, with Galois group G. Let $\rho : G \to GL_n(\mathbb{C})$ be a representation of G. The associated Artin L-function is defined as

$$L(s,\rho,L/K) := \prod_{\mathfrak{p} \in K} \det\left(I - (N\mathfrak{p})^{-s} \rho(\sigma_{\mathfrak{q}})\big|_{V^{I_{\mathfrak{q}}}} \right)^{-1}$$

   where $\mathfrak{q}$ is a prime ideal in L lying over prime ideal $\mathfrak{p}$ in K, $\sigma_{\mathfrak{q}}$ is the Frobenius automorphism associated to $\mathfrak{q}$ and $V^{I_{\mathfrak{q}}}$ is the complex vector space fixed by the inertia subgroup $I_{\mathfrak{q}}$.

   A conjecture of Artin states that for non-trivial irreducible representation ρ of $Gal(L/K)$, the associated Artin L-function $L(s,\rho,L/K)$ is entire. If the Artin conjecture is true, then these functions lie in the Selberg class.

5. Let f be a holomorphic newform of weight k to some congruence subgroup $\Gamma_0(N)$.

Suppose its Fourier expansion is given by

$$f(z) = \sum_{n=1}^{\infty} c(n) \exp(2\pi i n z).$$

Then, its normalized Dirichlet coefficients are given by

$$a(n) := c(n) n^{(1-k)/2},$$

and the associated L-function given by $L(s) := \sum_{n=1}^{\infty} a(n)/n^s$ for $\Re(s) > 1$ is an element in the Selberg class. It is also believed that the normalized L-function associated to a non-holomorphic newforms are elements in the Selberg class, but the Ramanujan hypothesis is yet to be proven in this case.

6. The L-function associated to the Rankin-Selberg convolution of any two holomorphic newforms is in the Selberg class.

The constants in the functional equation (2.1) depend on F, and although the functional equation may not be unique, we have some well-defined invariants, such as the degree d_F of F, which is defined as

$$d_F := 2 \sum_i \alpha_i.$$

The factor Q in the functional equation gives rise to another invariant referred to as the conductor q_F, which is defined as

$$q_F := (2\pi)^{d_F} Q^2 \prod_i {\alpha_i}^{2\alpha_i}. \tag{2.4}$$

It is an interesting conjecture that both the degree and the conductor for elements in the Selberg class are non-negative integers.

Conjecture 2 (see Selberg [40], Conrey-Ghosh [12]). *If $F \in \mathbb{S}$, then d_F and q_F are non-negative integers.*

There is recent progress towards the degree conjecture. In 1993, it was shown by Conrey-Ghosh [12] that for $F(s)$ in the Selberg class, if $d_F < 1$ then $d_F = 0$. This was proved using the fact that any non-trivial element in the Selberg class must satisfy a certain growth condition, which is captured by the degree. We give the proof of this statement here and shall use this idea to prove a more general statement in Chapter 3.

Theorem 2.1.1 (Conrey-Ghosh, 1992). *If $F(s) \in \mathbb{S}$, then $F = 1$ or $d_F \geq 1$.*

Proof. Suppose $d_F < 1$. It suffices to show that $d_F = 0$, since the only element in $\mathbb{S}$ with $d_F = 0$ is the constant function $F = 1$. Let

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s},$$

for $\Re(s) > 1$. Consider the function

$$f(x) := \sum_{n=1}^{\infty} a_F(n) e^{-nx}.$$

We know that

$$\sum_{n=1}^{\infty} a_F(n) e^{-nx} = \frac{1}{2\pi i} \int_{(2)} F(s)\Gamma(s) x^{-s} ds,$$

where the integral is on the vertical line $\Re(s) = 2$. By the growth of $F(s)$ and convexity conditions due to Phragmén-Lindelöf theorem, $F(s)$ has a polynomial growth in $|t|$ in vertical strips. Thus, moving the line of integration to the left and taking into account the possible pole at $s = 1$ of $F(s)$ and poles of $\Gamma(s)$ at $s = 0, -1, -2, ...$, we get that

$$f(x) = \frac{P(\log x)}{x} + \sum_{n=0}^{\infty} \frac{F(-n)}{n!}(-x)^n,$$

where P is a polynomial. Let the functional equation for F be given by

$$\psi(s) := \gamma_F(s)F(s) = \overline{\psi}(1-\overline{s}).$$

Then, we have

$$\sum_{n=0}^{\infty} \frac{F(-n)}{n!}(-x)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_F(n+1)F(n+1)}{\gamma_F(-n)n!} x^n,$$

which is an entire function in x since

$$\frac{\gamma_F(n+1)}{\gamma_F(-n)n!} \ll n^{-(1-d_F)n} A^n,$$

for some constant $A > 0$.

Hence, the function $f(x)$ is analytic in $\mathbb{C}\backslash\{x \leq 0 : x \in \mathbb{R}\}$. But, this function is also periodic with period $2\pi i$ and so it converges for all x. Thus, the function

$$f(z) = \sum_{n=1}^{\infty} a_F(n) e^{-nz}$$

is entire. Taking $z = -1$, we get that

$$\sum_{n=1}^{\infty} a_F(n) e^{n}$$

is convergent and thus, $a_F(n) = o(e^{-n})$. So, the coefficients have exponential decay and therefore, $a_F(n)n^k \ll 1$ for all $k \geq 1$. Hence, we have that

$$\sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}$$

is absolutely convergent for all values of s, and bounded on any right half plane. We further notice that for $\Re(s) = \sigma > 1$

$$F(1-s) = \frac{\psi(s)}{\gamma(1-s)} \sim \frac{\gamma(s)}{\gamma(1-s)}.$$

Using Stirlings formula, we get

$$\left|\frac{\gamma(s)}{\gamma(1-s)}\right| \sim c(\sigma) t^{d_F(\sigma-1/2)},$$

where $c(\sigma) > 0$ is a constant depending on σ. Since $F(s)$ is bounded on any right half plane, we get $d_F = 0$ as required. $\square$

More recently, Perelli and Kaczorowski showed that for $F(s) \in \mathbb{S}$, if $0 < d_F < 2$ then $d_F = 1$. In general, the degree conjecture is still open. But, a stronger version of this conjecture is expected to be true.

Conjecture 3 (Stronger degree conjecture)**.** *For $F(s) \in \mathbb{S}$, there exists a functional equation of the form* (2.1)*, where each $\alpha_i = 1/2$.*

It is known due to Conrey and Ghosh [12] that the functions of degree one in the Selberg class are the Riemann zeta-function $\zeta(s)$, Dirichlet L-functions $L(s, \chi)$ and their shifts $L(s + i\theta, \chi)$, where χ is non-principal primitive and $\theta \in \mathbb{R}$. For higher degrees, there is no such classification so far.

However, there are known examples of elements in the Selberg class with higher degrees. Dedekind zeta-functions attached to a number field $K/\mathbb{Q}$ denoted $\zeta_K(s)$ has degree equal to the degree of the field extension $[K : \mathbb{Q}]$. L-functions associated to holomorphic newforms (see Example 5) have degree 2. Moreover, L-functions associated to non-holomorphic newforms, if in the Selberg class, would also have degree 2. The Rankin-Selberg L-function of normalized eigenforms are elements of the Selberg class of degree 4.

Apart from the alternate interpretation of degree in terms of the growth of the function (as used in the proof above), there is another interpretation in terms of the zero-distribution.

Let $F(s) \in \mathbb{S}$. Define

$$N(T, F) := \#\{ \text{ zeroes of } F(s) \text{ in the region } 0 < \Re(s) < 1 \text{ and } |\Im(s)| < T\}.$$

We have the asymptotic formula of $N(T, F)$ in terms of the degree of F given by

$$N(T, F) \sim \frac{d_F}{\pi} T \log T \tag{2.5}$$

as $T \to \infty$. We will revisit this topic in Chapters 3 and 4.

2.2 The Generalized Riemann hypothesis

From the Euler-product (2.2) representation of $F \in \mathbb{S}$, we conclude that $F(s)$ has no zeroes in the region $\Re(s) > 1$. Moreover, the functional equation (2.1) gives rise to zeroes of $F(s)$ corresponding to the poles of the Γ-factors. More precisely, we get that all the zeroes in the region $\Re(s) < 0$ are given by

$$s = -\frac{m + \beta_j}{\alpha_j},$$

for all non-negative integers m and α_j, β_j as in (2.1). These zeroes are called the trivial zeroes of $F(s)$. Thus, we have the notion of the critical strip $0 < \Re(s) < 1$ where all the other zeroes lie. These zeroes are called the non-trivial zeroes of $F(s)$. Due to the symmetric nature of the functional equation, it is expected that elements in the Selberg class must satisfy the analogue of the Riemann hypothesis.

Conjecture 4 (The Generalized Riemann hypothesis)**.** *There are no zeroes of* $F(s) \in \mathbb{S}$

in the region $\Re(s) > 1/2$.

Each axiom in the definition of the Selberg class is vital for expecting the Riemann hypothesis to hold.

1. The Dirichlet series representation of $F \in \mathbb{S}$ must be of the form $\sum_n a_n/n^s$. We do not include the generalized Dirichlet series given by $F(s) = \sum_n a_n e^{-\lambda_n s}$ in the Selberg class. For instance, $\zeta(s+1/2)$ is not a member of the Selberg class because it would violate the Riemann Hypothesis.

2. The condition that $F(s) \in \mathbb{S}$ has a possible pole at $s = 1$ is necessary because of the expectation that if $F(s) \in \mathbb{S}$ has a pole, then we should be able to write it as a product of the Riemann zeta-function and another entire L-function. Hence, we expect poles to only arise from the Riemann zeta-function. Therefore, if $F(s)$ has finitely many poles, we should be able to factorize it as product of shifted Riemann-zeta functions to account for the poles at points other than $s = 1$. For $a \in \mathbb{C}$ satisfying $\Re(a) \neq 0$, $\zeta(s+a)$ violates the Riemann hypothesis, because we would then have infinitely many zeroes on $\Re(s) = 1/2 - a$.

3. In the Euler product (2.2), the condition $\theta < 1/2$ is important because we expect the Euler factors to be convergent in the region $\Re(s) > 0$. For example, suppose the Euler factor corresponding to the prime $p = 2$ is given by

$$F_2(s) = \left(1 - \frac{2}{2^s}\right)^{-1} = \sum_{m=0}^{\infty} \frac{2^m}{2^{ms}}.$$

   Here we have $\theta = 1$ and clearly see that $F_2(s)$ has poles on $\Re(s) = 1$ and hence not analytic on $\Re(s) > 0$.

2.3 Selberg's Conjectures

The elements in the Selberg class are not closed under linear combination. But, the Selberg class is closed under multiplication and forms a semi-group with respect to multiplication i.e., if $F, G \in \mathbb{S}$, then $FG \in \mathbb{S}$.

We call the fundamental elements in $\mathbb{S}$ with respect to multiplication as the primitive elements.

Definition 2.3.1. *$F \in \mathbb{S}$ is said to be a primitive element if any factorization $F = F_1F_2$ with $F_1, F_2 \in \mathbb{S}$ implies that either $F_1 = 1$ or $F_2 = 1$.*

In other words, an element in $\mathbb{S}$ is primitive if it cannot be further factorized into non-trivial elements in $\mathbb{S}$.

Using the characterization of degree in (2.5), we have that if $F \in \mathbb{S}$ has a factorization $F = F_1F_2$, with $F_1, F_2 \in \mathbb{S}$, then

$$N(T, F) = N(T, F_1) + N(T, F_2).$$

Taking $T \to \infty$, we conclude that

$$d_F = d_{F_1} + d_{F_2}.$$

We also know from Theorem 2.1.1 that non-trivial elements in $\mathbb{S}$ cannot have degree < 1. Therefore, we cannot factorize an element $F \in \mathbb{S}$ indefinitely. So, every element in the Selberg class can be factorized into primitive elements.

Proposition 2.3.2. *Every element $F \in \mathbb{S}$ can be factorized into primitive elements in $\mathbb{S}$.*

It is still unknown whether the above factorization is unique.

Conjecture 5 (Unique Factorization in $\mathbb{S}$)**.** *Every element $F \in \mathbb{S}$ can be uniquely factorized into primitive elements.*

From the above discussion, it is clear that every element $F \in \mathbb{S}$ with degree $d_F = 1$ is a primitive element. Thus, the Riemann zeta function and Dirichlet L-functions are all primitive elements in the Selberg class. M. R. Murty [36] produced primitive elements of degree 2 associated to holomorphic newforms.

Selberg's conjectures claim that these distinct elements in $\mathbb{S}$ do not interact with each other. In particular, the primitive elements are in some sense orthogonal to each other.

Conjecture 6 (Selberg's conjectures)**.** *In [40], Selberg made the following conjectures.*

1. *Conjecture A - Let $F \in \mathbb{S}$. There exists a constant n_F such that*

$$\sum_{p \leq x} \frac{|a_F(p)|^2}{p} = n_F \log\log x + O(1). \tag{2.6}$$

2. *Conjecture B - Let $F, G \in \mathbb{S}$ be primitive elements. Then*

$$\sum_{p \leq x} \frac{a_F(p)\overline{a_G(p)}}{p} = \begin{cases} \log\log x + O(1), & \text{if } F = G, \\ O(1) & \text{otherwise.} \end{cases}$$

Conjecture B is known as Selberg's orthogonality conjecture.

It is easy to verify Conjecture A in particular cases. For instance, Conjecture A clearly holds for the Riemann zeta-function. Moreover, using the prime number theorem for arithmetic progression, one can verify Conjecture A for Dirichlet L-functions. Conjecture B can also be verified in the case of Dirichlet L-functions using the orthogonality relations for characters.

In the view of Proposition 2.3.2, it is easy to see that Conjecture B implies Conjecture

A. Let $F \in \mathbb{S}$ has a factorization into primitive elements given by

$$F(s) = F_1(s)F_2(s)\cdots F_m(s),$$

where $F_k(s)$ is primitive for all $1 \le k \le m$. Then, we have

$$\sum_{p \le x} \frac{|a_F(p)|^2}{p} = \sum_{1 \le j \le k \le m} \sum_{p \le x} \frac{a_{F_j}(p)\overline{a_{F_k}(p)}}{p}.$$

By Conjecture B, the above sum is of the form

$$m \log\log x + O(1),$$

where m is the number of factors in the factorization of $F(s)$ into primitive elements.

Selberg [40] noted that there are connections between these conjectures and several other conjectures like the Sato-Tate conjecture, Langlands conjectures etc. Conrey and Ghosh [12] showed that Conjecture B implies the unique factorization into primitives in $\mathbb{S}$ as shown in the proposition below. In [37], M.R. Murty proved that Conjecture B implies the strong Artin's conjecture. More precisely, he showed that for any irreducible representation ρ of $Gal(L/K)$ of degree n, there exists an irreducible cuspidal automorphic representation π of $GL_n(\mathbb{A}_{\mathbb{Q}})$, such that $L(s, \rho, L/K) = L(s, \pi)$. Since $L(s, \pi)$ are known to be entire, so is $L(s, \rho, L/K)$.

Proposition 2.3.3. *Conjecture B implies that every element $F \in \mathbb{S}$ has unique factorization into primitive elements.*

Proof. Suppose $F \in \mathbb{S}$ has two different factorizations into primitives, say,

$$F(s) = \prod_{j=1}^{m} F_j(s) = \prod_{k=1}^{r} G_k(s).$$

We can further assume that no F_j is same as G_k. Since

$$\sum_{j=1}^{m} a_{F_j}(p) = \sum_{k=1}^{r} a_{G_k}(p),$$

multiplying both sides by $\overline{a_{F_1}(p)}/p$ and summing over $p < x$, we get

$$\sum_{j=1}^{m} \sum_{p \le x} \frac{a_{F_j}(p)\overline{a_{F_1}(p)}}{p} = \sum_{k=1}^{r} \sum_{p \le x} \frac{a_{G_k}(p)\overline{a_{F_1}(p)}}{p}. \tag{2.7}$$

Now, Conjecture B implies that the LHS of (2.7) is unbounded where as the RHS is bounded as x tends to infinity, which leads to a contradiction. $\square$

By a similar argument as above, we also conclude the following.

Proposition 2.3.4. *An element $F \in \mathbb{S}$ is a primitive element if and only if $n_F = 1$, where n_F is given by* (2.6)*.*

The Selberg class is designed to model the class of L-functions satisfying the Riemann hypothesis. So, one might ask whether the analogue of prime number theorem is true for the elements in the Selberg class. Recall that the prime number theorem for natural numbers follows from the fact that $\zeta(s)$ does not vanish on the vertical line $\Re(s) = 1$. So, the following statement can be considered as the analogue to prime number theorem in the Selberg class.

Conjecture 7 (Generalized prime number theorem)**.** *If $F \in \mathbb{S}$, then $F(s) \neq 0$ for $s = 1 + it$ for any $t \in \mathbb{R}$.*

Unfortunately, this is still open. Recently, Kaczorowski and Perelli [24] proposed a refined version of the prime number theorem for the Selberg class. But, the above conjecture can be shown assuming the Conjecture B.

Proposition 2.3.5. *Conjecture B implies Conjecture 7.*

Proof. We use the following lemmas.

Lemma 2.3.6. *If $F \in \mathbb{S}$ has a pole or a zero at $s = 1 + i\theta$ for $\theta \in \mathbb{R}$, then*

$$\sum_{p \le x} \frac{a_p}{p^{1+i\theta}}$$

is unbounded as x tends to ∞.

Proof. If $F(s)$ has a pole or zero of order $m \neq 0$ at $1 + i\theta$, then we have

$$F(s) \sim c(s - (1 + i\theta))^m,$$

near $1 + i\theta$. Writing $s = \sigma + it$ and taking log to get

$$\log F(s) \sim m \log(\sigma - 1),$$

near $s = 1 + i\theta$. Moreover, from the Euler product, we have for $\sigma > 1$,

$$\log F(s) = \sum_p \frac{a_F(p)}{p^s} + O(1).$$

Thus, we get

$$\sum_p \frac{a_F(p)}{p^s} \sim m \log(\sigma - 1),$$

as $\sigma \to 1^+$. Assume the function

$$S(x) = \sum_{p \le x} \frac{a_F(p)}{p^{1+i\theta}}$$

is bounded. Then, we have

$$\begin{aligned}\sum_p \frac{a_F(p)}{p^s} &= \int_1^\infty x^{1-\sigma} dS(x) \\ &= (\sigma - 1)\int_1^\infty S(x)x^{-\sigma}\,dx \ll 1,\end{aligned}$$

which is a contradiction. □

There are many more interesting consequences of the orthogonality Conjecture B. Using a similar argument as in Proposition 2.3.3, one can prove that the Conjecture B implies that if $F \in \mathbb{S}$ has a pole at $s = 1$, it must be coming the Riemann-zeta function. More precisely,

Lemma 2.3.7. *If $F(s) \in \mathbb{S}$ has a pole of order m at $s = 1$, then Conjecture B implies that $\zeta(s)^m$ divides $F(s)$.*

Proof. Since Conjecture B implies unique factorization into primitive elements in $\mathbb{S}$, it suffices to show that if $F \in \mathbb{S}$ is a primitive element with a pole at $s = 1$, then it is $\zeta(s)$. From Proposition 2.3.4 we know that $n_\zeta = 1$ and $n_F = 1$. If $F \neq \zeta$, then Conjecture B implies that

$$\sum_{p \le x} \frac{a_F(p)}{p} \ll 1,$$

which is a contradiction. □

This expectation that every pole comes from $\zeta(s)$ can be thought of as the amerlioration of the Dedekind's conjecture, which states that every Dedekind zeta-function $\zeta_K(s)$ must factorize through $\zeta(s)$.

