GLOBAL WELL-POSEDNESS AND SCATTERING OF BESOV DATA FOR THE
ENERGY-CRITICAL NONLINEAR SCHRÖDINGER EQUATION

by

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# Abstract

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We examine the Defocusing Energy-Critical Nonlinear Schrödinger Equation in dimension 3. This equation has been studied extensively when the initial data is in the critical homogeneous Sobolev space $\dot{H}^1$, and a satisfactory theory is given in the work of Colliander, Keel, Sataffilani, Takaoka and Tao. We extend the analysis of this equation to include infinite energy data $u_0 \in \dot{B}^1_{2,q}$ ($2 \leq q \leq \infty$) that can be decomposed as a finite energy component (a part in $\dot{H}^1$) and a small Besov part, with the size of the energy part depending on the size of the Besov part. If $2 \leq q < \infty$, the solution is shown to scatter. For $q = \infty$, the solution is shown to be globally well-posed. Traditionally, the well-posedness theory has been studied in Strichartz spaces, but we use more subtle spaces to deal with the high frequencies that arise from the Besov data, $X^q(I)$. These spaces are variants of bounded variation spaces and satisfy a duality that allows us to recover the traditional multilinear estimate along with a Strichartz variant that allows for extracting smallness by shrinking the time interval.

We also discuss a conjecture that all data $u_0 \in \dot{B}^1_{2,q}$ for $2 \leq q < \infty$ evolve to a global solution that scatters and we discuss the next steps to proving this.

# Acknowledgements

# Contents

# Chapter 1

# Introduction

## 1.1 Basic Theory for the Nonlinear Schrödinger Equation

We consider the Cauchy problem for the defocusing, energy-critical Nonlinear Schrödinger Equation (NLS) in dimension 3,

$$(NLS) \begin{cases} i\partial_t u + \triangle u = |u|^4 u \\ u_{t=0} = u_0. \end{cases} \tag{1.1.1}$$

Equation (1.1.1) is a Hamiltonian equation with Hamiltonian given by

$$E(u(t)) := \int \left(\frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{6}|u(t,x)|^6\right)dx.$$

The energy-critical NLS is where we will focus our attention, but for the moment, let us consider the general Defocusing Nonlinear Schrödinger Equation with a nonlinearity of degree $p$ in dimension $d$,

$$\begin{cases} i\partial_t u + \triangle u = |u|^{p-1}u \\ u_{t=0} = u_0. \end{cases} \tag{1.1.2}$$

Equation (1.1.2) is invariant under time translations: if $u(t,x)$ is a solution to Equation (1.1.2) , then so is $u(t+\tau)$ for fixed $\tau$. Noether's theorem then tells us that the Hamiltonion is conserved. Similarly, Equation (1.1.2) is space translation invariant which leads to conservation of the momentum $P(u) := \int_{\mathbb{R}^d} 2Im(\bar{u}\nabla u)dx$ and phase rotation invariant $(u \to e^{i\theta}u)$, which leads to conservation of mass $\int_{\mathbb{R}^d} |u|^2 dx$. The equation also enjoys a time-reversal symmetry: if $u(t,x)$ is a solution, then so is $\overline{u(-t,x)}$. This allows us to extend the time domain of the solution to Equation (1.1.2) from $[0,T)$ to $(-T,T)$. The results in this thesis are presented with positive time intervals to make the exposition smoother, however all of the results can be extended to negative times.

For reasons discussed below, we usually take the initial data $u_0$ in a homogeneous Sobolev space. We will define these spaces now.

**Definition 1.1.1.** *(Sobolev Spaces) For $s \geq 0$, the homogeneous Sobolev space $\dot{H}^s$ is defined as the closure of the Schwartz functions under the norm*

$$\|f\|_{\dot{H}^s_x(\mathbb{R}^3)} = \||\nabla|^s f\|_{L^2_x(\mathbb{R}^3)}.$$

If $u(t,x)$ is a solution to (1.1.2), then $u^\lambda := \lambda^{-\frac{2}{p-1}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ is also a solution to (1.1.2). We use a superscript here to denote the scaled version of $u$ instead of the usual subscript because we will be making use of subscripts throughout the document to denote Littlewood-Paley projections (see Equation (1.1.4)). It can be shown that

$$\|u^\lambda\|_{\dot{H}^s_x(\mathbb{R}^d)} = \lambda^{-\frac{2}{p-1} - s + \frac{3}{2}} \|u\|_{\dot{H}^s_x(\mathbb{R}^d)}. \tag{1.1.3}$$

So if $s = \frac{d}{2} - \frac{2}{p-1}$, the $\dot{H}^s_x(\mathbb{R}^d)$ norm is preserved under scaling. In particular, Equation (1.1.1) preserves the $\dot{H}^1_x(\mathbb{R}^d)$ norm of a solution and this is why we call Eequation (1.1.1) energy-critical.

**Definition 1.1.2.** *For the function $f : \mathbb{R}^3 \to \mathbb{C}$, we define the space Fourier transform of $f$ to be the function $\hat{f} : \mathbb{R}^3 \to \mathbb{C}$, $\hat{f}(\xi) = (\mathscr{F}(f))(\xi) := \int_{\mathbb{R}^3} e^{-2\pi i x \cdot \xi} f(x) dx.$*

Throughout this thesis, subscripts denote Littlewood-Paley projections in space onto dyadic frequencies. Let $\varphi(\xi)$ be a $C^\infty$ function which is 1 on the ball of radius 1 and decreases monotonically to zero outside a ball of radius 2. Let $\lambda$ be a dyadic number, i.e.; $\lambda = 2^k$ for some integer $k$. Then we define

$$u_\lambda(t,x) := \mathscr{F}^{-1}[\mathscr{F}(u)(\varphi(\frac{\xi}{\lambda}) - \varphi(\frac{2\xi}{\lambda}))]. \tag{1.1.4}$$

Throughout this thesis, if a sum that is written with lower bound $N$ $\sum_N$, or if a sum is to be taken 'over dyadic numbers,' it is to be interpreted in the following way:

$$\sum_N F(N) := \sum_{k=-\infty}^{\infty} F(2^k). \tag{1.1.5}$$

From Littlewood-Paley theory (see [44] for example), we can write

$$\|f\|_{L^p_x(\mathbb{R}^d)} \quad \sim_{s,d} \quad \left( \sum_N \|f_N\|^2_{L^p_x(\mathbb{R}^d)} \right)^{\frac{1}{2}} \tag{1.1.6}$$

$$\|f\|_{\dot{H}^s_x(\mathbb{R}^d)} \quad \sim_{s,d} \quad \left( \sum_N N^{2s} \|f_N\|^2_{L^2_x(\mathbb{R}^d)} \right)^{\frac{1}{2}}, \tag{1.1.7}$$

where $f_N$ denotes the Littlewood-Paley projection of $f$ onto the $N^{th}$ dyadic frequency and $a \sim_\alpha b$ means $a \leq C(\alpha) b$ for some constant $C(\alpha)$. We can generalize these spaces by changing $\ell^2$ to $\ell^p$ for any $p \in [2, \infty]$.

**Definition 1.1.3.** *(Besov Spaces) For $s \geq 0$ and $1 \leq p < \infty$, $1 \leq q \leq \infty$, the homogeneous Besov space $\dot{B}^s_{p,q}$ is defined as the closure of the Schwartz functions under the norm*

$$\|f\|_{\dot{B}^s_{p,q}} = \left( \sum_N N^{qs} \|f_N\|^q_{L^p_x(\mathbb{R}^3)} \right)^{\frac{1}{q}},$$

*where $f_N$ is the Littlewood-Paley projection onto the $N^{th}$ frequency and the sum is over all dyadic numbers N.*

Clearly we have the continuous embeddings for $2 < p < q \leq \infty$,

$$\dot{H}^1 \subset \dot{B}^1_{2,p} \subset \dot{B}^1_{2,q}.$$

By Plancherel's theorem, we can see that

$$||f||_{\dot{H}^1_x} \sim \left( \sum_N N^{2s} ||f_N||^2_{L^2_x} \right)^{\frac{1}{2}}, \tag{1.1.8}$$

and so $\dot{H}^1_x = \dot{B}^1_{2,2}$. For a more detailed analysis of Besov spaces, see [24].

The following definition of solutions to Equation (1.1.1) requires the spaces $X^q(I)$. We postpone the definition of these spaces to Section 2.1 because they require some technicalities.

**Definition 1.1.4.** *Consider Equation (1.1.1) with $u_0 \in \dot{B}^1_{2,q}$. A function $u : I \times \mathbb{R}^3 \to \mathbb{C}$ on a non-empty time interval I containing 0 is a solution to (1.1.1) if it belongs to $L^\infty_t \dot{B}^1_{2,q}(K \times \mathbb{R}^3) \cap X^q(K)$ for every compact interval $K \subset I$ and obeys the Duhamel formula*

$$u(t) = e^{it\triangle}u_0 - i \int_0^t e^{i(t-s)\triangle}(|u|^4 u)(s)ds \tag{1.1.9}$$

*for all $t \in I$. We refer to the interval I as the lifespan of u. We say that u is a maximal-lifespan solution if the solution cannot be extended to any strictly larger interval. The solution u is global if $I = \mathbb{R}$.*

*We denote by $\Gamma$ the Duhamel mapping*

$$\Gamma(u) := e^{it\triangle}u_0 - i \int_0^t e^{i(t-s)\triangle}(|u|^4 u)(s)ds, \tag{1.1.10}$$

*and by $\mathscr{I}$, the multilinear mapping*

$$\mathscr{I}(u) := \int_0^t e^{i(t-s)\triangle}(|u|^4 u)(s)ds. \tag{1.1.11}$$

**Definition 1.1.5.** *Let s be the critical Sobolev exponent for (1.1.1). Given a global solution u to Eequation (1.1.2), we say the solution **scatters in** $\dot{B}^1_{2,q}(\mathbb{R}^3)$ if there exists unique asymptotic states $u_\pm \in \dot{B}^1_{2,q}(\mathbb{R}^3)$ such that*

$$\lim_{t \to \pm\infty} ||u(t) - e^{it\triangle} u_\pm||_{\dot{B}^1_{2,q}(\mathbb{R}^3)} = 0. \tag{1.1.12}$$

*A solution **scatters** in $\dot{H}^1(\mathbb{R}^3)$ if the above definition is satisfied with all instances of $\dot{B}^1_{2,q}(\mathbb{R}^3)$ changed to $\dot{H}^1(\mathbb{R}^3)$.*

If the time interval $I$ can be inferred from context (usually in this case $I = \mathbb{R}$), we let $||u||_{q,r} = ||u||_{L^q_t L^r_x(I \times \mathbb{R}^3)}$.

Equation (1.1.2) has been studied extensively in a variety of settings. We will name a few that are particularly relevant to this thesis. In [1], Cazenave and Weissler proved local well-posedness results for equation (1.1.2) in critical Sobolev settings. For the energy-critical equation (equation (1.1.1)) we see a particularly beautiful evolution of the theory. In [4], Bourgain proved global well-posedness and scattering for (1.1.1) for arbitrary data assuming radial symmetry. He did this using an induction on energy strategy that he pioneered and which we outline now. It is known that if $u$ is a solution to (NLS) on the time interval $[t_0, T)$ and $||u||_{L^{10}_t L^{10}_x([t_0,T) \times \mathbb{R}^3)} < \infty$ for all $T > t_0$, then $u$ scatters (See [37] for example). For this reason, we say the $L^{10}_t L^{10}_x$ norm is a scattering norm for the Energy Critical Nonlinear Schrödinger Equation. For every energy $E \geq 0$, we define $M(E) := \sup_{I=[t_0,T],||u_0||_{\dot{H}^1} \leq E} ||u||_{L^{10}_t L^{10}_x([t_0,T) \times \mathbb{R}^3)}$. If $M(E)$ is finite for all $E > 0$, then $u$ scatters. In [4], Bourgain proves $M(E) \leq C(E, \eta, M(E - \eta^4))$ for $\eta = \eta(E)$ which is bounded away from zero. It can be shown that this implies $M(E)$ is finite for all $E > 0$ and hence shows scattering (for radial, finite-energy data).

In [15], Colliander, Keel, Staffilani, Takaoka and Tao use the induction on energy framework to make the important advance of removing the radial assumption and proving global well-posedness and scattering for (1.1.1) with global bounds. Profile decomposition (see Chapter 5) results were obtained for the $L^2$-critical NLS in [31], [30], [35] and for the $\dot{H}^1$-critical NLS (Equation (1.1.1)) in [36]. These profile decomposition results along with the rigidity arguments developed by Bourgain, the authors of [15] and Kenig and Merle [33] [34] led to the latter two authors producing a "road map" to prove global well-posedness using these tools, which is described by Kenig in [28]. Steps in the analysis involve local theory, profile decomposition, perturbation theory, and Morawetz-type estimate, along with boundedness of solutions in the data space. This is sufficient to prove global well-posedness (and scattering in most cases). These techniques were further developed by Visan [17] [18], Ryckman-Visan [16], Tao, Killip. There is also an extensive presentation of the technique in Visan's Oberwolfach notes [10]. This technique is only possible when solutions remain bounded in the initial data space. For Equation (1.1.2), this means we are restricted to the $L^2$-critical and $\dot{H}^1$-critical equations, since the $L^2$ norm and $\dot{H}^1$ norms of solutions to Equation (1.1.1) are guaranteed to be bounded. However, if we assume solutions are in $L^\infty_t \dot{H}^s$ for $s \in (0,1)$, then we may try to use this road map to prove well-posedness for the $\dot{H}^s$-critical NLS. Indeed, this was done for $s = \frac{1}{2}$ by Kenig and Merle in [33]. In [39], Jason Murphy proved global well-posedness and scaterring (with this boundedness assumption) for $s \in (0,1)$. The super-critical regime ($s > 1$) was also attacked using this approach by Killip and Visan in [19]. In Chapter 6, devoted to future projects, we give an outline to proving global well-posedness for any data in the Besov space $\dot{B}^1_{2,q}$ assuming the bound $||u||_{L^\infty_t \dot{B}^1_{2,q}(I \times \mathbb{R}^3)}$ for any solution $u$ to (NLS) on interval $I$.

In the $L^2$-critical theory, Tao, Visan, Killip and Zhang used similar machinery to prove global well-posedness and scattering for radial data ( [61] [62]). and Dodson removed the radial data in his groundbreaking work in [40], [41], [42] and [43].

The Hamiltonian

$$E(u(t)) := \int (\frac{1}{2}|\nabla u(t,x)|^2 - \frac{1}{6}|u(t,x)|^6) dx$$

gives rise to the focusing Nonlinear Schrödinger Equation in dimension three,

$$\begin{cases} i\partial_t u + \triangle u = -|u|^4 u \\ u_{t=0} = u_0. \end{cases} \tag{1.1.13}$$

This equation is more subtle than the defocusing case. In [54], it is shown that

$$W(x) = \left(1 + \frac{1}{3}|x|^2\right)^{-\frac{1}{2}}$$

is a stationary solution to Equation (1.1.13) which has infinite scattering norm. In fact, it is conjectured that $W$ is a minimal counterexample to global spacetime bounds. See [55] for more details.

## 1.2  Infinite Energy Solutions

Most of the work on Equation (1.1.1) has been in the realm of finite energy solutions. Ie; the initial data considered has energy $E(u_0) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + \frac{1}{6}|u(t,x)|^6\right) dx < \infty$. It is easy to see why. Conservation of the Hamiltonian, along with the positive sign separating the terms gives us that the $\dot{H}^1$ norm of a solution to (1.1.1) is bounded. This is the toe-hold that seems crucial for proving a well-posedness theory. A similar pattern can be seen for the mass-critical case; very little work has been done on this equation without assuming $u_0 \in L^2_x$. This is similar in the theory of the wave equations and other dispersive equations. There are notable exceptions even for (1.1.1). We will see one such result below. But first, let us state precisely the main theorem in [15].

**Theorem 1.2.1.** *For any $u_0$ with $||u_0||_{\dot{H}^1} < \infty$, there exists a global solution $u \in C^0_t(\dot{H}^1_x) \cap L^{10}_{t,x}$ to Equation (1.1.1) that is unique in $C^0_t(\dot{H}^1_x)$ such that*

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} |u(t,x)|^{10} dx dt \le C(E(u_0)) \tag{1.2.1}$$

*for some constant $C(E(u_0))$ that depends only on the energy.*

In addition, if $u$ is a solution to the energy-critical Nonlinear Schrödinger Equation, with $||u||_{L^{10}L^{10}(\mathbb{R}\times\mathbb{R}^3)} < \infty$, then $u$ scatters in the energy space. Thus, any finite-energy data evolves to a global solution that scatters.

We can then ask the following questions:

1. What is the largest space $\mathscr{G}$ of initial data that will guarantee that solutions to Equation (1.1.1) are globally well-posed?

2. What is the largest space for initial data $\mathscr{S}$ that will guarantee that solutions to Equation (1.1.1) scatter?

Theorem 1.2.1 tells us that $\dot{H}^1 \subset \mathscr{S} \subset \mathscr{G}$.

In [8], Planchon shows that initial data of the form $u_0(x) = \frac{\varepsilon_0}{|x|^{\frac{1}{2}}}$ produces a self-similar solution $u(t,x) = \frac{1}{\sqrt{t}^{\frac{1}{2}}} U\left(\frac{x}{\sqrt{t}}\right)$ for every positive $\varepsilon_0$. This solution converges weakly to the initial data as $t \to 0$. The initial condition $\frac{1}{|x|^{\frac{1}{2}}}$ is in $\dot{B}^1_{2,\infty}$. It is also shown that for general data $u_0$ with $||u_0||_{\dot{B}^1_{2,\infty}} < \varepsilon$, solutions evolve globally, thus global well-posedness holds for a class of infinite-energy solutions. This is also an important example of global well-posedness for self-similar solutions. However, these self-similar solutions cannot scatter. Thus, if $B^q_\varepsilon := \{u_0 : ||u_0||_{\dot{B}^1_{2,q}} < \varepsilon\}$, for $2 \le q \le \infty$ then for some $\varepsilon$ small enough:

$$B^\infty_\varepsilon \cup \dot{H}^1 \subset \mathscr{G} \quad \text{and} \quad B^\infty_\varepsilon \not\subset \mathscr{S}.$$

In the setting on nonlinear Wave equations, Tao has shown that for a logarithmically supercritical nonlinearity, one can obtain solutions that are infinite in the critical space. In particular, in three spatial dimensions, for the scalar field $u : I \times \mathbb{R}^3 \to \mathbb{R}$, if we consider the nonlinear wave equation

$$\Box u = |u|^4 u, \tag{1.2.2}$$

where $\Box u = -\partial_{tt} u + \triangle u$, we see that (1.2.2) is energy-critical. If we modify the nonlinearity logarithmically and consider the equation

$$\Box u = |u|^4 u \log(2 + u^2), \tag{1.2.3}$$

then (1.2.3) is energy super-critical and the following result holds.

**Theorem 1.2.2.** *[63] Let $u_0$, $u_1 \in C^\infty(\mathbb{R}^3)$ be any spherically symmetric smooth initial data. Then there is a unique global smooth solution to (1.2.3) with initial position $u(0,x) = u_0(x)$ and initial velocity $\partial_t u(0,x) = u_1(x)$.*

This result differs from the main result of this thesis (see below) in that the super-criticality of the theorem comes from the equation as opposed to the initial data, but both theorems show that there is some maneuverability when it comes to exploring dispersive equations past the critical regime.

## 1.3 Main Result and Further Conjecture

Our goal is to show that there is a class of data with less regularity than $\dot{H}^1$ that scatters and a class of data with even less regularity that evolves globally. Although we are not able to show that all Besov data evolves globally, we are able to expand the current set of initial data to include small perturbations (in the Besov space) of $\dot{H}^1$ data. The precise statement of the theorem requires the definition of the spaces $X^q(I)$, see Section 2.1.

**Theorem 1.3.1.** *Let $u_0 \in \dot{B}^1_{2,q}$ with $2 \le q \le \infty$ and $u_0 = v_0 + w_0$, $v_0 \in \dot{H}^1$ and $w_0 \in \dot{B}^1_{2,q}$ with $||w_0||_{\dot{B}^1_{2,q}} < \min\left(\frac{1}{2}, \varepsilon_0\left(||v||_{L^{10}_t L^{10}_x(\mathbb{R}\times\mathbb{R}^3)}, ||v_0||_{\dot{H}^1}\right)\right)$, where $v$ is the unique solution to Equation (1.1.1) emerging from initial data $v_0 \in \dot{H}^1$. There exists a unique global solution $u(t,x)$ to Equation (1.1.1) which satisfies:*

*For $2 \le q < \infty$,*

$$u \in C^0_t \dot{B}^1_{2,q}(\mathbb{R}^+ \times \mathbb{R}^3) \cap X^q(\mathbb{R}^+),$$

*and u scatters in $\dot{B}^1_{2,q}$. For $q = \infty$,*

$$u \in L^\infty_t \dot{B}^1_{2,\infty}(\mathbb{R}^+ \times \mathbb{R}^3) \cap X^\infty(\mathbb{R}^+)$$

*and u converges weakly to $u_0$ in $\dot{B}^1_{2,\infty}$ as $t \to 0$.*

In Lemma 2.1.9, we will show that $X^2 \subset L^{10}L^{10}$. Since $\dot{H}^1 = \dot{B}^1_{2,2}$, Theorem 1.3.1 is at least as strong as Theorem 1.2.1.

**Corollary 1.3.2.** *Let $u_0 \in \dot{H}^1$. There exists a unique solution $u(t,x)$ to Equation (1.1.1) for all time with $u \in C^0_t \dot{H}^1(\mathbb{R}^+ \times \mathbb{R}^3) \cap X^q(\mathbb{R}^+)$. Furthermore, u scatters in $\dot{H}^1$.*

Theorem 1.3.1 allows us to say for $q \in [2,\infty)$,

$$\begin{aligned} B^\infty_{\varepsilon_0} + \dot{H}^1 &\subset \mathscr{G}, \\ B^q_{\varepsilon_0} + \dot{H}^1 &\subset \mathscr{S}. \end{aligned}$$

In particular, Theorem 4.3.1 generalizes the result of [15] and [8]. A further question is the exact nature of $\mathscr{G}$ and $\mathscr{S}$.

**Conjecture 1.3.3.** *For $q \in [2,\infty)$,*

$$\begin{aligned} \dot{B}^1_{2,\infty} &\subset \mathscr{G} \\ \dot{B}^1_{2,q} &\subset \mathscr{S}, \end{aligned}$$

where $q \in [2,\infty)$. As of this time, we are not able to prove this conjecture, but in Section 6 we outline a procedure that may produce a partial result.

## 1.4 The Limitations of Strichartz Spaces and the Introduction of $U^p - V^p$ Spaces

To obtain well-posedness, a fixed-point argument has traditionally been used when studying dispersive equations. Strichartz spaces are perfect for running fixed-point arguments when dealing with data in $\dot{H}^s$ because of the Strichartz estimates below.

**Definition 1.4.1.** *The space-time Lebesgue space $L^q_t L^r_x(I \times \mathbb{R}^3)$ is called a **Strichartz space** (for regularity s) if $2 \le q, r \le \infty$ and $(q,r)$ satisfy the admissibility condition*

$$\frac{2}{q} + \frac{3}{r} = \frac{d}{2} - s. \tag{1.4.1}$$

*We call such pairs **admissible**. If $s = 1$ we use the terms **energy-admissible Strichartz space** and we call $(q,r)$ an **energy-admissible pair**. If $s = 0$ we use the terms **mass-admissible Strichartz space** and **mass-admissible pair**.*

Strichartz spaces work well with solutions to the linear Schrödinger equation

$$i\partial_t u + \triangle u = 0 \tag{1.4.2}$$

In particular, we have the following Strichartz estimates. The nonendpoint cases were developed in [51], [52], [48] and [50]. The endpoint cases were proved in [53]. The following is taken from [44].

**Lemma 1.4.2.** *Let $u_0 \in L^2$ and let $(q,r)$ and $(\tilde{q}, \tilde{r})$ be mass-admissible pairs. Then*

$$||e^{it\triangle}u_0||_{L_t^q L_x^r(I \times \mathbb{R}^3)} \leq ||u_0||_{L_x^2}, \tag{1.4.3}$$

$$||\int_{\mathbb{R}} e^{-is\triangle}F(s)ds||_{L_x^2(\mathbb{R}^3)} \lesssim_{\tilde{q},\tilde{r}} ||F||_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^3)}, \tag{1.4.4}$$

$$||\int_{t'<t} e^{i(t-t')\triangle}F(t')dt'||_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^3)} \lesssim_{q,r,\tilde{q},\tilde{r}} ||F||_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^3)}. \tag{1.4.5}$$

Using Sobolev embedding, we can tailor these estimates for the energy-critical setting. They are presented in this form for referencing in later sections as it turns out it is easier to use the $L^2$ estimates directly for our purposes. These estimates along with the Duhamel formula Equation (1.1.9), Hölder's inequality and the Sobolev Embedding Theorem allow one to prove the multilinear estimate

$$||u||_{10,10} \lesssim ||u_0||_{\dot{H}^1} + ||u||_{10,10}^5, \tag{1.4.6}$$

for solutions $u$ to (NLS), with initial data $u_0 \in \dot{H}^1$. This estimate is the key to the fixed point argument and it is not hard to show local well-posedness in a Strichartz space using it. However, Lemma 1.4.2 does not generalize to Besov data and that makes proving mutlilinear estimates similar to those in Equation (1.4.6) impossible (in a Strichartz space). We would like to find a space-time space $X$ where this is possible. In particular, we would like to show the following multilinear estimate for solutions to (NLS) with initial data $u_0 \in \dot{B}_{2,q}^1$:

$$||u||_{X(I\times\mathbb{R}^3)} \lesssim ||u_0||_{\dot{B}_{2,q}^1} + ||u||_{X(I\times\mathbb{R}^3)}^5. \tag{1.4.7}$$

In Section 2.1, we will properly introduce new spaces that are equipped to handle the frequency refinements that we need to deal with Besov initial data. These spaces are variants of the bounded variation spaces $V^p$ (also introduced in Section 2.1). They are robust enough for both Strichartz-type estimates and the multilinear estimates that we require. $U^p$ and their dual $V^{p'}$ spaces were introduced to the theory of dispersive equations by Hadac, Herr, Koch, Tataru and Tzvetkov in [13], [9], [12], [11], [22].

**Remark 1.4.3.** *There are two properties of Strichartz spaces that we will highlight. Strichartz spaces enjoy a 'fungibility' property in the following sense. If we know that $||u||_{L^q L^p([0,T]\times\mathbb{R}^3)} < \infty$, then for every $\varepsilon > 0$, we can find $0 < T' < T$ such that $||u||_{L^q L^p([0,T']\times\mathbb{R}^3)} < \varepsilon$. Fungibility is related to absolute continuity in the following way. For a norm $||\cdot||_{X(I\times\mathbb{R}^3)}$ and a function $u$, we define a measure $\mu_f$ by $\mu_f(I) = ||u||_{X(I\times\mathbb{R}^3)}$. Then the norm is said to be fungible if for every $u \in X(I \times \mathbb{R}^3)$, $\mu_f$ is absolutely continuous with respect to the Lebesgue measure. This property of Strichartz spaces is used implicitly throughout the theory of nonlinear Schrödinger equations and other dispersive equations where Lebesgue spaces are used in a fixed-point argument. Fungibility seems to be a bit of a misnomer, but I will use that term as it is embedded in the literature to date.* [1]

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[1] I am grateful to Rowan Killip for helping me understand the nuances of fungibility.

*The second property is that we have a choice of exponents given in Lemma 1.4.2, equations (1.4.4) and (1.4.5) (we may choose admissible $(\tilde{q}', \tilde{r}')$). This is known as Strichartz flexibility, and helps when proving multilinear estimates like Equation (1.4.6). The spaces that we will be using in our analysis are lacking both of these properties and therein lies most of the difficulty.*

## 1.5 Summary of Proof

Theorem 1.3.1 is proved with a stability argument. We summarize it now. In Section 2.1, we introduce our function spaces and give some basic statements about them. In Section 4.2, we prove well-posedness theorems. In particular, Theorem 4.2.1 shows that Equation (1.1.1) is globally well-posed in $X^q$, for small initial data in $\dot{B}^1_{2,q}$ for $2 \le q \le \infty$. This implies that $w_0$ evolves to a global solution, $w(t,x)$ which we take as fixed. To prove this, we rely heavily on the multilinear estimates in Chapter 3. There are two multilinear estimates we will need. The first, Prop. 3.2.2 gives us the inequality

$$||u||_{X^q} \le ||u_0||_{\dot{B}^1_{2,q}} + ||u||^2_{X^q}||u||^3_{X^\infty}, \tag{1.5.1}$$

for solutions $u$ to (NLS), with $2 \le q \le \infty$. This is proved in Chapter 3, by considering the cases $q = 2$ and $q = \infty$ separately proved in Prop. 3.1.1 and Prop. 3.1.2 and then interpolating between them to complete the proof of Prop. 3.2.2. Although the technique to prove this estimate is similar in the $q = 2$ case and the $q = \infty$ case, there are technical subtleties that arise. In particular, there are two duality arguments that are needed, Lemma 2.3.1 and Lemma 2.3.2 which are proved in Section 2.3. The other multilinear estimate we prove is Prop. 3.2.3. The estimate is

$$||u||_{X^q} \le ||u_0||_{\dot{B}^1_{2,q}} + ||u||^4_{X^q}||u||_{10,10} \tag{1.5.2}$$

for solutions $u$ to (NLS) with $2 \le q \le \infty$. The benefit of this second estimate is the ability to extract smallness on the right-hand side by shrinking the time interval (see Remark 1.4.3). However, this estimate is only useful if $u \in L^{10}_t L^{10}_x$. Thus, both these estimates are crucial in our analysis. To prove the multilinear estimates Equation (1.5.4) and Equation (1.5.5) , a bilinear Strichartz estimate, Proposition 2.2.4 is required which is a variant on the classical bilinear estimate used for Strichartz spaces, developed by Bourgain [3]. The proof of these multilinear estimates become quite technical, as the spaces that we are working with require a frequency decomposition. In particular, we must decompose each instance of $u$ and $\bar{u}$ in $\int_0^t e^{i(t-s)\triangle} \left(|u|^4 u\right)_N (s)ds$ using Littlewood-Paley theory and sum over the dyadic projections. The sum is split up into 5 sub-sums according to the relative size of $N$ compared to the frequency of each instance of $u$ and $\bar{u}$. Each sub-sum requires a different analysis, making the proof quite involved.

