Unique Reconstruction of a Potential from the Dirichlet to Neumann Map in Locally CTA Geometries

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Abstract

Let (M^3, g) be a compact smooth Riemannian manifold with smooth boundary and suppose that U is an open set in M such that $g|_U$ is Euclidean. Let $\Gamma = U \cap \partial M$ be connected and suppose that U is the convex hull of Γ. We will study the uniqueness of an unknown potential for the Schrödinger operator $-\triangle_g + q$ from the associated Dirichlet to Neumann map, Λ_q. Indeed, we will prove that if the potential q is a priori explicitly known in U^c then one can uniquely reconstruct q from Λ_q. We will also give a reconstruction algorithm for the potential. More generally we will also discuss the cases where Γ is not connected or $g|_U$ is conformally transversally anisotropic and derive the analogous result.

Contents

Chapter 1

Introduction

1.1 Problem Formulation

The main aim of this dissertation is to study the Calderón problem in Riemannian geometries. In 1980, Calderón [4] proposed the following question: Can the conductivity of an unknown medium be determined from voltage and current measurments on the boundary? Since then this question has been of considerable interest in the area of inverse problems and there is a rich literature of results. Let us assume that Ω is a domain in $\mathbb{R}^n$ with C^∞ boundary. Under the assumption of no sources or sinks in the domain we can represent the current flow in a body through the following elliptic partial differential equation:

$$\left\{ \begin{array}{ll} \nabla \cdot (\gamma \nabla u) = 0 & \text{for } x \in \Omega \\ u = f & \text{for } x \in \partial\Omega \end{array} \right\}$$

Here γ is a positive definite matrix-valued function representing the conductivity tensor for the medium and f is a given voltage on the boundary. The normal component of the current flux at the boundary is given by the expression:

$$\Lambda_\gamma f := \gamma \nabla u \cdot \nu|_{\partial\Omega}$$

where ν represents the outward pointing unit normal vector on $\partial\Omega$. The problem of Calderón asks whether the knowledge of Λ_γ uniquely determines the conductivity tensor γ. While the initial formulation of the problem assumed a scalar (isotropic) conductivity, much current research considers the case of anisotropic conductivities. This is motivated by applications to medical imaging: muscle or heart tissues have conductivity that depends not only on the location, but also on the direction (for instance along or across the muscle fiber). Since the conductivity equation above corresponds to a Laplace Beltrami equation in a Riemannian manifold, to bring to bear essential tools from Differential Geometry, it is useful to reformulate the question as follows. Let (M, g) be a compact smooth Riemannian manifold with smooth boundary and consider harmonic functions u with prescribed Dirichlet data on ∂M:

$$\left\{ \begin{array}{ll} \triangle_g u = 0 & \text{for } x \in M \\ u = f & \text{for} x \in \partial M \end{array} \right\}$$

For $f \in H^{\frac{1}{2}}(\partial M)$ the above partial differential equation is known to have a unique solution $u \in H^1(M)$. Inspired by the physical intrepretation of the problem we define the Dirichlet to Neumann map Λ_g as a bilinear functional as follows:

$$(\Lambda_g f, h) = \int_M \langle du, dv \rangle_g dV_g$$

where $v \in H^1(M)$ satisfies $v|_{\partial M} = h$ and $u \in H^1(M)$ satisfies $-\triangle_g u = 0$ with $u|_{\partial M} = f$. In fact we note that for smoother data $f \in H^{\frac{3}{2}}(\partial M)$:

$$\Lambda_g f = \partial_\nu u|_{\partial M}.$$

The Calderón problem can then be reformulated as follows: Does the knowledge of the Dirichlet to Neumann map Λ_g uniquely determine the unknown metric g? There is an immediate obstruction to uniqueness (first noted by Luc Tartar [28]) due to diffeomorphisms that fix the boundary as can be seen from the following Lemma [5].

Lemma 1.1.1. *If $F : M \to M$ is a diffeomorphism such that $F|_{\partial M} = I$ then:*

$$\Lambda_{F^* g} = \Lambda_g$$

Here $F^* g$ denotes the pullback of the metric g.

We can now state the Calderón conjecture in Riemannian geometries as follows:

Conjecture 1.1.2. *Let (M, g) denote an n-dimensional Riemannian manifold with smooth boundary with $n \geq 3$. Then Λ_g uniquely determines g up to diffeomorphisms that fix the boundary.*

In [24] it is showed that Λ_g is an elliptic pseudo-differential operator and that the symbol of Λ_g determines the metric and its jet at the boundary. This yields the solution of the Calderón problem in the case where the metric is real analytic, assuming some mild topological assumptions on the manifold.The general case of smooth manifolds is a major open problem.

There is a simpler version of Conjecture 1.1.2 that is concerned with determining the conformal factor when the conformal class of the manifold is known. The standard starting point when one wishes to determine the conformal factor is the observation that:

$$\triangle_{cg} u = c^{-\frac{n+2}{4}} (\triangle_g + q_c)(c^{\frac{n-2}{4}})$$

where $q_c = c^{\frac{n-2}{4}} \triangle_{cg}(c^{-\frac{n-2}{4}})$. It can be shown that if $c|_{\partial M} = 1$ and $\partial_\nu c|_{\partial M} = 0$ we have:

$$\Lambda_{cg} = \Lambda_{g,-q_c}$$

where $\Lambda_{g,q}$ in general denotes the Dirichlet to Neumann map for the Schrödinger equation:

$$\left\{ \begin{array}{ll} (-\triangle_g + q)u = 0 & \text{for } x \in M \\ u = f & \text{for} x \in \partial M \end{array} \right\}$$

We will assume that 0 is not a Dirichlet eigenvalue for this equation (the assumption can easilty be removed by using all Cauchy data pairs). Then, $\Lambda_{g,q} : H^{\frac{1}{2}}(\partial M) \to H^{-\frac{1}{2}}(\partial M)$ can be defined weakly through the bilinear pairing:

$$(\Lambda_{g,q} f, h) = \int_M \langle du, dv \rangle_g dV_g + \int_M quv dV_g$$

where $v \in H^1(M)$ satisfies $v|_{\partial M} = h$ and $u \in H^1(M)$ satisfies $(-\triangle_g + q)u = 0$ with $u|_{\partial M} = f$. As before we note that for $f \in H^{\frac{3}{2}}(\partial M)$ we have that:

$$\Lambda_{g,q} f = \partial_\nu u|_{\partial M}.$$

Thus one can observe that the Calderón problem in the conformal setting can be posed as follows:

Conjecture 1.1.3. *Let (M, g) be a smooth compact Riemannian manifold with smooth boundary and let $q \in L^\infty(M)$ be an unknown bounded function. Then the knowledge of $\Lambda_{g,q}$ will uniquely determine q.*

Remark 1.1.4. Henceforth, for the sake of brevity, we will use Λ_q to denote $\Lambda_{g,q}$. Here the reader should note that the geometry g is a priori known and q is unknown. Note also that once q is determined from the Dirichlet to Neumann map one can recover the corresponding conformal factor.

This conjecture was solved for Euclidean geometries by John Sylvester and Gunther Uhlmann in 1987 [19]. We will discuss their methods in the subsequent section. In two dimensions the problem was first solved by Adrian Nachman [16] for potentials that arise from positive functions c as above. A different proof was given by Buckgheim in 2008 that works for general potentials [26]. For general smooth Riemannian manifolds this is still a difficult open question. The most general result obtained in this direction is the uniqueness of the potential for conformally cylindrical geometries. These are geometries where the manifold has a product structure $M = \mathbb{R} \times M_0$ with (M_0, g_0) being called the transversal manifold and the metric takes the form:

$$g = c(t, x)(dt^2 + g_0(x)).$$

D. Dos Santos Ferreira, C. E. Kenig, M. Salo and G. Uhlmann proved in 2009 that the potential q can be uniquely determined from the Dirichlet to Neumann map in CTA geometries under some geometric restrictions on the transversal manifold (M_0, g_0). In 2013, their result was strenghtend by assuming weaker restrictions on the transversal manifold.

Let us now formulate in more detail some of the main problems of interest in this thesis and mention the key result. We are mostly interested in the role of special hypersurfaces in an arbitrary manifold (M, g). Suppose we have a surface Γ embedded in M. We ask whether one can construct special solutions to a Schrödinger equation which concentrate on Γ in some sense and whether these solutions can then help recover the unknown potential q on Γ. Towards this end we first recall the following definition:

Definition 1.1.5. An embedded surface $\Gamma \subset M$ is **umbilic** if there exists a conformal factor

c such that $h = cg_\Gamma$. Here h denotes the second fundamental form on Γ and g_Γ is the first fundamental form.

We propose the following conjecture:

Conjecture 1.1.6. *Let (M, g) be a three dimensional compact Riemannian manifold with smooth boundary. Let $\Gamma \subset M$ be umbilic. Then the knowledge of Λ_q will yield the knowldge of $\int_\Gamma qf(z)d\sigma_{g_\Gamma}$. Here $f(z)$ is an arbitrary Holomorphic function on Γ in the induced isothermal coordinates and g_Γ denotes the induced metric on Γ.*

Chapter 2 is mostly concerned with this conjecture. We will start by explaining the reason behind the natural appearance of umbilic surfaces in studying concentrating solutions and then propose a geometry that will be appropriate to study. This geometry will be a generalization of the CTA geometries. We will then present the significant partial results we have obtained on this problem, which we believe constitute a key step towards settling the full conjecture.

As mentioned above, umbilic surfaces appear to naturally arise in studying concentrating solutions to the Schrödinger equation. In order to use this idea and some of the tools in chapter 2 to solve an inverse problem we propose a geometry where there are subdomains which contain a large class of umbilic surfaces. This is what we call a Locally Euclidean geometry and it is the basic assumption for proving the main theorems in the thesis. More precisely, we will prove the following theorems:

Theorem 1.1.7. *Let (Ω^3, g) denote a compact smooth Riemannian manifold with smooth boundary. Let $U \subset \Omega$ be an open subset such that $\Gamma = U \cap \partial\Omega$ is non-empty, connected and strictly convex. Suppose U can be covered with a coordinate chart in which $g|_U$ is the Euclidean metric and that U is the convex hull of Γ. Suppose q is a smooth function and let us assume that $q - q_*$ is compactly supported in U where q_* is a globally known smooth function. Then the knowledge of Λ_q will uniquely determine $q|_U$.*

If U is not connected, we have the following result:

Theorem 1.1.8. *Let (Ω^3, g) denote a compact smooth Riemannian manifold with smooth boundary. Let $U \subset \Omega$ be an open subset such that $\Gamma = U \cap \partial\Omega$ is non-empty, and strictly convex. Let $\Gamma = \cup_{i=1}^{l} \Gamma^i$ where Γ^i denotes the connected components of Γ. Suppose q is a smooth function and suppose $q - q_*$ is compactly supported in U where q_* is a globally known smooth function. Furthermore suppose $g|_U$ is the Euclidean metric. Let U^i denote the convex hull of Γ_i and let us assume that $U^i \cap U^j = \emptyset \ \forall i, j$ and $U = \cup_{i=1}^{l} U^i$. Then the knowledge of Λ_q will uniquely determine q.*

More generally, we have the following result which does not require g to be Euclidean in U:

Definition 1.1.9. Let (M, g) be a smooth Riemannian manifold with smooth strictly convex boundary. M is called **simple** if for any two points in M there exists a unique length minimizing geodesic.

Definition 1.1.10. Let $U \subset M$ and $f \in C^1(M)$. We say the geodesic ray transform on M is locally injective with respect to U if the following holds:

$$\int_\gamma f = 0$$

for all geodesics γ with end points on $\partial M \cap U$ implies that

$$f|_U \equiv 0$$

Theorem 1.1.11. *Let (Ω^3, g) denote a compact smooth Riemannian manifold with smooth boundary with $\Omega = I \times \Omega_0$. Let U_0 be an open subset such that $\Gamma_0 = U_0 \cap \partial\Omega_0$ is non-empty and that U_0 is the convex hull of Γ_0. Suppose that $(\partial I \times U_0) \cup (I \times \partial U_0)$ is connected, and let $U = I \times U_0$. Assume that U_0 is simple and that the geodesic ray transform of Ω_0 is locally injective with respect to U_0. Suppose $q \in C^1(\Omega)$ is a function that is explicitly known in U^c, and that U is conformally transversally anisotropic. Then the knowledge of Λ_q uniquely determines q.*

1.2 Background

In this section we will explain some of the main methodologies in the literature regarding Conjecture 1.1.3. We will be closely following [4,5,18,19]. Note that heuristically speaking, Conjecture 1.1.3 is simpler to study as the unknown function q is a zeroth order term in the Schrödinger equation. A key starting point in this direction is the so called Alessandrini identity which uses integration by parts to relate the boundary data to the unknown potentials in the interior. More precisely, we have the following:

Lemma 1.2.1 (Alessandrini Identity)**.**

$$\int_{\partial M} f_1(\Lambda_{q_1} - \Lambda_{q_2})f_2 = \int_M (q_1 - q_2)u_1 u_2$$

where u_i solves the equation $(-\triangle_g + q_i)u_i = 0$ with $u_i|_{\partial M} = f_i$ for $i \in \{1, 2\}$.

The Alessandrini identity suggests that one method for proving Conjecture 1.1.3 is to show that the product of solutions to the Schrödinger equation are dense in $L^2(M)$. Indeed this idea is key to the majority of results in this direction. For simplicity we will assume that the manifold is Euclidean and explain the methodology in establishing Conjecture 1.1.3 for Euclidean geometries. The key ingredient in the majority of results in this direction is the construction of Complex Geometric Optic solutions (CGO).

An observation which goes back to Calderón's pioneering paper is that in Euclidean space, functions of the form $u(x) = e^{ix\cdot\zeta}$ with $\zeta \in \mathbb{V}$ where $\mathbb{V} = \{\zeta \in \mathbb{C}^n | \zeta \cdot \zeta = 0\}$ solve $\triangle u = 0$. The set of these complex frequencies $\zeta = \zeta_R + i\zeta_I$ must satisfy:

$$|\zeta_R| = |\zeta_I| = \tau$$

and:

$$\zeta_R \cdot \zeta_I = 0.$$

Starting from these complex exponential solutions to the Laplacian equation one can prove that for sufficiently large $|\tau|$ there exist corresponding solutions to the Schrödinger equation $(-\triangle + q)u = 0$ of the form $u = e^{i\zeta\cdot x}(e^{-i\xi\cdot x} + r)$ where $\|r\|_{L^2(M)} \leq \frac{C}{|\tau|}$ (for some constant C depending on the size of the domain). Here $\xi \in \mathbb{C}^n$ is chosen such that $(\zeta - \xi) \cdot (\zeta - \xi) = 0$. Let us explain this in more detail by proving the following two lemmas:

Lemma 1.2.2 (Basic Carleman Estimate)**.** *Let $q \in L^\infty(\Omega)$. For $|\zeta|$ large enough we have the following estimate:*

$$\|e^{-ix\cdot\zeta}(-\triangle + q)(e^{ix\cdot\zeta}v)\|_{L^2(\Omega)} \geq C|\zeta|\|v\|_{L^2(\Omega)}$$

for all $v \in C_c^\infty(\Omega)$.

Proof. Without loss of generality we can consider an isometry such that in the new coordinate system $(x_1, x_2, ..., x_n)$ we have $ix \cdot \zeta = \frac{-i\tau}{\sqrt{2}}(x_1 + ix_2) = -is(x_1 + ix_2)$ (where $s = \frac{\tau}{\sqrt{2}}$). Note that in that case we have:

$$\|e^{-ix\cdot\zeta}(-\triangle)(e^{ix\cdot\zeta}v)\|^2_{L^2(\Omega)} = \|e^{-sx_1}(-\triangle)(e^{sx_1}\hat{v})\|^2_{L^2(\Omega)}$$

where $\hat{v} = e^{isx_2}v$. Hence:

$$\|e^{-ix\cdot\zeta}(-\triangle)(e^{ix\cdot\zeta}v)\|^2_{L^2(\Omega)} = \|(-\triangle - s^2)\hat{v}\|^2_{L^2(\Omega)} + 4s^2\|\partial_1\hat{v}\|^2_{L^2(\Omega)} \geq Cs^2\|\hat{v}\|_{L^2(\Omega)} \geq Cs^2\|v\|_{L^2(\Omega)}$$

Here we are using a Poincare inequality in the x_1 direction. The corresponding estimate for the Schrödinger equation easily follows by taking the parameter s to be much larger than $\|q\|_{L^\infty(\Omega)}$. □

Lemma 1.2.3. *Let (Ω, g) be a compact domain in $\mathbb{R}^n$ and assume that $\zeta \in \mathbb{V}$. For any $f \in L^2(\Omega)$ and $|\zeta|$ sufficiently large, there exists solutions $r \in H^1(\Omega)$ to $e^{-ix\cdot\zeta}(-\triangle+q)(e^{ix\cdot\zeta}r) = f$ such that the following estimate holds: $\|r\|_{L^2(\Omega)} \leq \frac{C}{|\zeta|}$.*

Proof. This is a rather standard proof about deducing surjectivity for some operator T from the knowledge of injectivity and closed range for its adjoint operator T^*. Let $P(\cdot) = e^{-ix\cdot\zeta}(-\triangle + q)(e^{ix\cdot\zeta}\cdot)$ and define $\mathbb{D} = P^*C_c^\infty(\Omega)$ as a subspace of $L^2(\Omega)$. Consider the linear functional $L : \mathbb{D} \to \mathbb{C}$ through:

$$L(P^*v) = \langle v, f\rangle \qquad \forall v \in C_c^\infty(\Omega).$$

This is well-defined since any element of $\mathbb{D}$ has a unique representation as P^*v with $v \in C_c^\infty(\Omega)$ by the Carleman estimate. Also using the Cauchy-Schwarz inequality and the Carleman estimate from Lemma 1.2.2:

$$|L(P^*v)| \leq \|v\|_{L^2}\|f\|_{L^2} \leq C\tau^{-1}\|f\|_{L^2}\|P^*v\|_{L^2}.$$

Thus L is a bounded linear operator on $\mathbb{D}$. Extend L by continuity to the closure $\bar{\mathbb{D}}$ and finally extend to all of $L^2(\Omega)$ through the Hahn-Banach theorem. Thus we obtain a bounded linear operator $\hat{L} : L^2(\Omega) \to \mathbb{C}$ with $\hat{L}|_{\mathbb{D}} = L$. Furthermore:

$$\|\hat{L}\| \leq C\tau^{-1}\|f\|_{L^2}.$$

Now by the Riesz representation therorem we deduce that there exists a unique $r \in L^2(\Omega)$ such that $\hat{L}(w) = \langle w, r\rangle \quad \forall w \in L^2(\Omega)$ and such that $\|r\| \leq C\tau^{-1}\|f\|_{L^2}$. Note that:

$$\langle v, Pr\rangle = \langle P^* v, r\rangle = \hat{L}(P^* v) = L(P^* v) = \langle v, f\rangle.$$

Hence $P_\tau r = f$ in the weak sense.

□

This construction can be used to prove the uniqueness result for $n \geq 3$ as follows. Suppose that $\Lambda_{q_1} = \Lambda_{q_2}$. Let $\xi \in \mathbb{R}^n$ be an arbitrary vector. Choose unit vectors ω_1 and ω_2 in $\mathbb{R}^n$ such that $\{\omega_1, \omega_2, \xi\}$ forms an orthogonal set and let $\zeta = s(\omega_1 + i\omega_2)$. One can use the lemmas above to construct $u_1(x) = e^{ix\cdot\zeta}(e^{ix\cdot\xi} + r_1)$ solving $(-\triangle + q_1)u_1 = 0$ with $\|r_1\|_{L^2(\Omega)} \leq \frac{C}{s}$ and $u_2(x) = e^{ix\cdot\zeta}(1 + r_2)$ solving $(-\triangle + q_2)u_2 = 0$ with $\|r_2\|_{L^2(\Omega)} \leq \frac{C}{s}$. Then if $f_i = u_i|_{\partial\Omega}$ for $i \in \{1, 2\}$ we have:

$$0 = \int_{\partial\Omega} f_1(\Lambda_{q_1} - \Lambda_{q_2})f_2 = \int_\Omega (q_1 - q_2)u_1 u_2 = \int_\Omega (q_1 - q_2)(e^{-ix\cdot\xi} + r_1)(1 + r_2).$$

By taking the limit $|\tau| \to \infty$ and using the fact that $\|r_i\|_{L^2(\Omega)} \leq \frac{C}{|\tau|}$ for $i \in \{1, 2\}$ we can conclude that:

$$\int_\Omega (q_1 - q_2)(x)e^{-ix\cdot\xi} \equiv 0.$$

Let us extend $(q_1 - q_2)$ to all of $\mathbb{R}^n$ by setting it equal to 0 outside Ω. Then the previous statement yields that the Fourier transform of $q_1 - q_2$ must vanish everywhere. This shows that $q_1 \equiv q_2$.

Motivated by the success of complex exponential solutions in Euclidean geometries one would like to generalize them to Riemannian space.Here we will present the main ideas in constructing these CGO solutions and explain why these will lead us to CTA geometries. We will be closely following [18]. A natural generalization of the Euclidean CGO harmonic functions $e^{ix\cdot\zeta}$ with $\zeta \in \mathbb{V}$ is to consider the WKB ansatz $u = e^{\tau\Phi}v$ with Φ a smooth complex-valued function. One would then like v to formally solve the conjugated Laplacian equation:

$$e^{-\tau\Phi}\triangle_g(e^{\tau\Phi}v) = 0.$$

Note that:

$$e^{-\tau\Phi}\triangle_g(e^{\tau\Phi}v) = \tau^2(\langle d\Phi, d\Phi\rangle_g)v + \tau[2\langle d\Phi, dv\rangle_g + (\triangle_g\Phi)v] + (\triangle_g v).$$

Formally solving in powers of τ we obtain a complex Eikonal equation for the phase function Φ and a complex transport equation for v as follows:

$$\langle d\Phi, d\Phi\rangle_g = 0$$

and

$$2\langle d\Phi, dv\rangle_g + (\triangle_g\Phi)v = 0.$$

Note that if these two equations are satisfied, $e^{\tau\Phi}v$ would be an approximate solution to $(\triangle_g + q)u = 0$. We would like to produce exact solutions of the form $e^{\tau\Phi}(v + r)$ with $\|r\|_{L^2(M)} \leq \frac{C}{|\tau|}$. Furthermore for the density arguments for products of solutions method to work one needs to produce similar solutions for large parameters of τ of both signs. In [5] these ideas are made precise through the use of Carleman estimates. Let us explain some of the key ideas here.