Now, we are ready to prove the proposition. Since Conjecture B implies unique factorization, it is enough to show the non-vanishing of $F(s)$ on $\Re(s) = 1$ for primitive elements $F \in \mathbb{S}$. Since $\zeta(s)$ do not vanish on $\Re(s) = 1$, using Lemma 2.3.7, we can further assume that $F(s)$ is entire. This implies that $F(s + i\alpha) \in \mathbb{S}$ for any $\alpha \in \mathbb{R}$.

Now, if F has a zero at $s = 1 + i\theta$, Lemma 2.3.6 implies that

$$\sum_{p \le x} \frac{a_F(p)}{p^{1+i\theta}}$$

is unbounded as $x \to \infty$. But the Conjecture B applied to $\zeta(s)$ and $F(s + i\theta)$ yields

$$\sum_{p \le x} \frac{a_F(p)}{p^{1+i\theta}} \ll 1,$$

which leads to a contradiction. □

In [24], Kaczorowski and Perelli proved the Conjecture 7 with an assumption weaker than Conjecture A. This weaker assumption is often called the Normality conjecture, where the Conjecture A is assumed with a weaker error term.

2.4 The Lindelöf Hypothesis

We say that a function $F(s)$, analytic in the strip $\sigma_1 \le \Re(s) \le \sigma_2$ with finitely many poles in this region, is of finite order in this strip, if there exists a constant $c > 0$ such that

$$F(s) \ll |t|^c \text{ as } t \to \infty.$$

This order is different from the usual complex Ritt order (see Steuding [43, p.36]). One can make similar definition of finite order in a right half plane, say $\Re(s) > \sigma_1$. Any Dirichlet series is of finite order in its half-plane of convergence (conditional convergence).

The order of growth of meromorphic function is an important theme. As we shall see, any Dirichlet series satisfying a Riemann type functional equation is of finite order in any half-plane $\Re(s) > \sigma$ for any $\sigma \in \mathbb{R}$. First, we make the following definition.

Definition 2.4.1. *Let $F(s) \in \mathbb{S}$. Define*

$$\mu_F(\sigma) := \limsup_{|t|\to\infty} \frac{\log|F(\sigma+it)|}{\log|t|}.$$

Since $F(s)$ is absolutely convergent on $\Re(s) > 1$, we immediately obtain that $F(s)$ is bounded in the region $\Re(s) > 1+\epsilon$ for any $\epsilon > 0$. Hence, $\mu_F(\sigma) = 0$ for $\sigma > 1+\epsilon$. The order of growth in the left half plane $\Re(s) < 0$ is obtained using the functional equation

$$F(s) = \frac{\gamma(1-\overline{s})}{\gamma(s)}\overline{F(1-\overline{s})},$$

where the gamma-factor is given by

$$\gamma(s) = Q^s \prod_{j=1}^{m} \Gamma(\alpha_j s + \beta_j).$$

Applying Stirling's formula, we get for $t \geq 1$, uniformly in σ,

$$\frac{\gamma(1-\overline{s})}{\gamma(s)} = \left(\alpha Q^2 t^{d_F}\right)^{1/2-\sigma-it} \exp\left(itd_F + \frac{i\pi(\beta - d_F)}{4}\right)\left(\omega + O\left(\frac{1}{T}\right)\right), \tag{2.8}$$

where

$$\alpha := \prod_{j=1}^{k} \alpha_j^{2\alpha_j} \text{ and } \beta := 2\sum_{j=1}^{k}(1-2\beta_j).$$

Recall the Phragmén-Lindelöf theorem given by

Theorem 2.4.2 (Phragmén-Lindelöf)**.** *Let $f(s)$ be analytic in the strip $\sigma_1 < \Re(s) \leq \sigma_2$ with $f(s) \ll \exp(\epsilon|t|)$. If*

$$|f(\sigma_1 + it)| \ll |t|^{c_1} \textit{ and}$$

$$|f(\sigma_2 + it)| \ll |t|^{c_2},$$

then

$$|f(\sigma+it)| \ll |t|^{c(\sigma)},$$

uniformly in $\sigma_1 \leq \sigma \leq \sigma_2$*, where* $c(\sigma)$ *is linear in* σ *with* $c(\sigma_1)=c_1$ *and* $c(\sigma_2)=c_2$*.*

Using the Phragmén-Lindelöf theorem and (2.8), we get the following upper bounds on the growth of an element in $\mathbb{S}$.

Theorem 2.4.3. *Let* $F \in \mathbb{S}$*. Uniformly in* σ*, as* $|t| \to \infty$*,*

$$F(\sigma+it) \sim |t|^{(1/2-\sigma)d_F}|F(1-\sigma+it)|.$$

We also have

$$\mu_F(\sigma) \leq \begin{cases} 0 & \textit{if } \sigma > 1, \\ \frac{1}{2}d_F(1-\sigma) & \textit{if } 0 \leq \sigma \leq 1, \\ d_F(\frac{1}{2}-\sigma) & \textit{if } \sigma < 0. \end{cases}$$

Lindelöf conjectured that the order of growth of the Riemann zeta-function is much smaller than what the Phragmén-Lindelöf principle gives. In fact, he conjectured that $\zeta(s)$ is bounded on $\sigma > 1/2$. This statement is known to be false. But, a weaker version would state that $\mu_\zeta(1/2)=0$. In other words,

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \ll |t|^{\epsilon},$$

for any $\epsilon > 0$. This is known as the Lindelöf hypothesis. Note that, the Phragmén-Lindelöf theorem only implies that $|\zeta(1/2+it)| \ll_\epsilon |t|^{1/4+\epsilon}$ for any $\epsilon > 0$. Any improvement on the constant 1/4 is called the phenomena of "breaking convexity". The best known improvement on this constant is replacing 1/4 with 9/56. This is due to Bombieri and Iwaniec [7] using Weyl's method of estimating exponential sums, which was earlier incorporated by Hardy and Littlewood to attack the same problem.

More general statement of the Lindelöf hypothesis on the Selberg class is given by

Conjecture 8 (Generalized Lindelöf hypothesis)**.** *For $F \in \mathbb{S}$ and any $\epsilon > 0$,*

$$\left| F\left(\frac{1}{2} + it\right) \right| \ll |t|^{\epsilon}.$$

It is known due to Littlewood that the Riemann hypothesis implies the Lindelöf hypothesis. Further, Conrey and Ghosh [13] showed that the Generalized Riemann hypothesis implies the Generalized Lindelöf hypothesis. They also computed the implied constant in terms of the invariants arising from the functional equation of $F(s)$. Moreover, the Lindelöf hypothesis itself has many interesting consequences. The most prominent one is in the context of value distribution of L-functions.

For $F \in \mathbb{S}$, let $N_F(\sigma, T)$ denote the number of zeroes of $F(s)$ in the region

$$\left\{ s \in \mathbb{C} : \Re(s) > \sigma, |\Im(s)| < T \right\}.$$

The Lindelöf hypothesis for Riemann zeta-function implies the density hypothesis, which states that for $\sigma > 1/2$,

$$N_\zeta(\sigma, T) \ll T^{2(1-\sigma)}.$$

In case of the Selberg class, the Lindelöf hypothesis implies a statement regarding the zero-distribution of L-functions, which we call the zero hypothesis.

The classical result on zero density estimate due to Bohr and Landau [6] states that most of the zeroes of $\zeta(s)$ are clustered near the 1/2-line, i.e., they showed that

$$N_\zeta(\sigma, T) \ll T^{4\sigma(1-\sigma)}, \tag{2.9}$$

for $\sigma > 1/2$.

More recently, we have the following density theorem due to Perelli and Kaczorowski

[24] for the Selberg class.

Theorem 2.4.4 (Density Theorem). *For $F \in \mathbb{S}$,*

$$N_F(\sigma, T) \ll_\epsilon T^{c(1-\sigma)},$$

for $\sigma > 1/2$ and $c = 4d_F + 12$.

The above zero-density estimate suggests that the number of zeroes close to the vertical line $\Re(s) = 1$ is very small. In general, we formulate the zero hypothesis, which claims that for $F \in \mathbb{S}$ all the zeroes are clustered near the 1/2-line.

Conjecture 9 (Zero Hypothesis). *For $F \in \mathbb{S}$, there is a positive constant c such that for $\sigma > 1/2$,*

$$N_F(\sigma, T) \ll T^{1-c(\sigma-1/2)+\epsilon}.$$

Chapter 3

The Lindelöf class

The Selberg class, as described in Chapter 2, comprises of a general class of L-functions satisfying axioms, which are expected of functions similar to the Riemann zeta-function. Despite its generality, the Selberg class $\mathbb{S}$ has several limitations. For instance, it is not closed under addition. This is due to the rigidity of the functional equation and the Euler product, which are crucial in the definition of $\mathbb{S}$. The functional equation provides some satisfactory answers to the value distribution, in particular the zero distribution of L-functions. But, since a linear combination of L-functions may not satisfy a functional equation, the question of their value distribution, which is of significance, is not answered by studying the Selberg class.

Moreover, some naturally occurring L-functions such as the Epstein zeta function, do not always have an Euler product and hence are not members of the Selberg class. This raises the question of studying a larger class of L-functions, which would contain linear combinations of known L-functions, i.e., closed under addition and also include some naturally occurring L-functions, which are not part of the Selberg class.

This motivated V. Kumar Murty [34] to introduce a class of L-functions defined based on growth conditions. As we shall see in Chapter 4 and 5, this larger class of L-functions seems to successfully find common patterns in its value distribution as in the case of the

Selberg class. Additionally, this class of L-functions also has a richer algebraic structure.

In the following section, we carefully define some growth parameters and introduce a class of L-functions namely the class $\mathbb{M}$.

3.1 The growth parameters

Definition 3.1.1. The class $\mathbb{T}$. *Define the class* $\mathbb{T}$ *to be the set of functions* $F(s)$ *satisfying the following conditions:*

1. **Dirichlet series** - *For* $\sigma > 1$, $F(s)$ *is given by the absolutely convergent Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}.$$

2. **Analytic continuation** - *There exists a non-negative integer* k, *such that* $(s-1)^k F(s)$ *is an entire function of order* ≤ 1.

3. **Ramanujan hypothesis** - $|a_F(n)| = O_\epsilon(n^\epsilon)$ *for any* $\epsilon > 0$.

For the class $\mathbb{T}$, we define growth parameters μ and μ^*, which would be analogous to the definition of μ_F in 2.4.1 of Chapter 2.

Definition 3.1.2. *Let* $F \in \mathbb{T}$ *be entire. Define* $\mu_F(\sigma)$ *as*

$$\mu_F(\sigma) := \begin{cases} \inf\left\{\lambda \geq 0 : |F(s)| \leq (|s|+2)^\lambda, \ \textit{for all } s \textit{ with } Re(s) = \sigma\right\}, \\ \\ \infty, \ \textit{if the infimum does not exist.} \end{cases} \tag{3.1}$$

Also define:

$$\mu_F^*(\sigma) := \begin{cases} \inf\left\{\lambda \geq 0 : |F(\sigma+it)| \ll_\sigma (|t|+2)^\lambda\right\}, \\ \\ \infty, \ \textit{if the infimum does not exist.} \end{cases} \tag{3.2}$$

In the definition of $\mu_F^*(\sigma)$, the implied constant depends on both F and σ, where as in $\mu_F(\sigma)$, the constant only depends on F and is independent of σ.

We further extend the definition of μ_F and μ_F^* to all the elements in class $\mathbb{T}$ as follows. Suppose $F \in \mathbb{T}$ has a pole of order k at $s = 1$. Consider the function

$$G(s) := \left(1 - \frac{2}{2^s}\right)^k F(s). \tag{3.3}$$

Clearly, $G(s)$ is an entire function and belongs to $\mathbb{T}$. We define

$$\mu_F(\sigma) := \mu_G(\sigma),$$
$$\mu_F^*(\sigma) := \mu_G^*(\sigma).$$

Intuitively, $\mu_F^*(\sigma)$ does not see how $F(s)$ behaves close to the real axis. It is only dependent on the growth of $F(s)$ on $\Re(s) = \sigma$ and $\Im(s) \ll T$ for arbitrary large T. On the other hand, $\mu_F(\sigma)$ captures an absolute bound for $F(s)$ on the entire vertical line $\Re(s) = \sigma$.

It follows from the definition that

$$\mu_F^*(\sigma) \leq \mu_F(\sigma)$$

for any σ. From the above definition, we immediately conclude the following.

Proposition 3.1.3. *Let $F \in \mathbb{T}$. For $\sigma > 1 + \epsilon$,*

$$\mu_F^*(\sigma) = 0$$
$$\mu_F(\sigma) \ll_{F,\epsilon} 1.$$

for any $\epsilon > 0$.

Proof. Since $F \in \mathbb{T}$, it is given by a Dirichlet series $F(s) = \sum_n a_n/n^s$, which is absolutely convergent for $\sigma > 1$ and hence bounded in the region $\sigma \geq 1+\epsilon$, with the bound depending on F and ϵ, but independent of σ. Hence, we have the proposition. $\square$

If $F \in \mathbb{S}$, by the functional equation (2.1), using Stirling's formula, we have (see [34], Sec.2.1])

$$\mu_F^*(\sigma) \leq \frac{1}{2} d_F(1-2\sigma) \text{ for } \sigma < 0. \tag{3.4}$$

Using the Phragmén-Lindelöf theorem, we deduce that

$$\mu_F^*(\sigma) \leq \frac{1}{2} d_F(1-\sigma) \text{ for } 0 < \sigma < 1.$$

The same results hold for μ_F upto a constant depending on F. To see this, we use the functional equation for F,

$$F(s) = \Delta_F(s)\overline{F(1-\overline{s})},$$

where

$$\Delta_F(s) := \omega Q^{1-2s} \prod_{j=1}^{k} \frac{\Gamma(\alpha_j(1-s)+\overline{\beta}_j)}{\Gamma(\alpha_j s+\beta_j)}.$$

Using the Stirling's formula, we get

Lemma 3.1.4. *For $F \in \mathbb{S}$ and $t \geq 1$, uniformly in σ,*

$$\Delta_F(\sigma+it) = \left(\alpha Q^2 t^{d_F}\right)^{1/2-\sigma-it} \exp\left(itd_F + \frac{i\pi(\beta - d_F)}{4}\right)\left(\omega + O\left(\frac{1}{T}\right)\right),$$

where

$$\alpha := \prod_{j=1}^{k} \alpha_j^{2\alpha_j} \text{ and } \beta := 2\sum_{j=1}^{k}(1-2\beta_j).$$

From the above Lemma 3.1.4, we conclude that

$$\mu_F(\sigma) \leq \frac{1}{2} d_F(1-2\sigma) + O(1) \text{ for } \sigma < 0. \tag{3.5}$$

and

$$\mu_F(\sigma) \leq \frac{1}{2} d_F(1-\sigma) + O(1) \text{ for } 0 < \sigma < 1.$$

Thus, for $F \in \mathbb{S}$, these parameters $\mu_F(\sigma)$ and $\mu_F^*(\sigma)$ are well-defined. We imitate this behaviour of μ and μ^* to introduce a growth condition. This leads to the definition of class $\mathbb{M}$.

Definition 3.1.5. The class $\mathbb{M}$. *Define the class $\mathbb{M}$ (see [34], Sec.2.4) to be the set of functions $F(s)$ satisfying the following conditions:*

1. **Dirichlet series** - *$F(s)$ is given by a Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{a_F(n)}{n^s},$$

which is absolutely convergent in the right half plane $\Re(s) > 1$.

2. **Analytic continuation** - *There exists a non-negative integer k such that $(s-1)^k F(s)$ is an entire function of order ≤ 1.*

3. **Growth condition** - *The quantity $\frac{\mu_F(\sigma)}{(1-2\sigma)}$ is bounded for $\sigma < 0$.*

4. **Ramanujan hypothesis** - $|a_F(n)| = O_\epsilon(n^\epsilon)$ *for any $\epsilon > 0$.*

Notice that in the condition of analytic continuation, we have to force the complex order to be ≤ 1. In case of the Selberg class, this condition is implicit due to the functional equation.

3.2 The invariants c_F and c_F^*

From the observation (3.4) and (3.5), we define the following invariants for $\mathbb{M}$, which would be considered analogous to the notion of degree in $\mathbb{S}$.

Definition 3.2.1. *For $F \in \mathbb{M}$, define*

$$c_F := \limsup_{\sigma<0} \frac{2\mu_F(\sigma)}{1-2\sigma},$$
$$c_F^* := \limsup_{\sigma<0} \frac{2\mu_F^*(\sigma)}{1-2\sigma}.$$

By the growth condition, c_F and c_F^* are well-defined in $\mathbb{M}$. Moreover, since $\mu_F^*(\sigma) \leq \mu_F(\sigma)$ for all $\Re(s) = \sigma$, we have

$$c_F^* \leq c_F.$$

Note that c_F and c_F^* are ≥ 0. Using the Phragmén-Lindelöf theorem, we have

$$\mu_F(\sigma) \leq \frac{1}{2}c_F(1-\sigma),$$
$$\mu_F^*(\sigma) \leq \frac{1}{2}c_F^*(1-\sigma)$$

for $0 < \sigma < 1$. We mention a few examples.

Proposition 3.2.2. *Any Dirichlet polynomial F belongs to $\mathbb{M}$ and $c_F = c_F^* = 0$.*

Proof. Suppose

$$F(s) = \sum_{n=1}^{m} \frac{a_n}{n^s}.$$

Let $s = \sigma + it$. We can write

$$F(s) = a_1 + 2^{-\sigma}a_2 2^{-it} + 3^{-\sigma}a_3 3^{-it} + ... + m^{-\sigma}a_m m^{-it}.$$

Therefore, for some large k, there exists a constant $M > 0$, such that

$$|F(s)| \leq Mm^{-\sigma},$$

when $\sigma < -k$. By changing the base, we have

$$Mm^{-\sigma} = M(|\sigma|+2)^{|\sigma|\log_{(|\sigma|+2)} m},$$

for $\sigma < -k$. By (3.1), we get

$$\mu_F(\sigma) \leq |\sigma| \log_{(|\sigma|+2)} m.$$

Therefore, we conclude,

$$\limsup_{\sigma<0} \frac{\mu_F(\sigma)}{1-2\sigma} = 0.$$

Hence, $c_F = 0$ which implies $c_F^* = 0$. □

Proposition 3.2.3. *Any Dirichlet series F which is absolutely convergent on the whole complex plane has $c_F^* = 0$.*

Proof. By (3.2), we can conclude that if a Dirichlet series F is absolutely convergent on the vertical line $\Re(s) = \sigma$, then $\mu_F^*(\sigma) = 0$. Since, $\mu_F^*(\sigma) = 0$ for all σ, we have $c_F^* = 0$. □

Proposition 3.2.4. *The Riemann zeta-function $\zeta(s)$ is in $\mathbb{M}$ and $c_\zeta = c_\zeta^* = 1$.*

Proposition 3.2.5. *If F is an element in the Selberg class, then F also belongs to $\mathbb{M}$. Moreover, $c_F = c_F^*$ and is given by the degree of F.*

The proof of Proposition 3.2.4 and 3.2.5 will follow from Proposition 3.2.13.

Proposition 3.2.6. *Linear combinations of elements in the Selberg class $\mathbb{S}$ are in $\mathbb{M}$.*

We shall see this later, when we prove that $\mathbb{M}$ forms a ring.