In Theorem 4.2.1, we show that if the initial data is small enough, the data will evolve globally under Equation 1.1.1. In particular, if we let $w$ denote the solution to Equation 1.1.1 with initial data $w_0$ (see Theorem 1.3.1), then $w$ evolves globally. With solution $w$ fixed, we let $e = e(t,x) = |w + \tilde{u}|^4(w + \tilde{u}) - |w|^4 w - |\tilde{u}|^4 \tilde{u}$. If $\tilde{u}$ is a solution to

$$(\tilde{NLS}) \begin{cases} i\partial_t \tilde{u} + \triangle \tilde{u} = |\tilde{u}|^4 \tilde{u} + e \\ \tilde{u}_{t=0} = v_0 \in \dot{H}^1, \end{cases} \tag{1.5.3}$$

then $u(t,x) = \tilde{u}(t,x) + w(t,x)$ is a solution to (NLS). Thus, to show that (NLS) is globally well-posed, it suffices to show that Equation (1.5.3) is globally well-posed. In Section 4.2, we show that Equation (1.5.3) is locally well-posed. If the maximal time of existence $T^*$ is finite, we know that $||u||_{X^q[0,T^*)} = \infty$. We assume this and seek a contradiction. The contradiction comes from our stability theorem, Theorem 4.2.6 in Section 4.2. Indeed, if we consider the solution to (NLS) $v = v(t,x)$ emerging from initial data $v_0$, Theorem 4.2.6 tells us that if $||w_0||_{\dot{B}^1_{2,q}}$ is small enough, then

$$||v - \tilde{u}||_{X^q([0,T^*))} < \infty. \tag{1.5.4}$$

From [15], we know that $||v||_{L^{10}_t L^{10}_x (\mathbb{R}^+ \times \mathbb{R}^3)} < \infty$. In Section 4.2, Lemma 4.2.2, we show that it is also true that $v = v(t,x)$ has global bounds in the space $X^q$,

$$||v||_{X^q(\mathbb{R}^+)} < \infty. \tag{1.5.5}$$

Since $||w||_{X^q(\mathbb{R}^+)}$ is bounded, Theorem 4.2.1, Equation (1.5.4) and Equation (1.5.5) imply $||u||_{X^q([0,T^*))} < \infty$, which gives us our desired contradiction. Theorem 4.2.6 is essentially the statement that initial data will evolve under (NLS) and a perturbation of (NLS) in a way that keeps the two solutions close when measured in the $X^q$ norm.

To prove the Stability Theorem 4.2.6 for small times, a continuity argument is used by employing the multilinear estimates in Chapter 3. To extend this to large times, an induction argument is used. This technique of using perturbations to expand the domain of data that is well-posed dates back to Bourgain's work in [5]. In [59], Germain uses this strategy on the semilinear wave equation to obtain global well-posedness using variants of Lorentz-Besov spaces.

For $q < \infty$, if $\lim_{t \to \infty} || \int_t^\infty e^{-is\triangle} \left( |u|^4 u \right) (s) ds ||_{\dot{B}^1_{2,q}} = 0$, this suffices to show scattering, as done in Section 4.4. This usually straightforward argument is made complicated by the fact that the $X^q$ spaces are not "fungible" (See Remark 1.4.3). This difficulty is overcome by using another multilinear lemma, Lemma 4.4.1.

## 1.6   Physical Motivation

The Nonlinear Schrödinger Equation (1.1.2) describes a broad array of phenomena, depending on the dimension d and the power of the nonlinearity p. It models propagation of light in nonlinear optical fibers and is an important model in the theory of Bose-Einstein condensates [57], [58]. The Nonlinear Schrödinger Equation arises naturally as the description for envelope dynamics of a quasi-monochromatic plane wave propagating in a weakly nonlinear dispersive medium when dissipative processes are negligible. Sulem and Sulem's work in [56] gives a detailed analysis of this, which we describe now in summary.

We consider a scalar nonlinear wave equation

$$L(\partial_t, \nabla)u + G(u) = 0,$$

with dispersion relation $L(-i\omega, ik) = 0$, where $\omega$ is the frequency and $\mathbf{k}$ is the wave vector. This equation admits approximate monochromatic wave solutions $u = \varepsilon \psi e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$, with constant, small amplitude $\varepsilon \psi$. If we consider

a nonlinear medium responding adiabatically to a finite wave amplitude, the nonlinearity affects the dispersion relation and the frequency $\omega = \omega(\mathbf{k})$ must be replaced by $\Omega = \Omega(\mathbf{k}, \varepsilon^2 |\psi|^2)$. Furthermore, we must replace our space and time variables with the slow variables $\mathbf{X} = \varepsilon \mathbf{x}$ and $T = \varepsilon t$, the derivatives being replaced by $\partial_t + \varepsilon \partial_T$ and $\partial_{\mathbf{X}} + \varepsilon \nabla$, where $\nabla$ is now the slow spatial variable gradient. This leads to the weakly nonlinear dispersion relation

$$\left[ \omega + i\varepsilon \partial_T - \Omega(\mathbf{k} - i\varepsilon \nabla, \varepsilon^2 |\psi|^2) \right] \psi = 0.$$

Expanding to second order in the small variable $\varepsilon$ leads to

$$i\left(\partial_T + \mathbf{v}_g \cdot \nabla\right) \psi + \varepsilon \left\{ \nabla \cdot (\mathscr{D} \nabla \psi) + \gamma |\psi|^2 \psi \right\} = 0, \tag{1.6.1}$$

where $\mathbf{v}_g = \nabla_{\mathbf{k}} \omega$ is the group velocity and $\mathscr{D} = \left( \frac{1}{2} \frac{\partial \omega}{\partial k_j \partial k_\ell} \right)$, with $j, \ell = 1, \cdots, d$ defined as half the Hessian matrix of the frequency, both evaluated at the wave vector $\mathbf{k}$. $\gamma = \frac{\partial \Omega}{\partial (|\psi|^2)}$, evaluated at $|\psi|^2 = 0$ and wave vector $\mathbf{k}$.

If we view Equation (1.6.1) as an initial value problem in time and rewrite the equation in moving reference frame, we get the Nonlinear Schrödinger Equation

$$i\frac{\partial \psi}{\partial \tau} + \nabla \cdot (\mathscr{D} \nabla \psi) + \gamma |\psi|^2 \psi = 0,$$

where the derivatives are taken with respect to the variable of the moving reference frame.

As the above discussion elucidates, there is certainly ample physical motivation for analyzing the Nonlinear Schrödinger Equation, however it should be noted that there is a further reason for considering the Energy Critical Nonlinear Schrödinger Equation (NLS). The subtleties of the equation are brought to the forefront when studying dispersive equations at the critical scaling. In particular, the general Nonlinear Schrödinger Equation is locally well-posed if the initial data $u_0$ is taken in Sobolev space $\dot{H}^s$ with $s \geq s_c$ and $0 \leq s_c \leq 1$, where $s_c$ is the critical scaling (see Section 1.1). See [2] for a systematic study of the well-posedness theory for the Nonlinear Schrödinger Equation in the subcritical and critical regimes. In the supercritical regime $s < s_c$, Equation (1.1.2) is ill-posed [60].

## 1.7  Future Directions

We discuss three directions to take further research.

i) Is it possible to prove an analogous result in the $L^2$ setting? In particular, in the $L^2$-critical setting, can we expand the class of initial data that evolves to global solutions from $L^2$ to $L^2 + \dot{B}_{2,q}^0$ for $2 \leq q \leq \infty$ and can we expand the class of initial data that evolves to scattering solutions from $L^2$ to $L^2 + \dot{B}_{2,q}^0$ for $2 \leq q < \infty$? More generally, we may ask the question: If we have a sufficiently robust well-posedness theory and we assume the bound $\|u\|_{L_t^\infty \dot{H}^s} < \infty$ (which may come naturally from the equation or may be taken as a hypothesis), can we apply the techniques of this document and extend the space of initial data that scatter to include Besov data? In other words, for what regularity and dimension can we use this technique to prove global well-posedness and scattering for Besov data?

ii) Theorem 4.3.1 requires that the size of the "Besov part" $w_0$ is bounded as a function of the "energy part" $v_0$.

In [59], in the context of the semilinear wave equation, Germain removes this boundedness restriction for certain initial data. For the sake of completeness, we state this result now.

**Theorem 1.7.1.** *[59] Let $d = 6$. There exists $\varepsilon > 0$ such that the Cauchy problem (NLW) has a global solution $u$ provided the initial data $(u_0, u_1)$ can be written*

$$u_0(x) = v_0(x) + \frac{c_0}{|x|^2} \quad and \quad u_1(x) = v_1(x) + \frac{c_1}{|x|^3},$$

*with $(v_0, v_1) \in \dot{H}^1 \times L^2$, $c_1 < \varepsilon$ and $c_2 < \varepsilon$. Furthermore, $u$ is unique in the set $\mathscr{E} = \{u, d_X(u, \mathscr{S}) < \varepsilon_1\}$ where $\varepsilon_1 > 0$.*

(NLW) refers to the Nonlinear Wave Equation

$$(NLW) \begin{cases} \Box u + |u|u = 0 \\ u_{t=0} = u_0 \\ \partial_t u_{t=0} = u_1. \end{cases} \tag{1.7.1}$$

We can ask whether this Theorem has an analogous counterpart in the setting of the Nonlinear Schrödinger equation.

iii) Finally, a useful step towards proving Conjecture 1.3.3 is to assume that solutions are bounded in time in $\dot{B}^1_{2,q}$ for $2 \leq q \leq \infty$ and to prove the conjecture in that setting. We state this more precisely as the following conjecture:

**Conjecture 1.7.2.** *Assume solutions to Equation (1.1.1) with data $u_0 \in \dot{B}^1_{2,q}$, $q \leq \infty$ evolve with the condition $u \in L^\infty_t \dot{B}^1_{2,q}$. Let $u_0 \in \dot{B}^1_{2,q}$, $q < \infty$. There exists a unique solution $u(t,x)$ to Equation (1.1.1) for all time with $u \in L^\infty_t \dot{B}^1_{2,q}([0,\infty) \times \mathbb{R}^3) \cap X^q([0,\infty))$. If $q < \infty$, then $u$ also scatters.*

One way to prove this conjecture might be to follow the aaaaap outlined by Kenig and Merle (see Chapter 1.1). The multilinear estimates from Chapter 3, the stability theory developed in Chapter 4 and the profile decomposition result in Chapter 5 are some of the components necessary for the road map. We discuss how we can apply the theory from the road map to prove Conjecture 1.7.2 in Chapter 6.

# Chapter 2

# Linear Estimates

## 2.1 Function Spaces

Here we define the function spaces $U^p$, $V^p$ as well as some variants that will be use in establishing our well-posedness theory. The general theory of $U^p$, $V^p$ spaces was developed by Hadac, Herr, Koch, Tataru and Tzvetkov in [13], [9], [12], [11], [22]. The following results are adapted from [9]. We limit our introduction to $L^2$-based spaces, since these are all we require in our analysis, but we can replace $L^2$ by any Banach space and still allow for a consistent theory. Some basic proofs in this section are omitted as they can be found in the original works mentioned above and require an amount of technical detail that is unnecessary in this presentation.

For interval $I$, we let $\mathscr{Z}$ be the set of finite partitions $\{t_i\}_{i=1}^n$. If the interval $I = (\alpha, \beta)$, with $-\infty \leq \alpha$ and $\beta \leq \infty$ is open, we define the set of partitions by the requirement $\alpha < t_0 < \ldots < t_n < \beta$. If the interval is half open or closed, a similar definition applies with the partition including the endpoints that are contained in the interval.

**Definition 2.1.1.** *Let $I$ be an interval, $1 \leq p < \infty$ and $v : I \to L^2$. We define*

$$||v||_{V^p(I)} = \max \left\{ ||v||_{L^\infty_t L^2_x (I \times \mathbb{R}^3)}, \sup_{\tau \in \mathscr{Z}} \left( \sum_{i=1}^{n-1} ||v(t_{i+1}) - v(t_i)||_{L^2_x}^p \right)^{\frac{1}{p}} \right\}, \tag{2.1.1}$$

*and the space $V^p = V^p(I)$ to be the space of functions $v$ for which this norm is finite. We omit $I$ where it is implied by the context.*
*We define $V^P_{RC}$ to be the closed subspace of $V^p$ consisting of right-continuous functions.*

$V^p$ is a Banach space. It is the dual space to the space $U^{p'}$ which we define now. Here, $(p, p')$ are Hölder duals $\frac{1}{p} + \frac{1}{p'} = 1$.

**Definition 2.1.2.** *A p-atom is a right continuous step function,*

$$a(t) = \sum_{i=1}^n \phi_i \chi[t_i, t_{i+1})(t), \tag{2.1.2}$$

*where $\{t_i\}_{i=1}^n \in \mathscr{Z}$, $t_{n+1} = \beta$, $\{\phi_i\}_{i=1}^n$ are $L^2$ functions such that $\sum\limits_{i=1}^{n} ||\phi_i||_{L_x^2}^p \leq 1$, and with the condition that the function $a(t)$ must be zero in a neighbourhood of the endpoint $\alpha$.*

*We define the atomic space $U^p(I)$ as the space consisting of functions $u = \sum\limits_{j=1}^{\infty} \lambda_j a_j$, where $\{a_j\}_{j=1}^{\infty}$ is a sequence of p-atoms and $\{\lambda_j\}_{j=1}^{\infty}$ a summable sequence of complex numbers, $\sum\limits_{j=1}^{\infty} |\lambda_j|$. We define the associated norm by*

$$||u||_{U^p(I)} := \inf \left( \sum_{j=1}^{\infty} |\lambda_j| \text{ such that } u = \sum_{j=1}^{\infty} \lambda_j a_j, \ \lambda_j \in \mathbb{C} \text{ and } a_j \text{ are p-atoms} \right). \tag{2.1.3}$$

**Lemma 2.1.3.** *[9] Let $2 \leq p < q < \infty$. Then the following continuous embeddings hold:*

$$\begin{aligned}
U^p &\subset & V_{RC}^p \subset V^p \tag{2.1.4} \\
V_{RC}^p &\subset & U^q. \tag{2.1.5}
\end{aligned}$$

Let $1 < p < \infty$ and $p'$ be Hölder duals ($\frac{1}{p} + \frac{1}{p'} = 1$). Then $V^{p'}$ is the dual space of $U^p$ in the sense that there exists a bilinear form

$$B : U^p \times V^{p'} : (u,v) \rightarrow B(u,v) \tag{2.1.6}$$

such that the mapping

$$V^{p'} \ni v \rightarrow (u \rightarrow B(u,v)) \in (U^p)^* \tag{2.1.7}$$

is a surjective isometry.

The following proposition gives an integral representation of our bilinear form. This will be useful for proving the duality statements we will need for our multilinear estimates.

**Proposition 2.1.4.** *Let $u \in V^1$ be absolutely continuous on compact intervals with $\lim\limits_{t\to\infty} u(t) = 0$ and $v \in V^p$, then*

$$B(u,v) = -\int_{-\infty}^{\infty} \langle u'(t), v(t) \rangle dt. \tag{2.1.8}$$

**Definition 2.1.5.** *For $2 \leq p \leq \infty$ we let $U_{\triangle}^p L^2$ (resp. $V_{\triangle}^p L^2$) be the spaces of all functions $u : \mathbb{R} \to L^2$ such that $t \to e^{-it\triangle} u(t)$ is in $U^p$ (resp. $V^p$), with norms*

$$\begin{aligned}
||u||_{U_{\triangle}^p} &= & ||e^{-it\triangle} u||_{U^p} \\
||u||_{V_{\triangle}^p} &= & ||e^{-it\triangle} u||_{V^p}.
\end{aligned}$$

**Remark 2.1.6.** *The space $U_\triangle^p$ is an atomic space with atoms $a = \sum\limits_{i=1}^{n} \chi_{[t_i,t_{i+1})} e^{it\triangle} \phi_i$.*

For $1 \le p \le \infty$, the $V_\triangle^p$ norm is preserved under the linear evolution of data in $L^2$. In particular, it is easy to show that

$$||e^{it\triangle} u_0||_{V_\triangle^p} = ||u_0||_{V^2} = ||u_0||_{L_x^2}. \tag{2.1.9}$$

For a solution $u$ to to (1.1.1), the $V_\triangle^p$ norm is a measure of how far the solution is from the corresponding linear evolution. Similar constructions of spaces go back to [6], where the author introduces $X^{s,b}$ spaces. We skip the definition of $X^{s,b}$ space and define the homogeneous Besov refinement of such spaces, $\dot{X}^{s,b,q}$ by the norm

$$||u||_{\dot{X}^{s,b,q}} = \left( \sum_N N^{sq} ||(1+|\tau|^2)^{\frac{b}{2}} \mathscr{F}_{t,x}(u)||_{L_t^2 L_x^2}^q \right)^{\frac{1}{q}}, \tag{2.1.10}$$

where $\mathscr{F}_{t,x}$ is the space-time Fourier transform. If we define $X_\triangle^{s,b,q}$ by

$$||u||_{X_\triangle^{s,b,q}} = ||e^{-it\triangle} u||_{X^{s,b,q}}, \tag{2.1.11}$$

then we have the continuous embeddings ( [9]):

$$\dot{X}_\triangle^{0,\frac{1}{2},1} \subset U_\triangle^2 \subset V_{\triangle,RC}^2 \subset \dot{X}_\triangle^{0,\frac{1}{2},\infty}. \tag{2.1.12}$$

See [7] for a survey on $X^{s,b}$ spaces. $X^{s,b}$ spaces and their variants work well in subcritical settings, but for critical settings, $U_\triangle^p$ and $V_\triangle^p$ are successful replacements. Since our data is in Besov spaces, we introduce the following refinement:

**Definition 2.1.7.** *For $2 \le q \le \infty$, let $X^q(I)$ be the space of functions $u : I \times \mathbb{R}^3 \to \mathbb{C}$ such that for every dyadic N, $e^{-it\triangle} u_N \in V_{RC}^2$, equipped with the norm*

$$||u||_{X^q(I)} = \left( \sum_N N^q ||u_N||_{V_\triangle^2(I)}^q \right)^{\frac{1}{q}},$$

*for $2 \le q < \infty$ and with the norm*

$$||u||_{X^\infty(I)} = \sup_N \left( N ||u_N||_{V_\triangle^2(I)} \right),$$

*for $q = \infty$. This is the space where we will develop our well-posedness theory for data $u_0 \in \dot{B}_{2,q}^1$.*

**Lemma 2.1.8.** *For $2 \le q < p \le \infty$ and $I \subset \mathbb{R}$, we have the following continuous embeddings:*

$$X^q \subset X^p \subset L_t^\infty \dot{B}_{2,p}^1. \tag{2.1.13}$$

*Proof.* The first embedding just follows from $\ell^q \subset \ell^p$.

We now prove the second embedding. Interchanging the sum and supremum, we have

$$
\begin{aligned}
\|u\|_{L_t^\infty \dot{B}_{2,p}^1} &= \operatorname*{ess\,sup}_t \left( \left( \sum_N N^p \|u_N\|_{L_x^2}^p \right)^{\frac{1}{p}} \right) \\
&\leq \left( \sum_N N^p \operatorname*{ess\,sup}_t \|u_N\|_{L_x^2}^p \right)^{\frac{1}{p}}.
\end{aligned}
$$

Since $e^{it\triangle}$ is unitary, by Definition 2.1.1 and Definition 2.1.5, $\operatorname*{ess\,sup}_t \|u_N\|_{L_x^2} = \left\| e^{-it\triangle} u_N \right\|_{L_t^\infty L_x^2} \leq \|u_N\|_{V_\triangle^2}$. Thus, by Definition 2.1.7, we have

$$
\|u\|_{L_t^p \dot{B}_{2,\infty}^1} \leq \left( \sum_N N^p \|u_N\|_{V_\triangle^2}^p \right)^{\frac{1}{p}} = \|u\|_{X^p}. \tag{2.1.14}
$$

$\square$

**Lemma 2.1.9.** *We have the following continuous embedding:*

$$
X^2 \subset L_t^{10} L_x^{10}. \tag{2.1.15}
$$

*Proof.* By the Littlewood-Paley inequality (see [44]) and rewriting the space-time Lebesgue integral, we have

$$
\begin{aligned}
\|u\|_{10,10} &\sim \left\| \left( \sum_N |u_N|^2 \right)^{\frac{1}{2}} \right\|_{10,10} \\
&= \left( \int_{\mathbb{I} \times \mathbb{R}^3} \left( \sum_N |u_N|^2 \right)^5 dt\, dx \right)^{\frac{1}{10}},
\end{aligned}
$$

where the sum is over all dyadic numbers N. Expanding the fifth power of the sum, we have $\left( \sum_N |u_N|^2 \right)^5 = \sum_i |u_{i_1}|^2 \cdots |u_{i_5}|^2$, where $u_{i_k} = u_M$ for some $M$ and $\sum_i$ is a sum over $i_1, i_2, i_3, i_4, i_5$. Then by switching the sum and the integral, and using Hölder's inequality, we see

$$
\begin{aligned}
\|u\|_{10,10} &\sim \left( \int_{\mathbb{R} \times \mathbb{R}^3} \sum_i |u_{i_1}|^2 \cdots |u_{i_5}|^2 dt\, dx \right)^{\frac{1}{10}} \\
&= \left( \sum_i \int_{\mathbb{R} \times \mathbb{R}^3} |u_{i_1}|^2 \cdots |u_{i_5}|^2 dt\, dx \right)^{\frac{1}{10}} \\
&\leq \left( \sum_i \|u_{i_1}\|_{10,10}^2 \cdots \|u_{i_5}\|_{10,10}^2 \right)^{\frac{1}{10}}
\end{aligned}
$$

Now since $\left(\sum_N |u_N|^2\right)^5 = \sum_i |u_{i_1}|^2 \cdots |u_{i_5}|^2$, we have $\left(\sum_N \|u_N\|_{10,10}^2\right)^5 = \sum_i \|u_{i_1}\|_{10,10}^2 \cdots \|u_{i_5}\|_{10,10}^2$, and so

$$
\begin{aligned}
\|u\|_{10,10} &\lesssim \left(\left(\sum_N \|u_N\|_{10,10}^2\right)^5\right)^{\frac{1}{10}} \\
&= \left(\sum_N \|u_N\|_{10,10}^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Then by Proposition 2.2.2 (see below) and Lemma 2.1.3, we have

$$
\begin{aligned}
\|u\|_{10,10} &\lesssim \left(\sum_N N^2 \|u_N\|_{U_\triangle^{10}}^2\right)^{\frac{1}{2}} \\
&\lesssim \left(\sum_N N^2 \|u_N\|_{V_\triangle^2}^2\right)^{\frac{1}{2}} \\
&= \|u\|_{X^2}.
\end{aligned}
$$

$\square$

**Definition 2.1.10.** *We define the space $Y^q(I)$ as the space of functions $u : I \times \mathbb{R}^3 \to \mathbb{C}$, equipped with the norm* $\|u\|_{Y^q(I)} = \left(\sum_N N^{-q} \|u_N\|_{V_\triangle^2}^q\right)^{\frac{1}{q}}$, *where the sum is taken over all dyadic numbers $N$.*

This space will only be used in the duality estimate Lemma 2.3.2. If the interval $I$ is clear from context, we may omit it and write $\|u\|_{Y^q}$.

## 2.2   Stricharz Estimates

This subsection is devoted to generalized Strichartz estimates for the adapted function spaces introduced in Section 2.1. The following lemma is a direct consequence of the definition of our function spaces $X^q$, Definition 2.1.7.

**Lemma 2.2.1.** *For $2 \le q \le \infty$, we have the isometry*

$$
\|e^{it\triangle} u_0\|_{X^q(I)} = \|u_0\|_{\dot{B}_{2,q}^1}. \tag{2.2.1}
$$

*Proof.* By Definition 2.1.5 and Definition 2.1.7,

$$\|e^{it\triangle}u_0\|_{X^q(I)} = \left(\sum_N N^q \|e^{it\triangle}(u_0)_N\|_{V_\triangle^2}^q\right)^{\frac{1}{q}}$$

$$= \left(\sum_N N^q \max\left\{\|e^{-it\triangle}e^{it\triangle}(u_0)_N\|_{L_t^\infty L_x^2}^q, \sup_{\{t_i\}_{i=1}^n \in \mathscr{Z}}\left(\sum_{i=1}^n \|e^{it_{i+1}\triangle}e^{-it_{i+1}\triangle}(u_0)_N - e^{it_i\triangle}e^{-it_i\triangle}(u_0)_N\|_{L_x^2}^2\right)^{\frac{q}{2}}\right\}\right)^{\frac{1}{q}}.$$

Since $\|e^{it_{i+1}\triangle}e^{-it_{i+1}\triangle}(u_0)_N - e^{it_i\triangle}e^{-it_i\triangle}(u_0)_N\|_{L_x^2} = 0$ and $\|e^{-it\triangle}e^{it\triangle}(u_0)_N\|_{L_t^\infty L_x^2} = \|(u_0)_N\|_{L_x^2}$, we have

$$\|e^{it\triangle}u_0\|_{X^q(I)} = \left(\sum_N N^q \|(u_0)_N\|_{L_x^2}^q\right)^{\frac{1}{q}}$$

$$= \|u_0\|_{\dot{B}_{2,q}^1}.$$

$\square$

The linear propagator is an isometric mapping from $\dot{B}_{2,q}^1$ to $X^q$. This is in stark contrast to the behaviour of the mapping from Sobolev spaces to Strichartz spaces, which allows for smallness by shrinking the time interval. See Remark 1.4.3.

The next proposition relates the space-time Lebesgue norm to the $U_\triangle^p$ norm, while keeping track of scaling. Using the Littlewood-Paley projection $u_\lambda$ given in Equation (1.1.4), we have:

**Proposition 2.2.2.** *Let $\frac{3}{2} - \frac{2}{q} - \frac{3}{r} > 0$ and $I \subset \mathbb{R}$. The inequality*

$$\|u_\lambda\|_{q,r} \leq \lambda^{\frac{3}{2} - \frac{2}{q} - \frac{3}{r}} \|u_\lambda\|_{U_\triangle^q} \tag{2.2.2}$$

*holds.*

*Proof.* Due to the atomic structure of the spaces $U_\triangle^p$, it suffices to show Equation (2.2.2) for atoms. ie; we let $u_\lambda(t,x) = \sum_{i=1}^n \chi_{[t_i, t_{i+1})}(t)e^{it\triangle}\phi_i$, with $\sum_{i=1}^n \|\phi_i\|_{L_x^2}^q = 1$ and $\{t_i\}_{i=1}^n \in \mathscr{Z}$. It is clear from Definition 2.1.5 that $\|u_\lambda\|_{U_\triangle^q} = 1$. Thus, it will suffice to show that $\|u_\lambda\|_{q,r} \leq \lambda^{\frac{3}{2} - \frac{2}{q} - \frac{3}{r}}$.

By splitting up the time integration, we see

$$\|u_\lambda\|_{L_t^q L_x^r(I \times \mathbb{R}^3)}^q = \int_{\mathbb{R}} \|u_\lambda\|_{L_x^r(\mathbb{R}^3)}^q (t)dt$$

$$= \sum_{i=1}^n \int_{t_i}^{t_{i+1}} \|u_\lambda\|_{L_x^r(\mathbb{R}^3)}^q (t)dt$$

$$= \sum_{i=1}^n \int_{t_i}^{t_{i+1}} \|e^{it\triangle}\phi_i\|_{L_x^r(\mathbb{R}^3)}^q (t)dt.$$

Let $\tilde{r}$ be such that $(q,\tilde{r})$ is an $L^2-admissible$ pair, ie; $\frac{2}{q}+\frac{3}{\tilde{r}}=\frac{3}{2}$. Since $\frac{3}{2}-\frac{2}{q}-\frac{3}{r}>0, r>\tilde{r}$, we can use the Sobolev embedding theorem $\dot{W}^{k,\tilde{r}}\subset L^r$, where, because $q,\tilde{r}$ are an admissible pair, we can express $k$ as $k=\frac{3}{2}-\frac{2}{q}-\frac{3}{r}$. Since $\phi$ is localized in frequency to $\lambda$, the Sobolev Embedding Theorem tells us that $||e^{it\triangle}\phi_i||_{L^r_x(\mathbb{R}^3)}\leq\lambda^{\frac{3}{2}-\frac{2}{q}-\frac{3}{r}}\leq||e^{it\triangle}\phi_i||_{L^{\tilde{r}}_x(\mathbb{R}^3)}$. Then by Lemma 1.4.2,

$$
\begin{aligned}
||u_\lambda||^q_{L^q_t L^r_x(I\times\mathbb{R}^3)} &\leq \sum_{i=1}^n \int_{t_i}^{t_{i+1}} \lambda^{\frac{3q}{2}-\frac{2q}{q}-\frac{3q}{r}}||e^{it\triangle}\phi_i||^q_{L^{\tilde{r}}_x(\mathbb{R}^3)}(t)dt \\
&= \sum_{i=1}^n \lambda^{\frac{3q}{2}-2-\frac{3q}{r}}||e^{it\triangle}\phi_i||^q_{L^q_t L^{\tilde{r}}_x(I_i\times\mathbb{R}^3)} \\
&\leq \lambda^{\frac{3q}{2}-2-\frac{3q}{r}}\sum_{i=1}^n ||\phi_i||^q_{L^2_x(\mathbb{R}^3} \\
&\leq \lambda^{\frac{3q}{2}-2-\frac{3q}{r}}.
\end{aligned}
$$

Taking the $q^{th}$ root of both sides, we have our desired result. □

To prove the multilinear estimates in Chapter 3, we require a bilinear Strichartz estimate for Strichartz spaces. Multilinear refinements of this kind were first proved by Bourgain in [3] and used extensively in the literature thereafter, for example in [15]. The following can be found in [10]. Lemma 2.2.3 below is a version of the bilinear Strichartz estimate that we will use to prove Proposition 2.2.4, which is a refinement adapted to our function spaces.

**Lemma 2.2.3.** *(Bilinear Strichartz Estimate in Strichartz Spaces) Let $M\leq N$ be dyadic frequency scales. Then*

$$||e^{it\triangle}f_N e^{it\triangle}g_M||_{L^2_t L^2_x(\mathbb{R}\times\mathbb{R}^3)}\lesssim MN^{-\frac{1}{2}}||f||_{L^2_x(\mathbb{R}^3)}||g||_{L^2_x(\mathbb{R}^3)}. \tag{2.2.3}$$

Similar to [12], we have the following Bilinear Strichartz estimate which is the analogous result to the previous lemma, but refined for $V^2_\triangle$.