Let us assume $\phi : M \to \mathbb{R}$ is a $C^2(M)$ function. Constructing solutions of the form $u = e^{\tau\phi}v$ to $(-\triangle_g + q)u = 0$ requires a solvability operator for the conjugated Laplacian $P_\tau(\cdot) = e^{-\tau\phi}\triangle_g(e^{\tau\phi}\cdot)$ with appropriate bounds in terms of $|\tau|$. Indeed we would like to have the following:

$$\forall\tau \geq \tau_0, \forall f \in L^2(M) \exists r \in L^2(M) \text{ solving } P_\tau r = f \text{ such that } \|r\|_{L^2(M)} \leq \frac{C}{|\tau|}.$$

Similar to the Euclidean case described above, this can be achieved through a Hahn-Banach argument provided that we have the following coercivity estimate:

$$\|P_\tau w\|_{L^2(M)} \geq C|\tau|\|w\|_{L^2(M)} \;\; \forall w \in C_c^\infty(M), \;\; \forall|\tau| > |\tau_0|.$$

Note that the formal adjoint of P_τ with respect to the $L^2(M)$ inner product is given by $P_\tau^*(\cdot) = P_{-\tau}(\cdot)$. To prove the Conjecture 1.1.3 through the use of density arguments would require CGO solutions for both signs of τ as $|\tau|$ approaches infinity. At this point we will use the Weyl semiclassical calculus to obtain a necessary condition for achieving this estimate for large τ. Let us define $h = \frac{1}{\tau}$ and define $P_h(\cdot) := e^{-\frac{\phi}{h}}(-h^2\triangle_g)(e^{\frac{\phi}{h}}\cdot)$ and note that we would like to prove estimates of the following form for $|h|$ small enough:

$$\|P_h w\|_{L^2(M)} \geq Ch^2\|w\|_{L^2(M)} \ \ \forall w \in C_c^\infty(M), \ \ \forall |h| < |h_0|$$

Note that:

$$P_h(\cdot) = \langle d\phi, d\phi\rangle_g \cdot + h[2\langle d\Phi, d\cdot\rangle_g + (\triangle_g\Phi)\cdot] + h^2(\triangle_g \cdot).$$

Hence:

$$P_h = A + B$$

Where A and B are the formally symmetric and antisymmetric operators (with respect to the standard $L^2(M)$ inner product) given by:

$$Av = -h^2\triangle_g v - |d\phi|_g^2 v$$

$$Bv = h(2\langle d\Phi, dv\rangle_g + (\triangle_g\Phi)v).$$

Furthermore the principal symbols $a(x,\xi)$ and $b(x,\xi)$ of A and B are given by:

$$a(x,\xi) = |\xi|_g^2 - |d\phi|_g^2$$

$$b(x,\xi) = 2\langle d\Phi, \xi\rangle_g.$$

We have $p(x,\xi) = a(x,\xi) + ib(x,\xi)$. Recall that the set $p(x,\xi) = 0$ denotes the characteristic variety of the pesudo differential operator P_h. Using the Hilbert space structure of $L^2(M)$ we observe that:

$$\|P_h v\|_{L^2(M)}^2 = \|Av\|_{L^2(M)}^2 + \|Bv\|_{L^2(M)}^2 + ([A,B]v, v)_{L^2(M)}.$$

The only negative contribution in the above sum can come from the commutator term $[A,B]$. Heuristically, at the level of principal symbols we would like the principal symbol of $[A,B]$ to be positive when the symbols $a(x,\xi)$ and $b(x,\xi)$ vanish. This would ensure that any negative contribution from the commutator term can be absorbed into the positive terms $\|Av\|^2_{L^2(M)}$ and $\|Bv\|^2_{L^2(M)}$. Thus we have arrived at a necessary condition for proving the coercivity estimates above which states that the principal symbol of $[A,B]$ should be non-negative along the bi-characteristics of P_h. Let us recall the following useful lemma:

Lemma 1.2.4. *The principal symbol of* $[A,B]$ *is given by* $ih\{a,b\}(x,\xi)$. *Here* $\{a,b\}$ *denotes the Poisson bracket of the symbols* a *and* b.

For a proof of this lemma we refer the reader to [18]. Hence the necessary condition for the coercivity estimate above can be rewritten as:

$$ih\{a,b\}(x,\xi) \geq 0 \ \ \forall x,\xi \ \text{ such that } \ a(x,\xi) = b(x,\xi) = 0.$$

Recall that we need the above necesssary condition to hold for both signs of h with h sufficiently small. Thus we arrive at the following necessary condition known as the limiting Carleman condition:

$$\{a,b\}(x,\xi) = 0 \ \ \forall x,\xi \ \text{ such that } \ a(x,\xi) = b(x,\xi) = 0.$$

We call a phase function satisfying the above condition to be a Limiting Carleman Weight (LCW). It should be clear that the CGO method of proving uniqueness could only work for geometries that allow the existence of LCW's and thus characterization of such geometries would describe the manifolds for which one can hope to generalize the product of solutions method.

Remark 1.2.5. Note that the aformentioned condition is not sufficient for getting a coercivity estimate. Once this condition is checked we would need a more detailed analysis to establish the coercivity estimate. This is merely a necessary condition. In the cases where the Poisson bracket term $i\{a,b\}(x,\xi) > 0$ we would have the well known Hörmander hypoellipticity condition. In such a case we can actually prove a stronger coercivity estimate for P_h for $0 < h < h_0$ by using the easy Garding inequality:

$$\|P_h w\|_{L^2(M)} \geq Ch\|w\|_{L^2(M)} \ \ + Ch^{\frac{1}{2}}\|Dw\|_{L^2(M)} \forall w \in C_c^\infty(M), \ \ \forall 0 < h < h_0.$$

Let us continue with the characterization of geometries that allow existence of LCWs. Towards this end, we will first mention a useful lemma as follows:

Lemma 1.2.6 (see [5]).

$$\{a, b\}(x, \xi) = 4D^2\phi(\xi, \xi) + 4D^2\phi(\nabla\phi, \nabla\phi).$$

Proof. We will fix a point p in M and consider the normal local coordinates near p and consider a neighbordhood of p in T^*M. Then:

$$\{a, b\}(x, \xi)|_p = \sum_{i,j=1}^{n} (\partial_{\xi_i} a \partial_{x_j} b - \partial_{\xi_i} b \partial_{x_j} a)|_p = \sum_{i,j=1}^{n} (4\phi_{x_i x_j} \xi_i \xi_j + 4\phi_{x_i x_j} \phi_{x_i} \phi_{x_j})|_p$$

Here we are implicitly using the fact that $g^{ij}|_p = \delta^{ij}$ and $\partial_k g^{ij} = 0$. The lemma follows immediately. □

The following lemmas help characterize all geometries admitting Limiting Carleman weights. For the sake of brevity we will omit the proofs as they are straightforward. They can be found in [18].

Lemma 1.2.7. *If ϕ is an LCW in (M, g) then ϕ is an LCW for (M, cg) for any smooth positive function c. In other words LCW's only depend on the conformal class of (M, g).*

Lemma 1.2.8. *If ϕ is a smooth function in (M, g) and if $\tilde{g} = |\nabla\phi|_g^2 g$ then $|\nabla_{\tilde{g}}\phi|_{\tilde{g}} = 1$.*

Lemma 1.2.9. *Let ϕ be a smooth function in (M, g) and suppose $|\nabla\phi|_g = 1$. The following conditions are equivalent:*

$$\phi \quad \textit{is an LCW.}$$

$$D^2\phi \equiv 0.$$

$$\nabla\phi \ \textit{ is parallel.}$$

It can easily be seen from these lemmas that existence of LCWs implies that the manifold is CTA. Thus the product of solutions method can only work for CTA manifolds. In what follows here we will briefly explain how one can go about proving Conjecture 1.1.3 for CTA geometries. Let us recall that in CTA geometries the manifold has a product structure

$M = \mathbb{R} \times M_0$ with (M_0, g_0) being called the transversal manifold and the metric takes the form:

$$g = c(t,x)(dt^2 + g_0(x)).$$

In this setting $\phi(t,x) = t$ would then be a Limiting Carleman Weight. For proofs of the following lemmas we refer the reader to [5]. Our goal here is merely to show how the method of products of solutions can be applied in CTA geometries.

Lemma 1.2.10. *Let (M,g) denote a CTA geometry and let $\phi(t,x) = t$. Then the following estimate holds for $|\tau|$ large enough:*

$$\|P_\tau w\|_{L^2(M)} := \|e^{-\tau\phi} \triangle_g (e^{\tau\phi w})\|_{L^2(M)} \geq C|\tau| \|w\|_{L^2(M)} \ \forall w \in C_c^\infty(M), \ \ \forall |\tau| > |\tau_0|.$$

Using the injectivity estimate above together with the Hahn-Banach theorem we can construct a solvability operator with appropriate gains for the conjugated Laplacian. Indeed we have the following:

Lemma 1.2.11. *Let (M,g) be a CTA geometry and assume that $\phi(t,x) = t$. For any $f \in L^2(M)$ and $|\tau|$ sufficiently large, there exist solutions $r \in H^1(M)$ to $e^{-\tau\phi}(-\triangle + q)(e^{\tau\phi} r) = f$ such that the following estimate holds: $\|r\|_{L^2(M)} \leq \frac{C}{|\tau|}$.*

Let us now show how the above lemma can help us construct CGO solutions in CTA geometries. Indeed let us start with the ansatz $v = e^{i\tau\psi} a$. Note that:

$$P_\tau v = e^{i\tau\psi}(\tau^2 \langle d\Phi, d\Phi\rangle_g a + \tau[2\langle d\Phi, da\rangle_g + (\triangle_g \Phi)a] + (\triangle_g a)).$$

Here $\Phi(t,x) = t + i\psi(t,x)$ where ψ is a real valued smooth function. Proceeding with the same methodology as in the Euclidean case let us choose Φ and a such that:

$$\langle d\Phi, d\Phi\rangle_g = 0$$

$$2\langle d\Phi, da\rangle_g + (\triangle_g \Phi)a = 0.$$

Let us consider the transversal manifold (M_0, g_0). Consider a small extension into a bigger manifold $(\hat{M}_0, \hat{g}_0)$ where we extend the metric smoothly to the exterior. Pick a point p in $\hat{M}_0 \setminus M_0$ and consider the normal coordinates around p. In these coordinates we can write

$g_0 = dr^2 + \rho(x) d\Omega^2$. In fact one can solve the complex eikonal and transport equations by choosing:

$$\psi(x) = r$$

$$a = \rho(x)^{-\frac{1}{4}} f(t+ir)$$

Here $f(t+ir)$ is any holomorphic function of $z = t+ir$. The construction lemma above can then be used to show existence of solutions of the form $u_k = e^{(-1)^k \tau \Phi}(\rho(x)^{-\frac{1}{4}} f(t+ir) + r_k)$ solving $(-\triangle_g + q_i)u_i = 0$ such that $\|r_k\|_{L^2(M)} \leq \frac{C}{\tau}$ for $k \in \{1,2\}$. One can use the Alessandrini identity with these CGO solutions to deduce that the geodeisc ray transform of $q_1 - q_2$ must vanish for all geodesics in (M_0, g_0). Thus the uniqueness of the potential in CTA geometries can be proven as long as the geodesic ray transfrom is invertible for the transversal manifold (M_0, g_0). Examples of such manifolds are simple manifolds (see Definition 1.1.9) or manifolds which are globally foliated by strictly convex hypersurfaces. [3]

1.3 Motivation

As explained in the previous section most approaches to the Calderón problem use the product of complex exponential solutions to $\triangle_g u = 0$ with high frequencies of opposing signs. As can be seen above this limits the geometry to conformally cylindrical manifolds. One of the goals of this thesis is to introduce a different approach in studying the Calderón problem via complex exponential solutions for one family of parameters $\tau_n \to \infty$. We will not rely on the density of products of solutions and will only require the coercivity estimate for P_τ for τ large enough. This results in a wider class of phase functions which must satisfy:

$$D^2\phi(\nabla\phi, \nabla\phi) + D^2\phi(X, X) \geq 0 \ \ \forall X \ \text{ with } \ \langle X, d\phi\rangle_g = 0, |X|_g = |d\phi_g|. \qquad (*)$$

Let us briefly explain the main idea here. Rather than pairing two exact solutions at the boundary and using the Alessandrini identity we will construct an exact solution to the Schrödinger equation and pair it with an approximate harmonic function at the boundary. More precisely suppose that one has a family of solutions to $(-\triangle_g + q)u = 0$ of the form $u = e^{\tau\Phi}(v + r)$ with Φ solving the complex eikonal equation and v solving the complex transport equations. We would additionally require that $\phi := \Re(\Phi)$ satisfies the Hörmander hypoellipticity condition above (*). Let us also assume that through the knowledge of Λ_q one might be able to obtain the boundary values of $u|_{\partial M}$. This is indeed a reasonable assumption as several such reconstruction algorithms are available in the literature [15]. Let us pick the explicit function $w = e^{-\tau\Phi}v$ and use Green's identity:

$$I_\tau = \int_{\partial M} w\partial_\nu u - \int_{\partial M} u\partial_\nu w = \int_M w\triangle_g u - \int_M u\triangle_g w.$$

Hence:

$$I_\tau = \int_M qwu - \int_M e^{\tau\Phi}(v+r)\triangle_g(e^{-\tau\Phi}v).$$

Let $J_\tau = \int_M ve^{\tau\Phi}\triangle_g(e^{-\tau\Phi}v)$. Then:

$$I_\tau + J_\tau = \int_M qv^2 + \int_M qvr - \int_M re^{\tau\Phi}\triangle_g(e^{-\tau\Phi}v).$$

But notice that:

$$e^{\tau\Phi}\triangle_g(e^{-\tau\Phi}v) = \tau^2\langle d\Phi, d\Phi\rangle_g v - \tau[2\langle d\Phi, dv\rangle_g + (\triangle_g\Phi)v] + \triangle_g v.$$

Hence:

$$\|e^{\tau\Phi}\triangle(e^{-\tau\Phi}v)\|_{L^2(M)} \leq C$$

with C independent of τ and therefore using Cauchy-Schwarz we see that:

$$|\int_M re^{\tau\Phi}\triangle(e^{-\tau\Phi}v)| \leq \frac{C}{\tau}.$$

Similarly we notice that:

$$|\int_M qvr| \leq \frac{C}{\tau}.$$

Thus:

$$I_\tau + J_\tau = \int_M qv^2 + O(\frac{1}{\tau}).$$

This simple analysis shows that it may be possible to obtain integrals of the unknown function q against a family of explicitly known functions v^2 without limiting the geometry to CTA geometries. The key steps in the success of this approach would be to better understand solutions to the complex eikonal equation, the complex transport equation and to achieve a boundary reconstruction algorithm for the special solutions u.

Another point of interest in this thesis is to understand the geometry of two dimensional surfaces $\Gamma \subset M$ such that Λ_q might yield some information about the potential q on Γ. This is motivated by [12] in which the authors construct families of solutions to the Schrödinger equation such that they concentrate on a plane in an appropriate sense. This subsequently gives the knowledge of the Radon transform of the unknown potential q on 2-planes or on spheres. Similarly in [24] one can obtain Radon transforms of the unknown potential q on the surfaces $\mathbb{R}\times\gamma$ where γ denotes any geodesic on the transversal manifold. We believe that umbilic surfaces are the only natural candidates for which this might be possible. Incidentally we know that planes and spheres in Euclidean space and the special surfaces in [24] are all umbilic.

Remark 1.3.1. We would like to point out that the relevance of umbilic surfaces in studying Calderón problem has been noted before. We are merely rediscovering them using a different approach. Indeed, in Lemma 3.1 [5] it is shown that the level sets of Limiting Carleman Weights are umbilic hypersurfaces.

It should be noted that umbilic surfaces are conformally invariant and that in general a generic Riemannian manifold does not admit any umbilic surfaces. Before we begin our analysis as motivation behind conjecture 1.1.6 let us recall the construction of Fermi coordinates near a surface [26]. Let (x_1, x_2) denote a local coordinate system on Γ and let N denote the normal unit vector field on Γ. Define a local diffeomorphism $Z : \Gamma \times \mathbb{R} \to M$ through:

$$Z(x', x_3) = Exp_{x'}(x_3 N(x')).$$

Gauss's Lemma implies that:

$$Z^* g = g_{x_3} + dx_3^2$$

where g_{x_3} is the induced metric on $\Gamma_{x_3} = \{x_3 = const\}$ smoothly depending on x_3. Furthermore we can show that:

$$g = \mathring{g} - 2x_3 \mathring{h} + O(x_3^2)$$

Where $\mathring{g}$ denotes the induced metric on Γ and $\mathring{h}$ is the second fundamental form on Γ.

Motivated by the ideas above one is interested in constructing a family of approximate solutions to $(-\triangle_g + q)u = 0$ of the form $e^{\tau\phi}v$ such that v is concentrated on Γ. Indeed we would like to construct functions v such that:

$$\|e^{-\tau\phi}\triangle_g(e^{\tau\phi}v)\|_{L^2(M)} \leq C$$

where C is independent of τ and τ is large (for both signs of τ). By concentration on the surface Γ we mean that:

$$\|v\|_{L^2(M)} \asymp 1$$

and

$$v dV_g \to d\mu_\Gamma$$

where the convergence is a weak convergence and μ_Γ is a measure on the surface Γ.

For simplicity let us assume that $\tau > 0$ is large. We seek a formal power series and try the ansatz $v = e^{\tau\Theta}a$. Note that we have:

$$e^{-\tau\phi}(\triangle_g)(e^{\tau\phi}v) = e^{\tau\Theta}e^{-\tau\Phi}(\triangle_g)(e^{\tau\Phi}a)$$

Here $\Phi = \phi + \Theta$. We would like to solve the complex eikonal equation locally by setting $\langle d\Phi, d\Phi\rangle_g = 0$ up to an arbitrary order on Γ and solving as a power series in x_3. In order to make this approximate solution concentrate on Γ we need $e^{\tau\Theta}$ to have Gaussian type decay away from the surface Γ. Therefore we set:

$$\Phi(x_1, x_2, x_3) = \sum_{k=0}^{N} \Phi_k(x_1, x_2)x_3^k$$

such that:

$$\langle d\Phi, d\Phi\rangle_g = (\partial_3\Phi)^2 + g_{x_3}^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi \asymp^N 0 \quad \text{on } \Gamma.$$

Note that this implies that:

$$\Phi_1(x) = i|d\Phi_0|_{g_0} := i\psi_1.$$

Let us first assume that $|d\Phi_0| \neq 0$. Since the anstaz must be concentrated on Γ (in the above sense) we need Φ_0 to be a real function. This implies that:

$$\Phi_2 = -\frac{\langle d\phi, d\psi_1\rangle_{g_0}}{\psi_1}.$$

We can similarly solve for all Φ_k for $k \geq 3$. Recall that v needs to be concentrated on Γ. This implies that $\Re(\Phi_2)$ must be negative for $\tau > 0$ and positive for τ negative. Hence noting the fact that Φ_2 is uniquely determined in terms of Φ_0 in this case it is impossible to achieve concentrating solutions. Thus it is imperative to start with:

$$|d\Phi_0|_{g_0} \equiv 0.$$

In this case we see that $\Phi_1 \equiv 0$. Furthermore we know that:

$$\partial_3((\partial_3\Phi)^2 + g_{x_3}^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi)|_{x_3=0} = 0.$$

This implies that:

$$\mathring{h}^{\alpha\beta}\partial_\alpha\Phi_0\partial_\beta\Phi_0 = 0.$$

From this we will easily deduce that Γ must be an umbilic surface.

1.4 Outline of the Thesis

In chapter 2, motivated by our discussion in this section we will study the Calderón problem in manifolds which are **foliations by umbilic surfaces**. This is a natural generalization to CTA gemetries and we hope to convey some of the main reasons behind why we think working in these geometries might be beneficial. At the moment Conjecture 1.1.6 is beyond the scope of this thesis but we hope some of the techniques in chapter 2 might be a step forward in that direction.

Chapter 3 works with geometries that contain a CTA subdomain. More precisely we will prove Theorem 3.1.1 and Theorem 3.6.5 which are the main results of this thesis. Essentially this is an instance where there is a large family of umbilic surfaces present in a manifold and that is suffcient to construct the unknown potential q from the DN map.

Remark 1.4.1. Chapters 2 and 3 are self-contained and there will therefore be some small overlaps.

Chapter 2

Umbilically Foliated Geometries

2.1 Problem Formulation

In this section, motivated by our remarks in the introductory section of the thesis we will study the problem of uniqueness of the potential in the Schrödinger equation from the Dirichlet to Neumann map for geometries which are foliations by umbilic surfaces. More precisely, let (Ω, g) be a three dimensional compact Riemannian manifold with smooth boundary and $\Omega \subset\subset \mathbb{R}^3$. we will assume that there exists global coordinates such that $g = e^{\alpha(x)} dx_1 \otimes dx_1 + e^{\alpha(x)} dx_2 \otimes dx_2 + e^{\beta(x)} dx_3 \otimes dx_3$ where $\alpha, \beta : \mathbb{R}^3 \to \mathbb{R}$ are smooth functions. Note that this implies that the hypersurfaces $\{x_3 = c\}$ are umbilic. To see this, note that the induced metric g_{x_3} on the $\{x_3 = c\}$ level sets are given by $g_{x_3} = e^{\alpha}(dx_1^2 + dx_2^2)$. We know that $h_{x_3} = -(\frac{1}{2})\partial_N g_{x_3}$ (see [26]) where h_{x_3} denotes the induced second fundamental form and N is the unit normal vector field on the surface $\{x_3 = c\}$. One can therefore see that h_{x_3} is a scalar multiple of g_{x_3}. Thus we can observe that this metric would be a good model to study foliations by umbilic surfaces.

The Dirichlet to Neumann map for the Laplacian operator is defined as follows: $\Lambda_0 : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is defined weakly through the bilinear pairing:

$$(\Lambda_0 f, h) = \int_\Omega \langle du, dv \rangle_g dV_g$$

where $v \in H^1$ satisfies $v|_{\partial\Omega} = h$ and $u \in H^1$ satisfies $-\triangle_g u = 0$ with $u|_{\partial\Omega} = f$. Similarly we will define the Dirichlet to Neuman map Λ_q for Schrödinger operator as follows:

$$(\Lambda_q f, h) = \int_\Omega \langle du, dv\rangle_g dV_g + \int_\Omega quv dV_g$$

where $v \in H^1$ satisfies $v|_{\partial\Omega} = h$ and $u \in H^1$ satisfies $(-\triangle_g + q)u = 0$ with $u|_{\partial\Omega} = f$. Note that throughout this paper we will assume that 0 is not a Dirichlet eigenvalue for $-\triangle_g + q$.

Before we begin our analysis, we want to argue that without loss of generality one can assume that $\beta = 0$ everywhere. The main point is the observation that in 3 dimensions, the Laplace Beltrami operator transforms under conformal scalings of the metric by:

$$\triangle_{cg} u = c^{-\frac{5}{4}}(\triangle_g + q_c)(c^{\frac{1}{4}}u)$$

$$q_c = c^{\frac{1}{4}}\triangle_{cg} c^{-\frac{1}{4}}.$$

Henceforth we will take $\beta \equiv 0$.

2.2 Carleman Estimates

Recall that we are considering Riemannian metrics of the form:

$$g = e^{\alpha(x)}dx_1 \otimes dx_1 + e^{\alpha(x)}dx_2 \otimes dx_2 + dx_3 \otimes dx_3.$$

Definition 2.2.1. We call a smooth positive function $e^\alpha : \Omega \to \mathbb{R}$ admissible if there exists a Conformal map $F(z)$ such that the following inequality holds on Ω:

$$\Re(\partial(\frac{e^\alpha}{F'(z)})) \leq 0.$$

Definition 2.2.2. A smooth function ϕ is called a Carleman weight if there exists h_0 such that the following estimate holds:

$$\|e^{\frac{\phi}{h}}\triangle_g e^{-\frac{\phi}{h}}u\|_{L^2(\Omega)} \geq \frac{C}{h}\|u\|_{L^2(\Omega)}$$

$\forall 0 < h < h_0$ and $u \in C_c^\infty(\Omega)$.