Another example of an L-functions in $\mathbb{M}$, which is not constructed from linear combination of elements in $\mathbb{S}$ are the translates of Epstein zeta-functions. For a given real positive definite $n \times n$-matrix T, the Epstein zeta-function is defined as (see [17], [18])

$$\zeta(T,s) := \sum_{\mathbf{0}\neq\mathbf{v}\in\mathbb{Z}^n} (\mathbf{v}^t T\mathbf{v})^{-s}.$$

This series is absolutely convergent for $\Re(s) > n/2$. It can be analytically continued to $\mathbb{C}$ except for a simple pole at $s = n/2$ with residue

$$\frac{\pi^{n/2}}{\Gamma(n/2)\sqrt{\det T}}.$$

Moreover, it satisfies a functional equation. Let

$$\psi(T,s) := \pi^{-s/2}\Gamma(s)\zeta(T,s).$$

Then,

$$\psi(T,s) = (\det T)^{-1/2}\psi\left(T^{-1}, \frac{n}{2} - s\right).$$

Thus, the function $\zeta(T, s+n/2-1)$ is an element in $\mathbb{M}$. The growth condition is satisfied because of the functional equation and further we have $c_{\zeta(T,s+n/2-1)} = c^*_{\zeta(T,s+n/2-1)} = 2$.

Proposition 3.2.7. *If $F(s)$ belongs to $\mathbb{M}$, then all its translates given by $F(s)+a$ also belong to $\mathbb{M}$. Moreover, they have the same values of c_F and c^*_F.*

In order to ensure that $\mathbb{M}$ is closed under addition, we need to establish that these invariants c_F and c^*_F in fact satisy an ultrametric inequality.

Proposition 3.2.8 ([34], Prop. 1)**.** *For $F, G \in \mathbb{M}$,*

$$c_{FG} \leq c_F + c_G \quad \textit{and} \quad c_{F+G} \leq \max(c_F, c_G).$$

Similarly,

$$c^*_{FG} \leq c^*_F + c^*_G \quad \textit{and} \quad c^*_{F+G} \leq \max(c^*_F, c^*_G).$$

In fact, if $c_F > c_G$ (resp. $c_F^* > c_G^*$), then

$$c_{F+G} = c_F \ (\textit{resp. } c_{F+G}^* = c_F^*).$$

Proof. The proof follows immediately from the definition of μ_F and μ_F^*. This is because, for any fixed σ and $|t| > 1$, we have

$$|F(\sigma + it)| \ll_\sigma |t|^{\mu_F^*(\sigma)+\epsilon},$$

for any $\epsilon > 0$. Therefore, for any $F, G \in \mathbb{M}$, we have

$$|FG(\sigma + it)| \ll_\sigma |t|^{\mu_F^*(\sigma)+\mu_G^*(\sigma)+\epsilon}.$$

Similarly,

$$\begin{aligned} |(F+G)(\sigma + it)| &\ll_\sigma |t|^{\mu_F^*(\sigma)} + |t|^{\mu_G^*(\sigma)+\epsilon} \\ &\ll_\sigma |t|^{\max(\mu_F^*(\sigma), \mu_G^*(\sigma))+\epsilon}. \end{aligned}$$

Incorporating this into the definition of c_F^*, we are done. Proof of the ultra-metric inequalities for c_F follows from a similar argument by replacing μ_F^* with μ_F. □

As mentioned in the Example 4 above, for $F \in \mathbb{S}$, c_F and c_F^* coincide with the degree d_F of F. Since $d_{FG} = d_F + d_G$, we immediately get for $F, G \in \mathbb{S}$,

$$\begin{aligned} c_{FG} &= c_F + c_G, \\ c_{FG}^* &= c_F^* + c_G^*. \end{aligned}$$

The degree conjecture, as described in Conjecture 2 of Chapter 2, claims that the degree of any element in $\mathbb{S}$ must be a non-negative integer. The question arises if we can make similar claims about these invariants c_F and c_F^*. As it turns out, c_F can take non-integer values, in fact one can manufacture functions in $\mathbb{M}$ with any arbitrary non-negative value of c_F. But, we expect the analogue of the degree conjecture to be true for the invariant c_F^*.

Conjecture 10. *Let $F \in \mathbb{M}$, then c_F^* is always a non-negative integer.*

In this direction imitating the argument of Conrey-Ghosh [12], in which they showed that the degree d_F in the Selberg class cannot take non-integer values between 0 and 1, we get the following result.

Proposition 3.2.9 (Corrected version of [34], Prop. 3)**.** *Suppose $F \in \mathbb{M}$. Then $c_F < 1$ implies $c_F^* = 0$.*

Proof. Let $F \in \mathbb{M}$. For $\sigma > 1$, let $F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}$. Then we know that,

$$f(x) := \sum_{n=1}^{\infty} a_F(n) e^{-nx} = \frac{1}{2\pi i} \int_{(2)} F(s)\Gamma(s) x^{-s} ds,$$

where the integration is on the line $Re(s) = 2$.

By the growth condition and convexity, F has a polynomial growth in $|t|$ in vertical strips. Thus, moving the line of integration to the left and taking into account the possible pole at $s = 1$ of $F(s)$ and poles of $\Gamma(s)$ at $s = 0, -1, -2, ...$, we find that

$$f(x) = \frac{P(\log x)}{x} + \sum_{n=0}^{\infty} \frac{F(-n)}{n!} (-x)^n, \tag{3.6}$$

where P is a polynomial. By the definition of c_F, we have

$$|F(-n)| \ll n^{\frac{1}{2} c_F (1+2n) + \epsilon}.$$

Using Stirling's formula, i.e,

$$n! \sim n^{n-1/2} e^{-n} (2\pi)^{-1/2},$$

we get

$$\left|\frac{F(-n)}{n!}(-x)^n\right| \ll n^{\frac{c_F+1}{2}+\epsilon}\left(\frac{e|x|n^{c_F}}{n}\right)^n.$$

If $c_F < 1$, then the series in equation (3.6) converges absolutely for all values of x. Hence, the function $f(x)$ is analytic in $\mathbb{C}\backslash\{x \leq 0 : x \in \mathbb{R}\}$. But, this function is also periodic with period $2\pi i$ and so it converges for all x. Thus, the function

$$f(z) = \sum_{n=1}^{\infty} a_F(n)e^{-nz}$$

is entire. Taking $z = -1$, we get that

$$\sum_{n=1}^{\infty} a_F(n)e^{n}$$

is convergent and thus, $a_F(n) = o(e^{-n})$. So, the coefficients have exponential decay and therefore, $a_F(n)n^k \ll 1$ for all $k \geq 1$. Hence, we have that

$$\sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}$$

is absolutely convergent for all values of s. Therefore, $\mu_F^*(\sigma) = 0$ for all σ and hence, $c_F^* = 0$. □

We can completely characterize all $F \in \mathbb{M}$ such that $c_F^* = 0$. These are precisely all the functions whose Dirichlet series is convergent on the whole complex plane, up to a Dirichlet polynomial. We invoke the following theorem of Landau to prove the result.

Theorem 3.2.10 (Landau, [20] Chapter VII, sec. 10, Thm. 51)**.** *Let $F(s)$ be an entire function. Suppose $F(s)$ has a Dirichlet series representation*

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

which is absolutely convergent for $\Re(s) > 1$. *Also, suppose that*

$$a_n = O(n^{\epsilon})$$

for all positive ϵ. *If*

$$F(s) = O(|t|^{\beta}) \quad (\beta > 0)$$

uniformly in the half plane $\Re(s) > \eta$, *then the Dirichlet series is convergent in the half plane* $\Re(s) > \eta_1$, *where*

$$\eta_1 = \begin{cases} \frac{\eta+\beta}{1+\beta}, & \text{if } \eta+\beta > 0 \\ \eta+\beta, & \text{if } \eta+\beta < 0. \end{cases}$$

Using the above Theorem 3.2.10, we get the following result classifying all elements in $\mathbb{M}$ with $c_F^* = 0$.

Proposition 3.2.11. *Suppose* $F \in \mathbb{M}$ *and let*

$$H(s) = 1 - \frac{2}{2^s}.$$

If $c_F^* = 0$, *then the Dirichlet series given by*

$$H(s)^k F(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s},$$

is absolutely convergent on the whole complex plane, where k *is the order of the possible pole of* $F(s)$ *at* $s = 1$.

Proof. Let $F \in \mathbb{M}$. Suppose σ_a is the abscissa of absolute convergence for the Dirichlet series associated with F. If the Dirichlet series is not convergent on the whole complex plane, then we have $\sigma_a > -\infty$. Note that

$$G(s) := F(s + \sigma_a - 1)$$

is in $\mathbb{M}$ whose abscissa of absolute convergence is $\sigma_a = 1$. Therefore, without loss of generality we assume that the Dirichlet series $F(s) = \sum_{n=1}^{\infty} a_n/n^s$ has abscissa of absolute convergence $\sigma_a = 1$.

Since $c_F^* = 0$, we can choose a $\sigma_1 < 0$ such that

$$F(\sigma_1 + it) = O((|t| + 2)^{\epsilon(1/2-\sigma_1)}),$$

for any $\epsilon > 0$. Using the Phragmén-Lindelöf theorem, we get

$$F(\sigma + it) = O((|t| + 2)^{\epsilon(1/2-\sigma)}) \tag{3.7}$$

for $\sigma_1 < \sigma < 1$. Let σ be fixed. For any $\epsilon_1 > 0$, choose $\epsilon = \frac{2\epsilon_1}{(1-2\sigma)}$ in (3.7) to get

$$|F(\sigma + it)| = O((|t| + 2)^{\epsilon_1}).$$

Suppose $F(s)$ has a pole of order k at $s = 1$. Define

$$G(s) := \left(1 - \frac{2}{2^s}\right)^k F(s).$$

$G(s)$ is analytic on the whole complex plane and $G \in \mathbb{M}$ with the Dirichlet series representation

$$G(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

with abscissa of absolute convergence at $\sigma_a = 1$. Also $c_G^* = c_F^* = 0$ by the definition of μ_F^*.

Moreover, for $\sigma_1 \ll 0$ and any $\epsilon > 0$, we have

$$|G(\sigma + it)| = O((|t|)^\epsilon)$$

uniformly for $\sigma > \sigma_1$. By Theorem 3.2.10, we conclude that the Dirichlet series representation of

$$G(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

converges in the half plane $\Re(s) > \sigma_1 + \epsilon$. Therefore, the abscissa of absolute convergence is $\sigma_0 < \sigma_1 + 1 + \epsilon < 1$, which contradicts the assumption that $\sigma_0 = 1$.

Hence $\sigma_a = -\infty$ and the Dirichlet series representation of

$$G(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

is convergent on the whole complex plane and the proposition follows. □

Next, we show that the analogue of the degree conjecture given by Conjecture 10 holds between 0 and 1, i.e., c_F^* does not take non-integer values between 0 and 1.

Proposition 3.2.12. *Suppose $F(s) \in \mathbb{M}$. Then $c_F^* < 1$ implies $c_F^* = 0$.*

Proof. Let $c_F^* < 1$. Without loss of generality, we assume

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

has the abscissa of absolute convergence $\sigma_0 = 1$. If F has a pole of order k at $s = 1$, we consider

$$G(s) := \left(1 - \frac{2}{2^s}\right)^k F(s).$$

As discussed in the proof of Proposition 3.2.11,

$$G(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

also has abscissa of absolute convergence at $\sigma_0 = 1$. Moreover, $c_G^* = c_F^*$.

Hence, for any $\epsilon > 0$, we have a $\sigma_1 < 0$, such that

$$|G(\sigma_1 + it)| = O((|t| + 2)^{c_F^*(1/2-\sigma_1)+\epsilon}).$$

Using Phragmén-Lindelöf theorem on the strip $\sigma_1 < \Re(s) < 1$, we get

$$|G(\sigma + it)| = O((|t| + 2)^{c_F^*(1/2-\sigma)+\epsilon}) \tag{3.8}$$

for $\sigma > \sigma_1$. By Theorem 3.2.10, we conclude that the Dirichlet series of $G(s)$ converges in the half plane $\Re(s) > \sigma + c_F^*(\frac{1}{2} - \sigma) + \epsilon_1$ for any $\sigma < 0$ and $\epsilon_1 > 0$. Since $c_F^* < 1$, choosing $\sigma \ll 0$ such that $\sigma - c_F^*\sigma < -1$, we get the convergence of the Dirichlet series of $G(s)$ on the half plane

$$\Re(s) > -1 + \frac{1}{2}c_F^*.$$

Therefore, the Dirichlet series converges on $\Re(s) > -\frac{1}{2}$. Since the abscissa of absolute convergence for the Dirichlet series of $G(s)$ is $\sigma_0 = 1$, we know that the abscissa of convergence is ≥ 0, which leads to a contradiction. $\square$

We finally show that these invariants c_F and c_F^* in fact coincide with the degree of the Selberg class.

Proposition 3.2.13. *Let $F \in \mathbb{M}$, and suppose that F has a functional equation of the Riemann type as in the Selberg class. Then $c_F = d_F$, where d_F is the degree of F in the sense of the Selberg class.*

Proof. For simplicity, assume F has only one Γ-factor. We have

$$Q^s\Gamma(as + b)F(s) = wQ^{1-\bar{s}}\Gamma(a(1 - \bar{s}) + b)F(1 - \bar{s}),$$

where a, Q are non-negative real numbers, $b \in \mathbb{C}$ with $\Re(b) \geq 0$ and $|w| = 1$. We know from (3.4) that $c_F \leq d_F$. So we only need to show that c_F is at least d_F. We shall first show it for the Riemann-zeta function and use this template to prove it in general. Substituting $s = 1 - 2k$ for any integer $k > 0$ in the functional equation for $\zeta(s)$ gives

$$\pi^{-(1-2k)/2}\zeta(1-2k)\Gamma\left(\frac{1-2k}{2}\right) = \pi^{k}\zeta(2k)\Gamma\left(\frac{1+2k}{2}\right).$$

Thus, we get

$$\zeta(1-2k) = \frac{\pi^{s-1/2}\zeta(2k)\Gamma(k+1/2)}{\Gamma(1/2-k)}.$$

Repeatedly using the identity $\Gamma(s+1) = s\Gamma(s)$, we have

$$|\zeta(1-2k)| = \frac{\Gamma(2k)\zeta(2k)}{2^{2k-1}\pi^{2k}}.$$

Since $\zeta(2k)$ is bounded, using Stirling's formula we conclude that

$$|\zeta(1-2k)| \sim (2k)^{2k-\frac{1}{2}}\,A,$$

where $A \ll k^k$. Thus, we conclude that $c_\zeta = 1$ simply by considering the values taken by ζ on the real line.

Imitating this proof in general, substitute $s = 1 - ck$ for any integer $k > 0$ and any positive constant c, such that $a(1-ck)+b$ is not an integer. The functional equation gives

$$Q^{1-ck}\Gamma(a+b-ack)F(1-ck) = wQ^{ck}\Gamma(ack+b)F(ck).$$

Thus, we get

$$F(1-ck) = \frac{wQ^{2ck-1}\Gamma(ack+b)F(ck)}{\Gamma(a+b-ack)}.$$

Repeatedly using the identity $\Gamma(s+1) = s\Gamma(s)$ and then using Stirling's formula, we get

$$|F(1-ck)| \sim |ack+b|^{2ack+c'}.A,$$

where $A \ll k^k$ and c' is a constant independent of k. Thus, we conclude that $c_F = 2a$, which is the degree of F. $\square$

Moreover, if $F \in \mathbb{M}$ satisfies a functional equation of the Riemann type, then c_F^* is also the same as the degree d_F of F. This is an immediate consequence of the Lemma 3.1.4.

In view of this, one may wonder how closely the invariants c_F and c_F^* are related in $\mathbb{M}$. Suppose F satisfies a functional equation, i.e., there exists real numbers $Q, \alpha_i > 0$ and complex numbers β_i, ω with $\Re(\beta_i) > 0$ and $|\omega| = 1$ such that

$$F(1-s) = \frac{\omega Q^{2s-1} F(s) \prod_i \Gamma(\alpha_i s + \beta_i)}{\prod_i \Gamma(\alpha_i(1-s) + \beta_i)}.$$

Then, the contribution of the Γ-factors, by the Stirling's formula gives us that $c_F = c_F^*$ (see Proposition 3.2.13). However, in the absence of a functional equation, c_F and c_F^* can be very different. For $F \in \mathbb{M}$, it can happen that c_F^* is 0 and c_F is arbitrarily large.

We exhibit examples below of Dirichlet series which are absolutely convergent on the whole complex plane and hence having $c_F^* = 0$, yet taking large values of c_F.

Let

$$F(s) = \sum_{n=1}^{\infty} \frac{e^{-n}}{n^s}.$$

This Dirichlet series is absolutely convergent on the whole complex plane. But at the negative integers we have,

$$F(-k) = \sum_{n=1}^{\infty} e^{-n} n^k \sim \int_1^{\infty} e^{-t} t^k dt = \Gamma(k) + O(1).$$

Using Stirling's formula, we conclude that $c_F = 1$.

Moreover, $F^r(s) \in \mathbb{M}$ satisfying $c_{F^r}^* = 0$ and $c_{F^r} = r$ for any integer $r > 0$.

In fact, if we start with the Dirichlet series $F(s) = \sum_{n=1}^{\infty} e^{-n^r} n^{-s}$, by a similar argument we see that, for any real $r > 0$, $c_F = r$ but $c_F^* = 0$.

A more intricate question is to classify all elements $F \in \mathbb{M}$ with $c_F^* = 1$. We have not been able to answer this question yet. In case of the Selberg class, as shown by Conrey and Ghosh in [12], the functions with degree 1 are precisely the Riemann zeta-function $\zeta(s)$ and all vertical shifts of Dirichlet L-functions $L(s + i\theta, \chi)$, where χ is a primitive character and $\theta \in \mathbb{R}$. The conductor q_F as given in Chapter 2, (2.4) plays the crucial role of the modulus of the character χ. An analogous notion of conductor for elements in the class $\mathbb{M}$ can be defined as follows.

For $F \in \mathbb{M}$ and $\Re(s) = \sigma < 0$, define

$$q_F(\sigma) := \inf \left\{ \mu \in \mathbb{R} : \mu \geq (2\pi)^{c_F^*} \left| \frac{F(s)^{\sigma - 1/2}}{(|s| + 2)^{c_F^*}} \right|, \text{ for sufficiently large } |t|, \text{ where } s = \sigma + it \right\}.$$

If the infimum in the above definition does not exist, we set $q_F(\sigma) = \infty$.

Define

$$q_F := \limsup_{\sigma < 0} \, q_F(\sigma).$$

Remark 1. *Note that in case of the Selberg class $\mathbb{S}$, q_F captures the conductor as defined in Chapter 2,* (2.4)*. If the functional equation for $F \in \mathbb{S}$ is given by*

$$F(s) = \Delta_F(s) \overline{F(1 - \overline{s})},$$

then using Lemma 3.1.4, we have

$$\left| \Delta_F(\sigma + it) \right| = \left| \left(\alpha Q^2 t^{d_F} \right)^{1/2 - \sigma} \right|.$$

Now taking large negative values for σ, the above definition of q_F gives

$$q_F = (2\pi)^{d_F} \alpha Q^2,$$

which coincides with the definition of conductor for the Selberg class.

We expect that a better understanding of this invariant q_F would provide us with further insight into the structure of $\mathbb{M}$, specifically, in understanding the classification of elements in $\mathbb{M}$ with $c_F^* = 1$.

3.3 The generalized Lindelöf hypothesis

The class $\mathbb{M}$ defined above is closed under addition and does not satisfy any symmetry, since there is no functional equation. We do not expect that the Riemann hypothesis is satisfied by elements in $\mathbb{M}$. For example, we may consider any Dirichlet polynomial, say

$$F(s) := 1 - \frac{2}{2^s}.$$

The zeroes of F all lie on the line $\{\Re(s) = 1\}$, which violates the Riemann hypothesis. But we expect the analogue of the Lindelöf hypothesis to be true for $\mathbb{M}$. Recall that the Riemann hypothesis implies the Lindelöf hypothesis. Since the critical strip for elements in $\mathbb{M}$ is not between 0 and 1, we have to carefully find a region and formulate the Lindelöf hypothesis.