**Proposition 2.2.4.** *(Refined Bilinear Strichartz Estimate) Let $I\subset\mathbb{R}$, $\beta\leq\lambda$ be dyadic frequency scales, $u_\lambda,v_\beta\in V^2_\triangle$, then we have the bilinear estimate,*

$$||u_\lambda v_\beta||_{L^2_t L^2_x(I\times\mathbb{R}^3)}\leq\beta\lambda^{-\frac{1}{2}}||u_\lambda||_{V^2_\triangle}||v_\beta||_{V^2_\triangle}. \tag{2.2.4}$$

*Proof.* We in fact prove something slightly stronger. We will show

$$||u_\lambda v_\beta||_{L^2_t L^2_x(I\times\mathbb{R}^3)}\leq\lambda^{-\frac{1}{2}}\beta||u_\lambda||_{U^4_\triangle}||v_\beta||_{U^4_\triangle}. \tag{2.2.5}$$

This, together with the continuous embedding $V^2_\triangle\hookrightarrow U^4_\triangle$ will give us (2.2.4).
Similar to the proof of Lemma 1.4.2, we use the atomic structure of $U^4_\triangle$ and it will suffice to show Equation (2.2.5) for $u_\lambda(t,x)=\sum_{i=1}^n\chi_{[t_i,t_{i+1})}(t)e^{it\triangle}\phi_i$, with $\sum_{i=1}^n ||\phi_i||^4_{L^2_x}=1$ and $v_\beta(t,x)=\sum_{j=1}^m\psi_{[t_j,t_{j+1})}(t)e^{it\triangle}\psi_j$, with $\sum_{j=1}^m ||\psi_j||^4_{L^2_x}=1$. I.e. we must show

$$||u_\lambda v_\beta||_{L_t^2 L_x^2(I \times \mathbb{R}^3)} \leq \lambda^{-\frac{1}{2}}\beta. \tag{2.2.6}$$

Without loss of generality, we can assume the atoms have the same partition $\{t_i\} \in \mathscr{Z}$.

$$
\begin{aligned}
||u_\lambda v_\beta||_{2,2}^2 &= \int_{\mathbb{R}} ||u_\lambda v_\beta||_{L_x^2}^2(t)dt \\
&= \int_{\mathbb{R}} ||\sum_{i=1}^n \chi_{[t_i,t_{i+1})}(t)e^{it\triangle}\phi_i \sum_{i=1}^n \chi_{[t_i,t_{i+1})}(t)e^{it\triangle}\psi_i||_{L_x^2}^2(t)dt.
\end{aligned}
$$

The cross terms have disconnected support in time and we continue,

$$
\begin{aligned}
&\leq \int_{\mathbb{R}} ||\sum_{i=1}^n \chi_{[t_i,t_{i+1})}(t)e^{it\triangle}\phi_i e^{it\triangle}\psi_i||_{L_x^2}^2(t)dt \\
&= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} ||e^{it\triangle}\phi_{i-1}e^{it\triangle}\psi_{i-1}||_{L_x^2}^2(t)dt \\
&= \sum_{i=1}^n ||e^{it\triangle}\phi_{i-1}e^{it\triangle}\psi_{i-1}||_{L_t^2 L_x^2([t_{i-1},t_i) \times \mathbb{R}^3)}^2.
\end{aligned}
$$

We may assume $\phi_i$ and $\psi_i$ have the same frequency support as $u_\lambda$ and $v_\beta$. Then by Lemma 2.2.3 and Cauchy-Schwarz,

$$
\begin{aligned}
&\leq \sum_{i=1}^n (\lambda\beta^{-\frac{1}{2}})^2 ||\phi_{i-1}||_{L_x^2}^2 ||\psi_{i-1}||_{L_x^2}^2 \\
&\leq (\lambda\beta^{-\frac{1}{2}})^2 (\sum_{i=1}^n ||\phi_{i-1}||_{L_x^2}^4)^{\frac{1}{2}} (\sum_{i=1}^n ||\psi_{i-1}||_{L_x^2}^4)^{\frac{1}{2}} \\
&\leq (\lambda\beta^{-\frac{1}{2}})^2.
\end{aligned}
$$

Thus,

$$||u_\lambda v_\beta||_{L_t^2 L_x^2(I \times \mathbb{R}^3)} \leq \lambda^{-\frac{1}{2}}\beta ||u_\lambda||_{U_\triangle^4} ||v_\beta||_{U_\triangle^4},$$

and the proposition follows.                                                                                 $\square$

We can use Proposition 2.2.2 to show that $X^2$ embeds continuously into the Lebesgue space $L_t^{10}L_x^{10}$.

## 2.3   Duality Lemmas

The following duality arguments will be crucial for our multilinear estimates. They allow us the same maneuverability as the Strichartz estimates in Lemma 1.4.2 do when trying to prove well-posedness for finite

energy data.

**Lemma 2.3.1.** $||\int_{t_0}^{t} e^{i(t-t')\triangle}F(u)dt'||_{U_\triangle^2} = \sup\limits_{||v||_{V_\triangle^2}=1} |\int\limits_{(t,x)\in\mathbb{R}\times\mathbb{R}^3} F(u)vdtdx|.$

*Proof.* By Theorem 2.2.4, Equation (2.1.6) and The Fundamental Theorem of Calculus,

$$
\begin{aligned}
||\int_{t_0}^{t} e^{i(t-t')\triangle}F(u)dt'||_{U_\triangle^2} &= ||\int_{t_0}^{t} e^{-it'\triangle}F(u)dt'||_{U^2} \\
&= \sup_{||v||_{V^2}=1} |B(\int_{t_0}^{t} e^{-it'\triangle}F(u)dt', v)| \\
&= \sup_{||v||_{V^2}=1} |\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \left(\frac{d}{dt}\int_{t_0}^{t} e^{-it'\triangle}F(u)dt'\right)\bar{v}dxdt| \\
&= \sup_{||v||_{V^2}=1} |\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} e^{-it\triangle}F(u)\bar{v}dxdt| \\
&= \sup_{||v||_{V_\triangle^2}=1} |\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} F(u)\bar{v}dxdt|,
\end{aligned}
$$

where the last line follows from definition. □

Lemma 2.3.1 will suffice to prove the multilinear estimate we require for $q = \infty$. For $q < \infty$, we require another duality lemma. The proof below is adapted from the proof of Prop. 2.11 in [13]. Refer to Definition 2.1.10 for the definition of $Y^2$.

**Lemma 2.3.2.** $||\mathscr{I}u||_{X^2} = \sup\limits_{||v||_{Y^2}=1} |\int(|u|^4uv)dtdx|$

*Proof.* For ease of reading, we let $f(s) = (|u|^4u)(s)$. So we are trying to prove

$$||\mathscr{I}(u)||_{X^2} = \sup_{||v||_{Y^2}=1} |\int f(t,x)v(t,x)dtdx|. \tag{2.3.1}$$

By Definition 2.1.7, the duality of $\ell^2$ and Lemma 2.1.3,

$$
\begin{aligned}
||\mathscr{I}(u)||_{X^2} &= \left(\sum_N N||\int_{s<t} e^{i(t-s)\triangle}(f(s))_N ds||_{V_\triangle^2}\right) \\
&= \sup_{||b||_{\ell^2}=1} \sum_N b_N N||\int_{s<t} e^{i(t-s)\triangle}(f(s))_N ds||_{V_\triangle^2} \\
&\leq \sup_{||b||_{\ell^2}=1} \sum_N b_N N||\int_{s<t} e^{i(t-s)\triangle}(f(s))_N ds||_{U_\triangle^2}.
\end{aligned}
$$

Note that $\{b_N\}$ denotes an arbitrary $\ell^2$ sequence with norm 1. In particular, the subscript does not represent a Littlewood-Paley projection. By Lemma 2.3.1 and Plancherel's theorem,

$$||\mathscr{I}(u)||_{X^2} \;=\; \sup_{||b||_{\ell^2}=1} \sum_N b_N N \left[ \sup_{||v^N||_{V^2_\triangle}=1} \left| \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} f(t,x)(v(t,x))_N dxdt \right| \right].$$

Note that for each $N$, we must take the supremum over unit sized functions in $V^2_\triangle$. We denote this with a superscript $N$ so that it is not mistaken for a Littlewood-Paley projection. Define $\tilde{v}_b(t,x) = \sum_N b_N N v_N^N(t,x)$. Then

$$||\mathscr{I}(u)||_{X^2} \;\leq\; \sup_{||b||_{\ell^2}=1} \left( \sup_{||v||_{V^2_\triangle}=1} \left| \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} f(t,x)\tilde{v}_b(t,x)dxdt \right| \right). \qquad (2.3.2)$$

Our Lemma then follows by showing $\tilde{v}_b$ does indeed have $Y^2$ norm equal to 1 (although we will not argue for why this is the most general form of a function with $Y^2$ norm equal to 1). Indeed,

$$||\tilde{v}_b||_{Y^2} \;=\; \left( \sum_N N^{-1} ||(\tilde{v}_b)_N||_{V^2_\triangle} \right)^{\frac{1}{q}}$$

$$=\; \left( \sum_N N^{-1} || \left( \sum_M b_M M v_M^M(t,x) \right)_N ||_{V^2_\triangle} \right)^{\frac{1}{q}}.$$

Because we are taking the $N^{th}$ Littlewood-Paley projection, the sum collapses to just the $N^{th}$ term.

$$||\tilde{v}_b||_{Y^2} \;=\; \left( \sum_N N^{-1} ||b_N N v_N^N||_{V^2_\triangle} \right)^{\frac{1}{q}}$$

$$\leq\; \sup_N ||v_N^N||_{V^2_\triangle} \left( \sum_N b_N \right)^{\frac{1}{q}}$$

$$\leq\; \sup_N ||v_N^N||_{V^2_\triangle}.$$

For every $\varepsilon$, there exists a $\bar{N}$ such that

$$\sup_N ||v_N^N||_{V^2_\triangle} \leq ||v_{\bar{N}}^{\bar{N}}||_{V^2_\triangle} + \varepsilon \leq \sup_M ||v_M^{\bar{N}}||_{V^2_\triangle} + \varepsilon.$$

Since $||f_N||_{V^2_\triangle} \leq ||f||_{V^2_\triangle}$,

$$\sup_N ||v_N^N||_{V^2_\triangle} \leq ||v^{\bar{N}}||_{V^2_\triangle} + \varepsilon \leq 1 + \varepsilon,$$

and we can conclude $||\tilde{v}_b||_{Y^2} \leq 1$.

$\square$

# Chapter 3

# Multilinear Estimates

In this chapter we prove two multilinear estimates that will be fundamental in Section 4.2: Prop. 3.2.2 and Prop. 3.2.3, which are both given in Section 3.2. These estimates allow us to use fixed point arguments to obtain a local solution. For ease of reading, Prop. 3.2.2 will be split up in the following way. Prop. 3.1.1 focuses on the $q = \infty$ case, Prop. 3.1.2 focuses on the $q = 2$ case and Lemma 3.2.1 allows us to interpolate between the those cases to obtain the result in the cases $2 < q < \infty$. These are presented and proven in Section 3.1.

## 3.1 Multilinear Estimates in $X^\infty$ and $X^2$

**Proposition 3.1.1.** *Let $u \in X^\infty(I)$ for the time interval $I = [t_0, t_1)$. Then*

$$||\int_{t_0}^{t} e^{i(t-s)\triangle} \left( |u|^4 u \right)(s) ds||_{X^\infty(I)} \lesssim ||u||_{X^\infty(I)}^5. \tag{3.1.1}$$

**Proposition 3.1.2.** *Let $u \in X^2(I)$ for the time interval $I = [t_0, t_1)$. Then*

$$||\int_{t_0}^{t} e^{i(t-s)\triangle} \left( |u|^4 u \right)(s) ds||_{X^2(I)} \lesssim ||u||_{X^\infty(I)}^3 ||u||_{X^2(I)}^2. \tag{3.1.2}$$

We now prove Prop. 3.1.1.

*Proof.* We suppress the interval $I$ in what follows for ease of reading. Recall subscripts N refer to the Littlewood-Paley projection onto the dyadic frequency N. Using the Duhamel representation of the solution (Equation (1.1.9)), the triangle inequality and Lemma 2.3.1,

$$\left\| \int_{t_0}^{t} e^{i(t-s)\triangle} \left(|u|^4 u\right)(s)ds \right\|_{X^\infty} \quad = \quad \sup_N N \left\| \int_{t_0}^{t} e^{i(t-s)\triangle} ((|u|^4 u)_N)(s)ds \right\|_{V^2_\triangle}$$

$$\leq \quad \sup_N N \left\| \int_{t_0}^{t} e^{i(t-s)\triangle} ((|u|^4 u)_N)(s)ds \right\|_{U^2_\triangle}$$

$$= \quad \sup_N \left( \left| \sup_{\|v\|_{V^2_\triangle} =1} N \int_{I\times\mathbb{R}^3} (|u|^4 u)_N v dx dt \right| \right).$$

We may use Plancherel's theorem to transfer the Littlewood-Paley projection onto the solitary $v$ term. Therefore,

$$\left\| \int_{t_0}^{t} e^{i(t-s)\triangle} \left(|u|^4 u\right)(s)ds \right\|_{X^\infty} \leq \sup_N \left( \left| \sup_{\|v\|_{V^2_\triangle} =1} N \int_{I\times\mathbb{R}^3} (|u|^4 u) v_N dx dt \right| \right). \tag{3.1.3}$$

We use Plancherel's Theorem and decompose each factor of the nonlinearity into their Littlewood-Paley frequency projections and sum over all dyadic numbers for each factor. We then use Plancherel's Theorem once more to obtain

$$N \left| \int (|u|^4 u) v_N dx dt \right| \quad = \quad N \left| \int (\hat{u} * \hat{\bar{u}} * \hat{u} * \hat{\bar{u}} * \hat{u}) \hat{v}_N d\xi dt \right| \tag{3.1.4}$$

$$= \quad N \left| \int ((\hat{u} \sum_{\lambda_1} \chi_{\lambda_1}) * (\hat{\bar{u}} \sum_{\lambda_2} \chi_{\lambda_2}) * (\hat{u} \sum_{\lambda_3} \chi_{\lambda_3}) * (\hat{\bar{u}} \sum_{\lambda_4} \chi_{\lambda_4}) * (\hat{u} \sum_{\lambda_5} \chi_{\lambda_5})) \hat{v}_N d\xi dt \right|$$

$$= \quad N \left| \sum_{\lambda_1,\ldots,\lambda_5} \int (\hat{u}\chi_{\lambda_1} * \hat{\bar{u}}\chi_{\lambda_2} * \hat{u}\chi_{\lambda_3} * \hat{\bar{u}}\chi_{\lambda_4} * \hat{u}\chi_{\lambda_5}) \hat{v}_N d\xi dt \right|$$

$$= \quad N \left| \sum_{\lambda_1,\ldots,\lambda_5} \int u_{1,\lambda_1} u_{2,\lambda_2} u_{3,\lambda_3} u_{4,\lambda_4} u_{5,\lambda_5} v_N dx dt \right|,$$

where $\sum \chi_{\lambda_i}$ is a dyadic partition of unity. $\sum_{\lambda_1,\ldots,\lambda_5}$ is a sum ranging over all dyadic numbers $\lambda_i$. $u_{i,\lambda_i} := (u_i)_{\lambda_i}$ is the Littlewood-Paley projection onto the $\lambda_i^{th}$ frequency, where for convenience we write $u_i$ for either $u$ or $\bar{u}$ (this convenience becomes clear shortly). Due to symmetry, we may also assume $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_5$. We will break the sum into cases based on the position of $N$ with respect to all $\lambda_i$. Note that in each case, the largest two frequencies must be comparable or the support of the integral will be null. For example, if we assume $N < \lambda_1 < \ldots < \lambda_5$, then by Plancherel,

$$\int u_1 \cdots u_5 v_N dx dt = \int \widehat{u_1 \cdots u_4} \hat{u}_5 d\xi dt$$

$$= \int \left( \left( \left( (\hat{v}_N * \hat{u}_1) * \hat{u}_2 \right) * \hat{u}_3 \right) \hat{u}_4 \right) \hat{u}_5 d\xi dt$$

$$= \int \left( \int \left( \int \left( \int \left( \int \hat{v}_N(\xi - \mu - \eta - \beta - \alpha)\hat{u}_1(\alpha)d\alpha \right) \hat{u}_2(\beta)d\beta \right) \hat{u}_3(\eta)d\eta \right) \hat{u}_4(\mu)d\mu \right) \hat{u}_5(\xi)d\xi dt.$$

This integral is null unless

$$\frac{1}{2}\lambda_5 \leq |\xi| \leq 2\lambda_5, \quad \frac{1}{2}\lambda_4 \leq |\mu| \leq 2\lambda_4, \quad \frac{1}{2}\lambda_3 \leq |\eta| \leq 2\lambda_3, \quad \frac{1}{2}\lambda_2 \leq |\beta| \leq 2\lambda_2, \quad \frac{1}{2}\lambda_1 \leq |\alpha| \leq 2\lambda_1,$$
$$|\xi - \eta - \beta - \alpha| \leq 2\lambda_1,$$

which implies $\lambda_5 \leq 20\lambda_4$.

We decompose the sum on the right-hand side of (3.1.4) into five terms as follows:

$$\sum_{\lambda_1,\ldots,\lambda_5} \int u_{1,\lambda_1} u_{2,\lambda_2} u_{3,\lambda_3} u_{4,\lambda_4} u_{5,\lambda_5} v_N dx dt \quad \lesssim \quad \sum_1 \int u_{1,\lambda_1} u_{2,\lambda_2} u_{3,\lambda_3} u_{4,\lambda_4} u_{5,\lambda_5} v_N dx dt$$
$$+ \ldots +$$
$$\sum_5 \int u_{1,\lambda_1} u_{2,\lambda_2} u_{3,\lambda_3} u_{4,\lambda_4} u_{5,\lambda_5} v_N dx dt,$$

where

$$(\text{Master Sum}) \quad \sum_1 = \sum_{N < \lambda_1 < \infty} \sum_{\lambda_1 < \lambda_2 < \infty} \sum_{\lambda_2 < \lambda_3 < \infty} \sum_{\lambda_3 < \lambda_4 \sim \lambda_5 < \infty} \tag{3.1.5}$$

$$\sum_2 = \sum_{-\infty < \lambda_1 < N} \sum_{N < \lambda_2 < \infty} \sum_{\lambda_2 < \lambda_3 < \infty} \sum_{\lambda_3 < \lambda_4 \sim \lambda_5 < \infty}$$

$$\sum_3 = \sum_{-\infty < \lambda_2 < N} \sum_{N < \lambda_3 < \infty} \sum_{\lambda_3 < \lambda_4 \sim \lambda_5 < \infty} \sum_{-\infty < \lambda_1 < \lambda_2}$$

$$\sum_4 = \sum_{-\infty < \lambda_3 < N} \sum_{N < \lambda_4 \sim \lambda_5 < \infty} \sum_{-\infty < \lambda_2 < \lambda_3} \sum_{-\infty < \lambda_1 < \lambda_2}$$

$$\sum_5 = \sum_{-\infty < \lambda_4 < N \sim \lambda_5} \sum_{-\infty < \lambda_3 < \lambda_4} \sum_{-\infty < \lambda_2 < \lambda_3} \sum_{-\infty < \lambda_1 < \lambda_2}$$
.

For example, $\sum_1$ refers to the sum of frequencies such that $N < \lambda_1 < \ldots < \lambda_4$ and $\lambda_4 \sim \lambda_5$ in the sense that $\lambda_4 \leq \lambda_5 \leq 20\lambda_4$. We sum over $\lambda_4$ and $\lambda_5$ first, denoting by $\sum_{\lambda_3 < \lambda_4 \sim \lambda_5 < \infty}$, the sum over all dyadic numbers $\lambda_4$ that are between $\lambda_3$ (fixed in this first sum) and positive infinity. We continue in this way, however we do not need to distinguish between $\lambda_1 < \ldots < \lambda_4 < N < \lambda_5$ and $\lambda_1 < \ldots < \lambda_5 < N$, because in both cases, $\lambda_5 \sim N$. The order of these sums is important. The index $N$ must appear as an upper or lower bound in the last sum (or last two sums) in each case. Notice that in $\sum_5$, we do not need to sum over $\lambda_5$ as it is comparable to $N$ which is fixed.

By the triangle inequality, we have

$$N\left|\int (|u|^4 u)v_N dxdt\right| \leq \mathscr{S}_1 + \mathscr{S}_2 + \mathscr{S}_3 + \mathscr{S}_4 + \mathscr{S}_5,$$

where

$$\mathscr{S}_i = N\left|\sum_i \int u_{1,\lambda_1} u_{2,\lambda_2} u_{3,\lambda_3} u_{4,\lambda_4} u_{5,\lambda_5} v_N dxdt\right|.$$

We now bound $\mathscr{S}_i$ using Hölder's inequality, Proposition 2.2.2 and Lemma 2.1.3 . We begin with $\mathscr{S}_1$:

$$
\begin{aligned}
\mathscr{S}_1 &\leq N\sum_1 ||u_{1,\lambda_1}||_{6,6} \cdots ||u_{5,\lambda_5}||_{6,6}||v_N||_{6,6}\\
&\leq N\sum_1 (\lambda_1 \cdots \lambda_5 N)^{\frac{2}{3}} ||u_{1,\lambda_1}||_{U_\triangle^6} \cdots ||u_{5,\lambda_5}||_{U_\triangle^6}||v_N||_{U_\triangle^6}\\
&\leq N\sum_1 N^{\frac{2}{3}}(\lambda_1 \cdots \lambda_5)^{-\frac{1}{3}}(\lambda_1||u_{1,\lambda_1}||_{V_\triangle^2}) \cdots (\lambda_1||u_{5,\lambda_5}||_{V_\triangle^2})||v_N||_{V_\triangle^2}.
\end{aligned}
$$

We continue,

$$
\begin{aligned}
\mathscr{S}_1 &\leq N^{\frac{5}{3}}(\sup_{\lambda_1}\lambda_1||u_{1,\lambda_1}||_{V_\triangle^2}) \cdots (\sup_{\lambda_5}\lambda_5||u_{5,\lambda_5}||_{V_\triangle^2})||v_N||_{V_\triangle^2}\sum_1 (\lambda_1 \cdots \lambda_5)^{-\frac{1}{3}}\\
&= N^{\frac{5}{3}}||v_N||_{V_\triangle^2}(\sup_{\lambda}\lambda||u_\lambda||_{V_\triangle^2})^5 \sum_1 (\lambda_1 \cdots \lambda_5)^{-\frac{1}{3}} = N^{\frac{5}{3}}||v_N||_{V_\triangle^2}||u||_{X^\infty}^5 \sum_1 (\lambda_1 \cdots \lambda_5)^{-\frac{1}{3}}.
\end{aligned}
$$

where $\sum_1$ is defined in (3.1.5). To evaluate the sum $\sum_1 (\lambda_1 \cdots \lambda_5)^{-\frac{1}{3}}$, we write

$$\sum_{\lambda_3 < \lambda_4 \sim \lambda_5 < \infty} (\lambda_1 \cdots \lambda_5)^{-\frac{1}{3}} \lesssim \sum_{\lambda_3 < \lambda_5 < \infty} (\lambda_1 \lambda_2 \lambda_3)^{-\frac{1}{3}}\lambda_5^{-\frac{2}{3}} \lesssim (\lambda_1 \lambda_2 \lambda_3)^{-\frac{1}{3}}\lambda_3^{-\frac{2}{3}},$$

where we have used that for $\alpha < 0$, $\sum_{\lambda_i < \lambda_j < \infty} \lambda_j^\alpha \lesssim \lambda_i^\alpha$.

Consequently,

$$
\begin{aligned}
\sum_1 (\lambda_1 \cdots \lambda_5)^{-\frac{1}{3}} &\leq \sum_{N < \lambda_1 < \infty}\sum_{\lambda_1 < \lambda_2 < \infty}\sum_{\lambda_2 < \lambda_3 < \infty} (\lambda_1 \lambda_2 \lambda_3)^{-\frac{1}{3}}\lambda_3^{-\frac{2}{3}}\\
&\leq \sum_{N < \lambda_1 < \infty}\sum_{\lambda_1 < \lambda_2 < \infty} (\lambda_1 \lambda_2)^{-\frac{1}{3}}\lambda_2^{-\frac{3}{3}}\\
&\leq \sum_{N < \lambda_1 < \infty} (\lambda_1)^{-\frac{1}{3}}\lambda_1^{-\frac{4}{3}} \leq N^{\frac{5}{3}}||v_N||_{V_\triangle^2}||u||_{X^\infty}^5 N^{-\frac{5}{3}}.
\end{aligned}
$$

Returning to $\mathscr{S}_1$, we have

$$\mathscr{S}_1 \lesssim ||v_N||_{V_\triangle^2}||u||_{X^\infty}^5. \tag{3.1.6}$$

We proceed with the second sum. The technique is similar, but requires more finesse in choosing the right exponents to guarantee we are summing over dyadic numbers with negative exponents at each step. By Hölder's inequality, Proposition 2.2.2 and Lemma 2.1.3,

$$
\begin{aligned}
\mathscr{S}_2 \;\leq\;& N\sum_2 \|u_{1,\lambda_1}\|_{p,p}\|u_{2,\lambda_2}\|_{q,q}\cdots\|u_{5,\lambda_5}\|_{q,q}\|v_N\|_{q,q} \\[4pt]
\leq\;& N\sum_2 \lambda_1^{\frac{3}{2}-\frac{5}{p}}(\lambda_2\cdots\lambda_5 N)^{\frac{3}{2}-\frac{5}{q}}\|u_{1,\lambda_1}\|_{U_\triangle^p}\|u_{2,\lambda_2}\|_{U_\triangle^q}\cdots\|u_{5,\lambda_5}\|_{U^q}\|v_N\|_{U_\triangle^q} \\[4pt]
\leq\;& N\sum_2 N^{\frac{3}{2}-\frac{5}{q}}\lambda_1^{\frac{3}{2}-\frac{5}{p}-1}(\lambda_2\cdots\lambda_5)^{\frac{3}{2}-\frac{5}{q}-1}(\lambda_1\|u_{1,\lambda_1}\|_{V_\triangle^2})\cdots(\lambda_5\|u_{5,\lambda_5}\|_{V_\triangle^2})\|v_N\|_{V_\triangle^2} \\[4pt]
\leq\;& N^{\frac{3}{2}-\frac{5}{q}+1}\|v_N\|_{V_\triangle^2}\|u\|_{X^\infty}^5\sum_2 \lambda_1^{\frac{3}{2}-\frac{5}{p}-1}(\lambda_2\cdots\lambda_5)^{\frac{3}{2}-\frac{5}{q}-1},
\end{aligned}
$$

where we require $\frac{1}{p}+\frac{5}{q}=1$ for Hölder's inequality, $p,q>\frac{10}{3}$ for Proposition 2.2.2, $p,q>2$ for embedding (2.1.4) and $\frac{3}{2}-\frac{5}{p}>1$ for summability, as $\lambda_1$ is summed from negative infinite and hence requires a positive exponent. It suffices to choose $p=20$ and $q=5\times\frac{20}{19}$. We sum over $\lambda_5,\lambda_4,\lambda_3,\lambda_2,\lambda_1$, respectively:

$$
\begin{aligned}
\sum_2 \lambda_1^{\frac{1}{4}}(\lambda_2\cdots\lambda_5)^{-\frac{9}{20}} \;\lesssim\;& \sum_{-\infty<\lambda_1<N}\sum_{N<\lambda_2<\infty}\sum_{\lambda_2<\lambda_3<\infty}\lambda_1^{\frac{1}{4}}(\lambda_2\lambda_3)^{-\frac{9}{20}}\lambda_3^{-\frac{18}{20}} \\[4pt]
\leq\;& \sum_{-\infty<\lambda_1<N}\sum_{N<\lambda_2<\infty}\lambda_1^{\frac{1}{4}}\lambda_2^{-\frac{9}{20}}\lambda_2^{-\frac{27}{20}} \;\leq\; \sum_{-\infty<\lambda_1<N}(\lambda_1)^{\frac{1}{4}}N^{-\frac{36}{20}} \;\leq\; N^{-\frac{31}{20}},
\end{aligned}
$$

and so

$$
\mathscr{S}_2 \lesssim \|v_N\|_{V_\triangle^2}\|u\|_{X^\infty}^5. \tag{3.1.7}
$$

The third sum has both $\lambda_1$ and $\lambda_2$ summing from negative infinity and so $u_1$ and $u_2$ play the role that $u_1$ did in the previous sum. Choosing our exponents appropriately, by Hölder's inequality, Proposition 2.2.2 and Lemma 2.1.3,

$$
\begin{aligned}
\mathscr{S}_3 \;\leq\;& N\sum_3 \|u_{1,\lambda_1}\|_{20,20}\|u_{2,\lambda_2}\|_{20,20}\|u_{3,\lambda_5}\|_{\frac{40}{9},\frac{40}{9}}\cdots\|u_{5,\lambda_5}\|_{\frac{40}{9},\frac{40}{9}}\|v_N\|_{\frac{40}{9},\frac{40}{9}} \\[4pt]
\leq\;& N\sum_3 (\lambda_1\lambda_2)^{\frac{5}{4}}(\lambda_3\cdots\lambda_5 N)^{\frac{3}{8}}\|u_{1,\lambda_1}\|_{U_\triangle^{20}}\|u_{2,\lambda_2}\|_{U_\triangle^{20}}\|u_{3,\lambda_3}\|_{U_\triangle^{\frac{40}{9}}}\cdots\|u_{5,\lambda_5}\|_{U_\triangle^{\frac{40}{9}}}\|v_N\|_{U_\triangle^{\frac{40}{9}}} \\[4pt]
\leq\;& N\sum_3 N^{\frac{3}{8}}(\lambda_1\lambda_2)^{\frac{1}{4}}(\lambda_3\cdots\lambda_5)^{-\frac{5}{8}}(\lambda_1\|u_{1,\lambda_1}\|_{V_\triangle^2})\cdots(\lambda_1\|u_{5,\lambda_5}\|_{V_\triangle^2})\|v_N\|_{V_\triangle^2} \\[4pt]
\leq\;& N^{\frac{11}{8}}\|v_N\|_{V_\triangle^2}\|u\|_{X^\infty}^5\sum_3 (\lambda_1\lambda_2)^{\frac{1}{4}}(\lambda_3\cdots\lambda_5)^{-\frac{5}{8}}.
\end{aligned}
$$