We will sometimes interchange notation from h to $\tau = \frac{1}{h}$ and vice versa. One can prove these estimates either through Weyl symbol calculus or using integration by parts. We will proceed with the latter method. Let us start by defining the conjugated Laplacian operator P_ϕ as follows:

$$P_\phi(\cdot) = e^{\frac{\phi}{h}}(-h^2\triangle_g)(e^{-\frac{\phi}{h}}\cdot)$$

Lemma 2.2.3 (see [18]). *$P_\phi = A + iB$ where A and B are the formally self adjoint operators (in $L^2(\Omega)$):*

$$A = -h^2\triangle_g - |d\phi|_g^2$$
$$B = \frac{h}{i}(2\langle d\phi, d\cdot\rangle_g + \triangle_g\phi).$$

Proof.

$$\begin{aligned}
P_\phi u &= e^{\frac{\phi}{h}}(-h^2\triangle_g)(e^{-\frac{\phi}{h}}u) \\
&= |g|^{-\frac{1}{2}}e^{\frac{\phi}{h}}hD_j(e^{-\frac{\phi}{h}}|g|^{\frac{1}{2}}g^{jk}e^{\frac{\phi}{h}}hD_k(e^{-\frac{\phi}{h}}u)) \\
&= |g|^{-\frac{1}{2}}(hD_j + i\phi_{x_j})(|g|^{\frac{1}{2}}g^{jk}(hD_k + i\phi_{x_k})u) \\
&= -h^2\triangle_g u + h(2\langle d\phi, du\rangle_g + (\triangle_g\phi)u) - |d\phi|_g^2.
\end{aligned}$$

□

Lemma 2.2.4 (see [18]). *Let $\hat{g} = cg$ be a conformal rescaling of the metric. Then ϕ is a Carleman weight with respect to g iff ϕ is a Carleman weight with respect to $\hat{g}$.*

Proof. Let us assume that ϕ is a Carleman weight with respect to g. Then:

$$e^{\frac{\phi}{h}}(-h^2\triangle_{\hat{g}})(e^{-\frac{\phi}{h}}u) = e^{\frac{\phi}{h}}(-h^2c^{-\frac{5}{4}}\triangle_g)(c^{\frac{1}{4}}e^{-\frac{\phi}{h}}u) - h^2q_cc^{-1}u$$

where:

$$q_c = c^{\frac{1}{4}} \triangle_{cg} c^{-\frac{1}{4}}$$

Now note that $c(x) > 0$ for all $x \in \Omega$ and $\|q_c\|_{L^\infty} < \infty$. Therefore :

$$\|e^{\frac{\phi}{h}}(-h^2\triangle_{\hat{g}})(e^{-\frac{\phi}{h}}u)\|_{L^2(\hat{g})} \gtrapprox h\|u\|_{L^2} - h^2\|q_c c^{-1}\|_{L^\infty}\|u\|_{L^2}.$$

The result follows for h small enough.

∎

The next result is well-known. We include a proof to make the presentation self-contained.

Lemma 2.2.5 (Poincare Inequality)**.** *If $\alpha \in \mathbb{R}^n$ is a unit vector, we have the following Poincare inquality in the unbounded strip $S = \{a \leq x \cdot \alpha \leq b\}$:*

$$\|u\|_{L^2(S)} \leq (b-a)\|\alpha \cdot Du\|_{L^2(S)} \qquad\qquad \forall u \in H_0^1(S).$$

Proof. Since $C_c^\infty(S)$ is dense in $H_0^1(S)$ it suffices to prove the lemma for smooth test functions of compact support. Furthermore due to the rotation and translational invariance of the inequality we will assume without any loss in generality that $\alpha = (1, 0, ..., 0)$ is the unit vector in the x_1 direction and that $S = \{0 < x_1 < a\}$ we will write $x = (x_1, x')$ with $x' \in \mathbb{R}^{n-1}$.

$$|u(x_1, x')| = |\int_0^{x_1} \partial_1 u \, dx_1| \leq \int_0^a |\partial_1 u| dx_1$$
$$|u(x_1, x')|^2 \leq a(\int_0^a |\partial_1 u|^2 dx_1)$$
$$\int_0^a u^2 dx_1 \leq a^2(\int_0^a |\partial_1 u|^2 dx_1)$$
$$\|u\|_{L^2}^2 \leq a^2\|\partial_1 u\|_{L^2}^2.$$

∎

Remark 2.2.6. One can strengthen the constant in Poincare lemma above to get the following inequality (see [18]):

$$\|u\|_{L^2(S)} \leq \frac{b-a}{\sqrt{2}}\|\alpha \cdot Du\|_{L^2(S)} \qquad \forall u \in H_0^1(S).$$

Lemma 2.2.7. *Consider* (Ω, g) *with* $g = e^{\alpha(x)}dx_1 \otimes dx_1 + e^{\alpha(x)}dx_2 \otimes dx_2 + dx_3 \otimes dx_3$ *and suppose that* $\partial_1(e^{-\alpha}) \geq 0 \quad \forall x \in \Omega$ *. Then* $\phi(x) = x_1$ *is a Carleman weight with respect to* g*.*

Proof. First note that it suffices to show that $\phi = x_1$ is a Carleman weight with respect to $\hat{g} = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + \Theta(x)dx_3 \otimes dx_3$ where $\Theta(x) = e^{-\alpha(x)}$ and $\partial_1\Theta \geq 0$. Thus we need to show that:

$$\|P_\phi u\|_{L^2(\hat{g})} \gtrapprox h\|u\|_{L^2(\hat{g})} \qquad \forall h \leq h_0 \qquad u \in C_c^\infty(\Omega)$$

Recall that $P_\phi u = Au + iBu$ where:

$$A = -h^2\triangle_g - |d\phi|_g^2.$$

$$B = \frac{h}{i}(2\langle d\phi, d\cdot\rangle_g + \triangle_g\phi).$$

Hence:

$$\|P_\phi u\|_{L^2}^2 = \|Au\|_{L^2}^2 + \|Bu\|_{L^2}^2 + (i[A,B]u, u)_{L^2}.$$

Now note that:

$$i[A,B] = -2h^3[\triangle_{\hat{g}}, \partial_1].$$

Also recall that, modulo first order terms in u we have:

$$\triangle_{\hat{g}} = \hat{g}^{ij}\partial_{ij}^2 + C^i\partial_i.$$

where $C^i(x)$ with $i \in \{1,2,3\}$ are bounded functions depending on g and we are using the Einstein summation convention.Thus:

$$\|P_\phi u\|_{L^2}^2 \gtrapprox \|Bu\|_{L^2}^2 + 2h^3 \int_\Omega (\partial_1 \Theta)\Theta^{-1}|\partial_3 u|^2 + h^3 \int_\Omega Cu^2.$$

$$Bu = \frac{h}{i}(2\partial_1 u + (\triangle_{\hat{g}}\phi)u) = \frac{h}{i}(\frac{2}{\Theta^{\frac{1}{4}}}\partial_1(\Theta^{\frac{1}{4}}u)).$$

Thus using Lemma 2.2.5 :

$$\|Bu\|_{L^2}^2 \gtrapprox h^2\|u\|_{L^2}^2.$$

Finally choosing h_0 small enough and noting that $\int_\Omega (\partial_1\Theta)\Theta^{-1}|\partial_3 u|^2 \geq 0$ we obtain the desired estimate.

□

The previous lemma yields a sufficient condition on a metric in a specific coordinate system for which an explicit Carleman weight can be constructed. We would now like to show that given no constraints on α (namely $\partial_1\alpha \leq 0$) we would still be able to derive an explicit Carleman weight $\phi(x) = \Re(f(z))$for some holomorphic function $f(z)$ where $z = x_1 + ix_2$. The trick would be to use the conformal structure of the $\{x_3 = \mathit{const}\}$ hypersurfaces and to choose a suitable conformal diffeomorphism, for which, the auxiliary condition $\partial_1\alpha \leq 0$ would be satisfied.

Theorem 2.2.8. *Consider (Ω, g) with $\Omega \subset\subset \mathbb{R}^3$ and $g = e^{\alpha(x)}dx_1 \otimes dx_1 + e^{\alpha(x)}dx_2 \otimes dx_2 + dx_3 \otimes dx_3$ with $e^\alpha(x)$ admissible. There exists a conformal function $G(z)$ such that $\phi(x) = \Re(G(z))$ is a Carleman weight with respect to g. In other words:*

$$\|e^{\frac{\phi}{h}}\triangle_g e^{-\frac{\phi}{h}}u\|_{L^2(\Omega)} \geq \frac{C}{h}\|u\|_{L^2(\Omega)}$$

$\forall h \le h_0$ *and* $u \in C_c^\infty(\Omega)$.

Proof. Let $\Phi(x) = G(z)$. we define $w = G(z)$ and use the local diffeomorphisms $(z, x_3) \to (w, x_3)$ and Lemma 2.2.7 to get a sufficient condition for $\Phi(x)$ to be a Carleman weight. Note that $z = G^{-1}(w) = F(w)$ and:

$$g = e^{\alpha \circ F} |F'(w)|^2 dw \otimes d\bar{w} + dx_3 \otimes dx_3.$$

It is sufficient to require that:

$$\partial_{w_1}(e^{\alpha \circ F}|F'(w)|^2) \le 0.$$

Simplifying the above condition we deduce that:

$$\begin{aligned}
&\partial_{w_1}(e^{\alpha \circ F}|F'(w)|^2) = (\partial_w + \partial_{\bar{w}})(e^{\alpha \circ F}|F'(w)|^2) \\
&= F'(\bar{w})\partial_w(e^{\alpha \circ F}F'(w)) + F'(w)\partial_w(e^{\alpha \circ F}F'(\bar{w})) \\
&= |F'(w)|^2\partial_w e^{\alpha} + |F'(w)|^2\partial_{\bar{w}} e^{\alpha} + e^{\alpha}F''(w)F'(\bar{w}) + e^{\alpha}F''(\bar{w})F'(w) \\
&= |F'(w)|^2 F'(w)\partial_z e^{\alpha} + |F'(w)|^2 F'(\bar{w})\partial_{\bar{z}} e^{\alpha} + e^{\alpha}F''(w)F'(\bar{w}) + e^{\alpha}F''(\bar{w})F'(w) \\
&= |F'(w)|^2(F'(w)\partial_z e^{\alpha} + F'(\bar{w})\partial_{\bar{z}} e^{\alpha} + e^{\alpha}\frac{F''(w)}{F'(w)} + e^{\alpha}\frac{F''(\bar{w})}{F'(\bar{w})}) \\
&= |F'(w)|^2(F'(w)\partial_z e^{\alpha} + F'(\bar{w})\partial_{\bar{z}} e^{\alpha} - e^{\alpha}\frac{G''(z)}{G'(z)} - e^{\alpha}\frac{G''(\bar{z})}{G'(\bar{z})}) \le 0.
\end{aligned}$$

Thus we arrive at the following sufficient condition:

$$\Re(\partial_z(\frac{e^{\alpha}}{G'(z)}) \le 0$$

which clearly holds due to the admissibility condition of the metric.

□

Remark 2.2.9. The preceding theorem yields a Carleman weight $\phi(x) = \Re(f(z))$. In what shall follow only the holomorphic structure of the phase function matters. Thus without loss of generality we will take $\phi(x) = \Re(z) = x_1$ as our Carleman weight.

2.3 Complex Exponential Solutions

In this section we will utilize Theorem 2.2.8 to construct a family of solutions to the Schrödinger operator $-\triangle_g + q$. The main idea here is that one can consider P_ϕ as a bounded linear operator and note that $P_\phi^* = P_{-\phi}$. In principle, Theorem 2.2.8 yields injectivity and closed range for P_ϕ. Elementary functional analysis on Hilbert spaces then yields surjectivity for $P_\phi^* = P_{-\phi}$. We will now make these arguments more precise and in particular derive a Green's function for the conjugated Laplacian in our admissible geometry. This is a rather standard construction and we will be closely following [17].

Lemma 2.3.1. *Let $f \in L^2(\Omega, g)$. There exists a unique function $r := H_\tau f \in \{v \in L^2 : \triangle_g v \in L^2\}$ such that $P_\tau r = e^{-\tau\phi}\triangle_g e^{\tau\phi} r = f$ on Ω with r orthogonal (with respect to the L^2 inner product) to $\Sigma = \left\{v \in L^2 : \triangle_g(e^{\tau\phi}v) = 0\right\}$. Furthermore for τ large enough, r satisfies the estimate:*

$$\|r\|_{L^2(\Omega)} \leq C\tau^{-1}\|f\|_{L^2(\Omega)}.$$

Proof. This is a rather standard proof about deducing surjectivity for some operator T from the knowledge of injectivity and closed range for the adjoint operator T^*. Let $P_\tau = e^{-\tau\phi}\triangle_g e^{\tau\phi}$ and define $\mathbb{D} = P_\tau^* C_c^\infty(\Omega)$ as a subspace of L^2. Consider the linear functional $L : \mathbb{D} \to \mathbb{C}$ through:

$$L(P_\tau^* v) = \langle v, f\rangle \qquad \forall v \in C_c^\infty(\Omega).$$

This is well-defined since any element of $\mathbb{D}$ has a unique representation as $P_\tau^* v$ with $v \in C_c^\infty(\Omega)$ by the Carleman estimate. Also using Cauchy-Schwarz and the Carleman estimate:

$$|L(P_\tau^* v)| \leq \|v\|_{L^2}\|f\|_{L^2} \leq C\tau^{-1}\|f\|_{L^2}\|P_\tau^* v\|_{L^2}.$$

Thus L is a bounded linear operator on $\mathbb{D}$. Extend L by continuity to the closure $\bar{\mathbb{D}}$ and finally extend to all of $L^2(\Omega)$ through projection operator $\pi : L^2(\Omega) \to \bar{\mathbb{D}}$. Thus we obtain a bounded linear operator $\hat{L} : L^2(\Omega) \to \mathbb{C}$ with $\hat{L}|_{\mathbb{D}} = L$. Furthermore:

$$\|\hat{L}\| \leq C\tau^{-1}\|f\|_{L^2}.$$

Now by the Riesz representation therorem we deduce that there exists a unique $r \in L^2(\Omega)$ such that $\hat{L}(w) = \langle w, r\rangle \quad \forall w \in L^2(\Omega)$ and such that $\|r\| \leq C\tau^{-1}\|f\|_{L^2}$. Note that:

$$\langle v, P_\tau r\rangle = \langle P_\tau^* v, r\rangle = \hat{L}(P_\tau^* v) = L(P_\tau^* v) = \langle v, f\rangle.$$

Hence $P_\tau r = f$ in the weak sense and by construction we have r orthogonal to Σ.

□

We will now proceed to construct a family of solutions to the Schrödinger equation with specific estimates. Let us define $\Phi_0 = z = x_1 + ix_2 = \phi_0 + i\psi_0$

Lemma 2.3.2. $\langle d\Phi_0, d\Phi_0\rangle_g = 0$.

Proof. This is a direct result of the special geometry and can easily be checked in the local coordinate system.

□

Recall that the ∂,$\bar{\partial}$ operators are defined as follows:

$$\partial = \frac{1}{2}(\partial_1 - i\partial_2).$$

$$\bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2).$$

With these notations we have:

$$e^{-\tau\Phi_0}\triangle_g e^{\tau\Phi_0} v = \tau[4e^{-\alpha}\bar{\partial}v] + \triangle_g v.$$

Lemma 2.3.3. *There exists a solution to* $-\triangle_g u_0 = 0$ *of the form* $u_0 = e^{\tau\Phi_0}(v_0 + r_0)$ *with* $v_0(x) = f(z)\chi(x_3)$ *and* $\|r_0\|_{L^2} \leq \frac{C}{\tau}$ *for* τ *large enough.*

Proof. We are interested in solving $P_\tau r_0 = -P_\tau v_0$ and this can be done using Lemma 2.3.1. The estimate easily follows since $\|P_\tau v_0\|_{L^2} = \|\triangle_g v_0\|_{L^2} < \infty$.

□

Lemma 2.3.4. *Let $q \in L^\infty(\Omega)$. There exists a solution u_1 to $(-\triangle_g + q)u_1 = 0$ of the form $u_1 = e^{\tau\Phi_0}(v_0 + r_1)$ with $v_0(x) = f(z)\chi(x_3)$ and $\|r_1\|_{L^2} \leq \frac{C}{\tau}$ for τ large enough.*

Proof. Consider the equation:

$$e^{-\tau\phi_0}(-\triangle_g + q)(e^{\tau\phi_0}r) = -e^{-\tau(\phi_0-\Phi_0)}e^{-\tau\Phi_0}(-\triangle_g + q_1)(e^{\tau\Phi_0}v_0) =: f.$$

Note that by construction we have:

$$\langle d\Phi_0, d\Phi_0\rangle_g = 0$$

and:

$$2\langle d\Phi_0, dv_0\rangle_g + (\triangle_g\Phi_0)v_0 = 0.$$

Hence we can immediately conclude that $\|e^{-\tau\Phi_0}(-\triangle_g + q_1)(e^{\tau\Phi_0}v_0)\|_{L^2(\Omega)} \leq C$ for some constant C.

Motivated by Lemma 2.3.1 we make the ansatz $r = H_\tau\tilde{r}$ to obtain:

$$(-I + qH_\tau)\tilde{r} = f.$$

But $H_\tau : L^2(\Omega) \to L^2(\Omega)$ is a contraction mapping for τ large enough with $\|H_\tau\| \leq \frac{C}{\tau}$ and thus for sufficiently large τ the inverse map $(-I + qH_\tau)^{-1} : L^2(\Omega) \to L^2(\Omega)$ exists and it is given by the following infinite Neumann series:

$$(-I + qH_\tau)^{-1} = -\sum_{j=0}^{\infty}(qH_\tau)^j.$$

Hence:

$$\|(-I + qH_\tau)^{-1}\|_{L^2(\Omega)\to L^2(\Omega)} \leq C.$$

So we deduce that if :

$$r = H_\tau(-I + qH_\tau)^{-1}f.$$

then if we choose $r_1 = e^{\tau(\phi_0 - \Phi_0)}r$ we have that $u_1 = e^{\tau\Phi_0}(v_0 + r_1)$ solves $(-\triangle_g + q_1)u_1 = 0$ and furthermore:

$$\|r_1\|_{L^2(\Omega)} \leq \frac{C}{\tau}.$$

□

In the next section we will use these complex exponential solutions to the Schrödinger equation to obtain some partial knowledge about the potential function from the DN map.

2.4 Partial Information About The Potential

We have constructed special complex exponential solutions to $(-\triangle_g + q)u_1 = 0$ of the form $u_1(x) = e^{\tau\Phi_0}(v_0 + r_1)$ with $v_0(x) = f(z)\chi(x_3)$ and $\|r_1\|_{L^2(\Omega)} \leq C\tau^{-1}$. By asymptotically analyzing these special solutions on the boundary of the domain we will in fact be able to determine some partial information about the potential q in the manifold.

Theorem 2.4.1. *Let (Ω, g) denote the admissible geometry. Suppose $q \in C^1(\Omega)$. Then the knowledge of the complex exponential solutions at the boundary will yield the following linear data about the potential q:*

$$\int_{\Gamma_t} qf(z)d\sigma_{g_t}$$

$$\int_{\Gamma_t} qf(\bar{z})d\sigma_{g_t}$$

where $\Gamma_t = \{x_3 = t\}$ and g_t denotes the induced metric on Γ_t. Namely, the potential will be recovered up to the kernel of functions orthogonal to holomorphic and anti holomorphic functions.

Proof. Let $u_1(x) = e^{\tau\Phi_0}(v_0 + r_1)$ and take $w(x) = e^{-\tau\Phi_0}$ Using Green's's Identity we deduce that:

$$
\begin{aligned}
I_\tau &= \int_{\partial\Omega} w\partial_\nu u_1 - \int_{\partial\Omega} u_1\partial_\nu w = \int_\Omega w\triangle_g u_1 - \int_\Omega u_1\triangle_g w \\
&= \int_\Omega q u_1 w - \int_\Omega u_1 \triangle_g w \\
&= \int_\Omega q u_1 w \\
&= \int_\Omega q v_0 + \int_\Omega q r_1.
\end{aligned}
$$

Furthermore using the estimate on the error term r_1 and the Cauchy-Schwarz inequality we obtain :

$$\int_\Omega q r_1 \le C\tau^{-1}.$$

Thus by taking χ_0 to be a non negative function of compact support and defining $\chi(x_3) = \lambda\chi_0(\lambda x_3)$ and taking the limit as τ and λ approach ∞ we conclude the proof.

□

Corollary 2.4.2. *Let (Ω, g) denote the admissible geometry. Suppose $q \in C^1(\Omega)$ and that Λ_q is known. Then the knowledge of the complex exponential solutions at the boundary will yield the following linear data about the potential q:*

$$\int_{\Gamma_t} \partial_3(qe^\alpha)f(z)dx$$

$$\int_{\Gamma_t} \partial_3(qe^\alpha)f(\bar{z})dx.$$

Theorem 2.4.3. *Let (Ω, g) denote the admissible geometry. Suppose $q \in C^1(\Omega)$ and that Λ_q is known. Then the knowledge of Dirichlet to Neumann map, Λ_q, will yield the following linear data about the potential q:*

$$\int_{\Gamma_t} \partial_3(qe^\alpha)\zeta f(z)dx$$

$$\int_{\Gamma_t} \partial_3(qe^\alpha)\bar{\zeta} f(\bar{z})dx$$

where ζ is the unique solution to $\bar{\partial}\zeta = e^\alpha$ with ζ orthogonal to all holomorphic functions $h(z)$.