In case of the Selberg class $\mathbb{S}$, the Lindelöf hypothesis is equivalent to saying that if $F \in \mathbb{S}$

$$\mu_F(\sigma) = 0,$$

for $\sigma > 1/2$. To find a region for the statement of Lindelöf hypothesis in $\mathbb{M}$, we invoke Carlson's theorem [10] on the mean-square of Dirichlet series on vertical lines.

Theorem 3.3.1 (Carlson)**.** *Let $F(s)$ given by $\sum_n a_n/n^s$ is absolutely convergent on some right half plane $\Re(s) > \sigma_a$, such that it is analytic on $\Re(s) > \sigma_1$ except for finitely many poles in this region. Further, let $F(s)$ has finite order for $\sigma > \sigma_1$ and*

$$\limsup_{T\to\infty} \frac{1}{T} \int_{-T}^{T} |F(\sigma + it)|^2 \, dt < \infty, \tag{3.9}$$

for $\sigma > \sigma_1$. Then,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |F(\sigma+it)|^2 \, dt = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}}$$

for $\sigma > \sigma_1$, and uniformly in any strip $\sigma_1 + \epsilon \le \sigma \le \sigma_2$.

Carlson [11] further showed that if $F(s)$ be as above with the abscissa of absolute convergence $\sigma_0(F) = 1$, then the condition (3.9) holds for

$$\sigma > \max\left\{\frac{1}{2}, 1 - \frac{1-\sigma}{1+2\mu_F(\sigma)}\right\}. \tag{3.10}$$

For $F \in \mathbb{S}$, if the Lindelöf Hypothesis is true, by Carlson's theorem we get

$$\lim_{T\to\infty} \frac{1}{T} \int_{1}^{T} |F(\sigma+it)|^2 dt \ll 1,$$

for $\sigma > 1/2$.

In order to formulate the Lindelöf hypothesis for $\mathbb{M}$, we first define the region of absolute convergence of the mean square of L-functions.

Definition 3.3.2. *For $F \in \mathbb{M}$, define*

$$\sigma_m(F) := \inf\left\{\sigma : \lim_{T\to\infty} \frac{1}{T} \int_{1}^{T} |F(\sigma+it)|^2 dt \ll 1\right\}.$$

The generalized Lindelöf hypothesis can be stated as follows.

Conjecture 11 (The Generalized Lindelöf Hypothesis)**.** *For $F \in \mathbb{M}$, $\mu_F(\sigma_m) = 0$.*

In case of the Selberg class, unconditionally it is known that

$$\sigma_m(F) \le 1 - 1/(2d_F),$$

where d_F is the degree of $F \in \mathbb{S}$. On assuming the Lindelöf hypothesis and using Carlson's theorem, we have $\sigma_m(F) = 1/2$. So, at present, the above analogue of the

Lindelöf hypothesis is a weaker generalization of the Lindelöf hypothesis for the Selberg class.

3.4 The Lindelöf class

The class $\mathbb{M}$ defined in section 3.1 clearly forms a ring. Now we are ready to define the Lindelöf class.

Let $\mathbb{M}_0$ and $\mathbb{M}_0^*$ be the subsets of $\mathbb{M}$ with $c_F = 0$ and $c_F^* = 0$, respectively. Note that both $\mathbb{M}_0$ and $\mathbb{M}_0^*$ form subrings of $\mathbb{M}$, which follows immediately from Proposition 3.2.8. Therefore, all the non-zero elements of $\mathbb{M}_0$ (resp. $\mathbb{M}_0^*$) form a multiplicatively closed set. Call it $\mathbb{M}_{00}$ (resp. $\mathbb{M}_{00}^*$).

We define the Lindelöf class [34] by localizing at these sets,

$$\mathfrak{L} = \mathbb{M}_{00}^{-1}\mathbb{M}.$$
$$\mathfrak{L}^* = {\mathbb{M}_{00}^*}^{-1}\mathbb{M}.$$

These define classes of L-functions, which when multiplied with some entire Dirichlet series is analytic in the region $\Re(s) > 1$ and have a Dirichlet series representation. Since a Dirichlet polynomial is entire and has many zeroes, so elements of these classes $\mathfrak{L}$ and $\mathfrak{L}^*$ may have many poles. But for any $F \in \mathfrak{L}$ (resp. $\mathfrak{L}^*$), one can always find a right half plane, where F does not have a pole. Note that $\mathbb{S} \subset \mathbb{M} \subset \mathfrak{L}^* \subset \mathfrak{L}$ and $\mathbb{S} \cap \mathbb{M}_0 = \{1\}$. Further since $\mathbb{M}$ is a ring, the group ring $\mathbb{C}[\mathbb{S}]$ is contained in $\mathbb{M}$.

We extend the definition of the function c_F (resp. c_F^*) on $\mathfrak{L}$ (resp. $\mathfrak{L}^*$). If $G \in \mathfrak{L}$ (resp. $\mathfrak{L}^*$), then $G = F/\mu$, where $\mu \in \mathbb{M}_{00}$ (resp. $\mathbb{M}_{00}^*$) and $F \in \mathbb{M}$. We define $c_G = c_F$ (resp. $c_G^* = c_F^*$). Note that this definition is compatible with the definition of c_F based on the growth condition. We first check that the function c_F (resp. c_F^*) is well-defined on $\mathfrak{L}$ (resp. $\mathfrak{L}^*$). If we write $G = H/\nu$ for some other $H \in \mathbb{M}$ and $\nu \in \mathbb{M}_{00}$, and suppose $c_H > c_F$, then $c_{F\nu - H\mu} = c_H$. But, by definition $c_{F\nu - H\mu} = 0$, which leads to a contradiction. Thus, c_F is well-defined on $\mathfrak{L}$. Moreover, we also have that a non-zero F is a unit of $\mathfrak{L}$ if and only if $c_F = 0$. We use the same argument to show that c_F^* is well-defined on $\mathfrak{L}^*$.

3.5 Ring theoretic properties of $\mathbb{M}$, $\mathfrak{L}$ and $\mathfrak{L}^*$

In the previous section, we defined the ring $\mathbb{M}$ as the class of L-functions satisfying some growth conditions. We are interested in understanding the algebraic properties of this ring. Since, these rings consist of meromorphic functions, $\mathbb{M}, \mathfrak{L}$ and $\mathfrak{L}^*$ clearly form integral domains. We further show that the ring $\mathbb{M}$ is non-Noetherian and non-Artinian. Although, we know that for a commutative ring, non-Noetherian implies non-Artinian, it is worthwhile proving them separately in our case, since the construction of sequence of ideals sheds some light on the algebraic nature of these rings.

Proposition 3.5.1. *$\mathbb{M}, \mathfrak{L}$ and $\mathfrak{L}^*$ are non-Artinian.*

Proof. Consider $F \in \mathbb{M}$ with $c_F^* > 0$. The ideal $\langle F \rangle$ is a non-trivial proper ideal of $\mathbb{M}$. We have the following strictly decreasing sequence of ideals in $\mathbb{M}$ given by

$$\langle F \rangle \supsetneq \langle F^2 \rangle \supsetneq \langle F^3 \rangle \cdots . \tag{3.11}$$

Thus, $\mathbb{M}$ is not Artinian. Moreover, $\langle F \rangle \subset \mathbb{M}$ generates a non-trivial ideal $I \subset \mathfrak{L}$ and $I^* \in \mathfrak{L}^*$ and we have strictly decreasing sequence of ideals.

$$\langle I \rangle \supsetneq \langle I^2 \rangle \supseteq \langle I^3 \rangle \cdots ,$$
$$\langle I^* \rangle \supsetneq \langle (I^*)^2 \rangle \supsetneq \langle (I^*)^3 \rangle \cdots .$$

Hence, we conclude that $\mathfrak{L}$ and $\mathfrak{L}^*$ are non-Artinian. $\square$

Now we show that $\mathbb{M}$ is non-Noetherian.

Lemma 3.5.2. *If $F(s) \in \mathbb{M}$ and $F(s)$ is entire, then $F_1(s) := F(rs) \in \mathbb{M}$ for any integer $r \geq 1$. Moreover, $c_F = c_{F_1}$.*

Proof. We clearly have the properties (1),(2),(4) in the definiton of $\mathbb{M}$ for F_1. We only need to prove the growth condition, which can be realised by

$$c_{F_1} = \limsup_{\sigma<0} \frac{2\mu_{F_1}(\sigma)}{1-2\sigma} = \limsup_{\sigma<0} \frac{2\mu_F(\sigma)}{1-2\sigma} = c_F.$$

□

Lemma 3.5.3. *If $F(s) \in \mathbb{M}$ and $F(s)$ is entire, then $F_2(s) := F(s+s_0) \in \mathbb{M}$ for* $\Re(s_0) > 0$.

Proof. Again, properties (1),(2) and (4) in the definition of $\mathbb{M}$ hold clearly. We only need to check the growth condition.

$$c_{F_1} = \limsup_{\sigma<0} \frac{2\mu_{F_1}(\sigma)}{1-2\sigma} = \limsup_{\sigma<0} \frac{2\mu_F(\sigma)}{1-2\sigma} = c_F.$$

□

Proposition 3.5.4. $\mathbb{M}$ *is non-Noetherian.*

Proof. As earlier, we prove this by explicitly constructing a strictly increasing infinite chain of ideals. Let I_s be as in the proof of Proposition 3.5.1. Define

$$K_1 := \cap_{k=1}^{\infty} I_{-2k}.$$

We know that $L(s,\chi) \in K_1$, where χ is any even primitive Dirichlet character, because it vanishes at all negative even integers. By Lemma 3.5.3, $L(s+2n,\chi) \in \mathbb{M}$ and it vanishes on all even negative integers except $\{-2k : k \leq n, k \in \mathbb{N}\}$. We define

$$K_i := \cap_{k=i}^{\infty} I_{-2k}.$$

Clearly, $K_1 \subseteq K_2 \subseteq K_3 \subseteq$ ..... Since we have exhibited that the function $L(s+2n,\chi)$ belongs to K_{n+1} but not K_n. Thus, we get

$$K_1 \subsetneq K_2 \subsetneq K_3 \subsetneq ...$$

Therefore, $\mathbb{M}$ is non-Noetherian. $\square$

The lemmas 3.5.2 and 3.5.3 show that shifts of entire functions in $\mathbb{M}$ also lie in $\mathbb{M}$. Recall that in $\mathbb{S}$, vertical shifts of entire functions in $\mathbb{S}$ lie in $\mathbb{S}$. In case of $\mathbb{M}$, we no longer have to restrict ourselves to only vertical shifts.

We are not yet able to show that $\mathfrak{L}$ and $\mathfrak{L}^*$ are non-Noetherian, although we expect it to be true. Note that the ring $\mathbb{C}[\mathbb{S}]$ is a subring of $\mathbb{M}$. We expect that $\mathbb{C}[\mathbb{S}]$ is also non-Noetherian. Indeed we show below that this is a consequence of Selberg's orthonormality conjecture. This was already known due to Molteni [32].

We first show that distinct elements in the Selberg class are linearly independent over $\mathbb{C}$. This was already observed by Selberg in [40] and Bombieri-Hejhal in [8].

Lemma 3.5.5. *Distinct elements in $\mathbb{S}$ are linearly independent over $\mathbb{C}$.*

Proof. For $F \in \mathbb{S}$ with the Dirichlet series representation $F(s) = \sum_n a_F(n)/n^s$ for $\Re(s) > 1$, the coefficients $a_F(n)$ give a multiplicative arithmetical function, i.e. they satisfy $a_F(mn) = a_F(m)a_F(n)$ for $(m,n) = 1$. For a multiplicative function $f : \mathbb{N} \to \mathbb{C}$, we define its p-component $(f)_p$ as $(f)_p(m) = f(p^m)$. We say that two multiplicative functions f, g are equivalent if $(f)_p = (g)_p$ for all but finitely many p. It is known that non-equivalent multiplicative functions are linearly independent over $\mathbb{C}$ (see Lemma 1 of [33]). In order to prove that distinct elements in the Selberg class are linearly independent over $\mathbb{C}$, we invoke the strong multiplicity one theorem due to Murty-Murty [38], which states that

Theorem 3.5.6 (Strong multiplicity one)**.** *Suppose $F, G \in \mathbb{S}$ with the Euler product given by*

$$F(s) = \prod_p F_p(s), \quad G(s) = \prod_p G_p(s),$$

for $\Re(s) > 1$. If $F_p = G_p$ for all but finitely many primes p, then $F = G$.

From the above Theorem 3.5.6 we conclude that if $F, G \in \mathbb{S}$ with coefficients $a_F(n)$ and $b_F(n)$ are equivalent as multiplicative functions, then $F = G$. Moreover, the linear independence of non-equivalent multiplicative functions leads to the linear independence of distinct elements in $\mathbb{S}$. □

Proposition 3.5.7. *Selberg's orthonormality Conjecture B (see Chapter 2, Conjecture 2.6) implies that $\mathbb{C}[\mathbb{S}]$ is non-Noetherian.*

Proof. We show that Selberg's Conjecture B implies that distinct primitive elements in the Selberg class are algebraically independent.

We know that there are infinitely many primitive elements in $\mathbb{S}$, say $L(s, \chi)$ for different non-principal primitive Dirichlet characters χ, all of which have degree 1 and hence primitive in the Selberg class. Using this, we shall conclude that $\mathbb{C}[\mathbb{S}]$ is non-Noetherian. Recall that Selberg's orthonormality conjecture implies that the factorization into primitive elements in the Selberg class is unique (see Chapter 2, Proposition 2.3.3). Suppose, $F_1, F_2, ..., F_n$ are distinct primitive elements in $\mathbb{S}$ satisfying a polynomial $P(x_1, x_2, ..., x_n) \in \mathbb{C}[x_1, x_2, ..., x_n]$. By linear independence of distinct elements in $\mathbb{S}$, we conclude that not all terms in the polynomial expansion of $P(F_1, ..., F_n)$ are distinct. Thus, we have relations of the form

$$F_1^{a_1} F_2^{a_2} ... F_n^{a_n} = F_1^{b_1} F_2^{b_2} ... F_n^{b_n}, \tag{3.12}$$

where not all the a_i's are the same as the b_i's. But, both left hand side and right hand side in (3.12) are elements in the Selberg class. This contradicts the unique factorization. $\square$

Since $\mathbb{C}[\mathbb{S}] \subseteq \mathbb{M}$ and we know that $\mathbb{M}$ is non-Noetherian, the above proposition can be thought of as some indicative evidence towards the validity of unique factorization into primitives in the Selberg class.

3.5.1 Primitive elements in $\mathbb{M}$ and $\mathfrak{L}^*$

The notion of primitive elements in $\mathbb{S}$ can be extended to the class $\mathbb{M}$ and $\mathfrak{L}^*$.

Definition 3.5.8. *We say that an element $F \in \mathfrak{L}^*$ (resp. $\mathbb{M}$) is primitive if $c_F^* > 0$ and $F = F_1F_2$ implies that either $c_{F_1}^* = 0$ or $c_{F_2}^* = 0$.*

Note that every element $F \in \mathfrak{L}^*$ (resp. $\mathbb{M}$) with $c_F^* = 1$ is primitive, which directly follows from Theorem 3.2.12. But, this is not quite true for $\mathfrak{L}$. Moreover, we could have some Dirichlet series $F(s)$ absolutely convergent on the whole complex plane with $c_F > 0$, which can be written as a product of infinitely many Dirichlet series $\prod_i F_i(s)$ each of which are absolutely convergent on $\mathbb{C}$ and $c_{F_i} > 0$ for each i. Hence, we avoid defining the notion of primitive elements for the class $\mathfrak{L}$.

We now show that every element in $\mathbb{M}$ and $\mathfrak{L}^*$ can be written as a product of primitive elements. We follow the same argument as in the case of the Selberg class $\mathbb{S}$.

Proposition 3.5.9. *Let $F \in \mathfrak{L}^*$ (resp. $\mathbb{M}$), then $F(s)$ can be be written as a finite product of primitive elements in $\mathfrak{L}^*$ (resp. $\mathbb{M}$).*

Proof. Let $F \in \mathfrak{L}^*$ (resp. $\mathbb{M}$) and suppose

$$F = F_1F_2,$$

where $F_1, F_2 \in \mathfrak{L}^*$ (resp. $\mathbb{M}$). By properties of c_F^*, we know that $c_F^* \leq c_{F_1}^* + c_{F_2}^*$. If $c_{F_i}^* < 1$, then by Proposition 3.2.12, $c_{F_i}^* = 0$. Hence, we cannot factorize F indefinitely into non-units (i.e. elements with $c^* > 0$). Therefore, $F(s)$ has a factorization into primitive elements. □

With the above notion of primitivity, we can ask whether this factorization is unique.

Conjecture 12. *Every element $F \in \mathfrak{L}^*$ (resp. $\mathbb{M}$) can be uniquely factorized into primitive elements.*

Assuming unique factorization, we conclude the algebraic independence of distinct primitive elements.

Proposition 3.5.10. *Conjecture 12 implies that linearly independent primitive elements in $\mathfrak{L}^*$ (resp. $\mathbb{M}$) are algebraically independent.*

Proof. The proof follows similar approach of Proposition 3.5.7. Let $F, G \in \mathfrak{L}^*$ be linearly independent primitive elements satisfying a polynomial equation $P(F, G) = 0$. From Proposition 3.2.8, we conclude that if the terms in the polynomial with largest c^* value, say d, must cancel each other. In other words, if $P(x, y) = \sum_{m,n} a_{m,n} x^m y^n$, then

$$\sum_{mc_F^* + nc_G^* = d} a_{m,n} F^m G^n = 0.$$

Since F and G are linearly independent, we also have that $d > 0$. Cancelling the common factors, we get an expression of the form

$$F^k = \sum_{m<k,n} b_{m,n} F^m G^n,$$

where each term in the RHS has a factor of G and hence $F^k/G \in \mathfrak{L}^*$ (resp. $\mathbb{M}$). This contradicts the unique factorization for F^k. Hence, F and G are algebraically independent. □

We now show that $\mathbb{M}, \mathfrak{L}$ and $\mathfrak{L}^*$ are closed under differentiation.

Proposition 3.5.11. *If $F \in \mathbb{M}$, then $F' \in \mathbb{M}$. This is also true if we replace $\mathbb{M}$ by $\mathfrak{L}$ or $\mathfrak{L}^*$. Moreover, $c_F \geq c_{F'}$ and $c_F^* \geq c_{F'}^*$.*

Proof. In order to show $F' \in \mathbb{M}$, the properties (1),(2) and (4) hold clearly. We only have to check the growth condition. Suppose $f(z)$ is a meromorphic function on $\mathbb{C}$. If f is bounded on the half plane $\{z : \Re(z) < -N_1\}$ by M, for some $N_1 > 0$, then $f'(z)$ is also bounded on the half-plane $\{z : \Re(z) < -N_2\}$, for some $N_2 > 0$. We can see this by using Cauchy's formula namely,

$$f'(a) = \frac{1}{2\pi i} \int_{C(\epsilon,a)} \frac{f(z)}{(z-a)^2}\, dz,$$

where $C(\epsilon, a)$ is the circle of radius ϵ centered at a and f is analytic in the interior of $C(\epsilon, a)$. Since $f(z)$ is bounded by M on $C(\epsilon, a)$, we get

$$|f'(a)| \leq \frac{M}{\epsilon}.$$

If we choose $N_2 > N_1 + 2$, for every point in $\{z : \Re(z) < -N_2\}$, we can set $\epsilon > 1$, and thus get $f'(z)$ is bounded by M in the region $\{z : \Re(z) < -N_2\}$.