We sum over $\lambda_1,\lambda_4,\lambda_5,\lambda_3,\lambda_2$, respectively:

$$
\begin{aligned}
\sum_3 (\lambda_1\lambda_2)^{\frac{1}{4}}(\lambda_3\cdots\lambda_5)^{-\frac{5}{8}} &\le \sum_{-\infty<\lambda_2<N}\sum_{N<\lambda_3<\infty}\sum_{\lambda_3<\lambda_4\sim\lambda_5<\infty} \lambda_2^{\frac{1}{2}}(\lambda_3\cdots\lambda_5)^{-\frac{5}{8}} \\
&\lesssim \sum_{-\infty<\lambda_2<N}\sum_{N<\lambda_3<\infty}\lambda_2^{\frac{1}{2}}(\lambda_3)^{-\frac{15}{8}} \le \sum_{-\infty<\lambda_2<N}\lambda_2^{\frac{1}{2}}N^{-\frac{15}{8}} \le N^{-\frac{11}{8}},
\end{aligned}
$$

and we obtain

$$
\mathscr{S}_3 \lesssim \|v_N\|_{V_\triangle^2}\|u\|_{X^\infty}^5. \tag{3.1.8}
$$

We continue with the fourth sum. By Hölder's inequality, Proposition 2.2.2 and Lemma 2.1.3,

$$
\begin{aligned}
\mathscr{S}_4 &\le N\sum_4 \|u_{1,\lambda_1}\|_{20,20}\|u_{2,\lambda_2}\|_{20,20}\|u_{3,\lambda_5}\|_{20,20}\|u_{4,\lambda_4}\|_{\frac{60}{17},\frac{60}{17}}\|u_{5,\lambda_5}\|_{\frac{60}{17},\frac{60}{17}}\|v_N\|_{\frac{60}{17},\frac{60}{17}} \\
&\le N\sum_4 (\lambda_1\lambda_2\lambda_3)^{\frac{5}{4}}(\lambda_4\lambda_5 N)^{\frac{1}{12}}\|u_{1,\lambda_1}\|_{U_\triangle^{20}}\|u_{2,\lambda_2}\|_{U_\triangle^{20}}\|u_{3,\lambda_3}\|_{U_\triangle^{20}}\|u_{4,\lambda_4}\|_{U_\triangle^{\frac{60}{17}}}\|u_{5,\lambda_5}\|_{U_\triangle^{\frac{60}{17}}}\|v_N\|_{U_\triangle^{\frac{60}{17}}} \\
&\le N\sum_4 N^{\frac{1}{12}}(\lambda_1\lambda_2\lambda_3)^{\frac{1}{4}}(\lambda_4\lambda_5)^{-\frac{11}{12}}(\lambda_1\|u_{1,\lambda_1}\|_{V_\triangle^2})\cdots(\lambda_1\|u_{5,\lambda_5}\|_{V_\triangle^2})\|v_N\|_{V_\triangle^2} \\
&\le N^{\frac{13}{12}}\|v_N\|_{V_\triangle^2}\|u\|_{X^\infty}^5\sum_4 (\lambda_1\lambda_2\lambda_3)^{\frac{1}{4}}(\lambda_4\lambda_5)^{-\frac{11}{12}}.
\end{aligned}
$$

We sum over $\lambda_1,\lambda_2,\lambda_4,\lambda_5,\lambda_3$, respectively:

$$
\begin{aligned}
\sum_4 (\lambda_1\lambda_2\lambda_3)^{\frac{1}{4}}(\lambda_4\lambda_5)^{-\frac{11}{12}} &\le \sum_{-\infty<\lambda_3<N}\sum_{N<\lambda_4\sim\lambda_5<\infty}\sum_{-\infty<\lambda_2<\lambda_3} \lambda_2^{\frac{1}{4}}(\lambda_2\lambda_3)^{\frac{1}{4}}(\lambda_4\lambda_5)^{-\frac{11}{12}} \\
&\le \sum_{-\infty<\lambda_3<N}\sum_{N<\lambda_4\sim\lambda_5<\infty} \lambda_3^{\frac{3}{4}}(\lambda_4\lambda_5)^{-\frac{11}{12}} \\
&\lesssim \sum_{-\infty<\lambda_3<N} \lambda_3^{\frac{3}{4}}N^{-\frac{22}{12}} \le N^{-\frac{13}{12}}.
\end{aligned}
$$

This gives us

$$
\mathscr{S}_4 \lesssim \|v_N\|_{V_\triangle^2}\|u\|_{X^\infty}^5. \tag{3.1.9}
$$

The fifth sum requires the Bilinear Strichartz Estimate, Proposition 2.2.4 in addition to Hölder, Proposition 2.2.2, and Lemma 2.1.3,

$$
\begin{aligned}
\mathscr{S}_5 &\leq N\sum_5 ||u_{4,\lambda_4} v_N||_{2,2} ||u_{1,\lambda_1}||_{\frac{54}{5},\frac{54}{5}} \cdots ||u_{3,\lambda_3}||_{\frac{54}{5},\frac{54}{5}} ||u_{5,\lambda_5}||_{\frac{9}{2},\frac{9}{2}} \\
&\leq N\sum_5 \lambda_4 N^{-\frac{1}{2}} (\lambda_1\lambda_2\lambda_3)^{\frac{28}{27}} \lambda_5^{\frac{7}{18}} ||u_{4,\lambda_4}||_{V_\triangle^2} ||v_N||_{V_\triangle^2} ||u_{1,\lambda_1}||_{U_\triangle^{\frac{54}{5}}} \cdots ||u_{3,\lambda_3}||_{U_\triangle^{\frac{54}{5}}} ||u_{5,\lambda_5}||_{U_\triangle^{\frac{9}{2}}} \\
&\leq N\sum_5 N^{-\frac{1}{2}} (\lambda_1\lambda_2\lambda_3)^{\frac{1}{27}} \lambda_5^{-\frac{11}{18}} (\lambda_1 ||u_{1,\lambda_1}||_{V_\triangle^2}) \cdots (\lambda_1 ||u_{5,\lambda_5}||_{V_\triangle^2}) ||v_N||_{V_\triangle^2} \\
&\leq N^{\frac{1}{2}} ||v_N||_{V_\triangle^2} ||u||_{X^\infty}^5 \sum_5 (\lambda_1\lambda_2\lambda_3)^{\frac{1}{27}} \lambda_5^{-\frac{11}{18}}.
\end{aligned}
$$

We sum over $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ respectively:

$$
\begin{aligned}
\sum_5 (\lambda_1\lambda_2\lambda_3)^{\frac{1}{27}} \lambda_5^{-\frac{11}{18}} &\leq \sum_{-\infty<\lambda_4<N\sim\lambda_5} \sum_{-\infty<\lambda_3<\lambda_4} \sum_{-\infty<\lambda_2<\lambda_3} \lambda_2^{\frac{1}{27}} (\lambda_2\lambda_3)^{\frac{1}{27}} \lambda_5^{-\frac{11}{18}} \\
&\leq \sum_{-\infty<\lambda_4<N\sim\lambda_5} \sum_{-\infty<\lambda_3<\lambda_4} \lambda_3^{\frac{1}{9}} \lambda_5^{-\frac{11}{18}} \\
&\leq \sum_{-\infty<\lambda_4<N\sim\lambda_5} \lambda_4^{\frac{1}{9}} \lambda_5^{-\frac{11}{18}} \lesssim \lambda_5^{\frac{1}{9}} \lambda_5^{-\frac{11}{18}} \leq N^{-\frac{1}{2}},
\end{aligned}
$$

and so

$$
\mathscr{S}_5 \lesssim ||v_N||_{V_\triangle^2} ||u||_{X^\infty}^5. \tag{3.1.10}
$$

Combining (3.1.6), (3.1.7), (3.1.8), (3.1.9), (3.1.10), we have

$$
N\left| \int (|u|^4 u) v_N \, dx dt \right| \lesssim ||v_N||_{V_\triangle^2} ||u||_{X^\infty}^5. \tag{3.1.11}
$$

Returning to inequality (3.1.3), we get

$$
|| \int_{t_0}^t e^{i(t-s)\triangle} \left( |u|^4 u \right)(s) ds ||_{X^\infty} \lesssim \sup_N \left( \sup_{||v||_{V_\triangle^2}=1} ||v||_{V_\triangle^2} ||u||_{X^\infty}^5 \right).
$$

Since $||v_N||_{V_\triangle^2} \leq ||v||_{V_\triangle^2}$, we have

$$
|| \int_{t_0}^t e^{i(t-s)\triangle} \left( |u|^4 u \right)(s) ds ||_{X^\infty} \lesssim ||u||_{X^\infty}.
$$

$\square$

We now prove Prop. 3.1.2.

*Proof.* We suppress the interval $I$ for ease of reading. Using the Duhamel representation of the solution, Lemma 2.3.2, and breaking the sum into its Littlewood-Paley frequency projections,

$$||\int_{t_0}^t e^{i(t-s)\triangle}\left(|u|^4 u\right)(s)ds||_{X^2} = \sup_{||v||_{Y^2}=1}|\int_{I\times\mathbb{R}^3}(|u|^4 uv)dxdt|$$

$$= \sup_{||v||_{Y^2}=1}|\sum_{\lambda_i,N}\int_{I\times\mathbb{R}^3}u_{1,\lambda_1}\cdots u_{5,\lambda_5}v_N dtdx|,$$

where the sum $\sum_{\lambda_i,N}$ is over all dyadic scales $\lambda_i$, $i=1,\ldots,5$ and N. Notice that our duality lemma in this case, Lemma 2.3.2 works with the whole space $X^2$, not just $U^2_\triangle$. This is in contrast to our previous multilinear estimate, Prop. 3.1.1. Because of this, we must decompose $v$ into frequency projections $v_N$ and sum over them as well.

We break up the sum into five sub-sums, based on the relative position of the various $\lambda_i$ and $N$:

$$\sum_{\lambda_i,N}\int_{I\times\mathbb{R}^3}u_{1,\lambda_1}\cdots u_{5,\lambda_5}v_N dtdx \;\lesssim\; \sum_1\int_{I\times\mathbb{R}^3}u_{1,\lambda_1}\cdots u_{5,\lambda_5}v_N dtdx$$
$$+\ldots+$$
$$\sum_5\int_{I\times\mathbb{R}^3}u_{1,\lambda_1}\cdots u_{5,\lambda_5}v_N dtdx,$$

where

$$\sum_1 = \sum_{-\infty<\lambda_5(\sim\lambda_4)<\infty}\sum_{-\infty<\lambda_3<\lambda_5}\sum_{-\infty<\lambda_2<\lambda_3}\sum_{-\infty<\lambda_1<\lambda_2}\sum_{-\infty<N<\lambda_1} \qquad (3.1.12)$$

$$\sum_2 = \sum_{-\infty<\lambda_5(\sim\lambda_4)<\infty}\sum_{-\infty<\lambda_3<\lambda_5}\sum_{-\infty<\lambda_2<\lambda_3}\sum_{-\infty<N<\lambda_2}\sum_{-\infty<\lambda_1<N}$$

$$\sum_3 = \sum_{-\infty<\lambda_5(\sim\lambda_4)<\infty}\sum_{-\infty<\lambda_3<\lambda_5}\sum_{-\infty<N<\lambda_3}\sum_{-\infty<\lambda_2<N}\sum_{-\infty<\lambda_1<\lambda_2}$$

$$\sum_4 = \sum_{-\infty<\lambda_5(\sim\lambda_4)<\infty}\sum_{-\infty<N<\lambda_5}\sum_{-\infty<\lambda_3<N}\sum_{-\infty<\lambda_2<\lambda_3}\sum_{-\infty<\lambda_1<\lambda_2}$$

$$\sum_5 = \sum_{-\infty<N(\sim\lambda_5)<\infty}\sum_{-\infty<\lambda_4<N}\sum_{-\infty<\lambda_3<\lambda_4}\sum_{-\infty<\lambda_2<\lambda_3}\sum_{-\infty<\lambda_1<\lambda_2}.$$

Note that in each case, the largest two frequencies must be comparable or the integral will be 0.

We can treat $\sum_1$ through $\sum_4$ in a similar way. For $i\in\{1,2,3,4\}$, we must bound $\sup_{||v||_{Y^2}=1}|\sum_i\int u_{1,\lambda_1}\cdots u_{5,\lambda_5}v_N dtdx|=:\mathscr{S}_i$. Again, we require the more refined Strichartz estimate in this case, Bilinear Strichartz Proposition 2.2.4. By Hölder, Proposition 2.2.4, Proposition 2.2.2 and Lemma 2.1.3,

$$\mathscr{S}_i \;=\; \sup_{||v||_{Y^2}=1} |\sum_i \int_{I\times\mathbb{R}^3} u_{1,\lambda_1}\cdots u_{4,\lambda_4}u_{5,N}v_N dtdx|$$

$$\leq \sup_{||v||_{Y^2}=1}\sum_i ||u_{5,\lambda_5}v_N||_{2,2}||u_{1,\lambda_1}||_{\frac{54}{5},\frac{54}{5}}||u_{2,\lambda_2}||_{\frac{54}{5},\frac{54}{5}}||u_{3,\lambda_3}||_{\frac{54}{5},\frac{54}{5}}||u_{4,\lambda_4}||_{\frac{9}{2},\frac{9}{2}}$$

$$\leq \sup_{||v||_{Y^2}=1}\sum_i N\lambda_5^{-\frac{1}{2}}(\lambda_1\lambda_2\lambda_3)^{\frac{28}{27}}\lambda_4^{\frac{7}{18}}||u_{5,\lambda_5}||_{V_\triangle^2}||v_N||_{V_\triangle^2}||u_{1,\lambda_1}||_{U_\triangle^{\frac{54}{5}}}\cdots||u_{3,\lambda_3}||_{U_\triangle^{\frac{54}{5}}}||u_{4,\lambda_4}||_{U_\triangle^{\frac{9}{2}}}$$

$$\leq \sup_{||v||_{Y^2}=1}\sum_i N^2(\lambda_1\lambda_2\lambda_3)^{\frac{1}{27}}\lambda_5^{-\frac{19}{9}}(N^{-1}||v_N||_{V_\triangle^2})(\lambda_1||u_{1,\lambda_1}||_{V_\triangle^2})\cdots(\lambda_5||u_{5,\lambda_5}||_{V_\triangle^2})$$

Grouping the terms and pulling out the supremum over $\lambda_i||u_{i,\lambda_i}||_{V_\triangle^2}$ for $i=1,2,3$, we have

$$\mathscr{S}_i \;\leq\; \sup_{||v||_{Y^2}=1}\Bigg(\sup_{\lambda_1}\lambda_1||u_{1,\lambda_1}||_{V_\triangle^2})\cdots(\sup_{\lambda_3}\lambda_3||u_{3,\lambda_3}||_{V_\triangle^2})(\sup_N N^{-1}||v_N||_{V_\triangle^2})\times$$

$$\times\sum_i N^2(\lambda_1\lambda_2\lambda_3)^{\frac{1}{27}}\lambda_5^{-\frac{19}{9}}(\lambda_4||u_{4,\lambda_4}||_{V_\triangle^2})(\lambda_5||u_{5,\lambda_5}||_{V_\triangle^2})\Bigg).$$

We observe $\sup_N N^{-1}||v_N||_{V_\triangle^2}\leq ||v||_{Y^2}$ (this is just the embedding $\ell^2\subset\ell^\infty$). We then sum over $\lambda_1,\lambda_2,\lambda_3$ and N in the order (dependent on $i$) prescribed by Equation (3.1.12), analogous to the sums in Prop. 3.1.1. In each case ($i=1,2,3,4$), we are left with the final sum which is over $\lambda_4\sim\lambda_5$,

$$\mathscr{S}_i \;\lesssim\; \sup_{||v||_{Y^2}=1}||u||_{X^\infty}^3||v||_{Y^2}\sum_{-\infty<\lambda_4(\sim\lambda_5)<\infty}(\lambda_4||u_{4,\lambda_4}||_{V_\triangle^2})(\lambda_5||u_{5,\lambda_5}||_{V_\triangle^2})$$

$$\lesssim\; ||u||_{X^\infty}^3\sum_{-\infty<\lambda_4(\sim\lambda_5)<\infty}(\lambda_4||u_{4,\lambda_4}||_{V_\triangle^2})(\lambda_5||u_{5,\lambda_5}||_{V_\triangle^2}).$$

By Cauchy-Schwarz, we have

$$\mathscr{S}_i \;\lesssim\; ||u||_{X^\infty}^3(\sum_{-\infty<\lambda_4<\infty}\lambda_4^2||u_{4,\lambda_4}||_{V_\triangle^2}^2)^{\frac{1}{2}}(\sum_{-\infty<\lambda_5<\infty}\lambda_5^2||u_{5,\lambda_5}||_{V_\triangle^2}^2)^{\frac{1}{2}}$$

$$=\; ||u||_{X^\infty}^3||u||_{X^2}^2.$$

The last sum can be treated in the following way. Letting $\mathscr{S}_5 := \sup_{||v||_{Y^2}=1}|\sum_5\int_{I\times\mathbb{R}^3}u_{1,\lambda_1}\cdots u_{5,\lambda_5}v_N dtdx|$, by Hölder, Proposition 2.2.4, Proposition 2.2.2 and Lemma 2.1.3,

$$
\begin{aligned}
\mathscr{S}_5 &= \sup_{||v||_{Y^2}=1} |\sum_{1} \int_{I \times \mathbb{R}^3} u_{1,\lambda_1} \cdots u_{4,\lambda_4} u_{5,N} v_N \, dx dt| \\
&\leq \sup_{||v||_{Y^2}=1} \sum_{5} ||u_{4,\lambda_4} v_N||_{2,2} ||u_{1,\lambda_1}||_{\frac{54}{5},\frac{54}{5}} ||u_{2,\lambda_2}||_{\frac{54}{5},\frac{54}{5}} ||u_{3,\lambda_3}||_{\frac{54}{5},\frac{54}{5}} ||u_{5,\lambda_5}||_{\frac{9}{2},\frac{9}{2}} \\
&\leq \sup_{||v||_{Y^2}=1} \sum_{5} \lambda_4 N^{-\frac{1}{2}} (\lambda_1 \lambda_2 \lambda_3)^{\frac{28}{27}} \lambda_5^{\frac{7}{18}} ||u_{4,\lambda_4}||_{V^2_\triangle} ||v_N||_{V^2_\triangle} ||u_{1,\lambda_1}||_{U^{\frac{54}{5}}_\triangle} \cdots ||u_{3,\lambda_3}||_{U^{\frac{54}{5}}_\triangle} ||u_{5,\lambda_5}||_{U^{\frac{9}{2}}_\triangle} \\
&\leq \sup_{||v||_{Y^2}=1} \sum_{5} N^{-\frac{1}{9}} (\lambda_1 \lambda_2 \lambda_3)^{\frac{1}{27}} (N^{-1} ||v_N||_{V^2_\triangle}) (\lambda_1 ||u_{1,\lambda_1}||_{V^2_\triangle}) \cdots (\lambda_5 ||u_{5,\lambda_5}||_{V^2_\triangle}).
\end{aligned}
$$

Pulling out the supremum's over $\lambda_i ||u_{i,\lambda_i}||_{V^2_\triangle}$, for $i = 1, 2, 3, 4$ and summing over $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, as prescribed by Equation (3.1.12), we have

$$
\begin{aligned}
\mathscr{S}_5 &\leq \sup_{||v||_{Y^2}=1} (\sup_{\lambda_1} \lambda_1 ||u_{1,\lambda_1}||_{V^2_\triangle}) \cdots (\sup_{\lambda_4} \lambda_4 ||u_{4,\lambda_4}||_{V^2_\triangle}) \sum_{5} N^{-\frac{1}{9}} (\lambda_1 \lambda_2 \lambda_3)^{\frac{1}{27}} (N^{-1} ||v_N||_{V^2_\triangle}) (\lambda_5 ||u_{5,\lambda_5}||_{V^2_\triangle}) \\
&\leq \sup_{||v||_{Y^2}=1} ||u||^4_{X^\infty} \sum_{-\infty < N(\sim \lambda_5) < \infty} (N^{-1} ||v_N||_{V^2_\triangle}) (\lambda_5 ||u_{5,\lambda_5}||_{V^2_\triangle}).
\end{aligned}
$$

We apply Cauchy-Schwarz and obtain

$$
\begin{aligned}
\mathscr{S}_5 &\leq \sup_{||v||_{Y^2}=1} ||u||^4_{X^\infty} (\sum_{-\infty < N < \infty} N^{-2} ||v_N||^2_{V^2_\triangle})^{\frac{1}{2}} (\sum_{-\infty < \lambda_5 < \infty} \lambda_5^2 ||u_{5,\lambda_5}||^2_{V^2_\triangle})^{\frac{1}{2}} \\
&\leq \sup_{||v||_{Y^2}=1} ||u||^4_{X^\infty} ||u||_{X^2} ||v||_{Y^2} \\
&\leq ||u||^4_{X^\infty} ||u||_{X^2}.
\end{aligned}
$$

Thus,

$$
\sup_{||v||_{Y^2}=1} |\sum_{(\lambda_i, N)} \int_{I \times \mathbb{R}^3} u_{1,\lambda_1} \cdots u_{5,\lambda_5} v_N \, dx dt| \leq ||u||^4_{X^\infty} ||u||_{X^2}. \tag{3.1.13}
$$

This is in fact stronger than we need for our Lemma, since $X^2$ embeds continuously into $X^\infty$. Combining these cases, we have $|| \int_{t_0}^{t} e^{i(t-s)\triangle} (|u|^4 u)(s) ds ||_{X^2} \lesssim ||u||^3_{X^\infty} ||u||^2_{X^2}$. $\qquad\square$

## 3.2   Multilinear Estimates in $X^q$ and $L^{10} L^{10}$

We require the following multilinear interpolation result for the multilinear estimates in the range $(2, \infty)$. This lemma is taken from [23] and can also be found in [24]. The proof can be found in those sources as well. If $A$

and $B$ are a compatible interpolation pair, we let $[A,B]_s$ denote the complex interpolation pair derived from $A$ an $B$. See [24] and [23] for more details.

**Lemma 3.2.1.** *Let $(A_i, B_i)$, $i = 1, 2, \ldots, n$ and $(A, B)$ be interpolation pairs. Let $L(x_1, \ldots, x_n)$, $x_i \in A_i \cap B_i$, be a multilinear operation defined in the direct sum $\oplus_{i=1}^n (A_i \cap B_i)$ with values in $A \cap B$ and such that*

$$||L(x_1, \ldots, x_n)||_A \leq M_0 \prod_1^n ||x_i||_{A_i} \tag{3.2.1}$$

*and*

$$||L(x_1, \ldots, x_n)||_B \leq M_1 \prod_1^n ||x_i||_{B_i}. \tag{3.2.2}$$

*Then if $C = [A, B]_s$ and $C_i = [A_i, B_i]_s$, we have*

$$||L(x_1, \ldots, x_n)||_C \leq M_0^{1-s} M_1^s \prod_1^n ||x_i||_{C_i}, \tag{3.2.3}$$

*for $0 \leq s \leq 1$.*

**Proposition 3.2.2.** *For $2 \leq q \leq \infty$, let $u \in X^q(I)$ for the time interval $I = [t_0, t_1)$. Then*

$$|| \int_{t_0}^t e^{i(t-s)\triangle} \left( |u|^4 u \right)(s) ds ||_{X^q(I)} \lesssim ||u||_{X^\infty(I)}^3 ||u||_{X^q(I)}^2. \tag{3.2.4}$$

*Proof.* We apply Lemma 3.2.1. Any two of $X^\infty \subset X^q \subset X^2$ are compatible interpolation pairs. Furthermore, (by [38], Theorem 9.3.2 for example) we have that for some $s$, $[X^2, X^\infty]_s = X^q$. From the proof of Prop. 3.1.1, we see that the operator $\mathscr{I}(u) := \int_{t_0}^t e^{i(t-s)\triangle}(|u|^4 u)(s) ds$ can be extended to a bounded multilinear operator (which by abuse of notation, we also refer to as $\mathscr{I}$),

$$\mathscr{I} : X^\infty \times X^\infty \times X^\infty \times X^\infty \times X^\infty \to X^\infty \tag{3.2.5}$$

$$\mathscr{I}(u_1, u_2, u_3, u_4, u_5) = \int_{t_0}^t \left( e^{i(t-s)\triangle} u_1 \bar{u}_2 u_3 \bar{u}_4 u_5 \right)(s) ds. \tag{3.2.6}$$

Similarly, we can extend $\mathscr{I}$ to a multilinear mapping defined on $X^2 \times X^2 \times X^\infty \times X^\infty \times X^\infty$,

$$\mathscr{I} : X^2 \times X^2 \times X^\infty \times X^\infty \times X^\infty \to X^2 \tag{3.2.7}$$

$$\mathscr{I}(u_1, u_2, u_3, u_4, u_5) = \int_{t_0}^t \left( e^{i(t-s)\triangle} u_1 \bar{u}_2 u_3 \bar{u}_4 u_5 \right)(s) ds. \tag{3.2.8}$$

By Prop. 3.1.1 and Prop. 3.1.2 we have the estimates

$$||\mathscr{I}(u_1,\ldots u_5)||_{X^\infty} \leq M_0 ||u_1||_{X^\infty} ||u_2||_{X^\infty} ||u_3||_{X^\infty} ||u_4||_{X^\infty} ||u_5||_{X^\infty} \tag{3.2.9}$$

$$||\mathscr{I}(u_1,\ldots u_5)||_{X^2} \leq M_1 ||u_1||_{X^2} ||u_2||_{X^2} ||u_3||_{X^\infty} ||u_4||_{X^\infty} ||u_5||_{X^\infty}. \tag{3.2.10}$$

If we take $L = \mathscr{I}$, $A = A_i = X^\infty$, for $i = 1,\ldots,5$, $B = B_1 = B_2 = X^2$ and $B_3 = B_4 = B_5 = X^\infty$, then we may apply Prop. 3.1.1. In particular, for $q \in (2,\infty)$, if $s$ is such that $X^q = [X^2, X^\infty]_s$, then from Lemma 3.2.1 we obtain the estimate

$$||\mathscr{I}(u_1,\ldots,u_5)||_{X^q} \leq M_0^{1-s} M_1^s ||u_1||_{X^\infty} ||u_2||_{X^\infty} ||u_3||_{X^\infty} ||u_4||_{X^q} ||u_5||_{X^q}. \tag{3.2.11}$$

$$\square$$

In Chapter 4 we will prove some well-posedness results. These will require a standard fixed-point argument, which will require smallness in some norm. In Theorem 4.2.1, this requirement is satisfied by the assumed smallness of the initial data. In Theorem 4.2.2 and Theorem 4.2.4, we use the fact that nonlinearity contains terms that are in $L^{10}L^{10}$. We require a multilinear estimate with one of the factors in $L^{10}L^{10}$, so that by shrinking the time interval, we can ensure these terms are small. See also Remark 1.4.3. This multilinear estimate will also be required for the stability theorem, Theorem 4.2.6 for similar reasons.

**Proposition 3.2.3.** *For $2 \leq q \leq \infty$, let $u \in X^q(I) \cap L_t^{10} L_x^{10}(I \times \mathbb{R}^3)$ for the time interval $I = [t_0, t_1)$. Then*

$$|| \int_{t_0}^t e^{i(t-s)\triangle} \left(|u|^4 u\right)(s)ds ||_{X^q(I)} \lesssim ||u||_{L_t^{10} L_x^{10}(I \times \mathbb{R}^3)} ||u||_{X^\infty(I)}^2 ||u||_{X^q(I)}^2. \tag{3.2.12}$$

*Proof.* The proof is similar to the proof of Prop. 3.2.2. This time we will not split up the proof into two lemmas. We drop the interval for ease of reading and begin with $q = \infty$.

**q = ∞:**

By Lemma 2.3.1 and decomposing the each function into its Littlewood-Paley projections as in Prop. 3.1.1 and Prop. 3.1.2, we have

$$
\begin{aligned}
|| \int_{t_0}^t e^{i(t-s)\triangle} \left(|u|^4 u\right)(s)ds ||_{X^\infty} &\leq \sup_N \left( \sup_{||v||_{V_\triangle^2}=1} N \left| \int_{I \times \mathbb{R}^3} |u|^4 u v_N dx dt \right| \right) \\
&\leq \sup_N \left( \sup_{||v||_{V_\triangle^2}=1} N \sum_{\lambda_i} \left| \int u u_{1,\lambda_1} u_{2,\lambda_2} u_{3,\lambda_3} u_{4,\lambda_4} v_N dx dt \right| \right),
\end{aligned}
$$

where the sum is over all dyadic numbers $\lambda_i$, $i = 1,2,3,4$, and $u_i$ denotes either $u$ or $\bar{u}$. Notice we do not break the first $u$ into frequencies. We break up the sum into parts depending on the relationship of $N$ compared to $\lambda_i$. This time we will only show the cases where $N > \lambda_i$ for $i = 1,2,3,4$ and where $N < \lambda_i$ for $i = 1,2,3,4$.