Proof. Let us define

$$\Phi_\epsilon(x) = \sum_0^N \Phi_k(x)\epsilon^k$$

with $\epsilon = \frac{1}{\sqrt{\tau}}$ and $N > 4$ where $\Phi_0 = z$, $\Phi_1 = ix_3$ and the functions Φ_k are chosen such that:

$$\langle d\Phi_\epsilon, d\Phi_\epsilon \rangle_g = O(\epsilon^{-N})$$

Indeed, one can iteratively solve equations of the form

$$4\bar{\partial}\Phi_n + F(\Phi_1, ..., \Phi_{n-1}) = 0$$

to obtain the next term Φ_n from the previous ones. For the sake of uniqueness we will always solve $\bar{\partial}^{-1}$ by imposing orthogonality to holomorphic functions. Define

$$v_\epsilon(x) = \sum_0^N v_k(x)\epsilon^k$$

with $v_0(x) = f(z)\chi(x_3)$,

$$v_1 = -\frac{i}{4}(\partial_3\zeta)v_0 - \frac{i}{2}\zeta\partial_3 v_0$$

and we inductively solve for v_k through:

$$2\langle d\Phi_\epsilon, dv_\epsilon \rangle_g + (\triangle_g \Phi_\epsilon)v_\epsilon = O(\epsilon^{-N})$$

This will yield equations of the type:

$$4\bar{\partial}v_n + F(\Phi_1, ..., \Phi_n, v_0, v_1, ..., v_{n-1}) = 0$$

Let us now define $P_{\Phi_\epsilon} = e^{-\tau\Phi_\epsilon}\triangle_g e^{\tau\Phi_\epsilon}$ and note that we have:

$$\|P^*_{\Phi_\epsilon} u\|_{L^2(\Omega)} \geq \tau \|u\|_{L^2(\Omega)}$$

for all $u \in C_c^\infty(\Omega)$ and $\tau > \tau_0$.
Theorem 2.3.1 implies that there exists solutions to $(-\triangle_g + q)u_\epsilon = 0$ of the form:

$$u_\epsilon = e^{\tau\Phi_\epsilon}(v_\epsilon + r_\epsilon)$$

where

$$\|r_\epsilon\|_{L^2} \leq \frac{C}{\tau}.$$

We now use the Green's's identity with $w = e^{-\tau\Phi_\epsilon} v_\epsilon$;

$$I_\epsilon = \int_{\partial\Omega} w\partial_\nu u_\epsilon - \int_{\partial\Omega} u_\epsilon \partial_\nu w = \int_\Omega w \triangle_g u_\epsilon - \int_\Omega u_\epsilon \triangle_g w.$$

The result follows once we take limit $\tau \to \infty$ □

Remark 2.4.4. One might consider the idea of further refining the data by taking $\epsilon = \tau^{-\beta}$ where $\beta \to 0$. The main obstacle in pursuing this idea is that the Carleman estimate

$$\|P^*_{\Phi_\epsilon} u\|_{L^2(\Omega)} \geq \tau \|u\|_{L^2(\Omega)}$$

will <u>not</u> hold for $\beta < \frac{1}{2}$ unless $\partial_1 \alpha \equiv 0$. Thus the idea can not be generalized any further unless we are in the transversally anistropic geometries.

Remark 2.4.5. Consider the surfaces $\Gamma_0 = \{x_3 = 0\}$ and let us consider an isothermal coordinate system on this hypersurface. One important feature of the admissible geometry is that this isothermal coordinate system will remain isothermal on all the following leaves Γ_t under the flow generated by the vector field ∂_3. This observation along with the fact that any holomorphic function $u(x) = f(z)$ solves $\triangle_g u = 0$ suggests that the conformal structure of the Γ_t hypersurfaces will be crucial in analyzing uniqueness of the potential from the DN map in this geometry. Furthermore the fact that in our local coordinate system we have the orthogonality $\langle \partial_3, \partial_1 \rangle_g = 0$ and $\langle \partial_3, \partial_2 \rangle_g = 0$ suggests that it might be possible to localize information on Γ_t hypersurfaces. Indeed both these observations are explicitly utilized in obtaining the linear data I_τ in Theorem 2.4.1.

The methodology used to obtain the data I_τ is not robust with respect to tilting the hypersurfaces Γ_τ and pursuing a similar type of analysis. The main reason is the difficulty in solving the eikonal and complex transport equations under such a local purtubation of the hypersurface. Indeed all the special structure of the geometry will be lost if one were to pursue such an idea. Therefore we need to continue with construction of solutions to the Schrödinger equation that concentrate on these special hypersurfaces and yield more information. With this viewpoint in mind, one might be tempted to consider the classical approaches of deriving uniqueness of potential in two dimensional manifolds and use it in this three dimensional geometry. Indeed, in two dimensions, there are two fundamental methods of obtaining a uniqueness result from the DN map. Adrian Nachman [15] used the factorization of conjugated Laplacian along with $\bar{\partial}$ methods to obtain information on low frequency special solutions. Buckgheim [2] used complex phases with critical points inside the domain to deduce information about the potential at a point inside the domain. The difficulties in proving global uniqueness in two dimenions stems from the fact that the inverse problem is no longer overdetermined. Indeed a simple count of the Schwartz kernel of the Dirichlet to Neumann data in n dimension shows that Λ_q depends on $2(n-1)$ variables whereas q depends on n variables. This is the main reason for us to expect that in our geometry, studying large complex frequency behaviour would be sufficient and that we would not need to concern ourselves too deeply with lower frequency information of the special solutions. Let us consider the conjugated Laplacian operator $P_\Phi = e^{-\tau\Phi}\triangle_g e^{\tau\Phi}$. To construct exponential harmonic functions of the form $u = e^{\tau\Phi}(v+r)$ with the error term r being small enough for large frequencies τ we would need to pick $\phi = \Re(\Phi)$ to be a Carleman weight. Furthermore following the WKB method here we deduce that Φ should solve an eikonal type equation and that v should solve a complex transport equation:

$$\langle d\Phi, d\Phi\rangle_g = 0$$

$$2\langle d\Phi, dv\rangle_g + (\triangle_g\Phi)v = 0.$$

Indeed our construction of the special solutions derived in section 3 revolved around solving these equations explicitly subject to the additional condition that $\phi = \Re(\Phi)$ be a Carleman weight. Solving the eikonal equation beyond the real analytic case can prove extremely difficult. The complex transport equation may not even be locally solvable for some phase functions and indeed a Nirenberg-Treves condition needs to be satisfied for local solvability[23].

2.5 Complex Gaussian Solutions

Partial information about the potential is based on high frequency analysis of complex exponential solutions to the Schrödinger operator which are highly oscillatory in the induced isothermal coordinates on each leaf Γ_t. We will now introduce a class of complex exponential solutions that are highly oscillatory in z and localized in x_3. This method resembles the Wave Packet Transform (also called short time Fourier Transform) analysis first introduced by A.Córdoba and C. Fefferman [27].

We will write ∂^{-1} and $\bar{\partial}^{-1}$ for the solid Cauchy transforms:

$$\partial^{-1} f(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f(w)}{\bar{w}-\bar{z}} dw \wedge d\bar{w}$$

and

$$\bar{\partial}^{-1} f(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f(w)}{w-z} dw \wedge d\bar{w}.$$

Lemma 2.5.1. *Let $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ Then:*

$$\int_{\mathbb{R}} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}.$$

Proof. This is a standard computation using contour integration and the Cauchy-Goursat Theorem. Consider the contour of integration γ defined by the horizontal segment $\{-R < x < R\}$, slant line $\{\theta = \arg(\sqrt{\alpha})\}$ and the two arcs at $\{|x| = R\}$ and perform the contour integration of the function $g(z) = e^{-z^2}$ on γ. By taking the limit as $R \to \infty$ the lemma follows.

□

Recall that

$$e^{-\tau\Phi}\triangle_g(e^{\tau\Phi}a) = \tau^2\langle d\Phi, d\Phi\rangle_g a + \tau[2\langle d\Phi, da\rangle_g + (\triangle_g\Phi)a] + \triangle_g a.$$

We shall make the ansatz

$$\Phi(x) = \sum_{k=0}^{N} \Phi_k(x_1, x_2) x_3^k.$$

and

$$a(x) = \sum_{k=0}^{N} a_k(x_1, x_2) x_3^k.$$

Define

$$\Theta(x) = \sum_{k=1}^{N} \Phi_k(x_1, x_2) x_3^k.$$

We first choose $\Phi(x)$ such that

$$\langle d\Phi, d\Phi \rangle_g = 0 \text{ to } N^{th} \text{ order on } \Gamma_0.$$

To this end we will consider the eikonal equation as a transport equation in ∂_3 and try to solve this system by assignning Φ_0 arbitrarily.

Note that

$$\begin{aligned} \langle d\Phi, d\Phi \rangle_g &= g^{jk} \partial_j \Phi \partial_k \Phi = (\partial_3 \Phi)^2 + \langle d\Phi, d\Phi \rangle_{g_{x_3}} \\ &= (\partial_3 \Phi)^2 + e^{-\alpha(x)} (\partial_1 \Phi)^2 + e^{-\alpha(x)} (\partial_2 \Phi)^2 \end{aligned}$$

where g_{x_3} denotes the induced metric on Γ_t hypersurfaces.

Let us assume for a moment that α is real analytic. In that case using the Cauchy-Kovalevskaya Theorem, we can solve the above partial differential equation by taking any Φ_0 such that $\langle d\Phi_0, d\Phi_0 \rangle_{g_0} \neq 0$ and construct $\Phi_k(x)$ for $k > 0$. Indeed if one picks Φ_0 such that $\langle d\Phi_0, d\Phi_0 \rangle_{g_0} = 0$ then one can immediately deduce that $\Phi_1(x) = 0$ and thus we arrive at a constraint on the metric in order to continue solving for $\Phi_k(x)$ with $k > 1$. Interestingly this condition imposes that the surface Γ_0 should be umbilical. Indeed in the Euclidean space $\mathbb{R}^3$, such solutions (subject to $\langle d\Phi_0, d\Phi_0 \rangle_{g_0} = 0$) to the eikonal equation exist. For example, one can take $\Phi(x) = z + \frac{x_3^2}{\bar{z}}$.

With this observation in mind, we now return to solving the eikonal equation subject to taking $\Phi_0(x) = z$

Since

$$\langle d\Phi, d\Phi \rangle_g|_{\Gamma_0} = 0.$$

we obtain

$$\Phi_1(x_1, x_2) = 0.$$

Furthermore

$$\partial_3 \langle d\Phi, d\Phi \rangle_g|_{\Gamma_0} = 0$$

is automatically satisfied because Γ_0 is umbilical. We stress that this is crucial otherwise the construction would break down at this step of the iteration. The next term

$$\partial_{33} \langle d\Phi, d\Phi \rangle_g|_{\Gamma_0} = 0$$

yields

$$\Phi_2^2 + e^{-\alpha}\bar{\partial}\Phi_2 = 0.$$

We will take

$$\Phi_2(x_1, x_2) = \frac{1}{\zeta(x_1, x_2) + L}$$

where $\bar{\partial}\zeta = e^{\alpha}$ and $L \le L_0$ is a sufficiently large negative number.

Following the iteration we note that for $k \ge 3$ we have that :

$$\partial_3^k \langle d\Phi, d\Phi \rangle_g|_{\Gamma_0} = 0.$$

This yields:

$$k\Phi_2\Phi_k + 2e^{-\alpha}\bar{\partial}\Phi_k = F$$

where F is a function of the previous terms $\Phi_0, \Phi_1, ..., \Phi_{k-1}$ Thus we can solve iteratively for all Φ_k with $k \le N$.

We will now proceed with the same idea to solve the complex transport equation:

$$2\langle d\Phi, da\rangle_g + (\triangle_g \Phi)a = 0 \text{ to } N^{th} \text{ order on } \Gamma_0.$$

Define a_0 through

$$a_0 = \frac{f(z)}{\sqrt{\zeta + L}}.$$

Note that

$$2\langle d\Phi, da\rangle_g + (\triangle_g \Phi)a = 2\partial_3 d\Phi \partial_3 a + 2\langle d\Phi, da\rangle_{g_{x_3}} + (\triangle_g \Phi)a.$$

The first iteration yields that

$$(2\langle d\Phi, da\rangle_g + (\triangle_g \Phi)a)|_{\Gamma_0} = 0.$$

which clearly holds because of the choice of a_0 and since $\Phi_1 \equiv 0$. One can now iteratively solve for the functions $a_k(x_1, x_2)$ as the k^{th} iteration yields that:

$$\partial_3^k (2\langle d\Phi, da\rangle_g + (\triangle_g \Phi)a)|_{\Gamma_0} = 0.$$

This implies that:

$$e^{-\alpha}\bar{\partial} a_k + k\Phi_2 a_k = G$$

where G is a function of all the previous terms Φ_i and a_j. Thus we have been able to construct the functions:

$$\Phi(x) = \sum_{k=0}^{N} \Phi_k(x_1, x_2) x_3^k$$

and

$$a(x) = \sum_{k=0}^{N} a_k(x_1, x_2) x_3^k$$

such that the eikonal and complex transport equations vanish up to N^{th} order on Γ_0 and furthermore we have that:

$$a_0 = \frac{f(z)}{\sqrt{\zeta + L}}$$

$$\Phi_0(x_1, x_2) = z$$

$$\Phi_1(x_1, x_2) = 0$$

$$\Phi_2(x_1, x_2) = \frac{1}{\zeta(x_1, x_2) + L}.$$

Theorem 2.5.2. *Let (Ω, g) denote the admissible geometry and suppose $N > 4$. There exists solutions u to $-\triangle_g u = 0$ of the form $u(x) = e^{\tau\Phi_0}(e^{\tau\Theta}v(x) + r)$ where :*

$$\Theta(x) = \sum_{k=1}^{N} \Phi_k(x_1, x_2)x_3^k$$

$$v(x) = \tau^{\frac{1}{4}} a(x)\chi_0(\frac{x_3}{\delta})$$

$$a(x) = \sum_{k=0}^{N} a_k(x_1, x_2)x_3^k$$

and such that for τ large enough we have:

$$\|r\|_{L^2(\Omega)} < C\tau^{-1}$$

Here $\chi_0(x)$ is a smooth function with $\chi_0 = 1$ for $|x| \leq \frac{1}{4}$ and $\chi_0 = 0$ for $|x| \geq \frac{1}{2}$.

Proof. Note that we are interested in solving a conjugated Laplacian type equation of the following form:

$$P_{\Phi_0} r(x) = -P_{\Phi_0}(e^{\tau\Theta}v(x)).$$

But

$$P_{\Phi_0}(e^{\tau\Theta}v(x)) = e^{\tau\Theta}P_{\Phi}v(x).$$

Now because of our construction of the power series functions $\Theta(x)$ and $a(x)$ we note that we can choose $\delta < 1$ small enough such that $Re(\Theta) \leq -Cx_3^2$ and therefore:

$$|e^{\tau\Theta}| \leq e^{-C\tau x_3^2}.$$

Note that:

$$P_\Phi v(x) = \tau^{\frac{1}{4}}(C\tau^2 x_3^N \chi_0 + C\tau x_3^N \chi_0 + C\chi_0).$$

and thus by choosing $N > 4$ we obtain that:

$$\|e^{\tau\Theta} P_\Phi v(x)\|_{L^2(\Omega)} \leq C.$$

Finally using the solution operator for the conjugated Laplacian P_{Φ_0} constructed in Theorem 2.3.1 we deduce the estimate on the error term.

□

Corollary 2.5.3. *Let (Ω, g) denote the admissible geometry and suppose $N > 4$. There exists solutions u to $(-\triangle_g + q)u = 0$ of the form $u(x) = e^{\tau\Phi_0}(e^{\tau\Theta}v(x) + r)$ where :*

$$\Theta(x) = \sum_{k=1}^{N} \Phi_k(x_1, x_2)x_3^k$$

$$v(x) = \tau^{\frac{1}{4}} a(x)\chi_0(\frac{x_3}{\delta})$$

$$a(x) = \sum_{k=0}^{N} a_k(x_1, x_2)x_3^k$$

and such that for τ large enough we have:

$$\|r\|_{L^2(\Omega)} < C\tau^{-1}$$

Here $\chi_0(x)$ is a smooth function with $\chi_0 = 1$ for $|x| \leq \frac{1}{4}$ and $\chi_0 = 0$ for $|x| \geq \frac{1}{2}$.Furthermore the trace of these solutions on the boundary can be obtained from the knowledge of the Dirichlet to Neumann map Λ_q.

2.6 Boundary Reconstruction of CGOs

In the previous sections we constructed special complex exponential solutions to $(-\triangle_g + q)u = 0$ that concentrate on the umbilic surfaces $\{x_3 = c\}$. If one can determine the boundary values of these special solutions $u|_{\partial\Omega}$ then we can deduce partial information about $q|_{x_3=c}$. For the general geometries considered in Conjecture 1.1.6, it is not clear whether the boundary values of the solutions we have constructed can be determined from the DN map. Here, we will describe the major difficulty in achieving this boundary reconstruction. To this end let us explain the standard method of obtaining the boundary values of CGO solutions in Euclidean geometries and in CTA geometries. Let us first consider the Euclidean space. We will closely follow the approach in [15] here. Recall that the CGO solutions to $(-\triangle + q)u = 0$ are simply solutions of the form $u = e^{i\zeta\cdot x}v$ with $\zeta \in \mathbb{V}$ and v solving $e^{-i\zeta\cdot x}(-\triangle + q)(e^{-i\zeta\cdot x}v) = 0$ in Ω. The CGO solutions can be generalized to the entire space by considering the conjugated Laplacian operator globally in $\mathbb{R}^n$. For $\zeta \in \mathbb{V}$:

$$g_\zeta(x) := \frac{1}{(2\pi^n)} \int \frac{e^{ix\cdot\xi}}{\xi^2 + 2\zeta\cdot\xi} d\xi$$

is a tempered distribution and we have:

$$(-\triangle - 2i\zeta\cdot\nabla)g_\zeta(x) = \delta(x).$$

One can use this distribution to construct the Fadeev Green's function for the Laplace operator by defining:

$$G_\zeta(x) = e^{i\zeta\cdot x}g_\zeta(x).$$

Recall that L^2_δ denotes the weighted Hilbert space:

$$L^2_\delta(\mathbb{R}^n) = \{f \in L^1_{loc}(\mathbb{R}^n) : \|f\|_\delta = \sqrt{\int (1+|x|^2)^\delta |f(x)|^2}\}.$$

Lemma 2.6.1 (see [15])**.** *(a) For all $\zeta \in \mathbb{V}$ and $|\zeta| \geq a$:*

$$\|g_\zeta * f\|_{\delta-1} \leq \frac{c(\delta,a)}{|\zeta|}\|f\|_\delta$$

for $0 < \delta < 1$.

(b) For $|\zeta| > c(\delta,a)\|q(x)(1+x^2)^\delta\|_{L^\infty}$ there exists a unique solution of $(-\triangle + q)u = 0$ with $e^{-i\zeta\cdot x}u - 1 \in L_{-\delta}$. Furthermore:

$$\|e^{-i\zeta\cdot x}u - 1\|_{-\delta} \leq \frac{c(\delta,a)}{|\zeta|}\|q\|_\delta$$

The approach in [15] is then to set up a novel boundary value problem in the exterior region Ω^c (where q is known) as follows:

$$\left\{\begin{array}{ll}(-\triangle_g + q)u_o = 0 & \text{for } x \in \Omega^c \\ e^{-i\zeta\cdot x}u_o - 1 \in L_{-\delta} & \text{for } x \in \Omega^c \\ \partial_\nu u_o|_{\partial\Omega} = \Lambda_q(u_o|_{\partial\Omega}) & \text{for } x \in \partial\Omega\end{array}\right\}$$

One then shows that this exterior problem has a unique solution and that this solution recovers the boundary values of the solutions u (constructed in Lemma 2.6.1 (b)) in the exterior of Ω. Indeed most boundary reconstruction methods in inverse problems rely on a similar method where an exterior PDE is set up and a one-one correspondence is shown to hold between solutions to this exterior PDE and global CGO solutions.

We can now explain why similar ideas would break down in our admissible geometry. Indeed let us assume that there exists an exterior pde whose solutions will turn out to be the CGO solutions discussed in the previous sections. Let us first recall that CGO solutions will be solutions of the form $u = e^{\tau\phi}(v + r)$. Motivated by the existing ideas for boundary determination we will assume that we can construct globally defined CGO solutions. As can be seen from Lemma 2.3.1, existence of these solutions will require a coercivity estimate for $P_{-\tau}(\cdot)$. Hence as a consequence we must have:

$$\|P_{-\tau}v\|_X \geq \tau\|v\|_Y \qquad \forall v \in C_c^\infty(\Omega) \quad (*)$$

for some choice of Banach norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. From the coercivity estimates for $P_{-\tau}$ we can construct a Fadeev type global Green's function for the conjugated Schrödinger equation that will imply existence of solutions to $(-\triangle_g + q)u = 0$ with $u = e^{\tau\phi}(v + r)$ such that $\|r\|_{\tilde{X}} \leq \frac{C}{\tau}\|P_\tau v\|_{\tilde{Y}}$ for some choice of Banach spaces $\tilde{X}$ and $\tilde{Y}$.

Let us now assume that there exists some exterior PDE of the following type that will determine the special solutions $u = e^{\tau\phi}(v + r)$ (with v satisfying $P_\tau v \in \tilde{Y}$):

$$\left\{\begin{array}{ll}(-\triangle_g + q)u_o = 0 & \text{for } x \in \Omega^c \\ e^{-\tau\phi}u_o - v \text{ satisfies an appropriate exterior condition} & \text{for } x \in \Omega^c \\ \partial_\nu u_o|_{\partial\Omega} = \Lambda_q(u_o|_{\partial\Omega}) & \text{for } x \in \partial\Omega\end{array}\right\}$$

We are going to explain that uniqueness arguments for exterior pdes of the above type will always imply that the phase function ϕ must be a limiting Carleman weight. To see this let us simply choose $v \in C_c^\infty(\Omega)$. Note that $P_\tau v \in \tilde{Y}$ and so the global PDE must have a unique solution. However $v \equiv 0$ for all $x \in \Omega^c$. Hence the only choice for the unique solution must be $u_o \equiv 0$. This implies that the special solution must also vanish inside Ω, that is to say $u = e^{\tau\phi}(v + r) = 0$. Hence we must have:

$$\|P_\tau v\|_{\tilde{Y}} \geq \frac{C}{\tau}\|v\|_{\tilde{X}}. \quad (**)$$

Combining (*) and (**) we deduce that ϕ must be an LCW and we are back to CTA geometries! This heuristic argument suggests that, surprisingly, the standard approach in boundary determination also limits us to CTA geometries. A new boundary determination argument may require deeper analysis of the Fadeev type Green's functions for conjugated Laplacian.

In the next chapter, we will consider geometries with a subdomain that contain a large class of umbilic surfaces and prove a uniqueness theorem for the potential from the knowledge of the Dirichlet to Neumann map for the Schrödinger operator. We will impose a restriction on the potential function (see Theorem 3.1.1) that allows us to determine the special solutions at the boundary and obtain the uniqueness theorem.

Chapter 3

Locally Euclidean Geometries

3.1 Problem Formulation

In this chapter, we will prove a uniqueness theorem for potentials in geometries that are essentially Euclidean in a subdomain. More specifically suppose that (Ω^3, g) is a compact smooth three dimensional Riemannian manifold. As before, we define the Dirichlet to Neuman map $\Lambda_q : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ for a Schrödinger operator as follows:

$$(\Lambda_q f, h) = \int_\Omega \langle du, dv\rangle_g dV_g + \int_\Omega quv dV_g$$

where $v \in H^1(\Omega)$ satisfies $v|_{\partial\Omega} = h$ and $u \in H^1(\Omega)$ satisfies $(-\triangle_g + q)u = 0$ with $u|_{\partial\Omega} = f$. We will assume that 0 is not a Dirichlet eigenvalue for $-\triangle_g + q$. Λ_q is a self adjoint bounded linear Elliptic Pseudo Differential Operator [4]. Note that if $f \in H^{\frac{3}{2}}(\partial\Omega)$ then $\Lambda_q f = \partial_\nu u|_{\partial\Omega}$ where ν is the outward normal unit vector field on the boundary of the domain.