Now, if $F(z) \in \mathbb{M}$, then by the growth condition we know that for any $\epsilon > 0$,

$$g(z) := \frac{F(z)}{z^{c_F(1-2\sigma)+\epsilon}}$$

is bounded in the half-plane $\{z : \Re(z) < -N_1\}$, for some $N_1 > 0$. Therefore, we have for some $N_2 > 0$, in the half-plane $\{z : \Re(z) < -N_2\}$,

$$\left| \left(\frac{F(z)}{z^{c_F(1-2\sigma)+\epsilon}} \right)' \right| = \left| \frac{F'(z) z^{c_F(1-2\sigma)+\epsilon} - \left(z^{c_F(1-2\sigma)+\epsilon}\right)' F(z)}{z^{2c_F(1-2\sigma)+2\epsilon}} \right| < M$$

$$\implies |F'(z)||z|^{c_F(1-2\sigma)+\epsilon} < M|z|^{2c_F(1-2\sigma)+2\epsilon} + \left| \left(z^{c_F(1-2\sigma)+\epsilon}\right)' \right| |F(z)|.$$

If we fix σ, then

$$|F'(z)||z|^{c_F(1-2\sigma)+\epsilon} < M'|z|^{2c_F(1-2\sigma)+2\epsilon}.$$

Hence,

$$|F'(z)| < M''|z|^{c_F(1-2\sigma)+\epsilon}.$$

Thus, we get the growth condition on $F'(z)$. Moreover, we also conclude that if $F \in \mathbb{M}$, then $c_{F'} \leq c_F$ and $c^*_{F'} \leq c^*_F$. The proof of the statement for $\mathfrak{L}$ (resp. $\mathfrak{L}^*$) follows by proving the fact that the derivative of a unit in $\mathfrak{L}$ (resp. $\mathfrak{L}^*$) is also a unit in $\mathfrak{L}$ (resp. $\mathfrak{L}^*$). Then, for any $F \in \mathfrak{L}$ (resp. $\mathfrak{L}^*$), we can find a unit $\nu \in \mathfrak{L}$ (resp. $\mathfrak{L}^*$), such that $\nu F \in \mathbb{M}$, after which we can use the above argument to say $(\nu F)' \in \mathfrak{L}$ (resp. $\mathfrak{L}^*$). But $(\nu F)' = \nu' F + \nu F'$ and thus $F' \in \mathfrak{L}$ (resp. $\mathfrak{L}^*$).

□

3.6 Ideals in $\mathfrak{L}$ and $\mathfrak{L}^*$

As a first step to understanding the ideal theory of $\mathfrak{L}$ (resp. $\mathfrak{L}^*$), we construct some non-trivial ideals of $\mathfrak{L}$ (resp. $\mathfrak{L}^*$.) We use the following proposition for the construction.

Proposition 3.6.1. *Let $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a non-constant analytic function, which converges on the whole complex plane. Then $F(s)$ cannot have zeroes on any arithmetic progression,*

$$S = \{a, a-d, a-2d, ...\}, \tag{3.13}$$

where $d \in \mathbb{R}^+$ and $a \in \mathbb{C}$.

Proof. First, we show that if

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

is convergent on the whole complex plane, it cannot have zeroes on non-positive integers. In other words, it cannot vanish on $\{0, -1, -2, -3...\}$. Since, $\sum_{n=1}^{\infty} a_n n^{-s}$ is entire, we have

$$|a_n| \ll \frac{1}{n^k}. \tag{3.14}$$

for all $k \in \mathbb{N}$. Define

$$f(x) := \sum_{n=1}^{\infty} a_n x^n.$$

By the root test and using (3.14), we conclude that the power series $f(x)$ defines an analytic function on the whole complex plane. Consider the Taylor series expansion around $x = 1$, given by,

$$f(x) = \sum_{n=0}^{\infty} b_n (x-1)^n, \tag{3.15}$$

where,

$$b_n = \frac{f^{(n)}(1)}{n!}.$$

Now, suppose $F(s)$ takes zeroes on all non-positive integers, we have

$$F(-k) = \sum_{n=1}^{\infty} a_n n^k = 0,$$

for all $k \in \mathbb{N} \cup \{0\}$. In particular, we get $b_0 = F(0) = 0$ and $b_1 = F(-1) = 0$. In general,

$$b_k = \sum_{n=k}^{\infty} \binom{n}{k} a_n.$$

We write it as,

$$\sum_{n=k}^{\infty} \binom{n}{k} a_n = \sum_{n=1}^{\infty} \frac{n(n-1)...(n-k+1)}{k!} a_n, \tag{3.16}$$

where the first $(k-1)$-terms of the right hand side of (3.16) are 0. Moreover, each term in the right hand side is a polynomial in n of degree k. Since, $\sum_{n=1}^{\infty} a_n n^k$ is absolutely convergent for all k, we can rearrange the terms in the summation. Thus, we get,

$$b_k = \sum_{n=1}^{\infty} \binom{n}{k} a_n = c_0 \sum_{n=1}^{\infty} a_n + c_1 \sum_{n=1}^{\infty} n a_n + ... + c_k \sum_{n=1}^{\infty} n^k a_n,$$

where c_i's are some real constants. Therefore,

$$b_k = c_0 F(0) + c_1 F(-1) + ... + c_k F(-k) = 0,$$

for all k. Hence, $f(x)$ is identically zero, which leads to $F(s)$ being identically 0.

We deduce the general case of an arithmetic progression S as in the statement of the proposition, by considering

$$F_1(s) := F(sd + a) = \sum_{n=1}^{\infty} \frac{a_n}{n^a} \frac{1}{n^{sd}}.$$

This function is no longer a standard Dirichlet series but it converges for all $s \in \mathbb{C}$ and vanishes at all non-positive integers. Now, consider the series

$$f(x) := \sum_{n=1}^{\infty} \frac{a_n}{n^a} x^{n^d}.$$

Choosing the principal branch of log, the function $f(x)$ is well-defined and absolutely convergent on $\mathbb{C} \backslash \mathbb{R}_{\leq 0}$. The rest of the proof is similar as above. $\square$

Consider the ideal in $\mathbb{M}$ given by,

$$I_S = \cap_{s \in S} I_s,$$

where I_s is as in Proposition 3.5.1 and S is an arithmetic progression as in (3.13). I_S is non-empty since it contains elements in $\mathbb{S}$. By Proposition 3.6.1, I_S does not contain any Dirichlet series which converges on the whole of $\mathbb{C}$. By definition,

$$\mathfrak{L} \text{ (resp. } \mathfrak{L}^*) = \mathbb{M}_{00}^{-1}\mathbb{M} \text{ (resp. } {\mathbb{M}_{00}^*}^{-1}\mathbb{M}),$$

where $\mathbb{M}_{00}$ and $\mathbb{M}_{00}^*$ consist of Dirichlet series which are convergent on $\mathbb{C}$, up to a Dirichlet polynomial(by Proposition 3.2.11). Hence, I_S generates a non-trivial ideal in $\mathfrak{L}$ (resp. $\mathfrak{L}^*$).

3.7 Topology on $\mathfrak{L}$ and $\mathfrak{L}^*$

Let $F, G \in \mathfrak{L}$ (resp. $\mathfrak{L}^*$). Define an equivalence relation

$$F \sim G, \text{ if } F = \mu G + \nu$$

for $\mu, \nu \in \mathfrak{L}$ (resp. $\mathfrak{L}^*$) such that $c_\mu = c_\nu = 0$ (resp. $c_\mu^* = c_\nu^* = 0$). In other words, each of the equivalence class consists of linearly dependent elements in $\mathfrak{L}$ (resp. $\mathfrak{L}^*$).

Call the equivalence class associated to F as T_F (resp. T_F^*). Then clearly

$$\mathfrak{L} = \sqcup_\alpha T_{F_\alpha},$$
$$\mathfrak{L}^* = \sqcup_\beta T_{F_\beta}^*$$

where the indices α and β run over all the equivalence classes in $\mathfrak{L}$ and $\mathfrak{L}^*$ respectively. Note that the c and c^* values are common among the elements in the same equivalence class. We define

$$c_{T_F} := c_F,$$
$$c_{T_F^*} := c_F^*.$$

for the equivalence class associated to F.

If we consider the space $\{T_{F_\alpha}\}$ (resp. $\{T_{F_\alpha}^*\}$), then we define following distance function.

$$d(T_F, T_G) := \begin{cases} |c_{T_F} - c_{T_G}|, & \text{if } c_{T_F} \neq c_{T_G} \\ \left(\inf\left\{c_H : F' = G' + H, F' \in T_F \text{ and } G' \in T_G\right\}\right)^{-1} & \text{if } c_{T_F} = c_{T_G}. \end{cases}$$

Analogously, we define the distance function $d^*(F, G)$ by replacing c with c^* in the definition.

The function $d(.,.)$ (resp. $d^*(.,.)$) gives a well defined function from the space $\{T_{F_\alpha}\}$ (resp. $\{T^*_{F_\alpha}\}$) to the set of non-negative real numbers. In fact, from the properties of c_F and c^*_F, we see that these distance functions define an ultrametric on the space $\{T_{F_\alpha}\}$ and $\{T^*_{F_\alpha}\}$.

Let $F, G \in \mathfrak{L}$, we say that $F \sim_k G$ if $c_F = c_G$ and $\mu F + G = H$, for some unit μ in $\mathfrak{L}$ (resp. $\mathfrak{L}^*$) and $H \in \mathfrak{L}$ (resp. $\mathfrak{L}^*$) with $c_H \leq k$. We immediately conclude that the relation $\sim_k$ is an equivalence relation. This follows from the fact that d defines an ultrametric and all balls in an ultrametric are open and closed, and each point inside the ball is the center. Analogously, we have equivalence relations $\sim_k$ in $\mathfrak{L}^*$.

Many interesting questions arise pertaining to understanding these spaces of equivalence classes. This gives rise to a front for studying geometric and topological nature of space of L-functions.

1. For a given c_{T_F} are there countably many equivalence classes, each containing a primitive element, up to shifts i.e., we avoid counting classes containing $F(s)$ and $F(s + a)$ separately. This is the generalized version of the similar question in the Selberg class, i.e., for a fixed degree, are there countably many primitive elements in the Selberg class up to vertical shifts? Although, it is clear for degree 1 elements in the Selberg class, it is not known in general. In our case, we do not know the answer to this even for equivalence classes with $c_{T_F} = 1$ or $c^*_{T_F} = 1$.

2. If F and G belong to different equivalence classes, are they algebraically indepen-

dent?

3. Since the distance functions $d(.,.)$ and $d^*(.,.)$ define metric spaces on the space $\{T_{F_\alpha}\}$ and $\{T^*_{F_\alpha}\}$, what can we say about the boundary elements in the completion of these metric spaces. In other words, what are the limiting elements arising from a family of L-functions?

Chapter 4

Value Distribution and Uniqueness Theorems

The study of value distribution deals with the zero-distribution of L-functions, i.e., all the points in the complex plane where $L(s) = 0$. More generally, we study all the points with $L(s) = a$, which are often called the a-values of the L-function. In their value distribution, L-functions encode properties of the underlying arithmetic and algebraic structure, for which no known elementary or algebraic methods work. For instance, the Riemann hypothesis, which is a statement about the zero-distribution of the Riemann zeta-function in the critical strip has implications on the distribution of prime numbers. Moreover, the special values of certain Dirichlet L-functions at $s = 1$, give us information regarding how far the ring of integers of a quadratic extension is from having unique factorization into primes. This can be seen using the Dirichlet class number formula. There are many unsolved conjectures on the value distribution of certain L-functions. For example, the Birch and Swinnerton-Dyer conjecture claims that the L-function attached to an elliptic curve E over $\mathbb{Q}$, has a zero at $s = 1$ of order equal to the rank of the Mordell-Weil group of E. Hence, the study of the value distribution for L-functions is crucial.

In this chapter, we produce some results on the value distribution of elements in the Lindelöf class, more precisely, giving bounds on the number of points where an L-function satisfies $L(s) = a$. Moreover, we address the more interesting question on how many values an L-function can share with another L-function or a meromorphic function without being equal.

We define the a-values of an L-function as follows.

Definition 4.0.1. *Let $L \in \mathbb{M}$, then the a-values of $L(s)$ ignoring multiplicity is defined as the set*

$$L^{-1}(a) := \{s \in \mathbb{C} : L(s) = a\}.$$

Selberg introduced the study of these sets and called them the a-values of an L-function.

4.1 Number of zeroes

Definition 4.1.1. *Let $F \in \mathbb{M}$ with abscissa of absolute convergence $\sigma_a(F)$. Define*

$$N_F(T) := \#\Big\{s : F(s) = 0,\, \sigma_0(F) - 1 < |\Re(s)| < \sigma_0(F),\, |\Im(s)| < T\Big\},$$
$$N_F(a,T) := \#\Big\{s : F(s) = a,\, \sigma_0(F) - 1 < |\Re(s)| < \sigma_0(F),\, |\Im(s) < T|\Big\}.$$

Here, $N_F(T)$ and $N_F(a,T)$ counts the number of zeroes and c-values in the critical strip according to multiplicities. For $F \in \mathbb{S}$, using the Riemann-von Mangoldt-type formula, it is known that the following holds.

Proposition 4.1.2. *For $F \in \mathbb{S}$ with a Riemann-type functional equation as in* (2.1) *and a any complex number, we have*

$$N_F(a,T) = \frac{d_F}{\pi} T \log T + O(T), \tag{4.1}$$

where d_F is the degree of F.

Notice that the degree of an L-function in $\mathbb{S}$ can be retrieved from the number of a-values in the critical strip. This motivates the definition of an analogous notion of degree in $\mathbb{M}$ arising from the number of zeroes.

Definition 4.1.3. *For $F \in \mathbb{M}$, define*

$$\delta_F := \limsup_{T\to\infty} \frac{2\pi}{T\log T} N_F(T).$$

Clearly for $F \in \mathbb{S}$, $\delta_F = d_F$. We first show that this function $\delta_F < \infty$. For this purpose, we first make the following definitions.

Let f be a meromorphic function. Denote by $n(f,r)$ the number of poles of $f(s)$ in the disc $|s| < r$ counting multiplicities and denote by $n(f,a,r)$ the number of a-values of f in $|s| < r$, counting multiplicities. Indeed

$$n(f,a,r) = n\left(\frac{1}{f-a}, r\right). \tag{4.2}$$

Lemma 4.1.4. *For $F \in \mathbb{M}$, $\delta_F < \infty$. In fact $\delta_F < A\, c_F^*$, where A is a constant depending on σ_0, the abscissa of absolute convergence of F.*

Proof. Without loss of generality, we assume that $\sigma_0 = 1$. From the definition of μ_F^* and the Phragmén-Lindelöf theorem, we know that for $\sigma < 0$,

$$\mu_F^*(\sigma) \leq \frac{1}{2} c_F^*(1-2\sigma). \tag{4.3}$$

Define

$$G(s) := F(s + (1 + iT)).$$

Using Jensen's theorem, we know that if $G(0) \neq 0$,

$$\int_0^R \frac{n(G,0,r)}{r}\,dr = \frac{1}{2\pi}\int_0^{2\pi} \log|G(re^{i\theta})|\,d\theta - \log|G(0)|. \tag{4.4}$$

Note that

$$N_F(T+1) - N_F(T) \leq n(G,2) \leq \int_0^3 \frac{n(G,0,r)}{r}\,dr,$$

since $n(G,r)$ is an increasing function. From the growth parameters, we have

$$|G(3e^{i\theta})| = |F(1+3e^{i\theta}+iT)| \ll (|s|+2)^{\mu_F^*(-2)+\epsilon},$$

for any $\epsilon > 0$. Taking log and using (4.3), we get for large T,

$$\frac{\log|F(1+3e^{i\theta}+iT)|}{\log T} \leq \frac{5}{2}c_F^* + \epsilon,$$

for $\epsilon > 0$ arbitrarily small. Thus, we have for large T,

$$N_F(T+1) - N_F(T) \leq \left(\frac{5}{2}c_F^* + \epsilon\right)\log T.$$

Hence, we have

$$N_F(T) \leq \frac{5}{2}c_F^* T \log T.$$

Therefore,

$$\delta_F \leq (2\pi)\frac{5}{2}c_F^*.$$

In general, if the abscissa of absolute convergence of F is $\sigma_0 < 1$, then by the same argument, we have

$$N_F(T) \le \left(\frac{7-2\sigma_0}{2}\right) c_F^* T \log T,$$
$$\delta_F \le 2\pi \left(\frac{7-2\sigma_0}{2}\right) c_F^*.$$

Thus, we have that δ_F is bounded and hence well-defined. □

By a similar argument, using Jensen's theorem, we also get that if $N_F(\sigma_1, \sigma_2, T)$ be the number of zeroes of $F(s)$ in the region $\{s \in \mathbb{C}, \sigma_1 < \Re(s) < \sigma_2, |\Im(s)| < T\}$, then

$$N_F(\sigma_1, \sigma_2, T) = O(T \log T). \tag{4.5}$$

Moreover, if the leading coefficient of the Dirichlet series is $\neq 0$, then we can find a right half plane where $F(s) \neq 0$ and thus take σ_2 to be infinity in (4.5).

In Lemma 4.1.4, we can replace $N_F(T)$ with $N_F(a, T)$, because counting a-values of $F(s) \in \mathbb{M}$ is same as counting the zeroes of $F(s) - a$, which is also in $\mathbb{M}$. Moreover $c_F^* = c_{F-a}^*$, so the same upper bound for $N_F(T)$ holds for $N_F(a, T)$.

4.2 The class $\mathbb{M}^1$

An interesting question would be to ask if we can always guarantee that $\delta_F > 0$? This is indeed the case if F satisfies a functional equation of the Riemann type (2.1). In general, this seems to be a difficult question to answer. As we shall see, the growth condition in $\mathbb{M}$ promises the existence of infinitely many T's such that $N(T+1) - N(T)$ is at least of the order $\log T$, but fails to answer anything about how often such T's occur. However, if F satisfies a functional equation, it provides some equidistribution to the rate at which the number of zeroes appear in the critical strip. In view of this, we define a class of L-functions smaller than $\mathbb{M}$, forcing a stronger growth condition.

Definition 4.2.1. The class $\mathbb{M}^1$. *The class $\mathbb{M}^1$ consists of functions $F(s) \in \mathbb{M}$ satisfying the following condition.*

Stronger growth condition - *There exist $\epsilon, \delta > 0$ depending on F such that*

$$\limsup_{|s|\to\infty} \left\{ s : |\arg(s)| > \pi/2 + \delta, |F(s)| \gg_{\sigma} (|s| + 2)^{\mu_F^*(\sigma)-\epsilon} \right\} = \infty. \tag{4.6}$$

It is not apriori clear if $\mathbb{M}^1 = \mathbb{M}$. In fact, we do not know if $\mathbb{M}^1$ is closed under addition. But, $\mathbb{S} \subset \mathbb{M}^1$. Moreover, if we assume Selberg's orthogonality conjecture, then from algebraic independence of primitive elements (see Chapter 3, Proposition 3.5.7) and the fact that distinct elements in $\mathbb{S}$ cannot share many zeroes, we have that $\mathbb{C}[\mathbb{S}] \subseteq \mathbb{M}^1$.

We first establish that the number of zeroes up to height T for functions in $\mathbb{M}^1$ is of the order $T \log T$. We will prove a modified version of this, where instead of counting zeroes in the critical strip, we count the zeroes in a disc of radius T around 0. Note that, in the case of Selberg class, this count is almost identical since there are no non-trivial zeroes outside the critical strip. In view of this, we make the following definitions.

Let $n(f, a, r)$ and $n(f, r)$ be as in (4.2).