<u>Case 1 :</u> $N < \lambda_i, i = 1,2,3,4$. I.e. we consider the sum

$$\sum_1 = \sum_{N<\lambda_1<\infty} \sum_{\lambda_1<\lambda_2<\infty} \sum_{\lambda_2<\lambda_3\sim\lambda_4<\infty} . \tag{3.2.13}$$

By Hölder's inequality, Proposition 2.2.2 and Lemma 2.1.3,

$$
\begin{aligned}
\mathscr{S}_1 \;\; := \;\; & \sup_N \left( \sup_{||v||_{V_\triangle^2}=1} N\sum_1 \left| \int u u_{2,\lambda_2} u_{3,\lambda_3} u_{4,\lambda_4} v_N dx dt \right| \right) \\
\leq \;\; & \sup_N \left( \sup_{||v||_{V_\triangle^2}=1} N\sum_1 ||u||_{10,10} \prod_{i=1}^4 ||u_{i,\lambda_i}||_{\frac{50}{9},\frac{50}{9}} ||v_N||_{\frac{50}{9},\frac{50}{9}} \right) \\
\leq \;\; & \sup_N \left( \sup_{||v||_{V_\triangle^2}=1} N\sum_1 ||u||_{10,10} \prod_{i=1}^4 \lambda_i^{\frac{6}{10}} ||u_i||_{U_\triangle^{\frac{50}{9}}} N^{\frac{6}{10}} ||v_N||_{U_\triangle^{\frac{50}{9}}} \right) \\
\leq \;\; & \sup_N \left( \sup_{||v_N||_{V_\triangle^2}=1} N\sum_1 ||u||_{10,10} \prod_{i=1}^4 \lambda_i^{\frac{6}{10}} ||u_i||_{V_\triangle^2} N^{\frac{6}{10}} ||v_N||_{V_\triangle^2} \right) .
\end{aligned}
$$

By Definition 2.1.7, we have

$$
\begin{aligned}
\mathscr{S}_1 \;\; \leq \;\; & ||u||_{10,10} ||u||_{X^\infty} \sup_N \left( \sup_{||v||_{V_\triangle^2}=1} ||v_N||_{V_\triangle^2} N\sum_1 \prod_{i=1}^4 \lambda_i^{-\frac{4}{10}} N^{\frac{6}{10}} \right) \\
\leq \;\; & ||u||_{10,10} ||u||_{X^\infty}^4 \sup_N \left( N^{\frac{16}{10}} \sum_1 \prod_{i=1}^4 \lambda_i^{-\frac{4}{10}} \right),
\end{aligned}
$$

where we have used $||v_N||_{V_\triangle^2} \leq ||v||_{V_\triangle^2}$. We sum in order, $\lambda_4, \lambda_3, \lambda_2, \lambda_1$, as in the proof of Prop. 3.1.1, and we obtain

$$
\begin{aligned}
\mathscr{S}_1 \;\; \lesssim \;\; & ||u||_{10,10} ||u||_{X^\infty}^4 \sup_N \left( N^{\frac{16}{10}} N^{-\frac{16}{10}} \right) \\
\leq \;\; & ||u||_{10,10} ||u||_{X^\infty}^4 .
\end{aligned}
$$

<u>Case 2 :</u> $N > \lambda_i, i = 1,2,3,4$. I.e. we consider the sum

$$\sum_4 = \sum_{-\infty<\lambda_3<N\sim\lambda_4} \sum_{-\infty<\lambda_2<\lambda_3} \sum_{-\infty<\lambda_1<\lambda_2} . \tag{3.2.14}$$

By Hölder's inequality, Proposition 2.2.2, Proposition 2.2.4 and Lemma 2.1.3,

$$\mathscr{S}_4 \; := \; \sup_N \left( \sup_{||v||_{V_\triangle^2}=1} N \sum_4 \int u u_{1,\lambda_1} u_{2,\lambda_2} u_{3,\lambda_3} u_{4,\lambda_4} v_N \right)$$

$$\leq \; \sup_N \left( \sup_{||v||_{V_\triangle^2}=1} N \sum_4 ||u||_{10,10} ||u_{2,\lambda_2} v_N||_{2,2} ||u_{1,\lambda_1}||_{15,15} ||u_{3,\lambda_3}||_{15,15} ||u_{4,\lambda_4}||_{\frac{15}{4},\frac{15}{4}} \right)$$

$$\leq \; \sup_N \left( \sup_{||v||_{V_\triangle^2}=1} N \sum_4 ||u||_{10,10} \lambda_2 ||u_{2,\lambda_2}||_{V_\triangle^2} N^{-\frac{1}{2}} ||v_N||_{V_\triangle^2} \lambda_1^{\frac{7}{6}} ||u_{1,\lambda_1}||_{U_\triangle^{15}} \lambda_3^{\frac{7}{6}} ||u_{3,\lambda_3}||_{U_\triangle^{15}} \lambda_4^{\frac{1}{6}} ||u_{4,\lambda_4}||_{U_\triangle^{\frac{15}{4}}} \right)$$

$$\leq \; \sup_N \left( \sup_{||v||_{V_\triangle^2}=1} N \sum_4 ||u||_{10,10} \lambda_2 ||u_{2,\lambda_2}||_{V_\triangle^2} N^{-\frac{1}{2}} ||v_N||_{V_\triangle^2} \lambda_1^{\frac{7}{6}} ||u_{1,\lambda_1}||_{V_\triangle^2} \lambda_3^{\frac{7}{6}} ||u_{3,\lambda_3}||_{V_\triangle^2} \lambda_4^{\frac{1}{6}} ||u_{4,\lambda_4}||_{V_\triangle^2} \right).$$

We have $||v_N||_{V_\triangle^2} \leq ||v||_{V_\triangle^2}$, and so by Definition 2.1.7, we have

$$\mathscr{S}_4 \; \leq \; \sup_N \left( \sup_{||v||_{V_\triangle^2}=1} N \sum_4 ||u||_{10,10} ||v||_{V_\triangle^2} (\lambda_1 ||u_{1,\lambda_1}||_{V_\triangle^2})(\lambda_2 ||u_{2,\lambda_2}||_{V_\triangle^2})(\lambda_3 ||u_{3,\lambda_3}||_{V_\triangle^2})(\lambda_4 ||u_{4,\lambda_4}||_{V_\triangle^2}) N^{-\frac{1}{2}} \lambda_1^{\frac{1}{6}} \lambda_3^{\frac{1}{6}} \lambda_4^{\frac{-5}{6}} \right)$$

$$\leq \; ||u||_{10,10} ||u||_{X^\infty}^4 \sup_N \left( N \sum_4 N^{-\frac{1}{2}} \lambda_1^{\frac{1}{6}} \lambda_3^{\frac{1}{6}} \lambda_4^{\frac{-5}{6}} \right).$$

We sum in order, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, and we obtain

$$\mathscr{S}_4 \; \lesssim \; ||u||_{10,10} ||u||_{X^\infty}^4 \sup_N \left( N^{\frac{1}{3}} N^{-\frac{1}{3}} \right)$$

$$\leq \; ||u||_{10,10} ||u||_{X^\infty}^4.$$

We have shown that

$$|| \int_{t_0}^t e^{i(t-s)\triangle} \left( |u|^4 u \right)(s) ds ||_{X^\infty} \lesssim ||u||_{X^\infty}^4 ||u||_{L_t^{10} L_x^{10}}. \tag{3.2.15}$$

**q = 2**:

Using the Duhamel representation of the solution, Lemma 2.3.2, and decomposing the terms of the sum into their Littlewood-Paley frequency projections,

$$||\int_{t_0}^{t} e^{i(t-s)\triangle}\left(|u|^4 u\right)(s)ds||_{X^2} = \sup_{||v||_{Y^2}=1} |\int_{I\times\mathbb{R}^3} (|u|^4 uv)dxdt| \tag{3.2.16}$$

$$= \sup_{||v||_{Y^2}=1} |\sum_{\lambda_i,N} \int_{I\times\mathbb{R}^3} uu_{1,\lambda_1}u_{2,\lambda_2}u_{3,\lambda_3}u_{4,\lambda_4}v_N dxdt|,$$

where the sum $\sum_{\lambda_i,N}$ is over all dyadic scales for $\lambda_i$, $i=1,\ldots,4$ and N.

We break up the sum based on the relative position of the various $\lambda_i$ and $N$. Note that in each case, the largest two frequencies must be comparable or the integral will be 0. Again, we will only show the cases where $N > \lambda_i$ for $i=1,2,3,4$ and where $N < \lambda_i$ for $i=1,2,3,4$.

<u>Case 1 :</u> $N < \lambda_i$, $i=1,2,3,4$. We consider the sum

$$\sum_1 := \sum_{-\infty<\lambda_4(\sim\lambda_3)<\infty}\sum_{-\infty<\lambda_2<\lambda_4}\sum_{-\infty<\lambda_1<\lambda_2}\sum_{-\infty<N<\lambda_1} \tag{3.2.17}$$

We must bound $\sup_{||v||_{Y^2}=1} |\sum_1 \int u_{1,\lambda_1}\cdots u_{4,\lambda_4}v_N dxdt| =: \mathscr{S}_1$. By Hölder, Proposition 2.2.2 and Lemma 2.1.3,

$$\mathscr{S}_1 = \sup_{||v||_{Y^2}=1} |\sum_1 \int u_{1,\lambda_1}\cdots u_{4,\lambda_4}v_N dxdt|$$

$$\leq \sup_{||v||_{Y^2}=1} \sum_1 ||u||_{10,10}\prod_{i=1}^4 ||u_{i,\lambda_i}||_{\frac{50}{9},\frac{50}{9}}||v_N||_{\frac{50}{9},\frac{50}{9}}$$

$$\leq \sup_{||v||_{Y^2}=1} \sum_1 ||u||_{10,10}\prod_{i=1}^4 \lambda_i^{\frac{6}{10}}||u_i||_{U_{\triangle}^{\frac{50}{9}}}N^{\frac{6}{10}}||v_N||_{U_{\triangle}^{\frac{50}{9}}}$$

$$\leq \sup_{||v||_{Y^2}=1} \sum_1 ||u||_{10,10}\prod_{i=1}^4 \lambda_i^{-\frac{4}{10}}\left(\lambda_i||u_i||_{V_{\triangle}^2}\right)N^{\frac{6}{10}}||v_N||_{V_{\triangle}^2}.$$

By Definition 2.1.7, we have

$$\mathscr{S}_1 \leq ||u||_{10,10}\sup_{||v||_{Y^2}=1}\sup_N\left(N^{-1}||v_N||_{V_{\triangle}^2}\right)||u||_{\dot{X}^\infty}^2\sum_1(\lambda_1\lambda_2\lambda_3\lambda_4)^{-\frac{4}{10}}\left(\lambda_3||u_3||_{V_{\triangle}^2}\right)\left(\lambda_4||u_4||_{V_{\triangle}^2}\right)N^{\frac{16}{10}}.$$

By the continuous embedding $\ell^2 \subset \ell^\infty$, we have that $\sup_N\left(N^{-1}||v_N||_{V_{\triangle}^2}\right) \leq ||N^{-1}||v_N||_{V_{\triangle}^2}||_{\ell^2} = ||v||_{Y^2}$. We sum over $N$, $\lambda_1$, $\lambda_2$ and apply Cauchy-Schwarz to obtain

$$\mathscr{S}_1 \lesssim \sup_{||v||_{Y^2}=1} ||u||_{10,10} ||v||_{Y^2} ||u||_{X^\infty}^2 \sum_{-\infty<\lambda_3\sim\lambda_4<\infty} \left(\lambda_3 ||u_3||_{V_\triangle^2}\right) \left(\lambda_4 ||u_4||_{V_\triangle^2}\right) \tag{3.2.18}$$

$$\lesssim ||u||_{10,10} ||u||_{X^\infty}^2 \left(\sum_{-\infty<\lambda_3<\infty} \left(\lambda_3 ||u_3||_{V_\triangle^2}\right)^2\right)^{\frac{1}{2}} \left(\sum_{-\infty<\lambda_4<\infty} \left(\lambda_4 ||u_4||_{V_\triangle^2}\right)^2\right)^{\frac{1}{2}} \tag{3.2.19}$$

$$\leq ||u||_{10,10} ||u||_{X^\infty}^2 ||u||_{X^2}^2. \tag{3.2.20}$$

$$\tag{3.2.21}$$

<u>Case 2 :</u> $N > \lambda_i$, $i = 1,2,3,4$. I.e. we consider the sum

$$\sum_4 := \sum_{-\infty<N(\sim\lambda_4)<\infty} \sum_{-\infty<\lambda_3<N} \sum_{-\infty<\lambda_2<\lambda_3} \sum_{-\infty<\lambda_1<\lambda_2}. \tag{3.2.22}$$

Letting $\mathscr{S}_4 := \sup_{||v||_{Y^2}=1} |\sum_4 \int uu_{1,\lambda_1}\cdots u_{4,\lambda_4} v_N dt dx|$, by Hölder, Proposition 2.2.4, Proposition 2.2.2 and Lemma 2.1.3,

$$\begin{aligned}
\mathscr{S}_4 &= \sup_{||v||_{Y^2}=1} |\sum_4 \int uu_{1,\lambda_1}\cdots u_{4,\lambda_4} v_N dx dt| \\
&\leq \sup_{||v||_{Y^2}=1} \sum_4 ||u||_{10,10} ||u_{2,\lambda_2} v_N||_{2,2} ||u_{1,\lambda_1}||_{15,15} ||u_{3,\lambda_3}||_{15,15} ||u_{4,\lambda_4}||_{\frac{15}{4},\frac{15}{4}} \\
&\leq ||u||_{10,10} \sup_{||v||_{Y^2}=1} \sum_4 N^{-\frac{1}{2}} \lambda_2 \lambda_1^{\frac{7}{6}} \lambda_3^{\frac{7}{6}} \lambda_4^{\frac{1}{6}} ||u_{2,\lambda_2}||_{V_\triangle^2} ||v_N||_{V_\triangle^2} ||u_{1,\lambda_1}||_{U_\triangle^{15}} ||u_{3,\lambda_3}||_{U_\triangle^{15}} ||u_{4,\lambda_4}||_{U_\triangle^{\frac{15}{4}}} \\
&\leq ||u||_{10,10} \sup_{||v||_{Y^2}=1} \sum_4 N^{-\frac{1}{2}} \lambda_2 \lambda_1^{\frac{7}{6}} \lambda_3^{\frac{7}{6}} \lambda_4^{\frac{1}{6}} ||u_{2,\lambda_2}||_{V_\triangle^2} ||v_N||_{V_\triangle^2} ||u_{1,\lambda_1}||_{V_\triangle^2} ||u_{3,\lambda_3}||_{V_\triangle^2} ||u_{4,\lambda_4}||_{V_\triangle^2}.
\end{aligned}$$

Summing over $\lambda_1$, $\lambda_2$ and $\lambda_3$, and applying Cauchy-Schwarz, we have

$$\begin{aligned}
\mathscr{S}_4 &\lesssim ||u||_{10,10} \sup_{||v||_{Y^2}=1} (\sup_{\lambda_1} \lambda_1 ||u_{1,\lambda_1}||_{V_\triangle^2})\cdots(\sup_{\lambda_3} \lambda_3 ||u_{3,\lambda_3}||_{V_\triangle^2}) \sum_4 N^{\frac{1}{2}} \lambda_1^{\frac{1}{6}} \lambda_3^{\frac{1}{6}} \lambda_4^{-\frac{5}{6}} (N^{-1}||v_N||_{V_\triangle^2})(\lambda_4 ||u_{4,\lambda_4}||_{V_\triangle^2}) \\
&\leq ||u||_{10,10} \sup_{||v||_{Y^2}=1} ||u||_{X^\infty}^3 \sum_{-\infty<N(\sim\lambda_5)<\infty} (N^{-1}||v_N||_{V_\triangle^2})(\lambda_4 ||u_{4,\lambda_4}||_{V_\triangle^2}) \\
&\lesssim ||u||_{10,10} \sup_{||v||_{Y^2}=1} ||u||_{X^\infty}^3 \left(\sum_{-\infty<N<\infty} \left(N^{-1}||v_N||_{V_\triangle^2}\right)^2\right)^{\frac{1}{2}} \left(\sum_{-\infty<\lambda_4<\infty} \left(\lambda_4 ||u_{4,\lambda_4}||_{V_\triangle^2}\right)^2\right)^{\frac{1}{2}},
\end{aligned}$$

and so,

$$\mathscr{S}_4 = ||u||_{10,10} \sup_{||v||_{Y^2}=1} ||u||_{X^\infty}^3 ||u||_{X^2} ||v||_{Y^2} \leq ||u||_{10,10} ||u||_{X^\infty}^3 ||u||_{X^2}.$$

Thus,

$$\sup_{||v||_{Y^2}=1} |\sum_4 \int uu_{1,\lambda_1}\cdots u_{4,\lambda_4} v_N dx dt| \leq ||u||_{10,10} ||u||_{X^\infty}^3 ||u||_{X^2}. \tag{3.2.23}$$

This is in fact stronger than we need for our Lemma, since $X^2$ embeds continuously into $X^\infty$. Combining (3.2.18) and (3.2.23), by (3.2.16) we have

$$|| \int_{t_0}^{t} e^{i(t-s)\triangle} \left( |u|^4 u \right)(s) ds||_{X^2} \lesssim ||u||_{10,10} ||u||^2_{X^\infty} ||u||^2_{X^2}. \tag{3.2.24}$$

In a similar manner to Prop. 3.2.2, we use Lemma 3.2.1 to interpolate between Equation (3.2.15) and Equation (3.2.24). This gives us for all $2 \le q \le \infty$,

$$|| \int_{t_0}^{t} e^{i(t-s)\triangle} \left( |u|^4 u \right)(s) ds||_{X^q} \lesssim ||u||_{10,10} ||u||^2_{X^\infty} ||u||^2_{X^q}. \tag{3.2.25}$$

$\square$

**Remark 3.2.4.** *By Lemma 2.1.9, we see that Prop. 3.2.3 is stronger than Prop. 3.2.2 in the case $q = 2$.*

We will need a lemma that tells us for a given $u \in L_t^{10} L_x^{10}(I \times \mathbb{R}^3)$, how many sub-intervals $I_k \subset I$ will suffice so that on each interval, we have $||u||_{L_t^{10} L_x^{10}(I_k \times \mathbb{R}^3)} < \eta$.

**Lemma 3.2.5.** *Given $\eta$, we can partition the time intervals $I$ into $\frac{||u||^{10}_{L_t^{10} L_x^{10}(I_k \times \mathbb{R}^3)}}{\eta^{10}}$ many sub-intervals $I_k$ so that on each interval, $||u||_{L_t^{10} L_x^{10}(I_k \times \mathbb{R}^3)} < \eta$.*

*Proof.* Let $n$ be the number of intervals required. We will assume for simplicity that there is no remainder when we divide up our intervals.

$$||u||_{L_t^{10} L_x^{10}(I \times \mathbb{R}^3)} = \left( \sum_{k=1}^{n} \int_{I \times \mathbb{R}^3} |u|^{10} \chi_{(I_k \times \mathbb{R}^3)} \right)^{\frac{1}{10}} = \left( \sum_{k=1}^{n} \int_{I_k \times \mathbb{R}^3} |u|^{10} \right)^{\frac{1}{10}} = n^{\frac{1}{10}} \eta, \tag{3.2.26}$$

so $n = \frac{||u||^{10}_{L_t^{10} L_x^{10}(I \times \mathbb{R}^3)}}{\eta^{10}}$. $\square$

# Chapter 4

# Global Well-posedness and Scattering

## 4.1 Continuity of Solutions with Respect to Time

The proofs of Prop. 3.1.1, Prop. 3.1.2 can be used to prove that solutions to (NLS) are continuous in time in $X^q$. In particular, for $q = 2$ and $q = \infty$, we can see that the nonlinear term $\int_0^t e^{-i(t-s)\triangle} \left( |u|^4 u \right)(s)ds$ can be bounded by a sum of products of $u_N$ in a collection of Lebesgue spaces. For example, let us assume that $u$ is a solution on the interval $[0,T)$. Then we know $||u||_{X^\infty_{[0,T)}} < \infty$. If we look at the proof of Prop. 3.1.1, we see that in the case $N < \lambda_i$ for all $i$,

$$\mathscr{S}_1 \leq N \sum_1 ||u_{1,\lambda_1}||_{L_t^6,L_x^6([0,T)\times\mathbb{R}^3)} \cdots ||u_{5,\lambda_5}||_{L_t^6 L_x^6([0,T)\times\mathbb{R}^3)} ||v_N||_{L_t^6 L_x^6([0,T)\times\mathbb{R}^3)} \lesssim ||u||^5_{X^\infty[0,T)}. \qquad (4.1.1)$$

Since the right side of the above equation is bounded, by the fungibility of Lebesgue spaces we know that for every $\varepsilon > 0$, there exists a time $\bar{T}$ such that

$$N \sum_1 ||u_{1,\lambda_1}||_{L_t^6 L_x^6([0,\bar{T})\times\mathbb{R}^3)} \cdots ||u_{5,\lambda_5}||_{L_t^6 L_x^6([0,\bar{T})\times\mathbb{R}^3)} ||v_N||_{L_t^6 L_x^6([0,\bar{T})\times\mathbb{R}^3)} \leq \varepsilon, \qquad (4.1.2)$$

and so $\mathscr{S}_1 \leq \varepsilon$. A similar statement can be made for $\mathscr{S}_i$ for $i = 2,3,4,5$ and a similar argument can be made in the case $q = 2$ by examining the proof of Prop. 3.1.2. For $2 < q < \infty$, we must decompose the solution into high and low frequencies.

We now discuss the linear term. For $u_0 \in \dot{B}^1_{2,q}$, if $2 \leq q < \infty$, the linear term $e^{it\triangle}u_0$ is continuous in time in $\dot{B}^1_{2,q}$. Indeed,

$$||e^{it\triangle}u_0 - u_0||_{\dot{B}^1_{2,q}} = \left( \sum_N N^q ||(e^{-it|\xi|^2} - 1)(\hat{u}_0)_N||^q_{L^2} \right)^{\frac{1}{q}},$$

so for any $\bar{N}$, we can find a $\bar{t} > 0$ such that for $N < \bar{N}$ and $0 < t < \bar{t}$, $|e^{itN^2} - 1| < \varepsilon$ and continuity follows. However, this relies on the fact that $N||(e^{it|\xi|^2} - 1)(\hat{u}_0)_N||_{L^2}$ decays in N. For $q = \infty$, we do not have this decay and in fact we cannot have continuity. Indeed, for any $t > 0$, for any $\varepsilon > 0$, there exists an $N$ such that $|e^{itN^2} - 1| > \varepsilon$. We do however have weak continuity in the case $q = \infty$, as can be shown by a similar argument.

**Lemma 4.1.1.** *(Continuity) For $2 \leq q < \infty$, solutions to (NLS) are continuous in time in $X^q$ and in $\dot{B}^1_{2,q}$. For $q = \infty$, solutions to (NLS) are weakly continuous in time in $X^q$ and in $\dot{B}^1_{2,q}$. Furthermore, the nonlinear term $\|\int_0^t e^{-i(t-s)\triangle} \left(|u|^4 u\right)(s)ds\|_{X^q[0,T)}$ converges to 0 as $t \to 0$ for $2 \leq q \leq \infty$.*

*Proof.* Let $u$ be a solution to (NLS) on the interval $[0, \tilde{T})$. We real the Duhamel form of the solution

$$u(t,x) = e^{it\triangle}u_0 - i \int_0^t e^{i(t-s)\triangle}(|u|^4 u)(s)ds.$$

Since the linear Schrödinger solution $e^{it\triangle}u_0$ is continuous in time for $2 \leq q < \infty$ and weakly continuous for $q = \infty$, we just need to show that for every $\varepsilon > 0$, there exists a time $0 < \bar{T} < \tilde{T}$ such that

$$\|\int_0^t e^{-i(t-s)\triangle}\left(|u|^4 u\right)(s)ds\|_{X^q} < \varepsilon. \tag{4.1.3}$$

We begin with the case $q = \infty$. From the proof of Prop. 3.1.1, we see that for small $T' > 0$,

$$\|\int_0^t e^{i(t-s)\triangle}(|u|^4 u)(s)ds\|_{X^\infty_{[0,T')}} \;\leq\; \sum_{i=1}^5 N \sum_i \|u_{1,\lambda_1}\|_{L_t^{q_1},L_x^{r_1}([0,T')\times\mathbb{R}^3)} \cdots \|u_{5,\lambda_5}\|_{L_t^{q_5}L_x^{r_5}([0,T')\times\mathbb{R}^3)} \|v_N\|_{L_t^{q_6}L_x^{r_6}([0,T')\times\mathbb{R}^3)}$$

$$\leq \;\|u\|^5_{X^\infty[0,T')} < \infty,$$

for $(q_k, r_k) = (q_{k,i}, r_{k,i})$ and $\|v\|_{V^2_\triangle} \leq 1$. We can then find $0 < \bar{T} < T'$ that satisfies

$$\sum_{i=1}^5 N \sum_i \|u_{1,\lambda_1}\|_{L_t^{q_1},L_x^{r_1}([0,\bar{T})\times\mathbb{R}^3)} \cdots \|u_{5,\lambda_5}\|_{L_t^{q_5}L_x^{r_5}([0,\bar{T})\times\mathbb{R}^3)} \|v_N\|_{L_t^{q_6}L_x^{r_6}([0,\bar{T})\times\mathbb{R}^3)} \leq \varepsilon, \tag{4.1.4}$$

and so Equation (4.1.3) is true for $q = \infty$.

For the case $q = 2$, from the proof of Prop. 3.1.2 we see that for small enough $T' > 0$,

$$\|\int_0^t e^{i(t-s)\triangle}(|u|^4 u)(s)ds\|_{X^2_{[0,T')}} \;\lesssim\; \sum_{i=1}^5 \sum_i \|u_{1,\lambda_1}\|_{L_t^{q_1},L_x^{r_1}([0,T')\times\mathbb{R}^3)} \cdots \|u_{5,\lambda_5}\|_{L_t^{q_5}L_x^{r_5}([0,T')\times\mathbb{R}^3)} \|v_N\|_{L_t^{q_6}L_x^{r_6}([0,T')\times\mathbb{R}^3)}$$

$$\leq \;\|u\|^5_{X^2[0,T')} < \infty,$$

for $(q_k, r_k) = (q_{k,i}, r_{k,i})$ and $\|v\|_{Y^2} \leq 1$. We can then find a $\bar{T} > 0$ that satisfies

$$\sum_{i=1}^5 \sum_i \|u_{1,\lambda_1}\|_{L_t^{q_1},L_x^{r_1}([0,\bar{T})\times\mathbb{R}^3)} \cdots \|u_{5,\lambda_5}\|_{L_t^{q_5}L_x^{r_5}([0,\bar{T})\times\mathbb{R}^3)} \|v_N\|_{L_t^{q_6}L_x^{r_6}([0,\bar{T})\times\mathbb{R}^3)} \leq \varepsilon, \tag{4.1.5}$$

and so Equation (4.1.3) holds for $q = 2$.

Let $2 < q < \infty$. Let $u_{N \leq \bar{N}}$ be the frequency projection of $u$ onto dyadic frequencies less than or equal to $\bar{N}$, where $\bar{N}$ will be chosen later, and $u_{N > \bar{N}}$ be defined similarly. Then for small enough $T'$,

$$||\int_0^t e^{i(t-s)\triangle}(|u|^4 u)(s)ds||_{X_{[0,T')}^q} \leq ||\int_0^t e^{i(t-s)\triangle}(|u_{N\leq\bar{N}}+u_{N>\bar{N}}|^4(u_{N\leq\bar{N}}+u_{N>\bar{N}}))(s)ds||_{X_{[0,T')}^q}$$

$$\leq \sum_i ||\int_0^t e^{i(t-s)\triangle}(u_{i,1}u_{i,2}u_{i,3}u_{i,4}u_{i,5})(s)ds||_{X_{[0,T')}^q},$$

where $u_{i,j} \in \{u_{N\leq\bar{N}}, \bar{u}_{N\leq\bar{N}}, u_{N>\bar{N}}, \bar{u}_{N>\bar{N}}\}$ for all $i, j$. For each term in the sum with at least one high frequency function, we use Prop. 3.2.2 to bound it by

$$||\int_0^t e^{i(t-s)\triangle}(u_{i,1}u_{i,2}u_{i,3}u_{i,4}u_{i,5})(s)ds||_{X_{[0,T')}^q} \leq ||u_{i,1}||_{X_{[0,T')}^q}||u_{i,2}||_{X_{[0,T')}^q}||u_{i,3}||_{X_{[0,T')}^q}||u_{i,4}||_{X_{[0,T')}^q}||u_{i,5}||_{X_{[0,T')}^q}$$

$$\leq ||u||_{X_{[0,T')}^q}^4 ||u_{N>\bar{N}}||_{X_{[0,T')}^q}.$$

Since $||u||_{X_{[0,T')}^q} = \left(\sum_N N^q ||u_N(t)||_{V_\triangle^2 [0,T')}^q\right)^{\frac{1}{q}} < \infty$, we can choose $\bar{N}$ so that

$$||u||_{X_{[0,T')}^q}^4 ||u_{N>\bar{N}}||_{X_{[0,T')}^q} < \frac{\varepsilon}{100}. \tag{4.1.6}$$

The only other terms we need to bound are the terms with only low frequency components. But by Definition 2.1.7, we see that $u_{N<\bar{N}} \in X_{[0,T')}^2$. Because $\ell_2 \subset \ell_q$, we can use the proof of Prop. 3.1.2 again to bound these terms. We suppress the index $i$, so that $u_j = u_{i,j}$:

$$||\int_0^t e^{i(t-s)\triangle}(u_1 u_2 u_3 u_4 u_5)(s)ds||_{X_{[0,T')}^q}$$

$$\leq ||\int_0^t e^{i(t-s)\triangle}(u_1 u_2 u_3 u_4 u_5)(s)ds||_{X_{[0,T')}^2}$$

$$\leq \sum_{i=1}^5 \sum_i ||u_{1,\lambda_1}||_{L_t^{q_1},L_x^{r_1}([0,T')\times\mathbb{R}^3)} \cdots ||u_{5,\lambda_5}||_{L_t^{q_5}L_x^{r_5}([0,T')\times\mathbb{R}^3)}||v_N||_{L_t^{q_6}L_x^{r_6}([0,T')\times\mathbb{R}^3)}$$

$$\leq ||u_{N<\bar{N}}||_{X_{[0,T')}^2}^5$$

$$< \infty.$$

By the fungibility of the Lebesgue spaces, we can then choose $\bar{T}$ appropriately small so that $||\int_0^t e^{i(t-s)\triangle}(u_{i,1}u_{i,2}u_{i,3}u_{i,4}u_{i,5})(s)ds||_{X_{[0,\bar{T})}^q} < \frac{\varepsilon}{100}$.

Combining these estimates, we have that for $\bar{T}$ small enough, Equation (4.1.3) holds for $2 < q < \infty$.