We will assume that there exists a Euclidean subdomain U in Ω and study the question of determining the unknown potential q in this region. To overcome the difficulties in the reconstruction at the boundary described in the previous chapter, we will assume that the unknown part of the potential function is supported in this Euclidean subset. More precisely, we have the following theorem:

Theorem 3.1.1. *Let (Ω^3, g) denote a compact smooth Riemannian manifold with smooth boundary. Let $U \subset \Omega$ be an open subset such that $\Gamma = U \cap \partial\Omega$ is non-empty, connected and* *strictly convex. Suppose that U can be covered with a coordinate chart in which $g|_U$ is the Euclidean metric and that U is the convex hull of Γ. Let q be a smooth function and suppose that $q - q_*$ is compactly supported in U where q_* is a globally known smooth function. Then the knowledge of Λ_q will uniquely determine q everywhere.*

Let us make a few remarks before we begin the proof. Note that one may argue that since q is supported in U and since the metric g is known everywhere in Ω we might be able to determine the Dirichlet to Neumann map for the Schrödinger equation in the smaller domain (U, e) from the DN map in (Ω, g). This is however not immediate as the set of solutions to $(-\triangle_g + q)u = 0$ in (Ω, g) is not the same as the set of solutions in (U, e). It is possible though to use a density argument and a quantitative version of Runge approximation as discussed in [21] to conclude uniqueness of the potential q in U. The proof will not be constructive however and an altogether different approach is probably needed to give a reconstruction algorithm. Motivated by this, we approach the question quite differently. Uniqueness of potential is proved and we sketch out a reconstruction algorithm as well. The key in our reconstruction algortihm is to construct a symmetric Fadeev type Green's function [17] for the Laplacian operator and use a strong unique continuation argument. [22]

3.2 Carleman estimates

Definition 3.2.1 (Carleman Weight)**.** A smooth function ϕ is called a Carleman weight with respect to (Ω_1, g) if there exists $h_0 > 0$ such that the following estimate holds:

$$\|e^{\frac{\phi}{h}}\triangle_g(e^{-\frac{\phi}{h}}u)\|_{L^2(\Omega_1)} \geq \frac{C}{h}\|u\|_{L^2(\Omega_1)} + C\|Du\|_{L^2(\Omega_1)}$$

$\forall 0 < h < h_0$ and $u \in C_c^\infty(\Omega)$.

Let us extend the manifold Ω to a slightly larger manifold Ω_1. We extend q to all of Ω_1 by setting it equal to zero outside Ω and extend g smoothly to Ω_1 such that $g|_{U_1}$ is Euclidean.

Here U_1 denotes the extension of U to the larger manifold Ω_1. Note that U_1 has a foliation by a family of planes $\mathbb{A} = \{\Pi_t\}_{t \in I}$. We start by taking a fixed plane $\Pi \in \mathbb{A}$. A local coordinate system (x_1, x_2, x_3) can be constructed in U_1 such that $\Pi = \{x_3 = 0\}$ with (x_1, x_2) denoting the usual cartesian coordinate system on the plane Π and ∂_3 denoting the normal flow to this plane. We can assume that support of q lies in the compact set $V \subset\subset \{-t_1 \leq x_3 \leq t_2\}$ with $t_1, t_2 > 0$. In this framework $U_1 = \cup_{c=-t_1-\delta_1}^{c=t_2+\delta_2} \{x_3 = c\}$ with $\delta_i > 0$ for $i \in \{1, 2\}$.

Definition 3.2.2. Let us define two smooth functions $\omega : \Omega_1 \to \mathbb{R}$ and $\tilde{\omega} : \Omega_1 \to \mathbb{R}$ as follows:

- Let $\omega : \Omega_1 \to \mathbb{R}$ be any smooth function such that $d\omega \neq 0$ everywhere in Ω_1 and $\omega(x) \equiv x_3$ for $x \in U_1$.
- Let $\tilde{\omega} : \Omega_1 \to \mathbb{R}$ be any smooth function such that $\tilde{\omega}(x) \equiv x_2$ for $x \in U_1$.

Definition 3.2.3. Let us define two globally defined $C^{k-1}(\overline{\Omega_1})$ functions $\chi_0 : \Omega_1 \to \mathbb{R}$ and $F_\lambda : \mathbb{R} \to \mathbb{R}$ as follows:

$$\chi_0(x) = \left\{ \begin{array}{lr} 1, & \text{for } -t_1 < x_3 < t_2 \\ (1 - (\frac{x_3 - t_2}{\delta_2})^{8k})^k, & \text{for } t_2 \leq x_3 \leq t_2 + \delta_2 \\ (1 - (\frac{x_3 + t_1}{\delta_1})^{8k})^k, & \text{for } -t_1 - \delta_1 \leq x_3 \leq -t_1 \\ 0 & \text{otherwise} \end{array} \right\}$$

$$F_\lambda(x) = \left\{ \begin{array}{lr} 0, & \text{for } -t_1 < x < t_2 \\ e^{\lambda(\frac{x-t_2}{\delta_2})^2} (\frac{x-t_2}{\delta_2})^{2k}, & \text{for } t_2 \leq x \\ e^{\lambda(\frac{x+t_1}{\delta_1})^2} (\frac{x+t_1}{\delta_1})^{2k}, & \text{for } x \leq -t_1 \end{array} \right\}$$

We will use the functions defined above to construct an appropriate global Carleman weight in the entire domain (Ω_1, g). The key idea here is to start with a local Limiting Carleman weight (for example the function $\phi(x) = x_1$) over the support of the potential q and extend it smoothly to the manifold in a way that it will satisfy the Hörmander hypoellipticity condition in one direction. We believe this is the first time that the idea of extending a Limiting Carleman Weight is being implemented to solve an inverse problem.

Lemma 3.2.4. *Let* $\tilde{\phi}_0(x_1, x_2, x_3) = x_1\chi_0(x) + (F_\lambda \circ \omega)(x)$ *where* $k \geq 1$ *is an arbitraty integer and* $\lambda(\Omega_1, k, ||g_{ij}||_{C^2})$ *is sufficiently large. Then the Hörmander hypo-ellipticity condition is satisfied in* Ω_1*, that is to say:*

$$D^2\tilde{\phi}_0(X, X) + D^2\tilde{\phi}_0(\nabla\tilde{\phi}_0, \nabla\tilde{\phi}_0) \geq 0$$

whenever $|X| = |\nabla\tilde{\phi}_0|$ *and* $\langle\nabla\tilde{\phi}_0, X\rangle = 0$.

Proof. The proof will be divided into three parts. We will consider the the three regions $A_1 = \{-t_1 \leq x_3 \leq t_2\}$, $A_2 = \{t_2 \leq x_3 \leq t_2 + \delta_2\} \cup \{-t_1 - \delta_1 \leq x_3 \leq -t_1\}$ and $A_3 = \Omega_1 \setminus (A \cup B)$ and prove the inequality holds in all these regions. Recall that the metric is Euclidean on U_1 which implies that both A_1 and A_2 are Euclidean. Let us first consider A_1. Note that in this region $\tilde{\phi}_0(x_1, x_2, x_3) = x_1$ and since the metric is Euclidean in this region we deduce that $D^2\tilde{\phi}_0(X, Y) \equiv 0$ for all X, Y and hence the Hörmander condition is satisfied.

Let us now focus on the region denoted by A_3. Notice that in this region we have $\tilde{\phi}_0 = F_\lambda(\omega(x))$. Therefore the level sets of $\tilde{\phi}_0(x)$ will simply be the level sets $\{\omega(x) = c\}$.

$$D^2\tilde{\phi}_0(X, X) = \langle D_X\nabla\tilde{\phi}_0, X\rangle.$$

Since $|X| = |F'_\lambda(\omega)||\nabla\omega|$ we obtain the following estimate:

$$D^2\tilde{\phi}_0(X, X) \leq C|F'_\lambda(\omega)|^3$$

where it is important to note that the constant C is independent of λ. Furthermore we have:

$$D^2\tilde{\phi}_0(\nabla\tilde{\phi}_0, \nabla\tilde{\phi}_0) = \frac{1}{2}\nabla\tilde{\phi}_0(|\nabla\tilde{\phi}_0|^2).$$

Since $\tilde{\phi}_0 = F_\lambda(\omega(x))$:

$$D^2\tilde{\phi}_0(\nabla\tilde{\phi}_0, \nabla\tilde{\phi}_0) = \frac{1}{2}(F'(\omega)^3\nabla\omega(|\nabla\omega|^2) + 2F'(\omega)^2F''(\omega)|\nabla\omega|^4).$$

One can easily check that for $x \in A_3$:

$$|F_\lambda'(\omega)| \leq \left\{ \begin{array}{ll} C\lambda e^{\lambda(\frac{\omega-t_2}{\delta_2})^2} & \text{for } t_2+\delta_2 \leq x_3 \\ C\lambda e^{\lambda(\frac{\omega+t_1}{\delta_1})^2}, & \text{for } x_3 \leq -t_1-\delta_1 \end{array} \right\}$$

$$F_\lambda''(\omega) \geq \left\{ \begin{array}{ll} C\lambda^2 e^{\lambda(\frac{\omega-t_2}{\delta_2})^2} & \text{for } t_2+\delta_2 \leq x_3 \\ C\lambda^2 e^{\lambda(\frac{\omega+t_1}{\delta_1})^2}, & \text{for } x_3 \leq -t_1-\delta_1 \end{array} \right\}$$

Thus we can easily conclude that for λ large enough the Hörmander hypoellipticity condition is satisfied in this region. Let us now turn our attention to the transition region $x \in A_2$. Recall that the metric g is flat in A_2. We will actually prove the stronger claims:

(1) $D^2\tilde{\phi}_0(\nabla\tilde{\phi}_0, \nabla\tilde{\phi}_0) \geq 0$

(2) $D^2\tilde{\phi}_0(X, X) \geq 0$ for all X with $\langle \nabla\tilde{\phi}_0, X\rangle = 0$.

The idea is that near the $x_3 = 0$ hypersurface the convexity of $x_3^{2k}e^{\lambda x_3^2}$ yields the Hormander Hypo Ellipticity. Furthermore away from this surface a suitable choice of λ large enough will yield non-negativity as well thus completing the proof. We will now make these statements more precise as follows:

$$F_\lambda'(x) = \left\{ \begin{array}{ll} (\frac{x_3-t_2}{\delta_2})^{2k-1} e^{\lambda(\frac{x_3-t_2}{\delta_2})^2} (\frac{2k+2\lambda(\frac{x_3-t_2}{\delta_2})^2}{\delta_2}), & \text{for } t_2 \leq x_3 \leq t_2+\delta_2 \\ (\frac{x_3+t_1}{\delta_1})^{2k-1} e^{\lambda(\frac{x_3+t_1}{\delta_1})^2} (\frac{2k+2\lambda(\frac{x_3+t_1}{\delta_1})^2}{\delta_1}), & \text{for } -t_1-\delta_1 \leq x_3 \leq -t_1 \end{array} \right\}$$

$$F_\lambda''(x) = (\frac{x_3-t_2}{\delta_2})^{2k-2} e^{\lambda(\frac{x_3-t_2}{\delta_2})^2}((\frac{(2k)(2k-1)}{\delta_2^2} + \frac{8\lambda k+2\lambda}{\delta_2^2}(\frac{x_3-t_2}{\delta_2})^2 + \frac{4\lambda^2}{\delta_2^2}(\frac{x_3-t_2}{\delta_2})^4)$$

for $t_2 \leq x_3 \leq t_2+\delta_2$ and:

$$F_\lambda''(x) = (\frac{x_3+t_1}{\delta_1})^{2k-2} e^{\lambda(\frac{x_3+t_1}{\delta_1})^2}((\frac{(2k)(2k-1)}{\delta_1^2} + \frac{8\lambda k+2\lambda}{\delta_1^2}(\frac{x_3+t_1}{\delta_1})^2 + \frac{4\lambda^2}{\delta_1^2}(\frac{x_3+t_1}{\delta_1})^4)$$

for $-t_1-\delta_1 \leq x_3 \leq -t_1$.

Note that:

$$D^2\tilde{\phi}_0(\nabla\tilde{\phi}_0, \nabla\tilde{\phi}_0) = (\partial_3\tilde{\phi}_0)^2\partial_{33}\tilde{\phi}_0 + 2\partial_1\tilde{\phi}_0\partial_3\tilde{\phi}_0\partial_{13}\tilde{\phi}_0.$$

So:

$$D^2\tilde{\phi}_0(\nabla\tilde{\phi}_0, \nabla\tilde{\phi}_0) \geq |\partial_3\tilde{\phi}_0|(|\partial_3\tilde{\phi}_0|\partial_{33}\tilde{\phi}_0 - 2|\chi_0\chi_0'|).$$

$$|\partial_3\tilde{\phi}_0| = |x_1\chi_0' + F'(x_3)| \geq |F'(x_3)| - |x_1||\chi_0'|.$$

Using the Cauchy-Schwarz inequality we see that:

$$|F_\lambda'(x)| \geq \left\{ \begin{array}{ll} \frac{4}{\delta_2}\sqrt{\lambda k}(\frac{x_3-t_2}{\delta_2})^{2k} & \text{for } t_2 \leq x_3 \leq t_2+\delta_2 \\ \frac{4}{\delta_2}\sqrt{\lambda k}(\frac{x_3+t_1}{\delta_1})^{2k} & \text{for } -t_1-\delta_1 \leq x_3 \leq -t_1 \end{array} \right\}$$

And:

$$|x_1||\chi_0'| \leq \left\{ \begin{array}{ll} C(\Omega)k^2|(\frac{x_3-t_2}{\delta_2})|^{8k-1} & \text{for } t_2 \leq x_3 \leq t_2+\delta_2 \\ C(\Omega)k^2|(\frac{x_3+t_1}{\delta_1})|^{8k-1}, & \text{for } -t_1-\delta_1 \leq x_3 \leq -t_1 \end{array} \right\}$$

Hence we can conclude that:

$$|\partial_3\tilde{\phi}_0| \geq \left\{ \begin{array}{ll} \frac{4}{\delta_2}\sqrt{\lambda k}(\frac{x_3-t_2}{\delta_2})^{2k} - C(\Omega)k^2|(\frac{x_3-t_2}{\delta_2})|^{8k-1} & \text{for } t_2 \leq x_3 \leq t_2+\delta_2 \\ \frac{4}{\delta_2}\sqrt{\lambda k}(\frac{x_3+t_1}{\delta_1})^{2k} - C(\Omega)k^2|(\frac{x_3+t_1}{\delta_1})|^{8k-1}, & \text{for } -t_1-\delta_1 \leq x_3 \leq -t_1 \end{array} \right\}$$

and therefore for λ sufficiently large we obtain that:

$$|\partial_3\tilde{\phi}_0| \geq \left\{ \begin{array}{ll} \frac{2}{\delta_2}\sqrt{\lambda k}(\frac{x_3-t_2}{\delta_2})^{2k} & \text{for } t_2 \leq x_3 \leq t_2+\delta_2 \\ \frac{2}{\delta_2}\sqrt{\lambda k}(\frac{x_3+t_1}{\delta_1})^{2k}, & \text{for } -t_1-\delta_1 \leq x_3 \leq -t_1 \end{array} \right\}$$

Now:

$$\partial_{33}\tilde{\phi}_0 = x_1\chi_0' + F''(x_3) \geq \frac{1}{2}F''(x_3).$$

Hence:

$$\partial_{33}\tilde{\phi}_0 \geq \left\{ \begin{array}{ll} \frac{2\lambda}{\delta_2^2}(\frac{x_3-t_2}{\delta_2})^{2k}, & \text{for } t_2 \leq x_3 \leq t_2+\delta_2 \\ \frac{2\lambda}{\delta_1^2}(\frac{x_3+t_1}{\delta_1})^{2k}, & \text{for } -t_1-\delta_1 \leq x_3 \leq -t_1 \end{array} \right\}$$

Hence combining the above we see that for λ sufficiently large we have that:

$$D^2\tilde{\phi}_0(\nabla\tilde{\phi}_0, \nabla\tilde{\phi}_0) \geq 0.$$

Let us now analyze the term $D^2\tilde{\phi}_0(X,X)$ for all X with $\langle\nabla\tilde{\phi}_0, X\rangle = 0$

Note that $d\tilde{\phi}_0(X) = 0$ implies that:

$$X \in \text{span}\{\partial_2, \partial_3\tilde{\phi}_0\partial_1 - \partial_1\tilde{\phi}_0\partial_3\}.$$

but since g is Euclidean in this region we have the following:

$$D^2\tilde{\phi}_0(\partial_2, X) = 0.$$

Now:

$$D^2\tilde{\phi}_0(\partial_3\tilde{\phi}_0\partial_1 - \partial_1\tilde{\phi}_0\partial_3, \partial_3\tilde{\phi}_0\partial_1 - \partial_1\tilde{\phi}_0\partial_3) = (\partial_1\tilde{\phi}_0)^2\partial_{33}\tilde{\phi}_0 - 2\partial_1\tilde{\phi}_0\partial_3\tilde{\phi}_0\partial_{13}\tilde{\phi}_0.$$

So:

$$D^2\tilde{\phi}_0(\partial_3\tilde{\phi}_0\partial_1 - \partial_1\tilde{\phi}_0\partial_3, \partial_3\tilde{\phi}_0\partial_1 - \partial_1\tilde{\phi}_0\partial_3) = \chi_0^2(x_1\chi_0'' + F'') - 2\chi_0\chi_0'(x_1\chi_0' + F').$$

Using the Cauchy-Schwarz inequality again and by looking at the sign of the x_3 we can get the following inequalities:

$$-2x_1|\chi_0'|^2\chi_0 - 2\chi_0\chi_0'F' \geq 0$$
$$F'' + x_1\chi_0'' \geq \frac{F''}{2} \geq 0.$$

and thus by combining the above inequalites we obtain that:

$$D^2\tilde{\phi}_0(\partial_3\tilde{\phi}_0\partial_1 - \partial_1\tilde{\phi}_0\partial_3, \partial_3\tilde{\phi}_0\partial_1 - \partial_1\tilde{\phi}_0\partial_3) \geq 0.$$

□

We will now provide a suitable modification of the well known fact that the **strict** Hörmander Hypo-Ellipticity yields a global Carleman estimate.

Lemma 3.2.5. *Let (Ω_1, g) be a compact smooth Riemannian manifold with smooth boundary and suppose $\psi \in C^2(\Omega_1)$ is such that $d\psi \neq 0$ and the Hormander HypoEllipticity condition is satisfied:*

$$D^2\psi(X, X) + D^2\psi(\nabla\psi, \nabla\psi) \geq 0$$

whenever $|X| = |\nabla\psi|$ and $\langle\nabla\psi, X\rangle = 0$. Then there exists constants $C(\Omega, g)$ and $h_0 > 0$ such that for all $v \in C_c^\infty(\Omega_1)$ and all $0 < h < h_0$ the following estimate holds:

$$\|e^{\frac{\psi}{h}}\triangle_g(e^{-\frac{\psi}{h}}v)\|_{L^2(\Omega_1)} \geq \frac{C}{h}\|v\|_{L^2(\Omega_1)} + C\|Dv\|_{L^2(\Omega_1)}.$$

Remark 3.2.6. In general there is a rather standard technique of proving these estimates either through integration by parts or semiclassical calculus. We will employ the former method due to its simplicity. In cases where

$$D^2\psi(X, X) + D^2\psi(\nabla\psi, \nabla\psi) > 0$$

whenever $|X| = |\nabla\psi|$ and $\langle \nabla\psi, X\rangle = 0$ one can refer to [9] for proving this estimate where in fact we would get a stronger gain in terms of h. Similarly in the case where

$$D^2\psi(X,X) + D^2\psi(\nabla\psi, \nabla\psi) = 0$$

whenever $|X| = |\nabla\psi|$ and $\langle \nabla\psi, X\rangle = 0$ one can refer to [5] or [18] for a proof. In our setting we are in an intermediate case and thus need to adjust the arguments.

Proof. It suffices to prove the claim for the renormalized metric $\hat{g} = |\nabla^g\psi|_g^2 g$. To see this let us assume that $c = |\nabla^g\psi|_g^{-2}$ and that ψ is a Carleman weight with respect to $\hat{g}$. But then using the transformation property of the Laplace Beltrami operator under conformal changes of metric we deduce that:

$$e^{\frac{\psi}{h}}(-h^2\triangle_g)(e^{-\frac{\psi}{h}}v) = e^{\frac{\psi}{h}}(-h^2 c^{-\frac{5}{4}}\triangle_{\hat{g}})(c^{\frac{1}{4}}e^{-\frac{\psi}{h}}v) - h^2 q_c c^{-1} v$$

where:

$$q_c = c^{\frac{1}{4}}\triangle_{c\hat{g}} c^{-\frac{1}{4}}$$

Now note that $c(x) > 0$ for all $x \in \Omega$ and $\|q_c\|_{L^\infty} < \infty$. Therefore :

$$\|e^{\frac{\psi}{h}}(-h^2\triangle_g)(e^{-\frac{\psi}{h}}v)\|_{L^2(g)} \gtrapprox h\|v\|_{L^2} + h^2\|Dv\|_{L^2} - h^2\|q_c c^{-1}\|_{L^\infty}\|v\|_{L^2}.$$

The claim will clearly follow for h small enough.

Let $P_\psi := e^{\frac{\psi}{h}}(-h^2\triangle_{\hat{g}})e^{-\frac{\psi}{h}} = A + B$ where A and B are the formally symmetric and anti-symmetric operators (in $L^2(\Omega_1, \hat{g})$):

$$A = -h^2\triangle_{\hat{g}} - 1$$

$$B = h(2\langle d\psi, d\cdot\rangle_{\hat{g}} + \triangle_{\hat{g}}\psi).$$

Hence:

$$\|P_\psi v\|^2_{L^2(\hat{g})} = \|Av\|^2_{L^2(\hat{g})} + \|Bv\|^2_{L^2(\hat{g})} + ([A,B]v,v)_{L^2(\hat{g})}.$$

Now note that:

$$[A,B] = -2h^3[\triangle_{\hat{g}}, \langle d\psi, d\cdot\rangle_{\hat{g}}] + h^3 X$$

where $X = f^i(x)\partial_i + h(x)$ is a smooth vector field.