Definition 4.2.2. *The integrated counting function is defined as*

$$N(f, a, r) := \int_0^r \Big(n(f, a, t) - n(f, a, 0) \Big) \frac{dt}{t} + n(f, a, 0) \log r, \tag{4.7}$$

$$N(f, r) := \int_0^r \Big(n(f, t) - n(f, 0) \Big) \frac{dt}{t} + n(f, 0) \log r. \tag{4.8}$$

We now show that for $F \in \mathbb{M}^1$ with $c_F^* > 0$, the function $N(F, 0, r)$ is in fact $\Omega(r \log r)$.

Proposition 4.2.3. *Let $F \in \mathbb{M}^1$ and $c_F^* > 0$, then*

$$N(F, 0, r) = \Omega(r \log r)$$

Proof. If F has a pole of order k at $s = 1$, we define

$$G(s) := (s-1)^k F(s).$$

Note that $G(s)$ is entire and also satisfies the growth conditions of F. By Hadamard product factorization, we have

$$G(s) = s^m e^{As+B} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{s/\rho},$$

where ρ runs over the zeros of G and m, A, B are constants. We use the following result (see [19], p. 56, Remark 1), which states that if

$$H(s) = \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{s/\rho}$$

and $\sum_\rho 1/|\rho|^2$ is bounded, then

$$\log|H(s)| \ll |s| \int_0^{|s|} \frac{n(t,0,H)}{t^2} dt + |s|^2 \int_{|s|^2}^\infty \frac{n(t,0,H)}{t^3} dt. \tag{4.9}$$

Recall that, by Jensen's theorem we have $N(T+1,0,G) - N(T,0,G) = O(\log T)$. Hence,

$$\sum_{G(\rho)=0} \frac{1}{|\rho|^2}$$

is bounded. Applying (4.9) to G, we have

$$\log|G(s)| \ll |s| \int_0^{|s|} \frac{n(t,0,G)}{t^2} dt + |s|^2 \int_{|s|^2}^\infty \frac{n(t,0,G)}{t^3} dt + O(|s|). \tag{4.10}$$

If we assume that $N(T,0,F) = o(T\log T)$, then the RHS of (4.10) is $o(T\log T)$. To see this, we first note that if $N(T,0,G) = o(T\log T)$, then $n(T,0,G) = o(T\log T)$. Suppose $n(T,0,G)$ is not $o(T\log T)$, then there exist infinitely many T such that $n(T,0,G) \ll$

$T \log T$. But, that implies

$$N(2T, 0, G) = \int_0^{2T} \frac{n(t, 0, G)}{t} dt \geq \int_T^{2T} \frac{n(t, 0, G)}{t} dt \gg T \log T,$$

which contradicts the assumption that $N(T, 0, F) = o(T \log T)$. Therefore, it follows that the RHS of (4.10) is $o(T \log T)$.

But the strong growth condition implies that we can find s, with $|s|$ arbitrarily large such that the LHS of (4.10) is $\Omega(T \log T)$. This leads to a contradiction. □

It is possible to prove Proposition 4.2.3 without the strong growth condition if the function $F \in \mathbb{M}$ respects some nicer value distribution.

Proposition 4.2.4. *Let $F \in \mathbb{M}$ satisfy one of the following conditions.*

1. **Distribution on vertical lines**: *There exist $\sigma < 0$ and $\epsilon > 0$ such that*

$$\lim_{T \to \infty} \frac{1}{T} \; meas \Big\{ |t| < T : |F(\sigma + it)| \gg T^{\epsilon} \Big\} > 0.$$

2. **Equidistribution of zeroes**: *Let $N(L, 0, T) = \Omega(f(T))$ for $T \gg 1$, then*

$$N(L, 0, T+1) - N(L, 0, T) = O(f(T)/T).$$

Then, $N(T, 0, F) = \Omega(T \log T)$.

Proof. We invoke the following theorem of Landau (see [45], p. 56, sec. 3.9, Lemma α)

Theorem (Landau)**.** *If f is holomorphic in $|s - s_0| \leq r$ and $|f(s)/f(s_0)| < e^M$ in $|s - s_0| \leq r$, then*

$$\left| \frac{f'}{f}(s) - \sum_{|s_0 - \rho| < r/2} \frac{1}{(s - \rho)} \right| \leq \frac{4}{1 - 2\alpha^2} \frac{M}{r}, \tag{4.11}$$

where $|s - s_0| \leq \alpha r$ for any $\alpha < 1/2$ and ρ runs over the zeroes of f.

If $L \in \mathbb{M}$, then from Jensen's theorem, we have

$$\frac{L'}{L}(s) = \sum_{|s_0-\rho|<r/2} \frac{1}{(s-\rho)} + O(\log t),$$

where $s = \sigma + it$.

Let C_1, C_2, C_3 be circles with center $2+iT$ and radius $2-2\sigma_0, 2-\sigma_0$ and $1/2$ respectively, where $\sigma_0 < 0$. From (4.11), we have for $s \in C_2$

$$\frac{L'}{L}(s) = \sum_{\rho \in C_1} \frac{1}{(s-\rho)} + O(\log T).$$

Moreover, since $L(s)$ has a Dirichlet series representation on $Re(s) > 1$, we have that it is bounded above and below on C_3 and hence

$$\frac{L'}{L}(s) = O(1).$$

Define the function

$$g(s) = L(s) \prod_{\rho \in C_1} \frac{1}{s-\rho}.$$

If $N(L, 0, T) = o(T \log T)$, then the condition(2) implies that we have for $s \in C_3$

$$\left| \frac{g'}{g}(s) \right| = o(\log T).$$

Moreover, for $s \in C_2$, we have $|g'(s)/g(s)| = O(\log T)$. By Hadamard's three-circle theorem, we have for any circle C_4 with center $2+iT$ and radius $2-\sigma_0+\delta$,

$$\frac{g'}{g}(s) \ll (\log T)^{\alpha_1} o\left((\log T)^{\alpha_2}\right) = o(\log T), \tag{4.12}$$

where $s \in C_4$, $\delta > 0$, $0 < \alpha_1, \alpha_2 < 1$ and $\alpha_1 + \alpha_2 = 1$.

Now, consider the integral

$$\int_{\sigma_0+3\delta+iT}^{2+iT} \frac{g'}{g}(s)\,ds = \log L(2+iT) - \log L(\sigma_0+3\delta+iT)$$
$$- \sum_{\rho\in C_1} \log L(2+iT-\rho) - \log L(\sigma_0+3\delta+iT-\rho). \quad (4.13)$$

By (4.12), LHS of (4.13) is $o(\log T)$. But, by the growth condition we can choose T such that

$$\log|L(\sigma_0+iT)| = \Omega(\log T).$$

Thus, the RHS of (4.13) is $\Omega(\log T)$, because all the terms except $\log|L(\sigma+iT)|$ is $o(\log T)$. This is a contradiction.

Instead, if we assume condition(1) and suppose that $N(L,0,T) = o(T\log T)$. We have

$$\lim_{T\to\infty} \frac{1}{T}\ meas\Big\{|t| < T : N(L,0,T+1) - N(L,0,T) = \Omega(\log T)\Big\} = 0.$$

But condition(1) implies that we can find T and σ_0 such that $N(L,0,T+1)-N(L,0,T) = o(\log T)$ and $\log|L(\sigma_0+iT)| = \Omega(\log T)$. Now, we follow the same argument as before. $\square$

Note that, the best known result for the number of zeroes of a general Dirichlet series $F(s)$ with meromorphic continuation is due to Bombieri and Perelli [9], which states that

$$\limsup_{T\to\infty} \frac{N(T,0,L)+N(T,\infty,F)}{T^\delta} > 0,$$

for $\delta < 1$. In our case, enforcing a stronger growth condition ensures the number of zeroes to be $\Omega(T\log T)$.

4.3 Uniqueness results

For any two meromorphic functions f and g, we say that they share a value 'a' ignoring multiplicity(IM) if $f^{-1}(a)$ is same as $g^{-1}(a)$ as sets. We further say that f and g share a value a counting multiplicity(CM) if the zeroes of $f(x)-a$ and $g(x)-a$ are the same with multiplicity. Nevanlinna theory [39] establishes that any two meromorphic functions of finite order sharing five values IM must be the same. Moreover, if they share four values CM, then one must be a Möbius transform of the other. The numbers four and five are the best possible for meromorphic functions.

What if we replace meromorphic functions with L-functions? It was shown by M. Ram Murty and V. Kumar Murty [38] that if two L-functions in the Selberg class share 0 counting multiplicity, then they should be the same. For distinct $F, G \in \mathbb{S}$, define

$$D_{F,G}(T) = \sum_{\rho} |m_F(\rho) - m_G(\rho)|,$$

where ρ runs over all the non-trivial zeroes of F and G with $|\Im(\rho)| < T$ and $m_F(\rho)$ denote the order of the zero of F at ρ. Then, M. Ram Murty and V. Kumar Murty [38] showed that

$$\liminf_{T\to\infty} \frac{D_{F,G}(T)}{T} > 0.$$

Thus, distinct functions in $\mathbb{S}$ cannot share zeroes even up to an error term of $O(T)$.

Steuding [44] further showed that two L-functions in the Selberg class sharing two values IM, with some additional conditions should be the same. In 2011, Bao Qin Li [30] proved Steuding's result dropping the extra conditions. In a previous paper in 2010, Bao Qin Li [29] also showed that if a meromorphic function with finitely many poles and an L-function from the extended Selberg class share one value CM and another value IM, then they should be the same.

In this section, we establish all the above results in the more general setting of the class

$\mathbb{M}^1$. Moreover, we also show a different kind of uniqueness theorem involving derivatives of L-functions. Recall that the class $\mathbb{M}$ is closed under differentiation.

First, we set up some preliminaries and main results from Nevanlinna theory.

4.4 Nevanlinna Theory

Nevanlinna theory was introduced by R. Nevanlinna [39] to study the value-distribution of meromorphic functions. We recall some basic definitions and facts from this theory.

Let f be a meromorphic function. Recall the integrated counting function as defined in (4.7)

$$N(f,a,r) := \int_0^r \Big(n(f,a,t) - n(f,a,0)\Big)\frac{dt}{t} + n(f,a,0)\log r,$$
$$N(f,r) := \int_0^r \Big(n(f,t) - n(f,0)\Big)\frac{dt}{t} + n(f,0)\log r.$$

Define the proximity function by

$$m(f,r) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \tag{4.14}$$
$$m(f,a,r) := m\left(\frac{1}{f-a}, r\right), \tag{4.15}$$

where

$$\log^+ x = \max\{0, \log x\}.$$

This function $m(f,a,r)$ captures how close $f(s)$ is to the value 'a' on the circle $|s| = r$. The Nevanlinna characteristic function of f is defined by

$$T(f,r) := N(f,r) + m(f,r).$$

We also define

$$T(f,a,r) := N(f,a,r) + m(f,a,r).$$

We recall some basic properties of these functions.

1. If f and g are meromorphic functions, then

$$T(fg,r) \leq T(f,r) + T(g,r), \quad m(fg,r) \leq m(f,r) + m(g,r).$$

$$T(f+g,r) \leq T(f,r) + T(g,r) + O(1), \quad m(f+g,r) \leq m(f,r) + m(g,r) + O(1).$$

2. The complex order of a meromorphic function is given by

$$\rho(f) := \limsup_{r\to\infty} \frac{\log T(f,r)}{\log r}.$$

3. If $\rho(f)$ is finite, then we have the logarithmic derivative lemma (see [39], p. 370),

$$m\left(\frac{f'}{f}, r\right) = O(\log r). \tag{4.16}$$

The first main theorem of Nevanlinna states that $T(f,a,r)$ and $T(f,r)$ differ by a bounded quantity for any $a \in \mathbb{C}$.

Theorem 4.4.1. *[First Fundamental Theorem] Let f be a meromorphic function and let 'a' be any complex number. Then*

$$T(f,a,r) = T(f,r) + O(1),$$

where the error term depends on f and a and is independent of r.

This characteristic function $T(f,r)$ encapsulates in it information regarding the analytic behaviour of f. The Nevanlinna characteristic can also be bounded in terms of the value distribution of three or more values. This is often called the Second Fundamental theorem of Nevanlinna theory.

Theorem 4.4.2. *[Second Fundamental Theorem] Let f be a meromorphic function of finite order. Suppose a_j's are distinct complex values (including ∞) and $q \geq 3$, we have*

$$(q-2)T(f,r) \leq \sum_{j=1}^{q} \overline{N}\left(\frac{1}{f-a_j}, r\right) + O(\log r),$$

where $\overline{N}$ is the integrated counting function defined similarly as N, but ignoring multiplicity.

Nevanlinna also defined the quantity

$$\delta(f,a) := 1 - \limsup_{r\to\infty} \frac{N(f,a,r)}{T(f,r)},$$

called the deficiency of value 'a' in f. If the deficiency is positive, then there are relatively few a-values of f. The second fundamental theorem implies that $\sum_a \delta(f,a) \leq 2$, from which one can conclude that the deficiency is 0 for all but countably many $a \in \mathbb{C} \cup \infty$.

In 1999, Ye [48] calculated the above Nevanlinna functions for the Riemann zeta-function. One can extend his results to that of the functions in Selberg class (see Steuding [43], p 148-150).

For $F \in \mathbb{S}$,

$$T(F,r) = \frac{d_F}{\pi} r \log r + O(r),$$
$$\delta(F,\infty) = 1.$$

Note that, using (4.1) we have $\delta(F,a) = 0$ for all $a \in \mathbb{C}$. Moreover, using logarithmic derivative lemma and the fact that the only possible pole of $F \in \mathbb{S}$ is at $s = 1$, we also get the bounds

$$m(F, r) \leq \frac{d_F}{\pi} r \log r,$$
$$N(F, \infty, r) \ll \log r.$$

4.5 Main Theorems

Let $\mathbb{M}_1$ denote the class of L-functions as defined in Section 4.2. We first compute the Nevanlinna characteristic $T(F, r)$ for $F \in \mathbb{M}^1$ and use it to produce several uniqueness results.

Theorem 4.5.1. *If $L \in \mathbb{M}^1$, then*

$$T(L, r) = \Omega(r \log r).$$

Theorem 4.5.2. *If two L-functions $L_1, L_2 \in \mathbb{M}^1$ share a complex value 'a' counting multiplicity, then $L_1 = cL_2 - ca + a$ for some $c \in \mathbb{C}$.*

Denote $n(f, r)$ as the number of poles of f counting multiplicity in $|s| < r$ as defined earlier. We say that f and g share a complex value 'a' up to an error term $E(r)$, if

$$n\left(\frac{1}{(f-a)} - \frac{1}{(g-a)}, r\right) \leq E(r).$$

Similarly, denote $\overline{n}(f, r)$ as the number of poles of f ignoring multiplicity in $|s| < r$. We say that f and g share a complex value 'a' ignoring multiplicity, up to an error term $E(r)$, if

$$\overline{n}\left(\frac{1}{(f-a)} - \frac{1}{(g-a)}, r\right) \leq E(r).$$

Theorem 4.5.3. *Let f be any meromorphic function on $\mathbb{C}$ with finitely many poles and $L \in \mathbb{M}^1$ be such that they share one complex value counting multiplicity and another complex value ignoring multiplicity, up to an error term $o(r \log r)$, then $f = L$.*

Theorem 4.5.4. *Let f be any meromorphic function on $\mathbb{C}$ of order ≤ 1, with finitely many poles and $L \in \mathbb{M}^1$ be such that they share one complex value counting multiplicity and their derivatives, f' and L' share zeroes up to an error term $o(r \log r)$, then $f = \mu L$, where μ is a root of unity.*

Since $\mathbb{M}^1$ contains the Selberg class, we have, in particular, the following corollary.

Corollary 4.5.5. *Let f be a meromorphic function on $\mathbb{C}$ of order ≤ 1 with finitely many poles and L be an L-function in the Selberg class, such that they share one complex value counting multiplicity and their derivatives, f' and L' share zeroes up to an error term $o(r \log r)$, then $f = L$.*

4.6 Proof of the theorems

4.6.1 Proof of Theorem 4.5.1

Proof. We evaluate the Nevanlinna characteristics for L-functions in $\mathbb{M}^1$.

For $L \in \mathbb{M}^1$, we have

$$m(L, r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |L(re^{i\theta})| \, d\theta.$$

Since, $L(s)$ is bounded on $\sigma > 1$, we have

$$\frac{1}{2\pi} \int_{\theta; r \cos\theta > 1} \log^+ |L(re^{i\theta})| \, d\theta \ll 1.$$

Moreover, using the growth condition, we have

$$\begin{aligned}\frac{1}{2\pi}\int_{\theta; r\cos\theta<1}\log^+|L(re^{i\theta})|d\theta &\leq \frac{1}{2\pi}\int_{\theta; r\cos\theta<1}\mu_L(r\cos\theta)\,d\theta + O(r)\\ &\leq \frac{c_L}{\pi} r\log r + O(r).\end{aligned}$$

Thus, we conclude that

$$m(L,r) \leq \frac{c_L}{\pi} r\log r + O(r).$$

Since $L \in \mathbb{M}^1$ has only one possible pole at $s = 1$, we have for $r > 1$,

$$N(L,r) \leq \log r.$$

Hence,

$$T(L,r) \leq \frac{c_L}{\pi} r\log r + O(r).$$

In order to prove Theorem 4.5.1, we use Theorem 4.4.1, namely,

$$T(L,0,r) = T(L,r) + O(1).$$

We also know that

$$N(L,0,r) \leq T(L,0,r).$$

But, from Proposition 4.2.3 and 4.2.4, we have $N(L,0,r) = \Omega(r\log r)$. Therefore,

$$T(L,r) = \Omega(r\log r).$$

This proves Theorem 4.5.1. $\square$

4.6.2 Proof of Theorem 4.5.2

Proof. Suppose $L_1, L_2 \in \mathbb{M}$ share one complex value 'a', CM. Since L_1 and L_2 have only one possible pole at $s = 1$, we define F as

$$F := \frac{L_1 - a}{L_2 - a} H,$$

where $H = (s-1)^k$ is a rational function such that F has no poles or zeroes. Since, L_1 and L_2 have complex order 1, we conclude that F has order at most 1 and hence is of the form

$$F(s) = e^{ms+n}.$$

This immediately leads to $m = 0$, since L_1 and L_2 are absolutely convergent on $\Re(s) > 1$ and taking $s \to \infty$, $L_1(s)$ and $L_2(s)$ approach their leading coefficient. This forces

$$L_1(s) = cL_2(s) + d.$$

for some constants $c, d \in \mathbb{C}$. Moreover, since they share the a-value, $d = a - ca$. □

The above proof also goes through if we assume L_1 and L_2 share a-values CM, with an exception of a finite set.

4.6.3 Proof of Theorem 4.5.3

Proof. We argue similarly as in [29]. Suppose $L \in \mathbb{M}$ and a meromorphic function f shares a complex value 'a' CM and another complex value 'b' IM with an error term up to $o(r \log r)$. Consider the auxiliary function

$$G := \left(\frac{L'}{(L-a)(L-b)} - \frac{f'}{(f-a)(f-b)} \right) (f - L). \tag{4.17}$$

We first claim that $N(G, r) = o(r \log r)$. The only poles of the function G comes from the zeroes of denominators in (4.17) and the poles of f.

For any zero z of $L-a$ and $f-a$, $L'/(L-a)(L-b)$ and $f'/(f-a)(f-b)$ have the same principal part in the Laurent expansion at $s=z$, because L and f share the value a CM. Hence, every zero of $L-a$ is also a zero of G.

For zeroes of $L-b$ and $f-b$ in $|s|<r$, except for $o(r\log r)$ of them, $L'/(L-b)$ and $f'/(f-b)$ have a simple pole at those points which cancel with the zero of $(f-L)$. Thus, there are at most $o(r\log r)$ poles of G in $|s|<r$ coming from the zeroes of $L-b$ and $f-b$.