Continuity in the Besov space $\dot{B}_{2,q}^1$ follows from the continuous embedding $X^q \subset L_t^\infty \dot{B}_{2,q}^1$ from Lemma 2.1.8.  □

**Corollary 4.1.2.** *For $2 \leq q \leq \infty$ and $u \in X^q$, for every $\varepsilon$ there exists $T'$ such that*

$$|| \int_0^t e^{i(t-s)\triangle} \left( |u|^4 u \right) (s)||_{X^q[0,T']} < \varepsilon.$$  (4.1.7)

## 4.2   Local Well-posedness and Stability

**Theorem 4.2.1.** *(Small Data Global Well-Posedness) Let $u_0 \in \dot{B}^1_{2,q}$, for $2 \leq q \leq \infty$. There exists $\eta_0$ such that if $\eta < \eta_0$ and $||u_0||_{\dot{B}^1_{2,q}} < \eta$, then $u_0$ evolves to a unique solution $u(t,x)$ to Equation (1.1.1) which lies in $X^q([0,\infty)) \cap C^0_t \dot{B}^1_{2,q} \left( [0,\infty) \times \mathbb{R}^3 \right)$ for $q < \infty$ and in $X^\infty([0,\infty)) \cap L^\infty_t \dot{B}^1_{2,\infty} \left( [0,\infty) \times \mathbb{R}^3 \right)$ for $q = \infty$. Furthermore, $||u||_{X^q[0,\infty)} < 2\eta$.*

*Proof.* We run a fixed point argument in the ball $B = \{u : ||u||_{X^q(I)} < 2\eta\} \cap \{u : ||u||_{L^\infty_t \dot{B}^1_{2,q}(I \times \mathbb{R}^3)}\} < 2\eta$. Recall $\Gamma u = e^{it\triangle} u_o - i \int_0^t e^{i(t-s)\triangle} |u(s)|^4 u(s) ds$. By Lemma 2.2.1, Prop. 3.2.2,

$$||\Gamma u||_{X^q} \quad \leq \quad ||e^{it\triangle} u_0||_{X^q} + || \int_{T_0}^t e^{i(t-s)\triangle} |u(s)|^4 u(s) ds||_{X^q} \qquad (4.2.1)$$

$$\leq \quad \eta + ||u||^5_{X^q} \qquad (4.2.2)$$

$$\leq \quad \eta + (2\eta)^5. \qquad (4.2.3)$$

If $\eta_0$ is chosen small enough, then $(2\eta)^5 < \eta$ and we have

$$||\Gamma u||_{X^q} < 2\eta, \qquad (4.2.4)$$

By Lemma (2.1.8), $||\Gamma u||_{L^\infty_t \dot{B}^1_{2,q}} \leq ||u||_{X^q}$, and we see that $\Gamma$ maps $B$ to itself.

We now show that $\Gamma$ is a contraction on $B$. Consider $u, v \in B$. Since $\Gamma u$ and $\Gamma v$ have the same linear terms,

$$||\Gamma u - \Gamma v||_{X^q} = || \int_{T_0}^t e^{i(t-s)\triangle} \left( |u|^4 u - |v|^4 v \right) (s) ds||_{X^q}. \qquad (4.2.5)$$

Using Lemma 2.3.1, the algebraic inequality $||u|^4 u - |v|^4 v| \leq |u - v| \left( |u|^4 - |v|^4 \right)$ and Prop. 3.2.2, we have

$$||\Gamma u - \Gamma v||_{X^q} \quad \leq \quad ||u - v||_{X^q} \left( ||u||^4_{X^q} + ||v||^4_{X^q} \right)$$

$$\leq \quad ||u - v||_{X^q} \left( (2\eta)^4 + (2\eta)^4 \right).$$

By Lemma (2.1.8), we have the same bounds in the space $L^\infty_t \dot{B}^1_{2,q}$. This is a contraction if $\eta$ is chosen small enough. Continuity follows from Lemma 4.1.1.                                   $\square$

By Theorem 1.2.1, for $u_0 \in \dot{H}^1$, there exists a global solution $u$ to (NLS) with $||u||_{L_t^{10}L_x^{10}([0,\infty)\times\mathbb{R}^3)} < C(||u_0||_{\dot{H}^1})$. We will need global bounds for such solutions in the $X^q$ norm.

**Lemma 4.2.2.** *Let $u_0 \in \dot{H}^1$. Then there exists a unique global solution $u(t,x)$ to (NLS) for all time with $u \in C_t^0\dot{H}^1([0,\infty)\times\mathbb{R}^3)\cap X^q([0,\infty))$. Furthermore for any $2 \le q \le \infty$,*
$$||u||_{X^q([0,\infty))} \le C(||u_0||_{\dot{H}^1}, ||u||_{L_t^{10}L_x^{10}([0,\infty)\times\mathbb{R}^3)}).$$

*Proof.* Since $u_0 \in \dot{H}^1$, we can apply Theorem 1.2.1 and obtain a global solution to (NLS) which we call $v(t,x)$. The Theorem also gives us the bound $||v||_{10,10} < C(||u_0||_{\dot{H}^1})$. We define

$$(NLS') \begin{cases} i\partial_t u + \triangle u = |u|^4 v \\ u_{t=0} = u_0. \end{cases} \tag{4.2.6}$$

Notice that the last factor in the nonlinearity is our fixed solution $v$. We run a fixed point argument for $(NLS')$ in the ball $B = \{u : ||u||_{X^q(I)} < 2||u_0||_{\dot{H}^1}\} \cap \{u : ||u||_{L_t^\infty \dot{B}_{2,q}^1(I\times\mathbb{R}^3)} < 2||u_0||_{\dot{H}^1}\}$ where $I = [0,T)$ is an interval with endpoint to be chosen later. We recall $\Gamma u = e^{it\triangle}u_o - i\int_0^t e^{i(t-s)\triangle}|u(s)|^4 v(s)ds$. By Lemma (2.1.8) and Lemma 4.1.1, it suffices to show that $\Gamma$ maps $\{u : ||u||_{X^q(I)} < 2||u_0||_{\dot{H}^1}\}$ to itself. From Prop. 3.2.3,

$$\begin{aligned}
||\Gamma u||_{X^q(I)} &\le ||e^{it\triangle}u_0||_{X^q(I)} + ||\int_0^t e^{i(t-s)\triangle}(|u|^4 v)ds||_{X^q(I)} \\
&\le ||u_0||_{\dot{H}^1} + ||v||_{L_t^{10}L_x^{10}(I\times\mathbb{R}^3)}||u||_{X^q(I)}^4 \\
&\le ||u_0||_{\dot{H}^1} + ||v||_{L_t^{10}L_x^{10}(I\times\mathbb{R}^3)}16||u_0||_{\dot{H}^1}^3
\end{aligned}$$

Since $||v||_{L_t^{10}L_x^{10}(I\times\mathbb{R}^3)} < \infty$, we can choose $I$ to be small enough so that $||v||_{L_t^{10}L_x^{10}(I\times\mathbb{R}^3)} < \frac{1}{32||u_0||_{\dot{H}^1}^3}$. Then

$$||\Gamma u||_{X^q(I)} \le \frac{3}{2}||u_0||_{\dot{H}^1}, \tag{4.2.7}$$

and we see $\Gamma : X^q(I) \to X^q(I)$.

We now show that $\Gamma$ is a contraction on $B$. Consider $u_1, u_2 \in B$. Since $\Gamma u$ and $\Gamma v$ have the same linear terms,

$$||\Gamma u_1 - \Gamma u_2||_{X^q} = ||\int_0^t e^{i(t-s)\triangle}\left(|u_1|^4 v - |u_2|^4 v\right)(s)ds||_{X^q}. \tag{4.2.8}$$

Using Lemma 2.3.1, the algebraic inequality $||u_1|^4 v - |u_2|^4 v| \le |u_1 - u_2||v|\left(|u_1|^3 - |u_2|^3\right)$ and Prop. 3.2.2, we have

$$
\begin{aligned}
\|\Gamma u - \Gamma v\|_{X^q} &\leq \|u - v\|_{X^q} \|v\|_{L_t^{10} L_x^{10}(I \times \mathbb{R}^3)} \left( \|u_1\|_{X^q}^3 + \|u_2\|_{X^q}^3 \right) \\
&\leq \|u_1 - u_2\|_{X^q} \frac{1}{32 \|u_0\|_{\dot{H}^1}^3} \left( (2 \|u_0\|_{\dot{H}^1})^3 + (2 \|u_0\|_{\dot{H}^1})^3 \right) \\
&= \frac{1}{2} \|u_1 - u_2\|_{X^q}.
\end{aligned}
$$

We obtain a solution $u(t,x)$ to (NLS') on $I$ with $\|u\|_{X^q(I)} \leq 2\|u_0\|_{\dot{H}^1}$. Since $\|v\|_{L_t^{10} L_x^{10}(\mathbb{R} \times \mathbb{R}^3)}$ is finite and $\|u(t,x)\|_{L_\infty^t \dot{H}^1} \leq \|u_0\|_{\dot{H}^1}$, we can repeat this argument a finite number of times to get a global solution $u(t,x)$ to (NLS'), with $\|u\|_{X^q(\mathbb{R})} < C(\|u_0\|_{\dot{H}^1})$. But $v$ is such a solution, so by uniqueness $u = v$. Therefore $u(t,x)$ is in fact a solution to (NLS).

By Lemma 3.2.5, the number of intervals of appropriate size is $\dfrac{(32)^{10} \|v\|_{L_t^{10} L_x^{10}(\mathbb{R} \times \mathbb{R}^3)}^{10}}{\|u_0\|_{\dot{H}^1}^{30}}$.

Summing over the intervals, we obtain

$$
\begin{aligned}
\|u\|_{X^q(\mathbb{R})} &\leq \sum \|u\|_{X^q(I_k)} \\
&\leq 2\|u_0\|_{\dot{H}^1} \frac{(32)^{10} \|v\|_{L_t^{10} L_x^{10}(\mathbb{R} \times \mathbb{R}^3)}^{10}}{\|u_0\|_{\dot{H}^1}^{30}} \\
&\leq \frac{(32)^{11} \|v\|_{L_t^{10} L_x^{10}(\mathbb{R} \times \mathbb{R}^3)}^{10}}{\|u_0\|_{\dot{H}^1}^{29}}.
\end{aligned}
$$

$\square$

From Theorem 4.2.1, we know that there is a constant $\eta_0$ such that if $\|w_0\|_{\dot{B}_{2,q}^1} < \eta_0$, then $w_0$ evolves to a global solution $w(t,x)$ to Equation (1.1.1). We consider the solution $w$ fixed in the following.

Let $e = e(t,x) = |w + \tilde{u}|^4 (w + \tilde{u}) - |w|^4 w - |\tilde{u}|^4 \tilde{u}$. If $\tilde{u}$ is a solution to

$$
(\tilde{NLS}) \quad \begin{cases} i\partial_t \tilde{u} + \triangle \tilde{u} = |\tilde{u}|^4 \tilde{u} + e \\ \tilde{u}_{t=0} = v_0 \in \dot{H}^1, \end{cases} \tag{4.2.9}
$$

then $u(t,x) = \tilde{u}(t,x) + w(t,x)$ is a solution to (NLS):

$$
\begin{aligned}
i\partial_t u + \triangle u &= (i\partial_t \tilde{u} + \triangle \tilde{u}) + (i\partial_t w + \triangle w) \\
&= (|w + \tilde{u}|^4 (w + \tilde{u}) - |w|^4 w) + (|w|^4 w) \\
&= |w + \tilde{u}|^4 (w + \tilde{u}) \\
&= |u|^4 u.
\end{aligned}
$$

Thus, to show that (NLS) is globally well-posed for data described in Theorem 4.3.1, it suffices to show that Equation (4.2.9) admits a global solution. We begin with an appropriate local theory for Equation (4.2.9), but first a remark about continuity.

**Remark 4.2.3.** *By modifying the proof of Lemma (2.1.8) appropriately, we see that Lemma (2.1.8) is true with* (NLS) *replaced by* (ÑLS)*, and changing the nonlinearity appropriately.*

**Lemma 4.2.4.** *For* $2 \leq q \leq \infty$*, there exists a constant* $C = C(||\tilde{u}_0||_{\dot{H}^1})$ *and a time* $T_1$ *such that if* $||w_0||_{\dot{B}^1_{2,q}} < \tilde{C}$ *and* $0 < T < T_1$ *then* $v_0$ *evolves to a unique solution* $\tilde{u}(t,x)$ *to* (4.2.9) *which lies in* $X^q([0,T)) \cap C^0_t \dot{B}^1_{2,q}([0,T) \times \mathbb{R}^3)$ *for* $2 \leq q < \infty$ *and in* $X^\infty([0,T)) \cap L^\infty_t \dot{B}^1_{2,q}([0,T) \times \mathbb{R}^3)$ *for* $q = \infty$*. Furthermore,* $||\tilde{u}||_{X^q([0,T))} < 2||\tilde{u}_0||_{\dot{H}^1}$*.*

*Proof.* From Theorem 4.2.1, we see that there exists $C_1$ such that if $||w_0||_{\dot{B}^1_{2,q}} < C_1$, we can ensure a global solution $w(t,x)$ to (NLS), evolving from $w_0$. Let $w_0$ be such that $||w_0||_{\dot{B}^1_{2,q}} < \min\{C_1, \frac{1}{4}||\tilde{u}_0||_{\dot{H}^1}\}$. Then Theorem 4.2.1 guarantees that there exists global solution $w(t,x)$ to (NLS), evolving from $w_0$, with $||w||_{X^q([0,\infty))} < \frac{1}{2}||\tilde{u}_0||_{\dot{H}^1}$. We take this solution to be fixed. Let $I = [0,T)$, where we will decide on $T$ later in the proof. We run a fixed-point argument in the ball
$B = \{u : ||\tilde{u}||_{X^q(I)} < 2||\tilde{u}_0||_{\dot{H}^1}\} \cap \{u : ||u||_{L^\infty_t \dot{B}^1_{2,q}(I \times \mathbb{R}^3)} < 2||\tilde{u}_0||_{\dot{H}^1}\}$. Let

$$\tilde{\Gamma}\tilde{u} = e^{it\triangle}\tilde{u}_o - i\int_{T_0}^t e^{i(t-s)\triangle}\left(|\tilde{u}(s)+w(s)|^4(\tilde{u}(s)+w(s)) - |w(s)|^4 w(s)\right)ds. \tag{4.2.10}$$

From Remark 4.2.3 and Lemma (2.1.8), it suffices to show that $\tilde{\Gamma}$ maps $\{u : ||\tilde{u}||_{X^q(I)} < 2||\tilde{u}_0||_{\dot{H}^1}\}$ to itself.

For $\tilde{u} \in B$, since $||\tilde{u}||_{X^q(I)} = \left(\sum_N N^q ||\tilde{u}_N||^q_{V^2_\triangle(I)}\right)^{\frac{1}{q}} < 2||\tilde{u}_0||_{\dot{H}^1}$, and $||w||_{X^q(I)} < \frac{1}{2}||\tilde{u}_0||_{\dot{H}^1}$, there exists a dyadic frequency $M$ such that the projections of $\tilde{u}$ and $w$ onto frequencies greater than $M$, $\tilde{u}_{high} := \tilde{u}_{N \geq M}$ and $w_{high} := w_{N \geq M}$ satisfy

$$||\tilde{u}_{high}||_{X^q(I)} \leq \min\left\{\frac{1}{4}, \frac{1}{4}||\tilde{u}_0||_{\dot{H}^1}\right\}$$

$$||w_{high}||_{X^q(I)} \leq \min\left\{\frac{1}{4}, \frac{1}{4}||\tilde{u}_0||_{\dot{H}^1}\right\}.$$

We let $\tilde{u}_{low} := \tilde{u} - \tilde{u}_{high}$ and $w_{low} = w - w_{high}$. Then since these functions lack high frequencies, we see from Lemma 2.1.9,

$$||u_{low}||_{L^{10}_t L^{10}_x(I)} \lesssim ||\tilde{u}_{low}||_{X^2(I)} = \left(\sum_N N^2 ||\tilde{u}_{low}||^2_{V^2_\triangle}\right)^{\frac{1}{2}} < \infty,$$

$$||w_{low}||_{L^{10}_t L^{10}_x(I)} \lesssim ||w_{low}||_{X^2(I)} = \left(\sum_N N^2 ||w_{low}||^2_{V^2_\triangle}\right)^{\frac{1}{2}} < \infty.$$

Shrinking the time interval if needed, we let $T > 0$ be such that

$$\|u_{low}\|_{L_t^{10}L_x^{10}(I)} < \frac{1}{(4^5)(2^4)\|\tilde{u}_0\|_{\dot{H}^1}^3}$$

$$\|w_{low}\|_{L_t^{10}L_x^{10}(I)} < \frac{1}{(4^5)(2^4)\|\tilde{u}_0\|_{\dot{H}^1}^3}.$$

We must show that $\tilde{\Gamma}$ maps the ball B into itself. By the triangle inequality and Lemma 2.2.1:

$$\left\|\tilde{\Gamma}\tilde{u}\right\|_{X^q(I)} \leq \|\tilde{u}_o\|_{\dot{B}_{2,q}^1} + \left\|\int_{s<t} e^{i(t-s)\triangle}(|\tilde{u}(s)+w(s)|.^4(\tilde{u}(s)+w(s)) - |w(s)|^4w(s))ds\right\|_{X^q(I)} \quad (4.2.11)$$

From Definition 1.1.3 and Equation (1.1.6), $\|\tilde{u}_o\|_{\dot{B}_{2,q}^1} \leq \|\tilde{u}_o\|_{\dot{B}_{2,2}^1} \sim \|\tilde{u}_0\|_{\dot{H}^1}$.

We decompose $\tilde{u} = \tilde{u}_{low} + \tilde{u}_{high}$ and $w = w_{low} + w_{high}$ and expand the integrand into $4^5 - 2^5$ terms. Each term in the integrand can be written as $e^{i(t-s)\triangle}f_1f_2f_3f_4f_5$, where $f_i \in \{\tilde{u}_{low}, \tilde{u}_{high}, w_{low}, w_{high}\}$. We apply the triangle inequality to the second term on the right-hand side of Equation (4.2.11) and consider each of the $4^5 - 2^5$ terms, $\left\|\int_{s<t} e^{i(t-s)\triangle}f_1f_2f_3f_4f_5ds\right\|_{X^q(I)}$. For the terms where $f_i \in \{\tilde{u}_{high}, w_{high}\}$ for all $i$, by Prop. 3.2.2, Equation (4.2.11) and the embedding $X^q \subset X^\infty$ from Lemma 2.1.8,

$$\left\|\int_{s<t} e^{i(t-s)\triangle}f_1f_2f_3f_4f_5ds\right\|_{X^q(I)} \leq \|f_1\|_{X^\infty(I)}\|f_2\|_{X^\infty(I)}\|f_3\|_{X^\infty(I)}\|f_4\|_{X^q(I)}\|f_5\|_{X^q(I)}$$

$$\leq \|f_1\|_{X^q(I)}\|f_2\|_{X^q(I)}\|f_3\|_{X^q(I)}\|f_4\|_{X^q(I)}\|f_5\|_{X^q(I)}$$

$$\leq \frac{1}{4^5}\|\tilde{u}_0\|_{\dot{H}^1}.$$

Every other term has at least one $f_i \in \{\tilde{u}_{low}, w_{low}\}$. Without loss of generality, we will assume $f_5 \in \{\tilde{u}_{low}, w_{low}\}$. Then by Prop. 3.2.3, Equation (4.2.11) and Lemma 2.1.8,

$$\left\|\int_{s<t} e^{i(t-s)\triangle}f_1f_2f_3f_4f_5ds\right\|_{X^q(I)} \leq \|f_1\|_{X^\infty(I)}\|f_2\|_{X^\infty(I)}\|f_3\|_{X^q(I)}\|f_4\|_{X^q(I)}\|f_5\|_{10,10}$$

$$\leq \|f_1\|_{X^q(I)}\|f_2\|_{X^q(I)}\|f_3\|_{X^q(I)}\|f_4\|_{X^q(I)}\|f_5\|_{10,10}$$

$$\leq (2\|\tilde{u}_0\|_{\dot{H}^1})^4 \frac{1}{(4^5)(2^4)\|\tilde{u}_0\|_{\dot{H}^1}^3}$$

$$\leq \frac{1}{4^5}\|\tilde{u}_0\|_{\dot{H}^1}.$$

We have shown

$$\left\|\tilde{\Gamma}\tilde{u}\right\|_{X^q(I)} \leq \|\tilde{u}_o\|_{\dot{B}_{2,q}^1} + \frac{4^5 - 2^5}{4^5}\|\tilde{u}_0\|_{\dot{H}^1} < 2\|\tilde{u}_0\|_{\dot{H}^1}. \quad (4.2.12)$$

Using Lemma 2.3.1, the algebraic inequality $||u|^4u - |v|^4v| \leq |u-v|\left(|u|^4 - |v|^4\right)$ and the decomposition $\tilde{u} = \tilde{u}_{low} + \tilde{u}_{high}$, $\tilde{v} = \tilde{v}_{low} + \tilde{v}_{high}$, where the high and low projections satisfy the same bounds as in Equation

(4.2.11) and Equation (4.2.11), a similar argument to that above shows that $\tilde{\Gamma}$ is a contraction mapping on B:

$$\left\|\tilde{\Gamma}\tilde{u} - \tilde{\Gamma}\tilde{v}\right\|_{X^q(I)} \leq \frac{1}{2}\|u - v\|_{X^q(I)}. \tag{4.2.13}$$

$\square$

**Remark 4.2.5.** *The local theory, similar to that in [1] tells us that in particular, if $\|\tilde{u}\|_{X^q(I)}$ is finite on some time interval I, then the solution can be extended beyond I for some time.*

From Theorem 4.2.2, we see that if $u_0 \in \dot{H}^1$ evolves under (NLS), the solution $u$ is bounded in $X^q$ for any $2 \leq q \leq \infty$. For $u_0 \in \dot{H}^1$ evolving under the flow of $(\tilde{N}LS)$, we require similar bounds and in fact we can use a stability argument to obtain these bounds.

**Theorem 4.2.6.** *(Stability) Suppose we have a solution $\tilde{u}$ to the equation*

$$(\tilde{N}LS) \begin{cases} i\partial_t \tilde{u} + \triangle\tilde{u} = |\tilde{u}|^4 \tilde{u} + e \\ \tilde{u}_{t=0} = v_0 \in \dot{H}^1 \end{cases} \tag{4.2.14}$$

*on the time interval $I = [T_0, \bar{T})$ in the sense that for all $0 < T < \bar{T}$, $\tilde{u}$ solves (4.2.14) and $\tilde{u} \in X^q([T_0, T)) \cap L_t^\infty \dot{B}_{2,q}^1([T_0, T) \times (\mathbb{R}^3))$.*

*Let $v(t,x)$ be the unique solution to (NLS) with initial data $v_0$. There exists an $\varepsilon_0 = \varepsilon_0(\|v\|_{10,10}, \|v_0\|_{\dot{H}^1})$ such that for all $0 < \varepsilon < \varepsilon_0$, if*

$$\|\int_{T_0}^t e^{i(t-s)\triangle} e(s) ds\|_{X^q(I)} \leq 30\varepsilon^5 + 480\|v - \tilde{u}\|_{X^q(I)}^4 \varepsilon + 480\|v\|_{X^q(I)}^3 \|v\|_{L_t^{10}L_x^{10}(\mathbb{R}\times\mathbb{R}^3)} \varepsilon, \tag{4.2.15}$$

*then $\tilde{u}$ satisfies:*

$$\tilde{u} \in L_t^\infty \dot{B}_{2,q}^1(I \times \mathbb{R}^3)$$
$$\|v - \tilde{u}\|_{X^q(I)} \leq 1$$
$$\|\tilde{u}\|_{X^q(I)} \leq 1 + \|v\|_{X^q(I)}$$

*Proof.* Let $I = [T_0, T_1]$. We begin by assuming that $T_1$ is small enough so that $\|v\|_{L_t^{10}L_x^{10}(I\times\mathbb{R}^3)} < \delta$ for some $\delta$ to be chosen later.

Since $v$ solves (NLS) and $\tilde{u}$ solves $(\tilde{N}LS)$, we have

$$v(t,x) = e^{it\triangle} v_0 - i\int_{T_0}^t e^{i(t-s)\triangle} (|v|^4 v)(s) ds \tag{4.2.16}$$

$$\tilde{u}(t,x) = e^{it\triangle} v_0 - i\int_{T_0}^t e^{i(t-s)\triangle} (|\tilde{u}|^4 \tilde{u} + e)(s) ds,$$

and so

$$||v - \tilde{u}||_{X^q(I)} \leq ||\int_{T_0}^{t} e^{i(t-s)\triangle}((|v|^4 v) - (|\tilde{u}|^4 \tilde{u}))ds||_{X^q(I)} + ||\int_{T_0}^{t} e^{i(t-s)\triangle}e(s)||_{X^q(I)}.$$

By the algebraic inequality

$$
\begin{aligned}
||v|^4 v - |\tilde{u}|^4 \tilde{u}| &\leq |v - \tilde{u}| \left(|v|^4 + |\tilde{u}|^4\right) \\
&\leq |v - \tilde{u}| \left(|v|^4 + |v - \tilde{u} + v|^4\right) \\
&\leq |v - \tilde{u}| \left(|v|^4 + 16|v - \tilde{u}|^4 + 16|v|^4\right) \\
&\leq 17|v - \tilde{u}||v|^4 + 16|v - \tilde{u}|^5,
\end{aligned}
$$

Prop. 3.2.2 and Prop. 3.2.3,

$$||\int_{T_0}^{t} e^{i(t-s)\triangle}((|v|^4 v) - (|\tilde{u}|^4 \tilde{u}))ds||_{X^q} \leq 17||v - \tilde{u}||_{X^q}||v||^3_{X^q}||v||_{L_t^{10}L_x^{10}} + 16||v - \tilde{u}||^5_{X^q}. \tag{4.2.17}$$

By Equation (4.2.15) and Equation (4.2.17), we have

$$||v - \tilde{u}||_{X^q} \leq 17||v - \tilde{u}||_{X^q}||v||^3_{X^q}||v||_{L_t^{10}L_x^{10}} + 16||v - \tilde{u}||^5_{X^q} + 30\varepsilon^5 + 480||v - \tilde{u}||^4_{X^q}\varepsilon + 480||v||^3_{X^q}||v||_{L_t^{10}L_x^{10}}\varepsilon. \tag{4.2.18}$$

Let $A(T) = ||(v - \tilde{u})||_{X^q([T_0,T])}$. By Lemma 4.2.2, we have that $||v||_{X^q(I)} \leq C(||v||_{L_t^{10}L_x^{10}(\mathbb{R}\times\mathbb{R}^3)}, ||u_0||_{\dot{H}^1})$. We have shown:

$$A(T) \leq 17C^3 \delta A(T) + 16A(T)^5 + 30\varepsilon^5 + 480A(T)^4\varepsilon + 480C^3\delta\varepsilon. \tag{4.2.19}$$

If we choose $\delta$ so that $480C^3\delta = \frac{1}{100}$ and $\varepsilon$ small enough (we will choose an explicit $\varepsilon$ soon), a standard continuity argument shows that $A(T_1) < \varepsilon$. In other words, $||v - \tilde{u}||_{X^q([T_0,T_1])} < \varepsilon$.

Now we remove the smallness assumption above. We fix $\delta$ satisfying $480C^3\delta = \frac{1}{100}$. We decompose $I$ into intervals $I_k$ where on each $I_k$, $||v||_{L_t^{10}L_x^{10}(I_k\times\mathbb{R}^3)} < \delta$. From Lemma 3.2.5, there are $\frac{||v||^{10}_{L_t^{10}L_x^{10}(I\times\mathbb{R}^3)}}{\delta^{10}} =: n$ many intervals.

We use the triangle inequality and seek to bound each term in the sum:

$$||v - \tilde{u}||_{X^q(I)} \leq \sum_k ||v - \tilde{u}||_{X^q(I_k)}.$$

Let $I_k = [T_{k-1}, T_k]$, $A_k(T) = ||v - \tilde{u}||_{X^q([T_{k-1},T])}$ and $A_k = ||v - \tilde{u}||_{X^q([T_{k-1},T_k])}$. For our first interval $I_1$, the above shows that $A_1 \leq \varepsilon$. Let's consider our second interval $I_2$. By Equation (4.2.16) and our previous analysis ,

$$
\begin{aligned}
A_2(T) \quad &= \quad ||v - \tilde{u}||_{X^q([T_1,T])} \\[2mm]
&\leq \quad ||e^{i(T-T_1)\triangle}(v(T_1) - \tilde{u}(T_1))||_{X^q_{([T_1,T])}} + ||\int_{T_1}^{t} e^{i(t-s)}(|v|^4 v - |\tilde{u}|^4 \tilde{u} - e)(s)ds||_{X^q([T_1,T])} \\[2mm]
&\leq \quad ||e^{i(T-T_1)\triangle}(v(T_1) - \tilde{u}(T_1))||_{X^q_{([T_1,T])}} + 17C^3\delta A_2(T) + 16A_2(T)^5 + 30\varepsilon^5 + 480A_2(T)^4\varepsilon + 480C^3\delta\varepsilon,
\end{aligned}
$$

where the last line follows from a similar argument as above. Notice the linear terms differ for this interval. Let us consider the first term. By Lemma 2.2.1,

$$
\begin{aligned}
||e^{i(T-T_1)\triangle}(v(T_1) - \tilde{u}(T_1))||_{X^q([T_1,T])} \quad &= \quad ||v(T_1) - \tilde{u}(T_1)||_{\dot{B}^1_{2,q}} \\[2mm]
&\leq \quad ||v(t) - \tilde{u}(t)||_{L^\infty_t \dot{B}^1_{2,q}([T_0,T_1])} \\[2mm]
&\leq \quad ||v(t) - \tilde{u}(t)||_{X^q([T_0,T_1])}.
\end{aligned}
$$

We have shown that this is bounded by $\varepsilon$, so by Equations (4.2.20) and (4.2.20), we have

$$
A_2(T) \leq \varepsilon + 17C^3\delta A_2(T) + 16A_2(T)^5 + 30\varepsilon^5 + 480A_2(T)^4\varepsilon + 480C^3\delta\varepsilon. \tag{4.2.20}
$$

Taking $\varepsilon$ small enough, a continuity argument shows that $A_2 < 10\varepsilon$. We require a smaller $\varepsilon$ in this second interval, as the linear term is not zero. In fact, for each subsequent interval, we will need an exponentially smaller interval. For this interval, $\varepsilon < \frac{1}{100}$ will suffice, but for our choice to work on every subinterval, we take $\varepsilon < 10^{-n}$, where we recall $n = \frac{||v||^{10}_{L^{10}_t L^{10}_x(I\times\mathbb{R}^3)}}{\delta^{10}}$. Note that $\delta$ being fixed determines $n$ and this in turn determines our choice for $\varepsilon$.