Let us define the coordinate system (t, y_1, y_2) as follows: Define the normal vector field to the level sets of ψ and let the integral curves correspond to the coordinate t choosing $t = 0$ on one of these level sets. Furthermore let us consider smooth maps G_t to be smooth diffeomorphisms from the unit disk to the corresponding level set ψ_t smoothly depending on t. Note that in our coordinate system the pull back of the metric takes the following form :

$$g = dt \otimes dt + g_{\alpha\beta}(t,y)dy^\alpha \otimes dy^\beta.$$

Thus:

$$([A,B]v,v)_{L^2(\hat{g})} = -2h^3 \int \partial_t \hat{g}^{\alpha\beta}\partial_\alpha v \partial_\beta v + h^3 \int K(x)|v|^2.$$

Here, $K(x)$ denotes a continuous function on Ω_1. We now note that $-\partial_t \hat{g}^{\alpha\beta}$ denotes the inverse of the second fundamental form of the level sets of ψ with respect to the renormalized metric. Recall that if $\Gamma^{n-1} \subset M^n$ is an embedded nondegenerate hypersurface in M, then the second funamental form $h(X,Y)$ on Γ changes under conformal rescalings $\hat{g} = cg$ as follows:

$$\hat{h}(X,X) = \sqrt{c}(h(X,X) + \frac{1}{2}\frac{\nabla_N c}{c} g(X,X)).$$

Hence:

$$\hat{h}(X,X) = \sqrt{c}(D^2\psi(X,X) + D^2\psi(\nabla\psi,\nabla\psi)\frac{|X|^2}{|\nabla\psi|^2}).$$

Thus using the main assumption of the Lemma, we see that $-\partial_t \hat{g}^{\alpha\beta}$ is positive semi-definite and thus we can conclude that:

$$\|P_\psi v\|^2_{L^2(\hat{g})} \geq \|Av\|^2_{L^2(\hat{g})} + \|Bv\|^2_{L^2(\hat{g})} + ([A,B]v,v)_{L^2(\hat{g})}.$$

So:

$$\|P_\psi v\|^2_{L^2(\hat{g})} \geq \|Av\|^2_{L^2(\hat{g})} + \|Bv\|^2_{L^2(\hat{g})} + h^3 \int K(x)|v|^2. \; (*)$$

Note that:

$$Bv = h(2\langle d\psi, dv\rangle_{\hat{g}} + (\triangle_{\hat{g}}\psi)v) = h(2\partial_t v + (\triangle_{\hat{g}}\psi)v.)$$

The Poincare inequality implies that:

$$\|\partial_t v\|_{L^2(\Omega_1,\hat{g})} \geq C\|v\|_{L^2(\Omega_1,\hat{g})} \qquad \forall v \in H^1_0(\Omega_1).$$

Recall that the level sets of ψ are non-trapping since $d\psi \neq 0$ anywhere. Since we are working over a compact manifold we can use an integrating factor and use the Poincare inequality above to conclude that:

$$\|Bv\|_{L^2(\Omega_1,\hat{g})} \geq Ch\|v\|_{L^2(\Omega_1,\hat{g})} \qquad \forall v \in C^\infty_c(\Omega_1). \qquad (**)$$

Let us also observe that by integrating Av against $\delta h^2 v$ for some small δ independent of h we obtain the following estimate:

$$\|Av\|^2_{L^2(\hat{g})} \geq C\delta(h^4 \int |\nabla v|^2 - h^2 \int v^2). \qquad (***)$$

Combining (*),(**) and (***) yields the claim.

□

Corollary 3.2.7. *Let $\tilde{\phi}_0(x_1,x_2,x_3) = x_1\chi_0(x_3) + (F_\lambda \circ \omega)(x)$ as defined in the previous lemma with $k \geq 1$ arbitrary and λ sufficiently large and only depending on the domain (Ω_1, g) and on k. Then $\tilde{\phi}_0(x_1,x_2,x_3)$ is a Carleman weight in (Ω_1, g), that is to say there exists constants $h_0 > 0$ and $C(\Omega_1, g)$ such that the following estimate holds:*

$$\|e^{\frac{\tilde{\phi}_0}{h}} \triangle_g (e^{-\frac{\tilde{\phi}_0}{h}} v)\|_{L^2(\Omega_1)} \geq \frac{C}{h}\|v\|_{L^2(\Omega_1)} + C\|Dv\|_{L^2(\Omega_1)}$$

$\forall h \leq h_0$ and $v \in C^\infty_c(\Omega_1)$.

3.3 Complex Geometric Optics

In this section, we will utilize the above corollary to construct a family of solutions to the Schrödinger equation $(-\triangle_g + q)u = 0$. Before starting this construction let us remark that our complex exponential solutions will depend on fractional powers of the semiclassical symbol $\frac{1}{\tau}$. This is a new idea and the key reason for using this method is that it will unveil a new analytical method for reconstruction of potential that is different from the known geometric inversion methods for Euclidean geometries (conformally cylindrical manifolds) such as Radon transforms (geodesic ray transforms) [12] [5]. The arguments in the next section could be slightly simplified if we abandon these fractional powers. In that case in order to conclude the result we would need the local invertibility of Radon transform [13].

Let us start with the notion of complex exponential harmonic functions that concentrate (in some sense) on the plane $\Pi = \{x_3 = 0\}$ (see for example [12]). We recall that the plane Π is a fixed plane taken out of the foliation $\mathbb{A}$. Let $\epsilon(\tau, \beta) := \tau^{-\beta}$ for a fixed $0 < \beta < 1$. Choose $M > \frac{2}{\beta}$. We define $\Phi_\epsilon : V \to \mathbb{R}$ and $v_\epsilon : V \to \mathbb{R}$ as follows:

$$\Phi_\epsilon = \sum_{k=0}^{M} \Phi_k(x)\epsilon^k$$

$$v_\epsilon = \sum_{k=0}^{M} v_k(x)\epsilon^k$$

in such a way that:

$$\langle d\Phi_\epsilon, d\Phi_\epsilon \rangle_g = O(\epsilon^M) = O(\tau^{-2})$$

and:

$$2\langle d\Phi_\epsilon, dv_\epsilon \rangle_g + (\triangle_g \Phi_\epsilon)v_\epsilon = O(\epsilon^M) = O(\tau^{-2})$$

for $-t_1 \leq x_3 \leq t_2$. This is done through iterative determination of the coefficients as follows. We first choose Φ_0 and v_0:

$$\Phi_0 = z = x_1 + ix_2$$

$$v_0 = h(z)\chi(x_3)$$

where $h(z)$ is an arbitrary holomorphic function and χ is an arbitrary smooth function of compact support in $-t_1 \leq x_3 \leq t_2$. We impose the equations governing the terms Φ_k by requiring that:

$$4\bar{\partial}\Phi_k + \sum_{j=1}^{k-1} \langle d\Phi_{k-j}, d\Phi_j \rangle = 0 \qquad \forall k \leq M$$

since the metric is Euclidean for $-t_1 \leq x_3 \leq t_2$ we can in fact solve for an exact Φ_ϵ as follows:

$$\Phi_\epsilon = z + i\epsilon x_3 + \epsilon^2 \frac{\bar{z}}{4}.$$

Let us observe that $\Re(\Phi_\epsilon) = (1 + \frac{\epsilon^2}{4})x_1$. We will now rewrite the equations for v_ϵ :

$$2\bar{\partial}v_k + \sum_{j=1}^{2} \langle dv_{k-j}, d\Phi_j \rangle = 0.$$

Thus for $k \geq 1$:

$$2\bar{\partial}v_k + i\partial_3 v_{k-1} + \frac{1}{2}\partial v_{k-2} = 0 \qquad \forall k \leq M.$$

Let us now make a general remark about the form of v_k. Note that

$$v_0 = h(z)\chi(x_3)$$

and it is easy to see that we can take $v_1 = -\frac{i}{2}h(z)\chi'(x_3)\bar{z}$. In fact it is not hard to see using induction that: $v_k = (\frac{-i}{2})^k h(z)\chi^{(k)}(x_3)\frac{\bar{z}^k}{k!}$ modulo lower order terms in $\bar{z}$.More precisely it is possible to choose the v_k's such that:

$$v_k = \sum_{0}^{\lfloor \frac{k}{2} \rfloor} a_j \bar{z}^{k-j} h^{(j)}(z)\chi^{(k-2j)}(x_3) \qquad\qquad \forall k < M$$

with:

$$a_0 = (\frac{-i}{2})^k.$$

Definition 3.3.1. Let us define $\tilde{\Phi}_\epsilon : \Omega_1 \to \mathbb{C}$ through:

$$\tilde{\Phi}_\epsilon = \kappa(\tilde{\phi_0}(x) + i\tilde{\omega}(x) + i\epsilon\omega(x) + \epsilon^2(\frac{\tilde{\phi_0}(x) - i\tilde{\omega}(x)}{4})$$

Here, $\kappa = \frac{1}{1+\frac{\epsilon^2}{4}}$.

Notice that for $-t_1 \leq x_3 \leq t_2$ we have that $\tilde{\Phi}_\epsilon = \kappa\Phi_\epsilon$ and that $\Re(\tilde{\Phi}_\epsilon) = \tilde{\phi_0}$ for all $x \in \Omega_1$. Finally we note that for $-t_1 \leq x_3 \leq t_2$ we have:

$$\langle d\tilde{\Phi}_\epsilon, d\tilde{\Phi}_\epsilon \rangle_g = O(\epsilon^M) = O(\tau^{-2})$$

and:

$$2\langle d\tilde{\Phi}_\epsilon, dv_\epsilon \rangle_g + (\triangle_g \tilde{\Phi}_\epsilon)v_\epsilon = O(\epsilon^M) = O(\tau^{-2}).$$

Lemma 3.3.2. *Let $f \in L^2(\Omega_1, g)$. There exists a unique function $r := H_\tau f \in H^1(\Omega_1)$ such that $P_\tau r = e^{-\tau\tilde{\phi_0}}(\triangle_g - q_*)(e^{\tau\tilde{\phi_0}} r) = f$ on Ω_1 with r orthogonal (with respect to the L^2 inner product) to Σ. Here $\Sigma = \left\{ v \in L^2 : (\triangle_g - q_*)(e^{\tau\tilde{\phi_0}} v) = 0 \right\}$. Furthermore for τ large enough:*

$$\|H_\tau f\|_{L^2(\Omega_1)} \leq C\tau^{-1}\|f\|_{L^2(\Omega_1)}$$

where the constant C only depends on Ω_1.

Remark 3.3.3. This is a rather standard proof about deducing surjectivity for some operator T from the knowledge of injectivity and closed range for the adjoint operator T^*. We will closely follow the proofs provided in [17] and [18] here.

Proof. Let $P_\tau(\cdot) = e^{-\tau\tilde{\phi_0}}(\triangle_g - q_*)(e^{\tau\tilde{\phi_0}}\cdot)$ and define $\mathbb{D} = P_\tau^* C_c^\infty(\Omega_1)$ as a subspace of $L^2(\Omega_1)$. Here P_τ^* denotes the adjoint of P_τ with respect to the standard $L^2(\Omega_1)$ inner product. Consider the linear functional $L : \mathbb{D} \to \mathbb{C}$ through:

$$L(P_\tau^* v) = \langle v, f \rangle \qquad \forall v \in C_c^\infty(\Omega_1).$$

This is well-defined since any element of $\mathbb{D}$ has a unique representation as $P_\tau^* v$ with $v \in C_c^\infty(\Omega_1)$ by the Carleman estimate. Also using Cauchy-Schwarz and the Carleman estimate:

$$|L(P_\tau^* v)| \leq \|v\|_{L^2}\|f\|_{L^2} \leq C\tau^{-1}\|f\|_{L^2}\|P_\tau^* v\|.$$

for τ large enough with C depending only on Ω_1 and independent of the parameter β. Thus L is a bounded linear operator on $\mathbb{D}$. Extend L by continuity to the closure $\bar{\mathbb{D}}$ and finally extend to all of $L^2(\Omega_1)$ through projection operator $\pi_\tau : L^2(\Omega_1) \to \bar{\mathbb{D}}$. Thus we obtain a bounded linear operator $\hat{L} : L^2(\Omega_1) \to \mathbb{C}$ with $\hat{L}|_{\mathbb{D}} = L$. Furthermore:

$$\|\hat{L}\|_{L^2(\Omega_1)\to L^2(\Omega_1)} \leq C\tau^{-1}\|f\|_{L^2(\Omega_1)}.$$

Now by the Riesz representation therorem we deduce that there exists a unique $r \in L^2(\Omega_1)$ such that $\hat{L}(w) = \langle w, r\rangle \quad \forall w \in L^2(\Omega_1)$ and such that $\|r\| \leq C\tau^{-1}\|f\|_{L^2}$. Note that:

$$\langle v, P_\tau r\rangle = \langle P_\tau^* v, r\rangle = \hat{L}(P_\tau^* v) = L(P_\tau^* v) = \langle v, f\rangle.$$

Hence $P_\tau r = f$ in the weak sense and by construction we have r orthogonal to Σ.

□

Lemma 3.3.4. *Suppose $0 < \beta < 1$ is fixed and $\epsilon = \tau^{-\beta}$. There exists a family of functions u_ϵ solving $(-\triangle_g + q_*)u_\epsilon = 0$ on Ω_1 of the form*

$$u_\epsilon^0 = e^{\tau\tilde{\Phi}_\epsilon}(v_\epsilon + r_\epsilon^0)$$

where $\|r_\epsilon^0\|_{L^2(\Omega_1)} \leq \frac{C}{\tau}$ for τ large enough.

Proof. Let us first consider solving the equation

$$P_\tau r := e^{-\tau\tilde{\phi_0}}(\triangle_g - q_*)(e^{\tau\tilde{\phi_0}} r) = -e^{-\tau(\tilde{\phi_0} - \tilde{\Phi}_\epsilon)} e^{-\tau\tilde{\Phi}_\epsilon}(\triangle_g - q_*)(e^{\tau\tilde{\Phi}_\epsilon} v_\epsilon).$$

$$e^{-\tau\tilde{\Phi}_\epsilon}\triangle_g(e^{\tau\tilde{\Phi}_\epsilon} v_\epsilon) = \tau^2\langle d\tilde{\Phi}_\epsilon, d\tilde{\Phi}_\epsilon\rangle_g v_\epsilon + \tau[2\langle d\tilde{\Phi}_\epsilon, dv_\epsilon\rangle_g + (\triangle_g\tilde{\Phi}_\epsilon)v_\epsilon] + \triangle_g v_\epsilon.$$

Since v_ϵ is compactly supported in the region $-t_1 \leq x_3 \leq t_2$:

$$\langle d\tilde{\Phi}_\epsilon, d\tilde{\Phi}_\epsilon\rangle_g = O(\epsilon^M) = O(\tau^{-2})$$

and:

$$2\langle d\tilde{\Phi}_\epsilon, dv_\epsilon\rangle_g + (\triangle_g\tilde{\Phi}_\epsilon)v_\epsilon = O(\epsilon^M) = O(\tau^{-2}).$$

Hence we can immediately conclude that $\|e^{-\tau\tilde{\Phi}_\epsilon}(\triangle_g - q_*)(e^{\tau\tilde{\Phi}_\epsilon} v_\epsilon)\|_{L^2(\Omega_1)} \leq C$ for some constant C. Let

$$r = -H_\tau(e^{-\tau(\tilde{\phi_0} - \tilde{\Phi}_\epsilon)} e^{-\tau\tilde{\Phi}_\epsilon}(\triangle_g - q_*)(e^{\tau\tilde{\Phi}_\epsilon} v_\epsilon)).$$

Clearly $P_\tau r = -e^{-\tau(\tilde{\phi_0}-\tilde{\Phi}_\epsilon)}e^{-\tau\tilde{\Phi}_\epsilon}(\triangle_g - q_*)(e^{\tau\tilde{\Phi}_\epsilon}v_\epsilon)$ as desired.

Furthermore, since $\|e^{-\tau\tilde{\Phi}_\epsilon}(\triangle_g - q_*)e^{\tau\tilde{\Phi}_\epsilon}v_\epsilon\|_{L^2(\Omega_1)} \leq C$ and $\tilde{\phi_0} - \tilde{\Phi}_\epsilon$ is purely imaginary, we can use Lemma 3.3.2 to conclude that for τ sufficiently large: $\|r\|_{L^2(\Omega_1} \leq \frac{C}{\tau}$

We now choose $r_\epsilon^0 = e^{\tau(\tilde{\phi_0}-\tilde{\Phi}_\epsilon)}r$ to conclude the proof.

□

Lemma 3.3.5. *Let $q_1 \in L^\infty(\Omega_1)$. Suppose $0 < \beta < 1$ is fixed and $\epsilon = \tau^{-\beta}$. There exists a family of exact solutions u_ϵ^1 to $(-\triangle_g + q_1)u_\epsilon^1 = 0$ of the form $u_\epsilon^1 = e^{\tau\tilde{\Phi}_\epsilon}(v_\epsilon + r_\epsilon^1)$ where $\|r_\epsilon^1\|_{L^2(\Omega_1)} \leq \frac{C(\beta)}{\tau}$.*

Proof. Consider the equation:

$$-e^{-\tau\tilde{\phi_0}}\triangle_g(e^{\tau\tilde{\phi_0}}r) + q_1 r = -e^{-\tau(\tilde{\phi_0}-\tilde{\Phi}_\epsilon)}e^{-\tau\tilde{\Phi}_\epsilon}(-\triangle_g + q_1)(e^{\tau\tilde{\Phi}_\epsilon}v_\epsilon) =: f.$$

but since v_ϵ is compactly supported in $\{-t_1 \leq x_3 \leq t_2\}$:

$$e^{-\tau\tilde{\Phi}_\epsilon}\triangle_g(e^{\tau\tilde{\Phi}_\epsilon}v_\epsilon) = \tau^2\langle d\tilde{\Phi}_\epsilon, d\tilde{\Phi}_\epsilon\rangle_g v_\epsilon + \tau[2\langle d\tilde{\Phi}_\epsilon, dv_\epsilon\rangle_g + (\triangle_g\tilde{\Phi}_\epsilon)v_\epsilon] + \triangle_g v_\epsilon.$$

Since v_ϵ is compactly supported in the region $-t_1 \leq x_3 \leq t_2$:

$$\langle d\tilde{\Phi}_\epsilon, d\tilde{\Phi}_\epsilon\rangle_g = O(\epsilon^M) = O(\tau^{-2})$$

and:

$$2\langle d\tilde{\Phi}_\epsilon, dv_\epsilon\rangle_g + (\triangle_g\tilde{\Phi}_\epsilon)v_\epsilon = O(\epsilon^M) = O(\tau^{-2}).$$

Hence we can immediately conclude that $\|e^{-\tau\tilde{\Phi}_\epsilon}\triangle_g e^{\tau\tilde{\Phi}_\epsilon}v_\epsilon - qv_\epsilon\|_{L^2(\Omega_1)} \leq C$ for some constant C.

Motivated by Lemma 3.3.2 we make the ansatz $r = H_\tau\tilde{r}$ to obtain:

$$(-I + (q_1 - q_*)H_\tau)\tilde{r} = f.$$

But $H_\tau : L^2(\Omega_1) \to L^2(\Omega_1)$ is a contraction mapping for τ large enough with $\|H_\tau\| \leq \frac{C}{\tau}$ and thus for sufficiently large τ the inverse map $(-I + (q_1 - q_*)H_\tau)^{-1} : L^2(\Omega_1) \to L^2(\Omega_1)$ exists and it is given by the following infinite Neumann series:

$$(-I + (q_1 - q_*)H_\tau)^{-1} = -\sum_{j=0}^{\infty}(q_1 - q_*)(H_\tau)^j.$$

Hence:

$$\|(-I + (q_1 - q_*)H_\tau)^{-1}\|_{L^2(\Omega_1)\to L^2(\Omega_1)} \leq C.$$

So we deduce that if :

$$r = H_\tau(-I + (q_1 - q_*)H_\tau)^{-1} f.$$

then if we choose $r_\epsilon^1 = e^{\tau(\tilde{\phi_0} - \tilde{\Phi}_\epsilon)} r$ we have that $u_\epsilon^1 = e^{\tau\tilde{\Phi}_\epsilon}(v_\epsilon + r_\epsilon^1)$ solves $(-\triangle_g + q_1)u_\epsilon^1 = 0$ and furthermore:

$$\|r_\epsilon^1\|_{L^2(\Omega_1)} \leq \frac{C}{\tau}.$$

□

Definition 3.3.6. Let $\tilde{\psi_0}(x) = -x_1\chi_0(x) + (F_\lambda \circ \omega)(x)$. (Here, χ_0 and F_λ are the same functions as in Lemma 3.2.4)

Note that we have the following estimate as a result of Lemma 3.2.4:

$$\|e^{\frac{\tilde{\psi_0}}{h}} \triangle_g (e^{-\frac{\tilde{\psi_0}}{h}} v)\|_{L^2(\Omega_1)} \geq \frac{C}{h}\|v\|_{L^2(\Omega_1)} + C\|Dv\|_{L^2(\Omega_1)}$$

$\forall h \leq h_0$ and $v \in C_c^\infty(\Omega_1)$.

Thus we can state the following Lemma which is a direct parallel to Lemma 3.3.2:

Lemma 3.3.7. *Let $f \in L^2(\Omega_1, g)$. There exists a unique function $r := L_\tau f \in H^1(\Omega_1)$ such that $Q_\tau r := e^{-\tau\tilde{\psi_0}}(\triangle_g - q_*)(e^{\tau\tilde{\psi_0}} r) = f$ on Ω_1 with r orthogonal (with respect to the L^2 inner product) to $\tilde{\Sigma}$. Here $\tilde{\Sigma} = \left\{ v \in L^2 : \triangle_g(e^{\tau\tilde{\psi_0}} v) = 0 \right\}$. Furthermore for τ large enough, we have the estimate:*

$$\|L_\tau f\|_{L^2(\Omega_1)} \leq C\tau^{-1}\|f\|_{L^2(\Omega_1)}$$

where the constant C only depends on (Ω_1, g).

Definition 3.3.8. Let us define $\tilde{\Psi}_\epsilon : \Omega_1 \to \mathbb{C}$ through:

$$\tilde{\Psi}_\epsilon = \kappa(\tilde{\psi_0}(x) - i\tilde{\omega}(x) - i\epsilon\omega(x) + \epsilon^2(\frac{\tilde{\psi_0}(x) + i\tilde{\omega}(x)}{4})$$

Here, $\kappa = \frac{1}{1+\frac{\epsilon^2}{4}}$.