Since f has finitely many poles, we conclude that

$$N(G,r)=o(r\log r).$$

Moreover, since $L-a$ and $f-a$ share zeroes with multiplicity, we have an entire function which neither has a pole nor a zero given by

$$F:=\frac{L-a}{f-a}H,$$

where H is a rational function such that it cancels the poles of f. Hence, we have

$$F(s)=e^{g(s)}.$$

We prove that g is at most a linear function. By Theorem 4.4.2, we have

$$\begin{aligned} T(f,r) &< \overline{N}\left(\frac{1}{f-a},r\right)+\overline{N}\left(\frac{1}{f-b},r\right)+\overline{N}(f,r)+O(\log r)\\ &=\overline{N}\left(\frac{1}{L-a},r\right)+\overline{N}\left(\frac{1}{L-b},r\right)+\overline{N}(f,r)+o(r\log r)\\ &=O(r\log r). \end{aligned}$$

Hence, the complex order of f, given by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(f,r)}{\log r} \leq 1.$$

Thus, f is of order at most 1 and since L is of order 1, we conclude that g is linear.

In order to prove Theorem 4.5.3, it suffices to show that $G = 0$. We establish this by computing the Nevanlinna characteristic of G.

Since $g(s)$ is linear and H a rational function, $T(F,r) = O(r)$ and $T(H,r) = O(\log r)$. We thus have

$$m\left(\frac{f-L}{L-a}, r\right) \leq T\left(\frac{f-L}{L-a}, r\right) = T\left(\frac{H}{F} - 1, r\right) \leq O(r).$$

Similarly,

$$m\left(\frac{f-L}{f-a}, r\right) \leq T\left(\frac{f-L}{f-a}, r\right) = T\left(1 - \frac{F}{H}, r\right) \leq O(r).$$

Using the logarithmic derivative lemma (4.16), we have

$$m\left(\frac{f'}{f-b}, r\right) = O(\log r) \text{ and } m\left(\frac{L'}{L-b}, r\right) = O(\log r).$$

Therefore, we conclude

$$m(G,r) = O(r).$$

Since, $N(G,r) = o(r\log r)$, we get

$$T(G,r) = o(r\log r).$$

From Theorem 4.4.1, we have

$$T\left(\frac{1}{G}, r\right) = T(G,r) + O(1) = o(r\log r).$$

But, note that every zero of G is also a zero of $L-a$. Therefore,

$$N\left(\frac{1}{G}, r\right) \geq N(L-a, 0, r) = \Omega(r\log r).$$

This is a contradiction since

$$N\left(\frac{1}{G}, r\right) \leq T\left(\frac{1}{G}, r\right) = o(r\log r).$$

□

4.6.4 Proof of Theorem 4.5.4

Proof. Suppose f is a meromorphic function on $\mathbb{C}$ of order ≤ 1, with finitely many poles and $L \in \mathbb{M}$ such that they share a complex value 'a' counting multiplicity and their derivatives f' and L' share zeroes up to an error term $o(r\log r)$.

Since $L-a$ and $f-a$ share zeroes with multiplicity, we have an entire function which neither has zeroes nor poles given by

$$F := \frac{L-a}{f-a}H,$$

where H is a rational function such that it cancels the poles of f. Hence, we have

$$F(s) = e^{g(s)}.$$

Since f has complex order ≤ 1, we get $g(s)$ is linear.

Consider the auxiliary function

$$G(s) := \left(\frac{1}{(L-a)f'} - \frac{1}{(f-a)L'}\right)(f'-L')(f-L). \tag{4.18}$$

Now, we do a similar analysis as in the proof of Theorem 4.5.3. We first claim that $N(G,r) = o(r\log r)$. The only poles of G can arise from the zeroes of the denominator in (4.18).

For any zero z of $L-a$ and $f-a$, we have $L'/(L-a)L'f'$ and $f'/(f-a)f'L'$ have the same principal part in the Laurent expansion at $s=z$, because L and f share the value a CM. Hence, every zero of $L-a$ is also a zero of G.

For zeroes of L' and f' in $|s|<r$, except for $o(r\log r)$ of them, they are also zeroes $f'-L'$ of same multiplicity. Thus, there are at most $o(r\log r)$ poles of G in $|s|<r$ coming from the zeroes of L' and f'.

Since, f has finitely many poles, so does f' and hence we conclude

$$N(G,r) = o(r\log r).$$

In order to prove Theorem 4.5.4, it suffices to show that $G\equiv 0$. We establish this by computing the Nevanlinna characteristic of G.

Since $g(s)$ is linear and H a rational function, $T(F,r)=O(r)$ and $T(H,r)=O(\log r)$. We thus have

$$m\left(\frac{f-L}{L-a}, r\right) \le T\left(\frac{f-L}{L-a}, r\right) = T\left(\frac{H}{F}-1, r\right) \le O(r).$$

Similarly,

$$m\left(\frac{f-L}{f-a}, r\right) \le T\left(\frac{f-L}{f-a}, r\right) = T\left(1-\frac{F}{H}, r\right) \le O(r).$$

Note that

$$\frac{f'-L'}{f'} = 1-\frac{L'}{f'}.$$

Since $L(s)-a = (f(s)-a)\frac{e^{g(s)}}{H}$, taking derivatives we get

$$L'(s) = f'(s)\frac{e^{g(s)}}{H} + (f(s)-a)\left(\frac{e^{g(s)}}{H}\right)'.$$

Dividing by $f'(s)$, we have

$$\frac{L'}{f'}(s) = \frac{e^{g(s)}}{H} + \frac{(f(s)-a)}{f'}\left(\frac{e^{g(s)}}{H}\right)'.$$

We calculate the proximity function for the right hand side.

$$m\left(\frac{e^{g(s)}}{H}, r\right) \leq m(e^{g(s)}, r) + m(H, r) = O(r),$$

$$m\left(\frac{(f(s)-a)}{f'}\left(\frac{e^{g(s)}}{H}\right)', r\right) \leq m\left(\frac{(f(s)-a)}{f'}, r\right) + m\left(\left(\frac{e^{g(s)}}{H}\right)', r\right).$$

Using logarithmic derivative lemma (4.16), we get

$$m\left(\frac{(f(s)-a)}{f'}\left(\frac{e^{g(s)}}{H}\right)', r\right) = O(r).$$

Therefore, we have

$$m\left(\frac{f'-L'}{f'}, r\right) = O(r).$$

Similarly, we also get

$$m\left(\frac{f'-L'}{L'}, r\right) = O(r).$$

Hence, we conclude

$$T(G, r) = o(r\log r).$$

Now, we again proceed as in proof of Theorem 4.5.3. By Theorem 4.4.1, we have

$$T\left(\frac{1}{G}, r\right) = T(G, r) + O(1).$$

Moreover, every zero of $L-a$ is also a zero of G. Hence,

$$N\left(\frac{1}{G}, r\right) \geq N(L-a, 0, r) = \Omega(r \log r).$$

This contradicts the fact that

$$N\left(\frac{1}{G}, r\right) \leq T\left(\frac{1}{G}, r\right) = o(r \log r).$$

$\square$

Chapter 5

Universality and the Linnik-Ibragimov Conjecture

5.1 Voronin's universality theorem

In early twentieth century, Harald Bohr introduced geometric and probabilistic methods to the study of value distribution of the Riemann zeta-function. For this chapter, the probabilistic methods will be of significance.

For the Riemann zeta-function $\zeta(s)$, we know that if $\sigma_0 > 1$, then

$$|\zeta(s)| \leq \zeta(\sigma_0)$$

in the right half plane $\Re(s) \geq \sigma_0$. In other words, $\zeta(s)$ is bounded on any right half plane $\Re(s) > 1 + \epsilon$. The natural question to consider is what happens when σ_0 approaches 1 from the right.

In this regard, Bohr [2] proved that in any strip $1 < \Re(s) < 1 + \epsilon$, $\zeta(s)$ takes any non-zero complex value infinitely often. The main tool used by Bohr was the Euler product of $\zeta(s)$. Similar study in the critical strip is much more difficult. To tackle this problem, Bohr studied truncated Euler products

$$\zeta_M(s) := \prod_{p \le M} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Although $\zeta_M(s)$ do not converge in the critical strip as M tends to ∞, the mean of $\zeta_M(s)$ converges and approximates $\zeta(s)$ very well in this region. This was the revolutionary idea in the study of Bohr.

In [3], Bohr showed that for any $\sigma_0 \in (1/2, 1)$, the image of the vertical line $\{\Re(s) = \sigma_0\}$ given by

$$\left\{\zeta(s) : s = \sigma_0 + it, t \in \mathbb{R}\right\}$$

is dense in $\mathbb{C}$.

Later, Bohr and Jessen [4], [5] improved these results using probabilistic methods to prove the following remarkable limit theorem.

Theorem 5.1.1 (Bohr, Jessen)**.** *Let R be any rectangle in $\mathbb{C}$ with sides parallel to the real and imaginary axis. Let G be the half plane $\{\Re(s) > 1/2\}$ except for points $z = x+iy$ such that there is a zero of $\zeta(s)$ given by $\rho = x + i\tau$ with $y \le \tau$. For any $\sigma > 1/2$, the limit*

$$\lim_{T\to\infty} \frac{1}{T} meas\left\{\tau \in [0, T] : \sigma + i\tau \in G, \log \zeta(\sigma + i\tau) \in R\right\}$$

exists.

Later, Hattori and Matsumoto [21] identified the probability distribution given by the above limit theorem. Following Bohr's line of thought, Voronin [46] in 1972 proved the following generalization of Bohr's limit theorem.

Theorem 5.1.2 (Voronin)**.** *For any fixed distinct numbers $s_1, s_2, \cdots, s_n$ with $1/2 < \Re(s_j) < 1$ for $1 \le j \le n$, the set*

$$\left\{(\zeta(s_1 + it), \cdots, \zeta(s_n + it)) : t \in \mathbb{R}\right\}$$

is dense in $\mathbb{C}^n$*. Moreover, for any fixed number* s *with* $1/2 < \Re(s) < 1$*,*

$$\left\{(\zeta(s+it), \zeta'(s) \cdots, \zeta^{(n-1)}(s_n + it)) : t \in \mathbb{R}\right\}$$

is dense in $\mathbb{C}^n$*.*

Analogous limit and density theorems for other L-functions were obtained by Matsumoto [31], Laurinčikas [27], Šleževičienė [41] et al.

It is interesting to note that despite the density theorems, we do not understand the value distribution of $\zeta(s)$ on $\Re(s) = 1/2$. A folklore, yet unsolved conjecture is that the set of values of $\zeta(s)$ on $\Re(s) = 1/2$ is dense in $\mathbb{C}$. In this direction, Selberg showed that values "up to some normalization" of $\zeta(s)$ on the 1/2-line are normally distributed.

We now state the most remarkable result on the value distribution theory of Riemann zeta-function. In 1975, Voronin [47] proved a theorem for the Riemann zeta-function, which roughly says that any analytic function is approximated uniformly by shifts of the zeta function in the critical strip. This is called the Voronin universality theorem. More precisely,

Theorem 5.1.3 (Voronin)**.** *Let* $0 < r < \frac{1}{4}$ *and suppose that* $g(s)$ *is a non-vanishing continuous function on the disc* $\{|s| < r\}$*, which is analytic in the interior. Then, for any* $\epsilon > 0$*,*

$$\liminf_{T\to\infty} \frac{1}{T} \ meas \left\{|\tau| < T : |\zeta(s + \frac{3}{4} + i\tau) - g(s)| < \epsilon, |s| < r\right\} > 0.$$

We highlight the main ideas of the proof of Voronin's theorem and use similar ideas in the proofs of the later theorems in the chapter.

Proof. Let $K = \{s : |s| \leq r\}$. Since $g(s)$ is non-vanishing on K, we have $f(s) = \log g(s)$. When taking log, we will always assume the principal branch of logarithm. Let Ω be the set of sequences of real numbers indexed by primes, i.e. $\omega = (\omega_2, \omega_3, \cdots) \in \Omega$. Then, for any finite subset of primes M, we define the truncated Euler product given by

$$\zeta_M(s,\omega) := \prod_{p\in M}\left(1-\frac{\exp(-2\pi i\omega_p)}{p^s}\right)^{-1}.$$

Here, $\zeta_M(s,\omega)$ is non-vanishing and analytic on $\Re(s) > 0$. So, we can take log and have

$$\log\zeta_M(s,\omega) := -\sum_{p\in M}\log\left(1-\frac{\exp(-2\pi i\omega_p)}{p^s}\right).$$

The idea is to approximate $f(s)$ using rearrangements of the summation given above. Since f is uniformly continuous in the interior of K, there exists κ with $\kappa^2 < 1/4$ such that

$$\max_{|s|\le r}\left|f\left(\frac{s}{\kappa^2}\right)-f(s)\right| < \frac{\epsilon}{2}.$$

Now, $f(s/\kappa^2)$ is analytic and bounded in the region $R := \{|s| \le \kappa r\}$ and thus is an element in the Hardy space (Hilbert space with a specified norm).

Now, the key idea is to use Pechersky's generalization of Riemann's rearrangement theorem to Hilbert spaces.

Theorem 5.1.4 (Pechersky's rearrangement theorem). *Let $\{x_n\}$ be a sequence in a complex Hilbert space $\mathbb{H}$ satisfying*

1. $\sum_{n=1}^{\infty}\|x_n\|^2 < \infty$.

2. *For $0 \neq x \in \mathbb{H}$, the series $\sum_{n=1}^{\infty}|\langle x_n, x\rangle|$ diverges to infinity.*

Then, the set of all convergent series

$$\sum_{n=1}^{\infty} a_n x_n,$$

with $|a_n| = 1$ is dense in $\mathbb{H}$.

For $\omega \in \Omega$, let

$$u_k(s,\omega) := \log\left(1 - \frac{\exp(-2\pi i \omega_{p_k})}{p_k^{s+3/4}}\right).$$

Consider the series

$$\sum_{k=1}^{\infty} u_k(s,\omega).$$

By Pechersky's theorem 5.1.4, there exists a rearrangement of $\omega = (1/4, 2/4, 3/4, \cdots)$, such that

$$\sum_j u_{k_j}(s,\omega) = f(s/\kappa^2).$$

Thus, the tail of the series can be made as small as possible. Say for $j > y$, the above sum $\sum_{j>y} u_{k_j}(s,\omega) < \epsilon/2$. Take M a finite set of first n primes such that it contains all primes appearing in the set $\{p_{k_j}\}$ for $j \leq y$. Thus, we get

$$\max_{|s|<r} |\log \zeta_M(s + 3/4, \omega) - f(s)| < \epsilon/2.$$

The next key idea is to approximate $\log \zeta(s)$ using $\log \zeta_M(s)$.

For $E = \{s = \sigma + it + 3/4 : -\kappa r < \sigma \leq 2, |t| \leq 1\}$ and any $\delta > 0$, define

$$A_T := \left\{\tau \in [T, 2T] : \int_E |\zeta_M^{-1}(s + i\tau, \mathbf{0})\zeta(s + i\tau) - 1|^2 \, d\sigma \, dt < \delta^2\right\}.$$

Using the approximate functional equation, one can get that for sufficiently large T and M,

$$meas\ A_T > (1 - \delta)T.$$

Moreover, by continuity of $\log \zeta_M(s + 3/4, \omega)$, for any $\epsilon_1 > 0$, there exists a δ_1 such that if for some $\omega' \in \Omega$,

$$\|\omega_p - \omega'_p\| < \delta_1$$

for $p \in M$, then

$$\max_{|s| \leq \kappa r} \left| \log \zeta_M(s + 3/4, \omega) - \log \zeta_M(s + 3/4, \omega') \right| < \epsilon_1.$$

Let

$$B_T := \left\{ \tau \in [T, 2T] : \left\| \tau \frac{\log p}{2\pi} - \omega_p \right\| < \delta_1 \right\}.$$

Now using Weyl's refinement of Kronecker's approximation theorem on uniformly distributed curved mod1, it is possible to show that for T sufficiently large

$$meas\ B_T > cT,$$

for some positive constant c. Hence, we conclude

$$\frac{1}{T} meas\ (A_T \cap B_T) > 0,$$

and the theorem follows. □

5.2 Linnik-Ibragimov conjecture

After the result of Voronin, Bagchi [1] gave a proof of universality for the Riemann zeta-function $\zeta(s)$ and some other L-functions using probabilistic methods. Using Bagchi's technique, the universality property for many L-functions has been established, mainly due to the work of Laurančikas, Matsumoto, Steuding et al. In particular, we know that the universality property holds for elements in the Selberg class $\mathbb{S}$.

Theorem 5.2.1. *Let $L(s) \in \mathbb{S}$ with degree d_L. Let K be a compact subset of the strip*

$$1 - \frac{1}{d_L} < \Re(s) < 1,$$

with connected complement. Suppose $g(s)$ be any non-vanishing continuous function on K, which is analytic in the interior of K. Then, for any $\epsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\, meas \left\{ |\tau| < T : |L(s+i\tau) - g(s)| < \epsilon, s \in K \right\} > 0$$

It is important to note that the L-functions for which the universality property has been established is much larger than the Selberg class. In fact, L-functions such as the Hurwitz zeta function, Lerch zeta function or Matsumoto zeta functions are all known to be universal in a certain strip. In view of this, Linnik and Ibragimov conjectured the following.

Conjecture 13. *Let $F(s)$ have a Dirichlet series representation, absolutely convergent on $\Re(s) > 1$ and suppose $F(s)$ can be analytically continued to $\mathbb{C}$ except for a possible pole at $s = 1$ satisfying some "growth conditions", then $F(s)$ is universal in a certain strip.*

In this next section, we study the universality property for elements in $\mathbb{M}$.

5.3 Main results

Let $F \in \mathbb{M}$. For the purpose of studying the universality property in a vertical strip for $F(s)$, we can assume that $F(s)$ is entire. This is because if $F(s)$ has a pole of order k at $s = 1$, we study the function

$$G(s) := \left(1 - \frac{2}{2^s}\right)^k F(s).$$

If $G(s)$ is universal in a vertical strip, so is $F(s)$.

Therefore, without loss of generality, assume F is entire and given by $\sum_n a_n/n^s$ in the region $\Re(s) > \sigma_c$. Let σ_0 be the abscissa of absolute convergence of F. Define

$$\sigma_m := \inf\left\{\sigma > \sigma_c : \frac{1}{T}\int_{-T}^{T} |F(\sigma+it)|^2 dt \ll T\right\}.$$

We state the main results of this chapter below. Henceforth, for an open set $D \subseteq \mathbb{C}$, $H(D)$ denotes the space of holomorphic functions on D.

Theorem 5.3.1. *For $F \in \mathbb{M}$ with $c_F^* > 0$, let D denote the vertical strip*

$$\{s \in \mathbb{C} : \sigma_m < \Re(s) < \sigma_0\}.$$

There exists a sequence of functions F_n converging to F in $H(D)$, such that each of the F_n is universal in D.

Let $\mathfrak{K}$ be a compact set in the strip $\{s \in \mathbb{C} : \sigma_m < \Re(s) < \sigma_0\}$ with connected complement. We call a function $g(s)$ defined on $\mathfrak{K}$ as ϵ-approximable by $F(s) \in \mathbb{M}$ if for a given $\epsilon > 0$, we can find a $\delta > 0$ and a sequence F_n of universal functions converging to F in the strip as in Theorem 5.3.1, such that

$$\liminf_{T\to\infty} \frac{1}{T} meas\left\{\tau| < T : |F_{E_n}(s+i\tau) - g(s)| < \epsilon/2, s \in \mathfrak{K}\right\} > \delta,$$

for all $T > T_\epsilon$, for some large T_ϵ depending on T.