We proceed inductively. For $k \leq n$, we assume $A_m \leq 10^{(m-1)}\varepsilon$ for $m < k$ and are able to use this assumption to show that $A_k \leq 10^{k-1}\varepsilon$ by a continuity argument. Indeed, by Lemma 2.2.1,

$$
\begin{aligned}
A_k(T) \quad & \tag{4.2.21} \\[2mm]
&\leq \quad ||e^{i(T-T_1)\triangle}(v(T_{k-1}) - \tilde{u}(T_{k-1}))||_{X^q([T_{k-1},T])} + 17C^3\delta A_k(T) + 16A_k(T)^5 + 30\varepsilon^5 + 480A_k(T)^4\varepsilon + 480C^3\delta\varepsilon \\[2mm]
&\leq \quad ||v(t) - \tilde{u}(t)||_{X^q([T_{k-2},T_{k-1}])} + 17C^3\delta A_k(T) + 16A_k(T)^5 + 30\varepsilon^5 + 480A_k(T)^4\varepsilon + 480C^3\delta\varepsilon \\[2mm]
&\leq \quad 10^{k-2}\varepsilon + 17C^3\delta A_k(T) + 16A_k(T)^5 + 30\varepsilon^5 + 480A_k(T)^4\varepsilon + 480C^3\delta\varepsilon
\end{aligned}
$$

A standard continuity argument then shows that $A_k \leq 10^{k-1}\varepsilon$. We now show the steps of this process in detail. By Lemma 4.1.1, there exists $T > T_{k-1}$ such that $A_k(T) < 10^{k-1}\varepsilon$. For $T \in (T_{k-1}, T_k]$, we show that $A_k(T) < 10^{k-1}\varepsilon$ implies that $A_k(T) < \frac{1}{2}10^{k-1}\varepsilon$, so that by continuity, $A_k(T+\eta) < 10^{k-1}\varepsilon$ for some $\eta > 0$. This implies that the set $\{T : A_k(T) < 10^{k-1}\varepsilon\}$ is both open and closed in $(T_{k-1}, T_k]$, and hence must be the entire interval. This will prove that $A_k < 10^{k-1}\varepsilon$.

Assume $A_k(T) < 10^{k-1}\varepsilon$ and $10^n\varepsilon < \frac{1}{10}$. We bound each term on the RHS of Equation (4.2.21).

$$17C^3\delta A_k(T) \leq \frac{1}{100}A_k(T) < \frac{1}{100}10^{k-1}\varepsilon. \tag{4.2.22}$$

By assumption, $A_k(T) < 10^{k-1}\varepsilon < 10^n\varepsilon < \frac{1}{10}$.

$$16A_k(T)^5 < (16)(\frac{1}{10})^4 A_k(T) < \frac{1}{100}10^{k-1}\varepsilon. \tag{4.2.23}$$

$$480A_k(T)^4\varepsilon \leq 480(\frac{1}{10})^4\varepsilon < \frac{1}{100}10^{k-1}\varepsilon. \tag{4.2.24}$$

Certainly $30\varepsilon^5 < \frac{1}{100}10^{k-1}\varepsilon$ and $480C^3\delta\varepsilon < \frac{1}{100}\varepsilon$. Thus, by Equations (4.2.21), (4.2.22), (4.2.23) and (4.2.24), $A_k(T) < \frac{1}{2}10^{k-1}\varepsilon$. This guarantees $A_k < 10^{k-1}\varepsilon$ by continuity.

By Corollary 3.2.5, $10^n = 10^{\frac{||v||_{10,10}^{10}}{\delta^{10}}} = 10^{||v||_{10,10}^{10}||v||_{X^q(\mathbb{R})}(48000)^{10}} \leq 10^{\frac{||v||_{10,10}^{310}10^{710}}{||v_0||_{\dot{H}^1}^{29}}}$. Thus we require $\varepsilon < 10^{-\frac{||v||_{10,10}^{310}10^{710}}{||v_0||_{\dot{H}^1}^{29}}-1}$. Thus, if we choose $\varepsilon_0 = 10^{-\frac{||v||_{10,10}^{310}10^{710}}{||v_0||_{\dot{H}^1}^{29}}-1}$, then if $\varepsilon < \varepsilon_0$,

$$||v-\tilde{u}||_{X^q(I)} \quad \leq \quad \sum_{k=1}^{n}||v-\tilde{u}||_{X^q(I_k)} \leq \sum_{k=1}^{n}10^{k-1}\varepsilon \leq 10^n\varepsilon \leq 1,$$

$$||\tilde{u}||_{X^q(I)} \quad \leq \quad ||v-\tilde{u}||_{X^q(I)} + ||v||_{X^q(I)} \leq 1 + ||v||_{X^q(I)},$$

and by Lemma 2.2.1, $||\tilde{u}||_{L_t^\infty \dot{B}_{2,\infty}^1(I\times\mathbb{R}^3)} \leq ||\tilde{u}||_{X^q(I)} \leq 1 + ||v||_{X^q(I)}.$

□

## 4.3 Proof of Global Well-Posedness

For ease of reading, we restate Theorem 1.3.1 now.

**Theorem 4.3.1.** *Let $u_0 \in \dot{B}_{2,q}^1$ with $2 \leq q \leq \infty$ and $u_0 = v_0 + w_0$, $v_0 \in \dot{H}^1$ and $w_0 \in \dot{B}_{2,q}^1$ with $||w_0||_{\dot{B}_{2,q}^1} < \min\left(\frac{1}{2}, \varepsilon_0\left(||v||_{L_t^{10}L_x^{10}(\mathbb{R}\times\mathbb{R}^3)}, ||v_0||_{\dot{H}^1}\right)\right)$, where v is the unique solution to Equation (1.1.1) emerging from initial data $v_0 \in \dot{H}^1$. There exists a unique global solution $u(t,x)$ to Equation (1.1.1) which satisfies:*

*For $2 \leq q < \infty$,*

$$u \in C_t^0 \dot{B}_{2,q}^1(\mathbb{R}^+ \times \mathbb{R}^3) \cap X^q(\mathbb{R}^+),$$

*and u scatters in $\dot{B}_{2,q}^1$. For $q = \infty$,*

$$u \in L_t^\infty \dot{B}_{2,\infty}^1 (\mathbb{R}^+ \times \mathbb{R}^3) \cap X^\infty (\mathbb{R}^+)$$

*and u converges weakly to $u_0$ in $\dot{B}_{2,\infty}^1$ as $t \to 0$.*

In this section, we prove that data of the form described in Theorem 4.3.1 evolve globally under the evolution of (NLS).

*Proof.* Let $w_0$ satisfy $||w_0||_{\dot{B}_{2,q}^1} < \min\{C(||\tilde{u}_0||_{\dot{H}^1}), \eta_0, \frac{\varepsilon}{2}\}$, where $\eta_0$ is given in Theorem 4.2.1 and $C(||\tilde{u}_0||_{\dot{H}^1})$ is given in Lemma 4.2.4. In particular, by Theorem 4.2.1, since $||w_0||_{\dot{B}_{2,q}^1} < \eta_0$, $w_0$ evolves globally under (NLS) to a global solution $w(t,x)$, which satisfies the bound $||w||_{X^q[0,\infty)} < 2\min\{C(||\tilde{u}_0||_{\dot{H}^1}), \eta_0, \frac{\varepsilon}{2}\}$.

Recall, from Section 4.2, if $e = e(t,x) = |w + \tilde{u}|^4 (w + \tilde{u}) - |w|^4 w - |\tilde{u}|^4 \tilde{u}$, and $\tilde{u}$ is a solution to

$$(\tilde{NLS}) \begin{cases} i\partial_t \tilde{u} + \triangle \tilde{u} = |\tilde{u}|^4 \tilde{u} + e \\ \tilde{u}_{t=0} = v_0 \in \dot{H}^1, \end{cases} \tag{4.3.1}$$

then $u(t,x) = \tilde{u}(t,x) + w(t,x)$ is a solution to (NLS).

Lemma 4.2.4 tells us that since $||w_0||_{\dot{B}_{2,q}^1} < C(||\tilde{u}_0||_{\dot{H}^1})$, we have a local solution to $(\tilde{NLS})$. From the local theory, it follows that there is a maximal time of existence for this solution, $I$. We assume that $I$ is finite to obtain a contradiction. If we show that the error term $e$ satisfies the conditions for Theorem 4.2.6, then Theorem 4.2.6 tells us that $||\tilde{u}||_{X^q(I)}$ is bounded and thus from our local theory again, we know we may extend $I$ for some amount of time and this is a contradiction. Indeed, the error $e$ satisfies the requirements. From Definition 2.1.7, Lemma 2.1.3,

$$||\int_{T_0}^t e^{i(t-s)\triangle} e(s)ds||_{X^q(I)} \leq \left( \sum_N N^q ||\int_{T_0}^t e^{i(t-s)\triangle}(|w + \tilde{u}|^4 (w + \tilde{u}) - |w|^4 w - |\tilde{u}|^4 \tilde{u})_N ds||_{V_\triangle^2}^q \right)^{\frac{1}{q}}$$

$$\leq \left( \sum_N \int_{T_0}^t e^{i(t-s)\triangle}(|w + \tilde{u}|^4 (w + \tilde{u}) - |w|^4 w - |\tilde{u}|^4 \tilde{u})_N ds||_{U_\triangle^2} \right)^{\frac{1}{q}}.$$

By the series of algebraic inequalities,

$$\left| |w + \tilde{u}|^4 (w + \tilde{u}) - |w|^4 w - |\tilde{u}|^4 \tilde{u} \right| \leq 30 |w|^5 + 30 |\tilde{u}|^4 |w|$$

$$\leq 30 |w|^5 + 30 (|\tilde{u} - v| + |v|)^4 |w|$$

$$\leq 30 |w|^5 + 30 \left( 16 |\tilde{u} - v|^4 + 16 |v|^4 \right) |w|$$

$$= 30 |w|^5 + 480 |\tilde{u} - v|^4 |w| + 480 |v|^4 |w|,$$

Lemma 3.2.2 and Lemma 3.2.3, we obtain

$$|| \int_{T_0}^{t} e^{i(t-s)\triangle} e(s) ds ||_{X^q(I)} \leq 30 ||w||_{X^q(I)}^5 + 480 ||\tilde{u} - v||_{X^q(I)}^4 ||w||_{X^q(I)} + 480 ||v||_{X^q(I)}^3 ||w||_{X^q(I)} ||v||_{L_t^{10} L_x^{10}(I \times \mathbb{R}^3)}.$$

Since $||w_0||_{\dot{B}_{2,q}^1} < \frac{\varepsilon}{2}$, we have $||w||_{X^q(I)} < \varepsilon$ and so

$$|| \int_{0}^{t} e^{i(t-s)\triangle} e(s) ds ||_{X^q(I)} \leq 30\varepsilon^5 + 480 ||\tilde{u} - v||_{X^q(I)}^4 \varepsilon + 480 ||v||_{X^q(I)}^3 ||v||_{L_t^{10} L_x^{10}(I \times \mathbb{R}^3)} \varepsilon.$$

We reach our desired contraction and conclude $\tilde{u}$ solves (NLS) on $I = [0, \infty)$. Since $\tilde{u}$ evolves globally under $(\widetilde{NLS})$, $u$ evolves globally under (NLS). $\qquad \square$

## 4.4 Scattering

In this section, we prove that the solutions found scatter when $q < \infty$. There can be no scattering for $q = \infty$. Recall from Definition 1.1.5, that it suffices to show

$$\lim_{t \to +\infty} ||u(t) - e^{it\triangle} u_+||_{\dot{B}_{2,q}^1(\mathbb{R}^3)} = 0. \tag{4.4.1}$$

Since the linear operator is unitary, it suffices to show

$$\lim_{t \to +\infty} ||e^{-it\triangle} u(t) - u_+||_{\dot{B}_{2,q}^1(\mathbb{R}^3)} = 0. \tag{4.4.2}$$

If we define $u_+ = u_0 - i \int_0^\infty e^{-is\triangle} \left( |u|^4 u \right)(s) ds$, and recall from Duhamel's formula (1.1.9) that

$$e^{-it\triangle} u(t) = u_0 - i \int_0^t e^{-is\triangle} \left( |u|^4 u \right)(s) ds, \tag{4.4.3}$$

then

$$||e^{-it\triangle} u(t) - u_+||_{\dot{B}_{2,q}^1(\mathbb{R}^3)} = || \int_t^\infty e^{-is\triangle} \left( |u|^4 u \right)(s) ds||_{\dot{B}_{2,q}^1}, \tag{4.4.4}$$

and it suffices to show that

$$\lim_{t \to \infty} || \int_t^\infty e^{-is\triangle} \left( |u|^4 u \right)(s) ds||_{\dot{B}_{2,q}^1} = 0. \tag{4.4.5}$$

Before we proceed, let us pause to recall how scattering is shown when $u_0 \in \dot{H}^1$ (the case of $u_0 \in L^2$, etc. is similar). In this case, the space where the local theory is proved is $\dot{\mathscr{S}}^1(I)$, where

$$||u||_{\dot{\mathscr{S}}^0(I)} = \sup_{(q,r)} \left( \sum_N ||P_N u||^2_{L_t^q L_x^r(I\times\mathbb{R}^3)} \right), \tag{4.4.6}$$

the supremum taken over all pairs $(q,r)$ of admissible exponents, and

$$||u||_{\dot{\mathscr{S}}^1(I)} = ||\nabla u||_{\dot{\mathscr{S}}^0(I)}. \tag{4.4.7}$$

Let $u$ be the solution that evolves from $u_0$. If $||u||_{\dot{\mathscr{S}}^1([0,t))}$ is bounded independently of t, then $||u||_{\dot{\mathscr{S}}^1([0,\infty))} < \infty$. It can then be shown that $||\int_t^\infty e^{-is\triangle} \left(|u|^4 u\right)(s)ds||_{\dot{H}^1} < ||u||^5_{\dot{\mathscr{S}}^1([t,\infty))}$. It is clear now that as $t$ approaches infinity, the right side vanishes. Thus, scattering follows almost directly from bounds on the local-theory space $\dot{\mathscr{S}}^1(I)$.

Let us now return to the case $u_0 \in \dot{B}^1_{2,q}$. If we try to prove scattering in the analogous way, we fail. From Theorem 4.2.6 and Theorem 4.2.1, we see that $||u||_{X^q([0,t)}$ is bounded independently of t. Indeed,

$$||u||_{X^q(I)} \leq ||\tilde{u}||_{X^q(I)} + ||w||_{X^q(I)} \leq \frac{1}{10} + ||v||_{X^q(I)} + 2||w_0||_{\dot{B}^1_{2,q}}.$$

This implies $||u||_{X^q[0,\infty)}$ is also finite. However, if we examine Definition 2.1.7, we see that (because of the essential supremum) $||u||_{X^q([t,\infty))} < C$ does not imply $\lim_{t\to\infty} ||u||_{X^q([t,\infty))} = 0$. We must work harder. Furthermore, it is possible to show that $||\int_0^t e^{-is\triangle} \left(|u|^4 u\right)(s)ds||_{\dot{B}^1_{2,q}}$ is bounded independently of t, however this is not sufficient to show scattering, as the integral may not converge when we take $t$ to infinity. From Theorem 4.2.6, we see that $||u-v||_{X^q[0,t)}$ is bounded independently of t. $v$ solves (NLS) with initial data $v_0 \in \dot{H}^1$ and we have that $||v||_{L_t^{10}L_x^{10}(\mathbb{R}\times\mathbb{R}^3)}$ is bounded by Theorem 1.2.1. In fact, we can show for every N that $||(u-v)_N||_{L_t^{\frac{10}{3}} \times L_x^{\frac{10}{3}}([0,t)\times\mathbb{R}^3)}$ is bounded independently of t. On the surface, this looks like it will help us show the decay in time we are looking for, since $u$ will inherit the decay of $v$ in time, however this is also insufficient, as the time of decay might increase as the frequency increases.

Instead, we will show

$$||\int_t^\infty e^{-is\triangle} \left(|u|^4 u\right)(s)ds||_{\dot{B}^1_{2,q}} \leq \left( \sum_N N^q || \left(|u|^4 u\right)_N ||^q_{L_t^1 L_x^2[t,\infty)\times\mathbb{R}^3)} \right)^{\frac{1}{q}} \leq ||u||^5_{X^q[t,\infty)}. \tag{4.4.8}$$

The first inequality is just an application of Minkowski's Integral Inequality. The second inequality will take more work and will be proved in a similar manner to the multilinear estimates in Chapter 3. Inequality (4.4.8) suffices to show scattering. Indeed, since $||u||_{X^q([t,\infty))} < \infty$, we have that $\left( \sum_N N^q || \left(|u|^4 u\right)_N ||^q_{L_t^1 L_x^2[t,\infty)\times\mathbb{R}^3)} \right)^{\frac{1}{q}} < \infty$.

Thus $\lim_{t\to\infty} \left( \sum_N N^q || \left(|u|^4 u\right)_N ||^q_{L_t^1 L_x^2[t,\infty)\times\mathbb{R}^3)} \right)^{\frac{1}{q}} = 0$ and this implies $||\int_t^\infty e^{-is\triangle} \left(|u|^4 u\right)(s)ds||_{\dot{B}^1_{2,q}} \to 0$. This will conclude the proof of Therem 4.3.1. We note that this Lemma could appear in Chapter 3, but it was placed here to include the context with the lemma.

**Lemma 4.4.1.** *For $u \in X^q(I)$, for the time interval $I = [t_0, t_1)$,*

$$\left( \sum_N N^q || \left(|u|^4 u\right)_N ||^q_{L_t^1 L_x^2(I\times\mathbb{R}^3)} \right)^{\frac{1}{q}} \leq ||u||^5_{X^q(I)}. \tag{4.4.9}$$

*Proof.* The proof is similar to the proofs in Chapter 3. We use duality and frequency decomposition to reach our desired result. We break the proof up into the cases $q = \infty$ and $q = 2$ and then interpolate between these results. We drop the interval for ease of reading and begin with $q = \infty$.

**q $= \infty$:**

Using the duality $||f||_{L^1_t L^2_x(I \times \mathbb{R}^3)} = \sup\limits_{||v||_{L^\infty_t L^2_x(I \times \mathbb{R}^3)} = 1} \left| \int\limits_{I \times \mathbb{R}^3} (fv) \, dxdt \right|$, decomposing each function into its Littlewood-Paley projections as in Prop. 3.1.1, Prop. 3.1.2, etc., and Plancherel's Theorem, we have

$$
\sup_N \left( N || \left( |u|^4 u \right)_N ||_{L^\infty_t L^2_x(I \times \mathbb{R}^3)} \right) = \sup_N \left( \sup_{||v||_{L^\infty_t L^2_x(I \times \mathbb{R}^3)} = 1} N \left| \int\limits_{I \times \mathbb{R}^3} |u|^4 u v_N \, dxdt \right| \right)
$$

$$
\leq \sup_N \left( \sup_{||v||_{L^\infty_t L^2_x(I \times \mathbb{R}^3)} = 1} N \sum_{\lambda_i} \left| \int\limits_{I \times \mathbb{R}^3} u_{1,\lambda_1} u_{2,\lambda_2} u_{3,\lambda_3} u_{4,\lambda_4} u_{5,\lambda_5} v_N \, dxdt \right| \right),
$$

where the sum is over all dyadic numbers $\lambda_i$, $i = 1,2,3,4,5$, $u_i$ denotes either $u$ or $\bar{u}$ and $u_{i,\lambda_i} = (u_i)_{\lambda_i}$. For the sake of brevity, we will only show the cases where $N > \lambda_i$ for $i = 1,2,3,4,5$ and where $N < \lambda_i$ for $i = 1,2,3,4,5$.

<u>Case 1</u> : $N < \lambda_i$, $i = 1,2,3,4,5$. Ie; we consider the sum

$$
\sum_1 = \sum_{N < \lambda_1 < \infty} \sum_{\lambda_1 < \lambda_2 < \infty} \sum_{\lambda_2 < \lambda_3 < \infty} \sum_{\lambda_3 < \lambda_4 \sim \lambda_5 < \infty}. \tag{4.4.10}
$$

By Hölder's inequality, Proposition 2.2.2 and Lemma 2.1.3,

$$
\mathscr{S}_1 := \sup_N \left( \sup_{||v||_{L^\infty_t L^2_x(I \times \mathbb{R}^3)} = 1} N \sum_1 \left| \int\limits_{I \times \mathbb{R}^3} u_{1,\lambda_1} u_{2,\lambda_2} u_{3,\lambda_3} u_{4,\lambda_4} u_{5,\lambda_5} v_N \, dxdt \right| \right)
$$

$$
\leq \sup_N \left( \sup_{||v||_{L^\infty_t L^2_x(I \times \mathbb{R}^3)} = 1} N \sum_1 \prod_{i=1}^5 ||u_{i,\lambda_i}||_{5,10} ||v_N||_{\infty,2} \right)
$$

$$
\leq \sup_N \left( \sup_{||v||_{L^\infty_t L^2_x(I \times \mathbb{R}^3)} = 1} N \sum_1 \prod_{i=1}^5 \lambda_i^{\frac{4}{5}} ||u_{i,\lambda_i}||_{U^5_\triangle} ||v_N||_{\infty,2} \right)
$$

$$
\leq \sup_N \left( \sup_{||v_N||_{L^\infty_t L^2_x(I \times \mathbb{R}^3)} = 1} N \sum_1 \prod_{i=1}^5 \lambda_i^{-\frac{1}{5}} \left( \lambda_i ||u_{i,\lambda_i}||_{V^2_\triangle} \right) ||v_N||_{\infty,2} \right)
$$

$||v_N||_{\infty,2} \leq ||v||_{\infty,2}$, so by Definition 2.1.7, we have

$$\mathscr{S}_1 \leq ||u||_{X^\infty}^5 \sup_N \left( \sup_{||v||_{V_\triangle^2}=1} ||v||_{V_\triangle^2} N \sum_1 (\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)^{-\frac{1}{5}} \right)$$

$$\leq ||u||_{X^\infty}^5 \sup_N \left( N \sum_1 (\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)^{-\frac{1}{5}} \right).$$

We sum in order, $\lambda_5, \lambda_4, \lambda_3, \lambda_2, \lambda_1$, and we obtain

$$\mathscr{S}_1 \lesssim ||u||_{X^\infty}^5 \sup_N \left( NN^{-1} \right)$$

$$= ||u||_{X^\infty}^5.$$

<u>Case 2 :</u> $N > \lambda_i$, $i = 1, 2, 3, 4$ (and $N \sim \lambda_5$. I.e. we consider the sum

$$\sum_4 = \sum_{-\infty<\lambda_4<N\sim\lambda_5} \sum_{-\infty<\lambda_3<\lambda_4} \sum_{-\infty<\lambda_2<\lambda_3} \sum_{-\infty<\lambda_1<\lambda_2}. \tag{4.4.11}$$

By Hölder's inequality, Proposition 2.2.2, Proposition 2.2.4 and Lemma 2.1.3,

$$\mathscr{S}_4 := \sup_N \left( \sup_{||v||_{L_t^\infty L_x^2(I\times\mathbb{R}^3)}=1} N \sum_4 \left| \int_{I\times\mathbb{R}^3} u_{1,\lambda_1}u_{2,\lambda_2}u_{3,\lambda_3}u_{4,\lambda_4}u_{5,\lambda_5}v_N dxdt \right| \right)$$

$$\leq \sup_N \left( \sup_{||v||_{L_t^\infty L_x^2(I\times\mathbb{R}^3)}=1} N \sum_4 ||u_{1,\lambda_1}||_{6,\infty}||u_{2,\lambda_2}||_{6,\infty}||u_{3,\lambda_3}||_{6,\infty}||u_{4,\lambda_4}u_{5,\lambda_5}||_{2,2}||v_N||_{\infty,2} \right)$$

$$\leq \sup_N \left( \sup_{||v||_{L_t^\infty L_x^2(I\times\mathbb{R}^3)}=1} N \sum_4 (\lambda_1\lambda_2\lambda_3)^{\frac{7}{6}}\lambda_4\lambda_5^{-\frac{1}{2}}||u_{1,\lambda_1}||_{U_\triangle^6}||u_{2,\lambda_2}||_{U_\triangle^6}||u_{3,\lambda_3}||_{U_\triangle^6}||u_{4,\lambda_4}||_{V_\triangle^2}||u_{5,\lambda_5}||_{V_\triangle^2}||v_N||_{\infty,2} \right)$$

$$\leq \sup_N \left( \sup_{||v_N||_{L_t^\infty L_x^2(I\times\mathbb{R}^3)}=1} N||v_N||_{\infty,2} \sum_4 (\lambda_1\lambda_2\lambda_3)^{\frac{1}{6}}\lambda_5^{-\frac{3}{2}} \prod_{i=1}^5 \left( \lambda_i||u_{i,\lambda_i}||_{V_\triangle^2} \right) \right)$$

We recall $||v_N||_{\infty,2} \leq ||v||_{\infty,2}$. Taking the supremum over $\lambda_i||u_{i,\lambda_i}||_{V_\triangle^2}$, for $i = 1, 2, 3, 4, 5$, and pulling it out of the sum, by Definition 2.1.7, we have

$$\mathscr{S}_4 \leq ||u||_{X^\infty}^5 \sup_N \left( \sup_{||v||_{V_\triangle^2}=1} ||v||_{V_\triangle^2} N \sum_1 (\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)^{-\frac{1}{5}} \right)$$

$$\leq ||u||_{X^\infty}^5 \sup_N \left( N \sum_1 (\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)^{-\frac{1}{5}} \right).$$

We sum in order, $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$, and we obtain

$$
\begin{aligned}
\mathscr{S}_4 &\lesssim \|u\|_{X^\infty}^5 \sup_N \left( N N^{-1} \right) \\
&= \|u\|_{X^\infty}^5.
\end{aligned}
$$

$\underline{q=2}$:

Using the duality $\left( \sum_N N^2 \|f_N\|_{L_t^1 L_x^2(I\times\mathbb{R}^3)}^2 \right)^{\frac{1}{2}} = \sup_{\|v\|_{Z(I)}=1} \left| \int_{I\times\mathbb{R}^3} (fv)\, dxdt \right|$, where $\|v\|_{Z(I)} = \left( \sum_N N^{-2} \|v_N\|_{\infty,2}^2 \right)^{\frac{1}{2}}$,

decomposing the each function into its Littlewood-Paley projections as in Prop. 3.1.1, Prop. 3.1.2, etc., and Plancherel's Theorem, we have

$$
\begin{aligned}
\left( \sum_N N^2 \| \left( |u|^4 u \right)_N \|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)}^2 \right)^{\frac{1}{2}} &= \sup_{\|v\|_{Z(I)}=1} \left| \int_{I\times\mathbb{R}^3} |u|^4 u v_N\, dxdt \right| \\
&\leq \sup_{\|v\|_{Z(I)}=1} \sum_{\lambda_i} \left| \int_{I\times\mathbb{R}^3} u_{1,\lambda_1} u_{2,\lambda_2} u_{3,\lambda_3} u_{4,\lambda_4} u_{5,\lambda_5} v_N\, dxdt \right|,
\end{aligned}
$$

where the sum is over all dyadic numbers $\lambda_i$, $i=1,2,3,4,5$, $u_i$ denotes either $u$ or $\bar{u}$ and $u_{i,\lambda_i} = (u_i)_{\lambda_i}$. For the sake of brevity, we will only show the cases where $N > \lambda_i$ for $i=1,2,3,4,5$ and where $N < \lambda_i$ for $i=1,2,3,4,5$.