Notice that for $-t_1 \leq x_3 \leq t_2$ we have that $\tilde{\Psi}_\epsilon = -\kappa\Phi_\epsilon$ and that $\Re(\tilde{\Psi}_\epsilon) = -x_1$ for any $x \in \Omega_1$. Finally we note that for $-t_1 \leq x_3 \leq t_2$ we have:

$$\langle d\tilde{\Psi}_\epsilon, d\tilde{\Psi}_\epsilon \rangle_g = O(\epsilon^M) = O(\tau^{-2})$$

and:

$$2\langle d\tilde{\Psi}_\epsilon, dv_\epsilon \rangle_g + (\triangle_g \tilde{\Psi}_\epsilon)v_\epsilon = O(\epsilon^M) = O(\tau^{-2}).$$

Thus we can state the following corollary to Theorem 3.3.5:

Corollary 3.3.9. *Let $q_2 \in L^\infty(\Omega_1)$. Suppose $0 < \beta < 1$ is fixed and $\epsilon = \tau^{-\beta}$. There exists a family of exact solutions u_ϵ^2 to $(-\triangle_g + q_2)u_\epsilon^2 = 0$ of the form $u_\epsilon^2 = e^{\tau\tilde{\Psi}_\epsilon}(v_\epsilon + r_\epsilon^2)$ where $\|r_\epsilon^2\|_{L^2(\Omega_1)} \leq \frac{C(\beta)}{\tau}$.*

3.4 Proof of Uniqueness

Proof of Theorem 3.1.1. We start by assuming that q_1, q_2 are such that $\Lambda_{q_1} = \Lambda_{q_2}$. Choose an arbitrary $0 < \beta < 1$, set $\epsilon = \tau^{-\beta}$ and $M = \frac{2}{\beta}$. We use Green's identity pairing $u_\epsilon^1 = e^{\tau\tilde{\Phi}_\epsilon}(v_\epsilon + r_\epsilon^1)$ with $u_\epsilon^2 = e^{\tau\tilde{\Psi}_\epsilon}(v_\epsilon + r_\epsilon^2)$. Thus:

$$0 = \int_{\partial\Omega} u_\epsilon^1(\Lambda_{q_2} - \Lambda_{q_1})u_\epsilon^2 = \int_{\partial\Omega} u_\epsilon^1 \partial_\nu u_\epsilon^2 - \int_{\partial\Omega} u_\epsilon^2 \partial_\nu u_\epsilon^1 = \int_\Omega u_\epsilon^1 \triangle_g u_\epsilon^2 - \int_\Omega u_\epsilon^2 \triangle_g u_\epsilon^1.$$

Hence if we let $q = q_2 - q_1$ and invoke the equations $(-\triangle_g + q_i)u_\epsilon^i = 0$ for $i \in \{1, 2\}$ we have:

$$0 = \int_\Omega q u_\epsilon^1 u_\epsilon^2 = \int_V q u_\epsilon^1 u_\epsilon^2 = \int_V q(v_\epsilon + r_\epsilon^1)(v_\epsilon + r_\epsilon^2).$$

Thus using the Cauchy-Schwarz inequality and the fact that $\|r_\epsilon^i\|_{L^2(\Omega_1)} \leq \frac{C}{\tau}$ for τ large and for $i \in \{1, 2\}$ we have:

$$0 = \int_V q v_\epsilon^2 + O(\frac{1}{\tau}).$$

Let us recall that $v_\epsilon(x) = \sum_{k=0}^M v_k(x)\epsilon^k$ and in fact we have the following formulas:

$$v_k = \sum_0^{\lfloor \frac{k}{2} \rfloor} a_j \bar{z}^{k-j} h^{(j)}(z) \chi^{(k-2j)}(x_3) \qquad \forall k < M$$

with:

$$a_0 = (\frac{-i}{2})^k.$$

Note that $\int_\Omega q v_\epsilon^2$ is a polynomial of degree $2M$ in ϵ and therefore we can conclude that by taking the limit as $\tau \to \infty$ we can determine the coefficients of this polynomial up to the term with coefficient $\epsilon^{\frac{M}{2}}$. Taking note of the particular form of v_k's and focusing on the coefficient of ϵ^k for any $k < \frac{M}{2}$ we claim that the knowledge of $\Lambda_{q_1} = \Lambda_{q_2}$ yields the following integral data on Π:

$$0 = \int_\Pi (\partial_3^k q)\bar{z}^k h(z) \qquad\qquad 0 \le k \le \frac{M}{2}.$$

We will prove this using induction on k. Note that by looking at the coefficient of ϵ^0 in $\int_\Omega q v_\epsilon^2$ we obtain the knowledge of $\int_\Pi q h(z)^2$.By a simple perturbation argument and choosing $h(z) \to 1 + \eta h(z)$ with $\eta \to 0$ we arrive at the following information on the surface Π:

$$0 = \int_\Pi q h(z).$$

Let us assume that the claim holds for all $j \le k-1$. We will now prove the claim for $j = k$. Indeed note that the coefficient of ϵ^k in $\int_\Omega q v_\epsilon^2$ is equal to:

$$\int_\Omega q(\sum_{s=0}^{k} v_{k-s} v_s).$$

Let us choose $\chi(x_3) = t^{\frac{1}{2}} \chi_0(t x_3)$ where χ_0 is a non-negative smooth function with compact support with $\int |\chi_0|^2 dx_3 = 1$. Hence the coefficient of ϵ^k in $\int_\Omega q v_\epsilon^2$ is equal to:

$$\int_\Omega q h(z)^2 (\frac{-i}{2})^k \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \chi^{(j)}(x_3) \chi^{(k-j)}(x_3) \bar{z}^k + R.$$

and since

$$\sum_{j=0}^{k} \binom{k}{j} \chi^{(j)}(x_3) \chi^{(k-j)}(x_3) = (\chi^2)^{(k)}(x_3)$$

by taking the limit $t \to \infty$ and using the induction assumption we obtain the following data on Π:

$$0 = \int_\Pi (\partial_3^k q) \bar{z}^k h(z)^2$$

for all $k \geq 0$ and all holomorphic functions $h(z)$. By a simple perturbation argument and choosing $h(z) \rightarrow 1 + \eta h(z)$ with $\eta \rightarrow 0$ we arrive at the following information on the surface Π:

$$0 = \int_{\Pi} (\partial_3^k q) \bar{z}^k h(z)$$

for all $k \geq 0$. Finally by integrating this data in x_3 we obtain the knowledge of:

$$0 = \int_{\Pi} q \bar{z}^k h(z)$$

for all $0 \leq k \leq \frac{M}{2}$. Since M can be chosen to be as large as we require this yields the following data on Π:

$$0 = \int_{\Pi} q \bar{z}^k h(z)$$

for all $k \geq 0$.

As a final step we note that the subalgebra $\mathbb{A}$ generated by $\{z^k \bar{z}^j\}$ is separable and unital and therefore by Stone-Weistress theorem we deduce that $q|_{\Pi} = 0$

□

3.5 Reconstruction Algorithm

We next indicate how one may be able to make the above uniqueness proof constructive. In other words we would like to sketch out a reconstruction algorithm for q from the knowledge of Λ_q. One can immediately observe that the key in accomplishing this would be to construct special solutions u_ϵ^1 to $(-\triangle_g + q)u_\epsilon^1 = 0$ as before and show that we can determine the trace of these solutions on $\partial\Omega$ from the knowledge of Dirichlet to Neumann map Λ_q. Let us make a few remarks about the approach here. We will be closely following the approach in [16] and [17] for the boundary determination of special solutions to the Schrödinger equation but we have to make some fundamental changes as we are in a geometry where there is no global limiting Carleman weight. As explained in chapter 2, this issue appears to be a key limitation in the determination of boundary values of complex exponential solutions to Schrödinger equation from the DN map and thus we have to make adjustments to the existing arguments here. This will also be the key reason on why we have to assume that q is supported in U.

Let us recall that H_τ denotes the solution operator to

$$P_\tau v = e^{-\tau\tilde{\phi}_0}(\triangle_g - q_*)(e^{\tau\tilde{\phi}_0}v) = F.$$

Let $h_\tau(x;y)$ denote the kernel of H_τ. Thus:

$$P_\tau v = F \iff v(x) = \int_{\Omega_1} h_\tau(x;y)F(y)d\mu_y.$$

We will state the following theorem which will help us better understand the regularity of the solutions obtained through the above kernel. In particular the following lemma shows that $H_\tau : L^2(\Omega_1) \to H^2(\Omega_1)$ and that $H_\tau^* : L^2(\Omega_1) \to H^2(\Omega_1)$.

Lemma 3.5.1. *Let $f \in H^{-2}(\Omega_1)$. There exists a unique weak solution $\tilde{r} \in H_0^2(\Omega_1)$ to:*

$$P_\tau P_{-\tau}\tilde{r} = f$$

Furthermore $H_\tau f = P_{-\tau}\tilde{r}$.

Proof. Note that $P_\tau P_{-\tau}\tilde{r} = f$ implies that $(\triangle_g - q_*)(e^{2\tau\tilde{\phi}_0}(\triangle_g - q_*)(e^{-\tau\tilde{\phi}_0}\tilde{r})) = e^{\tau\tilde{\phi}_0}f = \hat{f}$. Let $\hat{r} = e^{-\tau\tilde{\phi}_0}\tilde{r}$. Then it suffices to show that there exists a unique $H_0^2(\Omega_1)$ solution to:

$$(\triangle_g - q_*)(e^{2\tau\tilde{\phi}_0}(\triangle_g - q_*)\hat{r}) = \hat{f}.$$

Note that weak solvability implies that:

$$B(\hat{r}, v) = \int_{\Omega_1} e^{2\tau\tilde{\phi}_0}((\triangle_g - q_*)\hat{r})((\triangle_g - q_*)v) = \int_{\Omega_1} \hat{f}v$$

for all $v \in H_0^2(\Omega_1)$. Note that B is a bilinear bounded operator on $H_0^2(\Omega_1) \times H_0^2(\Omega_1)$ and furthermore using the Poincare and Young inequalities we can obtain the coercivity estimate as well. Thus a simple application of the Lax-Milgram lemma yields the unique solvability of $P_\tau P_{-\tau}\tilde{r} = f$ in $H_0^2(\Omega_1)$. To see that $H_\tau f = P_{-\tau}\tilde{r}$ we note that for any $w \in \Sigma = \left\{ v \in L^2 : (\triangle_g - q_*)(e^{\tau\tilde{\phi}_0}v) = 0 \right\}$ we have:

$$(P_{-\tau}\tilde{r}, w)_{L^2} = 0$$

□

Let $l_\tau(x;y)$ denote the kernel of L_τ (see Lemma 3.3.7). Thus:

$$Q_\tau v = F \iff v(x) = \int_{\Omega_1} l_\tau(x;y)F(y)d\mu_y.$$

Let us now define a new right inverse for the operator $P_\tau = e^{-\tau\tilde{\phi_0}}(\triangle_g - q_*)(e^{\tau\tilde{\phi_0}}\cdot)$ through the formula

$$K_\tau = H_\tau + \pi_\tau L_\tau^*$$

Here $\pi_\tau : L^2(\Omega_1) \to L^2(\Omega_1)$ denotes the orthogonal projection operator of $L^2(\Omega_1)$ functions onto the set:

$$\Sigma = \left\{ v \in L^2 : (\triangle_g - q_*)(e^{\tau\tilde{\phi_0}}v) = 0 \right\}.$$

One can indeed show that

$$\pi_\tau : H^k(\Omega_1) \to H^k(\Omega_1)$$

For a more in depth analysis of this operator we refer the reader to [2] and [17].

Lemma 3.5.2.

$$K_\tau : L^2(\Omega_1) \to H^2(\Omega_1)$$
$$\|K_\tau\|_{L^2(\Omega_1)\to L^2(\Omega_1)} \leq \frac{C}{\tau}.$$

Proof. This is a direct consequence of Lemma 3.5.1. Note that:

$$K_\tau = H_\tau + \pi_\tau L_\tau^*.$$

Recall that $L_\tau : H^{-2}(\Omega_1) \to L^2(\Omega_1)$. By duality this implies that $L_\tau^* : L^2(\Omega_1) \to H^2(\Omega_1)$. Furthermore elliptic regularity implies that $H_\tau : L^2(\Omega_1) \to H^2(\Omega_1)$ and $\pi_\tau : H^2(\Omega_1) \to H^2(\Omega_1)$. Also note that $\|\pi_\tau\|_{L^2\to L^2} = 1$, $\|L_\tau^*\|_{L^2\to L^2} = \|L_\tau\|_{L^2\to L^2} \leq \frac{C}{\tau}$. The norm estimate follows immediately. □

We will now state a lemma that will be key in accomplishing the boundary determination of special solutions to Schrödinger equation.

Lemma 3.5.3.

$$K_\tau P_\tau v = v$$

for all $v \in C_c^\infty(V)$.

Proof.

$$\int_V k_\tau(x,y) P_\tau v(y) d\mu(y) = \int_V (P_{-\tau} k_\tau(x,y)) v(y) d\mu(y).$$

Now note that $K_\tau = H_\tau + \pi_\tau L_\tau^*$. Also note that $P_{-\tau} = Q_\tau$ for $y \in V$. Hence:

$$P_{-\tau} K_\tau(x,y) = P_{-\tau} h_\tau(x,y) + \pi_\tau(x,y).$$

Thus:

$$\int_V g_\tau(x,y) P_\tau v(y) d\mu(y) = \int_V h_\tau(x,y) P_\tau v(y) d\mu(y) + \int_V \pi_\tau(x,y) v(y) d\mu(y).$$

Let us note that for any $w \in L^2(\Omega_1)$ we have $w = (1-\pi_\tau)w + \pi_\tau w = P_{-\tau}\tilde{w} + \pi_\tau w$ for some $\tilde{w} \in H_0^2(\Omega_1)$ Hence:

$$\int_V h_\tau(x,y) P_\tau v(y) d\mu(y) = \int_{\Omega_1} h_\tau(x,y) P_\tau v(y) d\mu(y) = \int_{\Omega_1} h_\tau(x,y) P_\tau P_{-\tau} \tilde{v}(y) d\mu(y) = P_{-\tau}\tilde{v}(x).$$

Also:

$$\int_V \pi_\tau(x,y) v(y) d\mu(y) = \int_{\Omega_1} \pi_\tau(x,y) v(y) d\mu(y) = \pi_\tau v.$$

Hence:

$$\int_V k_\tau(x,y) P_\tau v(y) d\mu(y) = (P_{-\tau}\tilde{v})(x) + (\pi_\tau v)(x) = v(x).$$

□

Let us now define the operator $T_q : L^2(\Omega) \to H^1(\Omega_1)$ through:

$$T_q(F)(x) := \int_\Omega k_\tau(x;y)(q(y) - q_*(y)) F(y) d\mu_y.$$

Note that the Carleman estimate in the previous section yields that $T_q : L^2(\Omega) \to L^2(\Omega_1)$ is a contraction mapping with $\|T_q\| \leq \frac{C}{\tau}$ for τ large enough.Let $a_0 \in H^1(\Omega_1)$ be a solution to $P_\tau a_0 = 0$. Let us consider the integral equation

$$a_1(x) = a_0(x) + \int_{\Omega_1} k_\tau(x;y)(q(y) - q_*(y)) a_1(y) d\mu_y.$$

This integral equation has a unique solution $a_1 \in H^2(\Omega_1)$. Indeed the integral equation is equivalent to:

$$a_1 = a_0 + K_\tau((q - q_*)a_1).$$

but since $\|K_\tau\|_{L^2(\Omega_1)\to L^2(\Omega_1)} \leq \frac{C}{\tau}$ we see that $(I - K_\tau(q - q_*)) : L^2(\Omega_1) \to L^2(\Omega_1)$ is invertible with an explicit inverse in terms of a Neumann series and therefore $a_1 = (I - K_\tau(q - q_*))^{-1}a_0$. Furthermore it is clear that $a_1 \in H^2(\Omega_1)$.

Let $G_\tau(x;y) = e^{\tau\tilde{\phi_0}(x)}k_\tau(x;y)e^{-\tau\tilde{\phi_0}(y)}$ and $u_i = e^{\tau\tilde{\phi_0}}a_i$ for $i \in \{0, 1\}$

$$u_1(x) = u_0(x) + \int_\Omega G_\tau(x;y)(q(y) - q_*(y))u_1(y)d\mu_y.$$

Hence since $(-\triangle_g + q)u_1 = 0$ we have:

$$u_1(x) = u_0(x) + \int_V G_\tau(x;y)(\triangle_g - q_*)u_1(y)d\mu_y.$$

Note that $y \in V$ where V denotes the support of q. So for $x \in \Omega_1 \setminus \bar{\Omega}$ Green Identity implies that:

$$u_1(x) = u_0(x) + \int_{\partial V} G_\tau(x;y)\partial_\nu u_1(y)d\mu_y - \int_{\partial V} \partial_\nu G_\tau(x;y)u_1(y)d\mu_y + \int_V (\triangle_g - q_*)G_\tau(x;y))u_1(y)d\mu_y.$$

Recall that Lemma 3.5.3 implies that for $x \neq y$ and $y \in V$ we have $(\triangle_g - q_*)G_\tau(x;y) = 0$ (differentiation with respect to the y variable). Hence:

$$u_1(x) = u_0(x) + \int_{\partial V} G_\tau(x;y)\Lambda_q u_1(y)d\mu_y - \int_{\partial V} \partial_\nu G_\tau(x;y)u_1(y)d\mu_y.$$

For any $f \in H^{\frac{1}{2}}(\partial\Omega)$ let $R_q f \in H^1(\Omega)$ denote the solution operator to $(-\triangle_g + q)u = 0$ with $u|_{\partial\Omega} = f$. This solution exists and is unique since 0 is not a Dirichlet eigenvalue of $-\triangle_g + q$. Take $x \in \Omega_1 \setminus \bar{\Omega}$ and let $f := u_1|_{\partial\Omega}$. Then:

$$\int_{\partial V} G_\tau(x;y)\Lambda_q u_1(y)d\mu_y - \int_{\partial V} \partial_\nu G_\tau(x;y)u_1(y) = e^{\tau\tilde{\phi_0}(x)}T_q[e^{-\tau\tilde{\phi_0}}R_q(f)](x) = u_1(x) - u_0(x).$$

Define $\Gamma_\tau : H^{\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ through:

$$\Gamma_\tau f = Tr \circ [e^{\tau\tilde{\phi_0}} T_q[e^{-\tau\tilde{\phi_0}} R_q(f)]].$$

We will now state a unique continuation lemma that is a key step in determining the boundary values of special solutions.

Lemma 3.5.4. *For any $f \in H^{\frac{1}{2}}(\partial\Omega)$, $\Gamma_\tau f$ is known from the knowledge of Λ_q.*

Proof. It suffices to show that for any $f \in H^{\frac{1}{2}}(\partial\Omega)$ we can determine $\int_{\partial V} G_\tau(x;y)(\Lambda_q f)(y)d\mu_y - \int_{\partial V} \partial_\nu G_\tau(x;y)f(y)$ from the knowledge of Λ_q. We need to define some notation. Note that Π divides the manifold into two connected submanifolds. We will call the one intersecting the set $\{x_3 > 0\}$ to be W_u and the one intersecting the set $\{x_3 < 0\}$ to be W_l. Let us consider the manifold $W_u \setminus V$. Note that $\triangle_g(R_q f) = 0$ in $W_u \setminus V$. Furthermore we know the Dirichlet and Neumann data for $R_q f$ on Ω from Γ_q. We will now proceed to show that this "exterior" pde has a unique solution and thus conclude that $R_q f|_{\partial V}$ and $\partial_\nu R_q f|_{\partial V}$ can be determined from Λ_q. Indeed consider the pde $(\triangle_g - q_*)u = 0$ in $W_u \setminus V$ with $u|_{\partial\Omega\cap W_u} = f|_{\partial\Omega\cap W_u}$ and $\partial_\nu u|_{\partial\Omega\cap W_u} = \Lambda_q f|_{\partial\Omega\cap W_u}$. Suppose there are two solutions u_1, u_2 to this pde and consider $v = u_1 - u_2$. Since the metric g is Euclidean in a small neighborhood of V we can use CauchyKowalevski theorem to conclude that $v \equiv 0$ in a neighborhood of V. We can then use a unique continuation theorem for a second order elliptic operator to conclude that $v \equiv 0$ in $W_u \setminus V$ (See for instance[22]).

□

Observe that we have obtained the following boundary integral equation:

$$(\mathbf{I} - \mathbf{\Gamma}_\tau)(\mathbf{u_1}(\mathbf{x})|_{\partial\mathbf{\Omega}}) = \mathbf{u_0}(\mathbf{x})|_{\partial\mathbf{\Omega}}.$$

Thus in order to determine the boundary values of our special solutions to the Schrödinger equation it suffices to show that we can uniquely solve the above boundary integral equation. This will be accomplished through the following theorem.

Lemma 3.5.5. $\Gamma_\tau : H^{\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ *is compact. Furthermore* $\mathbb{N}(I - \Gamma_\tau) = \emptyset$.

Proof.

$$\Gamma_\tau = Tr \circ [e^{\tau\tilde{\phi_0}(x)} T_q [e^{-\tau\tilde{\phi_0}} R_q(f)]]$$

Note that $R_q : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ is a bounded linear operator.Secondly, $H^1 \subset\subset L^2$ and T_q is a bounded operator from L^2 to H^1. Furthermore $Tr : H^1(\Omega_1 \setminus \Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is bounded. Hence Γ_τ is compact.

To prove that the kernel is empty, let us suppose that $(I - \Gamma_\tau)f = 0$ Then: $\Gamma_\tau f = f$. Let $\rho = T_q(e^{-\tau\tilde{\phi_0}} R_q(f))$. Then:

$$\rho(x) = \int_\Omega k_\tau(x;y) e^{-\tau\tilde{\phi_0}(y)} (q(y) - q_*(y)) R_q(f)(y) d\mu_y.$$

Then:

$$e^{\tau\tilde{\phi_0}(x)} \rho(x) = \int_\Omega G_\tau(x;y)(q(y) - q_*(y)) R_q(f)(y) d\mu_y.$$

Hence:

$$(\triangle_g - q_*)(e^{\tau\tilde{\phi_0}} \rho) = (q - q_*) R_q(f) = (\triangle_g - q_*)(R_q(f)).$$

But by hypothesis we have that $\Gamma_\tau(f) = f$ so $e^{\tau\tilde{\phi_0}} \rho|_{\partial\Omega} = f$ Hence $e^{\tau\tilde{\phi_0}(x)} \rho(x) = R_q(f)(x)$ on Ω so:

$$\rho = T_q(\rho).$$

Finally since T_q is a contraction mapping for τ large enough, we deduce that $\rho = 0$ everywhere. Hence $f = 0$. ☐

3.6 Further Results

In this section we will state a few theorems which generalize Theorem 3.1.1. For the sake of brevity we will only indicate the key differences of the proofs to that of Theorem 3.1.1. As a first step in generalizing Theorem 3.1.1 recall that we had assumed Γ to be connected. This assumption can be relaxed significantly. Indeed one can strengthen the result by allowing Γ to have several components. In this case a more delicate Carleman weight needs to be constructed. This will be the main content of Theorem 3.6.1.

Finally we will seek to generalize the result further by assuming that the metric restricted to the subdomain U is conformally transversally anisotropic. These are manifolds (U, g) where $U = \mathbb{R} \times U_0$ and:

$$g = c(x)(dx_1^2 + g_0(x'))$$

Here, g_0 denotes the induced metric on the transversal submanifold U_0 which is independent of x_1. Indeed recall that a key step in proving Theorem 3.1.1 is the construction of a global phase function which is locally a limiting Carleman weight in U. In [5] it is shown that local existence of limiting Carleman weights restricts the geometry to CTA geometries. Hence it would seem natural to expect Theorem 3.1.1 to have a generalization in this setting. This will be the content of Theorem 3.6.5.

Theorem 3.6.1. *Let (Ω^3, g) denote a compact smooth Riemannian manifold with smooth boundary. Let $U \subset \Omega$ be an open subset such that $\Gamma = U \cap \partial\Omega$ is non-empty, and strictly convex. Let $\Gamma = \cup_{i=1}^{l} \Gamma^i$ where Γ^i denotes the connected components of Γ. Let us assume that U can be covered with coordinate charts in which $g|_U$ is the Euclidean metric. Let U^i denote the convex hull of Γ_i and let us assume that $U^i \cap U^j = \emptyset$ $\forall i, j$ and that $U = \cup_{i=1}^{l} U^i$. Suppose q is a smooth function and that $q - q_*$ is compactly supported in U where q_* is a globally known smooth function. Then the knowledge of Λ_q will uniquely determine q.*

Indeed one can see that the key in establishing this theorem is proving a similar Carleman estimate to that of Lemma 3.2.4. The rest of the techniques in the paper including the CGO solutions with fractionals powers of $\frac{1}{\tau}$ and the reconstruction algorithms would be exactly as before. Let us now give a sketch of how one can prove a Carleman estimate in this setting.

First, Let us extend the manifold Ω to a slightly larger manifold Ω_1. We extend q to all of Ω_1 by setting it equal to zero outside Ω and extend g smoothly to Ω_1 such that $g|_{U_1}$ is Euclidean. Here U_1 denotes the extension of convex hull of Γ to the larger manifold Ω_1.