Theorem 5.3.2. *Suppose $F \in \mathbb{M}$ with $c_F^* > 0$ and D be as in Theorem 5.3.1. Let K be a compact set in D with connected complement. Let $g(s)$ be a continuous function on K, holomorphic in the interior of K, and ϵ-approximable by $F(s)$. Then,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\left\{|\tau| < T : |F(s+i\tau) - g(s)| < \epsilon\right\} > 0.$$

5.4 Limit and Density Theorems

In this section, we introduce some tools and results to be used in the proof of Theoerm 5.3.1 and 5.3.2.

Let $H(G)$ be the space of functions analytic on G with the topology of uniform convergence on compact sets. Denote by $B(S)$ the class of Borel sets of the topological space S. Let $\gamma_m := \{s \in \mathbb{C} : |s| = 1\}$. Define the infinite dimensional torus

$$\Omega := \prod_{m=1}^{\infty} \gamma_m.$$

By inducing the product topology and pointwise multiplication, Ω forms a compact abelian group. Thus, we have the normalized Haar measure m_H on Ω, which generates a probability space $(\Omega, B(\Omega), m_H)$. Denote $\omega(m)$ as the projection of $\omega \in \Omega$. Then, $\{\omega(m) : m \in \mathbb{N}\}$ is a sequence of independent random variables defined on $(\Omega, B(\Omega), m_H)$.

We state the limit theorem due to Laurancikas, Steuding and Schwarz [25].

Theorem 5.4.1. *Let $f(s) = \sum_{m=1}^{\infty} a_m \omega(m) e^{-\lambda_m s}$, where $\omega \in \Omega$ and $\{\lambda_m\}$ be $\mathbb{Q}$-linearly independent. Let $D := \{s \in \mathbb{C} : \Re(s) > \sigma_m\}$ and $D_M := \{s = \sigma + it \in \mathbb{C} : \sigma_m < \sigma < \sigma_0, |t| < M\}$ satisfying the following properties.*

1. *$|F(s)| \ll |t|^{\alpha}$ for some $\alpha > 0$, where $\sigma_m < \sigma < \sigma_0$.*

2. *For $\sigma_m < \sigma < \sigma_0$*
$$\int_{-T}^{T} |F(\sigma + it)|^2 dt \ll T.$$

3. *$\lambda_m \geq c(\log m)^{\delta}$ for some $c, \delta > 0$.*

Then, the probability measure

$$\nu(F(s + i\tau) \in A), \text{ for } A \in B(H(D_M))$$

converges weakly to P_F as $T \to \infty$, where

$$P_F(A) := m_H(\omega \in \Omega : F(s,\omega) \in A) \textit{ for } A \in B(H(D_M)).$$

If $F \in \mathbb{M}$, then $\sum_n a_n/n^s = \sum_n a_n e^{-s\log n}$. Unfortunately, $\{\log n\}$ is not $\mathbb{Q}$-linearly independent. Thus, we do not have the limit theorem as above.

Consider a sequence $E = \{e_n\}$ with $0 < e_n < 1$. Define

$$F_E(s) := \sum_{n=1}^{\infty} a_n e^{-s\log n + e_n}.$$

Lemma 5.4.2. *If $F \in \mathbb{M}$, then $\sigma_c(F) = \sigma_c(F_E), \sigma_m(F) = \sigma_m(F_E)$ and $\sigma_0(F) = \sigma_0(F_E)$.*

Proof. Suppose the Dirichlet series of F given by $F(s) = \sum_n a_n/n^s$ is convergent in the region $\Re(s) > \sigma_c$. Note that

$$\left|\frac{a_n}{n^s} - \frac{a_n}{(n+e_n)^s}\right| \leq \left|\frac{a_n}{n^s}\right|\left|1 - \frac{n^s}{(n+e_n)^s}\right|.$$

Since $0 < e_n < 1$, we clearly have

$$\frac{1}{2} \leq \frac{n}{n+e_n} \leq 1.$$

Hence, for $\sigma > \sigma_c$, we have

$$\left|1 - \frac{n^s}{(n+e_n)^s}\right| \leq \frac{1}{2^{\sigma_c}}.$$

Thus,

$$|F(s) - F_E(s)| \ll |F(s)|, \tag{5.1}$$

for $\Re(s) > \sigma_c$. Therefore $\sigma_c(F_E) = \sigma_c(F)$. By the same argument, we conclude $\sigma_0(F_E) = \sigma_0(F)$.

We argue similarly for σ_m. Suppose $\Re(s) = \sigma > \sigma_m$. From $\sigma_c < \sigma_m < \sigma_0$, we know that the Dirichlet series of $F(s)$ and $F_E(s)$ are convergent. Moreover, from (5.1) we know that $|F_E(s)| \ll |F(s)|$. Therefore, we conclude

$$\int_{-T}^{T} |F_E(\sigma+it)|^2 dt \ll \int_{-T}^{T} |F(\sigma+it)|^2 dt \ll T.$$

$\square$

For every $n \in \mathbb{N}$ and given $0 < \delta_n < e^{-n}$, there exists $E = \{\epsilon_n\}$ such that $\epsilon_n < \delta_n$ and $\{\log(n+\epsilon_n)\}_{n=1}^{\infty}$ is $\mathbb{Q}$-linearly independent. Define

$$F_E(s) := \sum_{n=1}^{\infty} a_n e^{-s\log(n+\epsilon_n)}.$$

By our choice of E, we have σ_0, σ_m and σ_c of F_E is same as that of F. Moreover, $F_E(s)$ satisfies the Limit Theorem 5.4.1. We first show the universality property for F_E in the strip $\{s \in \mathbb{C} : \sigma_m < \Re(s) < \sigma_0\}$. We proceed as in [28].

Denote

$$D_W := \{s \in \mathbb{C} : \sigma_m < \Re(s) < \sigma_0, |\Im(s)| < W\}$$

The goal of this section is to show that the support of P_{F_E} is $H(D_W)$.

Let S be a separable topological space and P be a probability measure on $(S, B(S))$. Recall that the support of P in S is defined as

$$S_P := \bigcap_{\substack{P(C)=1,\\ C \text{ closed}}} C.$$

In other words,

$$S_P = \{x \in S : P(U) > 0, \text{ for any neighbourhood } U \text{ of } x\}.$$

Let X be an S-valued random element on a probability space $(\Omega, \mathfrak{F}, P)$, then the support of the random variable X, denoted by S_X, is the support of the distribution

$P(X \in A)$, where $A \in B(S)$.

We are now ready to prove the main lemma of this section. Let $\{\omega(m) : m \in \mathbb{N}\}$ be a sequence of independent random variables defined on $(\Omega, B(\Omega), m_H)$ as described above. For any such $\{w_m\}$, define

$$g_m(s) = a_m \omega(m) e^{-s \log(n+\epsilon_n)}$$

We have the following lemma.

Proposition 5.4.3. *Let $g_m(s)$ be as above. The set of all convergent series $\sum_m g_m(s)$ is dense in $H(D_W)$.*

Proof. We use the following theorem (see [26], Theorem 6.3.10).

Theorem 5.4.4. *Let D be a simply connected domain in $\mathbb{C}$. Let $\{f_m\}$ be a sequence in the space $H(D)$ satisfying the following properties.*

1. *If μ be a complex measure on $(\mathbb{C}, B(\mathbb{C}))$ with compact support contained in D such that*
$$\sum_{m=1}^{\infty} \int_{\mathbb{C}} f_m d\mu < \infty.$$
Then, $\int_{\mathbb{C}} s^r d\mu(s) = 0$ for all $r \in \mathbb{N} \cup 0$.

2. *The series $\sum_{m=1}^{\infty} f_n$ converges in $H(D)$.*

3. *For any compact $K \in D$,*
$$\sum_{n=1}^{\infty} \sup_{s \in K} |f_n(s)|^2 < \infty.$$
Then, the set of all convergent series
$$\sum_{m=1}^{\infty} b_n f_n,$$
with $|b_n| = 1$ for all n is dense in $H(D)$.

To prove the lemma, consider any complex measure μ on $(\mathbb{C}, B(\mathbb{C}))$ with compact support contained in D_W, such that

$$\sum_{m=1}^{\infty} \sup_{s\in K} |g_m(s)|^2 < \infty.$$

In order to conclude $\int_{\mathbb{C}} s^r d\mu(s) = 0$ for $r = 0, 1, 2, \cdots$, we first set some notations.

Let $F_E(s) = \sum_n a_n/(n+\epsilon_n)^s$ be as above with abscissa of absolute convergence σ_0. Denote

$$c_n := \frac{a_n}{(n+\epsilon_n)^{\sigma_0}}.$$

For any complex measure μ on $(\mathbb{C}, B(\mathbb{C}))$ with compact support on D_W, denote

$$\rho(s) := \int_{\mathbb{C}} e^{-sz} d\mu(z).$$

We use the following Lemma (see Lemma 5 of [28])

Lemma 5.4.5. *For any complex measure μ on $(\mathbb{C}, B(\mathbb{C}))$ with compact support on D_W, if*

$$\sum_{n=1}^{\infty} |c_n||\rho(\lambda_n)| < \infty,$$

then

$$\int_{\mathbb{C}} s^r d\mu(s) = 0,$$

for $r = 0, 1, 2, \cdots$

Using Lemma 5.4.5, we see that $g_m(s)$ satisfies all the conditions in the Theorem 5.4.4. Moreover, from [25], we also know that for almost all $\omega \in \Omega$,

$$\sum_{m=1}^{\infty} \frac{a_m \omega(m)}{(n+\epsilon_n)^s}$$

converges uniformly on compact sets on the right half plane $\{s : \Re(s) > \sigma_c\}$. Hence, applying the Theorem 5.4.4, we have the lemma. □

5.5 Proof of the main theorems

In this section, we use the tools introduced to prove Theorem 5.3.1 and 5.3.2. We first show that $F_E(s)$ is universal, where

$$F_E(s) := \sum_{n=1}^{\infty} \frac{a_n}{(n+\epsilon_n)^s}$$

and $\{\log(n+\epsilon_n)\}$ is $\mathbb{Q}$-linearly independent. To prove this, we use the following theorem (see Theorem 1.7.10 of [26]).

Theorem 5.5.1. *Let $\{X_n\}$ be a sequence of independent $H(D_W)$-valued elements such that $\sum_{m=1}^{\infty} X_m$ converges almost surely. Then, the support of the sum $\sum_{m=1}^{\infty} X_m$ is the closure of all $g \in H(D_W)$, which may be written as the convergent series*

$$g = \sum_{m=1}^{\infty} g_m,$$

for $g_m \in S_{X_m}$.

Using this theorem and Proposition 5.4.3, we show that P_{F_E} is supported on the whole of $H(D_W)$.

Proposition 5.5.2. *The support of the measure P_{F_E} is $H(D_W)$.*

Proof. The sequence $\{\omega(m)\}$ defined above is a sequence of independent random variables on the probability space $(\Omega, B(\Omega), m_H)$, where the support of every $\omega(m)$ is the unit circle. Therefore $\{a_m\omega(m)e^{s(-\log m+\epsilon_m)}\}$ is a sequence of $H(D_W)$-valued random elements and the support of each element is given by

$$\{g \in H(D_W) : g(s) = a_m b e^{s(-\log(m+\epsilon_m)}, \text{ with } |b| = 1\}.$$

Therefore, by Lemma 5.5.1, the support of $H(D_W)$-valued random element

$$F(s,w) := \sum_{m=1}^{\infty} a_m \omega(m) e^{-s\log(m+\epsilon_m)}$$

is the closure of the set of all convergent series $\sum a_m b_m e^{-s\log(m+\epsilon_m)}$, with $|b_m| = 1$. By Lemma 5.4.3, we are done. $\square$

We now show that $F_E(s)$ is universal.

Proposition 5.5.3. *Suppose F_E is defined as above. Let K be a compact set in $D = \{s \in \mathbb{C} : \sigma_m < Re(s) < \sigma_0\}$ with connected complement. Let $g(s)$ be a continuous function on K, holomorphic in the interior of K. Then, for any $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \quad meas \left\{ |\tau| < T : |F_E(s+i\tau) - g(s)| < \epsilon \right\} > 0.$$

Proof. Let K be a compact set in D with connected complement. Then, there exists a D_W such that $K \subseteq D_W$. We first show that vertical shifts of $F_E(s)$ approximate any polynomial $p(s)$ over D_W upto ϵ. Consider the following open set in $H(D_W)$.

$$G := \{f \in H(D_W) : |f(s) - p(s)| < \epsilon, \text{ for } s \in D_W\}.$$

Since G is open, by Lemma 5.5.2 and Theorem 5.4.1, we get

$$\liminf_{T\to\infty} \nu_T \left\{ \sup_{s\in K} |F_E(s+i\tau) - p(s)| < \epsilon \right\} \geq P_{F_E}(G) > 0.$$

We prove the assertion for any function $g(s)$ continuous on K and holomorphic in its interior by finding a polynomial approximation to $g(s)$ on K. $\square$

Next, we prove the Theorem 5.3.1. In other words, we show that any $F \in \mathbb{M}$ is a limit point of functions F_{E_m}, which are universal in the vertical strip $D = \{s \in \mathbb{C} : \sigma_m < \Re(s) < \sigma_0\}$. It suffices to show the following lemma.

Lemma 5.5.4. *For any given $\epsilon, T > 0$, there exists an $E = \{\epsilon_n\}$ such that*

$$|F(s) - F_E(s)| < \epsilon,$$

for $s \in D_T := \{s = \sigma + it \in \mathbb{C} : \sigma_m < \Re(s) < \sigma_0, |t| < T\}$.

Proof. For any fixed n and small $\delta_n > 0$, and $T > 0$, we can find $\epsilon_n > 0$, such that

$$1 - \delta_n \leq \left| \frac{n^s}{(n + \epsilon_n)^s} \right| \leq 1 + \delta_n,$$

for $s \in D_T$. Note that this is not true if we let $T \to \infty$. Choosing

$$\delta_n = \frac{\epsilon}{2^n} \left| \frac{n^{\sigma_m}}{a_n} \right|,$$

we have

$$\left| \frac{a_n}{n^s} \left(1 - \frac{n^s}{(n + \epsilon_n)^s} \right) \right| \leq \frac{\epsilon}{2^n}.$$

for all $s \in D_T$. By summing over all n, we conclude that

$$\left| \sum_{n=1}^{\infty} \frac{a_n}{n^s} - \sum_{n=1}^{\infty} \frac{a_n}{(n + \epsilon_n)^s} \right| < \epsilon,$$

for $s \in D_T$. Hence, we have the lemma. □

From Lemma 5.5.4, it is clear that for any $F \in \mathbb{M}$, we can find a sequence $\{E_n\}$, such that F_{E_n} converges to F and each $F_{E_n}(s)$ is universal is the strip $\sigma_m < \Re(s) < \sigma_0$. Hence, we have Theorem 5.3.1.

For Theorem 5.3.2, we suppose that for an $\epsilon > 0$, $g(s)$ is continuous on $K \subseteq D$ and analytic in the interior of K and ϵ-approximable by $F(s)$.

We can find a sequence F_{E_n} of functions universal in D such that $|F_{E_n}(s + it_0) - g(s)| < \epsilon/2$ for infinitely many t_0 and further there exists a $\delta > 0$ and $T_\epsilon > 0$ such that

$$meas\left\{|\tau| < T : |F_{E_n}(s+i\tau) - g(s)| < \epsilon/2\right\} \geq \delta T,$$

with $\delta > 0$ fixed, for all $T > T_\epsilon$

By Theorem 5.3.1, for any given $T > T_\epsilon$ we can find a subsequence F_{E_n} of functions universal in D such that

$$|F(s) - F_{E_n}(s)| < \epsilon/2$$

for $s \in D_T$.

By triangle inequality, we have

$$|F(s+i\tau) - g(s)| \leq |F(s+i\tau) - F_{E_n}(s+i\tau)| + |F_{E_n}(s+i\tau) - g(s)|.$$

Hence, we conclude for $T \gg T_\epsilon$ as chosen above,

$$meas\left\{|\tau| < T : |F(s+i\tau) - g(s)| < \epsilon\right\} \gg T.$$

This proves Theorem 5.3.2.

5.6 The Generalized Strong Recurrence

Bohr studied the notion of almost periodicity for Dirichlet series. Any analytic function $f(s)$ is said to be almost periodic in a vertical strip $\sigma_1 < \Re(s) < \sigma_2$, if for any $\epsilon > 0$, we can find a length $l > 0$, such that for any interval of length l, there exists a $\tau \in (\theta, \theta + l)$ satisfying

$$|f(s+i\tau) - f(s)| < \epsilon,$$

for a fixed s in the vertical strip $\sigma_1 < \Re(s) < \sigma_2$.

Bohr showed that any Dirichlet series is almost periodic in its half plane of absolute convergence. Moreover, he also discovered interesting connections between the Riemann hypothesis and the almost periodicity of L-functions. In fact, he showed that Riemann hypothesis for any Dirichlet L-function $L(s, \chi)$, where χ is a non-principal primitive character, is equivalent to almost periodicity of $L(s, \chi)$ in the right half plane $\Re(s) > 1/2$.

In 1981, Bagchi [1] observed that a similar notion of almost periodicity in the sense of universality is equivalent to the Riemann hypothesis. A refined version of Bagchi's theorem can be stated as follows.

Theorem 5.6.1 (Bagchi)**.** *Let $\theta \geq 1/2$. Then $\zeta(s)$ is non-vanishing in the half plane $\Re(s) > \theta$ if and only if, for $\epsilon > 0$, any z with $\theta < \Re(z) < 1$, and for any $0 < r < \min\{\Re(z) - \theta, 1 - \Re(z)\}$,*

$$\liminf_{T\to\infty} \frac{1}{T} \, meas \left\{ \tau \in [0, T] : \max_{|s-z|\leq r} |\zeta(s + i\tau) - \zeta(s)| < \epsilon \right\} > 0.$$

The above equivalent formulation of Riemann hypothesis is called the property of strong recurrence. We sketch a proof of the above theorem.

Proof. If the Riemann hypothesis is true, then we clearly have the strong recurrence as a consequence of Voronin's universality theorem.

On the other hand, if the strong recurrence is true, we will produce a large number of zeroes of $\zeta(s)$ in the vertical strip by appropriately choosing some subsets, which would violate the density theorem.

Suppose ρ is a zero of $\zeta(s)$ with $\Re(\rho) > \theta$. Consider a neighborhood around ρ, given by

$$H_\delta := \{s : |s - \rho| < \delta\}.$$

Assuming the strong recurrence, we can find $\tau > 0$ such that

$$|\zeta(s + i\tau) - \zeta(s)| < \epsilon,$$

for $s \in H_\delta$. Taking ϵ sufficiently small, Rouche's theorem ensures a zero of $\zeta(s)$ in the region $H_\delta + i\tau$. Moreover, the strong recurrence also ensures that the number of occurrences of τ is proportional to the vertical height T. Hence, we get the number of zeroes of $\zeta(s)$ in the region $\Re(s) > \theta$ and $|\Im(s)| < T$ to be $\Omega(T)$, which violates the density theorem 2.9,

$$N(\theta, T) \ll T^{4\theta(1-\theta)+\epsilon_1}.$$

$\square$

In class $\mathbb{M}$, although we do not have the Riemann hypothesis, we might expect to have a strong recurrence property.

We end by stating the analogue of Riemann hypothesis for the class $\mathbb{M}$ in terms of the strong recurrence property.

Conjecture 14 (Generalized Strong Recurrence)**.** *Let $F \in \mathbb{M}$. For any $\epsilon > 0$ and any z with $\sigma_m(F) < \Re(z) < \sigma_0(F)$, and for any $0 < r < \min\{\Re(z) - \sigma_m(F), \sigma_0(F) - \Re(z)\}$,*

$$\liminf_{T\to\infty} \frac{1}{T}\, meas \left\{\tau \in [0, T] : \max_{|s-z|\le r} |F(s + i\tau) - F(s)| < \epsilon\right\} > 0.$$

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