$\underline{\text{Case 1}}$: $N < \lambda_i$, $i=1,2,3,4,5$. I.e. we consider the sum

$$
\sum_1 = \sum_{-\infty < \lambda_4 \sim \lambda_5 < \infty} \sum_{-\infty < \lambda_3 < \lambda_5} \sum_{-\infty < \lambda_2 < \lambda_3} \sum_{-\infty < \lambda_1 < \lambda_2} \sum_{-\infty < N < \lambda_1}. \tag{4.4.12}
$$

By Hölder's inequality, Proposition 2.2.2, Proposition 2.2.4 and Lemma 2.1.3,

$$
\begin{aligned}
\mathscr{S}_1 &:= \sup_{\|v\|_{Z(I)}=1} \sum_1 \left| \int_{I\times\mathbb{R}^3} u_{1,\lambda_1} u_{2,\lambda_2} u_{3,\lambda_3} u_{4,\lambda_4} u_{5,\lambda_5} v_N\, dxdt \right| \\
&\leq \sup_{\|v\|_{Z(I)}=1} \sum_1 \|v_N\|_{\infty,2} \|u_{5,\lambda_5} u_{1,\lambda_1}\|_{2,2} \|u_{2,\lambda_2}\|_{6,\infty} \|u_{3,\lambda_3}\|_{6,\infty} \|u_{4,\lambda_4}\|_{6,\infty} \\
&\leq \sup_{\|v\|_{Z(I)}=1} \sum_1 \|v_N\|_{\infty,2} \lambda_5^{-\frac{1}{2}} \lambda_1 \|u_{5,\lambda_5}\|_{V_\triangle^2} \|u_{1,\lambda_1}\|_{V_\triangle^2} (\lambda_2 \lambda_3 \lambda_4)^{\frac{7}{6}} \|u_{2,\lambda_2}\|_{U_\triangle^6} \|u_{3,\lambda_3}\|_{U_\triangle^6} \|u_{4,\lambda_4}\|_{U_\triangle^6} \\
&\leq \sup_{\|v\|_{Z(I)}=1} \sum_1 \|v_N\|_{\infty,2} \left( \lambda_1 \|u_{1,\lambda_1}\|_{V_\triangle^2} \right) \left( \lambda_2 \|u_{2,\lambda_2}\|_{V_\triangle^2} \right) \left( \lambda_3 \|u_{3,\lambda_3}\|_{V_\triangle^2} \right) N \lambda_5^{-\frac{3}{2}} (\lambda_2 \lambda_3 \lambda_4)^{\frac{1}{6}} \|u_{4,\lambda_4}\|_{V_\triangle^2} \|u_{5,\lambda_5}\|_{V_\triangle^2}.
\end{aligned}
$$

Taking supremums out of the sum, by Definition 2.1.7, we have

$$\mathscr{S}_1 \quad \leq \quad ||u||_{X^\infty}^3 \sup_{||v||_{Z(I)}=1} \sup_N \left(N^{-1}||v_N||_{\infty,2}\right) \sum_1 N\lambda_5^{-\frac{3}{2}} \left(\lambda_2\lambda_3\lambda_4\right)^{\frac{1}{6}} ||u_{4,\lambda_4}||_{V_\triangle^2} ||u_{5,\lambda_5}||_{V_\triangle^2}.$$

We sum in order, $N, \lambda_1, \lambda_2, \lambda_3$. The embedding $\ell^2 \subset \ell^\infty$ then gives us

$$\mathscr{S}_1 \quad \leq \quad ||u||_{X^\infty}^3 \sup_{||v||_{Z(I)}=1} \left(\sum_N N^{-2}||v_N||_{\infty,2}^2\right)^{\frac{1}{2}} \sum_{-\infty<\lambda_4\sim\lambda_5<\infty} \lambda_5\lambda_4 ||u_{4,\lambda_4}||_{V_\triangle^2} ||u_{5,\lambda_5}||_{V_\triangle^2}.$$

Cauchy-Schwarz then gives us

$$\mathscr{S}_1 \lesssim ||u||_{X^\infty}^3 ||u||_{X^2}^2.$$

<u>Case 2</u>: $N > \lambda_i$, $i = 1,2,3,4$ (and $N \sim \lambda_5$). Ie; we consider the sum

$$\sum_4 = \sum_{-\infty<N\sim\lambda_5<\infty} \sum_{-\infty<\lambda_4<N} \sum_{-\infty<\lambda_3<\lambda_4} \sum_{-\infty<\lambda_2<\lambda_3} \sum_{-\infty<\lambda_1<\lambda_2}. \tag{4.4.13}$$

By Hölder's inequality, Proposition 2.2.2, Proposition 2.2.4 and Lemma 2.1.3,

$$\mathscr{S}_4 \quad := \quad \sup_{||v||_{Z(I)}=1} \sum_4 \left| \int_{I\times\mathbb{R}^3} u_{1,\lambda_1} u_{2,\lambda_2} u_{3,\lambda_3} u_{4,\lambda_4} u_{5,\lambda_5} v_N dx dt \right|$$

$$\leq \quad \sup_{||v||_{Z(I)}=1} \sum_4 ||v_N||_{\infty,2} ||u_{5,\lambda_5} u_{2,\lambda_2}||_{2,2} ||u_{1,\lambda_1}||_{6,\infty} ||u_{3,\lambda_3}||_{6,\infty} ||u_{4,\lambda_4}||_{6,\infty}$$

$$\leq \quad \sup_{||v||_{Z(I)}=1} \sum_4 ||v_N||_{\infty,2} \lambda_5^{-\frac{1}{2}} \lambda_2 ||u_{5,\lambda_5}||_{V_\triangle^2} ||u_{1,\lambda_1}||_{V_\triangle^2} \left(\lambda_1\lambda_3\lambda_4\right)^{\frac{7}{6}} ||u_{2,\lambda_2}||_{U_\triangle^6} ||u_{3,\lambda_3}||_{U_\triangle^6} ||u_{4,\lambda_4}||_{U_\triangle^6}$$

$$\leq \quad \sup_{||v||_{Z(I)}=1} \sum_4 \left(\lambda_1 ||u_{1,\lambda_1}||_{V_\triangle^2}\right) \left(\lambda_2 ||u_{2,\lambda_2}||_{V_\triangle^2}\right) \left(\lambda_3 ||u_{3,\lambda_3}||_{V_\triangle^2}\right) \left(\lambda_4 ||u_{4,\lambda_4}||_{V_\triangle^2}\right) \lambda_5^{-\frac{3}{2}} \left(\lambda_1\lambda_3\lambda_4\right)^{\frac{1}{6}} ||u_{5,\lambda_5}||_{V_\triangle^2} ||v_N||_{\infty,2}.$$

Taking supremums out of the sum, by Definition 2.1.7, we have

$$\mathscr{S}_4 \quad \leq \quad ||u||_{X^\infty}^4 \sup_{||v||_{Z(I)}=1} \sum_4 \lambda_5^{-\frac{3}{2}} \left(\lambda_1\lambda_3\lambda_4\right)^{\frac{1}{6}} ||u_{5,\lambda_5}||_{V_\triangle^2} ||v_N||_{\infty,2}.$$

We sum in order, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

$$\mathscr{S}_4 \quad \leq \quad ||u||_{X^\infty}^4 \sup_{||v||_{Z(I)}=1} \sum_{-\infty<N\sim\lambda_5<\infty} \lambda_5 N^{-1} ||u_{5,\lambda_5}||_{V_\triangle^2} ||v_N||_{\infty,2}.$$

Cauchy-Schwarz then gives us

$$\mathscr{S}_4 \lesssim \|u\|_{X^\infty}^4 \left( \sum_{-\infty < \lambda_5 < \infty} \lambda_5^2 \|u_{5,\lambda_5}\|_{V_\triangle^2}^2 \right)^{\frac{1}{2}} \left( \sum_{-\infty < N < \infty} N^{-2} \|v_N\|_{\infty,2}^2 \right)^{\frac{1}{2}}.$$

$\square$

# Chapter 5

# Wavelets and Profile Decomposition

In [28], Kenig outlines a procedure to prove global well-posedness and scattering for dispersive and wave equations. We will talk more about this in Section 6. An important part of this procedure is proving an appropriate profile decomposition. Profile decompositions measure the defect of compactness of embeddings. For example, from Lemma 1.4.2 and Sobolev embedding, we have the energy-critical Strichartz inequality

$$||e^{it\triangle}f||_{L_t^{10}L_x^{10}(\mathbb{R}\times\mathbb{R}^3)} \lesssim ||f||_{\dot{H}^1}.$$

However, the operator $e^{it\triangle} : \dot{H}^1 \to L_t^{10}L_x^{10}$ is far from compact. There is a group of non-compact symmetries consisting of space-translations, time-translations and scaling. Given a sequence $\{f_n\}$ in $\dot{H}^1$, we cannot extract a convergent subsequence from $\{e^{it\triangle}f_n\}$, however if we apply an appropriate member of the group to each term, we can collect "bubbles of concentration" with an error going to zero in a Strichartz space. This is the essence of the energy-critical profile decomposition given below.

Bahouri and GŚerard were the first to introduce a profile decomposition into the dispersive literature in the context of wave equations in [26]. In [36], Keraani proves a profile decomposition for $\dot{H}^1$ solutions to the linear Schrödinger Equation (1.4.2). A similar profile decomposition can be found in [10], which we state now for $d = 3$.

**Theorem 5.0.2.** *Let $\{f_n\}_{n\geq 1}$ be a sequence of functions bounded in $\dot{H}^1(\mathbb{R}^3)$. Passing to a subsequence if necessary, there exist $J^* \in \{0, 1, \dots\} \cup \{\infty\}$, functions $\{\phi^j\}_{j=1}^{J^*} \subset \dot{H}^1(\mathbb{R}^3)$, $\{\lambda_n^j\} \subset (,\infty)$, and $\{t_n^j, x_n^j\} \subset \mathbb{R}\times\mathbb{R}^3$ such that for each finite $0 \leq J \leq J^*$, we have the decomposition*

$$f_n = \sum_{j=1}^{J}(\lambda_n^j)^{-\frac{1}{2}}[e^{it_n^j\triangle}\phi^j](\frac{x - x_n^j}{\lambda_n^j}) + w_n^J \tag{5.0.1}$$

*with the following properties:*

$$\lim_{J\to J^*}\limsup_{n\to\infty}||e^{it\triangle}w_n^J||_{L_{t,x}^{10}(\mathbb{R}\times\mathbb{R}^3)} = 0 \tag{5.0.2}$$

$$\lim_{n\to\infty}[||\nabla f_n||_2^2 - \sum_{j=1}^{J}||\nabla\phi^j||_2^2 - ||\nabla w_n^J||_2^2] = 0 \tag{5.0.3}$$

$$\lim_{n \to \infty} \left[ ||f_n||_6^6 - \sum_{j=1}^{J} ||e^{it_n^j \triangle} \phi^j||_6^6 - ||w_n^J||_6^6 \right] = 0 \tag{5.0.4}$$

$$e^{-it_n^J \triangle} \left[ (\lambda_n^J)^{\frac{1}{2}} w_n^J (\lambda_n^J x + x_n^J) \right] \rightharpoonup 0 \tag{5.0.5}$$

*weakly in $\dot{H}^1(\mathbb{R}^3)$. Moreover, for each $j \neq k$ we have the following asymptotic decoupling of parameters:*

$$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{|x_n^j - x_n^k|^2}{\lambda_n^j \lambda_n^k} + \frac{|t_n^j (\lambda_n^j)^2 - t_n^k (\lambda_n^k)^2|}{\lambda_n^j \lambda_n^k} \to \infty \quad as \quad n \to \infty. \tag{5.0.6}$$

*Lastly, we may additionally assume that for each $j$ either $t_n^j \equiv 0$ or $t_n^j \to \pm\infty$.*

We would like to prove a profile decomposition result analogous to Theorem 5.0.2 for functions in $\dot{B}_{2,q}^1$ with the error term going to zero in the appropriate sense: $\lim_{J \to J^*} \limsup_{n \to \infty} ||e^{it\triangle} w_n^J||_{X^q(\mathbb{R} \times \mathbb{R}^3)} = 0$. Unfortunately, this will not work for our space $X^q$, since $||e^{it\triangle} f||_{X^q} = ||f||_{\dot{B}_{2,q}^1}$. The error cannot possibly go to zero in this sense. To proceed, a more refined space would have to be used. However, we can prove there is a profile decomposition for the embedding $\dot{B}_{2,q}^1 \hookrightarrow \dot{B}_{2,\infty}^1$.

**Theorem 5.0.3.** *Let $\{u_n\} \subset \dot{B}_{2,q}^1$, $||u_n||_{\dot{B}_{2,q}^1} \leq C$. Then up to a subsequence (which we still call $\{u_n\}$), there exists a family of profiles $\{\phi^l\}$ in $\dot{B}_{2,q}^1$ and sequences of scale-space indices $\{\lambda_l(n)\}_n$ for each $l > 0$ such that*

$$u_n = \sum_{l=1}^{L} \phi_{\lambda_l(n)}^l + r_{n,L} \tag{5.0.7}$$

*where*

$$\lim_{L \to +\infty} \left( \limsup_{n \to +\infty} ||r_{n,L}||_{\dot{B}_{2,\infty}^1} \right) = 0. \tag{5.0.8}$$

*The decomposition is asymptotically orthogonal in the sense that for any $k \neq l$,*

$$|j(\lambda_k(n)) - j(\lambda_l(n))| \to +\infty \text{ or } |k(\lambda_k(n)) - 2^{j(\lambda_k(n)) - j(\lambda_l(n))} k(\lambda_l(n))| \to +\infty \text{ as } n \to +\infty. \tag{5.0.9}$$

Suppose Theorem 4.3.1 can be shown in a space $Z$ (replace $X^q$ with $Z$ in the theorem) for which we have an appropriate Strichartz inequality and the refined Strichartz inequality

$$||e^{it\triangle} f||_{Z(\mathbb{R} \times \mathbb{R}^3)} \lesssim ||f||_{\dot{B}_{2,q}^1}^{\alpha} \sup_N ||e^{it\triangle} f_N||_{Z(\mathbb{R} \times \mathbb{R}^3)}^{\beta}. \tag{5.0.10}$$

We see that

$$
\begin{aligned}
||e^{it\triangle} f||_{Z(\mathbb{R} \times \mathbb{R}^3)} &\lesssim ||f||_{\dot{B}_{2,q}^1}^{\alpha} \sup_N ||e^{it\triangle} f_N||_{Z(\mathbb{R} \times \mathbb{R}^3)}^{\beta} \\
&\lesssim ||f||_{\dot{B}_{2,q}^1}^{\alpha} \sup_N ||f_N||_{\dot{B}_{2,\infty}^1} \\
&= ||f||_{\dot{B}_{2,q}^1}^{\alpha} ||f_N||_{\dot{B}_{2,\infty}^1}.
\end{aligned}
$$

Since $\{u_n\}$ is bounded in $\dot{B}_{2,q}^1$ and $||r_{n,L}||_{\dot{B}_{2,q}^1} \to \infty$, we would have $||e^{it\triangle} r_{n,L}||_{Z(\mathbb{R} \times \mathbb{R}^3)} \to \infty$ as we would want. Now Equation (5.0.10) is not an unreasonable goal. In [10], the author proves the following analogous result in the context of $\dot{H}^1$ which is stated here for $d = 3$.

**Lemma 5.0.4.** *Let $f \in \dot{H}^1(\mathbb{R}^3)$. Then*

$$||e^{it\triangle}f||_{L^{10}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \lesssim ||f||_{\dot{H}^1_x}^{\frac{1}{5}} \sup_N ||e^{it\triangle}f_N||_{L^{10}_{t,x}(\mathbb{R}\times\mathbb{R}^3)}^{\frac{4}{5}}. \tag{5.0.11}$$

We turn to proving Theorem 5.0.3. We will use a general approach using wavelets, which we will introduce now. See [40] for a introduction to wavelets. Some of the following is taken from [25].

A wavelet basis for a function space X is a finite set of "mother wavelets" $\{\psi^e\}$, together with their scaled and translated counterparts $\{\psi^e_\lambda\}$ such that any function $f \in X$ can be written as $f = \sum_{\lambda \in \nabla} c_\lambda \psi_\lambda$. Here $\nabla$ indexes translations by integers and dyadic scalings. ie; $\psi_\lambda = 2^{\frac{j}{2}} \psi(2^j \cdot -k)$ for some $j \in \mathbb{Z}$, $k \in \mathbb{Z}$. We will not need to keep track of which specific $j, k$ we are using in each instance and so we keep this suppressed in our notation. For further details on constructions of mother wavelets, see [40], [49] and [47].

In [25], the authors give a general method for proving profile decomposition results using wavelet bases. Wavelet bases are unconditional bases for Besov spaces in the sense that if X is a Besov space and $\{\psi_\lambda\}$ is a wavelet basis, then there exists a constant C such that for any finite subset $E \subset \nabla$ and coefficients that satisfy $|c_\lambda| < |d_\lambda|$ for all $\lambda$, we have $|| \sum_{\lambda \in E} c_\lambda \psi_\lambda ||_X \leq C|| \sum_{\lambda \in E} d_\lambda \psi_\lambda ||_X$.
Assume $f$ has the following wavelet decomposition $f = \sum_{\lambda \in \nabla} d_\lambda \psi_\lambda$. Define $E$ to be some ordering of $\nabla$ so that $d_{m+1} \leq d_m$ and $E_M$ to be the set obtained by removing the first $M$ elements from $E$. Define the nonlinear projector $Q_M$ so that $Q_M f := \sum_{\lambda \in E_M} d_\lambda \psi_\lambda$. To show that the embedding $X \subset Y$ has a profile decomposition, we require two assumptions.

**Assumption 1** The nonlinear projection satisfies

$$\lim_{M\to+\infty} \max_{||f||_X \leq 1} ||f - Q_M f||_Y = 0. \tag{5.0.12}$$

**Assumption 2** Consider a sequence of functions $(f_n)_{n>0}$ which are uniformly bounded in X and may be written as $f_n = \sum_{\lambda \in \nabla} c_{\lambda,n} \psi_\lambda$, and such that for all $\lambda$, the sequence $c_{\lambda,n}$ converges towards a finite limit $c_\lambda$ as $n \to +\infty$.
Then the series $\sum_{\lambda \in \nabla} c_\lambda \psi_\lambda$ converges in X with $|| \sum_{\lambda \in \nabla} c_\lambda \psi_\lambda ||_X \leq C \liminf_{n\to\infty} ||f_n||_X$, where C is a constant only depending on the space X and on the choice of the wavelet basis.

**Theorem 5.0.5.** *(General Profile Decomposition) [25] Consider two spaces with a continuous embedding $X \subset Y$ such that there exists a wavelet basis which is unconditional for both X and Y. Assume assumptions 1 and 2 hold for the embedding. Let $\{u_n\}$ be a bounded sequence in X. Then, up to a subsequence (which we'll still call $\{u_n\}$, there exists a family of profiles $\{\phi^l\}$ in X and sequences of scale-space indices $\{\lambda_l(n)\}_n$ for each $l > 0$ such that*

$$u_n = \sum_{l=1}^{L} \phi^l_{\lambda_l(n)} + r_{n,L} \tag{5.0.13}$$

*where*

$$\lim_{L\to+\infty} \left( \limsup_{n\to+\infty} ||r_{n,L}||_X \right) = 0. \tag{5.0.14}$$

*The decomposition is asymptotically orthogonal in the sense that for any $k \neq l$*

$$|j(\lambda_k(n)) - j(\lambda_l(n))| \to +\infty \ or \ |k(\lambda_k(n)) - 2^{j(\lambda_k(n)) - j(\lambda_l(n))} k(\lambda_l(n))| \to +\infty \ as \ n \to +\infty. \tag{5.0.15}$$

**Remark 5.0.6.** *From [25], we have that assumption 2 holds if X is a Besov space.*

**Remark 5.0.7.** *From [25], we have the following embeddings*

$$\dot{B}_{2,q}^{\frac{1}{2}} \hookrightarrow \dot{B}_{\frac{12}{5},2q}^{\frac{1}{4}} \hookrightarrow \dot{B}_{2,\infty}^{\frac{1}{2}} \tag{5.0.16}$$

**Lemma 5.0.8.** *Assumption 1 holds for the embedding $\dot{B}_{\frac{12}{5},2q}^{\frac{1}{4}} \hookrightarrow \dot{B}_{2,\infty}^{\frac{1}{2}}$.*

*Proof.* This follows from the representation of the Besov norm with wavelet coefficients.  □

**Lemma 5.0.9.** *Assume we have the continuous embeddings $X \subset Z \subset Y$. Assume assumption 1 holds for the embedding $Z \subset Y$, then assumption 1 holds for $X \subset Y$.*

*Proof.*

$$\max_{\|f\|_X \leq 1} \|f - Q_M f\|_Y \leq \max_{\|f\|_Z \leq 1} \|f - Q_M f\|_Y,$$

so

$$\lim_{M \to \infty} \max_{\|f\|_Z \leq 1} \|f - Q_M f\|_Y \to 0 \Rightarrow \lim_{M \to \infty} \max_{\|f\|_X \leq 1} \|f - Q_M f\|_Y \to 0.$$

□

The proof of Theorem 5.0.3 is now clear.

*Proof.* From [25], we know there exists a wavelet basis that is unconditional with respect to both $\dot{B}_{2,q}^{\frac{1}{2}}$ and $\dot{B}_{2,\infty}^{\frac{1}{2}}$.
Assumption 1 holds for the embedding $\dot{B}_{2,q}^{\frac{1}{2}} \subset \dot{B}_{2,\infty}^{\frac{1}{2}}$ by Remark 5.0.7, Lemma 5.0.8 and Lemma 5.0.9.
Assumption 2 holds for all Besov spaces by Remark 5.0.6 and by Theorem 5.0.5, this concludes the proof.  □

# Chapter 6

# Next Steps

In this chapter we consider a strategy to prove Conjecture 1.3.3 and also a conjecture that can be viewed as a preliminary result to Conjecture 1.3.3. We begin with this preliminary conjecture.

**Conjecture 6.0.10.** *Assume solutions to Equation (1.1.1) with data $u_0 \in \dot{B}_{2,q}^1$, $2 \leq q \leq \infty$ evolve with the condition $u \in L_t^\infty \dot{B}_{2,q}^1$. Let $u_0 \in \dot{B}_{2,q}^1$, $2 \leq q < \infty$. There exists a unique solution $u(t,x)$ to Equation (1.1.1) for all time with $u \in L_t^\infty \dot{B}_{2,q}^1(\mathbb{R}^+ \times \mathbb{R}^3) \cap X(\mathbb{R}^+)$. If $q < \infty$, then $u$ also scatters.*

The strategy to prove this is to use the road map outlined by Kenig and Merle [28] (see Section 1.1). We will in fact follow the procedure given by Visan in [10]. These Oberwolfach notes have been very useful to me throughout my PhD. For this technique, *we must assume that solutions to Equation (1.1.1) have their $\dot{B}_{2,\infty}^1$ norms bounded for all time.*

**Assumption 6.0.11.** *Solutions to Equation (1.1.1) with data $u_0 \in \dot{B}_{2,q}^1$ satisfy $||u||_{L_t^\infty \dot{B}_{2,q}^1 (\mathbb{R} \times \mathbb{R}^3)} < \infty$.*

Similar assumptions have been made when studying the Nonlinear Schrödinger Equation in a space where the critical Sobolev norm is not preserved (for example, $0 < s < 1$, [39]). See Section 1.1 for further discussion.

We will now outline the procedure as it relates to proving Conjecture 6.0.10. We require a number of Theorems, Propositions and Lemmas and if they require stating, we will list them as "statements" as they are unproven, however some of the results in this thesis may only require minor modifications.

We will require an appropriate local well-posedness theory. $X^q$ will not suffice, since it does not allow a profile decomposition theorem (see Section 5). We will call our required space $X$. We require the following for the space X:

1. A local well-posedness theorem analogous to Theorem 4.2.1.

2. A stability result analogous to Theorem 4.2.6.

3. A profile decomposition as outlined in Section 5.

4. A scattering condition: If $||u||_{X(\mathbb{R} \times \mathbb{R}^3)} < \infty$, then $u$ scatters (see Section 4.4).

Using the notation of [10], we define

$$L(E) := \sup\{||u||_{X(I)} : ||u||_{\dot{B}^1_{2,\infty}} \leq E\} \tag{6.0.1}$$

The set $\{E : L(E) < \infty\}$ is non-empty because of the small data theory and is open. Therefore there is some critical value $0 < E_c \leq \infty$ such that $L(E) < \infty$ for $E < E_c$ and $L(E) = \infty$ for $E \geq E_c$. We assume $E_c < \infty$, and prove that $L(E_c) < \infty$ for a contradiction.

To get this contradiction, we require the following Palais-Smale condition.

**Statement 6.0.12.** *Let $u_n : I_n \times \mathbb{R}^3 \to \mathbb{C}$ be a sequence of solutions to Equation (1.1.1) with $||(u_n)_0||_{\dot{B}^1_{2,\infty}} \to E_c$, for which there is a sequence of times $t_n \in I_n$ so that*

$$\lim_{n\to\infty} ||u_n||_{X([t_n,\infty))} = \lim_{n\to\infty} ||u_n||_{X((-\infty,t_n])} = \infty \tag{6.0.2}$$

*Then the sequence $u_n(t_n)$ has a subsequence that converges in $\dot{B}^1_{2,\infty}$ modulo scaling and spatial translations.*

*Proof.* We give a brief outline of the proof to highlight what lemmas need to be proved.

We apply the appropriate profile decomposition to the sequence $\{(u_n)_0$. We refer the reader to Section 5). In particular, we use the notation from Theorem 5.0.2 and the reader should refer to this theorem for any variables seen below.

It suffices to show that there is one profile. We assume there is more than one and reach a contradiction. It can be shown that each of the profiles is smaller than the critical size $E_c$ in the following sense: for some $\delta$,

$$\sup_j \limsup_{n\to\infty} ||e^{it_n^j \triangle}\phi^j||_{\dot{B}^1_{2,q}} \leq E_c - 2\delta, \tag{6.0.3}$$

where $\{\phi^j\}$ are the linear profiles associated to the sequence. Associated to our linear profiles $\phi^j$, we define nonlinear profiles according to the following rule. If $t_n^j \equiv 0$, we define $v^j : I^j \times \mathbb{R}^3 \to \mathbb{C}$ to be the maximal-lifespan solution to Equation (1.1.1) with initial data $v^j(0) = \phi^j$. If $t_n^j \to \pm\infty$, we define $v_n^j$ is also a solution to Equation (1.1.1) which scatters to $e^{it\triangle}\phi^j$ as $t \to \pm\infty$. These nonlinear profiles decouple in the following sense. We define $v_n^j := (\lambda_n^j)^{-\frac{1}{2}}[e^{it_n^j\triangle}\phi^j](\frac{x-x_n^j}{\lambda_n^j})v^j$.

**Statement 6.0.13.** *For $j \neq k$, $\lim_{n\to\infty} ||v_n^j v_n^k||_X = 0$.*

What this is telling us is essentially that since the linear profiles are orthogonal, the nonlinear profiles do not interact very much. This will lead to a contradiction, since the size of each profile is below the critical threshold (from Equation 6.0.3) and so the $X$-norm of these nonlinear profiles are bounded. The mild interaction from these nonlinear profiles cannot make the sum of these profiles too large in the $X$-norm. Indeed, from Equation (6.0.3) it can be seen that $||v^j||_X \lesssim_{E_c,\delta} 1$. We define

$$u_n^J := \sum_{j=1}^J v_n^j + e^{it\triangle}w_n^J. \tag{6.0.4}$$

This serves as an approximation of $u_n$. Statement 6.0.13 implies that

$$||u_n^J||_X \leq C_1. \tag{6.0.5}$$

An appropriate stability lemma gives us

$$||u_n - u_n^J||_X \leq C_2, \tag{6.0.6}$$

uniformly in n. This contradicts Equation (6.0.2) and so there must indeed be only one profile. □

The above is just a summary of the proof of course. There are a number of lemmas to be proved in order to close this argument, however this has now become standard in the theory and the necessary ingredients are listed above (1., 2., 3., 4.).

Theorem 6.0.12 implies the existence of a solution that is localized in both space and frequency. We make the following definition.

**Definition 6.0.14.** *We call a solution to Equation (1.1.1) almost periodic modulo symmetries if there exist functions $N : I \to \mathbb{R}^+$, $x : I \to \mathbb{R}^3$, and $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\int_{|x-x(t)| \geq C(\eta)/N(t)} |\nabla u(t,x)|^2 dx + \int_{|\xi| \geq C(\eta)N(t)} |\xi \hat{u}(t,\xi)|^2 d\xi \leq \eta \tag{6.0.7}$$

*for all $t \in I$ and $\eta > 0$.*

**Statement 6.0.15.** *If Conjecture 6.0.10 fails to be true, there exists $||u_0||_{\dot{B}^1_{2,\infty}} = E_c$ which evolves to a solution of Equation (1.1.1) such that*

$$||u||_{X([0,\infty))} = \infty, \tag{6.0.8}$$

*and is almost periodic modulo symmetries.*

*Proof.* If Conjecture 6.0.10 fails, then $0 < E_c < \infty$ and there is some sequence of initial data $\{(u_n)_0\}$ evolving to solutions $u_n : I_n \times \mathbb{R}^3 \to \mathbb{C}$ such that $||(u_n)_0||_{\dot{B}^1_{2,q}} \to E_c$ and $||u_n||_{X(I_n)} \to \infty$. By Statement 6.0.12, there is some sequence of times $\{t_n\}$ such that $u_n(t_n) \to \phi \in \dot{B}^1_{2,q}$ modulo scaling and spatial translations for some $\phi \in \dot{B}^1_{2,\infty}$. By a generalization of the Arzela-Ascoli theorem, it can be shown that $\phi$ evolves to an almost-periodic modulo symmetries solution $u$ that satisfies Equation (6.0.8). □

Now the game is to rule out such an enemy. This is where the general framework ends and we will require a more nuanced approach. In particular, in this case some variant of the Frequency-localized Interaction Morawetz Inequality will need to be used.

We now discuss an approach to prove Conjecture 1.3.3 directly. In particular, we examine initial data of the form $u_0 = w_0 + v_0$, where $w_0 \in \dot{B}^1_{2,q}$, $v_0 \in \dot{H}^1$, with no relationship between $||v_0||_{\dot{H}^1}$ and $||w_0||_{\dot{B}^1_{2,q}}$. We may take $||w_0||_{\dot{B}^1_{2,q}}$ to be small, but this will require $||v_0||_{\dot{H}^1}$ to be very large.

We recall that if $\tilde{u}$ is a solution to $i\partial_t \tilde{u} + \triangle \tilde{u} = |w + \tilde{u}|^4 (w + \tilde{u}) - |w|^4 w$, then $u = \tilde{u} + w$ is a solution to $(NLS)$. We look to use the road map on $(\tilde{NLS})$ directly. Notice that the nonlinearity contains terms with four Besov solutions (four instances of $w$ or $\bar{w}$). For this reason, we require a multilinear estimate:

**Statement 6.0.16.** *For $2 \leq q \leq \infty$, let $f_i \in Z^\infty(I)$ for $i = 1,2,3,4$ and $f_5 \in Z^2$, for the time interval $I = [t_0, t_1)$. Then*

$$|| \int_{t_0}^{t} e^{i(t-s)\triangle} (f_1 f_2 f_3 f_4 f_5)(s) ds ||_{Z^2(I)} \lesssim ||f_1||_{Z^\infty(I)} ||f_2||_{Z^\infty(I)} ||f_3||_{Z^\infty(I)} ||f_4||_{Z^\infty(I)} ||f_5||_{Z^2(I)}, \qquad (6.0.9)$$

where $Z^q$ are spaces that are adapted to data from $\dot{B}^1_{2,q}$ in a similar way that $X^q$ are, satisfying analogous results to those in Section 2 with the additional assumption that $Z^q$ satisfy the requirements 1., 2., 3., 4., (seen above) for the equation $(\tilde{NLS})$.

**Remark 6.0.17.** *Lemma 3.2.2 gives us a multilinear estimate in $X^q$ with three components in $X^\infty$ and two in $X^q$. It seems difficult to improve this to four components in $X^\infty$ and one in $X^q$, however this difficulty may just be technical in nature.*

This statement allows us to bound the $\dot{H}$ norm of any solution to $(\tilde{NLS})$. Indeed, let

$$|w + \tilde{u}|^4 (w + \tilde{u}) - |w|^4 w = \sum_i \tilde{u}_i^1 \cdots \tilde{u}_i^5,$$

where $\tilde{u}_i^j \in \{v, \bar{v}, w, \bar{w}\}$ for all $i, j$. Then by the Duhamel representation and Statement 6.0.16,

$$
\begin{aligned}
||\tilde{u}||_{L_t^\infty \dot{H}_x^1} \quad &\sim \quad ||\tilde{u}||_{L_t^\infty \dot{B}^1_{2,2}} \leq ||\tilde{u}||_{Z^2} \\
&\leq \quad ||u_0||_{\dot{H}^1} + \sum_i ||\tilde{u}_i^1||_{Z^\infty} ||\tilde{u}_i^2||_{Z^\infty} ||\tilde{u}_i^3||_{Z^\infty} ||\tilde{u}_i^4||_{Z^\infty} ||\tilde{u}_i^5||_{Z^2}.
\end{aligned}
$$

We place each instance of $w$ or $\bar{w}$ in the norm $|| \cdot ||_{Z^\infty}$. Note that by the embedding $Z^\infty \subset Z^2$, $||v||_{Z^\infty} \leq ||v||_{Z^2} < \infty$, so that $v$ and $\bar{v}$ may be placed in the $Z^\infty$ norm or the $Z^2$ norm.

This is not sufficient to use the road map technique, as this does not imply that $||\tilde{u}||_{L_t^\infty \dot{H}_x^1(I \times \mathbb{R}^3)}$ is bounded independently of the time interval $I$, however it does give us a toe-hold. We require the following statement.

**Statement 6.0.18.** *Solutions to $(\tilde{NLS})$ satisfy $||\tilde{u}||_{L_t^\infty \dot{H}_x^1(I \times \mathbb{R}^3)} < C$ independently of $I$.*

With this statement and an appropriate space $Z^q$, we may be able to prove Conjecture 1.3.3.

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