Note that $U_1 = \cup_1^l \mathbb{A}_i$ where $\mathbb{A}_i$ is a foliation by a family of planes $\mathbb{A}_i = \{\Pi_t^i\}_{t\in I}$. We start by taking a fixed family of planes $\Pi^i \in \mathbb{A}_i$ for all $1 \le i \le l$. A local coordinate system (x_1^i, x_2^i, x_3^i) can be constructed in each U_1^i such that $\Pi^i = \{x_3^i = 0\}$ with (x_1^i, x_2^i) denoting the usual Euclidean coordinate system on the plane Π^i and x_3^i denoting the normal flow to this plane. We can assume that within each component U^i the support of q lies in the compact set $V^i \subset\subset \{-t_1^i < x_3^i < t_2^i\}$ with $t_1^i, t_2^i > 0$. In this framework $U_1^i = \cup_{c=-t_1^i-\delta_1^i}^{c=t_2^i+\delta_2^i} \{x_3^i = c\}$ with $\delta > 0$. Let $\omega : \Omega_1 \to \mathbb{R}$ be any smooth function such that $d\omega \neq 0$ everywhere and $\omega(x) \equiv x_3^i$ for $x \in U_1^i$ for each $1 \le i \le l$. The existence of such ω is proved in Lemma 3.6.3. Let us define two globally defined $C^k(\overline{\Omega_1})$ functions $\chi_0 : \Omega_1 \to \mathbb{R}$ and $F_\lambda(x) : \mathbb{R} \to \mathbb{R}$ as follows:

$$\chi_0(x) = \begin{Bmatrix} 1, & \text{for } -t_1^i < x_3^i < t_2^i \\ (1 - (\frac{x_3^i - t_2^i}{\delta_2^i})^{8k})^k, & \text{for } t_2^i \le x_3^i \le t_2^i + \delta_2^i \\ (1 - (\frac{x_3^i + t_1^i}{\delta_1^i})^{8k})^k, & \text{for } -t_1^i - \delta_1^i \le x_3^i \le -t_1^i \\ 0 & \text{otherwise} \end{Bmatrix}$$

$$F_\lambda(x) = \begin{Bmatrix} 0, & \text{for } -t_1^i < x < t_2^i \\ e^{\lambda(\frac{x-t_2^i}{\delta_2})^2}(\frac{x-t_2^i}{\delta_2^i})^{2k}, & \text{for } t_2^i \le x \\ e^{\lambda(\frac{x+t_1^i}{\delta_1})^2}(\frac{x+t_1^i}{\delta_1^i})^{2k}, & \text{for } x \le -t_1^i \end{Bmatrix}$$

Motivated by Lemma 3.2.4 we will construct a global Carleman weight in Ω_1 in such a way that the phase function restricted to the support of q is equivalent to a limiting Carleman weight. In the transition regions we will use a convexification technique in order to make sure that the Hörmander hypo-ellipticity condition holds.

Lemma 3.6.2. *Let $\tilde{\phi}_0(x_1, x_2, x_3) = x_1^i \chi_0(x) + (F_\lambda \circ \omega)(x)$ where $k \ge 1$ is an arbitraty integer and $\lambda(\Omega_1, k, ||g_{ij}||_{C^2})$ is sufficiently large. Then the Hörmander hypo-ellipticity condition is satisfied in Ω_1, that is to say:*

$$D^2\tilde{\phi}_0(X, X) + D^2\tilde{\phi}_0(\nabla\tilde{\phi}_0, \nabla\tilde{\phi}_0) \ge 0$$

whenever $|X| = |\nabla\tilde{\phi}_0|$ and $\langle \nabla\tilde{\phi}_0, X\rangle = 0$.

Indeed a detailed look at the proof of Lemma 3.2.4 suggests that the key to proving Lemma 3.6.2 would be the existence of the function ω:

Lemma 3.6.3. *There exists a smooth function $\omega : \Omega_1 \to \mathbb{R}$ such that $d\omega \neq 0$ everywhere and $\omega(x) \equiv x_3^i$ for $x \in U_1^i$ for each $1 \leq i \leq l$.*

Before proving Lemma 3.6.3 let us recall Morse Lemma. This will be the key ingredient of the proof. Heuristically the idea is to start with an arbitrary smooth $\omega_0(x) \equiv x_3^i$ for $x \in U_1^i$ for each $1 \leq i \leq l$ and then pull out all the critical points to reach at the desired function ω.

Lemma 3.6.4 (Morse Lemma)**.** *Let b be a non-degenerate critical point of $f : \Omega_1 \to \mathbb{R}$. Then there exists a chart (x_1, x_2, x_3) in a neighborhood of b such that*

$$f(x) = f(b) - x_1^2 - x_2^2 - ... - x_\alpha^2 + x_{\alpha+1}^2 + ... + x_n^2$$

Here α is equal to the index of f at b.

Proof of Lemma 3.6.3. Define $\omega_0 : \Omega_1 \to \mathbb{R}$ such that $\omega_0(x) \equiv x_3^i$ for $x \in U_1^i$ for each $1 \leq i \leq l$. Let us remind the reader that the (x_1^i, x_2^i, x_3^i) are essentially the locally well defined Fermi coordinates near the planes Π^i. We know that a Generic smooth function is Morse and therefore it has isolated critical points. Thus by using a small C^∞ purturbation we can find a smooth function $\omega_1(x)$ such that $\omega_1(x) \equiv x_3^i$ for $x \in U_1^i$ for each $1 \leq i \leq l$ and $\omega_1(x)$ has isolated critical points and thus by compactness a finite number of isolated critical points b_k for $1 \leq k \leq L$. We will assume without loss of geneality that the index of these critical points is zero.

Since $\dim \Omega_1 = 3 > 2$, we can connect these critical points with points just outside the boundary by a family of disjoint paths that do not intersect U. We will denote these curves by γ_k.

Let V_k denote the neighborhood around b_k for which the Morse lemma holds. Choose h small enough such that the geodesic ball of radius h around b_k is inside V_k namely $B_{b_k}(h) \subset V_k$. Take

$$\omega_2(x) = \omega_1(x) + \epsilon(\alpha x_1 + \beta x_2 + \lambda x_3)\eta_k(x)$$

where η_k is a smooth function compactly supported in V_k and such that $\eta_k \equiv 1$ in the ball $B_{b_k}(\frac{h}{2})$. It is clear that for ϵ small enough we still have that $\omega_2(x) \equiv x_3^i$ for $x \in U_1^i$ for each $1 \leq i \leq l$. Furthermore we can see that for ϵ small enough the critical points of ω_2 outside V_k will remain the same and the critical point of ω_2 inside V_k must be in the ball $B_{b_k}(\frac{h}{2})$. Hence the critical point in V_k will 'move' from b_k to the point $(x_1, x_2, x_3) = (\frac{\epsilon\alpha}{2}, \frac{\epsilon\beta}{2}, \frac{\epsilon\lambda}{2})$. Since Ω_1 is compact, it is clear that we can 'move' the critcal points b_k along their respective curves γ_k and essentially construct a smooth function ω with $\omega(x) \equiv x_3^i$ for $x \in U_1^i$ for each $1 \leq i \leq l$ and such that $|d\omega|_g \neq 0$ anywhere in Ω_1.

□

With the proof of Lemma 3.6.3 complete we can deduce easily that Lemma 3.6.2 must also hold. This in turn implies that we can construct CGO solutions with fractional powers in the semiclassical parameter $\frac{1}{\tau}$ as in the previous section concentrating on the planes Π^i. One can also use the techniques in the previous sections to obtain the trace of these CGO solutions on the boundary through obtaining a Fredholm type boundary integral equation and thus give a reconstruction.

Let us now discuss the generalization to CTA geometries. In a sense this is the most general statement one could hope for, given the present tools in this paper. Before stating the Theorem, let us explain some notions. We will assume that $\Omega = I \times \Omega_0$ where $I = [a, b]$ is a compact interval. Let $U_0 \subset \Omega_0$ and define $U = I \times U_0$. Suppose that $(U, g|_U)$ is a CTA geometry, that is to say there exists a coordinate chart such that $g|_U = c(x)(dx_1^2 + g_0(x'))$. We have the following:

Theorem 3.6.5. *Let (Ω^3, g) denote a compact smooth Riemannian manifold with smooth boundary with $\Omega = I \times \Omega_0$. Let U_0 be an open subset such that $\Gamma_0 = U_0 \cap \partial\Omega_0$ is non-empty, strictly convex and that U_0 is the convex hull of Γ_0. Suppose that $(\partial I \times U_0) \cup (I \times \partial U_0)$ is connected, and let $U = I \times U_0$. Suppose q is a smooth function that is explicitly known in U^c, and that U can be covered with a coordinate chart in which $(U, g|_U)$ is conformally transversally anisotropic. Then the knowledge of Λ_q will uniquely determine q in U provided that U_0 is simple and the geodesic transform on Ω_0 is locally injective.*

Remark 3.6.6. The proof of this theorem is not given in its entirety as there will be several overlaps with the Euclidean case. In particular we will prove the Carleman estimate and construct the CGO solutions. We will also show how the CGO solutions will yield the uniqueness of potential. We will however omit the boundary reconstruction algorithm of the CGO solutions as that will be exactly as in the Euclidean case.

Remark 3.6.7. This proof can in turn be extended as in the Euclidean case to potentials known outside a multiply connected region to provide a generalization of Theorem 3.6.1.

Let us first observe that the Laplace operator in three dimensions transforms under the following law for conformal rescalings of the metric:

$$\triangle_{fg} u = f^{-\frac{5}{4}} (\triangle_g + q_f)(f^{\frac{1}{4}} u)$$

$$q_f = f^{\frac{1}{4}} \triangle_{fg} f^{-\frac{1}{4}}$$

Using this, we see that without loss of generality we can assume that in U the metric g takes the following form:

$$g = dx_1^2 + g_0(x')$$

In other words, without loss of generality we can assume $c(x) \equiv 1$. We will also assume without loss of generality that $q_* \equiv 0$.

First, Let us extend the manifold Ω_0 to a slightly larger manifold Ω_{01} and let $\Omega_1 = I \times \Omega_{01}$. We extend q to all of Ω_1 by setting it equal to zero outside Ω and extend g smoothly to Ω_1 such that $g|_{U_1} = dx_1^2 + g_0(x')$. Here $U_1 = I \times U_{10}$ denotes the extension of U to the larger manifold Ω_1. Set $\Gamma_1 = U_1 \cap \partial\Omega_{01}$ and note that U_{10} is the convex hull of Γ_1. ∂U_{10} consists of 4 curves. There will be two segments shared with $\partial\Omega_1$ and two geodesics γ_1 and γ_2. One should think of γ_2 as being above γ_1 according to the orientation of the manifold. Let us choose $\tilde{\gamma}_2$ to be a strictly convex curve just below γ_2 and take $\tilde{\gamma}_1$ to be a strictly concave curve just above γ_1. These curves will both exists as they can just be taken to be curves with sufficiently small constant mean curvatures. Existence of such curves will then be immediate as they are governed by second order ordinary differential equations. Let this region be denoted by V_0 and let $V := I \times V_0$. We note that the support of the potential q lies in the set V. We will construct a phase function that will essentially be a limiting Carleman weight in this region and then transitions to a globally well defined smooth function in such a way that the Hörmander hypoellipticity condition is satisfied. We will denote the region outside of V and above $I \times \tilde{\gamma}_2$ by W_u and the other remaining region outside of V and below $\tilde{\gamma}_1$ by W_l. Let us first construct the local Fermi coordinates (x_1, x_2, x_3) about the surface $I \times \tilde{\gamma}_2$ and (x_1, y_2, y_3) about $I \times \tilde{\gamma}_1$.

Note that near $I \times \tilde{\gamma}_2$ we have :

$$g = dx_1^2 + dx_3^2 + \rho(x_2, x_3)dx_2^2$$

and near $I \times \tilde{\gamma}_1$:

$$g = dx_1^2 + dy_3^2 + \tilde{\rho}(y_2, y_3)dy_2^2$$

The convexity conditions imply that locally near the two surfaces we have:

$$\partial_3 \rho < 0$$

and:

$$\partial_3 \tilde{\rho} > 0$$

Let δ_1, δ_2 be sufficiently small and define $\omega : \Omega_1 \to \mathbb{R}$ to be any smooth function such that $d\omega \neq 0$ everywhere, $\omega(x) \equiv x_3$ for $x \in W_u \cap \{0 \leq x_3 \leq \delta_2\}$ and $\omega(x) \equiv y_3$ for $x \in W_l \cap \{-\delta_1 \leq y_3 \leq 0\}$.

Define the two functions χ_0 and F_λ as follows:

$$\chi_0(x) = \left\{ \begin{array}{lr} 1, & \text{for } x \in V \\ (1 - (\frac{x_3}{\delta_2})^{8k})^k, & \text{for } 0 \leq x_3 \leq \delta_2 \\ (1 - (\frac{y_3}{\delta_1})^{8k})^k, & \text{for } -\delta_1 \leq y_3 \leq 0 \\ 0 & \text{otherwise} \end{array} \right\}$$

$$F_\lambda(x) = \left\{ \begin{array}{lr} 0, & \text{for } x \in V \\ e^{\lambda(\frac{\omega(x)}{\delta_2})^2}(\frac{\omega(x)}{\delta_2})^{2k}, & \text{for } x \in W_u \\ e^{\lambda(\frac{\omega(x)}{\delta_1})^2}(\frac{\omega(x)}{\delta_1})^{2k}, & \text{for } x \in W_l \end{array} \right\}$$

Lemma 3.6.8. *Let $\tilde{\phi}_0(x_1, x_2, x_3) = x_1\chi_0(x) + F_\lambda(x)$ where $k \geq 1$ is an arbitraty integer and $\lambda(\Omega_1, k, ||g_{ij}||_{C^2})$ is sufficiently large. Then the Hörmander hypo-ellipticity condition is satisfied in Ω_1, that is to say:*

$$D^2\tilde{\phi}_0(X, X) + D^2\tilde{\phi}_0(\nabla\tilde{\phi}_0, \nabla\tilde{\phi}_0) \geq 0$$

whenever $|X| = |\nabla\tilde{\phi}_0|$ *and* $\langle\nabla\tilde{\phi}_0, X\rangle = 0$.

Proof of Lemma 3.6.8. We will consider the the five regions V , $A = \{0 \leq x_3 \leq \delta_2\}$, $B = \{-\delta_1 \leq y_3 \leq 0\}$, $W_u \setminus A$ and $W_l \setminus B$ and prove the inequality holds in all these regions. Regions B and $W_l \setminus B$ can be handled in the exact same manner as regions A and $W_u \setminus A$ respectively so we only focus on the three regions V, A and $W_u \setminus A$. Let us first consider V. Note that in this region $\tilde{\phi}_0(x_1, x_2, x_3) = x_1$ and since the metric is conformally transversally anisotropic in this region we can use the result in [5] to deduce that the Hörmander condition is satisfied in A. This is due to the fact that x_1 is a limiting Carleman weight in this region.

The region denoted by $W_u \setminus A$. This region can be handled in the exact same manner as in Lemma 3.2.4 due to the convexification method implemented in the design of the function F_λ. We will now focus on the region denoted by A;
Recall that in Riemannian geometries:

$$Hess(f) := D^2 f = (\partial_j\partial_k f - \Gamma^l_{jk}\partial_l f)dx^j \otimes dx^k$$

Where Γ^l_{jk} denotes the Christoffel symbol defined through:

$$\Gamma^l_{jk} = \frac{1}{2}g^{lm}(\partial_k g_{mj} + \partial_j g_{mk} - \partial_m g_{jk})$$

Note that in the transition region A the only non-zero terms in Christoffel symbol are $\Gamma^2_{22}, \Gamma^2_{23}, \Gamma^3_{22}$. Also note that $\nabla\tilde{\phi}_0 = \partial_1\tilde{\phi}_0\partial_1 + \partial_3\tilde{\phi}_0\partial_3$. Hence:

$$D^2\tilde{\phi}_0(\nabla\tilde{\phi}_0, \nabla\tilde{\phi}_0) = (\partial_3\tilde{\phi}_0)^2\partial_{33}\tilde{\phi}_0 + 2\partial_1\tilde{\phi}_0\partial_3\tilde{\phi}_0\partial_{13}\tilde{\phi}_0$$

Indeed one can see that this is exactly the same expression as in the Euclidean setting and thus we have:

$$D^2\tilde{\phi}_0(\nabla\tilde{\phi}_0, \nabla\tilde{\phi}_0) \geq 0$$

Let us now analyze the term $D^2\tilde{\phi}_0(X, X)$ for all X with $\langle\nabla\tilde{\phi}_0, X\rangle = 0$
Note that $d\tilde{\phi}_0(X) = 0$ implies that:

$$X \in \text{span}\{\partial_2, Z\}$$

Where $Z = \partial_3\tilde{\phi}_0\partial_1 - \partial_1\tilde{\phi}_0\partial_3$. In this region we have the following:

$$D^2\tilde{\phi}_0(\partial_2, \partial_1) = 0$$

$$D^2\tilde{\phi}_0(\partial_2, \partial_2) = -\Gamma^3_{22}\partial_3\tilde{\phi}_0$$

$$D^2\tilde{\phi}_0(\partial_2, \partial_3) = 0$$

One should note that the convexity assumption on γ_2 implies that :

$$D^2\tilde{\phi}_0(\partial_2, \partial_2) \geq 0$$

Note that:

$$D^2\tilde{\phi}_0(\partial_3\tilde{\phi}_0\partial_1 - \partial_1\tilde{\phi}_0\partial_3, \partial_3\tilde{\phi}_0\partial_1 - \partial_1\tilde{\phi}_0\partial_3) = (\partial_1\tilde{\phi}_0)^2\partial_{33}\tilde{\phi}_0 - 2\partial_1\tilde{\phi}_0\partial_3\tilde{\phi}_0\partial_{13}\tilde{\phi}_0 - \Gamma^3_{33}(\partial_1\tilde{\phi}_0)^2\partial_3\tilde{\phi}_0$$

Again, the reader can see that this expression is exactly as in the Euclidean setting and thus the exact same argument applies here to coclude that:

$$D^2\tilde{\phi}_0(\partial_3\tilde{\phi}_0\partial_1 - \partial_1\tilde{\phi}_0\partial_3, \partial_3\tilde{\phi}_0\partial_1 - \partial_1\tilde{\phi}_0\partial_3) \geq 0$$

□

With the proof of Lemma 3.6.8 now complete, one can proceed with construction of the CGO solutions as follows. Let p be a point just outside Ω_{01} such that the geodesic γ emanating from p will be in the region between $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. Let us choose the normal coordinate system (r, θ) about p so that we have the following:

$$g = dx_1^2 + dr^2 + c(r, \theta)d\theta^2$$

Let us define $\Phi = \tilde{\phi}_0 + ir$. We also define $v_0 = c^{\frac{1}{4}}h(x_1 + ir)\chi(\theta)$ where h is an arbitrary holomorphic function and χ is an arbitrary function of compact support near γ. We have the following two Lemmas which are exact parallels to Lemma 3.3.4 and Lemma 3.3.5.

Lemma 3.6.9. *There exists a family of exact solutions u_0 to $-\triangle_g u_0 = 0$ of the form $u_0 = e^{\tau\Phi}(v_0 + r_0)$ where $\|r_0\|_{L^2(\Omega_1)} \leq \frac{C}{\tau}$.*

Lemma 3.6.10. *Let $q \in L^\infty(\Omega_1)$. There exists a family of exact solutions u_1 to $(-\triangle_g + q)u_1 = 0$ of form $u_1 = e^{\tau\Phi}(v_0 + r_1)$ where $\|r_1\|_{L^2(\Omega_1)} \leq \frac{C}{\tau}$.*

Remark 3.6.11. The proofs of Lemmas 3.6.9 and 3.6.10 are omitted as they are similar to Lemmas 3.3.4 and 3.3.5. The reader should also note that the proof of the reconstruction algorithm provided in the Euclidean setting can be duplicated here to obtain the trace of the CGO solutions provided above on $\partial\Omega$.

Proof of Theorem 3.6.5. Let $w = e^{-\tau\Phi}v_0$ and use the Green identity by pairing $u_1 = e^{\tau\Phi}(v_0 + r_1)$ with w. Thus:

$$I = \int_{\partial\Omega} w\partial_\nu u_1 - \int_{\partial\Omega} u_1\partial_\nu w = \int_\Omega w\triangle_g u_1 - \int_\Omega u_1\triangle_g w$$

Hence:

$$I = \int_\Omega qwu_1 - \int_\Omega e^{\tau\Phi}(v_0 + r_1)\triangle_g(e^{-\tau\Phi}v_1)$$

Let $J = \int_\Omega v_0 e^{\tau\Phi}\triangle_g(e^{-\tau\Phi}v_0)$. Then:

$$I + J = \int_\Omega qv_0(v_0 + r_1) - \int_\Omega r_1 e^{\tau\Phi}\triangle(e^{-\tau\Phi}v_0)$$

But notice that:

$$e^{\tau\Phi}\triangle(e^{-\tau\Phi}v_0) = \tau^2\langle d\Phi, d\Phi\rangle_g v_0 - \tau[2\langle d\Phi, dv_0\rangle_g + (\triangle_g\Phi)v_0] + \triangle_g v_0$$

Hence:

$$\|e^{\tau\Phi}\triangle(e^{-\tau\Phi}v_0)\|_{L^2(\Omega)} \leq C$$

and therefore using Cauchy-Schwarz we see that:

$$|\int_\Omega r_1 e^{\tau\Phi}\triangle(e^{-\tau\Phi}v_0)| \leq \frac{C}{\tau}$$

Similarly we notice that:

$$|\int_\Omega qv_0 r_1| \leq \frac{C}{\tau}$$

Thus:

$$I + J = \int_\Omega qv_0^2 + O(\frac{1}{\tau})$$

Thus similarly to the Euclidean setting we see that by choosing $\chi(\theta)$ approximating a delta distribution we can obtain the knowledge of following data:

$$\int_{I\times\gamma} qh(z)$$

By choosing $h(z) = e^{-i\lambda z}$ we see that we have the knowledge of the following data:

$$\int_\gamma \hat{q}(\lambda, r, \theta) e^{\lambda r} dr$$

where $\hat{q}$ denotes the fourier transform of q in x_1.

Let us note that the local geodesic transform on U is invertible since U is simple [24]. We will use this to conclude that q can be determined from the data above. Indeed suppose that we have a function f such that :

$$\int_\gamma f(\lambda, r, \theta) e^{\lambda r} dr = 0 \qquad (*)$$

Setting $\lambda = 0$ and using the injectivity of the geodesic ray transform we can conclude that $f(0, r, \theta) = 0$. Now differentiating (*) with respect to λ and setting $\lambda = 0$ yields that $\partial_\lambda f|_{\lambda=0} \equiv 0$. Repeating this argument yields that all derivatives of f must be zero. Since f represents the Fourier transform of a compactly supported function it must be real analytic and therefore it must vanish everywhere. Thus we can conclude that the integral data $\int_{I\times\gamma} qh(z)$ indeed reconstructs the potential q uniquely in V.

□

Remark 3.6.12. Using the Gaussian beam quasi mode construction in [3] the result in Theorem 3.6.5 can be strengthened by removing the simplicity restriction required on U_0 and instead imposing the injectivity of geodesic ray transform on U_0. Examples of such manifolds for which this injectivity is known, are simple manifolds, manifolds which are Foliations by strictly convex hypersurfaces [3][25][29].

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